Nonparametric Predictive Inference for Inventory Decisions

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A Thesis presented for the degree of
Doctor of Philosophy

Department of Mathematical Sciences
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England

April 2023

Dedicated to

My dear parents.

My beloved siblings.

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Abstract

In inventory theory, many scenarios have been studied with the aim of determining an optimal order strategy, typically with the aim to maximise expected profit. Traditionally, a stochastic model with a known probability distribution for random demand is assumed. In this thesis, an alternative approach to inventory problems is presented, with the aim of basing the order strategy on information in the form of previously observed demands, adding only quite minimal further assumptions. Nonparametric Predictive Inference (NPI) is used to predict a future demand given observations of past demands. NPI makes only a few modelling assumptions, which is achieved by quantifying uncertainty through lower and upper probabilities.

As the first use of NPI in inventory theory, the basic scenario of inventory for a single period is considered. We present NPI lower and upper probabilities for the event that the random profit achieved for one future period is non-negative, which can be used to determine an optimal inventory level. As second optimality criterion, we consider the NPI lower and upper expected profits for the next period. We also consider optimisation of a weighted average of the NPI lower and upper probabilities and expected profits.

We also develop the NPI method for two-period inventory problems, in which we choose to maximise expected profit as the optimal criterion for determining optimal inventory levels. We derive the optimal inventory level for the two-period model with a single order. We presume that we are filling the inventory for both periods at the same time. Therefore, the future demand will be a combination of the future demand for the first period and the future demand for the second period. We also derive the optimal inventory levels for both periods in the two-period independent demands model.

First, we determine the optimal inventory level for the second period, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimise over the first period.

Attention is also given to the situation of the two-period model with dependent demands. The NPI bootstrap (NPI-B) method is applied to deal with this model and the complexities in some of the inventory models. We study different strategies for the inventory levels to determine which one of those is optimal based on maximising the average profit.

The NPI method and the classical method are compared through simulations. Several cases are studied, some where the assumptions underlying the classical method are fully correct, so the classical method performs better; for a large number of observations, there is a tendency for the NPI to be close to the classical method. In the other cases where the assumed model is not well aligned with reality, the NPI method performs better.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification, and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

All praise is due to the Almighty Allah, I am truly grateful to Allah, my God, for everything you have given me. Without his great assistance, this work could not have been completed.

My main appreciation and thanks go to my supervisors, Prof. Frank Coolen and Dr. Tahani Coolen-Maturi, for their continuous support of my PhD study and research. I really appreciate their patience, kindness and valuable advice, whether in my academic or real life.

I would like to reserve my deepest and most special thanks to my parents for being my source of strength and confidence. My beloved mother Shikhah and my great father Omar, thank you. Without you both and your prayers, untiring support and care, I could not have reached this happy ending today.

I also owe thanks to my brothers, sisters, nephews and nieces for their pride in me, endless emotional support and encouragement. My heartfelt gratitude goes to my younger brother Yazeed for his enthusiasm and sound advice.

Thanks to the sponsors and supporters of my studies, King Saud University and the Saudi Arabian Cultural Bureau in London for facilitating my scholarship and offering me the opportunity to complete my studies abroad.

My final thanks go to Durham University, Mathematical Sciences Department, for the help and support they offered me. My appreciation also extends to my friends and to everyone who has assisted me or contributed to my educational progress in any way.

Contents

Chapter 1

Introduction

Inventory theory is one of the main areas in operations research, with a long history of contributions to the literature in which a variety of scenarios are studied. A long time ago, inventories were thought of as indicators of a nation's or an individual's riches and power. The wealth and power of a businessman or a country were measured by how much wheat, rice, cattle, etc. they had in their storehouses. Dealing with stock levels like that was simple. Because of the rapid advances and changes in product life these days, inventories are seen as a significant potential risk and need to be controlled by using scientific techniques instead of being regarded as a measure of wealth [66].

In 1913, Harris was the first researcher who presented and developed an inventory model [42]. Then, in 1951, the stochastic inventory theory was developed by Arrow et al. [2]. Their ground-breaking work in the field of inventory modelling has made it a significant part of operations research ever since. They made it clear how important it is to plan for unexpected demand, costs, and emergency stock. Some researchers, like Dvoretzky et al. [32, 33], made important contributions to the inventory theory's growth. Arrow et al. [4] have researched the origins of inventory theory's development as well as its nature and structure. Additionally, they examined a wide range of inventory models. The vast majority of the authors focused on continuous time models where stock levels are continuously monitored. The classical results in this field were provided by Hadley and Whitin [41], Nahmias [58] and Nahmias et al. [59]. In the subject of inventory modelling, Cheng et al. [15] and Andriolo et al. [1] addressed the fundamental problems of how much and when to order.

The classical approach is commonly used to study inventory problems in the operations research literature. This approach assumes that the random demand follows a known probability distribution, with variation in the number of periods, or decision moments, along with the related costs [54].

In this thesis, we propose a novel approach to inventory models using the Nonparametric Predictive Inference (NPI) method. Various nonparametric approaches for inventory management have been proposed in the literature. However, NPI method which we assume in this thesis, is new to inventory models. One of the nonparametric approaches is the minimax approach. Jagannathan [50] computed the optimal stocking quantity that will provide the maximum expected profit against the worst possible demand for stocking quantity using the minimax approach. Another nonparametric approach is based on a variant of a stochastic approximation algorithm based on censored demand samples. Using this approach, Burnetas and Smith [12] created an adaptive algorithm for ordering and pricing perishable goods. In 2007, Levi et al. [55] investigated single-period and dynamic inventory problems with zero setup cost from a nonparametric perspective and presented sample average approximation algorithms to solve them. Huh and Rusmevichientong [49] provided nonparametric prescriptions for potentially censored demand settings and analysed the performance of these policies via upper bounds or convergence of the prescribed decisions.

NPI [5, 17, 19, 20, 22] is a statistical framework based on Hill's assumption $A_{(n)}$ [44], as introduced in Section 1.3, with inferences explicitly in terms of one or more future observations. The explicitly predictive nature of NPI makes it particularly useful for a range of topics in operations research. For instance, the NPI method has been applied for queues [26] and age replacement issues [30]. In addition, NPI has been developed for a variety of data types and applications, including Bernoulli quantities [17], right-censored data [28, 29], ordinal data [27, 38], multinomial data [7, 21], for future order statistics [25] and reliability applications [24]. Recently, the NPI approach has been introduced to finance applications [14, 43].

This chapter is organised as follows. The motivation for the work in this thesis is given in Section 1.1. Section 1.2 presents an introduction to the mathematical concept of inventory. Section 1.3 provides a brief introduction to NPI. A detailed outline of this thesis is given in Section 1.4.

1.1 Motivation

In inventory theory, many scenarios have been studied to determine an optimal order strategy, typically with the aim to maximise expected profit. Traditionally, a fully known stochastic model is assumed, with a known probability distribution for the random demand. NPI has not yet been applied to inventory problems, so the main objective of this thesis is to develop and apply NPI methods to support inventory decisions. As such, it is a combination of modern statistical methods and more traditional operations research scenarios.

As a first step, we will consider the single-order inventory models to make inferences for only one future observation, as introduced in Chapter 2. We introduce NPI lower and upper probabilities for the event that the random profit is non-negative, and use these to determine an optimal inventory level. Also, we consider the optimality criteria by combining both lower and upper probabilities. In addition, we consider the NPI lower and upper expected profits for the next period, and we also combine both lower and upper expectations of the random profit for the next period. We study a model that is similar to the single-period model but with two demands; a single order for the two-period model is considered. We assume that we are filling the inventory for both periods at the same time to make inferences for one future observation, which is the sum of the future demands for the first and second periods.

In Chapter 3, we generalise the analysis to two-period independent demands model, where we determine the optimal inventory levels for both periods.

In order to deal with dependent demands for two periods, we rely on the NPI bootstrap (NPI-B) method, as shown in Chapter 4. In order to determine the optimal inventory levels, we use the average profit criterion.

In each chapter, we conduct simulations for different cases to evaluate the proposed approach and to compare its performance to that of the classical method for inventory models.

1.2 Inventory

We usually use the expression 'inventory' for the total number of goods or materials contained in a store or factory at any given time [52]. Common problems for an inventory system include the stocking of spare parts, perishable items and seasonal items. So, the inventory needs management to avoid problems and to calculate the needed costs. This must be accurately counted and valued at the end of each accounting period to determine a company's profit or loss [60].

More than one hundred years ago, the analysis of an inventory system appeared in the literature. In 1913, Harris is the first researcher who presented and developed an inventory model [42]. A lot of researchers have extended Harris's model with different types of demands and replenishment. According to Hadley and Whitin [41], the inventory models are classified based on the pattern of demand, which may be either deterministic or random over time. Deterministic demand occurs when we know the exact quantities needed over a period of time. However, the known demand may be constant or variable over time. Random demand occurs when the demand is not known within a period of time. Hence, forecasting the random demand is considered to be a major difficulty faced by the decision maker. Since the exact number of items the customers will buy during the period is uncertain, this type of demand could be stationary or non-stationary over time.

Inventory decisions may face some problems, for instance, single-period problems, two-period problems, or even multi-period problems. In this thesis, we only consider the single and two-period models.

1.2.1 Single period

One of the basic inventory problems that have been discussed frequently in the literature is the single-period inventory problem. Arrow et al. [2] and Dvoretzky et al. [32] were the first researchers who studied a single-period inventory control problem and proposed a newsvendor model. In the single period, the inventory will only be demanded for one-time duration and cannot be transferred to the next time duration [16]. According to Reid and Sanders [60], the single-period model is designed for goods that hold specific characteristics, such as being sold at their regular price only during a single time period.

Demand for these goods is highly variable but follows a known probability distribution, which may be either continuous or discrete. Furthermore, the value of the leftover inventory at the end of the season (salvage value) of these goods is less than their original cost, so the inventory owner loses money when they are sold for their salvage value.

For instance, imagine that one of these goods is a newspaper; a newsagent buys newspapers for a specific day from its wholesaler. The issue is deciding the number of newspapers that a newsagent should buy on a given day for his newsstand [51].

Hadley and Whitin [41] and Taha [65] presented the general single-period model with time independent costs and random demand. Mitra and Chatterjee [57] investigated and derived the optimal order quantity for the single-period newsvendor problem with stochastic end-of-season demand. This is the demand that occurs at the end of the season, which depends on the number of items left over at the end of selling season. In the classical model the leftover items are sold by predetermined salvage value and the profit will be calculated at the end of the season. While, in the end-of-season demand model the profit will be divided into partitions based on the number of items left over at the end of the selling season and the demand at this time.

1.2.2 Two period

A popular generalisation of this model is to consider a two-period problem in which the selling horizon is extended from one period to two periods, and a decision on the inventory level in each period is made before the demand is realized. The fundamental difference between the single-period inventory model and the two-period inventory model is that the two-period model may include stock left over from the previous period, which makes the optimal choice of inventory levels more complicated. Mills [56] was the first researcher who considered the inventory model with more than one period. Following Mills [56], several authors have explored two-period models, such as Bradford and Sugrue [11] who developed a model in which the second period demand depends on the first period demand. These authors determined a conditional order-up-to-level policy for the second period and an optimal order quantity for the first period, by using Bayesian updating whereby the first-period demands are used to update the prior parameters and revise the second-period demand forecast. The forecast is then incorporated into the model to derive optimal stocking policies which maximise expected profit over the two periods. Another important two-period model has been studied by Gurnani and Tang [40]. They assumed a two-period model with no demand in the first period, where at the end of the first period, exogenous information is collected, permitting one to update the initial forecast for the second period demand. They formulated the problem by using the bivariate normal distribution. A two-period model with uniformly distributed independent demands has been analysed by Hillier and Lieberman [48]. Cheaitou et al. [13] proposed a two-period production and inventory model with independent random demands for each period. In their model, a variety of salvage possibilities are available, as well as a variety of production techniques.

The purpose of studying inventory models is to find a single optimal inventory level for the single-period inventory model, and multiple optimal inventory levels for the two-period inventory model, when the demand is random and follows a continuous or discrete distribution. In our study, we consider a positive real-valued demand. There are different optimality criteria possible. Shih [63] investigated one of these criteria, which is to find the optimal order quantity that maximises the expected profit. Sankarasubramanian and Kumaraswamy [61] studied another criterion, namely maximisation of the probability that the profit is greater than or equal to zero. In this thesis, we consider these two criteria for the single-period model. While for the two-period model, we only consider maximising the expected profit criterion.

1.3 Nonparametric predictive inference (NPI)

As a measure of uncertainty, imprecise probability was proposed by Boole [10] in 1854, in which the uncertainties relating to events are measured using intervals rather than single numbers, as in classical probability [18]. For instance, the probability of an event A is $P(A) \in [0, 1]$. Several alternative approaches to quantifying uncertainty have been proposed in recent years, including Walley's imprecise probability theory [69] and Weichselberger's interval probability theory [72] which propose lower and upper probabilities instead of probabilities. These lower and upper probabilities are components of a statistical methodology known as Nonparametric Predictive Inference (NPI), which will be briefly discussed in this section and which is used in this thesis. So, Walley's imprecise probability theory and Weichselberger's interval probability theory are part of NPI.

Nonparametric Predictive Inference (NPI) is a frequentist statistical framework providing lower and upper probabilities for events involving future observations, depending on the $A_{(n)}$ assumption proposed by Hill [44, 45, 47]. We can summarise this assumption as follows: suppose that the random quantities $X_1, X_2, ..., X_n, X_{n+1}$ are exchangeable and real-valued. We assume that ties do not occur between $x_1, x_2, ..., x_n$, which denote the ordered observations of $X_1, X_2, ..., X_n$ so, $x_1 < x_2 < ... < x_n$. Also, we define $x_0 = 0$ and $x_{n+1} = \infty$ for ease of notation, where we set $x_0 = 0$ because we will work with non-negative random quantities. The n observations partition the real-line into $n+1$ intervals $I_j = (x_{j-1}, x_j)$ for $j = 1, 2, ..., n+1$. The assumption $A_{(n)}$ [46] is that the probability for the next observation X_{n+1} to fall in the open interval I_j is equal for all I_j, that is

$$P(X_{n+1} \in (x_{j-1}, x_j)) = \frac{1}{n+1}, \quad j = 1, 2, ..., n+1$$

Inferences based on $A_{(n)}$ are nonparametric and predictive, also called 'low structure inference', since these inferences are based on limited assumptions [39]. $A_{(n)}$ is not sufficient to provide precise probabilities for many events of interest; however, it can provide bounds for probabilities, which are lower and upper probabilities, depending on De Finetti's fundamental theorem of probability which states that the probability distribution of any exchangeable random variables is a mixture of independent and identically distributed sequences of random variables [31]. NPI uses imprecise probability [6], in which the lower and upper probabilities are the maximum lower bound and minimum upper bound, respectively, for the probability of the event of interest [72], based on the $A_{(n)}$ assumption. Augustin and Coolen [5] presented the following predictive lower and upper probabilities based on $A_{(n)}$:

The lower probability $\underline{P}(.)$ and the upper probability $\overline{P}(.)$ for the event $X_{n+1} \in B$ with $B \subset \mathbb{R}$, based on the intervals $I_j = (x_{j-1}, x_j)$ for $j = 1, 2, ..., n+1$, created by n real-valued non-tied observations, and the assumption $A_{(n)}$, are:

$$\underline{P}(X_{n+1} \in B) = \sum_{j=1}^{n+1} 1\{I_j \subseteq B\} P(X_{n+1} \in I_j) \tag{1.1}$$

$$\overline{P}(X_{n+1} \in B) = \sum_{j=1}^{n+1} 1\{I_j \cap B \neq \emptyset\} P(X_{n+1} \in I_j) \tag{1.2}$$

where $1\{A\}$ is an indicator function which is equal to 1 if event A occurs and 0 else. The lower probability in Equation (1.1) is obtained by summing only the probability masses assigned to intervals I_j that are necessarily within B, while the upper probability in Equation (1.2) is obtained by summing all the probability masses that can be in B, which is the case for the probability masses per interval I_j if the intersection of I_j and B is non-empty.

In this thesis, we focus on using NPI for inventory decisions. This requires a general concept for the quantification of uncertainty, as we will need a notation for probability mass assigned to intervals without further restrictions on the spread within the intervals. Such a partial specification of a probability distribution is called M-function [29], which is given by the following definition: A partial specification of a probability distribution for a real-valued random quantity X can be provided via probability masses assigned to intervals, with no restrictions as to where the probability mass falls in the interval. The probability mass assigned for a random quantity X to an interval (a, b) can be denoted by $M_X(a, b)$ and referred to as M-function value for X on (a, b). The concept of M-function is similar to that of Shafer's basic probability assignments [62]. Clearly, each M-function value should be in $[0, 1]$ and all masses must sum to one. It is important to emphasize that the different intervals for the M-function can overlap.

1.4 Outline of thesis

In this thesis, we present the NPI method for various types of inventory models. This thesis is structured as follows. In Chapter 2, we present the main idea of the single-order inventory models involving a single-period model and a single order for a two-period model. The classical inventory model with a single period is reviewed. We introduce NPI lower and upper probabilities for the event that the random profit is non-negative, and we use them to determine an optimal inventory level. Also, we consider an optimality criterion which combines both the lower and upper probabilities. In addition, we consider the NPI lower and upper expected profits for the next period, and we also combine both lower and upper expected profits for the next period. We investigate the performance of the NPI and classical methods for the single-period model via simulation study. Also, we study a model that is similar to the single-period model but with two demands; a two-period model with a single order is considered. We presume that we are filling the inventory for both periods at the same time. Therefore, the future demand will be a combination of the future demand for the first period and the future demand for the second period. We consider maximising the NPI lower and upper expected profits as the optimality criterion for this model. We investigate the performance of the NPI and classical methods via simulation study. Part of this chapter was presented online at the Institute of Mathematics and its Applications (IMA) and Operational Research (OR) Society Conference on Mathematics of Operational Research in April 2021. The single-period inventory model has been submitted for publication in the Journal of Operational Research Society, "Nonparametric Predictive Inference for the Single-Period Inventory Model".

Chapter 3 develops the NPI method for the two-period inventory model with independent demands. First, we determine the optimal inventory level for the second period, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimise over the first period. We chose to maximise expected profit as the optimal criterion for determining optimal inventory levels. We investigate the performance of the two-period model involving different assumptions via simulation studies. Part of this chapter was presented at the Operational Research Society's Annual Conference (OR64) at the University of Warwick, UK, in September 2022. A journal paper on this model is in preparation.

In Chapter 4, we introduce the NPI bootstrap method, which we indicate by NPI-B, as an alternative method to deal with inventory problems. We apply NPI-B to the single-order and two-period independent demands models, which we studied in Chapters 2 and 3. Also, we apply NPI-B to the two-period dependent demands model. We investigate the average profit criterion, in order to determine the optimal inventory levels.

Some basic assumptions are made for all inventory models analysed, which are as follows: the length of time between the placement and receipt of an order is zero, i.e. lead time is equal to zero. An order is placed at the beginning of a period. For two-period model, the periods are of equal and known lengths.

Some final remarks and conclusions are considered in Chapter 5. The appendix contains proofs for some properties. The calculations in this thesis were performed using the statistical software R version 3.6.3. The R codes will be available on the NPI website, www.npi-statistics.com.

Chapter 2

Single-order inventory models

2.1 Introduction

The single-order inventory models include both a single-period model and a single order for a two-period model. The single-period inventory model [53], is important from both theoretical and practical perspectives. An example of the single-period model is the "newsboy problem". The aim in studying the single-period inventory model is to determine the optimal inventory level that maximises the probability that the profit is greater than or equal to zero or maximises the expected profit. Many researchers considered the single-period model in their studies. Hadley and Whitin [41] introduce the newsboy problem and they used dynamic programming to solve the problem. The approximate solutions of the standard newsboy problem based on approximations for the Normal, Poisson and Gamma distributions are derived by Shore [64]. Walker [68] studied the single-period model with probabilistic demand, where the probability distribution is estimated from historical data. The single-period model can also be used to manage capacity and make booking decisions in service sectors such as airlines and hotels [71].

In this chapter, we maximise the probability that the profit is greater than or equal to zero and maximise the expected profit to determine the optimal inventory level for the single-period model. Also, we extend the single-period inventory model to two-period inventory model with a single order. The two-period inventory model allows a backlog of demand, where the items ordered before the first period can be sold in the second period [8]. We assume that we are simultaneously filling the inventory for the two periods. Therefore, the future demand is equal to the sum of the future demand for the first period and the future demand for the second period. We introduce NPI for selecting the optimal inventory level for a single order for two-period inventory models, where the inference is based on one future observation.

The rest of the chapter is organised as follows: Section 2.2, introduces the single-period inventory model. Section 2.3, provides brief reviews of the classical inventory model. In Section 2.4, we will introduce NPI for a single-period inventory model. In Section 2.5, comparison through the simulation study between the NPI method and the classical method for the single-period inventory model is presented. The NPI method for inventory decisions is presented in Section 2.6, where we introduce NPI for a single order for a two-period model to select the optimal inventory level that maximises the expected profit. In Section 2.7, comparison through the simulation study between the NPI method and the classical method for a single order for two-period model is presented. Section 2.8 presents the concluding remarks for this chapter.

2.2 Single-period inventory model

The single-period inventory problem considers the scenario in which an inventory is needed to satisfy demand during only a single period. This means that only once the level of inventory can be decided, and that demand cannot be satisfied in a later period [16]. According to Reid and Sanders [60], the single-period model is typically designed for goods with specific characteristics: they are sold at their regular price only during the single time period and the salvage value of goods left over at the end of the period is less than their original cost, so the inventory owner loses money when goods are sold for their salvage value. An example is a newsagent who orders newspapers for a specific day, where the problem for the newsagent is to decide how many newspapers to order [41, 51, 65]. In general, the demand, denoted by D, can be either a continuous or a discrete random quantity, in this thesis we restrict attention to continuous demand.

One can use a variety of optimality criteria for the inventory level, we will consider maximisation of the probability of non-negative profit and maximisation of the expected profit. We assume that the inventory level is y, we aim at determining the best value of y, which we denote by y^*. The costs considered in the basic single-order inventory model are holding cost h, which is the cost per unit for unsold items remaining at the end of the period. The total holding costs are equal to $h(y-D)^+$, where $(v)^+ := \max(0,v)$. Shortage cost s is the cost per unit of demand that cannot be met, we assume $s \geq 0$. The total shortage costs are $s(D-y)^+$. Purchasing cost c is the price per unit ordered, we assume $c > 0$. The total purchase costs are cy. Price p is the price per unit sold, we assume a selling price that is high enough to cover the costs of inventory. The total amount of money from sales is $p\min(D,y)$. We do not assume setup cost k in this model, which is, for example, the cost for delivery. According to Waters [70], these costs lead to the profit function

$$Pf(D,y) = p\min(D,y) - cy - h(y-D)^+ - s(D-y)^+ \tag{2.1}$$

that is:

$$Pf(D,y) = \begin{cases} pD - h(y-D) - cy & \text{when } D < y \\ py - s(D-y) - cy & \text{when } D > y \\ (p-c)y & \text{when } D = y \end{cases} \tag{2.2}$$

2.3 Classical inventory model

The classical model is a basic model for inventories, presented by Harris in 1913 [42]. Inventory models have been classified based on the pattern of demand, which may be either deterministic or random over time. Deterministic demand occurs when we know the exact quantities needed over a period of time. Random demand occurs when the demand is not known within a period of time. Forecasting the random demand tends to be a challenge for the decision maker [67]. In most of the literature on inventory problems, this difficulty is avoided by the assumption of a stochastic model for the random demand.

The aim of studying the inventory model is to fill the inventory at the optimal level that meets the demand and leads to maximum profit. There are a variety of optimality criteria for the inventory level, we will consider maximising the probability of non-negative profit in Section 2.3.1, and maximising the expected profit in Section 2.3.2.

2.3.1 Maximisation of the probability of non-negative profit

One may wish to order in such a way that the probability of a loss is minimal, hence to maximise the probability of a non-negative profit [61]. This criterion is straightforwardly

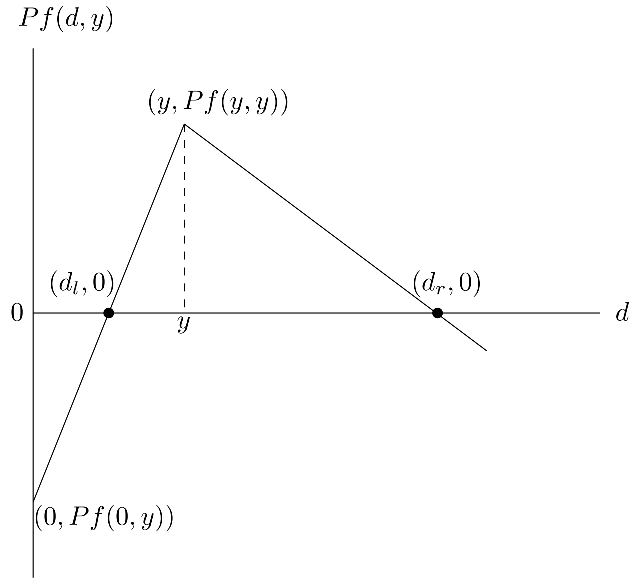


Figure 2.1: Profit for fixed inventory level y as function of demand d

adapted to maximisation of the probability that the profit exceeds a specific value other than zero, we do not address this further.

For fixed inventory level y, the profit function, given by Equation (2.1), is a function of the demand $D = d$, and it consists of two line segments as presented in Figure 2.1. From Equation (2.1), we find d_l and d_r such that $Pf(d_l, y) = Pf(d_r, y) = 0$. For $d < y$,

$$d_l = \frac{(c+h)y}{p+h} \tag{2.3}$$

and for $d > y$,

$$d_r = \frac{(p+s-c)y}{s} \tag{2.4}$$

Note that d_l and d_r are functions of y, we do not explicitly include this in the notation. The profit is greater than or equal to zero whenever $d_l \leq D \leq d_r$. So,

$$P(Pf(D,y) \geq 0) = P(d_l \leq D \leq d_r) = \int_{d_l}^{d_r} f_D(u)du \tag{2.5}$$

where $f_D(\cdot)$ is the probability density function (PDF) for the demand D. Let y^*_{CP} denote the optimal inventory level which maximises the probability that the profit is greater than or equal to zero. Setting the first derivative with respect to y of this probability equal to zero leads to

$$\frac{p+s-c}{s} f_D(d_r) - \frac{c+h}{p+h} f_D(d_l) = 0 \tag{2.6}$$

For any given probability distribution for D, Equation (2.6) provides the possible value(s) for y^*_{CP}, checking the second-order sufficient condition for optimality leads to the optimal inventory level.

Example 2.3.1 Suppose $D \sim N(\mu = 400, \sigma = 30), c = 20, h = 10, s = 20$ and $p = 50$. Our aim is to find the optimal inventory level that maximises the probability that the profit is greater than or equal to zero.

By substituting Equations (2.3) and (2.4) in Equation (2.6), we have

$$-2\sigma^2 \ln\left[\frac{(p+s-c)(p+h)}{s(c+h)}\right] = y^2\left[\left(\frac{c+h}{p+h}\right)^2 - \left(\frac{p+s-c}{s}\right)^2\right] + y\left[-2\mu\left(\frac{c+h}{p+h}\right) + 2\mu\left(\frac{p+s-c}{s}\right)\right]$$

$$\Longrightarrow \quad -6y^2 + 1600y + 2896.99 = 0$$

$$\text{so,} \quad y^*_{CP} = 268.47$$

◇

2.3.2 Maximisation of the expected profit

According to Shih [63] the expected profit for inventory level y is

$$E(Pf(D,y)) = \int_0^y (pu - h(y-u) - cy) f_D(u)du + \int_y^\infty (py - s(u-y) - cy) f_D(u)du \quad (2.7)$$

The optimal inventory level y^*_{CE}, which maximises the profit, is derived by setting the first derivative of this expected profit to zero, leading to the equation

$$P(D \leq y^*_{CE}) = \frac{p+s-c}{p+s+h} \quad (2.8)$$

As the second derivative of the expected profit is negative at all values of y with $f_D(y) > 0$, the value y^*_{CE} resulting from Equation (2.8) is the optimal inventory level when aiming at maximisation of the expected profit.

Example 2.3.2 Consider the same data as in Example 2.3.1. Our aim is to find the optimal inventory level that maximises the expected profit.

From Equation (2.8), $P(D \leq y^*_{CE}) = 0.63$ so, $y^*_{CE} = 409.96$. If we consider $s = 1$ then $y^*_{CE} = 400.75$. So, when the shortage cost decreases, the optimal inventory level decreases. Also, if we increase the purchase cost to be $c = 30$, the optimal inventory level becomes 400. So, when the purchase cost increases, the optimal inventory level decreases.

◇

2.4 NPI for single-period inventory model

In this section, we develop NPI for decision support in the single-period inventory model. We assume that data of demand in n previous periods are available, ordered as $d_1 < d_2 < ... < d_n$, and we consider the random demand D_{n+1} for the next single period for which the inventory decision is required. We assume that there is a known upper bound for the demand, denoted by d_u, which is logically greater than d_n, and that demand is positive. Based on these data, and setting $d_0 = 0$ and $d_{n+1} = d_u$, the assumption $A_{(n)}$ for D_{n+1} leads to:

$$P(D_{n+1} \in (d_{j-1}, d_j)) = \frac{1}{n+1} \quad \text{for} \quad j = 1, ..., n+1 \tag{2.9}$$

The essential step in developing NPI for the single-period inventory model, is the transfer of the partial probability distribution specification for D_{n+1} to the partial probability distribution specification for the profit, using the profit function given by Equation (2.1). This process is illustrated in Figure 2.2, and involves the profit $Pf(D_{n+1}, y)$ as function of the random demand D_{n+1} and fixed inventory level y. The overall aim is to determine an optimal value for y. Let $j_y \in \{1, ..., n+1\}$ be such that $y \in (d_{j_y-1}, d_{j_y})$. The probabilities for D_{n+1}, given in Equation (2.9), lead to the following M-function values for the random profit $Pf(D_{n+1}, y)$,

$$M(Pf(d_{j-1}, y), Pf(d_j, y)) = \frac{1}{n+1} \qquad \text{for } j \in \{1, ..., j_y - 1\} \tag{2.10}$$

$$M(\min[Pf(d_{j_y-1}, y), Pf(d_{j_y}, y)], Pf(y, y)) = \frac{1}{n+1} \qquad \text{for } j = j_y \tag{2.11}$$

$$M(Pf(d_j, y), Pf(d_{j-1}, y)) = \frac{1}{n+1} \qquad \text{for } j \in \{j_y + 1, ..., n+1\} \tag{2.12}$$

The transfers of the probabilities for D_{n+1}, as given in Equation (2.9), to M-function values for $Pf(D_{n+1}, y)$ for the cases in Equations (2.10) and (2.12) follow straightforwardly from Figure 2.2. Equation (2.11) is also best understood from this figure, where we should emphasize that since y divides the interval into two parts, the minimum is considered to cover all possible values of profit for $D_{n+1} \in (d_{j_y-1}, d_{j_y})$.

Section 2.4.1 presents the NPI lower and upper probabilities for the event that the profit over the single period considered is non-negative, and uses these to determine an optimal inventory level. This is followed by focus on the NPI lower and upper expected values for the profit, and their optimisation, in Section 2.4.2.

