The Geometry of Unipotent Deformations and Applications

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Abstract: This thesis studies primarily the local properties the unipotent connected component of the moduli space of Langlands parameters, the local rings of which give us Galois deformation rings, a crucial ingredient in the Taylor-Wiles-Kisin patching method that is used to prove global Langlands correspondences. We study first the simpler 'considerate' case to give a criterion for smoothness of the connected components when $G = \mathrm{GL}_n$. We also study the local rings of various unions of connected components to show that the Galois deformation rings are Cohen-Macaulay. We study further the Steinberg component in the case of 'extreme inconsiderateness' to show that the Steinberg component has at most rational singularities, so in particular is normal and Cohen-Macaulay. Finally, we give an application of the smoothness result, to give a freeness result of the module of certain Hida families of automorphic forms over its Hecke algebra, which in turn will give a multiplicity result for the Galois representations of these Hida families.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification. The material in Chapter 8 is based on joint work (equal contribution) with the author's supervisor (Jack Shotton).

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Never be afraid to trust an unknown future to a known God.

— Corrie ten Boom

So do not be afraid, you are of more value than many sparrows.

— Matthew 10v31

One ring to rule them all.

— from *The Fellowship of the Ring* by J.R.R.Tolkien

So long, and thanks for all the fish.

— from *The Hitchhiker's Guide to the Galaxy* by Douglas Adams

Dedicated to

The Lord Jesus, who has carried me since the day I was born, and to whom I owe everything in this Thesis

and

Shelly Funck, a woman with a dear passion for justice and the marginalised, who brings out the best qualities in me, and whose name I now wear proudly.

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Chapter 1

Introduction

In the late 1960's Robert Langlands first proposed the remarkable idea that there are certain similarities between two completely different areas of mathematics; automorphic forms from harmonic analysis, and Galois theory. The proposed idea, that there should be some 'correspondence' between the Hecke eigenforms for a connected reductive group G, and the continuous representations of Galois groups into its so-called 'Langlands dual group' $\hat{G}$, while to this day still largely unproven, has still inspired mathematicians to develop many fascinating ideas, from the earlier Galois deformation rings, complete local rings that parameterise 'deformations' of mod$l$ Galois representations which played such a crucial role in the proof of the modularity theorem by Andrew Wiles and Richard Taylor in 1995 [Wil95] and [TW95]; to the development of shtukas, the categorical Langlands programme and condensed maths in more recent years.

Since 1970, the above correspondence has taken on the name of the 'global Langlands conjecture' emphasising, that this correspondence occurs over a *global* number field (that is, a finite extension of $\mathbb{Q}$, or the function field of a smooth curve over a finite field); or (even better) over the adèles. But one can also formulate a 'local Langlands correspondence' over 'local' fields; that is, finite extensions of $\mathbb{Q}_l$ and of the field of Laurent series $\mathbb{F}_l((t))$. In fact, this local correspondence is an important area to study before one can even hope to understand the global correspondence.

Let F be a local p-adic field. and let G be a connected reductive algebraic group over F. The local Langlands conjectures (proven for GL_n by Harris and Taylor in [HT01]) stipulate the existence of a natural map, with finite fibres

$$\frac{\{\text{smooth irreducible representations of } G(F)\}}{\{\text{isomorphism}\}} \to \frac{\{L\text{-parameters of } {}^LG\}}{\{\hat{G} - \text{conjugacy}\}}.$$

Let l be a prime, different to p. Let $L \subset \bar{\mathbb{Q}}_l$ be an l-adic field, and $\mathcal{O}$ its ring of integers, with residue field $\mathbb{F}$. In recent years, by work of [BG19], [Hel21], [DHKM23], [Zhu21] and [FS21], there has been great interest in studying the properties of a moduli space of L-parameters $\mathrm{Loc}_{\hat{G},\mathcal{O}}$ and a closely related space, the moduli space of framed L-parameters, $\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}$. That is, an algebraic stack over $\mathcal{O}$, which is the the stackification of the prestack whose R-points (R an $\mathcal{O}$-algebra) are naturally identified with the $\hat{G}$-conjugacy classes of L-parameters, and a scheme whose R-points are the set of L-parameters respectively.

$$\mathrm{Loc}_{\hat{G},\mathcal{O}}(R) = \{L\text{-parameters of } \hat{G}, \text{ with } R\text{-coefficients}\}/\cong$$

$$\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}(R) = \{L\text{-parameters of } \hat{G}, \text{ with } R\text{-coefficients}\}$$

These spaces were created to have two properties. Firstly, they give the set of L-parameters on the right hand side of the local Langlands correspondence, a natural geometric structure, so that one may hope to better understand the correspondence by 'geometrising' it (see for example [FS21] or [Zhu21]). Secondly, (the completions of) the local rings of $\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}$ are the (framed) local Galois deformation rings from before, that play such an important role in the Taylor-Wiles method, as well as the later Taylor-Wiles-Kisin and Calegari-Geraghty methods. By studying such a moduli space, it is hoped to better understand Galois deformation rings.

To define an L-parameter, one needs the notion of an L-homomorphism. Let W_F be the Weil group of the field F, and for G a connected reductive group let $\hat{G}$ be the Langlands dual group. An L-homomorphism with R-coefficients is a homomorphism $\rho : W_F \to {}^LG(R) := \hat{G}(R) \rtimes W_F$, such that the projection onto the second factor gives the identity map on W_F. In this thesis, we reduce to the case where the action of W_F on $\hat{G}$ is trivial (this occurs, for example, when G is split), and so we may view L-homomorphisms as plain homomorphisms $W_F \to \hat{G}(R)$. Historically, there are multiple definitions of L-parameters, with varying degrees of usefulness. We interest ourselves in the moduli space of Bellovin and Gee [BG19] and make the following definition.

Definition 1.0.1. *A Langlands parameter is a Weil-Deligne representation (r, N), where $r : W_F \to {}^L G$ is an L-homomorphism with open kernel, and N is an element of Lie$(\hat{G})$ such that for any $g \in W_F$, Ad$(g)N = |g|N$, where $|.| : W_F \to F^\times \to \mathbb{R}^{\geqslant 0}$ is the valuation on W_F coming from local class field theory.*

It is known, as in Proposition 2.6 of [DHKM23], that this definition gives a good moduli space for Langlands parameters in characteristic 0, but in general the moduli spaces won't give deformation rings, because the way one relates Weil-Deligne representations to Galois representations utilises the exponential and logarithm maps, which may not exist in positive characteristic or be continuous in mixed characteristic, so in general one will need the moduli space $Z^1(W_F^0, \hat{G}(R))$ constructed in [DHKM23]. In Proposition 2.0.5, we show that when studying the unipotent component of $\mathrm{Loc}^{\square}_{\hat{G},\mathcal{O}}$, it is equivalent to study either moduli space whenever l is ${}^L G$-banal (we remark that this implies l is necessarily greater than the Coxeter number h_G), as then the exponential and logarithm maps present an isomorphism between our moduli space and the unipotent connected component of the moduli space of tame parameters seen in [DHKM23].

By Lemma 2.1.3 of [BG19], this moduli problem can be represented by an algebraic stack over $\mathbb{Q}_l$, $\mathrm{Loc}^{BG}_{G,\mathbb{Q}_l}$, which is a disjoint union of quotient stacks, indexed by the inertial type of the Weil Deligne representation. The moduli problem of framed L-parameters, $\mathrm{Loc}^{\square}_{G,\mathbb{Q}_l}$, can further be represented by an infinite disjoint union of affine varieties, indexed similarly by the inertial type.

In chapters 2, 3 and 4, we will denote by $\mathcal{O}$ a regular local ring of residue characteristic

l or 0. In these we seek to understand the geometry of the scheme studied in [Hel21], whose special fibre is precisely the scheme $Y_{L/L,\varphi,\mathcal{N}}$ of [BG19]. This is a reduced affine scheme of finite type $S_{G,\mathcal{O}}$, over the ring $\mathcal{O}$, whose R-points (R an $\mathcal{O}$-algebra) are given by

$$S_{G,\mathcal{O}}(R) = \{(\Phi, N) \in G(R) \times \mathfrak{g}(R) | \mathrm{Ad}(\Phi)N = qN\}.$$

This is naturally the space of framed unipotent Weil-Deligne representations over $\mathcal{O}$, with values in G (following Definition 2.1.2 of [BG19]). We remark that 'unipotent' here means that the Weil-Deligne representation (r, N) has $r : W_F \to {}^L G(R)$ factor through W_F/I_F where I_F is the inertia subgroup. We will in particular be interested in the case when $\mathcal{O}$ is the ring of integers in a finite extension of $\mathbb{Q}_l$, because the m_R-adic completion of the local rings, R, of the closed points of this scheme can be interpreted as local Galois deformation rings, for sufficiently large l (In fact, whenever the exponential and logarithm maps exist, which occurs as we shall see, whenever l is ${}^L G$-banal). We also note, that via Theorem 4.5 of [DHKM23], it is essentially sufficient to study $S_{G,\bar{\mathbb{Q}}_l}$ for various groups G to understand the geometry of any connected component of $\mathrm{Loc}^{\square}_{G,\bar{\mathbb{Q}}_l}$, so by restricting to this unipotent case, we do not lose generality in characteristic 0, or whenever l is ${}^L G$-banal.

In chapters 2 and 3, we describe a way of decomposing S_G that gives the irreducible components when $G = \mathrm{GL}_n$ that can be found in Proposition 2.1 of [Hel21]. (For the irreducible components of S_G more general G, see [Sho23]). Let $\mathcal{N} \subseteq \mathfrak{g}$ be the nilpotent cone inside the Lie algebra $\mathfrak{g}$. Let

$$p : S_{G,\mathcal{O}} \to \mathcal{N}$$

be the projection map onto the second factor. Let $C \subset \mathcal{N}_L$ be a G-conjugacy class inside $\mathcal{N}_L$. (We note that, in the case of GL_n, these can be characterised by partitions of n and in this situation we will denote the conjugacy class corresponding to λ by C_λ.) We remark, that because $S_{G,\mathcal{O}}$ is flat over $\mathcal{O}$, the irreducible components biject naturally with those of $S_{G,L}$. Then $\overline{p^{-1}(C)} \subseteq S_{G,\mathcal{O}}$ is a union of irreducible components of $S_{G,\mathcal{O}}$ (and in the case of $G = \mathrm{GL}_n$, is itself irreducible and all irreducible components arise in this way). In chapter 3, I expand on and generalise the results of Bellovin [Bel16] section 7.2 and Proposition 7.10 we prove theorems 3.0.1 and 3.0.3 which state:

Theorem 1.0.2. *1. Let $C_r \subseteq \mathcal{N}$ be the regular adjoint orbit, and $C_0 = \{0\} \subseteq \mathcal{N}$ be the zero conjugacy class, and let $X_0 = \overline{p^{-1}(C_0)}$ and $X_r = \overline{p^{-1}(C_r)}$ be the respective irreducible components of $S_{G,\mathcal{O}}$. Let Z is the centre of G, and assume it is smooth.*

Then X_0 is smooth over $\mathcal{O}$, and X_r is a disjoint union of $\pi_0(Z)$ smooth connected components.

2. Further, in the case $G = GL_n$, these are the only smooth irreducible components of $S_{G,\mathcal{O}}$

In chapter 4, we turn our interest to certain unions of the components of $S_{n,\mathcal{O}} = S_{\mathrm{GL}_n,\mathcal{O}}$. We will, for each partition p of n, define $X_{\leqslant p} := p^{-1}(\bar{C}_p)$. These varieties arise naturally as the support of certain patched modules. In this chapter, we conjecture that such varieties are Cohen-Macaulay, and prove it for the following dense subset of points, noted in the following theorem.

Theorem 1.0.3. *Let $X_{\leqslant p}^{\Phi\text{-}reg}$ be the open subscheme of $X_{\leqslant p}$ whose points (Φ, N) have Φ regular semisimple. Then $X_{\leqslant p}^{\Phi\text{-}reg}$ is Cohen-Macaulay. Further, the local ring at $P = (\Phi, N) \in X_{\leqslant p}^{\Phi\text{-}reg}$ is Gorenstein if and only if either:*

- *$p = 1 + 1 + ... + 1$, and so $X_{\leqslant p}$ is the unramified component of $S_{n,\mathcal{O}}$, or*
- *the inclusion $X_{\leqslant p} \hookrightarrow S_{n,\mathcal{O}}$ defines an isomorphism on stalks at P.*

In addition, we also prove some partial results towards removing the condition of Φ-regular semisimplicity.

In chapters 5 and 6 of this thesis, I apply the smoothness result of chapter 3 via the patching method, in a situation very similar to that studied in [Ger19]. Let

l be a prime and K a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$. Let F^+ be a totally real global number field, and consider an imaginary quadratic extension F of F^+. The Galois representations considered will correspond to certain Hida families of ordinary automorphic forms on a unitary algebraic group G_D/F^+, which is a unitary form of a unit group of a division algebra, D/F^+. We will define a certain space of Hida families of ordinary automorphic forms $S^{\mathrm{ord}}(U(l^\infty), L/\mathcal{O})_\mathfrak{m}$ for G_D with Hecke algebra $\mathbb{T}_\mathfrak{m}$, and a corresponding deformation ring $R_\mathcal{S}^{univ}$. We will then use the Taylor-Wiles patching method to deduce the following theorem:

Theorem 1.0.4. *The module $S^{ord}(U(l^\infty), L/\mathcal{O})_m^\vee[1/l]$ is a finite locally free $R_\mathcal{S}^{univ}[1/l]$ module.*

As a consequence, we can deduce that $R_\mathcal{S}^{univ}[1/l] \cong \mathbb{T}_\mathfrak{m}[1/l]$, and that the multiplicity of automorphic forms with a given characteristic zero Galois representation is constant along connected components of $R_\mathcal{S}^{univ}[1/l]$. In particular, one can extend any such multiplicity results from the classical case to the case of non-classical Hida families.

In Chapter 8, which is based on joint work with Jack Shotton, we return to the geometric properties of $\mathrm{Loc}^\square_{G,\mathcal{O}}$. In Chapters 3 and 4, we showed which irreducible components of S_G are smooth over $\mathcal{O}$, and studied the algebraic properties of various unions of irreducible components. However, these results all relied on the important condition that q is considerate towards $G_{/\mathcal{O}}$. Complications arise when studying $\mathrm{Loc}^\square_{G,\mathcal{O}}$ when 'q is inconsiderate'.

First, the isomorphism of Proposition 2.0.5 breaks down over the special fibre, so the model of Langlands parameters via Weil-Deligne representations is no longer the correct model, and we must instead study the moduli of tame Langlands parameters seen in [DHKM23] whose R-points are:

$$\mathrm{Loc}^{\square,\mathrm{tame}}_{\hat{G},\mathcal{O}}(R) = Z^1(W_F^0/I_F, \hat{G}(R)) = \{\Phi, \Sigma \in \hat{G}(R) \times \hat{G}(R) : \Phi\Sigma\Phi^{-1} = \Sigma^q\}.$$

The second problem is best demonstrated when Σ is regular unipotent (this is the analogous situation to when the nilpotent matrix N of $S_{G,\mathcal{O}}$ is regular nilpotent). Here, the matrix Φ, which forced to have eigenvalues in the ratio $q^{n-1} : q^{n-2} : ... : q^2 : q : 1$, is no longer regular semisimple along the special fibre. This leads to the special fibre containing singularities and complicates the behaviour of $\mathrm{Loc}^{\square}_{G,\mathcal{O}}$.

Thus, we restrict ourselves to the following special case. We define the Steinberg component to be the scheme theoretic closure in $\mathrm{Loc}^{\square,\mathrm{tame}}_{G,\mathcal{O}}$ of the open subset of the generic fibre

$$\{(\Phi, \Sigma) \in \mathrm{Loc}^{\square,\mathrm{tame}}_{G,L} : \Sigma \text{ is regular unipotent}\}.$$

This space is precisely the inconsiderate companion to the space of Chapter 3 Theorem 3.0.1. In this chapter, we study the geometry of this space in the 'extremely inconsiderate' case, when $q \equiv 1 \bmod l$. (We note that this is the same as the condition of 'quasi-banality' of [CHT08] Definition 5.1.1 when l is not-banal).

In this case, the reduction mod l of Φ is central in G and the $\pmod{l}$-reduction of the Steinberg component $\mathcal{X}_{St,\mathbb{F}}$ is closely related to the scheme over $\mathbb{F}$ whose R-points are:

$$X(R) = \{(M, N) \in \mathcal{N}(R) \times \mathcal{N}(R) : [M, N] = 0\}$$

This scheme is very non-singular, in contrast with Theorem 3.0.1 of Chapter 3. Consequently, we no longer expect the patched modules that arise on this space to be locally free coherent sheaves. This would, in general, lead to distinct $\bmod l$ Hecke-eigenforms with the same Hecke eigenvalues, (and consequently, the same Galois representation).

In Chapter 8, we use methods of Snowden, Vilonen and Xue, and Ngo ([Sno18],[VX16] and [Ngo18] resp.) to study the Steinberg irreducible component $\mathcal{X}_{St,\mathcal{O}}$ in the extremely inconsiderate setting. We will use the cohomological calculations of section 3 alongside a Lemma of Snowden (Lemma 2.1.4 of [Sno18]) to prove the theorem:

Theorem 1.0.5. *Suppose that l is sufficiently large (as defined in remark 8.4). Let $G = GL_3$ and let $\mathcal{X}_{St,\mathcal{O}}$ be the Steinberg component for G. There is a scheme $\mathcal{Y}_{\mathcal{O}}$*

smooth over $\mathcal{O}$ and a proper birational map

$$p : \mathcal{Y}_{\mathcal{O}} \to \mathcal{X}_{St,\mathcal{O}}$$

so that $\mathcal{X}_{St,\mathcal{O}}$ has resolution-rational singularities (in the terminology of [Kov22]). Further, $\mathcal{X}_{St,\mathcal{O}}$ is Cohen-Macaulay and the special fibre $\mathcal{X}_{St,\mathbb{F}}$ is reduced and normal.

We remark that Conjecture 8.4.4 suggests that l 'sufficiently large' means $l \geqslant 11$ in this context.

We also use these methods to both give equations for $\mathcal{X}_{St,\mathcal{O}}$ (see section 8.5) and to calculate the Weil-class group of $\mathcal{X}_{St,\mathcal{O}}$ (see section 8.6).

Chapter 2

Considerateness and the relation to the stack of L-parameters

Let $\mathcal{O}$ be a regular local ring, with residue field $\mathbb{F}$ of characteristic l or 0 and fraction field L. Let G be a split connected reductive algebraic group over $\mathcal{O}$ (note that for most of Chapters 2-4, we will consider $G = \mathrm{GL}_n$) and $\mathfrak{g}$ its Lie algebra. Throughout the paper, whenever l is in play, we will necessarily assume that $l > h_G$, where h_G is the Coxeter number of G.

Definition 2.0.1. *Let h_G be the Coxeter number of G. Let $q \in \mathcal{O}^\times$ be an element of $\mathcal{O}$ such that $q^k - 1$ is invertible in $\mathcal{O}$ for all $k \leqslant h_G$. When this occurs, we say that q is considerate towards G over $\mathcal{O}$.*

In applications, $\mathcal{O}$ will either be a field, or will be the ring of integers in some field extension of $\mathbb{Q}_l$. Notice that in this case, q being considerate towards G is equivalent to all $1, q, q^2, ..., q^{h_G}$ being distinct in the residue field k (in a sense, q is 'careful' where it treads around G).

We wish now to point out that this definition of 'considerateness' is very closely related to two other conditions.

Definition 2.0.2. *Let G be a split reductive group over a field L of characteristic l. We say:*

- *l is G-banal, if l divides the order of the finite group $G(\mathbb{F}_q)$.*

- *l is ${}^LG$-banal, if for any algebraically closed field E of characteristic l, then any $\phi \in Loc^{\square}_{\hat{G},E}$ can be 'Frobenius twisted' by some $g \in C_{\hat{G}}(\phi(I_F))$ (that is, the centraliser of the inertia subgroup) so that ϕ^g is a smooth point of $Loc^{\square}_{\hat{G},E}$.*

  The 'Frobenius twist' of a representation $\phi : W_F \to \hat{G}(L)$ by $g \in C_{\hat{G}}(\phi(I_F))$ is the representation $\phi^g : W_F \to \hat{G}(L)$ which is equal when restricted to the inertia subgroup, and for which $\phi^g(Frob) = \phi(Frob)g$.

Proposition 2.0.3. *Suppose that $\mathbb{F}$ is a field of positive characteristic $l > h_G$ and that G is a split reductive group. Then we have the following implications.*

- *If q is considerate towards $G_{/\mathbb{F}}$, then l is ${}^LG$-banal.*

- *If l is ${}^LG$-banal, then l is G-banal.*

- *if $G = GL_n$ or SL_n, then all concepts are equivalent.*

Proof. By definition, q is considerate towards $G_{/\mathbb{F}}$ when the order of q inside $\mathbb{F}$ is greater than the Coxeter number h. This is equivalent to $\prod_{n \leqslant h} \Phi_n(q) = 0$ inside $\mathbb{F}$ where Φ_n is the nth cyclotomic polynomial. This is the polynomial $\chi^*_{G,1}(q)$ of Theorem 5.7 of [DHKM23] (see definition B.3). Hence, by Theorems 5.6 and 5.7 of [DHKM23], it follows that this condition implies that l is ${}^LG$-banal.

That l is ${}^LG$-banal implies l is G-banal is a consequence of the Chevalley-Steinberg formula of Theorem 25a) [Ste16];

$$|G(\mathbb{F}_q)| = q^N \prod_d (q^d - 1)$$

where d ranges over the fundamental degrees of the Weyl group of G. If l divides $\prod_d (q^d - 1)$, then l certainly divides $\prod_{n \leqslant h} \Phi_n(q)$ as the Coxeter number is the highest fundamental degree. This shows the second statement by virtue of Theorem 5.7 of [DHKM23].

In the case $G = \mathrm{GL}_n$ or SL_n, we get $|\mathrm{SL}_n(\mathbb{F}_q)| = q^N \prod_{i=2}^n (q^i - 1)$. Hence, if l is G-banal, it follows that the order of q in $\mathbb{F}$ is at least n. Thus, q is considerate towards $G_{/\mathbb{F}}$. GL_n is similar. □

Remark. It is worth noting that Corollary 5.27 of [DHKM23] gives the criterion that G-banal and ${}^LG$-banal are equivalent concepts whenever G is unramified and has no exceptional factors (here, triality forms of type D_4 are also considered exceptional.), but this property does not hold in general (see, for example Remark 5.22 of [DHKM23]).

We make the following definition.

Definition 2.0.4. *We define the affine scheme $S_{G,\mathcal{O}}$ over $\mathcal{O}$ as the scheme whose R-points (R, an $\mathcal{O}$ algebra) are* $\{(\Phi, N) \in G(R) \times \mathfrak{g}(R) : Ad(\Phi)N = qN\}$

Corollary 5.4 of [Bel16] shows that this is a reduced scheme in characteristic zero, and hence is a variety when $\mathcal{O}$ is a field of characteristic zero. As discussed in the introduction, we may picture $S_{G,\mathcal{O}}$ as the moduli space of unipotent Weil-Deligne representations, (r, N) over $G(\mathcal{O})$. The unipotent condition is equivalent to that of $r(I_F) = 1$.

Proposition 2.0.5.

1. *Suppose q is considerate towards $G_{/\mathcal{O}}$. Then the natural map $p : S_G \to \mathfrak{g}$ factors through the nilpotent cone $\mathcal{N}_G$.*

2. *When G is split, and $l > h_G$ then $S_{\hat{G},\mathcal{O}}$ is isomorphic to a closed subscheme of the moduli space of tame parameters $Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}$ (See section 1.2 of [DHKM23] for a definition of this space).*

3. *Along with the conditions of part 2, assume l is ${}^LG$-banal. Then the closed subscheme that is the image of $S_{\hat{G},\mathcal{O}}$ inside $Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}$ is a connected component.*

Proof. Notice, that since $l > h_G$, in the notation of section 2.4 of [Cot22] l is very good. Hence, by Theorem 4.13 of [Cot22], we have an isomorphism of $\mathcal{O}$-algebras

$$\mathcal{O}[\mathfrak{g}]^G \to \mathcal{O}[\mathfrak{t}]^W$$

given by the restriction of functions on $\mathfrak{g}$ to $\mathfrak{t}$ where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, and W is the Weyl group.

By chapter 3 of [Hum90] (see table 1 of section 3.7, and table 2 of section 3.18) the generators of $R[\mathfrak{t}]^W$ are homogeneous of degree at most the Coxeter number h_G, and hence the same is true for $R[\mathfrak{g}]^G$. We note that while this reference restricts to the case of a field of characteristic zero, the results extend to $\mathcal{O}$, because $|W|$ is invertible inside $\mathcal{O}$, and $\mathcal{O}[\mathfrak{t}]^W$ is a free $\mathcal{O}$-module.

Let s be a generator of $\mathcal{O}[\mathfrak{g}]^G$, and $(\Phi, N) \in S_{G\mathcal{O}}(R)$ be an R-point. Then as s is G-invariant and homogeneous of degree at most the Coxeter number h_G, have $s(\mathrm{Ad}(\Phi)N) = s(qN)$ implies $s(N) = q^i s(N)$, for some $i \leqslant h_G$. As q is considerate towards $G_{/\mathcal{O}}$, we have that $q^i - 1$ is a non-zero divisor in $\mathcal{O}$, and hence that $s(N) = 0$. We then see that the image of N inside the GIT quotient $\mathfrak{g}//G$ is zero. Since l is very good, Theorem 4.12 of [Cot22] shows that N lies in the R-points of the nilpotent cone. Part 1 of the proposition follows.

When G is a split group, $Z^1 = Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}$ has a model as an affine scheme, flat over $\mathcal{O}$ (since $l \neq p$) with R-points equal to

$$Z^1(W_F^0/P_F, \hat{G})_{\mathcal{O}}(R) = \{(\phi, \sigma) \in \hat{G}(R)^2 : \phi\sigma\phi^{-1} = \sigma^q\}.$$

Since $l > h_G$, and we can invert by all primes $\leqslant h_G$, the exponential and logarithm maps of section 6 of [BDP17] are well defined polynomials, and thus we have an isomorphism between the nilpotent cone in $\mathcal{N}_G$ and unipotent cone $\mathfrak{U}_G$. Hence, we have a map

$$\begin{aligned} S_{\hat{G},\mathcal{O}} &\to Z^1(W_F^0/P_F\hat{G})_{\mathcal{O}} \\ (\phi, N) &\mapsto (\phi, \exp N) \end{aligned}$$

which is an isomorphism onto the closed subscheme of $Z^1(W_F^0/P_F\hat{G})_{\mathcal{O}}$ given by those elements (ϕ, σ) with $\sigma \in \mathcal{U} \subseteq \hat{G}$, where $\mathcal{U}$ is the unipotent cone.

For part 3, suppose l is ${}^LG$-banal. Let $\mathfrak{U}^+$ be the scheme-theoretic image of $Z^1(W_F^0/P_F\hat{G})_{\mathcal{O}}$ through the second projection onto $\hat{G}$. We note, that $\sigma \in \mathfrak{U}^+$ necessarily has σ conjugate to σ^q. Let $T \subset \hat{G}$ and $W = W_{\hat{G}}$ be a maximal split torus and the Weyl group of $\hat{G}$ respectively. Consider the map $\hat{G} \to \hat{G}//\hat{G} \cong T/W$. The image of $\mathfrak{U}^+$ through this map has image given by the scheme-theoretic union $S := \bigcup_{w \in W}\{\sigma \in T : \sigma^q = {}^w\sigma\}$, which is a finite flat scheme over $\mathcal{O}$. Thus, since the fibres of this map are conjugacy classes, they are connected, and hence, the connected components of $\mathfrak{U}^+$ are in bijection with those of S. If l is ${}^LG$-banal, then $Z^1_{\mathbb{F}}$ is reduced, and thus, so is $S_{\mathbb{F}}$. Hence, since S is finite flat over $\mathcal{O}$, we see that the connected components of the generic fibre are in natural bijection with those of the special fibre, and thus the same is true for Z^1. Hence, as $S_{\hat{G},\mathcal{O}}$ defines a connected component over the generic fibre, it is a connected component of Z^1. □

We will also need the following results.

Proposition 2.0.6. *1. The algebraic group G acts on S_G via the simultaneous conjugation*

$$g.(\Phi, N) = (g\Phi g^{-1}, Ad(g)N).$$

Assume now that q is considerate towards $G_{/\mathcal{O}}$.

2. *The scheme $S_{G,\mathcal{O}}$ is a complete intersection of relative dimension $\dim G$ over $\mathcal{O}$.*

3. *The scheme $S_{G,\mathcal{O}}$ is flat over $\mathcal{O}$.*

4. *Define the second projection map $p : S_G \to \mathcal{N}_G$ as earlier. If C is a $G(L)$ conjugacy class inside $\mathcal{N}_{G,L} \subseteq \mathcal{N}_G$, then the closed subscheme $X_C := \overline{p^{-1}(C)} \subset S_G$ is a union of irreducible components, and we have $S_G = \bigcup_C X_C$.*

5. *If in addition $G = GL_n$, the X_C are irreducible components of $S_{n,\mathcal{O}} := S_{GL_n,\mathcal{O}}$, and these can be naturally identified with partitions of n. We call the component corresponding to the partition p, X_p.*

6. *The scheme $S_{G,\mathcal{O}}$ is reduced.*

Proof. 1. This is clear.

2. As $S_{G,\mathcal{O}}$ is isomorphic to the fibre over 0 of the map $G \times \mathfrak{g} \to \mathfrak{g}$ given by $(g, N) \mapsto \mathrm{Ad}(g)N - qN$, we see that each irreducible component is of dimension at least $\dim(G) + \dim(\mathcal{O})$. To show equality, suppose Y is some irreducible component with larger dimension. Then there is some prime $v \in \mathrm{Spec}(\mathcal{O})$ with residue field $k(v)$ for which the fibre Y_v of Y over v has dimension strictly greater than $\dim(G)$. But since q is considerate over the residue field, it is considerate over $k(v)$, and thus, the morphism $p : S_{G,k(v)} \to \mathfrak{g}_{k(v)}$ factors through the nilpotent cone $\mathcal{N}_{k(v)}$. For each G-conjugacy class C inside $\mathcal{N}_{k(v)}$, choose a closed point $J \in C$. Then the fibres of the map $p^{-1}(C) \to C$ over any closed point x are a Torsor over the centraliser $C_G(J)$, which is a smooth group scheme, because $l > h_G$. We remark that the map $p^{-1}(C) \to C$ is flat with smooth fibres, and thus is smooth, and open. One calculates via orbit-stabiliser that the dimension of $p^{-1}(C)$ is

$$\begin{aligned}\dim(p^{-1}(C)) &= \dim(C) + \dim(C_G(J)) \\ &= \dim(\mathrm{Orb}_G(J)) + \dim(\mathrm{Stab}_G(J)) \\ &= \dim(G)\end{aligned}$$

and thus that $S_{G,k(v)}$ is the set theoretic union of dimension $\dim(G)$ locally closed subschemes. It follows that every irreducible component of $S_{G,\mathcal{O}}$ is at most dimension $\dim(G)+\dim(\mathcal{O})$, and thus that $S_{G,\mathcal{O}}$ is a complete intersection.

3. Let $R = \mathcal{O}_{S_{G,\mathcal{O}}}(S_{G,\mathcal{O}})$. From the previous part, all irreducible components have the same dimension $\dim(G) + \dim(\mathcal{O})$. As $S_{G,\mathcal{O}} = \mathrm{Spec}(R)$ is Cohen-Macaulay, for any prime ideal $\mathcal{P} \trianglelefteq \mathcal{O}$, the unmixedness theorem tells us that all associated primes of $R/\mathcal{P}$ have the same height as $\mathcal{P}$. Since no irreducible component is contained inside the fibre over any prime $v \in \mathrm{Spec}(\mathcal{O})$, this shows that any such associated prime P_a has $P_a \cap \mathcal{O} = \mathcal{P}$. It follows that any element $\mathfrak{m}_{\mathcal{O}} \backslash \mathcal{P}$ is

a non-zero divisor in $R/\mathcal{P}$. Via induction, we can then show that any regular sequence in $\mathcal{O}$ is regular in R.

Let w be a maximal ideal of R, and R_w the localisation. Take any regular sequence of $\mathcal{O}$, extend it to a regular sequence of R_w, and let A be the subring of R generated by this regular sequence. Then R_w is finitely generated as an A_w-module and since R_w is Cohen-Macaulay, it follows from the miracle flatness theorem that R_w is flat over A_w, and hence is flat over $\mathcal{O}$. As every closed point localisation of R is flat over $\mathcal{O}$, it follows that R is flat over $\mathcal{O}$.

4. As $S_{G,\mathcal{O}}$ is flat over $\mathcal{O}$, the irreducible components of $S_{G,\mathcal{O}}$ are exactly those of the open subscheme $S_{G,L}$. This then follows from the proof of part 2, after noticing that as sets, $\mathcal{N}_L = \bigcup_C C_L$.

5. For $G = \mathrm{GL}_n$, recall that the centraliser $C_{\mathrm{GL}_n}(J)$ is irreducible. Then the map $p^{-1}(C) \to C$ is flat with irreducible smooth fibres, and thus is smooth, and open. Since centralisers inside GL_n are irreducible, C is irreducible, and p is open, by [Sta23, Lemma 004Z], it follows that $p^-1(C)$ is irreducible, and thus so is X_C. The final claim follows from the theory of Jordan normal forms.

6. By the previous part, $S_{G,\mathcal{O}}$ is a complete intersection, so it satisfies Serre's condition S_1. It remains to show that it has Serre's condition R_0; that is, every irreducible component has a regular point on it. This follows because the map $p^{-1}(C) \to C$ is smooth whenever C is a conjugacy class inside $\mathcal{N}_L$.

$\square$

We note here that this directly generalises the results of Hartl, Hellman and Helm categorised in Proposition 2.1 of [Hel21], which proves the above in the case $G = GL_n$ over a field of characteristic 0.

2.1 Lemmas in commutative algebra and algebraic geometry

In the remaining part of this chapter we prove some lemmas from algebraic geometry and commutative algebra that we will need later

Lemma 2.1.1. *Let G be a smooth algebraic group over a scheme S, and let X be an S scheme. Suppose we have a morphism $m : G \times_S X \to X$ defining a group action of G on X. Then m is a smooth morphism.*

Proof. First, since G is smooth, we have that $G \to S$ is smooth. Hence the projection $p_X : G \times_S X \to X$ obtained by the base change of this map to X, is a smooth morphism. Now, consider the automorphism, ϕ of $G \times_S X$ given by $(g, x) \mapsto (g, g.x)$. as this is an isomorphism, it is a smooth morphism.

Now, observe that $m = p_X \circ \phi$ is a composite of smooth morphisms, and is hence smooth. □

Lemma 2.1.2. *Let P be one of the properties of local Noetherian rings: regular, local complete intersection, Gorenstein or Cohen Macaulay. Then for (A, m) a local Noetherian ring with maximal ideal m, A is P if and only if the m-adic completion $\hat{A}$ is P.*

Proof. For the properties Cohen Macaulay and regular, this is [Sta23, Lemma 07NX] and [Sta23, Lemma 07NY] respectively. For a local complete intersection, let $A = R/\langle x_1, ..., x_k\rangle$, with R local regular. Since $\hat{R}/x_1, ..., x_n \cong \hat{A}$, and by [Sta23, Lemma 07NV], it follows easily that A is a local complete intersection ring if and only if $\hat{A}$ is. To prove the statement for the Gorenstein property, notice that A is Cohen-Macaulay if and only if $\hat{A}$ is. Hence, after quotienting by a maximal length regular sequence $(\mathbf{x})$ in A, we see that it is sufficient to prove that $A/(\mathbf{x})$ is Gorenstein if and only if $\hat{A}/(\mathbf{x}) \cong (A/\hat{(}\mathbf{x}))$ is. But since these rings are zero dimensional (and are hence, Artinian), the natural inclusion $A/(\mathbf{x}) \hookrightarrow (A/\hat{(}\mathbf{x}))$ is an isomorphism. This proves the Lemma. □

Lemma 2.1.3. *Let P be one of the local properties: regular, local complete intersection, Gorenstein or Cohen-Macaulay. Let $f : X \to Y$ be a smooth morphism of schemes. Let $p \in X$. Then Y is P at $f(p)$ if and only if X is P at p.*

Proof. Suppose f has relative dimension n. Then by [Sta23, Lemma 054L] the map f factors locally through

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & \mathbb{A}^n_Y \arrow[d] \\ & Y \end{tikzcd}$$

with g étale. Thus, it suffices to prove the lemma in the case f étale, and in the case $\mathbb{A}^n_Y \to Y$. In the étale case, since étale morphisms induce isomorphisms on the completions of stalks, and by the previous lemma, for a Noetherian local ring, R is P if and only if the completion $\hat{R}$ is P, the result of the lemma follows in the étale case. In the affine case, it suffices to note that a local ring R is P if and only if $R[x]_x$ is P. $\square$

Lemma 2.1.4. *Suppose $(\mathcal{O}, \mathfrak{p}, \mathbb{F})$ is a regular local ring and R is a Noetherian local flat $\mathcal{O}$-algebra, with $\bar{R} = R/\mathfrak{p}$. Then R is Cohen Macaulay if and only if $\bar{R}$ is Cohen Macaulay.*

Proof. Suppose $\mathcal{O}$ has dimension d, and R has dimension n. Suppose R is Cohen Macaulay. Let $x_1, ..., x_d$ be a regular sequence for $\mathcal{O}$. Then this can be extended to a maximal regular sequence for R, $x_1, ..., x_d, x_{d+1}, ..., x_n$. We see immediately that since $\mathcal{O}$ is regular, that $x_{d+1}, ..., x_n$ is a regular sequence for $\bar{R}$ of length $n - d$, and since the dimension of this is also $n - d$, we see $\bar{R}$ is Cohen Macaulay.

