Applications of noncommutative intersection forms to linking

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Abstract: We construct a link homotopy invariant for three-component spherical link maps which is a generalisation of the Kirk invariant for two-component spherical link maps. We then construct an invariant for three-component annular link maps and establish that there is a relationship between the three-component annular link map invariant and the three-component link map invariant. We show the link map invariant can detect non-trivial three-component link maps which become trivial up to link homotopy when a component is removed. We establish that there exist link maps where each component has the same image in S^4 but are not link homotopic, unlike in the two-component case. Using the link map invariant we construct an invariant which is analogous to Milnor's triple linking number, and show that they can be used to distinguish different link maps. We provide a method for calculating our invariant for an infinite family of three-component annular link maps and deduce a computation for link maps and detect infinitely many three-component link maps which have vanishing Kirk invariants.

Next we prove that the Blanchfield form on a closed, orientable, connected three-manifold can be computed in terms of the intersection form of a four-manifold which it bounds. We aim to do this using the weakest assumptions possible, as similar results in the literature have been shown but proofs are often imprecise or make more assumptions than are needed.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

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Chapter 1

Introduction

The equivariant intersection form plays a large role in the study of surfaces in four-manifolds, as algebraic intersections often provide the simplest obstructions to an immersion being homotopic to an embedding or two immersed surfaces being homotopic. Their importance is also highlighted by the success of surgery theory, in both high dimensions and in the topological category for dimension four. This thesis is split into two main sections both of which involve equivariant intersection invariants: Chapter 2 through to Chapter 8 focuses on the study of link maps of spheres and annuli in four-manifolds (see Section 1.1 for details). In Chapter 9 we change our focus and study the Blanchfield form on closed orientable three-manifolds and its relationship to the intersections of surfaces in four-manifolds (see Section 1.2 for more details).

1.1 Link homotopy of surfaces

A continuous map

$$f = f_1 \sqcup \cdots \sqcup f_n : \coprod_{i=1}^{n} M_i^{q_i} \to N^q,$$

which keeps disjoint components in the domain disjoint in the image, i.e $f_i(M_i^{q_i}) \cap f_j(M_j^{q_j}) = \phi$, is called a *link map*. We consider link maps up to *link homotopy*, i.e. a homotopy through link maps. A spherical link map is a link map where $M_i^{q_i} = S^{q_i}$ for all i and $N^q = S^q$. Let $\mathrm{LM}^q_{q_1,\dots,q_n}$ be the set of such link maps up to link homotopy.

The classical study of link homotopy started with Milnor in his Bachelor's thesis [Mil54]. He made use of a quotient of the fundamental group of the complement of three disjoint circles in S^3, which we now call the Milnor group. Using this group Milnor classified three-component links in S^3 up to link homotopy. He did this by constructing a triple linking number which measures linking behaviour for links of three or more components. This number is only well defined modulo the greatest common divisor of the linking numbers between each pair of components. The triple linking number, along with the linking number of each two component sublink, classifies three-component links up to link homotopy.

Since then, work in higher dimensions has been studied. If we have a two component spherical link map $f : S^p \sqcup S^q \to \mathbb{R}^m$ such that $p, q \leq m-2$, then the homotopy class of the map $\phi : S^p \times S^q \to S^{m-1}$, defined by

$$\phi(x, y) = \frac{f(x) - f(y)}{|f(x) - f(y)|},$$

can be identified with an $\alpha(f)$ inside the stable homotopy group $\pi_{p+q}(S^{m-1})$. The following results about results have been proven by Massey and Rolfsen, and Levine respectively.

Theorem 1.1.1 ([MR85]). *The map $\alpha : \mathrm{LM}^m_{p.q} \to \pi_{p+q}(S^{m-1})$ is a group homomorphism, where the group structure on $\mathrm{LM}^m_{p.q}$ is connect sum and $2(p+q) \leq 3(m-2)$.*

Theorem 1.1.2 ([MR85]). *If $1 \leq p \leq m-2$ and $f \in \mathrm{LM}^m_{p,m-2}$ then if the codimension two component is embedded then $\alpha(f) = 0$*

This work will focus on the case of link maps of S^2 into S^4. By convention, when we say link maps we mean spherical link maps of S^2 in S^4 unless stated otherwise.

The study of link maps of S^2 inside S^4 began with Fenn and Rolfsen who showed there exists a link homotopically non-trivial link map $f : S^2 \coprod S^2 \to S^4$ which contains a self-intersection on each sphere [FR86]. Self-intersections are a necessary condition for generic smooth link maps of S^2 in S^4 to be non-trivial [BT99].