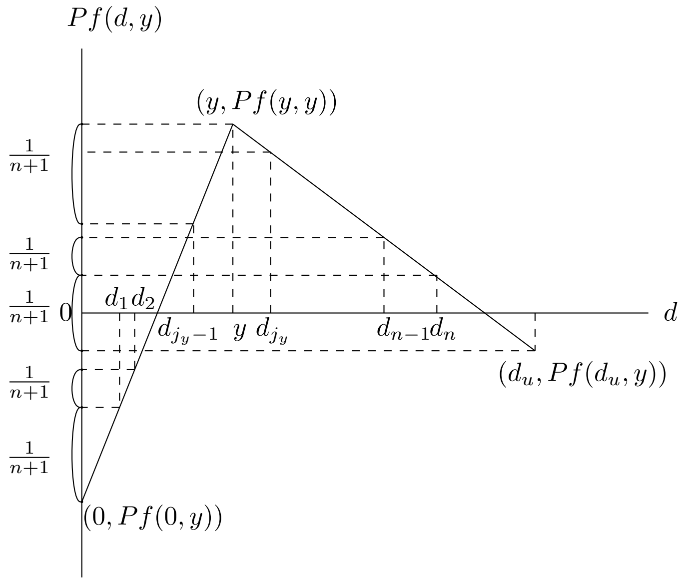


Figure 2.2: Single-period inventory - M-functions

2.4.1 NPI lower and upper probabilities for non-negative profit

In this section, we calculate the NPI lower and upper probabilities for the event that the profit corresponding to future demand is greater than or equal to zero, for given y. When aiming to maximise the NPI lower probability for non-negative profit, it is easily understood from Figures 2.1 and 2.2 that we should maximise the number of intervals (d_{j-1}, d_j), for $j = 1, ..., n+1$, that are entirely within $[d_l, d_r]$, where d_l and d_r are functions of the inventory level y, Equations (2.3) and (2.4). Clearly, the profit function is certainly positive for D_{n+1} in between two consecutive values d_{j-1} and d_j, of the data $d_1 < ... < d_n$, if both d_{j-1} and d_j are in between the two points, d_l and d_r, where the profit function is equal to zero. Hence, the lower probability for positive profit is just the number of intervals between consecutive d_j values that are entirely between d_l and d_r, where each such interval has been assigned probability mass $\dfrac{1}{n+1}$ based on the $A_{(n)}$ assumption for the future demand D_{n+1}. This number is easy to calculate as follows:

Let y_k be such that $Pf(d_k, y_k) = 0$ with $y_k \geq d_k$ for $k = 1, ..., n$. Hence, y_k is such that the increasing line segment of the function $Pf(D, y_k)$ is equal to 0 at $D = d_k$, so related to Figure 2.1 we have $d_l = d_k$. This leads to

$$y_k = \frac{(p+h)d_k}{c+h} \tag{2.13}$$

This value y_k fully specifies the profit function for given values of the costs.

Let d_k^r be such that $Pf(d_k^r, y_k) = 0$ and $d_k^r \geq y_k$, hence the decreasing line segment of the function $Pf(D, y_k)$ is equal to 0 at $D = d_k^r$. By Equations (2.3) and (2.4) we have

$$\begin{aligned} d_k^r =& \frac{(p+s-c)y_k}{s} \\ =& \frac{(p+s-c)(p+h)d_k}{s(c+h)} \end{aligned} \tag{2.14}$$

The NPI lower probability, $\underline{P}(Pf(D_{n+1}, y_k) \geq 0)$, is derived by counting the number of intervals (d_{j-1}, d_j) which are entirely contained in $[d_k, d_k^r]$, noting that each such interval has been assigned probability mass $\frac{1}{n+1}$ based on the $A_{(n)}$ assumption for the future demand D_{n+1}. Let n_k denote the number of d_j, for $j = 1, ..., n+1$, such that $d_k \leq d_j \leq d_k^r$. To summarise, we have Equations (2.13), (2.14) and

$$n_k = \#\{d_j : d_j \in [d_k, d_k^r], j \in \{1, ..., n+1\}\} \tag{2.15}$$

There are $(n_k - 1)^+$ intervals (d_{j-1}, d_j) which are entirely in $[d_k, d_k^r]$, so the NPI lower probability for the event $Pf(D_{n+1}, y_k) \geq 0$ is

$$\underline{P}(Pf(D_{n+1}, y_k) \geq 0) = \frac{(n_k - 1)^+}{n+1} \tag{2.16}$$

An optimal inventory level $y_{\underline{P}}^*$ which maximises $\underline{P}(Pf(D_{n+1}, y))$ is now derived by setting $y_{\underline{P}}^* = y_k$, with $k \in \{1, ..., n\}$ such that the corresponding value of n_k is maximal. Note that it is quite likely that there is not a unique inventory level that maximises $\underline{P}(Pf(D_{n+1}, y))$, not only because there may not be a unique value n_k leading to the maximum $\underline{P}(Pf(D_{n+1}, y))$, but also because values just less than $y_{\underline{P}}^*$ are likely to lead to the same value for this NPI lower probability for non-negative profit.

Deriving an optimal inventory level to maximise the NPI upper probability for the event $Pf(D_{n+1}, y) \geq 0$ can be done in a similar way as for the corresponding NPI lower probability. This upper probability is derived by counting the intervals (d_{j-1}, d_j) that have a non-empty intersection with $[d_l, d_r]$, which is again easily seen from Figures 2.1 and 2.2. Of course, this includes all the intervals that were included in the counts to derive the NPI lower probability, while some attention is needed for the intervals which contain either d_l or d_r. Assuming that $D_{n+1} = 0$ leads to negative profit, which is the case for any positive inventory level, y, then the interval (d_{j-1}, d_j) containing d_l must now be included in the count.

For the interval which contains d_r, we need to consider the value of the profit, for given y, at the end-point of demand d_u, since both d_r and d_u can be greater than each other. If d_u is large enough to have a negative profit at d_u for given y, then the interval (d_{j-1}, d_j) containing d_r must now be included in the count, but if the profit at d_u is non-negative, then that interval will already have been included in the count for the NPI lower probability. This leads to the following NPI upper probability for inventory levels y_k, for $k = 1, ..., n$, as defined in Equation (2.13),

$$\overline{P}(Pf(D_{n+1}, y_k) \geq 0) = \begin{cases} \underline{P}(Pf(D_{n+1}, y_k) \geq 0) + \frac{2}{n+1} & \text{if} \quad Pf(d_u, y_k) < 0 \\ \underline{P}(Pf(D_{n+1}, y_k) \geq 0) + \frac{1}{n+1} & \text{if} \quad Pf(d_u, y_k) \geq 0 \end{cases} \tag{2.17}$$

The optimal inventory level $y^*_{\overline{P}}$ which maximises $\overline{P}(Pf(D_{n+1}, y) \geq 0)$ is easily derived by taking the value y_k that maximises this upper probability over $k = 1, ..., n$. The same comment with regard to non-uniqueness of the optimal inventory level, as made above for the NPI lower probability, applies here. Furthermore, we should mention that, in this section, we have taken some liberties on mathematical accuracy for the sake of simplicity of the presentation: First, for the optimisation of the NPI upper probability, the arguments provided would actually require consideration of values $d_k - \epsilon$ instead of d_k, for very small $\epsilon > 0$, but the effect on the corresponding strategies y_k, and hence on the optimal inventory level, is neglectable. Secondly, we have not paid attention to situations where d_l or d_u could coincide with an observed value d_j. This is of no serious consequence when demand is a continuous random quantity; if one would want to model discrete demand then more care would be needed, but one would then also want to consider a different NPI approach than the one used in this chapter, this is not considered further here.

As optimisation of the NPI lower probability and the NPI upper probability for non-negative profit may not both lead to the same optimal inventory level, one may need to choose which of these two criteria to use. One could consider optimisation of the NPI lower probability a somewhat pessimistic perspective, with optimisation of the NPI upper probability reflecting a more optimistic point of view. However, one could also combine these two criteria using the Hurwicz criterion [3], which here implies maximisation of a weighted average of the NPI lower and upper probabilities. With $\omega \in [0,1]$, we define

$$H_{P,\omega}(Pf(D_{n+1}, y_k) \geq 0) = \omega \underline{P}(Pf(D_{n+1}, y_k) \geq 0) + (1-\omega)\overline{P}(Pf(D_{n+1}, y_k) \geq 0) \tag{2.18}$$

The parameter ω can be interpreted as an optimism-pessimism index. The choice of ω, like the overall choice of the optimality criteria, would be for the decision maker to make whether they want to be more pessimistic or more optimistic. So, ω is between 0 and 1, describing the level of optimism, with the remainder being pessimism. An ω of, say, 0.2 means that you are more pessimistic than optimistic. When $\omega = 0.1$, that means that you are even more pessimistic than when $\omega = 0.2$. Setting ω to 0.95 means that you are very optimistic, but a small amount of pessimism (5%) remains. Using Equations (2.16) and (2.17) we get

$$H_{P,\omega}(Pf(D_{n+1}, y_k) \geq 0) = \begin{cases} \dfrac{(n_k + 1 - 2\omega)^+}{n+1} & \text{if} \quad Pf(d_u, y_k) < 0 \\ \dfrac{(n_k - \omega)^+}{n+1} & \text{if} \quad Pf(d_u, y_k) \geq 0 \end{cases} \tag{2.19}$$

The optimal inventory level according to the Hurwicz criterion will be denoted by $y^*_{P,\omega}$. Example 2.4.1 illustrates the methods presented in this section, considering the optimisation of the NPI lower probability, the NPI upper probability, and the corresponding Hurwicz criterion.

Example 2.4.1 Consider an inventory system with the following costs: $p = 50, c = 20, h = 10, s = 20$, and assume that demand is known to be between $d_0 = 0$ and $d_u = 40$. Assume that there are $n = 5$ demand observations, with values $7.20, 12.50, 15.30, 22.60$, 35.40 and assume $\omega = 0.60$. Table 2.1 presents the results of the method presented in this section, specifying the NPI lower and upper probabilities and their weighted function for the event that the profit for the single period considered will be non-negative, together with the corresponding values of y_k and of the quantities d^r_k and n_k that are part of the computations as explained above. The NPI lower probability is maximal at $y_1 = 14.40$ and $y_2 = 25$, so either of these values can be chosen as $y^*_{\underline{P}}$. The NPI upper probability is maximal at $y^*_{\overline{P}} = y_1 = 14.40$. Maximisation of the Hurwicz criterion leads to $y^*_{P,\omega} = 14.40$ when $\omega = 0.60$. In general, for all $\omega \in [0,1)$, the the Hurwicz criterion is maximal at $y^*_{P,\omega} = 14.40$, while for $\omega = 1$ this criterion is the same as maximisation of the NPI lower probability, which was discussed above. ◇

k	y_k	d^r_k	n_k	$\underline{P}(Pf(D_6, y_k) \geq 0)$	$\overline{P}(Pf(D_6, y_k) \geq 0)$	$H_{P,\omega}(Pf(D_6, y_k) \geq 0)$
1	14.40	36.00	5	0.67	1.00	0.80
2	25.00	62.50	5	0.67	0.83	0.73
3	30.60	76.50	4	0.50	0.67	0.57
4	45.20	113.00	3	0.33	0.50	0.40
5	70.80	177.00	2	0.17	0.33	0.23

Table 2.1: NPI lower and upper probabilities and their weighted function for Example 2.4.1

2.4.2 NPI lower and upper expected profits

In this section, we present the NPI lower and upper expected profits for the next period, as function of the inventory level y. The derivations are based on the M-functions illustrated in Figure 2.2 and presented in Equations (2.10)-(2.12).

The NPI lower expected profit, denoted by $\underline{E}_{n+1}(y)$, is derived by assigning the probability masses $\frac{1}{n+1}$, according to the M-function values in Equations (2.10)-(2.12), to the minimum (or infimum) value for $Pf(D_{n+1}, y)$ per interval. With, as before, j_y such that $y \in (d_{j_y-1}, d_{j_y})$, this leads to

$$\begin{aligned}
\underline{E}_{n+1}(y) &= \sum_{j=1}^{j_y-1} M(Pf(d_{j-1}, y), Pf(d_j, y))Pf(d_{j-1}, y) \\
&\quad + M(\min[Pf(d_{j_y-1}, y), Pf(d_{j_y}, y)], Pf(y, y)) \min[Pf(d_{j_y-1}, y), Pf(d_{j_y}, y)] \\
&\quad + \sum_{j=j_y+1}^{n+1} M(Pf(d_j, y), Pf(d_{j-1}, y))Pf(d_j, y) \\
&= \frac{1}{n+1}\Bigg((j_y - 1)(-(c+h)y) + (p+h)\sum_{j=1}^{j_y-1} d_{j-1} + \min[-(c+h)y \\
&\quad + (p+h)d_{j_y-1}, (p-c+s)y - sd_{j_y}] + (n+1-j_y)(p-c+s)y - s\sum_{j=j_y+1}^{n+1} d_j\Bigg)
\end{aligned} \tag{2.20}$$

To determine an optimal inventory level, denoted by $y^*_{\underline{E}}$, which maximises $\underline{E}_{n+1}(y)$,

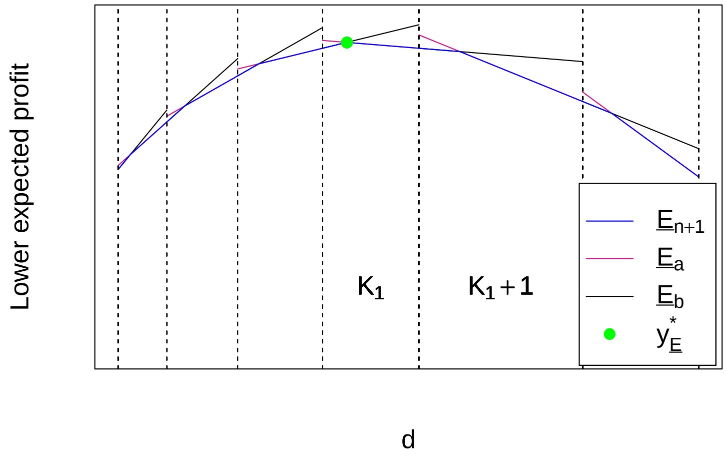


Figure 2.3: The NPI lower expectation function and the optimal value

we introduce the following two functions,

$$\underline{E}_a(y) = \frac{1}{n+1}\Bigg((j_y-1)(-(c+h)y) + (p+h)\sum_{j=1}^{j_y-1} d_{j-1} - (c+h)y + (p+h)d_{j_y-1} + (n+1-j_y)(p-c+s)y - s\sum_{j=j_y+1}^{n+1} d_j\Bigg) \tag{2.21}$$

and

$$\underline{E}_b(y) = \frac{1}{n+1}\Bigg((j_y-1)(-(c+h)y) + (p+h)\sum_{j=1}^{j_y-1} d_{j-1} + (p-c+s)y - sd_{j_y} + (n+1-j_y)(p-c+s)y - s\sum_{j=j_y+1}^{n+1} d_j\Bigg) \tag{2.22}$$

By Equation (2.20), $\underline{E}_{n+1}(y) = \min[\underline{E}_a(y), \underline{E}_b(y)]$. It is easy to verify that the function $\underline{E}_{n+1}$ is discontinuous at d_l, for all $l \in \{1, ..., n\}$, the proof of this property is given in Appendix A.1. We also note that $\underline{E}_a(y)$ and $\underline{E}_b(y)$ are linear functions in each interval $[d_{j_y-1}, d_{j_y}]$. We can show that $\underline{E}_a(y)$ is an increasing function in $[d_{j_y-1}, d_{j_y}]$ if and only if

$$j_y < \frac{(n+1)(p-c+s)}{p+h+s} =: K_1 \tag{2.23}$$

and $\underline{E}_a(y)$ is a decreasing function in $[d_{j_y-1}, d_{j_y}]$ if and only if $j_y > K_1$. $\underline{E}_b(y)$ is an increasing function in $[d_{j_y-1}, d_{j_y}]$ if and only if

$$j_y < \frac{c+h+(n+2)(p-c+s)}{p+h+s} =: K_1 + 1 \tag{2.24}$$

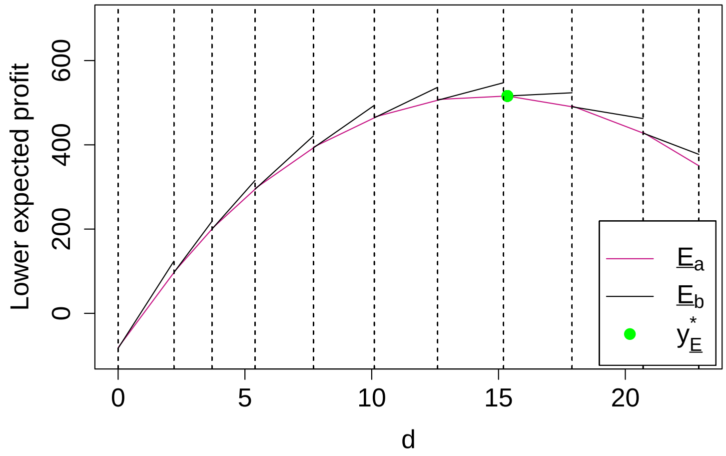


Figure 2.4: The functions $\underline{E}_a(y)$ and $\underline{E}_b(y)$ for Example 2.4.2

and $\underline{E}_b(y)$ is a decreasing function in $[d_{j_y-1}, d_{j_y}]$ if and only if $j_y > K_1 + 1$. This implies that the maximum value of $\underline{E}_{n+1}(y) = \min[\underline{E}_a(y), \underline{E}_b(y)]$ is at the intersection point of $\underline{E}_a(y)$ and $\underline{E}_b(y)$ in the single interval where $\underline{E}_a(y)$ decreases and $\underline{E}_b(y)$ increases. This leads to the optimal inventory level, which maximises the NPI lower expected profit,

$$y^*_{\underline{E}} = \frac{(p+h)d_{j_y-1} + sd_{j_y}}{p+h+s} \tag{2.25}$$

where $K_1 \leq j_y < K_1 + 1$. This is illustrated in Figure 2.3 and in the following example.

Example 2.4.2 Let the costs be $p = 103$, $c = 16$, $h = 20$ and $s = 7$, and suppose data consisting of $n = 9$ observations: $2.20, 3.70, 5.40, 7.70, 10.10, 12.60, 15.20, 17.90, 20.70$. We assume that the maximum possible value for the demand is $d_u = 22.90$.

We have $K_1 = 7.23$, so, in the first seven intervals of the partition of $[0, 22.90]$ created by the data, $\underline{E}_a(y)$ is an increasing function and thereafter it is a decreasing function, while $\underline{E}_b(y)$ is an increasing function in the first eight intervals, and a decreasing function thereafter. This is illustrated in Figure 2.4. So the minimum of these two functions reaches its maximum value at the intersection of these two functions in the interval $(15.20, 17.90)$, leading to $y^*_{\underline{E}} = 15.35$ with corresponding NPI lower expected profit $\underline{E}_{n+1}(y^*_{\underline{E}}) = 515.90$.

◇

Similarly, by assigning the probability masses $\frac{1}{n+1}$ in Equations (2.10)-(2.12) to the maximal value for $Pf(D_{n+1}, y)$ per interval, we derive the upper expected profit

$$\begin{aligned}
\overline{E}_{n+1}(y) &= \sum_{j=1}^{j_y-1} M(Pf(d_{j-1}, y), Pf(d_j, y))Pf(d_j, y) \\
&\quad + M(\min[Pf(d_{j_y-1}, y), Pf(d_{j_y}, y)], Pf(y, y))Pf(y, y) \\
&\quad + \sum_{j=j_y+1}^{n+1} M(Pf(d_j, y), Pf(d_{j-1}, y))Pf(d_{j-1}, y) \\
&= \frac{1}{n+1}\Bigg((j_y-1)(-(c+h)y) + (p+h)\sum_{j=1}^{j_y-1} d_j + (p-c)y \\
&\qquad + (n+1-j_y)(p-c+s)y - s\sum_{j=j_y+1}^{n+1} d_{j-1}\Bigg) \qquad (2.26)
\end{aligned}$$

We note that $\overline{E}_{n+1}(y)$ is a continuous function, the proof is provided in Appendix A.2. To derive the optimal inventory level which maximises $\overline{E}_{n+1}(y)$, denoted by $y^*_{\overline{E}}$, we use that $\overline{E}_{n+1}(y)$ is an increasing function over the interval $[d_{j_y-1}, d_{j_y}]$ if and only if

$$j_y < \frac{h+p+(n+1)(p-c+s)}{p+s+h} =: K_2 \qquad (2.27)$$

and $\overline{E}_{n+1}(y)$ is a decreasing function over the interval $[d_{j_y-1}, d_{j_y}]$ if and only if $j_y > K_2$. This implies that, as $\overline{E}_{n+1}(y)$ is a continuous function, so $y^*_{\overline{E}} = d_{l^*}$, with l^* the largest value in $\{1, 2, ..., n\}$ which is less than K_2. This is illustrated in the following example.

Example 2.4.3 Consider the same scenario as in Example 2.4.2, the aim is to find the optimal inventory level which maximises the NPI upper expected profit. The upper expected profit function $\overline{E}_{n+1}(y)$ is presented in Figure 2.5. We have $K_2 = 8.18$, so $y^*_{\overline{E}} = d_8$, that is the 8th ranked observation out of the 9 data observations, hence $y^*_{\overline{E}} = 17.90$ with the corresponding NPI upper expected profit $\overline{E}_{n+1}(y^*_{\overline{E}}) = 714.02$. ◇

As an alternative to maximising the NPI lower expected profit or the NPI upper expected profit, one can use the Hurwicz criterion and aim at maximising their weighted average,

$$H_{E,\omega}(y) = \omega\underline{E}_{n+1}(y) + (1-\omega)\overline{E}_{n+1}(y) \qquad (2.28)$$

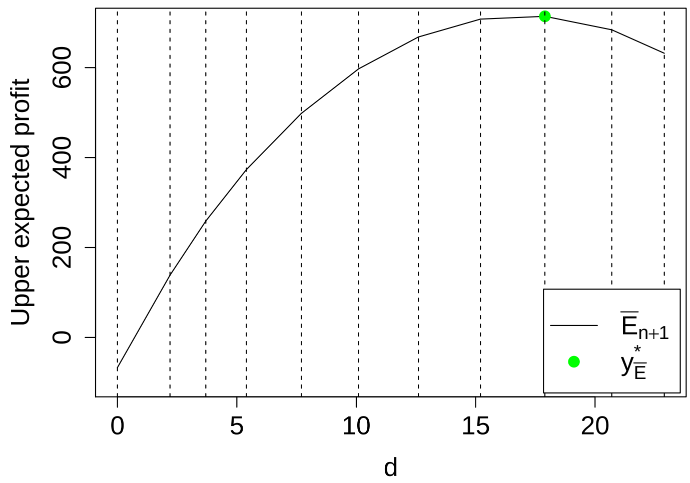


Figure 2.5: The NPI upper expectation function $\overline{E}_{n+1}(y)$

for $0 < \omega < 1$. Using Equations (2.20) and (2.26) leads to

$$\begin{aligned}H_{E,\omega}(y) = \frac{\omega}{n+1}\Big[&- (p-c)y + (p+h)(d_0 - d_{j_y-1}) - s(d_{n+1} - d_{j_y}) \\ &+ \min[-(c+h)y + (p+h)d_{j_y-1}, (p-c+s)y - sd_{j_y}]\Big] \\ &+ \frac{1}{n+1}\Big(y[(j_y-1)(-(c+h)) + (p-c) + (n+1-j_y)(p-c+s)] \\ &+ (p+h)\sum_{j=1}^{j_y-1} d_j - s\sum_{j=j_y+1}^{n+1} d_{j-1}\Big) \qquad (2.29)\end{aligned}$$

To determine an optimal inventory level, denoted by $y^*_{E,\omega}$, which maximises $H_{E,\omega}(y)$, we introduce the following two functions:

$$\begin{aligned}H_{\omega a}(y) = \frac{\omega}{n+1}\Big[&- (p-c)y + (p+h)(d_0 - d_{j_y-1}) - s(d_{n+1} - d_{j_y}) - (c+h)y \\ &+ (p+h)d_{j_y-1}\Big] + \frac{1}{n+1}\Big(y[(j_y-1)(-(c+h)) + (p-c) \\ &+ (n+1-j_y)(p-c+s)] + (p+h)\sum_{j=1}^{j_y-1} d_j - s\sum_{j=j_y+1}^{n+1} d_{j-1}\Big) \qquad (2.30)\end{aligned}$$

and

$$\begin{aligned}H_{\omega b}(y) = \frac{\omega}{n+1}\Big[&- (p-c)y + (p+h)(d_0 - d_{j_y-1}) - s(d_{n+1} - d_{j_y}) + (p-c+s)y - sd_{j_y}\Big] \\ &+ \frac{1}{n+1}\Big(y[(j_y-1)(-(c+h)) + (p-c) + (n+1-j_y)(p-c+s)] \\ &+ (p+h)\sum_{j=1}^{j_y-1} d_j - s\sum_{j=j_y+1}^{n+1} d_{j-1}\Big) \qquad (2.31)\end{aligned}$$

so $H_{E,\omega}(y) = \min[H_{\omega a}(y), H_{\omega b}(y)]$. The analysis to derive the optimal inventory level, $y^*_{E,\omega}$, is similar to that for optimising the NPI lower expected profit, with $H_{E,\omega}$ also discontinuous at d_l, for all $l \in \{1, ..., n\}$, the proof of this property is given in Appendix A.3, and $H_{\omega a}(y)$ and $H_{\omega b}(y)$ linear functions in each interval $[d_{j_y-1}, d_{j_y}]$. $H_{\omega a}(y)$ is an increasing function in the interval $[d_{j_y-1}, d_{j_y}]$ if and only if

$$j_y < \frac{(1-\omega)(p+h)+(n+1)(p-c+s)}{p+h+s} =: K_3 \tag{2.32}$$

and $H_{\omega a}(y)$ is a decreasing function in the interval $[d_{j_y-1}, d_{j_y}]$ if and only if $j_y > K_3$. Similarly, $H_{\omega b}(y)$ is an increasing function in the interval $[d_{j_y-1}, d_{j_y}]$ if and only if

$$j_y < \frac{\omega s+p+h+(n+1)(p-c+s)}{p+h+s} =: K_3+\omega \tag{2.33}$$

and $H_{\omega b}(y)$ is a decreasing function in the interval $[d_{j_y-1}, d_{j_y}]$ if and only if $j_y > K_3+\omega$. This leads to optimal inventory level

$$y^*_{E,\omega} = \frac{(p+h)d_{j_y-1}+sd_{j_y}}{p+h+s} \tag{2.34}$$

where j_y is the largest value in $\{1, \ldots, n\}$ for which $j_y < K_3+\omega$. This is illustrated in the following example.

Example 2.4.4 Consider again the same data as in Examples 2.4.2 and 2.4.3, and let $\omega = 0.70$. The aim is to find the optimal inventory level which maximises $H_{E,0.70}(y)$. We have $K_3 = 7.51$ so $j_y = 8$ and the maximum value of $H_{E,0.70}(y)$ is achieved at the intersection of $H_{\omega a}(y)$ and $H_{\omega b}(y)$ in the interval $(15.20, 17.90)$, leading to $y^*_{E,0.70} = 15.35$ with corresponding $H_{E,0.70}(y^*_{E,0.70}) = 573.57$. This is illustrated in Figure 2.6. ◇

2.5 Comparison of the NPI and classical methods for the single-period model

This section presents the results of simulation to investigate the performance of the NPI and classical methods for the single-period inventory problem. Our aim is effectively to check how close the classical method is to NPI when the distribution of the classical method is assumed to be known. Then, based on some assumptions, compare which method performs better than the other. We simulate n observations of demand from a Gamma distribution, since the demand is assumed to be positive in this thesis, we

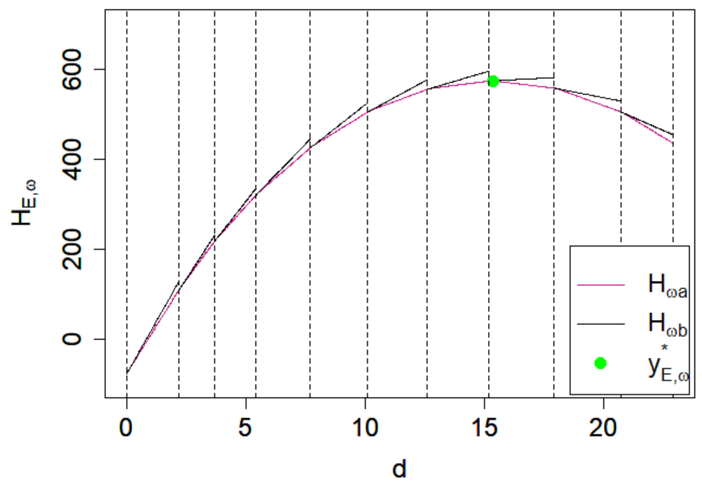


Figure 2.6: The functions $H_{\omega a}(y)$ and $H_{\omega b}(y)$ for Example 2.4.4

select the Gamma distribution in simulation settings as it is flexible in many shapes for positive real values. These n simulated data observations are used to determine the optimal inventory level y^* corresponding to one of the optimality criterion, lower and upper probabilities of non-negative profit, or the lower and upper expected profit. Then a value for one future observation D is simulated from the same underlying Gamma distribution, allowing the realised value of the profit function to be computed for the values of y^* and D.

The simulated future demand is compared with the optimal inventory level to study the performance of the methods, as follows:
If $D < y^*$, then the number of sales is D, then the profit is $pD - cy^* - h(y^* - D)$.
If $D > y^*$, then the number of sales is y^*, then the profit is $py - cy^* - s(D - y^*)$.
If $D = y^*$, then the profit is $(p - c)y^*$.

The difference between NPI and the classical method is that the classical method assumes the probability distribution of demand is fully known, while the NPI method only uses data, since the probability distribution of the demand is unknown.

We consider six different cases with regard to discrepancy between the model used for the data simulations, and the model assumed for the classical method to determine the optimal inventory level, which is compared to the optimal NPI inventory level. Each case is run 1000 times and we report the number of these runs in which the profit resulting from the NPI method is greater than the profit resulting from the classical method. In

Case	Simulation	Classical assumption
I	$D_i \sim$ Gamma$(3, 1)$	Gamma$(3, 1)$
II	$D_i \sim$ Gamma$(3, 1)$	Exp$(1/3)$
III	$D_i \sim$ Gamma$(3, 1)$	Exp$(1/2)$
IV	$D_i \sim$ Gamma$(3, \theta), \theta \sim$ Unif$(0, 2)$	Gamma$(3, 1)$
V	$D_i \sim$ Gamma$(3, 1)$	Exp(1)
VI	$D_i \sim$ Gamma$(3, 1)$	Exp(2)

Table 2.2: Simulation cases

the comparison, we only consider how often the profit is doing better in each run, but it could also be of interest to see by how much it is greater as a topic for future research.

The Gamma(k, θ) distribution, with shape parameter k and scale parameter θ, has probability density function $f(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{\frac{-x}{\theta}}$ for $x > 0$ and mean value $k\theta$. The Exponential(λ) distribution with rate λ has probability density function $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ and mean value $\frac{1}{\lambda}$. The cases considered are shown in Table 2.2, with first the model used for simulating the demands D specified, followed by the model assumed for the analysis according to the classical method. For the case where the Gamma scale parameter θ is simulated from the Uniform$(0, 2)$ distribution, one value is drawn and used for each run, so n observations are drawn using one specific value of θ, and a new value of θ is drawn for the next run.

Case I is the scenario where the model used for the classical analysis is exactly the same as the model used for the data generation. In Case II the model for the analysis is Exponential but with the same mean value as the Gamma distribution used to generate the data. The further cases have other discrepancies between these two models, set up in such a way that we expected that the classical method would perform more poorly for the later cases as the differences between the models increase.

We consider three different sample sizes, $n = 5, 50, 100$. For all cases, the costs used are $c = 20, p = 50, h = 10, s = 20$ and we use $\omega = 0.50$ for the Hurwicz criterion. As finite end-point for the support of the random demand we took $d_u = 15$; in the rare event that a simulated value in a run exceeds 15 we delete the value and draw a new one; this has no real impact on the methods as the probability to get a value which exceeds 15 is very small for all models considered.

	$n = 5$			$n = 50$			$n = 100$		
Case	$\underline{P}$	$\overline{P}$	$H_{P,\omega}$	$\underline{P}$	$\overline{P}$	$H_{P,\omega}$	$\underline{P}$	$\overline{P}$	$H_{P,\omega}$
I	384	385	384	436	436	436	486	486	486
II	384	385	384	436	436	436	486	486	486
III	551	557	551	704	704	704	710	710	710
IV	597	604	597	694	694	694	678	678	678
V	767	776	767	826	826	826	837	837	837
VI	822	831	822	889	889	889	896	896	896

Table 2.3: Number of cases out of 1000 where NPI-based profit is greater than the classical method for lower and upper probabilities and their weighted function

Tables 2.3 and 2.4 present the results from the simulation study. They provide the number of times, out of 1000 runs, in which the profit according to the NPI methods for the single-period model, are larger than for the corresponding classical method. Table 2.3 considers the probability of non-negative profit as optimality criterion, where $\underline{P}$, $\overline{P}$ and $H_{P,\omega}$ indicate that, for the NPI method, the lower probability, the upper probability, or the Hurwicz criterion was used, respectively. Table 2.4 considers the expected profit as optimality criterion, where $\underline{E}$, $\overline{E}$ and $H_{E,\omega}$ indicate that, for the NPI method, the lower expected profit, the upper expected profit, or the Hurwicz criterion was used, respectively.

