Generalised symmetries, anomalous magnetohydrodynamics and holography

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Abstract

In this thesis we study the finite temperature physics of a system which is afflicted by the Adler-Bell-Jackiw anomaly or, the chiral anomaly. The universality class of such systems are commonly referred to as the *chiral plasma*. Its weakly coupled physics is described by a theory of massless Dirac fermions coupled to dynamical electromagnetism and hence the universal symmetry structure of the *chiral plasma* is that of dynamical $U(1)$ Abelian gauge theory with charged matter. In this theory, the non-conservation of the axial current due to the chiral anomaly is given by a dynamical operator $f_{\mu\nu}\tilde{f}^{\mu\nu}$ constructed from the field-strength tensor. We attempt to describe this physics in a universal manner by casting this operator in terms of the 2-form current for the 1-form symmetry associated with magnetic flux conservation. The precise symmetry structure in encoded by the anomaly equation which can be formulated as the intertwining of these two currents. The sense in which this is universal is that it is preserved along RG flows. Utilising this universal structure we first perform a holographic investigation of this system and then construct a hydrodynamic effective action for it. This effective action can be understood as "an action" for *chiral magnetohydrodynamics*, which is devoted to understanding the long-distance, late-time behavior of such a system suffering from an ABJ anomaly.

To perform the holographic study, we first construct a dual bulk theory with the aformentioned symmetry breaking pattern and study some aspects of finite temperature anomalous magnetohydrodynamics. We explicitly calculate the charge susceptibility and the axial charge relaxation rate as a function of temperature and magnetic field and compare to recent lattice results. At small magnetic fields we find agreement with elementary hydrodynamics weakly coupled to an electrodynamic sector, but we find deviations at larger fields.

Next we consider chiral magnetohydrodynamics. Using the universal symmetry structure encoded in the anomaly we write down “effective actions” capturing the equilibrium physics and the physics of dissipation. We present Euclidean generating functional and dissipative action approaches to the dynamics and reproduce some aspects of known chiral MHD phenomenology from an effective theory viewpoint, including the chiral separation and magnetic effects. We also discuss the construction of the *non-invertible* axial symmetry defect operators in our formalism in real time.

Finally, to study the axial charge relaxation rate in the limit of vanishing magnetic field, we undertake a study to see if hydrodynamic fluctuations affect this rate. We compute the finite-frequency real-time topological susceptibility arising from magnetohydrodynamic fluctuations. We find that it vanishes at zero frequency, indicating that the axial charge dissipation rate vanishes at zero background magnetic field. This is probably suggestive of the fact that the symmetry structure encoded in the anomaly is protected by the non-invertible defect operators as 1-loop effects do not spoil it.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification, and it is the sole work of the author unless referenced to the contrary in the text.

Publications

The content of this thesis, specifically as presented in chapters 5 and 6, draws upon the following publications and pre-prints:

[1] A. Das, R. Gregory, and N. Iqbal, "Higher-form symmetries, anomalous magneto-hydrodynamics, and holography," SciPost Phys. 14 (5, 2022) 163, arXiv:2205.03619 [hep-th]

[2] A. Das, N. Iqbal, and N. Poovuttikul, "Towards an effective action for chiral magnetohydrodynamics," arXiv:2212.09787 [hep-th]

Additionally, the following forthcoming work has also contributed to chapter 7 of this thesis:

[3] A. Das, N. Iqbal, and N. Poovuttikul, "Hydrodynamic fluctuations and topological susceptibility in chiral magnetohydrodynamics," *to appear*

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"If I've seen further, it is by standing on the shoulders of giants."

– Sir Issac Newton

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"The most incomprehensible thing about the world is that it is comprehensible."

– Albert Einstein

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To my beloved Kushina

CHAPTER 1

Introduction

Quantum Field Theory (QFT) has been a vibrant field of research for several decades, standing as one of the crowning achievements of theoretical physics in the past century. Its applications are vast, spanning from high-energy physics, including the standard model, to low-energy physics, which encompasses our understanding of metals and semiconductors. It is a powerful mathematical framework that has revolutionized our understanding of the fundamental forces of nature, from the subatomic scale to the cosmological scale. It describes the interactions of quantum fields and particles in terms of exchange of energy, momentum, and other quantum numbers. One of the most significant achievements of QFT has been the development of the Standard Model of particle physics, which successfully describes the electromagnetic, weak, and strong forces and their associated particles, as well as the Higgs mechanism.

A major challenge in QFT is to understand the behavior of systems with strong coupling, where the interactions between the particles are so intense that standard perturbation techniques are not applicable. This is particularly important in the study of systems such as quark-gluon plasmas in nuclear physics, strongly correlated electron systems in condensed matter systems, and black holes in gravitational settings. One powerful tool in this context is the use of symmetries, which provide a means to simplify the problem by reducing the number of degrees of freedom or by mapping the problem to another, more tractable system. There are many pathways in which one can employ symmetries to study strongly interacting QFTs ranging from holography [3–5], conformal bootstrap [6,7], integrability [8] and techniques of scattering amplitude [9–11] to the use age-old techniques like hydrodynamics and effective field theories [12]. Here we will focus on the applicability of hydrodynamics and holography to understand strongly interacting systems using the knowledge of the global symmetries present in them. So for this let us discuss symmetries a bit.

1.1 EFT and Hydrodynamics

One of the most powerful applications of symmetries in QFT is the construction of effective field theories (EFTs), which provide a systematic way to describe low-energy phenomena in terms of a few relevant degrees of freedom. EFTs incorporate the symmetries of the underlying theory and allow for a consistent and predictive description of physical processes, even in the presence of strong coupling. This is owing to the universal nature of global symmetries since they are preserved along *renormalisation group* flows. Hydrodynamics is one such way to construct effective theories [12–16].

Hydrodynamics is the branch of physics that deals with the study of fluids, including their motion, behavior, and interactions with solid boundaries. It is governed by a set of equations known as the Navier-Stokes equations (dynamical equations of motion), which describe the conservation of mass, momentum, and energy in a fluid.

In conventional hydrodynamics, the fluid is treated as a continuum, and the macroscopic properties of the fluid, such as density, pressure, and velocity, are described in terms of spatially-averaged quantities. However, at the microscopic level, the fluid is composed of individual particles, which can be atoms, molecules, or even subatomic particles, depending on the system under consideration.

In the context of QFT, hydrodynamics can be applied to systems of strongly coupled quantum fields, where the interactions between the fields are so strong that they cannot be treated as independent entities. In the hydrodynamic regime, the system is described in terms of collective excitations and transport coefficients, such as viscosity, rather than individual particles. This can provide new insights into the behavior of the system, including the dynamics of energy and momentum transport and the response to external perturbations. Moreover, the construction of effective hydrodynamic actions allows for a systematic derivation of the equations of motion, incorporating both quantum and classical effects.

1.2 Holography

Holography in QFT is inspired by the AdS/CFT correspondence – a duality between a gravitational theory in a higher-dimensional Anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) on the boundary of that spacetime [5,17,18]. The idea of holography in QFT is that the information contained in a strongly coupled QFT can be encoded in a higher-dimensional gravitational theory, which is often easier to work with. This duality allows us to compute physical quantities, such as viscosity bounds, using tools from gravitational physics, which can be more tractable than the original QFT [19].

1.3 Symmetries

Symmetries play a central role in our understanding of any physical system, whether classical or quantum. They provide a powerful tool for simplifying and solving complex problems in theoretical physics. For instance, they help us to derive non-trivial constraints on strongly interacting QFTs which would otherwise be difficult to tackle using perturbative methods. They also help to explore and classify different phases of a theory, for instance, the Landau-Ginzburg theory makes use of symmetries to classify various phases of matter, for example, a *solid* is an object which breaks translational symmetry [20].

A symmetry is a transformation that leaves the physical properties of a system unchanged. The symmetry parameter which is associated to such a transformation plays a crucial role in the classification of symmetries. For instance, if this parameter is a global parameter, that is independent of spacetime coordinates then the corresponding symmetry is a *global symmetry*[1] else it is known as a *gauge symmetry*. The latter case should not be confused with *local symmetries* which act differently on different local degrees of freedom[2]. Gauge symmetries are not symmetries but

[1]There is a slight catch here regarding the word "global". Traditionally, "global" is used to mean that the symmetry transformation parameter is a constant in the spacetime as we define it here. For the more generalised notions symmetries [21], sometimes the symmetry parameters can have nontrivial dependence in spacetime. However, they will still be true symmetries that act nontrivially on the Hilbert space, rather than redundancies in the description. We will use the word "global" to mean both these cases – that is when the symmetry parameter is independent of spacetime and in the case where it depends upon spacetime but still acts non-trivially on the physical configurations.

[2]A simple example of a local symmetry is the following. Let us consider a classical ideal gas of N particles of mass m with positions $\vec{x}_i(t)$ and momenta $\vec{p}_i(t)$. The index $i = 1., \ldots, N$ labels the particles. Due to our idealisation approximation, the Hamiltonian contains only the

rather redundancies in the system. One way to think of these redundancies is to imagine relabelling your metre rods and clocks or, a coordinate transformation.

Similarly, if the symmetry parameter is continuous then we will have a *continuous symmetry* – which by Noether's theorem will give us a conserved current, else we will have what is called a *discrete symmetry*.

In a conventional setting whenever, we have a symmetry in the system, we have the concept of a *symmetry group* (or a *Lie group*) which acts on the fields living on the phase space (for classical systems) or on the Hilbert space (for quantum systems). These objects which transform under the group action are said to be *charged* under the symmetry transformation and hence are called *charged operators*. Since we know that the dynamical/evolution equations of a theory are second-order differential equations, the solutions to these equations will transform in irreducible representations of the corresponding symmetry group. Hence, considerations of symmetries places non-trivial constraints on the solution space leading to what are called *selection rules*. These selection rules are reflected in the quantisation of various charges or as non-trivial constraints on the correlation functions of the theory.

1.3.1 Symmetry breaking and Anomalies

Next let us look at the case how these symmetries can break in our system. We will discuss three types of breaking: *explicit breaking*, *spontaneous breaking* and *breaking by anomalies*. The first case happens when at the level of the action we have a term which does not obey the symmetry. When this happens the symmetry is entirely lost from the system and nothing much can be learnt. The second case happens when the action is invariant under the symmetry but atleast one stable state of the system, say the vacuum becomes invariant under the symmetry. An important consequence of this is the *Goldstone's theorem* which roughly states that for each spontaneously broken symmetry generator there exists a corresponding massless mode – called a *Goldstone mode* – in the spectrum which mediates long range interactions in the theory since it is massless [22, 23]. So, if we want to write down an effective field theory for such a system then in the long-wavelength limit the dynamical variables

kinetic energy of individual particles – as particles are non-interacting: $H = -\frac{1}{2m}\sum_{i=1}^{N}\vec{P}_i^2$. So, we see that the Hamiltonian is invariant under individual translations of each particle of the form: $\vec{x}_i(t) \to \vec{x}_i(t) + \vec{a}_i$. The local displacements, denoted by $\vec{a}_i$, may be different for each particle. For instance, we could consider the case in which $\vec{a}_i = 0$ with $i = 1, \ldots, N-1$ and $\vec{a}_N \neq 0$. So, the symmetry is a local one, rather than global [22].

are these *Goldstone modes*.

Now moving on to the third case, to be honest, this is actually not a symmetry breaking concept. To be precise, anomalies arise when the classical symmetry of a system does not survive the quantisation process. In other words, even though the action possess a symmetry the path-integral is not invariant under it which means the path-integral measure fails to remain invariant under the symmetry. Anomalies are non-perturbative phenomena in the theory and can be shown to be associated with topological properties of the differential operator in the theory. So, just like global symmetries their structure are also preserved along RG flows. Thus, they also help in deriving additional non-trivial constraints on correlation functions of the theory.

1.3.2 Generalised symmetries

In recent years, there has been a revolution in our understanding of the notion of symmetries and their generalisations following the seminal work [21]. These generalised notions of symmetries help in recasting various properties of gauge theories and also lead to the discovery of many novel phases of the theories. These symmetry structures are *generalised* in a way that they do not possess a *group* structure rather they have a *higher-group* (in some cases of higher-form symmetries) or *categorical* structure (in the case of non-invertible symmetries), where some of the *group* axioms are relaxed [24–29]. Following the previous discussion of the parameter associated to symmetry transformations, in the case of higher-form symmetries, this parameter will no longer be a 0-form but will be a p-form having additional spacetime indices. This is where the terminology – higher-form – comes from. In this reformulation, conventional symmetries are termed as 0-form symmetries.

There have been many applications of such generalised notions of symmetries. [30–35] makes use of the 1-form symmetry in QED to construct an effective hydrodynamic description of it stemming only from considerations of symmetries. In [36], the authors use 1-form symmetries to extend the conventional Landau-Ginzburg theory involving 0-form symmetries and develop a Landau-Ginzburg theory involving 1-form symmetries where the dynamical variable is an extended field called a *string field*.

The above discussion begs the following question: Can we learn something more about strongly interacting QFTs by applying the techniques of hydrodynamics and holography to these generalised symmetric principles? The objective of this thesis, in part, is to explore this question and provide evidence that supports an affirmative response to some degree. This thesis explores these generalised notions of symmetries to investigate quantum systems which suffer from anomalies involving dynamical fields.[3].

1.4 Outline of the thesis

This thesis is structured into several chapters, each addressing a distinct topic.

Chapter 2 begins with a review of ordinary symmetries in both classical and quantum field theories. We then explore the generalization of these symmetries to the 1-form case, followed by the more general p-form case with $p > 1$. The focus then shifts to an exploration of 1-form symmetries in Maxwell theory in $D = 4$. We also present Goldstone's theorem in the case of higher-form symmetries. The chapter concludes with a brief discussion on 't Hooft and ABJ anomalies.

Chapter 3 provides an overview of the framework for ordinary ideal hydrodynamics that involves conventional or 0-form symmetries. The discussion then extends to hydrodynamics with 1-form symmetries. We conclude the chapter with an exploration of anomalies and their implications for hydrodynamics.

Chapter 4 introduces the pertinent concepts related to holography that underpin this thesis. We first briefly review the holographic principle and then present the holographic dictionary. The chapter concludes with a very brief qualitative review of the membrane paradigm within the context of AdS/CFT. This chapter marks the conclusion of the background material required for subsequent computations in this thesis.

The following chapters present original contributions to the existing body of research.

In Chapter 5, we carry out a holographic examination of a chiral plasma at finite temperature. This system's weakly coupled dynamics can be described by a theory of massless Dirac fermion coupled to dynamical electromagnetism. We leverage the 1-form symmetry of QED to formulate the bulk holographic action that belongs to the same universality class of symmetries. From this bulk action, we calculate an

[3]Anomalies can be of two kinds: 't Hooft-kind and ABJ-kind. In the 't Hooft-kind, the non-conservation of the current is related to fixed non-dynamical sources which in principle can be turned off to still get a conserved current. On the contrary, in the ABJ-kind, the non-conservation of the current is related to dynamical operators which cannot be turned off. We discuss more on this later.

interesting observable of the boundary theory that captures the chiral decay rate by employing standard methods for computing quasi-normal modes in holography. According to the holographic dictionary, the lowest-lying quasi-normal mode in the bulk corresponds to the decay rate in the boundary. For smaller magnetic fields, this decay rate is proportional to the square of the magnetic field, which aligns with elementary hydrodynamic results. However, as we crank up the magnetic field, this quadratic relationship no longer holds.

Chapter 6 is devoted to the development of hydrodynamic effective field theories – capturing equilibrium and dissipation – which fall in the same universality class as that of dynamical QED at finite temperature. We start by identifying the global symmetries of this system, specifically the 1-form magnetic symmetry which is the global symmetry of QED. Additionally, there is a universal symmetry structure stemming from the ABJ anomaly, which links the non-conservation of the chiral current to the double-trace operator, $\star J \wedge \star J$, where J denotes the conserved 2-form current of the 1-form magnetic symmetry. Utilizing the constructed effective actions, we successfully reproduce well-known chiral MHD phenomenology, for example, the chiral separation effect and the chiral magnetic effect. The chapter concludes by demonstrating successful insertions of non-invertible defects [37, 38], into the dissipative action in real-time and into the equilibrium action in Euclidean signatures.

Chapter 7 investigates whether the chiral decay rate, known to be a function of the magnetic field as explored in Chapter 5, approaches zero in the limit of vanishing magnetic field. To estimate this rate, we employ the Kubo formula for the current density-current density correlation, where the current in question is the 2-form current associated with magnetic flux conservation, and examine if hydrodynamic fluctuations contribute to this decay rate. At the 1-loop level, we observe that this decay rate indeed remains zero as $\omega \to 0$. This finding implies that the non-invertible symmetry inherent in the chiral anomaly preserves the universal intertwining between the 1-form non-conserved chiral current and the 2-form conserved current.

Lastly, this thesis culminates with a succinct concluding chapter.

CHAPTER 2

Generalised global symmetries

2.1 Analysis of conventional symmetries

In this section we shall examine conventional symmetries in the context of quantum field theory. The idea is to express known results in this domain in terms of topology and differential forms. This way we shall provide a geometrical meaning to them. This will help us to generalise these concepts to the case of higher-form symmetries.

Here we will mostly follow the exposition given in [28] differing a bit at times as and when required. Contemporary reviews on this subject include [24–27, 29, 39].

2.1.1 Aspects of symmetries in classical field theory

One of the cornerstones of theoretical physics is the Noether's theorem, which posits a connection between continuous symmetries and conservation laws. A succinct derivation of this correlation can be illustrated as follows. Consider an action: $S[\Phi] = \int d^D x\, \mathcal{L}(\Phi)$, which subsumes a general ensemble of fields represented as Φs. Assume that under an infinitesimal transformation of these fields,

$$\Phi \to \Phi + \varepsilon_a \delta_a \Phi, \tag{2.1}$$

where ε_a is a collection of constant, global parameters[1], the action remains invariant. This suggests a symmetry within the context of the classical theory.

To locate the corresponding conserved current, we transform the global parameters ε_a to local ones, indicated by $\varepsilon_a \to \varepsilon_a(x)$. Under these circumstances, the transformation

$$\Phi \to \Phi + \varepsilon_a(x)\delta_a\Phi \tag{2.2}$$

is no longer a symmetry of the action. Note that, the variance in the action must incorporate the derivative of the parameters, noted as $\partial_\mu\varepsilon_a(x)$, which reinstates the invariance under the condition of a global transformation for constant parameters. From this, we can formulate the expression

$$\delta_\varepsilon S[\Phi] = \int d^Dx\, j^\mu_a \partial_\mu \varepsilon_a(x) \overset{\text{IBP}}{=} -\int d^Dx\, (\partial_\mu j^\mu_a)\varepsilon_a(x), \tag{2.3}$$

where the coefficients j^μ_a are arbitrary and in going to the second equality we have done integration by parts and have thrown away boundary terms. Now there is no reason to expect Eq. (2.3) to vanish.

The aforementioned local transformation in Eq. (2.2), can be interpreted simply as a sequence of arbitrary variations of the fields. In this instance, equation (2.3) vanishes when Φs are taken to be on-shell: $\delta_\Phi S[\Phi] = 0$, which is to say, in particular for the transformation in Eq. (2.2) we have,

$$\delta_\varepsilon S[\Phi] \overset{\text{EoMs}}{=} -\int d^Dx\, (\partial_\mu j^\mu_a)\varepsilon_a(x) = 0, \tag{2.4}$$

where EoMs stands for equations of motion.

Since $\varepsilon_a(x)$ are arbitrary, we infer that the coefficients j^μ_a in Eq. (2.3) are indeed the associated conserved currents[2],

$$\partial_\mu j^\mu_a = 0 \quad \text{implying,} \quad \frac{d}{dt}Q_a = 0, \tag{2.5}$$

where we have denoted the Noether charges as

$$Q_a(\Sigma) \equiv \int_\Sigma d^{D-1}x\, j^\mu \hat{n}_\mu = \int_{\Sigma_t} d^{D-1}\vec{x}\; j^0_a(\vec{x}, t), \tag{2.6}$$

[1]The index "a" denotes a collection of indices that may be either spacetime or internal.

[2]Here, we perform a slight abuse of notation. To be very precise, j^μ_as are actually current densities and not currents themselves. In the literature, one often refers to them as *Noether currents* and so, by an abuse of notation, we simply refer them as "currents".

where Σ is a $(D-1)$ sub-manifold and $\hat{n}^\mu$ is the unit normal vector to Σ_{D-1}. We get the second equality above when $\Sigma_{D-1} = \Sigma_t$ is a time slice. In this case, the charges are constant in time as given in Eq. (2.5). Now let us ask ourselves what does the charge $Q(\Sigma)$ counts. First of all note that, Σ can be any $(D-1)$ sub-manifold. However, when $\Sigma = \Sigma_t$ is a time slice then $Q(\Sigma_t)$ has a nice interpretation in the sense that it is the conserved particle number. So basically, $Q(\Sigma_t)$ counts the number of particles in the system which is conserved. Since particle worldlines cannot end in time, we can "catch" all the particles by integrating over a time slice, see Fig. 2.1.

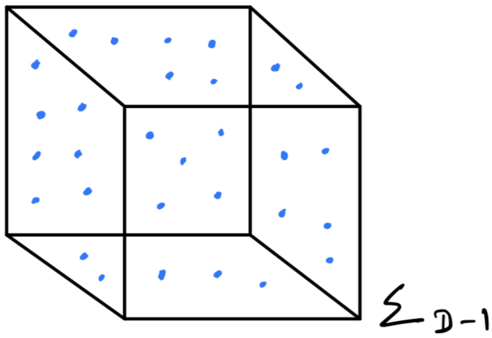

Figure 2.1: Here we have a $(D-1)$ spatial slice – a time slice – which catches all the particles in the system and upon integration, gives us the conserved particle number.

Charged local operators

Here we shall discuss the operators charged under ordinary symmetries, or, 0-form symmetries. These are called 0-form symmetries since the operators charged under ordinary symmetries are local operators supported on 0-dimensional spacetime points. We shall review these in more detail in a section below.

Consider the Hamiltonian picture where we can quantize on Σ_{D-1} and the corresponding operator's action on the associated Hilbert space $\mathcal{H}_{D-1}$ is performed by unitaries of the form,

$$U_g(\Sigma_{D-1}) = e^{i\lambda Q(\Sigma_{D-1})} = \exp\left(i\lambda \int_{\Sigma_{D-1}} d^{D-1}x\, j^\mu \hat{n}_\mu\right), \tag{2.7}$$

where $\hat{n}^\mu$ is the unit normal vector to Σ_{D-1} and $g = e^{i\lambda} \in G^{(0)}$ with $G^{(0)}$ being the 0-form symmetry group.

These operators obey the following composition rule owing to the group law inherited from the 0-form symmetry group $G^{(0)}$, [3]

$$U_{g_1}(\Sigma_{D-1})U_{g_2}(\Sigma_{D-1}) = U_{g_1 g_2}(\Sigma_{D-1}), \tag{2.8}$$

Let us denote the vacuum by $|0\rangle$ and say local operators $\mathcal{O}(x)$ act on the vacuum to create particle-states then U_g act on these local operators as,

$$U_g(\Sigma_{D-1})\mathcal{O}(x) = \mathcal{O}'(x)U_g(\Sigma_{D-1}), \tag{2.9}$$

which is directly analogous to the conjugation action that we have in QM:

$$U(\Sigma_t)\mathcal{O}(t,\vec{x})U^{-1}(\Sigma_t) = \mathcal{O}'(t,\vec{x}). \tag{2.10}$$

Furthermore, in QM, we know the unitary operator $U(t) \equiv U(\Sigma_t)$ acting on the Hilbert space at time t commutes with the Hamiltonian generating time evolution, that is: $U(t) = U(t') \quad \forall\, t, t'$. Thus for general Σ_{D-1}, we should have a similar result. We shall now show that [4],

conservation of $j^\mu \Leftrightarrow U_g(\Sigma_{D-1})$ depends on Σ_{D-1} upto homotopy

Consider,

$$\begin{aligned} U_g(\Sigma_{D-1})U_{g^{-1}}(\Sigma'_{D-1}) &= \exp\left[i\lambda\left(\int_\Sigma d^{D-1}x\, j^\mu \hat{n}_\mu - \int_{\Sigma'} d^{D-1}x\, j^\mu \hat{n}_\mu\right)\right] \\ &= \exp\left[i\lambda \int_{\bar{\Sigma}} d^D x\, \partial_\mu j^\mu\right] = \mathbb{1} \qquad \text{since} \quad \partial_\mu j^\mu = 0, \end{aligned} \tag{2.11}$$

where $\partial\bar{\Sigma} = \Sigma \cup \Sigma'_{\text{opp.}}$ and $\Sigma'_{\text{opp.}} = \Sigma'$ but with opposite orientation of the normal vector field (see Fig. 2.2).

[3] A non-rigorous, quick, way to see this is to note,

$$\begin{aligned} U_{g_1}U_{g_2} &= e^{i\lambda_1 Q}e^{i\lambda_1 Q} \simeq e^{i(\lambda_1+\lambda_2)Q}, \\ \text{now, } g_3 = g_1 g_2 &= e^{i\lambda_1}e^{i\lambda_2} = e^{i(\lambda_1+\lambda_2)}, \\ \text{implying, } U_{g_1}U_{g_2} &= U_{g_3}. \end{aligned}$$

[4] Σ_{D-1} is homotopic to Σ'_{D-1} if one can be smoothly deformed to the other.

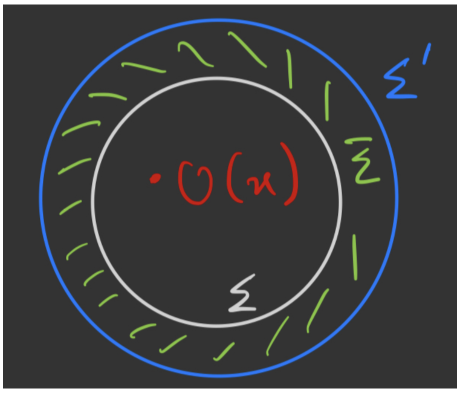

Figure 2.2: Here we have Σ_{D-1} homotopic to Σ'_{D-1}.

Thus, we get from above,

$$U_g(\Sigma_{D-1})U_{g^{-1}}(\Sigma'_{D-1}) = \mathbb{1}, \qquad U_g(\Sigma_{D-1}) \cong U_g(\Sigma'_{D-1}), \tag{2.12}$$

which means that $U_g(\Sigma_{D-1})$ is a topological charged operator. We shall revisit this topological property of the conserved charges in terms of *linking number* in a section below.

2.1.2 Aspects of symmetries in quantum field theory

Now let us move on to quantum analogues of symmetries. Continuous and discrete symmetries can be accommodated through unitary operators, while anti-unitary operators are solely for discrete symmetries. This is the content of the famous Wigner's theorem.

In the context of continuous symmetries, the corresponding transformation's unitary operator can be systematically constructed from the Noether charge, as:

$$U(\Sigma) = e^{i\varepsilon_a Q_a(\Sigma)}. \tag{2.13}$$

where Σ is a spatial manifold or a time slice. Note that, since Q_as are independent of the time slice Σ_{D-1}, the unitary operators $U(\Sigma)$s constructed from them as in Eq. (2.13) are also independent of the time slice Σ. As discussed above, since $Q(\Sigma)$ count particles in the system which is conserved, the above charged operators $U(\Sigma)$ are local operators which create and destroy particles at their point of insertions.

Now we shall move on to the concept of Ward identities. These identities are quantum equivalents of the classical conservation laws stated in Eq. (2.5). They generate relationships among the correlation functions of a theory and are derivable from the path integral. Consider the following partition function:

$$Z = \int \mathcal{D}\Phi \, e^{iS[\Phi]}. \tag{2.14}$$

The correlation functions can be expressed as:

$$\langle \Pi \rangle \equiv \frac{1}{Z} \int \mathcal{D}\Phi \, \Pi \, e^{iS[\Phi]}, \tag{2.15}$$

where Π symbolises a general product of fields, represented as $\Pi \equiv \prod_{j=1}^{N} \Phi(x_j)$. Since the fields are simply integration variables, we have the liberty to rename them or adjust these integration variables. Initially, the fields are renamed from Φ to Φ', then a variable change in the path integral is implemented, in accordance with Eq. (2.2). This leads to:

$$\begin{aligned}
\langle \Pi \rangle &= \frac{1}{Z} \int \mathcal{D}\Phi \, \prod_{j=1}^{N} \Phi(x_j) e^{iS[\Phi]} \\
&= \frac{1}{Z} \int \mathcal{D}\Phi' \, \prod_{j=1}^{N} \Phi'(x_j) e^{iS[\Phi']} \\
&= \frac{1}{Z} \int \mathcal{J} \, \mathcal{D}\Phi \left(\Pi + \sum_{j=1}^{N} \Phi(x_1) \ldots \varepsilon_a(x_j) \delta_a \Phi(x_j) \ldots \Phi(x_N) \right) e^{i(S+\delta S)} \\
&= \frac{1}{Z} \int (1 + i\delta\mathcal{A}) \, \mathcal{D}\Phi \left(\Pi + \sum_{j=1}^{N} \Phi(x_1) \ldots \varepsilon_a(x_j) \delta_a \Phi(x_j) \ldots \Phi(x_N) \right) (1 + i\delta S) e^{iS} \\
&= \frac{1}{Z} \int \mathcal{D}\Phi \left(\Pi + \sum_{j} \Phi(x_1) \ldots \varepsilon_a(x_j) \delta_a \Phi(x_j) \ldots \Phi(x_N) + i\delta S \, \Pi + i\delta\mathcal{A} \, \Pi + \cdots \right) e^{iS},
\end{aligned} \tag{2.16}$$

where we have neglected higher order terms like $\mathcal{O}(\varepsilon_a^2)$ and above, thereby treating $\varepsilon_a(x_j)$s as small parameters. The ellipses in the last line contain terms higher order in variation of the likes: $\mathcal{O}(\delta S \, \delta_a \Phi), \mathcal{O}(\delta\mathcal{A} \, \delta_a \Phi), \mathcal{O}(\delta\mathcal{A} \, \delta S)$ and above. $\mathcal{J}$ is the Jacobian for the transformation of the measure and $\mathcal{A}$ captures anomalies, if any, in the system. We shall discuss anomalies later in this chapter.

Now using the variation of action in the form provided in Eq. (2.3) we get:

$$\int d^D x\, \varepsilon_a(x) \left[\sum_j \delta^{(D)}(x - x_j) \langle \Phi(x_1) \ldots \delta_a \Phi(x_j) \ldots \Phi(x_N) \rangle - i \langle \partial_\mu j^\mu_a(x)\, \Pi \rangle + i \langle A_a(x)\, \Pi \rangle \right] = 0. \tag{2.17}$$

In the above equation, we have parameterised the anomaly as $\delta \mathcal{A} \equiv \int d^D x\, \varepsilon_a A_a(x)$. For now we can take $\mathcal{A} = 0$ or, $A_a = 0$, implying that there are no anomalies. The above equation, due to arbitrariness of $\varepsilon_a(x)$s, then gives the Ward identities as[5]:

$$i \langle \partial_\mu j^\mu_a(x)\, \Pi \rangle = \sum_j \delta^{(D)}(x - x_j) \langle \Phi(x_1) \ldots \delta_a \Phi(x_j) \ldots \Phi(x_N) \rangle, \tag{2.18}$$

Since Π does not depend on x^μ we can take the derivative out to get:

$$i \partial_\mu \langle j^\mu_a(x)\, \Pi \rangle = \sum_j \delta^{(D)}(x - x_j) \langle \Phi(x_1) \ldots \delta_a \Phi(x_j) \ldots \Phi(x_N) \rangle, \tag{2.19}$$

For simplicity let us take Π as a single field $\Phi(y)$. This simplifies Eq. (2.19) to:

$$\partial_\mu \langle j^\mu_a(x)\, \Phi(y) \rangle = -i \delta^{(D)}(x - y) \langle \delta_a \Phi(y) \rangle. \tag{2.20}$$

2.1.3 Spontaneous symmetry breaking of conventional symmetries

Now let us discuss spontaneous symmetry breaking (SSB) of conventional symmetries and the emergence of Goldstone modes. The Goldstone theorem articulates that when a continuous global symmetry undergoes spontaneous breaking, massless excitations, known as Goldstone bosons, manifest in the spectrum. Here we shall differ from the textbook treatment of the subject and derive Goldstone's theorem in two slightly different ways. This way we can generalise these derivations later to the case of higher-forms symmetries.

Let us consider the Ward identity Eq. (2.20). By applying a Fourier transform

[5] Here we have not been careful about time-ordering in the path-integral. This is fine for the Euclidean case and for the Lorentzian case we can analytically continue from the Euclidean result.

to x, we get the following:

$$\int d^D x\, e^{ipx} \partial_\mu \langle j_a^\mu(x)\, \Phi(y)\rangle = -i \int d^D x e^{ipx} \delta^{(D)}(x-y) \langle \delta_a \Phi(y)\rangle$$
$$\text{implying,} \quad -i \int d^D x\, e^{ipx} p_\mu \langle j_a^\mu(x)\, \Phi(y)\rangle = -i e^{ipy} \langle \delta_a \Phi(y)\rangle \tag{2.21}$$

where we have used: $p_\mu = i\partial_\mu$. Now let use $j_a^\mu(p) = \int d^D x\, e^{ipx} j_a^\mu(x)$ to write the above as,

$$p_\mu \langle j_a^\mu(p)\, \Phi(y)\rangle = e^{ipy} \langle \delta_a \Phi(y)\rangle$$
$$\text{implying,} \quad p_\mu \langle j_a^\mu(p)\, e^{-ipy} \Phi(y)\rangle = \langle \delta_a \Phi(y)\rangle. \tag{2.22}$$

Let us integrate both sides over y in Eq. (2.22) to get:

$$p_\mu \langle j_a^\mu(p) \Phi(-p)\rangle = \int d^D y \langle \delta_a \Phi(y)\rangle = \langle \delta_a \Phi(p=0)\rangle. \tag{2.23}$$

The term $\langle \delta_a \Phi(p=0)\rangle$ on the right hand side is an order parameter that depicts the different phases of the theory. The symmetric (or unbroken) phase corresponds to $\langle \delta_a \Phi(p=0)\rangle = 0$. On the other hand, if $\langle \delta_a \Phi(p=0)\rangle \neq 0$, spontaneous symmetry breaking has occurred.

In the case of a SSB phase, the correlation function $\langle j_a^\mu(p)\Phi(-p)\rangle$ must exhibit a pole at zero momentum. To see this from above, let us first note that, $\langle j_a^\mu(p)\Phi(-p)\rangle \sim c_1(p)\, p^\mu$.[6] Now, in SSB: $\langle \delta_a \Phi(p=0)\rangle = d_1 \neq 0$, where d_1 is some constant independent of the momentum p^μ. To satisfy this we must have $c_1(p) \sim \frac{1}{p^2}$. In other words, we have at zero momentum:

$$\langle j_a^\mu(p)\Phi(-p)\rangle \sim \frac{p^\mu}{p^2}. \tag{2.24}$$

This indicates the presence of massless physical excitations within the spectrum. Such excitations are referred to as Goldstone bosons. This proves the Goldstone's theorem stated above. The above proof can be found in [28].

Now let us look at an alternative approach to the above proof (see [40] and [41]). To proceed, let us first write some of the above expressions in the language of differential forms (see appendix A for a review).

[6]This is obtained by noting the index structure on the current given in left hand side.

Noether charges in the language of differential forms

In the language of differential forms, the Noether charges, as given in Eq. (2.6), can be expressed as an integral over a closed[7] $(D-1)$-dimensional sub-manifold Σ, formulated as:

$$Q(\Sigma) \;=\; \int_\Sigma \star j = \frac{1}{(D-1)!}\int_\Sigma j_\mu\, \epsilon^\mu{}_{\mu_1\ldots\mu_{D-1}} dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{D-1}}. \tag{2.25}$$

Now for the Ward identity, as in Eq. (2.18), consider integrating it over a D-dimensional manifold Ω_Σ whose boundary is the $(D-1)$-dimensional sub-manifold Σ, that is, $\partial\Omega_\Sigma = \Sigma$. As a result, the left-hand side of Eq. (2.20) can be written as:

$$\begin{aligned}\int_{\Omega_\Sigma}\langle d\star j\,\Phi(y)\rangle &= \int_\Sigma \langle \star j\,\Phi(y)\rangle \\ &= \langle Q(\Sigma)\Phi(y)\rangle,\end{aligned} \tag{2.26}$$

where we used Stokes' theorem in the first line. Given Eq. (2.20), we then get:

$$\langle Q(\Sigma)\Phi(y)\rangle = -i\int_{\Omega_\Sigma} d^Dx\,\delta^{(D)}(x-y)\langle\delta_a\Phi(y)\rangle. \tag{2.27}$$

Alternative proof of Goldstone's theorem

Now we are ready for the alternate proof of Goldstone's theorem. For this we will consider working with Euclidean path integrals[8].

Let us consider a theory with a global $U(1)$ symmetry, with associated conserved current j. From Eq. (2.20) we can write down its Ward identity in the presence of a charged operator $\mathcal{O}(x)$ with charge Q as[9]:

$$d\star j(x)\mathcal{O}(0) = iQ\,\mathcal{O}(0)\delta^{(D)}(x) \tag{2.28}$$

Let us now integrate both sides of this equation over a solid D-ball of radius R centered at the origin, as in Figure 2.3. We find the equation

$$\left(\int_{S^{D-1}(R)}\star j\right)\mathcal{O}(0) = iQ\,\mathcal{O}(0) \tag{2.29}$$

[7] We will see later why this property of *closedness* is needed.

[8] In this case, the Ward identity would be as given in Eq. (2.20) but the 'i' on the right hand side won't be there. Here, to simplify notation, we have dropped the angled brackets.

[9] The charged operator transforms under the symmetry as: $\mathcal{O}(x)\to e^{i\varepsilon Q}\mathcal{O}(x)$, where $x\in\Sigma$.

where on the left hand side we have used Stokes' theorem and hence now the integral is taken over the boundary of the D-ball, that is $\partial S^D(R) = S^{D-1}(R)$.

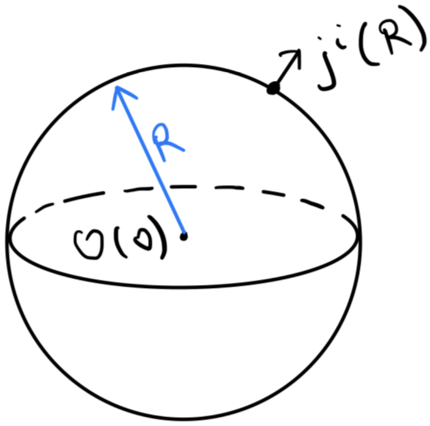

Figure 2.3: Charge operator defined on an S^{D-1} of radius R wrapping operator $\mathcal{O}(0)$ at the origin.

Finally, let us take the expectation value on both sides:

$$\left\langle \left(\int_{S^3(R)} \star j \right) \mathcal{O}(0) \right\rangle = iQ \left\langle \mathcal{O}(0) \right\rangle \tag{2.30}$$

Now, if we are in the SSB phase, then $\langle \mathcal{O}(0) \rangle \neq 0$, and the integral on the left hand side must be both nonzero and independent of the radius of the $(D-1)$-sphere R. Due to spherical symmetry, we see that the correlation of the local operator j on the $(D-1)$-sphere and $\mathcal{O}(0)$ must therefore depend on R as

$$\langle j^i(x) \mathcal{O}(0) \rangle \sim iQ\, n^i R^{-3} \tag{2.31}$$

where n^i is an outwardly pointing normal vector on the $(D-1)$-sphere. The R-dependence is fixed by demanding that the integral over the $(D-1)$-sphere results in an R-independent constant. Therefore, there is a power-law correlation in the theory and not an exponential one. This is a signature of *long range order* in the theory. Thus, we see the existence of a gapless excitation which is the Goldstone mode.

2.1.4 Linking number and finite transformations

The right hand side of Eq. (2.27) comprises the term $\int_{\Omega_\Sigma} d^D x\, \delta^{(D)}(x-y)$, which can be recognised as the intersection number of Ω_Σ and y. This intersection number is equivalent to the link number of Σ and y (see appendix J for a review):

$$\text{Link}(\Sigma, y) = \int_{\Omega_\Sigma} d^D x\, \delta^{(D)}(x-y), \tag{2.32}$$

The value of this link number is either 0 or 1, depending on whether y is inside the region Ω_Σ or not. Consequently, equation (2.27) can be expressed as:

$$\langle Q(\Sigma)\Phi(y)\rangle = -i\,\text{Link}(\Sigma, y)\langle\delta\Phi(y)\rangle. \tag{2.33}$$

The linking number defined in Eq. (2.32) is clearly topological since it is impervious to deformations of the surface Σ, given the deformations do not cross the point y. Furthermore, the charge $Q(\Sigma)$ can also be considered a topological invariant. We already saw this before but let us see this here in the language of differential forms.

This is evident when considering a deformation of the original region Ω_Σ to $\Omega'_{\Sigma'} = \Omega_\Sigma \cup \Omega_0$, such that $y \notin \Omega_0$. This leads to:

$$\begin{aligned}\langle Q(\Sigma+\partial\Omega_0)\,\Phi(y)\rangle &= \int_{\Omega_\Sigma\cup\Omega_0}\langle d\star j\Phi(y)\rangle = \int_{\Omega_\Sigma}\langle d\star j\Phi(y)\rangle + \int_{\Omega_0}\langle d\star j\Phi(y)\rangle \\ &= \int_{\Omega_\Sigma}\langle d\star j\Phi(y)\rangle = \langle Q(\Sigma)\Phi(y)\rangle. \end{aligned} \tag{2.34}$$

In the equation above, since $y \notin \Omega_0$, we have: $d\star j = 0$ inside the correlator in the last term of the first line. Thus, we get, $Q(\Sigma+\partial\Omega_0) = Q(\Sigma)$ as long as such a deformation doesn't hit any charged operators. Hence, the conservation law is converted into the fact that the operator $Q(\Sigma)$ is topological.

The finite form of relation Eq. (2.33) can also be expressed as follows:

$$\langle U(g,\Sigma)\Phi(y)\rangle = R(g)\langle\Phi(y)\rangle. \tag{2.35}$$

The above relationship holds if y and Σ are linked. Here, $U(g,\Sigma)$ is the unitary topological operator associated with the symmetry group g, and R is the representation in which the fields transform:

$$U(g,\Sigma) = e^{i\alpha_a Q_a}, \qquad R(g) = e^{\alpha_a t_a}. \tag{2.36}$$

In these expressions, t_as correspond to the generators in the representation that Φ transforms in. For infinitesimal parameters α_a, the relation in Eq. (2.33) is immediately restored with the identification $\delta_a\Phi \equiv t_a\Phi$. This is referred to as a 0-form symmetry, implying that the charged operators that transform under the symmetry are local operators $\Phi(y)$ supported at a point, i.e., within a 0-dimensional region. Alternatively, the transformation parameter is a closed 0-form, which is simply a constant (as can be seen from Eq. (2.1)).

Even in the context of discrete symmetries, where a conserved charge does not exist, we can define a topological unitary operator in the same way as in Eq. (2.35). In fact, consider the unitary operator for a discrete symmetry, $U(g)$ (devoid of any parameter), and corresponding acting it on a local operator:

$$\langle U(g)\Phi(y)U^{-1}(g)\rangle = R(g)\langle\Phi(y)\rangle. \tag{2.37}$$

In the above equation, the operator $U(g)$ is assumed to be defined at a time $y^0+\epsilon$, and the operator $U^{-1}(g)$ at the time $y^0-\epsilon$. Equal time is inferred in the limit as $\epsilon \to 0$. Additionally, a spatial slice can be associated with the operator $U(g)$. We then assume that $[U(g), p_\mu] = 0$, where p_μ is the generator of spacetime translations. This suggests that the spacetime region associated with $U(g)$ can be smoothly deformed into a closed one, through a series of transformations as illustrated in Fig. 2.4. Hence, the left-hand side of Eq. (2.37) can be expressed as:

$$\langle U(g)\Phi(y)U^{-1}(g)\rangle = \langle U(g,\Sigma)\Phi(y)\rangle, \tag{2.38}$$

when y and Σ are linked[10]. This leads us to conclude that the relation in Eq. (2.35) is applicable to discrete symmetries as well.

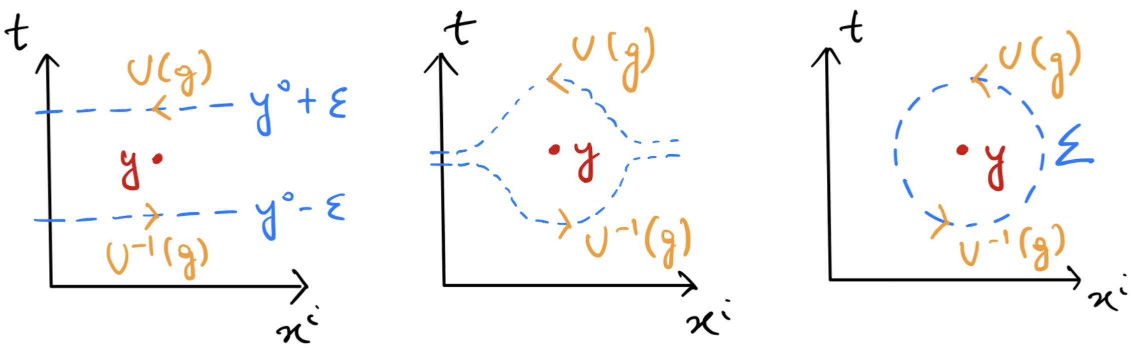

Figure 2.4: A sequence of deformations leading to the association of a closed surface Σ to the operator: $U(g,\Sigma)$.

[10]Note that, here the conjugation action as given in the first figure in Fig. 2.4 for two different time slices transforms into what is called the *lasso action* – as given in the third figure of Fig. 2.4 – when Σ encircles the point y and is compact. This is the content of Eq. (2.38).

The aforementioned construction offers a compelling viewpoint on symmetries in relativistic theories. Specifically, we can establish a direct correspondence between symmetry generators and topological operators:

$$\text{Symmetry Generator} \quad \Leftrightarrow \quad \text{Topological Operator}, \tag{2.39}$$

Here, the charged operators are associated with objects that have a nontrivial linkage with the topological operators. This perspective is insightful and paves the way for a natural extension of the concept of symmetry to encompass the higher-form case.

2.2 1-form symmetry

In the course of advancing our understanding of symmetries, we now venture into a more generalised framework, specifically targeting the case of a 1-form symmetry[11]. To pave the way, let's revisit the conventional understanding of symmetries, designated as a 0-form symmetry. In this context, the symmetry parameter – let us call it ξ_0 – adheres to the characteristics of a closed 0-form, that is, $d\xi_0 = 0$, as it is associated with a global symmetry. Translating this into the language of differential forms, the relation denoted by Eq. (2.3) can be reformulated as:

$$\delta S = \int_{\mathcal{M}_D} \star j \wedge d\xi_0, \tag{2.40}$$

Like before, in the above equation, we have taken the parameter ξ_0 to be a local function of spacetime, implying that, $d\xi_0$ is no more closed. Performing integration by parts and factoring in the equation of motion, this expression leads to the following expected relation:

$$d \star j = 0. \tag{2.41}$$

Now let us explore the following situation where a global symmetry is governed by a parameter that represents a closed 1-form, denoted as $\xi_1 = \xi_\mu dx^\mu$. Again as the symmetry is global we should have: $d\xi_1 = 0$. Extending the earlier expression Eq. (2.40) by removing the closeness of the parameter ξ_1, we get:

$$\delta S = \int_{\mathcal{M}_D} \star j \wedge d\xi_1, \tag{2.42}$$

[11] The discussion on p-form symmetries, with $p > 1$, will be undertaken shortly.

In this scenario, $\star j$ is a $(D-2)$-form, making j a 2-form and in components the conservation equation $d \star j = 0$ becomes:

$$\partial_\mu j^{\mu\nu} = 0. \tag{2.43}$$

Given that $\star j$ is a $(D-2)$-form, drawing parallels with the conventional scenario (as demonstrated in Eq. (2.25)), one can evidently define the charge within a closed Σ_{D-2} sub-manifold:

$$Q(\Sigma_{D-2}) \equiv \int_{\Sigma_{D-2}} \star j. \tag{2.44}$$

Let us pause and ask ourselves what does this operator count? It counts the conserved *string* number in the system, where strings are one dimensional extended objects which do not end in space or in time. So, an integral over a codimension-2 surface[12] is enough to "catch" all the strings, as shown in Fig. 2.5. In this case the charged operators – which we explore in the next section – create and destroy strings. Additionally, note that,, here the associated conserved 2-form current in Eq. (2.43) has one extra index compared to that of the usual 1-form current associated to conventional 0-form symmetries. This extra index can be intuitively understood as labelling the direction in which a string points in spacetime.

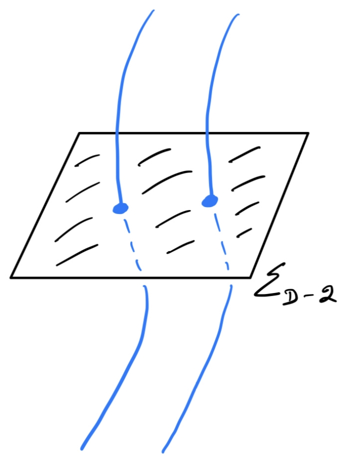

Figure 2.5: Integration over a $(D-2)$-manifold: Σ_{D-2}, counts the number of strings that cross it at an given instant of time.

Here also, if we move Σ_{D-2} up and down in time or left and right in space, or if we deform it to Σ'_{D-2} without hitting any charged operators, then owing to

[12]In a D-dimensional manifold, a sub-manifold of codimension p has dimensions $D-p$.

the conservation equation Eq. (2.43) we will get the same charge: $Q(\Sigma_{D-2})$. As discussed before, this means: $Q(\Sigma_{D-2})$ is topological.

Our next task is to find out the charged operators which transform under this symmetry. In other words, we seek objects that exhibit a nontrivial linking with the sub-manifold: Σ_{D-2}.

2.2.1 Operators charged under 1-form symmetry

To unravel the characteristics of charged entities under a 1-form symmetry, it is helpful to return to the foundational concepts of a 0-form symmetry. In this framework, an infinitesimal global transformation, governed by a constant parameter ξ, prompts a transformation in a local operator as:

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \xi\delta\Phi(x). \tag{2.45}$$

This equation encapsulates our initial assumption: the charged operators under the symmetry behave as local operators, supported on 0-dimensional regions of spacetime.

One natural question arises from this premise: can we conclusively ascertain that charged entities are inherently local operators? Using the Hilbert space perspective, the transformation expressed in Eq. (2.45) is seen to originate from the action of the charged operator, defined over a spatial slice of dimension $D-1 \equiv d$, at a fixed instant of time – a time slice. This comes from observing the fact that charged operators in this case capture the conserved particle number in the system, which is obtained by integrating over a codimension-1 manifold. So, from this perspective let us come up with a way to associate the parameters of transformations to manifolds. The concept of Poincaré duality relating manifolds[13] (see appendix J for a review) essentially provides this association.

Specifically, the Poincaré duality allows for a $(d-p)$-form to be associated with a p-dimensional manifold. The components of this Poincaré dual $(d-p)$-form are expressed as:

$$\xi_{i_{p+1}\dots i_d}(x) \equiv \frac{1}{p!(d-p)!}\int_{\Sigma_p} \epsilon_{i_1\dots i_p i_{p+1}\dots i_d}\,\delta^{(d)}(x-y)\,dy^{i_1}\wedge\cdots\wedge dy^{i_p}. \tag{2.46}$$

[13]Not to be confused with the Poincaré duality relating p-form to q-form fields in D dimensions. Though these two concepts are related they are not the same thing.

From this construction, the parameter of the global symmetry is recognised as the $(d-p)$-form, denoted $\xi_{d-p}(\Sigma_p)$, built from a sub-manifold of dimension p. Notably, the closed nature of ξ_{d-p} – which is required for the symmetry to be global – is implicit from the above construction. We shall see this below.

In the above language, the parameter marking the transformation of a conventional 0-from symmetry is identified, modulo a scaling factor, with the Poincaré dual of the spatial manifold Σ_d, essentially a 0-form constant. This can be seen by setting $p = d$ in (2.46):

$$\xi_0(x) = \frac{1}{d!} \int_{\Sigma_d} \epsilon_{i_1 \ldots i_d} \, \delta^{(d)}(x-y) \, dy^{i_1} \wedge \cdots \wedge dy^{i_d} = 1, \tag{2.47}$$

where the equalling of the above expression to unity follows from definition of the d-dimensional delta function.

Thus, we see that the parameters associated with conventional 0-form symmetries are closed 0-forms, supported on 0-dimensional regions of the manifold. Consequently, such parameters can be linked to the transformations of similar entities – specifically, entities supported on 0-dimensional regions of the manifold, identified as local operators.

Extension to 1-form symmetries

Now let us generalise the previous picture to the case of 1-form symmetries. Let us consider a sub-manifold with dimension $p = d-1$. The associated Poincaré dual for this manifold is the 1-form, $\xi_1(\Sigma_{d-1})$, which has the following components:

$$\xi_{i_d}(x) = \frac{1}{(d-1)!} \int_{\Sigma_{d-1}} \epsilon_{i_1 \ldots i_{d-1} i_d} \, \delta^{(d)}(x-y) \, dy^{i_1} \wedge \cdots \wedge dy^{i_{d-1}}. \tag{2.48}$$

Objects that transform under the 1-form symmetry are referred to as line operators because they have support specifically along a line. For any given operator situated along a line, denoted as $\mathcal{C}$, the transformation parameter (considering a constant factor that can be absorbed into the parameter) is represented as:

$$\int_{\mathcal{C}} \xi_1(\Sigma_{d-1}) = \int_{\mathcal{C}} \xi_i dx^i. \tag{2.49}$$

Delving deeper, the line operator undergoes an infinitesimal transformation, defined as:

$$W[\mathcal{C}] \to W'[\mathcal{C}] = W[\mathcal{C}] + \int_{\mathcal{C}} \xi_1(\Sigma_{d-1}) \delta W[\mathcal{C}], \tag{2.50}$$

An important aspect to note is that even when the line $\mathcal{C}$ is a closed loop, the integral $\int_{\mathcal{C}} \xi_1(\Sigma_{d-1})$ might not be zero, despite the condition $d\xi_1 = 0$. One might use the Stokes' theorem to transform the line integral into a surface integral with $\mathcal{C}$ acting as the boundary, as represented by $\int_{\mathcal{C}=\mathcal{S}} \xi_1(\Sigma_{d-1}) = \int_{\mathcal{S}} d\xi_1$. However, topological constraints might inhibit the application of the Stokes' theorem, especially when both $\mathcal{C}$ and Σ_{d-1} intersect.

Further investigation confirms that $\xi_1 = \xi_i dx^i$ is indeed closed, primarily because the boundary for Σ_{d-1} is absent[14]. This can be seen by setting $p = d-1$ in the following formula (see appendix J) relating the exterior derivative of ξ_1 with that of the boundary of the sub-manifold Σ_{d-1}:

$$d\xi_{d-p}(\Sigma_p) = (-1)^p \xi_{d-(p-1)}(\partial\Sigma_p). \tag{2.51}$$

We see from above that if $\partial\Sigma_{d-1} = \emptyset$ then $d\xi_1 = 0$.

Expanding our focus, we can redefine the charge manifold from Σ_{d-1} to Σ_{D-2} – now an arbitrary closed manifold in spacetime[15]. Additionally, the term "line operator" (a time-invariant operator acting on the Hilbert space) can be extended to include what is termed as a "defect" line. This term refers to an operator that not only spans spatial dimensions but also extends temporally. The symmetry transformation for such a defect line is:

$$W[\mathcal{C}] \to W'[\mathcal{C}] = W[\mathcal{C}] + \int_{\mathcal{C}} \xi_1(\Sigma_{D-2})\delta W[\mathcal{C}], \tag{2.52}$$

where the line $\mathcal{C}$ now extends along the time axis.

Given the above mathematical formulation, we can now extract the consequential Ward identities. Let us focus on the case of a single defect for simplicity. We have:

$$\begin{aligned} \langle W[\mathcal{C}]\rangle &= \int \mathcal{D}\Phi\, W[\mathcal{C}] e^{iS[\Phi]} \\ &= \int \mathcal{D}\Phi'\, W'[\mathcal{C}] e^{iS[\Phi']} \\ &= \int \mathcal{D}\Phi \left(W[\mathcal{C}] + \int_{\mathcal{C}} \xi_1(\Sigma_{D-2})\delta W[\mathcal{C}] \right) (1 + i\delta S)\, e^{iS[\Phi]}, \end{aligned} \tag{2.53}$$

where δS has been defined previously in (2.42). Note that, here to go to the second

[14] Σ_{d-1} cannot have a boundary else, upon integration over it, we won't catch all the strings in the system.

[15] Here we have relaxed the spatial condition on Σ_{D-2} but we still consider a closed manifold as the closedness is necesssary to associate a form to it by the Poincaré duality procedure.

equality above we have assumed $D\Phi = D\Phi'$. We believe that this is a reasonable assumption since we are dealing with 1-form symmetries and unlike the 0-form case where the violation of this equality – which is non-invariance of the integral measure – is a signature of the presence 0-form anomalies. However, here since Φ is a "local" collection of fields the transformation of the integral measure built out from it should not be able to diagnose 1-form anomalies. This is because, as discussed above, for 1-form symmetries the charged operators are line operators and not local operators. However, if we can write down a path integral in some kind of a loop space where the dynamical variables depend upon loops and hence so does the integral measure then probably in this case, the non-invariance of the "loop-integral-measure" might be a signature of the presence of 1-form anomalies. Such loop space path integrals (or actions) for 1-form symmetries have been considered in [36]. For details on higher-form anomalies see [42–44] and references therein.

To express this variation in components, we can represent it as:

$$\delta S = \int d^D x \, j^{\mu\nu} \partial_\mu \xi_\nu = -\int d^D x \, \xi_\nu \, \partial_\mu j^{\mu\nu}. \tag{2.54}$$

From the relationship outlined in (2.53), it becomes evident that,

$$\begin{aligned} i \int d^D x \, \xi_\nu(x) \langle \partial_\mu j^{\mu\nu}(x) \, W[\mathcal{C}] \rangle &= \int_{\mathcal{C}} dy^\nu \xi_\nu(y) \langle \delta W[\mathcal{C}] \rangle \\ &= \int d^D x \, \xi_\nu(x) \int_{\mathcal{C}} dy^\nu \, \delta^{(D)}(x-y) \langle \delta W[\mathcal{C}] \rangle, \\ i \int d^D x \, \xi_\nu(x) \left[\langle \partial_\mu j^{\mu\nu}(x) \, W[\mathcal{C}] \rangle - \int_{\mathcal{C}} dy^\nu \, \delta^{(D)}(x-y) \langle \delta W[\mathcal{C}] \rangle \right] &= 0. \end{aligned} \tag{2.55}$$

Now arbitrariness of $\xi_\nu(x)$ implies:

$$\langle \partial_\mu j^{\mu\nu}(x) \, W[\mathcal{C}] \rangle = -i \int_{\mathcal{C}} dy^\nu \, \delta^{(D)}(x-y) \langle \delta W[\mathcal{C}] \rangle, \tag{2.56}$$

This equation essentially is the Ward identity for a single line defect.