In 1988, Paul Kirk defined a two-component link map invariant

$$\sigma : \mathrm{LM}^4_{2,2} \to \mathbb{Z}[t] \oplus \mathbb{Z}[t] .$$

For each sphere he associated a polynomial in $\mathbb{Z}[t]$. This polynomial is constructed by taking the set of self-intersections I, and for each element $p \in I$ associating the polynomial $t^{|n_p|} - 1$, where $|n_p|$ is the absolute value of the linking number of the double point loop at p with the other sphere. Summing these polynomials over $p \in I$, taking the sign of the self-intersection into account, yields the invariant.

Schneiderman and Teichner showed that the Kirk invariant classifies two-component link maps up to link homotopy [ST17], computing the group $\mathrm{LM}^4_{2,2}$. Hence, the Kirk invariant is analogous to the linking number in the classical two-component case.

To generalise the Kirk invariant we will use $\mathbb{Z}[F/F_3]$ in place of $\mathbb{Z}[t]$. The group F/F_3 is the third lower central series quotient of the free group on two generators - F_3 is the third lower central series subgroup. This is also the free Milnor group on two generators as proven in Lemma 4.1.10. The group F/F_3 admits the presentation

$$F/F_3 \cong \left\langle\left\langle y, z, s \mid [x, s], [y, s], [x, y]\, s^{-1} \right\rangle\right\rangle ,$$

as shown by Section 4.1. It can also be described as the following central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow F/F_3 \longrightarrow \mathbb{Z}^2 \longrightarrow 0,$$

where the inclusion maps 1 to $[y, z]$. In this thesis, we construct a function

$$\widetilde{\sigma}^3 : \mathrm{LM}^4_{2,2,2} \to \widetilde{K},$$

where $\widetilde{K} := (\mathbb{Z}[F/F_3])^3 / \sim$ for some choice of equivalence relation, specified in Chapter 6. Unlike the Kirk invariant, $\widetilde{\sigma}^3$ is not a group homomorphism as $\mathrm{LM}^4_{2,2,2}$ cannot be turned into a group using connect sum as the operation is not well defined up to link homotopy, see Proposition 1.1.10. This invariant is constructed by first considering an invariant of based three-component link maps up to link homotopy, which takes values in $(\mathbb{Z}[F/F_3])^3$. These values are given by taking each component and computing the self-intersection number in the complement of the other two components. We then consider the effect of changing the basings of a link map on the invariant and determine the finest equivalence relation required to define an unbased invariant. The effect of changing the basing path transforms the group elements by multiplying group elements by the power of a commutator term.

This new invariant fits into a commutative diagram for each $i \in \{1, 2, 3\}$

$$\begin{tikzcd} \mathrm{LM}_{2,2,2} \arrow[r, "i"] \arrow[d, "\widetilde{\sigma}^3"] & \mathrm{LM}^4_{2,2} \arrow[d, "\sigma"] \\ \widetilde{K} \arrow[r, "p_i"] & (\mathbb{Z}[\mathbb{Z}])^2, \end{tikzcd}$$

where i is the map which forgets the ith component; σ is the Kirk invariant and the lower horizontal map, p_i, projects onto the other two components, and sets the ith meridian to 1. Hence, $\widetilde{\sigma}^3$ also contains all the information of the Kirk invariant for each two-component sublink.

In $(\mathbb{Z}[F/F_3])^3$, the group our based link map invaraint takes values in, the group generators we choose in each factor are different. In the first component we use the generators y, z and $s = [x, y]$; in the second factor we have generators z, x and $t = [z, x]$; and in the third factor we use the generators x, y and $u = [x, y]$. In each of these, x, y and z represent meridians of the first, second, and third sphere respectively. Using $\widetilde{\sigma}^3$ we prove the following.

Theorem 1.1.3. *There exists a three-component link map f with a choice of basing path such that*

$$\widetilde{\sigma}^3(f) = \left(z\,(s-1) + z^{-1}\left(s^{-1} - 1\right),\, 0,\, x\,(1-u) + x^{-1}\left(1 - u^{-1}\right)\right).$$

Furthermore, removal of any component gives a trivial two-component link map. But f is not link homotopically trivial.

