Generalised Symmetries and String Theory

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Abstract

In this thesis, we study the geometric origin of discrete higher-form symmetries and associated anomalies of d-dimensional quantum field theories in terms of defect groups via geometric engineering in M-theory and type IIB string theory by reduction on non-compact spaces X. As a warm-up, we analyze the example of 7d $\mathcal{N} = 1$ SYM theory, where we recover it from a mixed 't Hooft anomaly among the electric 1-form centre symmetry and the magnetic 4-form centre symmetry in the defect group. The case of 5-dimensional SCFTs from M-theory on toric singularities is discussed in detail. In that context, we determine the corresponding 1-form and 2-form defect groups and we explain how to determine the corresponding mixed 't Hooft anomalies from flux non-commutativity. For these theories, we further determine the $d+1$ dimensional Symmetry TFT, or SymTFT for short, by reducing the topological sector of 11d supergravity on the boundary ∂X of the space X. Central to this endeavour is a reformulation of supergravity in terms of differential cohomology, which allows the inclusion of torsion in the cohomology of the space ∂X, which in turn gives rise to the background fields for discrete symmetries.

We further extend our analysis to study the 1-form symmetries of 4-dimensional $\mathcal{N} = 2$ supersymmetric quantum field theories which arise from IIB on hypersurface singularities. The examples we discuss include a broad class of $\mathcal{N} = 2$ theories such as Argyres-Douglas and $D_p^b(G)$ theories. In our computation of the defect groups of hypersurface singularities, we rely on a fundamental result in singularity theory known as Milnor's theorem which establishes a connection between the topology of the hypersurface and the local behaviour of the singularity. For the $D_p^b(G)$ theories, in the simple case when $b = h^\vee(G)$, we use the BPS quivers of the theory to see that the defect group is compatible with a known Maruyoshi-Song flow. To extend to the case where $b \neq h^\vee(G)$, we use a similar Maruyoshi-Song flow to conjecture that the defect groups of $D_p^b(G)$ theories are given by those of $G^{(b)}[k]$ theories. In the cases of $G = A_n,\ E_6,\ E_8$ we cross-check our result by calculating the BPS quivers of the $G^{(b)}[k]$ theories and looking at the cokernel of their intersection matrix.

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List of Publications

This thesis contains material by the author that has appeared in the following publications:

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Contents

Nothing in physics seems so hopeful to as the idea that it is possible for a theory to have a high degree of symmetry was hidden from us in everyday life. The physicist's task is to find this deeper symmetry.

STEVEN WEINBERG

1 Introduction

Recently, there has been significant interest and progress in understanding the generalisation of ordinary symmetries in quantum field theories (QFTs). Although the idea is an old one explored in [5, 6] in the context of gauge symmetries and [7], the seminal work of Gaiotto, Kapustin, Seiberg, and Willett [8] has been the foundational paper initiating the recent progress on the topic. In ordinary global symmetries, the transformation laws are associated with the symmetry of scalar fields or zero forms. Generalised global symmetries go beyond this and include higher-dimensional charged objects, such as 't Hooft lines and Wilson lines in Yang-Mills theory[1]. These symmetries are characterised by transformations that change these higher-dimensional objects while leaving lower-dimensional physical observables invariant.

In QFTs, global symmetries have a fundamental role in understanding the properties of physical systems. An important example is that of Noether's theorem by Emmy Noether from 1915 which establishes a deep connection between symmetries and conservation laws. Noether's theorem states that for every continuous global symmetry in a physical system, there is a corresponding conserved quantity. Other important examples include a discovery in the 1960s, by theorists, including Philip Anderson, that QFT models with such global symmetries may exhibit spontaneous symmetry breaking. This happens when the equations of a theory are symmetric, but the ground state does not have the same symmetry. It was discovered by Robert J. Goldstone that, in the case of continuous symmetry, this symmetry breaking results in at least one spin-zero massless particle called a Goldstone boson. The latter is known as the Goldstone theorem. The Higgs mechanism, proposed by Peter Higgs and others, is an example of spontaneous symmetry breaking. Although in this example we have the breaking of a gauge symmetry, so the Goldstone boson can be

[1] However, note that the notion of symmetry can be introduced without reference to the objects charged under it. It is only possible to speak of the action of symmetry being faithful or not if one introduces the objects charged under it. In the examples considered in this thesis, the symmetries act faithfully.

set to zero by fixing the gauge (which results in a massive real scalar field and massive vector bosons). Recently, it was realised that these phenomena generalise to the breaking of continuous higher-form symmetries and in some cases, an analogue version of Goldestone's theorem in [8–10].

A further related application is given by anomalies in QFTs from the 1960s and 1970s, by physicists, such as James Bjorken, Sidney Coleman, and Curtis Callan. A famous example is that of the chiral anomaly, which specifically, involves the violation of a chiral symmetry in gauge theories. Recently further elaboration was given on our understanding of this anomaly in [11–13] through the application of generalised global symmetries. Namely, it was realised that the anomaly does not fully break this symmetry and a subset of it survives as a non-invertible symmetry. These non-invertible symmetries lead to selection rules, consistent with expectations with massless quantum electrodynamics and models of axions. Further, they provide an alternative explanation for the neutral pion decay.

As per the above examples we have seen many applications of ordinary symmetries and have started to discover their generalisations to higher-form and non-invertible symmetries. Therefore in this spirit, it is expected that we can also learn much physics from the study of generalised symmetries. Currently, the research on generalised symmetries is still in the early stages of understanding and uncovering the structure of these symmetries themselves. Alongside this, ongoing research aims to extend and explore the implications of generalised symmetries for physics. For example, some recent works on higher symmetries include [1–4, 14–48], and works on more generalised symmetries include [49–56] from a string theory perspective and [57–82] from a field-theoretic or mathematical perspective. Just to give a few examples on applications, their study has provided a deeper understanding of the following phenomena.

Just as in the case of ordinary symmetries, generalised symmetries have corresponding 't Hooft anomalies including mixed anomalies with other types of symmetries. They prevent us from gauging them and lead to 't Hooft anomaly-matching conditions. Examples of these applications are discussed for example in [2, 83]. They also lead to anomaly inflow mechanisms. Therefore, in addition, they provide a powerful framework for characterising and classifying topological phases of matter, such as topological insulators and superconductors, for example, see [84, 85]. There has also been a long interest in generalised global symmetries in the context of quantum gravity. There are conjectures that in a theory of quantum gravity of higher than two dimensions there are no such symmetries present, imposing a strong constraint in determining phenomenologically viable theories [86].

The formulation of generalised symmetries is naturally given in terms of topological operators. These operators are called topological as they remain unchanged under local perturbations to their shape. For example, in a d-dimensional QFT, there is a $(d-n-1)$-dimensional generalised symmetry operator that generates the n-form symmetry acting on n-dimensional defects. These topological structures offer a way to access non-perturbative information in QFTs.

Initially, these global symmetries were studied through the gauge theory description of field theories. However, not all theories have a known weakly-coupled gauge theory description and a corresponding gauge group. One alternative method to study higher-form symmetries is via geometric engineering in string or M-theory. This was initiated in [14, 24] and extended in [1, 2, 44, 47]. This setup is similar to the work of Witten [87]. There are also other similar methods such as brane construction in string theory and holography studied in [48, 88, 89]. These formulations are particularly useful for theories that cannot be realised perturbatively. Furthermore, they provide a systematic classification of certain types of generalised symmetries known as higher form symmetries and their anomalies, while the recent development in the field is demonstrating its great potential in the exploration of the other types.

In this framework, QFTs are obtained by dimensional reduction of string or M-theory on a space called the internal manifold or geometry. Continuous global symmetries are usually rather manifest in terms of the geometry, e.g. the R-symmetry of a 4d $\mathcal{N}=1$ SCFT is often encoded in some geometric isometry (such as in the setup of D3-branes probing Calabi-Yau cones), or the flavour symmetry in terms of non-compact divisors in a Calabi-Yau space. Discrete and continuous higher-form symmetries are encoded also in the topology of the internal space. The topological operators can be generated by wrapping branes on non-compact cycles of the geometry. The resulting higher symmetries may be partially screened by dynamical degrees of freedom obtained from branes wrapped on compact cycles. Through this generalised screening argument, unscreened operators generate a group called the defect group, which captures the electric/magnetic higher-form symmetries as subgroups [14]. Many methods have been developed in physics and applied from mathematics to determine the generalised symmetries for various geometries.

In geometric engineering, computing the defect group for discrete symmetries involves analysing the torsion part of the cohomology groups of the internal geometry, which are associated with the charges carried by the flux operators sourced by branes. The presence of torsional cycles in the internal manifold results in non-commuting fluxes [90, 91]. As a result, upon reduction on the geometry, there are mixed 't Hooft anomalies among different higher-form symmetry factors in the defect group.

These anomalies as well as 't Hooft anomalies associated with higher-form symmetries can also be studied using string theory and M-theory techniques. This is done through the construction of the so-called symmetry topological field theory (SymTFT) in one dimension higher than the engineered field theory [4, 92–94]. In the geometric engineering setup, the SymTFT is obtained by the reduction of the topological sector of M-theory on the link of the singularity of the internal geometry. This involves analysing the supergravity action in differential cohomology formulation to include torsional effects. The reduction results in topological couplings for the background fields of global symmetries.

Plan of the thesis. In this thesis, we focus on studying discrete higher-form symmetries using geometric engineering techniques in string theory and M-theory. Although the theories we consider are supersymmetric or may have conformal symmetry, the understanding and requirement of these properties are not needed for our purposes and are only a result of the specific geometries we consider.

The structure of this thesis is as follows. In the next section, we review the basic required mathematical and physics backgrounds needed for our analysis. This includes a short introduction to differential cohomology in section 2.1, a quick summary of the necessary background in string and M-theory in section 2.2, and an introduction to generalised symmetries in Section 2.3. Next, in section 3, we review the view of higher-form symmetries from the geometric engineering standpoint. This is done by first introducing the general setup and then giving the details of how specifically the symmetries arise in the construction in section 3.2.

As a simpler warm-up example, in section 4.1, we discuss the application of this framework to the case of 7-dimensional gauge theories with simple simply-laced Lie groups $G \in ADE$. This is done by looking at $\mathbb{C}^2/\Gamma_{ADE}$ compactifications in M-theory. In section 4.2, we set the stage for the analysis of the global structure of 5-dimensional SCFTs from M-theory on canonical singularities in Calabi-Yau three-folds. The anomalies of the higher symmetries are determined in these cases, including the mixed 0-1-form symmetry anomaly in the 5-dimensional theory. Moving on, section 5 focuses on the geometric engineering of IIB string theory on hypersurface singularities, resulting in 4-dimensional $\mathcal{N} = 2$ theories. This section includes the study of Argyres-Douglas theories and $D_p^b(G)$ theories, for which we derive 1-form symmetries using BPS quivers and Orlik's theorem. Finally, we conclude this thesis in section 6, where the main findings and contributions of the research are summarised.

2 Geometry, strings and quantum field theories

In this section, we review the relevant background material required for this thesis. We start by reviewing differential cohomology as it is an essential tool in our formulation. In the following subsection, we give a quick introduction to string theory and flux non-commutativity. Finally, in the last subsection, we review the concept of generalised symmetry which is one of the most important concepts needed for this thesis.

2.1 Differential cohomology

This thesis is concerned with the study of symmetries via geometric engineering. In this setup, the existence of non-trivial symmetries requires non-trivial topology, notably including torsional cycles, in string theory spacetime. Put differently, the symmetries are generated by higher-form gauge fields of non-trivial topology in string theory. The precise mathematical framework to discuss these fields is conjectured to be given by differential cohomology. This was originally introduced by Cheeger and Simons in mathematics literature [95]. There are alternative formulations given by Hopkins and Singer [96] and Deligne [97]. In this section, we briefly review the Cheeger and Simons formulation where part of the material is copied from the author's work in [4]. We refer the reader to [91, 96, 98–105] for further details.

It should be noted that differential cohomology has seen numerous applications within quantum field theory and string/M-theory. Some of the earlier works on the subject include [90, 91, 106–108]. For more recent examples of differential cohomology applications in formal high-energy physics that are of some relevance to this work see [48, 105, 109–117]. In this thesis, differential cohomology will be used to refine the notion of dimensional reduction (or KK-reduction) of supergravity theories, with the goal of providing a precise treatment of the effect of torsion cohomology classes in the compactification manifold.

The *differential cohomology group* of a space $\mathcal{M}$ is denoted by $\check{H}^p(\mathcal{M})$. Physically, it is the set of gauge inequivalent $U(1)$ $(p-1)$-form fields. Let us take the spacetime $\mathcal{M}$ to be a closed, connected and oriented manifold. Mathematically, Cheeger and Simons define it as follows.

Denote chains, cycles and boundaries on a space M by $C_*(M)$, $Z_*(M)$ and $B_*(M)$, respectively.

Definition 1. A degree p *Cheeger-Simons differential character*, also known as the *holonomy function*, on a manifold $\mathcal{M}$ is a homomorphism $\chi : Z_{p-1}(\mathcal{M}) \to U(1)$, for which there is a differential form $F \in \Omega^p(M)$ such that if $\Sigma = \partial B \in B_{p-1}(\mathcal{M})$ is the boundary of B, $\forall B \in C_p(\mathcal{M})$, then

$$\chi(\Sigma) = \exp\left(2\pi i \int_B F\right) . \tag{2.1}$$

The Cheeger-Simons differential characters form an Abelian group under multiplication $\chi(\Sigma)\chi(\Sigma') = \chi(\Sigma + \Sigma')$ denoted by $\check{H}^p(\mathcal{M})$. It is called the p-th differential cohomology group of $\mathcal{M}$.

The p-form F is identified with the physical *field strength* of the $(p-1)$-form gauge field.

The above definition depends on the choice of Σ and not B. Integrating over two boundary manifolds B and B' with $\Sigma = \partial B = \partial B'$, gives the same holonomy

$$\int_B F = \int_{B'} F' \mod 1 \,, \tag{2.2}$$

and so

$$0 = \int_{B-B'=\partial B} (F - F') = \int_B dF \mod 1 \,. \tag{2.3}$$

This implies $dF = 0$ or $F \in \Omega^p_{\text{closed}}$. In addition, observe that $F \in \Omega^p_{\mathbb{Z}}$ is an integrally quantised form

$$\int_{B\cup(-B')} F \in \mathbb{Z} \tag{2.4}$$

for two chains B and B' forming a cycle in $Z_p(\mathcal{M})$. The fact that F has integral periods encodes that the gauge group is $U(1)$ and not $\mathbb{R}$.

Given any $\chi \in \check{H}^p(\mathcal{M})$, then the field strength $F \in \Omega^p_{\mathbb{Z}}$ is uniquely determined by χ. Let B be an any p-chain in $\mathcal{M}$, and assume that $\chi(\Sigma) = \exp\left(2\pi i \int_B F\right)$ and

$\chi(\Sigma) = \exp\left(2\pi i \int_B F'\right)$. Then $\int_B (F - F') = 0$, so $F = F'$ as we have chosen an arbitrary chain B. This leads to the following definition.

Definition 2. The surjective map

$$R : \check{H}^p(\mathcal{M}) \to \Omega^p_{\mathbb{Z}}(\mathcal{M})\,. \tag{2.5}$$

defines a homomorphism called the *curvature map*.

Consider the case where $\Sigma \neq \partial B$ and rather $\Sigma \in C_{p-1}(\mathcal{M})$. Then, what is $\chi(\Sigma)$? To answer this question, redefine the holonomy function χ in an additive notation by taking

$$h = \frac{1}{2\pi i} log\chi : Z_{p-1}(\mathcal{M}) \to \mathbb{R}/\mathbb{Z}$$

so that

$$h(\partial B) = \int_B F \mod 1 \in \mathbb{R}/\mathbb{Z}\,. \tag{2.6}$$

Using the exact sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, we may arbitrarily lift $h \in \check{H}^p(\mathcal{M}) \subset \mathrm{Hom}(Z_{p-1}, \mathbb{R}/\mathbb{Z})$ to $\tilde{h} \in \mathrm{Hom}(Z_{p-1}(\mathcal{M}), \mathbb{R})$ along the quotient $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Further we may extend $\tilde{h}$ to chains by the map

$$\begin{array}{cccc} \tilde{h} : & C_{p-1}(\mathcal{M}) & \to & \mathbb{R} \\ & \Sigma & \mapsto & \int_\Sigma \tilde{h} \end{array} \tag{2.7}$$

Using Stoke's theorem we have $\tilde{h}(\partial B) = \int_{\partial B} \tilde{h} = \int_B \delta\tilde{h} = \delta\tilde{h}(B)$, so we may define the function $\delta\tilde{h} := \tilde{h} \circ \partial \in C^p(\mathcal{M}; \mathbb{R})$. From the exact sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, two lifts of h to $\tilde{h}$ differ by an element in $C^p(\mathcal{M}; \mathbb{Z})$. So there must be a unique integral cochain $a \in C^p(\mathcal{M}; \mathbb{Z})$ such that

$$\int_B \delta\tilde{h} = \tilde{h}(\partial B) = \int_B (F - a)\,. \tag{2.8}$$

Then,

$$\delta\tilde{h} = F - a\,. \tag{2.9}$$

Note that, $\delta a = \delta F - \delta^2\tilde{h} = 0$, and so $a \in Z^p(\mathcal{M})$ or $[a] \in H^p(\mathcal{M})$[1]. This leads to the following definition.

[1]Unless otherwise specified, all of our (co)homology groups are with integer coefficients, so $H^n(X) := H^n(X; \mathbb{Z})$.

Definition 3. The surjective map

$$I : \breve{H}^p(M) \to H^p(M) \tag{2.10}$$

defines a homomorphism that "forgets" the differential refinement, yielding back ordinary cohomology with coefficients in $\mathbb{Z}$. It is called the *charactristic class map*. The class $[a] = I(\chi)$ determines the topological class of χ and is called the *characteristic class* of χ.

Notation. In a more verbose notation, we can represent an element of $\breve{H}^p(\mathcal{M})$ as a triplet

$$\breve{a} = (a, \tilde{h}, F) \in Z^p(\mathcal{M};\mathbb{Z}) \times C^{p-1}(\mathcal{M};\mathbb{R}) \times \Omega^p_{\mathbb{Z}}(\mathcal{M})\,, \tag{2.11}$$

with the condition that $\delta\tilde{h} = F - a$. If a is trivial then the field $\breve{a}$ is called *topologically trivial.* If F is trivial, it is called a *flat* field. The notation $\breve{\cdot}$ stresses that there is generally no well-defined gauge field potential for a topologically non-trivial field.

Gauge redundancies. Again extending $\tilde{h}$ from cycles to chains, then $\tilde{h}$ in the triplet notation (2.11) has an ambiguity $\tilde{h} \to \tilde{h} + \delta c$ where $c \in C^{p-2}(\mathcal{M};\mathbb{R})$ and an ambiguity $\tilde{h} \to \tilde{h} + d$ where $d \in C^{p-1}(\mathcal{M})$. Thus, $\breve{a}$ has the ambiguity

$$(a, \tilde{h}, F) \to (a - \delta d, \tilde{h} + \delta c + d, F)\,. \tag{2.12}$$

The group $\breve{H}^p(\mathcal{M})$ sits at the center of the following commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{Tor}H^p(\mathcal{M}) \arrow[dr, "u"] & & \\
 & H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z}) \arrow[ur, "t"] \arrow[rr, "-\beta"] \arrow[dr, hook, "i"] & & H^p(\mathcal{M}) \arrow[dr, "\varrho"] & \\
 \frac{H^{p-1}(\mathcal{M};\mathbb{R})}{H^{p-1}_{\mathrm{Free}}(\mathcal{M})} \arrow[ur] \arrow[dr] & & \breve{H}^p(\mathcal{M}) \arrow[ur, "I"] \arrow[dr, "R"] & & H^p_{\mathrm{Free}}(\mathcal{M}) \\
 & \frac{\Omega^{p-1}(\mathcal{M})}{\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})} \arrow[ur, hook, "\tau"] \arrow[rr, "d_{\mathbb{Z}}"] \arrow[dr, "d"] & & \Omega^p_{\mathbb{Z}}(\mathcal{M}) \arrow[ur, "r"] & \\
 & & d\Omega^{p-1}(\mathcal{M}) \arrow[ur] & &
\end{tikzcd} \tag{2.13}$$

where all the diagonals are short exact sequences.

The maps i, I, τ, R are natural (that is, given a smooth map $f\colon \mathcal{M} \to \mathcal{M}'$ they commute with the pullback f^* of f). Let us now proceed to unpack the relevant information contained in the above diagram, and to provide some physical interpretation:

- The exactness of the central NW-SE diagonal in the diagram (2.13) demonstrates that flat elements of $\breve{H}^p(\mathcal{M})$ can be identified with elements in $H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z})$. Physically, the gauge-invariant information about a flat $(p-1)$-form gauge field is encoded in its holonomies around non-trivial $(p-1)$-cycles, which take values in $U(1) \cong \mathbb{R}/\mathbb{Z}$ and can be encoded in an element of $H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z})$.
- The exactness of the central SW-NE diagonal in the diagram (2.13) implies that topologically trivial elements of $\breve{H}^p(\mathcal{M})$ can be identified with globally defined $(p-1)$-forms on $\mathcal{M}$, up to additive shifts by closed $(p-1)$-forms with integral periods. In physics term, a topologically trivial $(p-1)$-form gauge field can be described globally by specifying a $(p-1)$-form. The shift by closed $(p-1)$-forms with integral periods is interpreted as a gauge transformation (a "large gauge transformation" if the $(p-1)$-form is closed but not exact, and a "small gauge transformation" if exact).
- Commutativity of the square on the RHS of the diagram (2.13) is the statement that, for any $\breve{a} \in \breve{H}^p(\mathcal{M})$,

  $$r\big(R(\breve{a})\big) = \varrho\big(I(\breve{a})\big) \ . \tag{2.14}$$

  The short exact sequence in the lower NW-SE diagonal of (2.13) comes from the isomorphism[2] $\Omega^p_{\mathbb{Z}}(\mathcal{M})/d\Omega^{p-1}(\mathcal{M}) \cong H^p_{\text{free}}(\mathcal{M})$ which is a by-product of de Rham's theorem. From the physics perspective, it is well-known that information about the topological aspects of a $(p-1)$-form gauge field configuration can be extracted from its field strength (for example, the integer charge of a monopole configuration for a $U(1)$ 1-form gauge field on $\mathcal{M} = S^2$ is extracted integrating the 2-form field strength on S^2). Crucially, however, the field strength encodes only $\varrho(I(\breve{a}))$ and not necessarily $I(\breve{a})$. To see this, let $I(\breve{a}) = [a] \in H^p(\mathcal{M})$ and embed $\mathbb{Z}$ into $\mathbb{R}$ to get $[a]_{\mathbb{R}} \in H^p(\mathcal{M};\mathbb{R})$. Then, for

[2] In this thesis, it is occasionally convenient to represent isomorphism using the equality symbol as it is common in the physics literature, even though it deviates from standard mathematical notation.

a de Rham cohomology class $[F]_{\rm dR} \in H^p_{\rm Free}(\mathcal{M}) \otimes \mathbb{R}$ of $F \in \Omega^p_{\mathbb{Z}}(\mathcal{M})$ we have $[F]_{\rm dR} = [a]_{\mathbb{R}}$. Thus, $[a]$ contains more information than $[F]_{\rm dR}$ at the differential level since $[a]_{\mathbb{R}}$ can be obtained from $[a]$ but the converse is not true. In particular, information about torsional components in $I(\breve{a})$ is lost in passing to $\varrho(I(\breve{a}))$.

- A flat element in $\breve{H}^p(\mathcal{M})$ is not necessarily topologically trivial. Suppose $\breve{a} \in \breve{H}^p(\mathcal{M})$ is flat; we aim to compute its characteristic class $I(\breve{a})$. From exactness of the NE-SW diagonal we know that $\breve{a} = i(u)$ for some $u \in H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z})$. Commutativity of the upper triangle in the diagram (2.13) gives us

$$I(\breve{a}) = I(i(u)) = -\beta(u) \ . \tag{2.15}$$

Here $\beta : H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z}) \to H^p(\mathcal{M})$ is the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$,

$$\ldots \to H^{p-1}(\mathcal{M}) \xrightarrow{\varrho} H^{p-1}(\mathcal{M};\mathbb{R}) \to H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} H^p(\mathcal{M}) \xrightarrow{\varrho} H^p(\mathcal{M};\mathbb{R}) \to \ldots \tag{2.16}$$

which is in general non-vanishing.

- We can also define the Bockstein homomorphism as

$$\beta : H^{p-1}(\mathcal{M};\mathbb{Z}_k) \to H^p(\mathcal{M})$$

via the embedding $\mathbb{Z}_k \in \mathbb{R}/\mathbb{Z} = U(1)$. This special case is important, especially when we consider a p-form discrete $\mathbb{Z}_k$ gauge field.

- A topologically trivial element in $\breve{H}^p(\mathcal{M})$ is not necessarily flat. Suppose $\breve{a} \in \breve{H}^p(\mathcal{M})$ is topologically trivial; we aim to compute its field strength $R(\breve{a})$. From the exactness of the central SW-NE diagonal we know that $\breve{a} = \tau([\omega])$ for some class $[\omega]$ in the quotient $\Omega^{p-1}(\mathcal{M})/\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})$. Commutativity of the lower triangle in the diagram (2.13) gives us

$$R(\breve{a}) = R(\tau([\omega])) = d_{\mathbb{Z}}[\omega] \ . \tag{2.17}$$

The symbol $d_{\mathbb{Z}}$ in the diagram denotes the standard de Rham differential on forms, which passes to the quotient of $\Omega^{p-1}(\mathcal{M})$ by $\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})$. The relation (2.17) is familiar in physics: if we have a topologically trivial $(p-1)$-form gauge field, described by the globally defined form ω in some gauge, its field strength is simply $d\omega$.

- An element $\breve{a} \in \breve{H}^p(\mathcal{M})$ can be both flat and topologically trivial. Such elements in $\breve{H}^p(\mathcal{M})$ are usually referred to as Wilson lines. A Wilson line in $\breve{H}^p(\mathcal{M})$ can be identified with an element in the quotient $H^{p-1}(\mathcal{M};\mathbb{R})/H^{p-1}_{\text{Free}}(\mathcal{M}) \cong H^{p-1}(\mathcal{M}) \otimes \mathbb{R}/\mathbb{Z}$. The latter is in turn isomorphic to
$$\frac{H^{p-1}(\mathcal{M};\mathbb{R})}{H^{p-1}_{\text{Free}}(\mathcal{M})} \cong \frac{\Omega^{p-1}_{\text{closed}}(\mathcal{M})}{\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})} \, , \tag{2.18}$$
which is a torus of dimension $b^{p-1} = \dim H^{p-1}(\mathcal{M};\mathbb{R})$.

Two differential cohomology classes $\breve{a}, \breve{b} \in \breve{H}^p(\mathcal{M})$ with $I(\breve{a}) = I(\breve{b})$ necessarily differ by a topologically trivial class. Exactness of the central NW-SE exact sequence in (2.13) then implies that $\breve{a} - \breve{b}$ can be represented by an element in $\Omega^{p-1}(\mathcal{M})/\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})$. We conclude that we can view $\breve{H}^p(\mathcal{M})$ as a fibration with basis the set of points in $H^p(\mathcal{M})$, and fiber isomorphic to $\Omega^{p-1}(\mathcal{M})/\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})$:

$$\begin{tikzcd} \Omega^{p-1}(\mathcal{M})/\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M}) \arrow[r] & \breve{H}^p(\mathcal{M}) \arrow[d] \\ & H^p(\mathcal{M}) \end{tikzcd} \; . \tag{2.19}$$

Concretely, if we pick some origin $\breve{\Phi}$ for the fiber on top of $I(\breve{\Phi})$, we can write the most general element $\breve{a}$ of the fiber as

$$\breve{a} = \breve{\Phi} + \tau([\omega]) \; , \tag{2.20}$$

where $\omega \in \Omega^{p-1}(X)$ is a differential form representing a class $[\omega]$ in the quotient of $\Omega^{p-1}(\mathcal{M})$ by $\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})$. As pointed out above, a different choice for ω in the same class $[\omega]$ is simply a gauge transformation.

Torsion Classes. Let us consider a torsion cohomology class $t \in H^p(\mathcal{M})$. It will be useful for us to choose a convenient origin $\breve{\Phi}$ for the fiber on top of t. By exactness of the long exact sequence (2.16), we have that if $t \in H^p(\mathcal{M})$ is torsion then there is some (not necessarily unique) $u \in H^{p-1}(\mathcal{M};\mathbb{R}/\mathbb{Z})$ such that $t = -\beta(u)$. Our choice for the origin of the fiber above t is $\breve{\Phi} = i(u)$. Commutativity of (2.13) ensures $I(\breve{\Phi}) = t$, confirming indeed that $\breve{\Phi}$ lies in the fiber on top of t. Moreover, the differential cohomology class $\breve{\Phi}$ is flat, $R(\breve{\Phi}) = 0$, as follows from exactness of the central NW-SE diagonal in (2.13).

Product structure in differential cohomology. There exists a bilinear product operation on differential cohomology classes,

$$\star: \quad \breve{H}^p(\mathcal{M}) \times \breve{H}^q(\mathcal{M}) \to \breve{H}^{p+q}(\mathcal{M}) \ . \tag{2.21}$$

The product $\star$ is natural and satisfies the following identities: for any $\breve{a} \in \breve{H}^p(\mathcal{M})$, $\breve{b} \in \breve{H}^q(\mathcal{M})$,

$$\breve{a} \star \breve{b} = (-)^{pq}\, \breve{b} \star \breve{a} \ , \qquad I(\breve{a} \star \breve{b}) = I(\breve{a}) \smile I(\breve{b}) \ , \qquad R(\breve{a} \star \breve{b}) = R(\breve{a}) \wedge R(\breve{b}) \ . \tag{2.22}$$

In the above relations, $\wedge$ is the standard wedge product of differential forms and $\smile$ is the standard cup product of cohomology classes.

The product of a topologically trivial (respectively flat) element in $\breve{H}^p(\mathcal{M})$ with any element in $\breve{H}^q(\mathcal{M})$ is again topologically trivial (respectively flat). More precisely, we have the identities

$$\tau([\omega]) \star \breve{b} = \tau([\omega \wedge R(\breve{b})]) \ , \qquad i(u) \star \breve{b} = i(u \smile I(\breve{b})) \ , \tag{2.23}$$

for any $\omega \in \Omega^{p-1}(\mathcal{M})$, $u \in H^{p-1}(\mathcal{M}; \mathbb{R}/\mathbb{Z})$, and $\breve{b} \in \breve{H}^q(\mathcal{M})$.[3] Recall that $[\omega]$ denotes the equivalence class of ω in $\Omega^{p-1}(\mathcal{M})/\Omega^{p-1}_{\mathbb{Z}}(\mathcal{M})$.

Fiber integration in differential cohomology. Given a locally trivial fiber bundle $\mathcal{M}$ with base $\mathcal{B}$ and closed fiber $\mathcal{F}$, we can define an integration over the fiber

$$\int_{\mathcal{F}}: \qquad \breve{H}^p(\mathcal{M}) \to \breve{H}^{p-\dim(\mathcal{F})}(\mathcal{B}) \, , \tag{2.24}$$

which we can characterise axiomatically. First, it is a natural group homomorphism that is compatible with taking the curvature and taking the characteristic class:

$$\int_{\mathcal{F}} R(\breve{a}) = R\left(\int_{\mathcal{F}} \breve{a}\right) \ , \qquad \int_{\mathcal{F}} I(\breve{a}) = I\left(\int_{\mathcal{F}} \breve{a}\right) \ . \tag{2.25}$$

(On the left hand side of these expressions we are using the usual notions of fiber integration of differential forms and cohomology classes.) It is also compatible with

[3] Using the fact that $\star$, $\wedge$, $\smile$ are graded commutative, these identities can also be written in the form

$$\breve{b} \star \tau([\omega]) = (-1)^q\, \tau([R(\breve{b}) \wedge \omega]) \ , \qquad \breve{b} \star i(u) = (-1)^q\, i(I(\breve{b}) \smile u) \ ,$$

for $\breve{b} \in \breve{H}^q(\mathcal{M})$, $\omega \in \Omega^{p-1}(\mathcal{M})$, $u \in H^{p-1}(\mathcal{M}; \mathbb{R}/\mathbb{Z})$.

the maps i and τ:

$$\int_{\mathcal{F}} i(u) = i\left(\int_{\mathcal{F}} u\right) , \qquad \int_{\mathcal{F}} \tau([\omega]) = \tau\left(\left[\int_{\mathcal{F}} \omega\right]\right) . \tag{2.26}$$

An important special case is when we take $\mathcal{B} = \mathrm{pt}$ and we identify the fiber $\mathcal{F}$ with $\mathcal{M}$ itself. One has $\breve{H}^0(\mathrm{pt}) \cong \mathbb{Z}$, $\breve{H}^1(\mathrm{pt}) \cong \mathbb{R}/\mathbb{Z}$, while $\breve{H}^p(\mathrm{pt})$ is trivial for $p \neq 0, 1$. We then have two non-trivial integration maps. The first is integer-valued and yields the so-called *primary invariant* of a differential cohomology class of degree $\dim(\mathcal{M})$,

$$\int_{\mathcal{M}} \breve{a} = \int_{\mathcal{M}} I(\breve{a}) = \int_{\mathcal{M}} R(\breve{a}) \in \mathbb{Z} , \qquad \breve{a} \in \breve{H}^{\dim(\mathcal{M})}(\mathcal{M}) . \tag{2.27}$$

The second integration operator is valued in $\mathbb{R}/\mathbb{Z}$ and yields the so-called *secondary invariant* of a differential cohomology class of degree $\dim(\mathcal{M}) + 1$,

$$\int_{\mathcal{M}} \breve{a} = \int_{\mathcal{M}} u \in \mathbb{R}/\mathbb{Z} , \quad \breve{a} \in \breve{H}^{\dim(\mathcal{M})+1}(\mathcal{M}) , \quad u \in H^{\dim(\mathcal{M})}(\mathcal{M}; \mathbb{R}/\mathbb{Z}) , \quad \breve{a} = i(u) . \tag{2.28}$$

We have used the fact that any element $\breve{a} \in \breve{H}^{\dim(\mathcal{M})+1}(\mathcal{M})$ is necessarily flat for dimensional reasons, and therefore can be written as $\breve{a} = i(u)$ for some $u \in H^{\dim(\mathcal{M})}(\mathcal{M}; \mathbb{R}/\mathbb{Z})$.