With this foundational understanding of the 1-form symmetry, we are now equipped to investigate further extensively the potential implications and consequences arising from this symmetry structure.

2.2.2 1-Form symmetry in $D = 4$

Probing further the properties and consequences of 1-form symmetries, we focus our attention on a particular case of $D = 4$ spacetime dimensions. First, we consider "true" line operators and not defect lines.

Within this framework, the conservation laws are expressed as:

$$\partial_\mu j^{\mu\nu} = \partial_0 j^{0\nu} + \partial_1 j^{1\nu} + \partial_2 j^{2\nu} + \partial_3 j^{3\nu} = 0. \tag{2.57}$$

A direct corollary of these conservation laws is the conservation of the respective charges,

$$Q^1 = \int dx^2 dx^3 j^{01}, \qquad Q^2 = \int dx^3 dx^1 j^{02}, \qquad Q^3 = \int dx^1 dx^2 j^{03}, \tag{2.58}$$

which are defined over spatial 2-dimensional sub-manifolds. Note the independence of Q^1 on x^1, Q^2 on x^2, and Q^3 on x^3. This stems directly from Eq. (2.57) with $\nu = 0$,

$$\partial_1 j^{10} + \partial_2 j^{20} + \partial_3 j^{30} = 0, \tag{2.59}$$

then integrating this expression over the corresponding 2-dimensional spaces and throwing away total derivative terms.

Now the Ward identity: Eq. (2.56) reads in $D = 4$:

$$\langle \partial_\mu j^{\mu\nu}(x)\, W[\mathcal{C}]\rangle = -i \int_{\mathcal{C}} dy^\nu\, \delta^{(4)}(x-y) \langle \delta W[\mathcal{C}]\rangle. \tag{2.60}$$

Let us choose $\nu = 3$ in the above equation to obtain:

$$\langle \partial_0 j^{03}(x)\, W[\mathcal{C}]\rangle + \langle \partial_1 j^{13}(x)\, W[\mathcal{C}]\rangle + \langle \partial_2 j^{23}(x)\, W[\mathcal{C}]\rangle = -i \int_{\mathcal{C}} dy^3\, \delta^{(4)}(x-y) \langle \delta W[\mathcal{C}]\rangle. \tag{2.61}$$

Upon integrating both sides across the interval $\int_{y^0-\epsilon}^{y^0+\epsilon} dx^0 \int dx^1 \int dx^2$, we arrive at[16]:

$$\langle [Q^3, W[\mathcal{C}]]\rangle = -i \int dx^1 dx^2 \int_{\mathcal{C}} dy^3 \delta^{(3)}(\vec{x}-\vec{y}) \langle \delta W[\mathcal{C}]\rangle. \tag{2.62}$$

We see that the integral on the right side captures the intersection between a two-dimensional plane (spanned by x^1 and x^2) and the curve $\mathcal{C}$ that extends in direction x^3. This is visually depicted in Fig. 2.6. A similar logic can be extrapolated to curves $\mathcal{C}$ aligned along other directions.

[16] Below on the left hand side we get expectation value of the commutator due to the time-ordering in the path-integral.

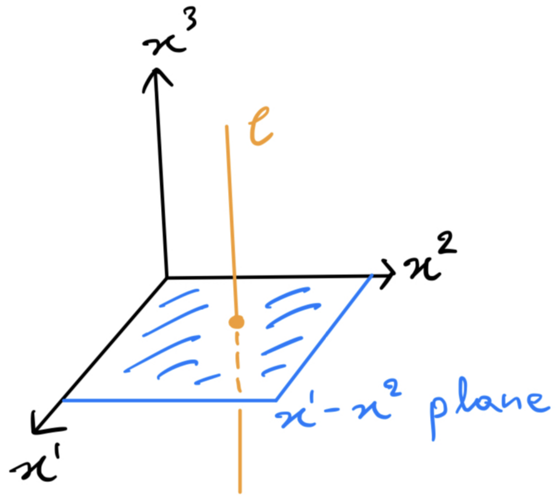

Figure 2.6: Intersection of a spatial 2D slice spanned along $x^1 - x^2$ directions with that of the line $\mathcal{C}$ extending along the third spatial direction x^3.

Moving on, to confer a topological significance to our charge, we revisit the Ward identity, Eq. (2.60)[17]. Upon integrating both sides over a region Ω_3, where $\partial\Omega_3 = S^2$, we get:

$$\int_{\Omega_3} (d\Omega_3)_\nu \langle \partial_\mu j^{\mu\nu}(x)\, W[\mathcal{C}]\rangle = -i \int_{\Omega_3} (d\Omega_3)_\nu \int_{\mathcal{C}} dy^\nu \delta^{(4)}(x-y) \langle \delta W[\mathcal{C}]\rangle. \tag{2.63}$$

where $(d\Omega_3)_\nu$ is the oriented element of integration on Ω_3.

Taking a closer look at the above equation, we can identify the intersection number as the link between the curve $\mathcal{C}$ and S^2,

$$\int_{\Omega_3} (d\Omega_3)_\nu \int_{\mathcal{C}} dy^\nu \delta^{(4)}(x-y) = \mathrm{Link}(S^2, \mathcal{C}). \tag{2.64}$$

Therefore, Eq. (2.63) becomes:

$$\langle Q(S^2) W[\mathcal{C}]\rangle = -i\mathrm{Link}(S^2, \mathcal{C})\langle \delta W[\mathcal{C}]\rangle, \tag{2.65}$$

which is the higher-form analogue of Eq. (2.33). The above topological nature is justified as long as $\mathcal{C}$ is infinitely extended or is a closed loop in spacetime so that it can link with the corresponding sub-manifold.

Note that, for a line aligned along direction x^3, the surface S^2 is embedded in a three-dimensional space defined by x^0–x^1–x^2.

[17] Now we shall consider the general case where $\mathcal{C}$ can align along any spacetime directions and not just the spatial ones.

2.2.3 q-form symmetries

In our quest to dig deeper into symmetries, we now turn our attention to the general scenario of a q-form symmetry. For a given $(q+1)$-form conserved current j, the associated conservation law is captured by the relation $d \star j = 0$. By taking this conservation equation and integrating it over a $(D-q)$-dimensional spatial region Ω_{D-q}, whose boundary is denoted as $\partial\Omega_{D-q} = \Sigma_{D-q-1}$ and then using Stokes', we get:

$$\int_{\Omega_{D-q}} d \star j = \int_{\Sigma_{D-q-1}} \star j. \tag{2.66}$$

From this relationship, it becomes evident that the charge, when associated with the q-form symmetry over the boundary Σ_{D-q-1}, is defined as:

$$Q(\Sigma_{D-q-1}) = \int_{\Sigma_{D-q-1}} \star j. \tag{2.67}$$

The $(D-q-1)$-dimensional manifold, represented by Σ_{D-q-1}, can be mapped to a complementary $D-1-(D-q-1) = q$-form via the Poincaré duality. This q-form then serves as the transformation parameter. In a quantum theoretical context, these symmetries correspond to operators that are supported on a q-dimensional manifold. Such a viewpoint not only enhances our understanding of symmetries but also elegantly intertwines topological constructs with conservation laws.

2.2.4 Higher-form symmetries in free Maxwell

The free Maxwell theory provides an illustrative example highlighting the significance of higher-form symmetries. The action is given as:

$$S[a] = \int -\frac{1}{2e^2} f \wedge \star f = \int d^D x \left(-\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} \right), \tag{2.68}$$

where a is the $U(1)$ gauge field which transforms as: $a \to a + d\Lambda$. $f = da$ is its field strength.

An important consideration arises when employing a compact gauge group $U(1)$ instead of its non-compact counterpart. With this choice, the gauge field a behaves as an angular variable. This angular behaviour imposes certain quantisation conditions, particularly the quantisation of the $U(1)$ charges or, the electric charges. This comes from invariance of the path-integral under *large gauge transformations* – gauge transformations which are not connected to the identity of the corresponding Lie group. We shall see this below.

Let us ask ourselves: what are the observables of the theory described by Eq. (2.68)? An intuitive response gravitates towards gauge-invariant objects, constructed from the field strength $f_{\mu\nu}$. These objects manifest as local operators. However, can one conceive extended gauge-invariant entities? The answer is yes and these are the *Wilson line* operators/defects defined as:

$$W_{q_e}[\mathcal{C}] \equiv \exp\left(iq_e \oint_{\mathcal{C}} a\right), \tag{2.69}$$

where q_e is the electric $U(1)$ charge.

As discussed before, the curve $\mathcal{C}$ should either be infinitely extended or be a closed-loop to maintain gauge invariance. The parameter q_e, symbolising the charge associated with the Wilson line, is an integer. Its integral nature is a testament to the compact nature of the underlying gauge group. A lucid pathway to discern this integral nature is achieved by visualising the temporal dimension as a unit circle S^1 with length L_0. Under this conception, a gauge function can be sculpted to envelop the unit circle:

$$\Lambda = 2\pi \frac{x^0}{L_0}. \tag{2.70}$$

Such a formulation leads to the compactness condition for the temporal component of the gauge field, a_0 as:

$$a_0 \sim a_0 + \frac{2\pi}{L_0}, \tag{2.71}$$

which is evident from the gauge transformation of a.

Now if the Wilson line in Eq. (2.69) is extended along the time direction, its invariance under large gauge transformations leads to the conclusion that q_e is quantised.

Proceeding further the line operator in Eq. (2.69) epitomizes the worldline trajectory of a probe charged particle – which has no dynamics. To see this let us evaluate the expectation value of the Wilson loop.

$$\langle W_{q_e}[\mathcal{C}]\rangle = \int \mathcal{D}a \, \exp\left(iq_e \oint_{\mathcal{C}} a\right) e^{iS[a]}. \tag{2.72}$$

Let us introduce a conserved current associated with a particle moving along a curve parametrized by $\vec{y}(x^0)$

$$j^0(x^0, \vec{x}) = q_e \delta^{(d)}(\vec{x} - \vec{y}(x^0)) \qquad \vec{j}(x^0, \vec{x}) = q_e \frac{d\vec{y}(x^0)}{dx^0} \delta^{(d)}(\vec{x} - \vec{y}(x^0)), \tag{2.73}$$

which can be succinctly written as,

$$j^\mu(x^0,\vec{x}) = q_e \frac{dy^\mu(x^0)}{dx^0}\delta^{(d)}(\vec{x}-\vec{y}(x^0)), \tag{2.74}$$

with the identification: $y^0 = x^0$. Thus, the Wilson line can be written as:

$$\begin{aligned} W_{q_e}[\mathcal{C}] = \exp\left(iq_e \oint_{\mathcal{C}} dy^\mu a_\mu(y)\right) &= \exp\left(iq_e \int dx^0 \frac{dy^\mu(x^0)}{dx^0} a_\mu(x^0,\vec{y})\right) \\ &= \exp\left(iq_e \int d^dx\, \delta^{(d)}(\vec{x}-\vec{y}(x^0)) \int dx^0 \frac{dy^\mu(x^0)}{dx^0} a_\mu(x^0,\vec{x})\right) \\ &= \exp\left(i\int d^Dx\, j^\mu a_\mu\right). \end{aligned} \tag{2.75}$$

Therefore, the expectation value of the Wilson loop corresponds simply to coupling the theory to non-dynamical charged matter, parametrised by the current j^μ,

$$\langle W_{q_e}[\mathcal{C}]\rangle = \int \mathcal{D}a\, e^{iS[a]+i\int d^Dx j^\mu a_\mu}. \tag{2.76}$$

Next let us move on to the equations of motion. From the action presented in Eq. (2.68) we get,

$$\frac{1}{e^2} d\star f = 0, \qquad df = 0. \tag{2.77}$$

When we dissect these equations into their component-wise representations, they take the form

$$\frac{1}{e^2}\partial_\mu f^{\mu\nu} = 0, \qquad \partial_{\mu_3}\left(\epsilon^{\mu_1\mu_2\mu_3\ldots\mu_D} f_{\mu_1\mu_2}\right) = 0. \tag{2.78}$$

In the above expressions, we have introduced the components of the dual field strength, a concept that can be explained as a $(D-2)$-form.

It is clear from the equations of motion that the underlying theory possesses two distinct conserved 2-form currents and hence two distinct higher-form symmetries. These can be categorized as the 1-form electric and the $(D-3)$-form magnetic symmetries. The conserved currents for these symmetries are defined as $j_e \equiv \frac{1}{e^2} f$ and $j_m \equiv \frac{1}{2\pi}\star f$.[18] The corresponding charges are:

$$Q_e(\Sigma_{D-2}) = \int_{\Sigma_{D-2}} \star j_e = \frac{1}{e^2}\int_{\Sigma_{D-2}} \star f \tag{2.79}$$

[18]Here the 2π is just by convention and why this convention is chosen will become clear in the upcoming section.

and for the magnetic symmetry:

$$Q_m(\Sigma_2) = \int_{\Sigma_2} \star j_m = \frac{1}{2\pi}\int_{\Sigma_2} \star(\star f) = \frac{1}{2\pi}\int_{\Sigma_2} -f. \tag{2.80}$$

For ease of reference, these symmetries are generally denoted as

$$U(1)_e^{(1)} \times U(1)_m^{(D-3)}. \tag{2.81}$$

We notice from above that there is no magnetic symmetry when $D = 2$. In scenarios where $D = 3$, the magnetic symmetry is seen as a standard 0-form symmetry. Interestingly, when $D = 4$, both the electric and magnetic symmetries are observed as 1-form symmetries.

To find the unitary operators that are responsible for generating these symmetries we exponentiate the associated charges. Referring to Eq. (2.79) and Eq. (2.80), these operators can be given as:

$$U_e(\alpha_e, \Sigma_{D-2}) = e^{i\alpha_e Q_e(\Sigma_{D-2})}, \qquad U_m(\alpha_m, \Sigma_2) = e^{i\alpha_m Q_m(\Sigma_2)}, \tag{2.82}$$

with the transformation parameters satisfying $\alpha_e \sim \alpha_e + 2\pi$ and $\alpha_m \sim \alpha_m + 2\pi$.

When we incorporate dynamical charged matter with charge $n > 1$, then the $U(1)_e^{(1)}$ symmetry breaks down to a $\mathbb{Z}_N$ 1-form symmetry. This breakdown of symmetry can be attributed to the screening of charges resulting from virtual pair creation[19]. To put this into context, let's consider the electric charge $Q_e(\Sigma_{D-2})$, which is defined over a closed surface Σ_{D-2}. This charge effectively perceives only the values of charges modulo N, since it can encircle a single member of a virtual pair, as depicted in Fig. 2.7. Consequently, the unitary operator $U_e(\alpha_e, \Sigma_{D-2})$ specified in Eq. (2.82) is mandated to be identity operator while acting on objects possessing charges in multiples of N, leading to

$$e^{i\alpha_e Q_e(\Sigma_{D-2})} = e^{i\alpha_e N k} = 1, \quad k \in \mathbb{Z}. \tag{2.83}$$

From this, one can infer that the transformation parameter can be expressed as $\alpha_e = \frac{2\pi}{N}$, suggesting that $e^{i\alpha_e}$ belongs to the $\mathbb{Z}_N$ group.

[19] In the next chapter, we will discuss a similar phenomenon where electric field gets screened and becomes short-ranged in a conducting medium like charged plasma. In this setting it is known as the *Debye screening*.

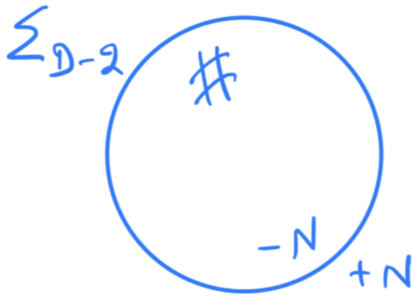

Figure 2.7: Screening of charges owing to virtual pair creation.

Another way to see the above is the following. Consider the finite version of the Ward identity as given in Eq. (2.35). In this equation, let us have: $\Sigma = \Sigma_{D-2}$ and the associated objects to be line operators instead of local operators on which the charged operators act in. This equation should then be understood as an operator equation, valid within general correlation functions provided that there are no other operator insertions that link non-trivially with Σ_{D-2} and $\mathcal{C}$. This can be understood as the result of shrinking the surface Σ_{D-2} to zero size on the line operator. This is allowed owing to the topological nature of the operator. Observe the left hand side figure given in Fig. 2.8. The link is established when the "green" circle shrinks to zero size on the "black" line operator.

Now due to the presence of charged matter in the theory, the Wilson lines can end on them. Thus, there is a way in which we can unlink Σ_{D-2} and $\mathcal{C}$, as depicted in Fig. 2.8. This is the case of *trivial* linking (see [45]) which means $U_e(\Sigma_{D-2})$ here essentially becomes the identity operator. Now from the definition of $U_e(\Sigma_{D-2})$ as given in Eq. (2.82) and noting that the charges are multiples of N, that is: $Q_e = Nk$ $(k \in \mathbb{Z})$, we get that: $\alpha_e = \frac{2\pi}{N}$ for $U_e(\Sigma_{D-2})$ to be topological.

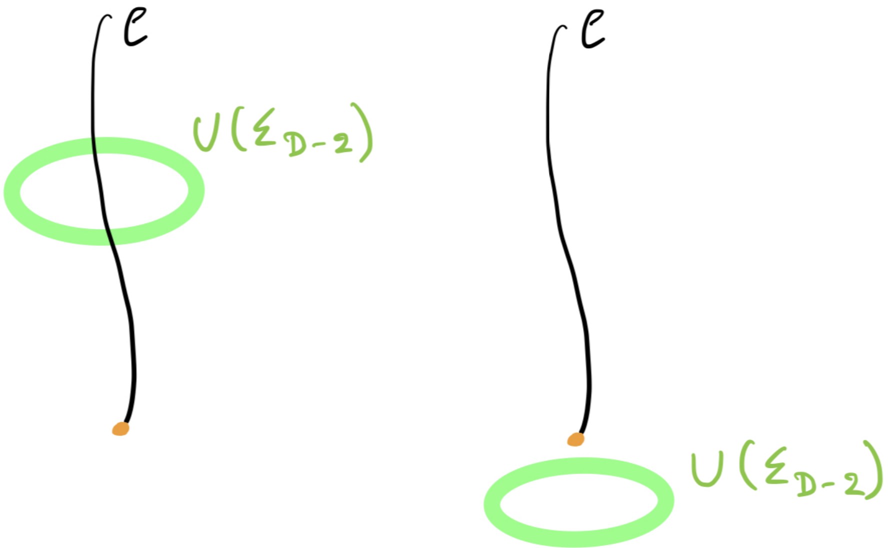

Figure 2.8: Topological operators wrapping line operators can unlink if the line operators end on charges. On the left we have a non-trivial linking and on the right a trivial linking. In this case, both sides have to match for $U(\Sigma_{D-2})$ to be topological implying Σ_{D-2} is the identity operator.

In an attempt to gain a deeper understanding of the magnetic symmetry in the system, it is helpful to express $\star f$ using a novel gauge field, denoted as $\tilde{a}$ such that $\star f = d\tilde{a}$. $\tilde{a}$ is a $D-3$ form. Now the idea is to express the action given in Eq. (2.68) with $\tilde{a}$ as the dynamical variable instead of a. This has been worked out in the appendix B and can be expressed as:

$$S[\tilde{a}] = \int d^D x \left(-\frac{e^2}{2} h \wedge \star h. \right) \tag{2.84}$$

where $h = d\tilde{a}$. Note that the coupling constant: e^2 now appears in the numerator of the action instead of the denominator. This action remains invariant under the associated gauge transformations, which can be presented as:

$$\tilde{a} \to \tilde{a} + d\lambda. \tag{2.85}$$

where λ is a $(D-4)$-form.

Beyond the local gauge-invariant observables derived from the field strength $*f$, its viable to look at gauge-invariant extended entities. An example of such an object is:

$$T_{q_m}[\Gamma_{D-3}] = \exp\left(i2\pi q_m \int_{\Gamma_{D-3}} \tilde{a} \right), \tag{2.86}$$

which are supported on a $(D-3)$-dimensional manifold denoted by Γ_{D-3}. Here, q_m represents the magnetic charge. These are the so called *'t Hooft operators*, which are the charged objects under the magnetic $(D-3)$-form symmetry.

Expanding on Maxwell in $D = 4$

In this section we shall expand the discussion of the previous section to the case $D = 4$. In this case, we encounter the higher-form symmetry of the form $U(1)_e^{(1)} \times U(1)_m^{(1)}$.[20]

Given this context, the conservation laws referenced in Eq. (2.78) simplify and are represented more succinctly as:

$$\partial_\mu f^{\mu\nu} = 0, \qquad \partial_\rho \left(\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu}\right) = 0. \tag{2.87}$$

Here, the conservation law given on the left is associated to the conservation of the electric field lines in the absence of charged matter, or in other words it states that electric field lines cannot end in absence of charges. The conservation law on the right hand side, which is nothing but the *Bianchi identity*, is associated to the conservation of magnetic field lines or in other words it states that magnetic fields lines do not end. This is just the Gauss' law[21].

Expanding further on the charges detailed in Eq. (2.79) and Eq. (2.80), we get:

$$Q_e(\Sigma_2) = \int_{\Sigma_2} \star j_e = \int_{\Sigma_2} \star f \tag{2.88}$$

and, similarly:

$$Q_m(\Sigma_2) = \int_{\Sigma_2} \star j_m = \frac{1}{2\pi} \int_{\Sigma_2} \star(\star f) = \frac{1}{2\pi} \int_{\Sigma_2} -f, \tag{2.89}$$

where we understand Σ_2 to be a closed manifold.

When we speak about charged operators within this context, they manifest as

[20] Since the dimension of the coupling constant e^2 is: $[e^2] = 4 - D$, it is dimensionless in $D = 4$ and hence, here we set it to 1 for simplicity.

[21] A curious thing to note is that, in the electric frame – where a was the dynamical variable – $d \star f = 0$ or conservation of j_e required the use of equations of motion but $df = 0$ or the conservation of j_m followed from the Bianchi identity. However, in the magnetic frame – where $\tilde{a}$ is the dynamical variable, one can readily see that this situation is exactly reversed. This is a general feature of the duality between these two frames where on-shell currents becomes topological currents and vice-verse while going between a frame and its dual.

Wilson and 't Hooft lines, expressed as[22]:

$$W_{q_e}[\mathcal{C}] = \exp\left(iq_e \oint_{\mathcal{C}} a\right), \qquad T_{q_m}[\mathcal{C}] = \exp\left(i2\pi q_m \oint_{\mathcal{C}} \tilde{a}\right). \tag{2.90}$$

With our deliberate inclusion of the 2π factor in the 't Hooft operator, we ensure that the magnetic charge, denoted as q_m, follows quantisation in a manner similar to the electric charge q_e, such that $q_m \in \mathbb{Z}$. The logic behind this becomes apparent when the Wilson line is taken along a closed curve. When considering this scenario, the curve can be envisioned as the boundary of two distinct surfaces, namely X_2 and X_2'. This relationship can be presented as:

$$W_{q_e}[\mathcal{C}] = \exp\left(iq_e \oint_{\mathcal{C}} a\right) = \exp\left(iq_e \int_{X_2} f\right) = \exp\left(iq_e \int_{X_2'} f\right), \tag{2.91}$$

where in the second equality we use Stokes' theorem.

Considering the orientation of the surfaces X_2 and X_2', the equation above leads to:

$$\begin{aligned} 1 &= \exp\left(iq_e \int_{\Sigma_2 = X_2 \cup X_2'} f\right) \\ &= \exp\left(i2\pi q_e Q_m(\Sigma_2)\right) = 1. \end{aligned} \tag{2.92}$$

Thus, it becomes evident that magnetic charges located within Σ_2, as quantified by $Q_m(\Sigma_2)$, assume integral values. Armed with this crucial insight, we can elucidate the periodic behavior of the field $\tilde{a}$, when considering large gauge transformations. Let us consider the following action:

$$S[f, \tilde{a}] = \int d^D x \left(-\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2}\tilde{a}_\mu \epsilon^{\mu\nu\rho\sigma}\partial_\nu f_{\rho\sigma}\right), \tag{2.93}$$

where $\tilde{a}$ can be treated as a Lagrange multiplier to enforce the closure of f.

Next consider this system placed in a manifold characterized by a periodic time structure, say $S^1 \times \Omega_3$, such that $\partial\Omega_3 = S^2$. A large gauge transformation of $\tilde{a}$ that

[22]Physically speaking, Wilson loop – where $\mathcal{C}$ is a closed loop – measures magnetic flux through an enclosed surface Σ which is enclosed by the loop. This scenario is like measuring the magnetic field lines poking through a surface which is enclosed by an electric current loop which in this case is the Wilson loop. Similarly, 't Hooft loop measures electric flux through surfaces they enclose.

winds the temporal direction: $\tilde{a}_0 \to \tilde{a}_0 + \lambda_0$, leads to:

$$\begin{aligned}
\delta S &= \int_0^{L_0} dx^0 \int d^3x \frac{1}{2}\lambda_0 \epsilon^{0ijk} \partial_i f_{jk} \\
&= \frac{1}{2}\lambda_0 \int_0^{L_0} dx^0 \int_{S^2} dS_i\, \epsilon^{ijk} f_{jk} \\
&= -\lambda_0 \int_0^{L_0} dx^0 \int_{S^2} d\vec{S} \cdot \vec{B} \\
&= -\lambda_0 L_0 2\pi \mathbb{Z}, \qquad (2.94)
\end{aligned}$$

where the last equality follows from the fact that $Q_m(\Sigma_2)$ considered in the previous paragraph are integral magnetic charges located within $\Sigma_2 = S^2$[23]. In other words, magnetic field lines poking through $\Sigma_2 = S^2$ is quantised as was seen above.

For the invariance of the path integral under such a large gauge transformation, we shoudl have $e^{i\delta S} = 1$. This in turn enforces a condition on λ_0 as:

$$\lambda_0 = \frac{n}{L_0}, \quad n \in \mathbb{Z}. \qquad (2.95)$$

Such large gauge transformations compactify the field $\tilde{a}$ as:

$$\tilde{a}_\mu \sim \tilde{a}_\mu + \frac{n_\mu}{L_\mu}. \qquad (2.96)$$

Applying the above large gauge transformation to the 't Hooft line as given in Eq. (2.90) leads to the quantisation of the magnetic charge: $q_m \in \mathbb{Z}$.

Moreover, by adopting $\Sigma_2 = S^2$ positioned within a purely spatial domain, the aforementioned charges can be understood to essentially capture the electric and magnetic fluxes:

$$Q_e(S^2) = \frac{1}{4} \int_{S^2} dS^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma} = \int_{S^2} d\vec{S} \cdot \vec{E} \qquad (2.97)$$

and, concurrently:

$$Q_m(S^2) = \frac{1}{4\pi} \int_{S^2} dS^{\mu\nu} f_{\mu\nu} = \frac{1}{2\pi} \int_{S^2} d\vec{S} \cdot \vec{B}. \qquad (2.98)$$

The charged objects under the above charges (objects that links with S^2 in $D = 4$) are line defects that are extended entirely along the time direction, that is, they correspond just to electric and magnetic charges at rest in space. Thus, at a given instant of time, these charges defined in Eq. (2.97) and Eq. (2.98) can detect the presence of electric and magnetic flux through the surface S^2.

[23] This can also be seen from considering the Dirac quantisation as can be seen in Eq. (L.3).

2.2.5 Abelian nature of p-form symmetries for $p \geq 1$

Let us consider the unitary topological operators associated with 1-form symmetry which are supported on $(D-2)$-manifolds: $U(g, \Sigma_{D-2})$. Then it is easy to see that they must form an Abelian group since we can pass $(D-2)$-dimensional surfaces past each other without intersection:

$$U^{(1)}(g', \Sigma'_{D-2})U^{(1)}(g, \Sigma_{D-2}) = U^{(1)}(g, \Sigma_{D-2})U^{(1)}(g', \Sigma'_{D-2}), \tag{2.99}$$

where Σ_{D-2} and Σ'_{D-2} are parallel space-like surfaces at different times, acting on the same Hilbert space. This argument can now be generalised to the case of p-form symmetries with $p \geq 2$. We know that for a p-form symmetry the the topological operators are supported on a Σ_{D-p-1} manifold. Now in D-dimensions we can pass $(D-p-1)$-dimensional surfaces past each other without intersection, for example consider parallel spacelike hypersurfaces of dimension equalling $D-p-1$ at constant time. The only obstruction to this arises when $p = 0$ as in this case infinitely extended $(D-1)$-dimensional surfaces cannot be taken past one another without having them intersect. Thus, conventional 0-form symmetries can be non-Abelian as we already know. However, for $p \geq 1$, p-form symmetries are always Abelian [21, 46, 47]. Now since p-form symmetries are always Abelian for $p > 0$, the only effective IR action we can write down is the kinetic term for the Goldstones[24]:

$$S = -\frac{1}{2g^2}\int F \wedge \star F \tag{2.100}$$

where $F = dA$ and above is the generalised Maxwell action.

2.2.6 Spontaneous symmetry breaking of higher-form symmetries

In this section we shall discuss spontaneous breaking of continuous 1-form symmetries [40, 47, 48]. For concreteness, we shall consider the example of QED in $D = 4$. For this we shall follow a derivation as given in [28]. Then, we shall present a general proof of Goldstone's theorem for SSB regarding continuous p-form symmetries with $p > 1$. This will be based on the derivation given in [40].

[24] If Goldstone bosons are the only massless fields in the IR

Goldstone mode for SSB of continuous 1-form symmetry

In our quest to understand the Goldstone excitations, we can employ a method analogous to the approach used for ordinary symmetries, detailed in Sec. (2.1.3).

We begin by considering the Ward identity as given in Eq. (2.56). For the sake of clarity and to maintain continuity in our discussion, we reproduce it here:

$$\langle \partial_\mu j^{\mu\nu}(x)\, W[\mathcal{C}]\rangle = -q_e \int_{\mathcal{C}} dy^\nu \delta^{(D)}(x-y)\langle W[\mathcal{C}]\rangle. \tag{2.101}$$

which is obtained by using $\langle \delta W[\mathcal{C}]\rangle = -iq_e\langle W[\mathcal{C}]\rangle$ in Eq. (2.56).

To analyze this expression in the frequency domain, we take its Fourier transform, by integrating over $\int d^D x\, e^{ipx}$ on both sides. This leads us to:

$$ip_\mu \langle j^{\mu\nu}(p) W[\mathcal{C}]\rangle = q_e f^\nu(p,\mathcal{C})\langle W[\mathcal{C}]\rangle, \tag{2.102}$$

after the following identification:

$$f^\nu(p,\mathcal{C}) \equiv \int_{\mathcal{C}} dy^\nu e^{ipy}. \tag{2.103}$$

The above Fourier transform is intriguing due to its unique properties. At the outset, it is generically non-vanishing at $p=0$. We can express this as:

$$f^\nu(0,\mathcal{C}) \equiv \int_{\mathcal{C}} dy^\nu \neq 0. \tag{2.104}$$

Additionally, the transform satisfies:

$$\begin{aligned} p_\nu f^\nu(p,\mathcal{C}) &= \int_{\mathcal{C}} dy^\nu p_\nu e^{ipy} \\ &= -i\int_{\mathcal{C}} dy^\nu \partial_\nu e^{ipy} = 0, \end{aligned} \tag{2.105}$$

which is valid for any closed curve $\mathcal{C}$. In the last equality above we have used Stokes' theorem.

Building upon this foundation, consider the expression Eq. (2.102) in the limit $p \to 0$:

$$\lim_{p\to 0} ip_\mu \langle j^{\mu\nu}(p) W[\mathcal{C}]\rangle = q_e f^\nu(0,\mathcal{C})\langle W[\mathcal{C}]\rangle. \tag{2.106}$$

In scenarios where $\langle W[\mathcal{C}]\rangle \neq 0$ – which is typically observed in a SSB phase of the theory – the correlation function $\langle j^{\mu\nu}(p) W[\mathcal{C}]\rangle$ should exhibit a pole at $p=0$. This can be seen by first noting that $\langle j^{\mu\nu}(p)W[\mathcal{C}] \sim c_1(p)[p^\mu f^\nu(p,\mathcal{C}) - p^\nu f^\mu(p,\mathcal{C})]$. This is obtained by noting the index structure of the 2-form current on the left hand side and recalling that it is anti-symmetric in its indices. Now, in SSB: $\langle \delta_a \Phi(p=0) \rangle = d_1 \neq 0$, where d_1 is some constant independent of the momentum p^μ. To satisfy this we must have $c_1(p) \sim \frac{1}{p^2}$. In other words, we have at zero momentum:

$$\langle j^{\mu\nu}(p) W[\mathcal{C}] \rangle \sim \frac{p^\mu f^\nu(p, \mathcal{C}) - p^\nu f^\mu(p, \mathcal{C})}{p^2}. \tag{2.107}$$

This means that there are massless/gapless modes in the spectrum. Their existence is a direct consequence of the spontaneous breaking of the continuous 1-form symmetry.

SSB of magnetic 1-form symmetry for QED in $D = 4$

In this section we shall see that for the case of Maxwell, in $D = 4$, the Goldstone modes arising due to the spontaneous breaking of the magnetic 1-form symmetry is precisely the photon. First let us note that the corresponding conserved current $j_{\rho\sigma} \sim \epsilon_{\mu\nu\rho\sigma} f^{\mu\nu}$ creates gapless excitations from the vacuum in the broken phase as:

$$|\mathrm{G}\rangle \sim j^{\mu\nu}(x)|0\rangle, \tag{2.108}$$

where $|\mathrm{G}\rangle$ stands for the state corresponding to the Goldstone mode.

To proceed, let us note the free field expansion in terms of ladder operators satisfying $\partial^2 a^\mu = 0$ [25]:

$$a^\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{\sqrt{2|\vec{p}|}} \sum_{\lambda=1}^{4} e^\mu_\lambda(p) \left[a_\lambda(p) e^{-ipx} + a^\dagger_\lambda(p) e^{ipx} \right], \tag{2.109}$$

where $e^\mu_\lambda(p)$ are four linearly independent polarization vectors – out of which not all are physical since some of them do not satisfy $\langle 0|\partial_\mu a^\mu|0\rangle = 0$ (negative norm states) and other states have zero norms (see [49]). We exclude these to get only two physical polarisations – say corresponding to $\lambda = 1, 2$ in the above equation – which is the correct number of physical degrees of freedom for the gauge field a^μ in

[25] We work in the Feynman gauge here.

4D.[26] Then, a single photon state is created by:

$$|\lambda, \vec{p}\rangle = a^\dagger_\lambda(p)|0\rangle, \quad \lambda = 1, 2, \tag{2.110}$$

with the ladder operators satisfying:

$$[a_\lambda(p), a^\dagger_{\lambda'}(p')] = (2\pi^3)\delta_{\lambda,\lambda'}\delta^{(3)}(\vec{p} - \vec{p}'), \quad \lambda, \lambda' = 1, 2. \tag{2.111}$$

Next let us compute $f^{\mu\nu}$ from Eq. (2.109). We have:

$$f^{\mu\nu} = \frac{i}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2|\vec{p}|}}\left[e^\nu_\lambda(p)\, p^\mu \left[-a_\lambda(p)e^{-ipx} + a^\dagger_\lambda(p)e^{ipx}\right] - (\mu \leftrightarrow \nu)\right],$$

$$\text{leading to, } \langle 0|f^{\mu\nu} = \frac{i}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2|\vec{p}|}} \langle 0|a_\lambda(p)\left[e^\mu_\lambda(p)\, p^\nu - e^\nu_\lambda(p)\, p^\mu\right] e^{-ipx}. \tag{2.112}$$

With the above set up, now let us compute the matrix element between $|\lambda', \vec{p}'\rangle$ and $|\mathrm{G}\rangle \sim \epsilon_{\mu\nu\rho\sigma} f^{\mu\nu}(x)|0\rangle$

$$\begin{aligned}
\langle \mathrm{G}|\, \lambda', \vec{p}'\rangle &= \langle 0|\epsilon_{\mu\nu\rho\sigma} f^{\mu\nu}(x)|\lambda', \vec{p}'\rangle \\
&= \frac{i}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2|\vec{p}|}}\epsilon_{\mu\nu\rho\sigma}\left[e^\mu_\lambda(p)\, p^\nu - e^\nu_\lambda(p)\, p^\mu\right] e^{-ipx}\langle 0|a_\lambda(p)\, a^\dagger_{\lambda'}(p')|0\rangle \\
&= \frac{i}{(2\pi)^3}\int \frac{d^3\vec{p}}{\sqrt{2|\vec{p}|}}\epsilon_{\mu\nu\rho\sigma}\left[e^\mu_\lambda(p)\, p^\nu - e^\nu_\lambda(p)\, p^\mu\right] e^{-ipx}(2\pi^3)\delta_{\lambda,\lambda'}\delta^{(3)}(\vec{p} - \vec{p}'), \\
&= \frac{i}{\sqrt{2|\vec{p}'|}}\epsilon_{\mu\nu\rho\sigma}\left[e^\mu_{\lambda'}(p')\, p'^\nu - e^\nu_{\lambda'}(p')\, p'^\mu\right] e^{-ip'x} \neq 0,
\end{aligned} \tag{2.113}$$

where we have normalised the vacuum as: $\langle 0|0\rangle \equiv 1$.

We see from above that the Goldstone mode has non-vanishing overlap with a single photon state. Thus, we see that the Goldstone excitation is the photon itself. This also justifies, robustly, why photon is massless in $D = 4$: since it is a Goldstone mode for the spontaneous breaking of the continuous magnetic 1-form symmetry associated with the conservation of magnetic flux.

[26] In D-dimensions, a gauge field a_μ will have $D - 2$ physical degrees of freedom. Compare this to the *graviton*'s, $h_{\mu\nu}$, degrees of freedom which equals $\frac{D(D-3)}{2}$. We see that for $D = 4$, both of these have two degrees of freedom.

Alternative proof for SSB of p-form symmetries

Now we will generalise the proof presented in Sec. 2.1.3 for the case of higher-form symmetry. As before we shall follow the discussion in [40]. Let us denote the charged operator under this symmetry to be: $\mathcal{W}[\mathcal{C}]$ which has support over a p-dimensional sub-manifold $\mathcal{C}$. The expectation value of this charged operator at long distances captures the phases of the theory (see appendix K for a discussion). If at long distances we have:

$$\langle \mathcal{W}[\mathcal{C}] \rangle \sim e^{-T_{p+1}\mathrm{Area}(\mathcal{C})}, \tag{2.114}$$

where Area($\mathcal{C}$) denotes area (or, volume) of a minimal $(p+1)$-dimensional hypersurface which is enclosed by $\mathcal{C}$ – then we are in the symmetric/unbroken phase. This is the *area law* criterion.

On the other hand if we have, at long distances:

$$\langle \mathcal{W}[\mathcal{C}] \rangle \sim e^{-T_p\mathrm{Perimeter}(\mathcal{C})}, \tag{2.115}$$

where Perimeter($\mathcal{C}$) denotes the volume of the $\mathcal{C}$ itself – then we are in the SSB phase. This is the *perimeter law* criterion.

Next say we are in the SSB phase. We can re-define $\mathcal{W}[\mathcal{C}]$ as:

$$\overline{\mathcal{W}}[\mathcal{C}] \equiv e^{T_p\mathrm{Perimeter}(\mathcal{C})}\mathcal{W}[\mathcal{C}], \tag{2.116}$$

where this re-definition removes the perimeter dependence. This operator satisfies the same Ward identity as was satisfied by its precursor. So, we have:

$$(d \star j)\overline{\mathcal{W}}[\mathcal{C}] = iQ\delta_{\mathcal{C}}(x)\overline{\mathcal{W}}[\mathcal{C}] \tag{2.117}$$

where j is a $(p+1)$ form which is the conserved current for the symmetry Q denotes the charge and $\delta_{\mathcal{C}}(x)$ is a $(D-p)$ form generalised delta-functional with support on $\mathcal{C}$.[27]

Next we take $\mathcal{C}$ to be a p dimensional plane with infinite extent and consider a $(D-p)$ dimensional ball B_{D-p} with radius R that intersects $\mathcal{C}$ at a single point[28],

[27] This generalised delta functional is defined as: $\int_{\mathcal{M}_D} A_p \wedge \delta^{(D-p)}(\mathcal{C}_p) = \int_{\mathcal{C}_p} A_p$ which basically projects the integration of A_p over $\mathcal{M}_D$ to over $\mathcal{C}_p \subset \mathcal{M}_D$ on which $\delta^{(D-p)}(\mathcal{C}_p) \equiv \delta_{\mathcal{C}}(x \in \mathcal{C}_p)$ is supported on (see [50] and appendix J for more details).

[28] Recall, from the discussion in appendix J that a p-dimensional sub-manifold and $(D-p)$-

as depicted in Fig. 2.9. The boundary of B_{D-p} is a $(D-p-1)$ sphere, that is: $\partial B_{D-p} = S^{D-p-1}$. This sphere wraps $\mathcal{C}$ and R is the perpendicular distance from $\mathcal{C}$ to S^{D-p-1}. Integrating both sides of Eq. (2.117) over B_{D-p} and using Stokes' theorem we get:

$$\overline{\mathcal{W}}[\mathcal{C}] \int_{S^{D-p-1}(R)} \star j = iQ\overline{\mathcal{W}}[\mathcal{C}]. \tag{2.118}$$

Let us take the vacuum expectation value of both sides of the above expression. Since we re-defined $\mathcal{W}[\mathcal{C}]$ in a way so as to remove the perimeter dependence, by construction $\overline{\mathcal{W}}[\mathcal{C}]$ will have a constant expectation value, say d_1 independent of R. So, the correlation function on the left hand side above should behave as,

$$\langle j^i(R)\overline{\mathcal{W}}[\mathcal{C}]\rangle = iQd_1 n^i R^{D-p-1} \tag{2.119}$$

where n^i is an outward pointing normal vector on S^{D-p-1}. The above equation shows the existence of a power law correlation implying the presence of long range order in the system. In other words, we have atleast one massless mode which is precisely the Goldstone mode arising out of the spontaneous breaking of the continuous p-form symmetry.

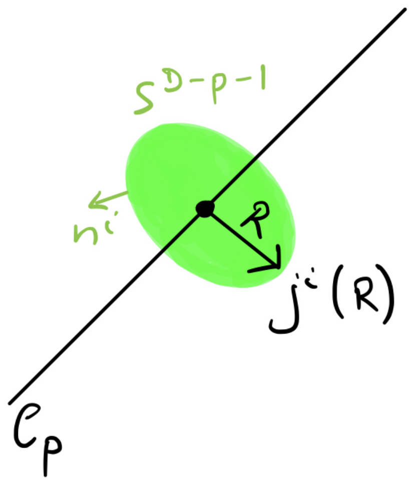

Figure 2.9: Intersection of B_{D-p} and $\mathcal{C}_p$ at a point. The boundary of B_{D-p} wis the sphere S^{D-p-1} which wraps $\mathcal{C}_p$.

dimensional sub-manifold can intersect at a point, a 0-dimensional sub-manifold, inside a D-dimensional manifold.

2.3 Anomalies

In this section, we consider an important novel possibility in the quantum theory which is absent in the classical theory, namely *anomalies*.

As alluded to already, around Eq. (2.16), an anomaly arises when the path integral measure no longer remains invariant under a classical symmetry. So in the quantum theory these transformations are not symmetries at all.

To give a concrete description of anomalies let us consider the integrated Ward identity as given in Eq. (2.17) but let us take $\langle \Pi \rangle = \langle 1 \rangle$ for simplicity. Then, we get,

$$\langle \partial_\mu j^\mu_a(x) \rangle = \langle A_a(x) \rangle. \tag{2.120}$$

Thus, the anomaly function $A_a(x)$, essentially quantifies the extent to which the current is no longer conserved. Now let us note some terminologies. When, $A_a(x)$ behaves as a fixed external field, or as a background field, then the resulting anomaly is termed as a *'t Hooft* anomaly. In this case, in some sense we can turn off this external field and still get a conserved current in the theory. On the contrary, if $A_a(x)$ behaves as a dynamical field, which can fluctuate, then we cannot simply turn it off since it now is a fluctuating dynamical operator which we have to integrate over in the path integral. In this case the resulting anomaly is termed as a *Adler-Bell-Jackiw* or ABJ *anomaly.*

A classic example is to consider a theory of massless Dirac fermions in $D = 4$. Let us couple it to electromagnetism, The action is:

$$S[a, \psi] = \int d^4x \left(-\frac{1}{4e^2} f^2 + \bar{\psi} \left(\not{\partial} - i \not{a} \right) \psi \right) \tag{2.121}$$

It is well known that, due to the fermions being massless, the chiral current: $j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$ is conserved classically. However, upon quantisation we get (see [51] for a detailed derivation of the anomaly in this theory),

$$\partial_\mu j^\mu_A = -k\, \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = k\, \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \tag{2.122}$$

where $k \equiv \frac{1}{16\pi^2}$ is the anomaly coefficient. In forms notation the above equation becomes: $d \star j_A \sim k\, f \wedge f \sim k\, (\star J \wedge \star J)$, with J being the conserved 2-form current associated with the conservation of magnetic flux. Thus, we have an anomaly.

Now let us freeze the electromagnetic sector, or let us not integrate over the gauge field a^μ in the path integral. Then, we can drop the Maxwell term in the above action. In this case, $F \wedge F$ on the right hand side of Eq. (2.122) behaves as a fixed external source which can be turned off to still get a conserved current. Conferring to our nomenclature: this is precisely a 't Hooft anomaly.

Let us contrast this case to the scenario when a^μ is no more a background gauge field but rather is a dynamical one and is to be integrated over the path integral. In this case, there is no way to get a conserved current since we cannot turn off a dynamical operator, so we get the famous ABJ anomaly [52].

2.3.1 Helicity and Linking

Next let us see what physical insight we can get from Eq. (2.122) in the case of an ABJ anomaly. Let us consider an integrated version of Eq. (2.122) over some 3D manifold $\mathcal{M}_3$,

$$\frac{\Delta Q_A}{\Delta t} = \int_{\mathcal{M}_3} d^3x \, \partial_t j_A^0 = k \, \Delta \left(\int_{\mathcal{M}_3} a \wedge da \right), \tag{2.123}$$

where Q_A is the axial charge defined as $Q_A \equiv \int_{\mathcal{M}_3} d^3x \, j_A^0$ and $\Delta\Big(\int \dots \Big)$ in the last equality denotes the change in the integral under consideration. The right hand side, in the last equality, of the equation is defined as the change in helicity associated with magnetic field lines. It is well known that helicity is a conserved quantity in ideal systems like in ideal charged fluids [53–55]. Helicity is related to the linking of magnetic field lines (see [53] for details, and appendix J for a brief overview). Now for the left hand side, we know that this integral counts the *chiral* particle number in the system. So, the content of Eq. (2.123) is: we can change the chirality or, helicity of our system provided we change the linking of closed magnetic loops along with it, see Fig. 2.10.

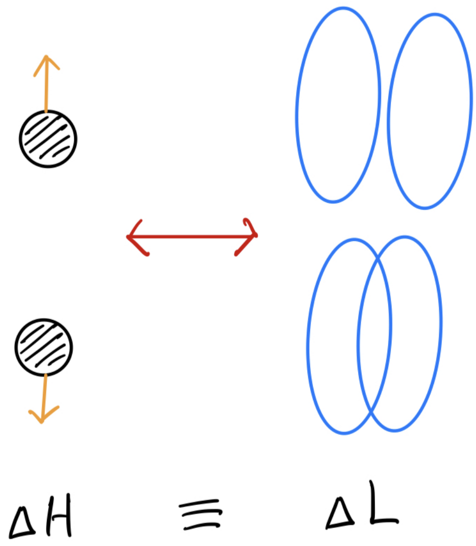

Figure 2.10: On the left hand side we have change in chirality/helicity, denoted by ΔH, and on the right hand side we have change in the linking of magnetic loops, denoted by ΔL.

So, this is the physical insight captured by the anomaly equation in Eq. (2.122). We refer the reader to Sec. 3.3 for applications of anomalies in hydrodynamics.

2.4 Non-invertible symmetries

Let us recall Wigner's theorem. It states that conventional symmetries or 0-form symmetries in quantum mechanics are implemented by (anti-)unitary operators. Such operators were constructed in for example Eq. (2.13) and being unitary these have inverses. In higher spacetime dimensions and for a general class of symmetries, this may not be always true. In particular, symmetries can be non-invertible – they are implemented by conserved operators without an inverse. However, these *non-invertible symmetries* lead to new conservation laws, selection rules, and dynamical constraints on RG flows. Next let us discuss one such non-invertible symmetry in the very familiar setting of QED in $D = 4$ couple to massless Dirac fermion.

2.4.1 Non-invertible chiral symmetry in $D = 4$

Note that, as discussed in Sec. 2.3, the chiral current: $j^\mu_{\rm A} = \bar{\psi}\gamma_5\gamma^\mu\psi$ obeys the ABJ anomaly equation,

$$d \star j^{\rm A} = \frac{1}{4\pi^2} f \wedge f, \tag{2.124}$$

which leads to the non-conservation of j_A at the quantum level. Does this mean there is no hope to have any symmetry principle here? It turns out (see [37, 38]) in this case there still exist a non-invertible symmetry. Let us see below how this is constructed and how it works out.

Let us first try to get a conserved current out of the anomaly equation. Consider the following re-definition of the chiral current,

$$\star\tilde{j}_A = \star j_A - \frac{1}{4\pi^2} a \wedge da, \tag{2.125}$$

where due to the anomaly we have the following conservation equation: $d \star \tilde{j}_A = 0$. However, this re-defined current is not gauge-invariant. Normally when such gauge non-invariant terms are there one tends to integrate these terms to make them gauge-invariant. So, consider the following operator – in flat spacetime – constructed out of the redefined current,

$$\hat{U}_\alpha(\mathbb{R}^3) = \exp\left[i\alpha \int_{\mathbb{R}^3} \star\tilde{j}_A\right] = \exp\left[i\alpha \int_{\mathbb{R}^3} \left(\star j_A - \frac{1}{4\pi^2} a \wedge da\right)\right] \tag{2.126}$$

This is a nice topological and gauge-invariant operator in flat spacetime. So, in some sense $U(1)_\mathrm{A}$ is still a symmetry in flat spacetime. This is reflected in the helicity conservation law regarding the scattering amplitudes of electrons and positrons – a selection rule that follows from the $U(1)_A$ symmetry (see Chapter 8 of [56]).

However, its topological nature is only true in flat spacetime as for generic spacetimes[29] the parameter α – which is the level for the Chern-Simons term $a \wedge da$ – is not properly quantised to make $\hat{U}_\alpha(\mathcal{M}_3)$ gauge-invariant under large gauge transformations and hence it is generally not well-defined (see appendix L for details regarding quantisation of Chern-Simons level).

Now let us provide an alternative viewpoint on the ABJ anomaly. We will see that the chiral symmetry is not totally broken by the ABJ anomaly. Rather, it turns into a non-invertible global symmetry.

[29] Even in flat spacetimes, if we have a magnetic monopole configuration, then it creates nontrivial topology, again leading to the violation of the $U(1)_\mathrm{A}$ symmetry.

2.4.2 Insights from fractional quantum hall effect

Following the construction in [37], let us focus on the case where the chiral rotation angle, α, is a fraction of the form:

$$\alpha = \frac{2\pi}{2N} \tag{2.127}$$

where N is any positive integer. So, under this we have: $\psi \to e^{\frac{i}{N}\gamma_5}\psi$. The naive operator $\hat{U}_{\frac{2\pi}{2N}}(\mathcal{M}_3)$ is still not gauge-invariant, because the Chern-Simons term is now of the form:

$$\frac{i}{4\pi N}\int_{\mathcal{M}_3} a \wedge da \tag{2.128}$$

which has a fractional level $1/N$.

Note that, condensed-matter theorists can immediately recognise the above CS term as the effective response action for the fractional quantum Hall (FQH) state in $1+2$D with filling fraction $\nu = 1/N$ (see [57]).

However, as we saw above, this CS term is not gauge invariant. This begs the following question: How can a realistic physical system – the fraction quantum hall system – be described by an action which is not gauge invariant? Again we will take inspiration from condensed matter to remedy this situation. To be more precise, the gauge-invariant action for the fractional quantum hall effect is given as:

$$\int_{\mathcal{M}_3} \frac{iN}{4\pi} A \wedge dA + \frac{i}{2\pi} A \wedge da\,, \tag{2.129}$$

where an additional dynamical $U(1)$ gauge field A has been introduced. This new action (2.129) is gauge-invariant because both the levels are now properly quantised (see appendix L).

Naively, one is tempted to integrate out A in (2.129) and find $A = -\frac{a}{N}$. Substituting this into (2.129) returns (2.128). Though this manipulation is not globally correct because both a and A are properly normalized gauge fields with quantized magnetic fluxes, i.e., $\oint da, \oint dA \in 2\pi\mathbb{Z}$, yet it provides a heuristic understanding of the relation between them. In any case, in the context of the FQH effect, (2.129) is the precise, gauge-invariant effective action.

Let us now return to our original setting: QED in $D = 4$. Motivated by the above discussion, let us consider the following operator [37, 38]:

$$\mathcal{D}_{\frac{1}{N}}(\mathcal{M}_3) = \int_{\mathcal{M}_3} [DA]_{\mathcal{M}_3} \exp\left[\int_{\mathcal{M}_3} i\left(\frac{2\pi}{2N} \star j_A + \frac{N}{4\pi} A \wedge dA + \frac{1}{2\pi} A \wedge da\right)\right], \tag{2.130}$$

where, we have introduced a new degree of freedom: A. It is an auxiliary 1-form gauge field. It only lives on the 3-manifold $\mathcal{M}_3$. Hence, it does not introduce any new asymptotic states in the theory and the theory is still QED in $D = 4$. Note that, the bulk physics away from the 3-manifold remains the same. Referring to our previous nomenclature, $\mathcal{D}_{\frac{1}{N}}(\mathcal{M}_3)$ can be viewed as a defect when $\mathcal{M}_3$ extends along the time direction. This defect operator does not have an inverse since, $\mathcal{D}_{\frac{1}{N}} \times \mathcal{D}_{-\frac{1}{N}} \neq \mathbb{1}$ (see Fig. (2.11)). To see this note that, $\mathcal{D}^{\dagger}_{\frac{1}{N}} = \mathcal{D}_{-\frac{1}{N}}$. Though naively one might think that the rational axial phase rotation with $\alpha = \frac{1}{N}$ can be undone by the negative of it, that is, for $\alpha = -\frac{1}{N}$, we show below that this is not the case in the sense that the operator $\mathcal{D}_{\frac{1}{N}}$ does not have an inverse. Consider the following multiplication,

$$\begin{aligned}\mathcal{D}_{\frac{1}{N}} \times \mathcal{D}_{-\frac{1}{N}} &= \int [DA]_{\mathcal{M}_3} [D\bar{A}]_{\mathcal{M}_3} \exp\left[\int_{\mathcal{M}_3} \left(\frac{iN}{4\pi} A \wedge dA - \frac{iN}{4\pi} \bar{A} \wedge \bar{A} + \frac{i}{2\pi}(A - \bar{A}) \wedge da\right)\right], \\ &\neq \mathbb{1},\end{aligned} \tag{2.131}$$

which shows that $\mathcal{D}_{\frac{1}{N}}$ is not unitary. An intuitive way to understand this is that Eq. (2.130) takes the form of an integral of unitaries, which is not unitary (for more details see [27]).

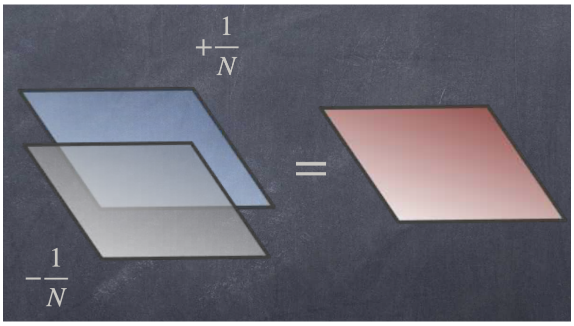

Figure 2.11: Non-invertibility of defect operators as $\mathcal{D}_{\frac{1}{N}}(\mathcal{M}_3) \times \mathcal{D}_{-\frac{1}{N}}(\mathcal{M}_3) \neq \mathbb{1}$.

The above operator $\mathcal{D}_{\frac{1}{N}}$ is gauge-invariant because both Chern-Simons terms have properly quantised levels. It can also be shown to be topological[30].

Now let us discuss about its action on charged operators. Let us take $\mathcal{M}_3 = S^3$ for simplicity. Now just like a usual 0-form defect operator, when this defect operator is collapsed onto a corresponding charged local operator, it also performs an axial

[30]Showing this is rather involved and uses a procedure called *half-gauging* and hence we do not review it here. Instead we refer the readers to the original articles: [37, 38]

rotation by an angle $\alpha = \frac{1}{N}$:

$$U_\alpha(S^3)\mathcal{O}(x) = e^{i\alpha Q}\mathcal{O}(x) \tag{2.132}$$

where S^3 wraps x.

So, the continuous 0-form axial symmetry is broken by the ABJ anomaly to a 0-form non-invertible symmetry where the defect opearators as in Eq. (2.130) are labelled by rationals.

In the construction above we only looked at $\alpha = \frac{1}{N}$. References [37,38] generalise the above construction for $\alpha = \frac{p}{N}$ with $\gcd(p, N) = 1$. Interestingly, [41,58] take a complimentary viewpoint to the above construction. Instead of a fractional rotation angle α, they consider a continuous angle and introduce a compact scalar field θ which is the additional degree of freedom living only on the defect manifold. This allows for the construction of a locally conserved current and to prove a Goldstone's theorem for this non-invertible symmetry.

Let us conclude this chapter by noting that such non-invertible symmetries are abundantly present in lower dimensional setting where they are better understood. For instance, any diagonally-invariant 2D RCFT admits *topological defect lines* (TDLs) associated to each primary operator [59]. These TDLs are also called *Verlinde lines* in this setting. These Verlinde lines can be both invertible and non-invertible. Consider, 2D Ising model which is the $\mathcal{M}(4,3)$ minimal model. It has a Verlinde line referred to as the N-line associated with the spin field primary. This Verlinde line is a non-invertible line which is responsible for the famous *Kramers-Wannier* duality in the Ising model [60].