As expected, the NPI methods perform worse than the classical methods in Case I, but for large n the performance of the NPI methods improves and the number of times it performs better than the classical method increases to close to 500. Case II is perhaps the most surprising as for optimising the probability of non-negative profit, the NPI method let to more profit in precisely the same numbers of cases for Case I, this is because that the classical optimal inventory levels that maximise the probability that the profit is greater than or equal to zero are equal when the mean is the same for Gamma and Exponential distribution; this general property is proven in Appendix A.4. Hence, the number of times that NPI leads to higher profit than the classical method for Case I is the same as Case II.

	$n=5$			$n=50$			$n=100$		
Case	$\underline{E}$	$\overline{E}$	$H_{E,\omega}$	$\underline{E}$	$\overline{E}$	$H_{E,\omega}$	$\underline{E}$	$\overline{E}$	$H_{E,\omega}$
I	469	393	413	487	456	455	485	496	485
II	426	391	400	401	397	392	411	410	411
III	524	505	511	560	547	545	553	556	553
IV	622	615	610	676	679	676	714	715	714
V	751	685	705	733	725	722	726	726	726
VI	835	771	794	810	804	804	805	806	805

Table 2.4: Number of cases out of 1000 where NPI-based profit is greater than the classical method for lower and upper expected profits and their weighted function

2.6 NPI for a single order for the two-period model

In this section, consider the case when we order once for two periods, i.e. we fill the inventory for the two periods together, so the future demand will be $D = D_{n+1} + D_{n+2}$, where D_{n+1} and D_{n+2} are the future demands for the first and second period, respectively. In the single-period model discussed in Sections 2.2-2.4, we order once for one period, while here we order once for two periods and we consider that there is no holding cost after the first period, so if any inventory remains at the end of the second period, it is disposed of. So, the profit function for this model will be the same as the profit function for the single-period inventory model, Equation (2.1).

In this model, we will only consider maximising the expected profit as the optimality criterion for the inventory level, since the other criterion discussed in Section 2.3.1, can be attractive in many situations, but it does not distinguish between actual levels of profit beyond whether it is non-negative or negative. Since the profit function for this model is the same as the profit function for the single-period inventory model, the optimal inventory level y^*_{CE} that maximises the expected profit for this model is the same as the optimal level for the single-period inventory model displayed in Section 2.3.2.

In Section 2.4, NPI was introduced for the single-period inventory model which considered only one future demand observation. NPI has been developed to deal with multiple future observations, say m future observations, with their interdependence explicitly taken into account, and based on repeated use of the assumptions $A_{(n)}, A_{(n+1)}, \ldots, A_{(n+m-1)}$

[44]. Here, we consider an inventory for two periods involving future demands, D_{n+1} and D_{n+2}, in which D_{n+2} depends on D_{n+1}.

Since we suppose a single order for the two-period model, we consider the random demand $D = D_{n+1} + D_{n+2}$ for the two periods for which the inventory decision is required. It is important to emphasize that the future demands D_{n+1} and D_{n+2} are assumed to come from the same data collection process as the n data observations. As before, we assume that data of demand in n previous periods are available, ordered as $d_1 < d_2 < ... < d_n$. We assume that there is a known upper bound for the demand, denoted by d_u, which is logically greater than d_n, and that demand is positive.

We link the observed demands and future demands via Hill's assumption $A_{(n)}$ [44], or more precisely, via consecutive application of $A_{(n)}, A_{(n+1)}, ..., A_{(n+m-1)}$. We refer to these generically as the $A_{(.)}$ assumptions, which can be considered as a post-data version of a finite exchangeability assumption for $n + m$ random quantities. The exchangeability assumption on demand is different from iid. In NPI, we only assume $A_{(n)}$ assumptions, which are directly related to the exchangeability assumption, and the multiple future periods are dependent. However, the iid assumes that each random variable has the same probability distribution as the others and all are mutually independent. The iid is a stronger assumption than exchangeability.

Let $O_i, i = 1, 2, ..., \frac{(n+m)!}{n!}$ be a specific ordering of the m future demands among n data observations. Then the $A_{(.)}$ assumptions lead to

$$P(O_i) = \frac{n!}{(n+m)!} \tag{2.35}$$

Equation (2.35) implies that all different orderings O_i, for $i = 1, 2, ..., \frac{(n+m)!}{n!}$, are equally likely. A suitable way to explain the $A_{(.)}$ assumptions with n data observations and m future observations is to think that the n revealed observations are randomly chosen from $n+m$ observations, which enable to make inferences about the m unrevealed observations. The ordering of the future observations among the observations is important, as the first future observation could be larger than the second one, and the second future observation could be larger than the first one.

To avoid huge analytic complexities, we restrict our focus in this section to the case where the number of observations is $n = 2$ and $m = 2$ future demands. For larger n, the NPI bootstrap approach has been used as an alternative approach; this will be shown in Chapter 4.

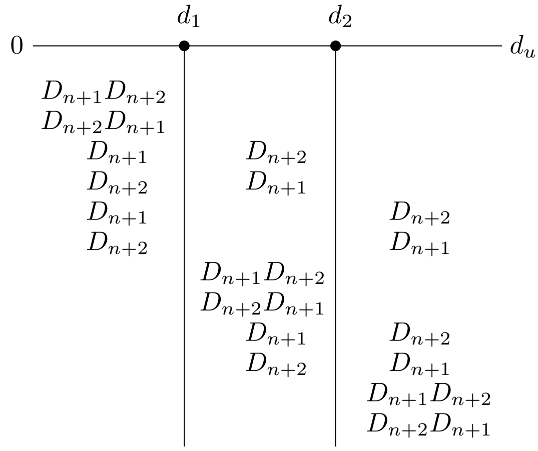


Figure 2.7: The different orderings of future observations when $n = m = 2$

D_{n+1}	D_{n+2}	$D = D_{n+1} + D_{n+2}$	$P(D)$
(d_0, d_1)	(d_0, d_1)	$(2d_0, 2d_1)$	$\frac{1}{12}$
(d_0, d_1)	(d_0, d_1)	$(2d_0, 2d_1)$	$\frac{1}{12}$
(d_0, d_1)	(d_1, d_2)	$(d_0 + d_1, d_1 + d_2)$	$\frac{1}{12}$
(d_1, d_2)	(d_0, d_1)	$(d_1 + d_0, d_2 + d_1)$	$\frac{1}{12}$
(d_1, d_2)	(d_1, d_2)	$(2d_1, 2d_2)$	$\frac{1}{12}$
(d_1, d_2)	(d_1, d_2)	$(2d_1, 2d_2)$	$\frac{1}{12}$
(d_0, d_1)	(d_2, d_u)	$(d_0 + d_2, d_1 + d_u)$	$\frac{1}{12}$
(d_2, d_u)	(d_0, d_1)	$(d_2 + d_0, d_u + d_1)$	$\frac{1}{12}$
(d_1, d_2)	(d_2, d_u)	$(d_1 + d_2, d_2 + d_u)$	$\frac{1}{12}$
(d_2, d_u)	(d_1, d_2)	$(d_2 + d_1, d_u + d_2)$	$\frac{1}{12}$
(d_2, d_u)	(d_2, d_u)	$(2d_2, 2d_u)$	$\frac{1}{12}$
(d_2, d_u)	(d_2, d_u)	$(2d_2, 2d_u)$	$\frac{1}{12}$

Table 2.5: Intervals for two future observations

Example 2.6.1 Assume $n = 2$ observations which create 3 intervals, consider $m = 2$ future observations, and set $d_0 = 0$ and $d_{n+1} = d_u$. Then the assumptions $A_{(2)}$ and $A_{(3)}$ imply that the next two observations, D_3, D_4, will fall in any one of these intervals with probability $\frac{1}{3}$ and $\frac{1}{4}$, respectively. This gives 12 equally likely orderings $O_i, i = 1, 2, ..., 12$, of the future demands, D_3, D_4 among the observations d_1 and d_2, as shown in Figure 2.7 and Table 2.5.

The 12 intervals for D_3 and D_4 will be combined into six overlapped intervals between seven observed demands with probability $\frac{2}{12} = \frac{1}{6}$ for each interval, which is shown in Table

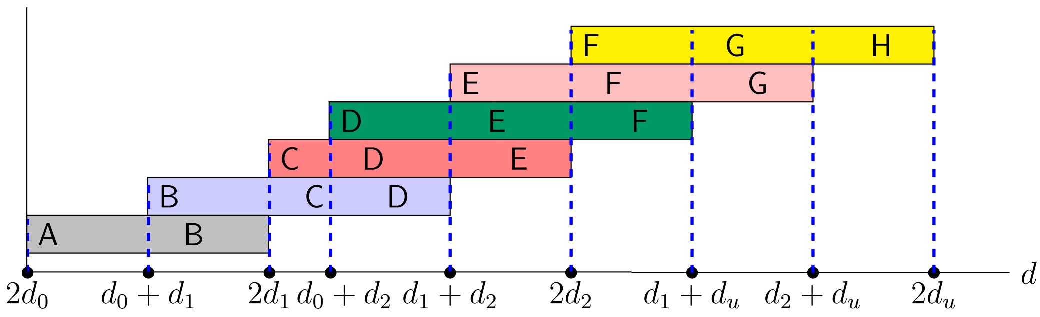


Figure 2.8: Overlapping intervals for a single order for two-period inventory model

2.5 and Figure 2.8. We place restrictions on the observed demands, which are $2d_1 < d_0+d_2$ and $2d_2 < d_1 + d_u$, this is just illustrating the approach for a specific scenario where the observed demand was ordered this way, and we assume the lower bound is $2d_0 = 0$, the upper bound is $2d_u$. ◇

Next, we illustrate the NPI lower and upper expected profits for the proposed model. The essential step in developing NPI for this model, is the transfer of the partial probability distribution specification for D to a partial probability distribution specification for the profit, using the profit function given by Equation (2.1). This process is similar to that for the single-period process presented in Section 2.4.2 and illustrated in Figure 2.2. It involves the profit $Pf(D, y)$ as function of the random demand D and fixed inventory level y. However, the probability masses in each interval in Figure 2.8 are not the same as in Section 2.4.2, so we will rely on the letters in Figure 2.8 to simplify the overlapping intervals and then to find the M-functions for this model. It is not easy to find general expressions for the lower and upper expected profits, similarly as was done in Section 2.4.2. So, we will deal with each interval separately to determine the optimal inventory level for each of them, then select the optimal one among them as the optimal inventory level for the proposed model.

Now we rely on Table 2.5 and Figure 2.8, to find the M-function values for the random profit $Pf(D, y)$. If y is in interval A, the M-function is as follows:

$$M(\min[Pf(2d_0, y), Pf(d_0 + d_1, y)], Pf(y, y)) = \frac{1}{6} \tag{2.36}$$

$$M(Pf(2d_j, y), Pf(2d_{j-1}, y)) = \frac{1}{6}, \quad \text{for } j \in \{2, 3\} \tag{2.37}$$

$$M(Pf(d_j + d_{j+1}, y), Pf(d_{j-1} + d_j, y)) = \frac{1}{6}, \quad \text{for } j \in \{1, 2\} \tag{2.38}$$

$$M(Pf(d_1 + d_u, y), Pf(d_0 + d_2, y)) = \frac{1}{6} \tag{2.39}$$

Equation (2.36) represents the probability masses that the future demand falls in interval A is equal to $\frac{1}{6}$. Equation (2.37) represents the probability masses in the red and yellow intervals, in this equation, we defined the profit, $Pf(2d_j, y)$, as the lower bound and $Pf(2d_{j-1}, y)$ as the upper bound, since whenever the demand is close to y, the profit becomes large, similarly for the other equations. Equation (2.38) represents the probability masses in the purple and pink intervals. Equation (2.39) represents the probability masses in the green interval.

The NPI lower expected profit, denoted by $\underline{E}_A$, is derived by assigning the probability masses $\dfrac{1}{6}$, according to the M-function values in Equations (2.36)-(2.39), to the minimum value for $Pf(D, y)$ per interval, which leads to

$$\underline{E}_A = \frac{1}{6}\Bigg[\min[-(c+h)y, (p-c+s)y - sd_1] + 5(p-c+s)y - 2s\sum_{j=2}^{3} d_j - s\sum_{j=1}^{2}(d_j + d_{j+1}) - s(d_1 + d_u)\Bigg] \tag{2.40}$$

To determine an optimal inventory level, $y^*_{\underline{E},A}$, which maximises $\underline{E}_A$ in the interval A, we introduce the following two functions,

$$\underline{E}_{1A} = \frac{1}{6}\Bigg[-(c+h)y + 5(p-c+s)y - 2s\sum_{j=2}^{3} d_j - s\sum_{j=1}^{2}(d_j + d_{j+1}) - s(d_1 + d_u)\Bigg] \tag{2.41}$$

and

$$\underline{E}_{2A} = \frac{1}{6}\Bigg[(p-c+s)y - sd_1 + 5(p-c+s)y - 2s\sum_{j=2}^{3} d_j - s\sum_{j=1}^{2}(d_j + d_{j+1}) - s(d_1 + d_u)\Bigg] \tag{2.42}$$

By Equation (2.40), $\underline{E}_A = \min[\underline{E}_{1A}, \underline{E}_{2A}]$. $\underline{E}_{1A}$ and $\underline{E}_{2A}$ are linear functions in the interval A, and they intersect at one point, which is the optimal value for the interval A. So, by equating $\underline{E}_{1A}$ and $\underline{E}_{2A}$, we have

$$y^*_{\underline{E},A} = \frac{sd_1}{p+h+s} \tag{2.43}$$

If y is in interval B, the M-function is as follows:

$$M(\min[Pf(d_0+d_1,y), Pf(2d_1,y)], Pf(y,y)) = \frac{2}{6} \tag{2.44}$$

$$M(Pf(2d_j,y), Pf(2d_{j-1},y)) = \frac{1}{6}, \quad \text{for } j \in \{2,3\} \tag{2.45}$$

$$M(Pf(d_1+d_u,y), Pf(d_0+d_2,y)) = \frac{1}{6} \tag{2.46}$$

$$M(Pf(d_2+d_u,y), Pf(d_1+d_2,y)) = \frac{1}{6} \tag{2.47}$$

Equation (2.44) represents the probability masses that the future demand falls in interval B is equal to $\frac{2}{6}$, since the interval B combined the grey and purple intervals. Equation (2.45) represents the probability masses in the red and yellow intervals. Equation (2.46) represents the probability masses in the green interval. Equation (2.47) represents the probability masses in the pink interval.

The NPI lower expected profit, denoted by $\underline{E}_B$, is derived by assigning the probability masses $\frac{1}{6}$, according to the M-function values in Equations (2.44)-(2.47), to the minimum value for $Pf(D,y)$ per interval, which leads to

$$\underline{E}_B = \frac{1}{6}\Bigg[2\min[-(c+h)y+(p+h)d_1, (p-c+s)y-2sd_1] + 4(p-c+s)y - 2s\sum_{j=2}^{3} d_j - s(d_1+d_3+d_2+d_3)\Bigg] \tag{2.48}$$

To determine an optimal inventory level, $y^*_{\underline{E},B}$, which maximises $\underline{E}_B$ in the interval B, we introduce the following two functions,

$$\underline{E}_{1B} = \frac{1}{6}\Bigg[2[-(c+h)y+(p+h)d_1] + 4(p-c+s)y - 2s\sum_{j=2}^{3} d_j - s(d_1+d_3+d_2+d_3)\Bigg] \tag{2.49}$$

and

$$\underline{E}_{2B} = \frac{1}{6}\Bigg[2[(p-c+s)y-2sd_1] + 4(p-c+s)y - 2s\sum_{j=2}^{3} d_j - s(d_1+d_3+d_2+d_3)\Bigg] \tag{2.50}$$

By Equation (2.48), $\underline{E}_B = \min[\underline{E}_{1B}, \underline{E}_{2B}]$. $\underline{E}_{1B}$ and $\underline{E}_{2B}$ are linear functions in the interval B, and they intersect at one point, which is the optimal inventory level for the interval B. So, by equating $\underline{E}_{1B}$ and $\underline{E}_{2B}$, we have

$$y^*_{\underline{E},B} = \frac{(p+h)d_1 + 2sd_1}{p+h+s} \tag{2.51}$$

Similarly, for the intervals, C, ..., H. So, if $y \in (x, z)$, the optimal inventory level for the lower expected profit is

$$y^*_{\underline{E}} = \frac{(p+h)x + sz}{p+h+s} \tag{2.52}$$

In general, for the lower expected profit, the optimal inventory level, $y^*_{\underline{E}_{\text{Once}}}$, for all different intervals is the one that corresponds to $\underline{E} = \max[\underline{E}_A, \underline{E}_B, ..., \underline{E}_H]$, as we consider maximising the expected profit as the optimality criterion for the inventory level.

Now, we consider the upper expected profit. The NPI upper expected profit for the interval A, $\overline{E}_A$, is derived by assigning the probability masses $\frac{1}{6}$, according to the M-function values in Equations (2.36)-(2.39), to the maximal value for $Pf(D, y)$ per interval, which leads to

$$\overline{E}_A = \frac{1}{6}\left[(p-c)y + 5(p-c+s)y - 2s\sum_{j=2}^{3} d_{j-1} - s\sum_{j=1}^{2}(d_{j-1} + d_j) - sd_2\right] \tag{2.53}$$

$\overline{E}_A$ is a linear function in the interval A. To find the optimal inventory level, which is denoted by $y^*_{\overline{E},A}$ for interval A, we need to find out if $\overline{E}_A$ is an increasing or decreasing function over the interval A. So, we derive the difference between the values of $\overline{E}_A$ of the upper and lower bounds of interval A, leading to the equation

$$\overline{E}_A(d_0 + d_1) - \overline{E}_A(2d_0) = \frac{d_1}{6}\left[6(p-c) + 5s\right] \tag{2.54}$$

If $\overline{E}_A(d_0 + d_1) - \overline{E}_A(2d_0) > 0$ in Equation (2.54), then $\overline{E}_A$ is an increasing function over the interval A and the optimal value is $y^*_{\overline{E},A} = d_0 + d_1$ which is the upper bound of interval A. While, if $\overline{E}_A(d_0+d_1) - \overline{E}_A(2d_0) < 0$ in Equation (2.54), the $\overline{E}_A$ is a decreasing function over A and the optimal value is $y^*_{\overline{E},A} = 2d_0$.

Similarly for interval B, the NPI upper expected profit for the interval B, $\overline{E}_B$, is derived by assigning the probability masses $\frac{1}{6}$, according to the M-function values in Equations (2.44)-(2.47), to the maximal value for $Pf(D, y)$ per interval, which leads to

$$\overline{E}_B = \frac{1}{6}\left[2(p-c)y + 4(p-c+s)y - 2s\sum_{j=2}^{3} d_{j-1} - s(d_1 + 2d_2)\right] \tag{2.55}$$

$\overline{E}_B$ is a linear function in the interval B. To find the optimal inventory level, which is denoted by $y^*_{\overline{E},B}$ for interval B, we need to find out if $\overline{E}_B$ is an increasing or decreasing function over the interval B. So, we derive the difference between the values of $\overline{E}_B$ of the upper and lower bounds of interval B, leading to the equation

$$\overline{E}_B(2d_1) - \overline{E}_B(d_0 + d_1) = \frac{d_1}{6}\left[6(p-c) + 4s\right] \tag{2.56}$$

If $\overline{E}_B(2d_1) - \overline{E}_B(d_0 + d_1) > 0$ in Equation (2.56), then $\overline{E}_B$ is an increasing function over the interval B and the optimal value is $y^*_{\overline{E},B} = 2d_1$ which is the upper bound of interval B. While, if $\overline{E}_B(2d_1) - \overline{E}_B(d_0 + d_1) < 0$ in Equation (2.56), the $\overline{E}_B$ is a decreasing function over B and the optimal value is $y^*_{\overline{E},B} = d_0 + d_1$.

Similarly, for the other intervals, C, ..., H. It is easy to show that the upper expected profit is a linear function over the intervals and increasing over some intervals and decreasing over others depending on the value of the cost parameters. The optimal value for the increasing function over the interval is obtained at the upper bound of that interval and for the decreasing function the optimal value over the interval is obtained at the lower bound of that interval.

In general, for the upper expected profit, the optimal inventory level, $y^*_{\overline{E}_{\text{Once}}}$, over all intervals, A-H, is the one that corresponds to $\overline{E} = \max[\overline{E}_A, \overline{E}_B, ..., \overline{E}_H]$, as we consider maximising the expected profit as the optimality criterion for the inventory level.

In the following example, we show how the proposed method works to find the optimal inventory levels that maximise the lower and upper expected profits for the model when we order once for both periods.

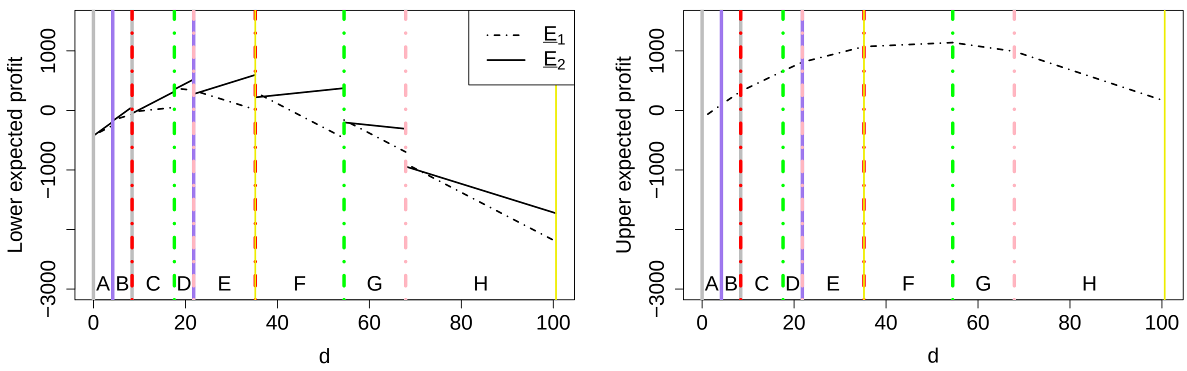
Example 2.6.2 Consider an inventory system with the following costs: $p = 70$, $c = 23$, $h = 17$, $s = 9$, and assume that demand is known to be between $d_0 = 0$ and $d_u = 50.30$. Assume that there are $n = 2$ demand observations with values $d_1 = 4.20, d_2 = 17.60$. Our aim is to find the optimal inventory level for the two-period model with a single order.

Table 2.6 presents the results of the method presented in this section, specifying the NPI lower and upper expected profits, together with the corresponding values of y. The corresponding functions $\underline{E}_1$ and $\underline{E}_2$ for all intervals A-H are given in Figure 2.9a, and the function $\overline{E}$ for all intervals A-H is given in Figure 2.9b.

The optimal inventory level for the lower expected profit is 17.99 which is in interval D, so the index of $y^*_{\underline{E}_{\text{Once}}}$ is $j_{y^*_{\underline{E}_{\text{Once}}}} = 4$. The optimal inventory level for the upper expected

y's interval	$y_{\underline{E}}$	$\underline{E}$	$y_{\overline{E}}$	$\overline{E}$
A=:(0, 4.20)	0.39	-404.25	4.20	98.10
B=:(4.20, 8.40)	4.59	-155.25	8.40	320.70
C=:(8.40, 17.60)	9.26	-16.80	17.60	661.10
D=:(17.60, 21.80)	17.99	368.90	21.80	810.20
E=:(21.80, 35.20)	23.06	304.95	35.20	1071.50
F=:(35.20, 54.50)	37.01	233.53	54.50	1139.05
G=:(54.50, 67.90)	55.76	-211.85	54.50	1139.05
H=:(67.90, 100.60)	70.97	-1015.98	67.90	991.65

Table 2.6: Inventory levels and the corresponding lower and upper expected profit



(a) Lower expected profit (b) Upper expected profit

Figure 2.9: Single order for two periods

profit is 54.50 which falls in interval F and G, so the index of $y^*_{\overline{E}_{\text{Once}}}$ is $j_{y^*_{\overline{E}_{\text{Once}}}} = 6$ or 7.$\diamond$

2.7 Comparison of the NPI and classical methods for the two-period model with a single order

In this section, we consider the same cases and procedures as in Section 2.5 and we consider the same cost parameters as in Example 2.6.2. Suppose the number of observation is $n = 2$. As finite end-point for the support of the random demand we took $d_u = 2.50$.

Table 2.7 presents the results from the simulation study. It provides the number of

Case	$\underline{E}$	$\overline{E}$
I	382	479
II	336	474
III	409	611
IV	508	586
V	844	734
VI	925	811

Table 2.7: Number of cases out of 1000 where NPI-based profit is greater than the classical method

times, out of 1000 runs, in which the profit according to the NPI method for the two-period model with a single order, is larger than the profit based on the classical method. This table considers the expected profit as optimality criterion, where $\underline{E}$ and $\overline{E}$ indicate that, for the NPI method, the lower expected profit and the upper expected profit was used, respectively.

It is obvious from the data that, for Cases I and II the classical method performs better than the NPI method, where the number of times that for profit based on the NPI method is greater than the profit based on the classical method, is less 500 out of 1000 simulations. While, for the other cases, the NPI method performs better than the classical method. It would be interesting to develop the method presented in this section for general n, we leave that for future research.

2.8 Concluding remarks

This chapter has introduced the NPI method for the single-order inventory models, with continuous random demand. To determine the optimal inventory level, several optimality criteria are used. We considered the single-period inventory problem then, we explored how to find the optimal inventory level y^*, which maximises the probability to get positive profit and maximises the expected profit.

Also, this chapter covered the model when we only ordered once for the two periods. We explored how to find the optimal inventory level y^*, which maximises the expected profit. To avoid huge analytic complexities, we restrict our focus in this model to the case where the number of observations is $n = 2$ and $m = 2$ future demands. For larger n, the NPI bootstrap approach has been used as an alternative approach; this will be shown in Chapter 4.

The performance of the classical method and NPI method was evaluated through simulation studies. We considered Gamma distribution to examine the general performance of the proposed methods. For the single-period model, some Cases (I and II) the classical method performs better than the NPI method, and when the number of observations increases NPI method gets close to the classical method. In other cases (III, IV, V and VI) the NPI method performs better than the classical method. Similarly, for the second model; two-period model with a single order, some Cases (I and II) the classical method performs better than the NPI method, and in other cases the NPI method performs better than the classical method. This has been done when the number of observations is $n = 2$, it will be of interest to generalise the number of observations for a single order for the two-period model as a topic for future research. Also, in the comparison of the simulation studies, we only consider how often the profit is doing better in each run, but it could also be of interest to see by how much it is greater as a topic for future research.

Chapter 3

Two-period independent demands model

3.1 Introduction

In this chapter, we extend the single-order inventory models presented in Chapter 2 to two-period independent demands inventory model. The two-period inventory model allows to order twice, once for the first period and once for the second period. Also, allows a backlog of demand, where the items ordered before the first period can be sold in the second period [8]. In this chapter, we will consider maximising the expected profit as the optimality criterion for the inventory level.

We apply NPI lower and upper expected profits for the two-period model to derive the optimal inventory levels. We start backward: first, we optimise over the second period only, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimise over the first period. This is effectively the way to solve a dynamic programming problem. Several examples are examined where models with and without order for the second period are assumed, and the best decision for ordering or not ordering for the second period is interpreted.

This chapter is organised as follows: in Section 3.2, we introduce a two-period independent demands model. First, we consider the second period, followed by the two periods. Section 3.3 considers the classical method for the two-period independent demands model. Section 3.4 presents the NPI approach for the two-period independent demands inventory model. The NPI lower and upper expected profits for the second period are given in Sections 3.5 and 3.6, respectively. In Sections 3.7 and 3.8, we consider the NPI lower and upper expected profits for the two periods, respectively. A simulation study to compare the NPI method and the classical method for the two-period model is presented in Section 3.9. Section 3.10 presents the concluding remarks for this chapter.

3.2 Independent demands for two-period model

In this section, we study the full separation of demands of the first and second periods, where the product for each period is the same and the leftover inventory from the first period can be used in the second period. So, the demands are independent, and the data only apply to each period independently. To illustrate this model, suppose we have observed demands for a product for Saturdays as the first period, and observed demands for the same product for Sundays as the second period, and we treat them as independent random quantities, then we will use NPI for next Saturday using Saturday data as the first period, and we use NPI for next Sunday using Sunday data as the second period.

In order to formulate the model, the following notations are introduced: n_i is the observations number for i^{th} period, $i = 1, 2$. The demand during the i^{th} period is D_i and the observed demand during the i^{th} period is $d_{i,j}$, $j = 1, ..., n_i$, for ease of notation we assume that the lower bound during the i^{th} period is $d_{i,0} = 0$ and the upper bound is $d_{i,u}$, which is logically greater than d_{i,n_i}, and that demand is positive. So, $d_{i,n_i+1} = d_{i,u}$. The random demand for the items in the i^{th} period is D_{i,n_i+1}. We assume that the inventory level at the start of the i^{th} period is y_i, we aim at determining the best value of y_i, which we denote by y_i^*.

This model considers fixed costs such as selling price p_i per unit in the i^{th} period. The total amount of money from sales in period i is $p_i \min(y_i, D_i)$. The setup cost is k_i per order in the i^{th} period, e.g. the cost for delivery. Holding costs for period i are h_i per unit, which is the cost for unsold items remaining at the end of the i^{th} period. The total holding costs are equal to $\sum_{i=1}^{2} h_i(y_i - D_i)^+$, where $(v)^+ := \max(0, v)$. We assume that some unmet demand from the first period can be met in the second period, so, a fixed proportion of demand remaining is α, $0 \leq \alpha \leq 1$. The selling price for this proportion of demand is p'_1 per unit, with $p'_1 < p_i$. The total amount of unsatisfied demand in the first period that can be met in the second period leads to income $\alpha p'_1(D_1 - y_1)^+$.

Parameter	Description
n_i	Number of observations for i^{th} period, $i = 1, 2$
$d_{i,j}$	Observed demands for the i^{th} period, $j = 1, ..., n_i$
D_{i,n_i+1}	Random demand for the i^{th} period
y_i	Inventory level at the start of the i^{th} period
p_i	Selling price per unit in the i^{th} period
k_i	Setup cost per order in the i^{th} period
h_i	Holding cost per unit for period i
s_i	Shortage cost during the i^{th} period per unit
c_i	Purchasing cost per unit for period i
α	Fixed proportion of demand remaining
p'_1	Selling price per unit in period 2 for demand from period 1

Table 3.1: Summary of notations

The purchasing cost is c_i per unit when the item is obtained from an external source at the i^{th} ordering cycle, we assume $c_i > 0$ and $c_i < p_i$. The total purchase cost for the second period will be affected by the stock level at the end of the first period $y_1 - D_1$. If $y_1 - D_1 > 0$, the total purchase costs for the second period are $c_2(y_2 - (y_1 - D_1))$. If $y_1 - D_1 < 0$, the total purchase costs are $c_2(y_2 + \alpha(D_1 - y_1))$. The shortage cost during the i^{th} period is s_i, which is the cost that occurs when the demand exceeds the available stock. The total shortage costs are equal to $\sum_{i=1}^{2} s_i(D_i - y_i)^+$. An overview of these notations is given in Table 3.1.

These costs lead to the profit function for the two-period inventory model as follows

$$\begin{aligned} Pf(D_1, D_2, y_1, y_2) &= p_1 \min(y_1, D_1) - c_1 y_1 - k_1 - h_1(y_1 - D_1)^+ - s_1(D_1 - y_1)^+ \\ &+ \alpha p'_1(D_1 - y_1)^+ + p_2 \min(y_2, D_2) - c_2(y_2 + \alpha(D_1 - y_1)^+ - (y_1 - D_1)^+) \\ &- k_2 - h_2(y_2 - D_2)^+ - s_2(D_2 - y_2)^+ \end{aligned} \tag{3.1}$$

In the following sections, we will study and investigate the performance of the classical method and the NPI method for the two-period model. First, we determine the optimal inventory level for the second period, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimise over the first period.