2.2 String theory and M-theory

In this section, we give a brief review of string theory. String theory, with over four decades of research, is a rich and expansive field of study that encompasses diverse areas of research, including string phenomenology, holography, and beyond. Our aim is not to give a pedagogical introduction to the topic, but rather focus on the relevant material relevant for our purposes and fix some notational conventions. The material here is based on the books [118–122] which are excellent resources for further details.

The idea of an extra dimension was first discussed by Kaluza and Klein in the 1920s in an attempt to unify electromagnetism and gravity by reducing 5-dimensional gravity to 4 dimensions. Their idea, known as compactification or KK reduction, is that the compact dimension is too small to be observed while its existence has consequences for the physics in the observable dimensions. Although their goal was ambitious, their observation was later generalised to define string theory in 10 dimensions which successfully unifies general relativity and quantum field theory. In these cases, the compactification is usually on manifolds called Calabi-Yau manifolds introduced in the 1980s as they are of phenomenological interest.

The ultimate goal in string theory is to understand how to construct theories which are viable phenomenologically. This however is a difficult problem to tackle, so as a more preliminary goal we use these theories as a tool for understanding quantum field theories. To do so, we take the large volume limit or the case of non-compact Calabi-Yau spaces where gravity decouples and is no longer dynamical but a background, so only the gauge dynamics become relevant. This construction, known as geometric engineering, was pioneered by Sheldon Katz, Albrecht Klemm, and Cumrun Vafa in the seminal papers [123, 124], where the authors laid the foundation for explaining the interplay between the field theory and the geometric properties of Calabi-Yau spaces.

Initially, five different string theories were realised, but later in the 1990s, it was made clear that these theories are related by dualities, which implies that they should not be regarded as distinct theories. They are known as type I, type IIA, type IIB, and heterotic $SO(32)$ and $E_8 \times E_8$ theories. There is in addition an 11-dimensional theory called M-theory discovered by Witten. This theory is related to string theory by dualities. The low energy limit of M-theory is given by a classical field theory called 11-dimensional supergravity. In this thesis, we are specifically interested in geometrically engineering QFTs from the low energy limit of M-theory and IIB, so we now look at these theories in more detail. The type IIB string theory has massless bosonic fields: a 2-form graviton G, NSNS 2-form B_2, dilaton ϕ and RR fields C_0, C_2 and C_4. In what's called a democratic formulation one includes the electromagnetic dual RR and NSNS fields. Recall that, two fields of degrees $(p+1)$ and $(8-p-1)$ are dual if their field strengths satisfy the relation $\star F_{p+2} = F_{8-p}$. So, there are in addition RR fields C_6, C_8 and NSNS field B_6, while the RR field C_4 is, in fact, self-dual with half the number of physical degrees of freedom. Type IIB also contains objects called Dp-branes charged magnetically under C_{7-p} and electrically charged under C_{p+1}. Dp-branes are important dynamical objects that when wrapped around the collapsed cycles of the Calabi-Yau give rise to gauge fields and other important objects in the geometrically engineered field theory.

In M-theory, the bosonic fields are a 2-form graviton and a 3-form field C_3. The magnetic dual to C_3 is C_6. The branes charged under these fields are called M2 and M5 branes. The low-energy effective action for M-theory has a topological part that can be written schematically in the form

$$e^{iS_{\text{top}}} = \exp\, 2\pi i \int_{\mathcal{M}_{11}} \left[-\frac{1}{6}\, C_3 \wedge G_4 \wedge G_4 - C_3 \wedge X_8 \right], \tag{2.29}$$

where $\mathcal{M}_{11}$ is 11d spacetime, G_4 is its field strength for C_3, and X_8 is an 8-form characteristic class constructed from the Pontryagin classes $p_i(T\mathcal{M}_{11})$, $i = 1, 2$, of the tangent bundle to $\mathcal{M}_{11}$,

$$X_8 = \frac{1}{192} \left[p_1(T\mathcal{M}_{11})^2 - 4\, p_2(T\mathcal{M}_{11}) \right]. \tag{2.30}$$

The expression (2.29) for the topological couplings can only be taken literally if the 3-form is topologically trivial, in which case C_3 is a globally defined 3-form on $\mathcal{M}_{11}$, and the integral in (2.29) can be understood as the standard integral of an 11-form. In topologically non-trivial situations, such as those studied in this work, we need to rewrite (2.29) in differential cohomology. This will be done in section 3.

2.2.1 Flux non-commutativity

In the last section, we introduced the fields in string theory. Next, we address an important question: How is the Hilbert space graded by electric and magnetic fluxes associated with these fields? This question was answered by Freed, Moore and Segal in [90, 103] and [91]. Their main observation was that in general the grading of the Hilbert space by electric fluxes and by magnetic fluxes do not necessarily induce a simultaneous grading. This section will provide a relevant summary of their work for the purposes of this thesis.

Consider a generalised Maxwell theory defined on d-dimensional spacetime of the form $M_d = Y \times \mathbb{R}$ which is closed, oriented and Spin. The gauge inequivalent classes of fields are elements $[\breve{A}] \in \breve{H}^l(M)$. For Maxwell's theory $l = 2$. To quantise the theory, we associate a Hilbert space $\mathcal{H}(Y_{d-1}) = L^2(\breve{H}^l(Y))$ to each $(d-1)$-dimensional manifold Y, where L^2 is the space of quadratically integrable functions. We represent a class of a state in the Hilbert space by $\psi([\breve{A}]) \in \mathcal{H}$.

By (2.19), the connected components of the space of the gauge fields $[\breve{A}] \in \breve{H}^l(Y)$ are labelled by the topological class of the magnetic flux $m \in H^l(Y)$. Thus, there is a natural grading of the Hilbert space by the magnetic flux

$$\mathcal{H} = \bigoplus_m \mathcal{H}\,. \tag{2.31}$$

At the same time, by the electro-magnetic duality of the generalised Maxwell theory, a completely equivalent theory can be realised based on the magnetic dual field $[\breve{A}_D] \in \breve{H}^{d-l}(Y)$. This implies that we can instead have a grading of the Hilbert space by the electric flux $e \in H^{d-l}(M)$

$$\mathcal{H} = \bigoplus_e \mathcal{H}_e \,. \tag{2.32}$$

Thus, a natural question is whether there is simultaneous grading by both electric and magnetic flux

$$\mathcal{H} \stackrel{?}{=} \bigoplus_{e,m} \mathcal{H}_{e,m} \,. \tag{2.33}$$

This question may be answered by just considering the theory based on $[\breve{A}]$, and then measuring the electric flux and checking the effect on the magnetic flux. This methodology is analogous to the measurement of position and momentum operators in quantum mechanics, where the non-commutativity of these operators can be established by measuring one and observing its effect on the other. One may assume that the magnetic flux can be measured via the integrals $\int F$ of the field strength of $\breve{A}$ on a l-closed cycle, and the electric flux via the integral $\int *F$ on a $(d-l)$-closed cycle. Although the two quantities appear to commute in this setup, the latter assumption may be premature. Specifically, it is important to note that e belongs to an abelian group, which can in general have nontrivial torsion subgroups. This observation requires a more refined definition given as follows.

Definition 4. A state $\breve{A} \in \breve{H}^l(Y)$ of a definite topological class of electric flux $e \in H^{d-l}(M)$, is an eigenstate under translation by flat characters $H^{l-1}(Y;\mathbb{R}/\mathbb{Z}) \subset \breve{H}^l(Y)$

$$\psi(\breve{A} + \breve{\phi}) = \exp\left(2\pi i \int_Y e\ \phi\right) \psi(\breve{A}) \tag{2.34}$$

for all $\phi \in H^{l-1}(Y;\mathbb{R}/\mathbb{Z})$ and $i(\phi) = \breve{\phi}$.

From the above definition, it follows that (2.32) is a grading of the Hilbert space in terms of the characters of the group of translations by $H^{l-1}(Y;\mathbb{R}/\mathbb{Z})$. While in the dual picture, the decomposition by magnetic flux (2.31) can be understood as the grading by the group of flat dual connections $H^{d-l-1}(Y;\mathbb{R}/\mathbb{Z})$.

It is now clear that (2.31) cannot hold. Consider, a state ψ of definite magnetic flux labelled by a fixed $m \in H^l(Y)$ in a given topological sector. Such a state cannot be an eigenstate under translation by flat characters. This is because shifting this state by a flat but topologically non-trivial character $H^{l-1}(Y;\mathbb{R}/\mathbb{Z}) \subset \breve{H}^l(Y)$ shifts the topological magnetic flux sector from m. Therefore, we see that (2.31) cannot hold as in general one cannot simultaneously measure both electric and magnetic flux.

Heisenberg group

A more systematic approach to demonstrate the non-commutativity between the subgroups of $\breve{H}^p(Y)$ and $\breve{H}^{d-l}(Y)$ is to define the Hilbert space in terms of the Heisenberg group. The key ideas relevant for us are discussed below. We refer the reader to [91, 103, 125, 126] for more details.

Definition 5. Let $\mathcal{W}$ be a central extension of the group G as given by the short exact sequence

$$1 \to U(1) \to \mathcal{W} \xrightarrow{\pi} G \to 0\,. \tag{2.35}$$

The central extensions of G by $U(1)$ are in one-to-one correspondence with continuous skew alternating bimultiplicative maps

$$s : G \times G \to U(1)\,, \tag{2.36}$$

which we require to be non-degenerate. The *Heisenberg group* is defined as the central extension $\mathcal{W}$, if for $\hat{G}$ the Pontryagin dual of G, the homomorphism

$$e : G \to \hat{G} \tag{2.37}$$

given by $e_{g^D}(g) = s(g^D, g)$ is an isomorphism.

We are interested in the case where $G = \breve{H}^l(Y) \times \breve{H}^{d-l}(Y)$. The standard definition of the Heisenberg group is given as follows. For each $\breve{B}$ denote the translation operator by $T_{\breve{B}}$ such that $T_{\breve{B}}\psi(\breve{A}) = \psi(\breve{B} - \breve{A})$ and the multiplication operator by $M_{\breve{B}^D}\psi(\breve{A}) = \langle \breve{B}^D|\breve{A}\rangle\, \psi(\breve{A})$. Then the Heisenberg group is

$$\mathcal{W} = \{zT_{\breve{B}}M_{\breve{B}^D} | z \in U(1), \breve{B} \in \breve{H}^l(Y), \breve{B}^D \in \breve{H}^{d-l}(Y)\}\,. \tag{2.38}$$

We have that $\pi(zT_{\breve{B}}M_{\breve{B}^D}) = (\breve{B}, \breve{B}^D) \in G$. It can be shown from this standard definition and the extension (2.35) that [91]

$$s\big((\breve{A}_1, \breve{A}_1^D), (\breve{A}_2, \breve{A}_2^D)\big) = \exp\Big[2\pi i\big(\, \langle \breve{A}_2|\breve{A}_1^D\rangle - \langle \breve{A}_1|\breve{A}_2^D\rangle\, \big)\Big]\,. \tag{2.39}$$

The pairing $\langle\cdot|\cdot\rangle$ is the natural pairing

$$\langle\cdot|\cdot\rangle : \breve{H}^{l}(Y) \times \breve{H}^{d-l}(Y) \to \mathbb{R}/\mathbb{Z} \tag{2.40}$$

defined by Poincare duality

$$\langle\breve{A}|\breve{A}^{D}\rangle := \int_{Y}^{\breve{H}} \breve{A} \star \breve{A}^{D} \,. \tag{2.41}$$

The non-commutativity between the subgroups of $\breve{H}^{l}(Y)$ and $\breve{H}^{(d-l)}(Y)$ can be quantified by considering the respective lifts of the groups $H^{l-1}(Y,\mathbb{R}/\mathbb{Z})$ and $H^{d-l-1}(Y,\mathbb{R}/\mathbb{Z})$. Take $\phi \in H^{l-1}(Y;\mathbb{R}/\mathbb{Z})$ such that $\Psi_{\phi}^{\text{electric}}$ is the unitary operator on $\mathcal{H}_e$ with eigenvalue $\exp\big(2\pi i \int_Y e\ \phi\big)$, and similarly take $\phi^D \in H^{d-l-1}(Y;\mathbb{R}/\mathbb{Z})$ with $\Psi_{\phi^D}^{\text{magnetic}}$ the unitary operator on $\mathcal{H}$ having an eigenvalue $\exp\big(2\pi i \int_Y m\ \phi\big)$. Then, (2.38) and the Heisenberg non-commutation (2.39) result in the commutator

$$[\Psi_{\phi}^{\text{electric}}, \Psi_{\phi^D}^{\text{magnetic}}] = \langle i(\phi)|i(\phi^D)\rangle \ , \tag{2.42}$$

where i is the map to differential cohomology classes and $t : H^{*}(Y;\mathbb{R}/\mathbb{Z}) \to \text{Tor}\, H^{*+1}(Y)$ both given in (2.13). (2.42) can be written in terms of the torsion pairing or the linking pairing T as[4]

$$\langle i(\phi)|i(\phi^D)\rangle = \exp[2\pi i T\big(t(\phi), t(\phi^D)\big)] \,. \tag{2.43}$$

The linking pairing is a map

$$T : \text{Tor}\, H^{l}(Y) \times \text{Tor}\, H^{d-l}(Y) \to \mathbb{R}/\mathbb{Z} \tag{2.44}$$

defined as

$$T(t(\phi), t(\phi^D)) = \int_Y t(\phi) \smile \beta^{-1}\big(u \circ t(\phi^D)\big) = \int_Y t(\phi) \smile \phi^D \tag{2.45}$$

where the second equality follows from the commutativity of (2.13).

From now we drop the superscripts "electric" or "magnetic" of the operators Ψ as the distinction is clear from the subscript. Note that the commutator (2.42) is

[4]In the cases of self-dual fields, the relation between $\langle\cdot|\cdot\rangle$ and T given in (2.42) is in general modified [103]. However, while we consider the F_5 self-dual field in IIB, for the geometries of interest in this case, it was shown in [24] that no modification is required.

defined multiplicatively as we are working with $U(1)$ group

$$[\Psi_\phi, \Psi\phi^D] = \Psi_\phi \Psi_{\phi^D} \Psi_\phi^{-1} \Psi_{\phi^D}^{-1} \,. \tag{2.46}$$

We can replace $U(1)$ with $\mathbb{R}/\mathbb{Z}$ and work additively instead of multiplicatively, in which case the non-commutativity can be written as

$$[\Psi_\phi, \Psi_{\phi^D}] = \Psi_\phi \Psi_{\phi^D} - \Psi_{\phi^D} \Psi_\phi \,. \tag{2.47}$$

In what follows we may replace the subscript ϕ in Ψ_ϕ with $t(\phi)$. Also, note that we might write the multiplicative commutation while we still write $\mathbb{R}/\mathbb{Z}$ or $\mathbb{Q}/\mathbb{Z}$ in defining the linking pairing, but it should be understood that to be precise one should write $U(1)$ instead.

The Hilbert space is defined as the irreducible representation of the Heisenberg group $\mathcal{W}$. This irrep is unique by the Stone-von Neumann theorem[5]. Thus, in the presence of torsional fluxes, we expect to have a grading of the Hilbert space

$$\mathcal{H} = \bigoplus_{(\alpha,\beta)} \mathcal{H}(\alpha,\beta) \qquad \text{with} \quad (\alpha,\beta) \in \frac{\breve{H}^l(Y) \times \breve{H}^{(d-l)}(Y)}{\mathrm{Tor}\left(\breve{H}^l(Y) \times \breve{H}^{(d-l)}(Y)\right)} \tag{2.48}$$

where each factor $\mathcal{H}(\alpha,\beta)$ is in turn a representation of the Heisenberg algebra (2.42).

M-theory

In the case of M-theory, the Hilbert space $\mathcal{H}(\partial\mathcal{M}_{11})$ should have a grading in terms of the *M-theory generalised cohomology theory group*, which we denote $\mathbb{E}_M(\partial\mathcal{M}_{11})$. The group $\mathbb{E}_M(\partial\mathcal{M}_{11})$ is expected to parametrise the flux sectors of M-theory. In this case, just as before the fluxes do not commute because the group $\mathbb{E}_M(\partial\mathcal{M}_{11})$ can contain a torsional part

$$\mathrm{Tor}\ \mathbb{E}_M(\partial\mathcal{M}_{11}) = \{x \in \mathbb{E}_M(\partial\mathcal{M}_{11})\colon nx = 0 \text{ for some } n \in \mathbb{Z}\} \tag{2.49}$$

The Heisenberg algebra of torsional fluxes is of the form

$$\Psi_x \Psi_y = T(x,y) \Psi_y \Psi_x \tag{2.50}$$

[5]This is a generalization of the usual Stone-von Neumann to infinite-dimensional groups (not locally compact groups). In this case, one needs to add the data of a polarisation which is physically given by a positive energy condition [103].

with $x, y \in \mathrm{Tor}\ \mathbb{E}_M(\partial\mathcal{M}_{11})$ and T modifying as

$$T\colon \mathrm{Tor}\ \mathbb{E}_M(\partial\mathcal{M}_{11}) \times \mathrm{Tor}\ \mathbb{E}_M(\partial\mathcal{M}_{11}) \to U(1) \tag{2.51}$$

The grading of the Hilbert space is

$$\mathcal{H} = \bigoplus_{\boldsymbol{\alpha}\in\mathbb{E}^o_M(\partial\mathcal{M}_{11})} \mathcal{H}(\boldsymbol{\alpha}) \qquad \text{with} \quad \mathbb{E}^o_M(\partial\mathcal{M}_{11}) \equiv \frac{\mathbb{E}_M(\partial\mathcal{M}_{11})}{\mathrm{Tor}\ \mathbb{E}_M(\partial\mathcal{M}_{11})}\,. \tag{2.52}$$

Differential cohomology is believed to be a suitable model of M-theory generalised theory.[6] Naively, one may assume the gauge equivalent classes of the 3-form C-field (and its dual 7-form) are differential cohomology classes. It turns out that this is not quite true, but is rather a "shifted" differential cohomology class [128]. This shift motivates what's called the E_8 model for the C-field [107, 129]. However, we will show that this shift does not contribute for the examples we discuss. Thus, we will take $\mathbb{E}_M(\partial\mathcal{M}_{11})$ to be $\breve{H}^4(\mathcal{M}_{11}) \times \breve{H}^7(\mathcal{M}_{11})$. We should note that it would be interesting to consider examples where this shift has non-trivial contributions.

Type IIB string theory

It is a widely accepted conjecture that the precise formulation of the effective description of type IIB string theory must be in K-theory rather than ordinary singular cohomology [130, 131]. In this formulation, the RR fields combine linearly into a single self-dual RR field classified by differential K-theory. As this is a self-dual field, it encodes both electric and magnetic fluxes. The relevant K-theory group in IIB is $K^1(\mathcal{M}_{10})$.

In the case of K-theory, there are again torsional subgroups and the flux operators Ψ_σ, with σ a torsional class in the first K-theory group of the boundary $K^1(\partial\mathcal{M}_{10})$, do not commute [91, 103]

$$\Psi_\sigma \Psi_{\sigma'} = e^{2\pi i T(\sigma,\sigma')} \Psi_{\sigma'} \Psi_\sigma, \tag{2.53}$$

where T is the linking pairing in $\partial\mathcal{M}_{10}$

$$T : \mathrm{Tor}\, K^1(\partial\mathcal{M}_{10}) \times \mathrm{Tor}\, K^1(\partial\mathcal{M}_{10}) \to \mathbb{Q}/\mathbb{Z}\,. \tag{2.54}$$

[6] [127] has also postulated a generalised cohomology theory for M-theory. It would be interesting to see if this more refined picture leads to any interesting consequences in field theory.

In this case, the non-commutativity of fluxes gives rise to the grading of the Hilbert space $\mathcal{H}_\alpha$ by the cosets

$$\alpha \in \frac{K^1(\partial\mathcal{M}_{10})}{\mathrm{Tor}\, K^1(\partial\mathcal{M}_{10})}\,. \tag{2.55}$$

In the examples we are interested in, the K-theory group reduces to ordinary cohomology. We show that for the geometries that we consider in this thesis

$$K^1(\mathcal{N}_9) = \sum_n H^{2n+1}(\mathcal{N}_9) \tag{2.56}$$

so, we will work on the cohomology formulation for simplicity.

More specifically, we consider the case of D3-branes which are charged under the self-dual field strength F_5 valued in $H^5(\mathcal{M})$. Then, (2.54) reduces to

$$T : \mathrm{Tor}\, H^5(\partial\mathcal{M}_{10}) \times \mathrm{Tor}\, H^5(\partial\mathcal{M}_{10}) \to \mathbb{Q}/\mathbb{Z}\,. \tag{2.57}$$

2.3 Generalised symmetries in quantum field theories

Now, we are about to shift our focus entirely. As we will discuss in the next section, symmetries of quantum field theories naturally arise from constructing them through string theory. Before delving into the specifics of how that works, we first need to introduce the concept of higher-form symmetries in QFTs. References on this topic include the generalised global symmetries paper [8] as well as the recent reviews [132, 133].

A theory has a global symmetry (in the absence of anomalies) if there is a G-bundle on the spacetime, where the theory displays an invariance under the global action of G on the fields. Given a G-bundle on the spacetime, we can introduce background 1-form fields or connections corresponding to this symmetry. Traditionally, a global symmetry corresponds to conserved quantities, for example through Noether's theorem in the Lagrangian formulation or evolution equation in the Hamiltonian formalism.

We may generalise from ordinary 0-form symmetry to p-form G symmetry, by replacing the 1-form background fields with $(p+1)$-form background fields. For distinct G, such fields are in bijective correspondence with homotopy classes of maps $[\mathcal{M}, B^{p+1}G]$ from $\mathcal{M}$ to the Eilenberg-MacLane space $K(G, p+1) \equiv B^{p+1}G$. The latter has the property that its only non-trivial homotopy group is $\pi_{p+1} = G$. As a consequence of this property, G is Abelian for $p > 0$, and for this case we have the isomorphism $[\mathcal{M}, B^{p+1}G] \cong H^{p+1}(\mathcal{M}; G)$. Thus, the background fields are locally elements of $H^{p+1}(\mathcal{M}; G)$.

Definition 6. For a theory defined on a space M, *a p-form symmetry* with group G is the data of the set of homotopy equivalent maps from M to an Eilenberg-MacLane space K(G,p+1).

It should be noted that the above definition does not distinguish between symmetry and its backgrounds, while such a distinction is made in the physics literature. Therefore, we will provide a reformulation of this definition in terms of topological operators below. More generally, there may be a combination of higher-form symmetries of various degrees. Then, we must construct a more general topological space by Postnikov towers. In simpler cases, these are given by direct products of Eilenberg-MacLane space, but in general, there may be non-trivial relations between the background fields leading to more generalised structures known as higher group symmetries. There may also be mixed 't Hooft anomalies between the different higher symmetries. We will discuss the latter case at the end of this section.

Next, let us give the definition of generalised global symmetries appearing in [8]. This is done by first reformulating the definition of 0-form symmetries and then generalising. A theory has an ordinary global symmetry if there is an operator $U_g(M)$ associated with a co-dimension one manifold $M \subseteq \mathcal{M}_D$ of D-dimensional spacetime $\mathcal{M}_D$ generating the symmetry with $g \in G$. The operators $U_g(M)$ are invariant under any homeomorphism of M to a manifold M' or, equivalently, we say $U_g(M)$ is a topological operator. $U_g(M)$ must also satisfy the group multiplication law, i.e. $U_g(M)\, U_{g'}(M) = U_{gg'}(M)$. We have both continuous and discrete global symmetries, where continuous symmetries have an associated current j which is a closed $D-1$-form. Then, the charge operator is constructed from j and the co-dimension one manifold M as

$$U_g(M) = e^{i\int_M j(g)}. \tag{2.58}$$

Both in the case of continuous and discrete symmetries $U_g(M)$ exist and act on local operators $O_i(p)$, defined at a point p which is surrounded by M, that is $U_g(M)\, O_i(p) = R_i^j(g)\, O_j(p)$, so $O(p)$ transform in the representation $R(g)$ of g.

Generalising, we can define higher or n-form symmetries as follows.

Definition 7. A theory has a *n-form G symmetry* if there is an operator $U_g(M)$ associated with a co-dimension $n+1$ manifold $M \equiv M_{(D-n-1)}$, with $g \in G$, such that

1. The operators $U_g(M)$ are topological;
2. The group multiplication law is satisfied $U_g(M)\, U_{g'}(M) = U_{gg'}(M)$;
3. The operator $U_g(M)$ acts on charged operators $O(L)$ defined on dimension n manifolds L which is surrounded by M, so $O(L)$ transform in the representation $R(g)$ of g, i.e. $U_g(M)\, O_i(L) = R_i^j(g)\, O_i(L)$.

Here, for $n \geq 1$, the n-form symmetry is Abelian or in other words, the operators must all commute with each other because of the high co-dimension and also $R(g)$ is just a phase. As with the ordinary symmetry when G is continuous, $U_g(M)$ can be written as in equation (2.58). Note that, when the group law is not satisfied, that is $U_g(M)\, U_{g'}(M) = w(g,g')\, U_{gg'}(M)$, then $w(g,g')$ is an anomalous phase and we say that the n-form symmetry has an anomaly. Thus such 't Hooft anomalies may be classified by group cohomology of the manifold M.

Importantly, the definition of n-form symmetry is valid abstractly independent of the existence of a Lagrangian. It is also valid whether the discrete or continuous symmetries are spontaneously broken or not. There are generalisations of higher-form symmetries to non-invertible symmetries if the group law is not satisfied. Then, the product of two operators is no longer given by another operator but a linear combination of a set of operators.

Example. To clarify the notion, let us look at the example of the 4d super Yang-Mills theory (SYM) with Lie algebra $\mathfrak{su}(N)$. Denote the topologically trivial gauge field by A with field strength F. Under electromagnetic duality, we have the dual field A^D with field strength $*F$. So, we have electrically charged objects under F and magnetically charged objects under $*F$. As discussed in [134], only a subset of these charged objects may be realised by the generalised Dirac quantisation condition. Thus, we are immediately faced with the important subtlety of choosing a gauge group for this theory. One may consider the theory based on the $SU(N)$ group, where there is no matter transforming under the $\mathbb{Z}_N$ centre of $SU(N)$. The topological surface operators $U_n(M)$, with $n = 0, 1, \cdots N-1$, generate the electric $\mathbb{Z}_N$ one-form symmetry. For this reason, the one-form symmetry is also known as the centre symmetry. The electric symmetry acts on Wilson lines W. They may be understood as the wordline of the electrically charged particle, defined as the holonomy of the connection A along the path γ

$$W(\gamma) = Tr_{\mathbf{R}} \exp\left(im \int_\gamma A(\gamma)\right) \tag{2.59}$$

where $m = 0, 1, \cdots N-1$ and $\mathbf{R}$ a representation of $SU(n)$ labelled by weight lattice of $\mathfrak{su}(N)$ quotient Weyl group. From the equation of motion, we see that this symmetry shifts the gauge field A, or equivalently the connection form A on the bundle over M, by a flat $\mathbb{Z}_N$ gauge connection. Therefore, we have

$$U_n(M)\, W(\gamma) = e^{2\pi i n m L(M,\gamma)}\, W(\gamma)$$

and the Wilson line picks up a phase under this symmetry, where $L(M,\gamma)$ is the linking of M and γ. Here there is no magnetic one-form symmetry since there are no 't Hooft lines in the fundamental representation but only in the tensor products of the adjoint which get screened by the matter in the adjoint.

Similarly, we can consider $4d$ Yang-Mills theories with gauge groups $PSU(N) \equiv SU(N)/\mathbb{Z}_N$. Then, there is a magnetic $\mathbb{Z}_N$ symmetry which acts on 't Hooft lines. The magnetic $\mathbb{Z}_N$ group is equal to the centre of the Langlands dual group of $PSU(N)$ which is isomorphic to $\pi_1(PSU(N))$. Therefore, we see that the various choices for the gauge groups result in distinct 1-form symmetries. Generally, given a YM theory based on Lie algebra $\mathfrak{g}$, the gauge group G is given by the quotient of the universal cover group $\tilde{G}$ of the Lie group for $\mathfrak{g}$ by any subset $H \subset C$ of the centre C of $\tilde{G}$, i.e. $\tilde{G}/H$.

In addition, for each choice of the gauge group, there may also be different choices of line operators that can be realised in the theory. This is a consequence of the generalised Dirac quantisation condition which states that electric and magnetic operators have non-trivial commutation relations and cannot be realised simultaneously in the theory. For example, consider the case where $\mathfrak{g} = \mathfrak{su}(2)$, resulting in two possibilities: $G = SU(2)$ or $(SU(2)/\mathbb{Z}_2) \simeq SO(3)$. In the latter case, there are two distinct choices for the set of mutually local line operators with one containing 't Hooft lines and the other dyonic lines.

More generally, this choice exists in theories of various dimensions that may include higher-dimensional extended operators. Each choice corresponds to a different theory with a specific spectrum of extended operators. Given that these extended operators possess charge under the higher-form symmetry, we can infer that, within a theory with a fixed local structure, there exist different choices of higher-form symmetries. This is often referred to as a choice of global structure.

The charged operators are also known as the defects in the theory. These defects can be screened by dynamical objects, such as 't Hooft lines that are screened by adjoint matter in $SU(N)$ Yang-Mills theory. The collection of all defects in the theory that are not screened by dynamical operators forms a group known as the defect group $\mathbb{D}$. The choice of a higher-form symmetry group can be understood as a selection of a Lagrangian (or maximal isotropic) subgroup $G \subset \mathbb{D}$ within the defect group. The significance of the concept of defect group will be important for our discussions in the upcoming sections.

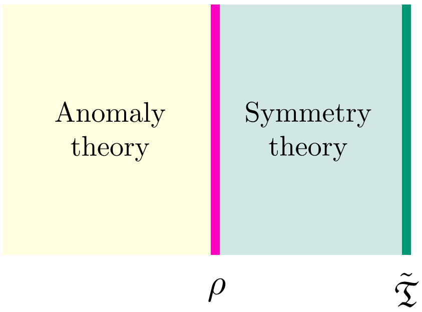
It is worth highlighting that certain theories, like 6-dimensional $\mathcal{N} = (2,0)$ superconformal field theories (SCFTs), do not have descriptions in terms of gauge theories. Nevertheless, even in these cases, there exist choices of higher symmetries. The study of symmetries in these theories can be approached by investigating their stringy origins as we will see in the upcoming section.

Let us note that throughout this thesis we approximate the cohomology theory to be the singular cohomology. This is rather an approximation and not the most precise framework when starting from string or M-theory. In line with that, our definition of higher symmetry is limited to p-form symmetry. It seems possible that dropping this assumption may result in a relation between the background fields of various relevant degrees leading to non-invertible categorical symmetries that might form higher structures.

2.3.1 Anomaly theory and symmetry TFT

A very useful way of organising these symmetries, that arises naturally in string theory, is in terms of the following construction [4, 92–94]. If the original theory $\mathfrak{T}$ is formulated on d dimensional spacetimes $\mathcal{M}_d$, we introduce a generically non-invertible topological $(d+1)$ dimensional quantum field theory (which in this thesis we will call the *symmetry theory*, or *symmetry TFT (SymTFT)* when we want to emphasise that it is topological[7]) with the property that it admits a non-topological theory $\tilde{\mathfrak{T}}$ as the theory of edge modes on manifolds with boundary (a relative theory, in the framework of [135]), and also a gapped interface ρ to the anomaly theory $\mathcal{A}$ of the theory $\mathfrak{T}$:

[7] In this thesis we will be considering cases with $d \in 2\mathbb{Z}+1$, so there are no local anomalies.



The anomaly theory is a well-understood object (see [136, 137] for reviews): it is an invertible theory that gives us a way of defining the phase of the partition function of $\mathfrak{T}$ by evaluating the partition function of $\mathcal{A}$ on a $d+1$ manifold with boundary (see [138] for the original discussion in the case of anomalies of fermions). The theory $\mathfrak{T}$, attached to its anomaly theory $\mathcal{A}$, arises when we collide ρ with $\tilde{\mathfrak{T}}$.