CHAPTER 3

Hydrodynamics

Let us say we are interested in the time-dependent, long distance and late time physics of a strongly interacting quantum field theory at finite temperature. A natural question to ask is: what are the degrees of freedom we need to keep track of to understand this low energy dynamics? To answer this question we need to understand on the evolution time scales of two kinds of quantities: non-conserved and conserved. Most quantities, which are non-conserved, evolve very rapidly, say with a time scale of the order of microscopic scales which in this case could be the temperature: $\omega_{\text{typical}} \sim \mathcal{O}(T)$. On the contrary, imagine creating a lump of conserved charges in the system say, a lump of $U(1)$ charges. Let us say this is represented by the blue blobs in the solution given in Fig. 3.1. This lump of conserved charges will evolve rather very slowly owing to their conservation: $\omega_{\text{conserved}} \sim \mathcal{O}(L^{-1})$, where L is the length scale associated with a blue blob.

Figure 3.1: Evolution of conserved charges in a system.

So, conserved quantities are crucial to understanding the late time, low energy dynamics of a system. These will be natural degrees of freedom of our low-energy effective field theory.

Let us begin by considering a system which has conserved stress-energy tensor, $T^{\mu\nu}$ and also enjoys a global $U(1)$ symmetry with j^μ as the associated conserved current. The conservation equations are,

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu j^\mu = 0, \tag{3.1}$$

Given these conserved quantities, there exists a well-established framework to construct the low-energy effective field theory, called *hydrodynamics*. First of all, note that, usual hydrodynamic variables are: T – temperature, $u^\mu = (u^t, \vec{u})$ – fluid velocity, normalised as $u^\mu u_\mu = -1$, and μ – chemical potential. Next we have constitutive relations, which express the above microscopic conserved quantities as functions of the above hydrodynamic variables. In the thermal equilibrium state of the system, the fluid variables are constant functions of the spacetime. Hydrodynamics is concerned with small fluctuations of the system around the thermal equilibrium, and hence we consider derivative expansions of the hydrodynamic variables. That is, we write the constitutive relations order-by-order in derivatives of T, u^μ, μ, in a specified derivative scheme. Due to the smallness of the fluctuations, zeroth-order term is greater than the first-order term and so on. The constitutive relations are[1]:

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu} + \mathcal{O}(\partial T, \partial u^\mu, \partial\mu) \qquad j^\mu = n u^\mu + \mathcal{O}(\partial T, \partial u^\mu, \partial\mu), \tag{3.2}$$

where ε is the energy density, p is the pressure and n is the charge density. When no deviations from thermal equilibrium are allowed, we are in the regime of *ideal hydrodynamics*. The leading order pieces in the above relations are the constitutive relations for *ideal hydrodynamics*. Conventionally, for *dissipative hydrodynamics*, one considers first-order derivative terms in the gradient expansion and the coefficients appearing here are called *transport coefficients*. Some examples are viscosities, resistivities, conductivities, etc. These transport coefficients have to be computed from the microscopics[2] and in this way dissipation can be interpreted as a trans-

[1]These so called constitutive relations are obtained by first decomposing the relevant tensors into irreducible tensor structures and then using hydrodynamic assumptions to express the coefficients of these irreducible tensor structures as functions of the fluid variables.

[2]These transport coefficients are computed using what are called *Kubo formula* (see appendix I) which relate these coefficients to correlators of operators in the microscopic theory.

fer of energy from the IR to the UV degrees of freedom through these transport coefficients.

Now the central idea of hydrodynamics is to use the conservation equations as equations of motion to study the evolution of the hydrodynamic variables. In this way, we can study perturbations away from equilibrium. Consider the following perturbations around the respective equilibrium values[3]: $T_0 \to T_0 + \delta T$, $\vec{u}_0 \to \delta\vec{u}_0$ and $\mu_0 \to \mu_0 + \delta\mu$. Now we consider plane wave solutions of the form, $\delta T, \delta\vec{u}, \delta\mu \sim e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. Plugging this into the Eq. (3.1) and going to momentum space gives us algebraic equations of a degenerate system of equations which can be solved by demanding vanishing of its determinant to obtain dispersion relations of the form, $f(\omega, \vec{k}) = 0$. Normally, these are of the following two kinds:

$$\omega(\vec{k}) = \pm v_s|\vec{k}| + \mathcal{O}(k^2) \qquad \omega(\vec{k}) = -iDk^2, \tag{3.3}$$

where the first mode is the sound mode with v_s being the speed of sound and the second mode is the diffusive mode with D being the diffusion constant (see Fig. 3.2).

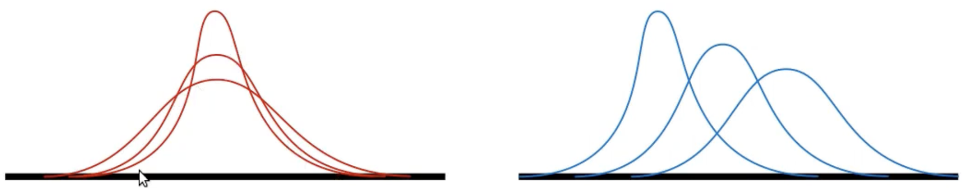

Figure 3.2: Two typical modes in hydrodynamics. The left hand side is the diffusive mode where a lump of charge slowly diffuses over time. The right hand side is a sound mode where the lump propagates in time and as it propagates it undergoes diffusion.

Based on the above discussion, we see that, hydrodynamics is a genuine effective field theory in the sense that it can be improved by an order-by-order in derivative expansion of the fluid variables. Furthermore, it is entirely dictated by the global symmetries of the system. The universality of hydrodynamics is based on the fact that, in the long wavelength limit, the microscopic details are washed out and the effective theory is fully described by the equations of state and a few parameters in

[3]Equilibrium values are denoted by the subscript 0. Note that, in equilibrium, the fluid velocity is chosen to be $\vec{u}_0 = 0$, that is, the fluid is at rest.

the gradient expansion: the transport coefficients. The validity regime of hydrodynamics is in time scales much larger than the mean free time and in length scales much larger than the mean free path.

For more details we refer the reader to [12] which is an excellent review on this subject.

In the next section we shall briefly review ordinary non-anomalous magnetohydrodynamics and then move on to the modern formulation of non-anomalous magnetohydrodynamics in terms of higher-form symmetries [30].

3.1 Brief review of ordinary MHD

Traditionally, if we couple the conservation equations given in Eq. (3.1) to Maxwell equations in some perturbative scheme then the resulting equations describe magnetohydrodynamics. The equations are:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}, \qquad \frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = -\sigma \vec{E} + 8k\mu_A \vec{B}, \tag{3.4}$$

where k is the anomaly coefficient, μ_A is the chiral chemical potential and the term $8k\mu_A\vec{B}$ is the anomaly-induced term which we can neglect in this chapter where we are focusing only on non-anomalous magnetohydrodynamics. $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively. σ denotes the conductivity.

Now let us analyse the electric and the magnetic fields. Note that, in a conducting medium, the electric field is short-ranged and gets screened. This is the phenomenon of *Debye screening*. Let us consider the local chemical potential, $\mu(t, \vec{x})$. Here this can be thought of as being the electric potential[4]. Hence, in equilibrium we have the electric field as gradient of the chemical potential, $E_\lambda \sim \partial_\lambda \mu \sim \mathcal{O}(\partial)$. Thus, this is how Debye screening is manifested that is in a gradient expansion the electric field doesn't appear at zeroth order in derivative but rather in first order in derivatives of μ. As far as the magnetic field is concerned it can be as larger as we want, that is, $B \sim \mathcal{O}(1)$. So, in this case, conventionally, the magnetic field serves as another hydrodynamic variable[5].

[4]This is because, the usual chemical potential is defined as: $\mu = \left(\frac{\partial E}{\partial N}\right)_{V,S}$ which comes from the first law of thermodynamics: $dE = TdS - pdV + \mu dN$. The definition suggests that, in equilibrium, at constant entropy and volume, μ is the energy cost reuqired to add an infinitesimal particle to the system. Now if we are dealing with electric charges, then $N = Q$ and μ is like the electric potential.

[5]Let us contrast this case to the case of an insulator where there are only dipoles but no

Now let us consider the familiar example of a relativistic system – e.g. an interacting complex scalar field – with an conventional $U(1)$ global symmetry. A microscopic Lagrangian for the system might take the form:

$$S = \int d^4x \left[-(\partial^\mu \phi)^*(\partial_\mu \phi) + V(\phi^* \phi)\right], \tag{3.5}$$

where we have frozen the electromagnetic sector and are only focusing on the charged matter sector. The dynamical case will be studied in the next section.

Within the hydrodynamic expansion, we can express the current operator $j^\mu(x)$ for the symmetry in thermal equilibirum to leading order in derivatives (and linear order in the chemical potential) as follows:

$$j^t = \chi\mu + \cdots \qquad j^i = -\sigma\left(\partial_i \mu - E_{i,\mathrm{ext}}\right) + \cdots \ . \tag{3.6}$$

Let us try to motivate these constitutive relations from elementary electrodynamics. First of all, we neglect stress-energy fluctuations thereby freezing the temperature T [6]. So, in thermal equilibrium $\mu(t, \vec{x})$ – the chemical potential is the basic degree of freedom. χ – the charge susceptibility – is a thermodynamic quantity. The equation for the time component of the current is just the definition of static charged susceptibility. The expression for j^i in terms of gradients of the chemical potential is called the Fick's law which states that the flow of current is proportional but opposite to the gradient of the of chemical potential. $E_{i,\mathrm{ext}}$ is an external applied electric field, and σ is a transport coefficient called the *conductivity*, which determines the amount of current flow in response to the applied electric field. This is the familiar Ohm's law.

Since σ determines the amount of current flow in response to an applied electric field. This leads to the Kubo formula (see appendix I) which determines σ in terms of a real-time current correlation function:

$$\sigma = \lim_{\omega \to 0}\left(\frac{1}{-i\omega} G^R_{j^x, j^x}(\omega, \vec{k} = 0)\right) \tag{3.7}$$

Furthermore, σ also determines the diffusion constant of the system: imposing

electric charges. Here, μ is no longer a relevant hydrodynamic variable. If we are interested in the hydrodynamics of such a system subject to external electric and magnetic fields, we are free to choose $B \sim \mathcal{O}(1)$ and $E \sim \mathcal{O}(1)$ in the derivative expansion, as due to the absence of electric charges the electric field is no more screened.

[6] As the energy density is a function of the temperature so freezing it implies freezing the temperature.

current conservation $\partial_\mu j^\mu = 0$ and setting $E_{i,\text{ext}} = 0$, we find the following dispersion relation for the diffusion of charge:

$$\mu(t,x) \sim \mu_0 e^{-i\omega t + ikx} \qquad \omega = -iDk^2 \qquad D = \frac{\sigma}{\chi} \ . \tag{3.8}$$

It is a non-trivial statement about hydrodynamics that the quantity obtained from the Kubo formula in Eq. (3.7) determines real-time dynamics as in Eq. (3.8).

3.2 Relativistic MHD and higher-form symmetry

We now turn to the main question of interest, the description of relativistic magnetohydrodynamics in terms of symmetry principles. For an illustrative microscopic description consider the quantum field theory of Maxwell electrodynamics in four dimensions, coupled to electrically charged matter, as described e.g. by the following action:

$$S = \int d^4x \left[-(D^\mu \phi)^* (D_\mu \phi) + V(\phi) - \frac{1}{4e^2} f^{\mu\nu} f_{\mu\nu} \right] \ , \tag{3.9}$$

with $D_\mu \phi = \partial_\mu \phi - ie\, a_\mu \phi$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$.

Note that if we place this theory at finite temperature, the degrees of freedom of a thermally excited plasma are electrically charged particles, interacting via electric and magnetic fields, see Fig. 3.3. We would like to understand the universal hydrodynamic theory describing the infrared finite-temperature physics. This framework is usually called relativistic magnetohydodynamics (see e.g. [61] for a review). As discussed in the previous section, it is traditionally constructed by considering Maxwell's equations coupled to a charge current that is assumed to be in thermal equilibirum following relations similar to Eq. (3.6).

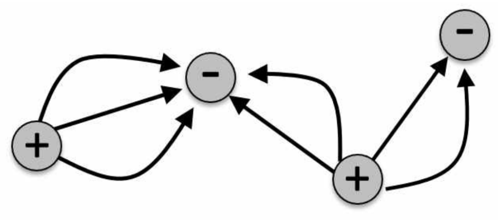

Figure 3.3: Interaction of charged particles in a plasma at fintite temperature.

Note however that such a construction relies on knowledge of the microscopic equations of motion, and implicitly requires the existence of a separation between the electromagnetic degrees of freedom a_μ and the thermalized ϕ degrees of freedom.

Such a separation may be well-justified if the electromagnetic coupling e is weak; however in this work we would like to study systems where e is generally $\mathcal{O}(1)$, and is not parametrically small in any sense.

More generally, it would be conceptually satisfying to have a construction of MHD that relies only on global symmetries and does not require any access to microscopic degrees of freedom such as a_μ, or treating the magnetic field as a hydrodynamic variable. Such a formulation is made possible by the understanding of higher-form global symmetries [21] in electrodynamics. Indeed, as described in the previous chapter, the global symmetry of Maxwell electrodynamics is a 1-form symmetry associated with the conservation of magnetic flux lines. This global symmetry results in a conserved current $J^{\mu\nu}$:

$$\partial_\mu J^{\mu\nu} = 0 \qquad J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} f_{\alpha\beta}. \tag{3.10}$$

This 1-form symmetry is the true global symmetry of electromagnetism, and is a useful starting point for an understanding of the phases of electrodynamics and will serve as an invaluable guiding principle to organise the low energy effective field theory.

In particular, it was shown in [30] that one indeed can reformulate MHD using this higher-form symmetry – i.e. magnetic flux conservation – as the organizing principle, resulting in a framework constrained only by effective theory and global symmetries. Here we present only the results of the construction, directing readers to [30] for a detailed discussion.

It is useful to consider coupling an external 2-form source $b_{\mu\nu}$ to (3.9) as

$$S \to S + \int d^4x\, b_{\mu\nu} J^{\mu\nu} \ . \tag{3.11}$$

Now let us write the 2-form current given in Eq. (3.10) as, $J^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho a_\sigma$ where a_λ is the gauge potential of electromagnetism. Next we put this expression of $J^{\mu\nu}$ in Eq. (3.12) and do an integration by parts so that the derivative now hits the $b_{\mu\nu}$. We get,

$$\Delta S[b] = \int d^4x\, a_\sigma \epsilon^{\mu\nu\rho\sigma}\partial_\rho b_{\mu\nu} = \int d^4x\, a_\sigma j^\sigma_{\text{ext}}, \tag{3.12}$$

where $j^\sigma_{\text{ext}} \equiv \epsilon^{\mu\nu\rho\sigma}\partial_\rho b_{\mu\nu}$ is interpreted as an external current such that the field strength db behaves like an external background electric charge density to which the system responds.

Now from (3.10) we see that in terms of the conventional electric and magnetic fields, we have

$$J^{ti} = B^i \qquad J^{ij} = \epsilon^{ijk} E^k \tag{3.13}$$

Next we consider fluctuations about the thermal state with no background magnetic field as in [62]. So, we can expand the magnetic flux current in constitutive relations in a higher-form analogue of Eq. (3.6), that is we add an extra index to the constitutive relations in Eq. (3.6):

$$J^{ti} = \Xi\mu^i \qquad J^{ij} = -\rho\left(\partial_i\mu_j - \partial_j\mu_i + (db)_{0ij}\right), \tag{3.14}$$

where we have worked only to linear order in the magnetic field, and have ignored the stress-energy tensor; the full construction can be found in [30]. The notation here has been picked to highlight the parallel with conventional hydrodynamics in (3.6), and we now unpack it again using concepts from elementary electrodynamics.

Here μ_i is a vector-valued chemical potential which can be thought of as the thermodynamic variable conjugate to magnetic flux. Ξ is a thermodynamic parameter that relates the conserved density B^i to its chemical potential: in conventional language it is the magnetic permeability μ [7].

b is an applied source as in (3.12). As shown above, db can be understood as an applied external electric charge current, $j^{\mu}_{\text{ext}} = \epsilon^{\mu\alpha\beta\gamma}\partial_\alpha b_{\beta\gamma}$, often called the "free charge current" in elementary electrodynamics.

Finally, ρ is a transport coefficient – it is precisely the *resistivity*. Now let us motivate this identification, for more details see [30]. Let us compare the second equation in Eq. (3.14) to the the second equation in Eq. (3.6). These two equations are reciprocal of each other in the sense that, in the former case: $\vec{j} \sim \sigma\vec{E}_{\text{ext}}$ and in the latter case: $\vec{E} \sim \rho\vec{j}_{\text{ext}}$. Thus, when interpreted as a response-source equation (see appendix I), the former implies that σ is the transport coefficient which determines the current flow response to applied external electric field – this is precisely the definition of conductivity. Transport coefficient ρ determines the response of the electric field to an applied external current density – this is precisely the definition of resistivity. However, as noted previously, the latter scenario is based only a symmetry-based approach but the former scenario makes sense in some sort of a weak coupling regime. This suggests that the Kubo formula of resistivity (see below)

[7] In fact, in elementary electrodynamics μ^i is precisely the field often called $\mathbf{H}^i$, i.e. the object whose curl is given by the free charge current. Also, recall, $\mathbf{B} = (\text{mag. permeability}) \times \mathbf{H}$. Here we choose to use the notation μ^i here to highlight the analogy with a conventional chemical potential.

in terms of correlations of the electric field is a universal formula unlike the Kubo formula for conductivity. This idea was noted in [30] and we explore it further in a recent work by computing both the above transport coefficients from classical lattice simulations [63].

Since ρ determines the response of the electric field to an applied external current density. Indeed, it can be obtained from the following Kubo formula (see [64]):

$$\rho = \lim_{\omega \to 0} \left(\frac{1}{-i\omega} G^R_{J^{xy}, J^{xy}}(\omega, \vec{k} = 0) \right) \tag{3.15}$$

Note that this is a correlation function for the electric field, as we have $J^{xy} = E^z$. Also, if we consider the equation of motion $\partial_\mu J^{\mu\nu} = 0$, then (setting the source $db = 0$) we find the following diffusive dispersion relation for the magnetic field

$$B_z(t, x) \sim B_0 e^{-i\omega t + ikx} \qquad \omega = -iDk^2 \qquad D = \frac{\rho}{\Xi}. \tag{3.16}$$

This is the familiar expression for magnetic diffusion in a plasma. As already noted in the previous section, this is a non-trivial statement about hydrodynamics that the quantity obtained from the Kubo formula in (3.15) determines real-time dynamics as in (3.16).

3.2.1 Phase structure of dynamical electromagnetism at finite temperature

In this section we shall describe the symmetries of non-anomalous dynamical electromagnetism at finite temperature. These ideas are mostly well-known, the precise explanation we present in terms of higher-form symmetry was explained in [65], following work in the hydrodynamic context from [33–35].

For this let us put the scalar QED action given in Eq. (3.9) at finite temperature that is we take the background spacetime to be of the form $\mathcal{M} \equiv S^1 \times \mathbb{R}^3$.

Next let us integrate out the matter. This will result in an effective action for $a_\mu = (a_\tau, a_i)$ where the Maxwell part will remain as it is with only the temporal component, denoted by Euclidean time τ, now running along the compactified thermal cycle with radius β.

Now let us ask: what are the symmetries of this effective action? To answer this, first let us obtain the effective action.

$$\int \mathcal{D}a \; \exp(i\Gamma[a]) = \int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^* \; \exp\left[i\int d^4x \left(-\frac{1}{4e^2}f^2 - \phi^*(D^2+m^2)\phi\right)\right], \quad (3.17)$$

where $f^2 = f^{\mu\nu}f_{\mu\nu}$ and m is the mass of the complex scalar field. Now since the original action is quadratic in ϕ we can evaluate the path integral exactly using the formula,

$$\int \mathcal{D}\phi\mathcal{D}\phi^* \; \exp\left[i\int d^4x \, (\phi^* M\phi + JM)\right] = \frac{\mathcal{N}}{\det M}\exp(iJM^{-1}J), \quad (3.18)$$

where $\mathcal{N}$ is a normalisation constant, M denotes the differential operator and J is an external source. So, for the scalar QED Lagrangian we find,

$$\int \mathcal{D}a \; \exp(i\Gamma[a]) = \mathcal{N}\int \mathcal{D}a \; \exp\left[i\int d^4x \left(-\frac{1}{4e^2}f^2\right)\right]\frac{1}{\det(-D^2-m^2)}. \quad (3.19)$$

The above equation will be satisfied if,

$$\exp\left[i\Gamma[a] + i\int d^4x \, \frac{f^2}{4e^2}\right] = \frac{\mathcal{N}}{\det(-D^2-m^2)},$$

$$\text{implying,} \;\; i\Gamma[a] + i\int d^4x \, \frac{f^2}{4e^2} = -\ln[\det(-D^2-m^2)] = -\mathrm{Tr}[\ln(-D^2-m^2)], \quad (3.20)$$

For simplicity let us just focus on evaluating the time component in the above determinant and consider: $a^\mu = (a_\tau, 0)$. Using the worldline formulation of QFT (see [66]), the above effective action can be written in terms of a path integral over paths as,

$$\Gamma[a] = -\ln[\det(-D^2-m^2)] = \int [dX] \; \exp(-mL(X)) + i\int_X a \quad (3.21)$$

where X denote a particle worldline and $L(x)$ is its length function. To evaluate this path integral we have to add particle and anti-particle contributions running along the directions of τ and opposite to τ respectively (see Fig. 3.4).

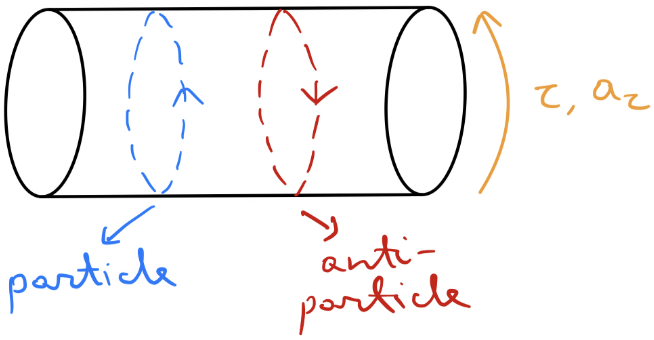

Figure 3.4: Particles and anti-particles running along and opposite to τ.

We get,

$$\Gamma[a]_\tau = e^{-m\beta + i a_\tau \beta} + e^{-m\beta - i a_\tau \beta} = e^{-m\beta}\cos(\beta a_\tau), \tag{3.22}$$

where the subscript τ denotes the temporal contribution to the effective action. Thus, the full effectve action becomes,

$$S[a] = \beta \int d^3x \left(c_1 (da_\tau)^2 + c_2 f_{ij} f^{ij} + c_3 \cos(\beta a_\tau) \right) \tag{3.23}$$

where the exterior derivative d is now for the three non-compact spatial directions, $f_{ij} = \partial_i a_j - \partial_j a_i$ and $i, j = 1, 2, 3$ denotes the spatial directions.

As we saw above, the cosine term comes from electric worldlines wrapping the finite temperature direction. This captures the physics of Debye screening and we can define the Debye mass as $c_3 \equiv e^{-m\beta}$. Note that a_τ is a periodic variable with periodicity β^{-1}. The 3D photon remains massless, while the time component of the photon picks up a mass due to the coupling to charged particles moving around the Euclidean time circle. This is precisely the Debye mass above. Now in 3D, the Poincaré dual of a 1-form is a scalar (see appendix B). So, we get $a_i \leftrightarrow \psi$ where ψ is a massless scalar and is precisely the Goldstone mode. The magnetic field is now $\mathcal{B}_i = \partial_i \psi$. Thus, ψ is the magnetic dual of the 3D photon. Being a Goldstone mode, the dual effective action, written in terms of ψ, will contain only derivative of ψ implying a usual shift symmetry for this mode: $\psi \to \psi + \Lambda_\tau(x^i)$.

Now let us see how $J^{\mu\nu}$ decomposes in the dimensionally reduced theory on

$S^1 \times \mathbb{R}^3$. On $\mathbb{R}^3$ we have,

$$U(1)^{(0)} \text{ 0-form symmetry } \to J^{i\tau} \to \mathcal{B}^i = J^{\tau i} \text{ is magnetic 3-vector,} \tag{3.24}$$

$$U(1)^{(1)} \text{ 1-form symmetry } \to J^{ij} \to \mathcal{E}^i = \frac{1}{2}\epsilon^{ijk} J_{jk} \text{ is electric 3-vector.} \tag{3.25}$$

Also, $J^{\tau i} = \epsilon^{ijk}\partial_j a_k$ and $J^{ij} = \epsilon^{ijk}\partial_k a_\tau$. Now, in equilibrium, to leading order in derivatives, $\mathcal{E}^i$ vanishes, due to Debye screening. As discussed above, the $U(1)^{(0)}$ symmetry is *spontaneously broken* in the normal phase[8] of the theory (see also [65]).

3.3 Anomalies in hydrodynamics

Now let us discuss some important applications of anomalies [67,68]. From Eq. (2.122), we have,

$$\frac{\partial \rho_A}{\partial t} = -8k\,\vec{E}\cdot\vec{B} \qquad (\text{as, } f_{\mu\nu}\tilde{f}^{\mu\nu} = 4\vec{E}\cdot\vec{B}) \tag{3.26}$$

where we have taken $\vec{j}_A = 0$ and ρ_A is the chiral charge density. Above is the local change rate in chirality. Chirality is defined in the following way: if a particle has its spin and momentum aligned then it is called a *right-handed* particle while if its spin and momentum are anti-aligned then it is called a *left-handed* particle.

Let us consider a Dirac sea of massless fermions with charges e and $-e$ in the presence of a background magnetic field, $\vec{B}$. In absence of external fields, chirality is conserved and we have two disconnected Fermi surfaces, consisting of left-handed and right-handed fermions. The chirality of the fermions can be changed by adiabatically by switching on external fields, in particular an electric field parallel to a magnetic field.

Consider Fig. 3.5. On the left image the system is chirally balanced and so the current generated by movement of right-handed particles along $\vec{B}$ is cancelled by the current generated by movement of left-handed particles opposite to $\vec{B}$.

Now let us consider turning on an electric field $\vec{E}$ parallel to $\vec{B}$. This would now lead to a change in chirality. The positive charges will move along $\vec{E}$ and the negative charges will move opposite to $\vec{E}$. Thus, due to this imbalanced chirality there will be a net electric current parallel to $\vec{B}$. This phenomenon of generation of

[8] In fact, the normal phase of the theory is defined as the phase where the $U(1)^{(0)}$ symmetry is spontaneously broken as in this case the magnetic field lines are *unscreened*. While in a superconducting phase these field lines are *confined*.

an electric current parallel to an external magnetic field is called the *chiral magnetic effect* (CME).

Let us consider the energy balance of this chirality change right-handed fermion requires removing a particle from the left-handed Fermi surface and adding it to the right-handed Fermi surface. The energy cost for this process will be: $\mu_A \, dN_A$, where μ_A is the chiral chemical potential and N_A is the chiral particle number. Now multiplying this energy by the rate of chirality change will give us the energy needed per unit time. This energy, if there are no losses in the system, will be equal to the power delivered by the CME current. This power will be equal to the product of the current with the electric field. Thus,

$$\int d^3x \, \vec{j}_V \cdot \vec{E} = \mu_A \frac{dN_A}{dt} = -8k\mu_A \int d^3x \, \vec{E} \cdot \vec{B} \tag{3.27}$$

Since $\vec{E}$ is parallel to $\vec{B}$ we get in the limit of $\vec{E} \to 0$,

$$\vec{j}_V = -8k\mu_A \vec{B}, \tag{3.28}$$

where $\vec{j}_V$ is called the density of the CME current[9].

[9]Note the minus sign above follows from our convention.

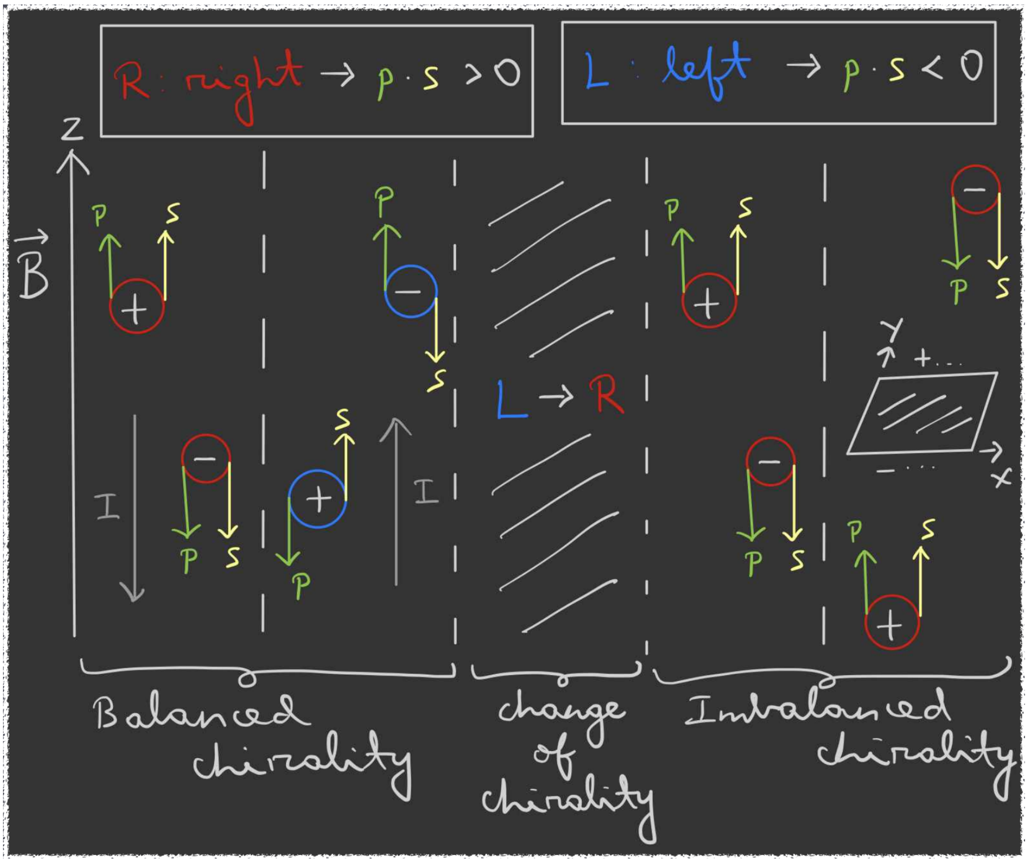

Figure 3.5: The chiral magnetic effect. The left hand side depicts a chirally balanced system and hence no net electric current generation. The right hand side depicts a chirally imbalanced system leading to the generation of an electric current parallel to the external magnetic field.

There is an analogous effect called the *chiral seperation effect* (CSE), which gives rise to an axial current parallel to an external magnetic field: $\vec{j}_A = -8k\mu_{\text{el}}\vec{B}$, where μ_{el} is the electric potential: $E_\lambda = \partial_\lambda \mu_{\text{el}}$.

CHAPTER 4

Holography

In this chapter we shall briefly review the AdS/CFT conjecture [5]. We assume the reader is familiar with it and hence we will be very brief and qualitative.

4.1 Holographic principle

Let us first state the holographic principle and explain heuristically what *holography* means. Consider an isolated system of mass E and entropy $\mathcal{S}_0$ in an asymptotically flat spacetime[1]. Let A be the area of a sphere that encloses the system. Let M_A be the mass of a Black Hole of horizon area A. Since the isolated system is not a black hole, for gravitational collapse to happen we must have $E \leq M_A$.

Let us add $(M_A - E)$ of energy to the system, keeping A fixed. Furthermore, let us say the final state is a black hole. By the second law of thermodynamics we

[1] An asymptotically flat spacetime is a spacetime which reduces to flat Minkowski spacetime when we take the radial coordinate $r \to \infty$.

have,

$$\Delta S \geq 0 \;\Rightarrow\; S_{\rm BH} - S_0 - \underbrace{S'}_{\substack{\text{entropy of}\\ \text{added energy}}} \geq 0,$$

$$\text{implying, } S_{\rm BH} \geq S_0 + \underbrace{S'}_{\substack{\text{entropy of}\\ \text{added energy}}} \tag{4.1}$$

Thus, Eq. (4.1) implies [69, 70],

$$S_0 \leq S_{\rm BH} = \frac{A}{4\hbar G_N}$$

If we assume that a black hole is the most massive object inside an area A (since the area A depends upon the mass M_A) then the above inequality implies that the maximal entropy inside a region boundary by area A, in a gravitational setting, is

$$S_{\rm max} = \frac{A}{4\hbar G_N} \tag{4.2}$$

Now let us recall the definition of von Neumann entropy:

$$S = -\text{Tr}(\rho \ln \rho)$$

where ρ denotes the density matrix for the system. For a system with an $N-$dimension Hilbert space, we have (since maximal entropy implies minimal information.):

$$S_{\rm max} = \ln N$$

Thus we get:

$$\rho_{\rm max} = \frac{1}{N}\mathbb{1} \tag{4.3}$$

as every state is equally distributed. Therefore, "the effective dimension" of the Hilbert space for a system inside a region of area A is bounded by (in the presence of gravity):

$$\ln N \leq \frac{A}{4\hbar G_N} = \frac{A}{4\, l_p^2} \tag{4.4}$$

where l_p is the Planck length.

Now in typical physical systems, we have: [dim $\sim e^{\#\,\text{of dofs}}$]

$$\#\,\text{of dofs} \sim \ln N \tag{4.5}$$

Then, the number of degrees of freedom (dofs) of any quantum gravity system inside area $A \leq \frac{A}{4\,lp^2}$ The bound is certainly violated in non-gravitational systems whose number of degrees of freedom (which is equal to $\ln N$) is proportional to the volume rather than the area. For example, consider a $3D$ lattice of spins with lattice spacing 'a'. Let the total volume be V and the non-hypotenuse length between two lattice sites be L. Then[2],

$$\begin{aligned}\#\,\text{of spins} &= \frac{V}{a^3} = \frac{A}{a^2}\cdot\frac{L}{a} \geq \frac{A}{l_p^2} \;\; (\text{for large } L)\\ N = 2^{V/a^3} &\Rightarrow S_{\max} = \frac{V}{a^3}\ln 2 \;\geq S_{\text{BH}} \;(\text{for large}\, V)\end{aligned} \tag{4.6}$$

Thus, we see that, *Quantum Gravity* leads to a huge reduction of the number of degrees of freedom. Quantum Field Theory has infinite degrees of freedom proportional to volume V but when it is coupled to gravity, then the number of degrees of freedom is reduced. According to the Holographic principle,

"*In Quantum Gravity, a region of boundary area A can be fully described by no more than* $\frac{A}{4\hbar G_N} = \frac{A}{l_p^2}$ *degrees of freedom, that is, 1 degree of freedom per Planck equation.*"

4.1.1 Renormalisation Group flow

In a local and well-defined quantum field theory, the behavior of observables is dependent on the specific energy scale under consideration. To rigorously examine how the couplings in the theory evolve when the energy scale μ changes, one can analyze the beta-function equation [71]:

$$\beta_g(g(\mu)) = \mu\partial_\mu g(\mu), \tag{4.7}$$

where g denotes the coupling parameter.

The key insight is that the beta-function equation Eq. (4.7) is entirely local with

[2]For a system of n spins, say spin $\frac{1}{2}$, we have : $N \sim 2^n$.

respect to the energy scale μ, and can be interpreted as a dynamical equation in an augmented dimension μ. This allows us to extend the system's dependence from the original spacetime coordinates (t, x) to an enlarged coordinate set (t, x, μ). In this framework, the beta-function equation serves as the governing equation in the extra μ-dimension. To better grasp this concept, one can think in terms of the Renormalization Group (RG) flow, particularly in the Wilsonian sense [72]. Altering the energy scale can be viewed as a coarse-graining process or, equivalently, as employing a different unit of measurement, as depicted in Fig. 4.1. By aligning system snapshots along the energy scale dimension, one effectively constructs a $(D+1)$-dimensional spacetime[3]. In this spacetime, as one moves from the boundary towards the infrared region, the length scale undergoes dilation, akin to the coarse-graining procedure.

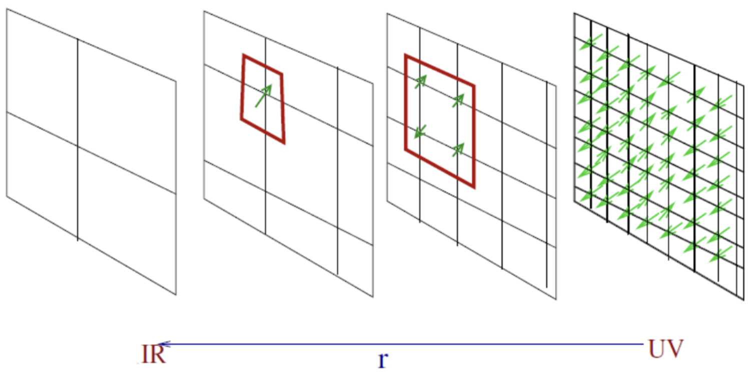

Figure 4.1: A pictorial representation of coarse-graining and re-interpretation of the energy scale as an extra dimension. Figure taken from [1, 2]

Now we will switch gears and present below the GKPW dictionary [17, 18]:

$$Z_{\text{grav}}[\phi_0^i(x); \partial\mathcal{M}] = \left\langle \exp\left(\sum_i \int d^d x\, \phi_0^i(x)\mathcal{O}^i(x)\right)\right\rangle_{\text{CFT on } \partial\mathcal{M}} \tag{4.8}$$

where i runs over all the light fields in the bulk effective field theory, and correspondingly over all the low-dimension local operators in CFT and Z_{grav} is the gravitational partition function in asymptotically AdS space. This dictionary can be expanded in to the following table:

[3] Say, if originally, (t, x) corresponded to a D dimensional spacetime

Table 4.1: The Holographic dictionary

Boundary		**Bulk**
CFT in $\mathbb{R}^{1,d-1}$	$\longleftrightarrow$	AdS_{d+1} gravity
Conformal Symmetry $SO(2,d)$	$\longleftrightarrow$	AdS_{d+1} isometry $SO(2,d)$
$U(1)$ global symmetry	$\longleftrightarrow$	$U(1)$ gauge symmetry
Global SUSY	$\longleftrightarrow$	Local SUSY
Representations of conformal group $SO(2,d)$	$\longleftrightarrow$	Representations of the isometry group $SO(2,d)$
Conformal local operators	$\longleftrightarrow$	Bulk fields
Scalar operators ($\hat{\mathcal{O}}$)	$\longleftrightarrow$	Scalar fields ($\phi(x)$)
Vector operators ($\hat{J}_\mu$)	$\longleftrightarrow$	Vector fields (A_M)
Tensor operators ($\hat{T}_{\mu\nu}$)	$\longleftrightarrow$	Tensor fields (h_{MN})
Deformations $\int d^dx\phi_0(x)\hat{\mathcal{O}}$	$\longleftrightarrow$	$\phi(x)$ in bulk (dual to $\hat{\mathcal{O}}$) has a boundary value ϕ_0 (which is the source for $\hat{\mathcal{O}}$ in the boundary theory)

4.2 Membrane paradigm

If we consider the AdS radius r as an energy scale within the framework of Wilsonian renormalization, it's intriguing to suggest that low-energy dynamics are encapsulated by the deep interior regions of the AdS spacetime, as depicted in Fig. 4.1.

In finite-temperature systems, the IR cutoff is set by the presence of a black hole horizon. Given that the Quantum Field Theory at long distances is described by hydrodynamic principles, it's tempting to consider, using holography, the area near the black hole horizon, often referred to as the "stretch horizon", as kind of a fluid. This fluid could potentially be identical to that in the dual QFT [73–75].

Upon more rigorous scrutiny of the fluid/gravity correspondence, it becomes evident that the notional fluid at the horizon and the fluid modeled by the dual Quantum Field Theory (QFT) are generally distinct entities [76–79]. This distinction arises because the hydrodynamic properties of the dual QFT are influenced not just by the stretch horizon, but also by the AdS spacetime. However, certain hydrodynamic characteristics of the so-called fictitious fluid at the horizon can be related to those in the dual QFT [75]. This correlation is possible due to the presence of conserved currents along the radial direction in AdS (along the r-direction). These currents facilitate the transfer of hydrodynamic information from the stretch horizon to the boundary, enabling a mapping to the dual QFT.

CHAPTER 5

Anomalous electrodynamics, 1-form symmetry and holography

In this chapter we will discuss the finite temperature physics of a magnetohydrodynamic *chiral* plasma, i.e. an electrodynamic plasma with an axial $U(1)_A$ current j_A that is afflicted by an Adler-Bell-Jackiw anomaly:

$$\partial_\mu j^\mu_A = -\frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} \ . \tag{5.1}$$

Here it is understood that this expression arises in a theory of dynamical electromagnetism, and $f_{\mu\nu}$ is the field strength of this fluctuating gauge field. We stress that the non-conservation of the axial current is given by a *dynamical* operator.

While the analysis presented here will be from a more formal, holographic, perspective, the system has clear phenomenological interest, with applications to baryon number violation [80–84], primordial magnetic fields [67, 85, 86], magnetised baryogenesis [87–89], and Dirac and Weyl semi-metals in condensed matter systems (for a review see [90]).

As described in [80], a quantity of interest in $U(1)$ anomalous processes is the relaxation rate of the chiral charge density j^0_A. This is what we seek to compute here. In our opinion a fully universal hydrodynamic treatment of this problem has not yet been given; indeed it is somewhat unclear whether one should exist.

We begin by carefully stating the problem and distinguishing it from the large existing literature on anomalous hydrodynamics. To orient ourselves, it is helpful to first imagine a weakly-coupled realization of the physics that we are interested in. Consider the following Lagrangian describing a massless Dirac fermion coupled to dynamical electromagnetism with photon a:

$$S[a,\psi] = \int d^4x \left(-\frac{1}{4e^2}f^2 + \bar{\psi}\left(\partial\!\!\!/ - i a\!\!\!/\right)\psi\right) \tag{5.2}$$

We will be interested in placing this system at finite temperature and understanding the hydrodynamic description.

What are the global symmetries of this system? It has a 1-form $U(1)^{(1)}$-symmetry associated with the conservation of magnetic flux:

$$\partial_\mu J^{\mu\nu} = 0 \qquad J^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}f_{\rho\sigma} \tag{5.3}$$

We will denote a p-form $U(1)$-symmetry by $U(1)^{(p)}$ [21].

The symmetry associated with vector phase rotations of the ψ field $\psi \to e^{i\alpha}\psi$ is gauged and does not correspond to a global symmetry. Classically, the theory appears to have a conventional (i.e. 0-form) $U(1)^{(0)}_A$-symmetry associated with $\psi \to e^{i\alpha\gamma^5}\psi$; however at the quantum mechanical level the conservation of the associated current is broken by the Adler-Bell-Jackiw anomaly Eq. (5.1). Note that the right-hand side of this expression is an *operator*, as f is a fluctuating dynamical field. This should be contrasted with the case of a 't-Hooft anomaly, where the right-hand side of a current-conservation equation involves a fixed external source that can be set to zero.

We are interested in understanding the realization of the symmetries at finite temperature. This is the domain of *hydrodynamics*, which describes how conserved quantities relax towards thermal equilibrium. Hydrodynamics in the presence of a 't-Hooft anomaly is a rich and well-studied field [91,92] (see also [90]). The situation with the anomaly above is somewhat different. As the right-hand side is a fluctuating operator, there is no longer a strictly universal sense in which the axial current is conserved. Thus it appears that the only true global symmetry of the system is the 1-form symmetry Eq. (5.3). A naive application of the conventional formalism of hydrodynamics applied to this system would then only involve a study of the 1-form symmetry in thermal equilibrium. Such an analysis was performed in [30], where it was shown that the resulting framework is essentially a reformulation of conventional relativistic magnetohydrodynamics, i.e. a description of an electrodynamic plasma (a holographic description of this plasma from this point of view was given in [31,62]). The only conserved quantity here is the usual magnetic flux, and this description of course makes no reference to the axial current whatsoever.

Nevertheless, to us this situation seems somewhat unsatisfactory; after all, from an applied viewpoint, it seems clear that the finite-temperature dynamics of Eq. (5.2) has a rich and physically relevant phenomenology. This physics is usually accessed by coupling the equations of (ungauged) hydrodynamics with a 't Hooft anomaly to weakly coupled electrodynamics "by hand" [85, 93]; in particular see [94] which constructs a hydrodynamic theory in a formal expansion in the anomaly coefficient. These constructions are not fully universal, and the domain of validity of the resulting theories is not entirely clear. In particular, recent work using classical lattice simulations [80, 81] that computes the charge relaxation rate Γ_A shows a disagreement with the predictions of the above hydrodynamic theories, suggesting that short-distance fluctuations play an important role that is not captured by the non-universal theories above.

It is difficult to come up with a universal hydrodynamic theory for this model. In fact, towards the end of our analysis, we will see that we have reasons to believe that such a universal hydrodynamic description might not exist for such a set up. For now we will explore this problem in a new way, by using holographic duality to explore aspects of the finite-temperature dynamics of a system in the same universality class as the weakly coupled theory described above. To construct our holographic dual theory, we must first carefully understand the symmetries in a manner that is independent of the description. One route to understand this is to note that the right-hand side of the expression above can be written in terms of the current $J^{\mu\nu}$ for the 1-form symmetry:

$$\partial_\mu j^\mu_A = k\,\epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \qquad k \equiv \frac{1}{16\pi^2} \tag{5.4}$$

We may now try to describe the dynamics of a system with a conserved 2-form current $J^{\mu\nu}$ and a 1-form axial current j^μ_A that satisfies the above non-conservation equation; this may be thought of as a kind of intertwining of the (genuine) 1-form symmetry and the (broken by the anomaly) 0-form symmetry. (To our knowledge, this particular intertwining does not appear to have a precise universal characterization in the field theory literature; however see further discussion in the conclusion).[1]

We will first construct a holographic model that possesses the above symmetries.

[1]The works: [37, 38] present a precise field-theoretical characterization of the ABJ anomaly in terms of non-invertible symmetries. In the next chapter, we shall use some of this technology to come up with an effective theory from chiral MHD.

We will then perform a preliminary investigation of the resulting holographic system. In particular, we study the system in the presence of a background magnetic field. We will explicitly compute the charge relaxation rate in this model and compare it both to elementary hydrodynamics with weakly coupled electromagnetism and to recent lattice results; we will find agreement with hydrodynamics at low magnetic fields, but disagreement at large magnetic fields; this suggests that UV fluctuations are important for a quantitative determination of this relaxation rate.

A short outline of the rest of this chapter is as follows. In Section 5.1 we review a simple hydrodynamic discussion of the charge relaxation rate. In Section 5.2 we introduce the holographic model that we will study in the remainder of the chapter. In Section 5.3 we place this model at finite temperature and study some aspects of static response (i.e. the analogue of the charge susceptibility). In Sections 5.4 and 5.5 we study finite-frequency response (both analytically at small frequencies and numerically) and we conclude with a brief discussion in Section 6.5.

5.1 Hydrodynamic calculation of relaxation rate

We begin our study by defining the relaxation rate that we will compute and using elementary physical arguments to understand what may control it; at the end we will compare the resulting physics to our holographic construction.

We first review the usual hydrodynamic computation of this charge relaxation rate. This is done in the usual framework of "chiral MHD". As described above, this means we assume a certain anomalous contribution to the dynamical electric current and couple it perturbatively to an MHD sector. We particularly highlight [94], where the authors perform a hydrodynamic study where the anomaly coefficient k is treated perturbatively; they compute the chiral charge relaxation rate Γ_A for small k. In this limit, they find that, $\Gamma_A \sim k^2 B^2$ with B being the magnetic field.

We review a similar calculation below: this is physically instructive but as discussed above is not truly universal. This calculation is a very slight generalization of the one presented in [80].

In our notation the anomaly takes the form

$$\partial_\mu j^\mu_A = -2k f_{\mu\nu}\tilde{f}^{\mu\nu} = k\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}, \tag{5.5}$$

where j^μ_A is the chiral current, $\tilde{F}_{\mu\nu}$ is the Hodge dual of the field strength $F_{\mu\nu}$. (In the case of a single Dirac fermion studied in [80] we have $2k = -\frac{e^2}{8\pi^2}$, where e is the electromagnetic coupling.)

For the homogeneous case (see [80]), we have $\vec{j}_A = 0$ and thus the anomaly equation becomes (for $j_A^0 \equiv \rho = \chi\mu_A$),

$$\chi \frac{d\mu_A}{dt} = -\frac{8k}{V} \int d^3x \ \vec{E} \cdot \vec{B}, \qquad (\text{as, } f_{\mu\nu}\tilde{f}^{\mu\nu} = 4\vec{E} \cdot \vec{B}) \tag{5.6}$$

where μ_A is the space-independent axial chemical potential, ρ the axial charge density, and χ the axial charge susceptibility, which in principle could depend on the temperature and background magnetic field.

Thus, we have the chiral relaxation rate as,

$$\frac{d\mu_A}{dt} = -\frac{8k}{\chi V} \int d^3x \ \vec{E} \cdot \vec{B}, \tag{5.7}$$

Following [80] we give below the chiral MHD equations as,

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}, \qquad \frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = -\sigma \vec{E} + 8k\mu_A \vec{B}. \tag{5.8}$$

where σ is the electric conductivity of the plasma, and we have assumed that the density of electric charge is zero, and the plasma has zero velocity. The last term: $8k\mu_A\vec{B}$, is the contribution from the chiral magnetic effect (CME)[2]. This system of equations is complemented by the anomaly equation above (Eq. (5.7)).

Now if we neglect the time-derivative of $\vec{E}$ in (5.8) we can express $\vec{E}$ in terms of $\vec{B}$ using (5.8). Then, we get for long-range fluctuations of the gauge fields (that is $\nabla \times \vec{B} \to 0$),

$$\frac{d\mu_A}{dt} = -\frac{8k}{\sigma\chi V} \int d^3x \ \left(8k\mu_A \vec{B}\right) \cdot \vec{B} = -\frac{64k^2 B^2}{\sigma\chi} \mu_A \equiv -\Gamma_A \mu_A, \tag{5.9}$$

where in the last equality Γ_A is defined as the rate of chirality non-conservation in the presence of an external homogeneous magnetic field $\vec{B}$. The solution of Eq.(5.9) goes as,

$$\mu_A(t) = e^{-\Gamma_A t} \mu_{A,0} \qquad (\text{where } \mu_{A,0} \text{ is an integration constant}) \tag{5.10}$$

[2]Note that there is a factor of 2 difference in the CME between the expression here and that of [80]. This is owing to the fact that in [80], μ_A couples to $\frac{j_A^0}{2}$ while here in the definition of μ_A, we have chosen it to couple to j_A^0.

From Eq.(5.9) we see that,

$$\Gamma_A = \frac{64k^2B^2}{\sigma\chi}. \tag{5.11}$$

i.e. the relaxation rate is quadratic in the magnetic field. We will compare this elementary discussion to an explicit holographic calculation later.

We note that in [80, 81], the authors perform a numerical lattice computation in determining the chiral charge relaxation rate Γ_A. They also found it to be quadratic in the magnetic field B. As mentioned above, it was observed that the pre-factor in $\Gamma_A \sim B^2$ is approximately 10 times that of the theoretical predictions of the same pre-factor from hydrodynamics. Calculating this pre-factor in a strongly coupled yet solvable holographic model and comparing it to the existing literature serves as a pragmatic motivation for this study.

5.2 Overview of holographic model

In this section, we will present a bulk holographic theory which realizes the pattern of symmetry non-conservation in Eq. (5.4). We will begin by presenting the bulk action and demonstrating the deformed Ward identity; in the next section, we will describe how we arrived at this theory from dualizing a different bulk action. (For a summary of our conventions and notation see Appendix A.)

5.2.1 Holographic bulk action

We desire a bulk theory with the following properties: it should have a bulk massless 2-form B_{MN}; as explained in detail in [62], this is associated with a global 1-form symmetry in the boundary, as B_{MN} is dual to the boundary 2-form current $J^{\mu\nu}$. The action should also have a vector field which we call E_M. E_M is dual to a vector operator representing the axial current j^μ_A on the boundary; this vector operator should be understood as the axial current, and it is not conserved. Thus, E_M should not enjoy a bulk gauge symmetry. (We will see in a later section that E_M is of the form $A_M - \partial_M\phi$, where A_M and ϕ enjoy (bulk) gauge symmetries in such a way that E_M is gauge-invariant). However, the divergence of j^μ_A on the boundary is not completely unconstrained; rather its divergence should be related to the a double-trace operator of the 2-form current $J_{\mu\nu}$ by the following anomaly equation

$$\partial_\mu j^\mu_A = k\,\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma} \tag{5.12}$$

where k is a parameter that should enter the bulk action.

We now present a bulk action which satisfies the above properties[3]:

$$S[E,B]=\int d^5x\sqrt{-g}\,\left[-\frac{1}{4}G^2-\frac{1}{12}H^2+16k^2\left(E\cdot H\right)^2-\frac{k}{3}\epsilon_{PQRMN}H^{PQR}E_LH^{LMN}\right]. \tag{5.13}$$

Here $G=dE$ and $H=dB$ are the field strengths of E and B respectively, and we have defined $H^2=H^{PQR}H_{PQR}$, $(E\cdot H)^2=E_LH^{LMN}E^PH_{PMN}$. The theory has an invariance under a 1-form gauge symmetry:

$$B\to B+d\Lambda \tag{5.14}$$

with Λ an arbitrary 1-form. E clearly enjoys no explicit gauge symmetry; note however that the "mass" terms for E have a specific structure, involving couplings to H that are parameterized by a single coupling k. We will show that this structure encodes the anomaly Eq. (5.12). This action should be understood as being correct to order $\mathcal{O}(E^2)$; as we will show, the anomaly structure Eq. (5.12) above is only correctly represented to that order. Below we will also present an algorithm that can be used to obtain an action that is correct to all orders in E, though we will not require it for our purposes.

Motivated by studies pertaining to similar anomalies, a bulk action involving anomaly-inspired mass terms for gauge fields was studied in [95] (see Eq.(30) in [95]). There, the anomaly is thought to be sourced by a dynamical non-Abelian gauge field, which does not have an associated 1-form symmetry. In our case, the dynamical gauge field is Abelian, and thus the non-conservation of the current is precisely related to a 2-form current with universal dynamics; thus our action takes a more constrained form (and describes somewhat different physics) compared to that in [95].[4]

The variation of the action above with respect to B results in the following

[3]In the action below one can certainly have higher order terms consistent with the symmetry structure described above but they won't contribute to the holographic calculation which we describe at linear order in k.

[4]We note that [95] also consider external vector Abelian magnetic fields; however these fields act as sources and are non-dynamical.

equations of motion:

$$
\begin{aligned}
&-\frac{1}{2}\partial_L\left(\sqrt{-g}H^{LMN}\right)-k\,\partial_L\left[\sqrt{-g}\;\epsilon^{LMNQR}E^P H_{PQR}\right]\\
&-\frac{k}{3}\partial_L\left[\sqrt{-g}\;H_{PQR}\left(E^L\epsilon^{PQRMN}+E^M\epsilon^{PQRNL}+E^N\epsilon^{PQRLM}\right)\right]=0. \qquad (5.15)
\end{aligned}
$$

and similarly, we have the following equations for the variation with respect to E:

$$
\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}G^{ML}\right)=-32k^2H^{LQR}E^P H_{PQR}+\frac{k}{3}\epsilon_{PQRMN}H^{PQR}H^{LMN}. \qquad (5.16)
$$

Note that if $k=0$ these equations of motion decouple into free Maxwell equations for E and B respectively. Here we work only to linearized order in E.

We would now like to interpret this bulk physics holographically. We begin with the 2-form B. As usual, we may construct the boundary 2-form current $J^{\mu\nu}$ by varying the action with respect to the boundary value of $B_{\mu\nu}$ (see appendix A for convention regarding the boundary current defined below)

$$
J^{\mu\nu}(x)=\,2\frac{\delta S}{\delta B_{\mu\nu}(\infty)} \qquad (5.17)
$$

The on-shell variation of the boundary value of the action may be reduced to a variation with respect to the radial derivative $\partial_r B_{\mu\nu}$, and we thus find

$$
\begin{aligned}
J^{\rho\sigma}(x)&=2\lim_{r\to\infty}\frac{\delta S}{\delta\left(\partial_r\left(B_{\rho\sigma}\right)\right)} \qquad (5.18)\\
&=\lim_{r\to\infty}\sqrt{-g}\left[-H^{r\rho\sigma}-2k\,\epsilon^{r\rho\sigma\mu\nu}E^\alpha H_{\alpha\mu\nu}-\frac{2k}{3}\left[H_{\alpha\beta\gamma}\{E^r\epsilon^{\alpha\beta\gamma\rho\sigma}+E^\rho\epsilon^{\alpha\beta\gamma\sigma r}+E^\sigma\epsilon^{\alpha\beta\gamma r\rho}\}\right]\right] \qquad (5.19)
\end{aligned}
$$

Here the first term is standard [62]; the others arise from the physics associated with the anomaly. We have omitted terms of order $\mathcal{O}(E^2)$; this is because in this work we will study only the linearized equations of motion of E.

Note that the radial component of the bulk wave equation Eq. (5.15) for B_2 ensures that we have

$$
\partial_\mu J^{\mu\nu}(x)=0 \qquad (5.20)
$$

i.e. that the 2-form current is conserved.

We now turn to E. We may construct the dual current as in Eq. (6.89):

$$
j_A^\mu=\frac{\delta S}{\delta E_\mu(\infty)}=-\sqrt{-g}G^{r\mu}\,(r\to\infty). \qquad (5.21)
$$

The last expression is standard for the boundary operator dual to a vector field.

Let us now understand the non-conservation of j_A^μ, i.e. let us derive Eq. (5.12) holographically at linear order in $\mathcal{O}(E)$. From the above expression for the 2-form current at the boundary let us now compute[5] $k\ J \wedge_4 J$:

$$\left[-\frac{k}{4}\epsilon_{\alpha\beta\mu\nu}J^{\alpha\beta}J^{\mu\nu}\right] = \left(\sqrt{-g}\right)^2 \left(\frac{8k^2}{\sqrt{-g}}E^\alpha H_{\alpha\mu\nu}H^{r\mu\nu} - \frac{k}{4}\epsilon_{\alpha\beta\mu\nu}H^{r\alpha\beta}H^{r\mu\nu}\right) \tag{5.22}$$

(Note the explicit appearance of factors of the determinant of the metric; this arises from the fact that from the point of view of the bulk $J^{\alpha\beta}$ is not a tensor, as can be seen from its definition in Eq. (5.18)). Now let us consider $L = r$ in Eq. (5.16),

$$32k^2 H^{r\mu\nu}E^\alpha H_{\alpha\mu\nu} - k\sqrt{-g}\epsilon_{\alpha\beta\mu\nu}H^{r\alpha\beta}H^{r\mu\nu} = -\partial_\sigma G^{\sigma r}, \qquad (\sigma \neq r) \tag{5.23}$$

Usually in AdS/CFT this component of the bulk Maxwell equations of motion is a radial constraint that enforces the conservation of the current j_A^μ; here we see that the current is instead not conserved. Using Eq. (5.22) we see that it is equal to

$$\partial_\sigma j_A^\sigma = \partial_\sigma\left(\sqrt{-g}G^{\sigma r}\right) = k\epsilon_{\alpha\beta\mu\nu}J^{\alpha\beta}J^{\mu\nu} \tag{5.24}$$

i.e. equivalent to Eq. (5.12) as desired.