3.2.1 Second period

Let the actual demand for the first period be denoted by d_1. We assume only the second period of the two-period model, based on the situation at the end of the first period, which contains the relevant information for the inventory resulting from the first period, so y_1 and D_1 are assumed to be known quantities. From Equation (3.1), the profit function for the second period when we order is given by:

$$Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1) = \alpha p'_1(d_1 - y_1)^+ + p_2 \min(y_2, D_2) - c_2(y_2 + \alpha(d_1 - y_1)^+ - (y_1 - d_1)^+) - k_2 - h_2(y_2 - D_2)^+ - s_2(D_2 - y_2)^+ \quad (3.2)$$

When we do not order for the second period, the inventory level at the beginning of the second period, y_2, depends on the inventory level at the end of the first period. If the inventory level at the end of first period is greater than zero, so, the inventory level for the second period is $y_2 = (y_1 - d_1)^+$, hence the profit function is:

$$Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = (y_1 - d_1)^+) = p_2 \min((y_1 - d_1)^+, D_2) - h_2((y_1 - d_1)^+ - D_2)^+ - s_2(D_2 - (y_1 - d_1)^+)^+ \quad (3.3)$$

If the inventory level at the end of the first period is less than zero, then the inventory level for the second period is zero and there is unmet demand equal to $d_1 - y_1$, hence the profit function is:

$$Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = 0) = -s_2 D_2 \quad (3.4)$$

3.2.2 Two periods

The profit function for the two periods when we order for the first and the second period, is given by Equation (3.1). However, when we order for the first period but do not order for the second period, then the inventory level for the second period is equal to the stock level at the end of the first period. So, when the stock level at the end of the first period is equal to $y_1 - D_1$, the inventory level for the second period is $y_2 = (y_1 - D_1)^+$. Hence, the profit function for this case is as follows:

$$Pf(D_1, D_2, y_1, y_2|y_2 = (y_1 - D_1)^+) = p_1 \min(y_1, D_1) - c_1 y_1 - k_1 - h_1(y_1 - D_1)^+ - s_1(D_1 - y_1)^+ + \alpha p'_1(D_1 - y_1)^+ + p_2 \min((y_1 - D_1)^+, D_2) - h_2((y_1 - D_1)^+ - D_2)^+ - s_2(D_2 - (y_1 - D_1)^+)^+ \quad (3.5)$$

While, when the inventory level at the end of the first period is less than zero, then the inventory level for the second period is zero and there is unmet demand equal to $D_1 - y_1$, hence the profit function is:

$$Pf(D_1, D_2, y_1, y_2|y_2 = 0) = p_1 \min(y_1, D_1) - c_1 y_1 - k_1 - h_1(y_1 - D_1)^+ \\ - s_1(D_1 - y_1)^+ - s_2 D_2 \qquad (3.6)$$

3.3 Classical method for two-period model

In this section, we consider maximising the expected profit, which was presented by Shih [63] as the optimality criterion for the inventory levels for the first and second period.

First, we maximise the expected profit for the second period to derive the optimal inventory level for the second period, y^*_{2CE}. With that optimal strategy for the second period, we then optimise over the first period.

We assume order for the second period, so, the inventory level for the second period, y_2, is greater than the inventory level at the end of the first period. From Equation (3.2), the expected profit function for the second period when we order is given by:

$$E(Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \\ \int_0^{y_2} \Big(\alpha p'_1(d_1 - y_1)^+ + p_2 u - c_2(y_2 + \alpha(d_1 - y_1)^+ - (y_1 - d_1)^+) - k_2 - \\ h_2(y_2 - u)\Big) f_{D_2}(u)du + \int_{y_2}^{\infty} \Big(\alpha p'_1(d_1 - y_1)^+ + p_2 y_2 - c_2(y_2 + \alpha(d_1 - y_1)^+ - \\ (y_1 - d_1)^+) - k_2 - s_2(u - y_2)\Big) f_{D_2}(u)du \qquad (3.7)$$

The optimal inventory level, y^*_{2CE}, for the second period, which maximises the expected profit, is derived by setting the first derivative of this expected profit function to zero, leading to the equation

$$P(D_2 \leq y^*_{2CE}) = \frac{p_2 + s_2 - c_2}{p_2 + s_2 + h_2} \qquad (3.8)$$

As the second derivative of the expected profit is negative at all values of y_2 with $f_{D_2}(y_2) > 0$, the value y^*_{2CE} resulting from Equation (3.8) is the optimal inventory level for the second period. This is effectively the same as the optimal value for the single-order inventory model.

Now, we will maximise the expected profit for the two-period inventory model for two different scenarios, namely with an order for each period, and ordering only for the first period.

3.3.1 Ordering for both periods

To find the optimal inventory level for this case, we will use the optimal inventory level for the second period, y^*_{2CE}, which is shown in Equation (3.8) to optimise the expected profit for the first period, $E(Pf^{1,2}(D_1, D_2, y_1, y_2))$.

From Equation (3.1), the expected profit function for the first period is given by:

$$E(Pf^{1,2}(D_1, D_2, y_1, y_2)) =$$
$$\int_0^{y_1} \Big(p_1u_1 - c_1y_1 - k_1 - h_1(y_1 - u_1) + c_2(y_1 - u_1)\Big) f_{D_1}(u_1)du_1 +$$
$$\int_{y_1}^{\infty} \Big(p_1y_1 - c_1y_1 - k_1 - s_1(u_1 - y_1) + \alpha p'_1(u_1 - y_1) - \alpha c_2(u_1 - y_1)\Big) f_{D_1}(u_1)du_1 \quad (3.9)$$

To find the optimal inventory level y^*_{1CE} that maximises the profit for the first period, we need to find the first derivative of Equation (3.9) with respect to y_1, and equate it to zero. This leads to

$$P(D_1 \leq y^*_{1CE}) = \frac{p_1 + s_1 - c_1 - \alpha(p'_1 - c_2)}{p_1 + s_1 + h_1 - c_2 - \alpha(p'_1 - c_2)} \quad (3.10)$$

As the second derivative of Equation (3.9) is negative at all values of y_1 with $f_{D_1}(y_1) > 0$, the value y^*_{1CE} resulting from Equation (3.10) is the optimal inventory level for the first period.

3.3.2 Ordering in the first period only

Since in this case we consider ordering for the first period only, so, the inventory level for the second period is either $y_2 = (y_1 - D_1)^+$ or $y_2 = 0$ with the remaining demand equal to $D_1 - y_1$. Our aim is to find the optimal inventory level for the first period.

When $y_2 = (y_1 - D_1)^+$, based on Equation (3.5), the expected profit function for the first period is given by:

$$E(Pf^1(D_1, D_2, y_1, y_2|y_2 = (y_1 - D_1)^+)) =$$
$$\int_0^{y_1} \Big(p_1u_1 - c_1y_1 - k_1 - h_1(y_1 - u_1)\Big) f_{D_1}(u_1)du_1 + \int_{y_1}^{\infty} \Big(p_1y_1 - c_1y_1 - k_1 -$$
$$s_1(u_1 - y_1) + \alpha p'_1(u_1 - y_1)\Big) f_{D_1}(u_1)du_1 \quad (3.11)$$

To find the optimal inventory level y^{*b1}_{1CE} that maximises the profit over both periods, we need to find the first derivative of Equation (3.11) with respect to y_1, and equate it to zero. This lead to

$$P(D_1 \leq y^{*b1}_{1CE}) = \frac{p_1 + s_1 - c_1 - \alpha p'_1}{p_1 + s_1 + h_1 - \alpha p'_1} \tag{3.12}$$

As the second derivative of Equation (3.11) is negative at all values of y_1 with $f_{D_1}(y_1) > 0$, the value y^{*b1}_{1CE} resulting from Equation (3.12) is the optimal inventory level for the first period when we just order for the first period.

When $y_2 = 0$ and the remaining demand from the first period is $D_1 - y_1$, based on Equation (3.6), the expected profit function for the first period is given by:

$$E(Pf^1(D_1, D_2, y_1, y_2|y_2 = 0)) = \int_0^{y_1} \Big(p_1 u_1 - c_1 y_1 - k_1 - h_1(y_1 - u_1)\Big) f_{D_1}(u_1) du_1 + \int_{y_1}^{\infty} \Big(p_1 y_1 - c_1 y_1 - k_1 - s_1(u_1 - y_1)\Big) f_{D_1}(u_1) du_1 \tag{3.13}$$

To find the optimal inventory level y^{*b2}_{1CE} that maximises the profit over both periods, we need to find the first derivative of Equation (3.13) with respect to y_1, and equate it to zero. This lead to

$$P(D_1 \leq y^{*b2}_{1CE}) = \frac{p_1 + s_1 - c_1}{p_1 + s_1 + h_1} \tag{3.14}$$

As the second derivative of Equation (3.13) is negative at all values of y_1 with $f_{D_1}(y_1) > 0$, the value y^{*b2}_{1CE} resulting from Equation (3.14) is the optimal inventory level for the first period when we just order for the first period.

3.4 NPI for two-period model with independent demands

In this section, we study NPI for the demands D_{1,n_1+1} and D_{2,n_2+1} in the two-period inventory model, in which we assume independence of these two demands. We also assume that demand data for one period does not contain information for the other period. Leftover items from the first period can be sold in the second period. For the previous two periods, we assume that data on demands are available and we set $d_{i,0} = 0$. The future demand in the first period is D_{1,n_1+1} and the ordered demand observations are $d_{1,1} < d_{1,2} < ... < d_{1,n_1}$. We assume that there is a known upper bound for the demand in period 1, $d_{1,n_1+1} = d_{1,u}$, so $d_{1,n_1} < d_{1,u}$. For the second period, D_{2,n_2+1} is the future demand, and the ordered observed demands in the second period are $d_{2,1} < d_{2,2} < ... < d_{2,n_2}$. We assume that there is a known upper bound for the demand in period 2, $d_{2,n_2+1} = d_{2,u}$, so $d_{2,n_2} < d_{2,u}$. In general, the assumption $A_{(n_i)}$ for D_{i,n_i+1} leads to:

$$P(D_{i,n_i+1} \in (d_{i,j-1}, d_{i,j})) = \frac{1}{n_i+1} \quad \text{for} \quad j = 1, ..., n_i+1 \quad \text{and} \quad i = 1, 2. \tag{3.15}$$

The essential step in developing NPI for the two-period independent demands inventory model, is the transfer of the partial probability distribution specification for the future demands to a partial probability distribution specification for the profit function. This process is illustrated in Figure 3.1, in which $i = 1$ when we study the first period and $i = 2$ for the second period, this is similar to the process used in Figure 2.2 in Chapter 2. The overall aim is to determine the optimal values for y_1 and y_2. Let $j_{i,y_i} \in \{1, ..., n_i+1\}$ be such that $y_i \in (d_{i,j_{i,y_i}-1}, d_{i,j_{i,y_i}})$. The $A_{(n_i)}$ assumption for D_{i,n_i+1}, given in Equation (3.15), implies the following M-function values for the random profit $Pf(D_{i,n_i+1}, y_i)$ as follows

$$M(Pf(d_{i,j-1}, y_i), Pf(d_{i,j}, y_i)) = \frac{1}{n_i+1} \quad \text{for } j \in \{1, ..., j_{i,y_i}-1\} \quad \text{and} \quad i = 1, 2 \tag{3.16}$$

$$M(\min[Pf(d_{i,j_{i,y_i}-1}, y_i), Pf(d_{i,j_{i,y_i}}, y_i)], Pf(y_i, y_i)) = \frac{1}{n_i+1} \quad \text{for } j = j_{i,y_i} \quad \text{and} \quad i = 1, 2 \tag{3.17}$$

$$M(Pf(d_{i,j}, y_i), Pf(d_{i,j-1}, y_i)) = \frac{1}{n_i+1} \quad \text{for } j \in \{j_{i,y_i}+1, ..., n_i+1\} \quad \text{and} \quad i = 1, 2 \tag{3.18}$$

In the following sections we will derive NPI lower and upper expected profits for the two-period model. First, we optimise over the second period only, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimise over the first period.

3.5 NPI lower expected profit for the second period

In this section, we optimise the profit for the second period, $Pf(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)$, based on the situation at the end of the first period, when the inventory level is greater than (or less than) the actual demand for the first period, $y_1 \geq d_1$ (or $y_1 < d_1$),

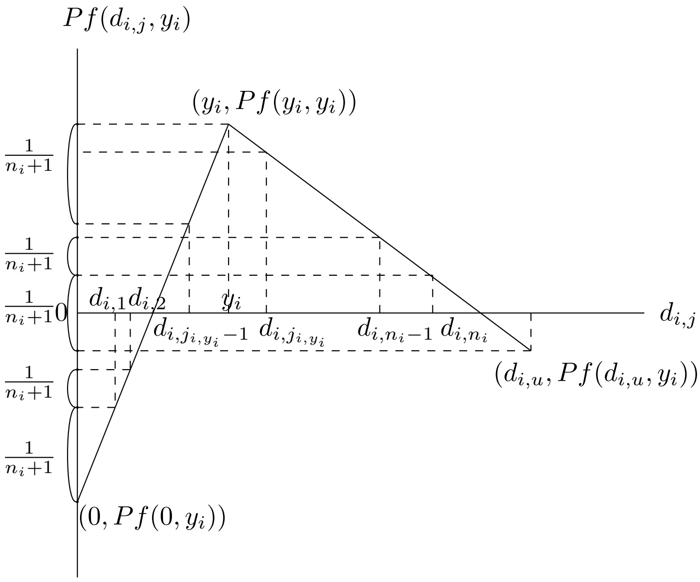


Figure 3.1: Two-period inventory - M-functions

with and without order for each inequality. We present the NPI lower expected profit for the second period, as function of the inventory level y_2. The derivations are based on the M-functions presented in Equations (3.16)-(3.18) and shown in Figure 3.1, with $i = 2$. The NPI lower expected profit, denoted by $\underline{E}(Pf(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1))$, is derived by assigning the probability masses $\frac{1}{n_2+1}$, according to the M-function values to the minimal values for $Pf(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)$ per interval, which leads to

$$\begin{aligned}
\underline{E}(Pf(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) &= \sum_{j=1}^{j_{2,y_2}-1} \Big(M(Pf(d_{2,j-1}, y_2), Pf(d_{2,j}, y_2)) \\
&\times Pf(d_{2,j-1}, y_2) \Big) + \Big[M(\min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)], Pf(y_2, y_2)) \\
&\times \min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)] \Big] + \sum_{j=j_{2,y_2+1}}^{n_2+1} \Big(M(Pf(d_{2,j}, y_2), Pf(d_{2,j-1}, y_2)) \\
&\times Pf(d_{2,j}, y_2) \Big) \\
&= \frac{1}{n_2+1} \Bigg[\sum_{j=1}^{j_{2,y_2}-1} Pf(d_{2,j-1}, y_2) + \min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)] \\
&+ \sum_{j=j_{2,y_2+1}}^{n_2+1} Pf(d_{2,j}, y_2) \Bigg] \qquad (3.19)
\end{aligned}$$

Next, we consider the NPI lower expected profit for the second period for the scenarios: with and without order in which $y_1 \geq d_1$, also, with and without order in which $y_1 < d_1$.

3.5.1 With order for the second period, with remaining stock from the first period

In this section, we assume that there is a remaining stock from the first period, $y_1 \geq d_1$. So, we derive the optimal inventory level that maximises the lower expected profit, $\underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1))$, under the assumption that we will order and that $y_1 \geq d_1$. By substituting Equation (3.2) in Equation (3.19) the lower expected profit is given by

$$
\begin{aligned}
&\underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[\sum_{j=1}^{j_{2,y_2}-1}\Big(p_2 d_{2,j-1} - c_2(y_2 - (y_1 - d_1)) \\
&\quad - k_2 - h_2(y_2 - d_{2,j-1})\Big) + \min[p_2 d_{2,j_{2,y_2}-1} - c_2(y_2 - (y_1 - d_1)) - k_2 - h_2(y_2 \\
&\quad - d_{2,j_{2,y_2}-1}), p_2 y_2 - c_2(y_2 - (y_1 - d_1)) - k_2 - s_2(d_{2,j_{2,y_2}} - y_2)] \\
&\quad + \sum_{j=j_{2,y_2}+1}^{n_2+1}\Big(p_2 y_2 - c_2(y_2 - (y_1 - d_1)) - k_2 - s_2(d_{2,j} - y_2)\Big)\Bigg] \\
&= \frac{1}{n_2+1}\Bigg[(j_{2,y_2} - 1)[-(c_2 + h_2)y_2 + c_2(y_1 - d_1) - k_2] + (p_2 + h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1} \\
&\quad + \min[(p_2 + h_2)d_{2,j_{2,y_2}-1} - (c_2 + h_2)y_2 + c_2(y_1 - d_1) - k_2, (p_2 - c_2 + s_2)y_2 \\
&\quad - s_2 d_{2,j_{2,y_2}} + c_2(y_1 - d_1) - k_2] + (n_2 + 1 - j_{2,y_2})[(p_2 - c_2 + s_2)y_2 + c_2(y_1 - d_1) - k_2] \\
&\quad - s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg]
\end{aligned}
\tag{3.20}
$$

To determine the optimal value of y_2, which maximises Equation (3.20), we introduce the following two functions:

$$
\begin{aligned}
&\underline{E}_a(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[(j_{2,y_2} - 1)[-(c_2 + h_2)y_2 \\
&\quad + c_2(y_1 - d_1) - k_2] + (p_2 + h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1} + (p_2 + h_2)d_{2,j_{2,y_2}-1} - (c_2 + h_2)y_2 \\
&\quad + c_2(y_1 - d_1) - k_2 + (n_2 + 1 - j_{2,y_2})[(p_2 - c_2 + s_2)y_2 + c_2(y_1 - d_1) - k_2] \\
&\quad - s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg]
\end{aligned}
\tag{3.21}
$$

and

$$\underline{E}_b(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[(j_{2,y_2} - 1)[-(c_2 + h_2)y_2 + c_2(y_1 - d_1) - k_2] + (p_2 + h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1} + (p_2 - c_2 + s_2)y_2 - s_2 d_{2,j_{2,y_2}} + c_2(y_1 - d_1) - k_2 + (n_2 + 1 - j_{2,y_2})[(p_2 - c_2 + s_2)y_2 + c_2(y_1 - d_1) - k_2] - s_2 \sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg] \tag{3.22}$$

Note that Equation (3.20) is the minimum of Equation (3.21) and Equation (3.22). We also note that $\underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1))$ is a discontinuous function at $d_{2,l}$, for all $l \in \{1, ..., n_2\}$; the proof of this property is given in Appendix A.5.

$\underline{E}_a(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1))$ and $\underline{E}_b(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1))$ are linear functions in each interval $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$. Equation (3.21) is an increasing in $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if

$$j_{2,y_2} < \frac{(n_2+1)(p_2 - c_2 + s_2)}{p_2 + h_2 + s_2} =: M_1 \tag{3.23}$$

and Equation (3.21) is a decreasing function in $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if $j_{2,y_2} > M_1$. Similarly, Equation (3.22) is an increasing in $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if

$$j_{2,y_2} < \frac{(n_2+1)(p_2 - c_2 + s_2) + p_2 + h_2 + s_2}{p_2 + h_2 + s_2} =: M_1 + 1 \tag{3.24}$$

and Equation (3.22) is a decreasing function in $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if $j_{2,y_2} > M_1 + 1$. This implies that the maximum value of Equation (3.20) is at the intersection point of Equation (3.21) and Equation (3.22) in the single interval where Equation (3.21) decreases and Equation (3.22) increases. This leads to the optimal inventory level for the second period, which maximises the NPI lower expected profit,

$$y^{*O}_{2\underline{E}} = \frac{(p_2 + h_2)d_{2,j_{2,y_2}-1} + s_2 d_{2,j_{2,y_2}}}{p_2 + h_2 + s_2} \tag{3.25}$$

where $M_1 \leq j_{2,y_2} < M_1 + 1$.

3.5.2 Without order for the second period, with remaining stock from the first period

In this section, we assume that there is a remaining stock from the first period, $y_1 \geq d_1$, so the inventory level for the second period is $y_2 = (y_1 - d_1)^+$. By substituting Equation

(3.3) into Equation (3.19) the lower expected profit under the assumption that there is no order and $y_1 \geq d_1$ is given by

$$\underline{E}(Pf^{-}(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = (y_1 - d_1)^{+})) = \frac{1}{n_2+1}\Bigg[(j_{2,y_2} - 1)(-h_2 y_2) + (p_2 + h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1} + \min[(p_2+h_2)d_{2,j_{2,y_2}-1} - h_2 y_2, (p_2+s_2)y_2 - s_2 d_{2,j_{2,y_2}}] + (n_2 + 1 - j_{2,y_2})[(p_2+s_2)y_2] - s_2 \sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg] \tag{3.26}$$

Since the condition in this section is not to order for the second period, the objective is not to determine an optimal inventory level for the second period. The goal is to decide which is better, to order or not to order for the second period. So, we need to compare the lower expected profit derived in Section 3.5.1, Equation (3.20), with the lower expected profit derived in this section, Equation (3.26). Then we find a threshold δ_1 such that if $y_2 = (y_1 - d_1)^{+} < \delta_1$, it is better to order for the second period in order to reach the optimal inventory level $y^{*O}_{2\underline{E}}$ in Equation (3.25); otherwise, it is better not to order for the second period. So,

$$\begin{aligned}&\underline{E}(Pf^{2}(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = y^{*O}_{2\underline{E}})) > \\ &\underline{E}(Pf^{-}(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = (y_1 - d_1)^{+})) \Longleftrightarrow \\ &y_1 - d_1 < \frac{1}{-(j_{2,y_1-d_1} - 1)(h_2+p_2+s_2) - n_2(c_2 - p_2 - s_2)}\Bigg[- y^{*O}_{2\underline{E}}[j_{2,y^{*O}_{2\underline{E}}}(h_2+p_2+s_2) \\ &- (c_2+h_2) - (n_2+1)(p_2 - c_2 + s_2)] + k_2(-n_2) + (p_2+h_2)\left(\sum_{j=1}^{j_{2,y^{*O}_{2\underline{E}}}-1} d_{2,j-1} \right. \\ &\left. - \sum_{j=1}^{j_{2,y_1-d_1}-1} d_{2,j-1}\right) - s_2\left\{\sum_{j=j_{2,y^{*O}_{2\underline{E}}}+1}^{n_2+1} d_{2,j} - \sum_{j=j_{2,y_1-d_1}+1}^{n_2+1} d_{2,j}\right\} + \min[(p_2+h_2)d_{2,j_{2,y^{*O}_{2\underline{E}}}-1} \\ &- (c_2+h_2)y^{*O}_{2\underline{E}} + c_2(y_1-d_1) - k_2, (p_2 - c_2 + s_2)y^{*O}_{2\underline{E}} - s_2 d_{2,j_{2,y^{*O}_{2\underline{E}}}} + c_2(y_1 - d_1) - k_2] \\ &- \min[(p_2+h_2)d_{2,j_{2,y_1-d_1}-1} - h_2(y_1-d_1), (p_2+s_2)(y_1-d_1) - s_2 d_{2,j_{2,y_1-d_1}}]\Bigg] =: \delta_1\end{aligned} \tag{3.27}$$

So, if $y_1 - d_1 < \delta_1$, it is better to order for the second period since then the expected profit when we order is larger than when we do not order. While, when $y_1 - d_1 > \delta_1$ it is better not to order for the second period since the expected profit when we do not order is larger than when we order.

In the following example, we illustrate how to find the optimal inventory level for the second period only, under the assumption that we will order and $y_1 \geq d_1$. Also, how to decide if it is better to order for the second period or not.

Example 3.5.1 Consider an inventory system with the following costs: $p_2 = 60, c_2 = 23, h_2 = 11, s_2 = 25, k_2 = 10$, and assume that demand is known to be between $d_{2,0} = 0$ and $d_{2,u} = 15$. Assume that there are $n_2 = 3$ demand observations, with values $d_{2,1} = 5.20, d_{2,2} = 9.10, d_{2,3} = 13.50$. Our aim is to find the optimal inventory level for the second period, given the situation at the end of the first period, under the assumption that we will order and $y_1 \geq d_1$. Also, we aim to decide if it is better to order for the second period or not.

In this example we consider the min bounds in Equation (3.27) as

$$\min[(p_2+h_2)d_{2,j_{2,y^{*O}_{2\underline{E}}}-1} - (c_2+h_2)y^{*O}_{2\underline{E}} + c_2(y_1-d_1) - k_2, (p_2-c_2+s_2)y^{*O}_{2\underline{E}} - s_2 d_{2,j_{2,y^{*O}_{2\underline{E}}}} + c_2(y_1-d_1) - k_2] = (p_2+h_2)d_{2,j_{2,y^{*O}_{2\underline{E}}}-1} - (c_2+h_2)y^{*O}_{2\underline{E}} + c_2(y_1-d_1) - k_2$$

and

$$\min[(p_2+h_2)d_{2,j_{2,y_1-d_1}-1} - h_2(y_1-d_1), (p_2+s_2)(y_1-d_1) - s_2 d_{2,j_{2,y_1-d_1}}] = (p_2+h_2)d_{2,j_{2,y_1-d_1}-1} - h_2(y_1-d_1)$$

hence

$$\delta_1 = \frac{1}{-(n_2+1)(c_2-p_2-s_2) - (j_{2,y_1-d_1})(p_2+s_2+h_2)} \Bigg[y^{*O}_{2\underline{E}}[(n_2+1)(p_2-c_2+s_2) - j_{2,y^{*O}_{2\underline{E}}}(h_2+p_2+s_2)] - k_2(n_2+1) + (p_2+h_2)\left(\sum_{j=1}^{j_{2,y^{*O}_{2\underline{E}}}} d_{2,j-1} - \sum_{j=1}^{j_{2,y_1-d_1}} d_{2,j-1} \right) - s_2 \left\{ \sum_{j=j_{2,y^{*O}_{2\underline{E}}}+1}^{n_2+1} d_{2,j} - \sum_{j=j_{2,y_1-d_1}+1}^{n_2+1} d_{2,j} \right\} \Bigg] \tag{3.28}$$

From Equations (3.23)-(3.25), the optimal inventory level for the second period is $y^{*O}_{2\underline{E}} = 10.25$ with $j_{2,y^{*O}_{2\underline{E}}} = 3$. We determine the threshold δ_1 based on Equation (3.28) to decide which is better, if we order for the second period or not.

- For $j_{2,y_1-d_1} = 1$, $\delta_1 = 7.44$ means that if $y_2 = (y_1-d_1)^+ \in (0, 5.20)$, then it is better to order for the second period since $y_1 - d_1 < 7.44$ in order to reach the optimal inventory level $y^{*O}_{2\underline{E}} = 10.25$.

- For $j_{2,y_1-d_1} = 2$, $\delta_1 = 9.53$ means that if $y_2 = (y_1 - d_1)^+ \in (5.20, 9.10)$, then it is better to order for the second period since $y_1 - d_1 < 9.53$ in order to reach the optimal inventory level $y^{*O}_{2\underline{E}} = 10.25$.

- for $j_{2,y_1-d_1} = 3$, $\delta_1 = 11.25$ means that if $y_2 = (y_1 - d_1)^+ \in (9.10, 13.50)$ and $y_1 - d_1 < 10.25$, then it is better to order for the second period in order to reach the optimal inventory level $y^{*O}_{2\underline{E}} = 10.25$. While, if $y_1 - d_1 \in (9.10, 13.50)$ and $y_1 - d_1 > 10.25$, then it is better not to order for the second period.

- For $j_{2,y_1-d_1} = 4$, $\delta_1 = 13.11$ means that if $y_2 = (y_1 - d_1)^+ \in (13.50, 15)$, then it is better not to order for the second period, since $y_1 - d_1 > 10.25$.

In general, in this example it is better to order for the second period for all $y_1 - d_1 < \delta_1 < y^{*O}_{2\underline{E}}$ and it is better not to order for the second period for all $y^{*O}_{2\underline{E}} < \delta_1 < y_1 - d_1$. However, if $y_1 - d_1$ falls within the third interval as $y^{*O}_{2\underline{E}}$, it is better to order for the second period as long as $y_1 - d_1 < y^{*O}_{2\underline{E}} < \delta_1$. ◇

3.5.3 With order for the second period, with the first period's demand not fully met

In this section, we derive the optimal inventory level for the second period which maximises the lower expected profit, $\underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$, under the assumption that we will order for the second period, with the first period's demand not fully met, $y_1 < d_1$. The lower expected profit for the second period only is derived by substituting Equation (3.2) in Equation (3.19), which leads to

$$\underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[\sum_{j=1}^{j_{2,y_2}-1}\Big(\alpha p_1'(d_1 - y_1) + p_2 d_{2,j-1}$$
$$- c_2(y_2 + \alpha(d_1 - y_1)) - k_2 - h_2(y_2 - d_{2,j-1})\Big) + \min[\alpha p_1'(d_1 - y_1) + p_2 d_{2,j_{2,y_2}-1}$$
$$- c_2(y_2 + \alpha(d_1 - y_1)) - k_2 - h_2(y_2 - d_{2,j_{2,y_2}-1}), \alpha p_1'(d_1 - y_1) + p_2 y_2 - c_2(y_2$$
$$+ \alpha(d_1 - y_1)) - k_2 - s_2(d_{2,j_{2,y_2}} - y_2)] + \sum_{j=j_{2,y_2}+1}^{n_2+1}\Big(\alpha p_1'(d_1 - y_1) + p_2 y_2 - c_2(y_2$$
$$+ \alpha(d_1 - y_1)) - k_2 - s_2(d_{2,j} - y_2)\Big)\Bigg]$$

$$= \frac{1}{n_2+1}\Bigg[(j_{2,y_2}-1)[-(c_2+h_2)y_2+\alpha(d_1-y_1)(p_1'-c_2)-k_2]+(p_2+h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1}$$
$$+\min[(p_2+h_2)d_{2,j_{2,y_2}-1}-(c_2+h_2)y_2+\alpha(d_1-y_1)(p_1'-c_2)-k_2,(p_2$$
$$-c_2+s_2)y_2-s_2d_{2,j_{2,y_2}}+\alpha(d_1-y_1)(p_1'-c_2)-k_2]+(n_2+1-j_{2,y_2})[(p_2$$
$$-c_2+s_2)y_2+\alpha(d_1-y_1)(p_1'-c_2)-k_2]-s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg] \tag{3.29}$$

To determine the optimal inventory level, $y^{*O}_{2\underline{E}}$, which maximises Equation (3.29), we introduce the following two functions:

$$\underline{E}_a(Pf^2(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1))=\frac{1}{n_2+1}\Bigg[(j_{2,y_2}-1)[-(c_2+h_2)y_2$$
$$+\alpha(d_1-y_1)(p_1'-c_2)-k_2]+(p_2+h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1}+(p_2+h_2)d_{2,j_{2,y_2}-1}-(c_2+h_2)y_2$$
$$+\alpha(d_1-y_1)(p_1'-c_2)-k_2+(n_2+1-j_{2,y_2})[(p_2-c_2+s_2)y_2+\alpha(d_1-y_1)(p_1'-c_2)$$
$$-k_2]-s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg] \tag{3.30}$$

and

$$\underline{E}_b(Pf^2(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1))=\frac{1}{n_2+1}\Bigg[(j_{2,y_2}-1)[-(c_2+h_2)y_2$$
$$+\alpha(d_1-y_1)(p_1'-c_2)-k_2]+(p_2+h_2)\sum_{j=1}^{j_{2,y_2}-1} d_{2,j-1}+(p_2-c_2+s_2)y_2-s_2d_{2,j_{2,y_2}}$$
$$+\alpha(d_1-y_1)(p_1'-c_2)-k_2+(n_2+1-j_{2,y_2})[(p_2-c_2+s_2)y_2+\alpha(d_1-y_1)(p_1'-c_2)$$
$$-k_2]-s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\Bigg] \tag{3.31}$$

Note that Equation (3.29) is the minimum of Equation (3.30) and Equation (3.31). We also note that $\underline{E}(Pf^2(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1))$ is a discontinuous function at $d_{2,l}$, for all $l\in\{1,...,n_2\}$; the proof of this property is given in Appendix A.6.

$\underline{E}_a(Pf^2(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1))$ and $\underline{E}_b(Pf^2(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1))$ are linear functions in each interval $[d_{2,j_{2,y_2}-1},d_{2,j_{2,y_2}}]$. Equation (3.30) is an increasing in $[d_{2,j_{2,y_2}-1},d_{2,j_{2,y_2}}]$ if and only if

$$j_{2,y_2}<\frac{(n_2+1)(p_2-c_2+s_2)}{p_2+h_2+s_2}=:M_2 \tag{3.32}$$

and Equation (3.30) is a decreasing function in $[d_{2,j_{2,y_2}-1},d_{2,j_{2,y_2}}]$ if and only if $j_{2,y_2}>M_2$.