We will argue that the picture that arises in string theory is the complementary one, in which we focus on the symmetry theory by sending ρ to infinity. More concretely, in this thesis we will consider singular string configurations, where we have a set of local degrees of freedom (often strongly coupled) living at the singular point of some non-compact cone X. We identify these local degrees of freedom with $\tilde{\mathfrak{T}}$. The choice of the actual symmetries of $\mathfrak{T}$ (which in our picture above would be associated with a choice of ρ), has been previously argued to live "at the boundary of X" [1, 24, 44], a behaviour that is also familiar in the context of holography [87]. The goal of this thesis is to sharpen this picture by giving a direct derivation of the symmetry theory from the string construction: we will see that we can obtain in a natural way a non-invertible topological theory encoding both the choices of symmetries for $\mathfrak{T}$ and their anomalies.

3 Higher-form symmetries from geometric engineering

As discussed in section 2.2, one of the natural approaches to constructing quantum field theories, particularly those with supersymmetry, involves exploring the low-energy limit of String/M/F-theory within the framework of dimensional reduction on Calabi-Yau spaces. In this section, we give the general philosophy behind the study of higher-form symmetries of quantum field theories obtained using geometric engineering. Our formulation is mainly based on the following works [1–4, 24, 44, 47, 50, 52] and other references given. More specifically, section 3.1 is a generalisation of the material discussed in the author's work [1]. Section 3.2 contains material from [1, 3, 24] and section 3.3 from [4].

3.1 Construction

In this work, we consider the SQFTs obtained from a higher dimensional theory in d dimensions via geometric engineering on backgrounds of the form

$$\mathcal{M}_m = \mathcal{M}_D \times \mathcal{V}_d \qquad d + D = m \tag{3.1}$$

where $\mathcal{V}_d$ is a local internal geometry and $\mathcal{M}_D$ is a D-dimensional space time manifold where the geometrically engineered D-dimensional quantum field theory $\mathfrak{T}_{\mathcal{V}_d}$ lives. We consider examples in the next sections where the higher dimensional theory is M-theory with $m = 11$ or type IIB string theory with $m = 10$.

For simplicity, we assume:[1]

- $\mathcal{V}_d$ is a supersymmetric background, therefore $\mathfrak{T}_{\mathcal{V}_d} \in \mathrm{SQFT}_D$; moreover

$$\mathcal{V}_d = \mathcal{C}(Y_{d-1})\,, \tag{3.2}$$

[1]Both assumptions can be dropped in principle, but we want to work in the simplest possible setup that highlights the features we want to study.

meaning that $\mathcal{V}_d$ is a metric cone over a $d-1$ dimensional manifold Y_{d-1};

- $\mathcal{M}_D$ is a closed spin manifold without torsion.[2]

Naively, this setup computes the partition function of the theory $\mathfrak{T}_{\mathcal{V}_d}$ on the manifold $\mathcal{M}_D$,

$$Z_{\mathfrak{T}_{\mathcal{V}_d}}(\mathcal{M}_D) \in \mathbb{C}\,. \tag{3.3}$$

However, in the presence of mixed 't Hooft anomalies for the factors in the defect group, not all fluxes can be diagonalised simultaneously, thus leading to several different choices. These choices are in one-to-one correspondence with the possible global structures of the quantum field theory $\mathfrak{T}_{\mathcal{V}_d}$.

The main feature of the geometric engineering limit is that the internal manifold is non-compact, and therefore $\mathcal{M}$ has a boundary at infinity

$$\partial\mathcal{M} = \mathcal{M}_D \times \partial\mathcal{V}_d\,. \tag{3.4}$$

If this is the case, as discussed in section 2.2.1, we can consider a Hamiltonian quantization viewing the direction normal to the boundary as time, and assign to this system a Hilbert space

$$\mathcal{H}(\partial\mathcal{M})\,. \tag{3.5}$$

The Hilbert space $\mathcal{H}(\partial\mathcal{M})$ should have a grading in terms of the generalised cohomology theory group $E(\partial\mathcal{M})$ associated with the higher dimensional theory under consideration. For example, as we noted in the previous section, for M-theory we have the M-theory generalised cohomology theory group $\mathbb{E}_M(\partial\mathcal{M})$ and for type IIB we have K-theory group $K^1(\partial\mathcal{M})$. In the examples we consider, there is a simple connection between the defect group $\mathbb{D}$ and $\mathrm{Tor}\, E(\partial\mathcal{M})$:

$$\mathrm{Tor}\, E(\partial\mathcal{M}) = \bigoplus_j H^{j+1}(\mathcal{M}_D) \otimes \mathbb{D}^{(j)} \tag{3.6}$$

where the sum over j depends on the specific generalised cohomology group we consider. This relation will be made clear in the next section.

[2] If $\mathcal{M}_D$ has torsion, there will be additional terms in our analysis below arising from the Künneth formula. It would be interesting to investigate the physical meaning of these terms in future works.

Recalling (2.42), in the presence of torsional fluxes, we expect to have a grading of the geometric engineering Hilbert space

$$\mathcal{H} = \bigoplus_{\boldsymbol{\alpha} \in E^o(\partial\mathcal{M})} \mathcal{H}(\boldsymbol{\alpha}) \qquad \text{with} \quad E^o(\partial\mathcal{M}) \equiv \frac{E(\partial\mathcal{M})}{\text{Tor } E(\partial\mathcal{M})} \tag{3.7}$$

where each factor $\mathcal{H}(\boldsymbol{\alpha})$ is in turn a representation of a Heisenberg algebra of torsional fluxes of the form

$$\Psi_x \Psi_y = T(x,y) \Psi_y \Psi_x \tag{3.8}$$

where

$$T \colon \text{Tor } E(\partial\mathcal{M}) \times \text{Tor } E(\partial\mathcal{M}) \to U(1) \tag{3.9}$$

is a perfect pairing. *This pairing encodes the mixed 't Hooft anomalies among the higher-form symmetries of the geometric engineering Hilbert space.* Abusing language, in light of (3.6) we will refer to these as *the mixed 't Hooft anomalies for the defect group.* Since the flux operators do not commute we cannot specify the asymptotic values for all fluxes simultaneously: two steps are required

1. We need to choose a maximally isotropic subgroup $L \subset \text{Tor } E(\partial\mathcal{M})$ of fluxes that can be simultaneously measured;

2. We need to choose a "zero flux" state, which corresponds to the unit eigenvalue

$$\Psi_x |0, L\rangle = |0, L\rangle \qquad \forall x \in L \tag{3.10}$$

Then we obtain a basis for the geometric engineering Hilbert space parametrised by

$$|f, L\rangle := \Psi_f |0, L\rangle \qquad\qquad f \in \mathsf{F}_L := \frac{\text{Tor } E(\partial\mathcal{M})}{L}\,. \tag{3.11}$$

A choice of background fluxes for the higher-form symmetries of this theory corresponds to fixing a state

$$|\{a_f\}\rangle = \sum_{f \in \mathsf{F}_L} a_f |f, L\rangle \tag{3.12}$$

whence the corresponding partition function is determined: the open manifold $\mathcal{M}$ can be viewed as an element $\langle\mathcal{M}|$ of $\mathcal{H}(\partial\mathcal{M})^* := \text{Hom}(\mathcal{H}(\partial\mathcal{M}), \mathbb{C})$ so the partition function is $\langle\mathcal{M}|\{a_f\}\rangle$ — see [24] for a more detailed version of this argument.

We stress that to fully specify a quantum field theory $\mathfrak{T}_{\mathcal{V}_d}$ these two steps are required. Indeed only in this case, do we end up with a partition function. Without specifying these details, the geometric engineering Hilbert space knows only about the whole set of possible theories with the same local dynamics but different global structures. When we choose the theory corresponding to the state $|0, L\rangle$, the defects with charges in L are non-genuine, while the ones with charges in F_L are the genuine ones (see section 3.3 of [24]). When we specify the state $|0, L\rangle$ this breaks the defect group $\mathbb{D}$ to the higher-form symmetry group of the genuine defects of the corresponding quantum field theory $\mathfrak{T}_{\mathcal{V}_d}$. The operators Ψ_f generate background flux for the higher-form symmetry associated with the genuine operators.

3.2 Branes and symmetries

In the previous section, we discussed the geometric origin of having different choices of higher symmetries in dimensional reduction. In this section, we aim to address the question of how these higher-form symmetries arise in the context of geometric engineering. The answer lies in the objects known as branes, which are objects in string theory that include D-branes in type II string theory and M-branes in M-theory.

Recall that, in geometric engineering we obtain a SQFT $\mathfrak{T}_{\mathcal{V}_d}$ by placing the higher dimensional theory on $\mathcal{M}_m = \mathcal{M}_D \times \mathcal{V}_d$ with $d + D = m$ and boundary $\partial\mathcal{M} = \mathcal{M}_D \times \partial\mathcal{V}_d$. For our purposes, it will be convenient to work with a smoothed-out and a compactified version of the geometry. Denote by $\tilde{\mathcal{V}}_d$ some smooth crepant resolution of $\mathcal{V}_d$ (it does not matter which one), and introduce $X_d := B_d \cap \tilde{\mathcal{V}}_d$, where B_d is a sufficiently large ball containing the exceptional set of $\tilde{\mathcal{V}}_d$. We have that $H_i(\tilde{\mathcal{V}}_d) = H_i(X_d)$, and since (topologically) $Y_{d-1} = \partial X_d = \partial\mathcal{V}_d$ we have a long exact sequence of the form

$$\ldots \to H_k(Y_{d-1}) \to H_k(X_d) \to H_k(X_d, Y_{d-1}) \xrightarrow{\partial} H_{k-1}(Y_{d-1}) \to \ldots \tag{3.13}$$

The physical interpretation of this long exact sequence in our physical context is as follows: p-branes are $(p+1)$-dimensional objects that, when wrapped around k-cycles in the internal geometry, leave behind $(p+1-k)$ residual dimensions that exist within the space $\mathcal{M}_D$. Branes wrapping cycles in $\mathcal{V}_d$ will map to branes wrapping cycles in X_d, as the two spaces are topologically equivalent. The particular cycles that they wrap around give rise to distinct objects within $\mathfrak{T}_{\mathcal{V}_d}$, as follows:

- p-branes wrapping the compact k-cycles in $H_k(X_d)$ result in the $(p + 1 - k)$-dimensional dynamical objects.

- p-branes wrapping non-compact k-cycles in the relative homology groups $H_k(X_d, Y_{d-1})$ that extend from the singularity to the boundary $\partial\mathcal{V}_d$ result in the $(p+1-k)$-dimensional line and surface defects.
- p-branes wrapping $(k-1)$-cycles in $H_{k-1}(\partial X)$ result in the $(p-k+2)$-dimensional line and surface operators generating the higher-form symmetries.

Consider a p-brane wrapping a $(k-1)$-cycle resulting in a $(p-k+2)$-dimensional generator and a p'-brane wrapping a k'-cycle in the geometry resulting in $(p'+1-k')$-dimensional defect. For this generator to act non-trivially on the defect, it is required that the generator and the defect live on subspaces in $\mathcal{M}_D$ that have non-trivial linking. So we must have

$$(p'+1-k') + (p-k+2) = D-1\,. \tag{3.14}$$

In addition, the two cycles k and k' must have non-trivial linking in the geometry and satisfy $k-1+k' = d-1$. The last two conditions imply we must have $p+p' = D+d-4$. This happens precisely when the p-brane and p'-brane are electromagnetic duals to each other in String/M-theory. As a result of this observation and comparison with the flux non-commutativity discussion, it has been conjectured that the defects and their generators are realised by electromagnetically dual branes [50, 52]. Let b_{k-1} denote the $(k-1)$-cycle wrapped by the p-brane, then the corresponding charge symmetry generator is given by exponentiation and integrating over b_{k-1} in the integral

$$\exp\left(2\pi i \int_{b_{k-1}\times M} \breve{F}_{p+2}\right)\,, \tag{3.15}$$

where $\breve{F}_{p+2}$ is the field associated to the the C_p field under which the p-brane is electrically charged. Let $\partial(a'_{k'}) = b'_{k'-1}$ be the $(k'-1)$-cycle wrapped by the p'-brane. The defect (charged object) is given by exponentiation and integrating the dual field, $\breve{F}_{p'+2}$, to $\breve{F}_{p+2}$ over $a'_{k'}$ in the integral

$$\exp\left(2\pi i \int_{a'_{k'}\times M'} \breve{F}_{p'+2}\right)\,, \tag{3.16}$$

coming from the WZ term in the p'-brane action.

Physically, as discussed in the section 2.3, we are only interested in the charges that survive 't Hooft screening by dynamical operators, mathematically this is encoded in the fact that we only care about the quotient $H_k(X_d, Y_{d-1})/H_k(X_d)$, or equivalently that we only need to know about the homology class of the intersection of the non-compact cycle with the boundary, namely $H_{k-1}(Y_{d-1})$. The long exact sequence does not necessarily truncate on $H_{k-1}(Y_{d-1})$ in general, so this statement would need correction in those cases in which it doesn't, but the truncation does take place in all cases of interest to us. In other words, the defect group given as $H_k(X_d, Y_{d-1})/H_k(X_d)$ is not simply given by $H_{k-1}(Y_{d-1})$ and may include further contributions. It would be interesting to investigate the physical meaning of this in future work.

Therefore, every p-brane wrapping a non-compact cycle could result in a defect group [14] for the discrete $(p+1-k)$-form global symmetry.

$$\mathbb{D} := \bigoplus_n \mathbb{D}^{(n)} \qquad \text{where} \quad \mathbb{D}^{(n)} := \bigoplus_{\substack{p \text{ branes and } k \text{ cycles} \\ \text{such that } p+1-k=n}} \left(\frac{H_k(X, \partial X)}{H_k(X)} \right) . \tag{3.17}$$

Notice that here we are including the non-torsional part in the definition of $\mathbb{D}^{(j)}$, while this is not always the convention. We stress here that to each free $\mathbb{Z}$ factor in the j-form defect group $\mathbb{D}^{(j)}$ there is a corresponding abelian $U(1)$ higher form symmetry which is shifted in degree by one.[3]

The discussion in this thesis is focused on the study of higher forms symmetries. However, as discussed in the previous section, there may be non-invertible symmetries where in comparison with higher-form symmetries the topological operators do not satisfy the group law. It should be noted that the branes also result in non-invertible and other categorical symmetries where the topological operators arise from considering more terms in the WZ action for the branes [49–55]. In addition, the backgrounds for the higher-form and non-invertible symmetries may be related to themselves or each other non-trivially. These relations result in categorical symmetries. There is some progress in understanding these symmetries from string theory but further investigation is needed to better understand how they arise. For example, two-group symmetries may arise by taking the internal space to have a non-isolated singularity [140].

A quick example

To clarify this statement let's briefly consider the example of geometric engineering of 4d $\mathcal{N}=4$ SYM with simple simply-laced Lie algebra $\mathfrak{g}_\Gamma$ where $\Gamma \subset SU(2)$ from

[3]This is ultimately related to the fact that the pairing for the free part of the defect group is the Poincaré pairing and not the linking pairing, and these are shifted by one in degree [139].

type IIB string theory. In this case, $\mathcal{V}_6 = \mathbb{C}^2/\Gamma \times T^2$ with boundary $\partial\mathcal{V}_6 = S^3/\Gamma \times T^2$. By the McKay correspondence, the relevant discrete groups Γ are in a one-to-one correspondence with the simple Lie algebras of ADE type $\mathfrak{g}_\Gamma$.

In this setup D3-branes wrapping non-compact 3-cycles in the geometry result in 1-form symmetry generators. The defect group associated with 1-form symmetry in this geometry is

$$\mathbb{D} = \mathrm{Tor}\left(\frac{H_3(X_6, Y_5)}{H_3(X_5)}\right) = H_2(X_5) = H_1^a(S^3/\Gamma) \oplus H_1^b(S^3/\Gamma)\,. \tag{3.18}$$

There are two factors associated with the two different 1-cycles $a, b \in H_1(T^2)$ in the torus T^2 which correspondingly we label by a and b. Physically, they correspond to the electric and a magnetic 1-form symmetry valued in the centre of G_Γ, the universal cover group of the Lie group for $\mathfrak{g}$

$$\mathbb{D} = Z(G_\Gamma)^{(1)}_{electric} \oplus Z(G_\Gamma)^{(1)}_{magnetic}\,. \tag{3.19}$$

However, these one-form symmetries of the geometric engineering Hilbert space have a mixed 't Hooft anomaly: the corresponding charge operators for self-dual fields on different cycles do not always commute as discussed in the last section. More precisely, for fluxes labelled by $(\omega_1 \otimes t_a)$ and $(\omega_2 \otimes t_b) \in H_2(\mathcal{M}_4) \otimes Z(G_\Gamma)$, we have

$$\Psi_a \Psi_b = \exp\left(2\pi i\, \mathsf{L}_\Gamma(t_a, t_b) \int_{\mathcal{M}_4} \omega_1 \wedge \omega_2\right) \Psi_b \Psi_a \tag{3.20}$$

where $\mathsf{L}_\Gamma(t_a, t_b)$ is a perfect pairing in $Z(G_\Gamma)$.

Specifying a state now selects a surviving subgroup of $\mathbb{D}$ which becomes the 1-form symmetry for the SYM quantum field theory. For instance, we can choose a maximal isotropic lattice L_a generated by D3-branes wrapping the cycle $t_a = t \otimes a$, with $t \in H_1(S^3/\Gamma)$. Then the state $|0, L_a\rangle$ corresponds to the theory $PSU(N) = SU(N)/\mathbb{Z}_N$ and the wrapped D3-branes on $l_b = l \otimes b$, with $l \in H_2(\mathbb{C}^3/\Gamma, S^3/\Gamma)$ are the genuine defects charged under the resulting magnetic 1-form symmetry. Conversely choosing to set to zero all D3 fluxes wrapping the a-cycle, we are selecting the state $|0, L_b\rangle$: we are preserving the electric 1-form symmetry, thus leading to the theory with gauge group $SU(N)$.

Figure 3.1: The cone $C(Y)$ over the link Y, and the deformation, shown on the right, to a long cylinder where the singularity is at the far end.

3.3 Symmetry TFTs and anomalies from M-theory

As discussed at the beginning of this section, we can construct the D-dimensional theories $\mathfrak{T}$ by reducing 11-dimensional spacetime on singular d-dimensional submanifolds. For concreteness, we will focus on M-theory on singular spaces with a single isolated singularity. The non-trivial local dynamics arise from massless M2 branes wrapping vanishing cycles at the singularity. Close to the singular point, the geometry will look like a real cone over some manifold Y with $\dim(Y) = d-1$. We can deform the cone into an infinitely long cigar, with the singularity at the tip, and Y as the base of the cylinder along the cigar, see figure 3.1. The information we are after is topological, so it is reasonable to expect that we can still obtain it from this deformed background (our results will support this expectation). If we now dimensionally reduce the M-theory action on Y we will obtain a theory on the remaining $d+1$ dimensions, which look like $\mathcal{M}_D \times \mathbb{R}_{\geq 0}$. Recalling the discussion of section 2.3.1, we claim that the topological sector – i.e. couplings that are metric independent – arising from this reduction on Y is precisely the symmetry theory for $\tilde{\mathfrak{T}}$.

Our methods do not require knowledge of a holographic dual, or of a weakly coupled description of the QFT. We find our results particularly illuminating in the case that the local degrees of freedom $\tilde{\mathfrak{T}}$ are those of a strongly coupled CFT without a Lagrangian description (generically we know little about such theories, so any additional information is useful), but we do not require conformality of $\mathfrak{T}$ either.

M-theory compactification on singular Calabi-Yau 2- and 3-folds gives rise to 7d super-Yang Mills (SYM) and 5d superconformal field theories (SCFTs), respectively. These theories have 1-form symmetries, and in the 5d case also 0-form symmetries. The 1-form symmetry in all these cases is discrete and is characterized in terms of the relative homology quotient of the Calabi-Yau, with respect to its boundary [1, 24, 44, 47]

$$\Gamma^{(1)} = \frac{H_2(X, \partial X)}{H_2(X)} \,. \tag{3.21}$$

To derive SymTFTs for global 1-form symmetries, one will have to incorporate their backgrounds $B_2 \in H^2(\mathcal{M}_D; \Gamma^{(1)})$ into the supergravity formalism. The torsional nature of these fields introduces various subtleties in the process. We will use differential cohomology to address these subtleties.[4]

We now give a very concrete example of how the concept of dimensional reduction is reformulated using differential cohomology. We begin with M-theory on a 5d space Y_5 (which in this paper will be a manifold linking the singular point of a non-compact Calabi-Yau three-fold) which has $H^2(Y_5) = \mathbb{Z}_n \oplus \mathbb{Z} = \langle t_2 \rangle \oplus \langle v_2 \rangle$, where t_2 is a torsional generator of the degree two cohomology group and v_2 is a free generator. The reduction for the latter is the standard KK-reduction. It is the torsion part that will most benefit from the uplift to differential cohomology. We denote the differential cohomological uplifts of t_2 and v_2 by $\breve{t}_2$ and $\breve{v}_2$, as discussed in 2.1.

We model the M-theory 3-form gauge field C_3 as a class $\breve{G}_4 \in \breve{H}^4(\mathcal{M}_{11})$ in (ordinary) differential cohomology.[5] In particular, we are implicitly restricting ourselves to situations in which the periods of the M-theory 4-form field strength are integrally quantised. As explained in [128], on certain spacetimes the periods must be half-integrally quantised. We argue in appendix C that this does not occur in the setups discussed in this work.[6]

As in the standard KK expansion, this has a decomposition in terms of differential cohomology classes along the internal space Y_5 (torsion and free), as well as external spacetime $\mathcal{M}_6$:

$$\breve{G}_4 = \breve{F} \star \breve{v}_2 + \breve{B}_2 \star \breve{t}_2 \,. \tag{3.22}$$

There is an extra term, discussed below, that we are ignoring here for simplicity. The CS term in the M-theory action is

$$\frac{S_{\text{top}}}{2\pi} = -\frac{1}{6} \int_{M_{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \,, \tag{3.23}$$

which upon inserting the decomposition (3.22) of $\breve{G}_4$ we integrate over $Y_5 \times \mathcal{W}_6$. The integration over the internal space Y_5 results in the SymTFT on $\mathcal{W}_6$. In ordinary cohomology the integral on Y_5 would pick out only the forms of degree 5, however

[4] In supergravity theories different prescriptions to incorporate torsion have been put forward [141, 142], but none that are mathematically entirely satisfactory or unambiguous.

[5] It should be noted that in the mathematical literature, this element of $\breve{H}^4(\mathcal{M}_{11})$ is sometimes denoted $\breve{C}_3$. We prefer the notation $\breve{G}_4$ to make manifest the degree of this differential cohomology class.

[6] We refer the reader to [143] for a model for the M-theory 3-form in terms of a shifted differential cohomology class, which can accommodate both integral and half-integral periods.

that would mean the purely torsional part $\int_{Y_5} \breve{t}_2 \star \breve{t}_2 \star \breve{t}_2$ would naively not contribute. Differential cohomology works differently, and as reviewed below $\int_{Y_5} \breve{t}_2 \star \breve{t}_2 \star \breve{t}_2$ can be non-vanishing.

Reformulating the problem in terms of differential cohomology on the link of the singularity involves some additional technical complications, but the effort pays off in a number of ways:

- Geometric engineering of QFTs corresponds to "compactification" of string and M-theory on non-compact spaces X_d. This can be mathematically challenging, in particular, it is difficult to define in a precise mathematical sense what one means by reducing the Chern-Simons action on a non-compact space X_d. In our approach we are instead reducing the Chern-Simons action on the closed manifold $Y_{d-1} = \partial X_d$, which is the boundary or link of the non-compact space X_d. This is a much better defined mathematical question, that can be clearly analysed using the formalism of differential cohomology.
- The effective field theory in $11 - d$ dimensions is most interesting when X_d is singular, so it becomes a non-abelian Yang-Mills theory in 7d (for $d = 4$) or a non-trivial interacting 5d SCFT (for $d = 6$). But it is precisely in this singular geometric regime that it is most difficult to pin down what one means by doing a geometric reduction of the effective action. By contract, our formalism is entirely agnostic about the singular structure of X_d, and can be applied without issues even when X_d is singular (in fact, it is arguably at the singular cone point in moduli space where it is most natural to apply our techniques!).
- Often, the analysis of reduction on singular spaces is done by removing these singularities, e.g. in 5d going to the Coulomb branch. It is well-documented, e.g. in the set of canonical singularities realizing 5d SCFTs from isolated hypersurface singularities (see [15, 144–146] for a discussion in the context of 5d SCFTs), that we can have terminal singularities that do not admit a Calabi-Yau (crepant) resolution. This is obviously a class of theories where many standard methods will fail. Another setting where field theory-inspired arguments (including those employed in [26]) are not extendable, is when the 5d SCFT may have a Coulomb branch, but does not admit a non-abelian gauge theory description.

  In contrast, our approach of deriving the SymTFT and thereby the anomaly of the QFT in terms of the reduction on the boundary is applicable in all those instances, and we will provide some examples of non-Lagrangian 5d SCFTs and their SymTFTs below.

- The approach uniformly encodes the entire SymTFT, for all symmetries arising from the compactification. E.g. in 5d we derive both the mixed 0-1-form symmetry anomalies as well as the 1-form symmetry $(B_2)^3$ anomaly [21] (see section 4.2.4).

In this thesis, we focus instead mostly on the torsional sector. Dealing with torsion cycles in supergravity has of course a history. One particularly promising framework was put forward in [141, 142]. In supergravity, form-fields are usually expanded in harmonic forms, which however do not capture the torsion parts $\mathrm{Tor}\, H^p(X_d)$. The key idea of these papers is to use non-harmonic forms to model classes in $\mathrm{Tor}\, H^p(X_d)$. More precisely, a non-trivial class in $\mathrm{Tor}\, H^p(X_d, \mathbb{Z})$ of order k is modeled by a $(p-1)$-form β_{p-1} and a p-form α_p subject to the condition

$$k\alpha_p = d\beta_{p-1}\,. \tag{3.24}$$

3.3.1 Differential cohomology formulation of M-theory

Now that we have clarified the details of the construction in the last section, let us look at the mathematical details. The topological action (2.29) is interpreted as the $\mathbb{R}/\mathbb{Z}$-valued secondary invariant of a differential cohomology class $\breve{I}_{12} \in \breve{H}^{12}(\mathcal{M}_{11})$,

$$\frac{S_{\mathrm{top}}}{2\pi} = \int_{\mathcal{M}_{11}} \breve{I}_{12} \mod 1\,, \tag{3.25}$$

where $\breve{I}_{12}$ is given by

$$\breve{I}_{12} = -\frac{1}{6}\,\breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 - \frac{1}{192}\,\breve{G}_4 \star \breve{p}_1(T\mathcal{M}_{11}) \star \breve{p}_1(T\mathcal{M}_{11}) + \frac{1}{48}\,\breve{G}_4 \star \breve{p}_2(T\mathcal{M}_{11})\,. \tag{3.26}$$

In the previous expression, $\breve{p}_i(T\mathcal{M}_{11}) \in \breve{H}^{4i}(\mathcal{M}_{11})$ denotes a differential refinement of the Pontryagin classes $p_i(T\mathcal{M}_{11}) \in H^{4i}(\mathcal{M}_{11})$ [95, 96, 147].

Within the formalism of differential cohomology we are allowed to consider products and $\mathbb{Z}$-linear combinations of differential cohomology classes, but multiplying by rational coefficients — such as the factor of $1/6$ in front of the $\breve{G}_4 \star \breve{G}_4 \star \breve{G}_4$ term in (3.26) — leads to a quantity which is not well defined in general. The fact that the particular combination $\breve{I}_{12}$ is nonetheless well-defined stems from the analysis of [128, 148], which demonstrates that the total topological action $e^{iS_{\mathrm{CS}}}$ is well-defined up to a sign, which cancels a potential sign problem in the definition of the Rarita-Schwinger determinant. This sign ambiguity arises if and only if the periods of the G_4 field strength are half-integrally quantised. As mentioned above, this does not occur for the setups discussed in this work, meaning that $e^{iS_{\mathrm{CS}}}$ is well-defined by itself.

3.3.2 Kaluza-Klein reduction in differential cohomology

Let us consider an 11d spacetime $\mathcal{M}_{11}$ that is the direct product of an "internal" manifold Y_n of dimension n, and an "external" spacetime $\mathcal{W}_{11-n}$ of dimension $11-n$,

$$\mathcal{M}_{11} = \mathcal{W}_{11-n} \times Y_n \ . \tag{3.27}$$

It is standard to consider the expansion of the M-theory 3-form onto harmonic forms on Y_n, to obtain massless $U(1)$ gauge fields on $\mathcal{W}_{11-n}$ of various p-form degrees. Our goal is to generalise this picture, by expanding the M-theory 3-form onto *all* cohomology classes of Y_n, both free and torsional.

On a factorised spacetime such as (3.27), it is natural to start from objects (differential forms, cohomology classes, differential cohomology classes) defined on the two factors, and combine them into objects on the total space. Let

$$p_{\mathcal{W}} : \mathcal{M}_{11} \to \mathcal{W}_{11-n} \ , \qquad p_L : \mathcal{M}_{11} \to Y_n \tag{3.28}$$

be the projection maps onto the two factors of $\mathcal{M}_{11}$. For notational simplicity, we henceforth omit the pullback maps $p^*_{\mathcal{W}}$ and p^*_L from various factorised expressions. For example,

$$\text{if } \lambda \in \Omega^r(\mathcal{W}_{11-n}) \text{ and } \omega \in \Omega^s(Y_n), \qquad \lambda \wedge \omega \text{ is shorthand for } p^*_{\mathcal{W}}(\lambda) \wedge p^*_L(\omega) \ , \tag{3.29a}$$

$$\text{if } a \in H^r(\mathcal{W}_{11-n}) \text{ and } b \in H^s(Y_n), \qquad a \smile b \text{ is shorthand for } p^*_{\mathcal{W}}(a) \smile p^*_L(b) \ , \tag{3.29b}$$

$$\text{if } \breve{a} \in \breve{H}^r(\mathcal{W}_{11-n}) \text{ and } \breve{b} \in \breve{H}^s(Y_n), \qquad \breve{a} \star \breve{b} \text{ is shorthand for } p^*_{\mathcal{W}}(\breve{a}) \star p^*_L(\breve{b}) \ . \tag{3.29c}$$

We observe that the naturality of the products $\smile$ and $\star$, together with (2.22), implies

$$R(\breve{a}\star\breve{b}) = R(\breve{a})\wedge R(\breve{b}) \ , \quad I(\breve{a}\star\breve{b}) = I(\breve{a}) \smile I(\breve{b}) \ , \quad \text{for } \breve{a} \in \breve{H}^r(\mathcal{W}_{11-n}),\ \breve{b} \in \breve{H}^s(Y_n) \ . \tag{3.30}$$

For each $p = 0, \dots, n$, $H^p(Y_n)$ is a finitely generated Abelian group. We take the generators of $H^p(Y_n)$ to be

$$\begin{aligned} \text{free generators of } H^p(Y_n)\text{:} \quad & v_{p(\alpha)} \ , \quad \alpha \in \{1, \dots b^p\} \\ \text{torsion generators of } H^p(Y_n)\text{:} \quad & t_{p(i)} \ , \quad i \in \mathcal{I}_p \ . \end{aligned} \tag{3.31}$$

The subscript p is a reminder that these are classes of degree p, while (α), (i) are labels that enumerate the generators. We define $b^p := \dim H^p(Y_n;\mathbb{R})$, the p-th Betti number of Y_n. For the torsion generators, the index set $\mathcal{I}_p$ is some finite set of labels, which can be specified more explicitly in concrete examples. Each torsional generator has a definite torsional order: the minimal positive integer $n_{(i)}$ such that $n_{(i)}\, t_{p(i)} = 0$ (no sum on i).

For simplicity we take

$$\mathrm{Tor}\, H^*(\mathcal{W}_{11-n}) = 0 \ . \tag{3.32}$$

By the Künneth formula, we may then expand a generic cohomology class $a_4 \in H^4(\mathcal{M}_{11})$ as

$$a_4 = \sum_{p=0}^{4} \sum_{\alpha_p=1}^{b^p} \sigma_{4-p}^{(\alpha_p)} \smile v_{p(\alpha_p)} + \sum_{p=0}^{4} \sum_{i_p \in \mathcal{I}_p} \rho_{4-p}^{(i_p)} \smile t_{p(i_p)} \ . \tag{3.33}$$

In the above expression, $\sigma_{4-p}^{(\alpha_p)}, \rho_{4-p}^{(i_p)} \in H^{4-p}(\mathcal{W}_{11-d})$.

Recall that the map I in (2.13) is surjective. This applies both for elements in $\breve{H}^*(\mathcal{W}_{11-n})$ and $\breve{H}^*(Y_n)$[7]. It follows that there exist differential cohomology classes $\breve{F}_{4-p}^{(\alpha_p)}, \breve{B}_{4-p}^{(i_p)} \in \breve{H}^{4-p}(\mathcal{W}_{11-n})$ and $\breve{v}_{p(\alpha_p)}, \breve{t}_{p(i_p)} \in \breve{H}^p(Y_n)$ such that

$$\sigma_{4-p}^{(\alpha_p)} = I(\breve{F}_{4-p}^{(\alpha_p)}) \ , \qquad \rho_{4-p}^{(i_p)} = I(\breve{B}_{4-p}^{(\alpha_p)}) \ , \qquad v_{p(\alpha_p)} = I(\breve{v}_{p(\alpha_p)}) \ , \qquad t_{p(i_p)} = I(\breve{t}_{p(i_p)}) \ . \tag{3.34}$$

[7]We are implicitly considering a Wick-rotated version of the theory, and we are taking external spacetime $\mathcal{W}_{11-n}$ to be a closed, connected, oriented manifold.