This shows that this holographic theory is in the correct universality class, by which we mean that it correctly links the non-conservation of the 1-form axial current with a bilinear constructed from the 2-form current $J^{\mu\nu}$. The fact that E has no gauge symmetry at all in the bulk is dual to the fact that its non-conservation is given in terms of a dynamical operator that cannot be turned off. We note also that the intermediate steps are somewhat complicated and rely on the detailed structure of the action Eq. (5.13). The reader who is willing to take this action as a given and is interested only in results can now skip to the next section, where we compute the holographic observables of interest.

In the remainder of this section we describe how we construct this action through bulk Poincaré duality.

[5]Note, the boundary metric is flat. Thus we have the following expression relating boundary and bulk Levi-Civita tensors, $\epsilon_{rabcd} = \sqrt{-g}\,\epsilon_{abcd}$.

5.2.2 Dualizing the action

Our approach to constructing the bulk action is essentially the bulk dual of the operation of "gauging a global $U(1)$ symmetry"; i.e. we begin by considering the very well-studied bulk action [95–97][6] for a theory with two 0-form global symmetries $U(1)_A \times U(1)_V$ with a mixed 't Hooft anomaly between them:

$$S_5\left[A_1, V_1\right] = \int_{\mathcal{M}^5} \left(-\frac{1}{2} F_2 \wedge \star F_2 - \frac{1}{2} G_2 \wedge \star G_2 - 4k\, A_1 \wedge F_2 \wedge F_2 - \frac{4k}{3} A_1 \wedge G_2 \wedge G_2\right). \tag{5.25}$$

Here A_1 and V_1 are two 1-form potentials which are holographically dual to the 0-form axial and vector currents respectively, and $F_2 = dV_1$ and $G_2 = dA_1$. The action is invariant (up to a boundary term) under the following gauge symmetries:

$$A_1 \to A_1 + d\Lambda_0 \qquad V_1 \to V_1 + d\lambda_0. \tag{5.26}$$

The boundary variation of the above gauge-transformation is nonzero, and it is well-understood [96] that this means that the dual field theory has a 't Hooft anomaly for the axial current:

$$\partial_\mu j^\mu_A = k\epsilon^{\mu\nu\rho\sigma}\left(F_{\mu\nu}F_{\rho\sigma} + \frac{1}{3} G_{\mu\nu}G_{\rho\sigma}\right) \tag{5.27}$$

where F and G are the field strengths of the fixed external sources for the vector and axial currents respectively.

We now want to study a field theory where we have "gauged" the global symmetry $U(1)_V$. In this operation the boundary gauge field V_1 will become dynamical, and we will thus lose the 0-form symmetry $U(1)_V$. However we expect to obtain a new 1-form symmetry (and 2-form current) associated with the conserved magnetic flux in our new $U(1)$ gauge theory; in holographic duality, we thus expect to obtain a new 2-form bulk field which we call B_2.

It is thus very natural to expect that the bulk operation equivalent to "gauging" on the boundary is to perform a *bulk* Poincaré duality on the bulk 1-form potential V_1, replacing it with a 2-form B_2. Similar operations have a long history in AdS/CFT, and may be viewed as a higher-form generalization of [99]; see [100, 101]

[6] We also found the exposition in [68, 98] useful.

for applications of such holographic operations in hydrodynamics. Also, see [102,103] for recent work in a similar holographic context.

We now describe the dualization process below, which proceeds essentially as it would in flat space. This is a well-posed operation, but to the best of our knowledge the details are not present in the literature for a non-linear action of the form Eq. (5.25) even in flat space. We will see some interesting wrinkles arising from the presence of the mixed Chern-Simons term.

Poincaré dualization

We follow the usual algorithm to dualize V_1, as can be found e.g. in Appendix B of [104] (see also [105]). The action does not depend on V_1 directly, but only on its field strength F_2; it is thus possible to treat F_2 as the dynamical variable rather than V_1. However we then need to impose its closure $dF_2 = 0$ through the use of a Lagrange multiplier B_2.

We construct the parent action S_{5p} by adding the Lagrange multiplier term's action (S_c) to the action S_5. We get for S_{5p},[7]

$$S_{5p}\,[A_1, F_2, B_2] = \underbrace{\int_{\mathcal{M}^5} -\frac{1}{2}F_2 \wedge \star F_2 - \frac{1}{2}G_2 \wedge \star G_2 - 4k\, A_1 \wedge F_2 \wedge F_2 - \frac{4k}{3}A_1 \wedge G_2 \wedge G_2}_{S_5}$$

$$+ \underbrace{\int_{\mathcal{M}^5} dB_2 \wedge F_2}_{S_c}, \tag{5.28}$$

where,

$$S_c = \int_{\mathcal{M}^5} dB_2 \wedge F_2 = \int_{\mathcal{M}^5} d(B_2 \wedge F_2) - B_2 \wedge dF_2 \tag{5.29}$$

Then, $\delta_{B_2} S_c = 0$ gives $dF_2 = 0$ (closure of F_2). Now imposing the equation of motion $\delta_{F_2} S_{5p} = 0$ yields,

$$\star F_2 = dB_2 - 8k\; (A_1 \wedge F_2), \tag{5.30}$$

The standard procedure is to now solve for dB_2 as a function of F_2 and eliminate the latter from the action entirely.

[7]In the action S_{5p}, F_2 is now a dynamical field.

Gauge symmetries of S_5

Before doing so, we discuss the symmetries: note that the realization of the 0-form gauge symmetry associated with A_1 has changed, as F_2 is now closed only on-shell. In the action as given in Eq. (5.28), consider instead the following gauge transformations,

$$A_1 \to A_1 + d\Lambda_0, \qquad B_2 \to B_2 + 8k\,\Lambda_0\,F_2\ . \tag{5.31}$$

It is easy to show that with the above gauge transformations, the action S_{5p} as given in Eq. (5.28) is gauge-invariant but the equation of motion Eq. (5.30) is not (off-shell). It fails to be gauge-invariant by a term $8k\Lambda_0\,dF_2$; since F_2 is an independent dynamical field now, it is not necessarily a closed 2-form unless we impose B_2's equations of motion. This may appear problematic: the action is gauge-invariant under the gauge transformations but the equations of motion are not off-shell gauge-invariant under the same gauge transformations. We shall remedy this below by introducing a new auxiliary field ϕ_0. The equations of motion of ϕ_0 shall serve as a constraint (which we refer to as a *gauge-invariant constraint*) for imposing the closure of $F_2 \wedge dF_2$.

Alternatively, we can introduce ϕ_0 by the following argument. Taking a step back, let us first consider variations of the action S_5 as given in Eq.(5.25) w.r.t. the gauge transformations as given in Eq.(5.26). We have,

$$\delta S_5 = 8k\,\Lambda_0 F_2 \wedge dF_2 \, + \, \frac{8k}{3}\,\Lambda_0\,G_2 \wedge dG_2 \, + \, \text{boundary terms}, \tag{5.32}$$

where the first two terms on the LHS vanish owing to the fact that both F_2 and G_2 in S_5 are closed 2-forms. This then ensures the invariance of S_5 (up to boundary terms) under gauge transformations (5.26). However, in the dualized form S_{5p}, F_2 is an arbitrary 2-form and not necessarily closed. So, due to the non-closure of F_2 we now may or may not have $F_2 \wedge dF_2 = 0$ (off-shell). Therefore, in addition to imposing the closure of F_2 by a Lagrange multiplier B_2 we have to add to S_{5p} another Lagrange multiplier to impose the constraint $F_2 \wedge dF_2 = 0$ (*gauge-invariant constraint*). From the degree of the term $F_2 \wedge dF_2$, it is clear that the Lagrange multiplier in this case would be a 0-form, say ϕ_0. Furthermore, as S_{5p} has to remain gauge-invariant under $A_1 \to A_1 + d\Lambda_0$, ϕ_0 has to be a gauge field with its own gauge transformation given as, $\phi_0 \to \phi_0 + \Lambda_0$ (by construction). Then, we have S_{5p} as,

$$S_{5p} = \int_{\mathcal{M}^5} -\frac{1}{2} F_2 \wedge \star F_2 - \frac{1}{2} G_2 \wedge \star G_2 - 4k\, A_1 \wedge F_2 \wedge F_2 - \frac{4k}{3} A_1 \wedge G_2 \wedge G_2 + dB_2 \wedge F_2$$
$$- \int_{\mathcal{M}^5} 8k\ \phi_0\, F_2 \wedge dF_2, \tag{5.33}$$

which can be re-written as,

$$S_{5p} = \int_{\mathcal{M}^5} -\frac{1}{2} F_2 \wedge \star F_2 - \frac{1}{2} G_2 \wedge \star G_2 - 4k\, (A_1 - d\phi_0) \wedge F_2 \wedge F_2 - \frac{4k}{3} A_1 \wedge G_2 \wedge G_2 + dB_2 \wedge F_2, \tag{5.34}$$

with $E_1 \equiv A_1 - d\phi_0$ being a vector field. Note that the emergence of the gauge-invariant field E_1 is precisely the structure anticipated earlier, which we now see emerges naturally when demanding off-shell gauge-invariance. We also note that the equations of motion of ϕ_0 are redundant – they follow automatically from the equations of A_1.[8]

Now let us give below the gauge transformations of A_1 and ϕ_0,

$$A_1 \to A_1 + d\Lambda_0, \qquad \phi_0 \to \phi_0 + \Lambda_0. \tag{5.35}$$

With these gauge transformations the F_2 equation of motion, $F_2 = -\star[dB_2 - 8k\ (E_1 \wedge F_2)]$, remains gauge-invariant even off-shell, and everything is consistent with the Poincaré dualization procedure. Clearly, S_{5p} is also invariant under (5.35) up to boundary terms.

The conclusion of the above discussion is that one should be careful while performing Poincaré dualization, as at times one may be required to impose a *gauge-invariant constraint* along with the usual closure constraint.

Inverse operation

Let us now proceed to eliminate F_2 from the action. We can now invert Eq. (5.30) to get F_2 in terms of E_1 and B_2 as below. See Appendix B.2 for details of this

[8]Another way to understand ϕ_0 is that $e^{i\phi_0}$ is an operator that is charged under the bulk 0-form "instanton" current $\star_5 F \wedge F$; in a conventional formalism where $F = dA$, this current is conserved identically. However, in this formalism its conservation must be enforced by ϕ_0's equations of motion.

calculation.

$$F_{MN} = -\frac{\tilde{c}_1}{6}\epsilon_{PQRMN}H^{PQR} + 8\tilde{c}_1 k\ E^P H_{PMN} + \frac{64}{3}\tilde{c}_1 k^2\ H^{PQR}E^L\epsilon_{PQRL[M}E_{N]}. \tag{5.36}$$

where $H_{PQR} = \partial_P B_{QR} + \partial_Q B_{RP} + \partial_R B_{PQ}$ (as, $H_3 = dB_2$) and $\tilde{c}_1 \equiv \frac{1}{1+64k^2E^2}$.

Note that, if in S_5 we have $k \to 0$, F_2's equations of motion become $F_2 = -\star dB_2$ and if we take $k \to 0$ in the above equation we get $F_2 = -\star dB_2$. F_{MN} above has been written in such a way so that it is manifestly anti-symmetric.

Now let us substitute Eq.(5.36) into Eq.(5.34) to obtain below the full non-linear bulk action,

$$\begin{aligned} S_{5p} = \int_{\mathcal{M}^5} \sqrt{-g}\, d^5x\ \Bigg[\Bigg\{ & -\frac{H^2}{12} - \frac{k}{3}\epsilon_{PQRMN}H^{PQR}E_D H^{DMN} + 16k^2\left((E\cdot H)^2 - \frac{2}{3}E^2H^2\right) \\ & -64k^3\epsilon_{PQRMN}E^P E_L H^{LQR}E_J H^{JMN} + 256k^4\left(E^2(E\cdot H)^2 - \frac{1}{3}E^4H^2\right)\Bigg\}\tilde{c}_1^2 \\ & + \left\{\frac{k}{3}\epsilon^{PQRMN}E_P G_{QR}G_{MN} - \frac{1}{4}G^2\right\}\Bigg]. \end{aligned} \tag{5.37}$$

Note that truncating the above action to $\mathcal{O}(E^2)$ results in the quadratic action in Eq.(5.13) above, which we will use for the remainder of this study, in which we consider only small fluctuations about equilibrium.

Now we give below the full 2-form current $J^{\mu\nu}$ obtained from the action above Eq.(5.37),

$$\begin{aligned} J^{\rho\sigma} &= 2\lim_{r\to\infty}\frac{\delta S_{5p}}{\delta\left(\partial_r B_{\rho\sigma}\right)} \\ &= \lim_{r\to\infty}\sqrt{-g}\Bigg[-H^{r\rho\sigma} - 2k\epsilon^{r\rho\sigma\mu\nu}E^\eta H_{\eta\mu\nu} - \frac{2k}{3}H_{\alpha\beta\gamma}\left(E^r\epsilon^{\alpha\beta\gamma\rho\sigma} + E^\rho\epsilon^{\alpha\beta\gamma\sigma r} + E^\sigma\epsilon^{\alpha\beta\gamma r\rho}\right) \\ &+ 64\left(k^2 + 32k^4E^2\right)E_\alpha\left(E^r H^{\alpha\rho\sigma} + E^\rho H^{\alpha\sigma r} + E^\sigma H^{\alpha r\rho}\right) - 8\left(1 + 32k^2E^2\right)k^2E^2H^{r\rho\sigma} \\ &- 256k^3E_\alpha E^\kappa H_{\kappa\beta\gamma}\left(E^r\epsilon^{\alpha\beta\gamma\rho\sigma} + E^\rho\epsilon^{\alpha\beta\gamma\sigma r} + E^\sigma\epsilon^{\alpha\beta\gamma r\rho}\right)\Bigg]\tilde{c}_1^2. \end{aligned} \tag{5.38}$$

Now one can, in principle, check that the anomaly structure of Eq.(5.12) can be obtained from the above 2-form current by performing an order-by-order (in k) comparison of coefficients on both sides of Eq.(5.12). We have checked this explicitly to $\mathcal{O}(k^3)$.

A note on the role of boundary terms

Often one needs to be careful while considering variations of a holographic bulk action like the one given in Eq. (5.37). This is because to have a well-defined variational principle for the bulk action one needs to add appropriate counter-terms to cancel off the boundary terms resulting from variation of the action. This goes under the scheme of *holographic renormalisation* (see [106] for details). In this work, though, we haven't been careful while considering variations of the action in Eq. (5.37), we give a qualitative argument below as to why addition of local counter-terms won't affect the general results obtained here. However, a careful detailed analysis of holographic renormalisation of the action in Eq. (5.37) is an interesting direction for a future work to rigorously justify the argument given below.

Usually, these boundary terms: δ(boundary) and hence the counter-terms are local function of sources. We saw above that the bulk action in Eq. (5.37) correctly reproduces the ABJ anomaly structure Eq. (5.12). Note that, as argued above, this anomaly is of the ABJ kind, that is, the right hand side of Eq. (5.12) – the operator J^2 – is a dynamical operator and hence is not a local function of the sources. On the contrary, if we were dealing with anomaly of the 't Hooft kind instead, then in that case, the operator appearing on the right hand side of the 't Hooft anomaly equation, say J^2, will now be a local function of the sources. Hence, the addition of local counter-terms, to get a well-defined variation principle, will affect anomaly of the 't Hooft kind but we believe they won't affect the anomaly of the ABJ kind.

5.3 Finite temperature physics: zero frequency

With the holographic action in hand, we will now study the plasma that is obtained from the realization of these symmetries at finite temperature. To heat up our system, we consider the background metric given by the usual planar black brane background

$$ds^2 = r^2 \left(-f(r)dt^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2 f(r)} \ , \tag{5.39}$$

where $f(r) = 1 - \left(\frac{r_h}{r}\right)^4$ and where we are working in units where the AdS radius $R = 1$. The Hawking temperature of the black brane is $T = \frac{r_h}{\pi}$. We are interested in the physics in the presence of a background magnetic field in the z direction, i.e. a configuration where $\langle J^{tz} \rangle = -b$. From the holographic dictionary Eq. (5.19), we see that this means that $H^{rtz} \neq 0$; solving the equations of motion Eq. (5.15) we see that the background profile is:

$$H^{rtz} = \frac{b}{r^3}, \qquad H_{rtz} = -\frac{b}{r}, \qquad E = 0. \tag{5.40}$$

Note that, here we are working in the so called *probe limit* – where we neglect the backreaction of the magnetic field onto the geometry. In other words, we assume the above background profile (Eq.(5.40)) doesn't affect the metric components given in Eq.(5.39). This is justified in the high-temperature limit; at low temperatures this backreaction cannot be neglected, and one would replace the background with a magnetic brane solution [107, 108]. Now we will begin our study by computing the static axial charge susceptibility χ in the presence of the background magnetic field. For a theory with a conserved charge, the susceptibility can be defined as:

$$\chi = \frac{\partial \langle j^t_A \rangle}{\partial \mu_A}, \tag{5.41}$$

$$\text{leading to, } \langle j^t_A \rangle = \chi \mu_A \quad \text{(in the linear regime)}, \tag{5.42}$$

where μ_A is the axial chemical potential. In the case of a non-conserved axial current the precise definition of the axial chemical potential as a dynamical hydrodynamic variable is somewhat more subtle (see e.g. [96]), but in thermal equilibrium it can be understood as the value of the axial source A_t, which coincides with E_t when all fields are static.

5.3.1 Susceptibility

In this section we shall consider the low frequency limit of Eq.(5.16) and compute the axial charge susceptibility in this model. From Eq.(5.16) we find,

$$\delta E_r = \frac{(i\omega r^2)\partial_r(\delta E_t) + 4kbf\partial_r(\delta B_{xy})}{r^2\omega^2 - 64b^2k^2 f}, \qquad \text{(with } L = r \text{ in Eq.(5.16))} \tag{5.43}$$

$$\partial_r^2(\delta E_t) + i\omega\partial_r(\delta E_r) + \left(\frac{3}{r}\right)\partial_r(\delta E_t) + \left(\frac{3}{r}\right) i\omega\delta E_r = \frac{64b^2k^2}{r^6 f}\delta E_t - \frac{4kbi\omega}{r^6 f}\delta B_{xy},$$
$$\text{(with } L = t \text{ in Eq.(5.16))} \tag{5.44}$$

Now plugging δE_r from Eq.(5.43) in Eq.(5.44) and taking the $\omega \to 0$ limit we obtain[9],

$$\partial_r^2\left(\delta E_t\right) + \left(\frac{3}{r}\right)\partial_r\left(\delta E_t\right) - \frac{64b^2k^2}{r^6 f}\delta E_t = 0\ . \tag{5.45}$$

General solution

Let us define the following dimensionless combination of temperature and background magnetic field for later convenience:

$$\zeta(b/T^2) \equiv \frac{\sqrt{r_h^4 - 64b^2k^2}}{r_h^2} = \left(\frac{b}{T^2\pi^2}\right)\sqrt{\frac{\pi^4}{(b^2/T^4)} - 64k^2}. \tag{5.46}$$

In the small magnetic field limit (with T fixed), that is $b \to 0$, we have,

$$\zeta(b/T^2) \underset{b\to 0}{\to} 1 - \frac{32k^2}{\pi^4}\left(\frac{b}{T^2}\right)^2 - \frac{512k^4}{\pi^8}\left(\frac{b}{T^2}\right)^4 + \mathcal{O}\left(\left(\frac{b}{T^2}\right)^6\right) \tag{5.47}$$

In the large magnetic field limit (with T fixed), that is $b \to \infty$, we have,

$$\zeta(b/T^2) \underset{b\to\infty}{\to} \frac{8ik}{\pi^2}\left(\frac{b}{T^2}\right) - \frac{i\pi^2}{16k}\left(\frac{b}{T^2}\right)^{-1} - \frac{i\pi^6}{4096k^3}\left(\frac{b}{T^2}\right)^{-3} + \mathcal{O}\left(\left(\frac{b}{T^2}\right)^{-5}\right) \tag{5.48}$$

Notice from the definition of ζ (in Eq.(5.46)), it appears that $\zeta = 0$ could be a point of non-analyticity for the susceptibility $\chi(\zeta)$; we shall show below that χ is actually a function of ζ^2 and not of ζ and hence is analytic at $\zeta = 0$.

Solving Eq.(5.45) analytically we find the general solution as,

$$\begin{aligned}\delta E_t(r)_{gen} = {}& d_1\ r_h^{1+\zeta}\ r^{-1-\zeta}\ {}_2F_1\left(-\frac{1}{4} - \frac{\zeta}{4}, \frac{1}{4} - \frac{\zeta}{4}; 1 - \frac{\zeta}{2}; \frac{r^4}{r_h^4}\right)\\ &+ d_2\ r_h^{1-\zeta}\ r^{-1+\zeta}\ {}_2F_1\left(-\frac{1}{4} + \frac{\zeta}{4}, \frac{1}{4} + \frac{\zeta}{4}; 1 + \frac{\zeta}{2}; \frac{r^4}{r_h^4}\right),\end{aligned} \tag{5.49}$$

where d_1 and d_2 are integration constants. From Eq.(5.49) we will fix them such that $\delta E_t(r)$ is regular near the horizon (or in the interior). The boundary condition

[9] Note that we could have obtained Eq.(5.45) by directly taking $\omega \to 0$ limit in Eq.(5.44). This is because in $\omega \to 0$ limit δE_r and δE_t decouple.

we seek is $\delta E_t(r=r_h)_{gen}=0$. We define

$$\begin{aligned}\delta E_t(r,r_h,b,k,d_3)\big|_p \equiv\ & r_h^{1+\zeta}\ r^{-1-\zeta}\ {}_2F_1\left(-\frac{1}{4}-\frac{\zeta}{4},\frac{1}{4}-\frac{\zeta}{4};1-\frac{\zeta}{2};\frac{r^4}{r_h^4}\right)\\ &+d_3\ r_h^{1-\zeta}\ r^{-1+\zeta}\ {}_2F_1\left(-\frac{1}{4}+\frac{\zeta}{4},\frac{1}{4}+\frac{\zeta}{4};1+\frac{\zeta}{2};\frac{r^4}{r_h^4}\right),\\ &(\text{such that } \delta(E_t)_{gen}=d_1\ \ \delta E_t(r,r_h,b,k,d_3)\big|_p \text{ and } d_3:=d_2/d_1)\end{aligned} \tag{5.50}$$

Then we evaluate $\delta E_t(r=r_h,r_h,b,k,d_3)\big|_p$ at the horizon and find,

$$\delta E_t(r=r_h,r_h,b,k,d_3)\big|_p=\frac{\Gamma\left(1-\frac{\zeta}{2}\right)}{\Gamma\left(\frac{3}{4}-\frac{\zeta}{4}\right)\Gamma\left(\frac{5}{4}-\frac{\zeta}{4}\right)}+\frac{d_3\ \Gamma\left(1+\frac{\zeta}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{\zeta}{4}\right)\Gamma\left(\frac{5}{4}+\frac{\zeta}{4}\right)} \tag{5.51}$$

We further fix d_3 as

$$d_3(r_h,b,k)=-\frac{\Gamma\left(1-\frac{\zeta}{2}\right)\Gamma\left(\frac{3}{4}+\frac{\zeta}{4}\right)\Gamma\left(\frac{5}{4}+\frac{\zeta}{4}\right)}{\Gamma\left(1+\frac{\zeta}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\zeta}{4}\right)\Gamma\left(\frac{5}{4}-\frac{\zeta}{4}\right)} \tag{5.52}$$

Note that, d_3 is chosen such a way that $\delta E_t(r=r_h,r_h,b,k,d_3)\big|_p=0$ or in other words, $\delta E_t(r=r_h)_{gen}=0$.[10]

Regular solution

After d_3 is fixed as above we obtain the following regular solution,

$$\begin{aligned}\delta E_t(r)=r_h^{1-\zeta}\ r^{-1-\zeta}\ \Gamma\left(1-\frac{\zeta}{2}\right)&\left[r_h^{2\zeta}\ {}_2\tilde{F}_1\left(-\frac{1}{4}-\frac{\zeta}{4},\frac{1}{4}-\frac{\zeta}{4};1-\frac{\zeta}{2};\frac{r^4}{r_h^4}\right)-\right.\\ &\left.2^{-\zeta}\ r^{2\zeta}\ \frac{\Gamma\left(\frac{3}{2}+\frac{\zeta}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{\zeta}{2}\right)}\ {}_2\tilde{F}_1\left(-\frac{1}{4}+\frac{\zeta}{4},\frac{1}{4}+\frac{\zeta}{4};1+\frac{\zeta}{2};\frac{r^4}{r_h^4}\right)\right],\end{aligned} \tag{5.53}$$

where ${}_2\tilde{F}_1$ is the regularized hypergeometric function.

Now we examine Eq.(5.53) in the vanishing magnetic field limit that is for $b=0$,

$$\delta E_t(r)\underset{b\to 0}{\to}-1+\frac{r_h^2}{r^2}+\mathcal{O}\left(\frac{1}{r^2}\right). \tag{5.54}$$

[10]Note that, $\delta E_t(r=r_h,r_h,b,k,d_3)\big|_p$ is a 'particular' solution (hence the notation $\delta E_t|_p$) which satisfies the boundary condition $\delta E_t(r=r_h)_{gen}=0$.

Now let us look at the boundary expansion of Eq.(5.53) $\left(\text{up to } \mathcal{O}\left(\frac{1}{r^2}\right)\right)$,

$$\delta E_t(r) \underset{r\to\infty}{\to} \left(-\frac{1}{r_h^4}\right)^{\frac{1}{4}-\frac{\zeta}{4}} r_h^{1-\zeta} \left(\left(-\frac{1}{r_h^4}\right)^{\frac{\zeta}{2}} r_h^{2\zeta} - \tan\left(\left(\frac{1}{4}+\frac{\zeta}{4}\right)\pi\right)\right)\left(\frac{\sqrt{\pi}\,\Gamma\left(1-\frac{\zeta}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{\zeta}{4}\right)\Gamma\left(\frac{5}{4}-\frac{\zeta}{4}\right)}\right)$$
$$+\frac{2}{r^2}\left(-\frac{1}{r_h^4}\right)^{\frac{3}{4}-\frac{\zeta}{4}} r_h^{5-\zeta} \left(\left(-\frac{1}{r_h^4}\right)^{\frac{\zeta}{2}} r_h^{2\zeta} - \cot\left(\left(\frac{1}{4}+\frac{\zeta}{4}\right)\pi\right)\right)\left(\frac{\sqrt{\pi}\,\Gamma\left(1-\frac{\zeta}{2}\right)}{\Gamma\left(-\frac{1}{4}-\frac{\zeta}{4}\right)\Gamma\left(\frac{3}{4}-\frac{\zeta}{4}\right)}\right). \tag{5.55}$$

Notice that the above expression is of the form, $A+Br^{-2}$. From this we can find the charge susceptibility as $\chi = -\frac{2B}{A}$ (using the definition of the current from Eq. (5.21)).

$$\chi(r_h, b) = -2r_h^2\left(\frac{\zeta^2-1}{16\pi^2}\cos\left(\frac{\zeta\pi}{2}\right)\Gamma^2\left(\frac{1-\zeta}{4}\right)\Gamma^2\left(\frac{1+\zeta}{4}\right)\right) = -2r_h^2\, g(\zeta) \equiv -2T^2\,\tilde{g}(b/T^2), \tag{5.56}$$

where[11] $\tilde{g}(b/T^2) = \pi^2 g(\zeta) \equiv \frac{\zeta^2-1}{16}\cos\left(\frac{\zeta\pi}{2}\right)\Gamma^2\left(\frac{1-\zeta}{4}\right)\Gamma^2\left(\frac{1+\zeta}{4}\right)$. Note that $g(\zeta)$ is manifestly an even function of ζ. Hence, $g(\zeta)$ is analytic as a function of ζ^2 and not ζ which we wanted to show (see Eq. (5.46)), and $g(\zeta)$ is analytic at $\zeta = 0$ (see Appendix C for further details on the $\zeta = 0$ case).

Note that $\lim_{\zeta\to\pm1} g(\zeta) = -1$. Furthermore, the vanishing magnetic field limit is $b = 0$ which corresponds to $\zeta = \pm1$ (from the definition of ζ). Hence, we find that in the vanishing magnetic field limit, $\chi(r_h, b=0) = 2r_h^2$, which is the usual charge susceptibility for the conventional black brane, and matches with what we would have gotten from computing the susceptibility from Eq.(5.54)).

In the field-theoretical study [80], the corresponding susceptibility was taken to be the free fermion result at zero magnetic field $\chi_s = \frac{1}{6}T^2$. Let us contrast this to the susceptibility obtained above in Eq.(5.56) from holography. A key difference is that the proportionality factor relating χ and T^2 is no longer a constant but a function of b/T^2, namely $\tilde{g}(b/T^2)$. (Presumably a similar effect would exists in a perturbative approach, where one would simply consider the effects of Landau levels on the charge susceptibility).

A plot of χ/T^2 as a function of kb/T^2 is shown in Figure 5.1.

[11] We have used $\Gamma(1+x) = x\Gamma(x)$ and $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$ to obtain Eq.Eq. (5.56) from Eq.(5.55).

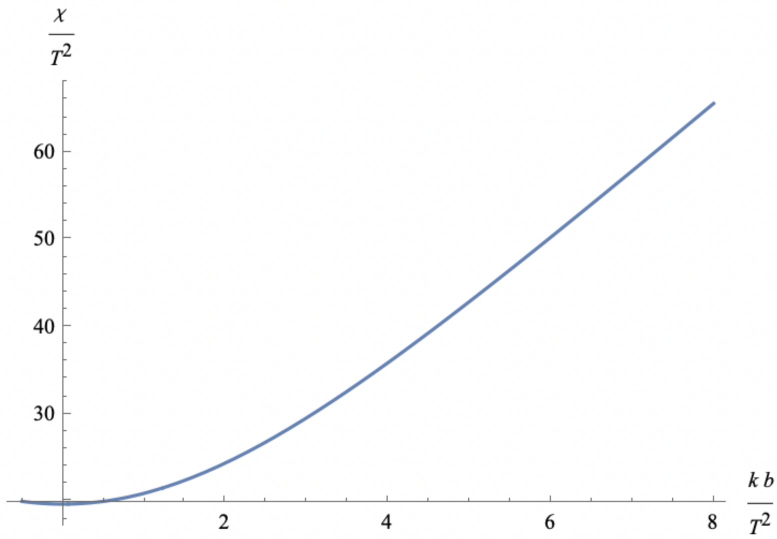

Figure 5.1: χ/T^2 as a function of kb/T^2

5.4 Hydrodynamic limit

We now solve the bulk equations of motion in a small frequency limit; we will see an analogue of the chiral magnetic effect appear in this limit, and we will also reproduce from the bulk certain aspects of the hydrodynamic calculation in Section 5.1.

Let us begin by noting from Eq. (5.19) that the equation of motion for the 2-form B_2 can be written as

$$\nabla_P \mathcal{H}^{PQR} = 0 \tag{5.57}$$

where the 3-form $\mathcal{H}_3$ is defined as

$$\begin{aligned} \mathcal{H}^{PQR} \equiv H^{PQR} &+ 2k \left[\epsilon^{PQRMN} E^L H_{LMN}\right] \\ &+ \frac{2k}{3}\left[H_{LMN}\left(E^P \epsilon^{LMNQR} + E^Q \epsilon^{LMNRP} + E^R \epsilon^{LMNPQ}\right)\right] \end{aligned} \tag{5.58}$$

with $H_3 = dB_2$. This general form – i.e. that the equation of motion can be written as the divergence of a 3-form $\mathcal{H}_3$ – follows from the fact that the action is a function of dB_2 alone.

We are now interested in solving these equations of motion in a hydrodynamic limit, i.e. with $\frac{\omega}{T} \to 0$. When taking a small frequency limit in AdS/CFT, it is useful to use the formalism of the membrane paradigm [75]. The usual infalling membrane boundary condition as applied to the modified field-strength $\mathcal{H}$ results in the following condition at the black hole horizon:

$$\sqrt{-g}\mathcal{H}^{rxy}(r_h) = \mathcal{H}_{txy}(r_h)\Sigma(r_h) \qquad \Sigma(r) \equiv \sqrt{\frac{-g}{-g_{rr}g_{tt}}} g^{xx}g^{yy} \tag{5.59}$$

See [62] for an application of these techniques to a minimally coupled 2-form B_2. Importantly, it is shown there that the quantity $\Sigma(r_h)$ can be understood as the conventional electric resistivity ρ.

We now study the consequences of this boundary condition for fluctuations about a background field configuration where $H_{rtz} \neq 0$ as in Eq. (5.40). We will study a configuration where the nonzero components of the fluctuations are $\mathcal{H}_{rxy}, \mathcal{H}_{txy}$, E_t and E_r.

We begin by writing out:

$$\mathcal{H}_{txy} = H_{txy} + 8k\sqrt{-g}E_t H^{rtz} \tag{5.60}$$

(where we have used an orientation in which $\epsilon_{txyrz} < 0$). The boundary condition Eq. (5.59) thus implies that at the horizon we have

$$\sqrt{-g}\mathcal{H}^{rxy}(r_h) = \Sigma(r_h)\left(H_{txy} + 8k\sqrt{-g}E_t H^{rtz}\right)\Big|_{r=r_h} \tag{5.61}$$

We would now like to propagate this information to the boundary, where it can be given an interpretation in the field theory. The equations of motion in the low-frequency limit take the form

$$\partial_r\left(\sqrt{-g}\mathcal{H}^{rxy}\right) = 0 \qquad \partial_r H_{txy} = 0 \tag{5.62}$$

where the former is the xy component of the diagonal equation of motion Eq. (5.57) and the latter is the Bianchi identity associated with $H_3 = dB_2$. Thus we can evaluate the expression above at the AdS boundary:

$$\sqrt{-g}\mathcal{H}^{rxy}(\infty) = \Sigma(r_h)\left(H_{txy}(\infty) + 8k\sqrt{-g}E_t(r_h)H^{rtz}(r_h)\right) \tag{5.63}$$

Now we note that $J^{xy} = -\lim_{r\to\infty}\sqrt{-g}\mathcal{H}^{rxy}$, and we thus find that

$$J^{xy} = -\Sigma(r_h)\left(H_{txy}(\infty) + 8k\sqrt{-g}E_t(r_h)H^{rtz}(r_h)\right) \tag{5.64}$$

Here J^{xy} may be understood as the electric field in the z direction $\mathcal{E}_z$; as explained before, $J^{tz} = -\sqrt{-g}H^{rtz}$ is the background magnetic field. Now H_{txy} is the applied source; let us set it to zero. We then find the following expression:

$$J^{xy} = 8k\Sigma(r_h)E_t(r_h)J^{tz} \tag{5.65}$$

Let us now pause to dissect this result. This appears reminiscent of the chiral magnetic effect; in the conventional language of electric and magnetic fields as in Eq. (7.1), J^{xy} is proportional to the electric field in the z direction; thus we see that in the absence of external sources, there is an electric field parallel to the applied magnetic field, where the constant of proportionality is $8k\Sigma(r_h)E_t(r_h)$. In comparing to field theory, we note that $\Sigma(r_h) = \rho$, i.e. the conventional electric resistivity in this theory.

Of course, if we are at precisely zero frequency, then we are required to have that $E_t(r_h) = 0$. This is completely consistent with the known physics of the chiral magnetic effect, which states that the *equilibrium* value of the chiral magnetic effect for the consistent vector current is zero [90].[12] This is thus an unexciting but expected result.

Now we should note however that in this work we are interested in small fluctuations around equilibrium. If $\omega \neq 0$, then it is no longer required that $E_t(r_h) = 0$ (indeed, in the conventional case of a massless gauge field, this quantity is no longer even gauge invariant). Let us instead allow $E_t(r) \neq 0$ and use the small frequency analysis above to compute the relaxation rate of the axial charge. Here we will make contact with the hydrodynamic calculation above, and we will thus study a situation where $E_t(\infty) = 0$, i.e. there is no axial source applied.

We first use Eq. (5.24) to write down

$$\partial_t j^t_A = 8kJ^{xy}J^{tz} \tag{5.66}$$

This equation holds at all r, (indeed, from above, all of the expressions in it are radially constant), and so we can evaluate it at the boundary to find:

$$-i\omega j^t_A = 64k^2\Sigma(r_h)(J^{tz})^2E_t(r_h) \tag{5.67}$$

[12]Here – as explained in detail in [90] – one must be careful about the distinction between the consistent and covariant currents; the covariant chiral magnetic effect is not zero, but the consistent one receives contributions both from the axial chemical potential and the value of the axial gauge field source, which precisely cancel in equilibrium.

Solving this for ω we find:

$$\omega = 64i \left(\frac{E_t(r_h)}{j_A^t} \right) \Sigma(r_h) \left(kJ^{tz} \right)^2 . \tag{5.68}$$

Thus we see that there is a diffusion pole, where the coefficient of the pole varies as the magnetic field squared. We stress that the approximation made was $\omega \to 0$; from above we see that this also requires that $kJ^{tz} \to 0$. Away from that limit, we expect to see deviations from the quadratic expression above.

Let us also examine the pre-factor of the expression in the limit $kJ^{tz} \to 0$. We see that the ratio of $E_t(r_h)$ and j_A^t appears. We can evaluate this from Eq.(5.54) (with the appropriate boundary conditions: $E_t(r_h) \neq 0$ and $E_t(\infty) = 0$) and Eq.(5.21) to get,

$$\frac{E_t(r_h)}{j_A^t} \underset{r\to\infty}{\to} -\frac{1}{2r_h^2} \equiv -\chi^{-1}. \tag{5.69}$$

We thus find (evaluating $\Sigma(r_h) = \frac{1}{r_h}$ from Eq. (5.39)):

$$\omega = -32ik^2 \left(\frac{b^2}{r_h^3} \right). \tag{5.70}$$

We should compare it to the expectation from elementary hydrodynamics given in Eq. (5.11). Putting in the holographic expressions for the field-theoretical quantities in Eq. (5.11) in the small magnetic field limit, using $\sigma = \frac{1}{\rho} = r_h$, and $\chi = 2r_h^2$, and $B = b$, we find that Eq. (5.11) becomes $\Gamma_A = \frac{32k^2b^2}{r_h^3}$, i.e. precisely the same as the pole exhibited above. This agreement is not surprising; indeed it can be seen that the derivation above parallels in the bulk the hydrodynamic calculation leading to Eq. (5.11).

However, here we see the limitations of the hydrodynamic calculation – in particular, we see explicitly in this holographic model that the analytic calculation is expected to break down if kJ^{tz} is not small; i.e. the result above is valid only in the small b limit. In the next section we explicitly compute the same relaxation rate numerically and compare with lattice results.

In this work we have neglected the backreaction of the charge degrees of freedom on the geometry. Our calculation is also entirely classical, in that we have ignored fluctuations, which in this framework are suppressed by $\frac{1}{N}$, with N a proxy for the number of field-theoretical degrees of freedom. It is reasonable to ask whether such effects will change the picture above. As the actual low-frequency calculation in the bulk essentially exactly parallels the hydrodynamic calculation (given in Sec.5.1), it seems reasonable to expect that such corrections would change individually the values of things like Γ_A and the resistivity ρ, but not change the *relationship* between them that we find here (see for instance Eq. (5.68)). This is broadly the expectation from the usual fluid-gravity correspondence. We note however that there are known examples in a similar hydrodynamic context where loop effects in the bulk can qualitatively change the infrared physics (see e.g. [109, 110]). Generally such effects can be anticipated on field-theoretical grounds, and we return to this issue in the conclusion.

5.5 Numerical results

In this section we calculate the quasi-normal modes of our system using standard holographic techniques.

5.5.1 Contributing equations of motion

From here on we shall work in ingoing Eddington-Finkelstein coordinates (r, v, x, y, z) rather than the Schwarzschild coordinates used above (see Appendix A for a brief review of the coordinate system). First let us give some useful expressions,

$$E^r = (E_v + r^2 f E_r), \qquad E^v = E_r,$$
$$H^{rxy} = \frac{f}{r^2}\partial_r (B_{xy}) - \frac{i\omega}{r^4} B_{xy}, \qquad H^{vxy} = \frac{1}{r^4}\partial_r (B_{xy}).$$

We will study finite frequency fluctuations about the equilibrium solution Eq. (5.40). Consider $H \to H_0 + \delta H$ and $E \to E_0 + \delta E$ with $E_0 = 0$ and $H_0 = H^{rvz}$, which are the background solutions Eq.(5.40) (in the ingoing coordinates). H_0 and E_0 are the background fields and δH and δE are the fluctuations of the fields that we are interested in.

Before proceeding to solve Eq.(5.15) let us first note that we are interested in wave-like solutions for δB and δE of the form $e^{-i\omega v}$. So, we are considering the corresponding wave vector to be of the form $p^\mu = (\omega, \vec{p}) = (\omega, 0)$, i.e. the only non-zero momentum is in the time direction. Furthermore, since the magnetic field is in the z-direction, the little group for fluctuations is $SO(2)$: the group of rotations in the $x - y$ plane. Therefore, we have the following contributing equations of motion in the relevant channels for the fluctuations: in the antisymmetric tensor channel (i.e. from the ($\rho\sigma = xy$) components of Eq.(5.15)),

$$\begin{aligned}
&-\left[\partial_v\left(\sqrt{-g}\ \delta H^{vxy}\right) + \partial_r\left(\sqrt{-g}\ \delta H^{rxy}\right)\right] \\
&-8k\ \left[\partial_v\left(\sqrt{-g}\ \delta E^v \epsilon^{rvzxy} H_{rvz}\right) + \partial_r\left(\sqrt{-g}\ \delta E^r \epsilon^{rvzxy} H_{rvz}\right)\right] = 0. \quad (\rho\sigma = xy)
\end{aligned} \tag{5.71}$$

Similarly, from Eq.(5.16) we have

$$64k^2H^2\ \delta E^v - 4k\epsilon_{rvzxy}H^{rvz}\ \delta H^{vxy} = -\frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}\ \delta G^{rv}\right), \tag{5.72}$$

$$64k^2H^2\ \delta E^r - 4k\epsilon_{rvzxy}H^{rvz}\ \delta H^{rxy} = -\frac{1}{\sqrt{-g}}\partial_v\left(\sqrt{-g}\ \delta G^{vr}\right). \tag{5.73}$$

In the above contributing equations of motion, we have not considered terms with $\partial_x,\ \partial_y,\ \partial_z$ as we have set spatial momenta to vanish. The vector channel involving δE^x, δH^{xyz} decouples and we will not consider it.

In these coordinates, the background solutions (5.40) become,

$$H^{rvz} = \frac{b}{r^3}, \qquad H_{rvz} = -\frac{b}{r}. \tag{5.74}$$

Now we solve for δE^r in Eq.(5.73) to obtain,

$$\delta E_r = \frac{i\omega r^4\ \partial_r(\delta E_v) + 64k^2b^2\delta E_v + 4kbr^2 f\ \partial_r(\delta B_{xy}) - 4kbi\omega\delta B_{xy}}{\tilde{\gamma}}, \tag{5.75}$$

where $\tilde{\gamma} := r^4\omega^2 - 64b^2k^2r^2f$.

Next we give Eqs.(5.71) and (5.72) in terms of the fields with all indices downstairs,

$$\begin{aligned}
&\partial_r^2\left(\delta B_{xy}\right)\left[-rf\right] + \partial_r\left(\delta B_{xy}\right)\left[-f - \frac{4r_h^4}{r^4} + \frac{2i\omega}{r}\right] + \delta B_{xy}\left[\frac{-i\omega}{r^2}\right] \\
&-\partial_r\left(\delta E_v\right)\left[\frac{8kb}{r}\right] + \delta E_v\left[\frac{8kb}{r^2}\right] \\
&-\partial_r\left(\delta E_r\right)\left[8kbrf\right] + \delta E_r\left[-8kbf - \frac{32kbr_h^4}{r^4} + \frac{8kbi\omega}{r}\right] = 0,
\end{aligned} \tag{5.76}$$

$$\frac{-64b^2k^2}{r^4}\delta E_r - \frac{4kb}{r^4}\partial_r\left(\delta B_{xy}\right) = \partial_r^2\left(\delta E_v\right) + \left(\frac{3}{r}\right)\partial_r\left(\delta E_v\right) + i\omega\partial_r\left(\delta E_r\right) + \left(\frac{3}{r}\right)i\omega\delta E_r. \tag{5.77}$$

If we substitute δE_r from Eq.(5.75) into Eqs.(5.76) and (5.77), then we obtain two coupled ODEs – these are somewhat lengthy so we do not present them explicitly, but we solve them numerically below.

5.5.2 Numerics

Now we shall solve Eq.(5.76) and Eq.(5.77) numerically using a mid-point shooting[13] method (see e.g. [111] for a discussion of the shooting method, with some previous applications to quasinormal modes in [112, 113]). Below we present some details of the boundary conditions; the reader interested only in the results can feel free to skip to the next section.

Logarithmic fall-off

B_{xy} has a logarithmic fall-off near the boundary, associated with the fact that the double-trace deformation associated with J^2 on the boundary is marginally (ir)relevant [62]. As explained in detail in that work, the correct boundary condition at the UV cut-off $u = u_\Lambda$ takes the form:

$$B_{xy}\left(u_\Lambda\right) - \frac{J}{\kappa} = 0, \tag{5.78}$$

where $J = u\partial_u B_{xy}$ and κ the double-trace coupling for J^2. The form of B_{xy} we take is,

$$B_{xy}(u) = d_0 + \sum_j d_j\ u^j + \ln(u)\left[d_0' + \sum_j d_j'\ u^j\right] \tag{5.79}$$

where the d_i are expansion coefficients. Using Eq. (5.78) we then find:

$$d_0 + \sum_j d_j\ u^j + \ln(u_\Lambda)\left[d_0' + \sum_j d_j'\ u^j\right] - \frac{1}{\kappa}\left[\sum_j j d_j\ u^j + d_0' + \sum_j d_j'\ u^j + \sum_j\ j\ u^j \ln(z) d_j'\right] = 0.$$

Now at $u = u_\Lambda$, (5.80) becomes (for $u \to 0$),

$$\begin{aligned} d_0 + \ln(u_\Lambda) d_0' - \frac{d_0'}{\kappa} &= 0 \\ d_0 = d_0'\left[\ln\left(e^{\frac{1}{\kappa}}/u_\Lambda\right)\right]. \end{aligned} \tag{5.80}$$

[13] In mid-point shooting method we numerically integrate the boundary solution from boundary and horizon solution from horizon and adjust these till they meet somewhere in the middle and this adjustment yields the QNM.

Thus, at the boundary $(u = 0)$ we obtain B_{xy} as,

$$B_{xy}(u) = d_0' \ln\left(ue^{\frac{1}{\kappa}}/u_\Lambda\right) + \sum_j d_j\ u^j + (\ln(u))\sum_j d_j'\ u^j,$$
$$B_{xy}(u) = d_0' \ln\left(u/u^*\right) + \sum_j d_j\ u^j + (\ln(u))\sum_j d_j'\ u^j. \tag{5.81}$$

where $u^* = u_\Lambda e^{-\frac{1}{\kappa}}$ is an RG-invariant combination of the double-trace coupling and the UV cutoff; this is the analogue of the Landau pole in regular QED, and by dimensional transmutation all physical results can depend on this alone (see [62,114] for a discussion in the holographic context).

Now since B_{xy} at the boundary has a logarithmic fall-off, the coupled nature of the equations of motion as given in Eq.(5.76) and Eq.(5.77) imply that E_v has the following form at the boundary:

$$E_v(u) = u^2\left(c_0 + \sum_j c_j\ u^j\right) + \ln(u)\left(\sum_j cc_j\ u^j\right). \tag{5.82}$$

The logarithm appearing in this boundary condition appears to follow from the fact that the axial current j^μ_A mixes with the 2-form current $J^{\mu\nu}$.

Next we present the numerical results; see Appendix D.1 for further details on the numerical implementation.

Hydrodynamic mode

The lowest quasinormal mode $\omega_l = -i\Gamma_A$ approaches the origin as b approaches zero. For small b (i.e. for $0 < b \leq 3$) it varies with b as

$$\Gamma_A(r_h, b) = 0.048\ \left(\frac{b^2}{r_h^3}\right). \tag{5.83}$$

where the prefactor was obtained from a numerical fit, and where we have restored r_h on dimensional grounds.

Next we move onto some higher values of b to show that away from a small neighbourhood of $b = 0$; the quadratic b^2 behaviour of no longer captures the full dependence and we obtain a more complicated function of b. We consider $0 < b \leq 6.5$ and obtain the behaviour shown in Fig.(5.2), where the orange curve is as given in Eq.(5.83). Note that both the curves above match till about $b \approx 3$.

Now let us compare the above result obtained from numerics to the result obtained in Section 5.4 using the membrane paradigm formalism (in the $kJ^{tz} \to 0$

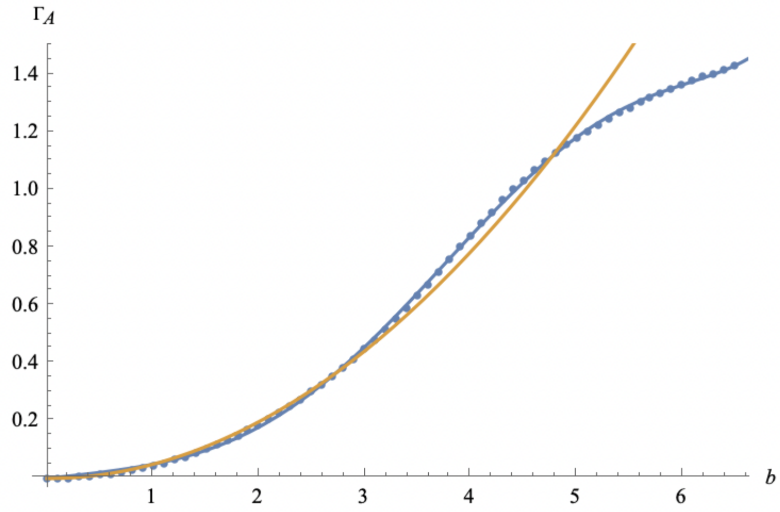

Figure 5.2: Γ_A vs b with $k = 0.0375$ and $r_h = 1$ (blue curve is numerics, orange curve is quadratic fit)

limit) in Eq.(5.70). Note that if we put $k = 0.0375$ in Eq.(5.70) then we find,

$$\Gamma_A(r_h, b) = 0.045 \left(\frac{b^2}{r_h^3}\right). \tag{5.84}$$

in approximate agreement (within 6%) with the small-frequency limit of the numerics. As mentioned in that section, the membrane paradigm analysis also agrees with elementary hydrodynamics arguments arising from treating electrodynamics perturbatively; thus we conclude that at small b the conventional chiral MHD approach from weakly gauged electrodynamics is valid.

However, from Fig.(5.2) we notice that for $b \gg 1$, the functional dependence of Γ_A on b is no longer quadratic, and is a non-trivial function of b. This function now appears to depend on UV physics, and is not simply determined by other thermodynamic quantities such as the susceptibility χ.

For example, we can try to improve the hydrodynamic result for Γ_A in Eq. (5.11) with the holographically determined susceptibility in Eq. (5.56). The resulting plot as a function of the magnetic field $B = b$ is shown in Figure 5.3. It appears barely different from the quadratic dependence as χ does not depend strongly on b. Indeed if we expand this improved hydrodynamic attempt at approximating $\Gamma_A^{\text{improved}}$ as a series in b we get,

$$\Gamma_A^{\text{improved}}(b) = 32\, k^2 \left(\frac{b^2}{r_h^3}\right) + \mathcal{O}(b^4) = 0.045\, b^2 + \mathcal{O}(b^4). \tag{5.85}$$

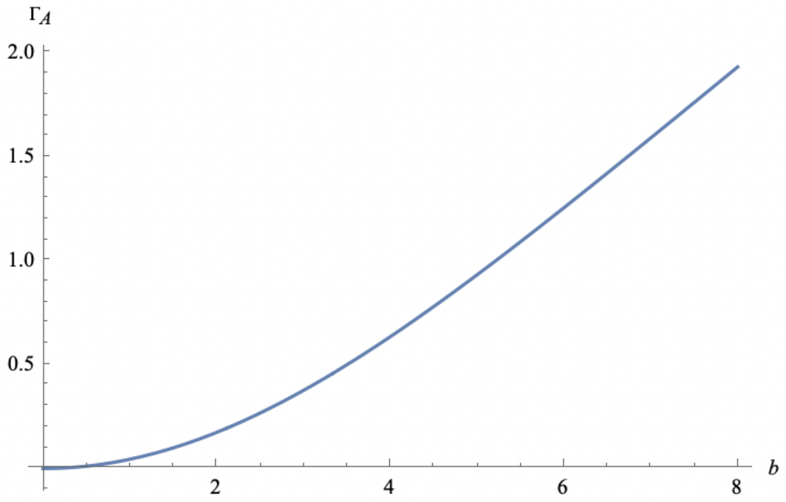

Figure 5.3: $\Gamma_A^{\text{improved}}$ from Eq. (5.11) (i.e. chiral MHD with weakly coupled electrodynamics) as a function of b with $k = 0.0375$ and $r_h = 1$; note it does not capture the non-trivial dependence on b seen in the numerical results of Figure 5.2
.

where in the second equality above we have put $k = 0.0375$ and $r_h = 1$ (for a comparison with the numerical parameters) and the coefficient of $\mathcal{O}(b^2)$ is 100 times that of the coefficient of $\mathcal{O}(b^4)$. Thus the dependence of the charge susceptibility on b is insufficient to account for the non-trivial dependence of $\Gamma_A(b)$, and it appears to not be determined by thermodynamic data.

Finally, in our numerical investigation we also observed many non-hydrodynamic gapped modes. These do not seem to have model-independent relevance, but we discuss them in Appendix D.2.

5.6 Discussion and outlook

In this chapter we have discussed a holographic model that is in the same universality class as a massless Dirac fermion coupled to QED at finite temperature, i.e. where axial charge is non-conserved due to an anomaly with a dynamical operator (involving a topological density constructed from the 2-form current associated with magnetic flux conservation) on the right hand side. We described the bulk dualization process by which we constructed the holographic model and computed some basic observables.

Perhaps our most significant result was explicit computation of the axial charge relaxation rate Γ_A in the presence of a background magnetic field; we found that due to the anomaly the axial charge density j^t_A is not conserved, and instead obeys an equation of the form $j^t_A \sim e^{-\Gamma_A t}$, where Γ_A was numerically found through solving the bulk equations of motion. It is a nontrivial function of the background magnetic field B and the temperature T, and can be seen in Figure 5.2. Note that at small magnetic field this relaxation rate is quadratic in the field B; indeed as the pole approaches the origin the pre-factor may be computed analytically from the small frequency limit of the bulk equations of motion. This pre-factor can also be obtained from elementary magneto-hydrodynamic arguments that essentially treat the anomaly coefficient perturbatively, as reviewed in Section 5.1. The resulting pre-factor agrees with our holographic computation, and we thus find that at small magnetic fields:

$$\Gamma_A^{\text{hol}}(B \to 0) \approx \gamma^{\text{MHD}} B^2 \tag{5.86}$$

At larger magnetic fields this is a non-trivial function of B and can only be obtained from a full holographic treatment. [14]

We now discuss the previous literature on this result. A lattice study of a field-theoretical model in the same universality class was recently performed in [80,81]; in particular, [80] studied the diffusion of the operator $F \wedge F$ (which can be related to the charge relaxation rate by a fluctuation-dissipation argument), and [81] directly measured Γ_A. Those works numerically observe an expression of the form:

$$\Gamma_A^{\text{lattice}}(B) \approx \gamma^{\text{lattice}} B^2, \tag{5.87}$$

Interestingly, those works found that $\gamma^{\text{lattice}}/\gamma^{\text{MHD}} \approx 10$, i.e. that the pre-factor obtained from the lattice differs from the hydrodynamic estimate by an order of magnitude [81]. In those works it was argued that this means that short-distance physics that is not taken into account in the hydrodynamic analysis is important. Interestingly, this is not what we find from our (UV-complete) holographic calculation; instead our holographic result precisely coincides with the hydrodynamic result at small magnetic fields, differing from it only at larger fields when the magnetic field *itself* probes UV scales.

It is interesting to speculate on the cause of this discrepancy between the lat-

[14] We note that Eq.(5.11) states that the relaxation rate vanishes in the limit of vanishing magnetic field. It is at the moment not clear to us whether this is an artifact of the classical description; for example it is possible that when one includes fluctuations there is a non-vanishing relaxation rate even at zero magnetic field. In principle this could be evaluated using an appropriate Kubo formula of the topological density; we take up this computation in chapter 7. and we thank Luca Delacrétaz for this comment [115].

tice and holography. An ingredient entering into the computation of $\gamma^{\rm MHD}$ is the resistivity of the electromagnetic sector; in holography it is very easy to see how this enters into the calculation and separate it from the anomalous dynamics but in a purely field-theoretical treatment it seems possible that uncertainties in this conductivity – a notoriously complicated quantity to calculate from first principles – could cloud this analysis, as was already suggested in [81]. It would be interesting to perform further tests of this hypothesis, perhaps by computing more observables from holography and the lattice and comparing them further.

If we take our results at face value, it suggests that for this observable, a hydrodynamic treatment of conventional MHD (treating the anomaly as a perturbation) is sufficient at weak magnetic fields, though it differs quantitatively from the true result at stronger fields.[15]

There are many directions for future research. Our bulk action Eq.(5.37) permits the explicit study of a strongly interacting system in the same universality class as the chiral plasma. It would be very interesting to understand other phenomena, e.g. if the instabilities (due to non-vanishing $\vec{p}$) [94] exist in this model, or the study of chiral magnetic waves [117].