Similarly, Equation (3.31) is an increasing in $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if

$$j_{2,y_2} < \frac{(n_2+1)(p_2-c_2+s_2)+p_2+h_2+s_2}{p_2+h_2+s_2} =: M_2+1 \tag{3.33}$$

and Equation (3.31) is a decreasing function in $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if $j_{2,y_2} > M_2+1$. This implies that the maximum value of Equation (3.29) is at the intersection point of Equation (3.30) and Equation (3.31) in the single interval where Equation (3.30) decreases and Equation (3.31) increases. This leads to the optimal inventory level for the second period, which maximises the NPI lower expected profit,

$$y^{*O}_{2\underline{E}} = \frac{(p_2+h_2)d_{2,j_{2,y_2}-1}+s_2 d_{2,j_{2,y_2}}}{p_2+h_2+s_2} \tag{3.34}$$

where $M_2 \leq j_{2,y_2} < M_2+1$. We notice that Equation (3.34) is equal to Equation (3.25), which means that the optimal inventory level when we order for the second period, $y^{*O}_{2\underline{E}}$, is not affected by the stock level at the end of the first period whether $y_1 \geq d_1$ or $y_1 < d_1$.

3.5.4 Without order for the second period, with the first period's demand not fully met

In this section, we suppose that there is no order for the second period, so the inventory level for the second period, y_2, is equal to the stock level at the end of the first period. As we suppose there is demand from the first period that is not fully met, $y_1 < d_1$, the inventory level for the second period is equal to zero with residual demand equal to d_1-y_1. We determine the lower expected profit, $\underline{E}(Pf^-(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1,y_2=0))$, under the assumption that there is no order and $y_1 < d_1$, by substituting Equation (3.4) into Equation (3.19) we have

$$\underline{E}(Pf^-(D_1,D_2,y_1,y_2|y_1=y_1,D_1=d_1,y_2=0)) = \frac{1}{n_2+1}\left[-s_2 d_{2,j_{2,y_2}} - s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j}\right] \tag{3.35}$$

Since the condition in this section is not to order for the second period, the objective is not to determine an optimal inventory level for the second period. The goal is to decide which is better, to order or not to order for the second period. So, we need to compare the lower expected profit derived in Section 3.5.3, Equation (3.29), with the lower expected profit derived in this section, Equation (3.35). Then we find a threshold δ_2 such that if the residual demand, $d_1 - y_1$, is less than δ_2, it is better to order for the second period in order to reach the optimal inventory level $y^{*O}_{2\underline{E}}$ in Equation (3.34) plus $d_1 - y_1$; otherwise, it is better not to order for the second period. So,

$$\begin{aligned}
&\underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = y^{*O}_{2\underline{E}})) > \\
&\underline{E}(Pf^-(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = 0)) \iff \\
&d_1 - y_1 < \frac{1}{\alpha n_2(c_2 - p_1')}\Bigg[- (j_{2,y^{*O}_{2\underline{E}}} - 1)[(h_2 + p_2 + s_2)y^{*O}_{2\underline{E}}] + n_2[(p_2 - c_2 + s_2)y^{*O}_{2\underline{E}} - k_2 \\
&\quad + \alpha(d_1 - y_1)(p_1' - c_2)] + (p_2 + h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\underline{E}}}-1} d_{2,j-1} - s_2\Bigg(\sum_{j=j_{2,y^{*O}_{2\underline{E}}}+1}^{n_2+1} d_{2,j} - \sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j} \\
&\quad - d_{2,j_{2,y_2}}\Bigg) + \min[(p_2 + h_2)d_{2,j_{2,y^*_2}-1} - (c_2 + h_2)y^{*O}_{2\underline{E}} - k_2, (p_2 - c_2 + s_2)y^{*O}_{2\underline{E}} - s_2 d_{2,j_{2,y^{*O}_{2\underline{E}}}} \\
&\quad + \alpha(d_1 - y_1)(p_1' - c_2) - k_2]\Bigg] =: \delta_2 \qquad (3.36)
\end{aligned}$$

So, if $d_1 - y_1 < \delta_2$, it is better to order for the second period since the expected profit when we order is larger than when we do not order. While, when $d_1 - y_1 > \delta_2$ it is better not to order for the second period since the expected profit when we do not order is larger than when we order.

3.6 NPI upper expected profit for the second period

In this section, we optimise the profit for the second period, $Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)$, based on the situation at the end of the first period, when the inventory level is greater than (or less than) the actual demand for the first period, $y_1 \geq d_1$ (or $y_1 < d_1$), with and without order for each inequality. We present the NPI upper expected profit for the second period, as function of the inventory level y_2. The derivations are based on the M-functions presented in Equations (3.16)-(3.18) and shown in Figure 3.1, with $i = 2$. The NPI upper expected profit, denoted by $\overline{E}(Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$, is derived by assigning the probability masses $\frac{1}{n_2+1}$, according to the M-function values to the maximal values for $Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)$ per interval, which leads to

$$\overline{E}(Pf(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \sum_{j=1}^{j_{2,y_2}-1} \Big(M(Pf(d_{2,j-1}, y_2), Pf(d_{2,j}, y_2))$$

$$\times Pf(d_{2,j}, y_2) \Big) + M(\min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)], Pf(y_2, y_2))Pf(y_2, y_2)$$

$$+ \sum_{j=j_{2,y_2+1}}^{n_2+1} M(Pf(d_{2,j}, y_2), Pf(d_{2,j-1}, y_2))Pf(d_{2,j-1}, y_2)$$

$$= \frac{1}{n_2+1} \left[\sum_{j=1}^{j_{2,y_2}-1} Pf(d_{2,j}, y_2) + Pf(y_2, y_2) + \sum_{j=j_{2,y_2+1}}^{n_2+1} Pf(d_{2,j-1}, y_2) \right] \tag{3.37}$$

Next, we consider the NPI upper expected profit for the second period for the scenarios: with and without order in which $y_1 \geq d_1$, also, with and without order in which $y_1 < d_1$.

3.6.1 With order for the second period, with remaining stock from the first period

In this section, we assume that there is a remaining stock from the first period, $y_1 \geq d_1$. So, we derive the optimal inventory level that maximises the upper expected profit, $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$, under the assumption that we will order and that $y_1 \geq d_1$. By substituting Equation (3.2) in Equation (3.37), the upper expected profit is given by

$$\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1} \Bigg[\sum_{j=1}^{j_{2,y_2}-1} \Big(p_2 d_{2,j} - c_2(y_2 - (y_1 - d_1))$$

$$- k_2 - h_2(y_2 - d_{2,j}) \Big) + (p_2 - c_2)y_2 + (y_1 - d_1)c_2 - k_2 + \sum_{j=j_{2,y_2}+1}^{n_2+1} \Big(p_2 y_2 - c_2(y_2$$

$$- (y_1 - d_1)) - k_2 - s_2(d_{2,j-1} - y_2) \Big) \Bigg]$$

$$= \frac{1}{n_2+1} \Bigg[j_{2,y_2}[-(p_2 + h_2 + s_2)y_2] + (p_2 + h_2)[y_2 + \sum_{j=1}^{j_{2,y_2}-1} d_{2,j}] + (n_2+1)[(p_2 - c_2$$

$$+ s_2)y_2 + c_2(y_1 - d_1) - k_2] - s_2 \sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j-1} \Bigg] \tag{3.38}$$

It is easy to show that $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is a continuous function; the proof of this property is given in Appendix A.7.

To determine the optimal inventory level, y^{*O}_{2E}, that maximises Equation (3.38), we use that $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is an increasing over the interval $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if

$$j_{2,y_2} < \frac{h_2 + p_2 + (n_2+1)(p_2 - c_2 + s_2)}{p_2 + s_2 + h_2} =: V_1 \tag{3.39}$$

and $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is a decreasing function over the interval $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if $j_{2,y_2} > V_1$. This implies that Equation (3.38) is maximised at $y^{*O}_{2E} = d_{2,l^*}$ with l^* the largest value in $\{1, 2, ..., n_2\}$ which is less than V_1.

3.6.2 Without order for the second period, with remaining stock from the first period

In this section, we assume that there is a remaining stock from the first period, $y_1 \geq d_1$, so the inventory level for the second period is $y_2 = (y_1 - d_1)^+$. The upper expected profit under the assumption that there is no order and $y_1 \geq d_1$ is given by,

$$\overline{E}(Pf^-(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = (y_1-d_1)^+)) = \frac{1}{n_2+1}\Bigg[j_{2,y_2}[-(p_2 + h_2 + s_2)y_2] + (p_2 + h_2)[y_2 + \sum_{j=1}^{j_{2,y_2}-1} d_{2,j}] + (n_2+1)[(p_2+s_2)y_2] - s_2 \sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j-1}\Bigg] \tag{3.40}$$

Since the condition in this section is not to order for the second period, the objective is not to determine the optimal inventory level for the second period. The goal is to decide which is better, to order or not to order for the second period. So, we need to compare the upper expected profit derived in Section 3.6.1, Equation (3.38), with the upper expected profit derived in this section, Equation (3.40). Then find a threshold τ_1 such that if $y_2 = (y_1 - d_1)^+ < \tau_1$, it is better to order for the second period in order to reach the optimal inventory level y^{*O}_{2E}; otherwise, it is better not to order for the second period. So,

$$\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = y^{*O}_{2E})) >$$
$$\overline{E}(Pf^-(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = (y_1-d_1)^+)) \Longleftrightarrow$$

j_{2,y_1-d_1}	τ_1
1	-30.77
2	-57.96
3	-265.63
4	146.62

Table 3.2: Results of Example 3.6.1

$$y_1 - d_1 < \frac{1}{-j_{2,y_1-d_1}(h_2+p_2+s_2)+h_2+p_2-(n_2+1)(c_2-p_2-s_2)} \Bigg[- y^{*O}_{2E}[j_{2,y^{*O}_{2E}}(h_2 + p_2 + s_2) + (p_2+h_2) + (n_2+1)(p_2-c_2+s_2)] - k_2(n_2+1) + (p_2+h_2)\left(\sum_{j=1}^{j_{2,y^{*O}_{2E}}-1} d_{2,j} - \sum_{j=1}^{j_{2,y_1-d_1}-1} d_{2,j}\right) - s_2\left\{\sum_{j=j_{2,y^{*O}_{2E}}+1}^{n_2+1} d_{2,j-1} - \sum_{j=j_{2,y_1-d_1}+1}^{n_2+1} d_{2,j-1}\right\}\Bigg] =: \tau_1 \tag{3.41}$$

So, if $y_1 - d_1 < \tau_1$, it is better to order for the second period since the expected profit when we order is larger than when we do not order. While, when $y_1 - d_1 > \tau_1$ it is better not to order for the second period since the expected profit when we do not order is larger than when we order. We illustrate how this section works in the following example.

Example 3.6.1 Consider an inventory system with the same data as in Example 3.5.1. Our aim is to find the optimal inventory level for the second period depending on the situation at the end of the first period, under the assumption that we will order and $y_1 \geq d_1$. Also, we aim to decide if it is better to order for the second period or not.

From Equation (3.39), the optimal inventory level for the second period depending on the situation at the end of the first period is $y^{*O}_{2E} = 13.50$ with $j_{2,y^{*O}_{2E}} = 3$. We determine the threshold τ_1 based on Equation (3.41) to decide which is better if we order for the second period or not. Table 3.2 presents the index, j_{2,y_1-d_1}, for the interval of the inventory level for the second period, $y_2 = (y_1 - d_1)^+$, as well as the τ_1 values.

Table 3.2 concludes that, for $j_{2,y_1-d_1} = 1, 2, 3$, it is better not to order for the second period since $y_1 - d_1 > \tau_1$. For $j_{2,y_1-d_1} = 4$, $y_2 = (y_1 - d_1)^+ \in (13.50, 15)$ so, $y_2 < \tau_1$ but $y_2 = (y_1 - d_1)^+ \geq y^{*O}_{2E} = 13.50$, so it is better not to order for second period. ◇

3.6.3 With order for the second period, with the first period's demand not fully met

In this section we derive the optimal inventory level for the second period which maximises the upper expected profit, $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$, under the assumption that we will order for the second period, with the first period's demand not fully met, $y_1 < d_1$. By substituting Equation (3.2) in Equation (3.37), the upper expected profit is given by

$$\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[\sum_{j=1}^{j_{2,y_2}-1}\Big(\alpha p_1'(d_1 - y_1) + p_2 d_{2,j} - c_2(y_2$$

$$+ \alpha(d_1 - y_1)) - k_2 - h_2(y_2 - d_{2,j})\Big) + \alpha p_1'(d_1 - y_1) + (p_2 - c_2)y_2 - \alpha(d_1 - y_1)c_2 - k_2$$

$$+ \sum_{j=j_{2,y_2}+1}^{n_2+1}\Big(\alpha p_1'(d_1 - y_1) + p_2 y_2 - c_2(y_2 + \alpha(d_1 - y_1)) - k_2 - s_2(d_{2,j-1} - y_2)\Big)\Bigg]$$

$$= \frac{1}{n_2+1}\Bigg[j_{2,y_2}[-(p_2 + h_2 + s_2)y_2] + (p_2 + h_2)[y_2 + \sum_{j=1}^{j_{2,y_2}-1} d_{2,j}] + (n_2+1)[(p_2 - c_2 + s_2)y_2$$

$$+ \alpha(d_1 - y_1)(p_1' - c_2) - k_2] - s_2\sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j-1}\Bigg] \tag{3.42}$$

It is easy to show that $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is a continuous function; the proof of this property is given in Appendix A.8.

To determine the optimal inventory level, y_{2E}^{*O}, that maximises Equation (3.42), we use that $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is an increasing over the interval $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if

$$j_{2,y_2} < \frac{h_2 + p_2 + (n_2+1)(p_2 - c_2 + s_2)}{p_2 + s_2 + h_2} =: V_2 \tag{3.43}$$

and $\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is a decreasing function over the interval $[d_{2,j_{2,y_2}-1}, d_{2,j_{2,y_2}}]$ if and only if $j_{2,y_2} > V_2$. This implies that Equation (3.42) is maximised at $y_{2E}^{*O} = d_{2,l^*}$ with l^* the largest value in $\{1, 2, ..., n_2\}$ which is less than V_2.

3.6.4 Without order for the second period, with the first period's demand not fully met

In this section there is no order for the second period, so the inventory level for the second period, y_2, is equal to the stock level at the end of first period. As we suppose there is demand from the first period that is not fully met, $y_1 < d_1$, the inventory level for the second period is equal to zero with residual demand equal to $d_1 - y_1$. We determine the upper expected profit, $\overline{E}(Pf^-(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = 0))$, under the assumption that there is no order and $y_1 < d_1$. By substituting Equation (3.4) into Equation (3.37) we have

$$\overline{E}(Pf^-(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = 0)) = \frac{1}{n_2 + 1}\left[- s_2 \sum_{j=j_{2,y_2}+1}^{n_2+1} d_{2,j-1}\right] \quad (3.44)$$

Since the condition in this section is not to order for the second period, the objective is not to determine the optimal inventory level for the second period. The goal is to decide which is better, to order or not to order for the second period. So, we need to compare the upper expected profit derived in Section 3.6.3, Equation (3.42), with the upper expected profit derived in this section, Equation (3.44). Then we find a threshold τ_2 such that if the residual demand, $d_1 - y_1$, is less than τ_2, it is better to order for the second period in order to reach the optimal inventory level y^{*O}_{2E} plus $d_1 - y_1$; otherwise, it is better not to order for the second period. So,

$$\overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = y^{*O}_{2E})) >$$
$$\overline{E}(Pf^-(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1, y_2 = 0)) \iff$$

$$d_1 - y_1 < \frac{1}{\alpha(n_2+1)(c_2 - p'_1)}\left[j_{2,y^{*O}_{2E}}[-(p_2 + h_2 + s_2)y^{*O}_{2E}] + (p_2 + h_2)\left(y^{*O}_{2E} + \sum_{j=1}^{j_{2,y^{*O}_{2E}}-1} d_{2,j} \right) \right.$$
$$\left. + (n_2+1)[(p_2 - c_2 + s_2)y^{*O}_{2E} - k_2] - s_2\left\{ \sum_{j=j_{2,y^{*O}_{2E}}+1}^{n_2+1} d_{2,j-1} - \sum_{j=j_{2,d_1-y_1}+1}^{n_2+1} d_{2,j-1} \right\} \right] =: \tau_2 \quad (3.45)$$

So, if $d_1 - y_1 < \tau_2$, it is better to order for the second period since the expected profit when we order is larger than when we do not order. While, when $d_1 - y_1 > \tau_2$ it is better not to order for the second period since the expected profit when we do not order is larger than when we order.

3.7 NPI lower expected profit for two periods

In this section, we optimise the NPI lower expected profit for the two-period model. The derivations are based on the M-functions presented in Equations (3.16)-(3.18) and shown in Figure 3.1.

The NPI lower expected profit, denoted by $\underline{E}(Pf(D_1, D_2, y_1, y_2))$, is derived by assigning the probability masses $\frac{1}{n_1+1}$ and $\frac{1}{n_2+1}$, according to the M-function values, to the minimal values for $Pf(D_1, D_2, y_1, y_2)$ per interval, which leads to

$$\begin{aligned}
\underline{E}(Pf(D_1, D_2, y_1, y_2)) &= \sum_{j=1}^{j_{1,y_1}-1} \Big(M(Pf(d_{1,j-1}, y_1), Pf(d_{1,j}, y_1)) Pf(d_{1,j-1}, y_1) \Big) \\
&+ \Big[M(\min[Pf(d_{1,j_{1,y_1}-1}, y_1), Pf(d_{1,j_{1,y_1}}, y_1)], Pf(y_1, y_1)) \\
&\times \min[Pf(d_{1,j_{1,y_1}-1}, y_1), Pf(d_{1,j_{1,y_1}}, y_1)] \Big] + \sum_{j=j_{1,y_1+1}}^{n_1+1} \Big(M(Pf(d_{1,j}, y_1), Pf(d_{1,j-1}, y_1)) \\
&\times Pf(d_{1,j}, y_1) \Big) + \sum_{j=1}^{j_{2,y_2}-1} M(Pf(d_{2,j-1}, y_2), Pf(d_{2,j}, y_2)) Pf(d_{2,j-1}, y_2) \\
&+ \Big[M(\min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)], Pf(y_2, y_2)) \\
&\times \min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)] \Big] + \sum_{j=j_{2,y_2+1}}^{n_2+1} \Big(M(Pf(d_{2,j}, y_2), Pf(d_{2,j-1}, y_2)) \\
&\times Pf(d_{2,j}, y_2) \Big) \\
&= \frac{1}{n_1+1} \Bigg[\sum_{j=1}^{j_{1,y_1}-1} Pf(d_{1,j-1}, y_1) + \min[Pf(d_{1,j_{1,y_1}-1}, y_1), Pf(d_{1,j_{1,y_1}}, y_1)] \\
&+ \sum_{j=j_{1,y_1+1}}^{n_1+1} Pf(d_{1,j}, y_1) \Bigg] + \frac{1}{n_2+1} \Bigg(\sum_{j=1}^{j_{2,y_2}-1} Pf(d_{2,j-1}, y_2) \\
&+ \min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)] + \sum_{j=j_{2,y_2+1}}^{n_2+1} Pf(d_{2,j}, y_2) \Bigg) \qquad (3.46)
\end{aligned}$$

We consider the NPI lower expected profit separately for the two different scenarios for the two periods: ordering for both periods and ordering in the first period only.

3.7.1 Ordering for both periods

Assume an order for both periods, and depending on the optimal inventory level $y^{*O}_{2\underline{E}}$ for the second period, given in Equation (3.25) in Section 3.5.1, we find the lower expected profit over both periods. Hence, we can get the optimal inventory level $y^{*O}_{1\underline{E}}$ for the first period. By substituting Equation (3.1) into Equation (3.46), we have the lower expected profit, $\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2 | y_2 = y^{*O}_{2\underline{E}}))$, under the assumption that we will order for the first and second period,

$$\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}})) = \frac{1}{n_1+1}\Bigg[(j_{1,y_1}-1)(-(c_1+h_1-c_2)y_1 - k_1)$$
$$+ (p_1+h_1-c_2)\sum_{j=1}^{j_{1,y_1}-1} d_{1,j-1} + \min[-(c_1+h_1-c_2)y_1 - k_1 + (p_1+h_1-c_2)d_{1,j_{1,y_1}-1},$$
$$(p_1 - c_1 + s_1 - \alpha p'_1 + \alpha c_2)y_1 - k_1 - (s_1 - \alpha p'_1 + \alpha c_2)d_{1,j_{1,y_1}}] + (n_1+1-j_{1,y_1})((p_1$$
$$- c_1 + s_1 - \alpha p'_1 + \alpha c_2)y_1 - k_1) - (s_1 - \alpha p'_1 + \alpha c_2)\sum_{j=j_{1,y_1}+1}^{n_1+1} d_{1,j}\Bigg]$$
$$+ \frac{1}{n_2+1}\Bigg((j_{2,y^{*O}_{2\underline{E}}}-1)[-(c_2+h_2)y^{*O}_{2\underline{E}} - k_2] + (p_2+h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\underline{E}}}-1} d_{2,j-1}$$
$$+ \min[(p_2+h_2)d_{2,j_{2,y^{*O}_{2\underline{E}}}-1} - (c_2+h_2)y^{*O}_{2\underline{E}} - k_2, (p_2-c_2+s_2)y^{*O}_{2\underline{E}} - s_2 d_{2,j_{2,y^{*O}_{2\underline{E}}}} - k_2]$$
$$+ (n_2+1-j_{2,y^{*O}_{2\underline{E}}})[(p_2-c_2+s_2)y^{*O}_{2\underline{E}} - k_2] - s_2\sum_{j=j_{2,y^{*O}_{2\underline{E}}}+1}^{n_2+1} d_{2,j}\Bigg) \tag{3.47}$$

To determine the optimal inventory level for the first period, $y^{*O}_{1\underline{E}}$, which maximises $\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$, we introduce two functions, $\underline{E}_a(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ and $\underline{E}_b(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ in which Equation (3.47) is the minimum of these two functions. We also note that $\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ is a discontinuous function at $d_{1,l}$, for all $l \in \{1, ..., n_1\}$; the proof of this property is given in Appendix A.9.

$\underline{E}_a(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ and $\underline{E}_b(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ are linear functions in each interval $[d_{1,j_{1,y_1}-1}, d_{1,j_{1,y_1}}]$. $\underline{E}_a(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ is an increasing function in $[d_{1,j_{1,y_1}-1}, d_{1,j_{1,y_1}}]$ if and only if

$$j_{1,y_1} < \frac{(n_1+1)(p_1 - c_1 + s_1 - \alpha p'_1 + \alpha c_2)}{h_1 + p_1 + s_1 + (\alpha-1)c_2 - \alpha p'_1} =: Z_1 \tag{3.48}$$

and $\underline{E}_a(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ is a decreasing function in $[d_{1,j_{1,y_1}-1}, d_{1,j_{1,y_1}}]$ if and only if $j_{1,y_1} > Z_1$. Similarly, $\underline{E}_b(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ is an increasing function in $[d_{1,j_{1,y_1}-1}, d_{1,j_{1,y_1}}]$ if and only if

$$j_{1,y_1} < \frac{(n_1+1)(p_1 - c_1 + s_1 - \alpha p'_1 + \alpha c_2) + h_1 + p_1 + s_1 + (\alpha-1)c_2 - \alpha p'_1}{h_1 + p_1 + s_1 + (\alpha-1)c_2 - \alpha p'_1} =: Z_1 + 1 \tag{3.49}$$

and $\underline{E}_b(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ is a decreasing function in $[d_{1,j_{1,y_1}-1}, d_{1,j_{1,y_1}}]$ if and only if $j_{1,y_1} > Z_1 + 1$. This implies that the maximum value of Equation (3.47) is at the

1st period	2nd period
$p_1 = 50, c_1 = 20, k_1 = 9, h_1 = 10, s_1 = 20$	$p'_1 = 30, \alpha = 0.7, p_2 = 60, c_2 = 23, k_2 = 10, h_2 = 11, s_2 = 25$
$n_1 = 2, d_{1,0} = 0, d_{1,u} = 11$	$n_2 = 3, d_{2,0} = 0, d_{2,u} = 15$
$d_{1,1} = 4.70, d_{1,2} = 8.90$	$d_{2,1} = 5.20, d_{2,2} = 9.10, d_{2,3} = 13.50$

Table 3.3: Inputs and data for Example 3.7.1

intersection point of $\underline{E}_a(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ and $\underline{E}_b(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ in the single interval where $\underline{E}_a(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ decreases and $\underline{E}_b(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}}))$ increases. This leads to the optimal inventory level for the first period, which maximises the NPI lower expected profit,

$$y^{*O}_{1\underline{E}} = \frac{1}{h_1 + p_1 + s_1 + (\alpha - 1)c_2 - \alpha p'_1}[(p_1 + h_1 - c_2)d_{1,j_{1,y_1}-1} + (s_1 - \alpha p'_1 + \alpha c_2)d_{1,j_{1,y_1}}] \tag{3.50}$$

where $Z_1 \leq j_{1,y_1} < Z_1 + 1$.

In the following example, we illustrate how to find the optimal inventory level, under the assumption that we will order for both periods.

Example 3.7.1 Consider an inventory system with two periods. The inputs and data for these periods are shown in Table 3.3. The objective is to find the optimal inventory level for the first period, $y^{*O}_{1\underline{E}}$, when there is an order for the second period.

We already computed the optimal inventory level for the second period in Example 3.5.1, which is $y^{*O}_{2\underline{E}} = 10.25$ and $j_{2,y^{*O}_{2\underline{E}}} = 3$. Now we find $y^{*O}_{1\underline{E}}$ using Equations (3.48) and (3.49), this leads to $Z_1 = 2.60$. So, $j_{1,y^{*O}_{1\underline{E}}} = 3$ which means that, the optimal inventory level for the first period is in $(d_{1,2}, d_{1,3}) = (8.90, 11)$. From Equation (3.50), the optimal inventory level is $y^{*O}_{1\underline{E}} = 9.51$ and $\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}})) = 139.79$. ◇

3.7.2 Ordering in the first period only

In this section, we consider the case with only an order for the first period, which means that there is a single order for two periods, and the stock level at the end of the first period is greater than zero. So, we suppose $y_1 = y^*_{\underline{E}_{\text{Once}}}, y_2 = (y_1 - D_1)^+$, in which $y^*_{\underline{E}_{\text{Once}}}$ is the optimal inventory level obtained in Section 2.6. We determine the lower expected profit, $\underline{E}(Pf^1(D_1, D_2, y_1, y_2|y_1 = y^*_{\underline{E}_{\text{Once}}}, y_2 = y^*_{\underline{E}_{\text{Once}}} - D_1))$, under the assumption that there is no order for the second period. By substituting Equation (3.5) in Equation (3.46) the lower expected profit is given by

$$\underline{E}(Pf^1(D_1, D_2, y_1, y_2|y_1 = y^*_{\underline{E}_{\text{Once}}}, y_2 = y^*_{\underline{E}_{\text{Once}}} - D_1)) =$$

$$\frac{1}{n_1+1}\Bigg[-(j_{1,y^*_{\underline{E}_{\text{Once}}}} - 1)[(c_1+h_1)y^*_{\underline{E}_{\text{Once}}} + k_1] + (p_1+h_1)\sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} d_{1,j-1} +$$
$$\min[(p_1+h_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} - (c_1+h_1)y^*_{\underline{E}_{\text{Once}}} - k_1, (p_1 - c_1 + s_1 - \alpha p_1')y^*_{\text{Once}} -$$
$$k_1 - (s_1 - \alpha p_1')d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}}] + (n_1 + 1 - j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1 - c_1 + s_1 - \alpha p_1')y^*_{\underline{E}_{\text{Once}}} -$$
$$k_1] - (s_1 - \alpha p_1')\sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j}\Bigg] + \frac{1}{n_2+1}\Bigg(-(j_{2,y_2} - 1)(h_2 y_2) +$$
$$(p_2+h_2)\sum_{l=1}^{j_{2,y_2}-1} d_{2,l-1} + \min[(p_2+h_2)d_{2,j_{2,y_2}-1} - h_2 y_2, (p_2+s_2)y_2 -$$
$$s_2 d_{2,j_{2,y_2}}] + (n_2 + 1 - j_{2,y_2})(p_2+s_2)y_2 - s_2\sum_{l=j_{2,y_2}+1}^{n_2+1} d_{2,l}\Bigg) \tag{3.51}$$

In Equation (3.51) we have $y_2 = y^*_{\underline{E}_{\text{Once}}} - D_1$ where D_1 is a random quantity which is assumed to be in the interval $(d_{1,j-1}, d_{1,j})$ for further analysis, so j_{2,y_2} is not exactly determined, hence we consider four cases according to different assumptions on this.

Case 1: Replace $D_1 \in (d_{1,j-1}, d_{1,j})$ by $D_1 = d_{1,j-1}$ and assume

$$\min[(p_2+h_2)d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}-1} - h_2(y^*_{\underline{E}_{\text{Once}}} - D_1), (p_2+s_2)(y^*_{\underline{E}_{\text{Once}}} - D_1) - s_2 d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}}]$$
$$= (p_2+h_2)d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}-1} - h_2(y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1})$$

Case 2:

Replace $D_1 \in (d_{1,j-1}, d_{1,j})$ by $D_1 = d_{1,j}$ and assume

$$\min[(p_2+h_2)d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}-1} - h_2(y^*_{\underline{E}_{\text{Once}}} - D_1), (p_2+s_2)(y^*_{\underline{E}_{\text{Once}}} - D_1) - s_2 d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}}]$$
$$= (p_2+h_2)d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1} - h_2(y^*_{\underline{E}_{\text{Once}}} - d_{1,j})$$

Case 3:

Replace $D_1 \in (d_{1,j-1}, d_{1,j})$ by $D_1 = d_{1,j-1}$ and assume

$$\min[(p_2+h_2)d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}-1} - h_2(y^*_{\underline{E}_{\text{Once}}} - D_1), (p_2+s_2)(y^*_{\underline{E}_{\text{Once}}} - D_1) - s_2 d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}}]$$
$$= (p_2+s_2)(y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1}) - s_2 d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}}$$

Case 4:

Replace $D_1 \in (d_{1,j-1}, d_{1,j})$ by $D_1 = d_{1,j}$ and assume

$$\min[(p_2+h_2)d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}-1} - h_2(y^*_{\underline{E}_{\text{Once}}} - D_1), (p_2+s_2)(y^*_{\underline{E}_{\text{Once}}} - D_1) - s_2 d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-D_1}}]$$
$$= (p_2+s_2)(y^*_{\underline{E}_{\text{Once}}} - d_{1,j}) - s_2 d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}}$$

These assumptions lead to heuristic approximations, $\underline{E}^1, \underline{E}^2, \underline{E}^3, \underline{E}^4$, for Equation (3.51). These are given below for Case 1 and Case 2, respectively, and similarly for Case 3 and Case 4,

$$\begin{aligned}
&\underline{E}^1(Pf(D_1,D_2,y_1,y_2|y_1 = y^*_{\underline{E}_{\text{Once}}}, y_2 = y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1})) = \frac{1}{n_1+1}\Bigg[-(j_{1,y^*_{\underline{E}_{\text{Once}}}} - 1)[(c_1\\
&\quad + h_1)y^*_{\underline{E}_{\text{Once}}} + k_1] + (p_1+h_1)\sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} d_{1,j-1} + \min[(p_1+h_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} - (c_1\\
&\quad + h_1)y^*_{\underline{E}_{\text{Once}}} - k_1, (p_1 - c_1 + s_1 - \alpha p'_1)y^*_{\underline{E}_{\text{Once}}} - k_1 - (s_1 - \alpha p'_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}}] + (n_1 + 1\\
&\quad - j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1 - c_1 + s_1 - \alpha p'_1)y^*_{\underline{E}_{\text{Once}}} - k_1] - (s_1 - \alpha p'_1)\sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j}\Bigg]\\
&\quad + \frac{1}{n_2+1}\Bigg(-(j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}} - 1)(y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1})h_2 + (p_2+h_2)\Bigg\{\sum_{l=1}^{j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}-1} d_{2,l-1}\\
&\quad + d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}-1}\Bigg\} - h_2(y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1}) + (n_2 + 1 - j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}})(p_2+s_2)(y^*_{\underline{E}_{\text{Once}}}\\
&\quad - d_{1,j-1}) - s_2 \sum_{l=j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}+1}^{n_2+1} d_{2,l}\Bigg)
\end{aligned} \tag{3.52}$$

$$\underline{E}^2(Pf(D_1,D_2,y_1,y_2|y_1=y^*_{\underline{E}_{\text{Once}}},y_2=y^*_{\underline{E}_{\text{Once}}}-d_{1,j}))=\frac{1}{n_1+1}\Bigg[-(j_{1,y^*_{\underline{E}_{\text{Once}}}}-1)[(c_1+h_1)y^*_{\underline{E}_{\text{Once}}}+k_1]+(p_1+h_1)\sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1}d_{1,j-1}+\min[(p_1+h_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1}-(c_1+h_1)y^*_{\underline{E}_{\text{Once}}}-k_1,(p_1-c_1+s_1-\alpha p'_1)y^*_{\underline{E}_{\text{Once}}}-k_1-(s_1-\alpha p'_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}}]+(n_1+1-j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1-c_1+s_1-\alpha p'_1)y^*_{\underline{E}_{\text{Once}}}-k_1]-(s_1-\alpha p'_1)\sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1}d_{1,j}\Bigg]+\frac{1}{n_2+1}\Bigg(-(j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1)(y^*_{\underline{E}_{\text{Once}}}-d_{1,j})h_2+(p_2+h_2)\Bigg\{\sum_{l=1}^{j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1}d_{2,l-1}+d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1}\Bigg\}-h_2(y^*_{\underline{E}_{\text{Once}}}-d_{1,j})+(n_2+1-j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}})(p_2+s_2)(y^*_{\underline{E}_{\text{Once}}}-d_{1,j})-s_2\sum_{l=j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}+1}^{n_2+1}d_{2,l}\Bigg) \quad (3.53)$$

Since the condition in this section is not to order for the second period, the objective is not to determine an optimal inventory level for the second period. The goal is to decide which is better, to order or not to order for the second period. So, we need to compare the lower expected profit given by Equation (3.47) in Section 3.7.1, with the heuristic approximations of the lower expected profits in this section, given by Equations (3.52)-(3.53) for Case 1 and Case 2, and similarly for Case 3 and Case 4. Then find a threshold such that, if the inventory level for the second period, if we do not order, is less than the threshold, it is better to order for the second period in order to reach the optimal inventory level $y^{*O}_{2\underline{E}}$ as given in Equation (3.25); otherwise, it is better not to order for the second period.