Suppose conversely, that $\bar{R}$ is Cohen Macaulay. Then a maximal regular sequence $\bar{y}_1, ..., \bar{y}_{n-d}$ for $\bar{R}$ can be lifted to a sequence $y_1, ..., y_{n-d}$ in R, such that $x_1, ..., x_d$, $y_1, ..., y_{n-d}$ is a regular sequence for R. The ring R is then Cohen Macaulay. $\square$

Lemma 2.1.5. *Let R be a finite local $\mathcal{O}$-algebra, and let x, $\bar{x}$ be prime ideals of R that give rise to the following commutative diagram.*

$$\begin{tikzcd} R \arrow[r, "x"] \arrow[dr, "\bar{x}"'] & \mathcal{O} \arrow[r, hook] \arrow[d, two heads] & L = \mathcal{O}[\frac{1}{l}] \\ & \mathbb{F} & \end{tikzcd}$$

Then

$$R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge} \cong R_x^{\wedge}$$

Proof. Notice that since $R\backslash x \supseteq R\backslash \bar{x} \cup \{\frac{1}{l}\}$, that $R_{\bar{x}}\left[\frac{1}{l}\right]_x \cong R_x$. Further, since R is of finite type over $\mathcal{O}$, we have $\bigcap_n \bar{x}^n = 0$, and thus we have an injection $R_{\bar{x}} \hookrightarrow R_{\bar{x}}^{\wedge}$. This gives us a local homomorphism inclusion

$$R_x = R_{\bar{x}}\left[\frac{1}{l}\right]_x \hookrightarrow R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x$$

We notice that $R_x/x \cong L$, and that

$$\left[R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x\right]/x \cong \left[\varprojlim_n (R/\bar{x}^n)/x\right][1/l] \cong \varprojlim_n (R/(x, l^n))[1/l] \cong (\varprojlim \mathcal{O}/l^n)[1/l] = L.$$

Thus, by [Sta23, Lemma 0394], we have that $R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge}$ is generated by the same topology as $R_x^{\wedge}$, and is a finite $R_x^{\wedge}$- module. It is now easy to see from looking at the residue field that the natural map

$$R_x^{\wedge} \to R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge}$$

is a surjection. It is also an injection, because the two rings have the same topology. In particular, if a sequence inside R_x converges to zero inside $R_{\bar{x}}^{\wedge}\left[\frac{1}{l}\right]_x^{\wedge}$, then it must converge to zero inside $R_x^{\wedge}$. This shows that the kernel is zero, and thus that the map is an isomorphism. □

Corollary 2.1.6. *Let Λ be a finite type $\mathcal{O}$-algebra, and let R_1, R_2 be finite type Λ-algebras, and let $R = R_1 \widehat{\otimes}_{\Lambda} R_2$. let $x \in \mathrm{Spec}(R[1/l])$ be a maximal ideal. Then $(R_1 \otimes_{\Lambda} R_2)_x^{\wedge} \cong R[1/l]_x^{\wedge}$. In particular, if $R_i[1/l]$ is regular for each i, then $R[1/l]$ is regular.*

Proof. Set $\bar{x}$ as the maximal ideal of $R_1 \otimes_{\Lambda} R_2$. Then for any x as above, we get a commutative diagram as in the statement of Lemma 2.1.5. Hence, by Lemma 2.1.5, we see that

$$(R_1 \otimes_\Lambda R_2)[1/l]^\wedge_x \cong ((R_1 \otimes_\Lambda R_2)\bar{x}^\wedge)[1/l]^\wedge_x \cong R[1/l]^\wedge_x.$$

To show the last part, it is sufficient to notice that since $R_1[1/l]\otimes_{\Lambda[1/l]} R_2[1/l], R[1/l]$ are finite type over L, they are x-adically separated, and thus are regular at x if and only if $R_1[1/l] \otimes_{\Lambda[1/l]} R_2[1/l]^\wedge_x, R[1/l]^\wedge_x$ are. Since $R_1[1/l] \otimes_{\Lambda[1/l]} R_2[1/l]$ is regular if and only if both $R_i[1/l]$ are, this completes the corollary. $\square$

Chapter 3

Smoothness results for X_p

In section 7.2 in [Bel16], Bellovin proves in the case where $\mathcal{O}$ is a field of characteristic 0, that the component X_n of $S_{\mathrm{GL}_n,\mathcal{O}}$ corresponding to the regular nilpotent orbit is smooth. The following theorem generalises this result to general connected reductive groups G, and more general regular local rings. Let $\mathcal{O}$ be a regular local ring with residue characteristic l or 0 as before. For general connected reductive groups G, we can generalise the decomposition of Proposition 2.0.6, to give $S_{G,\mathcal{O}} = \bigcup_C X_C$ where for an adjoint orbit, C, of the nilpotent cone $\mathcal{N}_G \subset \mathfrak{g}$, X_C is the closure $\overline{p^{-1}(C)}$ with $p : S_{G,\mathcal{O}} \to \mathcal{N}_G$ the natural G-equivariant projection. Note, that for more general groups G, these may not be irreducible. Indeed, if C is the regular nilpotent adjoint orbit of SL_2, then X_C is the union of two connected components. The following theorem shows that in C is a regular nilpotent conjugacy class, then X_C is smooth, and thus the connected components are the same as the irreducible components.

Theorem 3.0.1. *Let $G_{/\mathcal{O}}$ be a smooth reductive group with smooth centre, Z, and let $\mathfrak{g}$ be the Lie algebra of G, and suppose $q \in \mathcal{O}$ is considerate towards G over $\mathcal{O}$. Suppose that $C \subset \mathcal{N}_L$ is either the 0 or the regular nilpotent adjoint orbit. Then X_C is smooth over $\mathcal{O}$, and when C is the regular nilpotent orbit, X_C has the same number of connected components as Z.*

Proof. Consider first the case $C = 0$. Then $X_C = \{(\Phi, 0) \in S_{G,\mathcal{O}}\} \cong G$ via the map projecting to the Φ-coordinate. Since $G_{/\mathcal{O}}$ is smooth, this proves the theorem.

For the regular nilpotent case, note that X_C is flat and finitely generated over $\mathcal{O}$, so by [Sta23, Lemma 01V8] we have that $X_{\mathcal{O}}$ is smooth over $\mathcal{O}$ if and only if it is smooth over every localisation. It is therefore sufficient to prove the theorem after a localisation to a field. Without loss of generality, let $k = k(p)$ be a field for $p \in \mathrm{Spec}(\mathcal{O})$, and assume all subsequent schemes are schemes over k. Consider now, the case $C \subseteq \mathcal{N}$ is regular nilpotent adjoint orbit. Since qJ and J are conjugate, there is an element $\Phi_J \in G$ such that $\mathrm{Ad}(\Phi_J).J = qJ$. We claim that Φ_J is regular semisimple.

Since J is regular nilpotent, there is a unique Borel subgroup, B, such that $J \in \mathrm{Lie}(B)$. Let $\Pi = \{\alpha_1, ..., \alpha_h\}$ be the corresponding set of simple roots of G, and let $\{e_\alpha\} \in \mathfrak{g}$ be the set of eigenvectors of $\mathfrak{g}$ corresponding to the roots of G. We can write $J = \sum_{\alpha\in\Pi} c_\alpha e_\alpha \in \mathfrak{g}$ for $c_\alpha \neq 0$. Hence, we see

$$\sum_{\alpha\in\Pi} qc_\alpha e_\alpha = qJ = \mathrm{Ad}(\Phi_J)J = \sum_{\alpha\in\Pi} c_\alpha \alpha(\Phi_J)e_\alpha$$

and so $\alpha(\Phi_J) = q$ for every simple root α. If β is a positive root of G, we see that β is some positive combination of the α_i. Suppose $\beta = \sum_i m_i\alpha_i$. Then $\beta(\Phi_J) = q^{m_1+...+m_h}$. As q is considerate towards G over $\mathcal{O}$ (and hence is considerate towards G over k), we see that no $\beta(\Phi_J) = 1$. Hence Φ_J is regular semisimple by Lemma 12.2 of [Bor91].

Since Φ_J is regular semisimple, it is contained in a unique torus $T \subset G$. Consider the k-scheme

$$Y = Z\Phi_J \times \overline{T.J}.$$

We first claim that this is a subscheme of X_C. Let $(s\Phi_J, \mathrm{Ad}(t).J) \in Z\Phi_J \times T.J$. Then

$$\begin{aligned}
\mathrm{Ad}(s\Phi_J)(\mathrm{Ad}(t)J) &= \mathrm{Ad}(s\Phi_J t)J \\
&= \mathrm{Ad}(t\Phi_J s)J && \text{because } T \text{ is abelian} \\
&= \mathrm{Ad}(t)\mathrm{Ad}(\Phi_J)J
\end{aligned}$$

$$= \mathrm{Ad}(t)(qJ)$$
$$= q\mathrm{Ad}(t)J.$$

Hence, $Z\Phi_J \times T.N \subset X_C$. Since X_C is closed, we then see that the closure $\overline{Z\Phi_J \times T.J} = Z\Phi_J \times \overline{T.N} = Y \subset X_C$.

We now claim that Y is smooth over k. This is clear, because $Z_{/\mathcal{O}}$ is smooth by hypothesis and $\overline{T.J} = \mathrm{Span}(e_{\alpha_1}, ..., e_{\alpha_h})$ is isomorphic to affine space, $\mathbb{A}_k^h$. Define the morphism

$$f : G \times Y \to X_C$$
$$(g, (\Phi, N)) \mapsto (g\Phi g^{-1}, \mathrm{Ad}(g)N).$$

Consider the following commutative diagram

$$\begin{array}{ccc} G \times Y & \longrightarrow & X_C \\ \downarrow & & \downarrow \\ G \times Z\Phi_J & \longrightarrow & Z.G_{\Phi_J} \end{array}$$

where G_{Φ_J} denotes the conjugacy class of Φ_J in G, the vertical maps come from the “forget N" projections $(g, s\Phi_J, N) \in G \times Y \mapsto (g, s\Phi_J) \in G \times Z\Phi_J$ and $(\Phi, N) \in X_C \mapsto \Phi \in ZG_{\Phi_J}$ respectively and the horizontal maps are defined via the conjugation action of $g \in G$ on Y so that the diagram commutes, and is a pullback square. The bottom map, m, is flat with fibres isomorphic to $\mathrm{Stab}_G(\Phi_J)$, which is simply the Torus T, as Φ_J is regular semisimple. This shows that m is smooth. Hence, since the map f is the base change of m to X_C, by Proposition 10.1 of [Har77] we see that f is smooth.

Then by Lemma 2.1.3, since every point on $G \times Y$ is regular, this implies that its image in X_C is a smooth variety. To finish the proof, it is enough to show that this map is surjective. This is the same as saying that every pair $(\Phi, N) \in X_C$ is conjugate to something in Y.

Let $(\Phi', N) \in |X_C|$. Then there exists a regular nilpotent J' such that $\mathrm{Ad}(\Phi')J' = qJ'$. Then J' is conjugate to J by some element $g \in G_{/\mathcal{O}}$ (i.e. $\mathrm{Ad}(g)J' = J$). Then if $\Phi = g\Phi' g^{-1}$, we see $\mathrm{Ad}(\Phi)J = qJ$. By conjugating by an element of $\mathrm{Stab}_G(J)$, we can assume without loss of generality that Φ lies in T. Hence, $s = \Phi\Phi_J^{-1}$ is an element of $\mathrm{Stab}_T(J)$. We claim that $\mathrm{Stab}_T(J) = Z$. It is clear that there is a closed immersion $Z \subseteq \mathrm{Stab}_T(J)$, so we need only show this is surjective (as Z is smooth). Since $s \in \mathrm{Stab}_T(J)$ commutes with J, we see that $\mathrm{Ad}(s)J = J$, and thus $\sum_{\alpha\in\Pi} c_\alpha \alpha(s) e_\alpha = \sum_{\alpha\in\Pi} c_\alpha e_\alpha$. Since e_α form a basis of $\mathfrak{g}$, we see that $\alpha(s) = 1$ for each $\alpha \in \Pi$. Since this is a base, we see that $\beta(s) = 1$ for all roots β of G. Hence, s acts as the identity on the adjoint representation, and so lies in the centre $s \in Z$. Since $\mathrm{Ad}(g)N$ conjugates with Φ in the correct way, we see that N is a span of simple roots of G, and thus lies in $\overline{T.J}$. This shows that (Φ', N) is the image of $(g^{-1}, (A\Phi_J, \mathrm{Ad}(g)N)) \in G \times Y$. This proves the smoothness statement.

For the statement about the connected components, it suffices to notice that since G is connected, that the connected components of $G \times Y$ biject with those of Y, which in turn biject with the connected components of Z. Hence it suffices to show that there is a bijection between the connected components of $G \times Y$ and X_C. It is sufficient to show that the fibres of the G equivariant map $f : G \times Y \to X_C$ are connected. Since the action of G gives an isomorphism on fibres, it is sufficient to show that the fibres of $Y \subseteq X_C$ are connected. Let $P = (\Phi, N) \in Y$. Then $f^{-1}(P) = \{(g, \Phi', N') \in G \times Y : g\Phi' g^{-1} = \Phi \text{ and } \mathrm{Ad}(g)(N') = N\}$. Since $\Phi, \Phi' \in Z\Phi_J \subset T$ are regular semisimple, any $g \in G$ such that $g\Phi g^{-1} = \Phi'$ lies in the normaliser $N_G(T)$. Notice that for any simple root α of G, $\alpha(g\Phi g^{-1}) = \alpha(\Phi') = q = \alpha(\Phi)$. This implies that g must actually lie in $Z_G(T) = T$, and thus we get a well defined isomorphism

$$\begin{aligned} f^{-1}(P) &\leftrightarrow T \\ (g, \Phi', N') &\mapsto g \\ (g, \Phi, \mathrm{Ad}(g)^{-1}(N)) &\leftarrow\!\shortmid g \end{aligned}$$

Thus, since T is connected, so is $f^{-1}(P)$. This proves the final part of the theorem.

□

The conditions that G has smooth centre and that $q \in \mathcal{O}$ is considerate towards $G/\mathcal{O}$

are quite mild conditions. For example, if $\mathcal{O}$ is a field of characteristic 0 and q isn't a root of unity, q is automatically considerate. Further, when $q \in \mathbb{Z}$ is a prime power, if the residue characteristic, l, of $\mathcal{O}$ is larger than $q^{t(G)}$, then q is considerate. Since the centre of a reductive group G is smooth in large enough characteristic, this also shows that X_C is smooth over $\mathcal{O}$ with sufficiently large residue characteristic.

One may hope that the previous result holds for all components of S_G. i.e. that all components of S_G are smooth. When $G = \mathrm{GL}_2$, this is true since the only two components are those arising from the nilpotent conjugacy classes of $N = 0$, and $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and both cases studied in the previous theorem, (see also proposition 4.8.1 of [Pil08]). In [Bel16], Bellovin proves that this fails for GL_3, demonstrating that the component X_{21} is not smooth, and gives a description of all the points where singularities exist. Theorem 3.0.3 generalises these results, and shows us that, for $G = \mathrm{GL}_n$ and any partition $p \neq 1^n, n$, the component X_p is always singular.

We define some notation. For a an element of an $\mathcal{O}$-algebra R, and k a positive integer, define the $k \times k$ matrix,

$$M_k(a) = \begin{pmatrix} aq^{k-1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & aq & 0 \\ 0 & \dots & 0 & a \end{pmatrix}.$$

If k is a positive integer, and $\underline{b} = (b_1, ..., b_{k-1}) \in R^{k-1}$ are a $k-1$-tuple of elements of R, then set the $k \times k$ matrix

$$J_k(\underline{b}) = \begin{pmatrix} 0 & b_1 & & \\ & 0 & b_2 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}.$$

Lemma 3.0.2. *Let R be a finitely generated $\mathcal{O}$-algebra. Let $p = k_1 + k_2 + ... + k_m$ be a partition of n. For $a_i \in R^\times$, and $\underline{b}_i \in R^{k_i-1}$ the pair*

$$\left(\begin{pmatrix} M_{k_1}(a_1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & M_{k_{m-1}}(a_{m-1}) & 0 \\ 0 & \dots & 0 & M_{k_m}(a_m) \end{pmatrix}, \begin{pmatrix} J_{k_1}(\underline{b}_1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & J_{k_{m-1}}(\underline{b}_{m-1}) & 0 \\ 0 & \dots & 0 & J_{k_m}(\underline{b}_m) \end{pmatrix}\right) \in X_p(R).$$

Proof. When each of the vectors $\underline{b}_i$ lie in $R^\times$, the pair

$$(\Phi, N_\lambda) = \left(\begin{pmatrix} M(k_1,a_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M(k_m,a_m) \end{pmatrix}, \begin{pmatrix} \lambda J_{k_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda J_{k_m} \end{pmatrix} \right) \in p^{-1}(C_p)(R)$$

is inside $X_p(R)$. Hence, we obtain a morphism of schemes over R:

$$\pi' : \mathbb{G}_{m,R}^{n-m} \to p^{-1}(C_p)_R$$
$$(\underline{b}_1, \dots, \underline{b}_m) \mapsto \left(\begin{pmatrix} M(k_1,a_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M(k_m,a_m) \end{pmatrix}, \begin{pmatrix} J_{k_1}(\underline{b}_1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_{k_m}(\underline{b}_m) \end{pmatrix} \right)$$

which is an isomorphism onto it's scheme theoretic image and which extends naturally to a map $\pi : \mathbb{A}_R^{n-m} \to S_{n,R}$. Since the Zariski closure of $\mathbb{G}_{m,R}^{n-m}$ inside $\mathbb{A}_R^{n-m}$ is $\mathbb{A}_R^{n-m}$, we see that the Zariski closure of the image of π' inside $S_{n,R}$ is the image of π. Since $X_{p,R}$ is the Zariski closure of $p^{-1}(C_p)_R$, it follows that $X_{p,R}$ contains the image of π. The lemma then follows by looking at the R points of the image of π and $S_{n,R}$. $\square$

Theorem 3.0.3. *Let $G = GL_n$, and let p be a partition of n with $p \neq 1^n, n$. Then X_p is singular.*

Proof. Let $\mathbb{F}$ be the residue field of $\mathcal{O}$. Consider the following Cartesian diagram

$$\begin{array}{ccc} X_{p,\mathbb{F}} & \longrightarrow & \mathrm{Spec}(\mathbb{F}) \\ \downarrow & & \downarrow \\ X_{p,\mathcal{O}} & \longrightarrow & \mathrm{Spec}(\mathcal{O}) \end{array}$$

If the map $X_{p,\mathcal{O}} \to \mathrm{Spec}(\mathcal{O})$ were smooth, then by Proposition 10.1b) of [Har77] the map $X_{p,\mathbb{F}} \to \mathrm{Spec}(\mathbb{F})$ would also be smooth. Hence, without loss of generality, it suffices to show that $X_{p,\mathcal{O}}$ is singular when $\mathcal{O} = \mathbb{F}$ a field.

Choose any point $P = (\Phi_0, 0) \in X_p$, with Φ_0 semisimple. Define three subvarieties of S_n that contain P as follows.

1. Let $C = \mathrm{GL}_n.P$, be the GL_n-orbit of P.

2. Let D be the variety of diagonal matrices inside GL_n, seen as a subvariety of S_n via the inclusion $\Phi \mapsto (\Phi, 0)$.

3. Let $\mathcal{N}_0 = \{N \in \mathfrak{gl}_n : \Phi_0 N \Phi_0^{-1} = qN\}$ viewed as a closed subvariety of S_n via the inclusion $N \mapsto (\Phi_0, N)$.

Let $\mathbb{F}[\epsilon]$ be the ring of dual numbers. The first claim we make, is that the tangent space $T_P C$ can be identified with the elements of $X_p(k[\epsilon])$ that are $\mathrm{GL}_n(\mathbb{F}[\epsilon])$-conjugate to P, and have image P under the base change of the natural map $\mathrm{Spec}(\mathbb{F}) \to \mathrm{Spec}(\mathbb{F}[\epsilon])$ which sends $\epsilon \mapsto 0$. Note that we have a smooth surjective morphism $\mathrm{GL}_n \to C$, given by the conjugation action $g \mapsto g.P$, and so we have a surjection on the level of tangent spaces and a surjection $\mathrm{GL}_n(\mathbb{F}[\epsilon]) \to C(\mathbb{F}[\epsilon])$. This shows that any element of $C(\mathbb{F}[\epsilon])$ is conjugate to P via some element of $\mathrm{GL}_n(\mathbb{F})$. The rest of the claim is obvious.

Consider the tangent spaces of these varieties at P, $T_P C$,$T_P D$ and $T_P \mathcal{N}_0$. We claim that they form a direct sum inside $T_P S_n$. Let $P' = (\Phi', 0) \in T_P C \cap T_P D$. Then Φ' is a diagonal matrix in $\mathrm{GL}_n(\mathbb{F}[\epsilon])$, and is conjugate to Φ_0. Since diagonal matrices are only conjugate to each other if they share the same entries, this means that Φ' lies inside $\mathrm{GL}_n(\mathbb{F})$, and thus, $P' = P$. To show that $T_P \mathcal{N}_0$ intersects at the origin with $T_P C$ or $T_P D$, it suffices to notice that in either case, an element of $T_P C$ or $T_P D$ takes the form $P' = (\Phi', 0)$, while an element $P' \in T_P \mathcal{N}_0$ takes the form $P' = (\Phi_0, N) \in S_n(\mathbb{F}[\epsilon])$. For these to be equal, we must have $\Phi' = \Phi_0$ and $N = 0$, so $P' = P$. This proves the claim.

We split the proof of this theorem into two cases: the case where the parts of p are not all the same and the case where $p = k^m$ for integers $k, m > 1$ such that $km = n$. In both cases, the following strategy will be to count the number of linearly independent deformations in each of the subspaces of $T_P X_p$, $T_P C$, $T_P D \cap T_P X_p$ and $T_P \mathcal{N}_0 \cap T_P X_p$ and combine to give a lower bound on the dimension of $T_P X_p$, showing that $\dim_{\mathbb{F}} T_P > n^2 = \dim X_p$. This will prove the theorem.

Consider the case $p = (k_1, ..., k_m)$ with $k_1 \geqslant k_2 \geqslant ... \geqslant k_m$, not all equal. Consider the $n \times n$ diagonal matrix, $\Phi_0 = \mathrm{Diag}(q^{n-1}, ..., q, 1)$. Notice that Φ_0 has distinct eigenvalues, so that the stabiliser of $P = (\Phi_0, 0)$ is the n dimensional torus T_n.

By orbit-stabiliser, we then note that the orbit space must be $n^2 - n$ dimensional, and thus $\dim_{\mathbb{F}}(T_P C) \geqslant n^2 - n$. Consider now the deformations in $T_P\mathcal{N}_0$. Let $(\Phi_0, M\epsilon) \in X_p(\mathbb{F}[\epsilon]) \subseteq S_n(\mathbb{F}[\epsilon])$. The defining equation of $S_{n,\mathbb{F}}$ shows that all non-zero entries of M must lie on the off-diagonal. Further, to ensure $(\Phi_0, M\epsilon)$ lies on the component defined by p, one may choose, in accordance with Lemma 2.1.2, M as a block diagonal matrix, with blocks of size $k_1, k_2, ..., k_m$, each of the form

$$\begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & & \\ & & & 0 & * \\ & & & & 0 \end{pmatrix}$$

This leaves us with $\sum_i (k_i - 1) = n - m$ different non-zero entries of M, each of which defines a deformation, all of which are linearly independent, because they are inside $T_P(\mathrm{GL}_n \times \mathfrak{gl}_n) = \mathfrak{gl}_n^2$. Finally, consider the blocks of Φ defined by the partition p. For each $1 \leqslant i \leqslant m$, consider the matrix

$$E_i = \begin{pmatrix} I_{k_1} & & & & & \\ & I_{k_2} & & & & \\ & & \ddots & & & \\ & & & (1+\epsilon)I_{k_i} & & \\ & & & & \ddots & \\ & & & & & I_{k_m} \end{pmatrix} \in M_n(\mathbb{F}[\epsilon])$$

where I_k denotes the $k \times k$ identity matrix.

We consider the deformation $(\Phi E_i, 0)$ and note that this is contained in $X_p(\mathbb{F}[\epsilon])$ via Lemma 2.1.2, because we can split ΦE_i into block diagonal parts of sizes $k_1, ..., k_m$. This gives us m further deformations, which are similarly linearly independent because they are linearly independent inside $T_P(\mathrm{GL}_n \times \mathfrak{gl}_n)$. Finally, we note that we may reorder the blocks of the partition p, to give us the deformation $(\Phi E_{m+1}, 0)$ where

$$E_{m+1} = \begin{pmatrix} (1+\epsilon)I_{k_m} & \\ & I_{n-k_m} \end{pmatrix} \in M_n(R[\epsilon])$$

By the same reasoning, this deformation also lies on $X_p(\mathbb{F}[\epsilon])$, and since $k_m < k_1$, we see this adds a genuinely new deformation inside $T_P D$, because the deformations $\{(\Phi E_i) : 1 \leqslant i \leqslant m+1\}$ are all linearly independent in $T_P(\mathrm{GL}_n \times \mathfrak{gl}_n)$.

Piecing everything together, we have at least $(n^2 - n) + (n - m) + m + 1 = n^2 + 1 >$ $\dim(X_p)$ linearly independent deformations, which exceeds the dimension of the variety X_p. We conclude that $\dim_{\mathbb{F}} T_P X_p \geqslant \dim X_p$ and that P is a singular point.

Now, in the case $p = k^m$, we instead choose a point

$$(\Phi, 0) = \left(\begin{pmatrix} M(k, q^{k(m-1)-1}) & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & M(k, q^{k-1}) & 0 \\ 0 & \ldots & 0 & M(k, 1) \end{pmatrix}, 0 \right) \in X_p(R).$$

so that Φ is a diagonal matrix, with increasing powers of q going up the diagonal, with a single power of q repeated, that being q^{k-1}. Then the conjugation orbit is $n^2 - (n - 2 + 4) = n^2 - n - 2$ dimensional. The $T_P\mathcal{N}_0$-space deformations give us again, $(k-1)m$ deformations on the off-diagonal, and an additional two in the entries marked with a $\square$ below, appearing because of the repeated power of q in Φ

$$\left(\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \ldots & q^k & & & & \ldots \\ \ldots & & q^{k-1} & & & \ldots \\ \ldots & & & q^{k-1} & & \ldots \\ \ldots & & & & q^{k-2} & \ldots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ \ldots & 0 & * & \square & & \ldots \\ \ldots & & 0 & 0 & \square & \ldots \\ \ldots & & & 0 & * & \ldots \\ \ldots & & & & 0 & \ldots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right) \in X_p(R).$$

Each of these deformations lie inside $T_P X_p$ because they are conjugate inside $\mathrm{GL}_{n, \mathbb{F}[\epsilon]}$ to pairs in the form of Lemma 2.1.2.

Now if we define E_i as before, for $i \leqslant m$, we see by the lemma that $(\Phi E_i, 0) \in X_p(R)$ for each i, and this gives us another m deformations. Finally, let E_{m+1} be as follows:

$$E_{m+1} = \begin{pmatrix} I_{k(m-1)} & & & \\ & 1 + \epsilon & & \\ & & 1 & \\ & & & (1+\epsilon) I_{k-1} \end{pmatrix} \in M_n(\mathbb{F}[\epsilon]).$$

Then, because ΦE_{m+1} is conjugate to something of the form in Lemma 2.1.2, it lies inside $X_p(\mathbb{F}[\epsilon])$. Notice that the deformations ΦE_i for $i = 1, \ldots, m+1$ are linearly independent, because they are linearly independent inside $T_P(\mathrm{GL}_n) \supseteq T_P D$. This gives a total of $(n^2-n-2)+((k-1)m+2)+(m+1) = n^2-n+mk+1 = n^2+1 > n^2 = \dim(X_p)$ deformations, and shows that X_p is singular at $(\Phi, 0)$. $\square$

Chapter 4

Φ-Regular points of $X_{\leqslant p}$ are Cohen-Macaulay

In this chapter, we take a closer study of certain unions of irreducible components of $S_{n,\mathcal{O}}$ which appear as the support of certain maximal Cohen-Macaulay sheaves that appear as the outputs of patching functors.

4.1 Motivation

Let $F/\mathbb{Q}_l$ be a finite field extension as before. Let W_F be the Weil group of F and let I_F be the inertia subgroup.

Recall the dominance partial order on the set of partitions of n, which can be defined as follows: For p and q two partitions of n, we say $q \leqslant p$ if their corresponding nilpotent conjugacy classes C_q and C_p inside the nilpotent cone, $\mathcal{N}$, satisfy $C_q \subseteq \bar{C}_p$. Equivalently, if $p = (p_1, ..., p_k)$ and $q = (q_1, ..., q_m)$ and we adopt the conventions that $q_i = 0$ if $i > m$ and $p_i = 0$ if $i > k$, then $q \leqslant p$ if and only if for every $j \in \mathbb{N}$, $\sum_{i=1}^{j} q_i \leqslant \sum_{i=1}^{j} p_i$.We can make the following definition.

Definition 4.1.1. *For a given partition p of n, let $X_{\leqslant p} := \bigcup_{q \leqslant p} X_q \subseteq S_n$.*

We present a little motivation why these varieties are interesting to study.

Definition 4.1.2. *An inertial type is an isomorphism class of continuous representations $\tau : I_F \to GL(V)$ where V is a finite dimensional $E = \bar{\mathbb{Q}}_l$-vector space, that extends to a representation of W_F. A basic inertial type is an inertial type, that extends to an irreducible representation of W_F. Let $\mathcal{I}_0$ be the set of all basic inertial types.*

Let Part_n be the set of all partitions of n, and $\mathrm{Part} = \bigcup_n \mathrm{Part}_n$. In [Sho18] it is shown that there is a bijection between inertial types and the set $\mathcal{I}$ of all functions

$$\mathcal{P} : \mathcal{I}_0 \to \mathrm{Part}$$

of finite support, where and Part is the set of all partitions. We will denote the partition corresponding to $\tau \in \mathcal{I}_0$ by $\mathcal{P}_\tau$. For a partition $p \in \mathrm{Part}$, we say that the degree $\deg(p)$ is the number n that p partitions. We can extend deg to the set $\mathcal{I}$ by

$$\deg(\mathcal{P})(\tau_0) = \deg(\mathcal{P}(\tau_0))$$

and we can extend the dominance ordering on Part by saying that two inertial types $\mathcal{P}$ and $\mathcal{Q}$ have $\mathcal{P} \geqslant \mathcal{Q}$ if and only if they have the same degree, and if $\mathcal{P}(\tau_0) \geqslant \mathcal{Q}(\tau_0)$ for each $\tau_0 \in \mathcal{I}_0$.

Let $\bar{\rho} : G_F \to \mathrm{GL}_n(\bar{\mathbb{F}})$ be a representation with inertial type τ. Let $R^\square(\bar{\rho})$ be its framed deformation ring. and let $R^\square(\bar{\rho}, \tau)$ be the framed fixed inertial type deformation ring.

In chapter 6 of [EGS15] (see section 6.1 for full details), the notion of a patching functor (at least in the GL_2 case, though this notion can be generalised to more general connected reductive groups) is defined as an exact covariant functor M_∞ from the category of $K = \mathrm{GL}_n(\mathcal{O})$ representations on finite free $\mathcal{O}$-modules to the category of coherent sheaves on a certain space $X_\infty = \mathrm{Spec}(\widehat{\bigotimes}_v R_v^\square[[x_1, ..., x_h]])$ a finite product of local deformation rings, with certain properties. One of the properties we expect is that a certain K-representation $\sigma(\tau)$ (arising naturally from an inertial type τ) has the coherent sheaf $M_\infty(\sigma(\tau))$ supported on the closed subscheme $X_\infty(\tau)$, of points in X_∞ with inertial type $\leqslant \tau$. Further, $M_\infty(\sigma(\tau))$ is maximal Cohen-Macaulay over $X_\infty(\tau)$. We may hope then, since spaces arise as the supports of these patching functors, that the spaces $X_\infty(\tau)$ may be Cohen-Macaulay. This would happen if we can prove that each $X_{\leqslant p}$ is Cohen-Macaulay.

4.2 The main theorem

Let L be the fraction field of $\mathcal{O}$ as before. Let $\mathcal{N}_n \subseteq \mathfrak{gl}_n$ be the nilpotent cone. Recall there is a GL_n-equivariant morphism, given by the second projection, $p_2 : S_{\mathrm{GL}_n} \to \mathcal{N}_n$. For each partition p, we can find the locally closed subspace $C_p \subseteq \mathcal{N}_n$ given by the preimage of the conjugacy class given by p inside $(\mathcal{N}_n)_L$ through the flat morphism $\mathcal{N}_n \hookrightarrow (\mathcal{N}_n)_L$. Then $\bar{C}_p$ is a union of conjugacy classes in $\mathcal{N}_n$, and $\bar{C}_p = \bigcup_{q \leqslant p} C_q$. We may henceforth view $X_{\leqslant p} = p_2^{-1}(\bar{C}_p)$ as the preimage of $\bar{C}_p$ under the projection p_2. This is advantageous, as it shows us that any additional equations specifying $X_{\leqslant p}$ as a subspace of S_n need only have equations in the variables of N (namely, those equations that define the subvariety $\bar{C}_p$).

Definition 4.2.1. *We define $X_{\leqslant p}^{\Phi\text{-}reg} \subseteq X_{\leqslant p}$ to be the open subscheme over $\mathcal{O}$ defined as the complement of the equation Disc$(\chi_\Phi(X)) = 0$.*

Remark. Let $P \in |X_{\leqslant p}|$ lie in the fibre of a prime $\mathfrak{p} \in \mathrm{Spec}\ \mathcal{O}$ with residue field $K = k(\mathfrak{p})$ and separable closure K^{sep}, and suppose P corresponds to a (Gal_K-equivalence class of) pair of matrices $(\Phi, N) \in X_{\leqslant p}(K^{\mathrm{sep}})$. We notice that $P \in |X_{\leqslant p}^{\Phi\text{-reg}}|$ if and only if $\mathrm{Disc}(\chi_\Phi(X)) \notin \mathfrak{p}$, which occurs if and only if $\mathrm{Disc}(\chi_\Phi(X)) \neq 0$ inside the field $k(P)$, which is equivalent to the eigenvalues of Φ being distinct inside a separable closure $k(P)^{\mathrm{sep}}$, by virtue of $\mathrm{char}(k(P)) = 0$ or $l > n$.

Theorem 4.2.2. *Suppose that q is considerate towards GL_n over $\mathcal{O}$. Let p be a partition of n. Then $X_{\leqslant p}^{\Phi\text{-}reg}$ is Cohen-Macaulay.*

To approach this problem, we start by reducing the question to a ring R_P (to be defined) with which we can make explicit calculations.

Let $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}$, and let $K = k(\mathfrak{p})$. Choose a separable closure K^{sep} as before, and let $P \in |(X_{\leqslant p})|$ lie above $\mathfrak{p}$ correspond to a pair of matrices $(\Phi, N) \in X_{\leqslant p}^{\Phi\text{-reg}}(K^{\text{sep}})$. We may assume without loss of generality that $P = (\Phi, 0)$ with Φ semisimple. This is because the set of non-Cohen-Macaulay points is a closed subspace of $X_{\leqslant p}$. If $P = (\Phi, N) \in X_{\leqslant p}$ is a non-Cohen-Macaulay point, then the action of GL_n on $X_{\leqslant p}$ provides an isomorphism of local rings of any two points in the orbit of P. Thus, any point in the orbit of P is non-Cohen-Macaulay. Further, the semisimplification $(\Phi^{\text{s.s.}}, 0)$ is contained inside the closure of the orbit of P, and thus, $(\Phi^{\text{s.s.}}, 0)$ is also a non-Cohen-Macaulay point. As a consequence, if we show that every point $(\Phi, 0)$ with Φ semisimple is Cohen-Macaulay, we can deduce that $X_{\leqslant p}$ is Cohen-Macaulay, and thus we can reduce our attention to points of this form.

Let M be the stabiliser of Φ (necessarily M is of the form $M = \prod_{i=1}^{m} \mathrm{GL}_{k_i}$). We may assume that Φ has the form of a block diagonal matrix
$\Phi = \operatorname{Diag}(a_1 I_{k_1}, a_2 I_{k_2}, ..., a_m I_{k_m})$ where I_k are $k \times k$ identity matrices, and all the a_i are distinct with an ordering chosen such that $a_i/a_j = q$ inside K^{sep} implies that $j = i + 1$.

We set V_M to be the subscheme of $X_{\leqslant p}$, flat over $\mathcal{O}$ defined as $\{(\Phi, N) \in M \times \mathfrak{gl}_n : \Phi N \Phi^{-1} = qN \text{ and } N \text{ has conjugacy class} \leqslant p\}$. We now set $R_P := \mathcal{O}_{V_M, P}$ to be the local ring at P of this space.

Lemma 4.2.3. *Let $\mathcal{P}$ be one of the properties of local rings: smooth/ a local complete intersection/ Gorenstein/ Cohen-Macaulay. The scheme $X_{\leqslant p}$ is $\mathcal{P}$ at P if and only if R_P is $\mathcal{P}$ at P.*

Proof. We have a pullback diagram of $\mathcal{O}$-schemes

$$\begin{array}{ccc} \mathrm{GL}_n \times V_M & \longrightarrow & X_{\leqslant p} \\ \downarrow & & \downarrow \\ \mathrm{GL}_n \times M & \longrightarrow & \mathrm{GL}_n \end{array}$$

where the map horizontal maps are given by conjugation $(g, x) \mapsto gxg^{-1}$, and the vertical maps are 'forget the second coordinate'. Localising and completing along maximal ideals gives us a pushout diagram of complete local rings as follows:

$$\begin{array}{ccc} k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} R^\wedge_P & \longleftarrow & R^\wedge \\ \uparrow & & \uparrow \\ k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P & \longleftarrow & k[\mathrm{GL}_n]^\wedge_P \end{array}$$

with R the local ring of P on $X_{\leqslant p}$ Since this is a pushout diagram, the top map is smooth if the bottom map is smooth. We claim that the bottom map is smooth. Let $\mathcal{C}_\mathcal{O}$ be the category of complete Noetherian local $\mathcal{O}$-algebras with residue field k. We have $T := k[\mathrm{GL}_n]^\wedge_P \cong \mathcal{O}[[X_1, ..., X_{n^2}]]$ represents the functor on $\mathcal{C}_\mathcal{O}$ given by $A \in \mathcal{C}_\mathcal{O}$ maps to those elements of $\mathrm{GL}_n(A)$ which map to P in $\mathrm{GL}_n(k)$. This is the same as the set $P + \mathfrak{gl}_n(m_A)$, where m_A is the maximal ideal of A. likewise, $k[M]^\wedge_P \cong \mathcal{O}[[Y_1, ..., Y_{\dim M}]]$ represents the functor $A \mapsto P + \mathrm{Lie}(M)(m_A)$.

Consider $A = k[t]/t^2 \in \mathcal{C}_\mathcal{O}$, then the map of Zariski tangent spaces

$$\begin{aligned} [I + \mathfrak{gl}_n(m_A)] \times [P + \mathrm{Lie}(M)(m_A)] \to & P + \mathfrak{gl}_n(m_A) \\ (I + m, P + x) \mapsto & (I + x)(P + m)(I + x)^{-1} \\ & = P + [x, P] + m) \end{aligned}$$

is a surjection because $M = \mathrm{Stab}(P)$.

This provides us with an injection $m_T/m_T^2 \to m/m^2$ where m_R is the maximal ideal of $T = k[\mathrm{GL}_n]^\wedge_P$ and m is the maximal ideal of $k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P$. Let $T_1, ..., T_r$ be a set of elements of m such that they form a basis of $(m/m^2)/(m_R/m_R^2)$. Then, since T and $k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P$ are both power series rings, we see that $k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} k[M]^\wedge_P = R[[T_1, ..., T_r]]$. This shows that the bottom map is smooth, and hence that the top map is smooth.

As a result,

$$R^\wedge_P[[X_1, ..., X_{n^2}]] \cong k[\mathrm{GL}_n]^\wedge_I \widehat{\otimes} R^\wedge_P$$

is a power series ring in $R^\wedge$. Thus, if $\mathcal{P}$ is one of the properties in the lemma, we see that R is $\mathcal{P}$ if and only if $R^\wedge$ is $\mathcal{P}$ via Lemma 2.1.2, if and only if $R^\wedge_P[[X_1, ..., X_{n^2}]]$ is $\mathcal{P}$ if and only if R_P is $\mathcal{P}$. This completes the lemma. $\square$

Thus, to show that $X_{\leqslant p}^{\Phi\text{-reg}}$ is Cohen Macaulay at $P \in X_{\leqslant p}^{\Phi\text{-reg}}$ it suffices to show that R_P is Cohen-Macaulay. We now give an explicit description of R_P.