With the objects $\breve{F}^{(\alpha_p)}_{4-p}, \breve{B}^{(i_p)}_{4-p} \in \breve{H}_{4-p}(\mathcal{W}_{11-n})$ and $\breve{v}_{p(\alpha_p)}, \breve{t}_{p(i_p)} \in \breve{H}_p(Y_n)$ we can construct the following differential cohomology class

$$\breve{a}_4 = \sum_{p=0}^{4} \sum_{\alpha_p=1}^{b^p} \breve{F}^{(\alpha_p)}_{4-p} \star \breve{v}_{p(\alpha_p)} + \sum_{p=0}^{4} \sum_{i_p \in \mathcal{I}_p} \breve{B}^{(i_p)}_{4-p} \star \breve{t}_{p(i_p)} \ . \tag{3.35}$$

The salient property of $\breve{a}_4$ in (3.35) is that it represents a possible lift of a_4 in (3.33), in the sense that

$$I(\breve{a}_4) = a_4 \ . \tag{3.36}$$

This is verified using (3.34), the naturality of the differential cohomology product $\star$, and the second identity in (2.22).

The differential cohomology class $\breve{a}_4$ is not the most general class that reduces to a_4 under the action of I. From the discussion around (2.20), however, we know that any other class that reduces to a_4 must differ from $\breve{a}_4$ by a topologically trivial element of $\breve{H}^4(\mathcal{M}_{11})$, which can be represented by a globally defined 3-form. These considerations lead us to the following final form for the Ansatz for $\breve{G}_4$,

$$\breve{G}_4 = \sum_{p=0}^{4} \sum_{\alpha_p=1}^{b^p} \breve{F}^{(\alpha_p)}_{4-p} \star \breve{v}_{p(\alpha_p)} + \sum_{p=0}^{4} \sum_{i_p \in \mathcal{I}_p} \breve{B}^{(i_p)}_{4-p} \star \breve{t}_{p(i_p)} + \tau([\omega_3]) \ , \qquad \omega_3 \in \Omega^3(\mathcal{M}_{11}) \ . \tag{3.37}$$

The first two sums in (3.37) encode all topological information about $\breve{G}_4$, while the last term collects the topologically trivial part of $\breve{G}_4$.

The differential cohomology classes $\breve{F}^{(\alpha_p)}_{4-p}, \breve{B}^{(i_p)}_{4-p} \in \breve{H}^{4-p}(\mathcal{W}_{11-n})$ encode external gauge fields. More precisely, we have:

- The class $\breve{F}^{(\alpha_p)}_{4-p}$, of degree $(4-p)$, represents a $(3-p)$-form gauge field with gauge group $U(1)$, which restricts to a background field for a $U(1)$ $(2-p)$-form symmetry on the boundary;
- The class $\breve{B}^{(i_p)}_{4-p}$, of degree $(4-p)$, represents a discrete $(4-p)$-form gauge field with gauge group $\mathbb{Z}_{n_{(i_p)}}$, where $n_{(i_p)}$ is the torsion order of $t_{p(i_p)}$, which restricts to a background field for a $\mathbb{Z}_{n_{(i_p)}}$ $(3-p)$-form symmetry on the boundary.

The first case is familiar, but the second one requires some additional explanation. Notice in particular the difference in the relation between the differential cohomology class degree and the degree of the higher form symmetry on the boundary. Consider two classes $\breve{B}, \breve{B}' \in \breve{H}^{4-p}(\mathcal{W}_{11-d})$, such that $I(\breve{B}) = I(\breve{B}')$. Then

$I(\breve{B} - \breve{B}') = 0$, so by exactness of (2.13) there is some globally defined differential form b of degree $3-p$ such that $\breve{B}' = \breve{B} + \tau(\mathsf{b})$. By (2.23) and naturality of τ and R we then have that $\tau(\mathsf{b}) \star \breve{t}_{p(i_p)} = \tau(\mathsf{b} \wedge R(\breve{t}_{p(i_p)})) = 0$, since we have chosen $\breve{t}$ to be flat. This implies $\breve{B} \star \breve{t}_{p(i_p)} = \breve{B}' \star \breve{t}_{p(i_p)}$, so $\breve{B}^{(i_p)}_{4-p} \star \breve{t}_{p(i_p)}$ is fully determined by its cohomology class $I(\breve{B}^{(i_p)}_{4-p}) \smile I(\breve{t}_{p(i_p)})$ (given our canonical choice of $\breve{t}_{p(i_p)}$). This is an element of $H^{4-p}(\mathcal{W}_{11-d}) \otimes \operatorname{Tor} H^p(L_p)$, which by the universal coefficient theorem is isomorphic (since we are assuming $\operatorname{Tor} H^{4-p}(\mathcal{W}_{11-d}) = 0$) to $H^{4-p}(\mathcal{W}_{11-d}; \operatorname{Tor} H^p(L_p))$. So by this isomorphism, we can reinterpret $I(\breve{B}^{(i_p)}_{4-p}) \smile I(t_{p(i_p)})$ as a class in $H^{4-p}(\mathcal{W}_{11-d}; \mathbb{Z}_{n_{(i_p)}})$. But such a cohomology class is a map (up to homotopy) from $\mathcal{W}_{11-d}$ to the classifying space $K(\mathbb{Z}_{n_{(i_p)}}, 4-p) = B^{4-p}\mathbb{Z}_{n_{(i_p)}}$, which is the data that defines a principal bundle for a $(3-p)$-form symmetry. For instance, when $p = 3$ we have an ordinary (0-form) discrete symmetry, and the backgrounds for such symmetries are maps from $\mathcal{W}_{11-d}$ to $B\mathbb{Z}_{n_{(i_p)}}$, or equivalently elements of $H^1(\mathcal{W}_{11-d}; \mathbb{Z}_{n_{(i_p)}})$.

Integration on products. Finally, we want to integrate differential cohomology classes on product spaces. Assume that $\breve{a} \in \breve{H}^p(X)$, $\breve{b} \in \breve{H}^q(Y)$. Then

$$\int_{X\times Y} \breve{a} \star \breve{b} = (-1)^{(q-\dim(Y))\dim(X)} \left(\int_X \breve{a}\right) \star \left(\int_Y \breve{b}\right) . \tag{3.38}$$

For the Chern-Simons coupling (3.25) we have $p + q = \dim(X) + \dim(Y) + 1$. In this case:

$$\int_{X\times Y} \breve{a} \star \breve{b} = \begin{cases} \left(\int_X u\right)\left(\int_Y R(\breve{b})\right) & \text{if } p = \dim(X) + 1\,, \\ (-1)^p \left(\int_X R(\breve{a})\right)\left(\int_Y v\right) & \text{if } p = \dim(X)\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.39}$$

simply by taking into account that the integrals on the right hand side of (3.38) are only non-vanishing for very specific values of p and q, as explained above. Here we have used that in the first case $\breve{a}$ is flat for degree reasons, so there is some $u \in H^{\dim(X)}(X; \mathbb{R}/\mathbb{Z})$ such that $\breve{a} = i(u)$, and similarly $\breve{b} = i(v)$ in the second case.

Here we are particularly interested in those integrals over the internal space involving torsional elements $\breve{t}$ (we omit the subindices here for notational simplicity). First note that since we have chosen these torsional generators to be flat, $R(\breve{t}) = 0$, we have $\tau([\omega_3]) \star \breve{t} = \tau([\omega_3 \wedge R(\breve{t})]) = 0$, due to (2.23). So any integral involving the

$\breve{t}$ generators will be a topological invariant (including invariant under deformations of the connection), by virtue of being independent of $\tau([\omega_3])$.

This implies that when expanding $\breve{G}_4^3$ using (3.37) we have

$$\breve{G}_4^3 = \sum \text{monomials involving } \breve{t} \text{ and } \breve{v} + \sum \text{monomials involving } \breve{v} \text{ and } \tau([\omega_3])\,, \tag{3.40}$$

with no monomials involving both $\tau([\omega_3])$ and the torsional classes. The second class of monomials are accessible using the ordinary formalism based on differential forms, so we will not discuss them further; both because they are well-understood and because our interest is on discrete higher form symmetries, which arise from the torsional sector.

Now, regarding the first class of terms in (3.40), by (2.22) we have that $R(\breve{a}) = 0$, for all $\breve{a} = \breve{t} \star \breve{b}$ and any $\breve{b}$. (Note that by (2.22) $I(\breve{a})$ is automatically torsion if $I(\breve{t})$ is.) This implies that when doing the integration over the internal space Y_n, torsional elements $\breve{a} = \breve{t} \star \breve{b}$ only contribute if $\breve{t} \in \breve{H}^{n+1}(Y_n)$. By (3.39) this leads to effective actions on $\mathcal{W}_{11-d}$ which are primary invariants, not secondary ones. (Said more plainly: reducing the Chern-Simons term in 11d on the torsional sector leaves us with an ordinary integral of characteristic classes in $\mathcal{W}_{11-d}$.)

4 Symmetries and anomalies from M-theory

In the last section, we laid out the details of how to geometrically engineer symmetries and their anomalies in quantum field theories. We are now ready to apply these techniques to various examples and discover their applications. As a first example, we will discuss the case of 7d supersymmetric gauge theories. We will find agreement with the global structure obtained by considering Wilson and 't Hooft operators in the 7d side, and the global structure predicted by M-theory flux non-commutativity.

As more involved examples we will analyse the global structure of 5d SCFTs from M-theory on canonical CY singularities. As a result, we will obtain 5d defect groups of the form

$$\mathbb{D} = \left(\mathsf{Z}^{(1)}_{M2} \oplus \mathsf{Z}^{(2)}_{M5}\right) \oplus \left(\mathsf{Z}^{(-1)}_{M2} \oplus \mathsf{Z}^{(4)}_{M5}\right) \tag{4.1}$$

that often present very interesting global structures arising from the mechanism we have outlined in the last section. Here the superscript n in Z^n denotes the n-form symmetry, and Z is an abelian discrete group given by the torsional part of the cokernel of the intersection matrix of the corresponding CY. Physically this intersection matrix is associated with the Dirac pairing among the monopole strings and the BPS particles of the SCFT in a Coulomb phase and Z measures the 't Hooft charges of the defects. We will review some field theory results and then proceed to determine the corresponding defect groups from geometry first in general, and then focus on the case of toric canonical CY singularities, which are the main class of examples we study. The case of the 5d Yang-Mills theories with gauge algebra $\mathfrak{su}(p)$ and Chern-Simons level k will be studied in detail, as a consistency check for our methods. Some of the material on the discussion of the higher symmetries of 5d and 7d theories are adopted from [1]. Furthermore, following [149], we will derive the symmetry theory for these examples, and find that there are 't Hooft anomalies of purely higher form symmetries as well as mixed 't Hooft anomalies.

4.1 7d $\mathcal{N} = 1$ theories

We start with the case of seven-dimensional $\mathcal{N} = 1$ theory with gauge algebra $\mathfrak{g}_\Gamma$, with $\Gamma \subset SU(2)$ an ADE group.[1] Such theories can be engineered by considering M-theory on $\mathcal{M}_{11} = \mathcal{M}_7 \times \mathbb{C}^2/\Gamma$, with Γ a discrete subgroup of $SU(2)$.

From the field theory side, we expect to have a one-form symmetry associated with Wilson lines, and a $7 - 3 = 4$-form symmetry associated with 't Hooft surfaces. Or slightly more generally, we have electric charge operators of dimension $7 - 1 - 1 = 5$, associated with elements of $H^2(\mathcal{M}_7)$, measuring the flux that would be created by Wilson lines, and magnetic charge operators of dimension $7 - 4 - 1 = 2$, associated to elements of $H^5(\mathcal{M}_7)$, measuring the flux created by 't Hooft surfaces. It is useful to make this distinction since on manifolds of non-trivial topology it is possible to introduce the fluxes without introducing the extended operators themselves. These extended charge operators are valued on $Z(G_\Gamma)$, the centre of the universal cover of any gauge group with algebra $\mathfrak{g}_\Gamma$. For the ADE cases, we have $Z(G_\Gamma) = \Gamma^{\text{ab}} := [\Gamma, \Gamma]$, the abelianization of Γ (see table 4.1 below).

As discussed in the last section, not all such higher-form symmetries are present in any given theory simultaneously, though: since the Wilson line operators are not mutually local with respect to the 't Hooft surfaces it is not possible to construct charge operators measuring all such charges at the same time. What we can do instead is — as in [134, 151] — to choose a maximal set of mutually local Wilson/'t Hooft operators, and declare that these are the genuine ones.

We refer to the choice of p-form charge operators present in the theory as a choice of global form for the theory and an actual choice of flux for these operators as a background for the p-form symmetry in that theory. If we sum over fluxes in $H^2(\mathcal{M}_7; \Gamma^{\text{ab}})_m$ we would have the G_Γ/Γ theory ($SU(N)/\mathbb{Z}_N$, for instance, in this context the fluxes are often known as the generalised Stiefel-Whitney classes of the bundle), with a 4-form symmetry, while if we sum over $H^5(\mathcal{M}_7; \Gamma^{\text{ab}})_e$ instead we have the G_Γ theory ($SU(N)$, for instance) with a 1-form symmetry. We emphasise that in a purely perturbative presentation the notion of "sum over fluxes in $H^5(\mathcal{M}_7; \Gamma^{\text{ab}})_e$" is somewhat formal, as there are no fields in the Lagrangian that can detect these fluxes. Nevertheless, this choice of language becomes very natural from the string theory point of view (and also for four-dimensional theories with electromagnetic

[1] It is possible to consider non-simply-laced cases too in M-theory using frozen fluxes [150]. We assume that no such fluxes are present, but it would certainly be interesting to understand how the discussion gets modified in this case.

duality, although we will not consider such examples in this thesis), so we will still adopt it.

Let us now discuss how to reproduce these results from the M-theory perspective, along the lines of [24]. The key fact is that in the presence of torsion at infinity the boundary values for the F_4 and F_7 fluxes do not commute following section 2.2.1. The spaces in which we are engineering the seven-dimensional $\mathfrak{g}_\Gamma$ theory are non-compact, so strictly speaking they have no boundaries, but we will assume that whenever we have a space of the form $\mathcal{M}_{11} = \mathcal{M}_p \times \mathcal{C}(\mathcal{N}_{10-p})$, with $\mathcal{C}(\mathcal{N}_{10-p})$ asymptotically a cone over $\mathcal{N}_{10-p}$, then the right prescription for choosing asymptotic values for the fields is to quantise the theory on $\mathbb{R}_t \times \mathcal{M}_{10}$, with $\mathcal{M}_{10} \coloneqq \mathcal{M}_p \times \mathcal{N}_{10-p}$, and to choose a state in the Hilbert space associated to $\mathcal{M}_{10}$.

We are (thankfully) only interested in the grading of the Hilbert space of M-theory by topological class of the flux. In general, M-theory fluxes live on some cohomology theory $\mathbb{E}_M$, which is known not to be ordinary (singular, say) cohomology. A rather dramatic effect that can take us away from ordinary cohomology is that there is a shifted flux quantization condition of the G_4 field strength of the M-theory 3-form gauge field C_3 [128]:

$$\left[\frac{G_4}{2\pi}\right] - \frac{p_1(\mathcal{M}_{11})}{4} \in H^4(\mathcal{M}_{11}) \tag{4.2}$$

with $p_1(\mathcal{M}_{11})$ the first Pontryagin class of the tangent bundle of $\mathcal{M}_{11}$. Fortunately, this shifted quantization condition will not affect our discussion in any significant way, since $p_1(\mathcal{M}_p \times \mathcal{C}(\mathcal{N}_{10-p})) = p_1(\mathcal{M}_p) + p_1(\mathcal{C}(\mathcal{N}_{10-p}))$, which has legs either purely along $\mathcal{M}_p$ or $\mathcal{C}(\mathcal{N}_{10-p})$. The fluxes of interest to us, on the other hand, have legs along both components. (An exception to this statement are fluxes associated with (-1)-form symmetries that we will encounter below, but here we will not try to understand these in any detail.) Due to this fact we will use ordinary singular cohomology in our calculations below.

The screening argument we discussed in the last section gives the following defect group for the geometric engineering Hilbert space of this theory

$$\mathbb{D} = Z(G_\Gamma)^{(1)}_{M2} \times Z(G_\Gamma)^{(4)}_{M5} \tag{4.3}$$

Given flux operators $\Psi_{M2,a}$ with $a \in \operatorname{Tor} H^4(\mathcal{M}_{10})$ and $\Psi_{M5,b}$ with $b \in \operatorname{Tor} H^7(\mathcal{M}_{10})$ (measuring torsional M2 and M5 charge, respectively) we have from section 2.2.1 that

$$\Psi_{M2,a}\Psi_{M5,b} = e^{2\pi i T(a,b)}\Psi_{M5,b}\Psi_{M2,a} \tag{4.4}$$

where

$$T\colon \; \mathrm{Tor}\, H^4(\mathcal{M}_{10}) \times \mathrm{Tor}\, H^7(\mathcal{M}_{10}) \to \mathbb{Q}/\mathbb{Z} \tag{4.5}$$

is the linking pairing on $\mathcal{M}_{10}$. The space $\mathbb{C}^2/\Gamma$ is a cone over S^3/Γ, and Γ acts freely on the S^3, therefore in the case at hand $\mathcal{M}_{10} = \mathcal{M}_7 \times (S^3/\Gamma)$. Assuming that $\mathcal{M}_7$ has no torsion we can apply the Künneth formula

$$H^n(\mathcal{M}_7 \times (S^3/\Gamma)) = \sum_{i+j=n} H^i(\mathcal{M}_7) \otimes H^j(S^3/\Gamma)\,. \tag{4.6}$$

Since

$$H^\bullet(S^3/\Gamma) = \{\mathbb{Z}, 0, \Gamma^{\mathrm{ab}}, \mathbb{Z}\}\,, \tag{4.7}$$

this implies that

$$\mathrm{Tor}\, H^4(\mathcal{M}_{10}) = H^2(\mathcal{M}_7) \otimes \Gamma^{\mathrm{ab}} = H^2(\mathcal{M}_7; \Gamma^{\mathrm{ab}}) \tag{4.8}$$

and

$$\mathrm{Tor}\, H^7(\mathcal{M}_{10}) = H^5(\mathcal{M}_7) \otimes \Gamma^{\mathrm{ab}} = H^5(\mathcal{M}_7; \Gamma^{\mathrm{ab}})\,. \tag{4.9}$$

Writing, accordingly, $a = \alpha \otimes \ell_a$ and $b = \beta \otimes \ell_b$, with $\alpha \in H^2(\mathcal{M}_7)$, $\beta \in H^5(\mathcal{M}_7)$ and $\ell_i \in H^2(S^3/\Gamma) = \Gamma^{\mathrm{ab}}$, we have

$$T(a,b) = (\alpha \cdot \beta) T_\Gamma(\ell_1, \ell_2) \tag{4.10}$$

with T_Γ the linking form in S^3/Γ. The general form for T_Γ is given in [152] – see table 4.1. For instance, consider the case $\Gamma = \mathbb{Z}_N$, corresponding to the $\mathfrak{su}(N)$ theories in seven dimensions. We have $\Gamma^{\mathrm{ab}} = [\mathbb{Z}_N, \mathbb{Z}_N] = \mathbb{Z}_N$, with a linking form

$$T_\Gamma(1,1) = \frac{1}{N} \mod 1\,. \tag{4.11}$$

This implies that the flux operators in this theory will not commute

$$\Psi_\alpha \Psi_\beta = e^{2\pi i \ell^{-1}(\alpha\cdot\beta)} \Psi_\beta \Psi_\alpha\,, \tag{4.12}$$

indicating the defect group of this theory suffers from a mixed 't Hooft anomaly. Here $\ell^{-1} := T_\Gamma(\ell_1, \ell_2)$. In particular, this entails that some care is needed when choosing boundary conditions. Recalling the discussion in section 3.1, we may choose a basis

Γ	G_Γ	Γ^{ab}	T_Γ
$\mathbb{Z}_N$	$SU(N)$	$\mathbb{Z}_N$	$\frac{1}{N}$
$\mathrm{Dic}_{(4N-2)}$	$\mathrm{Spin}(8N)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$
$\mathrm{Dic}_{(4N-1)}$	$\mathrm{Spin}(8N+2)$	$\mathbb{Z}_4$	$\frac{3}{4}$
$\mathrm{Dic}_{(4N)}$	$\mathrm{Spin}(8N+4)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$
$\mathrm{Dic}_{(4N+1)}$	$\mathrm{Spin}(8N+6)$	$\mathbb{Z}_4$	$\frac{1}{4}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{2}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{1}{2}$
$2I$	E_8	0	0

Table 4.1: Linking pairings for G_Γ

of the Hilbert space that diagonalises Ψ_a, for instance.[2] Namely:

$$\Psi_a \left|b\right\rangle = e^{2\pi i T(a,b)} \left|b\right\rangle \tag{4.13}$$

with $b \in \mathrm{Tor}\, H^7(\mathcal{M}_{10})$. On the other hand, the states in this basis do not diagonalise Ψ_b

$$\Psi_{b'} \left|b\right\rangle = \left|b+b'\right\rangle \ . \tag{4.14}$$

There is analogously a basis of states $|a\rangle$, with $a \in \mathrm{Tor}\, H^4(\mathcal{M}_{10})$ that diagonalises the Ψ_b operators. The two choices for the basis are related by a discrete Fourier transform:

$$\left|a\right\rangle = \sum_b e^{2\pi i T(a,b)} \left|b\right\rangle \ . \tag{4.15}$$

For instance, consider choosing a state $|0, L_{M5}\rangle$ such that $\Psi_b\,|0, L_{M5}\rangle = |0, L_{M5}\rangle$ for all $b \in H^7(\mathcal{M}_{10})$. This corresponds to setting all M5-brane fluxes to 0, so that the M2 branes are genuine operators. In the seven-dimensional theory, we can interpret this choice as being in the $SU(N)$ theory, with no background fluxes for the 1-form symmetry of this theory, turned on, where the line operators coming from the wrapped M2 branes are genuine.

If, on the other hand, we choose our boundary conditions to be given by a state $|0, L_{M2}\rangle$ such that $\Psi_a\,|0, L_{M2}\rangle = |0, L_{M2}\rangle$ for all $a \in H^4(\mathcal{M}_{10})$, then our background will be in a superposition of all possible background fluxes for the M5-brane charge, since due to the properties of the Heisenberg algebra a change of basis from the

[2]We refer the reader to [153] for a clear illustration in a related context.

electric to the magnetic basis is a discrete Fourier transform:

$$|0, L_{M2}\rangle = \sum_{a \in H^4(\mathcal{M}_{10})} |a, L_{M5}\rangle \ . \tag{4.16}$$

In terms of the seven-dimensional theory, this implies being on a superposition of all possible values for the Stiefel-Whitney classes, or in other words choosing the $SU(N)/\mathbb{Z}_N$ global form for the theory, having gauged the one-form symmetry of the $SU(N)$ theory. Notice that, as a consequence of this gauging, the resulting theory has a magnetic $\mathbb{Z}_N^{(4)}$ higher symmetry. Other global forms for the gauge group are often possible, depending on the choice of Γ, the analysis of the possibilities is identical to the one in [134].

4.1.1 Choice of global structure from 8d

In this section, we will argue that reducing M-theory on $S^3/\Gamma = \partial(\mathbb{C}^2/\Gamma)$ leads to an eight-dimensional TFT which encodes both the choice of a global form discussed in the last section for the seven-dimensional theory (equivalently, the choice of its higher-form symmetries) and the anomalies of these higher-form symmetries.

We expect these two sectors of the eight-dimensional TFT to interact in interesting ways: recall that choosing a global form for the gauge group (which will be able to rephrase as a choice of boundary behaviour in the BF theory (4.17)) can also be understood as a gauging of the higher-form symmetries [109]. In the presence of 't Hooft anomalies this gauging procedure might be obstructed, or lead to less conventional symmetry structures (see for instance [58, 154] for systematic discussions). It would be very interesting to analyse this problem from our higher dimensional vantage point, where it requires the study of gapped boundary conditions of the TFTs we construct, but we will not do so in this thesis.

We start by discussing how to see the choice of global form in terms of a choice of boundary conditions for a gapped eight dimensional theory. As we just discussed in the last section the geometric origin of the 1-form and 4-form symmetries, and the fact that there is a choice to be made, can be traced back to the non-commutativity of the boundary values of F_4 and F_7 fluxes in the presence of torsion in the asymptotic boundary $\partial\mathcal{M}_{11}$ at infinity. We found that this is encoded in the non-commutativity relation (4.12). This is precisely the operator content and commutation relations of a BF theory with the following Lagrangian (see [87] for a derivation)

$$S_{\mathrm{Sym}} = \ell \int B_2 \wedge dC_5 \,. \tag{4.17}$$

For $|\ell| > 1$ this is a non-invertible theory, whose state space reproduces the choices of global structure expected from the field theory side [24, 87, 134, 153].

4.1.2 Symmetry TFT

In addition to the higher symmetries discussed above, the seven-dimensional theory has additionally a 2-form $U(1)$ instanton symmetry, with the generator the integral of the instanton density on a closed 4-surface. We will now describe how to obtain the eight-dimensional anomaly theory encoding the mixed 't Hooft anomaly between the 1-form centre symmetries and the 2-form $U(1)$ instanton symmetry. We will find that the symmetry theory on any (closed and torsion-free, for simplicity) $\mathcal{W}_8$ can be derived by taking the eleven-dimensional Chern-Simons part of the M-theory action on $\mathcal{W}_8 \times S^3/\Gamma$, and integrating over S^3/Γ.

We are assuming $\mathrm{Tor}(H^3(\mathcal{W}_8)) = 0$, so by the universal coefficient theorem $H^2(\mathcal{W}_8; \Gamma^{\mathrm{ab}}) = H^2(\mathcal{W}_8) \otimes H^2(S^3/\Gamma)$, and we can parametrise a generic element $a_4 \in H^4(M^{11})$ by

$$a_4 = \sigma_4 \smile 1 + \sum_i \rho_2^{(i)} \smile t_{2(i)} + \sigma_1 \smile \mathrm{vol}(S^3/\Gamma) \,. \tag{4.18}$$

where $\mathrm{vol}(S^3/\Gamma)$ is the generator of $H^3(S^3/\Gamma) = \mathbb{Z}$, $t_{2(i)}$ are the (torsional) generators of $H^2(S^3/\Gamma) = \Gamma^{\mathrm{ab}}$, and "1" is the generator of $H^0(S^3/\Gamma) = \mathbb{Z}$.

Given an element $a_4 \in H^4(\mathcal{M}^{11})$ written as (4.18) we can uplift to a differential cohomology class by using (3.37) to write

$$\breve{G}_4 = \breve{\gamma}_4 \star \breve{1} + \sum_i \breve{B}_2^{(i)} \star \breve{t}_{2(i)} + \breve{\xi}_1 \star \breve{v}_3 + \tau([\omega_3]) \,, \tag{4.19}$$

where $\omega_3 \in \Omega^3(\mathcal{M}^{11})$, $\breve{\xi}_1 \in \breve{H}^1(\mathcal{W}_8)$, $\breve{v}_3 \in \breve{H}^3(S^3/\Gamma)$, $I(\breve{\xi}_1) = \sigma_1$ and $I(\breve{v}) = \mathrm{vol}(S^3/\Gamma)$. We have $I(\breve{\gamma}_4) = \sigma_4$ and we choose $\breve{t}_{2(i)}$ such that $R(\breve{t}_{2(i)}) = 0$.

For the problem at hand, the terms in $\breve{I}_{12}$ in (3.26) originating from X_8 do not contribute for degree reasons. (They will play an important role in the case of 5d SCFTs below.) For simplicity of exposition, let us assume first that Γ^{ab} has a single generator. Substituting (4.19) into the Chern-Simons coupling (3.25) we obtain

$$\begin{aligned}\frac{S_{\text{top}}}{2\pi} &= -\frac{1}{6}\int_{M_{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \\ &= \frac{1}{2}\int_{\mathcal{W}_8} \breve{\gamma}_4^2 \star \breve{\xi}_1 \int_{S^3/\Gamma} \breve{v} - \frac{1}{2}\int_{\mathcal{W}_8} \breve{\gamma}_4 \star \breve{B}_2^2 \int_{S^3/\Gamma} \breve{t}_2^2 - \frac{1}{6}\int \tau(w_3)\,.\end{aligned} \tag{4.20}$$

The first term in this expression encodes a potential mixed anomaly between a (-1)-form symmetry and the 2-form instanton symmetry. The second term in (4.20) corresponds to a mixed 't Hooft anomaly between the centre 1-form symmetry $Z(G_\Gamma) = \Gamma^{\text{ab}}$ and the instanton 2-form symmetry. In what follows we will concentrate on this last term. To find the coefficient of this anomaly, we must evaluate the integral

$$\text{CS}[S^3/\Gamma, \breve{t}_2] = \frac{1}{2}\int_{S^3/\Gamma} \breve{t}_2 \star \breve{t}_2\,. \tag{4.21}$$

This $\mathbb{R}/\mathbb{Z}$-valued quantity is the spin Chern-Simons invariant, evaluated for the 3-manifold S^3/Γ and the flat connection $\breve{t}_2 \in \breve{H}^2(S^3/\Gamma)$. In general such Chern-Simons invariants also depend on the spin structure on the manifold. In our case, by construction, we have the spin connection induced on the boundary of the supersymmetric compactification of M-theory on $\mathbb{C}^2/\Gamma$.

Using $\int_{S^3/\Gamma} \breve{v} = 1$, and neglecting the τ term according to the general discussion of section 3.3.2, we compute the SymTFT:

$$\boxed{S_{\text{Sym}} = \frac{1}{2}\int_{\mathcal{W}_8} \breve{\gamma}_4 \star \breve{\gamma}_4 \star \breve{\xi}_1 - \text{CS}[S^3/\Gamma, \breve{t}_2]\int_{\mathcal{W}_8} \breve{\gamma}_4 \star \breve{B}_2 \star \breve{B}_2\,.} \tag{4.22}$$

The cases in which Γ^{ab} has two generators can be treated in a completely analogous way, yielding

$$S_{\text{Sym}} = \frac{1}{2}\int_{\mathcal{W}_8} \breve{\gamma}_4 \star \breve{\gamma}_4 \star \breve{\xi}_1 - \sum_{i,j} \text{CS}[S^3/\Gamma]_{ij} \int_{\mathcal{W}_8} \breve{\gamma}_4 \star \breve{B}_2^{(i)} \star \breve{B}_2^{(j)}\,. \tag{4.23}$$

Note that formally one would be tempted to write

$$\text{CS}[S^3/\Gamma]_{ij} = \frac{1}{2}\int_{S^3/\Gamma} \breve{t}_{2(i)} \star \breve{t}_{2(j)}\,. \tag{4.24}$$

The factor of $\frac{1}{2}$ makes the right hand side not well defined. Luckily (but unsurprisingly, given that our starting Chern-Simons coupling in M-theory action is well-defined [128]), due to the symmetry properties (2.22) of the Cheeger-Simons product it is only the sum $\mathrm{CS}[S^3/\Gamma]_{ij} + \mathrm{CS}[S^3/\Gamma]_{ji}$ that enters in the anomaly theory (4.23), and this sum is well defined:

$$\mathrm{CS}[S^3/\Gamma]_{ij} + \mathrm{CS}[S^3/\Gamma]_{ji} = \int_{S^3/\Gamma} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \,. \tag{4.25}$$

Similar remarks apply to the off-diagonal entries in the D_{2n} case in table 4.3 below. We provide a more systematic discussion of this issue at the end of this section.

Evaluation of the Chern-Simons invariant

Let us now discuss a convenient formalism to evaluate the CS invariant (4.21) (including the 1/2 prefactor), obtained by a a straightforward generalisation of a discussion in the three dimensional case by Gordon and Litherland [155].

Let Y_{d-1} be a closed, connected, oriented $(d-1)$-manifold, and suppose that Y_{d-1} can be realised as boundary of a d-manifold X_d. The long exact sequence in relative homology yields

$$\cdots \to H_{d-2}(X_d) \to H_{d-2}(X_d, Y_{d-1}) \to H_{d-3}(Y_{d-1}) \to H_{d-3}(X_d) \to \dots \,. \tag{4.26}$$

We now make the assumption

$$H_{d-3}(X_d) = 0 \,. \tag{4.27}$$

Using Poincaré duality in Y_{d-1} we have $H_{d-3}(Y_{d-1}) \cong H^2(Y_{d-2})$, and from (4.26) we get the exact sequence

$$H_{d-2}(X_d) \xrightarrow{A} H_{d-2}(X_d, Y_{d-1}) \xrightarrow{f} H^2(Y_{d-1}) \to 0 \,. \tag{4.28}$$

Notice in particular that the homomorphism f is surjective: any class in $H^2(Y_{d-1})$ can be lifted to an element in $H_{d-2}(X_d, Y_{d-1})$. Let us now consider a torsional class $a_2 \in H^2(Y_{d-1})$, satisfying $na_2 = 0$ for some positive integer n. We know that there exists an element $\kappa \in H_{d-2}(X_d, Y_{d-1})$ such that $f(\kappa) = a_2$. Since f is a homomorphism, $0 = na_2 = nf(\kappa) = f(n\kappa)$, i.e. $n\kappa \in \ker f$. Exactness of the sequence (4.28) implies that there exists an element $Z \in H_{d-2}(X_d)$ such that $A(Z) = n\kappa$.