For **Case 1**:

$$\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_1=y^{*O}_{1\underline{E}},y_2=y^{*O}_{2\underline{E}}))>$$
$$\underline{E}^1(Pf(D_1,D_2,y_1,y_2|y_1=y^*_{\underline{E}_{\text{Once}}},y_2=y^*_{\underline{E}_{\text{Once}}}-d_{j-1}))\iff$$

$$
\begin{aligned}
& y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} < \frac{n_2+1}{-j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}(h_2+p_2+s_2)+(n_2+1)(p_2+s_2)} \\
& \times \Bigg[\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) - \frac{1}{n_1+1}\Big\langle -(j_{1,y^*_{\underline{E}_{\text{Once}}}}-1)[(c_1+h_1)y^*_{\underline{E}_{\text{Once}}} \\
& + k_1] + (p_1+h_1) \sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} d_{1,j-1} + \min[(p_1+h_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} - (c_1+h_1)y^*_{\underline{E}_{\text{Once}}} \\
& - k_1, (p_1-c_1+s_1-\alpha p'_1)y^*_{\underline{E}_{\text{Once}}} - k_1 - (s_1-\alpha p'_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}}] + (n_1+1 \\
& - j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1-c_1+s_1-\alpha p'_1)y^*_{\underline{E}_{\text{Once}}} - k_1] - (s_1-\alpha p'_1) \sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j} \Big\rangle \\
& - \frac{1}{n_2+1}\Bigg((p_2+h_2)\Bigg\{ \sum_{l=1}^{j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}-1} d_{2,l-1} + d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}-1} \Bigg\} \\
& - s_2 \sum_{l=j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}+1}^{n_2+1} d_{2,l} \Bigg) \Bigg] =: \delta_3
\end{aligned} \tag{3.54}
$$

where the term $\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2 = y^{*O}_{2\underline{E}}))$ is given in Equation (3.47). So, if $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} < \delta_3$, it is better to order for the second period since the expected profit for the two periods, when we order for the second period, is larger than when we do not order for the second period. While, when $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} > \delta_3$, it is better not to order for the second period since the expected profit for the two periods, when we do not order for the second period, is larger than when we order for the second period.

For **Case 2**:

$$
\begin{aligned}
& \underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_1=y^{*O}_{1\underline{E}}, y_2=y^{*O}_{2\underline{E}})) > \\
& \underline{E}^2(Pf(D_1,D_2,y_1,y_2|y_1=y^*_{\underline{E}_{\text{Once}}}, y_2=y^*_{\underline{E}_{\text{Once}}}-d_j)) \iff
\end{aligned}
$$

$$
\begin{aligned}
y^*_{\underline{E}_{\text{Once}}} - d_{1,j} <{}& \frac{n_2+1}{-j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}(h_2+p_2+s_2)+(n_2+1)(p_2+s_2)} \\
& \times \Bigg[\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) - \frac{1}{n_1+1}\Big\langle -(j_{1,y^*_{\underline{E}_{\text{Once}}}}-1)[(c_1+h_1)y^*_{\underline{E}_{\text{Once}}} \\
& + k_1] + (p_1+h_1)\sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} d_{1,j-1} + \min[(p_1+h_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} - (c_1+h_1)y^*_{\underline{E}_{\text{Once}}} \\
& - k_1, (p_1-c_1+s_1-\alpha p_1')y^*_{\underline{E}_{\text{Once}}} - k_1 - (s_1-\alpha p_1')d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}}] + (n_1+1 \\
& - j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1-c_1+s_1-\alpha p_1')y^*_{\underline{E}_{\text{Once}}} - k_1] - (s_1-\alpha p_1')\sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j}\Big\rangle \\
& - \frac{1}{n_2+1}\Bigg((p_2+h_2)\Bigg\{\sum_{l=1}^{j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1} d_{2,l-1} + d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1}\Bigg\} \\
& - s_2\sum_{l=j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}+1}^{n_2+1} d_{2,l}\Bigg)\Bigg] =: \delta_4
\end{aligned}
\tag{3.55}
$$

where the term $\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2 = y^{*O}_{2\underline{E}}))$ is given in Equation (3.47). So, if $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} < \delta_4$, it is better to order for the second period since the expected profit for the two periods, when we order for the second period, is larger than when we do not order for the second period. While, when $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} > \delta_4$, it is better not to order for the second period since the expected profit for the two periods, when we do not order for the second period, is larger than when we order for the second period.

For **Case 3**:

$$
\begin{aligned}
&\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_1=y^{*O}_{1\underline{E}}, y_2=y^{*O}_{2\underline{E}})) > \\
&\underline{E}^3(Pf(D_1,D_2,y_1,y_2|y_1=y^*_{\underline{E}_{\text{Once}}}, y_2=y^*_{\underline{E}_{\text{Once}}}-d_{j-1})) \Longleftrightarrow
\end{aligned}
$$

$$y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} < \frac{n_2+1}{(1-j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}})(h_2+p_2+s_2)+(n_2+1)(p_2+s_2)}$$
$$\times \Bigg[\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) - \frac{1}{n_1+1}\Big\langle -(j_{1,y^*_{\underline{E}_{\text{Once}}}}-1)[(c_1+h_1)y^*_{\underline{E}_{\text{Once}}}$$
$$+k_1]+(p_1+h_1)\sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} d_{1,j-1} + \min[(p_1+h_1)d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} - (c_1+h_1)y^*_{\underline{E}_{\text{Once}}}$$
$$-k_1,(p_1-c_1+s_1-\alpha p_1')y^*_{\underline{E}_{\text{Once}}} - k_1 - (s_1-\alpha p_1')d_{1,j_{1,y^*_{\text{Once}}}}] + (n_1+1$$
$$-j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1-c_1+s_1-\alpha p_1')y^*_{\underline{E}_{\text{Once}}} - k_1] - (s_1-\alpha p_1')\sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j}\Big\rangle$$
$$-\frac{1}{n_2+1}\Bigg((p_2+h_2)\sum_{l=1}^{j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}-1} d_{2,l-1} - s_2[d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}}$$
$$+\sum_{l=j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}+1}^{n_2+1} d_{2,l}]\Bigg)\Bigg] =: \delta_5 \tag{3.56}$$

where the term $\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2 = y^{*O}_{2\underline{E}}))$ is given in Equation (3.47). So, if $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} < \delta_5$, it is better to order for the second period since the expected profit for the two periods, when we order for the second period, is larger than when we do not order for the second period. While, when $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} > \delta_5$, it is better not to order for the second period since the expected profit for the two periods, when we do not order for the second period, is larger than when we order for the second period.

For **Case 4**:

$$\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_1=y^{*O}_{1\underline{E}},y_2=y^{*O}_{2\underline{E}})) >$$
$$\underline{E}^4(Pf(D_1,D_2,y_1,y_2|y_1=y^*_{\underline{E}_{\text{Once}}},y_2=y^*_{\underline{E}_{\text{Once}}}-d_j)) \iff$$

1st period	2nd period
$p_1 = 70, c_1 = 23, k_1 = 19, h_1 = 17, s_1 = 9$	$p'_1 = 25, \alpha = 0.7, p_2 = 240, c_2 = 13, k_2 = 20, h_2 = 170, s_2 = 15,$
$n_1 = 2, d_{1,0} = 0, d_{1,u} = 50.30$	$n_2 = 4, d_{2,0} = 0, d_{2,u} = 100.40$
$d_{1,1} = 4.20, d_{1,2} = 17.60$	$d_{2,1} = 6.50, d_{2,2} = 20.60, d_{2,3} = 42.60, d_{2,4} = 70.20$

Table 3.4: Inputs and data for Example 3.7.2

$$y^*_{\underline{E}_{\text{Once}}} - d_{1,j} < \frac{n_2+1}{(1 - j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}})(h_2+p_2+s_2) + (n_2+1)(p_2+s_2)}$$
$$\times \Bigg[\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2 | y_2 = y^{*O}_{2\underline{E}})) - \frac{1}{n_1+1} \Big\langle - (j_{1,y^*_{\underline{E}_{\text{Once}}}} - 1)[(c_1+h_1) y^*_{\underline{E}_{\text{Once}}}$$
$$+ k_1] + (p_1+h_1) \sum_{j=1}^{j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} d_{1,j-1} + \min[(p_1+h_1) d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}-1} - (c_1+h_1) y^*_{\underline{E}_{\text{Once}}}$$
$$- k_1, (p_1 - c_1 + s_1 - \alpha p'_1) y^*_{\underline{E}_{\text{Once}}} - k_1 - (s_1 - \alpha p'_1) d_{1,j_{1,y^*_{\underline{E}_{\text{Once}}}}}] + (n_1 + 1$$
$$- j_{1,y^*_{\underline{E}_{\text{Once}}}})[(p_1 - c_1 + s_1 - \alpha p'_1) y^*_{\underline{E}_{\text{Once}}} - k_1] - (s_1 - \alpha p'_1) \sum_{j=j_{1,y^*_{\underline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j} \Big\rangle$$
$$- \frac{1}{n_2+1} \Bigg((p_2+h_2) \sum_{l=1}^{j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}-1} d_{2,l-1} - s_2 [d_{2,j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}-1}}$$
$$+ \sum_{l=j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}+1}^{n_2+1} d_{2,l}] \Bigg) \Bigg] =: \delta_6 \tag{3.57}$$

where the term $\underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2 | y_2 = y^{*O}_{2\underline{E}}))$ is given in Equation (3.47). So, if $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} < \delta_6$, it is better to order since the expected profit for the two periods, when we order for the second period, is larger than when we do not order for the second period. While, when $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} > \delta_6$, it is better not to order for the second period since the expected profit for the two periods, when we do not order for the second period, is larger than when we order for the second period. The following example illustrates how to decide whether to order for the second period or not.

Example 3.7.2 Consider an inventory system with two periods. The inputs and data for these periods are shown in Table 3.4. The objective is to decide if it is better to order for the second period or not. As defined in Example 2.6.2 in Section 2.6, the optimal inventory level for the two-period model with a single order is $y^*_{\underline{E}_{\text{Once}}} = 17.99$, $j_{y^*_{\underline{E}_{\text{Once}}}} = 4$.

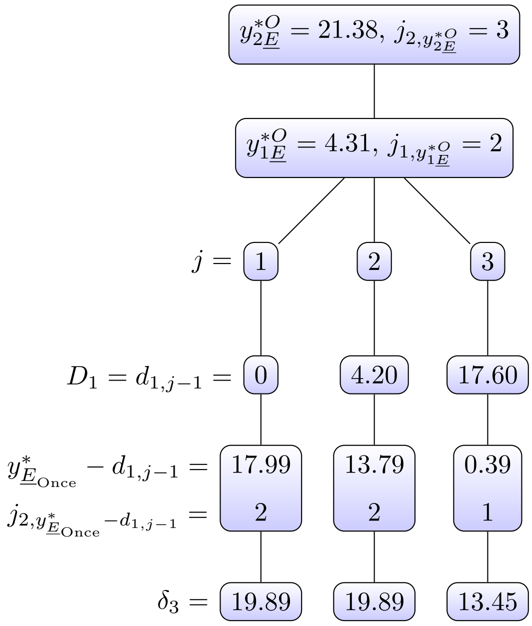


Figure 3.2: A decision tree for Example 3.7.2, **Case 1**

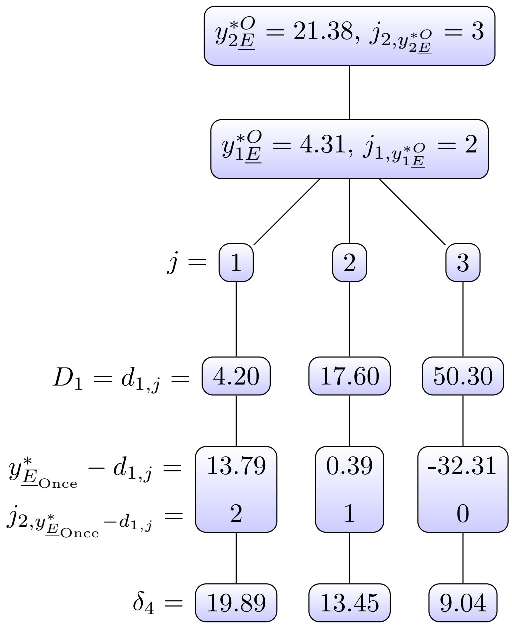


Figure 3.3: A decision tree for Example 3.7.2, **Case 2**

Figures 3.2-3.5 show the optimal inventory levels when we order for the first and second periods, $y^{*O}_{1\underline{E}}, y^{*O}_{2\underline{E}}$, the random demand at the end of the first period, D_1, the inventory level when we do not order for the second period, $y^*_{\underline{E}_{\text{Once}}} - D_1$ and the thresholds $\delta_3, ..., \delta_6$.

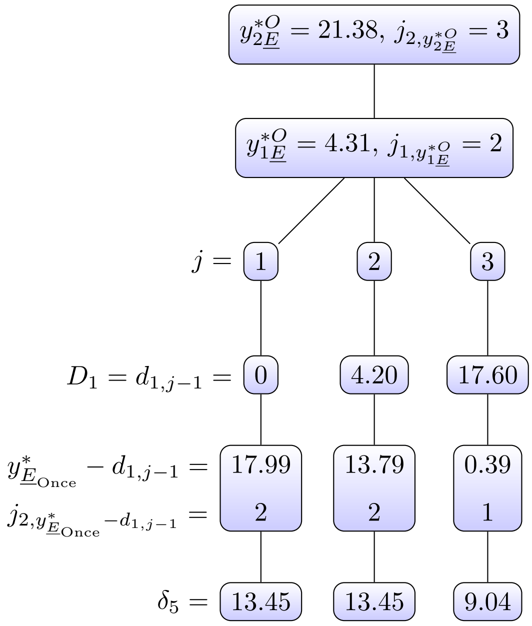


Figure 3.4: A decision tree for Example 3.7.2, **Case 3**

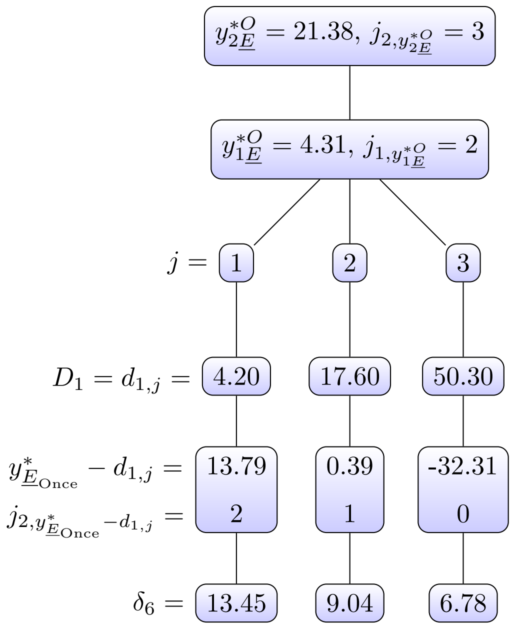


Figure 3.5: A decision tree for Example 3.7.2, **Case 4**

- For Case 1, when $j_{2,y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1}} = 2$, $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} = 17.99$ and $\delta_3 = 19.89$, it is better to order for the second period since $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} < \delta_3$ in order to reach the optimal inventory level $y^{*O}_{2\underline{E}} = 21.38$. Similarly, for other $j_{2,y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1}}$.

- For Case 2, when $j_{2,y^*_{\underline{E}_{\text{Once}}} - d_{1,j}} = 2$, $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} = 13.79$ and $\delta_4 = 19.89$, it is better to order for the second period since $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} < \delta_4$ in order to reach the optimal inventory level $y^{*O}_{2\underline{E}} = 21.38$. Similarly, for other $j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}$.

- For Case 3, when $j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}} = 2$, $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} = 17.99$ and $\delta_5 = 13.45$, it is better not to order for the second period since $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} > \delta_5$. Similarly, for other $j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j-1}}$.
- For Case 4, when $j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}} = 2$, $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} = 13.79$ and $\delta_6 = 13.45$, it is better not to order for the second period since $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} > \delta_6$. Similarly, for other $j_{2,y^*_{\underline{E}_{\text{Once}}}-d_{1,j}}$.

So, Figures 3.2 and 3.3 conclude that, for all j it is better to order for the second period since $y^*_{\underline{E}_{\text{Once}}} - d_{1,j-1} < \delta_3$ for Case 1 and $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} < \delta_4$ for Case 2. For Case 3, Figure 3.4 shows that it is better to order for the second period when $j = 3$. Figure 3.5 shows that it is better to order for the second period when $j = 2, 3$. ◇

3.8 NPI upper expected profit for the two periods

In this section, we present the NPI upper expected profit for the two-period model. The derivations are based on the M-functions presented in Equations (3.16)-(3.18) and shown in Figure 3.1.

The NPI upper expected profit, denoted by $\overline{E}(Pf(D_1, D_2, y_1, y_2))$, is derived by assigning the probability masses $\frac{1}{n_1+1}$ and $\frac{1}{n_2+1}$, according to the M-function values, to the maximal values for $Pf(D_1, D_2, y_1, y_2)$ per interval, which leads to

$$
\begin{aligned}
\overline{E}(Pf(D_1, D_2, y_1, y_2)) = & \sum_{j=1}^{j_{1,y_1}-1} \Big(M(Pf(d_{1,j-1}, y_1), Pf(d_{1,j}, y_1)) Pf(d_{1,j}, y_1) \Big) \\
& + \Big[M(\min[Pf(d_{1,j_{1,y_1}-1}, y_1), Pf(d_{1,j_{1,y_1}}, y_1)], Pf(y_1, y_1)) Pf(y_1, y_1) \Big] \\
& + \sum_{j=j_{1,y_1}+1}^{n_1+1} \Big(M(Pf(d_{1,j}, y_1), Pf(d_{1,j-1}, y_1)) Pf(d_{1,j-1}, y_1) \Big) \\
& + \sum_{j=1}^{j_{2,y_2}-1} \Big(M(Pf(d_{2,j-1}, y_2), Pf(d_{2,j}, y_2)) Pf(d_{2,j}, y_2) \Big) \\
& + \Big[M(\min[Pf(d_{2,j_{2,y_2}-1}, y_2), Pf(d_{2,j_{2,y_2}}, y_2)], Pf(y_2, y_2)) Pf(y_2, y_2) \Big] \\
& + \sum_{j=j_{2,y_2}+1}^{n_2+1} \Big(M(Pf(d_{2,j}, y_2), Pf(d_{2,j-1}, y_2)) Pf(d_{2,j-1}, y_2) \Big) \\
= & \frac{1}{n_1+1} \left[\sum_{j=1}^{j_{1,y_1}-1} Pf(d_{1,j}, y_1) + Pf(y_1, y_1) + \sum_{j=j_{1,y_1}+1}^{n_1+1} Pf(d_{1,j-1}, y_1) \right] \\
& + \frac{1}{n_2+1} \left(\sum_{j=1}^{j_{2,y_2}-1} Pf(d_{2,j}, y_2) + Pf(y_2, y_2) + \sum_{j=j_{2,y_2+1}}^{n_2+1} Pf(d_{2,j-1}, y_2) \right) \qquad (3.58)
\end{aligned}
$$

We consider the NPI upper expected profit separately for the two different scenarios for the two periods: ordering for both periods and ordering in the first period only.

3.8.1 Ordering for both periods

Assume an order for both periods, and depending on the optimal inventory level $y^{*O}_{2\overline{E}}$ for the second period, given in Section 3.6.1, we find the upper expected profit over both periods. Hence, we can get the optimal inventory level $y^{*O}_{1\overline{E}}$ for the first period. We derive the upper expected profit, $\overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2 | y_2 = y^{*O}_{2\overline{E}}))$, under the assumption that we will order for the first and the second period, by substituting Equation (3.1) in Equation (3.58),

$$\overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\overline{E}})) =$$
$$\frac{1}{n_1+1}\Big[(j_{1,y_1}-1)(-(c_1+h_1-c_2)y_1-k_1)+(p_1+h_1-c_2)\sum_{j=1}^{j_{1,y_1}-1}d_{1,j}+$$
$$(p_1-c_1)y_1-k_1+(n_1+1-j_{1,y_1})((p_1-c_1+s_1-\alpha p_1'+\alpha c_2)y_1-k_1)-$$
$$(s_1-\alpha p_1'+\alpha c_2)\sum_{j=j_{1,y_1}+1}^{n_1+1}d_{1,j-1}\Big]+\frac{1}{n_2+1}\Big((j_{2,y^{*O}_{2\overline{E}}}-1)[-(c_2+h_2)y^{*O}_{2\overline{E}}-$$
$$k_2]+(p_2+h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\overline{E}}}-1}d_{2,j}+(p_2-c_2)y^{*O}_{2\overline{E}}-k_2+(n_2+1-j_{2,y^{*O}_{2\overline{E}}})[(p_2-$$
$$c_2+s_2)y^{*O}_{2\overline{E}}-k_2]-s_2\sum_{j=j_{2,y^{*O}_{2\overline{E}}}+1}^{n_2+1}d_{2,j-1}\Big) \tag{3.59}$$

It is easy to show that $\overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\overline{E}}))$ is a continuous function; the proof of this property is given in Appendix A.10.

To determine the optimal inventory level, $y^{*O}_{1\overline{E}}$, that maximises Equation (3.59), we use that $\overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\overline{E}}))$ is an increasing over the interval $[d_{1,j_{1,y_1}-1},d_{1,j_{1,y_1}}]$ if and only if

$$j_{1,y_1}<\frac{h_1+p_1-c_2+(n_1+1)(p_1-c_1+s_1-\alpha p_1'+\alpha c_2)}{h_1+p_1+s_1+(\alpha-1)c_2-\alpha p_1'}=:V_3 \tag{3.60}$$

and $\overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\overline{E}}))$ is a decreasing function over the interval $[d_{1,j_{1,y_1}-1},d_{1,j_{1,y_1}}]$ if and only if $j_{1,y_1}>V_3$. This implies that the Equation (3.59) is maximised at $y^{*O}_{1\overline{E}}=d_{1,l^*}$ with l^* the largest value in $\{1,2,...,n_1\}$ which is less than V_3.

3.8.2 Ordering in the first period only

In this section, we consider the case with only an order for the first period, which means that there is a single order for two periods, and the stock level at the end of the first period is greater than zero. So, we suppose $y_1=y^*_{\overline{E}_{\text{Once}}}, y_2=(y_1-D_1)^+$, in which $y^*_{\overline{E}_{\text{Once}}}$ is the optimal inventory level obtained in Section 2.6. We determine the upper expected profit, $\overline{E}(Pf^1(D_1,D_2,y_1,y_2|y_1=y^*_{\overline{E}_{\text{Once}}},y_2=y^*_{\overline{E}_{\text{Once}}}-D_1))$ under the assumption that there is no order for the second period. By substituting Equation (3.5) in Equation (3.58), the upper expected profit is given by

$$\overline{E}(Pf^1(D_1, D_2, y_1, y_2 | y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - D_1)) =$$
$$\frac{1}{n_1+1}\Bigg[-(j_{1,y^*_{\overline{E}_{\text{Once}}}} - 1)[(c_1+h_1)y^*_{\overline{E}_{\text{Once}}} + k_1] + (p_1+h_1) \sum_{j=1}^{j_{1,y^*_{\overline{E}_{\text{Once}}}}-1} d_{1,j} +$$
$$(p_1 - c_1)y^*_{\overline{E}_{\text{Once}}} - k_1 + (n_1 + 1 - j_{1,y^*_{\overline{E}_{\text{Once}}}})[(p_1 - c_1 + s_1 - \alpha p_1')y^*_{\overline{E}_{\text{Once}}} -$$
$$k_1] - (s_1 - \alpha p_1') \sum_{j=j_{1,y^*_{\overline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j-1}\Bigg] + \frac{1}{n_2+1}\Bigg(-(j_{2,y_2} - 1)(h_2 y_2) +$$
$$(p_2 + h_2) \sum_{l=1}^{j_{2,y_2}-1} d_{2,l} + p_2 y_2 + (n_2 + 1 - j_{2,y_2})(p_2 + s_2)y_2 - s_2 \sum_{l=j_{2,y_2}+1}^{n_2+1} d_{2,l-1}\Bigg) \tag{3.61}$$

In Equation (3.61) we have $y_2 = y^*_{\overline{E}_{\text{Once}}} - D_1$ where D_1 is a random quantity which is assumed to be in the interval $(d_{1,j-1}, d_{1,j})$ for further analysis, so j_{2,y_2} is not exactly determined, hence we consider two cases according to different assumptions on this.

Case 1:

Replace $D_1 \in (d_{1,j-1}, d_{1,j})$ by $D_1 = d_{1,j-1}$.

Case 2:

Replace $D_1 \in (d_{1,j-1}, d_{1,j})$ by $D_1 = d_{1,j}$.

Considering these assumptions leads to heuristic approximations,
$\overline{E}^1(Pf(D_1, D_2, y_1, y_2 | y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - d_{1,j-1}))$ and
$\overline{E}^2(Pf(D_1, D_2, y_1, y_2 | y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - d_{1,j}))$, for Equation (3.61).

Since the condition in this section is not to order for the second period, the objective is not to determine an optimal inventory level for the second period. The goal is to decide which is better, to order or not to order for the second period. So, we need to compare the upper expected profit given by Equation (3.59) in Section 3.8.1, with the heuristic approximations of the upper expected profit in this section, $\overline{E}^1(Pf(D_1, D_2, y_1, y_2 | y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - d_{1,j-1}))$ and $\overline{E}^2(Pf(D_1, D_2, y_1, y_2 | y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - d_{1,j}))$. This will lead to find a threshold, such that, if the inventory level for the second period, if we do not order, is less than the threshold, it is actually better to order for the second period in order to reach the optimal inventory level $y^{*O}_{2\overline{E}}$ as given in Section 3.6.1;

otherwise, it is better not to order for the second period.

For **Case 1**:

$$\begin{aligned}
&\overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_1 = y^{*O}_{1\overline{E}}, y_2 = y^{*O}_{2\overline{E}})) > \\
&\overline{E}^1(Pf(D_1, D_2, y_1, y_2|y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - d_{j-1})) \Longleftrightarrow \\
&y^*_{\overline{E}_{\text{Once}}} - d_{1,j-1} < \frac{n_2+1}{(1 - j_{2,y^*_{\overline{E}_{\text{Once}}}-d_{1,j-1}})(h_2 + p_2 + s_2) + p_2 + n_2(p_2 + s_2)} \\
&\quad \times \Bigg[\overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\overline{E}})) - \frac{1}{n_1+1}\Bigg\langle - (j_{1,y^*_{\overline{E}_{\text{Once}}}} - 1)[(c_1 + h_1)y^*_{\overline{E}_{\text{Once}}} \\
&\quad + k_1] + (p_1 + h_1) \sum_{j=1}^{j_{1,y^*_{\overline{E}_{\text{Once}}}}-1} d_{1,j} + (p_1 - c_1)y^*_{\overline{E}_{\text{Once}}} - k_1 + (n_1 + 1 - j_{1,y^*_{\overline{E}_{\text{Once}}}})[(p_1 \\
&\quad - c_1 + s_1 - \alpha p_1')y^*_{\overline{E}_{\text{Once}}} - k_1] - (s_1 - \alpha p_1') \sum_{j=j_{1,y^*_{\overline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j-1} \Bigg\rangle - \frac{1}{n_2+1}\Bigg((p_2 \\
&\quad + h_2) \sum_{l=1}^{j_{2,y^*_{\overline{E}_{\text{Once}}}-d_{1,j-1}}-1} d_{2,l} - s_2 \sum_{l=j_{2,y^*_{\overline{E}_{\text{Once}}}-d_{1,j-1}}+1}^{n_2+1} d_{2,l-1}\Bigg)\Bigg] =: \tau_3 \qquad (3.62)
\end{aligned}$$

where the term $\overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\overline{E}}))$ is given in Equation (3.59). So, if $y^*_{\overline{E}_{\text{Once}}} - d_{1,j-1} < \tau_3$, it is better to order for the second period since the expected profit for the two periods, when we order for the second period, is larger than when we do not order. While, when $y^*_{\overline{E}_{\text{Once}}} - d_{1,j-1} > \tau_3$, it is better not to order for the second period since the expected profit for the two periods, when we do not order for the second period, is larger than when we order for the second period.

For **Case 2**:

$$\begin{aligned}
&\overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_1 = y^{*O}_{1\overline{E}}, y_2 = y^{*O}_{2\overline{E}})) > \\
&\overline{E}^2(Pf(D_1, D_2, y_1, y_2|y_1 = y^*_{\overline{E}_{\text{Once}}}, y_2 = y^*_{\overline{E}_{\text{Once}}} - d_j)) \Longleftrightarrow
\end{aligned}$$

$$
\begin{aligned}
y^*_{\overline{E}_{\text{Once}}} - d_{1,j} < {} & \frac{n_2+1}{(1-j_{2,y^*_{\overline{E}_{\text{Once}}}-d_{1,j}})(h_2+p_2+s_2)+p_2+n_2(p_2+s_2)} \\
& \times \Bigg[\overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2E})) - \frac{1}{n_1+1}\Big\langle -(j_{1,y^*_{\overline{E}_{\text{Once}}}}-1)[(c_1+h_1)y^*_{\overline{E}_{\text{Once}}} \\
& + k_1] + (p_1+h_1)\sum_{j=1}^{j_{1,y^*_{\overline{E}_{\text{Once}}}}-1} d_{1,j} + (p_1-c_1)y^*_{\overline{E}_{\text{Once}}} - k_1 + (n_1+1-j_{1,y^*_{\overline{E}_{\text{Once}}}})[(p_1 \\
& - c_1 + s_1 - \alpha p'_1)y^*_{\overline{E}_{\text{Once}}} - k_1] - (s_1-\alpha p'_1)\sum_{j=j_{1,y^*_{\overline{E}_{\text{Once}}}}+1}^{n_1+1} d_{1,j-1}\Big\rangle - \frac{1}{n_2+1}\Big((p_2 \\
& + h_2)\sum_{l=1}^{j_{2,y^*_{\overline{E}_{\text{Once}}}-d_{1,j}}-1} d_{2,l} - s_2 \sum_{l=j_{2,y^*_{\overline{E}_{\text{Once}}}-d_{1,j}}+1}^{n_2+1} d_{2,l-1}\Big)\Bigg] =: \tau_4
\end{aligned}
\tag{3.63}
$$

where the term $\overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2 = y^{*O}_{2E}))$ is given in Equation (3.59). So, if $y^*_{\overline{E}_{\text{Once}}} - d_{1,j} < \tau_4$, it is better to order for the second period since the expected profit for the two periods, when we order for the second period, is larger than when we do not order. While, when $y^*_{\overline{E}_{\text{Once}}} - d_{1,j} > \tau_4$, it is better not to order for the second period since the expected profit for the two periods, when we do not order for the second period, is larger than when we order for the second period. The following example illustrates how to decide whether to order for the second period or not.