This shows that the invariant can detect linking information which only occurs in three or more components. Using Theorem 1.1.3 we prove the following

Theorem 1.1.4. *For each $n \geq 3$ there exists link maps $f = f_1 \sqcup \cdots \sqcup f_n$ and $f' = f_1' \sqcup \cdots \sqcup f_n'$ such that for every i, $f_i(S^2) = f_i'(S^2)$, but f and f' are not link homotopic.*

This is done by precomposing the link map in Theorem 1.1.3 with a map which reflects the second sphere, reversing the orientation on the second sphere. Both maps evaluate to distinct elements of $\widetilde{K}$.

Since $\widetilde{\sigma}^3$ takes values in an orbit space, it can be difficult to tell whether one has two representatives of the same value in $\widetilde{K}$ or whether you have two distinct values. To resolve this, we construct invariants to help differentiate the different orbits of $\widetilde{K}$. Consider

$$v = \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a^v_{ijk} y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b^v_{ijk} z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c^v_{ijk} x^i u^j y^k \right)$$

and

$$w = \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a^w_{ijk} y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b^w_{ijk} z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c^w_{ijk} x^i u^j y^k \right)$$

where we have used a normal form of F/F_3 to describe each element of $\mathbb{Z}\left[F/F_3\right]$.

Then we show the following theorem holds.

Theorem 1.1.5. *Suppose v and w represent the same element of $\widetilde{K}$. Then for each a^v_{ijk} there exists an a^w_{lmp} such that $a^v_{ijk} = a^w_{lmp}$, $i = l$, $k = p$ and*

$$j \equiv m \mod \gcd(i,k).$$

Example 1.1.6. Let

$$v = \left(z^2s^2 + z^{-2}s^{-2} - 4zs - 4z^{-1}s^{-1} + 6,\, 0,\, 4xu + 4x^{-1}u^{-1} - x^2u^2 - x^{-2}u^{-2} - 6\right)$$

and

$$w = \left(z^2s + z^{-2}s - 4zs - 4z^{-1}s^{-1} + 6,\, 0, z^2u + z^{-2}u - 4zu - 4z^{-1}u^{-1} + 6\right).$$

Consider the term $a^v_{022} = 1$. In w there is no term a^w_{ijk} such that $i = 0, k = 2, j \equiv 0 \mod \gcd(i,k)$ and $a^w_{ijk} = 1$. Hence, v and w are distinct elements in $\widetilde{K}$.

In the above example v is in the image of $\widetilde{\sigma}^3$ but we were unable to determine whether w is.

This kind of indeterminacy is directly analogous to the triple linking number in the classical dimension. Next we go further and extract a stronger invariant. The naive approach would be to create an invariant listing the values of $k \mod \gcd(i,j)$ for each non-zero $a_{ijk} \neq 0$. However, the power of the commutator term changes simultaneously for all group elements, so we construct an invariant based on the total Milnor quotient in [DNOP20],

$$\overline{\mu} : \widetilde{K} \to \mathcal{A},$$

where $\mathcal{A}$ is a space of affine points, lines, and planes which measures how all the commutator terms change when we change basing paths. We show this invariant can be used to distinguish two link maps. See Section 6.3 for details on how $\mathcal{A}$ is defined.

Example 1.1.7. Let

$$v = (zs - z + z^{-1}s^{-1} - z^{-1}, 0, x - xu + x^{-1} - x^{-1}u^{-1})$$

and

$$w = (zs^2 - z + z^{-1}s^{-2} - z^{-1}, 0, x - xu^2 + x^{-1} - x^{-1}u^{-2}).$$

We cannot use Theorem 1.1.5 to distinguish between v and w, both of which are realised by link maps. However, we show that $\overline{\mu}$ can detect that these maps are different in Section 6.3.

We now consider n-component Annular link maps, which by definition are immersions

$$\coprod_{i=1}^{n} S^1 \times I \to B^3 \times I,$$

such that restricting to the boundary $\coprod_{i=1}^{n} S^1 \times \{j\} \to B^3 \times \{j\}$ is an embedding of n-component unlink of circles in B^3 when $j = \{0, 1\}$ where the ith component of maps onto the ith component of the unlink for each embedding. We consider the set of three-component annular link maps up to link homotopy, which we denote by $\mathrm{ALM}^4_{2,2,2}$. Using $\widetilde{\sigma}^3$ we prove the following proposition.

Proposition 1.1.8. *The group of n-component annular link map up to link homotopy is non-abelian when $n \geq 3$.*

This result was proven in [MY21] by studying the isomorphisms of the free Milnor group, which is isomorphic to the group of embedded annular link maps up to link homotopy, whereas our proof uses the self-intersections of annular link maps to prove the result.