From a field-theoretical point of view, it would be very interesting to go further than our holographic considerations and construct a true effective hydrodynamic theory for this system. Indeed this was one of the motivations for our construction of the holographic action Eq. (5.37), though it is sufficiently complicated that it does not shed much immediate light on how (or whether) an effective description could be computed. Indeed, the prospect of such an analysis is clouded by the fact that we are not aware of a completely universal field-theoretical description of the anomaly Eq. (5.1); conventional lore would tell us the axial symmetry is simply completely broken, though as we have argued this appears to miss important physics associated with the fact that it is broken not by a generic operator but rather by a topological density $\star J \wedge \star J$ constructed from the current of a 1-form symmetry. Indeed, along these lines recent work describes a novel higher group structure that is present when the axial symmetry is spontaneously broken [118–120]. It would be very interesting to understand whether such an analysis could be extended to the phase when the axial symmetry is unbroken or realized at finite temperature, as a first step towards constructing a hydrodynamic EFT. This is what we take up next in the following

[15] This is philosophically aligned with previous results [116] that argue that various transport coefficients are generically renormalized if the gauge fields sourcing the anomaly are dynamical.

chapter.

CHAPTER 6

EFT and Generalised symmetries

In this chapter, motivated by the results and discussion of the previous chapter, we discuss some progress towards formulating an effective theory of the chiral magnetohydrodynamic plasma. As discussed earlier, this can be understood as a finite-temperature system with a $U(1)_A$ current that is not conserved due to the Adler-Bell-Jackiw anomaly [121, 122], i.e. we have

$$d \star j_A = -\frac{1}{4\pi^2} F \wedge F \,, \qquad \partial_\mu j^\mu_A = -k\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{6.1}$$

if we consider ordinary QED with massless Dirac fermions[1] coupled to dynamical electromagnetism, we obtain this expression with $k = \frac{1}{16\pi^2}$.

In the case of an ABJ anomaly, naively one might assume that as the current is not conserved, the corresponding symmetry is simply explicitly broken and plays no role in constraining the dynamics. This statement is somewhat too fast: indeed, recent work [37, 38] has shown that in such a system, one can construct topological defect operators that count the axial charge, but these defect operators no longer obey a simple group composition law – in other words the symmetry becomes *non-invertible* (a partial list of references on non-invertible symmetries in higher dimensions are [45, 123–133]). This constitutes a precise non-perturbative characterization

[1] A single species of Dirac fermions also has a $U(1)^3_A$ 't Hooft anomaly which we will ignore here; it could be included by using the well-understood technology for the hydrodynamics of 't Hooft anomalies [91, 92].

of the manner in which the ABJ anomaly deforms the naive classical symmetry, and makes clear that – at least in the vacuum – a system with an ABJ anomaly is in a distinct universality class to one with no $U(1)$ symmetry at all. The understanding of the dynamical consequences of such a symmetry is still in its infancy; see e.g. [41] for an extension of Goldstone's theorem to this setting and [58] for gauging of such non-invertible symmetries.

The understanding of this non-invertible symmetry allows us to give a universal characterization of the chiral magnetohydrodynamic plasma: it is a system which realizes the non-invertible symmetry of [37,38] at finite temperature. In this chapter we make some attempts at describing the plasma from this point of view.

The existence of such non-invertible defect implies that there exist a conserved 2-form current $J^{\mu\nu}$ and a 1-form current j^{μ} which satisfy the following Ward identity[2]:

$$\partial_\mu J^{\mu\nu} = 0 \qquad \partial_\mu j^\mu = k\,\epsilon^{\mu\nu\rho\sigma} J_{\mu\nu} J_{\rho\sigma} \tag{6.2}$$

If the system admits a weakly coupled description in terms of a $U(1)$ photon whose field strength is $F_{\mu\nu}$, then $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}$; however we will not assume such a description in what follows. It is our understanding that any system which has vector and tensor fields that obey these two operator equations will allow the construction of the appropriate non-invertible defect operators; we will verify this in our constructions below. Our task is to understand how the symmetry structure Eq. (6.2) is realized in thermal equilibrium (and for small fluctuations around it).

Note on convention: In this chapter, we shall label the $U(1)$ gauge field as A^μ instead of a^μ and so the corresponding field strength will be labelled by $F = dA$ instead of $f = da$. This is because we will reserve the symbol a^μ to label an external gauge field which would couple to the dimensionally reduced chiral current j^μ on $\mathcal{M} = S^1 \times \mathbb{R}^3$. Furthermore, we shall label Dirac fermions by Ψ instead of ψ since the latter symbol will be used to label a 3D Goldstone mode. These notational changes are for this chapter only.

[2] Note that from here on we shall drop the subscript A from the non-conserved 1-form current j^μ since we shall be studying a general effective theory which is in the same universality class as that of QED at finite temperature.

6.1 Comparison to other approaches

The construction of the hydrodynamic description of a theory with an ABJ anomaly has a long history. For the convenience of the reader we briefly summarize some of this literature.

Early work in describing MHD in the presence of a finite chemical potential μ_A for the axial charge includes [86]. They show the generic existence of instabilities in the presence of finite μ_A.

Ref. [85] constructs a description of the axial charge density in terms of an effective dynamical axion θ.

They work in the limit where the fluid velocity is frozen $u^\mu = \delta^\mu_t$, and the presence of the dynamical axion means that the construction somewhat resembles an axial superfluid; in particular their equations of motion depend on spatial gradients of θ. A generalization of [85] to the case where the fluid velocity u^μ is dynamical can now be found in [134]. Upon taking the limit of small magnetic diffusivity (i.e. that controls the diffusion of 1-form current), they found that $\partial_i\theta$ dependence drop out of the equations of motion. This set of equations is what is generally referred to as *chiral magnetohydrodynamics*, and has numerous astrophysics applications. A relativistic extension to a generic spacetime can be found in [135].

An interesting construction of dissipative chiral MHD from first principles using an entropy current was performed in [94]. The consistency of their derivative expansion required the anomaly coefficient k in Eq. (6.1) to be a small parameter of order $\mathcal{O}(\partial^1)$. Another construction for the relativistic hydrodynamic description can be found in [93], where the effect of the anomaly on the electric charge current [91] is added to the Maxwell equation.

Finally, an equilibrium effective action for the theory with 't Hooft anomaly organised with $u^\mu, T \sim \mathcal{O}(\partial^0)$ with the background gauge fields be $\mathcal{O}(\partial^0)$ and field strengths be $\mathcal{O}(\partial^1)$ can be found in e.g. [116, 136, 137]. In [116] the $U(1)$ current is weakly gauged, resulting in a system with an ABJ anomaly, and it was found that most transport coefficients receive radiative corrections.

One major difference between our work and those earlier is that the bulk of the literature assumes the existence of a $U(1)^{(0)}_V$ vector electrical charge current, which is then coupled to dynamical electromagnetism in some manner, assuming some dynamics (e.g. Maxwell) for the electromagnetic sector. Philosophically, this can be thought of as weakly gauging a theory with a 't Hooft anomaly to convert it into an ABJ anomaly.

From a modern point of view, however, the introduction of a photon which then interacts strongly with the plasma seems like an unnecessary intermediate step. An alternative approach is to attempt to bypass the weakly gauged construction completely, and simply directly attempt to describe the global symmetry structure, analogous to what was done for pure MHD in [30]. Here we take some steps in that direction; i.e. we try construct an action-based approach to chiral MHD by realizing the symmetry structure Eq. (6.2) directly in a minimal fashion without constructing an electric charge current or coupling it to a Maxwell field.

Our work here has something of an exploratory character; we do not write down the most general actions possible, rather constructing the simplest actions that display the required physics. We also always work in a limit where the fluid velocity and temperature are frozen. Nevertheless, we will see that this is sufficient to reproduce many aspects of chiral MHD phenomenology.

We now present a brief summary of this chapter. In section 6.2 we present a brief overview of the Schwinger-Keldysh or Closed Time Path (CTP) formalism following [13]. In Section 6.3, we study the equilibrium sector of the hydrodynamic theory by placing the theory on $S^1 \times \mathbb{R}^3$ and constructing an equilibrium generating functional. We study the decomposition of the symmetry breaking pattern under dimensional reduction and present an algorithm (order by order in the anomaly coefficient k) to compute the part of the action that is not invariant under gauge transformations of the axial source. We demonstrate that this construction leads to the chiral separation effect.

After analysing the equilibrium sector, next we move onto the dissipative sector in Section 6.4. We construct a real time effective action using the Schwinger-Keldysh formalism. We do this by "gluing" together two independent theories for the 0-form and 1-form sectors in a way that preserves the anomaly structure, resulting in a dissipative action whose variation results in the expected equations of motion. A shortcoming with this construction (described in detail below) is that we are unable to preserve the so-called "diagonal shift" symmetry that is present in usual hydrodynamic actions describing 0-form symmetries in a normal phase. We conclude with a brief discussion in Section 6.5.

We also refer the reader to the following recent work [138], which takes a different approach towards constructing an Schwinger-Keldysh effective action of the chiral plasma.

6.2 Brief overview of Schwinger-Keldysh formalism

In this section we present a brief overview of the Schwinger-Keldysh formalism which we shall use to construct our effective field theory capturing dissipation. We shall follow closely the exposition as given in [13]. As we go along, we shall see the advantages of this approach of constructing effective theories in later parts of this section.

To begin, let us consider an initial state at time t_0 described by a density matrix ρ_0. Its time evolution is given as,

$$\rho(t) = U(t, t_i)\, \rho_0\, U^\dagger(t, t_i), \tag{6.3}$$

where $t > t_i$ and $U(t, t_i)$ is the unitary time evolution operator from t_i to t [3]. Here, we have suppressed all spatial dependence and we will do so in this section. Since the density matrix ρ_0 has two legs, its time evolution can be understood by considering two real time contours as given in the left hand side of Fig. 6.1. Thus, $\rho(t)$ can be understood as the path integrals corresponding to these two contours – one going forward in time from t_i to t under the action of U and one going backward in time from t to t_i under the action of $U^\dagger$.

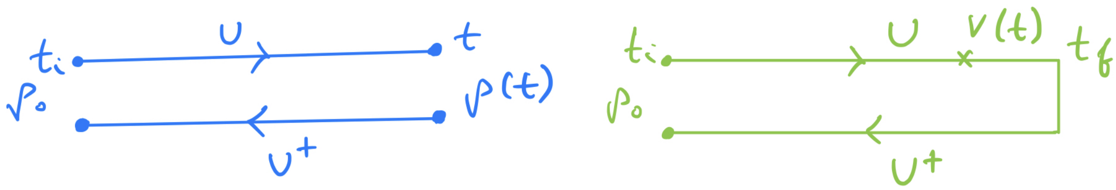

Figure 6.1: On the left hand side, we have $\rho(t)$ described as two path integrals with one describing a forward time evolution and the other describing a backward time evolution. On the right hand side, we have the path-integral representation of the expectation value with an operator insertion. This given correlator can be evaluated without having any backward time evolution.

Using the above path-integral representation for $\rho(t)$, we can now consider ex-

[3] Note that, here the time evolution of the density matrix is performed by the usual conjugation action carried out by the unitary operator U, but in an order opposite to the time evolution of operators in the Heisenberg picture. A quick way to see this oppositely ordered conjugation is by considering: $\rho_0 \equiv |\psi\rangle\langle\psi|$ which is nothing but the density matrix of a pure quantum state $|\psi\rangle$, and then performing its time evolution.

pectation value of the form,

$$\mathrm{Tr}(\rho(t)V) = \mathrm{Tr}(\rho_0 V(t)) \equiv \langle V(t)\rangle_{\rho_0}, \tag{6.4}$$

where the operator V is inserted at some time t in one of the contours of Eq. (6.3) and taking the trace amounts to joining the two contours at some time $t_f > t$. This resulting contour is sometimes called the *closed time path* (see right hand side of Fig. 6.1). Note that, here we only require a forward time evolution as the operator $V(t)$ is inserted along the forward-evolving contour. We can also consider more general correlation functions of the kind depicted in Fig. 6.2,

$$\langle W(t_4)V(t_2)W(t_3)V(t_1)\rangle_{\rho_0} = \mathrm{Tr}(\rho_0 W(t_4)V(t_2)W(t_3)V(t_1)), \tag{6.5}$$

where, $t_1 < t_2 < t_4 < t_3$ and computing this correlator involves, at certain path-integral segments, to go backward in time.

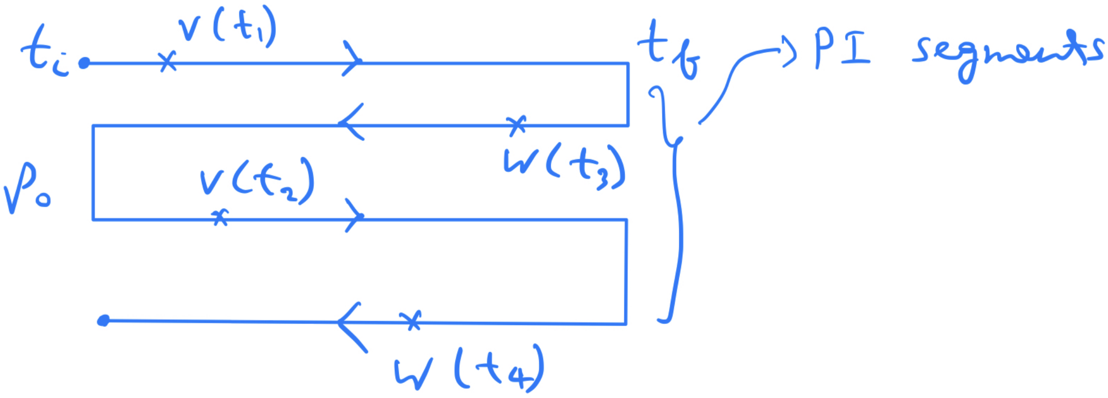

Figure 6.2: This is the path-integral representation of the correlation function $\langle W(t_4)V(t_2)W(t_3)V(t_1)\rangle_{\rho_0}$ with $t_1 < t_2 < t_4 < t_3$. Note that, here we see that backward time evolution needs to be taken into account to evaluate this given correlator.

Now let us discuss an ordering of general correlators in the above formalism. For this consider the following correlator (see Fig. 6.3),

$$\langle \mathcal{P}(V_1(t_1)W_1(t_2)V_1(t_3)W_2(t_4)V_2(t_5))\rangle_{\rho_0} = \langle \widetilde{\mathcal{T}}(W_2(t_4)V_2(t_5))\mathcal{T}(V_1(t_1)W_1(t_2)V_1(t_3))\rangle_{\rho_0}, \tag{6.6}$$

where $t_1 < t_5 < t_2 < t_4 < t_3$. The subscript '1' denotes operators inserted along the forward-evolving (or upper) contour and the subscript '2' denotes the operators inserted along the backward-evolving (or lower) contour. On the left hand side of the equation we have a path-ordering denoted by $\mathcal{P}$. On the right hand side of Eq. (6.6), we have made explicit that operators inserted on the upper segment are time-ordered (denoted by $\mathcal{T}$), while those on the lower segment are anti time-ordered (denoted by $(\widetilde{T})$), and the operators on the second segment always lie to the left of those on the first segment in the path-ordering. This is the prescription for ordering of general correlators in the Schwinger-Keldysh formalism.

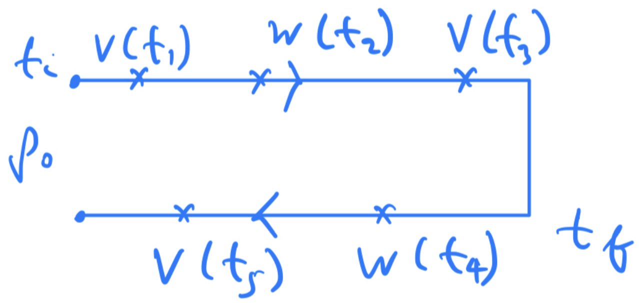

Figure 6.3: This is the path-integral representation of a correlator on the closed time path (CTP).

6.2.1 Connected correlators and Green's functions

Let us now discuss how to obtain connected correlators from a generating functional involving operators insertions on the CTP. First of all note that, on the CTP we will have two copies of the fields, leading to a doubling of fields of our system, since we have two real-time contours. With this in mind, consider the following generating functional of the fields[4],

$$\exp\left(W[\phi_{1i},\phi_{2i}]\right) = \mathrm{Tr}\left\{\rho_0\,\mathcal{P}\left(\exp\left[i\int dt\ \left(\mathcal{O}_{1i}(t)\phi_{1i}(t) - \mathcal{O}_{2i}(t)\phi_{2i}(t)\right)\right]\right)\right\}, \quad (6.7)$$

where $\mathcal{O}_i$ denote generic operators and ϕ_i their corresponding sources. The '$-$' sign on the right hand side in the above equation comes from the fact that on the lower segment on CTP there is a anti-time ordering leading to a reverse direction of the temporal integration. Note that, $\mathcal{O}_{1i}$ and $\mathcal{O}_{2i}$ refer to the same operator but on different segments of the CTP while ϕ_{1i} and ϕ_{2i} are different fields. For simplicity, we will take all sources ϕ_i to be real and all the operators $\mathcal{O}_i$ to be Hermitian and bosonic.

[4]Here we will take $t_i \to -\infty$ and $t_f \to \infty$ for convenience.

Now let us move onto another basis, which is convenient to use for certain calculations, called the $r-a$ basis. The relation between the $1-2$ and $r-a$ bases is the following,

$$\phi_{ri} = \frac{1}{2}(\phi_{1i} + \phi_{2i}), \tag{6.8}$$

$$\phi_{ai} = \phi_{1i} - \phi_{2i}, \tag{6.9}$$

where the r-type fields are physical and a-type fields are noise. In this basis, Eq. (6.7) can be written as,

$$\exp\left(W[\phi_{ri}, \phi_{ai}]\right) = \mathrm{Tr}\left\{\rho_0\, \mathcal{P}\left(\exp\left[i\int dt\ \left(\phi_{ai}(t)\mathcal{O}_{ri}(t) + \phi_{ri}(t)\mathcal{O}_{ai}(t)\right)\right]\right)\right\}, \tag{6.10}$$

We can now obtain path-ordered correlation functions by taking functional derivatives of W with respect to ϕ_is and then setting the sources to zero.

$$G_{\alpha_1\ldots\alpha_n}(t_1, \ldots, t_n) \equiv \frac{1}{i^{n_r}} \frac{\delta^n W}{\delta\phi_{\bar{\alpha}_1}(t_1)\ldots\delta\phi_{\bar{\alpha}_n}(t_n)}\bigg|_{\phi_{a,r}=0}, \tag{6.11}$$

$$= i^{n_a}\langle\mathcal{P}(\mathcal{O}_{\alpha_1}(t_1)\ldots\mathcal{O}_{\alpha_n}(t_n))\rangle, \tag{6.12}$$

where $\alpha_1, \ldots, \alpha_n \in (a, r)$, $\bar{\alpha} = r, a$ for $\alpha = a, r$. n_r, n_a denotes the number of r, a indices respectively in $\{\alpha_1, \ldots, \alpha_n\}$ and $n_r + n_a = n$. One can verify that know Green's functions like the advanced, retarded, symmetric Green's functions can now be expressed in terms of the $r-a$ Green's functions defined above as follows (see [13] for more details),

$$\begin{aligned} G_{ra}(x_1, x_2) &= G^R(x_1, x_2), \\ G_{ar}(x_1, x_2) &= G^A(x_1, x_2), \\ G_{rr}(x_1, x_2) &= G^S(x_1, x_2), \end{aligned} \tag{6.13}$$

where G^R, G^A and G^S denote the retarded, advanced and the symmetric Green's functions respectively.

6.2.2 KMS condition

In this subsection, we shall see how the Kubo-Martin-Schwinger (KMS) condition is implemented on the CTP. Let us begin by considering an observable A in the

Heisenberg picture,

$$A(t) = e^{itH} A(0) e^{-itH},$$

whose expectation value on a thermal state at temperature $T = \beta^{-1}$ is given as,

$$\langle A \rangle_\beta = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H} A), \qquad \text{with, } Z \equiv \mathrm{Tr}(e^{-\beta H}),$$

Now let us define the following Green's functions,

$$\begin{aligned} G_+^\beta(t, A, B) &\equiv \langle A_t B \rangle_\beta, \\ &= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H} e^{itH} A e^{-itH} B), \\ &= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H} A e^{-itH} B e^{itH}), \\ &= \langle A B_{-t} \rangle_\beta, \end{aligned} \tag{6.14}$$

where the third equality above follows from the cyclic property of the trace and noting that $[e^{-\beta H}, e^{\pm itH}] = 0$. Let us now define another similar Green's function,

$$\begin{aligned} G_-^\beta(t, A, B) &\equiv \langle B A_t \rangle_\beta, \\ &= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H} B e^{itH} A e^{-itH}), \\ &= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H} e^{-itH} B e^{itH} A), \\ &= \langle B_{-t} A \rangle_\beta, \end{aligned} \tag{6.15}$$

Thus, using the above we get,

$$G_+^\beta(t, A, B) - G_-^\beta(t, A, B) = \langle [A_t, B] \rangle. \tag{6.16}$$

Let us analytically continue to imaginary times with $z = t + is$ and $t, s \in \mathbb{R}$. Then, we get,

$$G_+^\beta(z, A, B) = \frac{1}{Z}\mathrm{Tr}(e^{i(z+i\beta)H} A e^{-izH} B), \tag{6.17}$$

$$G_-^\beta(z, A, B) = \frac{1}{Z}\mathrm{Tr}(B e^{izH} A e^{-i(z-i\beta)H}), \tag{6.18}$$

$$\text{implying, } G_+^\beta(z - i\beta, A, B) = G_-^\beta(z, A, B), \tag{6.19}$$

$$\text{or, for } z = t \in \mathbb{R} : \langle A_{t-i\beta} B \rangle_\beta = \langle B A_t \rangle_\beta, \tag{6.20}$$

where the last equation above is called the *KMS condition* and it reflects the periodicity of the Green's functions in imaginary time (for more details see [139]).

Now recall from Eq. (6.7) that,

$$e^{W[\phi_{1i},\phi_{2i}]} = \mathrm{Tr}\left\{\rho_0\, \widetilde{\mathcal{T}}\left(\exp\left[-i\int dt\, \mathcal{O}_{2i}(t)\phi_{2i}(t)\right]\right)\mathcal{T}\left(\exp\left[i\int dt\, \mathcal{O}_{1i}(t)\phi_{1i}(t)\right]\right)\right\}, \tag{6.21}$$

where we have expressed Eq. (6.7) in terms of time-ordered and anti time-ordered factors. Now, consider: $\rho_0 = \frac{1}{Z_0}e^{-\beta_0 H}$ with $Z_0 \equiv \mathrm{Tr}(e^{-\beta_0 H})$. Then, Eq. (6.21) becomes,

$$\begin{aligned} e^{W[\phi_{1i},\phi_{2i}]} &= \frac{1}{Z_0}\mathrm{Tr}\left\{e^{-\beta_0 H}\, \widetilde{\mathcal{T}}\left(\exp\left[-i\int dt\, \mathcal{O}_{2i}(t)\phi_{2i}(t)\right]\right)\mathcal{T}\left(\exp\left[i\int dt\, \mathcal{O}_{1i}(t)\phi_{1i}(t)\right]\right)\right\}, \\ &= \frac{1}{Z_0}\mathrm{Tr}\left\{e^{-(\beta_0-\theta)H}\widetilde{\mathcal{T}}\left(\exp\left[-i\int dt\, \mathcal{O}_{2i}(t)\phi_{2i}(t)\right]\right)e^{(\beta_0-\theta)H}e^{-\beta_0 H}e^{\theta H}\times \right. \\ &\left. \times\, \mathcal{T}\left(\exp\left[i\int dt\, \mathcal{O}_{1i}(t)\phi_{1i}(t)\right]\right)\right\}, \end{aligned} \tag{6.22}$$

where $\theta \in [0,\beta_0]$ is a constant and by the cyclic property of trace all the $e^{\pm\theta H}$ factors cancel off to identity and hence the inclusion of such factors above makes sense. Now, note that for arbitrary $a \in [-\beta_0,\beta_0]$, we have, for the anti time-ordered factor (inside thermal averages):

$$e^{-aH}\widetilde{\mathcal{T}}\left(\exp\left[i\int dt\, \mathcal{O}_{1i}(t)\phi_{1i}(t)\right]\right)e^{aH} = \widetilde{\mathcal{T}}\left(\exp\left[i\int dt\, \mathcal{O}_{1i}(t)\phi_{1i}(t-ia)\right]\right), \tag{6.23}$$

and similarly for the time-ordered factor. Using this, we can write Eq. (6.22) as,

$$\begin{aligned} e^{W[\phi_{1i},\phi_{2i}]} &= \frac{1}{Z_0}\mathrm{Tr}\left\{e^{-\beta_0 H}\mathcal{T}\left(\exp\left[i\int dt\, \mathcal{O}_{1i}(t)\phi_{1i}(t+i\theta)\right]\right)\times \right. \\ &\left. \widetilde{\mathcal{T}}\left(\exp\left[i\int dt\, \mathcal{O}_{2i}(t)\phi_{2i}(t-i(\beta_0-\theta))\right]\right)\right\}, \qquad (6.24) \\ &\equiv \exp\left[W_T[\phi_{1i}(t+i\theta),\phi_{2i}(t-i(\beta_0-\theta))]\right], \qquad (6.25) \end{aligned}$$

where since in the first equation above the time-ordering appears before the anti time-ordering, we have introduced a new notation to denote this as W_T. See Fig. 6.4 for integration contours defining W_T.

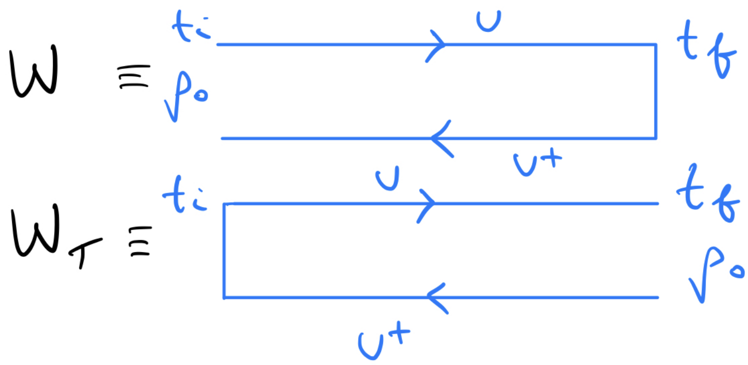

Figure 6.4: The top figure defines the integration contour for W and the bottom figure defines the integration contour for W_T.

Note that, Eq. (6.24) is the KMS condition now expressed in the of Schwinger-Keldysh formalism. As evident from Fig. 6.4, the KMS condition relates correlators with ρ_0 as the initial state to correlators with ρ_0 as the final state.

Now say that at the microscopic level the system has a discrete symmetry Θ which can include any combination of C, P, T such that $[\Theta, H] = 0$. Then, this can be combined with the KMS condition above to get the following constraint on W (for more detailes see [13]),

$$W[\phi_{1i}(x), \phi_{2i}(x)] = W[\tilde{\phi}_{1i}(x), \tilde{\phi}_{2i}(x)], \tag{6.26}$$

where we have restored the spatial dependence and $x^\mu = (t, \vec{x})$ and,

$$\begin{aligned} \tilde{\phi}_{1i}(x) &= \Theta\phi_{1i}(t - i\theta, \vec{x}), \\ \tilde{\phi}_{2i}(x) &= \Theta\phi_{2i}(t + i(\beta_0 - \theta), \vec{x}), \end{aligned} \tag{6.27}$$

for arbitrary constant: $\theta \in [0, \beta_0]$.

6.2.3 Application: a theory of diffusion

In this subsection, we shall apply the above formalism to construct a Schwinger-Keldysh effective theory to incorporate diffusion. Consider a system with $U(1)$ symmetry at fixed temperature which in turn implies that we are freezing the stress-tensor degrees of freedom. Thus, the only conserved current in this system will be the $U(1)$ current J^μ. In the Schwinger-Keldysh formalism we need to consider two copies of this current. Let us couple these currents to sources (or background gauge fields) and write down the generating functional.

$$\exp\left(W[A_{1\mu}, A_{2\mu}]\right) = \mathrm{Tr}\left\{\rho_0\, \mathcal{P}\left(\exp\left[i\int d^d x\, A_{1\mu} J_1^\mu - i\int d^d x\, A_{2\mu} J_2^\mu\right]\right)\right\}. \tag{6.28}$$

The advantage of the above generating functional W is that conservation of the currents J_1^μ and J_2^μ now boils down to invariance of the above path-ordered path integral under gauge transformations of the background gauge fields: $A_{1\mu}$ and $A_{2\mu}$.

$$W[A_{1\mu}, A_{2\mu}] = W[A_{1\mu} + \partial_\mu \lambda_1, A_{2\mu} + \partial_\mu \lambda_2], \tag{6.29}$$

for arbitrary functions $\lambda_{1,2}$ [5]. Now the goal is to write down an effective field theory (EFT) for collective variables associated with the currents: $J^\mu_{1,2}$ of the following form:

$$\exp\left(W[A_{1\mu}, A_{2\mu}]\right) = \int \mathcal{D}\phi_1\, \mathcal{D}\phi_2 \exp\left(iI_{\mathrm{EFT}}\left[\phi_{1,2}; A^\mu_{1,2}\right]\right), \tag{6.30}$$

such that Eq. (6.29) is satisfied and equations of motion for $\phi_{1,2}$ should be equivalent to conservation of currents $J^\mu_{1,2}$. For this to work, $\phi_{1,2}$ have to be scalar fields and they should appear with the external fields $A^\mu_{1,2}$ in the following combination:

$$B^{1,2}_\mu \equiv A^{1,2}_\mu + \partial_\mu \phi^{1,2}, \tag{6.31}$$

which implies that $\phi_{1,2}$ are Stueckelberg fields associated with the gauge symmetries as given in Eq. (6.29). These will serve as the dynamical hydrodynamic variables. Thus, we have for the effective action,

$$\exp\left(W[A_{1\mu}, A_{2\mu}]\right) = \int \mathcal{D}\phi_1\, \mathcal{D}\phi_2 \exp\left(iI_{\mathrm{EFT}}\left[B_{1\mu}, B_{2\mu}\right]\right). \tag{6.32}$$

Now $B^{1,2}_\mu$ and $I_{\mathrm{EFT}}[B^{1,2}_\mu]$ are invariant under the following gauge transformations,

$$A^{1,2}_\mu \to A^{1,2}_\mu - \partial_\mu \lambda^{1,2}, \qquad \phi^{1,2} \to \phi^{1,2} + \lambda^{1,2}, \tag{6.33}$$

Now we can define "off-shell" hydrodynamic currents $\hat{J}^\mu_{1,2}$ from the functional variation of this effective action,

$$\hat{J}^\mu_1(x) \equiv \frac{\delta I_{\mathrm{EFT}}}{\delta A_{1\mu}(x)}, \qquad \hat{J}^\mu_2(x) \equiv -\frac{\delta I_{\mathrm{EFT}}}{\delta A_{2\mu}(x)}, \tag{6.34}$$

[5] Here we assume that $\lambda_{1,2}$ vanish at spacetime infinities.

It then readily follows from Eq. (6.32) that,

$$\frac{\delta I_{\rm EFT}}{\delta\phi_{1,2}(x)} = -\partial_\mu \hat{J}^\mu_{1,2}(x) = 0, \tag{6.35}$$

Note that correlation functions of currents $J^\mu_{1,2}$ for the full theory Eq. (6.28) are given by those of the off-shell currents $\hat{J}^\mu_{1,2}$ in the EFT Eq. (6.32). For instance,

$$\langle \mathcal{P}\left(J^\mu_1(x)J^\nu_2(y)\right)\rangle = -\left.\frac{\delta W}{\delta A_{1\mu}(x)A_{2\mu}(y)}\right|_{A_1=A_2=0} = \int \mathcal{D}\phi_1\mathcal{D}\phi_2\, e^{I_{\rm EFT}[\partial_\mu\phi_1,\partial_\mu\phi_2]}\hat{J}^\mu_1(x)\hat{J}^\nu_2(y). \tag{6.36}$$

To describe diffusion let us now restrict to a system at finite temperature, and assume that the system is in a liquid phase. At this stage, there is still a distinction of a normal phase and a superfluid phase where the $U(1)$ symmetry is spontaneously broken. Interestingly, if one directly writes down the most general local derivative expansion of $I_{\rm EFT}\left[B^{1,2}_\mu\right]$, the theory describes a superfluid phase. To describe a normal phase one needs to impose a further symmetry. To understand this, note that, given the $U(1)$ symmetry, each local fluid element can have an independent phase rotation which should be independent of time, that is, phases of the form: $e^{i\lambda(\vec{x})}$. Now, physically we can view the system as a continuum of fluid elements, and interpret $B_{s\mu}$ with $(s = 1,2)$ as the "local" external sources for these fluid elements, which include not only external background gauge fields $A_{s\mu}$, but also contributions from dynamical variables ϕ_s. For example, we can define the local chemical potentials as,

$$\mu_s(x) = B_{s0}(x). \tag{6.37}$$

Now due to above spatial phase-rotation argument, to describe a system for which the $U(1)$ symmetry is not spontaneously broken, we require the EFT to be invariant under a time-independent, gauge transformations of $B^{1,2}_\mu$ which we will refer as the *chemical shift symmetry* [6],

$$B_{1i} \to B_{1i} - \partial_i\lambda(\vec{x}), \qquad B_{2i} \to B_{2i} - \partial_i\lambda(\vec{x}), \tag{6.38}$$

$$\text{or, equivalently: } \phi_r \to \phi_r - \lambda(\vec{x}), \qquad \phi_a \to \phi_a. \tag{6.39}$$

[6]Here i denotes the spatial coordinates.

To summarize, in order to write down the EFT for hydrodynamic variables corresponding to a conserved $U(1)$ current in a normal phase, we need to impose the following conditions on $I_{\text{EFT}}\left[B_\mu^{1,2}\right]$ (for more details see [13]):

- The KMS symmetry: $I_{\text{EFT}}\left[B_\mu^{1,2}\right] = I_{\text{EFT}}\left[\tilde{B}_\mu^{1,2}\right]$ where the relation between tilde and un-tilde fields is given in Eq. (6.27). This is also sometimes referred to as the *dynamical KMS symmetry.*
- Unitary constraints
- Rotation and translation symmetries
- chemical shift symmetry as given in Eq. (6.38)

With this, one can write down the final effective Lagrangian density at quadratic order in $B_{r,a}$, and to linear order in derivatives as,

$$\mathcal{L}_{\text{EFT}} = \frac{i\sigma}{\beta_0}(B_{ai})^2 + \chi B_{a0}B_{r0} - \sigma B_{ai}\partial_0 B_{ri}, \tag{6.40}$$

with $\sigma \geq 0$ and χ as constants. We can get the off-shell currents as,

$$\begin{aligned} \hat{J}_{r0} &= \chi\mu, \qquad \hat{J}_{ri} = \sigma(E_i - \partial_i\mu) + \frac{i\sigma}{\beta_0}B_{ai}, \\ \hat{J}_{a0} &= \chi B_{a0}, \qquad \hat{J}_{ai} = \sigma\partial_0 B_{ai}, \end{aligned} \tag{6.41}$$

where we have introduced the local chemical potential, $\mu \equiv B_{r0} = A_{r0}+\partial_0\phi_r$ and the background electric field, $E_i \equiv \partial_i A_{r0} - \partial_0 A_{ri}$. With this definitions, it is readily clear that the transports, σ is the electrical conductivity and χ is the charge susceptibility.

The equations of motion for $\phi_{r,a}$ lead to the current conservation: $\partial_\mu \hat{J}^\mu_{r,a} = 0$. Now let us turn off the unphysical sources: $A_{a\mu} = 0$ which leads to $\phi_a = 0$ from $\partial_\mu \hat{J}^\mu_a = 0$. We then find: $\hat{J}^\mu_a = 0$ and $B_{ai} = 0$. So, now the other conservation equation $\partial_\mu \hat{J}^\mu_r = 0$ becomes,

$$\partial_0 n - D\partial_i^2 n = -\sigma\partial_i E_i, \qquad \text{with, } D \equiv \frac{\sigma}{\chi}, \quad \text{and, } n \equiv \hat{J}^0_r, \tag{6.42}$$

where D is the diffusion constant, n is the number density and the above equation expresses the diffusion of the $U(1)$ charge.

Note that, in Eq. (6.41), at leading order in the a-field expansion $\hat{J}^\mu_r$ are expressed in terms of μ and E_i, that is, B_{ri} does not appear by itself. This is a consequence of the chemical shift symmetry Eq. (6.38) (or, equivalently Eq. (6.39)) which forbids presence of terms $\partial_i \phi_r$ and hence to all orders in derivatives (in the action, corresponding equations of motion and constitutive relations) and even at a nonlinear level; B_{ri} can only appear either with a time derivative $\partial_0 B_{ri} = -E_i + \partial_i \mu$ or through $F_{rij} \equiv \partial_i B_{rj} - \partial_j B_{ri} = \partial_i A_{rj} - \partial_j A_{ri}$. In the first of these expressions we will have terms like $\partial_0 \partial_i \phi_r$ which does not transform under the chemical shift (as $\lambda(\vec{x})$ does not depend upon time), and in the second of these expressions the $\partial_i \phi_r$ terms will cancel off due to anti-symmetrisation and commutativity of ordinary spatial derivatives.

Let us conclude this overview by noting that the Schwinger-Keldysh formalism is a very convenient for writing down hydrodynamic effective actions for the following reason. Say, we are interested in studying a system which has additional global symmetries along with the $U(1)$ symmetry that we discussed above. Then, one can think of gluing these additional symmetry sectors along with the $U(1)$ sector by adding auxiliary fields to the effective action in such a way that is dictated by the symmetries. As long as these additional terms do not introduce any couplings between the upper and the lower segments of the closed time path contours, the effective action will preserve the dynamical KMS symmetry: Eqs.(6.26)-(6.27) manifestly. Thus, we shall use this technology to construct effective hydrodynamic action which obey the symmetry structure between the 0-from and the 1-form sector as given in Eq. (6.2).

6.3 Equilibrium sector

In this section, we develop an action which realises the finite-temperature equilibrium sector of a hydrodynamic theory which is in the same universality class as that of quantum electrodynamics (QED) at finite temperature. Since, this action would describe the equilibrium sector, it should not contain any time derivatives of the fields in it, by definition. We first review the symmetry structure of our theory.

6.3.1 Symmetries

Consider a massless QED at finite temperature, whose weakly coupled physics is described by the following Lagrangian,

$$\mathcal{S}[A, \Psi, \overline{\Psi}] = \int d^4x \left(-\frac{1}{g^2} F^2 + \overline{\Psi}\, \gamma^\mu \left(\partial_\mu - i A_\mu\right) \Psi \right), \tag{6.43}$$

where A is the dynamical gauge field and Ψ is a massless Dirac fermion and $F = dA$. The above action has a $U(1)^{(0)}$ axial current, denoted by $j^\mu = \overline{\Psi}\gamma^\mu\gamma^5\Psi$ which is non-conserved, due to the ABJ anomaly, and a $U(1)^{(1)}$ 2-form current, denoted by $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ which is conserved due to the Bianchi identity. Furthermore, the non-conservation of the axial current obeys the following equation (see appendix F.1 for details on how non-invertible defect insertion is equivalent to saturating the anomaly equation as in Eq. (6.1)),

$$\partial_\mu j^\mu = k\, \epsilon_{\alpha\beta\rho\sigma} J^{\alpha\beta} J^{\rho\sigma}, \qquad \left(\text{where, } k \equiv \frac{1}{16\pi^2}\right). \tag{6.44}$$

Though we are inspired by massless QED, we will keep the constant k arbitrary in what follows. The partition function is a function of two sources a_μ and $b_{\mu\nu}$ via

$$Z[a,b] = \int D[A]D[\Psi]D[\bar{\Psi}] \exp\left(-\mathcal{S} + \int d^4x \left(j^\mu a_\mu + \frac{1}{2}J^{\mu\nu}b_{\mu\nu}\right)\right) \tag{6.45}$$

where $\epsilon^{\mu\nu\rho\sigma}(db)_{\nu\rho\sigma} \equiv j^\mu_{\text{ext}}$ can be thought of as an external current insert to the system.

From here on, we will be agnostic to the details of the matter field except that it has the same global symmetry as the above theory. The equilibrium effective action we construct will be based only on the data encoded in the background field a and b. This has the same spirit as [35, 136, 140] albeit in a more simplified metric and equations of state.

1-form symmetry in thermal equilibrium

Let us briefly review the notion of hydrodynamics with 1-form symmetries by considering ordinary MHD, i.e. a simple system in thermal equilibrium which has only a single 1-form symmetry (see [30, 65]). We performed this analysis already in Sec. 3.2.1 but we briefly mention some of it here to remind ourselves of the setting. The 1-form conservation equation takes the following form,

$$\partial_\mu J^{\mu\nu} = 0, \tag{6.46}$$

Since we are interested in finite temperature physics, let us put our theory on $S^1 \times \mathbb{R}^3$ and we shall denote the S^1 direction as τ (the Euclidean time). Now let us see how

$J^{\mu\nu}$ decomposes in the dimensionally reduced theory on $S^1 \times \mathbb{R}^3$. On $\mathbb{R}^3$ we have,

$$U(1)^{(0)} \text{ 0-form symmetry } \to J^{i\tau} \to \mathcal{B}^i = J^{i\tau} \text{ is magnetic 3-vector,} \tag{6.47}$$

$$U(1)^{(1)} \text{ 1-form symmetry } \to J^{ij} \to \mathcal{E}^i = \frac{1}{2}\epsilon^{ijk} J_{jk} \text{ is electric 3-vector.} \tag{6.48}$$

Now, in equilibrium, to leading order in derivatives, $\mathcal{E}^i$ vanishes and the $U(1)^{(0)}$ symmetry is actually *spontaneously broken* in the normal phase of the theory (see [65]). So, for this spontaneously broken symmetry we will have a Goldstone mode which we denote by ψ (this Goldstone mode may be thought of as the unscreened magnetic field in the plasma). Furthermore, due to the above symmetry breaking, ψ has a shift symmetry of the form,

$$\psi \to \psi + \Lambda_\tau(x^i), \tag{6.49}$$

where in the original theory, $\Lambda_\mu(x^i)$ is to be understood as a τ-independent 1-form symmetry parameter.

Now let us define the source for $U(1)^{(0)}$ to be $b_{i\tau}$. The transformation of $b_{i\tau}$ is as follows,

$$b_{i\tau} \to b_{i\tau} + \partial_i \Lambda_\tau, \tag{6.50}$$

The source for $U(1)^{(1)}$ is going to be b_{ij} with the following gauge transformation,

$$b_{ij} \to b_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i. \tag{6.51}$$

where $(i, j, k) = (x, y, z)$ to be considered in $\mathcal{M}_3 = \mathbb{R}^3$.

Let us now include the 0-form non-conserved axial current j^μ in our theory. It decomposes on $S^1 \times \mathbb{R}^3$ into j^τ and j^i. Note that the source for j^τ is a_τ and we define the source for j^i to be a_i which have the following gauge transformations,

$$a_\tau \to a_\tau, \tag{6.52}$$

$$a_i \to a_i + \partial_i \lambda. \tag{6.53}$$

Note that due to the anomaly, the equilibrium action will "strictly" not be gauge invariant with respect to the gauge transformations given in Eq. (6.53), but this gauge non-invariance will be quite constrained by the anomaly equation given in

Eq. (6.44) as we shall see below.[7]

The gauge transformations Eq. (6.49) and Eq. (6.50) together indicate that $B_i \equiv \partial_i \psi - b_{i\tau}$ is a gauge-invariant 3-vector. Furthermore, we also have the following gauge-invariant tensors: $h_{ijk} \equiv (db)_{ijk}$ and $H_{ij} \equiv (dB)_{ij}$ and $(da)_{ij}$. These will be the basic building blocks for our hydrodynamic equilibrium action.

6.3.2 Euclidean action

Let us develop an effective action which describes the equilibrium sector of our theory. For this, let us first note that, in our theory ψ is the only dynamical variable and a_μ, $b_{\mu\nu}$ are sources for the currents j^μ and $J^{\mu\nu}$ respectively. Furthermore, let us note that this equilibrium effective action should be separately C, P, T, CP and CPT invariant since the microscopic theory in Eq. (6.43) is invariant under each of these symmetries respectively. We tabulate in Table 6.1 the transformation of each of the sources under the discrete symmetries (see Appendix A.4 for details).

Table 6.1: Discrete Symmetry Table

Symm.	a_τ	a_i	b_{ij}	$b_{k\tau}$	∂_i
P	-1	$+1$	-1	$+1$	-1
T	$+1$	-1	$+1$	-1	$+1$
C	$+1$	$+1$	-1	-1	$+1$
CP	-1	$+1$	$+1$	-1	-1

With this we can write down an effective action (upto $\mathcal{O}(\partial^2)$) for the equilibrium sector. We focus on the state where a, b, B are small. The first few terms in this

[7]Note that in the dimensionally reduced 3d theory we formally have a new non-invertible symmetry arising from the non-conservation of the current j^i:

$$d \star j \sim k(\star J^{i\tau} \wedge \star J^{ij}), \tag{6.54}$$

where one views $J^{i\tau}$ and J^{ij} as currents for $U(1)^{(0)}$ and $U(1)^{(1)}$ in the 3d theory respectively. Similar non-invertible symmetries have recently been studied in [141].

expansion that preserved all the discrete symmetries are as follow

$$\begin{aligned}\mathcal{S}\left[\psi\right] = \int\limits_{\mathbb{R}^3} \Big[&\star\left(\frac{1}{2}\chi_A\left(a_\tau\right)^2\right) + \frac{1}{2}\chi_B\left(B\wedge\star B\right) + \frac{1}{2}\chi_C\, a_\tau\,\left(a\wedge da\right) + \frac{1}{2}\chi_G\, a_\tau\,\left(B\wedge dB\right) \\ &+\frac{1}{2}\chi_I\left(H\wedge\star H\right) + \frac{1}{2}\chi_K\left(da\wedge\star da\right) + \frac{1}{2}\chi_N\, a_\tau\left(\star h\right)\,\left(B\wedge da\right) \\ &+\frac{1}{2}\chi_O(\star h)\,\left(h\right) + \frac{1}{2}\chi_P\, a_\tau\left(\star h\right)\,\left(a\wedge dB\right)\Big]\,, \end{aligned} \tag{6.55}$$

The coefficient χ_A is the susceptibility of the axial charge sector and χ_B may be thought of as the susceptibility of the 1-form charge; physically it controls the amount of magnetic field produced in terms of a given 1-form chemical potential, which can be thought of as an applied external electric current [30]. In the above action, each of the coefficients should be allowed to be an *even* function of a_τ (and an arbitrary function of the 0^{th} order vector norm B^2). The extra explicit factors of a_τ render some of the coefficient functions odd under $a_\tau \to -a_\tau$ and guarantee the correct discrete transformation properties of the action. In the above action '$\star$' denotes the 3-dimensional Hodge dual; we reserve the notation '$\star_4$' for the 4-dimensional Hodge dual.

Finally, in this section we will seek to illustrate the minimum physics from imposing the anomaly constraint; let us then set all of the coefficients except χ_A, χ_B, χ_O to zero. Then we have,

$$\mathcal{S}\left[\psi\right] = \int\limits_{\mathbb{R}^3} d^3x\left[\frac{1}{2}\chi_A\left(a_\tau\right)^2 + \frac{1}{2}\chi_B\left(B_iB^i\right) + \frac{1}{12}\chi_O h_{ijk}h^{ijk}\right], \tag{6.56}$$

where we now further assume that the remaining χ_A, χ_B, χ_O are simply constants. This action, however, does not have the non-invertible symmetry as the insertion of symmetry defect operator is not topological, see Appendix F.1 for further details. It turns out that the above action has to be modified by adding terms that are not gauge-invariant under Eq. (6.53) in a very specific manner. We will see that this is sufficient to reproduce some of the chiral MHD phenomenology.

Gauge non-invariant term

If the action above is gauge-invariant under transformations of the axial source Eq. (6.53) and it will not yield the Ward identity in the dimensionally reduced theory i.e.

$$\partial_i j^i = 4k\,\epsilon_{ijk}J^{ij}J^{k\tau}\ . \tag{6.57}$$

To remedy this issue, we modify the action Eq. (6.56) into the following form

$$\mathcal{S}\left[\psi\right] = \int\limits_{\mathbb{R}^3} d^3x \left[\frac{1}{2}\chi_A \left(a_\tau\right)^2 + \frac{1}{2}\chi_B\left(B_i B^i\right) + \frac{1}{12}\chi_O\, h_{ijk}h^{ijk} + k\, a_i V^i\right], \tag{6.58}$$

where the V^i is an arbitrary vector that depends on a_τ, a^i, B^i, their derivatives and $h = db$. This is due to the fact that the action is invariant under the background gauge transformation of b_{ij} in Eq. (6.51). The currents in this theory can be written as

$$\begin{aligned} j^\tau &= \frac{\delta\mathcal{S}}{\delta a_\tau} = \chi_A\, a_\tau\,, & j^i &= \frac{\delta\mathcal{S}}{\delta a_i} = k\, V^i + k\,\frac{\delta V^j}{\delta a_i} a_j,\,, \\ J^{i\tau} &= \frac{\delta\mathcal{S}}{\delta b_{i\tau}} = -\chi_B B^i + k\,\frac{\delta V^j}{\delta b_{i\tau}} a_j\,, & J^{ij} &= \frac{\delta\mathcal{S}}{\delta b_{ij}} = -\epsilon^{ijk}\partial_k f, \end{aligned} \tag{6.59a}$$

Notice that the constitutive relation for J^{ij} is a total derivative of a 3-form $\mathcal{H}^{ijk} = \epsilon^{ijk} f$. This is due to the fact that $\mathcal{S}$ can only depends on the the total derivative of b_{ij}. A precise form of f in terms of V^i is

$$f = \epsilon^{ijk}\left[\frac{\chi_O}{12}h_{ijk} + \frac{k}{6}a_m\,\frac{\partial\left(V^m\right)}{\partial(\partial_k b_{ij})}\right]. \tag{6.59b}$$

To find a form of V^i which yield the Ward identity Eq. (6.57), we perform a transformation $a \to a + d\lambda$ in the action Eq. (6.58) to obtain the Ward identity

$$\partial_i j^i = k\,\partial_i\left[V^i + \frac{\delta V^j}{\delta a_i}a_j\right] \tag{6.60}$$

Demanding the r.h.s. of Eq. (6.60) to be the same as those in Eq. (6.57) and write J^{ij} in terms of f as in Eq. (6.59a), we find that

$$\partial_i\left[V^i + \frac{\delta V^j}{\delta a_i}a_j\right] = 4\epsilon_{ijk}J^{ij}J^{k\tau} = -8\left[\partial_i\left(J^{i\tau}f\right) - f\partial_i J^{i\tau}\right] = -8\partial_i(J^{i\tau}f) \tag{6.61}$$

where, to get the last equality, we use the conservation law $\partial_i J^{i\tau} = 0$ in Eq. (6.46) (upon dimensionally reduced on the thermal cycle). Substitute the form of $J^{i\tau}$ and f from Eq. (6.59a) and Eq. (6.59b) will provide us with a functional equation for V^i.

Finding solutions to this is a well-posed but complicated task; while it seems possible that exact expressions should exist for arbitrary k we have not been able to find them. To make progress, we thus consider a formal expansion in the anomaly coefficient k:

$$V^i = V^i_{(0)} + k\, V^i_{(1)} + \mathcal{O}(k^2)\,, \tag{6.62}$$

Solving Eq. (6.61), order by order in k, we find that

$$\begin{aligned} V^i_{(0)} &= \frac{2}{3}\chi_B\chi_O B^i \epsilon^{mpq} h_{mpq} = 4\chi_B\chi_O B^i |h|\,, \\ V^i_{(1)} &= 8\alpha a^i + 8\gamma\,(a\cdot B)\, B^i \end{aligned} \tag{6.63}$$

where we denote $|h| = \frac{1}{6}\epsilon^{ijk}h_{ijk}$, $(a\cdot b) = a_i B^i$, $\alpha \equiv (\chi_O^2\chi_B)|h|^2$ and $\gamma \equiv (\chi_B^2\chi_0)$. Thus the action for non-invertible symmetry can be written as

$$\begin{aligned} \mathcal{S}\,[\psi] = \int_{\mathbb{R}^3} d^3x \, \Big[& \frac{1}{2}\chi_A\,(a_\tau)^2 + \frac{1}{2}\chi_B\left(B_i B^i\right) + \frac{1}{12}\chi_O h^2 \\ & + 4k\, a_i \left\{ \chi_B\chi_O|h|B^i + 2k\left(\chi_O\chi_B\right)\left[\chi_O|h|^2 a^i + \chi_B\,(a\cdot B)\,B^i\right]\right\}\Big] \end{aligned} \tag{6.64}$$

Equations of motion

We first derive the equations of motion for the ψ field, $\frac{\delta\mathcal{S}}{\delta\psi} = 0$,

$$\partial_l B_l + 4k\,\chi_O\partial_l\left(|h|a_l\right) = 16k^2\,\chi_B\chi_O\partial_l\left[a_l\,(a\cdot B)\right]. \tag{6.65}$$

We note the curious fact that due to the explicit presence of a_i factor in the above equation, the equations of motion is no longer invariant under transformations of the axial source $a_i \to a_i + \partial_i\lambda$.

Such phenomena occur even in simpler systems; for example let us consider axion electrodynamics, whose action takes the form:

$$\mathcal{S}_{\text{axion}}[\theta, a] \sim \int d^4x\, \left[(d\theta - a)^2 + \theta F\wedge F + F^2 + F\wedge b\right] \tag{6.66}$$

where $F = da$ and $b_{\rho\sigma}$ is the source for the 2-form current $J^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

The equations of motion for F from the above action are,

$$d\star F = F d\theta, \tag{6.67}$$

Clearly, under a shift of the axion, $\theta \to \theta + \Lambda$, the above equations of motion is no longer gauge-invariant, and it cannot be made so without spoiling gauge-invariance of the dynamical $U(1)$ gauge symmetry. This is arising from the existence of the gauge non-invariant term $\theta F\wedge F$ in the action; here we see a similar phenomenon, except that we do not even have a degree of freedom analogous to θ in the action; rather the effect of the anomaly must be saturated using couplings to the axial source alone.

It seems possible that there is a more elegant way to couple a source to this current so that some sense of invariance under transformations of the source is preserved, perhaps using the non-invertible symmetry structure. See [58] for some recent work in this direction.

Physical consequences

In this section we interpret the result of the above section. Note that the magnetic field $\mathcal{B}$ is defined as, $\mathcal{B}^i = J^{i\tau}$ and the electric field $\mathcal{E}$ is defined as $\mathcal{E}_i = \frac{1}{2}\epsilon_{ipq}J^{pq}$. Now recall from Eq. (6.59a) that the form of J^{pq} is greatly constrained; as $J^{pq} = \frac{\delta \mathcal{S}}{\delta b_{pq}} = -\partial_m\left(\mathcal{H}^{mpq}\right)$ for appropriately chosen $\mathcal{H}^{mpq}$, we see that $\mathcal{E}_i \sim \partial_i\tilde{\alpha}$ where $\mathcal{H}^{mpq} = \tilde{\alpha}\,\epsilon^{mpq}$. In other words, invariance under 1-form transformations of the source b_{ij} Eq. (6.51) is an extremely fancy way to demonstrate the elementary fact that in equilibrium, the electric field is always the spatial gradient of a potential. As is conventional, we will call this potential the electric chemical potential $\mu_{\rm el}$.

Putting in the specific expressions from above, we find (at $\mathcal{O}(k)$),

$$\mathcal{E}_m = \partial_m\mu_{\rm el}, \tag{6.68}$$

$$\text{where, } \mu_{\rm el} \equiv -\frac{\chi_O}{2}|h| - 2k\chi_B\chi_O\left(a\cdot B\right), \tag{6.69}$$

$$\mathcal{B}_m = -\chi_B B_m - 4k\,\chi_B\chi_O|h|a_m. \tag{6.70}$$

Note that in the absence of the anomaly, $\mu_{\rm el}$ is simply proportional to h_{ijk}; this is the component of the applied source that corresponds to applying an external electric charge density $\rho_{\rm el}$ [30].

Using the above equations and Eq. (6.59a) we find (at $\mathcal{O}(k)$),

$$j^i = -8kJ^{i\tau}f = 4k\left[\chi_O\chi_B|h|B^i + 2k\chi_O^2\chi_Ba^i|h|^2 + 2k\chi_O\chi_B^2\left(a\cdot B\right)B^i\right] = 8k\mu_{\rm el}\mathcal{B}^i, \tag{6.71}$$

In other words, up to the order in k that we have been able to calculate, there is an axial current flow in the direction of the magnetic field, with coefficient precisely given by $k\mu_{\rm el}$. This is a well-known expression; it is the *chiral separation effect* [142–144], see also [90] for a review.

It is interesting to note that the coefficient of the chiral separation effect is given precisely by its value in the (ungauged) theory where we have a 't Hooft anomaly; the nonlinear terms in k conspire precisely to make this possible. We stress that in our starting action we have chosen a minimal set of terms Eq. (6.56) to explore the physics from the anomaly. It seems entirely possible that adding more terms will alter this relation, as is expected from [116], who found that including dynamical electromagnetism generically renormalizes all transport coefficients. We leave the investigation of this important issue to later work.

To conclude: here we constructed a Euclidean effective action that captures the simplest features of the axial anomaly. We demonstrated that it is possible (though somewhat cumbersome) to construct a generating function that saturates the anomaly equation; the resulting answers display the known physics of the chiral separation effect. We stress that we never weakly gauge electromagnetism; rather we simply directly discuss the universality class of the gauged theory.

6.4 Dissipative action

In this section, we move beyond the equilibrium construction of the previous section and construct a dissipative action that realizes this anomaly structure. We will use the recently constructed formalism for finite-temperature dissipative actions [14, 15, 145]; which we review in section 6.2 (see [13] for a detailed review of this technology). As we freeze the stress-energy sector we will require only a subset of the full technology. We will construct an effective action representing the symmetry structure described above, and we will see that already interesting physics appears at the first order in derivatives.

We note from the outset that while this does result in a useful action principle for obtaining the correct equations of motion, we do not currently feel that this is the most elegant formulation of the problem, for reasons explained below.

The strategy that we take is to first consider two theories which capture the dissipative dynamics of the charges associated with a conserved $U(1)$ 0-form symmetry (with an associated axial charge current j^μ) and a 1-form symmetry (with an associated magnetic flux current $J^{\mu\nu}$). We then "glue" these two theories together in a manner which results in the $U(1)$ 0-form symmetry being broken down in the manner described by the anomaly equation Eq. (6.2). We will see that the implementation of this structure is most convenient if we introduce some auxiliary fields; upon eliminating these fields we find Eq. (6.2) on-shell.