Example 3.8.1 Consider an inventory system with the same data as in Example 3.7.2, given in Table 3.4. Our aim is to decide if it is better to order for the second period or not.

As derived in Example 2.6.2 in Section 2.6, the optimal inventory level for the two-period model with a single order is $y^*_{\overline{E}_{\text{Once}}} = 54.50$, $j_{y^*_{\overline{E}_{\text{Once}}}} = 6$.

Figures 3.6 and 3.7 show the optimal inventory levels when we order for the first and second periods, y^{*O}_{1E} and y^{*O}_{2E}, the random demand at the end of the first period, D_1, the inventory level when we do not order for the second period, $y^*_{\overline{E}_{\text{Once}}} - D_1$ and the thresholds τ_3 and τ_4.

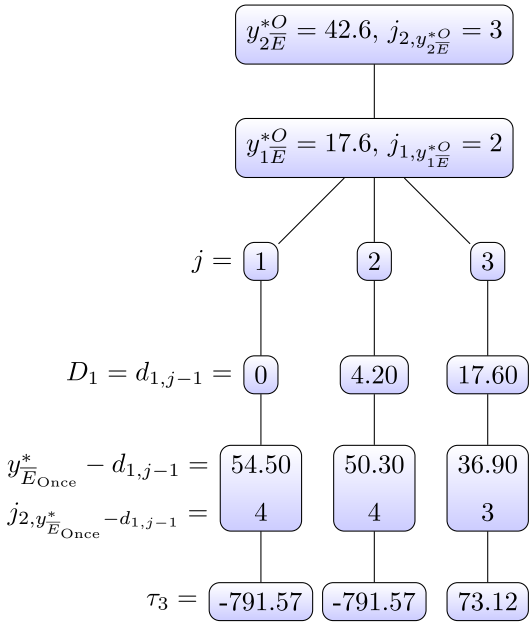


Figure 3.6: A decision tree for Example 3.8.1, **Case 1**

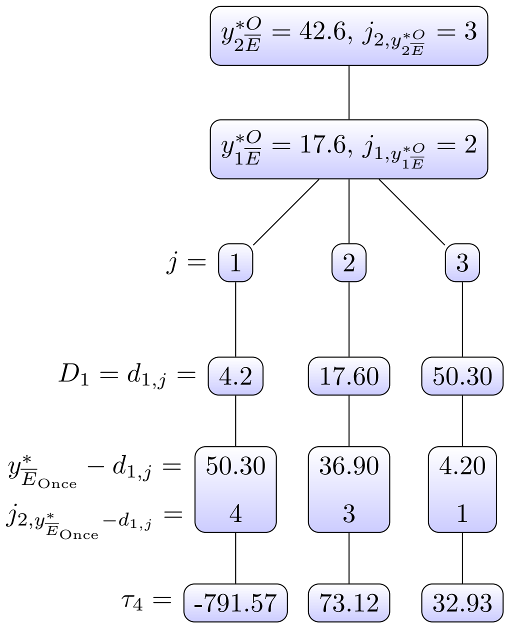


Figure 3.7: A decision tree for Example 3.8.1, **Case 2**

- For Case 1, when $j_{2,y^{*}_{\overline{E}_{\text{Once}}}-d_{1,j-1}} = 4$, $y^{*}_{\overline{E}_{\text{Once}}} - d_{1,j-1} = 54.50$ and $\tau_3 = -791.57$, it is better not to order for the second period since $y^{*}_{\overline{E}_{\text{Once}}} - d_{1,j-1} > \tau_3$. Similarly, for other $j_{2,y^{*}_{\overline{E}_{\text{Once}}}-d_{1,j-1}}$.

- For Case 2, when $j_{2,y^{*}_{\overline{E}_{\text{Once}}}-d_{1,j}} = 4$, $y^{*}_{\overline{E}_{\text{Once}}} - d_{1,j} = 50.30$ and $\tau_4 = -791.57$, it is better not to order for the second period since $y^*_{\underline{E}_{\text{Once}}} - d_{1,j} > \tau_4$. Similarly, for other $j_{2, y^*_{\underline{E}_{\text{Once}}} - d_{1,j}}$.

In general, for Case 1, it is better to order for the second period only when $j = 3$. For Case 2, it is better to order for the second period for $j = 2, 3$ since $y^*_{\overline{E}_{\text{Once}}} - d_{1,j} < \tau_4$. ◇

3.9 Comparison of the NPI and classical methods

In order to compare between the classical method and the NPI method, presented in this chapter, a simulation study is conducted for the two-period model when we order for the first and the second period. Our aim is effectively to check how close the classical method is to NPI when the distribution of the classical method is assumed to be known. Then, based on some assumptions, compare which method performs better than the other. We simulate n_1 observations for the demand of the first period and n_2 observations for the demand of the second period from a Gamma distribution, since the demand is assumed to be positive in this thesis, we select the Gamma distribution in simulation settings as it is flexible in many shapes for positive real values. Then, the n_1 and n_2 simulated data observations are used to derive the optimal inventory levels $y^{*O}_{1\underline{E}}$ and $y^{*O}_{2\underline{E}}$ corresponding to the lower expected profit criterion over both periods or $y^{*O}_{1\overline{E}}$ and $y^{*O}_{2\overline{E}}$ corresponding to the upper expected profit criterion over both periods. Then values for two future observations, D_{1,n_1+1} and D_{2,n_2+1}, are simulated from the same underlying distribution as the n_1 and n_2 simulated data observations, allowing the realised value of the profit function to be computed for the values of the optimal inventory levels.

We consider the same cases for simulation as presented in Section 2.5 with regard to discrepancies between the model used for the data simulations, and the model assumed for the classical method. The aim is to determine the optimal inventory levels for the classical and NPI methods, then calculate the profits based on the optimal inventory levels and future demands. Each case is run 1000 times and we report the number of these runs in which the profit resulting from the NPI method is greater than the profit resulting from the classical method.

The cases considered are given in Table 3.5, with first the model used for simulating the demands D_1 and D_2 specified, followed by the model assumed for the analysis according to the classical method. For the case where the Gamma scale parameter θ is simulated

Case	Simulation	Classical assumption
I	$D_1, D_2 \sim$ Gamma$(3, 1)$	Gamma$(3, 1)$
II	$D_1, D_2 \sim$ Gamma$(3, 1)$	Exp$(1/3)$
III	$D_1, D_2 \sim$ Gamma$(3, 1)$	Exp$(1/2)$
IV	$D_1, D_2 \sim$ Gamma$(3, \theta), \theta \sim$ Unif$(0, 2)$	Gamma$(3, 1)$
V	$D_1, D_2 \sim$ Gamma$(3, 1)$	Exp(1)
VI	$D_1, D_2 \sim$ Gamma$(3, 1)$	Exp(2)

Table 3.5: Simulation cases

from the Uniform(0, 2) distribution, one value is drawn and used for each run, so $n_1 + n_2$ observations are drawn using one specific value of θ, and a new value of θ is drawn for the next run.

Case I is the scenario where the model used for the classical analysis is exactly the same as the model used for the data generation. In Case II the model for the analysis is Exponential but with the same mean value as the Gamma distribution used to generate the data. The further cases have other discrepancies between these two models, set up in such a way that we expected that the classical method would perform more poorly for the later cases as the differences between the models increase.

For all cases, we consider three different sample sizes, $n_1 = 5, 50, 100$ and $n_2 = 10, 110, 550$. The costs used for the first period are the same as the cost for single-period model we have used in Section 2.5, with $k_1 = 19$, while the costs for the second period are $p'_1 = 30, c_2 = 25, p_2 = 100, h_2 = 15, s_2 = 30$ and $k_2 = 10$, we have chosen $\alpha = 0.60$.

As finite end-point for the support of the random demands for the first and second periods we took $d_{1,u} = d_{2,u} = 15$, since both are from the same distribution; in the rare event that a simulated value in a run exceeds 15, we delete the value and draw a new one; this has no real impact on the method as the probability to get values which exceed 15 is very small for all models considered.

Table 3.6 presents the results from the simulation study. It provides the number of times, out of 1000 runs, in which the profit according to the NPI method is larger than for the corresponding classical method. Table 3.6 considers the expected profit as optimality criterion, where $\underline{E}$ and $\overline{E}$ indicate that, for the NPI method, the lower expected profit or the upper expected profit was used, respectively.

	$n_1 = 5,$		$n_1 = 50,$		$n_1 = 100,$	
	$n_2 = 10$		$n_2 = 110$		$n_2 = 550$	
Case	$\underline{E}$	$\overline{E}$	$\underline{E}$	$\overline{E}$	$\underline{E}$	$\overline{E}$
I	368	112	448	425	468	483
II	461	121	669	664	752	751
III	408	331	471	464	461	462
IV	641	507	646	642	677	680
V	725	631	754	754	752	752
VI	871	797	892	890	889	889

Table 3.6: Simulation results for lower and upper expected profits (1000 runs)

	Single-period model		**Two-period model**			
Case	$y^*_{\underline{E}}$	$y^*_{\overline{E}}$	$y^{*O}_{1\underline{E}}$	$y^{*O}_{2\underline{E}}$	$y^{*O}_{1\overline{E}}$	$y^{*O}_{2\overline{E}}$
I, II, III, V, VI	2.64	2.99	7.03	3.12	15	3.67
IV	1.69	1.79	7.06	3.75	15	3.87

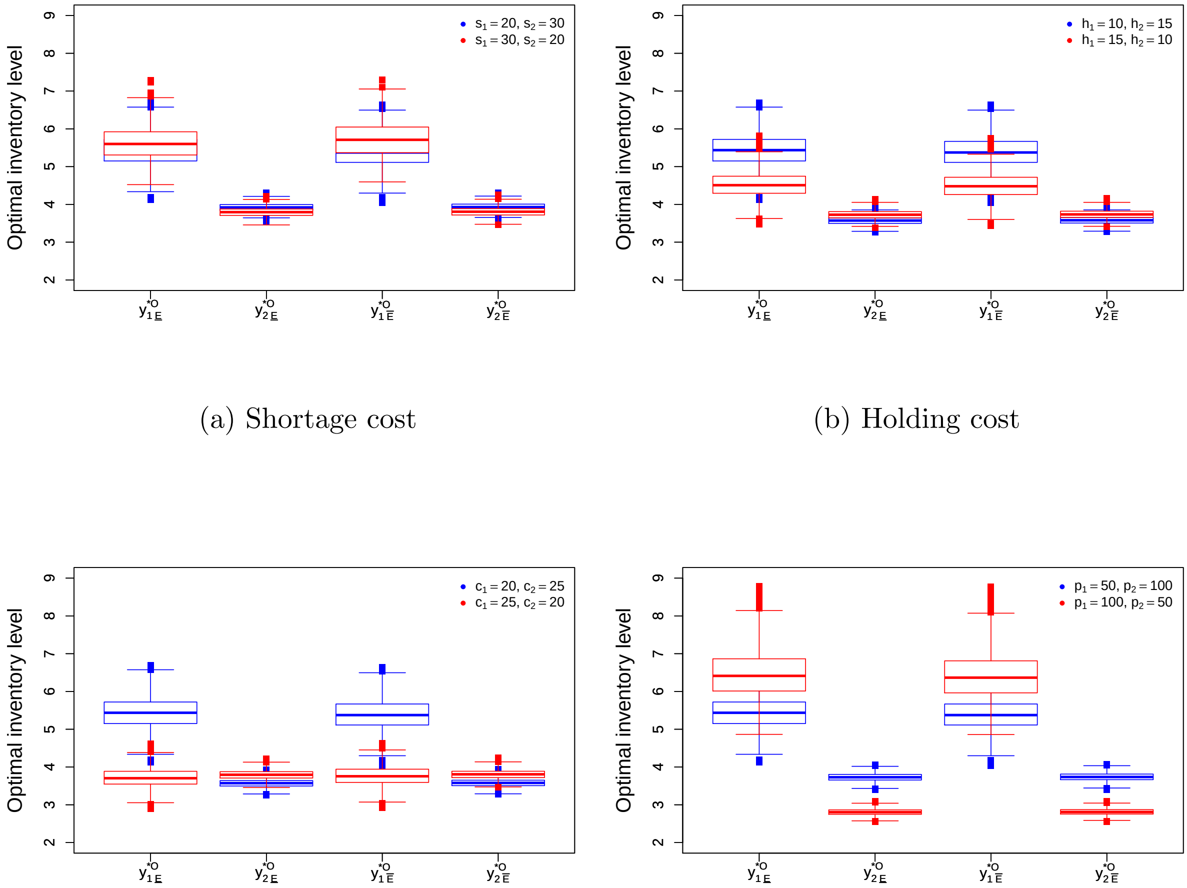
Table 3.7: Optimal inventory levels, $n_1 = 5, n_2 = 10$

As expected, the NPI method performs worse than the classical method in Cases I and III, but for large n_1 and n_2 the performance of the NPI methods improves and the number of times it performs better than the classical method increases to close to 500. For Cases IV-VI, the NPI method performs better than the classical method for all values of n_1 and n_2, with the number of times that the profit for the NPI method is greater than the profit for the classical method exceeding 500 out of 1000 simulations.

In Table 3.7, we compare the optimal inventory levels for the single-period model, which we studied in Section 2.4.2, with the optimal inventory levels for the two-period independent demands model. The results show that the optimal inventory level for the first period of the two-period model is higher than the optimal level for the single-period model. This is expected because in the two-period model we will order more for the first period, as leftover items can be used in the second period.

Next, we investigate the NPI lower and upper expected profits model when the cost parameters increase, we only considered Case I and $n_1 = 100, n_2 = 550$.

Figure 3.8a displays the difference in the optimal inventory levels of the lower and



(a) Shortage cost

(b) Holding cost

(c) Purchase cost

(d) Selling price

Figure 3.8: The optimal inventory levels of the lower and upper expected profits

upper expected profits when $s_1 = 20, s_2 = 30$ and $s_1 = 30, s_2 = 20$. For the first period, $y^{*O}_{1\underline{E}}$ and $y^{*O}_{1\overline{E}}$ increase when s_1 has been increased, so we will order more for the first period. While for the second period, $y^{*O}_{2\underline{E}}$ and $y^{*O}_{2\overline{E}}$ decrease when s_2 has been decreased, so we will order less for the second period.

For the holding cost, Figure 3.8b concludes that, when h_1 increased, $y^{*O}_{1\underline{E}}$ and $y^{*O}_{1\overline{E}}$ decreased, so we will order less for the first period. While for the second period, $y^{*O}_{2\underline{E}}$ and $y^{*O}_{2\overline{E}}$ increase when h_2 has been decreased, so we will order more for the second period.

Similarly, for the purchase cost and selling price. For the purchase cost, Figure 3.8c shows that, for the first period, $y^{*O}_{1\underline{E}}$ and $y^{*O}_{1\overline{E}}$ decrease when c_1 has been increased, so we will order less for the first period. While for the second period, $y^{*O}_{2\underline{E}}$ and $y^{*O}_{2\overline{E}}$ increase when c_2 has been decreased, so we will order more for the second period.

For the selling price, Figure 3.8d illustrates that, when p_1 is increased, $y^{*O}_{1\underline{E}}$ and $y^{*O}_{1\overline{E}}$ increase, so we will order more for the first period. While for the second period, $y^{*O}_{2\underline{E}}$ and $y^{*O}_{2\overline{E}}$ decrease when p_2 has been decreased, so we will order less for the second period.

In general, for any period, if the shortage cost or selling price increases, we will order more for that period. While, if the holding cost or purchase cost increases, we will order less for that period.

In this simulation study, we took into account the case when we order for both periods. It will be interesting to investigate the possibility of not ordering in either the first or the second period as a topic for future research. Using the same approach that is used in this chapter for more than two periods is not analytically feasible, hence a different method is therefore required; a possible alternative is briefly introduced in Chapter 4.

3.10 Concluding remarks

This chapter developed the NPI method for the single-order inventory models, presented in Chapter 2, to the two-period inventory model with independent demands.

We studied and investigated the performance of the classical method and the NPI method for the two-period model, considering different situations for ordering in the first and second periods. First, we determine the optimal inventory level for the second period, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimise over the first period, focusing on maximising the lower and upper expected profits.

The performance of the classical method and the NPI method was evaluated through simulation studies. We considered the two-period model when we ordered for the first and second period. We assumed continuous distributions to examine the general performance of the proposed methods. So, in some cases the classical method performs better than the NPI method. However, for the large number of observations, NPI gets close to the classical method, where this is based on correct assumptions. For other cases, the NPI approach tends to perform better than the classical method, depending on how far off the classical method's assumptions are from reality. Also, we discussed the impact of increasing the cost parameters on the NPI method. As a result, if the shortage cost or selling price has increased in any period, we will order more for that period. However, if the holding cost or purchase cost has increased in any period, we will order less for that period.

In the simulation study, we only considered the case when we ordered for both periods. It is also interesting to investigate the case when we do not order for either the first or second period. Also, the inventory model with more than two periods is an interesting idea for a future topic. But because the approach discussed in this chapter is hard to analyse for more than two periods, the next chapter briefly discusses a possible alternative.

Chapter 4

NPI bootstrap for inventory decisions

4.1 Introduction

In this chapter, we present the NPI bootstrap method, which we indicate by NPI-B, as an alternative method to deal with complexities in some of the inventory models, and then we discuss its use to predict the optimal inventory level.

As we have seen in Chapters 2 and 3, for some practical inventory problems, the analytic NPI approach is not appropriate. In this chapter, we show how NPI-B approach can be used instead to compare different strategies.

This chapter is organised as follows: In Section 4.2, we introduce NPI-B. Section 4.3 introduces NPI-B for the inventory models that we considered in Chapters 2 and 3. We will use maximisation of the average profit as a criterion to select the optimal inventory levels. In Section 4.4, we will present the NPI-B method for the two-period model with dependent demands. Section 4.5 presents the concluding remarks for this chapter.

4.2 NPI bootstap

Quantifying the variability of a sample estimate is an essential part of statistical inference. It is easy to make inference and draw conclusions and use a probability model in simple situations, but if the method's assumptions are wrong, it can lead to wrong conclusions in more complicated situations. To address this issue, Efron [34] devised a bootstrap method in 1979. This method makes fewer assumptions but requires more calculations, also, its application is uncomplicated with statistical software, so it is widely used in applied statistics [37]. Efron's bootstrap method has been applied to a range of topics, such as right-censored data [35] and bivariate data [36].

One of the bootstrap methods, known as nonparametric predictive inference bootstrap (NPI-B), is a computational version of NPI that is used to quantify uncertainty in statistical inference. The original investigation of the NPI-B method was presented by BinHimd and Coolen [9, 23]. To obtain an NPI-B sample, the procedure depends on selecting one interval randomly from the $n+1$ intervals created by the n original data observations, and from this interval, one future value is drawn uniformly. This value is added to the original data set, and continue to sample m future values in the same way in order to obtain the NPI-B sample. All possible orderings of the m future observations among the n original data observations are equally likely to appear in NPI-B [9, 23]. The assumptions are different for finite and infinite intervals, in which, for finite intervals, one observation is sampled uniformly from each chosen interval. However, it cannot be sampled uniformly from an open-ended interval. In this thesis, we are dealing with the demand, and we assumed upper and lower bounds, $[d_0, d_u]$, for the demand, so we use finite intervals, which we think is a suitable choice for our purposes here.

For one-dimensional real-valued data on a finite (bounded) interval, the NPI-B algorithm is as follows:

(i) The original n observations partition the intervals into $n+1$ intervals.

(ii) Choose one of the $n+1$ intervals at random, each with equal probability.

(iii) Uniformly sample one future value from this chosen interval.

(iv) Include this value in the data set: raise n to $n+1$.

(v) Repeat Steps (ii)-(iv), now with $n+1$ data, to get a further future value.

(vi) Do this $m-1$ times to get a NPI bootstrap sample of size m.

(vii) Perform Steps (ii)-(vi) in total N times to get a total of N NPI bootstrap samples of size m, where N is a chosen integer value.

In the following section, we introduce NPI-B for the inventory models.

4.3 NPI-B for inventory models

As we have seen in Chapters 2 and 3, for more interesting practical inventory problems, the analytic approach is not feasible. The NPI-B method can be applied to compare different order strategies. In this section, we introduce the NPI-B approach to derive the optimal inventory strategy for different scenarios, namely the single-order and the two-period independent demands inventory models which we also considered in Chapters 2 and 3.

4.3.1 NPI-B for the single-period inventory model

On the basis of the NPI-B, we compare different inventory levels, y, to select the optimal one based on maximisation of average profit. Here we will study different strategies for ordering $y \in (d_0, d_u)$ to determine which one is best based on the average profit criterion.

To sample an NPI-B sample of size m, we generate n observations from a continuous distribution as the original sample with support $[d_0, d_u]$. So, there are $n+1$ intervals between the data set values, $(d_0, d_1), (d_1, d_2), ..., (d_n, d_u)$. Choose one interval, and then sample the new value from this interval as the first value in the NPI-B sample. Then add this value to the data set so that it is observations $n+1$ and the intervals are $n+2$ intervals. Continue with this procedure to derive an NPI-B sample of size m. For the single-period model we took $m = 1$.

The process of choosing the optimal inventory level is based on calculating the profit or loss depending on y and the future observation, D_{n+1} which we derived by using NPI-B. We repeat this process N times, and we report the maximum average profit and the corresponding inventory level as the optimal value. In the following example, we illustrate this procedure.

Example 4.3.1 Assume $n = 5, 50, c = 20, p = 50, h = 10, s = 20$ and $d_u = 15$. We consider two different cases each case is run 1000 and 10,000 times and we report the optimal inventory levels that maximise the average profit resulting from the NPI method and the NPI-B method.

- **Case I:** $D_i \sim \text{Gamma}(3, 1)$.
- **Case II:** $D_i \sim \text{Exp}(\text{rate} = 1/5)$.

			NPI		NPI-B	
		Case	$y^*_{\underline{E}}$	Average profit	y^*	Average profit
$\boldsymbol{N = 1000}$	$n = 5$	I	3.29	36.61	3.24	11.16
		II	5.24	6.28	4.93	-0.86
	$n = 50$	I	3.08	35.89	3.19	32.16
		II	4.63	0.98	4.92	12.11
$\boldsymbol{N = 10,000}$	$n = 5$	I	3.21	36.04	3.38	13.30
		II	5.06	4.39	5.01	-2.84
	$n = 50$	I	3.18	35.41	3.24	34.17
		II	4.90	3.49	4.58	10.42

Table 4.1: Results of Example 4.3.1

Table 4.1 presents the optimal inventory levels for the single-period model based on the NPI and NPI-B methods for two different cases with $n = 5, 50$ and $N = 1000$ and $10,000$. We use Equation (2.25) to find $y^*_{\underline{E}}$ for NPI, but in NPI-B, we examine all values of y between $(d_0 = 0, d_u = 15)$, then we choose y^* as the one that corresponds to the maximum average profit. As a result, the NPI-B approach leads to similar optimal value as the full analytical method. While $y^*_{\underline{E}}$ and y^* seem quite close, the average profits differ quite a lot. This is due to the effect of future demand, which is sampled by using NPI-B in the profit function. $\diamond$

4.3.2 NPI-B for a single order for two-period model

In this section we compare different strategies for inventory level, y, to determine which one leads to the highest average profit. We restrict our focus in this section to the case where the number of observations is $n = 2$, we consider the case for a larger n in Section 4.4.

We follow the same procedure as in Section 4.3.1 to sample NPI-B, we suppose $m = 1$, since the future demand for this model is equal to $D = D_{n+1} + D_{n+2}$, as we have discussed in Section 2.6. Then we derive the profit based on y and D. We repeat this process N times, and report the maximum average profits and the corresponding inventory level as the optimal value. In the following example, we illustrate this procedure.

	Case	NPI $y^*_{\underline{E}_{\text{Once}}}$	NPI Average profit	NPI-B y^*	NPI-B Average profit
$N = 1000$	I	2.56	73.68	1.87	41.31
	II	2.52	42.30	1.40	19.49
$N = 10,000$	I	2.57	73.85	1.83	39.41
	II	2.57	44.65	1.39	20.42

Table 4.2: Results of Example 4.3.2

Example 4.3.2 Assume $n = 2, c = 23, p = 70, h = 17, s = 9$ and $d_u = 2.50$. We consider the same cases as shown in Example 4.3.1. Our aim is to derive the optimal inventory level by using NPI and NPI-B.

Table 4.2 presents the optimal inventory level for a single order for two-period model based on the NPI and NPI-B methods for two different cases with $n = 2$ and $N = 1000$ and $10,000$. We follow the procedure in Section 2.6 to find $y^*_{\underline{E}_{\text{Once}}}$ for NPI, but in NPI-B, we examine all values of y between $(d_0 = 0, d_u = 2.50)$ then we choose y^* as the value that correspond to the maximum average profit. As a result, the NPI-B approach leads to similar optimal value as the full analytical method. ◇

4.3.3 NPI-B for two-period model with independent demands

In this section we compare different strategies for inventory levels, (y_1, y_2), in which $y_1 \in (d_{1,0}, d_{1,u})$ and $y_2 \in (d_{2,0}, d_{2,u})$, to determine which one leads to the highest average profit. We follow the same procedure as in Section 4.3.1 to sample NPI-B sample but we suppose $m = 2$, since we are dealing with two-period model. Then we derive the total profit based on y_1, y_2, D_{1,n_1+1} and D_{2,n_2+1}. We repeat this process N times, and report the maximum average profits and the corresponding inventory levels as the optimal values. In the following example, we illustrate this procedure.

Example 4.3.3 Assume $n_1 = 10, n_2 = 27$. The cost parameters for the first period are $c_1 = 30, p_1 = 250, h_1 = 50, s_1 = 15$ and $k_1 = 19$. For the second period, the cost parameters are $p'_1 = 120, c_2 = 40, p_2 = 500, h_2 = 100, s_2 = 80$ and $k_2 = 70$, we suppose $\alpha = 0.60$.

Our aim is to derive the optimal inventory levels by using NPI and NPI-B. We consider

	Case	NPI			NPI-B		
		$y^{*O}_{1\underline{E}}$	$y^{*O}_{2\underline{E}}$	Average profit	y^*_1	y^*_2	Average profit
$\boldsymbol{N = 1000}$	I	4.63	4.31	1476.21	4.93	4.91	1454.80
	II	4.36	3.16	929.84	4.86	3.35	890.54
$\boldsymbol{N = 10,000}$	I	4.53	4.26	1474.99	4.90	4.90	1407.99
	II	4.52	3.13	933.54	4.87	3.26	912.86

Table 4.3: Results of Example 4.3.3

two different cases each case is run 1000 and 10,000 times and we report the optimal inventory levels that maximise the average profit resulting from the NPI method and the NPI-B method. Assume for Case I, $d_{1,u} = d_{2,u} = 15$ and for Case II, $d_{1,u} = 15$ and $d_{2,u} = 20$.

- **Case I:** $D_1, D_2 \sim \text{Gamma}(3, 1)$.
- **Case II:** $D_1 \sim \text{Gamma}(3, 1)$ and $D_2 \sim \text{Exp}(\text{rate} = 1/2)$.

Table 4.3 presents the optimal inventory levels for the two-period model based on the NPI and NPI-B methods for two different cases with $n_1 = 10, n_2 = 27$ and $N = 1000$ and $10,000$. We use Equations (3.25) and (3.50) to find $y^{*O}_{2\underline{E}}$ and $y^{*O}_{1\underline{E}}$ for NPI, respectively, but in NPI-B, we examine all values of y_1 and y_2 between $(d_0 = 0, d_{1,u} = 15)$ for Case I. For Case II, we examine all values of y_1 between $(d_0 = 0, d_{1,u} = 15)$ and y_2 between $(d_0 = 0, d_{2,u} = 20)$. Then we choose y^*_1 and y^*_2 as the values that correspond to the maximum average profit. As a result, the NPI-B approach leads to similar optimal values as the full analytical method. $\diamond$

4.4 NPI-B for two-period model with dependent demands

In the NPI approach for the single-order and two-period independent demands models studied in Chapters 2 and 3, we used the M-functions in order to assign the probability masses for the future demands within the intervals.

In this section, future demands are dependent, where we assume that the second future observation D_{n+2} is dependent on the first future observation D_{n+1}, in which the

$N = 1000$			$N = 10,000$		
y_1^*	y_2^*	Average profit	y_1^*	y_2^*	Average profit
5.74	3.71	168.20	5.82	3.64	163.43

Table 4.4: The optimal strategies for Example 4.4.1

future demands D_{n+i} are assumed to come from the same data collection process as the n data observations. When dealing with M-functions for dependent demands, this becomes complicated, as we discussed in Section 2.6. As a result, to avoid the analytic complexities, we will use the NPI-B approach to distinguish between different strategies.

We compare different strategies for inventory levels (y_1, y_2) in which $y_1, y_2 \in (d_0, d_u)$, to determine which one of those is optimal, based on maximising the average profit. This study covers two different models, one model assumes orders for the first period and second period. The other model assumes order only for the first period.

4.4.1 Ordering for both periods

In this section, we consider ordering for both periods, so we aim to determine the best strategies for ordering y_1 and y_2 in (d_0, d_u). To obtain an NPI-B sample, we follow a similar procedure to the one in Section 4.3.3. However, here we suppose dependent demands, D_{n+2} is dependent on D_{n+1}, so we sample the two future observations from the same data collection process as the n data observations. Then we derive the total profit based on y_1, y_2, D_{n+1} and D_{n+2}. We repeat this process N times, and report the maximum average profits and the corresponding inventory levels as the optimal values.

In the following examples, we compare different strategies and find the optimal inventory levels for these strategies by using the NPI-B method. We did not consider the NPI method for the two-period model with dependent demands when the number of observations is greater than 2, we leave this as a topic for future research.

Example 4.4.1 Assume the same number of observations for the two periods, $n = 3$ and $D \sim \text{Gamma}(3, \text{scale} = 1)$, in which the upper bound of the demand is $d_u = 6$. The cost parameters for the first period is as follow $c_1 = 1, p_1 = 40, h_1 = 12, s_1 = 15$, and $k_1 = 9$. For the second period, the cost parameters are $p_1' = 22, c_2 = 11, p_2 = 60, h_2 = 17, s_2 = 8$ and $k_2 = 7$, we suppose $\alpha = 0.60$. The results in this example are based on 1000 and

$N = 1000$			$N = 10,000$		
y_1^*	y_2^*	Average profit	y_1^*	y_2^*	Average profit
5.22	4.43	1484.39	5.24	4.32	1469.64

Table 4.5: The optimal strategies for Example 4.4.2

10,000 simulations.

Our aim is to compare different strategies and decide which one is the optimal based on NPI-B. From Equation (3.1), we calculate the profit or loss depending on y_1, y_2, D_4 and D_5 in which the future observations are the NPI-B sample.

Table 4.4 presents the optimal inventory levels for the first and second period in which $y_1, y_2 \in (0, 6)$ and the average profit based on NPI-B method. We examine all values of y_1 and y_2 between $(0, 6)$, then we choose y_1^* and y_2^* as the values that correspond to the maximum average profit. ◇

Example 4.4.2 Assume $n = 15, D \sim \text{Gamma}(3, \text{scale} = 1)$, in which the upper bound of the demand is $d_u = 10$. The cost parameters for the first period is as follow $c_1 = 20, p_1 = 250, h_1 = 50, s_1 = 15$, and $k_1 = 19$. For the second period, the cost parameters are $p'_1 = 120, c_2 = 40, p_2 = 500, h_2 = 100, s_2 = 80$ and $k_2 = 70$, we suppose $\alpha = 0.60$. The results in this example are based on 1000 and 10,000 simulations.

We follow the same procedure as in Example 4.4.1, to choose the optimal inventory levels.

Table 4.5 presents the optimal inventory levels for the first and second period in which $y_1, y_2 \in (0, 10)$ and the average profit. We examine all values of y_1 and y_2 between $(0, 10)$, then we choose y_1^* and y_2^* as the values that correspond to the maximum average profit. ◇

4.4.2 Ordering in the first period only

In this section, we suppose there is no order for the second period and the stock level at the end of the first period is greater than zero, so, the inventory level for the second period is $y_2 = (y_1 - D_{n+1})^+$. Our aim to determine the best strategies for ordering y_1 between d_0 and d_u.