The benefit of studying annular link maps instead of link maps is that the set of annular link maps up to link homotopy is a group under a natural stacking operation. This is a similar move Habegger and Lin made in [HL90] which allowed them to classify n-component classical links up to link homotopy. Combining our approach of studying intersections with Meilhan and Yasuhara's approach of studying the induced isomorphism, we construct an invariant of three-component annular link maps up to link homotopy giving a group homomorphism

$$\Theta : \mathrm{ALM}^4_{2,2,2} \to (\mathbb{Z}\left[F/F_3\right])^3 \rtimes \mathrm{Aut}(MF(3)),$$

where the semi-direct product records the self-intersection information on each component and the automorphism of the free Milnor group on three-generators, $\mathrm{Aut}(MF(3))$, which is induced by the annular link map. This map is an extension of the work by Meilhan and Yasuhara in [MY21] to the non-embedded case. Note that since annular link maps have a canonical choice of meridians and basing there is less indeterminacy in this invariant compared to $\widetilde{\sigma}_3$.

We show that Θ is a group homomorphism and there exists a closure map $\mathrm{ALM}^4_{2,2,2} \to \mathrm{LM}^4_{2,2,2}$ and establish the following theorem.

Theorem 1.1.9. *The following diagram commutes*

$$\begin{tikzcd}
\mathrm{ALM}^4_{2,2,2} \arrow[r, two heads] \arrow[d, "\Theta"] & \mathrm{LM}^4_{2,2,2} \arrow[r, two heads, "i"] \arrow[d, "\widetilde{\sigma}^3"] & \mathrm{LM}^4_{2,2} \arrow[d, "\sigma"] \\
\left(\mathbb{Z}\left[F/F_3\right]\right)^3 \rtimes \mathrm{Aut}(MF(3)) \arrow[r] & \widetilde{K} \arrow[r, "p_i"] & \left(\mathbb{Z}\left[\mathbb{Z}\right]\right)^2 .
\end{tikzcd}$$

We show that the subgroup of annular link maps whose closure is a Brunnian link map – link maps where each two component sublink is trivial up to link homotopy – is a normal subgroup, containing the subgroup of annular link maps link homotopic to a embedded annular link map.

Using the work of Fenn and Rolfsen [FR86], Brendle and Hatcher [BH08], and Benjamin Audoux, Paolo Bellingeri, Jean-Baptiste Meilhan, and Emmanuel Wagner [ABMW14], we show how to construct a class of link maps and compute $\widetilde{\sigma}^3$ on this class in terms of the Kirk invariants of the two-component sublinks. From this we prove the following proposition.

Proposition 1.1.10. *Connect sum on the group of n-component link maps does not give a well defined group structure for $n \geq 3$.*

Using techniques of Schneiderman and Teichner, we construct more non-trivial link maps and compute $\widetilde{\sigma}^3$ of the resulting link maps in Section 7.1.3.

1.1.1 Organisation of chapters on link homotopy of surfaces

In Chapter 2, we define the equivariant intersection form, geometric intersections invariant, and establish their relationship to each other.

In Chapter 3, three types of homotopy are defined which allow us to understand link homotopy in terms of a more rigid structure. We also discuss the results which have been achieved in the two-component case.

In Chapter 4, Milnor groups are defined and we revise commutator calculus. We then show that the Milnor group of the free group on two generators, F/F_3, is isomorphic to the integral Heisenberg group of matrices and compute its group homology.

In Chapter 5, we calculate the homology of the complement of generically immersed two-spheres in the four-sphere and show that the second homology is generated by Clifford tori around each self-intersection. Using Dwyer's theorem we show that the Milnor group of the fundamental group of the complement of two generically immersed disjoint spheres is isomorphic to F/F_3.

In Chapter 6, we give a link homotopy invariant for based three-component link maps of oriented two-spheres. We then remove the based restriction to create $\widetilde{\sigma}^3$. We then develop tools for being able to differentiate between values of $\widetilde{\sigma}^3$ since our invariant takes values in a quotient space. We show there exist examples of link maps with the same image in S^4 but are distinct. Additionally, we develop an invariant which highlights how analogous our invariant is to the triple linking number. We then construct an invariant which measures the size of the values of our invariant.

In Chapter 7, we construct an invariant for annular link maps and show how one can use annular link maps to create link maps and establish the existence of an annular link map invariant which records induced automorphism information and the intersection information.