The manifolds Y_{d-1} that we want to study are the link of a canonical Calabi-Yau singularity X_d^{singular}, so in our case there is a very natural family of choices of X_d: we can take any crepant resolution of X_d^{singular}. Relatedly, we refer to elements of

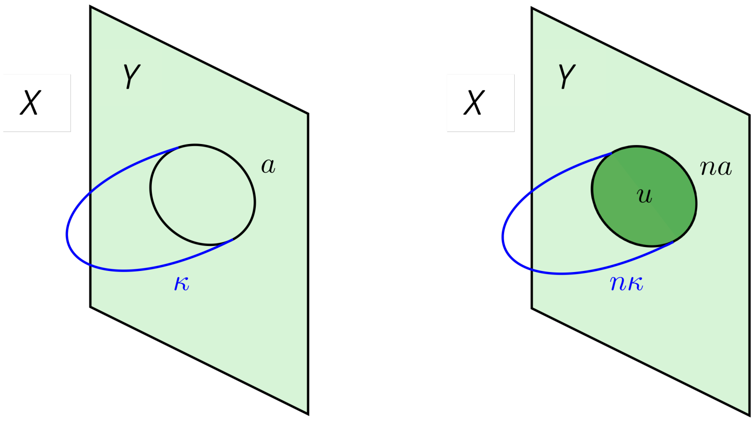


Figure 4.1: On the left: Under the assumption $H_{d-3}(X_d) = 0$, any $(d-3)$-cycle $a \in Z_{d-3}(Y_{d-1})$ in the link can be realised as boundary of a $(d-2)$-chain $\kappa \in C_{d-2}(X_d)$ in the bulk X_d. On the right: If a represents a torsional homology class, $na = \partial u$ for some $(d-2)$-chain $u \in C_{d-2}(Y_{d-1})$ in the link, which can also be naturally regarded as an element in $C_{d-2}(X_d)$. Combining the chains u and $n\kappa$ we get a cycle, $\partial(n\kappa - u) = 0$. This cycle can now be smoothly retracted to the interior of X_d, and can therefore be thought of as a compact cycle. Its homology class $[n\kappa - u]$ represents $Z \in H_{d-2}(X_d)$.

$H_p(X_d, Y_{d-1})$ as non-compact p-cycles in X_d, and to elements of $H_p(X_d)$ as compact p-cycles in X_d. The observations made so far can be summarised as follows:

- To every class $a_2 \in H^2(Y_{d-1})$ we can associate a non-compact $(d-2)$-cycle κ in X_d.
- To every torsional class $a_2 \in H^2(Y_{d-1})$ we can associate a compact $(d-2)$-cycle Z in X_d via the following relations,

$$na_2 = 0 \ , \qquad a_2 = f(\kappa) \ , \qquad A(Z) = n\kappa \ , \tag{4.29}$$

  where κ is a non-compact $(d-2)$-cycle in X_d.

Let us now analyse the map A in (4.28) in greater detail. By Lefschetz duality,

$$H_{d-2}(X_d, Y_{d-1}) \cong H^2(X_d) \ . \tag{4.30}$$

To proceed, we make the further assumption (that holds in all the cases in this thesis)

$$\text{Tor}\, H_1(X_d) = 0 \ . \tag{4.31}$$

The universal coefficient theorem then guarantees that

$$H^2(X_d) \cong \mathrm{Hom}(H_2(X_d), \mathbb{Z}) \ . \tag{4.32}$$

We may then recast (4.28) in the form

$$H_{d-2}(X_d) \xrightarrow{A} \mathrm{Hom}(H_2(X_d), \mathbb{Z}) \xrightarrow{f} H^2(Y_{d-1}) \to 0 \ . \tag{4.33}$$

The homomorphism A can be equivalently regarded as a bilinear $\mathbb{Z}$-valued pairing between $H_2(X_d)$ and $H_{d-2}(X_d)$,

$$A : H_{d-2}(X_d) \otimes H_2(X_d) \to \mathbb{Z} \ . \tag{4.34}$$

Indeed, A is identified with the intersection pairing of compact $(d-2)$-cycles and compact 2-cycles in X_d. Once we choose a basis for $H_2(X_d)$ and $H_{d-2}(X_d)$, the map A is represented by the intersection matrix $\mathcal{M}_{d-2,2}$.

Evaluation for S^3/Γ

To compute the SymTFT for the 7d theory (4.22) we need to evaluate the CS invariant. Let us now specialise to a 3d link Y_3, and fix a class $t_2 \in \mathrm{Tor}\, H^2(Y_3)$ such that $nt_2 = 0$. Let $Z \in H_2(X_4)$ be the compact 2-cycle in X_4 associated to t_2. By the discussion above, the linking pairing of (the Poincaré dual to) t_2 with itself can be computed as

$$\int_{Y_3} \breve{t}_2 \star \breve{t}_2 = \mathsf{T}_{Y_3}(\mathrm{PD}[t_2], \mathrm{PD}[t_2]) = \left[\frac{Z \cdot Z}{n^2}\right]_{\mathrm{mod}\ 1} . \tag{4.35}$$

In the above expression, $\cdot$ denotes the intersection pairing among compact 2-cycles in X_4. Our task is actually to compute a CS invariant of the form (4.21). In the Gordon-Litherland approach, this quantity is given by

$$\mathrm{CS}[Y_3, \breve{t}_2] = \frac{1}{2} \int_{Y_3} \breve{t}_2 \star \breve{t}_2 = \left[\frac{Z \cdot Z}{2\, n^2}\right]_{\mathrm{mod}\ 1} . \tag{4.36}$$

In particular, we apply this formalism to the case of interest $Y_3 = S^3/\Gamma$. The bulk X_4 can be chosen to be the resolved ALE space $\mathbb{C}^2/\Gamma$. We notice that the assumptions (4.27) and (4.31) are indeed satisfied. The intersection matrix $\mathcal{M}_{2,2}$ representing the map A equals minus the Cartan matrix of the Lie algebra $\mathfrak{g}_\Gamma$. For the ADE-singularities, the choice of central divisors which gives the centre of the

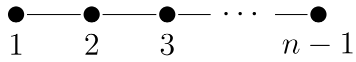
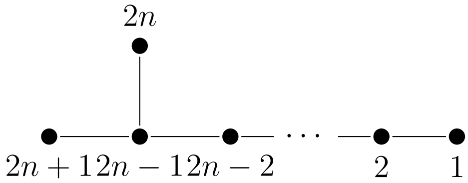
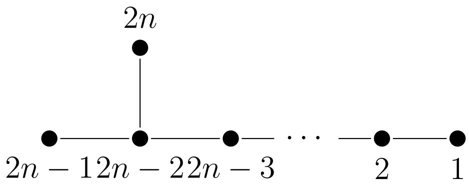
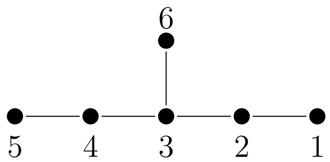
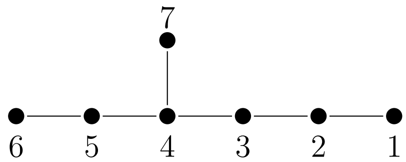
Γ	Dynkin diagram	Z
A_{n-1}		$\sum_{i=1}^{n-1} iS_i$
D_{2n}		$\sum_{i=1}^{2n-1}(1-(-1)^i)S_i$
D_{2n+1}		$\frac{1}{2}\sum_{i=1}^{2n-1}(1-(-1)^i)S_i\,,$ $\frac{1}{2}\sum_{i=1}^{2n-2}\left(1-(-1)^i\right)S_i + S_{2n}$
E_6		$\sum_{i=1}^{5} iS_i + S_{2n} + 3S_{2n+1}$
E_7		$S_1 + S_3 + S_7$

Table 4.2: The centre divisors Z written in terms of the compact curves associated to the simple roots for ADE-singularities. We omit the E_8 case as it has trivial centre symmetry.

gauge group has been identified in [47]. We list their results in table 4.2. Using these compact divisors as representatives of the torsional generators in order to be able to compare easily with known field theory results,[3] we obtain the results given in 4.3 for the spin Chern-Simons invariant $\mathrm{CS}[S^3/\Gamma, \breve{t}_2]$. It is a nice check of our formalism that the resulting coefficients in the anomaly theory perfectly reproduce the answer one gets from a pure field theory analysis [117]. (The A_{n-1} answer was also recently obtained in [26] from a related viewpoint.)

Relation to the Linking Pairing. Finally, we want to comment on the relationship between the Chern-Simons invariant $\mathrm{CS}[S^3/\Gamma, \breve{t}_2]$ and the linking pairing

[3]Note that there is a genuine ambiguity here: if t is a generator of $\mathbb{Z}_n$, so is kt for any k such that $\gcd(k, n) = 1$. We have $CS[S^3/\Gamma, k\breve{t}_2] = k^2 CS[S^3/\Gamma, \breve{t}_2] \bmod 1$, so this rescaling can potentially change the coefficient of the anomaly. This is why it is important to choose the right generator of the torsional group when comparing with field theory results, even though any choice of k is in principle physically allowed.

Γ	G_Γ	Γ^{ab}	$-\text{CS}[S^3/\Gamma, \breve{t}_2]$
A_{n-1}	$SU(n)$	$\mathbb{Z}_n$	$\frac{n-1}{2n}$
D_{2n}	$\text{Spin}(4n)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\frac{1}{4}\begin{pmatrix} n & n-1 \\ n-1 & n \end{pmatrix}$
D_{2n+1}	$\text{Spin}(4n+2)$	$\mathbb{Z}_4$	$\frac{2n+1}{8}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{5}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{3}{4}$
$2I$	E_8	0	0

Table 4.3: Γ of ADE type. G_Γ denotes the simply-connected gauge group in 7d, Γ^{ab} the abelianization of Γ. Finally $\text{CS}[S^3/\Gamma, \breve{t}_2]$ is the Chern-Simons invariant (4.21), which is closely related to the linking pairing, as explained in the main text. With these expressions, one can then evaluate the SymTFT for 7d SYM in (4.22).

$T(t_2, t_2)$. The relation is that $\text{CS}[S^3/\Gamma, \breve{t}_2]$ provides a quadratic refinement [96, 128] of the linking pairing:

$$\begin{aligned} T_{S^3/\Gamma}(\text{PD}[s_2], \text{PD}[t_2]) &= \int_{S^3/\Gamma} \breve{s}_2 \star \breve{t}_2 \\ &= \text{CS}[S^3/\Gamma, \breve{s}_2 + \breve{t}_2] - \text{CS}[S^3/\Gamma, \breve{s}_2] - \text{CS}[S^3/\Gamma, \breve{t}_2] \mod 1 \,, \end{aligned} \tag{4.37}$$

where $\breve{s}_2, \breve{t}_2 \in \breve{H}^2(S^3/\Gamma)$ are chosen to be flat: $R(\breve{s}_2) = R(\breve{t}_2) = 0$. The equality on the left can be proven as follows.

Since $\breve{s}_2$ and $\breve{t}_2$ are flat, we can use the commutativity of (2.13), and (2.23) to write

$$\breve{s}_2 \star \breve{t}_2 = -\breve{s}_2 \star i(\beta^{-1}(I(\breve{t}_2))) = -i\left(I(\breve{s}_2) \smile \beta^{-1}(I(\breve{t}_2))\right) \,, \tag{4.38}$$

where we have also used that $H^1(S^3/\Gamma; \mathbb{R}) = H^2(S^3/\Gamma; \mathbb{R}) = 0$, so the Bockstein map β in (2.13) is an isomorphism. For the integral of (4.38) we then have

$$\begin{aligned} \int_{S^3/\Gamma} i\left(I(\breve{s}_2) \smile \beta^{-1}(I(\breve{t}_2))\right) &= i\left(\int_{S^3/\Gamma} I(\breve{s}_2) \smile \beta^{-1}(I(\breve{t}_2))\right) \\ &= \int_{S^3/\Gamma} I(\breve{s}_2) \smile \beta^{-1}(I(\breve{t}_2)) \,, \end{aligned} \tag{4.39}$$

since $i : H^0(\mathrm{pt}; U(1)) \to \breve{H}^1(\mathrm{pt})$ is an isomorphism. The final expression in (4.39) is just the linking pairing T on S^3/Γ [156] (up to a sign convention)

$$\int_{S^3/\Gamma} \breve{s}_2 \star \breve{t}_2 = - \int_{S^3/\Gamma} I(\breve{s}_2) \smile \beta^{-1}(I(\breve{t}_2)) = T_{S^3/\Gamma}(\mathrm{PD}[s_2], \mathrm{PD}[t_2]) \mod 1\,. \quad (4.40)$$

This refinement of the linking pairing extends, in particular, the observation in [24] that the fractional instanton number for an instanton bundle in the presence of background 1-form flux is half of the linking pairing in S^3/Γ for the torsional class t_2 representing the 1-form flux background. The more refined statement that follows from our M-theory construction is instead:

$$n_{\mathrm{inst}} = -\mathrm{CS}[S^3/\Gamma, \breve{t}_2] \mod 1\,. \quad (4.41)$$

The discussion in [24] was specific to four dimensional theories on Spin manifolds, and the two statements agree on that class of manifolds (up to an overall sign that was chosen oppositely in [24]), but (4.41) gives the correct answer on non-Spin manifolds too.

4.2 5d superconformal field theories

The story is fairly similar for five-dimensional theories engineered from M-theory on singular Calabi-Yau threefolds, but the possibilities in geometry and field theory are much richer. For concreteness, we will focus on geometries of the kind $\mathcal{M}_5 \times \mathcal{V}_6$, where $\mathcal{M}_5$ is a closed Spin manifold without torsion, and $\mathcal{V}_6$ a non-compact Calabi-Yau threefold, given by the cone over some Sasaki-Einstein manifold Y_5. In order to make further progress below we will assume some additional conditions on $\mathcal{V}_6$, namely that $H_{2n+1}(\mathcal{V}_6) = 0$ for all n, and that $\mathrm{Tor}\, H_{2n}(\mathcal{V}_6) = 0$ for all n. Toric Calabi-Yau threefold varieties are an important class of examples that satisfy these requirements, and we will focus mostly on these below.

4.2.1 Field theory analysis

Let us first describe the expectations from field theory, in analogy with the seven-dimensional discussion above. In the five-dimensional theory, we will have line and surface operators, which we will call Wilson lines and 't Hooft surfaces, following the standard terminology in the cases with a Lagrangian description. The charge of these objects is measured on three-dimensional and two-dimensional surfaces linking the respective objects in $\mathcal{M}_5$. Equivalently, depending on the global structure that we choose for the theory, we have 1-form symmetries with background fluxes valued on $H^2(\mathcal{M}_5; \mathsf{Z})$, 2-form symmetries with background fluxes valued on $H^3(\mathcal{M}_5; \mathsf{Z})$, or combinations of both. Here Z is a group that in the cases with a Lagrangian is given by the subgroup of the universal cover of the gauge group that leaves all point operators invariant [134], as discussed in section 2.3. For instance, if we have a 5d Yang-Mills theory with algebra $\mathfrak{su}(N)$ and matter in the adjoint, we have that $\mathsf{Z} = \mathbb{Z}_N$.

As in the seven-dimensional case not all of these symmetries are simultaneously present in any given theory, and one cannot independently introduce background fluxes for all of them. Rather, we must choose a maximal mutually local set of extended operators, and introduce fluxes only for those. In the Lagrangian context, this choice is a choice of the global form for the gauge group. For instance, if the algebra is $\mathfrak{su}(N)$, a possible choice of global form is given by $SU(N)/\mathbb{Z}_N$, where we sum over all background fluxes in $H^2(\mathcal{M}_5; \mathbb{Z}_N)$ — that is, we sum over Stiefel-Whitney classes. Alternatively, we could sum over bundles in $H^3(\mathcal{M}_5; \mathbb{Z}_N)$. (As remarked above, the sum is not visible in the usual Lagrangian presentation.)

An interesting feature of five-dimensional theories is that instanton configurations behave very much like particles. In the presence of a Chern-Simons coupling (for $\mathfrak{su}(N)$ with $N > 2$)

$$S_{\text{CS}} = k \int \Omega_A \tag{4.42}$$

where Ω_A is the Chern-Simons form, these particles can potentially acquire a charge under the centre of the gauge group. If this happens, then the higher form symmetry of the $SU(N)$ theory can be (partially) broken, in the same way that ordinary matter in generic representations break the symmetry. Our task below will be to compute the charge of these particles under the centre of the $SU(N)$ gauge group, but the form of the coupling (4.42) suggests that the right answer will be that in the presence of such a coupling instanton particles acquire a charge k under the $\mathbb{Z}_N$ centre of the $SU(N)$ theory. One heuristic way to argue for this is that an instanton background becomes, in the point-like limit $\text{Tr}(F^2) = \delta^4(\vec{x})$, with $\vec{x}$ the directions transverse to the instanton, so the $\text{Tr}(A \wedge F^2)$ term in Ω_A becomes an integral of A over the worldline of the instanton particle, so the Chern-Simons level k can be identified with the charge of the particle. We will give below a more careful argument that shows that this is indeed the right result in the theories that we study. This implies that the centre of a $SU(N)$ theory at level k is broken down to $\mathbb{Z}_{\gcd(N,k)}$ due to the charge of the instanton particles.[4]

4.2.2 M-theory reduction

We now want to understand the previous gauge theory discussion in terms of the M-theory engineering of the relevant 5d theories. The line and surface operators of the five-dimensional operator will come from M2 and M5 branes wrapping suitable non-compact cycles in the internal toric Calabi-Yau threefold. As in the seven-dimensional case, we classify which of these operators can be simultaneously taken to be genuine by looking to a maximal choice of commuting fluxes on the boundary $\partial\mathcal{M}_{11}$ (which, recall, has topology $\mathcal{M}_5 \times Y_5$ in our case, with Y_5 a Sasaki-Einstein manifold). The non-trivial part of the flux commutation relations will come from the pairing

$$T\colon \ \mathrm{Tor}\, H^4(\partial\mathcal{M}_{11}) \times \mathrm{Tor}\, H^7(\partial\mathcal{M}_{11}) \to \mathbb{Q}/\mathbb{Z}\,. \tag{4.43}$$

We will show momentarily that Y_5 only has torsion in $H_1(Y_5) = H^4(Y_5)$ (or equivalently, by the universal coefficient theorem [157], in $H_3(Y_5) = H^2(Y_5)$). Together with the fact that $\partial\mathcal{M}_{11} = \mathcal{M}_5 \times Y_5$, with $\mathcal{M}_5$ torsion-free, this implies that we can use the Künneth formulas

$$\mathrm{Tor}\, H^4(\partial\mathcal{M}_{11}) = \Big(H^2(\mathcal{M}_5) \otimes \mathrm{Tor}\, H^2(Y_5)\Big) \oplus \Big(H^0(\mathcal{M}_5) \otimes \mathrm{Tor}\, H^4(Y_5)\Big)\,, \tag{4.44a}$$

$$\mathrm{Tor}\, H^7(\partial\mathcal{M}_{11}) = \Big(H^3(\mathcal{M}_5) \otimes \mathrm{Tor}\, H^4(Y_5)\Big) \oplus \Big(H^5(\mathcal{M}_5) \otimes \mathrm{Tor}\, H^2(Y_5)\Big)\,. \tag{4.44b}$$

Poincaré duality, together with the universal coefficient theorem, implies that $\mathrm{Tor}\, H^2(Y_5) = \mathrm{Tor}\, H^4(Y_5)$. For conciseness, let us define

$$\mathsf{Z} \coloneqq \mathrm{Tor}\, H_3(Y_5) = \mathrm{Tor}\, H^2(Y_5) = \mathrm{Tor}\, H^4(Y_5) = \mathrm{Tor}\, H_1(Y_5)\,. \tag{4.45}$$

(In Lagrangian theories Z will be the centre of the simply connected group with the given algebra.) The defect group for these geometries is

$$\mathbb{D} = \Big(\mathsf{Z}_{M2}^{(1)} \oplus \mathsf{Z}_{M5}^{(2)}\Big) \oplus \Big(\mathsf{Z}_{M2}^{(-1)} \oplus \mathsf{Z}_{M5}^{(4)}\Big)\,. \tag{4.46}$$

Under the assumption that $\mathcal{M}_5$ is torsion-free, the universal coefficient theorem implies that $H^i(\mathcal{M}_5) \otimes \mathsf{Z} = H^i(\mathcal{M}_5; \mathsf{Z})$, so the first terms on the right-hand side

[4] Denote by p the order of the surviving group. By Lagrange's theorem p divides N, so the subgroup is generated by N/p. For this element to leave a particle of charge k invariant we need that $kN/p \equiv 0 \bmod N$, or equivalently that p divides k.

are the ones we had anticipated from our field theory analysis above. We see that in the M-theory language these cohomology groups parametrise the flux operators measuring the fluxes created by M2 branes wrapping non-compact two-cycles in V, and M5 branes wrapping non-compact four-cycles in V, respectively, as one would have expected. The last two terms correspond to (-1)-form and 4-form symmetries, which are somewhat more exotic from the field theory point of view, and we will ignore them in our analysis. (See [158–160] for recent work exploring such symmetries from the field theory point of view.)

Next, we use Lefschetz duality [157] to rewrite

$$H_k(X_6, Y_5) = H_c^{n-k}(X_6)\,. \tag{4.47}$$

We will assume that X_6 is torsion-free (which is true, in particular, for toric varieties [161–163]), so by the universal coefficient theorem for cohomology we get

$$H_k(X_6, Y_5) = H_c^{n-k}(X_6) = \mathrm{Hom}(H_{n-k}(X_6), \mathbb{Z})\,. \tag{4.48}$$

If we now assume that $H_{2n+1}(X_6) = 0$ (which is again true in the special case of toric varieties [161–163]), the long exact sequence (3.13) reduces to

$$0 \to \mathrm{Hom}(H_0(X_6), \mathbb{Z}) \to H_5(Y_5) \to 0\,, \tag{4.49a}$$

$$0 \to H_4(Y_5) \to H_4(X_6) \xrightarrow{Q_4} \mathrm{Hom}(H_2(X_6), \mathbb{Z}) \xrightarrow{\partial_4} H_3(Y_5) \to 0\,, \tag{4.49b}$$

$$0 \to H_2(Y_5) \to H_2(X_6) \xrightarrow{Q_2} \mathrm{Hom}(H_4(X_6), \mathbb{Z}) \xrightarrow{\partial_2} H_1(Y_5) \to 0\,, \tag{4.49c}$$

$$0 \to H_0(Y_5) \to H_0(X_6) \to 0\,, \tag{4.49d}$$

where, the homomorphisms $Q_k\colon H_k(X_6) \to \mathrm{Hom}(H_{6-k}(X_6), \mathbb{Z})$ are given by partial evaluation of the intersection forms

$$q_k\colon H_k(X_6) \times H_{6-k}(X_6) \to \mathbb{Z} \tag{4.50}$$

with $k = 2, 4$. That is, $Q_k(x)(y) = q_k(x, y)$. Note that $Q_4 = Q_2^t$. It follows from these exact sequences that $H_0(Y_5) = H_5(Y_5) = H_0(X) = \mathbb{Z}$, and that

$$\begin{aligned} H_4(Y_5) &= \ker(Q_4), \quad H_3(Y_5) = \mathrm{coker}(Q_4), \\ H_2(Y_5) &= \ker(Q_2), \quad H_1(Y_5) = \mathrm{coker}(Q_2), \end{aligned} \tag{4.51}$$

so finally

$$\mathsf{Z} = \operatorname{Tor}\operatorname{coker}(Q_4) = \operatorname{Tor}\operatorname{coker}(Q_2)\,. \tag{4.52}$$

Having understood the space of charge operators for the five dimensional theory, we still need to find their commutation relations. This follows straightforwardly from (4.4), in a way very analogous to (4.10). Writing $a = \alpha \otimes \Sigma_a$ and $b = \beta \otimes D_b$, with $\alpha \in H^2(\mathcal{M}_5)$, $\beta \in H^3(\mathcal{M}_5)$, $\Sigma_a \in \operatorname{Tor} H^2(Y_5)$ and $D_b \in \operatorname{Tor} H^4(Y_5)$, we have

$$T(a,b) = (\alpha \cdot \beta) T_{Y_5}(\Sigma_a, D_b) \tag{4.53}$$

with T_{Y_5} the linking form in Y_5, a perfect pairing

$$T_{Y_5}\colon\ \operatorname{Tor} H^k(Y_5) \times \operatorname{Tor} H^{5-k+1}(Y_5) \to \mathbb{Q}/\mathbb{Z}\,. \tag{4.54}$$

We can derive T_{Y_5} from knowledge of the intersection matrix $Q_4 = Q_2^t$ as follows. Let $\sigma \in \operatorname{Tor} H_3(Y_5)$ and $\bar{\sigma} \in \operatorname{Tor} H_1(Y_5)$, and choose $\mu \in \operatorname{Hom}(H_2(Y_5), \mathbb{Z})$ and $\bar{\mu} \in \operatorname{Hom}(H_4(Y_5), \mathbb{Z})$ such that, $\partial_4 \mu = \sigma$ and $\partial_2 \bar{\mu} = \bar{\sigma}$. Then, for non-trivial $\operatorname{Tor} H_3(Y_5)$ and $\operatorname{Tor} H_1(Y_5)$, there are non-zero integers n and m such that $\partial(n\mu) = n\sigma = 0$ and $\bar{\partial}(m\bar{\mu}) = m\bar{\sigma} = 0$. Thus, we may pick $\nu \in H_4(X_6)$ and $\bar{\nu} \in H_2(X_6)$ such that, $Q_4 \nu = n\mu$ and $Q_2 \bar{\nu} = m\bar{\mu}$. The linking pairing is then[5] [164]

$$T_{Y_5}(\sigma, \bar{\sigma}) \equiv \frac{1}{nm} q(\nu, \bar{\nu}) \qquad \bmod 1\,. \tag{4.55}$$

This may be equivalently written as

$$T_{Y_5}(\sigma, \bar{\sigma}) \equiv q^{-1}(\mu, \bar{\mu}) \quad \bmod 1\,, \tag{4.56}$$

where $q^{-1}\colon\ \operatorname{Hom}(H_2(X_6), \mathbb{Z}) \times \operatorname{Hom}(H_4(X_6), \mathbb{Z}) \to \mathbb{Q}$. More explicitly, this means that, if $\alpha_i'^*$ is a generator of $\operatorname{Hom}(H_2(X_6), \mathbb{Z})$ and $\beta_j'^*$ is a generator of $\operatorname{Hom}(H_4(X_6), \mathbb{Z})$ such that $\partial \alpha_i'^*$ is the generator of $\operatorname{Tor} H_3(Y_5)$ and $\bar{\partial} \beta_j'^*$ is the generator of $\operatorname{Tor} H_1(Y_5)$ then, the linking number is just the $(i \times j)$th element of q^{-1}:

$$T_{Y_5}(\partial \alpha_i'^*, \bar{\partial} \beta_j'^*) = q^{-1}(\alpha_i'^*, \beta_j'^*) = q^{-1}_{ij} \quad \bmod 1\,. \tag{4.57}$$

The appendices contain various worked out examples of the application of this relation, which encodes the discrete mixed 't Hooft anomaly coefficients for defect

[5] Note that we use q and q^t interchangeably. It should be clear from the context which one we mean.

groups associated to the higher form symmetries of 5d SCFTs.

4.2.3 The case of toric Calabi-Yau varieties

An important special case of the previous discussion is that when $\mathcal{V}_6$ is a toric Calabi-Yau variety. (We refer the reader to [162, 165–167] for systematic reviews of toric geometry.)

The crepant resolution $\tilde{\mathcal{V}}_6$ is obtained by choosing a triangulation for the toric diagram. As mentioned above, the odd dimensional homology groups of a toric variety vanish, and the even homology groups for X_6 can be easily obtained by looking at the toric diagram for $\mathcal{V}_6$. Let I be the number of points in the interior of the diagram and B be the number of points on the edges of the diagram. Then the number of 4-cycles is I, and since $\mathcal{V}_6$ is connected the number of 0-cycles is 1. The Euler characteristic of the Calabi-Yau equals to the number of 2-dimensional faces of the resolved toric diagram [168], which is twice the area A of the toric diagram. The Euler characteristic also equals the number of even-dimensional cycles minus the number of odd-dimensional cycles. We know by Pick's theorem that $2A = 2I + B - 2$, so the number of 2-cycles is $I + B - 3$, and we have

$$H_0(X) = \mathbb{Z},\ H_2(X) = \mathbb{Z}^{(I+B-3)},\ H_4(X) = \mathbb{Z}^I,\ H_6(X) = 0\,. \tag{4.58}$$

One can also compute Q_4 (or equivalently Q_2) in toric varieties quite conveniently purely in terms of the toric data. What one needs to construct is the Mori cone of effective curves in the toric variety, and find their intersections with the compact divisors, which are manifest in the toric description as points in the interior of the toric diagram. Well-developed algorithms for doing this exist, reviewed for example in [165], and implemented for instance in Sage [169]. As an example, consider the Calabi-Yau cone over $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. This geometry can be alternatively described as the (real) Calabi-Yau cone over $Y^{2,0}$. Its toric diagram has external vertices $p_i = \{(1,0),(-1,0),(0,1),(0,-1)\}$, and an internal vertex at $t = (0,0)$. Its Mori cone is generated by two curves $\mathcal{C}_1, \mathcal{C}_2$ with intersection matrix with the toric divisors given by

$$\begin{array}{c|ccccc} & V(p_1) & V(p_2) & V(p_3) & V(p_4) & V(t) \\ \hline \mathcal{C}_1 & 1 & 0 & 1 & 0 & -2 \\ \mathcal{C}_2 & 0 & 1 & 0 & 1 & -2 \end{array} \tag{4.59}$$

We have included all toric divisors here, but the $V(p_i)$ divisors are non-compact. The divisor $V(t)$ is compact, on the other hand, so we find that

$$Q_2 = \begin{pmatrix} -2 & -2 \end{pmatrix} \tag{4.60}$$

and we predict that

$$\mathsf{Z} = \operatorname{Tor}\operatorname{coker}(Q_2) = \mathbb{Z}_2 \,. \tag{4.61}$$

We will see below that this result agrees with the expectation from field theory: the defect group in this case is

$$\mathbb{D} = \mathbb{Z}_2^{(1)} \oplus \mathbb{Z}_2^{(2)}. \tag{4.62}$$

All the results below can be derived using these methods, but in practice it is much more efficient to use instead a method introduced (to our knowledge) in [170], which avoids the need to introduce a triangulation for computing the Mori cone. Consider a toric Calabi-Yau cone with an isolated singularity, and v external vertices. In terms of the toric diagram, this means that there are no lattice points along the edges of the toric diagram. As argued in [170], one has that

$$H_i(Y_5) = H_i(B_3^L) \qquad \text{for } i \leq 2\,, \tag{4.63}$$

where B_3^L is a chain of lens spaces $L_{n_1}, \ldots, L_{n_v}$, joined at their torsion cycle, constructed as follows. For each external vertex p_i, $i \in \{1, \ldots, v\}$, construct the triangle T_i defined by the vertex and the two vertices adjacent to it, that is, the convex hull of $\{p_{i-1}, p_i, p_{i+1}\}$ (with $p_0 \coloneqq p_v$ and $p_{v+1} \coloneqq p_1$). Then $n_i = 2\text{Area}(T_i)$. Additionally, one can show that [170]

$$H_1(B_3^L) = \mathbb{Z}_{\gcd(n_1,\ldots,n_v)}\,, \tag{4.64}$$

so we find that in the toric case

$$\mathsf{Z} = \mathbb{Z}_{\gcd(n_1,\ldots,n_v)} \tag{4.65}$$

Coming back to our $\mathcal{C}_{\mathbb{R}}(Y^{2,0})$ example, we have four triangles, all of unit area. So

$$\mathsf{Z} = \mathbb{Z}_{\gcd(2,2,2,2)} = \mathbb{Z}_2 \,. \tag{4.66}$$

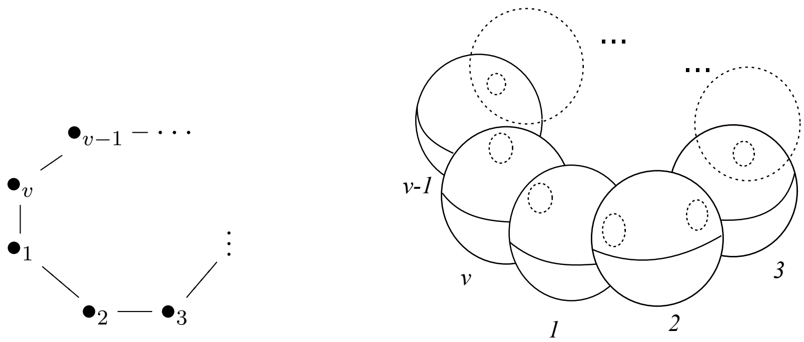


Figure 4.2: Schematic topology of B_3^L [170]

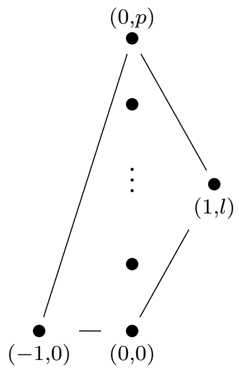


Figure 4.3: Toric diagram for $Y^{p,q}$. We have defined $l := p - q$.