We begin with the Lagrangian densities for the two subsectors for the 1-form and 0-form sectors respectively:

$$\mathcal{L}_0[a;\theta] = \frac{i\sigma}{\beta}A_{ai}^2 + \chi_A A_{a0}A_{r0} - \sigma A_{ai}\partial_0 A_{ri} \tag{6.72}$$

$$\mathcal{L}_1[b;\Phi] = \frac{i\rho}{\beta}G_{aij}^2 + \chi_B G_{a0i}G_{r0i} - \rho G_{aij}\partial_0 G_{rij} \tag{6.73}$$

where we have

$$A = a + d\theta\ , \qquad G = b + d\Phi\,. \tag{6.74}$$

where the time-component of the 1-form field Φ_t reduced to ψ in equilibrium configuration presented in Section 6.3. The Lagrangian Eq. (6.72) and Eq. (6.73) each describes the charge diffusion process of 0-form and 1-form symmetry respectively. It is also follows that this is the most general Lagrangian in each sector that preserved all C, P and T symmetry at first order in derivative and quadratic order in amplitude of A and G see e.g. [13, 146].

Here and throughout we use a notation where lowercase letters are applied sources and uppercase or greek letters are dynamical fields. For illustrative purposes we work with the simplest possible actions, i.e. only keeping terms to quadratic order in the fields. Eq. (6.72) describes the diffusive dynamics of an ordinary 0-form conserved charge in terms of the scalar Stuckelberg θ, the construction is reviewed in [13]. Eq. (6.73) has recently been constructed in [146] to describe diffusive dynamics of the magnetic field in terms of a vector Stuckelberg Φ; it should be clear that it is a 1-form generalization of Eq. (6.72).

In general one obtains the currents through functional differentiation with respect to the sources:

$$\delta S = \sum_{s=r,a}\int d^4x\left(j_s^{\mu}a_{\mu}^s + \frac{1}{2}J_s^{\mu\nu}\delta b_{\mu\nu}^s\right)\ . \tag{6.75}$$

As usual, the invariance of the action under the following combined transformations of the sources and dynamical fields:

$$a \to a + d\Lambda(x) \qquad \theta \to \theta - \Lambda(x) \tag{6.76}$$

$$b \to b + d\xi(x) \qquad \Phi \to \Phi - \xi(x) \tag{6.77}$$

(where Λ is a scalar and ξ a 1-form) implies conservation of the currents j^μ and $J^{\mu\nu}$ as defined in Eq. (6.75).

Let us briefly discuss the physical interpretation of the coefficients appearing in the action above. ρ is the electrical resistivity; as stressed in [30], in a universal formulation of magnetohydrodynamics, it is ρ that is a fundamental transport coefficient, and not the electrical conductivity. In particular, ρ can be matched to microscopics through the following Kubo formula, in terms of the retarded correlator of the 2-form current J^{xy}.

$$\rho = \lim_{\omega\to 0} \frac{1}{-i\omega} G^R_{xy,xy}(\omega) \tag{6.78}$$

σ is the conductivity of the $U(1)$ 0-form axial charge (and is unrelated to the vector electrical conductivity). χ_A and χ_B are charge susceptibility of 0-form and 1-form symmetry that appears in the zeroth derivative level of Eq. (6.55).

6.4.1 Combining theories using auxiliary fields

We would now like to glue these two theories together so that j^μ is no longer precisely conserved, but instead so that we find the following expression in the final combined theory:

$$\partial_\mu j^\mu = k\epsilon^{\mu\nu\rho\sigma} J_{\mu\nu} J_{\rho\sigma} \tag{6.79}$$

where this expression is now understood to hold on both legs of the doubled Schwinger-Keldysh contour, i.e. for the 1-type and 2-type fields individually. To do so, we introduce two new sets of auxiliary fields: two 2-forms $\Sigma^{r,a}$ and two 1-forms $C^{r,a}$. These fields are useful to "unwrap" the non-linearities that are present in the anomaly equation; to obtain the physical currents they should be eliminated, as we do explicitly below. We thus consider the following combined action:

$$\mathcal{L}[a,b;\theta,\Phi,\Sigma,C] = \mathcal{L}_0[a;\theta] + \mathcal{L}_1[b+\Sigma;\Phi] - \frac{1}{4}\left(\epsilon^{\mu\nu\rho\sigma}\Sigma^a_{\mu\nu} dC^r_{\rho\sigma} + \epsilon^{\mu\nu\rho\sigma}\Sigma^r_{\mu\nu} dC^a_{\rho\sigma}\right) + \mathcal{L}_{anom}[\theta,C], \tag{6.80}$$

where $\mathcal{L}_{anom}[\theta,C]$ takes the form:

$$\mathcal{L}_{anom}[\theta,C] = -k\left(\theta_1 \epsilon^{\mu\nu\rho\sigma} dC^1_{\mu\nu} dC^1_{\rho\sigma} - \theta_2 \epsilon^{\mu\nu\rho\sigma} dC^2_{\mu\nu} dC^2_{\rho\sigma}\right) \tag{6.81}$$

Note the direct coupling to the Stuckelberg field θ clearly breaks its shift symmetry. It will often be useful for us to rewrite this action in the $r-a$ basis:

$$\mathcal{L}_{anom}[\theta,C] = k\left(\theta^a\left(\epsilon^{\mu\nu\rho\sigma} dC^r_{\mu\nu} dC^r_{\rho\sigma} + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} dC^a_{\mu\nu} dC^a_{\rho\sigma}\right) + 2\theta^r \epsilon^{\mu\nu\rho\sigma} dC^a_{\mu\nu} dC^r_{\rho\sigma}\right) \tag{6.82}$$

Varying the action with respect to $\theta^{1,2}$, we find:

$$\partial_\mu j_1^\mu = -k\epsilon^{\mu\nu\rho\sigma} dC^1_{\mu\nu} dC^1_{\rho\sigma} \tag{6.83}$$

We now note that the Σ fields couple to the 1-form sector as a direct shift of the external 2-form source b, i.e. always in the combination $b+\Sigma$. Thus the equations of motion for the auxiliary field $\Sigma^{r,a}$ are determined by the variation of $\mathcal{L}_1$ with respect to b: $\delta_\Sigma \mathcal{L}_1 = \delta_b \mathcal{L}_1$. This results in the following equations of motion:

$$\frac{\delta \mathcal{L}_1}{\delta b^a_{\mu\nu}} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(dC^r)_{\rho\sigma} \tag{6.84}$$

(and similarly for $r \leftrightarrow a$). However the left-hand side is by construction the 2-form current J^r Eq. (6.75). Thus we see that the role of the Σ fields is to simply to precisely correlate the $C^{r,a}$ fields with the 2-form currents as:

$$J^{\mu\nu}_{r,a} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} dC^{r,a}_{\rho\sigma} \tag{6.85}$$

Inserting this into Eq. (6.83) we find exactly the desired expression Eq. (6.79); thus this construction always relates the non-conservation of the axial current with the magnetic flux in the correct fashion.

A modern understanding of Eq. (6.79) is that it permits the construction of topological defect operators that measure the axial charge [37,38]; in Appendix F.2 we verify that such defect operators can be constructed in this theory (indeed we have one such operator living on each of the legs of the Schwinger-Keldysh contour).

We note some facts about this construction:

1. The structure does not depend on the precise form of the 0-form and 1-form theories $\mathcal{L}_0$ and $\mathcal{L}_1$, but only on their invariances under symmetries.

2. As none of the new terms we have added introduce any couplings between the two legs of the time contour, the action is automatically invariant under the so-called *KMS symmetry*; this acts on all fields and sources ϕ as

$$\phi_a(x) \to \Theta\phi_a(x) + i\beta\Theta\partial_t\phi_r(x) \tag{6.86}$$

$$\phi_r(x) \to \Theta\phi_r(x) \tag{6.87}$$

   with Θ an anti-unitary symmetry representing time-reversal.

3. The action has a rather undesirable feature: it is not invariant under the "diagonal shift" symmetry in the scalar sector. To be more precise, in an action-based formulation of hydrodynamics one typically requires that the action be invariant under shifting the r-type Stuckelberg θ^r by an arbitrary

spatially dependent phase, i.e.

$$\theta^r(t, x^i) \to \theta^r(t, x^i) + \lambda(x^i) \tag{6.88}$$

where $\lambda(x^i)$ is an arbitrary function of space [14,15,147]. Invariance under this symmetry is generally required to forbid arbitrary spatial gradients of $\partial_i\theta^r$ in the action or constitutive relations; if this diagonal symmetry is broken then one should add such spatial gradients to the action, and we then generically end up in a superfluid phase for the corresponding symmetry (in this case $U(1)_A$).

In our construction, the non-anomalous part of the action Eq. (6.72) is invariant under the "diagonal shift" symmetry, but the anomalous part Eq. (6.82) is *not*. In the dual variables that we are using, it is not straightforward to make the action invariant under this diagonal shift. This is undesirable: in the case of the 't Hooft anomaly, the interplay between the diagonal shift and the realization of the anomaly plays a very important role [16,148,149].

At the moment, we are unclear on the precise implications of breaking this diagonal shift symmetry. We will proceed with this action and find physically very reasonable results; however, as there is no symmetry preventing us from adding $\partial_i\theta^r$ terms to the action we cannot in good-faith call this an *effective* field theory; rather it is simply an action which one can use to obtain a consistent set of equations of motion. Given the unclear formal status of chiral MHD, this still seems to be of value, and we leave to the future a more refined understanding of the interplay of the diagonal shift symmetry and the non-invertible character of the axial anomaly.

6.4.2 Chiral MHD phenomenology

We now study some simple consequences of varying this action. From Eq. (6.75) above we have from the magnetic sector

$$J_r^{0i} = \chi_B \tilde{G}_{r0i} \qquad J_r^{ij} = \frac{2i\rho}{\beta}\tilde{G}_{aij} - \rho\partial_0\tilde{G}_{rij} \tag{6.89}$$

$$J_a^{0i} = \chi_B \tilde{G}_{a0i} \qquad J_a^{ij} = \rho\partial_0\tilde{G}_{aij} \tag{6.90}$$

where we have defined a shifted G which takes into account fluctuations of the new auxiliary field Σ:

$$\tilde{G} \equiv G + \Sigma = b + d\Phi + \Sigma \ . \tag{6.91}$$

Similarly, in the axial charge sector we have:

$$j_r^0 = \chi_A A_{r0} \qquad j_r^i = \frac{2i\sigma}{\beta} A_{ai} - \sigma\partial_0 A_{ri} \tag{6.92}$$

$$j_a^0 = \chi_A A_{a0} \qquad j_a^i = -\sigma\partial_0 A_{ai} \tag{6.93}$$

Finally, varying the action with respect to C^a, we find the following expression for Σ:

$$\partial_{[\rho}\Sigma^r_{\mu\nu]} = 8k\partial_{[\rho}\theta^r dC^r_{\mu\nu]} \tag{6.94}$$

When studying classical equations of motion it is self-consistent to set all a-type fields to zero after variation of the action; we have done this above. In the remainder of this section we will thus omit the "r" superscript on all quantities; everything that remains is an r-type field.

As usual, we now define the axial chemical potential to be

$$\mu_A = A_0 = (\partial_0\theta + a_0) \tag{6.95}$$

It is also convenient to define the following vector "chemical potential" for the 1-form charge:

$$\mu_i = \tilde{G}_{0i} = (b + \Sigma + dA)_{0i} \tag{6.96}$$

so that we can write

$$\partial_0(b + \Sigma + dA)_{ij} = h_{0ij} + (d\Sigma)_{0ij} + \partial_i\mu_j - \partial_j\mu_i \tag{6.97}$$

where $h = db$. We can then write the currents as

$$J^{0i} = \chi_B \mu^i \tag{6.98}$$

$$J^{ij} = -\rho(\partial_i\mu_j - \partial_j\mu_i + (d\Sigma)_{0ij}) \tag{6.99}$$

where we have set the sources $h = db = 0$.

We now see the first effect of the anomaly; the Σ field is now contributing to the spatial components J^{ij}; in a conventional formulation of the theory this component of the 2-form current is the electric field. We may explicitly find expressions for the currents by using Eq. (6.94) and Eq. (6.85) to eliminate dC and Σ. To this order in derivatives this is a linear set of equations that can in principle be straightforwardly solved; in practice the expressions are somewhat cumbersome.

We present first the answer assuming the the system is spatially homogenous

$(\partial_i = 0)$. We set to zero all sources except for the axial source a_t. We then find:

$$(d\Sigma)_{0ij} = 8k(\partial_0\theta)\epsilon_{ijk}J^{0k} \tag{6.100}$$

which then leads to the following expressions for the currents

$$\begin{aligned} j^0 &= \chi_A \mu_A \qquad & j^i &= 0 \\ J^{0i} &= \chi_B \mu^i \qquad & J^{ij} &= -8k\rho\left(\mu_A - a_t\right)\epsilon^{ijk}J^{0k} \end{aligned} \tag{6.101}$$

Let us examine the expression for J^{ij}; we see that the same transport coefficient ρ that determines the resistivity determines the strength of the electric field $\mathcal{E}^i \sim k\rho(\mu_A - a_t)\mathcal{B}^i$. This is a manifestation of the chiral magnetic effect. In a more conventional weakly-gauged description, this arises from considering the vector current $j^i_{\rm el} \sim k\mu_A B^i$ and relating it to the electric field through the electrical conductivity $\mathcal{E}^i = \sigma_{\rm el} j^i_{\rm el}$. However, in a formulation of MHD based only around symmetries, it is difficult to give a precise meaning to either $j^{\rm el}$ or $\sigma_{\rm el}$ [30]; here we see (as expected) that in this dual language the CME is controlled by ρ instead.

The equations of motion arise from varying the action with respect to θ and Φ, and are the (non)-conservation of the respective currents:

$$\partial_\mu j^\mu = k\epsilon^{\mu\nu\rho\sigma}J_{\mu\nu}J_{\rho\sigma} \qquad \partial_\mu J^{\mu\nu} = 0 \tag{6.102}$$

Putting in the constitutive relations for the currents, we now find that μ_i is constant in time, but that μ_A necessarily evolves according to the following equation:

$$\dot{\mu}_A = -64k^2\frac{\rho}{\chi_A}(J^{0i})^2(\mu_A - a_t) \tag{6.103}$$

Thus we find that μ_A relaxes towards the externally applied a_t with an exponential decay, i.e.

$$\mu_A - a_t \sim e^{-\Gamma_A t} \qquad \Gamma_A = 64k^2\frac{\rho}{\chi_A}(J^{0i})^2\,. \tag{6.104}$$

This equilibrium configuration is consistent with what one would get from gauging procedure shown in Eq. (E.1). It also allows us to make sense of the constitutive relation in Eq. Eq. (6.101) as a small expansion around equilibrium configuration in the late time limit where $e^{-\Gamma_A t} \ll 1$. This kind of procedure is common in the study of hydrodynamics with weakly broken global symmetry, see e.g. [150, 151] for recent discussions. The presentation of the decay rate Γ_A suggests a useful formula for it in the small J^{0i} limit, i.e.

$$\Gamma_A = c(J^{0i})^2 \qquad c = \frac{32k^2}{\chi_A}\left(\lim_{\omega\to 0}\frac{1}{-i\omega}G^R_{xy,xy}(\omega)\right) \tag{6.105}$$

where we have used the Kubo formula for ρ Eq. (6.78).

We see that as $t \to \infty$, an equilibrium configuration can have a nonzero value of J^{0i} (as dictated by the unbroken 1-form symmetry), but that μ_A will always be equal to a_t, and that even homogenous fluctuations around this value are damped. Of course for an ordinary conserved current fluctuations of μ obey a diffusion equation and are undamped in the homogenous limit.

6.4.3 Spatial derivatives

We now allow nonzero spatial dependence, i.e. we allow $\partial_i \neq 0$. It was demonstrated above that the equilibrium state takes the form:

$$\mu_A = a_t \qquad J^{0i} = \chi_B \mu^i \tag{6.106}$$

where all the fields are constant in space and time. We now consider linearized perturbations $\mu^i \to \mu^i + \delta\mu^i$, $\mu_A \to \mu_A + \delta\mu_A$ around this background configuration. We find the following expressions for the currents:

$$J^{0i} = \chi_B(\mu^i + \delta\mu^i) \qquad J^{ij} = -\rho\left(\partial_i\delta\mu_j - \partial_j\delta\mu_i\right) - 8k\rho\chi_B\delta\mu_A\epsilon^{ijk}\mu_k \tag{6.107}$$

$$j^0 = \chi_A\delta\mu_A \qquad j^i = -\sigma\partial_i\delta\mu_A \tag{6.108}$$

where as before we have assumed that the external sources have vanishing field strength, i.e. $da = db = 0$.

These expressions are what one might expect on physical grounds; in particular we record the expression for J^{ij} in the conventional language of the electric field $\mathcal{E}^i = \frac{1}{2}\epsilon^{ijk}J_{jk}$:

$$\mathcal{E} = \rho(\chi_B^{-1}\nabla\times\mathcal{B} - 8k(\mu_A - a_t)\mathcal{B}) \tag{6.109}$$

(where the first term receives contributions only from the fluctuations in $\mathcal{B}$, whereas the second receives contributions from fluctuations in μ_A and the background in $\mathcal{B}$).

We note that the equations of motion can be written only in terms of μ_A, and do not require explicit mention of the Stuckelberg field θ. This happens because θ enters only through $\partial_t\theta$, and spatial gradients $\partial_i\theta$ do not appear; this can be traced back to the precise form of the expression Eq. (6.94) and Eq. (6.99). We note that this was not obviously guaranteed. In the usual formulation of effective actions for hydrodynamics this property is enforced by the diagonal shift symmetry. As we noted previously, our system does not have this symmetry, and it seems possible that at higher orders in non-linearities and/or derivatives spatial gradients of the Stuckelberg field will appear. We leave this possibility for later investigation.

As a simple application we study the dispersion relations in this framework. We orient the background field in the z direction μ^z and consider a fluctuation $\delta\mu^y$ with momentum q in the x direction[8], so that perturbations have the spacetime dependence $e^{-i\omega t+iqx}$. From Eq. (6.102) it is straightforward to find two modes $\omega_{1,2}(q)$. The expressions are somewhat cumbersome, so we record them in two limits. We assume $\frac{\rho}{\chi_B} < \frac{\sigma}{\chi_A}$, and we first present the "high" momentum limit:

$$\omega_1(q) = -iq^2\frac{\sigma}{\chi_A}\left(1 - \frac{64k^2\mathcal{B}^2\chi_B}{\rho\chi_A - \sigma\chi_B}\frac{1}{q^2} + \mathcal{O}(q^{-4})\right) \tag{6.111}$$

$$\omega_2(q) = -iq^2\frac{\rho}{\chi_B}\left(1 + \frac{64k^2\mathcal{B}^2\chi_B}{\rho\chi_A - \sigma\chi_B}\frac{1}{q^2} + \mathcal{O}(q^{-4})\right) \tag{6.112}$$

Here "high" means that $q^2 \gg \frac{k^2\mathcal{B}^2}{\sigma}$, i.e we are looking at scales higher than the scale determined by the anomaly. We see that in this regime the two modes are essentially those of the diffusion of conventional 0-form charge and that of magnetic field lines respectively, with the two diffusion constants set by $D_a = \frac{\sigma}{\chi_A}$ and $D_b = \frac{\rho}{\chi_B}$; i.e. in this regime we find the physics of the original non-anomalous model.

At low q we find instead the following interesting dispersion relations, which we expand in the first few orders in the spatial momentum q.

$$\omega_1(q) = \frac{-64i\mathcal{B}^2k^2\rho}{\chi_A} - iq^2\left(\frac{\rho}{\chi_B} + \frac{\sigma}{\chi_A}\right) + \mathcal{O}(q^4) \tag{6.113}$$

$$\omega_2(q) = -i\frac{\sigma}{64\mathcal{B}^2k^2\chi_B}q^4 + \mathcal{O}(q^6) \tag{6.114}$$

We see that the 0-form diffusive mode is now gapped in the IR, as we saw above in Eq. (6.104). The leading momentum-dependence of this mode now has a dissipative

[8]Had we set q to be in the direction parallel with μ^i, the two modes modes ω_1, ω_2 depends quadratically on q i.e.

$$\omega_1 = -\frac{32i\mathcal{B}^2k^2\rho}{\chi_A} - iq^2\frac{\sigma}{\chi_A}\,, \qquad \omega_2 = -iq^2\frac{\rho}{\chi_B}\,. \tag{6.110}$$

character, where the dissipation rate is interestingly controlled by the *sum* of the diffusion rates of the original magnetic and axial charge sector respectively.

The second "subdiffusive" mode – which at high momenta becomes the diffusion of magnetic flux –interestingly starts at $\mathcal{O}(q^4)$ unlike the usual diffusion where $\omega \sim -iq^2$ commonly found in hydrodynamic systems. It should be noted here that the modes $\omega \sim -iq^4$ has been observed in various anisotropic systems with intricate symmetry structure such as a system with 't Hooft anomaly at strong mangetic field [152], systems with conserved multipole moment [153,154] and easy-axis Heisenberg spin chain with integrability-breaking perturbation [155] to name a few. Its physical origin seems to be tied to these somewhat exotic symmetry patterns and deserves further investigation.

We should also mentioned that, often in the chiral MHD literature, the system exhibits various kinds of interesting instabilities which have potential applications in astrophysical plasma. However, our focus is on the perturbation around equilibrium configuration and study how the system relaxed back to equilibrium. Had we chosen to perform a perturbation around constant $\mu \neq 0$ and $a_t = 0$, we will also find unstable solution as those in e.g. [86].

6.5 Discussion and outlook

In this chapter we documented some progress towards a description of chiral MHD that relies only on the global symmetries. One of our results was be an expression for the low-field limit of the chiral charge relaxation rate Γ_A. This suggests that in the limit of small $\mathcal{B}$ field, this expression is universal, and takes the form:

$$\Gamma_A = c\mathcal{B}^2 \qquad c = \frac{64k^2}{\chi_A} \lim_{\omega \to 0} \frac{1}{-i\omega} G^R_{\mathcal{E}^z,\mathcal{E}^z}(k \to 0, \omega) \tag{6.115}$$

where we have rephrased Eq. (6.104) in terms of conventional electric and magnetic fields.

Formulas of this sort are well-known (see e.g. [80,156]), but are usually presented in terms of the electrical conductivity σ_{el} instead, which makes sense only in a weakly-gauged description. Note that as ρ and not σ_{el} enters into this derivation, in principle this relation makes sense even when the electrodynamic sector is strongly coupled. Due to the issues with universality described earlier, we cannot claim that we have shown that this formula universally describes the decay rate. It is however in agreement with holographic results exhibited in [156], and it would be very interesting to compare it to recent lattice results [80, 81], where it may be possible to independently measure ρ and χ_A on the lattice. As described in those works, at the moment lattice computations for this prefactor are in disagreement with elementary hydrodynamic arguments based around a gauged vector current, and one might imagine that our more universal treatment is of value.

This expression suggests that as $\mathcal{B} \to 0$ the relaxation rate Γ_A vanishes. At this point we should note that we have been working in a classical theory and have not included any sort of fluctuations, i.e. we have essentially interpreted the operator $\langle J\tilde{J}\rangle$ to be $\langle J\rangle\langle\tilde{J}\rangle$ [9]. This is clearly only an approximation, and it seems possible that incorporating fluctuations could result in a $\mathcal{B}$-independent contribution to Γ_A. This would then become the dominant effect at small $\mathcal{B}$. However, as we will see in the next chapter this will not be the case and the chiral decay rate will actually vanish in the limit of vanishing magnetic field at 1-loop.

We conclude with a brief symmary of our results. Had our symmetry simply be a product of explicitly broken 0-form $U(1)$, with a lifetime $(\Gamma_A)^{-1}$ and unbroken 1-form $U(1)$ symmetry, one would expect the theory in the deep IR at $t \gg (\Gamma_A)^{-1}$ to only consist of a 1-form symmetry, i.e. to be indistinguishable from those in [30]. We show that the theory with a system with ABJ anomaly is differ from an ordinary MHD both in and out of equilibrium.

In the equilibrium sector, the physics that we get is that of the *chiral separation effect* (CSE). Surprisingly, the formula that describes the CSE on our case exactly matches with the one present in Eq.(4.15) of [90], where in the latter it was derived in a weakly-coupled way with non-dynamical electromagnetism. We see that if we correct for the electric and magnetic fields to $\mathcal{O}(k)$, then the functional form of the CSE remains the same. However, we have not presented the most general equilibrium action in the sense that it can have more terms in it and then one needs to check if the functional form of the CSE still remains same or if it receives correction (as shown in [116]) even after addition of more terms to the equilibrium action. We shall return this exercise in future. However, the crucial point to note here is that we get this chiral MHD phenomenology even though we never make any reference to electromagnetism.

In the dissipative sector, we derive the physics of the *chiral magnetic effect*. The coefficients appearing in this effect can be derived from Kubo formulae, as given

[9] We note that in the presence of some limit (e.g. large N) allowing classicality we can consider such an approximation.

in [30], and in this sense they are somewhat universal. However, the dissipative action is not invariant under a diagonal shift symmetry. We shall return to this point later in future to resolve this issue with the dissipative action. We also found that, while the density in ordinary MHD relaxed to equilibrium through diffusion process with $\omega \sim -iq^2$, a theory with ABJ anomaly relaxed to the equilibrium configuration through subdiffusion process with $\omega \sim -iq^4$. While this is not the first time that such a mode is found, it would be interesting to investigate whether or not it is signature of non-invertible symmetry of this type.

CHAPTER 7

Chiral decay rate in vanishing magnetic field

This chapter is concerned with fluctuation-driven effects in chiral magnetohydrodynamics. We begin by briefly stating the problem. A chiral plasma belongs to the same universality class (in the context of global symmetry) as that of massless Dirac fermions coupled to dynamical QED at finite temperatures (see chapter 5 and [156]), a fact which has been iterated many times in the thesis now. Let us reacll the symmetry structure of the chiral plasma,

$$\partial_\mu J^{\mu\nu} = 0, \qquad J^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}, \tag{7.1}$$

$$\partial_\mu j^\mu_A = k\,\epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \tag{7.2}$$

We can define the electric and magnetic fields, in the usual way as in electrodynamics, in terms of the components of the 2-form current as follows: $J^{0i} = B_i$. Thus, $E_l = \frac{1}{2}\epsilon_{ijl} J^{ij}$.

Let us denote the topological density by the operator

$$Q(x) = \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} \ . \tag{7.3}$$

The non-conservation equation for the axial charge then reads

$$\partial_\mu j^\mu_A = -kQ(x) \tag{7.4}$$

This non-conservation equation leads one to expect that in a thermal state the axial charge should decay exponentially in time as $n_A \sim e^{-\Gamma_A t}$. In a formal limit where the anomaly coefficient k is taken to be small, the decay rate for the axial charge is given by the following formula (see Eq.(2.8) in [80])[1]:

$$\Gamma_A = \lim_{\Omega\to 0} \frac{k^2}{\chi_A \Omega} \mathrm{Im}\, G^R_{QQ}(\Omega, \vec{p} = 0) \tag{7.5}$$

where χ_A is the axial charge susceptibility and $G^R_{QQ}(\Omega, \vec{p})$ is the retarded correlation function of the topological density[2]. This expression is perturbative in k but makes no assumptions on the dynamics of the degrees of freedom entering the topological charge density. Let us now explore some implications of this formula.

First, we note that in elementary language, $Q(x) = 8\vec{E}\cdot\vec{B}$. Let us consider a state with a background magnetic field $\vec{B} \neq 0$ pointing along the z axis, and study only the fluctuations of the electric field about the equilibrium (i.e. assuming that $\vec{B}$ does not fluctuate). We then find the following expression for the axial charge decay rate, in terms of the retarded correlation function of the electric field operator E.

$$\Gamma_A = \frac{64k^2B^2}{\chi_A} \lim_{\Omega\to 0} \frac{1}{\Omega} \mathrm{Im}\, G^R_{E_z,E_z}(\Omega) \tag{7.6}$$

This formula was derived from an effective theory for chiral MHD in the previous chapter. In particular, we note that the resistivity ρ of the plasma is defined in terms of a Kubo formula for the electric field, as explained in [30]. Thus this expression states that the axial charge relaxation rate is:

$$\Gamma_A = \frac{64k^2B^2\rho}{\chi_A} \,. \tag{7.7}$$

This expression can be understood from elementary arguments involving a quasiparticle description [80], and has also been verified in a holographic model in chapter 5. It has also been subject to numerical investigation in classical simulations [80, 81], which displayed a robust linear scaling of the decay rate with B^2, though the prefactor is at the moment poorly understood.[3]

Importantly, this expression states that as the magnetic field B is taken to zero, the lifetime of the axial charge is arbitrarily long. Indeed it is this parametric separation of scales that allowed the construction of the effective theories in the

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[1] This expression can be derived from the so called *memory matrix formalism* in hydrodynamics. See [157] for a review.

[2] We are grateful to L. Delacretaz for suggesting this route for calculation [115]

[3] This will be discussed in work to appear with A. Florio.

previous chapter and in [138], and the fact that this separation is in principle possible implicitly lies behind the extensive phenomenological literature on this subject.

Upon reflection, however, this is a somewhat strong statement – in particular, the right hand side of Eq. (7.5) does not obviously appear to vanish at zero B. One may imagine that the thermal *fluctuations* of the topological density $\vec{E} \cdot \vec{B}$ would create a nonzero decay rate which would become the dominant decay channel when the applied magnetic field is sufficiently small. This would lead to a crossover between the classical result Eq. (7.5) at moderately strong B and some nonzero fluctuation-driven effect at small B. If the decay rate does not vanish at zero B, it would have significant consequences: it would mean that at sufficiently long time scales the axial charge simply does not exist as a hydrodynamic degree of freedom. In particular, it would mean that the effective field theories described in [158] are unstable towards the inclusion of fluctuations.

One might be tempted to argue that the right-hand side of Eq. (7.5) is likely to vanish as follows: the infrared limit of the correlation function is presumably related to the integral of the Euclidean correlation function of $Q(\tau, \vec{x})$ over all Euclidean spacetime. But we know that Q is a total derivative:

$$Q = \partial_\mu K^\mu \qquad K^\mu \equiv \epsilon^{\mu\nu\rho\sigma} a_\nu f_{\rho\sigma} \tag{7.8}$$

and thus the integral will receive contributions only from field configurations with nontrivial topological structure. However in an Abelian gauge theory with vanishing background B field there are no $U(1)$ instantons, and thus the integral is zero. The argument is somewhat heuristic and it seems to us that it depends sensitively on boundary conditions at infinity. We were unable to formulate a completely satisfactory version of this argument, and indeed the art of extrapolating real-time dynamical physics from Euclidean non-perturbative data is quite subtle (see e.g. [159]).

In this work we thus directly compute the leading contribution to the decay rate Γ_A from thermal fluctuations by evaluating the Kubo formula Eq. (7.5) in the state with zero background magnetic field. In particular, we will evaluate the contribution to the retarded correlation function $Q(x) = 8\vec{E} \cdot \vec{B}$ arising from a one-loop calculation where the propagating degrees of freedom are diffusive MHD waves, the leading low-frequency degrees of freedom in the MHD plasma. Importantly, we will demonstrate explicitly that these fluctuations do result in a nontrivial real-time correlation function $G^R_{QQ}(\Omega)$ for the topological density – which we calculate as a function of frequency Ω – but that this function vanishes quickly enough at small frequency that it does not result in non-vanishing fluctuation-driven decay rate. This is consistent with the heuristic argument above, and (to this order) is consistent with safe use of the effective hydrodynamic description.

7.1 Fluctuations

In this section, we compute the finite-frequency real-time topological susceptibility arising from magnetohydrodynamic fluctuations. In particular, we are interested in computing the retarded correlation function

$$G^R_{QQ}(\Omega) = -i \int dt d^3x e^{i\Omega t}\, \mathrm{Tr}\left(e^{-\beta H}[Q(\vec{x},t), Q(0)]\right)\theta(t)\ . \tag{7.9}$$

where the operator $Q = 8\vec{E}\cdot\vec{B}$. We will now write the retarded correlation function above in terms of correlation function of $\vec{E}$ and $\vec{B}$; those can then be evaluated using an appropriate model for the dynamics.

A convenient way to proceed is to express the retarded correlation function in terms of the Euclidean finite-temperature correlation function $G^E_{QQ}(i\Omega_l)$, which is defined on a discrete set of Euclidean Matsubara frequncies $i\Omega_l = \frac{2\pi i\mathbb{Z}}{\beta}$. The retarded correlation function at real Ω is related to the Euclidean one by the usual formula

$$G^R_{QQ}(\Omega) = G^E_{QQ}(i\Omega_l = \Omega + i\epsilon) \tag{7.10}$$

The Euclidean correlator can be explicitly written as:

$$G^E_{QQ}(i\Omega_l) = \int_0^\beta d\tau \int d^3x e^{-i\Omega_l\tau}\langle Q(\tau,\vec{x})Q(0)\rangle \tag{7.11}$$

To proceed, we use $Q(x) = 8\vec{E}\cdot\vec{B}$. We also assume all correlations in the fluid are Gaussian. This is a reasonable starting point, as generally in hydrodynamics it is expected that the current densities are themselves weakly coupled at long distances so that a classical treatment is valid[4].

This assumption means that we can factorize the correlators in Euclidean space

[4]In low-dimensional hydrodynamics there are known examples where non-linearities are relevant in the IR (see e.g. [12] for a review) but to our knowledge this is not expected to be the case for (3+1)d MHD.

as follows:

$$\left\langle \left(\vec{E}\cdot\vec{B}\right)(x)\left(\vec{E}\cdot\vec{B}\right)(0)\right\rangle = \delta^{pq}\delta^{rs}\left[\langle E_p(x)E_r(0)\rangle\langle B_q(x)B_s(0)\rangle + \langle E_p(x)B_s(0)\rangle\langle B_q(x)E_r(0)\rangle\right], \tag{7.12}$$

Note that in this expression we have assumed that there is no background topological density, i.e. $\langle\vec{E}\cdot\vec{B}\rangle = 0$; this follows from CP invariance of the thermal state.

This can be conveniently interpreted as a Feynman diagram bubble evaluated in Euclidean spacetime, where the propagators of the bubble are the two-point correlators $\langle E_p(x)E_r(0)\rangle$ etc, see Fig. 7.1. Indeed, expressed in this form the problem has a great deal of formal similarity to the classic problem of evaluating (e.g.) a one-loop conductivity in terms of the propagators of the microscopic charged degrees of freedom – in both cases we are interested in determining the correlation function of an operator (i.e. Q) which is a bilinear in terms of fields with quadratic – and known – correlations (i.e. $\vec{E}$ and $\vec{B}$).

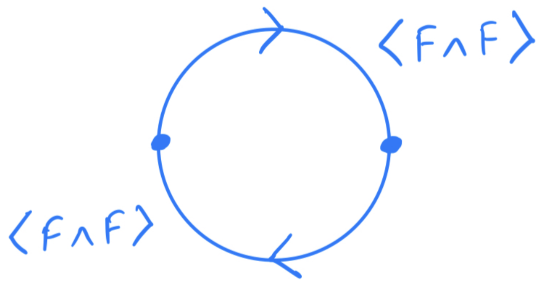

Figure 7.1: A bubble diagram representing the four-point function as given in Eq. (7.12).

Following the standard approach for that problem[5], it is convenient to write the expression in (Euclidean) frequency space, where we find that it becomes

$$G^E_{QQ}(i\Omega_l) = 64T\sum_{i\omega_m}\int\frac{d^3p}{(2\pi)^3}\left(G^E_{E_iE_j}(i\Omega_l+i\omega_m)G^E_{B_iB_j}(i\Omega_l) + G^E_{E_iB_j}(i\Omega_l+i\omega_m)G^E_{B_iE_j}(i\Omega_l)\right) \tag{7.13}$$

The above expression includes the same sum over the index structure that is present in Eq. (7.12).

We now note that we do not have access to the Euclidean correlation functions of the electric and magnetic fields $G^E_{E_iB_j}$ etc. in any suitable form. However from

[5]See e.g. [160] or a review in a holographic context in [161].

magnetohydrodynamics we do have access to the Lorentzian spectral densities for these correlations at small *real* frequencies[6]. It is thus convenient to rewrite this expression in terms of these spectral densities, which can be done using standard finite-temperature techniques (reviewed in Appendix G.1) to obtain:

$$G^E_{QQ}(i\Omega_l) = -64\int \frac{d^3p}{(2\pi)^3}\int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\left[\frac{f(\omega_1)-f(\omega_2)}{\omega_1-\omega_2-i\Omega_l}\right]\left(\rho_{E_iE_j}(\omega_1)\rho_{B_iB_j}(\omega_2)+\rho_{E_iB_j}(\omega_1)\rho_{B_iE_j}(\omega_2)\right) \tag{7.14}$$

where ρ_{EE} and ρ_{EB} are the spectral densities associated with the retarded correlation functions of $\vec{E}$ and $\vec{B}$, i.e.

$$\rho_{E_iE_j}(\omega,p) = -\frac{1}{\pi}\mathrm{Im}\, G^R_{E_iE_j}(\omega,p), \tag{7.15}$$

and similarly. Here $f(x)$ is the Bose distribution function[7]:

$$f(\omega) = \frac{1}{e^{\beta\omega}-1} \tag{7.16}$$

We have reduced the problem to evaluating integrals over these spectral densities. We now discuss these correlation functions.

7.1.1 Kinematics of plasma correlations

In the remainder of this section, we discuss the tensor structure of the correlators $\langle BB\rangle$, $\langle EE\rangle$ and $\langle EB\rangle$, expressing them in terms of scalar functions of momenta and frequencies and describing what is known about them from magnetohydrodynamics.

Tensor structure of correlators

In this work we are interested in fluctuations about the plasma at finite temperature with zero background magnetic field $\vec{B}_0 = 0$. We will restrict attention to a parity-invariant theory. Here we record the constraints on the correlation functions arising from parity invariance and magnetic flux conservation.

The most general possible tensor decomposition of the retarded correlation func-

[6]It is in general not trivial to analytically continue approximate expressions from real to Euclidean frequencies.

[7]Note a sight abuse of notation here: we also have labelled the field strngth of the $U(1)$ gauge field as $f^{\mu\nu}$. However, from the context it should be clear which f we are referring to.

tions is,

$$\begin{aligned}\langle E_i E_j\rangle &= A(\omega,|\vec{p}|)\delta_{ij} + X(\omega,|\vec{p}|)\frac{p_i p_j}{p^2} + U(\omega,|\vec{p}|)\epsilon_{ijk}\frac{p^k}{|\vec{p}|},\\ \langle B_i B_j\rangle &= C(\omega,|\vec{p}|)\delta_{ij} + Y(\omega,|\vec{p}|)\frac{p_i p_j}{p^2} + V(\omega,|\vec{p}|)\epsilon_{ijk}\frac{p^k}{|\vec{p}|},\\ \langle E_i B_j\rangle &= M(\omega,|\vec{p}|)\delta_{ij} + N(\omega,|\vec{p}|)\frac{p_i p_j}{p^2} + K(\omega,|\vec{p}|)\epsilon_{ijk}\frac{p^k}{|\vec{p}|},\end{aligned} \tag{7.17}$$

where we remind ourselves that p^2 above denote the square of the norm of the 3-vector: $\vec{p}$. At times we will use the short-hand notation for the above scalar functions, Z^ω to denote $Z(\omega,|\vec{p}|)$.

Noting that $\vec{E}$ is a vector and $\vec{B}$ a pseudo-vector under parity, the scalar coefficient functions U^ω, V^ω, M^ω and N^ω are all odd under parity and thus vanish in a parity-invariant state. Thus, A^ω, X^ω and C^ω, Y^ω will be expressed in terms of the diagonal components of the $\langle E_i E_j\rangle$ and $\langle B_i B_j\rangle$ correlators, respectively. On the contrary, K^ω will be expressed in terms of the off-diagonal components of the $\langle E_i B_j\rangle$.

To impose the constraints from conservation of $J^{\mu\nu}$, let us assume a plane wave basis of the form $e^{i(\vec{p}\cdot\vec{x}-\omega t)}$. Without loss of generality in this section we take the spatial momentum to be aligned along the z-direction, that is, $p_z \neq 0$ implying, $p^2 = p_z^2$ and $|\vec{p}| = p_z$. Now let us look at the various components of Eq. (7.1). Individually, the temporal and z components give: $J^{0z} = 0$. The remaining spatial components, that is for $i \in \{x, y\}$, we get:

$$J^{zi} = \frac{\omega}{p_z} J^{0i}, \tag{7.18}$$

Using Eq. (7.18) and the definition of $\vec{E}, \vec{B}$ in terms of the components of the 2-form current, we can find expressions for the scalar functions, given in Eq. (7.17), in terms of the relevant two-point functions of the form $\langle J^{\mu\nu} J^{\rho\sigma}\rangle$.

Let us first look at the $\langle B_i B_j\rangle$ correlator. Since $J^{0z} = 0$, we have $\langle B_z B_z\rangle = 0$. This implies, $C^\omega + Y^\omega = 0$. From the diagonal x, y-components, we get: $C^\omega = \langle B_i B_i\rangle$, where $i \in \{x, y\}$. Following a similar analysis for the $\langle E_i E_j\rangle$ correlator, we get from its diagonal z-component: $A^\omega + X^\omega = \langle E_z E_z\rangle$. Its remaining diagonal components give: $A^\omega = \langle E_i E_i\rangle$. Finally, we can use Eq. (7.18) to relate A^ω and C^ω

as follows,

$$C^{\omega} = \frac{p^2}{\omega^2} A^{\omega}. \tag{7.19}$$

The value of the scalar function K^{ω} is yet to be determined. This can be done by looking at the cross correlator: $\langle E_i B_j \rangle$, which we do next. First of all, note that for $i \neq j$ we have,

$$\langle E_i B_j \rangle = K \epsilon_{ijk} \frac{p^k}{|\vec{p}|} = \langle B_j E_i \rangle \,, \qquad \langle B_i E_j \rangle = -K \epsilon_{ijk} \frac{p^k}{|\vec{p}|} = -\langle E_i B_j \rangle \,.$$

Now since any correlator with B_z in it vanishes, as $J^{0z} = 0$, we have from above that any correlator with E_z in it should also vanish. This is because these correlators just differ from each other by a minus sign. So, for non-vanishing cross correlators of the form: $\langle E_i B_j \rangle$, we must have $i, j \neq z$. Choosing $i, j = x, y$, we get,

$$\langle E_x B_y \rangle = K^{\omega} = \left\langle J^{yz} J^{0y} \right\rangle = -\frac{\omega}{p_z} \left\langle J^{0y} J^{0y} \right\rangle = -\frac{\omega}{p_z} \langle B_y B_y \rangle = -\frac{\omega}{p_z} C^{\omega} = -\frac{p_z}{\omega} A^{\omega}, \tag{7.20}$$

where the third equality results from Eq. (7.18) and the last equality comes from Eq. (7.19).

Thus, to conclude, we have the following expressions for the correlators of $\vec{E}$ and $\vec{B}$:

$$\begin{aligned} \langle E_i E_j \rangle &= A(\omega, |\vec{p}|) \delta_{ij} + X(\omega, |\vec{p}|) \frac{p_i p_j}{p^2}, \\ \langle B_i B_j \rangle &= \frac{p^2}{\omega^2} A(\omega, |\vec{p}|) \left\{ \delta_{ij} - \frac{p_i p_j}{p^2} \right\}, \\ \langle E_i B_j \rangle &= -A(\omega, |\vec{p}|) \epsilon_{ijk} \frac{p^k}{\omega}. \end{aligned} \tag{7.21}$$

So far our considerations have been purely kinematical. We now turn to the specific dynamics of the finite-temperature plasma; now the low-frequency limits of these functions can be obtained from magnetohydrodynamics.

A general formulation of magnetohydrodynamics in terms of higher-form symmetry was given in [30]. In particular, the transverse channel A^{ω} contains the physics of diffusion of magnetic field lines, and the precise correlator needed was recorded in [62].

$$G^R_{J^{zx},J^{zx}}(\omega, p_z)_{\text{MHD}} = A(\omega, |\vec{p}|)_{\text{MHD}} = \frac{-i\omega^2\rho}{\omega + iDp^2}, \tag{7.22}$$

Here ρ is the resistivity of the plasma, and D the diffusion constant for magnetic field lines. It can be expressed in terms of the resistivity and magnetic permeability[8] Ξ of the plasma as

$$D = \frac{\rho}{\Xi} \; . \tag{7.23}$$

The longitudinal channel X appears only in the electric field channel and controls the physics of Debye screening. It is not expected to have any universal hydrodynamic structure, and is presumably analytic in frequency and momenta at low frequencies. We will see explicitly that it does not contribute at this order to the correlations of the topological density $J_{\mu\nu}\tilde{J}^{\mu\nu}$.

7.1.2 Computation of correlator

Now we are set to compute the factors in the momentum space version of Eq. (7.12). The first term in this factor which consists of same pairing correlators is given as,

$$\begin{aligned}\delta^{ij}\delta^{kl}\left[\langle E_iE_k\rangle_{\omega_1}\langle B_jB_l\rangle_{\omega_2}\right] &= \frac{p^2}{\omega_2^2}\delta^{ij}\delta^{kl}\left[\left\{A^{\omega_1}\delta_{ik} + X^{\omega_1}\frac{p_ip_k}{p^2}\right\}\left\{A^{\omega_2}\left(\delta_{jl} - \frac{p_jp_l}{p^2}\right)\right\}\right]\\ &= \frac{2p^2}{\omega_2^2}A^{\omega_1}A^{\omega_2},\end{aligned} \tag{7.24}$$

Similar computation for the second term which consists of cross corelators, gives,

$$\begin{aligned}\delta^{ij}\delta^{kl}\left[\langle E_iB_l\rangle_{\omega_1}\langle B_jE_k\rangle_{\omega_2}\right] &= \delta^{ij}\delta^{kl}\left\{-A^{\omega_1}\epsilon_{ilm}\frac{p^m}{\omega_1}\right\}\left\{-A^{\omega_2}\epsilon_{kjn}\frac{p^n}{\omega_2}\right\}\\ &= A^{\omega_1}A^{\omega_2}\frac{p_mp^n}{\omega_1\omega_2}\epsilon^{jkm}\epsilon_{kjn} = -\frac{2p^2}{\omega_1\omega_2}A^{\omega_1}A^{\omega_2}.\end{aligned} \tag{7.25}$$

Now we can compute the product of spectral densities as given in Eq. (7.14), using the above eqautions and Eq. (7.15).

$$\rho_{E_iE_j}(\omega_1)\rho_{B_iB_j}(\omega_2) = \frac{1}{\pi^2}\frac{2p^2}{\omega_2^2}a^{\omega_1}a^{\omega_2}, \qquad \rho_{E_iB_j}(\omega_1)\rho_{B_iE_j}(\omega_2) = -\frac{1}{\pi^2}\frac{2p^2}{\omega_1\omega_2}a^{\omega_1}a^{\omega_2}. \tag{7.26}$$

[8] The magnetic permeability can formally be thought of as the 1-form charge susceptibility, i.e. the thermodynamic quantity that measures the amount of magnetic field created by an applied field.

where $a^\omega = \mathrm{Im}\, A^\omega$. [9]

7.1.3 Correlation of topological density

We now compute the correlator of the topological density Eq. (7.14). More precisely, we will explicitly compute the following frequency-dependent quantity

$$\Gamma_A(\Omega) = \frac{k^2}{\chi_A \Omega} \mathrm{Im}\, G^R_{QQ}(\Omega, \vec{p} = 0) \tag{7.27}$$

Evaluated at $\Omega = 0$ this determines the decay rate of the axial charge, as explained in Eq. (7.5). However in this section we will compute its full frequency dependence.

From Eq. (7.14) we have,

$$\mathrm{Im}\, G^R_{QQ}(\Omega) = -64 \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \left[\frac{f(\omega_1) - f(\omega_2)}{\pi} \right] \delta(\omega_1 - \omega_2 - \Omega) \frac{2p^2}{\omega_1 \omega_2^2} (\omega_1 - \omega_2) a^{\omega_1} a^{\omega_2}, \tag{7.28}$$

To obtain the above we used $\Omega_l = -i\Omega + \varepsilon$ to go to real frequencies in Eq. (7.14) and used the identity,

$$\mathrm{Im} \left(\frac{1}{\omega_1 - \omega_2 - \Omega - i\varepsilon} \right) = \pi \delta(\omega_1 - \omega_2 - \Omega).$$

Next we evaluate the ω_1 integral and replace ω_2 by ω for notational simplicity. Then we have (see Eq. (7.5)):

$$\Gamma_A(\Omega) = -\frac{64k^2}{\chi_A} \frac{1}{\Omega} \left\{ \int_0^{\infty} \frac{4\pi p^2 dp}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{4\pi^3} \left[f(\omega + \Omega) - f(\omega) \right] \frac{2p^2}{(\omega + \Omega)\omega^2} \Omega\, a^{\omega+\Omega} a^{\omega} \right\}, \tag{7.29}$$

where we have changed to polar coordinates in momentum space using $d^3p = 4\pi p^2 dp$.

Given an explicit expression for a^ω, the above expression is in principle exact (assuming only Gaussian correlations in the plasma). To obtain an explicit answer, we now compute the contribution arising from hydrodynamic fluctuations alone, i.e. we set a^ω equal to its results from the MHD correlator from Eq. (7.22). In a given UV complete theory, this is not the full answer, as a^ω will not agree with the MHD

[9] This a^ω here should not be confused with the $U(1)$ gauge field denoted by the same letter a^μ. This is an abuse of notation but the context will make it clear what is being referred to.

result at high frequencies; however we expect that it should capture the dominant infrared contribution. Plugging in the MHD value Eq. (7.22) for a^ω, we find,

$$\Gamma_A(\Omega) = -\frac{64k^2}{\chi_A}\frac{1}{\Omega}\left\{\int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega}{4\pi^3}\left[f(\omega+\Omega)-f(\omega)\right]\frac{2p^2}{(\omega+\Omega)\omega^2}\Omega \right.$$
$$\left. \times \left(\frac{\omega^2\rho}{\omega^2+D^2p^4}\right)\left(\frac{(\omega+\Omega)^2\rho}{(\omega+\Omega)^2+D^2p^4}\right)\right\} \tag{7.30}$$

. Simplifying this result we find:

$$\Gamma_A(\Omega) = -\frac{16k^2\rho^2}{\pi^5\chi_A}\int_{-\infty}^{\infty} d\omega \left(\left[f(\omega+\Omega)-f(\omega)\right]\omega(\omega+\Omega)^2\int_0^\infty dp\frac{p^4}{[(\omega+\Omega)^2+D^2p^4][\omega^2+D^2p^4]}\right). \tag{7.31}$$

This integral is even in Ω, as can be seen from the change of variables $\omega \to -\omega$ and the identity $f(x)+f(-x) = -1$. This is expected from Eq. (7.5) and the fact that the imaginary part of the retarded correlator of the bosonic operator Q is an odd function of frequency.

Next, evaluating the p-integral we find

$$\Gamma_A(\Omega) = -\frac{16k^2\rho^2}{2\sqrt{2}\pi^4 D^{\frac{5}{2}}\chi_A}\int_{-\infty}^{\infty} d\omega\ \left[f(\omega+\Omega)-f(\omega)\right]\frac{\omega(\omega+\Omega)^2}{\Omega(2\omega+\Omega)}\left(\sqrt{|\omega+\Omega|}-\sqrt{|\omega|}\right). \tag{7.32}$$

We will first perform the remaining integral over ω for $\Omega > 0$. Let us denote the integrand in Eq. (7.32) by $L(\omega,\Omega)$.

Note that the integrand is now non-analytic as a function of ω and $\omega+\Omega$, arising from singularities in the integrand of Eq. (7.31) at small p when either of these frequencies vanish. We must thus separate the ω integral into the following three ranges

$$\omega \in (-\infty, -\Omega) \cup (-\Omega, 0) \cup (0, \infty)\ . \tag{7.33}$$

First let us perform the integration for $\omega \in (0,\infty)$, i.e. the integral of interest is:

$$I^+(\Omega) = \int_0^\infty d\omega\ \left[f(\omega+\Omega)-f(\omega)\right]\frac{\omega(\omega+\Omega)^2}{\Omega(2\omega+\Omega)}\left(\sqrt{\omega+\Omega}-\sqrt{\omega}\right). \tag{7.34}$$

We wish to extract the dependence on Ω in the limit that $\Omega \ll \beta$. This integral is convergent and can readily be done numerically; however obtaining an analytical handle on the small Ω limit of this integral is subtle. To see this note that expansion of the integrand in powers of Ω leads to an expression which is analytic in Ω. Naively, proceeding this way one is led to believe that Γ_A is also analytic in Ω. However, this is not the case; if we attempt to proceed naively, the integral over each term in the Taylor expansion in Ω fails to converge near $\omega = 0$, indicating that the integral itself not analytic as a function of Ω though the *integrand* is.

We thus need to carefully extract this non-analytic dependence of the decay rate on Ω. To do this we use the fact that there is a hierarchy of scales $\Omega \ll \beta^{-1}$ to introduce a cut-off Λ such that $\Omega \ll \Lambda \ll \beta^{-1}$. With this, we can separate the integral in Eq. (7.34) into IR, i.e. $\omega \in (0, \Lambda)$ and UV, i.e. $\omega \in (\Lambda, \infty)$ parts. The non-analytic dependence on Ω will come from the IR part. At the end of the calculation we will take $\Lambda \to 0$.

To do the IR part, we expand the integrand about $\beta \to 0$, and work to all orders in Ω. We then integrate the resulting expansion for the range: $\omega \in (0, \Lambda)$. We find the following result, which we have presented in a series expansion in Ω.

$$
\begin{aligned}
I^+(\Omega)_{\rm IR} = &\left(-\frac{\sqrt{\Lambda}}{2\beta} + \frac{\beta\Lambda^{5/2}}{120} - \frac{\beta^3\Lambda^{9/2}}{4320} + \mathcal{O}\left(\Lambda^{13/2}\right)\right)\Omega + \left(\frac{5}{6\beta} - \frac{\pi}{4\sqrt{2}\beta} - \frac{\sinh^{-1}(1)}{2\sqrt{2}\beta}\right)\Omega^{3/2} \\
&+ \left(\frac{1}{8\beta\sqrt{\Lambda}} + \frac{5\beta\Lambda^{3/2}}{288} - \frac{3\beta^3\Lambda^{7/2}}{4480} + \mathcal{O}\left(\Lambda^{11/2}\right)\right)\Omega^2 \\
&+ \left(-\frac{1}{24\beta\Lambda^{3/2}} - \frac{19\beta^3\Lambda^{5/2}}{28800} + \frac{\beta^5\Lambda^{9/2}}{24192} + \mathcal{O}\left(\Lambda^{13/2}\right)\right)\Omega^3 \\
&+ \left(\frac{139\beta}{10080} - \frac{\beta\pi}{192\sqrt{2}} - \frac{\beta\sinh^{-1}(1)}{96\sqrt{2}}\right)\Omega^{7/2} \\
&+ \left(\frac{11}{640\beta\Lambda^{5/2}} + \frac{5\beta}{1536\sqrt{\Lambda}} - \frac{13\beta^3\Lambda^{3/2}}{55296} + \mathcal{O}\left(\Lambda^{7/2}\right)\right)\Omega^4 + \mathcal{O}(\Omega^5), \qquad (7.35)
\end{aligned}
$$

Note the presence of a non-analytic series of terms starting at $\mathcal{O}(\Omega^{\frac{3}{2}})$. An interesting thing to note is that, in the above equation, the analytic terms in $\mathcal{O}$ depend upon the cut-off Λ, while the non-analytic pieces are independent of Λ, and are exact expressions which are functions of β. This suggests that the analytic pieces will receive contributions from the UV part while the non-analytic pieces are exact. Also, in the above equation there seems to be IR divergences in the analytic pieces about $\Lambda = 0$. As we will see below these will be cancelled from the UV part of the integral; of course the final answer cannot depend on Λ.

We now turn to the UV part of the integral $\omega \in (\Lambda, \infty)$. Here we can simply expand the integrand in powers of Ω; the integral over each term will converge, with any putative IR divergences cutoff by Λ. So, the UV limit of integration is simpler; expanding the integrand we find:

$$
\begin{aligned}
L^{+}(\omega,\Omega) &\xrightarrow{\Omega\to 0} -\frac{e^{\beta\omega}\beta\omega^{3/2}}{4(1-e^{\beta\omega})^2}\Omega + \frac{\beta\sqrt{\omega}}{64}\left(2\beta\omega\coth\left(\frac{\beta\omega}{2}\right)-5\right)\operatorname{csch}\left(\frac{\beta\omega}{2}\right)^2\Omega^2 \\
&-\frac{\beta^2\sqrt{\omega}}{768}\operatorname{csch}\left(\frac{\beta\omega}{2}\right)^4\left(4\beta\omega\left(2+\cosh\left(\beta\omega\right)\right)-15\sinh\left(\beta\omega\right)\right)\Omega^3 \\
&+\frac{\beta\left(4\beta^3\omega^3\left(11\cosh\left(\frac{\beta\omega}{2}\right)+\cosh\left(\frac{3\beta\omega}{2}\right)\right)-5\left(3+16\beta^2\omega^2+\left(-3+8\beta^2\omega^2\right)\cosh\left(\beta\omega\right)\right)\sinh\left(\frac{\beta\omega}{2}\right)\right)}{192\, e^{\frac{-5\beta\omega}{2}}\left(1-e^{\beta\omega}\right)^5\omega^{3/2}}\Omega \\
&+\mathcal{O}(\Omega^5).
\end{aligned}
\tag{7.36}
$$

The linear in Ω piece in the above expansion is free of any IR divergences and can be immediately integrated over the full range, $\omega\to(0,\infty)$. The term in Ω^2 is slightly more subtle; integrating it over $\omega\in(\Lambda,\infty)$ we find that the leading dependence on Λ is $-\frac{1}{8\beta\Lambda^{1/2}}$, which indeed precisely cancels the Λ-dependent divergent term in the quadratic piece in Eq. (7.35). We have explicitly verified that a similar cancellation takes place for the terms up to $\mathcal{O}(\Omega^4)$, and on general grounds it must happen to all orders in the Ω expansion. Next we can numerically integrate the UV part, term by term in the range $\omega\in(\Lambda,\infty)$ and finally take $\Lambda\to 0$.

Note that, the non-analytic pieces are controlled only by the IR integral while the analytic pieces received contribution both from the IR and the UV parts of the integral. Hence, in this sense, the non-analytic pieces are universal.