Consider the universal coordinates of V_M which (in block matrix form blocks of size $k_1, ..., k_m$:

$$\left(\begin{pmatrix} a_1(I_{k_1}+M_1) & 0 & ... & 0 \\ 0 & a_2(I_{k_2}+M_2) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_m(I_{k_m}+M_m) \end{pmatrix}, \begin{pmatrix} b_{1,1} & b_{1,2} & ... & b_{1,n} \\ b_{2,1} & b_{2,2} & ... & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & ... & b_{n,n} \end{pmatrix}\right)$$

Where each M_i is a $k_i \times k_i$ matrix, and each $b_{i,j}$ is a $k_i \times k_j$ matrix.

The equation $\Phi N = qN\Phi$ gives us the following for each (i, j)

$$a_i(I_{k_i} + M_i)b_{i,j} = qa_j b_{i,j}(I_{k_j} + M_j) = 0$$

which in turn give us

$$(a_i - qa_j)b_{i,j} + a_i M_i b_{i,j} - qa_j b_{i,j} M_j = 0$$

when $a_i - qa_j$ is non-zero in K^{sep}, it is invertible inside $\mathcal{O}_{\mathfrak{p}}$, Hence

$$b_{i,j} = -(a_i - qa_j)^{-1} a_i M_i b_{i,j} + (a_i - qa_j)^{-1} qa_j b_{i,j} M_j.$$

Let I be the ideal of R_P generated by the coordinates of $b_{i,j}$. Then we see from the above equation that $I = mI$ where m is the maximal ideal of R_P. Consequently by Nakayama's lemma, we see that $I = 0$.

Thus, $b_{i,j} = 0$ unless $j = i + 1$ and $a_i = qa_{i+1}$ in K^{sep}. When $a_i - qa_{i+1} \in \mathfrak{p}$, set $\pi = a_i^{-1}(a_i - qa_{i+1}) \in \mathfrak{p}$, then we get that the equations given by $\Phi N = qN\Phi$ give us exactly

$$(M_i b_{i,i+1} - b_{i,i+1} M_{i+1}) + \pi b_{i,j}(I + M_{i+1}) = 0$$

inside V_M. We will, from now on, write $N_i := b_{i,i+1}$

As a result, we get the following expression for R_P:

$$\frac{\mathcal{O}_{\mathfrak{p}}[M_1, ..., M_m, N_1, ..., N_{m-1}]}{\langle \{M_i b_{i,i+1} - b_{i,i+1} M_{i+1} + \pi N_i (I + M_{i+1}) : i < m\} \cup \{\text{some equations only in } N_i\} \rangle}$$

Where the equations in the coordinates of N_i are those that describe the conjugacy classes inside $\bar{C}_p$. As $\mathcal{O}_{\mathfrak{p}}$ is regular, and R_P is a Noetherian flat local $\mathcal{O}_{\mathfrak{p}}$-algebra, by Lemma 2.1.4 we see that R_P is Cohen Macaulay if and only if the ring

$$\bar{R}_P = \frac{K[M_1, ..., M_m, N_1, ..., N_{m-1}]}{\langle \{M_i b_{i,i+1} - b_{i,i+1} M_{i+1} : i < m\} \cup \{\text{some equations only in } N_i\} \rangle}$$

is Cohen Macaulay. Hence we reduce the problem to showing that $\bar{R}_P$ is Cohen Macaulay.

When $P = (\Phi, 0)$ is Φ -regular, the M_i and N_i are 1×1-matrices and thus commute, and so we can simplify even further. By setting $\lambda_i = M_i - M_{i+1}$, we see that that $\lambda_i N_i = 0$. We hence have reduced the problem to proving that this explicit $\bar{R}_P$ is Cohen-Macaulay, and have proven most of the following lemma:

Lemma 4.2.4. *For $S \subseteq \{1, ..., n-1\}$, define $N_S := \prod_{i \in S} N_i$. Let P be Φ-regular, and let $\bar{R}_P$ be as above. Then there exists a family $\mathcal{F}$ of subsets of $\{1, ..., n-1\}$ such that the local ring $\bar{R}_P$ has the following form:*

$$\bar{R}_P = \left(\frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_{n-1}]}{I_P} \right)_{\mathfrak{m}},$$

where

$$I_P := \langle \{\lambda_i N_i | 1 \leqslant i < n\} \cup \{N_i | a_i / a_{i+1} \neq q\} \cup \{N_S | S \in \mathcal{F}\} \rangle,$$

and $\mathfrak{m}$ is the maximal ideal $\langle \lambda_1, ..., \lambda_n, N_1, ..., N_{n-1} \rangle$. Furthermore, every set $S \in \mathcal{F}$ has empty intersection with the set $\{i | b_i \neq 0\}$.

Proof. We note that the only part left to prove is the statement about the remaining equations in the N_i that describe the conjugacy class of nilpotent matrix

$$\begin{pmatrix} 0 & N_1 & 0 & \dots & 0 \\ 0 & 0 & N_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & N_{n-1} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \bar{C}_p$$

in $\mathcal{N}_n$. By Lemma 4.3.1 in the next section, the equations that cut out W_p, defined as the closed subscheme of $\overline{C_p}$ with all non-zero entries on the off-diagonal, are given by products of the form

$$0 = \prod_{i \in S} N_i$$

for some set $S \subseteq \{1, ..., n-1\}$. The Lemma follows. $\square$

4.3 Calculations of the families $\mathcal{F}$ that appear for a given partition p

In this section, we study and calculate the equations that specify the union $X_{\leqslant p}$. We start off with a lemma.

Lemma 4.3.1. *Let $W^+ \cong \mathbb{A}^{n-1}_{\mathcal{O}}$ be the subscheme of the scheme M_n of $n \times n$ matrices over $\mathcal{O}$, consisting of matrices with entries only on the off-diagonal, so that*

$$W = \{ \begin{pmatrix} 0 & N_1 & & \\ & \ddots & \ddots & \\ & & \ddots & N_{n-1} \\ & & & 0 \end{pmatrix} : (N_1, ..., N_{n-1}) \in \mathbb{A}^{n-1} \}$$

Let W_p be the subscheme $W_p = (\overline{C_p} \cap W^+)^{red}$. Then W_p is cut out by squarefree products of the N_i.

Proof. Let $f = f(N_1, ..., N_{n-1})$ be a polynomial in the N_i such that $f = 0$. Since W_p is invariant under conjugation by the maximal torus T of GL_n, This action defines an action on f via $\lambda.f(N_1, ..., N_{n-1}) = f(\lambda_1 \lambda_2^{-1} N_1, ..., \lambda_{n-1} \lambda_n^{-1} N_{n-1})$, where $\lambda = (\lambda_1, ..., \lambda_n) \in T$, and we must have that $f(N) = 0$ implies $\lambda.f(N) = 0$. View f as a polynomial in N_i, and coefficients in the ring of polynomials $k[N_1, ..., \tilde{N_i}, ..., N_{n-1}]$ (where $\tilde{N_i}$ means ‘omit N_i’). Consider the action of $\lambda = (\lambda_1, ..., \lambda_n)$ where $\lambda_j = \alpha \in k^\times$ for all $j \leqslant i$ and $\lambda_j = 1$ for all $j > i$. Then this action preserves the coefficients of f, and multiplies the N_i^k term by α^k. We hence see that all the N_i-graded parts of f lie in the ideal. Since this is true for each i, we see that there are generating equations, $\{f_s : s \in I\}$ such that each f_s is a product of N_i’s, up to a constant coefficient, which we may forget without loss of generality. To prove that the generators are squarefree, it is sufficient to note that W_p is a reduced scheme. □

We now give a complete description of the families $\mathcal{F}$ that occur. They depend only on the partition p. We will denote the family obtained from R_P by $\mathcal{F}_p$, as this only depends on p.

Remark. Notice that as written in Lemma 4.2.4, $\mathcal{F}$ has no dependence on (a_i). If we wanted to we could change this, and include $\{i\} \in \mathcal{F}$ for each i such that $\{i | a_i/a_{i+1} \neq q\}$.

Let $T \subseteq S \subseteq \{1, ..., n-1\}$. Then $N_T | N_S$, so that we can enlarge $\mathcal{F}_p$ to make it an order ideal of $\mathcal{P}(\{1, .., n-1\})$. With this, we can observe that we have an order reversal, in that if q, p are partitions of n, and $q \leqslant p$, then $\mathcal{F}_q \supseteq \mathcal{F}_p$ (this happens, precisely because $C_q \subseteq \bar{C}_p$).

Proposition 4.3.2. *There is an algorithm to calculate $\mathcal{F}_p$ given a partition p of n.*

Proof. The algorithm consists of the following steps.

Step 1 Form the set $\mathcal{Q}$ of all 'minimal breaking' partitions $q = (q_1, ..., q_r)$ defined to be partitions of n such that there exists some integer s such that:

a) for every $j < s$, $\sum_{i=1}^{j} q_i \leqslant \sum_{i=1}^{j} p_i$.

b) $\sum_{i=1}^{s} q_i = \sum_{i=1}^{s} p_i + 1$

c) for each $i \in (s, r]$, $q_i = 1$.

Note that the minimal referred to here does not mean that q is minimal in the dominance order.

Step 2 For each minimal breaking partition $q = (q_1, ..., q_r)$, form the family $\mathcal{S}_q$ of all subsets of $\{1, ..., n-1\}$ that are a union of runs of length $q_1 - 1, q_2 - 1, ..., q_s - 1$ of the following form. To clarify, let $|a, q|$ be the set $\{a, a+1, a+2, ..., a+q-1\}$ (we call this a run of length q). The sets inside $\mathcal{S}_q$ are exactly those of the form $|a_1, q_{\sigma(1)} - 1| \cup |a_2, q_{\sigma(2)} - 1| \cup ... \cup |a_s, q_{\sigma(s)} - 1|$ with $a_{i+1} \geqslant a_i + q_{\sigma(i)}$ for every i, and some permutation $\sigma \in \mathrm{Sym}_s$.

Step 3 Take $\mathcal{F}_p$ to be the order ideal generated by the set $\bigcup_{q\in\mathcal{Q}} \mathcal{S}_q$.

It can be seen that this produces $\mathcal{F}_p$, since the equations $\mathcal{S}_q$ exclude, on the level of points, any nilpotent matrix in the conjugacy class defined by q. Since any partition $q' = (q'_1, ..., q'_{r'})$ such that q' is not dominated by p has some minimal breaking partition q such that $q \leqslant q'$, namely, if q' has s the smallest integer such that $\sum_{i=1}^{s} q'_i > \sum_{i=1}^{s} p_i$, then $q = (q'_1, ..., q'_{s-1}, \sum_{i=1}^{s} p_i + 1 - \sum_{i=1}^{s-1} q'_i, 1, ..., 1)$ does the job, we also see that any nilpotent matrix in the conjugacy class defined by q' is also excluded. Since each q is not dominated by p, this shows that any matrix in the conjugacy class defined by p is not excluded, and nor is any partition dominated by p. This shows that, at the level of points, these equations determine W_p. $\square$

We now present an example of this calculation in the case of $n = 6$ and $p = (4, 1, 1)$, and a diagram that shows $\mathcal{F}_p$ for each partition p of $n = 6$. On the left of the diagram are the partitions of 6, ordered according to the dominance order and on the right are the families $\mathcal{F}_p$, that correspond to p.

A brief remark about notation For clarity's sake, instead of usual set notation, I will denote the set containing the numbers $1, 3$ and 5 by the triple 135. Further, given sets 12,134,234, I will denote the order ideal $\mathcal{F} \subseteq \mathcal{P}(1, ..., 4)$ generated by $12, 134$ and 234 by angled bracket notation $\langle 12, 134, 234\rangle$. We note that $\langle\varnothing\rangle = \varnothing$.

Example 1. *Let* $n = 6$*, and* $p = (4, 1, 1)$*. Then the minimal breaking partitions of* p *are* $(5, 1), (4, 2)$ *and* $(3, 3)$*. Form* $\mathcal{S}_{(5,1)} = \{1234, 2345\}$*, the set of all runs of length 4. The set* $\mathcal{S}_{(4,2)} = \{1235, 1345\}$ *is the set of all sets containing a run of length 3 and a run of length 1, and the set* $\mathcal{S}_{(3,3)} = \{1245\}$ *is the only set that contains two runs of length 2. Thus, we see that* $\mathcal{F}_p = \langle 1234, 2345, 1235, 1345, 1245\rangle$*.*

$$\begin{tikzcd}
4 \arrow[d] & \langle \varnothing \rangle \arrow[d] & \\
31 \arrow[d] & \langle 123 \rangle \arrow[d] & \langle \text{all triples} \rangle \\
2^2 \arrow[d] & \langle 12, 23 \rangle \arrow[d] & \\
21^2 \arrow[d] & \langle 12, 13, 23 \rangle \arrow[d] & \langle \text{all pairs} \rangle \\
1^4 & \langle 1, 2, 3 \rangle &
\end{tikzcd}$$

Figure 4.1: A depiction of the families $\mathcal{F}_p$ that appear for each partition p of 4. On the left is the partitions of 4, ordered with respect to the dominance partial order, and on the right are the generators of the family $\mathcal{F}_p$.

$$\begin{tikzcd}
 & 6 \arrow[d] & \\
 & 51 \arrow[d] & \\
 & 42 \arrow[dl] \arrow[dr] & \\
3^2 \arrow[dr] & & 41^2 \arrow[dl] \\
 & 321 \arrow[dl] \arrow[dr] & \\
31^3 \arrow[dr] & & 2^3 \arrow[dl] \\
 & 2^2 1^2 \arrow[d] & \\
 & 21^4 \arrow[d] & \\
 & 1^6 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \langle \varnothing \rangle \arrow[d] & \\
 & \langle 12345 \rangle \arrow[d] & \\
 & \langle 1234, 2345 \rangle \arrow[dl] \arrow[dr] & \\
\langle 123, 234, 345 \rangle \arrow[dr] & & \langle \text{all quadruples} \rangle \arrow[dl] \\
 & \langle 123, 234, 345, 1245 \rangle \arrow[dl] \arrow[dr] & \\
\langle \text{all triples} \rangle \arrow[dr] & & \langle 12, 23, 34, 45 \rangle \arrow[dl] \\
 & \langle 12, 23, 34, 45, 135 \rangle \arrow[d] & \\
 & \langle \text{all pairs} \rangle \arrow[d] & \\
 & \langle 1, 2, 3, 4, 5 \rangle &
\end{tikzcd}$$

Figure 4.2: A depiction of the family $\mathcal{F}_p$ that corresponds to each partition p of 6 as with Figure 4.1

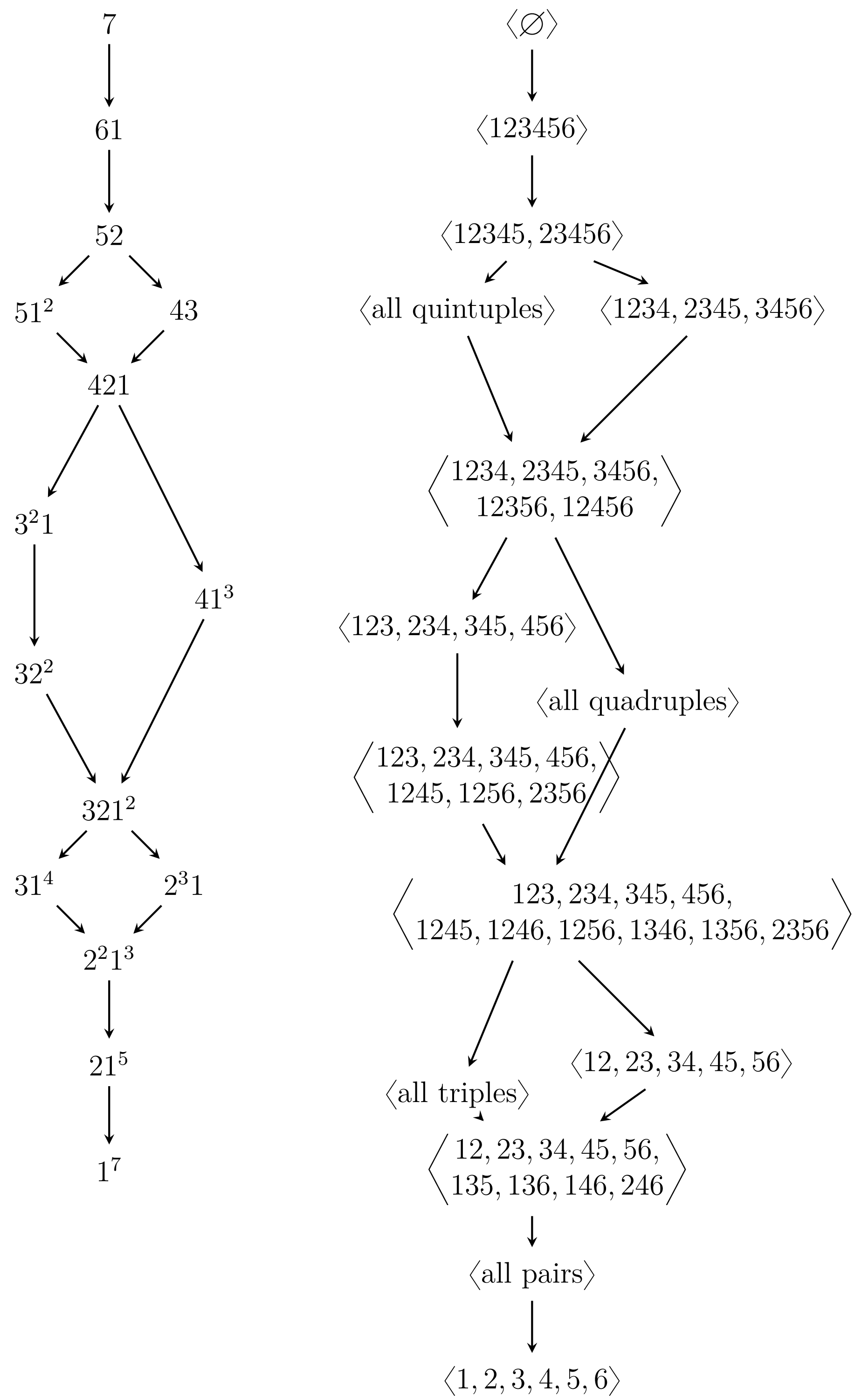


Figure 4.3: A depiction of the families $\mathcal{F}_p$ that appear for each partition p of 7.

4.4 Proof of Theorem 4.2.2

We prove a slight generalisation of Theorem 4.2.2:

Theorem 4.4.1. *Suppose K is a field, let $n \in \mathbb{N}$ and let $\mathcal{F} \subseteq \mathcal{P}(1, ..., n)$. Let*

$$R := K[\lambda_1, ..., \lambda_n, N_1, ..., N_n]/I$$

where I is the ideal generated by the set

$$\{\lambda_i N_i | 1 \leqslant i \leqslant n\} \cup \{N_S | S \in \mathcal{F}\}$$

where $N_S = \prod_{i \in S} N_i$ as before. Suppose $m \trianglelefteq R$ is the maximal homogeneous ideal. Then depth$(m, R) \geqslant n$.

Lemma 4.4.2. *Suppose R is a ring and $x \in R$ is a non-unit, such that for any $a \in R$, we have*

$$x^2 b = 0 \implies xb = 0.$$

Define

$$T := \frac{R[y]}{\langle xy \rangle}$$

Then $y - x$ is a non-unit and not a zero divisor of T.

Proof. There is a grading on T defined by $T_n = Ry^n$ for each $n \in \mathbb{N}$. Let $f \in T$. Since x is not a unit, the degree 0 part of $(y - x)f$ cannot be 1. Thus, $y - x$ is not a unit.

To show that it is a non-zero divisor, let $f \in T$ be such that $(y - x)f = 0$. Write $f = \sum_{i=0}^{n} a_i y^i$ for some $n \in \mathbb{N}$, and $a_i \in R$. Then:

$$0 = (y - x)f = a_n y^{n+1} + \sum_{i=0}^{n-1} (a_i - x a_{i+1}) y^{i+1} - x a_0 \tag{4.4.1}$$

$$= a_n y^{n+1} + \sum_{i=0}^{n-1} a_i y^{i+1} - x a_0 \tag{4.4.2}$$

$$= fy - x a_0. \tag{4.4.3}$$

Hence, $fy = xa_0 \in T_0$, and so $fy \in (\bigoplus_{i=1}^{n+1} T_i) \cap T_0 = 0$ and so $fy = 0$. So each of the constituents of the sum are zero too. Hence $a_0 y = 0$.

So we have an element $a_0 \in R$ such that $ya_0 = xa_0 = 0$ As $ya_0 = 0$ in T, we must have that $ya_0 \in \langle xy \rangle \trianglelefteq R[y]$. So $ya_0 = xyb$ for some $b \in R[y]$, and since $\deg(ya_0) = 1$, have $\deg(xb) = 0$, so we can choose $b \in R$. Hence, $(a_0 - xb)y = 0$ in $R[y]$, and so $a_0 = xb$ in R. Hence $0 = xa_0 = x^2 b$. So by hypothesis, $a_0 = xb = 0$. and so $f = \sum_{i=1}^{n} a_i y^i$.

Recall that $fy = 0$. So $fy = \sum_{i=1}^{n} a_i y^{i+1} = 0$. Then each of the terms $a_i = 0$ in S, so $a_i \in \langle xy \rangle$ in $R[y]$. So $f = \sum_{i=1}^{n} a_i y^i = 0$. This shows that $y - x$ is not a zero divisor. □

Lemma 4.4.3. *Let R be a ring, and J some finite indexing set, and $a_j \in R$ for $j \in J$. Let $T = R[x]/I$ where $I = \langle \{xa_j | j \in J\} \rangle$. Then x has the property that, for any $a \in T$*

$$x^2 a = 0 \implies xa = 0$$

Proof. First, we see that $R[x]$ is a graded ring, and I is a homogeneous ideal of degree 1, so T is also graded. Suppose $a \in T$ is such that $x^2 a = 0$. in S. We may lift a to $a' \in R[x]$, so that $x^2 a' \in I$. Then, for some $b_j \in R[x]$,

$$x^2 a' = \sum_{j \in J} x a_j b_j,$$

and so

$$xa' = \sum_{j \in J} a_j b_j.$$

Consider the degree zero part of xa'. Then

$$0 = \sum_{j \in J} a_j b_j(0),$$

where $b_j(d)$ denotes the degree d part of b_j. Therefore

$$\begin{aligned} xa' &= \sum_{d\geqslant 1}\sum_{j\in J} a_j b_j(d) x^d \\ &= \sum_{j\in J} x a_j c_j \in I \end{aligned}$$

with $c_j := \sum_{d\geqslant 1} b_j x^{d-1} \in R[x]$. Hence, $xa = 0$ in S. $\square$

Proof of Theorem 4.4.1. We show explicitly that the sequence $\{\lambda_i - N_i : i = 1, ..., n\}$ is a regular sequence. Let $J_i = \langle \lambda_1 - N_1, ..., \lambda_{i-1} - N_{i-1}\rangle$, and let

$$\begin{aligned} R_i &:= R/J_i \\ &\cong \frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_n]}{\langle \{N_S | S \in \mathcal{F}\} \cup \{\lambda_1 - N_1, ..., \lambda_{i-1} - N_{i-1}\}\rangle} \\ &\cong \frac{A[\lambda_i, N_i]}{\langle \{N_S | S \in \mathcal{F} \text{ and } i \in S\}, \lambda_i N_i\rangle} \\ &\cong \frac{(\frac{A[N_i]}{\langle N_i a_t | t\in T\rangle})[\lambda_i]}{\langle \lambda_i N_i\rangle} \end{aligned}$$

where T is some finite indexing set, $a_t \in A$ are some explicit elements of A, and

$$A = \frac{K[\lambda_1, N_1, ..., \lambda_{i-1}, N_{i-1}, \lambda_{i+1}, N_{i+1}, ..., \lambda_n, N_n]}{\langle \{\lambda_j N_j | j \neq i\} \cup \{\lambda_1 - N_1, ..., \lambda_{i-1} - N_{i-1}\} \cup \{N_S | S \in \mathcal{F} \text{ and } i \notin S\}\rangle}.$$

Now, since $B := \frac{A[N_i]}{\langle N_i a_t | t\in T\rangle}$ is of the form in Lemma 4.4.3, we know N_i is an element of B such that $N_i^2 a = 0 \implies N_i a = 0$, for $a \in B$. Hence, by Lemma 4.4.2, $\lambda_i - N_i$ is a non-unit, non-zero divisor in R_i. It then follows that $\lambda_1 - N_1, ..., \lambda_n - N_n$ is a regular sequence of length n. $\square$

We now prove Theorem 4.2.2.

Proof. Proof of Theorem 4.2.2 Recall from Lemma 4.2.4 that the local ring of a $P \in X_{\leqslant p}^{\Phi\text{-reg}}$ is of the following form:

$$\bar{R}_P = \frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_{n-1}]}{\langle \{\lambda_i N_i | 1 \leqslant i < n\} \cup \{N_i | a_i/a_{i+1} \neq q\} \cup \{\lambda_i | b_i \neq 0\} \cup \{N_S | S \in \mathcal{F}\}\rangle_m}$$

with $\mathcal{F}$ a family of subsets of $\{1, ..., n-1\}$.

We first can make a simplification. Notice that, by expanding $\mathcal{F}$ to include the sets $\{\{i\} | a_i/a_{i+1} \neq q\}$, we may assume without loss of generality that the second set of generators is empty. Reorder the i, so that $\{i | b_i \neq 0\} = \{k+1, k+2, ..., n-1\}$ for some k. Now, since for any $S \in \mathcal{F}$, $S \cap \{i | b_i \neq 0\} = \varnothing$, we can view $\mathcal{F}$ as a family of subsets of $\{1, ..., k\}$.Hence we see that

$$\bar{R}_P \cong \frac{K[\lambda_1, ..., \lambda_k, N_1, ..., N_k]}{\langle \{\lambda_i N_i | 1 \leqslant i \leqslant k\} \cup \{N_S | S \in \mathcal{F} \subseteq \mathcal{P}(\{1, ..., k\})\}\rangle}[N_{k+1}, ..., N_{n-1}, \lambda_n].$$

By Theorem 4.4.1, $\frac{K[\lambda_1, ..., \lambda_k, N_1, ..., N_k]}{\langle \{\lambda_i N_i | 1 \leqslant i \leqslant k\} \cup \{N_S | S \in \mathcal{F} \subseteq \mathcal{P}(\{1, ..., k\})\}\rangle}$ has a regular sequence of length k given by $\lambda_1 - N_1, ..., \lambda_k - N_k$. We can now extend this regular sequence by $N_{k+1}, ..., N_{n-1}, \lambda_n$ to get a regular sequence of length n in $m_P \trianglelefteq \bar{R}_P$. This shows that $\operatorname{depth}(m_P, \bar{R}_P) \geqslant n$. Further, since $\bar{R}_P$ is a local ring of a subvariety V of the affine variety Spec $\left(\frac{K[\lambda_1, ..., \lambda_n, N_1, ..., N_{n-1}]}{\langle \lambda_i N_i | 1 \leqslant i < n\rangle}\right)$ which has dimension n, we see

$$n \leqslant \operatorname{depth}(\bar{R}_P) \leqslant \dim(\bar{R}_P) \leqslant n$$

which implies equality throughout, Therefore $\bar{R}_P$ is Cohen-Macaulay of dimension n.

By the previous reductions, it follows that $X_{\leqslant p}^{\Phi\text{-reg}}$ is Cohen Macaulay. □

4.5 The Gorenstein condition

Once we know that our rings are Cohen-Macaulay, and we have a regular sequence for each of the rings, we can answer the question about when exactly the ring R_P is Gorenstein.

Theorem 4.5.1. *Suppose $P \in X_{\leqslant p}^{\Phi\text{-reg}}$. Then the local ring R_P is Gorenstein if and only if either:*

1. *$p = 1^n$; or*

2. Every component X_q that contains P, has $q \leqslant p$.

Proof. We prove that the rings in these two cases are Gorenstein first. In case 1, $X_{\leqslant p} \cong \mathrm{GL}_n$ is smooth, therefore is Gorenstein. In case 2, we notice that the natural inclusion map $X_{\leqslant p} \hookrightarrow S_n$ induces an isomorphism of local rings at P. Because S_n is a complete intersection, this implies that the local ring R_P is a complete intersection too, and thus is Gorenstein.

For the converse, suppose R_P is Gorenstein. Then R_P has type 1, ie, that

$$\dim(\mathrm{Ext}^{\dim R_P}(R_P/m, R_P)) = 1.$$

Consider the maximal regular sequence

$$(\mathbf{x}') = (\lambda_1 - N_1, ..., \lambda_k - N_k, N_{k+1}, ..., N_{n-1}, \lambda_n)$$

of $\bar{R}_P$ given in the previous section. Extend it by a regular sequence of $\mathcal{O}$ to a maximal regular sequence of R_P,

$$(\mathbf{x}) = (y_1, ..., y_{\dim \mathcal{O}}, \lambda_1 - N_1, ..., \lambda_k - N_k, N_{k+1}, ..., N_{n-1}, \lambda_n).$$

Consider the Artinian ring $R_0 := R_P/(\mathbf{x}) \cong \frac{K[N_1,...,N_{n-1}]}{\langle \{N_i^2 : 1 \leqslant i < n\} \cup \{N_S | S \in \mathcal{F}\} \rangle}$ with $\mathcal{F}$ as before. Let $\tilde{m}$ be the maximal ideal of R_0. By Lemma 3.1.16 of [BH93], we note that $\mathrm{Ext}^{\dim R_P}(R_P/m, R_P) \cong \mathrm{Hom}(R_0/\tilde{m}, R_0) \cong \mathrm{Soc}(R_0)$. We can describe the socle of R_0 as the span of those monomials corresponding to the maximal sets in the partially ordered set $\mathcal{T} = \{S \subseteq \{1, ..., n-1\} | N_S \neq 0\}$ (ordered by inclusion). So since R_0 has one-dimensional socle, we see that $\mathcal{T}$ has a unique maximal element.

Assume we are not in the case $p = 1^n$. Then each singleton $\{i\} \in \mathcal{T}$. And thus, since $\mathcal{T}$ has a unique maximal element, the union $\{1, 2, ..., n-1\} \in \mathcal{T}$. This shows that the family $\mathcal{F} = \mathcal{P}(1, ..., n-1) \backslash \mathcal{T}$ is empty, and thus, that R_P is isomorphic to the local ring of P in S_n. This shows the second condition. $\square$

4.6 The Cohen Macaulay-ness of non-Φ-regular points

When $P = (\Phi, N)$ is a non-Φ-regular point, we make the following conjecture.

Conjecture 4.6.1. *Let p be a partition of n. Then $X_{\leqslant p}$ is Cohen-Macaulay.*

In other words, we conjecture that Theorem 4.2.2 should be true, without the extra condition of Φ-regularity. One can prove this in a special case strong enough to prove the conjecture in the case $n = 3$.

Definition 4.6.2. *Let $\Phi \in GL_n$ be an $n \times n$ matrix, Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a list of non-negative integers that add up to n. We say that Φ has signature λ, if Φ has k distinct eigenvalues $a_1, ..., a_k$ where we require without loss of generality that these eigenvalues are ordered in such a way, that whenever $a_i/a_j = q$, then $j = i + 1$, and the generalised a_i-eigenspace is λ_i-dimensional.*

Note that Φ may not have a unique signature, because we only specify one property the ordering of the a_i should satisfy, which is not strong enough to specify uniqueness.

It should also be noted that Φ has signature $(1, 1, ..., 1)$ if and only if it is regular. Thus we have shown already that points $P = (\Phi, N)$ such that Φ has signature $(1, 1, ..., 1)$ are Cohen-Macaulay.

For the following result, we need a tool from commutative algebra called 'graded Hodge algebras'. We recall the definition and main result of these objects, and I refer the interested reader to [BH93].

Let H be a finite set. Set $\mathbb{N}^H$ as the set of monomials in the variables H. Notice that $\mathbb{N}^H$ naturally has a partial order on it defined by divisibility in the R-algebra $R[\mathbb{N}^H]$. An ideal of monomials is an order ideal $\Sigma \subseteq \mathbb{N}^H$ of the set of monomials, as ordered by divisibility. A generator of Σ is a minimal element, in the divisibility partial order. We call the set of monomials outside Σ the standard monomials.

Definition 4.6.3. *Let R be a ring and A an R-algebra. Let H be a partially ordered finite set, with an inclusion into A.*

We call A a graded Hodge algebra governed by Σ if the following axioms hold:

1. *A is a free R-module, which admits the set of standard monomials $\mathbb{N}^H \backslash \Sigma$ as a basis.*

2. *For any generator of $t \in \Sigma$, we can write t as a finite R-linear combination of standard monomials*
$$t = \sum_{s \in \mathbb{N}^H \backslash \Sigma} r_s s,$$
such that for any divisor $y \in H$ of t, and for any s that appears in the above sum, there is a divisor $z \in H$ of s for which $z < y$ in the partial order of H.

The equations found in axiom 2 are called the *straightening laws*. When all straightening laws are trivial (ie, the right hand side is 0) we call this a discrete graded Hodge algebra.

Let $\mathrm{Ind}(A) \subseteq H$ be the subset of H consisting of elements that appear on the right hand side in one of the straightening law equations. Let $h \in \mathrm{Ind}(A)$ be a minimal element under the ordering of H. Give A the filtration defined by $\mathrm{Fil}_n = \langle h^n \rangle$, and form the graded algebra
$$\mathrm{Gr}_h A := \bigoplus_n (\mathrm{Fil}_n / \mathrm{Fil}_{n+1}).$$

This is a new graded Hodge algebra, governed by the same data as A, but with every instance of h removed from the straightening laws (so $\mathrm{Ind}(\mathrm{Gr}_h A) \subseteq \mathrm{Ind}(A) \backslash \{h\}$).

Theorem 4.6.4. *Let H be a partial order, and Σ an order ideal in $\mathbb{N}^H$. Let A be a graded Hodge algebra with data (H, Σ).*

If $Gr_h A$ is Cohen-Macaulay, then so is A.

Proof. See the proof of Corollary 7.1.6 of [BH93]. $\square$

Corollary 4.6.5. *If the discrete Hodge algebra with data (H, Σ) is Cohen Macaulay, then so is any graded Hodge algebra with data (H, Σ).*

We can now continue with the following theorem.

Theorem 4.6.6. *Suppose that k_1, k_2, m are all non-negative integers, and that $m > 0$. Suppose that Φ is of signature $(k_2, 1^m, k_1)$. Then the local ring at a point $(\Phi, N) \in X_{\leqslant p}$ is Cohen-Macaulay.*

Proof. Let N_i, λ_i, $\nu_{i,j}$ and $\epsilon_{i,j}$ all be formal variables with appropriate indices. The local deformations at (Φ, N) take the form

$$\left(\begin{pmatrix} q^{m+1}(I_{k_2}+M_2) & & & & \\ & q^m(1+\lambda_{k_1+m}) & & & \\ & & \ddots & & \\ & & & q(1+\lambda_{k_1+1}) & \\ & & & & (I_{k_1}+M_1) \end{pmatrix}, \begin{pmatrix} 0_{k_2} & \underline{v}_2 & & & \\ & 0 & N_{k_1+m-1} & & \\ & & \ddots & \ddots & \\ & & & 0 & \underline{v}_1 \\ & & & & 0_{k_1} \end{pmatrix}\right)$$

where

$$M_1 = \begin{pmatrix} \lambda_{k_1} & \epsilon_{k_1,k_1-1} & \cdots & \epsilon_{k_1,2} & \epsilon_{k_1,1} \\ \epsilon_{k_1-1,k_1} & \lambda_{k_1-1} & \cdots & \epsilon_{k_1-1,2} & \epsilon_{k_1-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \epsilon_{2,k_1} & \epsilon_{2,k_1-1} & \cdots & \lambda_2 & \epsilon_{2,1} \\ \epsilon_{1,k_1} & \epsilon_{1,k_1-1} & \cdots & \epsilon_{1,2} & \lambda_1 \end{pmatrix}$$

is a $k_1 \times k_1$ matrix,

$$M_2 = \begin{pmatrix} \lambda_{k_1+m+k_2} & \nu_{k_2,k_2-1} & \cdots & \nu_{k_2,2} & \nu_{k_2,1} \\ \nu_{k_2-1,k_2} & \lambda_{k_1+m+k_2-1} & \cdots & \nu_{k_2-1,2} & \nu_{k_2-1,1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu_{2,k_2} & \nu_{2,k_2-1} & \cdots & \lambda_{k_1+m+2} & \nu_{2,1} \\ \nu_{1,k_2} & \nu_{1,k_2-1} & \cdots & \nu_{1,2} & \lambda_{k_1+m+1} \end{pmatrix}$$

is a $k_2 \times k_2$ matrix, $\underline{v}_2 = \begin{pmatrix} N_{n-1} \\ \vdots \\ N_{k_1+m} \end{pmatrix}$ is a k_2-dimensional column vector and $\underline{v}_1 = \begin{pmatrix} N_{k_1} & \cdots & N_1 \end{pmatrix}$ is a k_1-dimensional row vector.

Notice that because the N_i's are located on the block off-diagonal, there are $k_1 + (m-1) + k_2 = n-1$ in total.

In this case, the equations take the form:

1. $M_2\underline{v}_2 = \underline{v}_2\lambda_{k_1+m}$

2. $\lambda_{k_1+1}\underline{v}_1 = \underline{v}_1 M_1$

3. $\lambda_{i+1}N_{k_1+i} = N_{k_1+i}\lambda_i$ for $1 < i < m$

4. Some other equations in the variables N_i, $\underline{N}_1$ and $\underline{N}_m$ which depend only on the equations defining $\overline{C_p}$, the closure of the nilpotent conjugacy class of p. From section 4 of [Wey89], these equations are polynomials which are simply sums of square-free monomials.

We give our ring the structure of a graded Hodge algebra. Consider the generator set $H = \{\lambda_i, \nu_{i,j}, \epsilon_{i,j}, N_i\}$ and give H any partial order such that

- for any i, j, a, b, $N_i > \phi_j > \epsilon_a > \nu_b$
- $\phi_n > \phi_{n-1} > ... > \phi_{k_1+m+1} > ... > \phi_{k_1+2} > \phi_1 > \phi_2 > ... > \phi_{k_1} > \phi_{k_1+1}$

Now take $\Sigma \subset \mathbb{N}^H$ to be the order ideal generated by $\{\lambda_i + 1N_i : i > k_1\} \cup \{\lambda_i N_i : i \leqslant k_1\}$ and finally, we consider the straightening laws, for each generator in the above generating set:

$$\text{for } i \leqslant k_1;\ N_i\lambda_i = N_i\lambda_{k_1+1} - \sum_{j=1, j\neq i}^{k_1} N_j\nu_{j,i}$$

$$\text{for } k_1 < i < k_1+m;\ N_i\lambda_{i+1} = N_i\lambda_i$$

$$\text{for } i \geqslant k_1+m;\ N_i\lambda_{i+1} = N_i\lambda_{k_1+m} - \sum_{j=k_1+m+1, j\neq i}^{n} \epsilon_{i,j}N_j$$

It is readily checked that these equations do form a straightening law, due to our choice of order on the generating set, H.