$\mathfrak{su}(p)_k$ theory

We will now apply the previous results to a simple set of cases: $\mathfrak{su}(p)$ theories at Chern-Simons level k. It is well-known that $\mathfrak{su}(p)_k$ theories can be obtained by exploiting canonical CY singularities that are cones over Sasaki-Einstein manifolds of $Y^{p,q}$ type (see, for instance, [171] for a recent account and [172] for the original analysis of these geometries). Let us introduce for convenience $q := -k$, and we will assume $0 \leq q < p$. We show the resulting toric diagram in figure 4.3.

From our general discussion above, we need to compute Tor $H^2(Y^{p,q}) =$ Tor $H^4(Y^{p,q})$, together with the linking pairing, in order to determine the Heisenberg group encoding the higher symmetries of the theory. Whenever p and q are relatively prime, we have that [172] $Y^{p,q}$ is topologically $S^2 \times S^3$, so there is no torsion. So in these cases there is no choice of global structure for the field theory. More interesting is the case where $\gcd(p,q) \neq 1$. We can compute the relevant torsion groups following the general prescription in §4.2.3 as follows. Choose an ordering of the external points of the toric diagram in figure 4.3 such that adjacent points are consecutive.

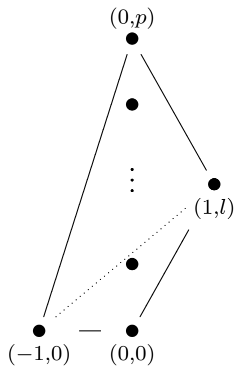


Figure 4.4: One of the triangles defined in the text.

For instance, choose

$$\{p_1, p_2, p_3, p_4\} = \{(-1,0), (0,0), (1,l), (p,0)\}\,. \tag{4.67}$$

Define now the triangles T_i, $i \in \{1,\ldots,4\}$, as the convex hull of $\{p_{i-1}, p_i, p_{i+1}\}$ (with $p_0 \coloneqq p_4$ and $p_5 \coloneqq p_1$). We show the triangle T_2 in figure 4.4 as an example. We have that

$$\text{Tor}\, H^2(Y^{p,q}) = \text{Tor}\, H^4(Y^{p,q}) = \mathbb{Z}_{\gcd(n_1,n_2,n_3,n_4)} \tag{4.68}$$

where n_i is defined as twice the area of T_i. It is elementary to show that $n_i = \{p, l, p, 2p-l\} = \{p, p-q, p, p+q\}$, which implies that

$$\mathsf{Z} = \text{Tor}\, H^2(Y^{p,q}) = \text{Tor}\, H^4(Y^{p,q}) = \mathbb{Z}_{\gcd(p,q)}\,. \tag{4.69}$$

We show in appendix B.2.1 that the linking pairing $T_{Y^{p,q}}\colon \text{Tor}\, H^2(Y^{p,q}) \times \text{Tor}\, H^4(Y^{p,q}) \to \mathbb{Q}/\mathbb{Z}$ is

$$T_{Y^{p,q}}(1,1) = -\frac{1}{\gcd(p,q)} \mod 1\,. \tag{4.70}$$

In the case that the Chern-Simons level k vanishes this leads to $\mathsf{Z} = \mathbb{Z}_p$, which is the expected result for pure $\mathcal{N}=1$ $\mathfrak{su}(p)_0$ theory in five dimensions. This theory admits a number of global variants, for instance, $SU(p)_0$ or $PSU(p)_0 \coloneqq SU(p)_0/\mathbb{Z}_p$. The classification of all such global forms proceeds just as in the case of $\mathfrak{su}(p)$ theories in four dimensions [134], so we will not delve into it further. The case with $k \neq 0$ is more subtle, and we turn to it now.

4.2.4 Symmetry TFTs for 5d SCFTs

The 1-form and 2-form symmetries in the 5-dimensional theory are not all realised simultaneously in a given field theory. As in the seven-dimensional case studied above, we will obtain a generically non-invertible BF sector in the symmetry theory when reducing on Y_5, and different choices of boundary conditions for the symmetry theory will determine which higher-form symmetries are actually realised. The derivation of the BF sector is very similar to the one in that case, so we will be brief. Consider the operators Ψ_{t_1} and Ψ_{t_3} wrapped on generators t_1 and t_3 of $H_1(Y_5)$ and $\mathrm{Tor}\, H_3(Y_5)$. They will lead to operators Ψ_{Σ_2} and Ψ_{Σ_3} in the effective six-dimensional symmetry theory, with a commutation relation

$$\Psi_{\Sigma_2}\Psi_{\Sigma_3} = e^{2\pi i \ell^{-1}\Sigma_2\cdot\Sigma_3}\Psi_{\Sigma_3}\Psi_{\Sigma_2} \tag{4.71}$$

on a spatial slice, where $\ell^{-1} \coloneqq T_{Y_5}(t_1, t_3)$. This is the content of a BF theory with action[6]

$$S_{\mathrm{BF}} = \ell \int B_2 \wedge dC_3\,. \tag{4.72}$$

The 1-form symmetries also participate in 't Hooft anomalies. Denote the background fields for the 1-form symmetry by $B_2 \in H^2(M_5; \Gamma^{(1)})$. From general field theory considerations, obtained by studying the Coulomb branch, there are two types of anomalies: the purely 1-form symmetry cubic anomaly (B^3) [21], and the mixed $U(1)_I$ and 1-form symmetry anomaly ($B^2 F_I$) [20]. The cubic 1-form symmetry anomaly was derived from field theory in the context of the SCFTs that have a Coulomb branch description as $\mathfrak{su}(p)_k$ [21]

$$\mathcal{A}_{B^3} = \frac{qp(p-1)(p-2)}{6\gcd(p,q)^3} B_2^3\,. \tag{4.73}$$

We are using conventions where the periods of B_2 are integrally quantised (as opposed to $2\pi/\gcd(p,q)$ quantised) and the 1-form symmetry in this case is $\Gamma^{(1)} = \mathbb{Z}_{\gcd(p,q)}$. This coupling corresponds to a 't Hooft anomaly for the 1-form symmetry, and therefore field theoretically obstructs its gauging. This will imply that (potentially) some asymptotic flux choice might be obstructed, and not all the global forms of the gauge group are allowed, unless a more complicated structure arises, which mixes the 1-form symmetry with other symmetries present in the theory. We

[6] We are abusing notation slightly here, and denote by B_2 both the continuous field that we use in writing the BF action and the discrete background for the 1-form symmetry, since they are identified at low energies.

plan to explore the deeper consequences of this coupling by using our methods in the future.

There is also a field-theoretic mixed anomaly between the instanton $U(1)_I$ and 1-form symmetry, determined in [20] for the $SU(2)_0$ theory using field theory arguments. In the IR for $\mathfrak{su}(p)_k$ it takes the form [20, 21]

$$\mathcal{A}_{FB^2} = \frac{p(p-1)}{2\,\mathrm{gcd}(p,q)^2} F_I B_2^2 \,. \tag{4.74}$$

(This contribution to the anomaly was also analysed in [26] using string theory methods, reaching a different conclusion. We believe that the discrepancy between their result and ours might be due to a different choice of torsional representative, see footnote 3 above.)

We will now derive these anomalies from the first principles using the differential cohomology approach developed in this thesis, being agnostic about whether this is a UV or IR computation. We will see that an essential contribution to these anomalies comes from the $C_3 \wedge X_8$ term in the M-theory effective action.

Link Reduction using differential cohomology

The integral cohomology of Y_5, the base of the toric Calabi-Yau cone X, takes the form

$$H^*(Y_5) = \left\{\mathbb{Z}, 0, \mathbb{Z}^{b^2} \oplus \mathrm{Tor}\, H^2(Y_5), \mathbb{Z}^{b^2} \oplus \mathrm{Tor}\, H^3(Y_5), \mathrm{Tor}\, H^2(Y_5), \mathbb{Z}\right\} \,. \tag{4.75}$$

For simplicity, we assume that the Betti number b^1 of Y_5 is zero. (This is true in all examples we study.) The expansion of $\breve{G}_4$ then reads

$$\begin{aligned} \breve{G}_4 = \breve{\gamma}_4 \star \breve{1} + \sum_{\alpha=1}^{b^2} \breve{F}_2^{(\alpha)} \star \breve{v}_{2(\alpha)} + \sum_{\alpha=1}^{b^2} \breve{\xi}_{1(\alpha)} \star \breve{v}_3^{(\alpha)} \\ + \sum_i \breve{B}_2^{(i)} \star \breve{t}_{2(i)} + \sum_m \breve{b}_1^{(m)} \star \breve{t}_{3(m)} + \sum_i \breve{\psi}_{0(i)} \star \breve{t}_4^{(i)} + \tau([\omega_3]) \,. \end{aligned} \tag{4.76}$$

The label α runs over generators of the free part of $H^2(Y_5;\mathbb{Z})$, the label i runs over generators of $\mathrm{Tor}\, H^2(Y_5;\mathbb{Z}_5)$, while the label m runs over generators of $\mathrm{Tor}\, H^3(Y_5;\mathbb{Z})$.

We can now consider the reduction of the G_4^3 coupling in M-theory. Using (4.76) and collecting all relevant terms, we arrive at

$$\begin{aligned}
&-\frac{1}{6}\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{G}_4 \star \breve{G}_4 \\
&= -\sum_{\alpha}\int_{\mathcal{W}_6} \breve{\gamma}_4 \star \breve{F}_2^{(\alpha)} \star \xi_{1(\alpha)} - \sum_{i,j,k}\left[\frac{1}{6}\int_{Y_5} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \star \breve{t}_{2(k)}\right]\int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{B}_2^{(k)} \\
&-\sum_{i,j,\alpha}\left[\frac{1}{2}\int_{Y_5} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \star \breve{v}_{2(\alpha)}\right]\int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{F}_2^{(\alpha)} \\
&-\sum_{i,\alpha,\beta}\left[\frac{1}{2}\int_{Y_5} \breve{t}_{2(i)} \star \breve{v}_{2(\alpha)} \star \breve{v}_{2(\beta)}\right]\int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{F}_2^{(\alpha)} \star \breve{F}_2^{(\beta)} \\
&+\sum_{m,n}\left[\frac{1}{2}\int_{Y_5} \breve{t}_{3(m)} \star \breve{t}_{3(n)}\right]\int_{\mathcal{W}_6} \breve{\gamma}_4 \star \breve{b}_1^{(m)} \star \breve{b}_1^{(n)} + \sum_{m,\alpha}\left[\int_{Y_5} \breve{t}_{3(m)} \star \breve{v}_3^{(\alpha)}\right]\int_{\mathcal{W}_6} \breve{\gamma}_4 \star \breve{b}_1^{(m)} \star \breve{\xi}_{1(\alpha)} \\
&-\sum_{i,j}\left[\int_{Y_5} \breve{t}_{2(i)} \star \breve{t}_4^{(j)}\right]\int_{\mathcal{W}_6} \breve{\gamma}_4 \star \breve{B}_2^{(i)} \star \breve{\psi}_{0(j)} - \sum_{\alpha,j}\left[\int_{Y_5} \breve{v}_{2(\alpha)} \star \breve{t}_4^{(j)}\right]\int_{\mathcal{W}_6} \breve{\gamma}_4 \star \breve{F}_2^{(\alpha)} \star \breve{\psi}_{0(j)} \ .
\end{aligned} \tag{4.77}$$

In the first term, we have used $\int_{Y_5} v_{2(\alpha)} \star v_3^{(\beta)} = \delta_\alpha^\beta$.

Next, let us consider the terms that originate from the $G_4 X_8$ coupling in M-theory. Recall that 11d spacetime is taken to be the direct product $\mathcal{M}_{11} = \mathcal{W}_6 \times Y_5$. As a result, at the level of cohomology classes with integer coefficients, one has[7]

$$\begin{aligned}
p_1(T\mathcal{M}_{11}) &= p_1(T\mathcal{W}_6) + p_1(TY_5) \\
p_2(T\mathcal{M}_{11}) &= p_2(T\mathcal{W}_6) + p_2(TY_5) + p_1(T\mathcal{W}_6) \smile p_1(TY_5) \,.
\end{aligned} \tag{4.79}$$

These relations imply

$$X_8 = -\frac{1}{96}\, p_1(T\mathcal{W}_6) \smile p_1(TY_5) \ . \tag{4.80}$$

Promoting integral cohomology classes to differential cohomology classes (the precise representative of p_1 one chooses is not important [174]) we can write the $G_4 X_8$

[7]The Whitney sum formula for Pontryagin classes in integral cohomology is [173]

$$p_q(A \oplus B) = \sum r_{2q-j}(A) \smile r_j(B) \,, \quad r_{2s} = p_s \,, \quad r_{2s+1} = \mathrm{Bock}(w_{2s}) \smile \mathrm{Bock}(w_{2s}) + p_s \smile \mathrm{Bock}(w_1) \,, \tag{4.78}$$

where A, B are O bundles, the w_i's are Stiefel-Whitney classes, and Bock is the Bockstein homomorphism associated with the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$. In appendix C we show $w_1(TY_5) = w_2(TY_5) = 0$, which implies that (4.79) holds at the level of cohomology with integral coefficients.

coupling in the form

$$
\begin{aligned}
-\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{X}_8 &= \frac{1}{96} \int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{p}_1(T\mathcal{W}_6) \star \breve{p}_1(TY_5) \\
&= \frac{1}{96} \sum_i \int_{Y_5} \breve{t}_{2(i)} \star \breve{p}_1(TY_5) \int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{p}_1(T\mathcal{W}_6) \ .
\end{aligned}
\tag{4.81}
$$

In the second step we used (4.76) and we observed that the only internal differential cohomology classes that can have a non-trivial pairing with $\breve{p}_1(TY_5)$ are the degree-2 torsional classes $\breve{t}_{2(i)}$.

To proceed, we make use of the following congruence for integral cohomology classes [175]

$$
p_1(T\mathcal{W}_6) \smile a_2 = 4\, a_2 \smile a_2 \smile a_2 \quad \text{mod } 24 \qquad \text{for any } a_2 \in H^2(\mathcal{W}_6; \mathbb{Z}) \ . \tag{4.82}
$$

This congruence can be derived using the Atiyah-Singer index theorem as follows [176]. We take external spacetime $\mathcal{W}_6$ to be a Spin manifold. Consider an arbitrary $a_2 \in H^2(\mathcal{W}_6; \mathbb{Z})$. There exists a line bundle with connection A on $\mathcal{W}_6$ such that its first Chern class equals a_2. Consider the Dirac operator on $\mathcal{W}_6$ twisted by this line bundle. The Atiyah-Singer theorem implies

$$
\begin{aligned}
\mathrm{Ind}(\not{D}_A) &= \int_{\mathcal{W}_6} \left[\frac{1}{6} F \wedge F \wedge F - \frac{1}{24} F \wedge p_1(T\mathcal{W}_6) \right] \\
&= \int_{\mathcal{W}_6} \left[\frac{1}{6} a_2 \smile a_2 \smile a_2 - \frac{1}{24} a_2 \smile p_1(T\mathcal{W}_6) \right] ,
\end{aligned}
\tag{4.83}
$$

where F is the curvature 2-form of the connection A, satisfying $[F]_{\mathrm{dR}} = \varrho(a_2)$. We conclude that $\int_{\mathcal{W}_6} [4\, a_2 \smile a_2 \smile a_2 - a_2 \smile p_1(T\mathcal{W}_6)] \in 24\, \mathbb{Z}$, which is equivalent to (4.82).

Relation (4.82) then implies

$$
\begin{aligned}
\int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{p}_1(T\mathcal{W}_6) &= \int_{\mathcal{W}_6} B_2^{(i)} \smile p_1(T\mathcal{W}_6) = 24\, M^{(i)} + 4 \int_{\mathcal{W}_6} B_2^{(i)} \smile B_2^{(i)} \smile B_2^{(i)} \\
&= 24\, M^{(i)} + 4 \int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} \ ,
\end{aligned}
\tag{4.84}
$$

where we have used the fact that $\int_{\mathcal{W}_6} \breve{a}_6 = \int_{\mathcal{W}_6} I(\breve{a}_6)$ for any $\breve{a}_6 \in \breve{H}^6(\mathcal{W}_6)$, together with (2.22), $I(\breve{B}_2^{(i)}) = B_2^{(i)}$, and $I(\breve{p}_1(T\mathcal{W}_6)) = \breve{p}_1(T\mathcal{W}_6)$. The quantities $M^{(i)}$ are unspecified integers, encoding the ambiguity in the mod 24 congruence (4.82). There is no summation on the repeated label i in (4.84). Inserting (4.84) into (4.81), we arrive at

$$-\int_{\mathcal{M}_{11}} \breve{G}_4 \star \breve{X}_8 = \frac{1}{24} \sum_i \int_{Y_5} \breve{t}_{2(i)} \star \breve{p}_1(TY_5) \int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} \star \breve{B}_2^{(i)} + \sum_i M^{(i)} \int_{Y_5} \breve{t}_{2(i)} \star \frac{\breve{p}_1(TY_5)}{4} \,. \tag{4.85}$$

In appendix C we show that $p_1(TY_5)/4$ is an integral class. Although we have no general proof, we also find that in all of our examples the quantity multiplying $M^{(i)}$ is an integer, so we will drop the last term in what follows, and focus on the terms in the symmetry theory that contain only the fields $\breve{B}_2^{(i)}$ and $\breve{F}_2^{(\alpha)}$.

Notice that there is an ambiguity in the definition of the differential cohomology classes $\breve{v}_{2(\alpha)}$ associated to the free part of $H^2(Y_5;\mathbb{Z})$, which can be shifted by integral multiples of the differential cohomology classes $\breve{t}_{2(i)}$ associated to $\operatorname{Tor} H^2(Y_5;\mathbb{Z})$, $\breve{v}_{2(\alpha)} \to \breve{v}_{2(\alpha)} + m_{(\alpha)}{}^{(i)}\, \breve{t}_{2(i)}$, with $m_{(\alpha)}{}^{(i)} \in \mathbb{Z}$. Our choices are such that for the examples in the thesis we have

$$\int_{Y_5} \breve{t}_{2(i)} \star \breve{v}_{2(\alpha)} \star \breve{v}_{2(\beta)} = 0 \,. \tag{4.86}$$

Combining (4.77) and (4.85), we obtain the following anomaly couplings in the symmetry TFT:

$$\boxed{S_{\rm Sym} = \sum_{i,j,k} \Omega_{ijk} \int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{B}_2^{(k)} + \sum_{i,j,\alpha} \Omega_{ij\alpha} \int_{\mathcal{W}_6} \breve{B}_2^{(i)} \star \breve{B}_2^{(j)} \star \breve{F}_2^{(\alpha)} \,,} \tag{4.87}$$

where the $\mathbb{R}/\mathbb{Z}$-valued quantities Ω_{ijk}, $\Omega_{ij\alpha}$ are CS invariants defined by

$$\begin{aligned} \Omega_{ijk} &= -\frac{1}{6} \int_{Y_5} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \star \breve{t}_{2(k)} + \frac{1}{24}\, \delta_{i,j}\, \delta_{i,k} \int_{Y_5} \breve{t}_{2(i)} \star \breve{p}_1(TY_5) \\ \Omega_{ij\alpha} &= -\frac{1}{2} \int_{Y_5} \breve{t}_{2(i)} \star \breve{t}_{2(j)} \star \breve{v}_{2(\alpha)} \,. \end{aligned} \tag{4.88}$$

As we demonstrate in appendix C, for the setups of interest in this work G_4 is integrally quantised, and therefore the 11d couplings in the M-theory effective action are guaranteed to be well-defined. It follows that the CS invariants (4.88) are also well-defined. Let us emphasise, however, that the two terms in Ω_{ijk} with $i = j = k$ are not separately well-defined, in general.

The CS invariants Ω_{ijk}, $\Omega_{ij\alpha}$ are defined purely in terms of the link geometry Y_5. In order to evaluate them for a given Y_5, however, it can be convenient to resort to a computation in the bulk of the Calabi-Yau X_6, using an extension of the Gordon-Litherland formalism discussed in section 4.1.2. Let $n_{(i)}$ denote the torsional degree of $t_{2(i)} \in H^2(Y_5)$, and let $Z_{(i)}$ be the compact divisor in the bulk associated to $t_{2(i)}$. We also associate a non-compact divisor $D_{(\alpha)}$ to the non-torsional classes $v_{2(\alpha)} \in H^2(Y_5)$, which correspond to flavour symmetries. With this notation, the invariants (4.88) can be computed as

$$\begin{aligned} \Omega_{ijk} &= \left[-\frac{1}{6} \frac{Z_{(i)} \cdot Z_{(j)} \cdot Z_{(k)}}{n_{(i)}\, n_{(j)}\, n_{(k)}} + \frac{1}{24} \delta_{i,j}\, \delta_{i,k} \frac{Z_{(i)} \cdot p_1(TX_6)}{n_{(i)}} \right]_{\text{mod } 1} , \\ \Omega_{ij\alpha} &= \left[-\frac{1}{2} \frac{Z_{(i)} \cdot Z_{(j)} \cdot D_{(\alpha)}}{n_{(i)}\, n_{(j)}} \right]_{\text{mod } 1} , \end{aligned} \tag{4.89}$$

where $\cdot$ denotes intersection of divisors in X_6.

$\mathfrak{su}(p)_k$ theory

This general approach can be exemplified for all toric Calabi-Yau cones, in particular, the SCFTs with $\mathfrak{su}(p)_k$ IR description, which have from field theory analysis, the anomalies in (4.73) and (4.74). As discussed in the last section, the Sasaki-Einstein link is given by $Y^{p,q}$, and the Calabi-Yau has a simple toric description. In this subsection, we generalise the definition of this toric diagram to the one given in 4.5 with the toric diagram $\mathfrak{su}(p)_k$ given by figure 4.3, as this is more convenient for calculations. Now the external vertices that determine the toric fan are

$$w_0 = (0,0)\,, \qquad w_p = (0,p)\,, \qquad w_x = (-1, k_x)\,, \qquad w_y = (1, k_y)\,, \tag{4.90}$$

where the CS-level $k = -q$ is determined by

$$q = p - (k_x + k_y)\,. \tag{4.91}$$

Comparing with the notation before $l = k_x + k_y$. There are linear relations among these non-compact divisors D_{w_i}, and the instanton $U(1)$ is identified with

$$D_I = D_{w_x}\,. \tag{4.92}$$

The compact divisors are

$$S_a = (0,a)\,, \qquad a = 1, \cdots, p-1\,. \tag{4.93}$$

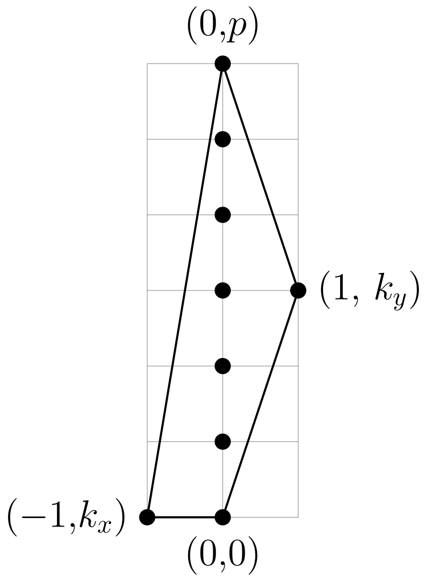


Figure 4.5: The toric diagram for the 5d SCFT realization of $\mathfrak{su}(p)_k$. The example shown is $p = 6$, $q = 6 - (k_x + k_y) = 3$, i.e. $SU(6)_3$, which has $\mathbb{Z}_3$ 1-form symmetry.

As shown in [47] the centre symmetry generator of the gauge theory $SU(p)$ is obtained by taking the linear combination

$$Z = \sum_{a=1}^{p-1} a S_a \,. \tag{4.94}$$

This compact divisor is also identified with the compact divisor associated to the generator of Tor $H^2(Y_5)$ according to the discussion in section 4.1.2.

We will also need an explicit expression for $p_1(TX_6) = -c_2(TX_6 \otimes \mathbb{C})$, with $TX_6 \otimes \mathbb{C}$ the complexification of the tangent bundle of the toric Calabi-Yau X_6. Since TX_6 is a complex vector bundle we have $TX_6 \otimes \mathbb{C} = TX_6 \oplus \overline{TX_6}$, so $c(TX_6 \otimes \mathbb{C}) = c(TX_6)c(\overline{TX_6})$. For a toric variety X_6 with divisors D_i we have [162]

$$c(TX_6) = \prod_{i=1}^{n} (1 + D_i) \tag{4.95}$$

so

$$c(TX_6 \otimes \mathbb{C}) = c(TX_6)c(\overline{TX_6}) = \left(\prod_{i=1}^{n} (1 + D_i) \right) \left(\prod_{i=1}^{n} (1 - D_i) \right) = \prod_{i=1}^{n} (1 - D_i^2) \,. \tag{4.96}$$

and therefore $p_1(TX_6) = \sum_i D_i^2$.

With this information at hand, it is easy to compute the anomaly coefficients using the formulae (4.89), specialised in the case of one torsion generator of order $\gcd(p,q)$ and one free generator. We find results compatible with the empirical formulas

$$\begin{aligned} &Z \cdot Z \cdot Z = p\,(p-1)\,(p^2 + p\,q - 2\,q) \ , \qquad Z \cdot p_1 = 4\,p\,(p-1) \ , \\ &Z \cdot Z \cdot D_I = -p\,(p-1) \ , \end{aligned} \tag{4.97}$$

which we conjecture hold in general. We have also verified in a large class of examples that we always have $Z \cdot D_I \cdot D_I = 0$, in accordance to the general claim (4.86), and that

$$\int_{Y_5} \breve{t}_2 \star \frac{\breve{p}_1(TY_5)}{4} = \left[\frac{Z \cdot p_1}{4\gcd(p,q)} \right]_{\mathrm{mod}\ 1} = 0 \ . \tag{4.98}$$

This condition guarantees that the terms in (4.85) not fixed by the mod 24 congruence (4.82) can indeed be safely dropped.

Assuming the validity of (4.97) it is straightforward to verify that

$$-\frac{1}{6}\,Z{\cdot}Z{\cdot}Z + \frac{1}{24}\gcd(p,q)\,Z{\cdot}p_1 = \frac{q\,p\,(p-1)\,(p-2)}{6} - \gcd(p,q)^3 (p-1)\,\frac{P\,(P+1)\,(P-1)}{6} \ , \tag{4.99}$$

where $P = p/\gcd(p,q)$. Plugging (4.97) in (4.89), and using (4.99), we find that the action for the symmetry TFT contains the terms

$$S_{\mathrm{Sym}} = \int_{\mathcal{W}_6} \left[\frac{q\,p\,(p-1)\,(p-2)}{6\gcd(p,q)^3}\,B_2^3 + \frac{p\,(p-1)}{2\gcd(p,q)^2}\,B_2^2\,F_I \right] . \tag{4.100}$$

This result is in perfect agreement with the field theory results (4.73) and (4.74). It may be worth noting that the result is well-defined because it is invariant under shifts of B_2 by $\gcd(p,q)$ times an arbitrary integral class. For example, if we perform the shift $B_2 \to B_2 + \gcd(p,q)\,b_2$, the extra terms generated by the B_2^3 term are

$$\begin{aligned} &\frac{q\,p\,(p-1)\,(p-2)}{6\gcd(p,q)^3}\,3\gcd(p,q) \int_{\mathcal{W}_6} B_2^2\,b_2 \in \mathbb{Z} \ , \\ &\frac{q\,p\,(p-1)\,(p-2)}{6\gcd(p,q)^3}\,3\gcd(p,q)^2 \int_{\mathcal{W}_6} B_2\,b_2^2 \in \mathbb{Z} \ , \\ &\frac{q\,p\,(p-1)\,(p-2)}{6\gcd(p,q)^3}\gcd(p,q)^3 \int_{\mathcal{W}_6} b_2^3 \in \mathbb{Z} \ . \end{aligned} \tag{4.101}$$

Similar remarks apply to the $B_2^2\,F_I$ term.

5 Symmetries from type IIB

In this section, we will play the same game of geometrically engineering symmetries as in the previous section, but with a new focus on IIB theory instead of M-theory. We will look at the geometrical engineering of 4d $\mathcal{N} = 2$ theories. These include the examples of generalised Argyres-Douglas theories known as Cecotti-Neitzke-Vafa $(\mathfrak{g}, \mathfrak{g}')$. Other classes of examples include $D_p^b(G)$ theories whose defect groups we identify with those of $G^{(b)}[k]$ theories. We find that these theories contain defect groups of 1-form symmetries listed in tables 5.1 and 5.4 with mixed 't Hooft anomalies between electric and magnetic factors. The material on $(\mathfrak{g}, \mathfrak{g}')$ and $G^{(b)}[k]$ theories are based on [2] and [3], respectively. We discuss that the Dirac pairing of the BPS quivers is related to the intersection matrix of the 2-cycles and 4-cycles in the geometry encoding the 1-form symmetries.

5.1 4d $\mathcal{N} = 2$ theories

Consider 10-dimensional type IIB superstring theory on

$$\mathcal{M}_4 \times \mathbf{X}_6 \,, \tag{5.1}$$

where $\mathbf{X}_6$ is a local Calabi-Yau threefold. Via geometric engineering, this gives rise to a broad class of $\mathcal{N} = 2$ theories

$$\mathfrak{T}_{\mathbf{X}_6} \in \mathrm{SQFT}_{3+1}^{\mathcal{N}=2} \tag{5.2}$$

coupled to a four-manifold $\mathcal{M}_4$. Notice that $\mathfrak{T}_{\mathbf{X}_6}$ is not necessarily conformal: for instance if one considers the IIB geometry

$$\mathbf{X}_6 \equiv \left\{ e^z + e^{-z} + p_{\mathfrak{g}}(x, y) + u^2 = 0 \right\} \subset \mathbb{C}^4 \,, \tag{5.3}$$

the corresponding $\mathfrak{T}_{\mathbf{X}_6}$ is the $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theory with simple simply-laced gauge algebra $\mathfrak{g}$.

One generic feature of the class of $\mathfrak{T}_{\mathbf{X}_6}$ theories that we study is that non-local BPS dyons become simultaneously massless. Argyres and Douglas gave the original examples of this phenomenon in [177, 178]. Since the massless degrees of freedom are mutually non-local, the corresponding dynamics cannot be described by a conventional Lagrangian.[1] Moreover, by the scale invariance of the corresponding Seiberg-Witten (SW) geometry, the theories are argued to be $\mathcal{N} = 2$ superconformal.

Argyres-Douglas theories can be realised in type IIB superstrings on isolated hypersurface singularities [179, 180]. This perspective allows us to compute the corresponding spectrum of BPS states from the bound states of D3 branes on vanishing special Lagrangian 3-cycles [181]. The same geometric construction can be generalised to more general hypersurface singularities at finite distance in moduli space[2] — which translates to the requirement that the corresponding 2d (2,2) Landau-Ginzburg worldsheet theory has central charge $\hat{c} < 2$ [181, 193]. Each such model is characterised by a quasihomogeneous polynomial

$$f(\lambda^{w_i} X_i) = \lambda^d f(X_i) \tag{5.4}$$

where d and w_i are positive integers known respectively as degree and weights of the singularity, and the corresponding geometry is given by $\mathbf{X}_6 \coloneqq \{f(X_i) = 0\}$. The $\mathbb{C}^*$ action in (5.4) plays the same role of the scale invariance for the SW geometry: it is well-known that this can be exploited to compute the dimensions of the various Coulomb branch operators of the SCFT [181]. These singularities are at a finite distance in moduli space provided $\sum_i w_i > d$ which is the singularity theory translation of the condition $\hat{c} < 2$ [194, 195].

The non-trivial local degrees of freedom of such SCFTs arise from massless D3 branes wrapped on the vanishing three-cycles at the singular point of $\mathbf{X}_6$. However, as discussed in section 3.1 in the presence of a nontrivial defect group at the horizon of the IIB compactification, extra information is required to fully specify the theory and its partition function on $\mathcal{M}_4$. Our main result in this section is to determine the defect group for these theories, and the corresponding Heisenberg algebra of non-commuting fluxes. We find that many of these theories admit different inequivalent global structures and hence distinct partition functions on four-manifolds

[1] At least at the IR fixed point, but this does not exclude the existence of non-conformal Lagrangian theories in the same universality class. We will study some examples of this phenomenon below.
[2] See [182–190] for more examples of this kind, as well as [191, 192] for interesting generalizations beyond the class of hypersurface singularities.

with nontrivial intersecting 2-cycles.[3] While this does not affect the superconformal index for these models [199–205] which corresponds to the partition function on $S^1 \times S^3$, it does affect the lens space index or more complicated partition functions (see e.g. [206–209] for some interesting examples of $\mathcal{N} = 2$ backgrounds that would be interesting to couple to AD SCFTs).

The interesting part of the defect group for theories of this class is given by

$$\mathbb{D}^{(1)} = \mathbb{Z}^{\kappa} \oplus \bigoplus_{i=1}^{n} \left(\mathbb{Z}_{m_i} \oplus \mathbb{Z}_{m_i} \right) \tag{5.5}$$

where n is the rank of the SCFT and κ is the rank of its flavour symmetry group. The m_i's are positive integers that we determine below — if for some i the corresponding m_i equals 1, the corresponding factor is trivial and the summand is dropped above. The torsional groups in parenthesis are non-trivially paired, meaning that the corresponding charge operators form a non-commuting Heisenberg algebra.