We may treat the remaining two pieces of the integral in Eq. (7.33) in the same way; we leave the details of these integrals to the Appendix H.1. Now gathering everything together we find the Ω-dependence of the integral to be:

$$
\Gamma_A(\Omega)=\frac{16k^2\rho^2}{2\sqrt{2}\pi^4 D^{\frac{5}{2}}\chi_A}\left\{\frac{\pi}{2\sqrt{2}\beta}\Omega^{3/2}-\frac{0.3236}{\sqrt{\beta}}\Omega^2+\frac{\pi\beta}{96\sqrt{2}}\Omega^{7/2}-0.00518\,\beta^{3/2}\Omega^4+\mathcal{O}(\Omega^6)\right\},
\tag{7.37}
$$

where the numerical coefficients of the analytic pieces are obtained by numerical integration and hence are approximate values. On the contrary, the numerical coefficients of the non-analytic pieces are exact values which do not receive UV corrections, and their dependence on the IR data ρ and D is expected to be universal.

We now recall that our computation above assumed $\Omega>0$. Note that, due to the non-analytic dependence on Ω, strictly speaking we do not know the behavior for $\Omega<0$. We explicitly compute the 1-loop integral for $\Omega<0$ in Appendix H.1.2

and show that $\Gamma_A(-\Omega) = \Gamma_A(\Omega)$ as required. So, for $\Omega \in \mathbb{R}$, the decay rate is given as,

$$\Gamma_A(\Omega) = \frac{16k^2\rho^2}{2\sqrt{2}\pi^4 D^{\frac{5}{2}}\chi_A}\left\{\frac{\pi}{2\sqrt{2}\beta}|\Omega|^{3/2} - \frac{0.3236}{\sqrt{\beta}}\Omega^2 + \frac{\pi\beta}{96\sqrt{2}}|\Omega|^{7/2} - 0.00518\,\beta^{3/2}\Omega^4 + \mathcal{O}(\Omega^6)\right\}, \tag{7.38}$$

which is now manifestly even.

Finally, we may note that Eq. (7.38) implies that $\Gamma_A(\Omega \to 0) = 0$. As claimed, the 1-loop decay rate itself vanishes in the vanishing magnetic field limit. We discuss the implications of this result further in the conclusion.

7.2 Discussion and outlook

Above we presented an explicit calculation of the real-time topological susceptibility – i.e. the retarded correlation function G^R_{QQ} of the operator $Q = \vec{E}\cdot\vec{B}$ – arising from hydrodynamic fluctuations about an equilibrium with vanishing magnetic field $\vec{B} = 0$ in a magnetohydrodynamic plasma.

In the presence of a finite B field a classical calculation leads to $\Gamma_A \sim \frac{B^2\rho}{\chi_A}$ at zero frequencies, as shown in Eq. (7.7). The goal of this calculation was to determine the leading fluctuation-induced contribution to the decay rate at vanishing B field. If this is non-zero, then strictly speaking in the infrared limit axial charge should not be considered a hydrodynamic variable.

Importantly, however, we found that the resulting correlation function vanishes at low frequencies, as shown explicitly in Eq. (7.37). This has the immediate implication that in a chiral plasma, the decay rate of axial charge (as computed to one-loop order in hydrodynamics) remains zero if background magnetic field is zero, i.e. the classical analysis here is trustworthy. In our analysis we have also computed the first few terms in a small frequency expansion in Ω; it is interesting to note that the presence of gapless diffusive modes results in a non-analytic dependence on Ω, though this dependence begins at $\mathcal{O}(\Omega^{\frac{3}{2}})$ and so is soft enough not to contribute to the decay rate itself.

It is interesting to compare this to corresponding results for a *non*-Abelian plasma, where the quantity analogous to Eq. (7.5) is the Chern-Simons diffusion rate, i.e. the low-frequency limit of the correlation function of the non-Abelian topological density $\mathrm{Tr}(f^a_{\mu\nu}\tilde{f}^{a\mu\nu})$. This quantity has been extensively studied both at weak-coupling [162] and from holography [163], and is certainly not zero.

A universal way to understand the difference between the Abelian and non-Abelian case is the following: in the Abelian case studied here the topological density in question can be understood as a bilinear in a conserved 2-form current $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}$, i.e. the anomaly equation Eq. (7.2) reads:

$$\partial_\mu j^\mu_A = k\, \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu} J^{\rho\sigma} \tag{7.39}$$

This 2-form current is associated with the continuous $U(1)$ 1-form symmetry that protects magnetic flux conservation in electrodynamics [21]. The presence of this continuous 1-form symmetry gives a great deal of extra structure to this problem. This structure is not present for non-Abelian gauge theory, where at most we have a discrete 1-form symmetry. At a calculational level, it was the presence of this continuous symmetry current (and its subsequent realization in thermal equilibrium) which allowed us to obtain non-trivial constraints on the infrared physics from magnetohydrodynamics.

More generally, it has recently been shown that a precise characterization of the anomaly Eq. (7.39) is possible in terms of *non-invertible symmetries* [37, 38]: i.e. there still exist topological operators that count axial charge, but these operators no longer obey a standard group composition law, and there is no longer a simple conserved current. Our understanding of the dynamical consequences of such non-invertible symmetries is still in its infancy. However it seems that one way to understand the calculation above is that the finite temperature dynamics of a charge density protected by a non-invertible symmetry is somewhat constrained – for example, it will not relax to nothingness unless an external magnetic field is applied. This result is philosophically consistent with [41], which showed that at zero temperature a form of Goldstone's theorem applies to such non-invertible symmetries in that there is a protected gapless mode when the symmetry is spontaneously broken.

This suggests that the calculation above could be reorganized to make the role played by the non-invertible symmetry more manifest. One possible way to do so would be to use at one-loop level the effective theories constructed in [138, 158], which realize the symmetries more directly. It would be very interesting to obtain a robust argument for the vanishing of this relaxation rate to all loop order in hydrodynamics. Another direction for future work is to compare our results for $G^R_{QQ}(\omega)$ to real-time lattice computations such as those in [80, 81], where one might hope that the non-analytic dependence on Ω in Eq. (7.37) – which are in principle fully determined by hydrododynamic data – could be verified from the lattice. We hope to return to this in the future.

CHAPTER 8

Conclusion

In this thesis, we conducted a thorough exploration of the finite temperature physics of a system afflicted by the Adler-Bell-Jackiw anomaly or chiral anomaly. We focused on systems belonging to the universality class of the chiral plasma. A defining feature of these systems is the non-conservation of the axial current due to the chiral anomaly, as described by a dynamical operator $f_{\mu\nu}\tilde{f}^{\mu\nu}$ derived from the field-strength tensor. Seeking a universal framework for this physics, we reformulated this operator in terms of the 2-form current associated with magnetic flux conservation.

Utilizing this universal structure, we undertook a holographic study of the system, constructing a dual bulk theory that exhibited the described symmetry structure. In this study, we probed various aspects of finite temperature anomalous magnetohydrodynamics, such as charge susceptibility and axial charge relaxation rate, as functions of temperature and magnetic field. Our findings in the small magnetic field regime aligned with basic hydrodynamics weakly coupled to an electrodynamic sector, but notable deviations emerged at larger fields.

Furthermore, we examined chiral magnetohydrodynamics in light of the universal symmetry structure embedded in the anomaly. We formulated ”effective actions” to capture both the equilibrium physics and the physics of dissipation. From these effective actions, we reproduced some familiar aspects of chiral MHD phenomenology, including chiral separation and magnetic effects. Our formalism also facilitated the construction of non-invertible axial symmetry defect operators in real time.

In addition, we studied the axial charge relaxation rate in the limit of vanishing magnetic field. Our analysis revealed that it vanished at zero frequency, suggesting that the axial charge dissipation rate disappears at zero background magnetic field. This outcome might indicate that the symmetry structure encoded in the anomaly is shielded by the non-invertible defect operators, as 1-loop effects do not disrupt it.

Overall, this research represents a modest step in the ongoing quest to better understand the complex physics of systems affected by the chiral anomaly. The insights gained through holography, hydrodynamics, and effective field theory approaches contribute to the existing body of knowledge on the dynamics and universal features of chiral systems and their late-time behavior. We hope the results and methods presented here may prove helpful for further investigations in this intriguing field.

APPENDIX A

Notations, Conventions and Discrete symmetries

A.1 Notations and Conventions

We work in natural units with $c = \hbar = 1$ and our metric signature is mostly plus: $(-, +, +, \cdots\cdots, +)$.

For boundary indices we have $\mu\nu\rho\sigma\lambda\cdots$ and for bulk indices we have $MNPQR\cdots$. We have for the epsilon symbol, $\tilde{\epsilon}_{0123\cdots} = +1$ and for the volume form, $\epsilon = +\sqrt{-g}\, d^D x$, in D spacetime dimensions.

In ingoing EF coordinates (r, v, x, y, z) we have,

$$ds_5^2 = -r^2 f(r) dv^2 + 2dvdr + r^2 \left(dx^2 + dy^2 + dz^2\right), \tag{A.1}$$

$$\sqrt{-g} = r^3, \tag{A.2}$$

$$\epsilon_{rvzxy} = r^3 \tilde{\epsilon}_{rvzxy} = r^3, \tag{A.3}$$

$$\epsilon^{rvzxy} = -r^{-3}, \tag{A.4}$$

where $f \equiv f(r) := \left(1 - \frac{r_h^4}{r^4}\right)$ and $\partial_r f(r) = \frac{4r_h^4}{r^5}$.

A.2 Conventions regarding differential forms

Let A be a p-form and B be a q-form,

$$(A \wedge B)_{p+q} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\mu_{p+1} \ldots \mu_{p+q}]} \tag{A.5}$$

where $[\ldots]$ in the subscript denotes complete anti-symmetrisation of the involved indices.

The exterior derivative is given as,

$$(dA)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \tag{A.6}$$

Let us give below some more identities involving differential forms:

$$d(A_p \wedge B_q) = dA_p \wedge B_q + (-1)^p A_p \wedge dB_q \tag{A.7}$$

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p \tag{A.8}$$

$$A_p \wedge \star B_p = B_p \wedge \star A_p \tag{A.9}$$

The square of the Hodge star acting on a p form in D dimensions on a metric with s minus signs in its eigenvalues is

$$\star^2 = (-1)^{s+p(D-p)} \tag{A.10}$$

Also,

$$\star A_p = \frac{1}{p!} \epsilon^{\nu_1 \ldots \nu_p}{}_{\mu_1 \ldots \mu_{D-p}} A_{\nu_1 \ldots \nu_p} \tag{A.11}$$

which gives,

$$\star \epsilon = (-1)^s, \qquad \star 1 = \epsilon = \sqrt{-g}\, d^D x \tag{A.12}$$

Next we record a few useful identities relating differential forms to their components, starting with:

$$A_p \wedge \star A_p = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} A^{\mu_2 \cdots \mu_p} \epsilon, \tag{A.13}$$

Integrating we find,

$$\int_{\mathcal{M}_D} A_p \wedge \star A_p = \frac{1}{p!} \int_{\mathcal{M}_D} d^D x \sqrt{-g} A_{\mu_1 \cdots \mu_p} A^{\mu_1 \cdots \mu_p}. \tag{A.14}$$

We will also sometimes use expressions of the following form,

$$\int_{\mathcal{M}_5} H_3 \wedge F_2 = \frac{(-1)^s}{2!3!} \int_{\mathcal{M}_5} d^5x \sqrt{-g} \epsilon^{\mu\nu\rho\alpha\beta} H_{\mu\nu\rho} F_{\alpha\beta} \tag{A.15}$$

where s is the number of minus signs in the metric.

A.3 Conventions regarding definition of the boundary current

For this let us consider free Maxwell action in D spacetime (bulk) dimensions as given below,

$$S = \frac{1}{2} \int_{\mathcal{M}_D} F_{p+1} \wedge \star F_{p+1} = \frac{1}{2} \int_{\mathcal{M}_D} \frac{1}{(p+1)!} \left(F_{p+1}\right)^2 \tag{A.16}$$

where $\left(F_{p+1}\right)^2 \equiv F^{\mu_1 \dots \mu_{p+1}} F_{\mu_1 \dots \mu_{p+1}}$. The boundary dual of the above theory has a magnetic $(D-p-3)$-form symmetry, $J_{D-p-2} \equiv \star \left. F_{p+1} \right|_{r \to \infty}$. So, in $D = 5$ spacetime (bulk) dimensions we have the usual story that $J_2 \equiv \star \left. F_2 \right|_{r \to \infty}$ (with r being the holographic radial coordinate).

Now let us Poincaré dualize the above action to get,

$$S_{\text{dual}} = \frac{1}{2} \int_{\mathcal{M}_D} H_{D-p-1} \wedge \star H_{D-p-1} = \frac{1}{2} \int_{\mathcal{M}_D} \frac{1}{(D-p-1)!} \left(H_{D-p-1}\right)^2 \tag{A.17}$$

where, $H_{D-p-1} = dB_{D-p-2}$ with B_{D-p-2} being the Lagrange multiplier to enforce the closure of F_{p+1} during the dualization procedure. The dualization gives, $H_{D-p-1} = \star F_{p+1}$ which in turn implies that,

$$J_{D-p-2} = \left. H_{D-p-1} \right|_{r \to \infty} . \tag{A.18}$$

The AdS/CFT dictionary we need to define the boundary current is,

$$\left\langle \exp\left(\frac{1}{p!} \int b_{\mu_1 \dots \mu_p} J^{\mu_1 \dots \mu_p} \right) \right\rangle_{\text{CFT}} = \mathcal{Z}_{\text{grav}}[B_{\mu_1 \dots \mu_p}(r \to \infty) = b_{\mu_1 \dots \mu_p}]$$

$$\text{leading to, } J^{\mu_1 \dots \mu_p} = p! \lim_{r \to \infty} \frac{\delta S_{\text{bulk}}}{\delta \left(\partial_r B_{\mu_1 \dots \mu_p} \right)}, \tag{A.19}$$

Note that there is a factor of $p!$ in the definition of the boundary current in A.19. This factor is needed to get A.18. Let us show this below.

Now let us obtain the boundary current from S_{dual} using A.19.

$$\frac{\delta S}{\delta\left(\partial_r B_{\mu_1\cdots\mu_{D-p-2}}\right)} = \frac{1}{2(D-p-1)!}\frac{\partial\left(H_{D-p-1}\right)^2}{\partial\left(\partial_r B_{\mu_1\cdots\mu_{D-p-2}}\right)} = \frac{1}{2(D-p-1)!}2H^{r\mu_2\cdots\mu_{D-p-2}}\left(D-p-1\right)$$
$$= \frac{1}{(D-p-2)!}H^{r\mu_2\cdots\mu_{D-p-1}} \tag{A.20}$$

Thus, we see that to get A.18, we should have the following normalization in the boundary current definition,

$$J^{\mu_2\cdots\mu_{D-p-1}} = (D-p-2)!\lim_{r\to\infty}\frac{\delta S_{\text{bulk}}}{\delta\left(\partial_r B_{\mu_2\cdots\mu_{D-p-1}}\right)}, \tag{A.21}$$

So, A.21 explains the factor of $p!$ in A.19.

A.4 Discrete symmetries

Here we record some background on the construction of Table 6.1 of discrete symmetries. Note that a^μ is the axial source and hence it is a pseudo-vector and has the same transformation under discrete symmetry as that of $\overline{\Psi}\gamma^\mu\gamma^5\Psi$. Next let us look at $b_{\mu\nu}$. Since $b_{\mu\nu}$ is the source for the 2-form current, it couples in the action as $\epsilon^{\mu\nu}b_{\mu\nu}F_{\rho\sigma}$, where $F_{\rho\sigma}$ is the field strength of the vector gauge field. So, b_{ij} couples to $F^{k\tau}$ and $B_k(=b_{k\tau})$ couples to F^{ij}. $F^{\mu\nu}$ being a 2-index object we will have the following transformations of its components under discrete symmetries, $F^{k\tau}$ will transform as $\overline{\Psi}\sigma^{k\tau}\Psi$ and F^{ij} will transform as $\overline{\Psi}\sigma^{ij}\Psi$. Once we have identified the above discrete transformations, we may now use standard results (e.g. those in [164]).

APPENDIX B

Poincaré duality and Inverse operation

B.1 Poincaré Duality

As a first example consider Maxwell's action in D spacetime dimensions (see for example [105]). The action is given as,

$$S = \frac{1}{2}\int_{\mathcal{M}^D} F_{p+1}(A_p) \wedge \star F_{p+1}(A_p), \tag{B.1}$$

where $F_{p+1} = dA_p$ is a $(p+1)$ form and $A_p \in \bigwedge^p \left(\mathcal{M}^D\right)$. The EOMs are,

$$dF_{p+1} = 0, \tag{B.2}$$

$$d \star F_{p+1} = 0. \tag{B.3}$$

where Eq. (B.2) is the Bianchi Identity and Eq. (B.3) is the source part of EOM let us call it 's-EOM'. Note that here F_{p+1} is a closed $(p+1)$ form.

We are interested in dualizing F_{p+1} (the field strength) in Eq. (B.1). Naively, note that if we have,

$$H_{D-p-1} := \star F_{p+1}, \tag{B.4}$$

$$\text{leading to, } \star H_{D-p-1} = \star(\star F_{p+1}) = \pm F_{p+1}, \tag{B.5}$$

$$\text{thereby EOMs, } d \star H_{D-p-1} = 0, \ and \ dH_{D-p-1} = 0 \tag{B.6}$$

where locally due to Poincare lemma we can have $H_{D-p-1} = dB_{D-p-2}$ where $B_{D-p-2} \in \bigwedge^{D-p-2}(\mathcal{M}^{\mathcal{D}})$. Note the above interchanges the Bianchi Identity and s-EOM. In the above, naively we see that a p form potential A_p is dual to a $D-p-2$ form potential B_{D-p-2}. Let us do a small counting argument. The no. of linearly independent p forms in D dimension is $\frac{D!}{(D-p)!p!}$. However, if we are looking at physical DoFs, then the physical DoFs of A_p is $\frac{(D-2)!}{(D-p-2)!p!}$ (as A_p has $D-2$ polarisations in D dimensions). Now note that, the physical DoFs of a p form and its dual $D-p-2$ form are equal.

B.1.1 Poincaré Dualization

Consider an arbitrary (not necessarily closed) $(p+1)$ form F_{p+1}. Take $B_{D-p-2} \in \bigwedge^{D-p-2}(\mathcal{M}^D)$ which will play the role of a Lagrange multiplier to impose the closure constraint of F_{p+1}. Now consider the action,

$$S_c = (-1)^p \int_{\mathcal{M}^D} dB_{D-p-2} \wedge F_{p+1}, \tag{B.7}$$

where the $(-1)^p$ infront of the integral is present by convention.

We can add S_c to a parent action S_p given as,

$$S_p = \int_{\mathcal{M}^D} \frac{1}{2} F_{p+1} \wedge \star F_{p+1} + (-1)^p dB_{D-p-2} \wedge F_{p+1}. \tag{B.8}$$

Note that we can rewrite S_c as,

$$S_c = (-1)^p \int_{\mathcal{M}^D} d(B_{D-p-2} \wedge F_{p+1}) + (-1)^{D-p-1} B_{D-p-2} \wedge dF_{p+1}, \tag{B.9}$$

$$\text{leading to, } S_c = \int_{\mathcal{M}^D} (-1)^{D-1} B_{D-p-2} \wedge dF_{p+1} \text{ (ignoring surface term)}, \tag{B.10}$$

which gives for B_{D-p-2}'s EOM,

$$dF_{p+1} = 0 \;\Rightarrow F_{p+1} = dA_p \text{ (locally by Poincare lemma)}. \tag{B.11}$$

Then (locally) due to Poincare lemma we can re-write S_p as,

$$S_p = \int_{\mathcal{M}^D} \frac{1}{2} (dA_p) \wedge (\star dA_p) + (-1)^p dB_{D-p-2} \wedge dA_p, \tag{B.12}$$

which gives EOMs as,

$$d\star F_{p+1} = 0,\ (A_p's\ \text{EOM}) \tag{B.13}$$

$$dF_{p+1} = 0.\ (B'_{D-p-2}s\ \text{EOM}) \tag{B.14}$$

Now in order to obtain the dual description of Eq. (B.1) we vary S_p as in Eq. (B.8) w.r.t. F_{p+1}. Varying S_p as in Eq. (B.8) w.r.t. F_{p+1} we obtain,

$$(\star F_{p+1})_{\mu_1\cdots\mu_{D-p-1}}\ dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p+1}}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{D-p-1}}$$
$$+(-1)^p(dB_{D-p-2})_{\mu_1\cdots\mu_{D-p-1}}\ dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{D-p-1}}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p+1}} = 0$$

leading to, $(\star F_{p+1})_{\mu_1\cdots\mu_{D-p-1}}\ dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p+1}}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{D-p-1}}$

$$=(-1)^{p+1}(-1)^{(p+1)(D-p-1)}(dB_{D-p-2})_{\mu_1\cdots\mu_{D-p-1}}$$
$$dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{p+1}}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{D-p-1}}$$

leading to, $\star F_{p+1} = (-1)^{(p+1)(D-p)}dB_{D-p-2} = (-1)^{(p+1)(D-p)}H,$ (B.15)

where $H_{D-p-1} := dB_{D-p-2}$ (field strength for B_{D-p-2}). Now one can see that,

$$F_{p+1} = (-1)^p(\star H_{D-p-1}), \tag{B.16}$$

leading to, $S_D = \dfrac{1}{2}\displaystyle\int_{\mathcal{M}^D} H_{D-p-1}(B_{D-p-2})\wedge\star H_{D-p-1}(B_{D-p-2})$

(replacing H_{D-p-1} in Eq. $(B.8)$), (B.17)

where, S_D is the dual action to S and now clearly the Bianchi Identity and s-EOM are interchanged. Note that in the whole Poincare Dualization procedure it is the field strength F_{p+1} which is dualized to H_{D-p-1} and not the potential A_p.

So, the theories given by S and S_D are therefore equivalent and we can only consider potentials upto rank $\frac{D}{2}-1$ (as, A_p is dual to B_{D-p-2} and so, $D-p-2=p$ which gives, $p=\frac{D}{2}-1$). The above discussion on Poincare Dualization suggests that if one considers S_p (as given in Eq. (B.12)) for the action of Maxwell EM in $D=4$ then one can recover both pair of EOMs from S_p as presented in Eqns. (B.13) and (B.14).

B.1.2 Toy Example $S_3:\ A_1\wedge F_2$

Now we consider another action S (in $D=3$) and we want to look at its dual description. As before we shall dualize the field strength and not the potential.

Consider the following action,

$$S_3 = \int_{\mathcal{M}^3} \frac{1}{2} F_2 \wedge \star F_2 + \frac{1}{2} G_2 \wedge \star G_2 + k A_1 \wedge F_2, \tag{B.18}$$

where $F_2 = dV_1$ and hence is a closed 2 form, A_1 and V_1 are a 1 forms, $G_2 = dA_1$ and k is a constant scalar. Now again a naive look tells us that V_1 shall have a corresponding dual potential in the dual description which will be a 0 form say θ_0 (as, $p \leftrightarrow D - p - 2$ and $D = 3, p = 1$ for this case). Notice that S_3 in this case is at the level of S as given in Eq. (B.1).

Now proceeding as before, let us define a parent action which shall contain a Lagrange multiplier (which shall be a 0 form in this case) to ensure the closure of an arbitrary (not necessarily closed) 2 form F_2. We have,

$$S_{3p} = \int_{\mathcal{M}^3} \frac{1}{2} F_2 \wedge \star F_2 + \frac{1}{2} G_2 \wedge \star G_2 + k A_1 \wedge F_2 + (-1)^1 d\theta_0 \wedge F_2. \tag{B.19}$$

We define S_c as,

$$S_c = \int_{\mathcal{M}^3} (-1)^1 d\theta_0 \wedge F_2. \tag{B.20}$$

Notice that,

$$(-1) d\theta_0 \wedge F_2 = (-1) d(\theta_0 \wedge F_2) + (-1)^0 \theta_0 \wedge dF_2$$

Hence, we can re-write S_c as (ignoring the surface term),

$$S_c = \int_{\mathcal{M}^3} \theta_0 \wedge dF_2$$

where varying the above S_c w.r.t. θ_0 gives, $dF_2 = 0$ (closure of F_2 through θ_0).

Now let us vary S_{3p} w.r.t. F_2. Then $\delta_{F_2} S_{3p} = 0$ gives,

$$\begin{aligned}
&(\star F_2)_{\mu_1}\, dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\mu_1} + (-1)(d\theta_0)_{\mu_1}\, dx^{\mu_1} \wedge dx^{\nu_1} \wedge dx^{\nu_2} \\
&+ k(A_1)_{\mu_1}\, dx^{\mu_1} \wedge dx^{\nu_1} \wedge dx^{\nu_2} = 0 \\
\text{leading to, } &(\star F_2)_{\mu_1}\, dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\mu_1} = (d\theta_0)_{\mu_1}\, dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\mu_1} \\
&- k(A_1)_{\mu_1}\, dx^{\nu_1} \wedge dx^{\nu_2} \wedge dx^{\mu_1} \\
\text{leading to, } &\star F_2 = d\theta_0 - k A_1, \qquad \text{(B.21)} \\
\text{leading to, } &F_2 = -(\star d\theta_0 - k(\star A_1)). \qquad \text{(B.22)}
\end{aligned}$$

Let us define $B_2 := \star d\theta_0$ which implies, $\star B_2 = -d\theta_0$. Now let us compute the dual action S_{3D} by putting the obtained values of F_2 and $\star F_2$ in S_{3p},

$$\begin{aligned} S_{3D} = & \int_{\mathcal{M}^3} \frac{1}{2} \left[(k(\star A_1) - B_2) \wedge ((-B_2) - kA_1) \right] + kA_1 \wedge (k(\star A_1) - B_2) \\ & + \star B_2 \wedge (k(\star A_1) - B_2) + \frac{1}{2} G_2 \wedge \star G_2 \\ = & \int_{\mathcal{M}^3} \frac{1}{2} \left[kA_1 \wedge (k(\star A_1) - B_2) + \ast B_2 \wedge (k(\star A_1) - B_2) \right] + \frac{1}{2} G_2 \wedge \star G_2 \\ = & \int_{\mathcal{M}^3} \frac{1}{2} G_2 \wedge \star G_2 + \frac{1}{2} (d\theta_0 - kA_1) \wedge \star (d\theta_0 - kA_1). \end{aligned} \tag{B.23}$$

B.2 Inverse operation

In terms of tensor-index notation, Eq.(5.30) is,

$$F_{MN} = -\frac{1}{2} \epsilon_{PQRMN} \; (\partial^P B^{QR}) + 4k \; \epsilon_{PQRMN} \; A^P F^{QR},$$

$$\text{leading to, } \left[\delta^Q_M \delta^R_N - 4k \; \epsilon_{IJKMN} \; g^{JQ} g^{KR} A^I \right] F_{QR} = -\frac{1}{2} \epsilon_{IJKMN} \; \partial^I (B^{JK}), \tag{B.24}$$

$$\text{leading to, } \mathcal{O}^{QR}_{\;\;MN} \; F_{QR} = -\frac{1}{2} \epsilon_{IJKMN} \; \partial^I (B^{JK}), \tag{B.25}$$

where $\mathcal{O}^{QR}_{\;\;MN} \equiv \delta^Q_M \delta^R_N - 4k \; \epsilon_{IJKMN} \; g^{JQ} g^{KR} A^I$.

Now the task is to invert $\mathcal{O}^{QR}_{\;\;MN}$ to obtain $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ such that $\mathcal{O}^{QR}_{\;\;MN} \; (\mathcal{O}^{-1})^{MN}_{\;\;LK} = \delta^Q_L \delta^R_K$.[1] Then, $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ would enable us to write F_2 in terms of B_2 and A_1 and their derivatives which is what we are after.

Note that we are not assuming that $\mathcal{O}^{QR}_{\;\;MN}$ or $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ is anti-symmetric at this point. However, ultimately they will be contracted with F_{QR} (for instance, see Eq.(B.24)), and their symmetric parts would cancel out and things will turn out to be consistent.

Let us consider the most general $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ possible (arranged in ascending powers of A_1) and then we shall demand it to be $\mathcal{O}^{QR}_{\;\;MN}$'s inverse. The most general expression for $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ is,

$$\begin{aligned} (\mathcal{O}^{-1})^{MN}_{\;\;LK} \; = \; & c_0 \; \delta^M_L \delta^N_K \; + \; \bar{c}_0 \; \delta^M_K \delta^N_L \; + \; 4c_1 k \; \epsilon_{PIJLK} \; A^P \; g^{IM} g^{JN} \; + \; 16 c_2 k^2 \delta^M_L A^N A_K \\ & + \; 16 \bar{c}_2 k^2 \delta^N_L A^M A_K + \; 16 c_2' k^2 \delta^N_K A^M A_L \; + \; 16 \tilde{c}_2 k^2 \delta^M_K A^N A_L, \end{aligned} \tag{B.26}$$

where $c_0, \; \bar{c}_0, \; c_1, \; c_2, \; \bar{c}_2, \; c_2', \; \tilde{c}_2$ are coefficients to be determined by demanding that

[1] Note that $\delta^Q_L \delta^R_K F_{QR} = F_{LK}$. So, in the space of 2 forms $\delta^Q_L \delta^R_K$ is the identity operator.

$(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ is the inverse of $\mathcal{O}^{QR}_{\;\;MN}$ (their subscript is numbered as per the powers of A_1 they are coefficients of). Note that we cannot have any more powers of A_1 in the above expression (in the sense of anti-symmetric indices of A_1) as when they would be contracted with ϵ_{PIJQR} they would cancel. Note that, every term other than terms whose coefficients are c_0 and $\tilde{c}_0$ have to come with some powers of k otherwise on $k \to 0$ limit they would not give the proper inverse of the $\mathcal{O}^{QR}_{\;\;MN}\Big|_{k\to 0} = \delta^Q_M \delta^R_N$ as they would survive the $k \to 0$ limit.

Now demanding, $\mathcal{O}^{QR}_{\;\;MN}\,(\mathcal{O}^{-1})^{MN}_{\;\;LK} = \delta^Q_L \delta^R_K$ we get the following equation,

$$
\begin{aligned}
&(c_0 + 32c_1k^2A^2)\delta^Q_L\delta^R_K \;+\; (\bar{c}_0 - 32c_1k^2A^2)\delta^Q_K\delta^R_L \;+\; (c_1 - c_0 + \bar{c}_0)4k\,\epsilon_{PIJLK}A^P\; g^{IQ}g^{JR} \\
&+\; (c_2 - 2c_1)16k^2\,\delta^Q_L A^R A_K \;+\; (\bar{c}_2 + 2c_1)16k^2\,\delta^R_L A^Q A_K \;+\; (c_2^{'} - 2c_1)16k^2\,\delta^R_K A^Q A_L \\
&+\; (\tilde{c}_2 + 2c_1)16k^2\,\delta^Q_K A^R A_L = \delta^Q_L\delta^R_K
\end{aligned}
$$

So the above equation is satisfied if,

$$
c_1 = \frac{1}{1+64k^2A^2},\; c_0 = \frac{1+32k^2A^2}{1+64k^2A^2},\; \bar{c}_0 = \frac{32k^2A^2}{1+64k^2A^2}, \tag{B.27}
$$

$$
c_2 = c_2^{'} = 2c_1 = \frac{2}{1+64k^2A^2},\; \bar{c}_2 = \tilde{c}_2 = -2c_1 = \frac{-2}{1+64k^2A^2}. \tag{B.28}
$$

One can readily check that with the above values for the coefficients $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ as given in Eq.(B.26) is the inverse of $\mathcal{O}^{QR}_{\;\;MN}$ (and also gives the correct inverse in the $k \to 0$ limit when $c_0 \to 1$ and only the term with c_0 as the coefficient survives).

Next we multiply, Eq.(B.25) with $(\mathcal{O}^{-1})^{MN}_{\;\;LK}$ to get Eq.(5.36),

For completeness here we express the differential forms that make up S_{5p} in terms of their components (up to quadratic orders in E),

$$
-\frac{1}{2}F_2 \wedge \star F_2 \to \left[\frac{H^2}{12} - 48k^2\,(E\cdot H)^2 + \frac{32}{3}k^2H^2E^2 + \frac{2k}{3}\epsilon_{PQRLK}H^{PQR}E_MH^{MLK}\right]\tilde{c}_1^2, \tag{B.29}
$$

$$
+H_3 \wedge F_2 \to \left[-\frac{H^2}{6} - \frac{64}{3}k^2H^2E^2 - \frac{2k}{3}\epsilon_{PQRLK}H^{PQR}E_MH^{MLK} + 32k^2\,(E\cdot H)^2\right]\tilde{c}_1^2, \tag{B.30}
$$

$$
-4kE_1 \wedge F_2 \wedge F_2 \to \left[32k^2\,(E\cdot H)^2 - \frac{k}{3}\epsilon_{PQRLK}H^{PQR}E_MH^{MLK}\right]\tilde{c}_1^2, \tag{B.31}
$$

$$
-\frac{1}{2}G_2 \wedge \star G_2 \to -\frac{1}{4}G^2, \tag{B.32}
$$

and when expanded in powers of small k, $\tilde{c}_1^2 = 1 - 128\,k^2E^2 + \mathcal{O}(k^4)$.

APPENDIX C

$\zeta \to 0$ and hypergeometric differential equation

Note that if we naively put $\zeta = 0$ in (5.49), then we do not get two linearly independent solutions at $\zeta = 0$. This is related to the structure of the Riemann differential equation about the point $\zeta = 0$. Here we give the general solution to Eq.(5.45) in the limit of $\zeta \to 0$[1],

$$\delta E_t(r)_{gen}|_{\zeta\to 0} = \frac{r_h}{r}\Bigg[m_1\ {}_2F_1\left(-\frac{1}{4},\frac{1}{4};1;\frac{r^4}{r_h^4}\right) + m_2\ \Bigg\{{}_2F_1\left(-\frac{1}{4},\frac{1}{4};1;\frac{r^4}{r_h^4}\right)\ln\left(\frac{r^4}{r_h^4}\right) + \sum_{i=0}^{\infty}\frac{(a)_i\,(b)_i}{(c)_i\,i!}\left(\frac{r^4}{r_h^4}\right)^i\left(\psi\left(i-\frac{1}{4}\right)+\psi\left(i+\frac{1}{4}\right)-2\psi\left(1+i\right)\right)\Bigg\}\Bigg], \tag{C.1}$$

where $a = -\frac{1}{4},\ b = \frac{1}{4},\ c = 1,\ \psi(x) \equiv \frac{d}{dx}\ln(\Gamma(x))$ is the digamma function, m_1 and m_2 are integration constants, and $(a)_i$ is the rising Pochhammer symbol defined as,

$$(a)_i := \begin{cases} 1, & i = 0 \\ a(a+1)\cdots\cdots(a+i-1), & i > 0 \end{cases}$$

[1]For more information see section 15.10 of [165].

APPENDIX D

Shooting method and Gapped modes

D.1 Implementation of numerics

In this appendix we aim to find a searching condition for the numerical implementation of our quasinormal modes. To do this we perform a simple linear algebra exercise. Suppose we have N fields $\Phi^I(r)$. Let $\Phi^I_a(r)$ be a basis for the solutions that are ingoing at horizon ($a \in \{1, \cdots\cdots, N\}$). Let $\Phi^I_\alpha(r)$ be a basis for the AdS boundary solutions with an appropriate boundary condition ($\alpha \in \{1, \cdots\cdots, N\}$). Let r^* be the matching point. We wish to know when $\exists$ C_a and D_α such that,

$$\sum_a C_a \Phi^I_a(r^*) = \sum_\alpha D_\alpha \Phi^I_\alpha(r^*), \tag{D.1}$$

$$\sum_a C_a \Phi'^I_a(r^*) = \sum_\alpha D_\alpha \Phi'^I_\alpha(r^*), \tag{D.2}$$

which implies that there exists a solution that satisfies both sets of boundary conditions.

We can frame the above as a linear algebra problem. Consider $X^A_a := \left\{\Phi^I_a(r^*), \Phi'^I_a(r^*)\right\}$ (with $A \in \{1, \cdots\cdots, 2N\}$) and view X^A_a as a subspace of $\mathbb{R}^{2N}$. Similarly, consider $Y^A_\alpha := \left\{\Phi^I_\alpha(r^*), \Phi'^I_\alpha(r^*)\right\}$. We wish to know when the subspaces X^A_a and Y^A_α overlap. This condition will then imply the existence of a solution that would satisfy both sets of boundary conditions Eq. (D.1) and Eq. (D.2). Let us consider $\left(Y^\perp\right)^A_b$ that satisfies,

$$\sum_A \left(Y^{\perp}\right)_b^A Y_\alpha^A = 0, \tag{D.3}$$

where $b \in \{1, \cdots\cdots, N\}$.

Then we want,

$$\sum_A \left(Y^{\perp}\right)_b^A X_a^A = 0, \tag{D.4}$$

to have a non-trivial solution. Eq.(D.4) will have a non-trivial solution if and only if,

$$\det_{a,b}\left[\sum_A \left(Y^{\perp}\right)_b^A X_a^A\right] = 0. \tag{D.5}$$

Eq.(D.5) is the QNM searching condition that we have been looking for. Next we perform the numerics[1] with the mid-point shooting method. For the matching point, we have, $y_m = 0.6$. The numerical parameters used are,

Tolerance (t_m)	Horizon radius (r_h)	UV cut-off (u_Λ)
0.1	1	0.1
Double-trace coupling (κ)	RG parameter (u^*)	Anomaly coefficient (k)
$-1/\ln(10)$	1	0.0375

D.2 Non-hydrodynamic (gapped) modes and quasi-normal mode table

We observe from the numerics that, there exists a higher (generically) complex non-hydrodynamic mode $\forall\ b \in [0.00001, 20]$. This mode seems to be independent of b as it exists $\forall\ b \in [0.00001, 20]$ and has the value,

$$\Gamma^{\text{cplx}}_{\text{non-hydro},0} = \pm 0.965 - 1.736i, \tag{D.6}$$

[1]An order of convergence of 10^{-11} and lower has been treated as zero in the numerics. The numerical results that are presented have been verified (up to very slight variations) for the matching point in the range $y_m \in [0.2, 0.8]$. We have also dropped a few terms in the UV and the IR expansions of the fields and have verified the robustness of the numerical results.

However, $\forall\ b \in (0, 15.3]$, $\Gamma^{\text{cplx}}_{\text{non-hydro},0}$ is not the lowest QNM and for $\forall\ b \geq 15.4$, $\Gamma^{\text{cplx}}_{\text{non-hydro},0}$ becomes the lowest QNM.

We also give below some more generically complex non-hydro (gapped) modes which exist $\forall\ b \in [0.00001, 20]$,

$$\Gamma^{\text{cplx}}_{\text{non-hydro},1} = \pm 3.359 - 3.697i, \tag{D.7}$$
$$\Gamma^{\text{cplx}}_{\text{non-hydro},2} = \pm 5.334 - 5.663i, \tag{D.8}$$
$$\Gamma^{\text{cplx}}_{\text{non-hydro},3} = \pm 7.175 - 6.797i, \tag{D.9}$$
$$\Gamma^{\text{cplx}}_{\text{non-hydro},4} = \pm 8.96 - 8.604i, \tag{D.10}$$
$$\Gamma^{\text{cplx}}_{\text{non-hydro},5} = \pm 10.537 - 7.337i, \tag{D.11}$$
$$\Gamma^{\text{cplx}}_{\text{non-hydro},6} = \pm 13.102 - 5.463i, \tag{D.12}$$

Table D.1: Lowest QNM (Γ_A) vs magnetic field (b)

S.No	b	$-i\Gamma_A$
0	10^{-5}	$-2.3721962 \times 10^{-12}\ i$
1	0.1	$-0.000450031\ i$
2	0.2	$-0.0018005\ i$
3	0.3	$-0.00405253\ i$
4	0.4	$-0.00720806\ i$
5	0.5	$-0.0112698\ i$
6	0.6	$-0.0162414\ i$
7	0.7	$-0.0221275\ i$
8	0.8	$-0.0289337\ i$
9	0.9	$-0.036667\ i$
10	1.0	$-0.0453354\ i$
11	1.1	$-0.0549485\ i$
12	1.2	$-0.0655176\ i$
13	1.3	$-0.0770557\ i$
14	1.4	$-0.0895777\ i$
15	1.5	$-0.103101\ i$
16	1.6	$-0.117644\ i$
17	1.7	$-0.13323\ i$
18	1.8	$-0.149884\ i$
19	1.9	$-0.167633\ i$
20	2.0	$-0.186508\ i$
21	2.1	$-0.206544\ i$
22	2.2	$-0.227778\ i$
23	2.3	$-0.250252\ i$
24	2.4	$-0.27401\ i$
25	2.5	$-0.299101\ i$
26	2.6	$-0.325573\ i$
27	2.7	$-0.353477\ i$
28	2.8	$-0.382864\ i$
29	2.9	$-0.41378\ i$
30	3.0	$-0.446263\ i$
31	3.1	$-0.480341\ i$
32	3.2	$-0.516016\ i$
33	3.3	$-0.553261\ i$

APPENDIX E

Equilibrium configuration

E.1 Equilibrium configuration from gauging procedure

In this section, we will analyse the equilibrium configuration from the point of view of gauging the the anomalous $U(1)$ symmetry. We will show that the gauging procedure put constraints on the equilibrium parameter of the ungauged theory. These kind of constaints are well-known but there are subtleties upon turning on the background fields gauge fields which play crucial roles in the perturbation around equilibrium configuration considered in the next section.

In the simplest example of a theory with anomaly-free 0-form $U(1)$ global symmetry. The equilibrium partition function can be constructed in terms of thermodynamic quantities and background metric $g_{\mu\nu}$ and background gauge fields a_μ as in [136, 140] namely[1]

$$W_0 = -\log Z_0 = \int d^4x\sqrt{g}\left(p(T, g_{\mu\nu}, \mu, f_{\mu\nu}) + \text{higher-derivative terms}\right) \tag{E.1}$$

where $\mu/T = \log\left(\exp\left(\int_0^{1/T} d\tau a_\tau\right)\right)$ is the $U(1)$ holonomy around the thermal cycle τ and $f = da$ is the field strength. From this, one can write the local expression for

[1] In the convention of [136, 140], the field strength is treated as first derivative quantity. Here, however, we will treat it as zeroth derivative quantity as in [166]

chemical potential as

$$\mu = u^\mu \left(a_\mu + \partial_\mu \theta\right) \tag{E.2}$$

where u^μ is the unit vector along the thermal S^1 direction which, for flat space, is nothing but $u^\mu = \delta^\mu_\tau$. While it is common to choose a gauge where $\mu = u^\mu a_\mu$, it is possible to turn have nonzero chemical potential without external gauge field a_τ by choosing the parameter $\theta = \mu\tau$ with a singularity at $\tau = 0 \sim \beta$. This distinction is important as the chemical potential and background a_τ corresponds to different quantities when the $U(1)$ is the axial global symmetry. In this framework, the $U(1)$ current can be written as

$$j^\mu = \frac{1}{\sqrt{g}} \frac{\delta W}{\delta a_\mu} = \rho u^\mu + \nabla_\nu M^{\mu\nu} \tag{E.3}$$

where $\rho = \partial p/\partial\mu$ and $M^{\mu\nu} = \partial p/\partial f_{\mu\nu}$. In this configuration there is no relation between ρ and $M^{\mu\nu}$ except that they depends on the arbitrary function p.

The story is quite different upon promoting the background a_μ to the dynamical gauge field A_μ. Upon doing this the partition function becomes

$$-\log Z_{\text{gauged}}[b] = W_0 + W_{EM} + \int d^4x A_\mu j^\mu_{\text{ext}}\,, \qquad W_{EM} = \frac{1}{4e^2}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{E.4}$$

with $F = dA$ and $\star j_{ext} = db$ is the external current. The equations of motion for A_μ implies that there is an additional relation

$$\frac{\delta}{\delta A_\mu}(W_0 + W_{EM}) + j^\mu_{\text{ext}} = 0\,, \qquad \rho u^\mu + \nabla_\nu M^{\mu\nu}_{\text{gauged}} + j^\mu_{\text{ext}} = 0 \tag{E.5}$$

where $M^{\mu\nu}_{\text{gauged}} = \delta(W_0 + W_{EM})/\delta F_{\mu\nu}$. We now see that there is a new relation between ρ and $M^{\mu\nu}_{\text{gauged}}$ which does not exist before gauging. In a state where the electric field vanishes, this is nothing but the charge neutrality condition in plasma. It should also be noted that this form of the Eq. (E.5) is applicable for arbitrarily nonlinear form of W_{EM} which may or maynot be the Maxwell action. One may view this relation as a generalisation of Guass and Ampere's law.

E.1.1 Gauging $U(1)$ symmetry in a theory with mixed anomaly

We will now consider how a constraint such as Eq. (E.5) is modified when the ungauged theory has a mixed anomaly. We shall now consider a theory with mixed anomaly between vector $U(1)$ and axial $U(1)$ which yield the Ward identity Eq. (6.1) upon gauging. The ungauged theory partition function consist of

$$W_0 = W_{\rm inv}[\mu_A, \mu_v, f_A, f_v] + W_{\rm anom}[a, v, \mu_A, \mu_v, f_A, f_v] \tag{E.6}$$

where $\mu_A, a, f_A = da$ and $\mu_v, v, f_v = dv$ are chemical potential, background gauge field and field strength of the axial and vector $U(1)$ global symmetry respectively. This action is not invariant with respect to the background gauge transformation and so do the *consistent* currents obtained via varying W_0 with respect to the the background gauge field. The invariant partition function W_{cov} can be made out of W_0 by attaching the 4d theory to a 5d bulk with the Chern-Simons term I_{CS} which satisfy $dI_{CS} = kf_A \wedge f_v \wedge f_v$, see e.g. [116,167] for a modern summary. The *covariant* currents can be obtained via the usual variation

$$j^\mu_{a,cov} = j^\mu_A + j^\mu_{A,BZ}\,, \qquad j^\mu_A = \frac{\delta W_0}{\delta a_\mu}\,, \qquad j^\mu_{A,BZ} = \frac{\delta I_{CS}}{\delta a_\mu} \tag{E.7}$$

and similar expression for $j^\mu_{v,cov}$ obtain by simply chaning $a \to v$. The expression for $j^\mu_{A,cov}$ and $j^\mu_{v,cov}$ is well-known in the literature in the scheme where f_A, f_v are treated as first derivative derivative quantities. In our case, where we treat f as a zeroth derivative quantity and focus on the case of space with $u^\mu = \delta^{\mu\tau}$ and T be a constant, we find that

$$j^i_{v,\rm cov} = 8k\mu_a \mathcal{B}^i + \partial_j M^{ij}\,, \qquad j^i_{v,BZ} = 8ka_t\mathcal{B}^i\,, \qquad M^{ij} = \epsilon^{ijk}\frac{\delta W_{inv}}{\delta \mathcal{B}_k} \tag{E.8}$$

where $\mathcal{B}^i = \epsilon^{ijk}(f_v)_{jk}$ is the background magnetic field at this stage.

Upon gauging the vector $U(1)$ in 4d theory describes by W_0 i.e. promoting the background v to the dynamical gauge field V and add the source term and kinetic term as in Eq. (E.4), the equations of motion for A_V implies that there is a condition on the consistent current

$$j^i_v + j^i_{\rm ext} = 0 \tag{E.9}$$

We find that the spatial component becomes

$$\partial_j M^{ij}_{\rm gauge} + 8k(\mu_A - a_t)\mathcal{B}^i + j^i_{\rm ext} = 0 \tag{E.10}$$

where $M^{ij}_{\rm gauge} = \delta(W_{\rm inv} + W_{EM})/\delta F_{V,ij}$ with $F_V = dV$ be the dynamical field strength.

In the case where the partition function is dominated by magnetic field

$$W_{inv} + W_{EM} = \int d^4x \left(\frac{1}{2\chi_B} \mathcal{B}_i \mathcal{B}^i + \text{subleading terms} \right) \tag{E.11}$$

with χ_B be the susceptibility of the 1-form $U(1)$ density. In such a state, we find that, in the absence of the source.

$$\epsilon^{ijk} \partial_j \mathcal{B}_k + 8\chi_B k(\mu_A - a_t)\mathcal{B}^i = 0 \tag{E.12}$$

The relation Eq. (E.12) must be satisfied for any theory with ABJ anomaly in a homogeneous configuration and it has very simple solutions. Upon contracting with $\mathcal{B}_i$ one finds that this relation implies

$$\mu_A - a_t = 0\,, \quad \mathcal{B}^i \neq 0 \qquad \text{or} \qquad \mu_A \neq a_t\,, \quad \mathcal{B}^i = 0\,. \tag{E.13}$$

The choice where $\mu_A - a_t = 0$ is the only one that allows finite magnetic field and will be the equilibrium configuration of focus in the remaining of this work.

One may wonder, from the definition of chemical potential in Eq. (E.2) why is it not possible for us to perform a background gauge transformation $a \to a + d\lambda$ to guarantee that μ is always a_t. There are at two ways to argue why is is not automatically satisfied. First of all, the current j^μ which coupled to a is not a conserved current due to the r.h.s. of Eq. (6.1) and thus the source a and $a + d\lambda$ are not equivalent. Second, one can see from a Landau-level calculation for Weyl fermion, see e.g. [90]. There, the source a_t indicates the difference in energy at the tips of the right-handed and left-handed Weyl cone while the chemical potential μ is conjugate to the difference of occupation number between left- and right-handed fermions.

APPENDIX F

Defect operator insertions

F.1 Defect operator insertion in equilibrium

In this section, we outline how the non-invertible defect leads to the Ward identity in Eq. (6.2). The non-invertible codimension-1 defect defined in [37,38] is on a closed surface Σ can be written as

$$\hat{\mathcal{D}}_{\frac{p}{N}}(\Sigma) = \exp\left[i\int_{\Sigma}\left(\frac{2\pi p}{2N}\star j + \mathcal{A}^{N,p}[\star_4 J/N]\right)\right] \tag{F.1}$$

with N and p mod N are two coprime and the 2-form J is the conserved current associated to the 1-form global symmetry. $\mathcal{A}^{N,p}$ is the Lagrangian density of the minimal TQFT with $\mathbb{Z}_N$ 1-form global symmetry with the 1-form $\mathbb{Z}_N$ anomaly parametrised by an integer p. In a particular case when $p = 1$, we have

$$\exp\left[i\int_{\Sigma}\mathcal{A}^{N,1}[\star_4 J/N]\right] = \int D[\bar{a}]\exp\left[i\int_{\Sigma}\left(\frac{N}{4\pi}\bar{a}\wedge d\bar{a} + \frac{1}{2\pi}\bar{a}\wedge\star_4 J\right)\right] \tag{F.2}$$

This describes the fractional quantum Hall system as integrating out $\bar{a}$ yield $Nd\bar{a} + \star_4 J = 0$ and resulting in Chern-Simons term with fractional coefficient p/N. The Chern-Simons living on the defect has a $\mathbb{Z}_N$ anomaly [21] which can be cancelled when attached to a bulk TQFT on $\mathcal{M}$, such that $\partial\mathcal{M} = \Sigma$, with the following action [168]

$$\mathcal{S}_{\text{bulk}} = \int_{\mathcal{M}}\left(\frac{N}{4\pi}B\wedge B + \frac{N}{2\pi}B\wedge dc\right) \tag{F.3}$$

with c is the 1-form $U(1)$ gauge field whose equation of motion forced B to be the a $\mathbb{Z}_N$ 2-form gauge field (to be identified with $NB = \star_4 J$). The minimal TQFT $\mathcal{A}^{N,p}$ is then defined as a theory living on $\Sigma = \partial\mathcal{M}$ with coefficient of $B \wedge B$ in Eq. (F.3) from $N/4\pi$ to $Np/4\pi$. The fusion algebra, which shows the non-invertibility $\hat{\mathcal{D}}_{\frac{p}{N}} \times \hat{\mathcal{D}}_{-\frac{p}{N}} \neq \mathbb{1}$ can be found in [37, 38]. An alternative description for $\hat{\mathcal{D}}$ with the arguements p/N extended to $U(1)$ valued can also be found in [41, 58].

A theory is said to have non-invertible global symmetry of this type when the operator $\hat{\mathcal{D}}_{\frac{p}{N}}(\Sigma)$ is topological. That is, one can continuously deform the surface Σ to Σ' without changing the partition function. As a consequence, if we consider two nearby surface Σ, Σ' which enclosed a small fluid element living in a small (and topologically trivial) m_4 such that $\partial m_4 = \Sigma \cup \Sigma'$, we find that $\hat{\mathcal{D}}(\Sigma)$ and $\hat{\mathcal{D}}(\Sigma')$ are identical if only (consider $p = 1$ for simplicity)

$$\begin{aligned}
1 &= \exp\left[\left(\int_\Sigma - \int_{\Sigma'}\right)\left(\frac{2\pi}{2N} \star j + \mathcal{A}^{N,1}[\star_4 J/N]\right)\right] \\
&= \int D[\bar{a}] \exp\left[\int_{m_4}\left(\frac{\pi}{N} d \star j + \frac{N}{4\pi} d\bar{a} \wedge d\bar{a} + \frac{1}{2\pi} d\bar{a} \wedge \star_4 J\right)\right] \\
&= \exp\left[\int_{m_4}\left(\frac{\pi}{N} d \star j - \frac{1}{4\pi N} J \wedge J\right)\right]
\end{aligned} \tag{F.4}$$

Converting the term in the parenthesis in components, we find the Ward identity Eq. (6.2).

To analyse the equilibrium of a theory with non-invertible symmetry, we can put it on a manifold $S^1 \times \mathbb{R}^3$ as in Section 6.3. In this case, let us consider a local fluid element in m_4 which also contain the thermal cycle S^1. The topological condition Eq. (F.4) implies the (dimensionally reduced) Ward identity Eq. (6.57)[1]. Notice that, had the equilibrium partition function only consist of susceptibility of 0-form and 1-form global symmetry and thus described by the action

$$\mathcal{S} = \int d^3x \left[\frac{1}{2}\chi_A (a_\tau)^2 + \frac{1}{2}\chi_B (B_i B^i)\right] . \tag{F.5}$$

i.e. when $\chi_{\mathcal{O}}$ in Eq. (6.56) is turned off, the topological property Eq. (F.4) is trivially satisfied. This is because both j^i and $\epsilon_{ijk} J^{i\tau} J^{jk}$ vanishes identically. However, as one turned on $\chi_{\mathcal{O}}$ in Eq. (6.56), then $j^i = 0$ but $\epsilon_{ijk} J^{i\tau} J^{jk} = 2\chi_B^i \partial_i f \neq 0$ which means that the non-invertible defect is not topological (see Eq. (6.59) for the notation).

[1] It is possible that, upon dimensionally reduced on the thermal S^1, the non-invertible defect in Eq. (F.1) will give rises to codimension-0 and codimension-1 defect in three dimensions similar to those in [141]. This is an interesting future direction, however we will only consider the consequence of Eq. (F.4) at the level of the Ward identity in this work.

Thus, the action with nonzero χ_A, χ_B and $\chi_{\mathcal{O}}$ has to be modified in a nontrivial way as demonstrated in Section 6.3.2.

F.2 Defect operator insertion in dissipative theory

F.2.1 Defect operator insertion

In this section we shall discuss defect insertions in the dissipative action given in 6.80. Following [13] we have, $(1,2)$ as the two degrees of freedom in *Schwinger-Keldysh* formalism. Also, in the $r-a$ basis, the "r" type fields are somewhat like the physical fields and the "a" type fields are somewhat like noise. So, to go from here to the equilibrium phase we neglect the time derivatives and put $\phi_a = 0$. The basic transformation among the two bases is,

$$\phi_r = \frac{1}{2}\left(\phi_1 + \phi_2\right), \qquad \phi_a = \left(\phi_1 - \phi_2\right), \tag{F.6}$$

$$\phi_1 = \phi_r + \frac{1}{2}\phi_a, \qquad \phi_2 = \phi_r - \frac{1}{2}\phi_a. \tag{F.7}$$

Dissipative Action

Let us consider the dissipative acion in the main text, i.e. Eq. (6.80)

$$\mathcal{L}[a,b;\theta,\Phi;\Sigma,C] = \mathcal{L}_{\text{MHD}} + \mathcal{L}_a - 2\left(\Sigma_a \wedge dC_r + \Sigma_r \wedge dC_a\right) + \mathcal{L}_{\text{anom}}[\theta, C]. \tag{F.8}$$

where a, b denote external sources, θ, Φ denote dynamical fields and Σ, C are auxiliary fields or Lagrange multipliers. The 1-form currents as follows,

$$j_r = \frac{\delta \mathcal{S}}{\delta a_a}, \qquad j_a = \frac{\delta \mathcal{S}}{\delta a_r}, \tag{F.9}$$

where $\mathcal{S}$ is now to be understood as the dissipative action. Similarly, we obtain the 2-from currents as,

$$J_r = \frac{\delta \mathcal{L}}{\delta \Sigma_a} = dC_r\,, \qquad J_a = \frac{\delta \mathcal{L}}{\delta \Sigma_a} = dC_a\,. \tag{F.10}$$

This implies in terms of these new C_r and C_a fields, the currents J_r and J_a are now identically conserved.

The part of the action that involves the axial charge fluctuation can be written in the $1,2$ basis as follows

$$\mathcal{L}_a = \frac{i\sigma}{\beta} A_{ai}^2 + \chi_A A_{a0} B_{r0} - \sigma A_{ai} A_{ri,0}, \tag{F.11}$$

$$\begin{aligned} &= \frac{i\sigma}{\beta}\left[A_{1i}^2 + A_{2i}^2 - 2A_{1i}A_{2i}\right] + \frac{\chi_A}{2}\left[A_{1t}^2 - A_{2t}^2\right] \\ &- \frac{\sigma}{2}\left[A_{1i}\left(A_{1i,t} + A_{2i,t}\right) - A_{2i}\left(A_{1i,t} + A_{2i,t}\right)\right], \end{aligned} \tag{F.12}$$

where $A = a + d\theta$. Similar decomposition can also be done in for the MHD part i.e.

$$\mathcal{L}_{\text{MHD}} = \frac{i\rho}{\beta_0} \tilde{G}_{aij}^2 + \chi_B \tilde{G}_{a0i} \tilde{G}_{r0i} - \sigma \tilde{G}_{aij} \tilde{G}_{rij,0}. \tag{F.13}$$

where $\tilde{G} = b + d\Phi + \Sigma$, as well as the Lagrange multiplier

$$\Sigma_a \wedge dC_r + \Sigma_r \wedge dC_a = \Sigma_1 \wedge dC_1 - \Sigma_2 \wedge dC_2\,. \tag{F.14}$$

For this action to be compatible with the non-invertible defect, we have to add additional terms $\mathcal{L}_{\text{anom}}$ of the following form

$$\mathcal{L}_{\text{anom}} = -4K\left(\theta_1 dC_1 \wedge dC_1 - \theta_2 dC_2 \wedge dC_2\right)\,. \tag{F.15}$$

At this stage, K can be any function of thermodynamic quantities which may or may not has to do with the constant $k = 1/16\pi^2$ in the Ward identity. Here, we will show that, for the defect insertion to be consistent, the function K must be a constant and equal to k.