Utilising Corollary 4.6.5, it can be seen that this ring is Cohen-Macaulay if the corresponding discrete graded Hodge algebra (with the same data) is. However, since the discrete graded Hodge algebra $R_0 = \mathcal{O}[\lambda_1, ..., \lambda_n, N_1, ..., N_n]/I$ with $I = \langle\{\lambda_i Ni : i \leqslant k_1\} \cup \{\lambda_i + 1N_i : i > k_1\}\rangle + J$ with J an ideal generated by squarefree monomials in the N_i is of the form in Theorem 4.4.1, it follows that R is Cohen-Macaulay. □

Corollary 4.6.7. *Let p be a partition of 3. Then $X_{\leqslant p}$ is a Cohen Macaulay variety.*

Proof. The cases $p = 3$ and $p = 1^3$ are a complete intersection and a smooth variety respectively. This leaves only $p = 21$. Let $P = (\Phi, N) \in X_{\leqslant 21}$. Then Φ can have signature $(1, 1, 1)$, $(2, 1)$, $(1, 2)$ or 3. The case $(1, 1, 1)$ is the Φ-regular case, so is CM by Corollary 12. The signature (3) case also follows because P is only on the component X_{1^3}, which is smooth, ergo Cohen-Macaulay. The cases $(2, 1)$ and $(1, 2)$ are covered by Theorem 4.6.6. □

Chapter 5

Automorphic forms for unitary groups

We now turn to an application of the smoothness result found in chapter 3. In this chapter, we define the space of ordinary automorphic forms, and the Hecke algebra attached to it. We then state a freeness result, and prove it in the Chapter 6 of this thesis.

Let l be a prime. Suppose F^+ is a totally real number field with an imaginary quadratic extension F, such that for any prime v of F^+ that lies above l, then v splits in F. We will also make the rather strong assumption that $F : F^+$ is an unramified extension. Let S_l be the set of all primes of F^+ that lie above l. Let G_{F^+} and G_F be the absolute Galois groups of F^+ and F respectively. Let L be a finite extension of $\mathbb{Q}_l$ with ring of integers $\mathcal{O}$, and residue field k. Let $\bar{L}$ be a choice of algebraic closure. We will assume that L is large enough that it contains all of the embeddings $F \hookrightarrow \bar{L}$ lie inside L. Let $c \in \mathrm{Gal}(F : F^+) = G_{F^+}/G_F$ be the unique non-trivial element, given by complex conjugation. For $a \in F$, we will denote $c(a)$ by $\bar{a}$ when convenient.

5.1 Unitary groups

Consider D/F a central simple algebra of F-dimension n^2, and let S_D be a finite set of primes of F^+ that split in F. Suppose that

- D splits at places w of F that do not lie above some place in S_D;
- There is an isomorphism $D^{op} \cong D \otimes_{F,c} F$ of F-algebras;
- The intersection $S_D \cap S_l = \varnothing$;
- For all places w of F above some place in S_D, D_w is a division algebra;
- Either n is odd, or n is even and $\frac{n}{2}[F^+ : \mathbb{Q}] + \#S_D \equiv 0 \pmod 2$.

By [HT01] section 3.3 we can find an involution of the second kind on D, that is, because of the condition that either n is odd, or n is even with $\frac{n}{2}[F^+ : \mathbb{Q}] + \#S_D \equiv 0 \pmod 2$, we may construct a map

$$* : D \to D$$

such that:

- $*$ is an F^+ linear anti-automorphism of D;
- $(a^*)^* = a$ for all $a \in D$;
- When restricted to F, $*$ coincides with complex conjugation.

In addition, we assume that this involution of the second kind is positive, that is, for any $\gamma \in D \backslash \{0\}$,

$$\mathrm{tr}_{F:\mathbb{Q}}[\mathrm{tr}_{D/F}(\gamma\gamma^*)] > 0.$$

Such an involution gives rise to a Hermitian form $\langle , \rangle : D \times D \to D$ given by $\langle x, y \rangle = x^* y$, and by [HT01] we may find such an involution such that the Hermitian form is non-degenerate. We make the assumption that the involution has this property.

Let $\mathcal{O}_D$ be an order in D, such that $\mathcal{O}_D^* = \mathcal{O}_D$, and such that for any split prime v of F^+, $\mathcal{O}_{D,v}$ is a maximal order of D_v. Such an order exists by section 3.3 of [CHT08]. Define the unitary group over $\mathcal{O}_{F^+}$, whose R-points (R an $\mathcal{O}_{F^+}$-algebra) are given by $G_D = \{g \in (\mathcal{O}_D \otimes_{\mathcal{O}_{F^+}} R)^\times : g^* g = 1\}$. Then G_D is an algebraic group over $\mathcal{O}_{F^+}$. By the positivity condition, we have that at each infinite place v of F^+, that $G_{D,v} \cong U(n)$.

For each prime v of F^+ that splits in F, choose a prime $\tilde{v}$ of F lying above v. This choice allows us to give an isomorphism $i_{\tilde{v}} : G_D(F_v^+) \to D \otimes_F F_{\tilde{v}}$, which restricts to an isomorphism $G_D(\mathcal{O}_{F^+,v}) \cong \mathcal{O}_{D,\tilde{v}}$ as in section 3.3 of [CHT08]. Note that when $v \notin S_D$ is split in F with w lying above v, G_D is split, so that $G_D(F_v^+) \cong (D \otimes_F F_{\tilde{v}})^\times \cong \mathrm{GL}_n(F_w)$. If T is a set of primes of F^+ that splits in F, set $\tilde{T} = \{\tilde{v} | v \in T\}$.

5.2 Automorphic forms of G_D

We define the automorphic forms for G_D as in [Gro99] and [CHT08].

Recall from the classification of representations of algebraic groups that finite dimensional simple modules for a reductive group G over a field L are uniquely determined by the highest weight in the character group of a maximal torus $T_G \subseteq G$ $X(T_G) := \mathrm{Hom}(T_G, \mathbb{G}_m)$. Recall further, that there is a unique simple module with highest weight λ if and only if λ is dominant.

In the case of GL_n, the weights are naturally in correspondence with $\mathbb{Z}^n$, and the dominant weights are $\mathbb{Z}^n_+ := \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n : \lambda_i \geqslant \lambda_{i+1} \forall i\}$. We set the L-vector space W_λ to be the irreducible representation of weight λ. We will need to choose a $\mathcal{O}$ lattice of W_λ. For λ a dominant weight, we do this as in [Ger19] by setting ξ_λ the representation $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/\mathcal{O}}$, for B_n a choice of Borel with maximal torus $T_n \subset \mathrm{GL}_n$, and w_0 the longest element of the Weyl group. We denote by M_λ the representation given by the $\mathcal{O}$-points of ξ_λ, so that $W_\lambda \cong M_\lambda \otimes_{\mathcal{O}} L$.

Let $L : \mathbb{Q}_l$ be the finite field extension defined before, with ring of integers $\mathcal{O}$. The finite dimensional algebraic representations in L vector spaces of $\mathrm{Res}_{\mathbb{Q}}^{F^+} G_D \otimes \mathbb{Q}_l \cong \prod_{w \in \tilde{S}_l} \mathrm{Res}_{\mathbb{Q}_l}^{F_{\tilde{v}}}(\mathrm{GL}_n)$ are characterised by the sequence of dominant weights, one for each embedding corresponding to $w \in \tilde{S}_l$. We define the set as $W = (\mathbb{Z}_+^n)^{\mathrm{Hom}(F^+,L)}$. For each $\mu \in W$, we can now define the algebraic representation of $G_D/\mathcal{O}_{F^+}$ with highest weight μ by $M_\mu = \bigotimes_{\tau \in \mathrm{Hom}(F^+,L),\mathcal{O}} M_{\lambda_\tau}$, and $W_\mu = M_\mu \otimes_{\mathcal{O}} L$.

For each $v \in S_D$, choose a finite-free $\mathcal{O}$-module representation $\rho_v : G_D(\mathcal{O}_{F^+,v}) \to \mathrm{GL}(M_v)$. Set $M_{\{\rho_v\}} = \bigotimes_{v \in S_D} M_v$. We set $M_{\mu,\{\rho_v\}} = M_\lambda \otimes M_{\{\rho_v\}}$.

Definition 5.2.1. *Let $\lambda = (\mu, \{\rho_v\})$ be as above. We define the space of automorphic forms for G_D of weight λ with A-coefficients $S_\lambda(A)$, where A is an $\mathcal{O}$-algebra or $\mathcal{O}$-module, as the space of functions*

$$f : G_D(F^+)\backslash G_D(\mathbb{A}_{F^+}^{\infty)} \to M_\lambda \otimes_{\mathcal{O}} A$$

such that there is an open compact subgroup

$$U \subset G_D(\mathbb{A}_{F^+}^{\infty,S_l}) \times G_D(\mathcal{O}_{F^+,l})$$

with

$$u \cdot f(gu) = f(g)$$

for all $g \in G_D(\mathbb{A}_{F^+}^{\infty)}$ and $u \in U$ where $u\cdot$ denotes the action of u on M_λ factoring through $\prod_{v \in S} G_D(F_v^+)$.

Notice that $S_\lambda(A)$ is a smooth representation of $G_D(\mathbb{A}_{F^+}^{\infty)}$, under the action $(hf)(g) = h \cdot f(gh^{-1})$ (again, the $\cdot$ action acting through the representation of $G_D(F_l^+) \times \prod_{v \in S_D} G_D(F_v^+)$ on M_λ). We denote by $S_\lambda(U, A) = S_\lambda(A)^U$ the invariants under this action.

5.3 Hecke Operators

For much of the next two chapters, the argument will be a slight adaptation on that in [Ger19]. As such, the details can be found in sections 2 and 4 of [Ger19], so this will just highlight the definitions and results needed, and refer to [Ger19] for the proofs, which we will adapt into this case. Let T be a finite set of places of F^+ containing $S_D \cup S_l$ such that every place in T splits in F, and let $\tilde{T}$ be a set of primes of F above those in T as defined before. Fix an open compact subgroup $U = \prod_v U_v$ of $G_D(\mathbb{A}_{F^+}^{\infty})$, such that for any split place v outside T, $U_v \cong \mathrm{GL}_n(\mathcal{O}_{F,\tilde{v}})$ via the map i_v, and such that for any place of F^+, v, inert in F, suppose U_v is hyperspecial. Suppose further that U is sufficiently small, that is, there is a place v such that U_v contains no non-identity roots of unity. We define the Hecke operators on the subspace $S_\lambda(U, A)$.

Hecke operators at unramified places Let v be a place of F^+ split in F and $w = \tilde{v}$ be a place in F. Let ϖ_w be a uniformiser. We can define the Hecke operators as the double coset operators:

$$T_p^{(i)} = \left[i_v^{-1} \left(\mathrm{GL}_n(\mathcal{O}_{F,w}) \begin{pmatrix} \varpi_w I_i & 0 \\ 0 & I_{n-i} \end{pmatrix} \mathrm{GL}_n(\mathcal{O}_{F,w}) \right) \times U^v \right]$$

Hecke operators at places dividing l At places dividing the residual characteristic of $\mathcal{O}$, we set $\alpha_{\tilde{v}}^{(i)} = \begin{pmatrix} \varpi_{\tilde{v}} I_i & 0 \\ 0 & I_{n-i} \end{pmatrix}$, and define

$$U_{\mu,\tilde{v}}^{(i)} = (w_0\mu_v)(\alpha_{\tilde{v}}^{(i)})^{-1}[U\alpha_{\tilde{v}}^{(i)}U]$$

where w_0 is the longest element of the Weyl group of GL_n, and $\mu \in W$, with μ_v the dominant weight for the embedding $F^+ \hookrightarrow L$.

We make the following adjustment to the group U.

Definition 5.3.1. *For v a place of F^+ above l, and b a positive integer, let $I^b(\tilde{v})$ be the set of matrices in $GL_n(F^{\tilde{v}})$ which are upper triangular unipotent mod $\tilde{v}^b$. Define $U(l^b) = \prod_{v \in S_l} I^{b,c}(\tilde{v}) \times U^l$.*

In the case with the group $U(l^b)$, further define the following diamond operators:

Definition 5.3.2. *Let T_n be the maximal torus inside GL_n as before. For $v \in S_l$, and $u \in T_n(\mathcal{O}_{F_{\tilde{v}}})$, define $\langle u \rangle$ as the operator*

$$[U(l^b)uU(l^b)]$$

on $S_\lambda(U(l^b), A)$. For $u \in T_n(\mathcal{O}_{F^+,l}) = \prod_{v \in S_l} T_n(\mathcal{O}_{F_v}) \cong \prod_{v \in S_l} T_n(\mathcal{O}_{F_{\tilde{v}}})$, define $\langle u \rangle = \prod_{v \in S_l} \langle u_{\tilde{v}} \rangle$.

Define the Hecke algebra $\mathbb{T}^T = \mathbb{T}^T(U(l^b), A)$ as the A-subalgebra of $\mathrm{End}(S_\lambda(U(l^b), A))$ generated by all the operators

$$\{T_{\tilde{v}}^{(i)}, (T_{\tilde{v}}^{(n)})^{-1}) | v \text{ split in } F \text{ outside of } T\} \cup \{U_{\mu,\tilde{v}}^{(i)} | v \in S_l\} \cup \{\langle u \rangle | u \in T_n(\mathcal{O}_{F^+,l})\}.$$

Notice that the map $u \mapsto \langle u \rangle$ defines a group homomorphism

$$T_n(\mathcal{O}_{F^+,l}) \to \mathbb{T}^T(U(l^b), A)^\times$$

which factors through $T_n(\mathcal{O}_{F^+,l}/l^b) = \prod_{v \in S_l} T_n(\mathcal{O}_{F^+,v}/v^b)$.

5.4 Big ordinary Hecke algebras and the action of Λ

From this point on, we wish to focus on the cases where $A = \mathcal{O}, L/\mathcal{O}$, or is a finite module $\mathcal{O}/\pi^n\mathcal{O}$.

Recall from Hida theory, as explained fully in section 2.4 of [Ger19], that for any place $v \in S_l$, and any i, the operator $e_v^{(i)} := \lim_{n\to\infty}(U_{\mu,\tilde{v}}^{(i)})^{n!}$ is a projection on

$S_\lambda(U, A)$. We can further define the projection $e = \prod_{v,i} e_v^{(i)}$. We define the ordinary submodule $S_\lambda^{\mathrm{ord}}(U, A) := e.S_\lambda(U, A)$ as the image of this projection. Notice, since all the Hecke operators commute, that this is a Hecke invariant submodule. We also define $\mathbb{T}^{T,\mathrm{ord}}(U(l^b), A) = e\mathbb{T}^T(U(l^b), A)$.

Definition 5.4.1. *Let T_n be the maximal torus of GL_n as before. For $b \geqslant 1$, let $T_n(l^b)$ be the kernel of $T_n(\mathcal{O}_{F^+,l}) \to T_n(\mathcal{O}/l^b)$.*

We define the following algebras,

$$\Lambda_b = \mathcal{O}[[T_n(l^b)]] = \varprojlim_{b' \geqslant b} \mathcal{O}[T_n(l^b)/T_n(l^{b'})]$$

$$\Lambda = \mathcal{O}[[T_n(l)]] = \mathcal{O}[[T_n(l^1)]]$$

$$\Lambda^+ = \mathcal{O}[[T_n(\mathcal{O}_{F^+,l})]] = \varprojlim_{b' \geqslant b} \mathcal{O}[T_n(\mathcal{O}_{F^+,l})/T_n(l^{b'})].$$

We denote by a_N the kernel of the map $\Lambda \to \mathcal{O}[T_n(l)/T_n(l^N)]$. Notice that, since U is sufficiently small, $S_\lambda^{\mathrm{ord}}(U(l^{b,c}), A)$ is a free Λ/a_b-module, through the action of $T_n(\mathcal{O}_{F^+,l})$, and hence we have an inclusion $\Lambda/a_b \hookrightarrow \mathbb{T}(U(l^b), L/\mathcal{O})$ by Proposition 2.5.3 of [Ger19].

Infinite level

We need to consider the big ordinary Hecke algebra. Set

$$\mathbb{T}^{T,\mathrm{ord}}(U(l^\infty), A) = \varprojlim_{b>0} \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,b}), A)$$

and

$$S^{\mathrm{ord}}(U(l^\infty), A) = \varinjlim_{b>0} S^{\mathrm{ord}}(U(l^{b,b}), A).$$

Notice that because of the inclusions $\Lambda/a_b \hookrightarrow \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,c}), L/\mathcal{O})$, we get an inclusion $\Lambda \hookrightarrow \mathbb{T}^{T,\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})$, and we see that $S^{\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})$ is a discrete Λ-module, so its Pontryagin dual is a compact Λ-module. (and in fact is finite free, by Proposition 2.5.3 of [Ger19] since we assume $U(l)$ is sufficiently small.)

We can now give a statement of a theorem that can be proved by the application Theorem 3.0.1. Under certain hypotheses (to be determined in chapter 6) we have Theorem 6.4.3, which states: The $\mathbb{T}^{T,\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})$-module $S^{\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})^{\vee}$ is locally free over the generic fibre $\mathbb{T}^{T,\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})[1/l]$.

As a consequence, the multiplicity of $S^{\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})^{\vee}$ is the same at every characteristic zero point of $\mathbb{T}^{T,\mathrm{ord}}(U(l^{\infty}), L/\mathcal{O})$, and thus, we expect the multiplicity of non-classical points (those corresponding to Hida families of ordinary automorphic forms) is the same as at classical modular forms.

Chapter 6

Galois representations and deformation rings

6.1 Local deformation rings

We now define a deformation problem. Let $v \in S_D$ with residue field of size q_v. We say that an n-dimensional representation $\rho : G_{F_v^+} \to \mathrm{GL}_n(A)$ is Steinberg if the map $R_{\bar{\rho}}^{\square} \to A$ determined by ρ factors through $\mathcal{O}_{\hat{X}_n}$.

We note that this is equivalent to the statement, that the representation ρ lies on the irreducible component $X_n(A)$ of S_{GL_n}, which in the case when $A = L$ is a characteristic 0 field, the Weil-Deligne representation obtained from ρ, $WD(\rho) = (r, N)$, then r is unramified and the eigenvalues of $r(\mathrm{Frob}_{q_v})$ are in the ratio $q_v^{n-1} : q_v^{n-2} : ... : q_v : 1$. Note that this definition puts ρ on the irreducible component X_n of S_n.

Let $\mathcal{C}_{\mathcal{O}}$ be the category of Artinian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$, as in Mazur. For each $v \in S_D$, and Steinberg representation $\bar{\rho}_v : G_{F,\tilde{v}} \to \mathrm{GL}_n(\mathbb{F})$ define a functor

$$\begin{aligned} D_{\bar{\rho}_v}^{n,\square} : \mathcal{C}_{\mathcal{O}} &\to \mathcal{S}et \\ A &\mapsto \{\text{Steinberg liftings of } \bar{\rho}_v \text{ to } A\} \end{aligned}$$

This functor is pro-representable by the complete Noetherian local ring $R_v^{\square,\mathrm{st}} := \mathcal{O}_{X_n,\bar{\rho}}$. We notice that when we view X_n as a scheme over L, Theorem 3.0.1 tells us, since q is not a root of unity in L, that any localisation of $R_v^{\square,\mathrm{st}}[1/l]$ is a regular ring. This shows us that $R_v^{\square,\mathrm{st}}[1/l]$ is regular.

For ρ a deformation of $\bar{\rho}_v$ to A, we say that ρ is of type X_n if the map $R_{\bar{\rho}}^{\square} \to A$ defined by ρ factors through $R_v^{\square,st}$.

We recall the definition of $\tilde{r}$-discrete series found in section 2.4.5 in [CHT08].

Let $\tilde{r}_v : G_{F,\tilde{v}} \to \mathrm{GL}_d(\mathcal{O})$ be a representation such that:

1. $\tilde{r}_v \otimes k$ is absolutely irreducible (k the residue field of $\mathcal{O}$;
2. Every irreducible subquotient of $\tilde{r}_v|_{I_{\tilde{v}}}$ is absolutely irreducible;
3. For each $i = 0, ..., m$, $\tilde{r} \otimes k \not\cong \tilde{r} \otimes k(i)$.

For R an $\mathcal{O}$ algebra, we say a representation $\rho : G_{F,\tilde{v}} \to \mathrm{GL}_{md}(R)$ is $\tilde{r}$-discrete series if there is an decreasing filtration $\{\mathrm{Fil}^i\}$ of ρ by R-direct summands such that

1. $\mathrm{gr}^i\rho \cong \mathrm{gr}^0\rho(i)$ for $i = 0, ..., m-1$
2. $\mathrm{gr}^0\rho|_{I,\tilde{v}} \cong \tilde{r}|_{I,\tilde{v}} \otimes_{\mathcal{O}} R$.

Proposition 6.1.1. *Suppose $l > h_G$. Let $\tilde{r}$ be a rank d representation as above, and let n be an integer with $d|n$. Let $X_{\tilde{r},n}$ be the moduli space, defined over $\mathcal{O}$, of framed $\tilde{r}$-discrete series representations of rank n. Then the base change, $(X_{\tilde{r},n})_L$, to L is smooth over L.*

Proof. Let $S_{\tilde{r}}$ be the moduli prestack over $\mathcal{O}$ of n-dimensional $\tilde{r}$-discrete representations, so that the stackification $S_{\tilde{r}}^{st} \cong [X_{\tilde{r}}/\mathrm{GL}_n]$ and let $S_{\mathbb{1}}$ be the prestack of $m := n/d$-dimensional $\mathbb{1}$-discrete series representations. Let $S_{\tilde{r}}^{\mathrm{WD}}$ be the prestack over L whose groupoid over R consists of objects (ρ', N) where ρ' is a rank $n = dm$ $\tilde{r}$-discrete series representation with open kernel, and N is an element of $\mathrm{End}_R(R^n)$

such that $\rho' N \rho'^{-1} = q^\nu N$. Define $S_{\mathbb{1}}^{\mathrm{WD}}$ analogously. Recall that there is a morphism $S_{\tilde{r}}^{\mathrm{WD}} \to S_{\tilde{r}}$ given by (ρ', N) is sent to the unique representation ρ given by $g \mapsto \rho(g)\exp(t_l(g))$ for $g \in I$ and $\rho(\mathrm{Frob}) = \rho'(\mathrm{Frob})$. Recall that this is an isomorphism on the base change to L.

Then we have an morphism of prestacks $S_{\mathbb{1}}^{\mathrm{WD}} \to S_{\tilde{r}}^{\mathrm{WD}}$ given by the morphism $(\rho', N) \mapsto (\rho', N) \otimes \tilde{r}$. We claim that this is an isomorphism. By an exercise in Clifford theory and by assumptions on $\tilde{r}$, $\tilde{r}|_I$ can be written as a direct sum of pairwise non-isomorphic absolutely irreducible I-representations $\tau \oplus \tau^{\mathrm{Frob}} \oplus, ..., \oplus \tau^{\mathrm{Frob}^{k-1}}$ for some $k \in \mathbb{N}$. As ρ' is $\tilde{r}$-discrete series in characteristic zero, we see that $\rho'|_I \cong m(\tau \oplus \tau^{\mathrm{Frob}} \oplus, ..., \oplus \tau^{\mathrm{Frob}^{k-1}})$. Let $V_{\tilde{r}}(R) = \mathrm{End}_{R[I]}(\tilde{r}^m)$ be the the space of I-equivariant maps of any representation in $S^{\mathrm{WD}_{\tilde{r}}}(R)$, and define $V_{\mathbb{1}}(R) = \mathrm{End}_{R[I]}(\mathbb{1}^m)$ similarly. Note that the map

$$V_{\mathbb{1}}(R) \to V_{\tilde{r}}(R) \tag{6.1.1}$$

$$N \mapsto N \otimes \mathrm{id}_{\tilde{r}} \tag{6.1.2}$$

is injective, and hence is isomorphic onto its image. We claim that if $(\rho', N) \in S_{\tilde{r}}^{\mathrm{WD}}(R)$, then N is in the image of this map.

First, note that N is I-equivariant. We calculate using Schur's lemma that $V_{\tilde{r}}(R) \cong kM_m(R)^k$, since each τ^{Frob^i} is absolutely irreducible, and we see the above map corresponds to the diagonal map $\Delta : M_m(R) \to M_m(R)^k$.

The space $V_{\tilde{r}}(R)$ has a natural action of Frobenius on it, and under this action $N = (N_1, ..., N_k) \in M_m(R)^k$ has $Frob.(N_1, ..., N_k) = q(N_1, ..., N_k)$. Notice that Frob induces an isomorphism of the underlying spaces $\tau^m \to (\tau^{\mathrm{Frob}})^m$, which gives us a commutative diagram

$$\begin{tikzcd}
\tau^m \arrow[r, "\mathrm{Frob}"] \arrow[d, "N_1"] & (\tau^{\mathrm{Frob}})^m \arrow[d, "qN_2"] \\
\tau^m \arrow[r, "\mathrm{Frob}"] & (\tau^{\mathrm{Frob}})^m
\end{tikzcd}$$

Hence, we see $(qN_2, ..., qN_k, qN_1) = q(N_1, ..., N_{k-1}, N_k)$, and thus N lies in the image of the diagonal map. This proves the claim.

Let $\chi_{\tilde{r}} = \hom_I(\tau, \tilde{r})$. Notice that this is an unramified character. We claim that $(\mathrm{Hom}_I(\tau, \_) \otimes \chi_{\tilde{r}}^{-1}, \Delta^{-1}) : S_{\tilde{r}}^{\mathrm{WD}} \to S_{\mathbb{1}}^{\mathrm{WD}}$ is an inverse defining the equivalence.

For $(\Theta, N) \in S_{\tilde{r}}^{\mathrm{WD}}(R)$, the previous claim gives us an isomorphism on the N-part of the stacks $S_{\tilde{r}}^{\mathrm{WD}}(R)$, so we focus on the representation part. Since $\theta|_I$ acts through a finite quotient, and R is an algebra over a characteristic 0-field, we have that Θ is semisimple and hence we get a decomposition of I-representations:

$$M \cong \bigoplus_{i=0}^{k-1} \mathrm{Hom}_I(\tau^{\mathrm{Frob}^i}, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \tau^{\mathrm{Frob}^i}$$

for some positive integer k. Since each τ^{Frob^i} occurs in Θ with equal multiplicity, we see that each $\mathrm{Hom}_I(\tau^{\mathrm{Frob}^i}, \Theta) \cong \mathrm{Hom}_I(\tau, \Theta)$, and thus,

$$\Theta \cong \mathrm{Hom}_I(\tau, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \bigoplus_{i=0}^{m-1} \tau^{\mathrm{Frob}^i} \cong \mathrm{Hom}_I(\tau, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \tilde{r}$$

As I representations. To see an isomorphism on the level of W_F-representations, notice that we have an unramified character χ defined over an algebraic closure $\bar{L}$ such that for each i $\mathrm{gr}^i\Theta \cong \tilde{r} \otimes \chi(i)$. Then

$$\mathrm{Hom}_{\bar{L}[I]}(\tau, \mathrm{gr}^m(\Theta)) \cong \mathrm{Hom}_{\bar{L}[I]}(\tau, \tilde{r} \otimes \chi) \cong \chi_{\tilde{r}} \otimes \chi(i).$$

Since $\tilde{r}(i) \not\cong \tilde{r}$ for each $1 \leqslant i \leqslant m$, $\Theta = \bigoplus_i \mathrm{gr}^i(\tilde{r})$,so we get a $\bar{L}[W_F]$ isomorphism

$$\Theta \otimes \bar{L} \cong (\mathrm{Hom}_I(\tau, \Theta) \otimes \chi_{\tilde{r}}^{-1} \otimes \tilde{r}) \otimes_L \bar{L}$$

Finally, since $\tilde{r}$ is absolutely irreducible, this can be upgraded to an isomorphism L-vector spaces. Hence, the composite $S_{\tilde{r}}^{\mathrm{WD}}(R) \to S_{\mathbb{1}}^{\mathrm{WD}}(R) \to S_{\tilde{r}}^{\mathrm{WD}}(R)$ is the identity.

To show $S_{\mathbb{1}}^{\mathrm{WD}}(R) \to S_{\tilde{r}}^{\mathrm{WD}}(R) \to S_{\mathbb{1}}^{\mathrm{WD}}(R)$ is the identity, let $\rho \in S_{\mathbb{1}}(R)$. Then the natural map

$$\rho \to \mathrm{Hom}_I(\tau, \rho \otimes \tilde{r}) \tag{6.1.3}$$

$$v \mapsto \{w \mapsto v \otimes w\} \tag{6.1.4}$$

defines an I isomorphism. So we need only check that $\rho \otimes \chi_{\tilde{r}}$ and $\mathrm{Hom}_I(\tau, \rho \otimes \tilde{r})$ have the same action of Frobenius. This can be checked again, by looking at the character $\mathrm{gr}_i(\rho)$. Hence, we have exhibited an equivalence of categories $S_{\mathbb{1}} \leftrightarrow S_{\tilde{r}}$.

Given a choice of Frobenius, Frob, and a topological generator of the tame inertia group, s, we can explicitly write an isomorphism of stacks

$$S_{\mathbb{1}}^{st} \cong [X_m/\mathrm{GL}_m]$$
$$\rho \mapsto (\rho(\mathrm{Frob}), \log(\rho(s)))$$
$$\rho_\Phi(\mathrm{Frob}^n x) = \Phi^n \exp(N t_l(x)) \leftarrow\!\shortmid (\Phi, N)$$

As $(X_m)_L$ is a smooth scheme, it shows that $S_{\mathbb{1}}^{st}[1/l]$ is a smooth stack, and because the isomorphism of prestacks gives an isomorphism through the stackification, it gives us that $S_{\tilde{r}}^{st}[1/l]$ and $X_{\tilde{r},n}$ are smooth. $\square$

In light of this proposition, if $\bar{\rho} : G_{F,\tilde{v}} \to \mathrm{GL}_n(\mathbb{F})$ is $\tilde{r}$-discrete series, we let $R_v^{\square,\tilde{r}}$ be the universal lifting ring of $\tilde{r}$-discrete series representations. By the proposition, $R_v^{\square,\tilde{r}}[1/l]$ is regular at every maximal ideal.

For $v \in S_l$, Let $\bar{I}_{\tilde{v}}$ be the inertia subgroup of $G_{F,\tilde{v}}^{\mathrm{ab}}$, and let $\bar{I}_{\tilde{v}}(l)$ be the pro-l part. As in chapter 3 of [Ger19] we can define a lifting $\Lambda_{\tilde{v}} : \mathcal{O}[[\bar{I}_{\tilde{v}}(l)]]$-algebra $R_v^{\triangle}$. This is the quotient of the universal lifting ring $R_v^{\square}$ of pairs $(\rho, \{\chi_i\})$, such that a morphism $r : R_v^{\square} \to A$ corresponding to representation $\rho : G_v \to \mathrm{GL}_n(A)$ and a sequence of characters $\chi_i : I_{\tilde{v}}$ factors through $R_v^{\triangle}$ if and only if ρ is $\mathrm{GL}_n(\mathcal{O})$-conjugate to an upper triangular representation with diagonal characters equal to $\chi_1, ..., \chi_n$ when restricted to inertia.

Lemma 6.1.2. *Suppose that $\bar{\rho}_v : G_{F,\tilde{v}} \to GL_n(\mathbb{F})$ is an ordinary Galois representation with diagonal characters $\bar{\chi}_1, \bar{\chi}_2, ..., \bar{\chi}_n$, such that for no pair $i < j$ is $\chi_i = \varepsilon\chi_j$, with ε the cyclotomic character, then $R_v^{\triangle}[1/l]$ is formally smooth.*

Proof. We see that the dimension of $R_v^{\triangle}[1/l]$ is $n^2 + [F_v : \mathbb{Q}_l]\frac{n(n+1)}{2}$. For any choice of closed point x of $\mathrm{Spec} R_v^{\triangle}[1/l]$, part 1 of Lemma 3.2.3 of [Ger19] tells us that the dimension of the tangent space of $R_w^{\triangle,ar}[1/l]$ is $n^2 + [F_w : \mathbb{Q}_l] + \dim H^2(G_{F_w}, \mathrm{Fil}^0\mathrm{ad}(V_x))$. From part 3 of Lemma 3.2.3, we also see that if the diagonal characters of $\bar{\rho}$, $(\bar{\chi}_i)$

have $\chi_i/\chi_j \neq \epsilon$ for every pair $i < j$, then $\dim H^2(G_{F_w}, \mathrm{Fil}^0\mathrm{ad}(V_x)) = 0$. Hence, the ring $R_v^{\triangle}[1/l]$ is regular. □

6.2 Local-Global compatibility

We start by introducing the group $\mathcal{G}_n$ from [CHT08], defined as the group scheme that is the semi-direct product of $\mathrm{GL}_n \times \mathrm{GL}_1$ with $C_2 = \{1, j\}$ where j acts as

$$j(g,\mu)j^{-1} = (\mu(g^{-1})^{\mathrm{T}}, \mu).$$

By Lemma 2.1.1 of [CHT08], we have that representations $r : G_{F^+} \to \mathcal{G}_n(R)$ such that $r^{-1}(\mathrm{GL}_n(R) \times \mathrm{GL}_1(R)) = G_F$ correspond with pairs (ρ, χ), where ρ is an n-dimensional representation of G_F, and χ is a character of G_{F^+}, such that $\rho^c \cong \chi\rho^\vee$, and $c \in G_{F^+}$ is sent to j.

For brevity, whenever we have a homomorphism $r : G_{F^+} \to \mathcal{G}_n(R)$, and a subgroup $H \subset G_{F^+}$, we use $r|_{G_F}$ to mean the restriction, followed by the projection to GL_n. Typically, H will be the subgroup G_F or its localisations G_{F_w}.

Proposition 6.2.1. *Suppose that $\mathfrak{m} \trianglelefteq \mathbb{T}^{T,ord}(U(l^\infty), \mathcal{O})$ is a maximal ideal, with residue field $\mathbb{F}$. Then there is a unique continuous semisimple representation*

$$\bar{r}_{\mathfrak{m}} : G_F \to GL_n(\mathbb{F})$$

such that:

1. $$\bar{r}^c \cong \bar{r}_{\mathfrak{m}}^\vee(1-n);$$

2. *For any place v of F^+, outside T, $\bar{r}_{\mathfrak{m}}|_w$ is unramified;*

3. *If further, v splits as $v = ww^c$ in F, then the characteristic polynomial of $\bar{r}_{\mathfrak{m}}(Frob_w)$ is*

$$X^n - T_w^{(1)}X^{n-1} + ... + (-1)^j N(w)^{\frac{j(j-1)}{2}} T_w^{(j)} X^{n-j} + ... + (-1)^n N(w)^{\frac{n(n-1)}{2}} T_w^{(n)}$$

modulo $\mathfrak{m}$*;*

4. *Let* $\tilde{r}_{\tilde{v}} : G_F \to GL_{m_v}(\mathcal{O})$ *be as in section 3.2 of [CHT08] (note: this is constructed from the smooth representation* $\rho_v : G_D(F_v^+) \to GL(M_v)$ *via the Jacquet-Langlands and local Langlands correspondences). If* $v \in S_D$ *and* $U_v = G_D(\mathcal{O}_{F^+,v})$*, then* $\bar{r}_{\mathfrak{m}}|_{G_{F,v}}$ *is* $\tilde{r}_{\tilde{v}}$*-discrete series.*

Proof. Apart from statement 4, this is Propositions 2.7.3 in [Ger19], so we prove only this part. By the argument of Proposition 2.7.3 in [Ger19], the maximal ideals of $\mathbb{T}$ are in bijection with those of $\mathbb{T}/m_\Lambda$. Hence, this proposition follows immediately from the classical situation. The proof of this can be found in Proposition 3.4.2 of [CHT08], which proves the proposition. □

Proposition 6.2.2. *If* $\mathfrak{m}$ *is non-Eisenstein, that is,* $\bar{r}_{\mathfrak{m}}$ *is irreducible, then* $\bar{r}_{\mathfrak{m}}$ *can be extended to a representation* $\bar{r}_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbb{F})$*, and this representation can be lifted to a representation*

$$r_{\mathfrak{m}} : G_{F^+} \to \mathcal{G}_n(\mathbb{T}^{T,ord}(U(l^\infty), \mathcal{O})_{\mathfrak{m}})$$

1. *For* $\nu : \mathcal{G}_n \to GL_1$*, the second projection,* $\nu \circ r_{\mathfrak{m}} = \epsilon^{1-n}\delta_{F/F^+}^{\mu_{\mathfrak{m}}}$*. where* ϵ *is the cyclotomic character,* δ_{F/F^+} *is the non-trivial character of* G_{F^+}/G_F*, and* $\mu_m \in \mathbb{Z}/2$*;*

2. *For any place* $v \notin T$ *of* F^+*,* $\bar{r}_{\mathfrak{m}}|_{\tilde{v}}$ *is unramified;*

3. *If further,* v *splits as* $v = ww^c$ *in* F*, then the characteristic polynomial of* $\bar{r}_{\mathfrak{m}}(Frob_w)$ *is*

$$X^n - T_w^{(1)}X^{n-1} + ... + (-1)^j N(w)^{\frac{j(j-1)}{2}} T_w^{(j)} X^{n-j} + ... + (-1)^n N(w)^{\frac{n(n-1)}{2}} T_w^{(n)};$$

4. *If* $v \in S_D$*, then* $r_{\mathfrak{m}}|_{G_{F,\tilde{v}}}$ *is* $\tilde{r}_{\tilde{v}}$*-discrete series.*

Proof. As the previous proposition, apart from statement 4, this is Proposition 2.7.4 in [Ger19], so we prove only this final statement. By the proof of Proposition 2.7.4

of [Ger19], we may find a sequence of maximal ideals $\mathfrak{m}_b \subset \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,b}), \mathcal{O})$ such that $\mathbb{T}_{\mathfrak{m}} = \varprojlim_b \mathbb{T}^{T,\mathrm{ord}}(U(l^{b,b}), \mathcal{O})_{\mathfrak{m}_b}$, and we define $r_{\mathfrak{m}} = \varprojlim_b r_{\mathfrak{m}_b}$. By Lemma 3.4.4 of [CHT08], each $r_{\mathfrak{m}_b}|_{G_{F,\tilde{v}}}$ is $\tilde{r}_{\tilde{v}}$-discrete series, and so now it remains to show that $r_m|_{G_{F,\tilde{v}}}$ is too. Since for each $b > c$ each $r_{\mathfrak{m}_b} \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c} = r_{\mathfrak{m}_c}$, if follows that the filtration, Fil^i_b on $r_{\mathfrak{m}_b}$ descends to a filtration $\mathrm{Fil}^i_b \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$ on $r_{\mathfrak{m}_c}$, and that the graded parts have $[\mathrm{gr}^i(r_{\mathfrak{m}_b})] \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c} \cong \mathrm{gr}^i[r_{\mathfrak{m}_b}) \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c}]$. It follows that $\mathrm{Fil}^i_b \otimes \mathbb{T}^{T,\mathrm{ord}}(U(l^{c,c}), \mathcal{O})_{\mathfrak{m}_c}$ is a defining filtration on $r_{\mathfrak{m}_c}$. From Lemma 2.4.25 of [CHT08], such a filtration is unique, so we have a compatible system of filtrations on the $r_{\mathfrak{m}_b}$ which lift to a filtration on $r_{\mathfrak{m}}|_{G_{F,\tilde{v}}}$. We see from compatibility that $\mathrm{gr}_i(r_{\mathfrak{m}}) = \varprojlim_b \mathrm{gr}_i(r_{\mathfrak{m}_b})$, and so it is easy to check that $r_{\mathfrak{m}}|_{G_{F,\tilde{v}}}$ is $\tilde{r}_{\tilde{v}}$-discrete series.