The free factor of $\mathbb{D}^{(1)}$ in equation (5.5) corresponds to the continuous zero-form symmetries of these SCFTs. For such zero-form symmetries, this group can enhance to become non-Abelian — see [210, 211] for conditions about the enhancement of the flavour symmetries in terms of the corresponding categories of BPS states. It is also interesting to remark that in the language of those papers, the Grothendieck group of the cluster category associated with the BPS quiver precisely coincides with the full $\mathbb{D}^{(1)}$ we compute here, referred to as the 't Hooft group in [212, 213].

As there are no known Lagrangian descriptions of most SCFTs (with the exception of some cases; we will come back to these momentarily) it might seem hard to find the 1-form symmetries for these theories using purely field theoretical tools. However, as we explain below — extending the results of the previous section and [14, 24, 44] to the four-dimensional setting — there is a way of rephrasing the results from the IIB analysis in purely field theoretical terms. We find that the role played by the unscreened part of the centre of the gauge group in the analysis in [134] is played in the non-Lagrangian setting in this thesis by $\mathrm{Tor}(\mathrm{coker}\,\mathsf{Q})$, with Q the BPS quiver for the theory [214, 215]. The free part of the group $\mathrm{coker}\,\mathsf{Q}$ coincides with the factor $\mathbb{Z}^{\kappa}$, hence $\mathbb{D}^{(1)} = \mathrm{coker}\,\mathsf{Q}$: it is natural to expect that this field theoretical formulation will be general, even in the absence of a simple IIB construction, and indeed this follows from the analysis done in [139, 213].

[3] See [196–198] for work on the interplay between partition functions of various $\mathcal{N} = 2$ SCFTs and the theory of 4-manifold invariants.

As discussed in section 2.2.1, the F_5 flux is self-dual, so flux operators labelled by it do not commute among themselves. Recall, the commutation relations (2.57) between these operators

$$\Psi_\sigma \Psi_{\sigma'} = e^{2\pi i T(\sigma,\sigma')} \Psi_{\sigma'} \Psi_\sigma \tag{5.6}$$

with $T(\sigma, \sigma')\colon \operatorname{Tor} H^5(\mathcal{N}_9) \times \operatorname{Tor} H^5(\mathcal{N}_9) \to \mathbb{Q}/\mathbb{Z}$ the linking number between the cohomology classes (we will discuss this linking number in more detail below). This implies for the grading of Hilbert space $\mathcal{H}(\mathcal{N}_9)$ by fluxes, and thus for the available choices of boundary conditions, is that there is no zero flux eigenstate $|\mathbf{0}\rangle$ such that $\Psi_\sigma |\mathbf{0}\rangle = |\mathbf{0}\rangle$ for all $\sigma \in \operatorname{Tor}(H^5(\mathcal{N}_9))$[4].

The best that we can do when choosing boundary conditions is to choose a maximally commuting set L of operators, and impose that our boundary state is neutral under these. In detail, we define $s(\sigma, \sigma') := e^{2\pi i T(\sigma,\sigma')}$, and define a maximal isotropic subgroup $L \subset \operatorname{Tor} H^5(\mathcal{N}_9)$ to be a maximal set such that $s(\sigma, \sigma') = 1$ for all $\sigma, \sigma' \in L$. This implies that the subgroup generated by the operators $\{\Psi_\sigma | \sigma \in L\}$ is abelian, and provides a maximal set of commuting observables. Once we choose L, there is a unique state $|\mathbf{0}; L\rangle$ in the Hilbert space $\mathcal{H}(\mathcal{N}_9)$ such that $\Psi_\sigma |\mathbf{0}; L\rangle = |\mathbf{0}; L\rangle$ for all $\sigma \in L$. We can interpret this state as follows. Define

$$F_L := \frac{\operatorname{Tor} H^5(\mathcal{N}_9)}{L} \tag{5.7}$$

and choose a representative $\mathbf{f}$ of each coset. Then the states $|\mathbf{f}; L\rangle := \Psi_{\mathbf{f}} |\mathbf{0}; L\rangle$ are eigenvectors of the flux operators in L:

$$\Psi_\sigma |\mathbf{f}; L\rangle = s(\sigma, \mathbf{f}) |\mathbf{f}; L\rangle \ , \tag{5.8}$$

so we find that the $|\mathbf{f}; L\rangle$ are the states with definite flux in L. In particular, $|\mathbf{0}; L\rangle$ can be interpreted as a state with zero flux in L. Once L is chosen this state is unique, so the non-canonical nature of the choice of boundary condition reduces to the absence of a canonical choice for L in $H^5(\mathcal{N}_9)$.

5.1.1 The choice of global form in 4d

The partition function $Z_{\mathfrak{T}_{\mathbf{X}_6}}(\mathcal{M}_4)$ of $\mathfrak{T}_{\mathbf{X}_6}$ should be fully determined by the data of the string background. This is indeed true, but as discussed in section 3.1 in the case that $\mathfrak{T}_{\mathbf{X}_6}$ has non-trivial higher symmetries, we must make additional choices to fix the partition function.

[4] We show in appendix A that the K-theory group in this example reduces to singular cohomology.

The higher n-form symmetries in the four-dimensional theory act on the n-dimensional defects of the theory [5]. In IIB, these defects arise from Dp-branes wrapping k-cycles on $\mathbf{X}_6$, where $n = p - k + 1 \leq 4$. More concretely, any background for the higher form symmetries introduces monodromies for these defects, so in the string theory construction such a background must be realised with a choice of background fluxes. The latter is then providing the stringy realization of higher-form symmetries. Therefore, in order to fully specify the theory $\mathfrak{T}_{\mathbf{X}_6}$, we need to specify the boundary conditions for the fluxes and understand the Hilbert space that type IIB string theory associates to the boundary $\mathcal{N}_9$.

In this section, we focus on the case $\mathcal{N}_9 = \mathcal{M}_4 \times Y_5$, with $\operatorname{Tor} H^\bullet(\mathcal{M}_4) = 0$ so by the Künneth formula we have

$$\operatorname{Tor} H^5(\mathcal{N}_9) = \bigoplus_{n=0}^{4} H^n(\mathcal{M}_4) \otimes \operatorname{Tor} H^{5-n}(Y_5)\,. \tag{5.9}$$

In the cases of interest to us, we additionally have that Y_5 is simply connected [216], so $\operatorname{Tor} H_1(Y_5) = \operatorname{Tor} H^4(Y_5) = 0$. The universal coefficient theorem [157] additionally implies that $\operatorname{Tor} H^2(Y_5) = \operatorname{Tor} H_1(Y_5) = 0$, so the only possible non-trivial torsion lives in $H^3(Y_5)$:

$$\operatorname{Tor} H^5(\mathcal{N}_9) = H^2(\mathcal{M}_4) \otimes \operatorname{Tor} H^3(Y_5) \cong H^2(\mathcal{M}_4; \operatorname{Tor} H^3(Y_5))\,. \tag{5.10}$$

As will be clarified in the next section, the factor $H^3(\mathcal{Y}_5)$ is what we defined in (5.5) to be the defect group $\mathbb{D}$.

In principle we should now classify all the $L \subset \operatorname{Tor} H^5(\mathcal{N}_9)$ for every $\mathcal{N}_9$, but there is a class of such isotropic subgroups which is particularly interesting in the context of four-dimensional physics on $\mathcal{M}_4$. Assume that we fix a Y_5, or equivalently its cone X_6. Then there is a subclass of the possible L of the form

$$L = L_5 \otimes H^2(\mathcal{M}_4) \tag{5.11}$$

that can be defined uniformly for every $\mathcal{M}_4$.[6] Here L_5 is a maximal isotropic subgroup of $\operatorname{Tor} H^3(Y_5)$. The theories defined by such choices are sometimes called

[5] In general n-form symmetries act on m-dimensional defects with $m \geq n$. However, such examples do not arise from the analysis in this thesis.

[6] At least for those $\mathcal{M}_4$ without torsion. We do not know if a similar canonical subset of choices exists if we allow for torsion in $\mathcal{M}_4$.

"genuine" four-dimensional theories.[7]

The choices of global structure for the genuine $\mathfrak{T}_{\mathbf{X}_6}$ theories are thus the choices of maximal isotropic $L_5 \subset \mathrm{Tor}\, H^3(Y_5)$, with $\partial\mathbf{X}_6 = Y_5$. Once we have such an L_5 we have a choice for the 2-surface operators generating the 1-form symmetries of $\mathfrak{T}_{\mathbf{X}_6}$: they come from the reduction of the Ψ_σ flux operators in the IIB theory. And relatedly, introducing background fluxes for F_5 at infinity will introduce background fluxes for the 1-form symmetries in the four-dimensional theory on $\mathcal{M}_4$.

5.2 1-form symmetries from BPS quivers

Our discussion so far has been fairly general and has not required us to make use of the fact that the $\mathfrak{T}_{\mathbf{X}_6}$ theories preserve $\mathcal{N} = 2$. In fact, in addition to being $\mathcal{N} = 2$ supersymmetric the theories that we will be discussing have the nice property that their BPS spectrum can be generated from a BPS quiver [214, 215] (we refer the reader unfamiliar with BPS quivers to these papers for reviews), and this leads to a reformulation of the answer that we just found in terms of screening of line operators, generalizing to our current context the discussions in [14, 134].

Recall that each node in the BPS quiver represents a BPS building block, and the arrows encode how they can be recombined. From the IIB perspective, the nodes in the quiver represent D3 branes wrapped on generators of a basis of $H_3(\mathbf{X}_6)$, and the arrows in the quiver encode the intersection numbers of the corresponding 3-cycles. BPS states in $\mathfrak{T}_{\mathbf{X}_6}$ can be obtained from D3 branes wrapping supersymmetric compact cycles in $\mathbf{X}_6$, and such D3 branes can always be constructed by taking a combination of generators with the right total charge, and recombining them.[8]

We can connect our discussion in the previous section to the formulation in terms of BPS quivers as follows. Take a small 7-dimensional sphere S^7 around the origin in $\mathbb{C}^4$. Our "boundary at infinity" Y_5 is homotopy equivalent to the intersection $\mathcal{Y}_5 = S^7 \cap \{P_{(\mathfrak{g},\mathfrak{g}')} = \epsilon\}$, where $P_{(\mathfrak{g},\mathfrak{g}')}$ is the polynomial defining the hypersurface singularity, and we take ϵ small but non-vanishing in order to make the interior of $\mathcal{Y}_5$ smooth. Finally, introduce $\mathcal{X}_6$ to be the (smooth) interior of $\mathcal{Y}_5$, namely the intersection of a ball B^8 (such that $\partial B^8 = S^7$) with $P_{(\mathfrak{g},\mathfrak{g}')} = \epsilon$.

[7] Choices of boundary conditions outside this class will depend on specific features of $\mathcal{M}_4$. This is perfectly fine from the IIB point of view, but the four-dimensional interpretation of the resulting theories is slightly less conventional. We refer the reader to [24] for a more detailed discussion of this point.

[8] More formally, we have that the derived category of A-branes on X_6 is isomorphic to the derived category of representations of the quiver. We refer the reader to [217] for a review of this approach.

Since $\partial\mathcal{X}_6 = \mathcal{Y}_5$ there is a long exact sequence in homology of the form

$$\ldots \to H_n(\mathcal{Y}_5) \to H_n(\mathcal{X}_6) \to H_n(\mathcal{X}_6, \mathcal{Y}_5) \to H_{n-1}(\mathcal{Y}_5) \to \ldots \tag{5.12}$$

where $H_n(\mathcal{X}_6, \mathcal{Y}_5)$ denotes relative homology, and the maps in the same degree are the obvious ones.

We are interested in $\mathrm{Tor}\, H^3(\mathcal{Y}_5) = \mathrm{Tor}\, H_2(\mathcal{Y}_5)$. It is a classical result of Milnor (see theorems 5.11 and 6.5 of [216]) that $\mathcal{X}_6$ has the homotopy type of a bouquet of three-spheres, and in particular $H_2(\mathcal{X}_6) = 0$. The long exact sequence in homology above then implies that

$$H_2(\mathcal{Y}_5) = \frac{H_3(\mathcal{X}_6, \mathcal{Y}_5)}{H_3(\mathcal{X}_6)} \tag{5.13}$$

where the embedding of $H_3(\mathcal{X}_6)$ into $H_3(\mathcal{X}_6, Y_5)$ is the natural one. This equation has a natural interpretation in terms of four-dimensional field theory, just as discussed in section 3.2, as follows. The numerator denotes the homology class of 3-cycles in $\mathcal{X}_6$, including those that extend to the boundary. If we wrap D3 branes on these cycles we obtain lines in the four-dimensional theory. The D3 branes wrapping compact 3-cycles in $\mathcal{X}_6$ give dynamical lines, while the ones extending to the boundary give line defects. The denominator includes the dynamical lines only, so (5.13) is saying that in order to understand the global structure of the theory, we need to consider the line defects modulo the dynamical excitations, or in other words the unscreened part of the line defect charge, as in [14, 134]. Thus, it is clear that $H_2(\mathcal{Y}_5)$ is the defect group.

It is convenient to rephrase the previous discussion in the language of cohomology groups. Lefschetz duality implies that $H_3(\mathcal{X}_6, \mathcal{Y}_5) \cong H^3(\mathcal{X}_6)$, and the universal coefficient theorem then implies that $H_3(\mathcal{X}_6, \mathcal{Y}_5) \cong \mathrm{Hom}(H_3(\mathcal{X}_6), \mathbb{Z})$. On the other hand, the embedding of $H_3(\mathcal{X}_6)$ into $\mathrm{Hom}(H_3(\mathcal{X}_6), \mathbb{Z})$ is given simply by the partial evaluation of the intersection form $q\colon H_3(\mathcal{X}_6) \times H_3(\mathcal{X}_6) \to \mathbb{Z}$. That is, given any element $x \in H_3(\mathcal{X}_6)$ we have an embedding $Q\colon H_3(\mathcal{X}_6) \to \mathrm{Hom}(H_3(\mathcal{X}_6), \mathbb{Z})$ given by $Q(x) = q(x, \cdot)$. We can thus rewrite (5.13) as $H_2(\mathcal{Y}_5) = \mathrm{coker}(q)$, or equivalently

$$H_2(\mathcal{Y}_5) = \mathrm{coker}(Q)\,. \tag{5.14}$$

Now, Q is an integer-valued antisymmetric matrix, so there is a change of basis to Q (that is, an integer matrix P with $\det(P) = \pm 1$ such that $Q = P^t \mathsf{Q} P$) with (see theorem IV.1 in [218])

$$
\mathsf{Q} = \begin{pmatrix}
0 & r_1 & & & & & & & & \\
-r_1 & 0 & & & & & & & & \\
& & 0 & r_2 & & & & & & \\
& & -r_2 & 0 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & 0 & r_n & & & \\
& & & & & -r_n & 0 & & & \\
& & & & & & & 0 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & 0
\end{pmatrix} \tag{5.15}
$$

and $r_i \in \mathbb{Z}$, such that $r_i \mid r_{i+1}$. Without loss of generality we can choose $r_i > 0$. Since P is invertible we have $\text{coker}(Q) = \text{coker}(\mathsf{Q})$. Let us focus on a single 2×2 block in Q of the form

$$
\mathsf{Q}_i = \begin{pmatrix} 0 & r_i \\ -r_i & 0 \end{pmatrix} \tag{5.16}
$$

with $r_i > 0$. We denote the generators of $H_3(\mathcal{X}_6)$ on this subspace a, b, and the dual elements $a^*, b^* \in \text{Hom}(H_3(\mathcal{X}_6))$. We have $\mathsf{Q}_i(a) = r_i b^*$ and $\mathsf{Q}_i(b) = -r_i a^*$. This implies that $\text{coker}(\mathsf{Q}_i) = \mathbb{Z}_{r_i} \oplus \mathbb{Z}_{r_i}$. We thus have

$$
\text{coker}(\mathsf{Q}) = \mathbb{Z}^\kappa \oplus \sum_{i=1}^{n} \mathbb{Z}_{r_i} \oplus \mathbb{Z}_{r_i} \tag{5.17}
$$

with

$$
\kappa = \text{rank } (H_3(\mathcal{X}_6)) - 2n = \text{rank } F
$$

is the factor corresponding to the rank of 0-form flavour symmetry F of the theory, while n is its rank (i.e. the dimension of the Coulomb branch of the SCFT). This determines $\text{Tor}\, H_2(\mathcal{Y}_5)$ as an Abelian group

$$
\text{Tor}\, H_2(\mathcal{Y}_5) = \sum_{r_i > 1} \mathbb{Z}_{r_i} \oplus \mathbb{Z}_{r_i} \,. \tag{5.18}
$$

The case $r_i = 1$ is trivial; we choose to exclude it from the sum.

In order to understand the global structure of the Argyres-Douglas theories we need a final piece of additional information, the linking pairing between elements in

$H_2(\mathcal{Y}_5)$. Let us focus again on a single block Q_i, with $r_i > 1$. The linking form T_i on $[\mathrm{Tor}\, H_2(\mathcal{Y}_5)]_i$ (that is, the i-th block of $\mathrm{Tor}\, H_2(\mathcal{Y}_5)$) is related very simply to Q_i [219]:

$$T_i = \mathsf{Q}_i^{-1} = \begin{pmatrix} 0 & -\frac{1}{r_i} \\ \frac{1}{r_i} & 0 \end{pmatrix} \mod 1\,. \tag{5.19}$$

The final answer from our analysis is thus quite straightforward. Recall from (5.11) that we are after maximal isotropic subgroups of $\mathrm{Tor}\, H_2(\mathcal{Y}_5)$, where the commutation relations are determined by the linking form T. From the form above, the problem then reduces to the classification of the maximal isotropic sublattices for each block of Q_i, that is the maximal isotropic sublattices of $\mathbb{Z}_{r_i} \oplus \mathbb{Z}_{r_i}$ with the pairing (5.19). (This problem is isomorphic to the problem of determining the global forms of the $\mathcal{N} = 4$ $\mathfrak{su}(r_i)$ theory, studied in [134].)

As an example, assume that

$$\mathsf{Q} = \begin{pmatrix} 0 & 2 & & \\ -2 & 0 & & \\ & & 0 & 2 \\ & & -2 & 0 \end{pmatrix}\,. \tag{5.20}$$

As we will show below, $\mathcal{T}[A_4, D_6]$ is of this type. Each 2×2 block is of the form

$$\mathsf{Q}_i = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \tag{5.21}$$

leading to a contribution to the torsion of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, so in total $\mathrm{Tor}\, H_2(\mathcal{Y}_5) = \mathbb{Z}_2^4$. For each block, we have three maximal isotropic subgroups, which in this case comprise a single element. They are $\{(1,0)\}$, $\{(0,1)\}$ and $\{(1,1)\}$. (At this point we encourage the reader to compare with the global forms of the $\mathfrak{su}(2)$ theory in [134].) So, we find that there are in total $3 \times 3 = 9$ possible choices for the global form of the $\mathcal{T}[A_4, D_6]$ theory.

5.3 1-form symmetries from from Orlik's theorem

Let Y_5 be, as above, a 5d manifold homotopic to the boundary at infinity. We model it by $f^{-1}(0) \cap S^7$, namely the intersection of the hypersurface $V = \{x = (x_1, x_2, x_3, x_4) \mid f(x) = 0\}$ and the 7-sphere S^7 inside $\mathbb{C}^4$, where $f = f(x)$ is a quasi-smooth [220] weighted homogeneous polynomial with weights $w = (w_1, w_2, w_3, w_4)$

and of total degree d, with isolated singularities at the origin. All of the examples that we will discuss in this section are of this type. We have that [221, 222]

$$H_2(Y_5) = \mathbb{Z}^\kappa \oplus \sum_{i=1}^{4} \mathbb{Z}_{r_i}^{2g_i} , \tag{5.22}$$

where

$$\kappa = \sum (-1)^{4-s} \frac{u_{i_1} \cdots u_{i_s}}{v_{i_1} \cdots v_{i_s} \operatorname{lcm}(u_{i_1} \cdots u_{i_s})} \qquad u_i = \frac{d}{\gcd(d, w_i)} \qquad v_i = \frac{w_i}{\gcd(d, w_i)} \tag{5.23}$$

and the sum is taken over all the 16 subsets $\{i_1, ..., i_s\}$ with s elements of the index set $\{1, 2, 3, 4\}$. Moreover,

$$r_i = \gcd(w_1, .., \hat{w}_i, ..., w_4) \tag{5.24}$$

and

$$2g_i = -1 + \sum_{j \neq i} \frac{\gcd(d, w_j)}{w_j} - d \sum_{j<k\,,\, j,k \neq i} \frac{\gcd(w_j, w_k)}{w_j w_k} + d^2 \frac{r_i}{w_1 ... \hat{w}_i ... w_4} , \tag{5.25}$$

and the notation $\hat{w}$ means omit w.

5.4 $(\mathfrak{g}, \mathfrak{g}')$ Argyres-Douglas theories

We now want to apply the ideas developed in the previous sections to the particular case of the Argyres-Douglas theories. The BPS quivers for these theories are known [182, 223], so in principle, we already have all the information that we need at hand, but in order to give general results it is more convenient to use results by Boyer, Galicki and Simanca [222] that we now summarise. An additional benefit is that these results also apply to some examples beyond the $(\mathfrak{g}, \mathfrak{g}')$ theories that will be of interest below, and whose BPS quiver has not appeared previously in the literature.

Within the class of theories engineered by type IIB superstrings on hypersurface singularities, a subset of geometries that naturally generalises the original examples by Argyres and Douglas are the Cecotti-Neitzke-Vafa $(\mathfrak{g}, \mathfrak{g}')$ SCFTs, or $\mathfrak{T}[\mathfrak{g}, \mathfrak{g}']$ for short [182]. These are also known as the generalised Argyres-Douglas theories of type $(\mathfrak{g}, \mathfrak{g}')$ [186]. Consider a background of the form $\mathcal{M}_4 \times \mathbf{X}_6$, where $\mathcal{M}_4$ is some arbitrary closed four-manifold, which we will always assume to be closed Spin without torsion and $\mathbf{X}_6$ is a non-compact Calabi-Yau threefold with an isolated

(g, g')	$\mathrm{Tor}\, H_2(Y_{(\mathfrak{g},\mathfrak{g}')})$	
(A_n, A_m)	0	
(A_{n-1}, D_{m+1})	$\mathbb{Z}_2^{\gcd(n,m)-1}$	if $2 \nmid n$
	$\mathbb{Z}_2^{\gcd(n,m)-2}$	if $2 \mid m$ and $\gcd(n, 2m) \mid m$
	0	otherwise
(A_{n-1}, E_6)	0	if $12 \mid n$
	$\mathbb{Z}_2^2$	if $6 \mid n$
	$\mathbb{Z}_3^2$	if $4 \mid n$
	0	otherwise
(A_{n-1}, E_7)	0	if $18 \mid n$
	$\mathbb{Z}_2^6$	if $9 \mid n$
	$\mathbb{Z}_3^2$	if $6 \mid n$
	0	otherwise
(A_{n-1}, E_8)	0	if $30 \mid n$
	$\mathbb{Z}_2^8$	if $15 \mid n$
	$\mathbb{Z}_3^4$	if $10 \mid n$
	$\mathbb{Z}_5^2$	if $6 \mid n$
	0	otherwise
(D_{n+1}, D_{m+1})	$\mathbb{Z}_2^{\gcd(n,m)}$	if $2 \mid m$ and $(2\gcd(n,m)) \mid n$
	$\mathbb{Z}_2^{\gcd(n,m)}$	if $2 \mid n$ and $(2\gcd(n,m)) \mid m$
	$\mathbb{Z}_2^{\gcd(n,m)-1}$	if $2 \mid n$ and $2 \nmid m$
	$\mathbb{Z}_2^{\gcd(n,m)-1}$	if $2 \nmid n$ and $2 \mid m$
	0	otherwise
(D_{n+1}, E_6)	$\mathbb{Z}_2^6$	if $12 \mid n$
	0	if $4 \mid n$ or $6 \mid n$
	$\mathbb{Z}_4^2$	if $3 \mid n$
	$\mathbb{Z}_3^2$	if $2 \mid n$
	0	otherwise
(D_{n+1}, E_7)	$\mathbb{Z}_2^6$	if $18 \mid n$
	0	if $9 \mid n$ or $6 \mid n$
	$\mathbb{Z}_3^2$	if $3 \mid n$
	0	otherwise
(D_{n+1}, E_8)	$\mathbb{Z}_2^8$	if $30 \mid n$
	0	if $15 \mid n$ or $10 \mid n$ or $6 \mid n$
	$\mathbb{Z}_3^4$	if $5 \mid n$
	$\mathbb{Z}_5^2$	if $3 \mid n$
	0	otherwise
(E_n, E_m)	0	

Table 5.1: Defect groups for the Argyres-Douglas theories. Whenever two cases overlap the earliest applicable one is the correct result. For instance, $\mathrm{Tor}(H_2(Y_{A_{11},E_6})) = 0$.

singularity, given by the hypersurface

$$P_{(\mathfrak{g},\mathfrak{g}')}(x,y,w,z) = P_{\mathfrak{g}}(x,y) + P_{\mathfrak{g}'}(w,z) = 0 \tag{5.26}$$

inside $\mathbb{C}^4$. Here

$$\begin{array}{c|c} \mathfrak{g} & P_{\mathfrak{g}}(x,y) \\ \hline A_n & x^2+y^{n+1} \\ D_n & x^2y+y^{n-1} \\ E_6 & x^3+y^4 \\ E_7 & x^3+xy^3 \\ E_8 & x^3+y^5 \end{array} \tag{5.27}$$

are such that $z^2+P_{\mathfrak{g}}(x,y)=0$ is the du Val singularity of type $\mathfrak{g}$. This space has an isolated singularity at $x=y=z=w=0$. The corresponding 2d $(2,2)$ LG theory has superpotential $W = P_{(\mathfrak{g},\mathfrak{g}')}(x,y,w,z)$. The resulting 2d worldsheet theory has a central charge $\hat{c} < 2$, in this case, [182], and these singularities are at a finite distance. It is believed that at low energies this configuration can be described by the $\mathfrak{T}[\mathfrak{g},\mathfrak{g}']$ four dimensional SCFT compactified on $\mathcal{M}_4$.

Recall that, as discussed in section 5.2, the defect groups of such hypersurface singularities are given by $H^2(\mathcal{Y}_5)$, which may be calculated using the results of section 5.3. We have summarised the defect groups of $(\mathfrak{g},\mathfrak{g}')$ Argyres-Douglas theories in table 5.1.

5.5 $D_p^b(G)$ theories

When we geometrically engineer a theory, such as in the example of the last section, it is typical to take the singular hypersurface to be embedded in $\mathbb{C}^4$. However, we can also consider the possibility that one or more of the directions of the hypersurface is represented by a $\mathbb{C}^*$ variable. An example of such a geometry is given by

$$\mathbf{X}_{6,\text{pure}} = \{\mathbf{x} \in \mathbb{C}^4 : U^2 + X^2 + Y^2 + e^z + e^{-z} = u\}, \tag{5.28}$$

where $\mathbf{x} = (U,X,Y,z)$ and u is a parameter that descends to a Coulomb branch parameter in the four-dimensional theory. It is known that this geometry engineers pure SU(2) super Yang-Mills (SYM) and the resulting BPS quiver is the $A(1,1)$ affine Dynkin quiver [182, 224].

One can consider generalisations of (5.28) to encounter more theories with affine quiver components. In particular, consider the geometry given by

$$\mathbf{X}_6 = \{\mathbf{x} \in \mathbb{C}^4 : U^2 + W_G(X,Y) + e^{pZ} + e^{-Z} = 0\}, \tag{5.29}$$

where W_G is the G-type Du Val singularity[9] and $p \in \mathbb{N}$. As the (X,Y) and Z terms aren't mixed, the resulting four-dimensional theory is factored in the same sense as (G, G') theories. The BPS quivers for the theories compactified on this geometry are simply given by $A(p,1) \boxtimes G$, where $\boxtimes$ is called the triangle tensor product of the two quivers (with the corresponding intersection matrix of the product quiver defined in the next paragraph).

By studying the light subcategory of these theories, it was shown in [185] that there exists a corner of parameter space where they simplify to SYM with gauge group G coupled to a superconformal system which we call the $D_p(G)$ theory. Decoupling the SYM factor, by taking the SYM coupling to zero, leaves us with the $D_p(G)$ theory with an enhanced flavour group containing G. In particular, the rank of the resulting theory's flavour group is given by

$$\text{rank } F = \text{rank } G + \sum_{d \in I_n^p} \varphi(d) = f(p; G), \tag{5.30}$$

where ϕ is the Euler totient function and I_n^p is a subset of divisors of p and $h^\vee(G)$. Furthermore, the effect of decoupling the G-SYM is manifest on both the BPS quiver and the engineering geometry. The geometry which engineers the $D_p(G)$ theory is obtained from (5.29) by simply dropping the e^{-Z} term while the BPS quiver reduces to $A(p,0) \boxtimes G$ [187], the intersection matrix of which can be usefully written as

$$B = (1-P) \otimes S_G + (1-P)^T \otimes S_G^T, \tag{5.31}$$

where P is the cyclic permutation matrix acting on p elements and S_G is the Stokes matrix of G [224].

[9]More precisely, it is the versal deformation of the minimal G-type Du Val singularity. Up to analytic isomorphism, it can be found by removing the squared term of each Du Val singularity.

5.5.1 Defect group and Maruyoshi-Song flows

If we use an engineering geometry of the form (5.29), we cannot use the methods of [222] to calculate the defect group as the geometry is not manifestly quasi-homogeneous. One can instead use the substitution $t = e^Z$ to obtain an equation that is quasi-homogeneous. These geometries coincide with those of (A_{p-1}, G) with some caveats. The most important of which is that the change of variable alters not only the geometry, but also the holomorphic top-form $\Omega^{\mathfrak{T}}$ of the theory $\mathfrak{T}$. In fact, the substitution is such that

$$\Omega^{D_p(G)} = \frac{\Omega^{(A_{p-1},G)}}{t}. \tag{5.32}$$

As we must have that $[\Omega^{\mathfrak{T}}] = 1$, this explicitly changes the scaling dimensions of the geometry. This shows that these two classes of theories are indeed different.

Motivated by this, one can then postulate that the defect groups of $D_p(G)$ and (A_{p-1}, G) theories have the same torsional part, and different flavour factors. However, as Orlik's algorithm [221] is typically formulated for surfaces in $\mathbb{C}^n$ we should be cautious[10].

Using (5.31) as the intersection form we can use the Smith normal form of B to read off the cokernel as described in section 5.2. Doing so for $3 \leq p \leq 30$ and $3 \leq \text{rank } G \leq 30$ we notice that the torsional part of the defect group is that of the (A_{p-1}, G) theory, as expected. Indeed, we have

$$\mathbb{D}_{D_p(G)} = \mathbb{Z}^{f(p;G)} \oplus \text{Tor } \mathbb{D}_{(A_{p-1},G)}. \tag{5.33}$$

This is a realisation of the Maruyoshi-Song (MS) flow from $D_p(G)$ to (A_{p-1}, G) [190, 225]. Furthermore, we already know the defect groups for the (A_{p-1}, G) theories, so we can simply read them from table 5.1.

Evidence of this MS flow can be seen at the level of BPS quivers. The $D_p(G)$ theory possesses a BPS quiver of size $p \cdot \text{rank } G$ given by $A(p, 0) \boxtimes G$. Deforming the theory by an MS term and triggering a flow using the principal nilpotent VEV of $\mathfrak{g}$ will reduce the size of the quiver in the IR to $|Q_{IR}| = (p-1)\,\text{rank } G$ [226]. Furthermore,

[10] If one assumes that the algorithm holds exactly in this case, one would use it to find the flavour ranks of the theories and conclude that they coincide. This is not the case! This hypothesis is tested in [3] by comparing with the results obtained from the BPS quivers of the theories.

Group	**Coxeter number** $h(G)$	**Characteristic polynomial of** Φ_G
A_n	$n+1$	$t^n + t^{n-1} + \ldots + t + 1$
D_n	$2n-2$	$(t+1)(t^{n-1}+1)$
E_6	12	$(t^2+t+1)(t^4-t^2+1)$
E_7	18	$(t+1)(t^6-t^3+1)$
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$
$A(p,1)$	–	$(t-1)(t^p-1)$

Table 5.2: The Coxeter numbers of simply-laced Dynkin groups and the characteristic polynomials of their Coxeter elements [227]. Additionally, we list the characteristic polynomial for the Coxeter element of $A(p,1)$ [187].

the IR quiver will have nullity given by

$$\dim(\ker B_{IR}) = \text{rank } F - \text{rank } G = \sum_{d \in I_n^p} \varphi(d). \tag{5.34}$$

This is a consequence of triggering the flow with the principal nilpotent VEV of $\mathfrak{g}$ instead of the full flavour symmetry $\mathfrak{f}$ of $D_p(G)$. Detaching one full G factor in the BPS quiver will break the $A(p,0)$ to A_{p-1} and leave us with $A_{p-1} \boxtimes G$, a possible BPS quiver for (A_{p-1}, G), which has the correct size to be a candidate for the IR theory. Let us check the rank of the flavour group for these theories.