Non-invertible defect operator insertion

Due to the doubling of the degrees of freedom we now have two defect operators constructed as in [37, 38]. Inserting the non-invertible defect turns the Schwingker-Keldysh generating function into

$$Z = \exp\left(-i\mathcal{S}\right)) \to Z' = \hat{\mathcal{D}}_1 \hat{\mathcal{D}}_2 \exp\left(i\mathcal{S}\right) \tag{F.16}$$

where $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$ are

$$\hat{\mathcal{D}}_1 = \int D[\bar{a}_1] \exp\left(\int_{\mathcal{M}} \left(\frac{2\pi}{2N} \star j_1 + \frac{N}{4\pi} \bar{a}_1 \wedge d\bar{a}_1 + \frac{1}{2\pi} \bar{a}_1 \wedge dC_1\right)\right), \tag{F.17}$$

$$\hat{\mathcal{D}}_2 = \int D[\bar{a}_2] \exp\left(\int_{\mathcal{M}} \left(\frac{2\pi}{2N} \star j_2 + \frac{N}{4\pi} \bar{a}_2 \wedge d\bar{a}_2 + \frac{1}{2\pi} \bar{a}_2 \wedge dC_2\right)\right), \tag{F.18}$$

where if $\mathcal{M} = \mathbb{R}^3$ then defect is inserted at $t = 0$ (temporal insertion) and if $\mathcal{M} = \mathbb{R}^{1,2}$ then defect is inserted at $z = 0$ (spatial insertion).

We shall assume that, apriori, to begin with, all fields are smooth across the defects. Let us first consider inserting the non-invertible defect at $z = 0$. The currents involved in this analysis are,

$$\begin{aligned} j_{rz} &= \frac{\delta \mathcal{S}}{\delta a_{az}} = \frac{2i\sigma}{\beta} A_{az} - \sigma \partial_t A_{rz}, \\ j_{az} &= \frac{\delta \mathcal{S}}{\delta a_{rz}} = \sigma \partial_t A_{az}, \end{aligned} \tag{F.19}$$

or in the 1,2 basis, we have

$$\begin{aligned} \star j_{1z} &= j_{rz} + \frac{1}{2} j_{az} = \frac{2i\sigma}{\beta} \left(A_{1z} - A_{2z}\right) - \sigma \partial_t \left(A_{2z}\right), \\ \star j_{2z} &= j_{rz} - \frac{1}{2} j_{az} = \frac{2i\sigma}{\beta} \left(A_{1z} - A_{2z}\right) - \sigma \partial_t \left(A_{1z}\right). \end{aligned} \tag{F.20}$$

Consider the equation of motion of θ_1, θ_2 in the presence of the non-invertible defect, we get

$$4K\, dC_s \wedge dC_s + \frac{2i\sigma}{\beta} \left[\theta_1 - \theta_2\right]_{,zz} + \frac{2i\sigma}{\beta} \left(\frac{2\pi}{2N}\right) \frac{d}{dz} \delta(z) + (\ldots) = 0\,, \tag{F.21}$$

where $s = 1, 2$. Here $(\ldots)$ includes terms with less than two z derivatives. Both equations yield the solution

$$\Delta(\theta_1 - \theta_2) \equiv (\theta_1 - \theta_2)\Big|_{z+\epsilon} - (\theta_1 - \theta_2)\Big|_{z-\epsilon} = -\frac{2\pi}{2N} \tag{F.22}$$

The equation of motion for $\bar{a}_1, \bar{a}_2$, we have

$$N d\bar{a}_s + dC_s\big|_{z=0} = 0, \tag{F.23}$$

which can be used to replaced $d\bar{a}_s$ in terms of dC_s. Finally, C_1's and C_2's equations of motion are

$$2d\Sigma_s + 8Kd\left(\theta_s dC_s\right) + (-1)^s \frac{d\bar{a}_s}{2\pi} \delta(z) = 0, \tag{F.24}$$

Combined all the equations of motion together, one finds that,

$$K = \frac{1}{16\pi^2}, \tag{F.25}$$

where note that, when $s = 1$ and $s = 2$, Eq. (F.24) is satisfied by the following conditions

$$\Delta\theta_1 = -\frac{\pi}{N}\,, \qquad \Delta\theta_2 = 0\,, \qquad \text{and} \qquad \Delta\theta_1 = 0\,, \qquad \Delta\theta_2 = \frac{\pi}{N}\,, \tag{F.26}$$

respectively.

So, in order for the theory to be compatible with non-invertible defect insertion we see that $K = k$.

Similar analysis can be done for the defect insertion localised in the time direction at $t = 0$, with $j^z_{r,a}$ in Eq. (F.19) replaced by $j^t_{r,a}$, and results in Eq. (F.25) without giving additional constraints.

APPENDIX G

Matsubara sums

G.1 Finite temperature conventions and Matsubara sums

Here we review some identities that are useful for performing the frequency integrals in the main text. All of these results are standard, and further background can be found e.g. in [169].

Consider a quantum field theory with a bosonic Hermitian operator $\mathcal{O}(t,\vec{x})$. We study the theory in the thermal state with temperature β^{-1}. There are various basic two-point functions for $\mathcal{O}$, including the Euclidean correlation function,

$$G^E(\tau,\vec{x}) \equiv \langle \mathcal{O}(\tau,\vec{x})\mathcal{O}(0)\rangle, \tag{G.1}$$

and the retarded real-time correlation function

$$G^R(t,\vec{x}) = -i\theta(t)\,\mathrm{Tr}\left(e^{-\beta H}[\mathcal{O}(t,\vec{x}),\mathcal{O}(0)]\right)\ . \tag{G.2}$$

The Euclidean correlation function in frequency space can be written in terms of the spectral density $\rho(\Omega)$:

$$G^E(i\omega_n) = \int \frac{d\Omega}{2\pi}\frac{\rho(\Omega)}{i\omega_n + \Omega} \tag{G.3}$$

We may also obtain the retarded correlator from the Euclidean one by evaluating the latter at a real frequency:

$$G^R(\Omega) = G^E(i\omega_l = \Omega + i\epsilon) \tag{G.4}$$

Inserting Eq. (G.3) into Eq. (G.4) and using the identity

$$\mathrm{Im}\left(\frac{1}{x - i\epsilon}\right) = \pi\delta(x) \tag{G.5}$$

we conclude that the imaginary part of $G^R(\omega)$ directly measures the spectral density.

$$\mathrm{Im}\, G^R(\omega) = -\pi\rho(\omega) \tag{G.6}$$

It is shown in [169] that for a bosonic operator $\rho(\omega)$ is an odd function of ω, and furthermore is positive for positive ω, i.e. $\omega\rho(\omega) > 0$.

G.1.1 Performing Matsubara sums

We will need to perform a loop sum over Euclidean frequencies. Here we review a standard trick to express such sums in terms of the corresponding spectral densities, following the discussion in [161]. Consider summing over a set of discrete Matsubara frequencies $i\Omega_m = \frac{2\pi m}{\beta}$, $m \in \mathbb{Z}$. We can express this in terms of a contour integral over a contour C in the complex ω plane, i.e.

$$T\sum_{i\omega_m} \to \frac{1}{2\pi i}\int_C d\omega \frac{1}{2}\coth\left(\frac{\beta\omega}{2}\right) \tag{G.7}$$

Here the hyperbolic function in the integrand has poles at each of the discrete Matsubara frequencies along the imaginary axis, and the contour C is a series of disjoint circles which encircles each of these poles.

To see an application of this, consider evaluating the following sum, where $i\Omega_l$ is a Matsubara frequency and $\omega_{1,2}$ are two real frequencies:

$$S(i\Omega_l, \omega_1, \omega_2) = T\sum_{i\omega_m} \frac{1}{i(\omega_m + \Omega_l) - \omega_1}\frac{1}{i\omega_m - \omega_2} \tag{G.8}$$

From above we see that this can be written as the following contour integral:

$$S(i\Omega_l, \omega_1, \omega_2) = \frac{1}{2\pi i}\int_C d\omega \frac{1}{2}\coth\left(\frac{\beta\omega}{2}\right)\frac{1}{\omega + i\Omega_l - \omega_1}\frac{1}{\omega - \omega_2} \tag{G.9}$$

Now consider deforming the contour C into two parallel lines, one running down the imaginary ω axis at infinitesimal real positive ω and the other running *up* the imaginary ω axis at infinitesimal real negative ω. We can now attempt to deform these lines away to infinity. At large $|\omega|$ the integrand behaves as $|\omega|^{-2}$. The contribution to the integrand at infinity can be neglected, and the full integral arises from the contribution at the non-Matsubara poles of the integrand, which appear only at $\omega = \omega_1 - i\Omega_l$ and $\omega = \omega_2$. Performing the integral by residues we find

$$S(i\Omega_l, \omega_1, \omega_2) = -\frac{1}{2}\left(\coth\left(\frac{\beta(\omega_1 - i\Omega_l)}{2}\right) - \coth\left(\frac{\beta\omega_2}{2}\right)\right)\frac{1}{\omega_1 - i\Omega_l - \omega_2} \tag{G.10}$$

which after some algebra can be seen to be equal to

$$S(i\Omega_l, \omega_1, \omega_2) = -\frac{f(\omega_1) - f(\omega_2)}{\omega_1 - i\Omega_l - \omega_2} \tag{G.11}$$

where we have used the fact that $e^{i\beta\Omega_l} = 1$ on a Matsubara frequency. Here $f(\omega)$ is the Bose distribution function:

$$f(\omega) = \frac{1}{e^{\beta\omega} - 1} \tag{G.12}$$

Let us now use this form to perform a frequency sum. In the bulk of the text we will find ourselves needing to calculate sums of the form

$$F(i\Omega_l) = T\sum_{i\omega_m} G_1^E(i\Omega_l + i\omega_m)G_2^E(i\Omega_l) \tag{G.13}$$

where here $G_{1,2}^E(i\Omega)$ are two (possibly different) Euclidean propagators. It is very convenient to express this in terms of the spectral densities $\rho_{1,2}(\omega)$ associated with these propagators. To do this, we first use Eq. (G.3) and then perform the sum over $i\omega_m$ using Eq. (G.11) to find

$$F(i\Omega_l) = -\int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\left(f(\omega_1) - f(\omega_2)\right)\frac{\rho_1(\omega_1)\rho_2(\omega_2)}{\omega_1 - i\Omega_l - \omega_2}\ . \tag{G.14}$$

This expression is used to obtain Eq. (7.14) in the main text.

APPENDIX H

Details of the loop integration

H.1 Details of the 1-loop integration

H.1.1 $\Omega > 0$

Here we give details of the remaining parts of the 1-loop integration with $\Omega > 0$.

As described around Eq. (7.33), the frequency integral in Eq. (7.32) must be split up into three parts

$$\omega \in (-\infty, -\Omega) \cup (-\Omega, 0) \cup (0, \infty) \ . \tag{H.1}$$

The last integral was performed in detail in the bulk of the text. In this Appendix we perform the other two using the same methods. We begin with $\omega \in (-\Omega, 0)$. To do this integral let us perform the following change of variable: $\omega \to -\omega$ and then perform the integration over the positive range: $\omega \in (0, \Omega)$. The integral of interest is:

$$I_1^-(\Omega) = \int_0^\Omega d\omega \ [f(\Omega - \omega) - f(-\omega)] \frac{\omega(\Omega - \omega)^2}{\Omega(2\omega - \Omega)} \left(\sqrt{\Omega - \omega} - \sqrt{\omega}\right), \tag{H.2}$$

To do the above integral, we expand the integrand about $\beta \to 0$ and then integrate term by term. We get,

$$I_1^-(\Omega) = \frac{(-2+\sqrt{2}\sinh^{-1}(1))}{2\beta}\Omega^{3/2} + \frac{\beta(-26+15\sqrt{2}\sinh^{-1}(1))}{1440}\Omega^{7/2} + \frac{\beta^3(214-105\sqrt{2}\sinh^{-1}(1))}{2419200}\Omega^{11/2}. \tag{H.3}$$

As expected the above integral is non-analytic in Ω and these pieces, as discussed before, do not receive any UV corrections.

Next let us move on to performing the integration over the range: $\omega \in (-\infty, -\Omega)$. As before let us do the variable change: $\omega \to -\omega$ and then the integration has to be performed over the range: $\omega \in (\Omega, \infty)$. The integral of interest is:

$$I_2^-(\Omega) = \int_\Omega^\infty d\omega\ [f(\Omega-\omega) - f(-\omega)]\, \frac{\omega(\Omega-\omega)^2}{\Omega(2\omega-\Omega)} \left(\sqrt{\omega-\Omega} - \sqrt{\omega}\right), \tag{H.4}$$

We can perform the above integration using the methods employed to do the integration in Eq. (7.34). We obtain the following results.

$$\begin{aligned} I_2^-(\Omega)_{\rm IR} = &\left(\frac{\sqrt{\Lambda}}{2\beta} - \frac{\beta\Lambda^{5/2}}{120} + \frac{\beta^3\Lambda^{9/2}}{4320} + \mathcal{O}\left(\Lambda^{13/2}\right)\right)\Omega + \left(\frac{1}{6\beta} - \frac{\pi}{4\sqrt{2}\beta} - \frac{\sinh^{-1}(1)}{2\sqrt{2}\beta}\right)\Omega^{3/2} \\ &+ \left(\frac{1}{8\beta\sqrt{\Lambda}} + \frac{5\beta\Lambda^{3/2}}{288} - \frac{3\beta^3\Lambda^{7/2}}{4480} + \mathcal{O}\left(\Lambda^{11/2}\right)\right)\Omega^2 \\ &+ \left(\frac{1}{24\beta\Lambda^{3/2}} + \frac{19\beta^3\Lambda^{5/2}}{28800} - \frac{\beta^5\Lambda^{9/2}}{24192} + \mathcal{O}\left(\Lambda^{13/2}\right)\right)\Omega^3 \\ &+ \left(\frac{43\beta}{10080} - \frac{\beta\pi}{192\sqrt{2}} - \frac{\beta\sinh^{-1}(1)}{96\sqrt{2}}\right)\Omega^{7/2} \\ &+ \left(\frac{11}{640\beta\Lambda^{5/2}} + \frac{5\beta}{1536\sqrt{\Lambda}} - \frac{13\beta^3\Lambda^{3/2}}{55296} + \mathcal{O}\left(\Lambda^{7/2}\right)\right)\Omega^4 + \mathcal{O}(\Omega^5). \end{aligned} \tag{H.5}$$

Comparing Eq. (H.5) with Eq. (7.35) we see that, for the analytic pieces: the terms with odd powers of Ω are of opposite signs and the terms with even powers of Ω have the same sign.

H.1.2 $\Omega < 0$

Here we work out the 1-loop integration with $\Omega < 0$. For simplicity, let us define $t = -\Omega$ with $t > 0$. From Eq. (7.32) we get,

$$\Gamma_A(-t) = -\frac{16k^2\rho^2}{2\sqrt{2}\pi^4 D^{\frac{5}{2}}\chi_A}\int_{-\infty}^{\infty} d\omega\;[f(\omega-t)-f(\omega)]\,\frac{\omega(\omega-t)^2}{(-t)(2\omega-t)}\left(\sqrt{|\omega-t|}-\sqrt{|\omega|}\right). \tag{H.6}$$

From the structure of the square-root above we see that the integral should be integrated over the following intervals,

$$\omega \in (-\infty, 0) \cup (0, t) \cup (t, \infty)$$

Contrast this to the intervals in the $\Omega > 0$ case. Both are mirror images of each other. Now let us do the integral over the range, $\omega \in (-\infty, 0)$. This integral, after a change of variable: $\omega \to -\omega$ becomes,

$$\begin{aligned}\Gamma_A(-t) &= \frac{16k^2\rho^2}{2\sqrt{2}\pi^4 D^{\frac{5}{2}}\chi_A}\int_{0}^{\infty} d\omega\;[f(-\omega-t)-f(-\omega)]\,\frac{\omega(\omega+t)^2}{t(2\omega+t)}\left(\sqrt{\omega+t}-\sqrt{\omega}\right)\\ &= -\frac{16k^2\rho^2}{2\sqrt{2}\pi^4 D^{\frac{5}{2}}\chi_A}\int_{0}^{\infty} d\omega\;[f(\omega+t)-f(\omega)]\,\frac{\omega(\omega+t)^2}{t(2\omega+t)}\left(\sqrt{\omega+t}-\sqrt{\omega}\right)\end{aligned} \tag{H.7}$$

where to get to the second equality we used the identity: $f(-x) = -1 - f(x)$. Note that the above integral is the same as the integral in Eq. (7.34).

Similarly, using the above identity, one can show that $\Gamma_A(-t)$ for $\omega \in (0, t)$ matches with the integral in Eq. (H.2) and $\Gamma_A(-t)$ for $\omega \in (t, \infty)$ matches with the integral in Eq. (H.4). Thus, we get: $\Gamma_A(\Omega) = \Gamma_A(-\Omega)$, as expected.

APPENDIX I

Linear response theory

In this appendix we shall briefly review *linear response theory* and *Kubo formulae*. We shall follow the exposition in [170] quite closely.

I.1 Sources in quantum mechanics

In the context of quantum mechanics, the observables of the system are represented as operators, denoted as $\mathcal{O}_i$. We will operate in the Heisenberg picture, where these operators are time-dependent, written as $\mathcal{O} = \mathcal{O}(t)$. Left on their own, the dynamics of these operators would be governed by a Hamiltonian $H(\mathcal{O})$. However, perturbing it is achieved by adding an extra term to the Hamiltonian, given by

$$H_{\text{source}}(t) = \phi_i(t)\mathcal{O}_i(t). \tag{I.1}$$

Here, the variables ϕ_i are *sources*. They represent external fields that are under our control and can be thought of as analogous to driving forces in classical mechanics.

I.2 Linear response

We are interested in understanding how our system reacts to the presence of a source. We want to understand how the correlation functions of the theory change when we introduce a source (or multiple sources) denoted by ϕ_i. We assume that the source is a small perturbation of the original system. This assumption provides a simple framework where progress can be made. Mathematically, this means that we assume the change in the expectation value of any operator is *linear* in the perturbing source. We can express this as follows:

$$\delta(\mathcal{O}_i(t)) = \int dt' \chi_{ij}(t;t')\phi_j(t') \tag{I.2}$$

Here, $\chi_{ij}(t;t')$ is referred to as a response function. A similar expression could be written for a classical dynamical system, where $\delta\langle\mathcal{O}_i\rangle$ would be replaced by $x_i(t)$, and ϕ would be substituted with the driving force $F_j(t)$. Then classical equations of motion would imply that the response function is the Green's function for the system. As a result, response functions are often referred to as Green's functions, and they are sometimes denoted as G rather than χ.

Moving forward, we will assume that our system is invariant under time translations. In this case, the response function can be written as follows:

$$\chi_{ij}(t;t') = \chi_{ij}(t-t') \tag{I.3}$$

We define the Fourier transform of the function $f(t)$ to be

$$f(\omega) = \int dt\, e^{i\omega t} f(t), \qquad f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega) \tag{I.4}$$

In particular, we adhere to the convention where the distinction between the two functions relies solely on their arguments.

Taking the Fourier transform of (Eq. (I.2)) yields,

$$\begin{aligned}\delta\langle\mathcal{O}_i(\omega)\rangle &= \int dt' \int dt e^{i\omega t}\chi_{ij}(t-t')\phi_j(t') \\ &= \int dt' \int dt e^{i\omega(t-t')}\chi_{ij}(t-t')e^{i\omega t'}\phi_j(t') \\ &= \chi^{ij}(\omega)\phi_j(\omega)\end{aligned} \tag{I.5}$$

This analysis reveals that the response is "localized" in frequency space. When you subject something to vibrations at frequency ω, its response occurs at the same frequency ω. Anything beyond this phenomenon falls within the realm of *nonlinear response.*

I.2.1 Analyticity and Causality

When dealing with a real source ϕ in conjunction with a Hermitian operator $\mathcal{O}$, which implies a real expectation value $\langle \mathcal{O} \rangle$, the associated function $\chi(t)$ is necessarily real. Let us explore the implications of this for the Fourier transform $\chi(\omega)$. It is helpful to introduce new notation for the real and imaginary components.

$$\begin{aligned}\chi(\omega) &= \mathrm{Re}\,\chi(\omega) + i\mathrm{Im}\,\chi(\omega)\\ &\equiv \chi'(\omega) + i\chi''(\omega)\end{aligned}$$

The response function $\chi(\omega)$ comprises both real and imaginary components, each with distinct interpretations. Let us examine each of these components in detail.

Imaginary Part: We can write an imaginary piece as

$$\begin{aligned}\chi'' &= -\frac{i}{2}[\chi(\omega) - \chi^*(\omega)]\\ &= -\frac{i}{2}\int_{-\infty}^{+\infty} dt\chi(t)[e^{i\omega t} - e^{-i\omega t}]\\ &= -\frac{i}{2}\int_{-\infty}^{+\infty} dte^{i\omega t}[\chi(t) - \chi(-t)]\end{aligned}$$

The imaginary component of $\chi(\omega)$ arises from the portion of the response function that is not invariant under time reversal, denoted by $t \to -t$. In essence, $\chi''(\omega)$ is indicative of the directionality of time. Given that microscopic systems are typically time-reversal invariant, the presence of the imaginary component $\chi''(\omega)$ can be attributed to the occurrence of dissipative processes.

The term $\chi''(\omega)$ is referred to as the *dissipative* or *absorptive* component of the response function, and is also commonly known as the *spectral function.* This component encodes information about the density of states within the system that participate in absorption processes.

It is important to note that $\chi''(\omega)$ exhibits the properties of an odd function,

$$\chi''(-\omega) = -\chi''(\omega)$$

Real part: Performing the same analysis as above shows us that

$$\chi'(\omega) = \frac{1}{2}\int_{-\infty}^{+\infty} dt\, e^{i\omega t}[\chi(t) + \chi(-t)]$$

The real component of the response function, irrespective of the direction of time, is termed the *reactive* portion of the response function. Characteristically, it behaves as an even function,

$$\chi'(-\omega) = \chi'(\omega)$$

Before proceeding further, it is necessary to briefly discuss the implications of reintroducing the labels i, j to the response functions. Analogous to the preceding analysis, the dissipative response function is found to originate from the anti-Hermitian component.

$$\chi''_{ij} = -\frac{i}{2}[\chi_{ij}(\omega) - \chi^*_{ji}(\omega)]$$

Causality

In accordance with the principle of causality, which posits that the past cannot be influenced, any response function must adhere to the following condition:

$$\chi(t) = 0 \text{ for all } t < 0$$

Consequently, χ is frequently designated as the causal Green's function or the *retarded Green's function*, occasionally represented by the notation $G_R(t)$. Let's examine the implications of this fundamental causality constraint for the Fourier expansion of χ.

$$\chi(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \chi(\omega)$$

When $t < 0$, the integral can be carried out by closing the contour in the upper-half plane, such that the exponent transforms into $-i\omega \times (-i|t|) \to -\infty$. Consequently, the result must be zero.

The value of the integral is dictated by the sum of the residues encompassed by the contour. Therefore, for the response function to become zero for all $t < 0$, it is necessary for $\chi(\omega)$ to lack poles in the upper-half plane. This implies that causality necessitates:

$$\chi(\omega) \quad \text{is analytic for } \operatorname{Im} \omega > 0$$

I.3 Kubo formula

Typically, in any given scenario, a source is only turned on for a single operator. Nonetheless, we might be interested in investigating how this source influences the expectation value of any other operator within the theory, denoted as $\langle \mathcal{O}_i \rangle$. By limiting ourselves to small values of the source, this can be approached using standard perturbation theory. To this end, we introduce the time evolution operator:

$$U(t, t_0) = T \exp\left(-i \int_{t_0}^{t} H_{\text{source}}(t')dt'\right)$$

which is designed to satisfy the operator equation $idU/dt = H_{\text{source}}U$. Subsequently, transitioning to the interaction picture, the states evolve as follows:

$$|\psi(t)\rangle_I = U(t, t_O)|\psi(t_0)\rangle_I$$

Typically, we will be operating within an ensemble of states characterized by a density matrix ρ. If, in the distant past (as $t \to \infty$), the density matrix is represented by ρ_0, then at a finite time, it evolves according to:

$$\rho(t) = U(t)\rho_0 U^{-1}(t)$$

where $U(t) = U(t, t_0 \to -\infty)$. Using this, we can determine the expectation value of any operator $\mathcal{O}_j$ when the sources ϕ are present. By employing first-order perturbation theory (as indicated from the third line below), we obtain:

$$\begin{aligned}
\langle \mathcal{O}_i(t)\rangle|_\phi &= \operatorname{Tr} \rho(t)\mathcal{O}_i(t) \\
&= \operatorname{Tr} \rho_0(t)U^{-1}(t)\mathcal{O}_i(t)U(t) \\
&\approx \operatorname{Tr} \rho_0(t) \left(\mathcal{O}_i(t) + i \int_{-\infty}^{t} dt'[H_{\text{source}}(t'), \mathcal{O}_i(t)] + \dots\right) \\
&= \langle \mathcal{O}_i(t)\rangle|_{\phi=0} + i \int_{-\infty}^{t} dt'[H_{\text{source}}(t'), \mathcal{O}_i(t)] + \dots
\end{aligned}$$

By incorporating our specific formulation of the source Hamiltonian, we can derive the variation in the expectation value, $\delta\langle \mathcal{O}_i\rangle = \langle \mathcal{O}_i\rangle_\phi - \langle \mathcal{O}_i\rangle_{\phi=0}$,

$$\begin{aligned}
\delta\langle \mathcal{O}_i\rangle &= i \int_{-\infty}^{t} dt' \langle [\mathcal{O}_j(t'), \mathcal{O}_i(t)]\rangle \phi_j(t') \\
&= i \int_{-\infty}^{+\infty} dt' \theta(t - t')\langle [\mathcal{O}_j(t'), \mathcal{O}_i(t)]\rangle \phi_j(t') \qquad \text{(I.6)}
\end{aligned}$$

where, in the second line, we have simply employed the step function to expand the time integration range to $+\infty$. By comparing this to our initial definition provided in Eq. (I.2), it becomes apparent that the response function in a quantum theory is expressed by the two-point function.

$$\chi_{ij}(t-t') = -i\theta(t-t')\langle[\mathcal{O}_i(t), \mathcal{O}_j(t')]\rangle \tag{I.7}$$

This is referred to as the *Kubo formula*. The step function in the above equation justifies the retarded nature of the correlators (see [170] for more details).

APPENDIX J

Linking number

J.1 Gauss linking number and magnetic helicity

Let us start by observing a nice relation between the *Biot-Savart* law for magnetostatics and linking number between loops[1] (see [171]). To focus on the Biot-Savart law for magnetostatics we need the following Maxwell equations:

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{B} = \mu_0 \vec{j}, \tag{J.1}$$

where the first equation is called the *Gauss' law* for magnetostatics and the second equation is called the *Ampere's law*.

Gauss' law above gives us $\vec{B}$ in terms of the vector potential: $\vec{B} = \nabla \times \vec{A}$. Plugging this gives us the Poisson equation[2]: $\nabla^2 \vec{A} = -\mu_0 \vec{j}$. Its solutions in Cartesian coordinates can be given as:

$$\vec{A}(\vec{r'}) = \frac{\mu_0}{4\pi} \int d^3 r \frac{\vec{j}(\vec{r})}{|\vec{r'} - \vec{r}|}. \tag{J.2}$$

[1] To be precise, this is called the Gauss' linking number.

[2] Recall, the vector calculus identity: $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ and use the Coulomb gauge: $\nabla \cdot \vec{A} = 0$.

Now taking the curl of the above equation gives us the Biot-Savart law,

$$\vec{B}(\vec{r}') = \frac{\mu_0}{4\pi}\int d^3r \frac{\vec{j}(\vec{r}) \times (\vec{r}' - \vec{r})}{|\vec{r}' - \vec{r}|^3} = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}} \frac{d\vec{r} \times (\vec{r}' - \vec{r})}{|\vec{r}' - \vec{r}|^3}, \tag{J.3}$$

where the last equality holds when we are interested in the magnetic field generated by a current flowing along a closed curve $\mathcal{C}$. In this case, we note that: $\vec{j}\, d^3r = \vec{j} A\, d\vec{r} = I\, d\vec{r}$, with A denotes the cross-sectional area of the wire[3].

Now consider a set up where there is another closed loop $\mathcal{C}'$ which is the boundary of a surface Σ', $\partial\Sigma' = \mathcal{C}'$, such that this loop does not intersect $\mathcal{C}$ (see Fig. J.1). We are interested in computing the following line integral along it:

$$\mathcal{G}_\ell = \oint_{\mathcal{C}'} \vec{B}(\vec{r}') \cdot d\vec{r}' = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}'} d\vec{r}' \cdot \oint_{\mathcal{C}} \frac{d\vec{r} \times (\vec{r}' - \vec{r})}{|\vec{r}' - \vec{r}|^3}, \tag{J.4}$$

where in the second equality we used the expression for $\vec{B}(\vec{r}')$ as obtained in Eq. (J.3). Now using Stokes' theorem, we can write the line integral as a surface integral as,

$$\mathcal{G}_\ell = \int_{\Sigma'} \left(\nabla' \times \vec{B}(\vec{r}')\right) \cdot d\vec{\Sigma}' = \mu_0 \int_{\Sigma'} \vec{j} \cdot d\vec{\Sigma}', \tag{J.5}$$

where in the last equality we have used the Ampere's law.

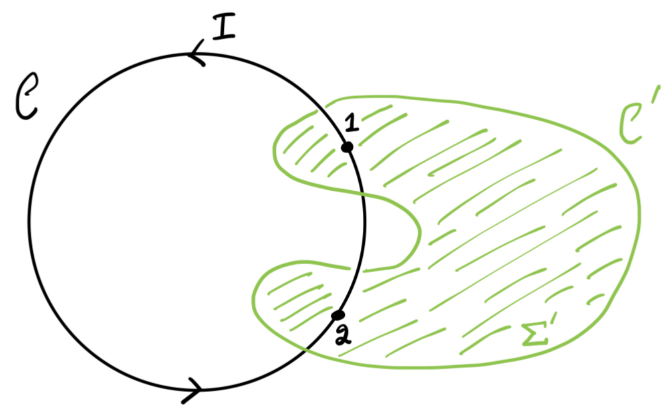

Figure J.1: In the above figure, the linking number between $\mathcal{C}$ and $\mathcal{C}'$ is 2 and hence we see that the current pokes through the surface Σ' twice at intersection points marked 1 and 2 on Σ'.

Let us now discuss an interesting consequence of the expression in Eq. (J.5). Let

[3]We are doing an abuse of notation here by calling both $\vec{j}$ and I as currents. To be precise, I is the current and $\vec{j}$ is called the current density.

us say that the closed curves $\mathcal{C}$ and $\mathcal{C}'$ link ℓ times. Then, the current $\vec{j}$ pokes through the surface Σ' exactly ℓ times, see Fig. J.1. Thus, in Eq. (J.5), $\mathcal{G}_\ell = \mu_0 I \ell$. Thus, from Eq. (J.4) we get an integral representation of the linking number between two arbitrary non-intersecting loops $\mathcal{C}$ and $\mathcal{C}'$:

$$\ell = \frac{1}{4\pi} \oint_{\mathcal{C}'} \oint_{\mathcal{C}} d\vec{r}' \cdot \left[\frac{d\vec{r} \times (\vec{r}' - \vec{r})}{|\vec{r}' - \vec{r}|^3} \right], \tag{J.6}$$

where the above integral is a topological invariant.

Our next task is to motivate helicity of magnetic field lines in terms of the Gauss linking number obtained above. To see this, let us parameterise our closed curves $\mathcal{C}$ and $\mathcal{C}'$ as follows. Let σ parameterise $\mathcal{C}$ and and we label points on $\mathcal{C}$ as $\vec{x}(\sigma)$. Similarly, for the curve $\mathcal{C}'$, let τ be the curve parameter and say points on it are labelled by $\vec{y}(\tau)$. Furthermore, let us define: $\vec{r} \equiv \vec{y} - \vec{x}$. Then the above linking number becomes[4]:

$$\ell_{12} = \frac{1}{4\pi} \oint_{\mathcal{C}} \oint_{\mathcal{C}'} \frac{d\vec{x}}{d\sigma} \cdot \left[\frac{d\vec{r}}{r^3} \times \frac{d\vec{y}}{d\tau} \right] d\sigma d\tau. \tag{J.7}$$

Now let us make the following assumption[5]: magnetic helicity sums the Gauss linking number over every pair of field lines within a volume. Note that, a magnetic field contains an infinite number of field lines, each with an infinitesimal flux and some of them may ergodically fill some finite volume in space without forming closed loops. However, to make our life simple let us say we can approximate the magnetic field in a closed volume V by closed flux tubes such that, $\vec{B}.\hat{n}|_{\partial V} = 0$, where $\hat{n}$ is the outward normal vector at the boundary ∂V. Let us say there are N flux tubes and each tube carries a flux Phi_i with $i = 1, \ldots, N$. Then, by our above assumption we get the helicity as:

$$\mathcal{H}_m = \sum_{i,j=1}^{N} \ell_{ij} \Phi_i \Phi_j, \tag{J.8}$$

If we take $N \to \infty$ and $\Phi_i, \Phi_j \to 0$ for $i, j = 1, \ldots, N$ then we get from Eq. (J.7)

[4]Here we associate a subscript "12" with ℓ to capture the orientation aspect of the linking number, implying $\ell_{12} = -\ell_{21}$.

[5]This is a classic result [172] which we try to motivate here.

and Eq. (J.8),

$$\mathcal{H}_m = \frac{1}{4\pi} \int \int \vec{B}(\vec{x}) \cdot \left[\frac{\vec{r}}{r^3} \times \vec{B}(\vec{y}) \right] d^3x \, d^3y \tag{J.9}$$

In the Coulomb gauge the vector potential in terms of the magnetic field is given as,

$$\vec{A}(\vec{x}) = \frac{1}{4\pi} \int \frac{\vec{r}}{r^3} \times \vec{B}(\vec{y}) \, d^3y \tag{J.10}$$

using which we get for $\mathcal{H}_m$,

$$\mathcal{H}_m = \int \vec{A}(\vec{x}) \cdot \vec{B}(\vec{x}) \, d^3x = \int \vec{A} \cdot (\nabla \times \vec{A}) \, d^3x \tag{J.11}$$

which is the definition of magnetic helicity. Thus, we see that our assumption indeed holds. In terms of differential forms, the above definition of helicity can be written as,

$$\mathcal{H}_m = \int A \wedge dA = \lim_{\substack{N \to \infty \\ \Phi_i, \Phi_j \to 0}} \sum_{i,j=1}^{N} \ell_{ij} \Phi_i \Phi_j \tag{J.12}$$

J.2 Generalised delta function

Consider a D-dimensional manifold $\mathcal{M}_D$ and a $p < D$ dimensional sub-manifold $\mathcal{C}_p$. We define a $D - p$ form δ-functional, $\delta^{(D-p)}\mathcal{C}_p$, for $\mathcal{C}_p$ as:

$$\int_{\mathcal{C}_p} A_p = \int_{\mathcal{M}_D} A_p \wedge \delta^{(D-p)}(\mathcal{C}_p), \tag{J.13}$$

J.2.1 δ-functional representation in flat space

In flat space, we have the following integral representation for $\delta^{(D-p)}(\mathcal{C}_p)$

$$\begin{aligned} \delta^{(D-p)}(\mathcal{C}_p) = {} & \frac{\epsilon_{m_1 \ldots m_p m_{p+1} \ldots m_D}}{p!(D-p)!} \times \\ & \times \left[\int_{\mathcal{C}_p} \delta p(x-y) \, \underbrace{dy^{m_1} \wedge \cdots \wedge dy^{m_p}}_{\text{coordinates on } \mathcal{C}_p} \right] \wedge dx^{m_{p+1}} \wedge \cdots \wedge dx^{m_D} \end{aligned} \tag{J.14}$$

We see from above that this is a way to associate $(D-p)$ form to a p-dimensional sub-manifold $\mathcal{C}_p$. This is the Poincaré duality relation between forms and manifolds.

Now let us justify the above integral representation below. Say,

$$D = 1 \Rightarrow p = 1 \Rightarrow \delta^{(0)}(C_1) = \int_{C_1} \delta(x-y)dy \tag{J.15}$$

Now,

$$\begin{aligned}\int_{\mathcal{M}_1} A_1\delta^{(0)}(C_1) &= \int_{\mathcal{M}_1} dx A_1(x) \int_{\mathcal{C}_1} \delta(x-y)dy \\ &= \int_{\mathcal{C}_1}\int_{\mathcal{M}_1} dx\delta(x-y)A_1(x)dy = \int_{\mathcal{C}_1} A_1(y)dy\end{aligned} \tag{J.16}$$

A little more rigorous justification is presented below.

From Eq. (J.14), consider[6]:

$$\begin{aligned}&\int_{\mathcal{M}_D} A_p \wedge \underbrace{\delta^{(D-p)}(\mathcal{C}_p)}_{\substack{\text{projects integration}\\ \text{of } A_p \text{ to } \mathcal{C}_p \subset \mathcal{M}_D}} \\ &= \# \int_{\mathcal{M}_D} A_{\mu_1\ldots\mu_p}(x)dx^{\mu_1} \wedge \cdots \wedge dx^{\mu^p} \wedge \epsilon_{m_1\ldots m_D} \int_{\mathcal{C}_p} \delta^{(p)}(x-y)dy^{m_1} \wedge \cdots \wedge dy^{m_p} \wedge \\ &\quad \wedge dx^{m_{p+1}} \wedge \cdots \wedge dx^{m_D} \\ &= \# \int_{\mathcal{C}_p} dy^{m_1} \wedge \cdots \wedge dy^{m_p} \wedge \int_{\mathcal{M}_D} A_{\mu_1\ldots\mu_p}(x)dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\delta^p(x-y)\times \\ &\quad \times \epsilon_{m_1\ldots m_D} dx^{m_{p+1}} \wedge \cdots \wedge dx^{m_D} \\ &= \# \int_{\mathcal{C}_p} (\ldots) \int_{\mathcal{M}_D} A_{\mu_1\ldots\mu_p}(x)\sqrt{-g}\ d^D x \delta^{\mu_1}_{m_1} \ldots \delta^{\mu_p}_{m_p} \delta^p x - y \\ &= \# \int_{\mathcal{C}_p} A_{m_1\ldots m_p}(y)dy^{m_1} \wedge \cdots \wedge dy^{m_p} \sim \int_{\mathcal{C}_p} A_p\end{aligned} \tag{J.17}$$

J.2.2 Exterior derivation of δ-functional

The exterior derivative of $\delta^{D-p}(\mathcal{C}_p)$ is given as:

$$d\delta^{D-p}(\mathcal{C}_p) = (-1)^p \delta^{D-(p-1)}(\partial \mathcal{C}_p) \tag{J.18}$$

[6] Below, we denote the combinatorial factor as the one which appears in Eq. (J.14) as #

Now let us prove the above. Consider,

$$
\begin{aligned}
& \int_{\mathcal{M}_D} A_{p-1} \wedge \delta^{D-(p-1)}(\partial\mathcal{C}_p) \\
& = \int_{\partial\mathcal{C}_p} = \int_{\mathcal{C}_p} dA_{p-1} = \int_{\mathcal{M}_D} dA_{p-1} \wedge \delta^{D-p}(\mathcal{C}_p) \\
\text{Now:} \quad & d(A_{p-1} \wedge \delta^{D-p}(p)) = dA_{p-1} \wedge \delta^{D-p}(\mathcal{C}_p) + (-1)^{p-1} A_{p-1} \wedge d\delta^{D-p}(\mathcal{C}_p) \\
& \Rightarrow \int_{\mathcal{M}_D} A_{p-1} \wedge \delta^{D-(p-1)}(\partial\mathcal{C}_p) \\
& = \int_{\mathcal{M}_D} d(A_{p-1} \wedge \delta^{D-p}) - (-1)^{p-1} \int_{\mathcal{M}_D} A_{p-1} \wedge d\delta^{D-p} \\
& = \underbrace{A_{p-1} \wedge \delta^{D-p} |_{\partial\mathcal{M}_D}}_{0 \text{ [as, } \delta^{D-p}|_{\partial\mathcal{M}_D} \to 0]} + (-1)^p \int_{\mathcal{M}_D} A_{p-1} \wedge d\delta^{D-p}(\mathcal{C}_p) \\
& \Rightarrow d\delta^{D-p}(\mathcal{C}_p) = (-1)^p \delta^{D-(p-1)}(\partial\mathcal{C}_p)
\end{aligned}
\tag{J.19}
$$

J.2.3 Intersection number

Let $\mathcal{C}_p$ and $\mathcal{S}_q$ be two sub-manifolds of $\mathcal{M}_D$ with $p \leq D$ and $q \leq D$. If $p+q \geq D$, then $\dim(\mathcal{C}_p \cap \mathcal{S}_q) = p+q-D$. Let us denote the intersecting surface as $I_{p+q-D} \equiv \mathcal{C}_p \cap \mathcal{S}_q$. If $p+q < D$, then $\mathcal{C}_p \cap \mathcal{S}_q$ can be thought of as a sub-manifold of $\mathcal{M}_{p+q}$. In this case, we have:

$$
I_{p+q} \equiv \mathcal{C}_p \cap \mathcal{S}_q \subseteq \mathcal{M}_{p+q} \subset \mathcal{M}_D \tag{J.20}
$$

Consider $p+q \geq D$. Then,

$$
\delta^{D-p+D-q}(I_{p+q-D}) = \delta^{D-p}(\mathcal{C}_p) \wedge \delta^{D-q}(\mathcal{C}_q) \tag{J.21}
$$

Now let us saturate the aforementioned inequality, that is, $p+q = D$. Then, $I_{p+q-D} = \mathcal{I}_0$, that is, $\mathcal{I}_0$ is the space of points.

Now, let us define intersection number of $\mathcal{C}_p$ and $\mathcal{S}_q$ (with $p+q=D$):

$$
I(\mathcal{C}_p, \mathcal{S}_q) = \int_{\mathcal{M}_D} \delta^D(\mathcal{I}_0) = \int_{\mathcal{M}_D} (\mathcal{C}_p) \wedge \delta^p(S_q) \tag{J.22}
$$

Note: by definition, $\int_{\mathcal{M}_D} \delta^D = 1$ and now if $\mathcal{I}_0 = \{p_1, p_2, \ldots, p_n\}$ where p_i is a point.

Then $\int_{\mathcal{M}_D} \delta^D(\mathcal{I}_0) = n = \#$ of points in $\mathcal{I}_0$[7].

Thus, $I(\mathcal{C}_p, \mathcal{S}_q)$ = number of points at which $\mathcal{C}_p$ and $\mathcal{S}_q$ intersect. Now, if $\mathcal{C}_p$ has a boundary $\partial\mathcal{C}_p$, then,

$$\text{Link}(\partial\mathcal{C}_p, \mathcal{S}_q) = I(\mathcal{C}_p, \mathcal{S}_q) \tag{J.23}$$

Note: $I(\mathcal{C}_p, \mathcal{S}_q)$ is defined only when $p + q = D \Rightarrow \text{Link}(\partial\mathcal{C}_p, \mathcal{S}_q)$ is defined only when: $(p-1) + q + 1 = D$ [8].

[7]Since $\int_{\mathcal{M}_D} A_0\delta^D(\mathcal{I}_0) = \int_{\mathcal{I}_0} A_0 = \sum_{i=1}^{n} 1 \cdot A_0 = n \cdot A_0$

[8]where $\dim\partial\mathcal{C}_p = p - 1$

APPENDIX K

Area, perimeter and phases

K.1 SSB and expectation values

In this section section, we explore the spontaneous symmetry breaking of higher-form symmetries, following the works of [21, 40, 47]. For specificity, we focus on 1-form symmetries in gauge theories, where the charged objects are represented by Wilson and 't Hooft lines. Here we follow the exposition as given in [28].

A crucial aspect of this discussion is identifying the vacuum expectation value of the Wilson loop as the order parameter for the electric 1-form symmetry, serving to differentiate between various phases. Given a closed curve $\mathcal{C}$, the expectation value $\langle W[\mathcal{C}]\rangle$ usually depends on geometric attributes, such as the area enclosed by $\mathcal{C}$ or its perimeter:

$$\langle W[\mathcal{C}]\rangle \sim e^{-\text{Area}[\mathcal{C}]} \quad \text{or} \quad \langle W[\mathcal{C}]\rangle \sim e^{-\text{Perimeter}[\mathcal{C}]}. \tag{K.1}$$

Distinct decay behaviors naturally indicate different phases. For a large loop $\mathcal{C}$, the area law decays much more quickly than the perimeter law, leading to an effective result of:

$$\langle W[\mathcal{C}]\rangle \sim e^{-\text{Area}[\mathcal{C}]} \to 0, \tag{K.2}$$

while, on the other hand:

$$\langle W[\mathcal{C}]\rangle \sim e^{-\text{Perimeter}[\mathcal{C}]} \neq 0. \tag{K.3}$$

Another commonly encountered decay behavior, even weaker than the perimeter law, is known as Coulomb behavior. This decay exhibits a scale-invariant dependence on the loop parameters. We will delve into this behavior in greater detail shortly.

Analogous to the case of ordinary symmetries, the perimeter and Coulomb laws are understood as indicating a non-zero value of the order parameter and are thus associated with phases where the 1-form symmetry is spontaneously broken. To gain a better understanding, let us discuss the relationship between the expectation value of the Wilson loop in Euclidean spacetime and the static potential between two charged probe particles. The concept involves considering a loop, as shown in Fig. K.1, with a straightforward physical interpretation: a pair of opposite charges is created in the distant past by a source that is slowly turned on, and the charges are then gently separated from each other at a distance R. After an extended time T, the pair is annihilated, again slowly.

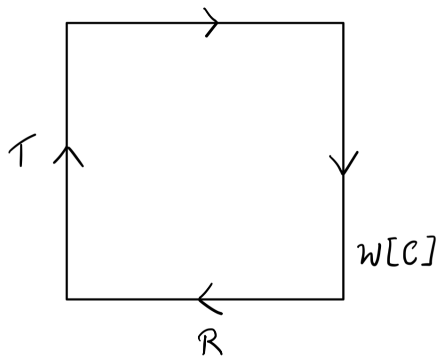

Figure K.1: Wilson loop describing a pair of static particles.

We are interested in the corresponding vacuum expectation value:

$$\langle W[\mathcal{C}]\rangle = \langle \mathrm{Tr} P e^{i\oint_{\mathcal{C}} a}\rangle, \tag{K.4}$$

computed in Euclidean spacetime, following the approach of [28,173]. As this object is gauge-invariant, we can select a convenient gauge. Let us choose the axial gauge, $a_0 = 0$, so that there is no contribution from the sections of $\mathcal{C}$ along the time direction T. Additionally, without loss of generality, let us consider that R is aligned with direction x^1. In this situation, the above expression simplifies to:

$$\langle W[\mathcal{C}]\rangle = \langle [Pe^{i\int_0^R dx^1 a_1(T,x^1,\ldots)}]^i{}_j [Pe^{i\int_R^0 dx^1 a_1(0,x^1,\ldots)}]^j{}_i\rangle. \tag{K.5}$$

To simplify the notation, let us define

$$\psi^i{}_j(T) \equiv [Pe^{i\int_0^R dx^1 a_1(T,x^1,\ldots)}]^i{}_j. \tag{K.6}$$

Using this, equation Eq. (K.5) becomes

$$\langle W[\mathcal{C}]\rangle = \langle \psi^i{}_j(T)\psi^{\dagger j}{}_i(0)\rangle. \tag{K.7}$$

Recalling that the Euclidean time T is related to the real time by $t \to -iT$, the time evolution is given by $\psi^i{}_j(T) = e^{HT}\psi^i{}_j(0)e^{-HT}$. Inserting a complete set of energy eigenstates $|n\rangle$ into Eq. (K.7), we get

$$\begin{aligned}\langle W[\mathcal{C}]\rangle &= \sum_n e^{-TE_n(R)}\langle \psi^i{}_j(0)|n\rangle\langle n|\psi^{\dagger j}{}_i(0)\rangle \\ &= \sum_n e^{-TE_n(R)}|\langle \psi^i{}_j(0)|n\rangle|^2.\end{aligned} \tag{K.8}$$

As $T \to \infty$, only the lowest-energy state contributes significantly,

$$\langle W[\mathcal{C}]\rangle \sim e^{-TE_0(R)}. \tag{K.9}$$

Because the charges are static, the energy reduces to the potential, and we finally obtain

$$V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln\langle W[\mathcal{C}]\rangle. \tag{K.10}$$

Thus, the behavior of the Wilson loop determines the form of the static potential between charges.

K.1.1 Phases of gauges theories

Let us discuss the typical behaviors of the expectation value of the Wilson loop associated with Fig. K.1.

Area law

Suppose it behaves according to the area law,

$$\langle W[\mathcal{C}]\rangle \sim e^{-\sigma TR}, \tag{K.11}$$

where σ is a dimensionful constant. As per Eq. (K.10), this results in a linear potential,

$$V(R) = \sigma R. \tag{K.12}$$

Therefore, the energy required to separate charges increases linearly with the distance R, leading to the confinement of the charges. In other words, the area law for $\langle W[\mathcal{C}]\rangle$ indicates confinement.

Perimeter law

Next, let us consider the perimeter law,

$$\langle W[\mathcal{C}]\rangle \sim e^{-\rho(T+R)}, \tag{K.13}$$

where ρ is a dimensionful constant. This behavior results in a constant potential,

$$V(R) = \rho. \tag{K.14}$$

As the energy cost to separate charges at a large distance is finite, this potential does not confine the charges. We express this by saying that the perimeter law corresponds to a *deconfining* phase.

Coulomb law

Lastly, let us consider the Coulomb or scale-invariant law. In this case, $\langle W[\mathcal{C}]\rangle$ decays more slowly than the perimeter law, depending on the dimensionless ratios T/R or R/T,

$$\langle W[\mathcal{C}]\rangle \sim e^{-\alpha\frac{T}{R}-\beta\frac{R}{T}}, \tag{K.15}$$

where α and β are dimensionless constants. The corresponding potential is

$$V(R) = \frac{\alpha}{R}, \tag{K.16}$$

which is precisely the Coulomb potential. Naturally, this also corresponds to a *deconfining* phase.

APPENDIX L

CS term quantisation and fraction quantum Hall effect

L.1 Quantisation of Chern-Simons level

Consider CS theory in $2+1$ dimensions[1]:

$$S_{CS} = \frac{k}{4\pi}\int_{\mathbb{R}^2\times S^1} d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{L.1}$$

Consider guage transformation:

$$A_\mu \to A_\mu + \partial_\mu \omega \Rightarrow \psi \to e^{ie\omega/\hbar}\psi \tag{L.2}$$

where, normally, we assume ω is single valued but only $e^{ie\omega/\psi}$ needs to be single valued.
$\Rightarrow \omega = \frac{2\pi\hbar\tau}{e\beta}$ makes the $e^{2\pi i\tau/\beta}$ single valued.

Let us consider a large guage transformation (not connected to the identity of Lie algebra) $A_0 \to A_0 + \frac{2\pi\hbar}{e\beta}$.

Now consider $\mathbb{R}^2 \to S^2$ and place a magnetic monopole inside it.

[1] In the expression below there is an additional factor of i coming from Wick rotation

Recall $F_{\tau i} = E_i$ and $F_{ij} = B_{ij}$. By Dirac quantisation condition, we get:

$$\frac{1}{2\pi}\int_{S^2} F_{12} = \frac{\hbar}{e} \tag{L.3}$$

Now evaluate S_{CS} on a configuration with $A_0 = a_0$ (constant periodic) and F_{12} satisfying Eq. (L.3).

$$\Rightarrow S_{CS} = \frac{k}{4\pi}\int_{S^2\times S^1} d^3x[A_0F_{12} + A_1F_{20} + A_2F_{01}] \tag{L.4}$$

Now, given our background $S^2 \times S^1$, we do not have τ dependence $\Rightarrow$ ∂_0 term vanish.

$$\begin{aligned} A_1F_{20} &= A_1(\partial_\tau a_0 - \cancelto{0}{\partial_0 A_2}) \text{ (where the first term is non-zero due to non-trivial topology.)} \\ &= -(\partial_\tau A_1)a_0 \text{ (after IBP)} \end{aligned} \tag{L.5}$$

$$\begin{aligned} \text{similarly, } A_2F_{01} &= A_2(\cancelto{0}{\partial_0 A_1} - \partial_1 a_0) \\ &= (\partial_1 A_2 a_0) \end{aligned} \tag{L.6}$$

$$\begin{aligned} \Rightarrow S_{CS} &= \frac{k}{4\pi}\int S^2 \times S^1 d^3x(2A_0F_{12}) \\ &= \frac{k}{2\pi}\int_{S^2\times S^1} d^3x A_0 F_{12} \\ &= k\int_{S^1} d\tau A_0\left[\frac{1}{2\pi}\int_{S^2} d^2x D_{12}\right] \\ &= \frac{k\hbar}{e}\int_{S^1} a_0 d\tau = \frac{k\hbar a_0}{e}\beta \end{aligned} \tag{L.7}$$

Now, under $a_0 \to a_0 + \frac{2\pi\hbar}{e\beta}$, we have

$$\underbrace{S_{CS} \to S_{CS} + \frac{2\pi\hbar^2 k}{e^2}}$$

fine as long as quantum partition function is guage invariant

$$\begin{aligned} &\Rightarrow e^{iS_{CS}/\hbar} \to e^{iS_{CS}/\hbar} \\ &\Rightarrow exp\left[\frac{2\pi i\hbar k}{e^2}\right] = 1 \\ &\Rightarrow \frac{\hbar k}{e^2} \in \mathbb{Z} \text{ with } \hbar = 1 = e(\text{natural units}) \\ &\Rightarrow k \in \mathbb{Z} \end{aligned} \tag{L.8}$$

Thus, the CS level k is integrally quantized. (L.9)

L.2 Effective action for fractional quantum Hall states

L.2.1 Exploration of Chern-Simons dynamics

In a $d = 2 + 1$ dimensional context, the simplest form of a topological field theory involves a dynamic gauge field denoted as A_μ associated with the group $U(1)$. It's important to emphasize that this is distinct from the gauge field related to electromagnetism, which let us continue to represent as a_μ. Instead, A_μ emerges as a result of the combined behavior of numerous underlying electrons.

We commonly think of gauge fields as representing massless degrees of freedom, at least in the classical context. Their dynamics are typically described using the Maxwell action, given by:

$$S_{\mathrm{Maxwell}}[A] = -\frac{1}{4g^2}\int d^3x\, F_{\mu\nu}F^{\mu\nu} \tag{L.10}$$

In this expression, $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and g^2 is a coupling constant. The corresponding equations of motion can be derived as $\partial_\mu F^{\mu\nu} = 0$. However, in $d = 2+1$ dimensions, there is only one permissible polarization. In summary, the $U(1)$ Maxwell theory in $d = 2 + 1$ dimensions characterizes a single massless degree of freedom.

However, as we have already discussed, there exists an alternative action that can be written for gauge fields in $d = 2 + 1$ dimensions. The Chern-Simons action,

$$S_{\mathrm{CS}}[a] = \frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{L.11}$$

As we discussed in the previous section, the coefficient k must be an integer (when measured in units of $e^2/\hbar$) if the emergent $U(1)$ symmetry is compact.

Let's explore how the Chern-Simons term influences both the classical and quantum dynamics. Consider the following action, which is a combination of the two terms we've discussed:

$$S = S_{\mathrm{Maxwell}} + S_{\mathrm{CS}} \tag{L.12}$$

The equation of motion for the field a_μ now becomes:

$$\partial_\mu F^{\mu\nu} + \frac{kg^2}{4\pi}\epsilon^{\nu\rho\sigma}F_{\rho\sigma} = 0 \tag{L.13}$$

This equation no longer describes a massless photon. Instead, any excitation in this system will decay exponentially over time. Solving these equations reveals that the addition of the Chern-Simons term effectively gives the photon a mass m. Equivalently, the spectrum now possesses an energy gap $E_{gap} = mc^2$. A quick calculation shows that it can be expressed as:

$$E_{gap} = \frac{kg^2}{2\pi} \tag{L.14}$$

As the coupling constant g^2 approaches infinity, the photon becomes infinitely massive, leaving us with no physical excitations. One might naturally question what the Chern-Simons theory describes, considering there are no propagating degrees of freedom in this case. The next section aims to address this question.

L.2.2 The effective theory for the Laughlin states

We are now prepared to present the effective theory for the fractional quantum Hall or, Laughlin states at filling fraction $\nu = 1/m$. These Hall states have an emergent, compact $U(1)$ gauge field A_μ. While this is a dynamical field, it should be included in our effective action. We can express the partition function as:

$$Z[a_\mu] = \int \mathcal{D}A_\mu e^{iS_{\text{eff}}[A;a]/\hbar} \tag{L.15}$$

Here, $\mathcal{D}A_\mu$ is a shorthand notation for all the standard gauge-fixing considerations needed to define a path integral for a gauge field.

Our next task is to formulate $S_{\text{eff}}[A;a]$. We need to introduce a coupling between A_μ and a_μ. Since a_μ must couple to the electron current j^μ, we must establish a relationship between A_μ and j^μ. The current can be expressed as:

$$j^\mu = \frac{e^2}{2\pi\hbar}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{L.16}$$

The conservation of this current, $\partial_\mu j^\mu = 0$, is basically an identity, when written in this form. This relationship implies that the magnetic flux of A_μ can be interpreted as the electric charge that couples to a_μ. The normalization comes directly from the compact emergent $U(1)$ gauge symmetry, which couples to particles with charge e. In this case, the minimal allowed flux is given by the Dirac quantization condition:

$$\frac{1}{2\pi}\int_{\mathbf{S^2}} F_{12} = \frac{\hbar}{e} \tag{L.17}$$

The relationship given in Eq. (L.16) ensures that the minimum charge is $\int j^0 = e$, as expected.

Let us propose the following effective action:

$$S_{\text{eff}}[A; a] = \frac{e^2}{\hbar} \int d^3x \frac{1}{2\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho - \frac{m}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \ldots \tag{L.18}$$

The first term is a "mixed" Chern-Simons term that arises from the $a_\mu j^\mu$ coupling; the second term is the simplest new term we can introduce. Following the same arguments we used earlier, the level must be an integer: $m \in \mathbb{Z}$. The ellipsis above represent additional, less relevant terms, including the Maxwell term from Eq. (L.10). At large distances, these terms will have no significant impact, so we will disregard them. The above action is the effective action for Laughlin states at filling fraction $\nu = 1/m$ (see [57] for more details).

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