$\square$

6.3 Global deformation rings

Let $F : F^+$, $T = S_l \coprod S_D \coprod R$, $\tilde{T}$ all be as before. Let $\bar{\rho} : G_F \to \mathrm{GL}(\mathbb{F})$ be a representation with local representations $\rho_w = \bar{\rho}|_{G_{F,w}}$, where w is a place of F. Assume that:

- the representation $\bar{\rho}$ is a irreducible automorphic representation, I.E., there is a non-Eisenstein maximal ideal $\mathfrak{m} \trianglelefteq \mathbb{T}^{T,\mathrm{ord}}(U(l^\infty, \mathcal{O}))$ so that $\bar{\rho} \cong \bar{r}_{\mathfrak{m}}$;
- the subgroup $\rho(G_{F^+(\zeta_l))}) \subset \mathcal{G}_n(\mathbb{F})$ is adequate in the sense of Definition 2.3 of [Tho12];
- the Level structure is minimal for $\bar{\rho}$;
- the representation $\bar{\rho}$ is unramified outside $\tilde{T}$;
- For each $v \in S_l$, have $\mathrm{Hom}_{G_{F,\tilde{v}}}(\bar{\rho}_{\tilde{v}}, \bar{\rho}_{\tilde{v}}\varepsilon) = 0$ for ε the cyclotomic character.

As $\bar{\rho} \cong \bar{r}_{\mathfrak{m}}$ is irreducible, via Proposition 6.2.2, ρ can be extended to a representation $\bar{\rho} : G_{F^+} \to \mathcal{G}_n(\mathbb{F})$ such that $\nu \circ \bar{\rho} = \epsilon^{1-n}\delta^{\mu_{\mathfrak{m}}}_{F:F^+}$, and we fix such an extension.

For each $v \in T$, define $R_v^\square$ as the framed deformation ring for $\bar{\rho}_{\tilde{v}}$. Set

$$R^{\text{loc}} := (\widehat{\bigotimes}_{\mathcal{O},v\in S_l} R_v^\triangle)\widehat{\otimes}_\mathcal{O}(\widehat{\bigotimes}_{\mathcal{O},v\in S_D} R_v^{\square,\tilde{r}_{\tilde{v}}})\widehat{\otimes}_\mathcal{O}(\widehat{\bigotimes}_{\mathcal{O},v\in R} R_v^\square)$$

to be the local deformation ring for $\bar{\rho}$. Our first observation, is that since each $R_v^\triangle$ is a $\Lambda_{\tilde{v}}$-module, we notice that R^{loc} inherits the structure of a $\widehat{\bigotimes}_{v\in S_l}\Lambda_{\tilde{v}} \cong \Lambda$-module. The isomorphism $\widehat{\bigotimes}_{v\in S_l}\Lambda_{\tilde{v}} \cong \Lambda$ is inherited from the group isomorphisms

$$T_n(\mathfrak{l}) \cong \prod_{v\in S_l} T_n\mathcal{O}_{F^+,v}(l) \cong \prod_{v\in S_l} T_n\mathcal{O}_{F,\tilde{v}}(l) \cong \prod_{v\in S_l} \bar{I}_{\tilde{v}}(l)^n$$

where the final isomorphism is given by local class field theory.

Notice, that by assumption on $\bar{\rho}$ and Lemma 6.1.2, that $R_v^\triangle[1/l]$ is smooth. We remark that $R_v^{\square,\tilde{r}}$ is the completion of a local ring on the moduli space of rank n framed $\tilde{r}$-discrete series representations, $X_{\tilde{r}}$. Since the map $X_{\tilde{r}} \to S_{\tilde{r}}$ given by 'forgetting the framing' is smooth, and the stack $S_{\tilde{r}}[1/l]$ is smooth over L by Proposition 6.1.1, we see that $\mathcal{O}_{X_{\tilde{r}},\bar{\rho}}[1/l]$ is regular, and hence, by an application of Lemma 2.1.5, we see that $R_v^{\square,\tilde{r}}[1/l]$ is regular.

Since the Level U is minimal, for ρ we have further, that $R_v^\square$ is regular for each $v \in R$. Hence, by Corollary 2.1.6, $R^{\text{loc}}[1/l]$ is regular.

Let $\mathcal{S}$ be the following tuple

$$\mathcal{S} = (F : F^+, T, \tilde{T}, \epsilon^{1-n}\delta_{F/F^+}^{\mu_m}, \{R_v^{\triangle,ar} : v \in S_l\}, \{R_v^{\square,st} : v \in S_D\}, \{R_v^\square : v \in R\})$$

and say that $\rho : G_{F^+} \to \mathcal{G}(A)$ is a lifting of $\bar{\rho}$ to $A \in \mathcal{C}_\Lambda$ of type $\mathcal{S}$ if:

1. $\rho|_{G_F}$ lifts $\bar{r}_m$;
2. ρ is unramified outside T;
3. For $v \in S_D$, ρ_v is $\tilde{r}$-discrete series and gives rise to the morphism $R_v^\square \to A$ which factors through $R_v^{\square,\tilde{r}}$;
4. For $v \in S_l$, the restriction ρ_v and the Λ-structure on A give a morphism $R_v^\square \otimes \Lambda \to A$ which factors through $R_v^\triangle$;

5. $\nu \circ \rho = \epsilon^{1-n} \delta_{F/F^+}^{\mu_m}$.

By Proposition 2.2.9 of [CHT08], we can construct the universal deformation ring, $R_{\mathcal{S}}^{\mathrm{univ}}$, and the universal lifting ring $R_{\mathcal{S}}^{\square}$.

Let $h_0 = [F^+ : \mathbb{Q}]\frac{n(n-1)}{2} + [F^+ : \mathbb{Q}]\frac{n(1-(-1)^{\mu_{\mathfrak{m}}-1})}{2}$, and let h be an integer larger than both h_0 and $\dim[H^1_{\mathcal{L}^\perp}(G_{F^+,T}, \mathrm{ad}\bar{\rho}(1))]$. (Here, $H^1_{\mathcal{L}^\perp}(G_{F^+,T}, \mathrm{ad}\bar{\rho}(1))$ is a particular subspace of the cohomology group $H^1(G_{F^+,T}, \mathrm{ad}\bar{\rho}(1))$ of the Galois group $G_{F^+,T}$ of the maximal extension of F^+ unramified outside of T, defined in Proposition 4.4 of [Tho12].)

After Thorne [Tho12], we will call a triple, $(Q, \tilde{Q}, \{\bar{\psi}_v\}_{v \in Q}$ a Taylor-Wiles triple if:

1. Q is a set of primes of F^+ which split in F;
2. for each $v \in Q$, $l | \mathrm{Nm}_{F^+}(v) - 1$
3. $|Q| = h$;
4. $\tilde{Q}$ is the set $\{\tilde{v} | v \in Q\}$;
5. for each $v \in Q$, $\bar{\rho}|_{G_v}$ splits as a direct sum into $\bar{s}_v \oplus \bar{\psi}_v$, with $\bar{\psi}$ the generalised eigenspace with eigenvalue $\bar{\alpha} \in \mathbb{F}$ of dimension d_v.

For any Taylor-Wiles set, Q, we can define a deformation problem $\mathcal{S}(Q)$, which is the same as $\mathcal{S}$, but in addition, we now allow $\rho_{\tilde{v}}$ for $v \in Q$ to ramify in the following way: $\rho_{\tilde{v}}$ splits as a direct sum $s \oplus \psi$, which lift $\bar{s}$ and $\bar{\psi}$ respectively, such that s is unramified, and $\psi|_{I_v} : I_v \to \mathrm{GL}_{d_v}$ factors through the scalar action on the underlying representation space. Using Proposition 2.2.9 in [CHT08] again, we can now take the universal deformation ring $R_{\mathcal{S}(Q)}^{\mathrm{univ}}$. Because stipulating that the local deformations at Taylor-Wiles primes are unramified is a closed condition, this presents us with a surjection $R_{\mathcal{S}(Q)}^{\mathrm{univ}} \twoheadrightarrow R_{\mathcal{S}}^{\mathrm{univ}}$. Further, we also have a natural map $R^{\mathrm{loc}} \to R_{\mathcal{S}(Q)}^{\mathrm{univ}}$ given by restrictions to the local subgroups at the level of functors.

Proposition 6.3.1. *For $N \in \mathbb{N}$, we can find a Taylor-Wiles triple $(Q_N, \tilde{Q}_N, \{\bar{\psi}_v\}_{v \in Q})$ such that for all $v \in Q_N$, $l^N || Nm_F(v) - 1$, and the global deformation ring $R^{\text{univ}}_{\mathcal{S}(Q)}$ can be topologically generated over R^{loc} by $h - h_0$ generators.*

Proof. This follows from Lemma 4.4 of [Tho12] applied in the case of Theorem 8.6. □

In light of this proposition, set $R_\infty = R^{\text{loc}}[[X_1, ..., X_h]]$, $R_N = R^{\text{univ}}_{\mathcal{S}(Q_N)}$ and $R_0 = R^{\text{univ}}_{\mathcal{S}}$ so that we have surjections $R_\infty \twoheadrightarrow R_N$.

We now define some important subgroups of $G_D(\mathbb{A}^{\infty)}_{F^+}$

Definition 6.3.2. *For $v \in Q_N$, suppose that $\bar{r}|_v = \bar{s} \oplus \bar{\psi}$, as before, with $\bar{\psi}$ a d_v dimensional semisimple unramified representation with all Frobenius eigenvalues equal. We take the group $U_i(\tilde{v})$ to be the subgroup of U_v of elements that take the form*

$$\begin{pmatrix} \varpi_{\tilde{v}}* & * \\ 0 & aI_{d_v} \end{pmatrix} (mod \tilde{v})$$

with $a = 1$ when $i = 1$, and arbitrary when $i = 0$. Set $U_i(Q) = U^v \times \prod_{v \in Q} U_i(\tilde{v})$

Set Δ_N be the maximal l-power quotient of $U_0(Q_N)/U_1(Q_N) \cong \prod_{v \in Q_N} k(\tilde{v})^\times$. We may view Δ_N as the maximal l-quotient of $\prod_{v \in Q_N} k(\tilde{v})^\times \cong (\mathbb{Z}/l^N)^q$. We claim there is an action of Δ_N on the ring $R^{\text{univ}}_{\mathcal{S}(Q)}$. The map, $\det \circ r^{\text{univ}}_N : I_{F,\tilde{v}} \to (R^{\text{univ}}_{\mathcal{S}(Q)})^\times$, given by the determinant of the universal deformation $r^{\text{univ}}_N := r^{\text{univ}}_{\mathcal{S}(Q_N),\bar{\rho}}$, factors through the kernel of $(R^{\text{univ}}_{\mathcal{S}(Q)})^\times \to \mathbb{F}^\times$, which is an abelian l-power group. By local class field theory, there is an isomorphism $I^{\text{ab}}_{F,\tilde{v}} \to \mathcal{O}^\times_{F,\tilde{v}}$, and the l-power quotient of this group is the l-power quotient of $k(\tilde{v})^\times$. We hence see that there is a map $\Delta_N \to (R^{\text{univ}}_{\mathcal{S}(Q_N)})^\times$ and thus a ring map $\Lambda[\Delta_N] \to R^{\text{univ}}_{\mathcal{S}(Q)}$, so that $R^{\text{univ}}_{\mathcal{S}(Q_N)}$ inherits the structure of a finitely generated $\Lambda[\Delta_N]$-algebra. Notice that if a_N is the augmentation ideal of $\Lambda[\Delta_N]$, then $R^{\text{univ}}_{\mathcal{S}(Q_N)}/a_N$ is the ring of the universal deformation ring which parametrises

Galois deformations of type $\mathcal{S}$. (These deformations are required to be unramified at places above Q_N.) Note, that by choice of Q_N, that $\Delta_N \cong (\mathbb{Z}/l^n\mathbb{Z})^h$.

As in Chapter 5, we can construct the Hecke algebra $\mathbb{T}_{N,1} := \mathbb{T}^{T\cup Q_N,\mathrm{ord}}(U_1(Q_N)(l^\infty),\mathcal{O})$ and through a map $\mathbb{T}^{T\cup Q_N,\mathrm{ord}}(U_1(Q_N)(l^\infty),\mathcal{O}) \to \mathbb{T}^{T,\mathrm{ord}}(U(l^\infty),\mathcal{O})$ we can lift our choice of maximal ideal $\mathfrak{m}$ to a maximal ideal $\mathfrak{m}_N \subset \mathbb{T}_{N,1}$. As in Proposition 6.2.2, we can construct a representation $r_{\mathfrak{m}_N} : G_{F^+} \to \mathcal{G}_n(\mathbb{T}_{N,1})$ which by the proof of Theorem 6.8 of [Tho12] gives us an $\mathcal{S}(Q_N)$-lifting of $\bar{\rho}$. Hence, we get a surjection $R^{\mathrm{univ}}_{\mathcal{S}(Q)} \twoheadrightarrow \mathbb{T}_{N,1}$ for each N.

6.4 Patching

We now define a module H_N over $\mathbb{T}^{T\cup Q_N,\mathrm{ord}}(U_1(Q_N)(l^\infty),\mathcal{O})_m$ for each set Q_N.

Define the space of automorphic forms $S^{\mathrm{ord}}(U_i(Q_N)(l^\infty),L/\mathcal{O})_m$ as before, and set $H_0 = S^{\mathrm{ord}}(U(l^\infty),L/\mathcal{O})_m^\vee$. In Proposition 5.9 of [Tho12], Thorne describes a projection Pr_v on $S^{\mathrm{ord}}(U_i(Q_N)(l^\infty),L/\mathcal{O})_m$, and in Theorem 6.8, modules

$$H_{i,N} := \prod_{v\in Q_N} \Pr_v[S^{\mathrm{ord}}(U_i(Q_N)(l^\infty),L/\mathcal{O})_m]^\vee$$

with the following properties:

Proposition 6.4.1. *[Tho12]*

1. *H_{1,Q_N} is a free $\Lambda[\Delta_{Q_N}]$-module, and restriction to $S^{ord}(U_0(Q_N)(l^\infty),L/\mathcal{O})_m$ gives an isomorphism $H_{1,Q_N}/\mathfrak{a}_N \cong H_{0,Q_N}$.*

2. *The map*
$$(\prod_{v\in Q_N} \Pr_{\tilde{v}})^\vee : H_{0,Q_N} \to H_0$$
is an isomorphism.

Theorem 6.4.2 (Patching)**.** *Let $R \twoheadrightarrow \mathbb{T}$ be a surjective Λ-algebra homomorphism, with $\mathbb{T}$ a finite Λ-algebra. Suppose we have the following data:*

1. *Integers* $t, h \geqslant 1$*;*

2. *a finite* $\mathbb{T}$*-module* H*;*

3. $S_N = \Lambda[(\mathbb{Z}/l^n\mathbb{Z})^h] \cong \Lambda[\Delta_{Q_N}]$ *with augmentation ideal* $\mathfrak{a}_N$*, with inverse limit* $S'_\infty := \varprojlim \Lambda[\Delta_{Q_N}] \cong \Lambda[[Y_1, ..., Y_h]]$ *;*

4. *a ring* $S_\infty = S'_\infty \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$*, where* $\mathcal{T} = \mathcal{O}[[X_1, ..., X_{|T|n^2}]]$

5. *For each* $N \geqslant 1$ *have*

   (a) $R_N \twoheadrightarrow \mathbb{T}_N$ *are* S_N*-algebra homomorphisms, such that reduction modulo* $\mathfrak{a}_N$ *reduces the map to* $R \twoheadrightarrow \mathbb{T}$*.*

   (b) *a finite* $\mathbb{T}_N$*-module* H_N*, which is finite and free over* S_N*, whose rank is independent of* N*;*

6. *An* S_∞*-algebra* R_∞ *such that* $R_\infty \twoheadrightarrow R_N$ *with kernel* $\ker(S_\infty \to S_N)R_\infty$*.*

Then there is an $R_\infty \otimes S_\infty$*-module* H_∞*, such that*

1. $H_\infty/\mathfrak{a}H_\infty \cong H$*,*

2. H_∞ *is a finite free* S_∞*-module.*

3. *The action of* S_∞ *on* H_∞ *factors through that of* R_∞*.*

Proof. The details of the Taylor-Wiles-Kisin patching method used here is essentially no different to chapter 4.3 of [Ger19]. One can also find details in chapter 8 of [Tho12], under the heading ‘another patching argument’.

However, I would also like to include a particularly neat explanation, via the notion of the ultrapatching functor developed by Scholze in [Sch18] and described in chapter 7.2. Define $\mathcal{I}_n = \ker(S_\infty \to S_n)$, and note that each $\mathcal{I}_n \subset \mathfrak{a}$ for every n, and because $\mathcal{I}_n$ form a chain with $\bigcap_n \mathcal{I}_n = 0$, for any open ideal J, they also satisfy the property that $\mathcal{I}_n \subset J$ for all but finitely many n. We can hence form the category of weak patching systems $\mathscr{wP}$ as in chapter 7.2. We claim that $\mathcal{R} = (R_n) \in \mathscr{wP} - \mathscr{Ring}$ is a patching algebra over R_0, and that $\mathcal{H}^{\square} = (H_n^{\square})$ is a patching $\mathcal{R}$-module over H. As we have a map $R_\infty \twoheadrightarrow R_n$ for each n, we get an S_∞-algebra surjection $R_\infty \twoheadrightarrow \mathcal{P}(\mathcal{R})$, and an R_∞-module $H_\infty^{\square} := \mathcal{P}(\mathcal{H})$. By part 4 of Proposition 7.2.4, we have that $H_\infty^{\square}/\mathfrak{a} \cong H$. Because each H_n is a free $\Lambda[\Delta_n]$-module, part 2 of Proposition 7.2.4 tells us that H_∞ is a free S_∞ module, and the fact that the action of S_∞ acts through R_∞ comes from the fact that it acts through $\mathcal{P}(\mathcal{R})$, and that the diagram

$$\begin{tikzcd} & S_\infty \arrow[dl] \arrow[dr] & \\ R_\infty \arrow[rr, two heads] & & \mathcal{P}(\mathcal{R}) \end{tikzcd}$$

commutes. □

Theorem 6.4.3. *The module $H_0[1/l]$ is a finite locally free $R_{\mathcal{S}}^{univ}[1/l]$-module.*

Proof. We calculate that $\dim(S_\infty) = \dim(\Lambda) + h + |T|n^2 = n[F^+ : \mathbb{Q}]n + h + |T|n^2$, and that

$$\begin{aligned} \dim(R_\infty) &= 1 + \sum_{v \in S_l} ([F_{\tilde{v}} : \mathbb{Q}_l]\frac{n(n+1)}{2} + n^2) + n^2|S_D \cup R| + h - h_0 \\ &= [F^+ : \mathbb{Q}]\frac{n(n+1)}{2} + |T|n^2 + h - h_0 \\ &= [F^+ : \mathbb{Q}]n + |T|n^2 + h - [F^+ : \mathbb{Q}]\frac{n(1-(-1)^{\mu_{\mathfrak{m}}-n})}{2} \end{aligned}$$

Consider the module $H_\infty^{\square}$. Since $H_\infty^{\square}$ is a finite free S_∞ module, and that the action of S_∞ factors through R_∞ we see that

$$\dim(S_\infty) = \mathrm{depth}_{S_\infty}(H_\infty^{\square}) \leqslant \mathrm{depth}_{R_\infty}(H_\infty^{\square}) \leqslant \dim(R_\infty)$$

and thus, the only possible way for this inequality to hold is if equality holds throughout, and $\mu_m \equiv n \bmod 2$, and $H_\infty^{\square}$ is a maximal Cohen-Macaulay R_∞ module. (We remark that in the terminology of chapter 7.2, that R_∞ is a covering of the patching algebra $\mathcal{R}$, and then that this argument becomes the same as that of Theorem 7.2.7.)

Now, consider the generic fibre. Let $m \subseteq R_\infty[1/l]$ be a maximal ideal. Since localisation commutes with tensor products, we see that

$$\left(\bigotimes_{\mathcal{O},v\in T} R_v\right)[1/l] \cong \bigotimes_{L,v\in T}(R_v[1/l]).$$

By Lemma 2.1.5, we see that

$$R_\infty[1/l]^\wedge_m = \left(\widehat{\bigotimes}_{\mathcal{O},v\in T} R_v\right)[1/l]^\wedge_m \cong \left(\bigotimes_{\mathcal{O},v\in T} R_v\right)[1/l]^\wedge_m$$

and so we see that $R_\infty[1/l]^\wedge_m$ is a power series ring tensor product of formally smooth rings. Since it is formally smooth, any finitely generated $R_\infty[1/l]_m$-module has finite projective dimension, and by the Auslander-Buchsbaum formula, is projective. This shows that $H^\square_\infty[1/l]_m$ is a free $R_\infty[1/l]_m$-module, this shows that $H^\square_\infty[1/l]$ is a locally finite free $R_\infty[1/l]$-module. It follows that $H_0[1/l]$ is a locally finite free $R^{univ}_{\mathcal{S}}[1/l]$-module. □

Corollary 6.4.4. $R^{univ}_{\mathcal{S}}[1/l] = \mathbb{T}[1/l]$.

Proof. Let I be the kernel of the surjection $R^{univ}_{\mathcal{S}}[1/l] \to \mathbb{T}[1/l]$. Choose any maximal ideal m of $R^{univ}_{\mathcal{S}}[1/l]$. Since localisation is an exact functor, we get a short exact sequence

$$0 \to I_m \to R^{univ}_{\mathcal{S}}[1/l]_m \to \mathbb{T}[1/l]_m \to 0.$$

Note that the action of $R^{univ}_{\mathcal{S}}[1/l]_m$ on $H_0[1/l]_m$ factors through $\mathbb{T}[1/l]_m$, so that I_m annihilates all of $H_0[1/l]_m$. Since this is a free module, this shows that I_m is trivial. Since this is true for every m, this shows that $\mathrm{Supp}(I) = \varnothing$ and hence $I = 0$. Hence the surjection above is an isomorphism $R^{univ}_{\mathcal{S}}[1/l] \cong \mathbb{T}[1/l]$.

□

Remark. We finally want to remark on an application of Theorem 6.4.3. Whenever M is a locally free coherent sheaf on a connected space X, the rank function

$$X \to \mathbb{N} \cup \{0\}$$

$$x \mapsto \mathrm{Rank}_x(M)$$

is locally constant. Therefore, the rank of a geometrically connected component can be calculated by calculating the rank at any special point $x \in X$. In our special case, the rank of the module $H_0[1/l]$ can be interpreted as the number of distinct automorphic forms with a given set of Hecke eigenvalues. Which can again, be interpreted as the multiplicity of the Galois representation determined by said Hecke eigenvalues inside the space of automorphic forms. We have shown that for these automorphic forms, the multiplicity is determined only by the connected component that the representation $\rho_{\mathfrak{m}}$ lies on. By Lemma 4.2 of [Ger19], we see that the minimal primes of $R_\infty[1/l]$ biject with the minimal primes of Λ, and thus we have a bijection with those of $R_{\mathcal{S}}^{univ}[1/l]$. Thus, if one could show that for each component of $\operatorname{Spec}\Lambda$, there is an automorphic form of some classical weight had multiplicity 1, then all the Hida families of forms would also have multiplicity 1. Thus these results have an application to 'multiplicity problems'.

Chapter 7

Ultrapatching

The aim of this chapter is to explain the concept of Ultrapatching, an idea first introduced by Scholze in [Sch18] which we utilised in Chapter 6 of this Thesis. The ideas are explained beautifully in a greater completion, though in slightly less generality (I.E. In the case $\Lambda = \mathbb{Z}_l$), than here by Manning in [Man].

7.1 Ultraproducts

We recall the following definitions.

Definition 7.1.1. *Let S be a set. A family $\mathcal{F} \subseteq \mathcal{P}(S)$ of subsets of S is called a filter if:*

1. *whenever $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$;*
2. *whenever $I, J \in \mathcal{F}$, then $I \cap J \in \mathcal{F}$.*

A filter $\mathcal{F}$ is called an ultrafilter if it is a maximal proper filter $\mathcal{F} \subsetneq \mathcal{P}(S)$, or equivalently, if for any $I \subseteq S$, exactly one of I and $S \backslash I$ is a member of $\mathcal{F}$.

Lemma 7.1.2. *If $\mathcal{F}$ is an ultrafilter on a set S and we have a finite partition $S = \coprod_{i=1}^{n} S_i$, then exactly one of the S_i lies inside $\mathcal{F}$.*

Proof. For each i, exactly one of S_i and $S\backslash S_i = \coprod_{j\neq i} S_j$ lies in $\mathcal{F}$. We cannot have $S\backslash S_i \in \mathcal{F}$ for every i, as otherwise the finite intersection $\varnothing = \bigcap_i S\backslash S_i \in \mathcal{F}$, which is impossible. So there is some $S_i \in \mathcal{F}$. Further, this S_i is unique because otherwise, there are $S_i, S_j \in \mathcal{F}$ distinct with the intersection $\varnothing = S_i \cap S_j \in \mathcal{F}$, which is again impossible. □

For any $s \in S$, we can write down an ultrafilter $\mathcal{F}_s$ defined by the property that $I \in \mathcal{F}_s$ if and only if $s \in I$. If $\mathcal{F}$ is such an ultrafilter, we call it a principal ultrafilter. When S is a finite set, all ultrafilters are principal as corollary of 7.1.2. When S is infinite, one can define the *cofinite filter* $\mathcal{F}_{cf}$, where $I \in \mathcal{F}_{cf}$ if and only if $S\backslash I$ is finite. When S is infinite, it is a routine application of Zorn's Lemma to show that non-principal ultrafilters exist.

From now on, we will wish to fix a non-principal ultrafilter of the infinite set $\mathbb{N}$.

Definition 7.1.3. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. We define the ultraproduct*

$$\prod^{\mathcal{F}} A_n := \{(a_n)_{n\in\mathbb{N}} : a_n \in A_n\}/$$

where two sequences $(a_n), (b_n)$ are equivalent if and only if $\{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{F}$.

Remarks. 1. When $A_n = \mathbb{R}$ for all n, the ultraproduct provides a model for the set of hyperreal numbers.

2. If the A_n lie inside a category, $\mathcal{C} = \mathcal{Ab}, \mathcal{Ring}, \mathcal{Mod}_R$, then the ultraproduct $\mathcal{A}$ will also be an object of the category $\mathcal{C}$.

3. For $f = (f_n) : (S_n) \to (T_n)$ a sequence of maps, we obtain a map of ultraproducts $\prod^{\mathcal{F}} f_n : \prod^{\mathcal{F}} S_n \to \prod^{\mathcal{F}} T_n$ given by $(s_n) \mapsto (f_n(s_n))$ on a representative. When (f_n) is a sequence of $\mathcal{C}$-morphisms, then $\prod^{\mathcal{F}} f_n$ also inherits the structure of an $\mathcal{C}$-morphism.

Lemma 7.1.4. *If A_n is a non-empty finite set for all n, and $|A_n| \leqslant M$ for some $M \in \mathbb{N}$, then the ultraproduct $\mathcal{A}$ is a finite set.*

Proof. We can give a finite partition of $\mathbb{N} = \coprod_{n=1}^{M} S_n$ where $S_n = \{i \in \mathbb{N} : |A_i| = n\}$. Then, by Lemma 7.1.2, there is a unique $S_n \in \mathcal{F}$. For each $i \in S_n$, label the elements $\{x_i^j : 1 \leqslant j \leqslant n\}$ and for $i \notin S_n$, fix some choice (not necessarily distinct) of x_i^j. Set $y_j = (x_i^j)_i \in \prod^{\mathcal{F}} A_i$. We claim that every sequence (z_i) is $\mathcal{F}$-equivalent to one of the y_j. Let $B_j = i \in S_n : z_i = x_i^j$. Then we have the partition $\mathbb{N} = \mathbb{N} \backslash S_i \cup \coprod_{j=1}^{n} B_j$, and so by Lemma 7.1.2, exactly one of the B_j or $\mathbb{N} \backslash S_n$ is in $\mathcal{F}$. As it is clearly not $\mathbb{N} \backslash S_n$, we have $B_j \in \mathcal{F}$ for some j and thus $z = (z_i) = y_j \in \prod^{\mathcal{F}} A_i$. We can hence conclude that $\mathcal{A}$ is finite. $\square$

Remark. We can actually conclude that $|\prod^{\mathcal{F}} A_i| = n$, in the notation of the above proof. This is because each of the y_j are necessarily distinct, due to the fact that $\{i \in \mathbb{N} : x_i^j = x_i^k\} \subset \mathbb{N} \backslash S_n \notin \mathcal{F}$ for j, k distinct. In the case of the constant sequence $\mathcal{M} = (M)_{n \in \mathbb{N}}$, we get a natural isomorphism $\prod^{\mathcal{F}}(\mathcal{M}) \cong M$ by choosing the representatives in the above proposition as constant sequences in the elements of M.

We will be especially interested in taking the ultraproduct of finite modules of a finite ring R. In this situation, the ultraproduct operation can be seen as a localisation functor.

Let R be a finite local ring with maximal ideal $\mathfrak{m}$, and let $\mathcal{R} = R^{\mathbb{N}}$. We have an injection from the set of filters $\mathcal{F}$ on $\mathbb{N}$ to the set of ideals of $\mathcal{R}$ given by

$$I_{\mathcal{F}} = \mathrm{Span}_{\mathcal{R}}[\mathbf{1}_S | S \in \mathcal{F}]$$

where the symbol $\mathbf{1}_S$ is an $\mathbb{N}$-sequence of 1's and 0's, equal to 1 at each $n \notin S$, and 0 for each $n \in S$. Whose image is exactly those ideals of $\mathcal{R}$ whose quotients are R-flat modules (R acts on $\mathcal{R}/I$ through the diagonal map $\Delta : R \to \mathcal{R}$). That the ideal relations imply the filter relations follows from the calculations:

- For $S' \supset S$ and $\mathbf{1}_S \in I_{\mathcal{F}}$, have $\mathbf{1}_{S'} = \mathbf{1}_{S'}\mathbf{1}_S$;
- For $S, T \in \mathcal{F}$, $\mathbf{1}_{S \cap T} = \mathbf{1}_S + \mathbf{1}_T - \mathbf{1}_S\mathbf{1}_T$.

Now, it follows that the set of ultrafilters $\mathcal{F}$ of $\mathbb{N}$ correspond exactly to those ideals I of $\mathcal{R}$ whose quotient $\mathcal{R}/I \cong R$.

Proposition 7.1.5. *Let R be a finite local ring. Suppose that for $\mathcal{F}$-many n, we have an exact sequence $0 \to A_n \to B_n \to C_n \to 0$ inside $\mathcal{M}od_R$. Then we obtain an exact sequence in the ultraproduct*

$$0 \to \prod^{\mathcal{F}} A_n \to \prod^{\mathcal{F}} B_n \to \prod^{\mathcal{F}} C_n \to 0$$

Proof. Since $\prod^{\mathcal{F}} M_n = (M_n)_{I_{\mathcal{F}}}$ is the localisation by the ideal $I_{\mathcal{F}}$, it follows that the ultraproduct is an exact functor. □

7.2 Ultrapatching

Let Λ be a (regular) local finite type $\mathbb{Z}_l$-algebra with maximal ideal $\mathfrak{l}$ complete with respect to the $\mathfrak{l}$-adic topology. Let $\mathbb{F} = \Lambda/\mathfrak{l}$. Let $S_\infty = \Lambda[[Y_1, ..., Y_h]]$ with augmentation ideal $\mathfrak{a} = \langle Y_1, ..., Y_h \rangle$.

We wish to fix a sequence of ideals $(\mathcal{I}_n)$ of S_∞ such that:

- $\mathcal{I}_n \subset \mathfrak{a}$ for each n;
- for any given open ideal $\mathcal{J} \subset S_\infty$, we require that $\mathcal{I}_n \subset \mathcal{J}$ for all but finitely many n.

Definition 7.2.1. *Let (M_n) be a sequence of finite type S_∞-modules.*

1. *We say that (M_n) is a* weak patching system *if for each n, $\mathcal{I}_n \subset \mathrm{Ann}(M_n)$, and if there is some $N \in \mathbb{N}$ such that each M_n is finitely generated as an S_∞-module by fewer than N elements.*

2. *If, in addition, for each n, there is a finite rank Λ-module M_0 and a Λ-module isomorphism $\alpha_n : M_n/\mathfrak{a} \to M_0$, we will call $((M_n), M_0, \alpha_n)$ (and by abuse of notation, (M_n)) a* patching system.

3. *Let (R_n) be a sequence of finite S_∞-algebras. We say that (R_n) is a* weak patching algebra *if it is a* weak patching system *viewed as a sequence of S_∞-modules. We further call it a* patching algebra *if there is a finite Λ-algebra R_0 and $\alpha_n : R_n/\mathfrak{a} \to R_0$ making $((R_n), R_0, \alpha_n)$ into a patching system.*

4. *If $\mathcal{R} = (R_n)$ is a weak patching algebra, we call a weak patching system $\mathcal{M} = (M_n)$ a* weak patching $\mathcal{R}$-module *if each M_n is an R_n-module.*

We can now define the category of weak patching systems, $\mathcal{wP}$ as the full subcategory of $\mathcal{Mod}_{S_\infty}^{\mathbb{N}}$ whose objects are weak patching systems. We define similarly the category of weak patching algebras $\mathcal{wP} - \mathcal{Ring}$, and for a weak patching algebra, $\mathcal{R}$, denote by $\mathcal{wP}_{\mathcal{R}}$ the category of weak patching $\mathcal{R}$-modules. When J is an ideal of S_∞, and $\mathcal{M} = (M_n) \in \mathcal{wP}$, denote $\mathcal{M}/J = (M_n/J) = (M_n \otimes_{S_\infty} S_\infty/J)$. This is also a weak patching system.

Definition 7.2.2. *We define the ultrapatching functor:*

$$\mathcal{P} : \mathcal{wP} \to \mathcal{Mod}_{S_\infty} \tag{7.2.1}$$

$$\mathcal{M} \mapsto \varprojlim_{J} \prod^{\mathcal{F}} \mathcal{M}/J \tag{7.2.2}$$

where the limit is taken over all open ideals J of S_∞.

We now want to include some properties of the patching functor.

Lemma 7.2.3. *There is a functor*

$$i : \mathcal{Mod}_{S_\infty}^{f.g.} \to \mathcal{wP}$$

$$M \mapsto (M/\mathcal{I}_n)_{n\in\mathbb{N}}$$

that acts as a right inverse to $\mathcal{P}$. I.e. $\mathcal{P} \circ i = id_{\mathcal{Mod}_{S_\infty}}$

Proof. That i is a functor is clear. For the result that it is a right inverse to $\mathcal{P}$, note that for any open ideal $J \trianglelefteq S_\infty$, $\mathcal{I} \subset J$ for all but finitely many n, and thus, $[M/\mathcal{I}_n]/J \cong M/J$ for those n. As M is finitely generated, M/J is a finite module.

Thus, When we take the ultraproduct, $\prod^{\mathcal{F}} i(M)/J = M/J$. As every S_∞-module is complete it follows that $M = \varprojlim_J M/J$. $\square$

From now, we will refer to the patching system $i(M)$ by $\underline{M}$.

Proposition 7.2.4. *1. If $\mathcal{M} = (M_n) \in \boldsymbol{w\mathcal{P}}$ has each M_n a free $S_\infty/\mathcal{I}_n$-module, then $\mathcal{P}(\mathcal{M})$ is a finite rank free S_∞-module.*

2. The functor $\mathcal{P}$ is a right exact additive functor.

3. For $\mathcal{M}$ a weak patching system, $\mathcal{P}(\mathcal{M})$ is a finitely generated S_∞-module.

4. When $\mathcal{M}$ is a patching system, then we have an isomorphism of Λ-modules $\alpha_\infty : \mathcal{P}(\mathcal{M})/\mathfrak{a} \to M_0$.

Proof. 1. Let J be any open ideal of S_∞. Because $\mathcal{I}_n \subset J$ for all but finitely many n, we see that M_n/J is a free S_∞/J-module for $\mathcal{F}$-many n. As there are only finitely many possible ranks, we can partition $\mathbb{N}$ into sets $S_k = \{n : M_n/J \text{ is free of rank } k\}$ and exactly one the $S_k \in \mathcal{F}$. It then follows, because S_∞/J^k is a finite set, that $\prod^{\mathcal{F}} M_n/J \cong S_\infty/J^k$. As this isomorphism is compatible with varying J, after taking the inverse limit we get $\mathcal{P}(\mathcal{M}) \cong \varprojlim S_\infty/J^k \cong S_\infty^k$ is a free S_∞-module.

2. That $\mathcal{P}$ is additive is obvious. For right exactness, note that tensor product is a right exact functor, and since all but finitely many of the M_n/J are finite modules (because all but finitely many , the ultraproduct of $\Lambda/\mathfrak{m}^k$-modules is an exact functor by Proposition 7.1.5. Finally, the direct limit functor is also exact, and hence the composition $\mathcal{P}$ is right exact.

3. If $\mathcal{M} = (M_n) \in \boldsymbol{w\mathcal{P}}$, then there is some $N \in \mathbb{N}$ such that S_∞^N surjects onto M_n for every n. Thus, we get a surjection in $\boldsymbol{w\mathcal{P}}$ $\underline{S_\infty^N} \twoheadrightarrow \mathcal{M}$. By right-exactness of $\mathcal{P}$, we hence see that there is a surjection $S_\infty^N = \mathcal{P}(\underline{S_\infty^N}) \twoheadrightarrow \mathcal{P}(\mathcal{M})$, and thus, $\mathcal{P}(\mathcal{M})$ is a finitely generated S_∞-module.

4. For a patching system $(\mathcal{M}, M_0, (\alpha_n))$ we have an isomorphism $(\alpha_n) : \mathcal{M}/\mathfrak{a} \to \underline{M_0}$ and thus a short exact sequence

$$0 \to \mathfrak{a}\mathcal{M} \to \mathcal{M} \to \underline{M_0} \to 0$$

Thus, after applying the patching functor, we get a short exact sequence

$$\mathcal{P}(\mathfrak{a}\mathcal{M}) \to \mathcal{P}(\mathcal{M}) \to M_0 \to 0$$

As $\mathcal{P}(\mathfrak{a}\mathcal{M}) = \mathfrak{a}\mathcal{P}(\mathcal{M})$, the result follows.

□

Remark. I wish to remark that although $\mathcal{P}$ is a right exact functor, it does not arise as the left adjoint of any functor $\mathscr{Mod}_{S_\infty} \to \mathscr{wP}$. Indeed, let $f : \mathbb{N} \to \mathbb{N}$ be any arithmetic function that diverges to infinity. Then consider the weak patching system $\mathcal{M} = (S_\infty/\mathfrak{m}^f(n))$. It follows easily that $\mathcal{P}(\mathcal{M}) \cong S_\infty$, for any sequence $f(n)$ as above. Letting $N \in \mathscr{Mod}_{S_\infty}$ and F be the supposed right adjoint of $\mathcal{P}$, we see that

$$N \cong \mathrm{Hom}_{S_\infty}(\mathcal{P}(\mathcal{M}), N) \cong \mathrm{Hom}_{\mathscr{wP}}(\mathcal{M}, F(N)).$$

Let K_n be the n-th entry of $F(S_\infty) \in \mathscr{wP}$, and let $I_n = \mathrm{Ann}(K_n)$. Let a_n be the smallest positive integer such that $\mathfrak{m}^{a_n} \subset I_n$, if this exists. otherwise, let $a_n = \infty$, and set $m_n = \min(n, a_n - 1)$. There are now two cases:

1. For some $N \in \mathbb{N}$, the set $A_N := \{n \in \mathbb{N} : m_n \leqslant N\} \in \mathcal{F}$.

2. For every $N \in \mathbb{N}$, $\mathbb{N}\backslash A_N \in \mathcal{F}$.

In the first case, the natural map $S_\infty \to K_n$ factors through $S_\infty/\mathfrak{m}^N$, for $\mathcal{F}$-many $n \in \mathbb{N}$, and thus, we exhibit non-zero maps $\mathrm{Hom}_{\mathscr{wP}}(\underline{S_\infty/\mathfrak{m}^N}, F(S_\infty))$, where $\underline{S_\infty/\mathfrak{m}^N}$ denotes the constant sequence. However, since $\mathcal{P}(\underline{S_\infty/\mathfrak{m}^N}) = S_\infty/\mathfrak{m}^N$, we see that

$$\mathrm{Hom}_{S_\infty}(\mathcal{P}(\underline{S_\infty/\mathfrak{m}^N}), S_\infty) = 0,$$

and thus, $F(S_\infty)$ cannot satisfy the first condition.