We can infer the flavour rank of the theory by inspecting the eigenvalues of the 2d monodromy $H = (S^{-1})^T S$ formed from the Stokes matrix S of the quiver [224]. For the (G, G') theories, we can write this as

$$H = \Phi_G \otimes \Phi_{G'}, \tag{5.35}$$

where Φ_G is the Coxeter element of the group G. The flavour rank of the theory is given by the number of eigenvalues of H that are equal to one. We know that the characteristic polynomial is given by

$$\det\left(H - t\,\mathrm{id}_{r(G)\times r(G')}\right) = \prod_{i=1}^{r(G)} \prod_{j=1}^{r(G')} \left(t - \lambda_i^G \lambda_j^{G'}\right), \tag{5.36}$$

where λ_i^G are the eigenvalues of Φ_G. The problem is now reduced to finding appropriate products of eigenvalues for $G = A_{p-1}$. We list the characteristic polynomials of the ADE group's Coxeter elements in table 5.2 from which we can infer the eigenvalues.

Condition (5.36) imposes a linear Diophantine equation for each choice of G' from which one can, in principle, establish the results. An easier method is to instead notice that the characteristic polynomial for $A(p,1)$ is simply that of A_{p-1} with two additional roots at $t=1$. However, no other characteristic polynomial listed in table 5.2 has a root at $t=1$, so no product with the additional roots will contribute to the flavour rank. We therefore have

$$F((A_{p-1},G)) = F(A(p,1)\boxtimes G), \tag{5.37}$$

where $F(\cdot)$ denotes the flavour rank of the theory. Noting that the additional flavour symmetries in the $D_p(G)$ theory (that is, the terms on the RHS of equation (5.34)) are exactly those flavour symmetries present in the $A(p,1)\boxtimes G$ theory, we have that the (A_{p-1},G) theory possesses exactly the desired flavour rank. Piecing these facts together, we see that this signals a possible MS flow

$$D_p(G) \longrightarrow (A_{p-1},G) \tag{5.38}$$

for any $p>1$ and simply-laced G. This is in agreement with the MS flow found in [190, 225].

5.5.2 Higher symmetries of $D_p^b(G)$ theories

In the previous section, we considered theories which are engineered by a surface including a Du Val singularity and an affine $(\mathbb{C}^*)$ part. We can further generalise these theories by replacing the singularity with a compound Du Val singularity [228]. We start with the $\mathbb{C}^4$ hypersurface defined by the vanishing locus of

$$W_G^{\text{comp.}}(u,x,y,z) = u^2 + W_G(x,y) + zg(u,x,y,z), \tag{5.39}$$

where $W_G(x,y)$ is the equation of the G-type Du Val singularity and g is an arbitrary polynomial in four variables. However, it is known that not all compound Du Val singularities are isolated [229] so requiring this to be the case greatly restricts the geometry. In fact, it can be shown that the only possible isolated geometries, in this case, are the ones listed in table 5.3 [230].

These geometries give rise to the $G^{(b)}[k]$ theories of [230] and taking $z=e^Z$ gives the $D_k^b(G)$ theory. The parameter b takes on physical meaning in the $G^{(b)}[k]$ theory where the holomorphic part of the Higgs field of the associated Hitchin system has

Group	b	**Singular hypersurface** $\mathcal{W} \subset \mathbb{C}^4$
A_n	$n+1$	$u^2 + x^2 + y^{n+1} + z^k = 0$
	n	$u^2 + x^2 + y^{n+1} + yz^k = 0$
D_n	$2n-2$	$u^2 + x^{n-1} + xy^2 + z^k = 0$
	n	$u^2 + x^{n-1} + xy^2 + yz^k = 0$
E_6	12	$u^2 + x^3 + y^4 + z^k = 0$
	9	$u^2 + x^3 + y^4 + yz^k = 0$
	8	$u^2 + x^3 + y^4 + xz^k = 0$
E_7	18	$u^2 + x^3 + xy^3 + z^k = 0$
	14	$u^2 + x^3 + xy^3 + yz^k = 0$
E_8	30	$u^2 + x^3 + y^5 + z^k = 0$
	24	$u^2 + x^3 + y^5 + yz^k = 0$
	20	$u^2 + x^3 + y^5 + xz^k = 0$

Table 5.3: The singular hypersurfaces which exhibit *isolated* compound Du Val singularities. The theories engineered from these geometries are called the $G^{(b)}[k]$ theories and taking $z \mapsto e^Z$ gives the $D_k^b(G)$ theories.

asymptotic form

$$\Phi_z \sim \frac{T}{z^{2+k/b}}, \tag{5.40}$$

where T is some regular semi-simple element of G.

Our goal is to compute the defect groups of both the $D_p^b(G)$ and the $G^{(b)}[k]$. Since we know that there is a Maruyoshi-Song flow for any $D_k^b(G)$ theory taking it to $G^{(b)}[k]$ triggered by the principal nilpotent VEV of G [190][11] the problem is greatly simplified. From this, we can find the defect groups of $D_k^b(G)$ theories from those of the $G^{(b)}[k]$ which in turn can be found easily via Orlik's theorem. The results are listed in table 5.4.

Furthermore, note that some of the theories discussed above have a Lagrangian description [231–233]. Thus, it is also possible to find the defect groups by looking at their gauge groups and matter contents. In fact, these cases including the $D_n^{(n)}[k]$ theories are discussed in [2].

[11]Indeed, the flow mentioned in the previous section is of this form.

Theory $\mathfrak{T}$	**Torsional part of defect group** Tor $\mathbb{D}_{\mathfrak{T}}$	
$A_{n-1}^{(n-1)}[k]$	0	
$D_n^{(n)}[k]$	$\mathbb{Z}_2^{\gcd(2k,n)-2}$	if $2 \mid \frac{n}{\gcd(k,n)}$
	0	otherwise
$E_6^{(9)}[k]$	$\mathbb{Z}_3^2$	if $9 \nmid k$ and $3 \mid k$
	0	otherwise
$E_6^{(8)}[k]$	$\mathbb{Z}_2^2$	if $8 \nmid k$ and $4 \mid k$
	0	otherwise
$E_7^{(14)}[k]$	$\mathbb{Z}_2^6$	if $2 \nmid k$ and $7 \mid k$
	0	otherwise
$E_8^{(24)}[k]$	$\mathbb{Z}_2^8$	if $24 \nmid k$ and $12 \mid 4$
	$\mathbb{Z}_2^4$	if $6 \mid k$
	$\mathbb{Z}_2^2$	if $3 \mid k$
	$\mathbb{Z}_3^4$	if $8 \mid k$
	0	otherwise
$E_8^{(20)}[k]$	$\mathbb{Z}_2^8$	if $20 \nmid k$ and $10 \mid k$
	$\mathbb{Z}_2^4$	if $5 \mid k$
	$\mathbb{Z}_5^2$	if $4 \mid k$
	0	otherwise

Table 5.4: The torsional parts of the defects groups for $G^{(b)}[k]$ theories. For theories where the cases overlap, the highest written condition takes priority.

6 Conclusion

In this thesis, we developed a powerful formulation to systematically derive higher-form symmetries of quantum field theories which have a string theory or M-theory construction. More specifically, we applied the framework of geometric engineering, which involves the dimensional reduction of string or M-theory on manifolds with non-trivial geometry. We explained how the defect groups of higher-form symmetries and 't Hooft anomalies associated with them arise by examining the torsion cycles in the cohomology groups of these manifolds.

Throughout this thesis, we focused on specific examples and constructions to illustrate the concepts and techniques involved in the study of higher-form symmetries. In the M-theory setup, as a first example, we discussed 7d supersymmetric gauge theories and found that they have an associated defect group with factors of 1-form symmetries that have a mixed 't Hooft anomaly. We further found that their corresponding symmetry TFT was obtained from link reduction. An important part of our construction of symmetry TFT was the consideration of differential cohomology classes in the reduction of the fluxes to include the consideration of torsion (co)-cycles that result in discrete symmetries. We found that the symmetry TFT of the 7d theory contains information about mixed 't Hooft anomaly between the 1-form centre symmetries and the 2-form $U(1)$ instanton symmetry.

Then, we looked at 5d SCFTs from M-theory on canonical CY singularities. It was found that they have 1-form and (-1)-form symmetries resulting from M2 branes wrapping non-compact cycles in the geometry, as well as 2-form and 4-form symmetries resulting from M5 branes wrapping non-compact cycles. Through analysis of flux commutativity, we further found that the 1-form and 2-form symmetries, as well as (-1)-form and 4-form symmetries, have mixed 't Hooft anomalies. This means that they cannot be all realised in the 5d theory and a choice of a global structure must be made. Moreover, we studied their Symmetry TFTs and found that they have a purely 1-form symmetry cubic anomaly (B^3), and the mixed $U(1)_I$ and 1-form symmetry anomaly.

In the case of IIB theory, we looked at many examples of 4-dimensional $\mathcal{N} = 2$ theories. There is no known Lagrangian description for many of these $\mathcal{N} = 2$ theories, which makes it essential to develop techniques to study these theories. A natural basic question in this context is to determine all the symmetries for these models. We took a first step in this direction and determined the 1-form symmetries of these theories using Orlik's theorem. For theories with a BPS quiver description, we found that the defect group is also given by Dirac pairing.

There are various applications to other geometric engineering setups. For instance, within the M-theory setting that we have discussed in this paper the natural extension is to consider reduction to 4d on G_2-holonomy manifolds, such as the ones proposed by Bryant and Salamon [234], and generalisations thereof, which model the confining-deconfining transition of SYM theories. The reduction on Calabi-Yau four- and five-folds should also result in interesting anomalies in 3d and 1d. Specialising to the case of elliptic Calabi-Yau n-folds, the results in M-theory have an uplift to F-theory, and thus anomalies in the context of even-dimensional QFTs, like 6d SCFTs (for a review see [235]), 4d $\mathcal{N} = 1$ SQFTs (for a review see [236]) and 2d (0,2) theories [237]. More generally, type IIB compactifications can yield supersymmetric gauge theories, which can have 1-form symmetries [2, 3, 22, 24, 27]. Given two dual construction of a theory, one using geometric engineering discussed here and another for example given by brane construction such as the setup given in [88], it would be interesting to match the symmetries across dualities for example by following [238].

The research presented in this thesis represents a significant contribution to the field of generalised global symmetries and their connection to geometric engineering. Furthermore, the insights gained from this work have implications for various branches of theoretical physics. There still remain many open questions and avenues for further exploration. The field is still in its early stages, and ongoing research is focused on uncovering the full structure and implications of these symmetries. Future investigations could delve into more complex or insightful scenarios. For example, by considering different geometrical and topological properties of spacetime, or by exploring the interplay between generalised symmetries and other fundamental aspects of theoretical physics.

In conclusion, the study of generalised global symmetries has enriched our understanding of quantum field theories. By combining the insights from string theory, we laid the groundwork for future advancements in this exciting field of research.

A K-theory groups for the boundary of threefold singularities

In this appendix, we will compute the K-theory groups of the manifold Y_5 at the boundary of an isolated hypersurface singularity. It is convenient to do so by computing the reduced Atiyah-Hirzebruch spectral sequence for homology (see remark 2 in pg. 351 of [239])

$$E_2^{p,q} = \tilde{H}_p(Y_5; K_q(\mathrm{pt})) \Longrightarrow \tilde{K}_{p+q}(Y_5)\,. \tag{A.1}$$

The second page of this spectral sequence is shown in figure A.1. Note that in writing that spectral sequence we are using $H^1(Y_5) = H^4(Y_5) = 0$.

The only potentially non-vanishing differential is indicated by d^3 in the drawing. This is the first non-vanishing differential, so it is a stable homology operation (see §4.L in [157] for a definition and proofs of some of the statements below) dual to a stable cohomology operation $d_3\colon H^0(Y_5) \to H^3(Y_5)$. Such operations are classified by $[K(\mathbb{Z}, 0), K(\mathbb{Z}, 3)] = [\mathbb{Z}, K(\mathbb{Z}, 3)] = H^3(\mathbb{Z}) = 0$. So there is no non-vanishing stable homology operations acting on these degrees, and the spectral sequence stabilises. There are no extension ambiguities either in going from the filtration to the K-theory group, so we conclude that $\tilde{K}_0(Y_5) = H_2(Y_5) = H^3(Y_5)$.

The relation between the reduced and non-reduced K-homology groups is $K_0(Y_5) = \mathbb{Z} \oplus \tilde{K}_0(Y_5)$ (see for instance eq. (1.5) in [240]), so $K_0(Y_5) = \mathbb{Z} \oplus \tilde{K}_0(Y_5)$. Note also that Y_5 admits a Spin structure (since its normal bundle in the Calabi-Yau cone X_6 is trivial, and X_6 is Spin), and in particular a Spin^c structure, or in other words it is K-orientable. So we can apply Poincaré duality, and $K_0(Y_5) = K^1(Y_5)$. We conclude that

$$K^1(Y_5) = H^3(Y_5) \oplus H^5(Y_5) \tag{A.2}$$

and similarly

$$K^0(Y_5) = H^0(Y_5) \oplus H^2(Y_5)\,. \tag{A.3}$$

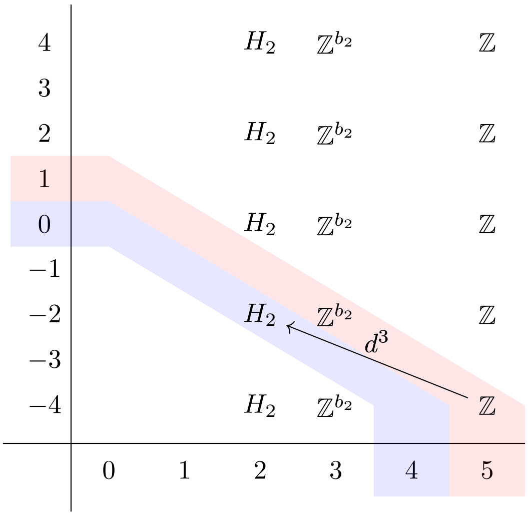


Figure A.1: Second page for the Atiyah-Hirzebruch spectral sequence for the reduced K-homology of the horizon manifold on an isolated hypersurface singularity. We have denoted $H_2 \coloneqq H_2(Y^5)$, $b_2 \coloneqq \mathrm{rk}(H_2 \otimes \mathbb{Q})$, and shown the only differential that might potentially be non-vanishing. The entries shaded in blue are those contributing to $K_0(Y_5)$, and those in pink are those contributing to $K_1(Y_5)$.

It is also clear that the K-theory groups of $\mathcal{M}_4$ agree with the formal sums of cohomology groups, since by the Chern isomorphism $K^i(\mathcal{M}_4) \otimes \mathbb{Q} \cong \bigoplus_n H^{2n+i}(\mathcal{M}_4; \mathbb{Q})$, and we are assuming that $\mathcal{M}_4$ has no torsion, so the relevant Atiyah-Hirzebruch spectral sequence has no non-vanishing differentials.

Finally, we can use the Künneth exact sequence in K-theory [241]

$$0 \to \bigoplus_{i+j=m} K^i(X) \otimes K^j(Y) \to K^m(X \times Y) \to \bigoplus_{i+j=m+1} \mathrm{Tor}_{\mathbb{Z}}(K^i(X), K^j(Y)) \to 0 \tag{A.4}$$

to assemble these results together, and prove the statement in the IIB discussion of section 2.2.1 that we can use ordinary cohomology for classifying IIB flux in these backgrounds.

B Toric computations

B.1 The Mori cone for $\mathcal{C}_{\mathbb{R}}(Y^{p,q})$

In this section we will study in detail the structure of the Mori cone for the Calabi-Yau cone over $Y^{p,q}$ [172], which we denote by $\mathcal{C}_{\mathbb{R}}(Y^{p,q})$. This Calabi-Yau threefold is toric, simplifying the relevant geometry analysis. We refer the reader to [162] for general background on toric geometry and [165–167] for introductions aimed at physicists. The computer algebra program SAGE contains very useful implementations of the toric algorithms that we use [169].

Define $l := p - q$. We can take the points in the toric diagram for $\mathcal{C}_{\mathbb{R}}(Y^{p,q})$ to be $P_1 = (-1, 1)$, $P_3 = (l, 0)$ and $I_i = (0, i)$, $i \in \{0, 1, \cdots, p\}$. We choose the triangulation as in figure B.1, that is, such that the 3-dimensional cones are of the form (P_1, I_k, I_{k+1}) and (P_3, I_k, I_{k+1}) with $k \in \{0, 1, 2, \cdots, p-1\}$.

We can construct a (non-minimal) basis of generating curves by taking intersections of toric divisors. The intersection numbers of the compact curves constructed in this way and the toric divisors are given in table B.1. The Mori cone is spanned by compact curves corresponding to 2-dimensional cones. Thus, the number of the generators of the Mori cone equals to the number of independent 2-cycles. From

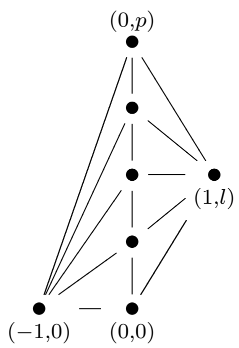


Figure B.1: Triangulation of $Y^{p,q}$ considered in the text.

Curve	P_1	P_3	I_0	I_1	I_2	$\cdots$	I_{k-1}	I_k	I_{k+1}	$\cdots$	I_{p-1}	I_p
$P_1 \cdot I_1$	0	0	1	-2	1	$\cdots$	0	0	0	$\cdots$	0	0
$\cdots$												
$P_1 \cdot I_k$	0	0	0	0	0	$\cdots$	1	-2	1	$\cdots$	0	0
$\cdots$												
$I_0 \cdot I_1$	1	1	$l-2$	$-l$	0	$\cdots$	0	0	0	$\cdots$	0	0
$I_k \cdot I_{k+1}$	1	1	0	0	0	$\cdots$	0	$l-2k-2$	$-l+2k$	$\cdots$	0	0
$\cdots$												
$I_{p-1} \cdot I_p$	1	1	0	0	$\cdots$	0	0	0	0	$\cdots$	$l-2p$	$-l+2(p-1)$

Table B.1: The intersection numbers of the $(2P-1)$ compact curves $P_1 \cdot I_i$, $I_i \cdot I_{i-1}$ and the $(P+3)$ points $P_1 = (-1,1)$, $P_3 = (l,0)$ and $I_i = (0,i)$, where $i \in \{0,1,\cdots,p\}$, and $k \in \{1,2,\cdots,(p-1)\}$. We have omitted the result for the curves $P_3 \cdot I_k$ as they give the same intersection numbers as $P_1 \cdot I_k$ for each fixed k.

our discussion in (4.58) we find that the number of independent compact 2-cycles is p, so this is the dimension of the Mori cone. We denote the Mori cone generators $C_1, \dots, C_p$. Any two curves are linearly equivalent iff their intersection with all toric divisors are the same, so the problem of determining the C_i reduces to finding a basis of linearly independent rows in table B.1. From the table we can deduce the equivalence relations

$$P_1 \cdot I_k \equiv P_3 \cdot I_k, \quad I_{k-1} \cdot I_k - I_k \cdot I_{k+1} \equiv (l-2k) P_1 \cdot I_k, \tag{B.1}$$

where $k \in \{1,2,\cdots,(p-1)\}$. Thus, we may choose the Mori cone generators to be

$$C_k = P_1 \cdot I_k, \quad C_p = I_0 \cdot I_1 \,. \tag{B.2}$$

B.2 Linking forms

B.2.1 $\mathcal{C}_{\mathbb{R}}(Y^{p,q})$

The intersection form Q_4 ($Q_2 = Q_4^T$) between 4-cycles (2-cycles) and 2-cycles (4-cycles) can be easily read from the Mori cone generators to be

$$Q_4 = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 \\ -l & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

for even l, where -2 in the last row is in the column $l/2$ of Q_4 or in a more compact notation

$$Q_4 = (q_{i,j}), \quad q_{i,j} = \begin{cases} \delta_{i,j-1} - 2\delta_{i,j} + \delta_{i,j+1}, & \text{for} \quad i,j \in \{1,2,\cdots,p-1\}, \\ -l & \text{for} \quad i = p \text{ and } j = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{B.3}$$

Now, we can calculate the homology groups of $Y^{p,q}$ using (4.51). We can easily determine the kernel and the cokernel of Q_4 by finding the Smith normal form of Q_4, which we call S_4. We find that S_4 has the form

$$S_4 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \gcd(p,q) \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \tag{B.4}$$

or in more compact notation

$$Q_4 = (s_{i,j}), \quad s_{i,j} = \begin{cases} \delta_{i,j}, & \text{for} \quad i,j \in \{1,2,\cdots,p-1\}, \\ \gcd(p,q)\delta_{i,p-1}\delta_{j,p-1}, & \text{for} \quad i = j = p-1, \\ 0, & \text{otherwise.} \end{cases} \tag{B.5}$$

Hence, the image of Q_4 is $\text{Im}(Q_4) = \mathbb{Z}^{p-2} + \text{gcd}(p,q)\mathbb{Z}$ and its kernel is zero. We have

$$\begin{aligned} H_3(Y^{p,q}) &= \text{coker}(Q_4) = \mathbb{Z} + \mathbb{Z}_{\text{gcd}(p,q)}, \\ H_4(Y^{p,q}) &= \ker(Q_4) = 0\,. \end{aligned}$$

Similarly, since $Q_2 = Q_4^T$ we find

$$H_2(Y^{p,q}) = \ker(Q_2) = \mathbb{Z}, \quad H_1(Y^{p,q}) = \text{coker}(Q_2) = \mathbb{Z}_{\text{gcd}(p,q)}\,.$$

We now want to compute the linking pairing

$$\mathsf{L}_{Y^{p,q}}: \ \text{Tor}\, H_{p-1}(Y^{p,q}) \times \text{Tor}\, H_{n-p-1}(Y^{p,q}) \to \mathbb{Q}/\mathbb{Z}\,. \tag{B.6}$$

In our case the only homology groups with non-trivial torsion are $H_3(Y^{p,q})$ and $H_1(Y^{p,q})$ so we may compute the linking of 3-cycles and 1-cycles as follows. From (4.57)

$$L(\partial\alpha_i^{\prime *}, \bar{\partial}\beta_j^{\prime *}) = q^{-1}(\alpha_i^{\prime *}, \beta_j^{\prime *}) = q_{ij}^{-1} \pmod 1\,. \tag{B.7}$$

We find

$$q_{i,k}^{-1} = \begin{cases} (i-j) + (p-j)c/2, & \text{for} \quad i \geq j\ ,\ j < p \text{ and } i < p-1 \\ (p-j)c/2, & \text{for} \quad i < j\ ,\ j < p \text{ and } i < p-1 \\ (p-2-j)/2 + (p-j)c/2, & \text{for} \quad j < p \text{ and } i = p-1 \\ -i/l - pc/(2l), & \text{for} \quad j = p \text{ and } i < p-1 \\ (1/l - p/(2l)) - pc/(2l), & \text{for} \quad j = p \text{ and } i = p-1\ , \end{cases} \tag{B.8}$$

such that $q^{-1}q = I$. All that remains is to find the generators $\alpha_i^{\prime *}$ and $\beta_j^{\prime *}$ defined above. This may be done by tracking how the generators in the basis defined by the matrix $q^{(T)}$ change as we switch basis by writing the matrix in its Smith normal form $S^{(T)}$. Given the form of our matrix q in (B.3)), for β_i^* and $\beta_j^{\prime *}$ the generators of $\text{Hom}(H_4(X_6), \mathbb{Z})$ in the q basis and the S basis, respectively, we find that, $\beta_{p-1}^{\prime *} = \beta_{p-1}^*$ where $\partial\beta_{p-1}^{\prime *}$ is the generator of $\text{Tor}\, H_1(Y)$. Similarly, for $\alpha_{\bar{i}}^*$ and $\alpha_{\bar{j}}^{\prime *}$ the generators of $\text{Hom}(H_2(X), \mathbb{Z})$ in the q^T basis, and the S^T basis, respectively, we find $\alpha_p^{\prime *} = p'\alpha_1^* + q'\alpha_p^*$ such that, $\partial\alpha_p^{\prime *}$ is the generator of $\text{Tor}\, H_3(Y^{p,q})$ where, $p' = \frac{p}{\text{gcd}(p,q)}$

Curve	a	b	I_0	I_1	I_2	$\cdots$	I_{k-1}	I_k	I_{k+1}	$\cdots$	I_{n-1}	I_n
$a \cdot I_1$	0	0	1	-2	1	$\cdots$	0	0	0	$\cdots$	0	0
$\cdots$												
$a \cdot I_k$	0	0	0	0	0	$\cdots$	1	-2	1	$\cdots$	0	0
$\cdots$												
$I_0 \cdot I_1$	1	1	-3	1	0	$\cdots$	0	0	0	$\cdots$	0	0
$I_k \cdot I_{k+1}$	1	1	0	0	0	$\cdots$	0	$-3-2k$	$2k+1$	$\cdots$	0	0
$\cdots$												
$I_{n-1} \cdot I_n$	1	1	0	0	$\cdots$	0	0	0	0	$\cdots$	$-2n-1$	$2n-1$

Table B.2: The intersection numbers of the $(2P-1)$ compact curves $a \cdot I_k$, $I_k \cdot I_{k-1}$ and the $(n+3)$ points $a = (-1, 0)$, $b = (1, -1)$ and $I_i = (0, i)$, where $i \in \{0, 1, 2, \cdots, n\}$ and $k \in \{0, 1, 2, \cdots, (n-1)\}$. We have omitted the result for the curves $b \cdot I_k$ and $a \cdot I_0$ as they give the same intersection numbers as $a \cdot I_k$ and $I_0 \cdot I_1$ for each fixed k, respectively.

and $q' = \frac{lp-l}{\gcd(p,q)}$. Therefore, the linking number is

$$\begin{aligned} L(\partial\alpha_p'^*, \bar{\partial}\beta_{p-1}'^*) &= L\big(p'\partial\alpha_1^* + q'\partial\alpha_p^*, \bar{\partial}\beta_{p-1}^*\big) \\ &= p'q_{1,p-1}^{-1} + q'q_{p,p-1}^{-1} \\ &= -\frac{1}{\gcd(p,q)} \pmod 1 , \end{aligned} \tag{B.9}$$

using (B.8) and the bilinearity of the linking pairing.

B.2.2 $\mathbb{C}^3/\mathbb{Z}_{2n+1}$

Let $a = (-1, 0)$, $b = (1, -1)$ and $I_i = (0, i)$ be the points on the toric diagram with $i = 0, 1, 2, \cdots, n$, and choose the triangulation such that the 3-dimensional cones are of the form (a, b, I_0), (a, I_k, I_{k+1}) and (b, I_k, I_{k+1}), where $k = 0, 1, 2, \cdots, n-1$. As before, from the toric diagram we have

$$H_2(X_6) = H_4(X_6) = \mathbb{Z}^n \,. \tag{B.10}$$

From the intersection numbers given in table B.2, we deduce the equivalence relations (by subtracting the two relevant rows in terms of k for the latter relation)

$$a \cdot I_i \equiv b \cdot I_i \,, \quad a \cdot I_0 \equiv I_0 \cdot I_1 \,, \quad I_{k+1} \cdot I_{k+2} - I_k \cdot I_{k+1} \equiv (3+2k)a \cdot I_k \,. \tag{B.11}$$

Therefore, we can choose the Mori cone generators C_k to be the rows of table given by the intersection numbers for $a \cdot I_k$

$$C_k = a \cdot I_k \,. \tag{B.12}$$

The intersection form is

$$q_{i,j} = \begin{cases} \delta_{i,j} - 2\delta_{i,j-1} + \delta_{i,j-2}\,, & \text{for} \quad i \in \{1, 2, ..n-1\}\,,\, j \in \{1, 2, ..n\} \\ -3\delta_{1,j} + \delta_{2,j}\,, & \text{for} \quad i = n\,,\, j \in \{1, 2, ..n\} \end{cases} \tag{B.13}$$

which has Smith normal form

$$S_{i,j} = \begin{cases} \delta_{i,j}\,, & \text{for} \quad i, j \in \{1, 2, ..n-1\} \\ 2n+1\,, & \text{for} \quad i, j = n \end{cases} \tag{B.14}$$

From this we find

$$\begin{aligned} H_1(Y_5) &= H_3(Y_5) = \text{coker}(Q) = \mathbb{Z}_{2n+1}, \\ H_2(Y_5) &= H_4(Y_5) = \ker(Q) = 0\,. \end{aligned} \tag{B.15}$$

Now, to find the linking number, we track the effect on the generators as we write Q in its Smith normal form. We find α_n^* and β_n^* to be the generators of $\text{Hom}(H_2(X), \mathbb{Z})$ and $\text{Hom}(H_4(X_6), \mathbb{Z})$, respectively such that, $\partial\alpha_n^*$ and $\partial\beta_n^*$ are the generators of $\text{Tor}\, H_3(Y_5)$ and $\text{Tor}\, H_1(Y_5)$, respectively. It can be shown that the inverse of $q_{i,j}$ is

$$-(2n+1)q_{i,j}^{-1} = \begin{cases} (2i-1)(n-j)\,, & \text{for} \quad j < n \text{ and } j+1 \geq i \\ (n-i+1)(2j+1)\,, & \text{for} \quad j+1 \leq i \\ (n-i+1)\,, & \text{for} \quad j = n \end{cases} \tag{B.16}$$

i.e. $q_{nn}^{-1} = -\frac{1}{2n+1}$, and so we have

$$L(\partial\alpha_n^*, \partial\beta_n^*) = q_{nn}^{-1} = -\frac{1}{2n+1} \mod 1\,. \tag{B.17}$$

C Integral quantisation of G_4-flux

In order to diagnose whether G_4 has integral or half-integral periods on a four-cycle $\mathcal{C}_4$, we can compute the integral of the fourth Stiefel-Whitney class of the tangent bundle $T\mathcal{M}_{11}$ on $\mathcal{C}_4$ [128],

$$\int_{\mathcal{C}_4} G_4 = \frac{1}{2}\int_{\mathcal{C}_4} w_4(T\mathcal{M}_{11}) \mod 1 \,, \tag{C.1}$$

where the pullback to $\mathcal{C}_4$ is implicit. In this work, we consider 11d spacetimes that are a direct product, $\mathcal{M}_{11} = \mathcal{W}_{11-n} \times L_n$, where $n = 3$ and $L_3 = S^3/\Gamma$ (with Γ and ADE subgroup of $SU(2)$), or $n = 5$ and L_5 a smooth Sasaki-Einstein manifold.

The total Stiefel-Whitney class splits as $w(T\mathcal{M}_{11}) = w(T\mathcal{W}_{11-n}) \smile w(TL_n)$. Possible contributions to $w_4(T\mathcal{M}_{11})$ are therefore of the form $w_{4-i}(T\mathcal{W}_{11-n}) \smile w_i(TL_n)$, $i = 0, 1, 2, 3, 4$. We observe that all the spaces L_n in this work are the base of a Calabi-Yau cone. In particular, each L_n is orientable and Spin, and therefore $w_1(TL_n) = 0 = w_2(TL_n)$. We also assume that external spacetime is orientable and Spin, so that $w_1(T\mathcal{W}_{11-d}) = 0 = w_2(T\mathcal{W}_{11-n})$. We conclude that terms of the form $w_{4-i}(T\mathcal{W}_{11-n}) \smile w_i(TL_n)$ with $i = 1, 2, 3$ cannot give any non-zero contributions to $w_4(T\mathcal{M}_{11})$. The remaining potential contributions thus have $i = 0$ or $i = 4$. To kill the contribution with $i = 0$ we assume that external spacetime satisfies $w_4(T\mathcal{W}_{11-n}) = 0$. What remains to be checked is whether $w_4(TL_n)$ is trivial. Clearly, we have $w_4(TL_3) = 0$ for dimensional reasons.[1] Next, we prove that $w_4(TL_5) = 0$ for any smooth Sasaki-Einstein five-manifold L_5. In fact, we can prove a stronger statement: $w_{i>0}(TL_5) = 0$.

Let us adopt the shorthand notation $w_i = w_i(TL_5)$. We have already observed $w_1 = 0 = w_2$. We also have (in general, from the Wu formula)

$$\mathrm{Sq}^1(w_2) = w_3 + w_1 \smile w_2 \,, \tag{C.2}$$

[1] For completeness, let us point out that also $w_3(S^3/\Gamma) = 0$. This follows directly from (C.2), recalling that w_1 and w_2 are zero because S^3/Γ is orientable and spin.

which implies $w_3 = \mathrm{Sq}^1(0) = 0$ in our case. We also have that the Wu class

$$\nu_4 = w_1^4 + w_2^2 + w_1 \smile w_3 + w_4 \tag{C.3}$$

necessarily vanishes on a 5-manifold for degree reasons (as it represents Sq^4 acting on $H^1(L_5;\mathbb{Z}_2)$, which vanishes by general properties of the Steenrod squares), so we conclude $w_4 = 0$. Finally, again from the Wu formula

$$\mathrm{Sq}^2(w_3) = w_2 \smile w_3 + w_1 \smile w_4 + w_5 \ , \tag{C.4}$$

which implies, given $w_1 = w_2 = w_3 = 0$, that $w_5 = 0$.

Let us conclude with further comments on $p_1 = p_1(TL_5)$. Note that on a Spin manifold we have [242]

$$\mathfrak{P}(w_2) = \rho_4(p_1) + \theta_2(w_1 \mathrm{Sq}^1(w_2) + w_4) \ , \tag{C.5}$$

so, since $w_1 = w_2 = w_4 = 0$, we learn from the analysis above that $\rho_4(p_1) = 0$, or equivalently that p_1 vanishes mod 4.

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