From Equation (3.5), we calculate the profit or loss depending on $y_1, y_1 - D_{n+1}, D_{n+1}$

	$N = 1000$		$N = 10,000$	
Example	y_1^*	Average profit	y_1^*	Average profit
4.4.1	5.99	143.60	5.99	136.62
4.4.2	8.09	1299.82	8.02	1290.55

Table 4.6: Optimal inventory level (no order for the second period)

	$N = 1000$		$N = 10,000$	
Example	With order	Without order	With order	Without order
4.4.1	5.74	5.99	5.82	5.99
4.4.2	5.22	8.09	5.24	8.02

Table 4.7: Optimal inventory level for the first period with and without order for the second period

and D_{n+2} in which the future observations are the NPI-B sample. We repeat this process N times, and report the maximum average profit and the corresponding inventory level as the optimal value.

In the following example, we find the optimal strategies when we do not order for the second period by maximising the average profit.

Example 4.4.3 Suppose we have the same data sets as in Examples 4.4.1 and 4.4.2. We will find the optimal inventory level for the first period y_1^* when we do not order for the second period.

Table 4.6 presents the optimal inventory level for the first period when we do not order for the second period. We examine all values of y_1 between $(0, 6)$ in Example 4.4.1 and all values of y_1 between $(0, 10)$ in Example 4.4.2, then we choose y_1^* as the value that correspond to the maximum average profit.

To summarise the results, we built Table 4.7. The table illustrates that in all Examples, the optimal inventory level when we order for the first period and do not order for the second period is higher than the optimal inventory level when we order for the first and second periods. $\diamond$

4.5 Concluding remarks

This chapter introduced the use of NPI-B method to distinguish between different order strategies for inventory models. We applied NPI-B to the single-order and two-period independent demands model. In order to find the optimal inventory levels, the average profit is maximised. Based on the numerical examples, the NPI-B approach leads to similar optimal values as the full analytical method.

Also, the NPI-B has also been applied to the two-period model with dependent demands when we order for the second period as well as no order for the second period. To find the optimal inventory levels, the average profit criterion is maximised. Based on the numerical examples, the optimal inventory level for the first period when we order only for the first period is higher than the optimal inventory level when we order for the first and second periods.

The NPI-B can also be applied to the multi-period model in order to compare various ordering strategies; we have not considered here and we leave it as a future topic.

Chapter 5

Conclusions

This chapter gives a brief summary of our most important results and suggests topics for future research. In this thesis, we have introduced nonparametric predictive inference (NPI) to support inventory decisions. First, we applied the NPI method only focusing on the single-period inventory model as discussed in Chapter 2. We explored how to find the optimal inventory level y^*, which maximises the probability of getting a positive profit and maximising the expected profit. We calculated the NPI lower and upper probabilities for the event that the profit of future demand is greater than, or equal to, zero. In addition, we studied the lower and upper expected profits for the next period. We also discovered optimality criteria that combine NPI lower and upper probabilities as well as NPI lower and upper expected profits.

In Chapter 2, the single order for two-period model is considered. In this model we assumed the number of observations is $n = 2$ and the future demand is $D = D_{n+1} + D_{n+2}$. We explored how to find the optimal inventory level, which maximises the expected profit. An investigation of the performance of the classical and NPI methods has been illustrated using different assumptions of the demand via simulation. In some cases, the assumptions underlying the classical method are entirely correct, so, the classical method performs better. In other cases, the NPI method performs better than the classical method. In the simulation studies, we only consider how often the profit is doing better in each run, but it could also be of interest to see by how much it is greater as a topic for future research.

In Chapter 3, we developed NPI for two independent future observations, D_{1,n_1+1} and D_{2,n_2+1}, for the inventory model. First, we derived the optimal inventory level for the second period, assuming there is a remaining stock (or shortage) from the first period, and with that optimal strategy for the second period, we then optimised over the first period. Various cases of ordering for the first and second periods have been studied.

An investigation of the performance of the classical and NPI methods has been illustrated using different assumptions of the demand via simulations. If the assumptions underlying the classical method are entirely correct, the classical method performs better. However, if the assumptions are incorrect, NPI can do better. Also, we discussed the impact of increasing the cost parameters on the NPI method. As a result, if the shortage cost or selling price has increased in any period, we will order more for that period. However, if the holding cost or purchase cost has increased in any period, we will order less for that period. In the simulation study, we only considered the case when we ordered for both periods. It is also interesting to investigate the case when we do not order for either the first or second period.

In Chapter 4, we consider NPI-B method, as an alternative method to deal with complexities in some of the inventory models, and then we discuss its use to predict the optimal inventory level. We applied NPI-B to the single-order and two-period independent demands model which we studied in Chapters 2 and 3. In order to find the optimal inventory levels, the average profit is maximised. Based on the numerical examples, the NPI-B approach leads to similar optimal values as the full analytical method.

Also, the NPI-B has also been applied to the two-period model with dependent demands when we order for the second period as well as no order for the second period. To find the optimal inventory levels, the average profit criterion is maximised. Based on the numerical examples, the optimal inventory level for the first period when we order only for the first period is higher than the optimal inventory level when we order for the first and second periods.

An important topic for future research is to extend the use of NPI-B to the multi-period inventory model. Another idea for future research is to explore models with some restrictions, such as storage capacity and varying order costs. Generally, considering inventory problems from a predictive perspective, in particular how to study the use of NPI-B for the multi-period inventory model, gives interesting new insights which may also have significant practical applications.

Appendix A

Proofs

A.1 Discontinuity proof for the lower expected profit function for the single-period model

From Equation (2.20)

$$\lim_{y\uparrow d_l} \underline{E}_{n+1}(y) = \frac{1}{n+1}\Bigg((l-1)(-(c+h)d_l) + (p+h)\sum_{j=1}^{l-1} d_{j-1} + \min[-(c + h)d_l + (p+h)d_{l-1}, (p-c+s)d_l - sd_l] + (n+1 - l)(p-c+s)d_l - s\sum_{j=l+1}^{n+1} d_j\Bigg)$$

and

$$\lim_{y\downarrow d_l} \underline{E}_{n+1}(y) = \frac{1}{n+1}\Bigg(l(-(c+h)d_l) + (p+h)\sum_{j=1}^{l} d_{j-1} + \min[-(c + h)d_l + (p+h)d_l, (p-c+s)d_l - sd_{l+1}] + (n-l)(p-c+s)d_l - s\sum_{j=l+2}^{n+1} d_j\Bigg)$$

hence

$$\lim_{y\uparrow d_l} \underline{E}_{n+1}(y) - \lim_{y\downarrow d_l} \underline{E}_{n+1}(y) = \frac{1}{n+1}\Bigg((p+h)(d_l - d_{l-1}) - s(d_{l+1} - d_l) + \min[-(c+h)d_l + (p+h)d_{l-1}, (p-c)d_l] - \min[(p-c)d_l, (p-c+s)d_l - sd_{l+1}]\Bigg)$$

So, $\underline{E}_{n+1}(y)$ is a discontinuous function at d_l, for any $l \in \{1, ..., n\}$, since

$$\lim_{y \uparrow d_l} \underline{E}_{n+1}(y) - \lim_{y \downarrow d_l} \underline{E}_{n+1}(y) \neq 0$$

The difference if $y \in (d_{l-1}, d_l)$ or $y \in (d_l, d_{l+1})$ is that there are some bounds of the lower expected profit in these intervals are different which are cause the discontinuity at d_l. When $y \in (d_{l-1}, d_l)$ these bounds are $\frac{1}{n+1}(\min[-(c+h)d_l + (p+h)d_{l-1}, (p-c+s)d_l - sd_l] + (p+h+s)d_l - sd_{l+1})$ while, when $y \in (d_l, d_{l+1})$, $\frac{1}{n+1}(\min[-(c+h)d_l + (p+h)d_l, (p-c+s)d_l - sd_{l+1}] + (p+h)d_{l-1})$. We have four different cases depending on the minimal values in each interval.

If the minimal equal to $-(c+h)d_l+(p+h)d_{l-1}$ for $y \in (d_{l-1}, d_l)$ and $-(c+h)d_l+(p+h)d_l$ for $y \in (d_l, d_{l+1})$, then there is a jump between the left and right hand limit which equal to: $\frac{1}{n+1}(s(d_{l+1} - d_l))$.

If the minimal equal to $-(c+h)d_l+(p+h)d_{l-1}$ for $y \in (d_{l-1}, d_l)$ and $(p-c+s)d_l - sd_{l+1}$ for $y \in (d_l, d_{l+1})$, then there is no different bounds of the lower expected profit so, there is no jump between the left and right hand limit.

If the minimal equal to $(p-c+s)d_l - sd_l$ for $y \in (d_{l-1}, d_l)$ and $-(c+h)d_l + (p+h)d_l$ for $y \in (d_l, d_{l+1})$, then there is a jump between the left and right hand limit which equal to: $\frac{1}{n+1}((p+h)(d_{l-1} - d_l) + s(d_{l+1} - d_l))$.

If the minimal equal to $(p-c+s)d_l - sd_l$ for $y \in (d_{l-1}, d_l)$ and $(p-c+s)d_l - sd_{l+1}$ for $y \in (d_l, d_{l+1})$, then there is a jump between the left and right hand limit which equal to: $\frac{1}{n+1}((p+h)(d_{l-1} - d_l))$.

A.2 Continuity proof for the upper expected profit function for the single-period model

From Equation (2.26)

$$\lim_{y \uparrow d_l} \overline{E}_{n+1}(y) = \lim_{y \downarrow d_l} \overline{E}_{n+1}(y)$$

where,

$$\lim_{y\uparrow d_l}\overline{E}_{n+1}(y) = \frac{1}{n+1}\Bigg(d_l[-l(p+s+h)+h+p+(p-c+s)(1+n)]+(p+h)\sum_{j=1}^{l-1}d_j - s\sum_{j=l+1}^{n+1}d_{j-1}\Bigg)$$

$$= \frac{1}{n+1}\Bigg(d_l[-l(p+s+h)+h+p+(p-c+s)(1+n)]+(p+h)(\sum_{j=1}^{l}d_j - d_l) - s(d_l+\sum_{j=l+2}^{n+1}d_{j-1})\Bigg)$$

$$= \frac{1}{n+1}\Bigg(d_l[-l(p+s+h)+h+p+(p-c+s)(1+n)]-(p+h+s)d_l + (p+h)\sum_{j=1}^{l}d_j - s\sum_{j=l+2}^{n+1}d_{j-1}\Bigg)$$

and

$$\lim_{y\downarrow d_l}\overline{E}_{n+1}(y) = \frac{1}{n+1}\Bigg(d_l[-(l+1)(p+s+h)+h+p+(p-c+s)(1+n)]+(p+h)\sum_{j=1}^{l}d_j - s\sum_{j=l+2}^{n+1}d_{j-1}\Bigg)$$

$$= \frac{1}{n+1}\Bigg(d_l[-l(p+s+h)-(p+s+h)+h+p+(p-c+s)(1+n)] + (p+h)\sum_{j=1}^{l}d_j - s\sum_{j=l+2}^{n+1}d_{j-1}\Bigg)$$

The difference between if y at the left of d_l and y at the right of d_l is that d_l will cost shortage cost since $y < d_l$, so the profit will be $py - cy - s(d_l - y)$, while, d_l will cost holding cost since $y > d_l$, so the profit will be $pd_l - cy - h(y - d_l)$. Hence, when we suppose $y = d_l$ we will find that there is no jump between the left and right hand limit.

A.3 Discontinuity proof of $H_{E,\omega}(E_{n+1}(y))$ for the single-period model

From Equation (2.29)

$$\lim_{y\uparrow d_l} H_{E,\omega}(E_{n+1}(y)) = \frac{\omega}{n+1}\Bigg[-(p-c)d_l + (p+h)(d_0 - d_{l-1}) - s(d_{n+1} - d_l) + \min[-(c + h)d_l + (p+h)d_{l-1}, (p-c+s)d_l - sd_l]\Bigg] + \frac{1}{n+1}\Bigg(y[(l-1)(-(c+h)) + (p-c) + (n+1-l)(p-c+s)] + (p+h)\sum_{j=1}^{l-1} d_j - s\sum_{j=l+1}^{n+1} d_{j-1}\Bigg)$$

and

$$\lim_{y\downarrow d_l} H_{E,\omega}(E_{n+1}(y)) = \frac{\omega}{n+1}\Bigg[-(p-c)d_l + (p+h)(d_0 - d_l) - s(d_{n+1} - d_{l+1}) + \min[-(c + h)d_l + (p+h)d_l, (p-c+s)d_l - sd_{l+1}]\Bigg] + \frac{1}{n+1}\Bigg(y[l(-(c+h)) + (p-c) + (n-l)(p-c+s)] + (p+h)\sum_{j=1}^{l} d_j - s\sum_{j=l+2}^{n+1} d_{j-1}\Bigg)$$

hence

$$\lim_{y\uparrow d_l} H_{E,\omega}(E_{n+1}(y)) - \lim_{y\downarrow d_l} H_{E,\omega}(E_{n+1}(y)) = \frac{\omega}{n+1}\Bigg[(p+h)(d_l - d_{l-1}) - s(d_{l+1} - d_l) + \min[-(c+h)d_l + (p+h)d_{l-1}, (p-c)d_l] - \min[(p-c)d_l, (p-c+s)d_l - sd_{l+1}]\Bigg] + \frac{1}{n+1}\Bigg((p+h+s)(y-d_l)\Bigg)$$

So, $H_{E,\omega}(E_{n+1}(y))$ is a discontinuous function at d_l, for any $l \in \{1, ..., n\}$, since

$$\lim_{y\uparrow d_l} H_{E,\omega}(E_{n+1}(y)) - \lim_{y\downarrow d_l} H_{E,\omega}(E_{n+1}(y)) \neq 0$$

A.4 Optimal inventory level of Gamma and Exponential distributions

The following proof is provided to illustrate that Exponential and Gamma distributions with the same expected value lead to the same optimal inventory level, y^*_{CP}, that maximises the probability that the profit is greater than or equal to zero.

Suppose $D \sim \text{Gamma}(\text{shape} = k, \text{scale} = \theta)$, the mean is equal to $k\theta$. By substituting Equations (2.3) and (2.4) in Equation (2.6), we have,

$$\left(\frac{p+s-c}{s}\right)\frac{1}{\Gamma k\theta^k}d_r^{k-1}e^{\frac{-d_r}{\theta}} = \left(\frac{h+c}{p+h}\right)\frac{1}{\Gamma k\theta^k}d_l^{k-1}e^{\frac{-d_l}{\theta}}$$

$$\left(\frac{p+s-c}{s}\right)\left(\frac{(p+s-c)y}{s}\right)^{k-1}e^{\frac{-y}{\theta}\left(\frac{p+s-c}{s}\right)} = \left(\frac{h+c}{p+h}\right)\left(\frac{(h+c)y}{p+h}\right)^{k-1}e^{\frac{-y}{\theta}\left(\frac{h+c}{p+h}\right)}$$

$$\left(\frac{p+s-c}{s}\right)^{k}e^{\frac{-y}{\theta}\left(\frac{p+s-c}{s}\right)} = \left(\frac{h+c}{p+h}\right)^{k}e^{\frac{-y}{\theta}\left(\frac{h+c}{p+h}\right)}$$

$$k\ln\left(\frac{p+s-c}{s}\right) - \frac{y}{\theta}\left(\frac{p+s-c}{s}\right) = k\ln\left(\frac{h+c}{p+h}\right) - \frac{y}{\theta}\left(\frac{h+c}{p+h}\right)$$

$$k\left[\ln\left(\frac{p+s-c}{s}\right) - \ln\left(\frac{h+c}{p+h}\right)\right] = \frac{y}{\theta}\left(\frac{(p+s-c)(h+p)-s(h+c)}{s(p+h)}\right)$$

$$\Rightarrow y^*_{CP} = k\theta\left(\frac{s(p+h)}{(p+s-c)(h+p)-s(h+c)}\right)\ln\left(\frac{(p+s-c)(p+h)}{s(h+c)}\right)$$

Similarly, for $D \sim \text{Exp}(\text{rate} = \lambda)$, as the Exponential distribution is a special case of Gamma, so when shape $= k = 1$, and scale $= \theta = \frac{1}{\lambda}$, we will get the results for the Exponential. So, Exponential and Gamma distributions with the same expected value lead to the same optimal inventory level.

A.5 Discontinuity proof for the lower expected profit function for the second period, with remaining stock from the first period

From Equation (3.20)

$$\lim_{y_2 \uparrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[(l-1)[-(c_2+h_2)d_{2,l} + c_2(y_1 - d_1) - k_2] + (p_2+h_2)\sum_{j=1}^{l-1} d_{2,j-1} + \min[(p_2+h_2)d_{2,l-1} - (c_2+h_2)d_{2,l} + c_2(y_1-d_1) - k_2, (p_2-c_2+s_2)d_{2,l} - s_2 d_{2,l} + c_2(y_1-d_1) - k_2] + (n_2+1-l)[(p_2-c_2+s_2)d_{2,l} + c_2(y_1-d_1) - k_2] - s_2\sum_{j=l+1}^{n_2+1} d_{2,j}\Bigg]$$

and

$$\lim_{y_2 \downarrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2 + 1} \Bigg[l[-(c_2 + h_2)d_{2,1} + c_2(y_1 - d_1) - k_2] + (p_2 + h_2) \sum_{j=1}^{l} d_{2,j-1} + \min[(p_2 + h_2)d_{2,l} - (c_2 + h_2)d_{2,1} + c_2(y_1 - d_1) - k_2, (p_2 - c_2 + s_2)d_{2,1} - s_2 d_{2,l+1} + c_2(y_1 - d_1) - k_2] + (n_2 - l)[(p_2 - c_2 + s_2)d_{2,1} + c_2(y_1 - d_1) - k_2] - s_2 \sum_{j=l+2}^{n_2+1} d_{2,j} \Bigg]$$

hence

$$\lim_{y_2 \uparrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) - \lim_{y_2 \downarrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2 + 1} \Bigg[(p_2 + h_2 + s_2)d_{2,l} - (p_2 + h_2)d_{2,l-1} - s_2 d_{2,l+1} + \min[(p_2 + h_2)d_{2,l-1} - (c_2 + h_2)d_{2,l} + c_2(y_1 - d_1) - k_2, (p_2 - c_2 + s_2)d_{2,l} - s_2 d_{2,l} + c_2(y_1 - d_1) - k_2] - \min[(p_2 + h_2)d_{2,l} - (c_2 + h_2)d_{2,1} + c_2(y_1 - d_1) - k_2, (p_2 - c_2 + s_2)d_{2,1} - s_2 d_{2,l+1} + c_2(y_1 - d_1) - k_2] \Bigg]$$

So, $\underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1))$ is a discontinuous function at $d_{2,l}$, for any $l \in \{1, ..., n_2\}$, since

$$\lim_{y_2 \uparrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) - \lim_{y_2 \downarrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2 | y_1 = y_1, D_1 = d_1)) \neq 0$$

A.6 Discontinuity proof for the lower expected profit function for the second period, with the first period's demand not fully met

From Equation (3.29)

$$\lim_{y_2 \uparrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[(l-1)[-(c_2+h_2)d_{2,l} + \alpha(d_1 - y_1)(p_1' - c_2) - k_2] + (p_2+h_2)\sum_{j=1}^{l-1} d_{2,j-1} + \min[(p_2+h_2)d_{2,l-1} - (c_2+h_2)d_{2,l} + \alpha(d_1-y_1)(p_1'-c_2) - k_2, (p_2 - c_2 + s_2)d_{2,l} - s_2 d_{2,l} + \alpha(d_1-y_1)(p_1'-c_2) - k_2] + (n_2+1-l)[(p_2-c_2+s_2)d_{2,l} + \alpha(d_1-y_1)(p_1'-c_2) - k_2] - s_2\sum_{j=l+1}^{n_2+1} d_{2,j}\Bigg]$$

and

$$\lim_{y_2 \downarrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg(l[-(c_2+h_2)d_{2,1} + \alpha(d_1 - y_1)(p_1' - c_2) - k_2] + (p_2+h_2)\sum_{j=1}^{l} d_{2,j-1} + \min[(p_2+h_2)d_{2,l} - (c_2+h_2)d_{2,1} + \alpha(d_1-y_1)(p_1'-c_2) - k_2, (p_2 - c_2 + s_2)d_{2,1} - s_2 d_{2,l+1} + \alpha(d_1-y_1)(p_1'-c_2) - k_2] + (n_2-l)[(p_2-c_2+s_2)d_{2,1} + \alpha(d_1-y_1)(p_1'-c_2) - k_2] - s_2\sum_{j=l+2}^{n_2+1} d_{2,j}\Bigg)$$

hence

$$\lim_{y_2 \uparrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) - \lim_{y_2 \downarrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[(p_2+h_2+s_2)d_{2,l} - (p_2+h_2)d_{2,l-1} - s_2 d_{2,l+1} + \min[(p_2+h_2)d_{2,l-1} - (c_2+h_2)d_{2,l} + \alpha(d_1-y_1)(p_1'-c_2) - k_2, (p_2-c_2+s_2)d_{2,l} - s_2 d_{2,l} + \alpha(d_1-y_1)(p_1'-c_2) - k_2] - \min[(p_2+h_2)d_{2,l} - (c_2+h_2)d_{2,1} + \alpha(d_1-y_1)(p_1'-c_2) - k_2, (p_2-c_2+s_2)d_{2,1} - s_2 d_{2,l+1} + \alpha(d_1-y_1)(p_1'-c_2) - k_2]\Bigg]$$

So, $\underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1))$ is a discontinuous function at $d_{2,l}$, for any $l \in \{1, ..., n_2\}$, since

$$\lim_{y_2 \uparrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) - \lim_{y_2 \downarrow d_{2,l}} \underline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) \neq 0$$

A.7 Continuity proof for the upper expected profit function for the second period, with remaining stock from the first period

From Equation (3.38)

$$\lim_{y_2 \uparrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2 + 1}\Bigg[l[-(p_2 + h_2 + s_2)d_{2,l}] + (p_2 + h_2)[d_{2,l} + \sum_{j=1}^{l-1} d_{2,j}] + (n_2 + 1)[(p_2 - c_2 + s_2)d_{2,l} + c_2(y_1 - d_1) - k_2] - s_2 \sum_{j=l+1}^{n_2+1} d_{2,j-1}\Bigg]$$

and

$$\lim_{y_2 \downarrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2 + 1}\Bigg[(l + 1)[-(p_2 + h_2 + s_2)d_{2,l}] + (p_2 + h_2)[d_{2,l} + \sum_{j=1}^{l} d_{2,j}] + (n_2 + 1)[(p_2 - c_2 + s_2)d_{2,l} + c_2(y_1 - d_1) - k_2] - s_2 \sum_{j=l+2}^{n_2+1} d_{2,j-1}\Bigg]$$

hence

$$\lim_{y_2 \uparrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) - \lim_{y_2 \downarrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = 0$$

A.8 Continuity proof for the upper expected profit function for the second period, with the first period's demand not fully met

From Equation (3.42)

$$\lim_{y_2 \uparrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[l[-(p_2 + h_2 + s_2)d_{2,l}] + (p_2 + h_2)[d_{2,l} + \sum_{j=1}^{l-1} d_{2,j}] + (n_2+1)[(p_2 - c_2 + s_2)d_{2,l} + \alpha(d_1 - y_1)(p_1' - c_2) - k_2] - s_2 \sum_{j=l+1}^{n_2+1} d_{2,j-1}\Bigg]$$

and

$$\lim_{y_2 \downarrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = \frac{1}{n_2+1}\Bigg[(l+1)[-(p_2 + h_2 + s_2)d_{2,l}] + (p_2 + h_2)[d_{2,l} + \sum_{j=1}^{l} d_{2,j}] + (n_2+1)[(p_2 - c_2 + s_2)d_{2,l} + \alpha(d_1 - y_1)(p_1' - c_2) - k_2] - s_2 \sum_{j=l+2}^{n_2+1} d_{2,j-1}\Bigg]$$

hence

$$\lim_{y_2 \uparrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) - \lim_{y_2 \downarrow d_{2,l}} \overline{E}(Pf^2(D_1, D_2, y_1, y_2|y_1 = y_1, D_1 = d_1)) = 0$$

A.9 Discontinuity proof for the lower expected profit function for the two-period model

From Equation (3.47)

$$\lim_{y_1 \uparrow d_{1,l}} \underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}})) = \frac{1}{n_1+1}\Bigg[(l-1)(-(c_1+h_1-c_2)d_{1,l}-k_1)$$
$$+ (p_1+h_1-c_2)\sum_{j=1}^{l-1} d_{1,j-1} + \min[-(c_1+h_1-c_2)d_{1,l} - k_1 + (p_1+h_1-c_2)d_{1,l-1}, (p_1$$
$$- c_1 + s_1 - \alpha p'_1 + \alpha c_2)d_{1,l} - k_1 - (s_1 - \alpha p'_1 + \alpha c_2)d_{1,l}] + (n_1+1-l)((p_1 - c_1 + s_1$$
$$- \alpha p'_1 + \alpha c_2)d_{1,l} - k_1) - (s_1 - \alpha p'_1 + \alpha c_2)\sum_{j=l+1}^{n_1+1} d_{1,j}\Bigg] + \frac{1}{n_2+1}\Bigg((j_{2,y^{*O}_{2\underline{E}}} - 1)[-(c_2$$
$$+ h_2)y^{*O}_{2\underline{E}} - k_2] + (p_2+h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\underline{E}}}-1} d_{2,j-1} + \min[(p_2+h_2)d_{2,j_{2,y^{*O}_{2\underline{E}}}-1} - (c_2+h_2)y^{*O}_{2\underline{E}}$$
$$- k_2, (p_2 - c_2 + s_2)y^{*O}_{2\underline{E}} - s_2 d_{2,j_{2,y^{*O}_{2\underline{E}}}} - k_2] + (n_2+1-j_{2,y^{*O}_{2\underline{E}}})[(p_2-c_2+s_2)y^{*O}_{2\underline{E}} - k_2]$$
$$- s_2 \sum_{j=j_{2,y^{*O}_{2\underline{E}}}+1}^{n_2+1} d_{2,j}\Bigg)$$

and

$$\lim_{y_1 \downarrow d_{1,l}} \underline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\underline{E}})) = \frac{1}{n_1+1}\Bigg[l(-(c_1+h_1-c_2)d_{1,l}-k_1)$$
$$+ (p_1+h_1-c_2)\sum_{j=1}^{l} d_{1,j-1} + \min[-(c_1+h_1-c_2)d_{1,l} - k_1 + (p_1+h_1-c_2)d_{1,l}, (p_1$$
$$- c_1 + s_1 - \alpha p'_1 + \alpha c_2)d_{1,l} - k_1 - (s_1 - \alpha p'_1 + \alpha c_2)d_{1,l+1}] + (n_1-l)((p_1 - c_1 + s_1$$
$$- \alpha p'_1 + \alpha c_2)d_{1,l} - k_1) - (s_1 - \alpha p'_1 + \alpha c_2)\sum_{j=l+2}^{n_1+1} d_{1,j}\Bigg] + \frac{1}{n_2+1}\Bigg((j_{2,y^{*O}_{2\underline{E}}} - 1)[-(c_2$$
$$+ h_2)y^{*O}_{2\underline{E}} - k_2] + (p_2+h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\underline{E}}}-1} d_{2,j-1} + \min[(p_2+h_2)d_{2,j_{2,y^{*O}_{2\underline{E}}}-1} - (c_2+h_2)y^{*O}_{2\underline{E}}$$
$$- k_2, (p_2 - c_2 + s_2)y^{*O}_{2\underline{E}} - s_2 d_{2,j_{2,y^{*O}_{2\underline{E}}}} - k_2] + (n_2+1-j_{2,y^{*O}_{2\underline{E}}})[(p_2-c_2+s_2)y^{*O}_{2\underline{E}} - k_2]$$
$$- s_2 \sum_{j=j_{2,y^{*O}_{2\underline{E}}}+1}^{n_2+1} d_{2,j}\Bigg)$$

hence

$$\lim_{y_1\uparrow d_{1,l}} \underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) - \lim_{y_1\downarrow d_{1,l}} \underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) = \frac{1}{n+1}\Big[(p_1+h_1+s_1-\alpha p_1'+(\alpha-1)c_2)d_{1,l} - (p_1+h_1-c_2)d_{1,l-1} - (s_1-\alpha p_1'+\alpha c_2)d_{1,l+1} + \min[-(c_1+h_1-c_2)d_{1,l}-k_1 + (p_1+h_1-c_2)d_{1,l-1}, (p_1-c_1+s_1-\alpha p_1'+\alpha c_2)d_{1,l}-k_1-(s_1-\alpha p_1'+\alpha c_2)d_{1,l}] - \min[-(c_1+h_1-c_2)d_{1,l}-k_1+(p_1+h_1-c_2)d_{1,l}, (p_1-c_1+s_1-\alpha p_1'+\alpha c_2)d_{1,l} - k_1-(s_1-\alpha p_1'+\alpha c_2)d_{1,l+1}]\Big]$$

So, $\underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2 = y^{*O}_{2\underline{E}}))$ is a discontinuous function at $d_{1,l}$, for any $l \in \{1,...,n_1\}$, since

$$\lim_{y_1\uparrow d_{1,l}} \underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) - \lim_{y_1\downarrow d_{1,l}} \underline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\underline{E}})) \neq 0$$

A.10 Continuity proof for the upper expected profit function for the two-period model

From Equation (3.59)

$$\lim_{y_1\uparrow d_{1,l}} \overline{E}(Pf^{1,2}(D_1,D_2,y_1,y_2|y_2=y^{*O}_{2\overline{E}})) = \frac{1}{n_1+1}\Big[(l-1)(-(c_1+h_1-c_2)d_{1,l}-k_1) + (p_1+h_1-c_2)\sum_{j=1}^{l-1} d_{1,j} + (p_1-c_1)d_{1,l} - k_1 + (n_1+1-l)((p_1-c_1+s_1-\alpha p_1' + \alpha c_2)d_{1,l}-k_1) - (s_1-\alpha p_1'+\alpha c_2)\sum_{j=l+1}^{n_1+1} d_{1,j-1}\Big] + \frac{1}{n_2+1}\Big((j_{2,y^{*O}_{2\overline{E}}}-1)[-(c_2+h_2)y^{*O}_{2\overline{E}} - k_2] + (p_2+h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\overline{E}}}-1} d_{2,j} + (p_2-c_2)y^{*O}_{2\overline{E}} - k_2 + (n_2+1-j_{2,y^{*O}_{2\overline{E}}})[(p_2-c_2+s_2)y^{*O}_{2\overline{E}} - k_2] - s_2\sum_{j=j_{2,y^{*O}_{2\overline{E}}}+1}^{n_2+1} d_{2,j-1}\Big)$$

and

$$\lim_{y_1 \downarrow d_{1,l}} \overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\overline{E}})) = \frac{1}{n_1+1}\Bigg[l(-(c_1+h_1-c_2)d_{1,l}-k_1)$$
$$+(p_1+h_1-c_2)\sum_{j=1}^{l} d_{1,j} + (p_1-c_1)d_{1,l} - k_1 + (n_1-l)((p_1-c_1+s_1-\alpha p_1'$$
$$+\alpha c_2)d_{1,l}-k_1) - (s_1-\alpha p_1'+\alpha c_2)\sum_{j=l+2}^{n_1+1} d_{1,j-1}\Bigg] + \frac{1}{n_2+1}\Bigg((j_{2,y^{*O}_{2\overline{E}}}-1)[-(c_2+h_2)y^{*O}_{2\overline{E}}$$
$$-k_2] + (p_2+h_2)\sum_{j=1}^{j_{2,y^{*O}_{2\overline{E}}}-1} d_{2,j} + (p_2-c_2)y^{*O}_{2\overline{E}} - k_2 + (n_2+1-j_{2,y^{*O}_{2\overline{E}}})[(p_2-c_2+s_2)y^{*O}_{2\overline{E}}$$
$$-k_2] - s_2\sum_{j=j_{2,y^{*O}_{2\overline{E}}}+1}^{n_2+1} d_{2,j-1}\Bigg)$$

hence

$$\lim_{y_2 \uparrow d_{1,l}} \overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\overline{E}}))$$
$$-\lim_{y_2 \downarrow d_{1,l}} \overline{E}(Pf^{1,2}(D_1, D_2, y_1, y_2|y_2 = y^{*O}_{2\overline{E}})) = 0$$

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