In the second case, the sequence m_n diverges to infinity on a subsequence in $\mathcal{F}$, and thus, setting $f(n) = m_n$ in the definition $\mathcal{M} = (S_\infty/\mathfrak{m}^{m_n})$ from before gives us $\mathcal{P}(\mathcal{M}) = S_\infty$, so $\mathrm{Hom}(\mathcal{P}(\mathcal{M}), S_\infty) = S_\infty$. However, each K_n does *not* annihilate $\mathfrak{m}^{m_n}$, so the only possible S_∞-homomorphism $S_\infty/\mathfrak{m}^{m_n} \to K_n$ is the zero map. Thus, $\mathrm{Hom}_{w\mathscr{P}}(\mathcal{M}, F(S_\infty)) = 0$, and so $F(S_\infty)$ cannot satisfy the second condition. It thus follows that There is no functor F that can be a right adjoint for $\mathcal{P}$.

Definition 7.2.5. *Let $\mathcal{R} = (R_n)$ be a weak patching algebra. We say a covering of $\mathcal{R}$ is a pair $(R_\infty, (\phi_n))$ consisting of a complete local S_∞-algebra R_∞, of dimension $\dim(S_\infty)$ and continuous surjective S_∞-algebra homomorphisms $\phi_n : R_\infty \to R_n$.*

Lemma 7.2.6. *Let $(R_\infty, (\phi_n))$ be a cover of a weak patching algebra $\mathcal{R}$. Then there is a surjection $R_\infty \twoheadrightarrow \mathcal{P}(\mathcal{R})$ of S_∞-algebras.*

Proof. As each ϕ_n is surjective, it follows, for any open ideal $J \trianglelefteq S_\infty$ that the composition $R_\infty \to R_n \to R_n/J$ is surjective, and thus, because the cardinalities (R_n/J) are finite and uniformly bounded, the map $R_\infty \to \prod^{\mathcal{F}} R_n/J$ is surjective. Hence, after we take the inverse limit, we get a surjection $R_\infty \to \mathcal{P}(\mathcal{R})$. □

Theorem 7.2.7. *Suppose we have the following data:*

- *A patching algebra $\mathcal{R}$ over R_0,*
- *A patching system $\mathcal{M}$, which is a patching module of $\mathcal{R}$.*
- *A cover R_∞ of $\mathcal{R}$.*

Suppose further that each M_n is free over $S_\infty/\mathcal{I}_n$ with rank bounded above by $N \in \mathbb{N}$. Then $M_\infty := \mathcal{P}(\mathcal{M})$ is a maximal Cohen Macaulay module over R_∞, and $\mathcal{P}(\mathcal{M})/\mathfrak{a} = M_0$.

Proof. As each M_n is free, Proposition 7.2.4 tells us that $\mathcal{P}(\mathcal{M})$ is a free S_∞-module. Further, since $\mathcal{M}$ is a patching module of $\mathcal{R}$, it follows that the action of $S_\infty \to \mathrm{End}(\mathcal{P}(\mathcal{M}))$ factors through $\mathcal{P}(\mathcal{R})$. Hence, As R_∞ is a cover of $\mathcal{R}$ it follows from the previous lemma that the action also factors through the S_∞-algebra homomorphism $R_\infty \to \mathcal{P}(\mathcal{R})$.

Hence, we get the following inequalities:

$$\dim(S_\infty) = \mathrm{depth}_{S_\infty}(M_\infty) \leqslant \mathrm{depth}_{R_\infty} \leqslant \dim(R_\infty)$$

but since $\dim(R_\infty) = \dim(S_\infty)$ by hypothesis, it follows we have equality throughout and thus that M_∞ is a maximal Cohen-Macaulay module of R_∞. The last part follows from part 4 of Proposition 7.2.4. $\square$

Chapter 8

The extremely inconsiderate Steinberg moduli space for GL_3

8.1 Introduction

The following chapter is the result of joint work with Jack Shotton.

Let F be a local field, with residue field $\mathbb{F}_q$ of order $q = p^r$ and let G be a reductive group defined over $\mathbb{Z}$, all as before.

Recall we have the moduli space of tame parameters of [DHKM23], $Z^1[W_F^\circ/P_F, G]_\mathcal{O}$. In the split case we have the model over $\mathbb{Z}(\frac{1}{q})$:

$$\mathrm{Loc}_{G,F}^{\square,t}(R) = \{(\Phi, \Sigma) \in G(R) \times G(R) : \Phi\Sigma\Phi^{-1} = \Sigma^q\}$$

In Chapters 2-4 of this thesis, we studied the geometry of the unipotent component of this space over the local ring $\mathbb{Z}_l$, for l a prime distinct from p. In particular, it was shown in Theorem 3.0.1 that when q is considerate towards $G/\mathbb{Z}_l$, (that is, when the order of q in $\mathbb{F}_l$ is greater than the Coxeter number h_G of G) then the Steinberg component, $\mathcal{X}_{St} \subseteq \mathrm{Loc}_{G,F}^{\square,t}$ is smooth. In this case, the smoothness of this variety gives rise to regular deformation rings, which consequently as shown in

Chapter 5, through the Taylor-Wiles-Kisin patching method, forces the maximal Cohen-Macaulay patched module M_∞ to be a free module. When this module is of rank one (which we expect), we gain access to a multiplicity 1 result for mod l automorphic forms.

This smoothness result only arises because along the Steinberg component, the Frobenius matrix Φ is regular semisimple, whose eigenvalues (in the case of $G = \mathrm{GL}_n$) lie in the ratio $1 : q : q^2 : ... : q^{h_G - 1}$. Consequently, when q is *inconsiderate* (that is, unsurprisingly, not considerate), we no longer get this regular semisimplicity property of Φ and so this smoothness result actually fails over $\mathbb{Z}_l$ in this case. As we will see in section 8.2, when $G = \mathrm{GL}_3$ and $q \equiv 1 \mod l$, (one can say that in this situation that q is *extremely inconsiderate* towards $G_{/\mathbb{Z}_l}$, though in the terminology of Definition 5.1.1 of [CHT08] it is called *quasi-banal*), then $\mathcal{X}_{St,\mathbb{F}}$ is closely related to the scheme over $\mathbb{F}$:

$$X(R) = \{(M, N) \in \mathcal{N}(R) \times \mathcal{N}(R) : [M, N] = 0\}$$

where $\mathcal{N}$ is the nilpotent cone inside $\mathfrak{gl}_3$. This space is very singular and so the patched modules M_∞ are usually not free. Consequently, this ought to lead to larger multiplicities of mod l Hecke eigenforms with a given Galois representation than we see in the characteristic zero case.

In this chapter, we study (using methods of Snowden, Vilonen and Xue, and Ngo; [Sno18], [VX16] and [Ngo18] respectively) the Steinberg irreducible component $\mathcal{X}_{St}$ in the extremely inconsiderate setting and show that $X = \mathcal{X}_{St,\mathbb{F}}$ and as a consequence $\mathcal{X}_{St}$ is reduced, normal, and has resolution-rational singularities (in the terminology of [Kov22]). We then use this to both give equations for X (see section 8.5) and to calculate the Weil-class group of X (see section 8.6).

8.2 Steinberg deformation rings; resolution

Let l and p be distinct primes and let q be a power of p. Let $n \geqslant 1$ be an integer. Let $\mathcal{O}$ be the ring of integers in a finite extension E of $\mathbb{Q}_l$, with residue field $\mathbb{F}$.

Let $G = \mathrm{GL}_{n,\mathcal{O}}$. Then we have the affine scheme $\mathcal{M}$ whose R-points, for an $\mathcal{O}$-algebra R, are given by

$$\mathcal{M}(R) = \{(\Phi, \Sigma) \in G(R) \times G(R) : \Phi\Sigma\Phi^{-1} = \Sigma^q\}.$$

From Corollary 2.4 and Proposition 2.7 of [DHKM23], this is an affine complete intersection over $\mathcal{O}$ of relative dimension $\dim G$, is flat over $\mathcal{O}$, and is reduced.

The irreducible components of $\mathcal{M}$ are in bijection with the q-stable conjugacy classes of Σ. Our aim is to study the component corresponding to the regular unipotent conjugacy class:

$$\mathcal{X}_{St} = \overline{\{(\Phi, \Sigma) \in \mathcal{M}(\bar{E}) : \Sigma \text{ regular unipotent}\}}.$$

As $\mathcal{X}_{St}$ is defined as the closure of an open subset of $\mathcal{M}(\bar{E})$, it is affine, reduced, and flat over $\mathcal{O}$; *a priori*, however, we do not have explicit equations for it.

Theorem 8.2.1. *Let $n = 2$ or 3. Suppose that $l > 2$ if $n = 2$, l is sufficiently large (as defined in remark 8.4, see also Conjecture 8.4.4) if $n = 3$, and that $q = 1 \mod l$. Then $\mathcal{X}_{St}$ is Cohen-Macaulay and $\mathcal{X}_{St} \otimes_{\mathcal{O}} \mathbb{F}$ is reduced, normal and has resolution-rational singularities (see [Kov22]).*

Remark. For $n = 2$ this is proved in [Sho16] by explicit calculation. We will include a different proof here, as an illustration of our methods for GL_3.

From now on, we assume:

Assumption 8.2.2. We have $l > n$ and $q \equiv 1 \mod l$.

On $\mathcal{X}_{St}$, the eigenvalues of Φ are in the ratio $1 : q : \ldots : q^{n-1}$. Since $q \equiv 1 \mod l$ and $l \nmid n$, there is then an isomorphism $\mathbb{G}_m \times \mathcal{X} \xrightarrow{\sim} \mathcal{X}_{St}$ where

$$\mathcal{X} = \{(\Phi, N) \in \mathcal{X}_{St} : \mathrm{tr}(\Phi) = 1 + q + \ldots + q^{n-1}.\}$$

It will be technically more convenient to work with $\mathcal{X}$.

As $l > n$, the logarithm map $\Sigma \mapsto \log(\Sigma - 1)$ is well defined for strongly unipotent Σ (that is, its characteristic polynomial is $(x-1)^n$). Hence, on $\mathcal{X}$ we may write $\Sigma = \exp(N)$ for a strongly nilpotent matrix N. On the special fibre $\mathcal{X}_{\mathbb{F}}$, Φ is strongly unipotent , so we may write $\Phi = \exp(M)$ for a strongly nilpotent matrix M.

We let $X = \mathcal{X}_{\mathbb{F}}^{red}$. Then the map

$$X \to \mathfrak{g}_{\mathbb{F}} \times \mathfrak{g}_{\mathbb{F}}$$
$$(\Phi, \Sigma) \mapsto (\log(\Phi), \log(\Sigma)$$

maps the open subset U of X of those (Φ, Σ) where Σ is regular unipotent isomorphically onto the locally closed subscheme of $\mathfrak{g}_{\mathbb{F}} \times \mathfrak{g}_{\mathbb{F}}$ of those pairs (M, N) of commuting strongly nilpotent matrices with N regular nilpotent. Hence, we see X is the Zariski closure in $\mathfrak{g}_{\mathbb{F}} \times \mathfrak{g}_{\mathbb{F}}$ of the set of pairs M, N of nilpotent matrices in $\mathfrak{g}(\bar{\mathbb{F}})$ such that $MN = NM$ and N is regular. because the above isomorphism extends to the closure.

8.2.1 Resolution of $\mathcal{X}$

Let $\mathcal{F}$ be the flag variety for $G = \mathrm{GL}_n$ over $\mathcal{O}$. We can write a flag $F \in \mathcal{F}(R)$ as $0 \subset F_{n-1} \subset \ldots \subset F_0 = R^n$ with the F_i projective R-modules such that $\mathrm{gr}_i(F_\bullet)$ are all projective. We define

$$\mathcal{Y} = \{(\Phi, N, F) \in G \times \mathfrak{g} \times \mathcal{F} : (\Phi - q^i)F_i \subset F_{i+1}, NF_i \subset F_{i+1}, \mathrm{ad}_\Phi(N) = qN\}.$$

Lemma 8.2.3. *The morphism $f : \mathcal{Y} \to \mathcal{X}$ given by forgetting F is a projective morphism that is an isomorphism over the open subset of $\mathcal{X}$ on which N is regular or l is invertible.*

Proof. The scheme $\mathcal{Y}$ is a closed subscheme of $\mathcal{X} \times \mathcal{F}$ and $\mathcal{F}$ is projective, thus $\mathcal{Y}$ is projective. Let U denote the open subset of $\mathcal{X}$ where l is invertible, or N is regular.

If l is invertible, it follows that q is not a root of unity and thus, that Φ is regular semisimple. If either Φ or N are regular, it follows that the flag defined above is unique and can be chosen algebraically as follows:

- When N is regular;

$$N \mapsto 0 \subset \ker(N) \subset \ker(N^2) \subset ... \subset \ker(N^n) = R^n.$$

- When l is invertible; $\Phi \mapsto F_\Phi$, where F_Φ is the flag with $F_{\Phi,i} = \bigoplus_{j=i}^n \ker(\Phi - q^i I_n)$.

Further, because $\mathrm{Ad}(\Phi)(N) = qN$, these flags agree when l is invertible and N is regular. This gives rise to a well defined inverse morphism $(f|_U)^{-1} : U \to \mathcal{Y}$ on the open subset U. $\square$

We will also require the scheme $\mathcal{Z}$ defined exactly as $\mathcal{Y}$ but without the closed condition $\mathrm{ad}_\Phi(N) = qN$. We thus have a closed embedding $\mathcal{Y} \hookrightarrow \mathcal{Z}$ fitting into the diagram below:

$$\begin{tikzcd}
\mathcal{Z} \arrow[dr, "\pi_{\mathcal{Z}}"'] & & \mathcal{Y} \arrow[ll, hook'] \arrow[dl, "\pi_{\mathcal{Y}}"] \arrow[r, "f"] & \mathcal{X} \\
& \mathcal{F} & &
\end{tikzcd}$$

We note the following facts.

Lemma 8.2.4. *1. The scheme $\mathcal{Z}$ is reduced and $\mathcal{O}$-flat.*

2. If $n \leqslant 3$, $\mathcal{Y}$ is $\mathcal{O}$-flat and is reduced along the special fibre.

Proof. 1. Let $b \in \mathcal{F}$ be a point. Then there is an open affine subscheme U of $\mathcal{F}$, with $U = \mathrm{Spec}(A)$, such that the projection $\mathrm{GL}_n \to \mathcal{F}$ has a section $\gamma : U \to \mathrm{GL}_n$. Notice that $\gamma \in G(A)$, so the universal pair (Φ, N) takes the form

$$(\gamma(\Phi_0 + M)\gamma^{-1}, \gamma N \gamma^{-1})$$

with $\Phi_0 = \mathrm{Diag}(q^{n-1}, ..., q, 1)$ and $M, N \in \mathfrak{n}$. It is now easy to see that $\mathcal{Z} \times_{\mathcal{F}} U \cong U \times \mathfrak{n}^2$. Thus $\mathcal{Z}$ is a vector bundle over $\mathcal{F}$ and hence is reduced and $\mathcal{O}$-flat.

2. When $n = 2$, $\mathcal{Y} = \mathcal{Z}$, so it follows from part 1. For $n = 3$, the argument of part 1 gives similarly that $\mathcal{Y} \times_{\mathcal{F}} U \cong U \times \mathcal{C}(\mathfrak{n})$, where

$$\mathcal{C}(\mathfrak{n}) = \{(M, N) \in \mathfrak{n}^2 | (\Phi_0 + M)N - qN(\Phi_0 + M) = 0\}$$

Let $M = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{pmatrix}$. Then the equation defining $\mathcal{C}(\mathfrak{n})$ reduces to $(q^2 - 1)e + af - dc = 0$. This equation is not divisible by a uniformiser of $\mathcal{O}$, so $\mathcal{C}(\mathfrak{n})$ and hence $\mathcal{Y}$ is $\mathcal{O}$-flat. To get the reducedness result, set $Y = \mathcal{Y}_{\mathbb{F}}$ and $C(\mathfrak{n}) = \mathcal{C}(\mathfrak{n})_{\mathbb{F}} = \mathbb{V}(ad - dc)$. As $af - dc \in \mathbb{F}[a, b, c, d, e, f]$ is homogeneous of degree 2 and has no linear factors (it would otherwise be the union of hyperplanes), we see $C(\mathfrak{n})$ is reduced and irreducible. Thus, as Y has $Y \times_F U \cong U \times C(\mathfrak{n})$ for some open cover $\{U\}$ of F, we see that Y is reduced.

□

Remark. For $n \geqslant 4$, it should be noted that the space $C(\mathfrak{n}^2)$ of commuting strictly upper triangular matrices becomes significantly more complicated. For example, when $n \geqslant 4$, $C(\mathfrak{n}^2)$ is not irreducible (see Example A of [Bas08], and for $n \geqslant 18$, it doesn't even have pure dimension (see Example D [Bas08]). Instead, we would need to take the irreducible component of $C(\mathfrak{n}^2)$ arising from the open subset of pairs (M, N) with at least one M, N regular nilpotent. Unfortunately, this space may also not behave very well. One can show that for $n \leqslant 5$ this irreducible component is Cohen-Macaulay, but as of yet we don't have results here in general (though this is a work in progress). One should also note, that there are other additional complications that arise for $n \geqslant 3$, which is that the cohomological calculations necessary for this method become increasingly complicated. Even with computer programmes, it is difficult to go much further beyond $n = 6$, and in addition, the best programmes utilise the BGG resolution, which works only in characteristic 0, so don't give effective lower bounds for the valid characteristics as in Theorem 8.2.1.

Recall that $X = \mathcal{X}_{\mathbb{F}}^{\mathrm{red}}$. Since Y is reduced, the morphism $Y \to \mathcal{X}_{\mathbb{F}}$ factors through X. This map we will now denote by $\bar{f}$. This gives us the following diagram on the reduced fibres:

$$\begin{tikzcd} Z \arrow[dr, "\pi"'] & & Y \arrow[ll, hook'] \arrow[dl, "\pi_Y"] \arrow[r, "\bar{f}"] & X \\ & F & & \end{tikzcd}$$

The proof of the next theorem occupies most of the rest of the paper.

Theorem 8.2.5. *Suppose that $n = 2$ and $l > 2$ or $n = 3$ and $l > 5$. Then*

1. *The morphism $f : Y \to X$ is a rational resolution of singularities (see Definition 8.2.6);*

2. *The variety X is Cohen-Macaulay, with $X = \operatorname{Spec}\Gamma(Y, \mathcal{O}_Y)$;*

3. *There is an isomorphism of canonical sheaves $\omega_X \cong f_*\omega_Y$.*

It is worth noting that the property of rational singularities is well understood only in zero characteristic, whereas we work purely in the positive and mixed characteristic case. So we recall a definition of Kovàcs (Definition 1.3 of [Kov22])

Definition 8.2.6. *We say an arbitrary X has resolution-rational singularities if:*

- *X is an excellent scheme that admits a dualising complex;*

- *There is an excellent scheme Y and a proper birational morphism $f : Y \to X$ which induces a isomorphisms*

$$\mathcal{O}_X \xrightarrow{\sim} Rf_*\mathcal{O}_Y$$

and

$$Rf_*\omega_Y \xrightarrow{\sim} \omega_X$$

in the derived category $\mathcal{D}_{\mathrm{coh}}(X)$*.*

Remark. 1. A definition of excellent scheme can be found here: [Sta23, Definition 07QT]. By Proposition [Sta23, Proposition 07QW], it follows that all of the schemes we consider (which are finite type schemes over either a finite field or a complete discrete valuation ring) are excellent. This is thus a property that will not feature much in what follows.

2. For a more detailed discussion of the multiple definitions for a scheme to have 'rational singularities' in positive/mixed characteristic and the various logical implications and equivalences, one can read [Kov22], of which Theorem 9.12 is particularly enlightening.

8.2.2 Proof of Theorem 8.2.1

In this section, we explain how to prove Theorem 8.2.1 from Theorem 8.2.5. As $\mathcal{X}_{St} = \mathcal{X} \times \mathbb{G}_m$, we need only prove the theorem for $\mathcal{X}$. Let $B = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and $A = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Thus we have a morphism $A \to B$ that we would like to show is an isomorphism.

By flat base change, $B \otimes_{\mathcal{O}} E = \Gamma(\mathcal{Y}_E, \mathcal{O}_{\mathcal{Y}_E}) = \Gamma(\mathcal{X}_E, \mathcal{O}_{\mathcal{X}_E}) = A \otimes_{\mathcal{O}} E$. Moreover, as $\mathcal{Y} \to \mathcal{X}$ is proper, B is a finite A-algebra.

Lemma 8.2.7. *We have* $B \otimes_{\mathcal{O}} \mathbb{F} = \Gamma(Y, \mathcal{O}_Y)$*.*

Proof. Using the short exact sequence $0 \to \mathcal{O}_{\mathcal{Y}} \xrightarrow{\times\varpi} \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_Y \to 0$ it suffices to show that $H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = 0$. Using that $\mathcal{X}$ is affine, this is equivalent to showing that $R^1 f_*(\mathcal{O}_{\mathcal{Y}}) = 0$. By Theorem 8.2.5, $R^i f_*(\mathcal{O}_Y) = 0$ for $i \geqslant 1$ and, in particular, $(R^1 f_*\mathcal{O}_{\mathcal{Y}}) \otimes \mathbb{F} = 0$ (using the short exact sequence again). But we also have $R^1 f_*\mathcal{O}_{\mathcal{Y}} \otimes E = 0$ by flat base change and the fact that $\mathcal{Y} \otimes \mathbb{Q}_p \to \mathcal{X} \otimes \mathbb{Q}_p$ is an isomorphism.

Now $R^1 f_* \mathcal{O}_{\mathcal{Y}}$ is a finitely-generated A-module (as f is proper) M such that $M \otimes_{\mathcal{O}} \mathbb{F} = M \otimes_{\mathcal{O}} E = 0$, from which it follows that $M = 0$. $\square$

Proposition 8.2.8. *The map $A \to B$ is an isomorphism.*

Proof. We know the proposition after inverting l. We claim that $A \to B$ is surjective. After $\otimes \mathbb{F}$, this follows as, by the previous lemma, $B \otimes \mathbb{F} = \Gamma(Y, \mathcal{O}_Y) = (A \otimes \mathbb{F}_p)^{red}$. But then the cokernel, a finite A-module, vanishes after $\otimes E$ and $\otimes \mathbb{F}$ and so must be zero, as at the end of the previous proof.

Now, if $A \to B$ is a surjective map of flat $\mathcal{O}$-algebras that is an isomorphism after inverting l, the kernel must have ϖ-torsion; but as A is ϖ-torsion free the kernel must vanish, whence the proposition. $\square$

Proof of Theorem 8.2.1. We have $\mathcal{X} = \mathrm{Spec}(A)$ with A ϖ-torsion free, so that ϖ is a regular element of A. By Proposition 8.2.8, $A \otimes_{\mathcal{O}} \mathbb{F} = B \otimes_{\mathcal{O}} \mathbb{F} = \Gamma(Y, \mathcal{O}_Y)$. Since Y is reduced, so is $A \otimes_{\mathcal{O}} \mathbb{F}$. By Theorem 8.2.5, $X = \mathrm{Spec}(A \otimes \mathbb{F})$ is Cohen–Macaulay, with resolution-rational singularities. Finally, in Lemma 8.6.1 we will show that the singular locus of X has codimension 2 which will show X to be normal by Serre's criterion. $\square$

Notice, that *a priori*, we did not assume that $\mathcal{X}_{\mathbb{F}}$ was reduced and instead worked with the reduction $X = (\mathcal{X}_{\mathbb{F}})^{\mathrm{red}}$. As a consequence of the isomorphism $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{\sim} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, we deduce above the isomorphism along the special fibres, which gives

$$H^0(\mathcal{X}_{\mathbb{F}}, \mathcal{O}_{\mathcal{X}_{\mathbb{F}}}) \xrightarrow{\sim} H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \otimes \mathbb{F} \xrightarrow{\sim} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \otimes \mathbb{F} \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y)$$

As Y is reduced, we can thus deduce as a result the corollary:

Corollary 8.2.9. *The scheme $\mathcal{X}_{\mathbb{F}} \cong X$ is reduced.*

8.3 Vector bundles on the flag variety

Our starting point is the following Theorem of Snowden ([Sno18]).

Lemma 8.3.1. *Let $f : Y \to X$ be a proper birational map of schemes over $\mathbb{F}$, with Y Cohen–Macaulay and X affine. Suppose that:*

- *the cohomology groups $H^i(Y, \mathcal{O}_Y) = H^i(Y, \omega_Y) = 0$ for each $i > 0$,*
- *the pullback map $f^* : H^0(X, \mathcal{O}_X) \hookrightarrow H^0(Y, \mathcal{O}_Y)$ is an isomorphism.*

Then X is Cohen-Macaulay, with $\omega_X \cong f_*(\omega_Y)$.

Proof. This is Lemma 2.1.4 of [Sno18]. □

We will apply this with X and Y the varieties over $\mathbb{F}$ introduced in section 8.2.1. The proof of the following theorem will occupy the next section.

Theorem 8.3.2. *Let $Y = \mathcal{Y}_{\mathbb{F}}, X = (\mathcal{X}_{\mathbb{F}})^{red}$ be as above and $f : Y \to X$ be the proper birational map from before. Suppose either $n = 2$ and $l > 2$ or $n = 3$ and $l > 5$, then the hypotheses of 8.3.1 hold. In particular, Theorem 8.2.5 holds.*

Let $\pi : Y \to F$ be the natural map to the flag variety F (over $\mathbb{F}$). We also write Z for the fibre over $\mathbb{F}$ of $\mathcal{Z}$ from above, so that Z is a vector bundle over F and

$$Y \hookrightarrow Z$$

is a closed immersion of varieties over F (if $n = 2$ then it is an isomorphism!). Since π is affine, for any coherent sheaf $\mathcal{V}$ on Z we have $H^i(Z, \mathcal{V}) = H^i(F, \pi_*\mathcal{V})$ (and similarly for sheaves on Y). This is the starting point of our analysis.

We now follow [VX16] closely. Working always over the field $\mathbb{F}$, we let $G = SL_n$ and take B to be the Borel subgroup of *lower* triangular matrices, with T the standard torus and U the unipotent radical. Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{n}$ be their Lie algebras. We choose the system of positive and simple roots in G corresponding to the *upper* triangular Borel subgroup; thus the roots in $\mathfrak{n}$ are negative roots. The representation $\mathfrak{g}$ is self-dual via the trace pairing and under this pairing $\mathfrak{b} = \mathfrak{n}^\perp$; thus $\mathfrak{n}^* \cong \mathfrak{g}/\mathfrak{b}$ as B-representations. We write ρ for half the sum of the positive roots.

As $F = G/B$, there is an equivalence of abelian categories between G-equivariant vector bundles on F and representations of B on finite-dimensional $\mathbb{F}$-vector spaces. We will abuse notation and conflate representations of B with the corresponding equivariant sheaves. From the adjoint action of B on $\mathfrak{g}$ we obtain sheaves $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{g}/\mathfrak{b}$, $\mathfrak{n}$. If $\chi \in X(T)$ then we denote by $\mathcal{O}(\chi)$ the corresponding equivariant line bundle on F; for V a G-equivariant vector bundle on F we write $V(\chi) = V \otimes_{\mathcal{O}_F} \mathcal{O}_F(\chi)$. If $\mathcal{F}$ is a sheaf on Z, we will denote $\mathcal{F}(\chi) := \mathcal{F} \otimes_{\mathcal{O}_Z} \pi^*\mathcal{O}_F(\chi)$.

Lemma 8.3.3. *For $\mathcal{F}$ an $\mathcal{O}_Z$-module and χ a character of $\mathfrak{g}$, we have the isomorphism*

$$\pi_*[\mathcal{F}(\chi)] \cong (\pi_*\mathcal{F})(\chi).$$

Proof. Using the fact that π_* preserves tensor product and for any $\mathcal{O}_F$-module $\mathcal{G}$, $\pi_*\pi^*\mathcal{G} \cong \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2) \otimes_{\mathcal{O}_F} \mathcal{G}$, we can make the following isomorphisms;

$$\begin{aligned}
\pi_*[\mathcal{F}(\chi)] &\cong \pi_*[\mathcal{F} \otimes_{\mathcal{O}_Z} \pi^*(\mathcal{O}_F(\chi))] \\
&\cong \pi_*\mathcal{F} \otimes_{\pi^*\mathcal{O}_Z} (\pi_*\pi^*[\mathcal{O}_F(\chi)]) \\
&\cong \pi_*\mathcal{F} \otimes_{\mathrm{Sym}(\mathfrak{g}/\mathfrak{b}^2)} \mathrm{Sym}(\mathfrak{g}/\mathfrak{b}^2)(\chi) \\
&\cong \pi_*\mathcal{F} \otimes_{\mathcal{O}_F} \mathcal{O}_F(\chi) \cong [\pi_*\mathcal{F}](\chi)
\end{aligned}$$

□

Proposition 8.3.4. *Let $\pi : Z \to F$ be as in the previous section. Note that $\pi : Z \to F$ is G-equivariant and so is its restriction to Y. We have the following isomorphisms of G-equivariant vector bundles on F:*

$$\pi_*\mathcal{O}_Z \cong \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2) \tag{8.3.1}$$

$$\pi_*\omega_Z \cong \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)(2\rho) \tag{8.3.2}$$

and, if $n = 3$, then

$$\pi_*\mathcal{I}_Y \cong \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)(\rho) \tag{8.3.3}$$

$$\pi_*\omega_Y \cong (\pi_*\mathcal{O}_Y)(\rho). \tag{8.3.4}$$

Proof. 1. The fibres of $\pi : Z \to F$ over a Borel β are equal to pairs of matrices $x, y \in \mathfrak{g}$ with x, y inside the derived subalgebra $[\beta, \beta]$ of $\beta \in F$. Thus, we can view Z as the total space of the equivariant G-bundle $\mathfrak{n}^2$. Therefore

$$Z = \underline{\mathrm{Spec}}_F(\mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)),$$

as $\mathfrak{g}/\mathfrak{b}$ is the dual of $\mathfrak{n}$. Therefore, $\pi_*\mathcal{O}_Z \cong \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)$.

2. Recall that $\omega_Z \cong \omega_{Z/F} \otimes_{\mathcal{O}_Z} \pi^*\omega_F$ because π is smooth and F is regular. We also recall from [Jan03] II. §4.2 that $\omega_F \cong \mathcal{O}(-2\rho)$. It remains to calculate $\omega_{Z/F}$. Notice that the relative tangent bundle has $T_{Z/F} = \pi^*((\mathfrak{n})^2)$, so $\omega_{Z/F} \cong \pi^* \det((\mathfrak{g}/\mathfrak{b})^2) = \pi^*(\mathcal{O}(4\rho))$. We hence see

$$\omega_Z \cong \pi^*(\mathcal{O}(-2\rho) \otimes \mathcal{O}(4\rho)) = \pi^*(\mathcal{O}(2\rho)).$$

As the functors $\pi_*\pi^*(\cdot)$ and $\cdot \otimes_{\mathcal{O}_F} \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)$ are isomorphic, we see that

$$\pi_*(\omega_Z) \cong \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)(2\rho).$$

3. We note that this lemma can also be found as Proposition 5.1 of [Ngo18] when the characteristic is zero. As we are working in characteristic l, we include the following proof.

Let U be any appropriate open subset of F for which $\pi^{-1}(U) \cong U \times \mathfrak{n}^2$, under the coordinates of $\mathfrak{n}^2$ defined by $(x, y) = ([x_{ij}], [y_{ij}])$ for $1 \leqslant i < j \leqslant 3$, the ideal $\mathcal{I}_Y(\pi^{-1}(U))$ is generated by the single element $x_{1,2}y_{2,3} - y_{1,2}x_{2,3}$, an element of weight ρ. Hence, we see that is a locally principal G-equivariant quasi-coherent sheaf on F. Hence, we see that in the category $\mathcal{R}ep_B$ it is generated as a $\mathrm{Sym}([\mathfrak{g}/\mathfrak{b}]^2)$-module (viewed as a ring with a B-representation structure) by a single element of weight ρ, and hence is isomorphic to $\mathrm{Sym}([\mathfrak{g}/\mathfrak{b}]^2)(\rho)$. Returning to $\mathcal{O}_F - \mathcal{mod}$ and upgrading to $\mathcal{O}_Z$-modules, we get the isomorphism of $\mathcal{O}_Z$-modules $\mathcal{I}_Y \cong \mathcal{O}(\rho)$.

4. Finally, $Y \subseteq Z$ is a closed subset of codimension 1 and since $Y \to F$ is a

complete intersection morphism, Y is Cohen-Macaulay. Hence we get

$$\omega_Y \cong \mathfrak{Ext}^1_Z(\mathcal{O}_Z/\mathcal{I}_Y, \omega_Z)$$

From the short exact sequence,

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$$

we get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{Hom}_Z(\mathcal{O}_Y, \omega_Z) \arrow[r] & \mathfrak{Hom}_Z(\mathcal{O}_Z, \omega_Z) \arrow[r] & \mathfrak{Hom}_Z(\mathcal{I}_Y, \omega_Z) \arrow[dll, out=-10, in=170] \\ & \mathfrak{Ext}^1_Z(\mathcal{O}_Y, \omega_Z) \arrow[r] & \mathfrak{Ext}^1_Z(\mathcal{O}_Z, \omega_Z) \arrow[r] & \dots \end{tikzcd}$$

As $\mathcal{O}_Z$ is the structure sheaf, we automatically get $\mathfrak{Hom}_Z(\mathcal{O}_Z, \omega_Z) = \omega_Z$ and $\mathfrak{Ext}^1_Z(\mathcal{O}_Z, \omega_Z) = 0$ by Proposition III.6.3 of [Har77]. As π is an affine morphism, the functor π_* is exact, hence we get an exact sequence

$$0 \to \pi_* \mathfrak{Hom}_Z(\mathcal{O}_Y, \omega_Z) \to \pi_*\omega_Z \to \pi_* \mathfrak{Hom}_Z(\mathcal{I}_Y, \omega_Z) \to \pi_*\omega_Y \to 0$$

We calculate for $U \subseteq F$ open

$$\begin{aligned} \pi_* \mathfrak{Hom}_Z(\mathcal{I}_Y, \omega_Z)(U) &\cong \mathrm{Hom}_{\mathcal{O}_Z|_{\pi^{-1}(U)}}(\pi^*(\mathcal{O}(\rho))|_{\pi^{-1}(U)}, \omega_Z|_{\pi^{-1}(U)}) \\ &\cong \mathrm{Hom}_{\mathcal{O}_F|_U}(\mathcal{O}(\rho)|_U, \pi_*\omega_Z|_U) \\ &\cong \pi_*\omega_Z(-\rho)(U) \end{aligned}$$

hence $\pi_* \mathfrak{Hom}_Z(\mathcal{I}_Y, \omega_Z) \cong \pi_*\omega_Z(-\rho)$ and

$$\begin{aligned} \pi_*\omega_Y &\cong \mathrm{coker}\left[\mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)(2\rho) \to \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)(\rho)\right] \\ &\cong \mathrm{coker}\left[\mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)(\rho) \to \mathrm{Sym}((\mathfrak{g}/\mathfrak{b})^2)\right](\rho) \\ &\cong \mathrm{coker}\left[\mathcal{I}_Y \to \mathcal{O}_Z\right](\rho) \\ &\cong \mathcal{O}_Y(\rho) \end{aligned}$$

This completes the proof.

□

In the following example, we will give a demonstration of how we wish to utilise Lemma 8.3.1, in a concrete example when $n = 2$. In this case, the idea of the proof is the same as that found in section 3.3 of [Sno18], but we include the proof here, to clearly demonstrate how the idea generalises for other connected reductive groups.

Proof of Theorem 8.3.2 when $n = 2$. Suppose that $G = \mathrm{GL}_2$, so that $F = \mathbb{P}^1$ and $Y = Z$ is the total space of the vector bundle $\mathfrak{n}^2$. As an ordinary line bundles on $\mathbb{P}^1$, we see that $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}^* \cong \mathcal{O}(2)$ and $\mathcal{O}(\rho) \cong \mathcal{O}(1)$ so that $H^1(\mathbb{P}^1, \mathrm{Sym}^r[(\mathfrak{g}/\mathfrak{b})^2]) = H^1(\mathbb{P}^1, \mathrm{Sym}^r[(\mathfrak{g}/\mathfrak{b})^2](2\rho)) = 0$ for every $r \geqslant 0$. Thus, the cohomology groups $H^i(Y, \mathcal{O}_Y) = H^i(F, \mathrm{Sym}[(\mathfrak{g}/\mathfrak{b})^2]) = 0$ and $H^i(Y, \omega_Y) = H^i(F, \mathrm{Sym}[(\mathfrak{g}/\mathfrak{b})^2](2\rho)) = 0$ for each $i > 0$. To show the theorem, it suffices now to show that $f^* : H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is an isomorphism.

For convenience, we set $R = H^0(X, \mathcal{O}_X)$ and $\tilde{R} = H^0(Y, \mathcal{O}_Y)$. We note that there is a surjection

$$H^0(F, \mathrm{Sym}[\mathfrak{g}^2]) = \mathbb{F}[\mathfrak{g}^2] \to R$$

because X is defined as a closed subscheme of $\mathfrak{g}^2$. Let I be the kernel of this surjection. The composite

$$H^0(F, \mathrm{Sym}[\mathfrak{g}^2]) \to R \xrightarrow{f^*} \tilde{R} = H^0(F, \mathrm{Sym}[(\mathfrak{g}/\mathfrak{b})^2])$$

is the morphism induced from the natural epimorphism of coherent sheaves $\mathfrak{g}^2 \to (\mathfrak{g}/\mathfrak{b})^2$.

Letting $S = \mathbb{F}[\mathfrak{g}]$ and S_+ be the irrelevant ideal of S, Proposition 2.1.5 of [Sno18] gives us that $\mathrm{Tor}^0(R, \mathbb{F}) = \tilde{R}/S_+\tilde{R} \cong H^0(\mathcal{O}_F) \oplus H^1(\mathfrak{b}^2)$ and $\mathrm{Tor}^1(R, \mathbb{F}) = I/S_+I \cong H^0(\mathfrak{b}^2) \oplus H^1(\Lambda^2[\mathfrak{b}^2])$. Because $\mathfrak{b} \cong \mathcal{O}(-1)^2$, this shows that $\tilde{R}/S_+\tilde{R}$ is 1-dimensional and further that I/S_+I is 6-dimensional. Hence, $\mathbb{F}[\mathfrak{g}^2] \to H^0(Y, \mathcal{O}_Y)$ is surjective and has kernel generated by 6 elements in degree 2. It thus follows that the map $H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is an isomorphism. Thus $X = \mathrm{Spec}(\Gamma(Y, \mathcal{O}_Y))$ is Cohen-Macaulay with dualising module $f_*\omega_Y$. □

Remark. It is possible to modify the above proof to work in the case $l = 2$ as well.

Proof of Theorem 8.3.2 when $n = 3$. For this proof, we will a priori assume that $H^i(Y, \mathcal{O}_Y) = H^i(Y, \omega_Y) = 0$ for all $i > 0$ and defer the proof of these calculations to section 3.4 (See Proposition 8.4.9). Thus, to prove the theorem, the only thing that remains to check is that the natural morphism $H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is an isomorphism. As the map $f : Y \to X$ is dominant, the above homomorphism is injective provided X is reduced. To show that $f_* : H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is surjective, define $S = \mathbb{F}[\mathfrak{g}^2] = \mathrm{Sym}[\mathfrak{g}^2]$, with irrelevant ideal S_+. Let $T = H^0(Z, \mathcal{O}_Z)$ for Z the total space of the vector bundle $[\mathfrak{g}/\mathfrak{b}]^2$ on G/B. Recall from Proposition 2.1.5 of [Sno18] that

$$T/S_+T = \mathrm{Tor}_0^S(T, \mathbb{F}) = \bigoplus H^i(F, \Lambda^i[\mathfrak{b}^2])[i]$$

By our calculations, we know that $H^i(F, \Lambda^i[\mathfrak{b}^2]) = 0$ unless $i = 0$, when $H^0(F, \Lambda^0[\mathfrak{b}^2]) = H^0(F, \mathbb{F}) = \mathbb{F}$. Thus, the map $S \to T$ is surjective and as Y is a closed subvariety of Z, that the composition map $S \to H^0(Y, \mathcal{O}_Y)$ is surjective. Since it factors through $H^0(X, \mathcal{O}_X)$, we see that the map $f_* : H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is surjective. This proves the isomorphism. □

8.4 Cohomological calculations for sheaves on the flag variety GL_3/B

Let k be an arbitrary field. Let G, B and T be GL_3, the Borel of *negative* roots and torus of diagonal matrices defined over k. Let $X(T)$ be the character lattice, $X^\vee(T)$ the cocharacter lattice and $\langle , \rangle : X(T) \times X^\vee(T) \to \mathbb{Z}$ the natural pairing. For a cocharacter $\lambda \in X(T)$, let $k_\lambda = k(\lambda)$ be the corresponding B-representation and recalling the equivalence of categories $\mathcal{Coh}_G(G/B) \simeq \mathcal{Rep}_B$, let $\mathcal{L}_\lambda$ be the corresponding G-equivariant line bundle on G/B. Let Φ^+ be the set of positive roots and W be the Weyl group of G. Recall the 'dot' action of W on $X(T)$ is defined through

$w.\lambda := w(\lambda) - \lambda$ where $w(\cdot)$ denotes the natural action of W on $X(T)$. We recall below the Borel-Weil-Bott Theorem.

Theorem 8.4.1 (The Borel-Weil-Bott Theorem)**.** *Consider $G_{/k}$ an algebraic group over a field k. Denote*

$$\overline{C}_{\mathbb{Z}} = \{\lambda \in X(T) : 0 \leqslant \langle \lambda + \rho, \beta^\vee \rangle \textit{ for all } \beta \in \Phi^+\}$$

when $\operatorname{char}(k) = 0$ *and*

$$\overline{C}_{\mathbb{Z}} = \{\lambda \in X(T) : 0 \leqslant \langle \lambda + \rho, \beta^\vee \rangle \leqslant l \textit{ for all } \beta \in \Phi^+\}$$

when $\operatorname{char}(k) = l$*. Then for $\lambda \in \bar{C}_{\mathbb{Z}}$ and $w \in W$, the Weyl group, $H^i(G/B, \mathcal{L}_{w\cdot\lambda}) = 0$ unless $i = l(w)$, in which case*

$$H^{l(w)}(G/B, \mathcal{L}_{w\cdot\lambda}) \cong H^0(G/B, \mathcal{L}_\lambda) \cong \operatorname{ind}_B^G(k_\lambda)$$

Proof. See section II, Corollary 5.5 of [Jan03]. □

We make the following calculations. We notice that by choosing l large enough, that all the potential supports of the $B-$representations lie within the region $\bigcup_{w \in W} w \cdot (\bar{C}_{\mathbb{Z}})$ where the Borel-Weil-Bott theorem applies. We will call this region the BWB locus. The content of the Borel-Weil-Bott theorem is that within the BWB-locus of $X(T)$ (shown inside the dashed blue line in figure 8.1 below), $C^i_{\mathbb{Z}}$ is the support of the functor H^i. The subset of the BWB-locus not contained in any $C^i_{\mathbb{Z}}$ (denoted by the solid black lines in figure 8.1) does not contribute to the cohomology groups at all.

It will also be useful to note that since $G/B \to \operatorname{Spec}(k)$ is a proper map, that we have an induced homomorphism of G-equivariant Grothendieck groups:

$$\begin{aligned} K_G(G/B) &\to K_G(*) \\ \mathcal{F} &\mapsto \sum_{i=0}^{\dim(G/B)} (-1)^i [H^i(G/B, \mathcal{F})] \end{aligned}$$

We will denote this map by χ, the Euler characteristic. Notice that $K_G(*) = K(\mathcal{Rep}_G)$ and when $G = \mathrm{GL}_3$ this is isomorphic to $\mathbb{Z}[x_1, x_2]$ generated by $x_1 = [V]$, the standard representation, and $x_2 = [V^*]$, the dual of V.

For E a B-representation, let

$$\mathrm{PSupp}(E) = \{\lambda \in X(T) : k_\lambda \text{ appears as a subquotient of } E\}$$

be the support of a B-representation, viewed as a multiset with multiplicity. Let $C^0_{\mathbb{Z}} = \bar{C}_{\mathbb{Z}} \cap X^+(T)$ and for $i > 0$ let $C^i_{\mathbb{Z}} = \coprod_{l(w)=i} w \cdot C^0_{\mathbb{Z}}$. Let $P^i(E)$ be the intersection of $\mathrm{PSupp}(E)$ with $C^i_{\mathbb{Z}}$ We define $\mathrm{PSupp}^i(E)$ to be the multiset image $\Theta(P^i(E))$ of the cardinality-preserving morphism of multisets

$$\begin{aligned} \Theta : \mathrm{PSupp}^i(E) &\to \mathbb{N}^{C^0_{\mathbb{Z}}} \\ \lambda &\mapsto H^i(k_\lambda) \end{aligned}$$

We also let

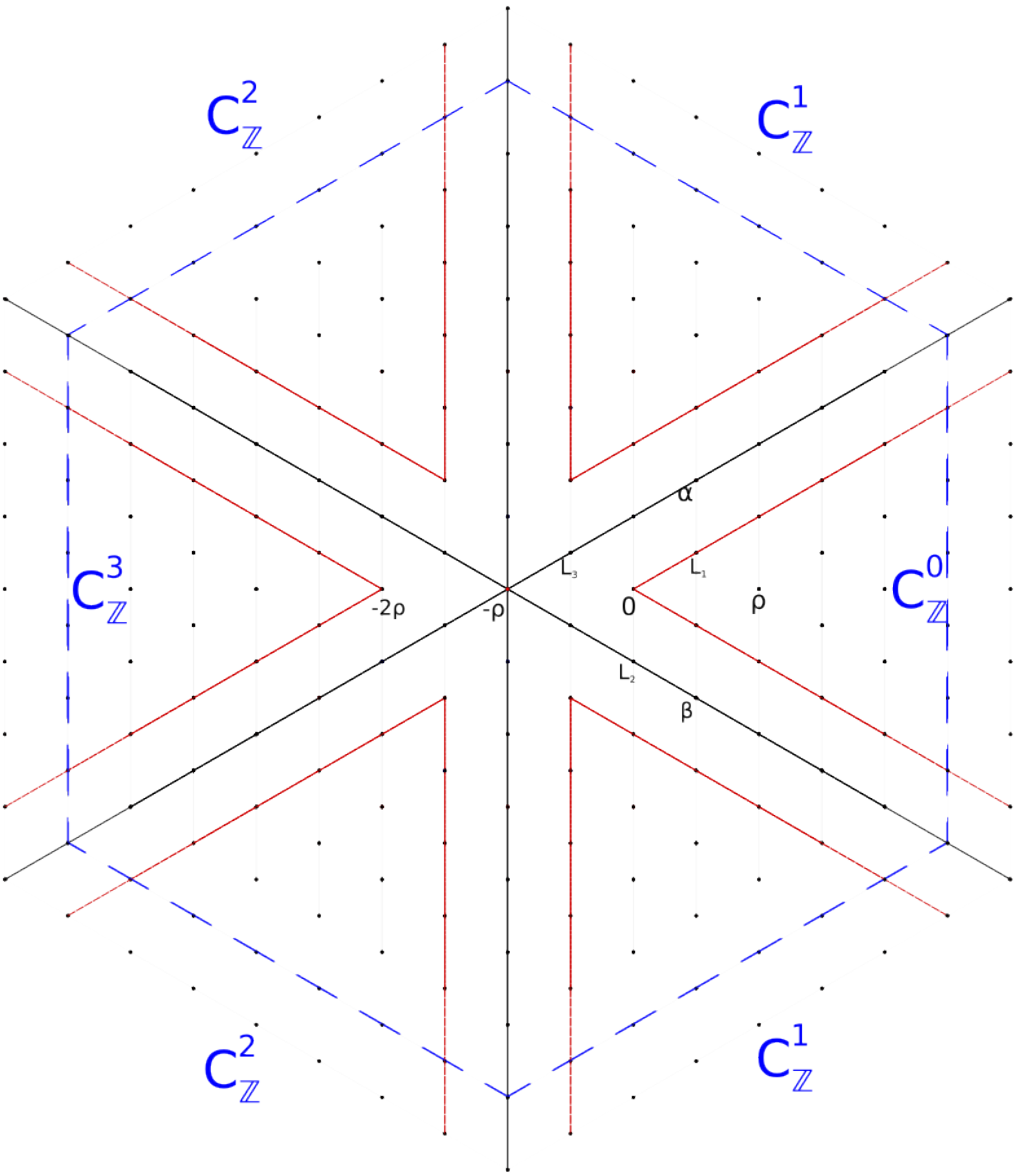
$$\mathrm{Supp}^i(E) = \{\lambda \in C^0_{\mathbb{Z}} : H^i(E) \text{ contains } V(\lambda) := H^0(k_\lambda) \text{ as a subrepresentation}\}$$

be the support of the G-representation $H^i(E)$, also considered as a multiset with multiplicity. Since we can take a composition series of a B-representation E; from which it follows that the cohomology group $H^i(E)$ is a subrepresentation of $\bigoplus_{\lambda \in \mathrm{PSupp}^i(E)} H^0(k_\lambda)$ provided that $\mathrm{PSupp}(E)$ is inside the BWB locus. It then follows, that $\mathrm{Supp}^i(E) \subseteq \mathrm{PSupp}^i(E)$ and hence, as a corollary, whenever $\mathrm{PSupp}^i(E) = \varnothing$, then $\mathrm{Supp}^i(E) = \varnothing$ and $H^i(E) = 0$.

We will also need the fact that whenever E is a B-representation and V a G-representation, then $H^i(V \otimes E) \cong V \otimes H^i(E)$ for all i. We further record the fact that $\mathrm{PSupp}(\mathfrak{b}) = \{0, 0, -\rho, -\alpha, -\beta\}$.

Figure 8.1 (found below) shows the BWB-locus when $p = 7$ inside the blue dashed region and shows the different regions $C^i_{\mathbb{Z}}$ inside the regions bounded by the red lines, as well as the weights, L_1, L_2, L_3, $\alpha, \beta, \rho, -\rho$ and -2ρ.

Figure 8.1: A depiction of the root space of $\mathfrak{sl}_3$. Equivalently, each vertex in the lattice corresponds to a unique SL_3-equivariant line bundle of the flag variety $\mathcal{F} = \mathrm{GL}_3/B$.



Calculating $H^i(\Lambda^j[\mathfrak{b} \oplus \mathfrak{b}])$

Calculations for $H^i(\mathfrak{b}), H^i(\mathfrak{b} \otimes \mathfrak{b})$ and $H^i(\mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{b})$, from which some of our results below can be deduced, can be found in [VX16] in the case of $k = \mathbb{C}$. We include the proofs of these calculations anyway, because we are in particular interested in which finite characteristics l these calculations occur within the BWB locus, to show that they are valid in characteristic l.

Remark. During the proofs of each calculation, we will proceed with the calculation as though the characteristic were not important and only thereafter will we discuss how large the characteristic $l = \mathrm{char}(k)$ needs to be to make the BWB locus large enough to contain all the representations inside.

Calculation 8.4.2. Let $n = 3$ and suppose that the characteristic $l = \mathrm{char}(k) \geqslant 7$. Then the G-representations $H^i(\Lambda^j \mathfrak{b})$ are tabulated as follows:

j \ i	0	1	2	3
0	k	·	·	·
1	·	·	·	·
2	·	k	·	·
3	·	·	k	·
4	·	·	·	·
5	·	·	·	k

Table 8.1: The cohomology of $H^i(\Lambda^j \mathfrak{b})$

Proof. We now calculate the cohomology groups of $\Lambda^j \mathfrak{b}$.

0. When $j = 0$, $\Lambda^j \mathfrak{b} = k$ and $H^i(k) = 0$ unless $i = 0$, in which case $H^0(\Lambda^0 \mathfrak{b}) = \mathbb{F}$.

1. When $j = 1$, we note that $\mathrm{PSupp}(\mathfrak{b})$ has $\mathrm{PSupp}^2(\mathfrak{b}) = \mathrm{PSupp}^3(\mathfrak{b}) = \varnothing$, so $H^2(\mathfrak{b}) = H^3(\mathfrak{b}) = 0$.

   Further, there is a short exact sequence

$$0 \to \mathfrak{b} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{b} \to 0$$

   which gives a long exact sequence in cohomology

$$
\begin{tikzcd}
0 \arrow[r] & H^0(\mathfrak{b}) \arrow[r] & H^0(\mathfrak{g}) \arrow[r] & H^0(\mathfrak{g}/\mathfrak{b}) \arrow[dll] \\
 & H^1(\mathfrak{b}) \arrow[r] & H^1(\mathfrak{g}) \arrow[r] & \dots
\end{tikzcd}
$$

As $\mathfrak{g}$ is a G-representation, we see that $H^0(\mathfrak{g}) = \mathfrak{g}$ and $H^1(\mathfrak{g}) = 0$. The module $\mathfrak{g}/\mathfrak{b}$ is unsupported outside $\mathrm{PSupp}^0(\mathfrak{g}/\mathfrak{b}) = \{\rho\}$, thus $H^0(\mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$. Since the support of $\mathfrak{b}$ is $\mathrm{PSupp}(\mathfrak{b}) = \{0, 0, -\rho, \alpha, -\beta\}$, we see that $H^0(\mathfrak{b})$ and $H^1(\mathfrak{b})$ are both subrepresentations of a trivial module, we see that they are both zero.

2. When $j = 2$, consider the exact sequence

$$0 \to \mathfrak{b} \otimes \mathfrak{b} \to \mathfrak{g} \otimes \mathfrak{b} \to \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b} \to 0$$

which gets a long exact sequence with parts

$$H^i(\mathfrak{g} \otimes \mathfrak{b}) \to H^i(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}) \to H^{i+1}(\mathfrak{b} \otimes \mathfrak{b}) \to H^{i+1}(\mathfrak{g} \otimes \mathfrak{b}).$$

As $H^i(\mathfrak{b}) = 0$ for all i, we get isomorphisms $H^i(\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \cong H^{i+1}(\mathfrak{b} \otimes \mathfrak{b})$ for all i. Thus, $H^0(\mathfrak{b} \otimes \mathfrak{b}) = 0$ and $H^3(\mathfrak{b} \otimes \mathfrak{b}) = H^2(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}) = 0$, since $\mathrm{PSupp}^2(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}) = \varnothing$.

For $H^2(\mathfrak{b} \otimes \mathfrak{b})$, consider first the exact sequence

$$0 \to \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to 0$$

giving rise to the long exact sequence

$$0 \to H^0(\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to H^0(\mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b}) \to H^0(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to H^1(\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to H^1(\mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b}).$$

Notice that $H^1(\mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b}) \cong \mathfrak{g} \otimes H^1(\mathfrak{g}/\mathfrak{b}) = 0$ and $H^0(\mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b}) \cong \mathfrak{g} \otimes H^0(\mathfrak{g}/\mathfrak{b}) = \mathfrak{g} \otimes \mathfrak{g}$ so that $H^1(\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = \mathrm{coker}(\mathfrak{g} \otimes \mathfrak{g} \to H^0(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}))$.

One can calculate that $\mathrm{PSupp}^0(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = \{2\rho, \rho+\alpha, \rho+\alpha, \rho+\beta, \rho+\beta\}$ which implies $H^1(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b})$ is a quotient of a non-trivial representation, which implies that $0 \notin \mathrm{Supp}^2(\mathfrak{b} \otimes \mathfrak{b})$. But since we can directly check that $\mathrm{PSupp}^2(\mathfrak{b} \otimes \mathfrak{b}) =$

$\{0,0\}$, we see that $H^2(\mathfrak{b}\otimes\mathfrak{b})=0$.

To get $H^1(\mathfrak{b}\otimes\mathfrak{b})$, it now suffices to consider the additive function $\chi(\mathfrak{b}\otimes\mathfrak{b})=4[0]-4[0]-4[0]+2[0]+2[0]-[0]=-[0]$. Hence, $\sum_j(-1)^j[H^j(\mathfrak{b}\otimes\mathfrak{b})]=-[0]$ implies that $H^1(\mathfrak{b}\otimes\mathfrak{b})=k$. As $\Lambda^2\mathfrak{b}$ is a direct summand of $\mathfrak{b}\otimes\mathfrak{b}$, it follows that $H^i(\Lambda^2\mathfrak{b})=0$ when $i\neq 1$. Again, as

$$\chi(\Lambda^2\mathfrak{b})=[0]-4[0]+2[0]=-[0]$$

We see that $H^1(\Lambda^2\mathfrak{b})=k$.

3. When $j=3$, Notice that there is a B-equivariant pairing $\Lambda^3\mathfrak{b}\otimes\Lambda^2\mathfrak{b}\to\Lambda^5\mathfrak{b}\cong k(-2\rho)$ which gives rise to the isomorphism $\Lambda^3\mathfrak{b}\cong[\Lambda^2\mathfrak{b}]^\vee\otimes k(-2\rho)$. Hence, by Serre duality, (identifying $\omega_F\cong k(-2\rho)$) we see

$$H^i(\Lambda^3\mathfrak{b})\cong H^i([\Lambda^2\mathfrak{b}]^\vee\otimes\omega_F)\cong H^{3-i}(\Lambda^2\mathfrak{b})^\vee$$

. The result then follows from the result for part 2.

4. When $j=4$, we see from the same pairing $\Lambda^4\mathfrak{b}\otimes\mathfrak{b}\to\omega_F$ that $H^i(\Lambda^4\mathfrak{b})\cong H^{3-i}(\mathfrak{b})^\vee=0$.

5. Finally, when $j=5$, we see that $\Lambda^5\mathfrak{b}\cong k(-2\rho)$. So $H^i(\Lambda^5\mathfrak{b})=0$ unless $i=3$, in which case $H^3(\Lambda^5\mathfrak{b})=k$.

In calculations $0-2$, all representations are subquotients of $\mathfrak{g}\otimes\mathfrak{g}$, and in $3-5$, we used the pairings $\Lambda^i\mathfrak{b}\otimes\Lambda^{5-i}\mathfrak{b}\to k(-2\rho)$ to calculate these cohomologies. Notice, that all of these representations are in the BWB locus when the characteristic $l=\mathrm{char}(k)\geqslant 7$. This proves the claim on the characteristic. □

Calculation 8.4.3. 1. Suppose $l=\mathrm{char}(k)\geqslant 11$. Then the B-representation $\mathfrak{b}\otimes\Lambda^2\mathfrak{b}$ has cohomology $H^i(\mathfrak{b}\otimes\Lambda^2\mathfrak{b})=\mathfrak{g}^2\oplus k$ when $i=2$ and vanishes otherwise.

2. Suppose $l=\mathrm{char}(k)\geqslant 11$. Then the cohomology group $H^3(\mathfrak{b}\otimes\Lambda^3\mathfrak{b})=0$.

3. Suppose $l = \mathrm{char}(k) \geqslant 11$. Then the cohomology group $H^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) \leqslant \mathfrak{g}^2$.

Proof. 1. The short exact sequence

$$0 \to \Lambda^2\mathfrak{b} \otimes \mathfrak{b} \to \Lambda^2\mathfrak{b} \otimes \mathfrak{g} \to \Lambda^2\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to 0$$

along with the fact $H^i(\mathfrak{g} \otimes \Lambda^2\mathfrak{b}) = \mathfrak{g} \otimes H^i(\Lambda^2\mathfrak{b})$ gives rise to exact sequences in cohomology,

$$0 \to H^0(\Lambda^2\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}) \to H^1(\Lambda^2\mathfrak{b}\otimes\mathfrak{b}) \to \mathfrak{g} \to H^1(\Lambda^2\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}) \to H^2(\Lambda^2\mathfrak{b}\otimes\mathfrak{b}) \to 0,$$

$H^2(\Lambda^2\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}) \cong H^3(\Lambda^2\mathfrak{b}\otimes\mathfrak{b})$, and $H^0(\Lambda^2\mathfrak{b}\otimes\mathfrak{b}) = 0$. As $\mathrm{PSupp}^2(\Lambda^2\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}) = \varnothing$, we see that $H^3(\Lambda^2\mathfrak{b} \otimes \mathfrak{b}) = 0$.

It follows from looking at the supports of $\Lambda^2\mathfrak{b}\otimes\mathfrak{b}$ and $\Lambda^2\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}$ that: $H^0(\Lambda^2\mathfrak{b}\otimes \mathfrak{g}/\mathfrak{b}) = H^1(\Lambda^2\mathfrak{b} \otimes \mathfrak{b}) \cong k^n$, $H^1(\Lambda^2\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}^3 \otimes k^{n+1}$ and $H^2(\Lambda^2\mathfrak{b} \otimes \mathfrak{b}) = \mathfrak{g}^2 \oplus k^{n+1}$ with n some integer $0 \leqslant n \leqslant 7$.

The B-representation $\Lambda^2\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}$ is a direct summand of $\mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}$, which fits into the short exact sequence

$$0 \to \mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to 0$$

which after taking cohomology gives an exact sequence

$$0 \to H^0(\mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to \mathfrak{g} \otimes H^0(\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$$

which implies that $H^1(\mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{b}) = H^0(\mathfrak{b} \otimes \mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b})$ is at most the non-trivial G-representation $\mathfrak{g}$. Hence, as $H^0(\Lambda^2\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = k^n$ is a direct summand of $\mathfrak{g}$, it follows that $H^0(\Lambda^2\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = H^1(\Lambda^2\mathfrak{b} \otimes \mathfrak{b}) = 0$, giving the result.

To get the bound on the characteristic for which this calculation is valid, we note that the modules required for this calculation are subquotients (in $\mathcal{Rep}_B$) of: $k, \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}$, and $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$; which all have their potential supports contained inside the BWB locus when $l \geqslant 11$. Hence $l \geqslant 11$ is sufficient.

2. Using that $H^i(\Lambda^3\mathfrak{b}) = k$ only when $i = 2$ and vanishes otherwise, we can gain

from the short exact sequence $0 \to \Lambda^3\mathfrak{b} \otimes \mathfrak{b} \to \Lambda^3\mathfrak{b} \otimes \mathfrak{g} \to \Lambda^3\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b} \to 0$ an exact sequence

$$\mathfrak{g} \to H^2(\Lambda^3\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \to H^3(\Lambda^3\mathfrak{b} \otimes \mathfrak{b}) \to 0$$

The potential support $\mathrm{PSupp}^2(\Lambda^3\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) \subseteq \{0,0\}$ This allows us to see that $H^2(\Lambda^3\mathfrak{b} \otimes \mathfrak{g}/\mathfrak{b}) = k^n$ for some $0 \leqslant n \leqslant 2$. We thus see that $H^3(\Lambda^3\mathfrak{b} \otimes \mathfrak{b}) = k^n$.

There is an exact sequence given by

$$0 \to \Lambda^3\mathfrak{b} \to \Lambda^3\mathfrak{g} \to \Lambda^2\mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b} \to \mathfrak{g} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \to \mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b}) \to 0$$

Let $W = \Lambda^3\mathfrak{g}/\Lambda^3\mathfrak{b}$, and let $K = \ker(\mathfrak{g} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \to \mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b}))$. Then we have an exact sequence

$$0 \to \Lambda^3\mathfrak{b} \otimes \mathfrak{b} \to \Lambda^3\mathfrak{g} \otimes \mathfrak{b} \to W \otimes \mathfrak{b} \to 0$$

which, since $H^i(\mathfrak{b}) = 0$ for all i, induces isomorphisms $H^i(W \otimes \mathfrak{b}) \cong H^i(\Lambda^3\mathfrak{b} \otimes \mathfrak{b})$ for all i. The complex induces a S.E.S

$$0 \to W \otimes \mathfrak{b} \to \Lambda^2\mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b} \to K \otimes \mathfrak{b} \to 0$$

from which we get, because $H^i(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}) = 0$ for $i \neq 0$, an isomorphism $H^1(\mathfrak{b} \otimes C) \to H^2(W \otimes \mathfrak{b})$. Finally. considering the S.E.S

$$0 \to K \otimes \mathfrak{b} \to \mathfrak{g} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{b} \to \mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{b} \to 0$$

gives as part of the long exact sequence in cohomology

$$H^0(\mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{b}) \to H^1(K \otimes \mathfrak{b}) \to \mathfrak{g} \otimes H^1(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{b})$$

Recall that there is a surjection $H^0(\mathfrak{g}/\mathfrak{b} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \twoheadrightarrow H^1(\mathfrak{b} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}))$, and that we can calculate $H^0(\mathfrak{g}/\mathfrak{b} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})) = [3\rho] \oplus [2\rho + \alpha] + [2\rho + \beta] + 3[2\rho] \oplus 2[\rho + \alpha] + 2[\rho + \beta]$. Hence, (as all morphisms split in $\mathcal{Rep}_G$) $H^1(\mathfrak{b} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}))$ is a subrepresentation of $H^0(\mathfrak{g}/\mathfrak{b} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})$. It thus follows that $\mathfrak{g} \otimes H^1(\mathfrak{b} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}))$ is a subrepresentation of

$$H^0(\mathfrak{g} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{g}/\mathfrak{b})$$

Notice, that any $\lambda \in \mathrm{PSupp}(\mathfrak{g} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{g}/\mathfrak{b})$ satisfies

$$\langle \lambda, \rho \rangle \geqslant \langle \lambda, \alpha \rangle + \langle \lambda, \rho \rangle + \langle \lambda, -\rho \rangle = 1$$

from which it follows that 0 is not in $\mathrm{Supp}(\langle \lambda, \alpha \rangle)$, and thus that $H^0(\mathfrak{g} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{g}/\mathfrak{b})$ contains no trivial factor. Similarly, when we look at $H^0(\mathfrak{b} \otimes \mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b}))$, we see that any $\lambda \in \mathrm{PSupp}(\mathfrak{b} \otimes \mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b})$ satisfies

$$\langle \lambda, \rho \rangle \geqslant 3 + \langle \lambda, -\rho \rangle = 1.$$

and thus, that $H^0(\mathfrak{b} \otimes \mathrm{Sym}^3(\mathfrak{g}/\mathfrak{b}))$ contains no trivial factors. It thus follows that $H^1(K \otimes \mathfrak{b})$, and hence $H^3(\Lambda^3\mathfrak{b} \otimes \mathfrak{b})$, contains no trivial factors. This shows that $H^3(\Lambda^3\mathfrak{b} \otimes \mathfrak{b}) = 0$.

To get the bound on the characteristic for which this calculation is valid, we note that the modules required for this calculation are subquotients (in $\mathcal{Rep}_B$) of: $k, \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ and $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$; which all have their potential supports contained inside the BWB locus when $l \geqslant 11$.

3. Observe that $\mathrm{PSupp}^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) = \{(\rho, 2), (0, 17)\}$ and $\mathrm{PSupp}^2(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) = \{\rho + \alpha, \rho + \beta, (\rho, 8), (0, 28)\}$. We have a Koszul complex given by

$$0 \to \Lambda^2\mathfrak{b} \to \Lambda^2\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b} \to \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \to 0$$

Let $C = \Lambda^2\mathfrak{g}/\Lambda^2\mathfrak{b}$ so that we have two short exact sequences

$$0 \to \Lambda^2\mathfrak{b} \to \Lambda^2\mathfrak{g} \to C \to 0$$

and

$$0 \to C \to \mathfrak{g} \otimes \mathfrak{g}/\mathfrak{b} \to \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \to 0.$$

The first S.E.S gives us $H^0(W) = \Lambda^2\mathfrak{g} \oplus k$ and $H^i(W) = 0$, for $i > 0$, and the second gives us $H^0(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})) = L(2\rho)\oplus\mathfrak{g}$ and $H^i(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})) = 0$ for $i>0$.

Let $C = \Lambda^2\mathfrak{g}/\Lambda^2\mathfrak{b}\otimes\Lambda^2\mathfrak{b} = W\otimes\Lambda^2\mathfrak{b}$. Then the S.E.S

$$0\to\Lambda^2\mathfrak{b}\otimes\Lambda^2\mathfrak{b}\to\Lambda^2\mathfrak{g}\otimes\Lambda^2\mathfrak{b}\to C\to 0$$

gives us an isomorphism $H^2(C)\cong H^3(\Lambda^2\mathfrak{b}\otimes\Lambda^2\mathfrak{b})$. We can calculate

$$\mathrm{PSupp}^2(C) = \{2[\rho], 14[0]\}.$$

From the short exact sequence

$$0\to C\to\Lambda^2\mathfrak{g}\otimes W\to W\otimes W\to 0$$

We get

$$0\to H^0(C)\to\Lambda^2\mathfrak{g}\otimes\Lambda^2\mathfrak{g}\oplus\Lambda^2\mathfrak{g}\to H^0(W\otimes W)\to H^1(C)\to 0$$

and the isomorphism $H^1(W\otimes W)\cong H^2(C)$.

The short exact sequence

$$0\to\mathfrak{b}\otimes W\to\mathfrak{g}\otimes W\to\mathfrak{g}/\mathfrak{b}\otimes W\to 0$$

give us exact sequences

$$0\to H^0(\mathfrak{b}\otimes W)\to\mathfrak{g}\otimes H^0(W)\to H^0(\mathfrak{g}/\mathfrak{b}\otimes W)\to H^1(\mathfrak{b}\otimes W)\to 0$$

and $H^1(W\otimes\mathfrak{g}/\mathfrak{b})\cong H^2(W\otimes\mathfrak{b})$. Notice that $\mathrm{PSupp}^2(W\otimes\mathfrak{b}) = \{0,0\}$, so that $H^1(W\otimes\mathfrak{g}/\mathfrak{b}) = k^i$ is a trivial representation. On the other hand, the S.E.S

$$0\to W\otimes\mathfrak{g}/\mathfrak{b}\to\mathfrak{g}\otimes\mathfrak{g}/\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}\to\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})\otimes\mathfrak{g}/\mathfrak{b}\to 0$$

gives us

$$0\to H^0(W\otimes\mathfrak{g}/\mathfrak{b})\to\mathfrak{g}\otimes H^0(\mathfrak{g}/\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b})\to H^0(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})\otimes\mathfrak{g}/\mathfrak{b})\to H^1(W\otimes\mathfrak{g}/\mathfrak{b})\to 0$$

and because $W\otimes\mathfrak{g}/\mathfrak{b}$ has no potential support in C^2 or C^3, and $H^i(\mathfrak{g}/\mathfrak{b}\otimes\mathfrak{g}/\mathfrak{b}) = 0$ for all $i>0$, it follows that $H^i(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b})\otimes\mathfrak{g}/\mathfrak{b}) = 0$ whenever

$i > 0$, and hence we deduce $H^0(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{g}/\mathfrak{b}) = \chi(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes \mathfrak{g}/\mathfrak{b}) = [3\rho] + [2\rho + \alpha] + [2\rho + \beta] + 3[2\rho] + 2[\rho + \alpha] + 2[\rho + \beta]$ has no trivial part. Therefore, we see that $H^1(W \otimes \mathfrak{g}/\mathfrak{b}) = 0$

Then, we can look at the S.E.S

$$0 \to W \otimes W \to W \otimes \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{g} \to W \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes W \to 0$$

which gives us an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(W \otimes W) \arrow[r] & \mathfrak{g} \otimes H^0(W \otimes \mathfrak{g}/\mathfrak{b}) \arrow[r] & H^0(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes W) \arrow[dll, out=-10, in=170] \\ & & H^1(W \otimes W) \arrow[r] & 0 \end{tikzcd}$$

It is easily seen that $0 \notin \mathrm{PSupp}^0(\mathrm{Sym}^2(\mathfrak{g}/\mathfrak{b}) \otimes W)$, so that $H^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) \cong H^1(W \otimes W)$ has no trivial component.

Thus, $H^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) \leqslant \mathfrak{g}^2$.

To get the bound on the characteristic for which this calculation is valid, we note that the modules required for this calculation are subquotients (in $\mathcal{Rep}_B$) of: $k, \mathfrak{g}, \mathfrak{g}\otimes\mathfrak{g}, \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ and $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$; which all have their potential supports contained inside the BWB locus when $l \geqslant 11$.

□

Remark. The last part of this proof only proves a weaker statement than that we wish to prove.

Using the programs of Hemelsoet and Voorhaar [HV21], available at `https://github.com/RikVoorhaar/bgg-cohomology`, it is easy to verify that $H^3(\Lambda^4(\mathfrak{b} \oplus \mathfrak{b})) = 0$ over a field of characteristic zero. In particular, as $\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}$ is a direct summand of $\Lambda^4(\mathfrak{b} \oplus \mathfrak{b})$, it shows that $H^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) = 0$ vanishes in characteristic zero.

This implies that these cohomology groups vanish in sufficiently large positive characteristic; however, this bound is *not effective* since their algorithm relies on the BGG resolution, which only exists in characteristic zero.

We however, can make the following conjecture for an effective bound:

Conjecture 8.4.4. *When* $l = \mathrm{char}(k) \geqslant 11$*, the cohomology* $H^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) = 0$.

We believe this ought to be sufficient, as any argument as in Calculation 8.4.3 needs only to consider subquotients of $\mathfrak{g}^{\otimes n}$ up to $n = 4$, which is contains in the BWB locus for characteristic larger than 10.

We now wish to calculate the cohomology groups $H^i(\Lambda^j[\mathfrak{b} \oplus \mathfrak{b}])$.

Calculation 8.4.5. Suppose that $l = \mathrm{char}\, k$ is sufficiently large so that $H^3(\Lambda^2\mathfrak{b} \otimes \Lambda^2\mathfrak{b}) = 0$ ($l \geqslant 11$ is conjectured large enough).

The G-module $H^i(\Lambda^j[\mathfrak{b} \oplus \mathfrak{b}])$ is calculated as in the following table:

i / j	0	1	2	3
0	k	$\cdot$	$\cdot$	$\cdot$
1	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2	$\cdot$	k^3	$\cdot$	$\cdot$
3	$\cdot$	$\cdot$	$\mathfrak{g}^4 \oplus k^4$	$\cdot$
4	?	?	?	0

Table 8.2: The cohomology of $H^i(\Lambda^j[\mathfrak{b} \oplus \mathfrak{b}])$. Dots indicate trivial modules

Proof. 1. When $j = 0$, we see that $\Lambda^j(\mathfrak{b} \oplus \mathfrak{b}) = k$. Thus, $H^i(k)$ is trivial unless $i = 0$.

2. When $j = 1$, we see $\Lambda^1(\mathfrak{b} \oplus \mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{b}$. Hence the result follows from the vanishing $H^i(\mathfrak{b}) = 0$.

3. From the isomorphism $\Lambda^2(\mathfrak{b} \oplus \mathfrak{b}) \cong (\Lambda^2\mathfrak{b})^{\oplus 2} \oplus \mathfrak{b} \otimes \mathfrak{b}$ and Calculation 8.4.2, $H^i(\Lambda^2(\mathfrak{b} \oplus \mathfrak{b})) = H^i(\Lambda^2\mathfrak{b})^{\oplus 2} \oplus H^i(\mathfrak{b} \otimes \mathfrak{b}) = k^3$ when $i = 1$ and is trivial for all other i.

4. Notice $\Lambda^3(\mathfrak{b} \oplus \mathfrak{b}) = [\Lambda^3\mathfrak{b} \oplus \Lambda^2\mathfrak{b} \otimes \mathfrak{b}]^{\oplus 2}$. From Calculation 8.4.2, $H^i(\Lambda^3\mathfrak{b}) = k$ when $i = 2$ and is trivial otherwise. From Calculation 8.4.3 $H^i(\Lambda^2\mathfrak{b} \otimes \mathfrak{b}) = \mathfrak{g}^2 \oplus k$ when $i = 2$ and 0 otherwise, thus $H^i(\Lambda^3[\mathfrak{b} \oplus \mathfrak{b}]) = \mathfrak{g}^4 \oplus k^4$ when $i = 2$ and is zero for all other i.

5. We also need to calculate $H^3(\Lambda^4[\mathfrak{b}\oplus\mathfrak{b}]) = H^3(\Lambda^4\mathfrak{b})^2\oplus H^3(\Lambda^3\mathfrak{b}\otimes\mathfrak{b})^2\oplus H^3(\Lambda^2\mathfrak{b}\otimes\Lambda^2\mathfrak{b})$. From Calculation 8.4.2 we know that $H^3(\Lambda^4\mathfrak{b}) = H^0(\mathfrak{b})^\vee = 0$. From Calculation 8.4.3 we know that $H^3(\Lambda^2\mathfrak{b}\otimes\Lambda^2\mathfrak{b}) = H^3(\Lambda^3\mathfrak{b}\otimes\mathfrak{b}) = 0$. Thus, we get $H^3(\Lambda^4[\mathfrak{b}\oplus\mathfrak{b}]) = 0$

We simply remark that the conditions on the lower bound for the characteristic is simply that bound given by combining those of Calculations 8.4.2 and 8.4.3. □

Calculating $H^i(\Lambda^j[\mathfrak{b}\oplus\mathfrak{b}](\rho))$ and $H^i(\Lambda^j[\mathfrak{b}\oplus\mathfrak{b}](2\rho))$

Later, to give the result $Y \to X$ is a rational resolution of singularities, we will also need cohomology groups of $\mathcal{I}_Y \cong \mathcal{O}_Z(\rho)$ (see for example Theorem 3.1 of [Ngo18], and corollary 4.3 where Ngo proves the vanishing of these groups in characteristic 0). Further, as $\omega_Y \cong \mathcal{O}_Y(\rho)$, and we wish to calculate this sheaf, we will need also the cohomologies of $\mathcal{O}_Z(\rho)$ and $\mathcal{O}_Z(2\rho)$. For these, we must calculate $H^{k+i}(\Lambda^i\mathfrak{b}(\rho))$, $H^{k+i}(\Lambda^i\mathfrak{b}(2\rho))$, $H^{k+i}(\Lambda^i[\mathfrak{b}\oplus\mathfrak{b}](\rho))$, and $H^{k+i}(\Lambda^i[\mathfrak{b}\oplus\mathfrak{b}](2\rho))$ for $i \geqslant 0$ and $k > 0$.

Calculation 8.4.6. Suppose $l = \mathrm{char}(k) \geqslant 7$. Then we have the following: $H^i(\Lambda^k\mathfrak{b}(2\rho)) = 0$ whenever $i > 0$ and $H^i(\Lambda^k\mathfrak{b}(\rho)) = 0$ whenever $i > 1$.

Further,

$$\begin{aligned}
H^0(k(2\rho)) &= V(2\rho)\\
H^0(\mathfrak{b}(2\rho)) &= V(2\rho)^2 \oplus V(\rho+\alpha) \oplus V(\rho+\beta)\\
H^0(\Lambda^2\mathfrak{b}(2\rho)) &= V(2\rho) \oplus V(\rho+\alpha) \oplus V(\rho+\beta) \oplus \mathfrak{g}^3\\
H^0(\Lambda^3\mathfrak{b}(2\rho)) &= V(\rho+\alpha) \oplus V(\rho+\beta) \oplus \mathfrak{g}^3 \oplus k\\
H^0(\Lambda^4\mathfrak{b}(2\rho)) &= \mathfrak{g} \oplus k^2\\
H^0(\Lambda^5\mathfrak{b}(2\rho)) &= k
\end{aligned}$$

We also have

$$\begin{aligned}
H^0(k(\rho)) &= \mathfrak{g}\\
H^0(\mathfrak{b}(\rho)) &= \mathfrak{g}^2 \oplus k
\end{aligned}$$

$$H^1(\mathfrak{b}(\rho)) = 0$$
$$H^0(\Lambda^2\mathfrak{b}(\rho)) = \mathfrak{g} \oplus k$$
$$H^1(\Lambda^2\mathfrak{b}(\rho)) = 0$$

Proof. First, note that $\Lambda^k\mathfrak{b}(2\rho)$ has potential support contained in $\{0, \alpha, \beta, \rho, \rho + \alpha, \rho + \beta, 2\rho\}$, which has empty intersection with the regions $C^i_{\mathbb{Z}}$ for $i > 0$ and $\Lambda^k\mathfrak{b}(\rho)$ has potential support contained in $\{-\rho, -\beta, -\alpha, 0, \alpha, \beta, \rho\}$ which has empty intersection with the regions $C^i_{\mathbb{Z}}$ when $i = 2, 3$. This gives us the first part of the calculation. For $H^0(\Lambda^k\mathfrak{b}(2\rho))$, we simply notice that this is equal to the Euler characteristic $\chi(\Lambda^k\mathfrak{b}(2\rho))$, which gives us the next part of the result.

It is readily seen that $\mathfrak{b}(\rho)$ has potential support only in $C^0_{\mathbb{Z}}$, so the Euler characteristic gives us $H^0(\mathfrak{b}(\rho)) = \mathfrak{g}^2 \oplus k$.

To get $H^1(\Lambda^2(\mathfrak{b})(\rho))$, notice that $\mathrm{PSupp}^1(\Lambda^2(\mathfrak{b})(\rho)) = \{0, 0\}$ has only trivial potential support. Additionally, we have a short exact sequence:

$$0 \to \mathfrak{b} \otimes \mathfrak{b}(\rho) \to \mathfrak{g} \otimes \mathfrak{b}(\rho) \to \mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}(\rho) \to 0$$

from which we obtain an exact sequence

$$H^0(\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}(\rho)) \to H^1(\mathfrak{b} \otimes \mathfrak{b}) \to \mathfrak{g} \otimes H^1(\mathfrak{b}(\rho)) = 0$$

As $\mathrm{Supp}^0(\mathfrak{g}/\mathfrak{b}(\rho)) \subset \{(\rho, 3), (\rho + \alpha, 3), (\rho + \beta, 3), (2\rho, 2)\}$ it follows that $H^1(\mathfrak{b} \otimes \mathfrak{b}(\rho))$ has no trivial factors and as $\Lambda^2\mathfrak{b}(\rho)$ is a direct summand thereof, it follows that $H^1(\Lambda^2\mathfrak{b}(\rho)) = 0$. The Euler characteristic then gets us $H^0(\Lambda^2\mathfrak{b}(\rho)) = \mathfrak{g} \oplus k$.

To get the bound on the characteristic for which this calculation is valid, we note that the only modules required for this calculation (other than $\Lambda^i(\mathfrak{b})(\rho), \Lambda^i(\mathfrak{b})(2\rho)$ which as already discussed lie in the BWB locus for $l \geqslant 7$) are $\mathfrak{b} \otimes \mathfrak{b}(\rho), \mathfrak{g} \otimes \mathfrak{b}(\rho)$ and $\mathfrak{g}/\mathfrak{b} \otimes \mathfrak{b}(\rho)$. which all have their potential supports contained inside the BWB locus when $l \geqslant 7$. Therefore $l \geqslant 7$ is sufficient for all the above calculations.

□

Calculation 8.4.7. Suppose that $l = \mathrm{char}(k) \geqslant 7$. Then the cohomology groups $H^{k+i}(\Lambda^k[\mathfrak{b} \oplus \mathfrak{b}](\lambda))$ vanish for $\lambda = \rho, 2\rho$ whenever $i > 0$, or when $i = 0$ and $k > 0$.

Proof. We start with the assumption $i > 0$.

0. When $k = 0$, $\Lambda^0(\mathfrak{b} \oplus \mathfrak{b})(\lambda) = k(\lambda)$, so $H^i(k_\lambda) = 0$, because λ is in the dominant Weyl chamber.

1. When $k = 1$, $H^i([\mathfrak{b} \oplus \mathfrak{b}](\lambda)) = H^i(\mathfrak{b}(\lambda))^2 = 0$ for all $i > 0$ by Calculation 8.4.6.

2. When $k = 2$, if μ is a weight of $\Lambda^2[\mathfrak{b} \oplus \mathfrak{b}](\lambda)$, then the inner product $\langle \rho, \mu \rangle \geqslant \langle \rho, -2\rho + \lambda \rangle = -4 + \langle \rho, \lambda \rangle \geqslant -2$, because $\lambda = \rho$ or 2ρ. Hence, the support of $\Lambda^2[\mathfrak{b} \oplus \mathfrak{b}](\lambda)$ does not intersect $C^2_{\mathbb{Z}}$ or $C^3_{\mathbb{Z}}$, so $H^k(\Lambda^2[\mathfrak{b} \oplus \mathfrak{b}](\lambda)) = 0$ for $k = 2, 3$.

3. When $k = 3$, we see that $\Lambda^3[\mathfrak{b} \oplus \mathfrak{b}](\lambda) = \Lambda^3\mathfrak{b}(\lambda)^2 \oplus \Lambda^2\mathfrak{b} \otimes \mathfrak{b}(\lambda)^2$. As every weight, μ of $\mathfrak{b}$ satisfies $\langle \rho, \mu \rangle \geqslant -2$, every weight μ of $\Lambda^2\mathfrak{b}$ satisfies $\langle \rho, \mu \rangle \geqslant -3$, and every weight μ of $\Lambda^3\mathfrak{b}$ satisfies $\langle \rho, \mu \rangle \geqslant -4$, we see that every weight μ of $\Lambda^3[\mathfrak{b} \oplus \mathfrak{b}](\lambda) = \Lambda^3\mathfrak{b}(\lambda)^2 \oplus \Lambda^2\mathfrak{b} \otimes \mathfrak{b}(\lambda)^2$ has

$$\langle \rho, \mu \rangle \geqslant \min(-2 - 3, -4) + \langle \rho, \lambda \rangle \geqslant -5 + \langle \rho, \lambda \rangle.$$

   As $\lambda = \rho$ or 2ρ, we get that $\langle \rho, \lambda \rangle = 2$ or 4. Thus, every weight μ of $\Lambda^3[\mathfrak{b} \oplus \mathfrak{b}](\lambda)$ has $\langle \rho, \mu \rangle \geqslant -3$. On the other hand, if μ is a weight inside $C^3_{\mathbb{Z}}$, we see that $\langle \rho, \mu \rangle \leqslant \langle \rho, -2\rho \rangle = -4$. Thus, we see that there can be no intersection of the potential support $\mathrm{PSupp}(\Lambda^3[\mathfrak{b} \oplus \mathfrak{b}](\lambda))$ with $C^3_{\mathbb{Z}}$, so $H^3(\Lambda^3[\mathfrak{b} \oplus \mathfrak{b}](\lambda)) = 0$.

Now assume $i = 0$. Then:

4. When $k = 1$, $H^1(\Lambda^1[\mathfrak{b} \oplus \mathfrak{b}](\lambda)) = H^1(\mathfrak{b}(\lambda))^2 = 0$ by Calculation 8.4.6.

5. When $k = 2$, $H^2(\Lambda^2[\mathfrak{b} \oplus \mathfrak{b}](\lambda)) = H^2(\Lambda^2(\mathfrak{b})(\lambda))^2 \oplus H^2(\mathfrak{b} \otimes \mathfrak{b}(\lambda))$. Calculation 8.4.6 tells us that $H^2(\Lambda^2(\mathfrak{b})(\lambda)) = 0$. If μ is a weight of $\mathfrak{b} \otimes \mathfrak{b}(\lambda)$, then we have

$$\langle \rho, \mu \rangle \geqslant \langle \rho, -\rho - \rho + \lambda \rangle \geqslant -2 - 2 + 2 = -2$$

   Which leaves $\mathfrak{b} \otimes \mathfrak{b}(\lambda)$ unsupported in $C^2_{\mathbb{Z}}$ and thus, $H^2(\mathfrak{b} \otimes \mathfrak{b}(\lambda)) = 0$.

6. When $k = 3$, we note that $H^3(\Lambda^3[\mathfrak{b}\oplus\mathfrak{b}](\lambda)) = H^3((\Lambda^3(\mathfrak{b})(\lambda))^2 \oplus H^3(\Lambda^2\mathfrak{b}\otimes\mathfrak{b}(\lambda))^2$. Calculation 8.4.6 tells us $H^3((\Lambda^3(\mathfrak{b})(\lambda)) = 0$, so we turn our attention to $H^3(\Lambda^2\mathfrak{b}\otimes\mathfrak{b}(\lambda))$. Again, if μ is a weight of $\Lambda^2\mathfrak{b}\otimes\mathfrak{b}(\lambda)$, we see

$$\langle\rho,\mu\rangle \geqslant \langle\rho, -\rho-\alpha-\rho+\lambda\rangle \geqslant -3-2+2 = -3$$

Which shows that since any $\mu \in C^3_{\mathbb{Z}}$ has $\langle\rho,\mu\rangle \leqslant -4$, that $\Lambda^2\mathfrak{b}\otimes\mathfrak{b}(\lambda)$ is unsupported in $C^3_{\mathbb{Z}}$. This gives $H^3(\Lambda^2\mathfrak{b}\otimes\mathfrak{b}(\lambda)) = 0$ and the result.

We now discuss the characteristic $l = \mathrm{char}(k)$. Because Calculation 8.4.6 is valid for $l \geqslant 7$, so are the subcalculations 0, 1 and 4. As $\Lambda^2(\mathfrak{b}\oplus\mathfrak{b})$ and $\Lambda^3(\mathfrak{b}\oplus\mathfrak{b})$ lie in the BWB for $l \geqslant 7$, subcalculations 2, 3, 5, and 6 are valid here too. □

8.4.1 Results for the cohomology of sheaves on Z and Y.

Now, set the field $k = \mathbb{F}$, assume that the characteristic $l \geqslant 11$ is large enough so that $H^3(\Lambda^2\mathfrak{b}\otimes\Lambda^2\mathfrak{b}) = 0$ and return to the previous situation of Section 8.3. We now use the above results to calculate cohomology groups of particular sheaves on Z and Y.

Lemma 8.4.8. *Let $\lambda = 0, \rho$ or 2ρ. Then the cohomology groups $H^i(Z, \mathcal{O}_Z(\lambda)) = 0$ whenever $i > 0$.*

Proof. Because the map $\pi : Z \to F$ is affine, from Lemma 8.3.3 we have the following isomorphisms of cohomology groups:

$$H^i(Z, \mathcal{O}_Z(\lambda)) = H^i(F, \pi_{Z,*}[\mathcal{O}_Z(\lambda)]) = H^i(F, \mathrm{Sym}([\mathfrak{g}/\mathfrak{b}]^2)(\lambda))$$

So the question reduces to the calculation of the cohomology groups $H^i(F, \mathrm{Sym}^n[\mathfrak{g}/\mathfrak{b}^2])$.

As in section 4 of [VX16], one can consider the Koszul complex of $0 \to \mathfrak{b}^2 \to \mathfrak{g}^2 \to \mathfrak{g}/\mathfrak{b}^2 \to 0$ giving us:

$$\ldots \to \Lambda^i[\mathfrak{b}^2]\otimes\mathrm{Sym}^{n-i}[\mathfrak{g}^2] \to \ldots \to \mathrm{Sym}^n[\mathfrak{g}^2] \to \mathrm{Sym}^n[\mathfrak{g}/\mathfrak{b}^2] \to 0$$

Since $\otimes \mathcal{O}_f(\lambda)$ defines an equivalence of categories, we thus also have an exact sequence

$$\ldots \to \Lambda^i[\mathfrak{b}^2] \otimes \mathrm{Sym}^{n-i}[\mathfrak{g}^2](\lambda) \to \ldots \to \mathrm{Sym}^n[\mathfrak{g}^2](\lambda) \to \mathrm{Sym}^n[\mathfrak{g}/\mathfrak{b}^2](\lambda) \to 0.$$

From this complex, we can produce an injective resolution for each module to give a double complex, from which we get a spectral sequence, $E^r_{p,q}$. Since the rows of the double complex are exact (coming from exactness of the above complex) this shows us $E^\infty_{p,q} = 0$ for all p, q. Hence, the spectral sequence gives us the structure of an increasing filtration F^p on $H^k(\mathrm{Sym}^n[\mathfrak{g}/\mathfrak{b}^2](\lambda))$ and surjective maps

$$H^{k+i}(\Lambda^i[\mathfrak{b}^2](\lambda)) \otimes \mathrm{Sym}^{n-i}[\mathfrak{g}^2] \to \mathrm{gr}^i[H^k(\mathrm{Sym}^n[\mathfrak{g}/\mathfrak{b}^2](\lambda))]$$

onto the graded parts.

By Calculation 8.4.5 in section 3.3 in the case of $\lambda = 0$ and Calculation 8.4.7 in the cases $\lambda = \rho$ or 2ρ, we see that $H^{k+i}(\Lambda^i[\mathfrak{b}^2]) = 0$ for any $k > 0$ and thus each graded part gr^i is also 0, so we get $H^k(\mathrm{Sym}[\mathfrak{g}/\mathfrak{b}^2](\lambda)) = 0$ for all $k > 0$. □

Note from Proposition 8.3.4 that $\mathcal{I}_Y \cong \mathcal{O}_Z(\rho)$ and $\omega_Z \cong \mathcal{O}_Z(2\rho)$. Hence, we also immediately get $H^i(\omega_Z) = H^i(\mathcal{I}_Y(\rho)) = H^i(\mathcal{I}_Y) = 0$ whenever $i > 0$.

Proposition 8.4.9. *The cohomology groups $H^i(Y, \mathcal{O}_Y) = H^i(Y, \omega_Y) = 0$ for all $i > 0$.*

Proof. From the short exact sequences of coherent sheaves on Z:

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Z \to \mathcal{O}_Y \to 0$$

and

$$0 \to \mathcal{I}_Y(\rho) \to \mathcal{O}_Z(\rho) \to \omega_Y \to 0,$$

we get long exact sequences in cohomology which, along with the previous lemma show that $H^i(Z, \mathcal{O}_Y) = H^i(Y, \mathcal{O}_Y) = H^i(Y, \omega_Y) = 0$ for all $i > 0$. □

Proposition 8.4.10. *We have the following isomorphisms in degree 0 cohomology:* $H^0(Z, \mathcal{O}_Z) = \mathrm{Sym}(\mathfrak{g}^2)$ *and* $H^0(Z, \mathcal{O}_Z(\lambda)) = \mathrm{Sym}(\mathfrak{g}^2) \otimes H^0(k(\lambda))$ *when* $\lambda = \rho$ *or* 2ρ*.*

Proof. To calculate $H^0(Z, \mathcal{O}_Z(\lambda))$, recall that Calculations 8.4.5 and 8.4.7 give $H^k(\Lambda^k[\mathfrak{b} \oplus \mathfrak{b}](\lambda)) = 0$ unless $k = 0$ in the cases $\lambda = 0$ and $\lambda = \rho$ or 2ρ respectively. Hence, the prior spectral sequence in the proof of Lemma 8.4.8 induces an isomorphism for each n,

$$H^0(\Lambda^0[\mathfrak{b} \oplus \mathfrak{b}](\lambda)) \otimes \mathrm{Sym}^n[\mathfrak{g}^2] \xrightarrow{\sim} H^0(\mathrm{Sym}^n[(\mathfrak{g}/\mathfrak{b})^2](\lambda))$$

Which shows $H^0(\mathrm{Sym}^n[(\mathfrak{g}/\mathfrak{b})^2](\lambda)) = \mathrm{Sym}^n[\mathfrak{g}^2] \otimes H^0(k(\lambda))$.

Thus, $H^0(Z, \mathcal{O}_Z)$ is the graded ring $\mathrm{Sym}(\mathfrak{g}^2)$ and $H^0(Z, \mathcal{O}_Z(\lambda))$ the graded $H^0(Z, \mathcal{O}_Z)$ module $\mathrm{Sym}(\mathfrak{g}^2) \otimes H^0(\mathbb{F}(\lambda))$. $\square$

8.4.2 Summary

We summarise the results of Calculations 8.4.2, 8.4.3, 8.4.5, 8.4.6 and 8.4.7 in characteristic $\mathrm{char}(k) = l \geqslant 11$ or $\mathrm{char}(k) = 0$ below:

i	0	1	2	3
$H^i(\Lambda^0\mathfrak{b})$	k	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^1\mathfrak{b})$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^2\mathfrak{b})$	$\cdot$	k	$\cdot$	$\cdot$
$H^i(\Lambda^3\mathfrak{b})$	$\cdot$	$\cdot$	k	$\cdot$
$H^i(\Lambda^4\mathfrak{b})$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^5\mathfrak{b})$	$\cdot$	$\cdot$	$\cdot$	k
$H^i(\Lambda^0[\mathfrak{b}\oplus\mathfrak{b}])$	k	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^1[\mathfrak{b}\oplus\mathfrak{b}])$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^2[\mathfrak{b}\oplus\mathfrak{b}])$	$\cdot$	k^3	$\cdot$	$\cdot$
$H^i(\Lambda^3[\mathfrak{b}\oplus\mathfrak{b}])$	$\cdot$	$\cdot$	$\mathfrak{g}^4\oplus k^4$	$\cdot$
$H^i(\Lambda^4[\mathfrak{b}\oplus\mathfrak{b}])$	?	?	?	$\leqslant \mathfrak{g}^2$
$H^i(\Lambda^0\mathfrak{b}(\rho))$	$\mathfrak{g}$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^1\mathfrak{b}(\rho))$	$\mathfrak{g}^2\oplus k$	0	$\cdot$	$\cdot$
$H^i(\Lambda^2\mathfrak{b}(\rho))$	$\mathfrak{g}\oplus k$	0	$\cdot$	$\cdot$
$H^i(\Lambda^0\mathfrak{b}(2\rho))$	$V(2\rho)$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^1\mathfrak{b}(2\rho))$	$V(2\rho)^2\oplus V(\rho+\alpha)\oplus V(\rho+\beta)$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^2\mathfrak{b}(2\rho))$	$V(2\rho)\oplus V(\rho+\alpha)\oplus V(\rho+\beta)\oplus\mathfrak{g}^3$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^3\mathfrak{b}(2\rho))$	$V(\rho+\alpha)\oplus V(\rho+\beta)\oplus\mathfrak{g}^3\oplus k$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^4\mathfrak{b}(2\rho))$	$\mathfrak{g}\oplus k^2$	$\cdot$	$\cdot$	$\cdot$
$H^i(\Lambda^5\mathfrak{b}(2\rho))$	k	$\cdot$	$\cdot$	$\cdot$

Table 8.3: Cohomology G representations of vector bundles on the flag variety F.

8.5 Equations for X

Assume from now on that $l = \mathrm{char}(\mathbb{F})$ is sufficiently large (in accordance with Remark 8.4).

Proposition 8.5.1 (Proposition 2.1.5 of [Sno18])**.** *Let $\mathcal{F}$ be a scheme over $\mathbb{F}$ and*

$$0 \to \mathcal{A} \to \mathcal{E} \to \mathcal{B} \to 0$$

be a short exact sequence of vector bundles on $\mathcal{F}$, with $\mathcal{E}$ a free bundle. Notice that TotalSpace($\mathcal{E}$) $= \mathcal{F}\times V$ for some vector space V. Let $S = \mathrm{Sym}(V^*)$ be a graded ring.

Assume that $H^i(\mathcal{F}, Sym(\mathcal{B})) = 0$ for each $i > 0$. Let T be the graded ring $H^0(\mathcal{F}, \mathrm{Sym}(\mathcal{B})) = H^0(Z, \mathcal{O}_Z)$ with grading given by the grading on $\mathrm{Sym}(\mathcal{B})$. Then we have an isomorphism of graded $\mathbb{F}$-vector spaces

$$\mathrm{Tor}_n^S(T,\mathbb{F}) \cong \bigoplus H^{i-n}(\mathcal{F},\Lambda^i\mathcal{A})[i]$$

where the suffix $-[i]$ indicates the grading.

Remark. We remark that there is a map $S \to T$. We define I as the kernel and the T' as the image of this map. Set S_+ as the irrelevant ideal of S. Then it follows that $\mathrm{Tor}_1^S(T', \mathbb{F}) = I/S^+I$ and $\mathrm{Tor}_0^S(T', \mathbb{F}) = T'/S^+T'$. If $T/S^+T = \mathbb{F}$, then $S \to T$ is surjective and $T = T'$, so we get $\mathrm{Tor}_1^S(T, \mathbb{F}) = I/S^+I$.

Thus, from the calculations in section 3, we see the following

Theorem 8.5.2. *Suppose the characteristic l of $\mathbb{F}$ is large enough that $H^3(\Lambda^4(\mathfrak{b} \oplus \mathfrak{b})) = 0$ (by Conjecture 8.4.4 $l \geqslant 11$ is good enough). The homogeneous ideal $I \trianglelefteq S$ is generated by 3 polynomials in degree 2 and 36 polynomials in degree 3.*

Further, we can list these polynomials as:

- *In degree 2:*
$$tr(M^2), tr(N^2), tr(MN);$$
- *In degree 3: $tr(M^3), tr(N^3)$, and all the entries of*
$$M^2N, N^2M, NM^2, MN^2,$$
noting that we have the relations $tr(M^2N) = tr(NM^2)$ and similarly with $tr(N^2M) = tr(MN^2)$.

Proof. Recall that Z is the total space of the vector bundle $[\mathfrak{g}/\mathfrak{b}]^2$ on F and there is a trivial (viewed only as an $\mathcal{O}_F$-module) vector bundle $\mathfrak{g}^2$ with a short exact sequence;

$$0 \to \mathfrak{b}^2 \to \mathfrak{g}^2 \to [\mathfrak{g}/\mathfrak{b}]^2 \to 0$$

as in Proposition 8.5.1. We may hence take V^* as $\mathfrak{g}^2$ (viewed as a finite dimensional $\mathbb{F}$-representation). We set $T = H^0(Z, \mathcal{O}_Z) = H^0(F, \mathrm{Sym}([\mathfrak{g}/\mathfrak{b}]^2))$ and $S = \mathbb{F}[\mathfrak{g}^2]$. First, we note from Proposition 8.5.1 that $\mathrm{Tor}_0^S(T, \mathbb{F}) = T/S^+T = \bigoplus_i H^i(F, \Lambda^i\mathfrak{b}^2)[i]$ and by Calculation 8.4.5 this is simply $\mathbb{F}$. Thus, from the remark following Proposition

8.5.1 and Calculation 8.4.5 we get the isomorphisms:

$$\begin{aligned} I/S^+I &\cong \bigoplus_i H^{i-1}(F, \Lambda^i[\mathfrak{b} \oplus \mathfrak{b}])[i] \\ &= 0[1] \oplus \mathbb{F}^3[2] \oplus [\mathfrak{g}^4 \oplus \mathbb{F}^4][3] \oplus H^3(\Lambda^4[\mathfrak{b} \oplus \mathfrak{b}])[4] \end{aligned}$$

Hence we see that there are 3 degree 2 equations, 36 in degree 3, and no equations in degree 4. Because $S = \mathrm{Sym}((\mathfrak{g}^*)^2)$ and $\mathfrak{g} = \mathfrak{gl}_3$, we see that $S = \mathbb{F}[M, N]/\langle \mathrm{tr}(M), \mathrm{tr}(N) \rangle$. Since $\mathrm{tr}(M^2), \mathrm{tr}(MN)$ and $\mathrm{tr}(N^2)$ are all linearly independent, we see that they span the degree 2 equations. We have the following 'obvious' equations in degree 3, given by $\mathrm{tr}(M^3)$; $\mathrm{tr}(N^3)$; M^2N,MNM,NM^2; and MN^2, NMN, N^2M.

It can be shown via computer calculations, that any 2 of the matrices M^2N, MNM and NM^2 span a 17-dimensional vector space. Thus, we see that

M^2M, N^2M, N^2M, MN^2, $\mathrm{tr}(M^3)$ and $\mathrm{tr}(N^3)$ all span a 36 dimensional $\mathbb{F}$-subspace of $S/\langle \mathrm{tr}(M^2), \mathrm{tr}(MN), \mathrm{tr}(N^2) \rangle$.

It then follows that these polynomials generate the ideal I. □

Theorem 8.5.3. *Let the characteristic $l = \mathrm{char}(\mathbb{F})$ be as in Theorem 8.5.2. The scheme X is a closed subspace of $\mathbb{A}^{18}_{\mathbb{F}}$, which we view as pairs of 3×3-matrices (M, N); and is cut out by the ideal generated by:*

- *$tr(M), tr(N)$ in degree 1;*
- *$tr(M^2), tr(MN), tr(N^2), MN - NM$ in degree 2;*
- *and $tr(M^3), tr(N^3), M^2N, MN^2$ in degree 3*

Proof. The morphism $Y \to Z$ is a closed immersion, given by a sheaf of ideals $\mathcal{I}_Y$, so we have a surjection $H^0(Z, \mathcal{O}_Z) \twoheadrightarrow H^0(Y, \mathcal{O}_Y)$ whose kernel is generated by the polynomial entries of $MN - NM$. From the previous theorem, it now follows that

X, which is equal to $\operatorname{Spec}(H^0(Y, \mathcal{O}_Y))$ is the closed subspace of $\mathbb{A}^{18}$ given by

$$\mathbb{V}\begin{bmatrix} \operatorname{tr}(M), \operatorname{tr}(N), \operatorname{tr}(M^2), \operatorname{tr}(MN), \operatorname{tr}(N^2), MN - NM, \\ \operatorname{tr}(M^3), \operatorname{tr}(N^3), M^2N, N^2M, NM^2, MN^2 \end{bmatrix}.$$

From the commutation relation we can simplify the generators down to

$$\mathbb{V}\begin{bmatrix} \operatorname{tr}(M), \operatorname{tr}(N), \operatorname{tr}(M^2), \operatorname{tr}(MN), \operatorname{tr}(N^2), MN - NM, \\ \operatorname{tr}(M^3), \operatorname{tr}(N^3), M^2N, N^2M \end{bmatrix}$$

which proves the theorem. □

We can now present equations for the affine scheme $\mathcal{X}_{St,\mathcal{O}}$.

Theorem 8.5.4. *Let $\mathcal{O}$, and l be sufficiently large as before. The affine scheme $\mathcal{X}_{St,\mathcal{O}}$ is isomorphic to the Zariski-closed subset of $(\Phi, N, d) \in GL_3 \times \mathfrak{gl}_3 \times \mathbb{G}_m$ given by:*

- $\Phi N \Phi^{-1} = qN$,
- *N is strongly nilpotent, (I.E. the characteristic polynomial $\chi_N(x) = x^3$),*
- *the characteristic polynomial of Φ is $\chi_\Phi(x) = (x - q^2)(x - q)(x - 1)$,*
- *$tr(\Phi N) = 0$,*
- $N(\Phi - qI)(\Phi - I) = 0$,
- $N^2(\Phi - I) = 0$.

Proof. First, we note that the copy of $\mathbb{G}_m$ comes from the isomorphism, $\mathcal{X}_{St} \cong \mathcal{X} \times \mathbb{G}_m$. We henceforth consider the equations in $\mathcal{X}$, the Steinberg component with fixed trace.

That $\Phi N = qN\Phi$ on $\mathcal{X}$ is clear. Because we have the proper, surjective birational map $\mathcal{Y} \to \mathcal{X}$, it follows that for any R-point, $(\Phi, N) \in \mathcal{X}(R)$, there is a decreasing flag F_i (not necessarily unique) such that $(\Phi - q^i)F_i \subseteq F_{i+1}$ and $NF_i \subseteq F_{i+1}$. by these relations, we see $N^3F_0 \subseteq F_3 = 0$ $N^2(\Phi - I)F_0 \subseteq F_3 = 0$ and $N(\Phi - qI)(\Phi - I)F_0 \subseteq F_3 = 0$ and $(\Phi - q^2I)(\Phi - qI)(\Phi - I)F_0 \subseteq F_3 = 0$, so $(\Phi - q^2I)(\Phi - qI)(\Phi - I) = N(\Phi - qI)(\Phi - I) = N^2(\Phi - I) = N^3 = 0$. Further, by the existence of such a flag, we can see that $N\Phi$ is strictly upper-triangular inside the corresponding Borel subalgebra. Hence $\mathrm{tr}(N\Phi) = 0$, and all the above equations are satisfied on $\mathcal{X}_{\mathcal{O}}$.

Along the special fibre, we can equate $q = 1$, to get the equations:

- $\Phi N = N\Phi$,
- N is strongly nilpotent, (I.E. the characteristic polynomial $\chi_N(x) = x^3$),
- the characteristic polynomial of Φ is $\chi_\Phi(x) = (x - 1)^3$ (so Φ is strongly unipotent),
- $\mathrm{tr}(\Phi N) = 0$,
- $N(\Phi - I)^2 = 0$,
- $N^2(\Phi - I)$.

As Φ is strongly unipotent along the special fibre, there is a matrix M such that $\Phi = \exp(M)$, and $M = \log(\Phi)$ is strongly nilpotent. Note that $\log(\Phi) = (\Phi - I) - \frac{1}{2}(\Phi - I)^2$ because all higher terms identically vanish. The statements M, N are strongly nilpotent are equivalent to $\mathrm{tr}(M) = \mathrm{tr}(M^2) = \mathrm{tr}(M^3) = 0$ and $\mathrm{tr}(N) = \mathrm{tr}(N^2) = \mathrm{tr}(N^3) = 0$ respectively, because 2 and 3 are invertible in characteristic l. That Φ and N commute is equivalent to $MN = NM$. We also have

$$MN^2 = \log(\Phi)N^2 = (\log(\Phi) = ((\Phi - I) - \frac{1}{2}(\Phi - I)^2)N^2 = (I - \frac{1}{2}(\Phi - I))(\Phi - I)N^2$$

and

$$(\Phi - I)N^2 = (I + \frac{1}{2}M)MN^2$$

which shows that $(\Phi - I)N^2 = 0$ if and only if $MN^2 = 0$. Similarly, one can show that $(\Phi - I)^2N = 0$ if and only if $M^2N = 0$. The equalities

$$\mathrm{tr}(MN) = \mathrm{tr}(\log(\Phi)N)$$

$$
\begin{aligned}
&= \mathrm{tr}[((\Phi - I) - \frac{1}{2}(\Phi - I)^2)N] \\
&= \mathrm{tr}(\Phi N) + \mathrm{tr}(N) + \frac{1}{2}\mathrm{tr}((\Phi - I)^2 N)
\end{aligned}
$$

show that $\mathrm{tr}(MN) = 0$ is equivalent to $\mathrm{tr}(\Phi N) = 0$ when $\mathrm{tr}(N) = 0$ and $(\Phi - I)^2 N = 0$. We can then conclude that the equations along the special fibre above equivalent to the equations in Theorem 8.5.3. Hence, this equations cut out the special fibre X. It follows that these equations cut out the affine scheme $\mathcal{X}_{St,\mathcal{O}}$ as claimed. □

8.6 The Picard group

In this section, we compute the Picard group of X and the class of the canonical divisor. For $n = 2$ this gives another perspective on the calculations of [Man21]. We might hope for similar automorphic applications for $n = 3$, but there appear to be issues with finding a Hecke-equivariant pairing on the spaces of automorphic forms (we thank Jeff Manning for explaining this point to us).

Recall the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{p} & X \\
\downarrow{\scriptstyle \pi} & & \\
F & &
\end{array}
$$

where the map p is proper. Let U be the open subscheme of X defined as the locus of points $(M, N) \in X(\bar{\mathbb{F}})$ such that either M or N is regular nilpotent. If $V = p^{-1}(U)$ then $p|_V : V \to U$ is an isomorphism. This is because a regular nilpotent M (or N) is contained in a unique Borel subalgebra. Thus p is a birational equivalence.

Lemma 8.6.1. *The map $\pi : Y \to F$ is a fibre bundle with fibres isomorphic to*

$$C(\mathfrak{n}) = \{(M, N) \in \mathfrak{n} \times \mathfrak{n} : [x, y] = 0\}$$

The complement $Y \backslash V$ has codimension 1 in Y and the codimension of $X \backslash U$ inside X is $\geqslant 2$. When $n = 3$, the open subscheme of $(M, N) \in C(\mathfrak{n})$ given by those points *with one of M, N regular nilpotent has*

$$C(\mathfrak{n})^{reg} \cong \mathbb{A}^2 \backslash \{(0,0)\} \times \mathbb{G}_m \times \mathbb{A}^2.$$

Proof. We note that the fibres of $Y \to F$ are all isomorphic to the closed subscheme $C(\mathfrak{n}) = \{(M,N) \in \mathfrak{n} \times \mathfrak{n} : [x,y] = 0\}$. We also notice that $Y \to F$ is locally isomorphic to $C(\mathfrak{n}) \times F \to F$. We write

$$C(\mathfrak{n}) = \left\{ \left(\begin{pmatrix} 0 & a & b \\ & 0 & c \\ & & 0 \end{pmatrix}, \begin{pmatrix} 0 & d & e \\ & 0 & f \\ & & 0 \end{pmatrix} \right) : af = cd \right\}.$$

Notice that M is regular if and only if $ac \neq 0$, and N is regular if and only if $df \neq 0$. We then have that the fibres of $p|_V$ are isomorphic to the scheme

$$\{(a,b,c,d,e,f) \in \mathbb{A}^6 : af = dc\} \backslash \mathbb{V}(ac) \cap \mathbb{V}(df).$$

We see that $\mathbb{V}(ac, df)$ is codimension 1 inside Y. To show that U is codimension 2 inside X, let $Z = X \backslash U$. Consider Z° as the dense open subset of Z with $M \neq 0$ and let $(M,N) \in Z^\circ$. Then the matrix M is necessarily conjugate to $M_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and has stabiliser

$$\mathrm{Stab}(M_0) = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & a \end{pmatrix} \in \mathrm{GL}_3(\mathbb{F}) \right\}$$

which is 5 dimensional. By the orbit-stabiliser theorem, the conjugacy class of M is 4-dimensional and the space of non-regular nilpotent matrices that commute with M_0 is

$$\left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & e \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{F}) : be = 0 \right\}$$

is 2 dimensional. All fibres of Z° are isomorphic to this space, hence we see that Z° and thus Z has dimension 6. As the dimension of X is the same as that of Y, which is 8, the claim follows.

To prove the final claim, we take the map

$$\begin{aligned}\mathbb{A}^2\backslash\{(0,0)\} \times \mathbb{G}_m \times \mathbb{A}^2 &\to C(\mathfrak{n})^{\text{reg}}\\ (a,d,\lambda,b,e) &\mapsto (a,b,\lambda a,d,e,\lambda d)\end{aligned}$$

with inverse defined by $\frac{f}{d} \mapsto \lambda$ when $d \neq 0$, and $\frac{c}{a} \mapsto \lambda$ when $a \neq 0$. This is well defined since $\frac{c}{a} = \frac{f}{d}$ when $da \neq 0$, (so the definitions are compatible) and because we must always have at least one of a, d non-zero because one of M, N is regular nilpotent. □

Theorem 8.6.2. *Let $E \to \mathcal{F}$ be a Zariski-locally trivial fibration of varieties over a field $\mathbb{F}$ with connected fibre C. Assume that $\mathcal{F}$ and C are smooth over the base field. Then there is an exact sequence*

$$\Gamma(C, \mathcal{O}_C^\times) \to \operatorname{Pic}(\mathcal{F}) \to \operatorname{Pic}(E) \to \operatorname{Pic}(C) \to 0.$$

Proof. As $\mathcal{F}$ and C are smooth, this theorem follows from Theorem 5 of [Mag75] along with the remark that follows. □

Corollary 8.6.3. *The Weil divisor class group $Cl(X) \cong \operatorname{Pic}(U) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.*

Proof. As the codimension of $X\backslash U$ in X is at least 2 and U is a regular integral scheme, we have isomorphisms $\operatorname{Cl}(X) \to \operatorname{Cl}(U) \to \operatorname{Pic}(U) \cong \operatorname{Pic}(V)$. Since V is a fibre bundle over F with fibres $C(\mathfrak{n})^{\text{reg}}$ is as in Theorem 8.6.2, we see that

$$\Gamma(C(\mathfrak{n})^{\text{reg}}, \mathcal{O}^\times_{C(\mathfrak{n})^{\text{reg}}}) \to \operatorname{Pic}(F) \to \operatorname{Pic}(V) \to \operatorname{Pic}(C(\mathfrak{n})^{\text{reg}}) \to 0.$$

Because the equivalence of categories between $\mathcal{Coh}^G(G/B)$ and $B-\text{rep}$ is rank preserving, we see $\operatorname{Pic}(F) \cong X^*(T) \cong \mathbb{Z}^2$ is generated by the weights L_1 and L_2 and since $C(\mathfrak{n})^{\text{reg}}$ is an open subscheme of $\mathbb{A}^5$, we get $\operatorname{Pic}(C(\mathfrak{n})^{\text{reg}}) = 0$. We have that

$$\Gamma(C(\mathfrak{n})^{\text{reg}}, \mathcal{O}^\times_{C(\mathfrak{n})^{\text{reg}}})/\mathbb{F}^\times \cong \mathbb{F}[\lambda^{\pm 1}]^\times/\mathbb{F}^\times \cong \mathbb{Z},$$

is generated by $\lambda = \frac{c}{a} = \frac{f}{d}$, which is mapped to the weight of $\frac{c}{a}$ (note that the weight of $\frac{c}{a}$ is equal to that of $\frac{f}{d}$), which is $(L_2 - L_3) - (L_1 - L_2) = 2L_2 - L_1 - L_3 = 3L_2$. (Strictly, we haven't checked that this is precisely where λ gets sent, or whether it gets sent to $-\frac{c}{a}$. This map arises from a boundary map in cohomology, so is necessarily *not* explicit.) It follows that $\mathrm{Pic}(V) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, with $\mathbb{Z}$ generated by L_1 and $\mathbb{Z}/3\mathbb{Z}$ generated by L_2. $\square$

Corollary 8.6.4. *The canonical sheaf has* $[\omega_X] = \rho = 2L_1 + L_2 \in Cl(X)$.

Proof. By Theorem 8.2.5, we see that $\omega_X = f_*\omega_Y$. From restriction on V, we see that $\omega_U = \omega_X|U = \omega_Y = \mathcal{O}_Y(\rho) = L_1 - L_2 + L_2 - L_3 = 2L_1 + L_2 \in \mathrm{Pic}(U) = Cl(X)$. $\square$

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