In Chapter 8, we discuss generalising $\widetilde{\sigma}^3$ to establish an n-component link map invariant.

1.2 Noncommutative Blanchfield pairings

In Chapter 9 we change topics and consider the computation of Blanchfield pairings. The *linking form* on a closed, connected, orientable three manifold M is a bilinear form

$$\mathrm{lk} : TH_1(M;\mathbb{Z}) \times TH_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z},$$

where $TH_1(M;\mathbb{Z})$ is the torsion part of the first homology. This form was first defined by Seifert in 1935 [Sei35] and since then has been a large object of study in low dimensional topology, particularly in the study of knot theory. It has been well documented that we can compute the linking form of three manifolds in terms of a four-manifold which it bounds [CFH16; GL78; Fri16].

Consider the cover of M associated to the kernel of a group epimorphism $v : \pi_1(M) \to \pi$, whose group of deck transformations is π. The chain complex of the cover is defined to be $C_*(M; \mathbb{Z}\pi)$, which comes equipped with a left action of $\mathbb{Z}\pi$. This has corresponding homology $H_*(M; \mathbb{Z}\pi)$. Let R be a ring with involution and $\rho : \pi \to U_n(R)$ where $U_n(R)$ is the group of unitary $n \times n$ matrices extending linearly to a representation $\rho : \mathbb{Z}\pi \to M_{n\times n}(R)$. We have the chain complex

$$C_*(M; R^n) := R^n \otimes_\rho C_*(M; \mathbb{Z}\pi)$$

with corresponding homology $H_*(M; R^n)$; we define such homology groups in generality in Chapter 2. On these homology groups we can define a generalisation of the linking form known as the Blanchfield form

$$\mathrm{Bl} : T^S \times T^S \to R_S/R,$$

where $S \subset R$ is a multiplicative subset which allows us localise on R (specifically the pair (R, S) satisfy the left Ore condition, see Chapter 9 for details) to create R_S and $T^S \subset H_1(M; R^n)$ is the S-*torsion* submodule i.e. $\{x \in H_1(M; R^n) \mid \exists s \in S \text{ such that } sx = 0\}$. The Blanchfield form is known to be sesquilinear and hermitian, which we prove in Corollary 9.3.8. Definition 9.2.7 and Proposition 9.2.10 give criterion for the Blanchfield pairing to be non-singular.

It is folklore that we should be able to calculate the Blanchfield form of a three-manifold in terms of the intersection form of a compact, connected, orientable four-manifold W, with $\partial W = M$. However, proofs of this result, both in the literature and unpublished make one or more of the following simplifying assumptions: $H_1(M; R^n)$ is torsion; $H_1(W; R^n) = 0$; $n = 1$; the ring R is commutative; and $H_2(W; R^n)$ is free. Some examples in the literature making these restrictions are the following:

- Anthony Conway, in his paper [Con18] , uses commutative coefficients – a localisation of the integral Laurent polynomials; the first homology of the four-manifold is trivial; the rank n is equal to one; and the second homology of the four-manifold is free.

- Maciej Borodzik and Stefan Friedl in [BF15] use commutative coefficients; they assume that the first homology of the four-manifold is torsion free; the rank of the representation is equal to one; and the second homology of the four-manifold is free.

- Cameron Gordon & Richard Litherland in [GL78], while working with the linking form and not the general Blanchfield form, assumed that the first homology of the four-manifold was trivial and that first homology of the three-manifold only has torsion and that the second homology of the four-manifold is free.

This list is by no means exhaustive. However, we aim to make the weakest/smallest amount of assumptions possible, as we cannot find a proof in full generality in the literature. Of the five simplifications in the previous paragraph we make one weakened assumption in that direction; we require that $H_1(W; R^n)$ is *S-torsion free* i.e for all non zero $x \in H_1(W; R^n)$ and for all $s \in S$, $sx \neq 0$. An assumption of this kind turns out to be necessary; consider the linking form on a lens space L with fundamental group $\mathbb{Z}/n^2$ which is the boundary of a rational homology four-ball B. From the long exact sequence of the pair we have

$$\begin{tikzcd}
H_2(B;\mathbb{Z}) \arrow[r, "q"] \arrow[d, "\cong"] & H_2(B,L;\mathbb{Z}) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_1(L;\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & H_1(B;\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z}/n \arrow[r] & \mathbb{Z}/n^2 \arrow[r] & \mathbb{Z}/n \arrow[r] & 0.
\end{tikzcd}$$

It is well known that the linking form on a lens space is not identically zero so we cannot compute the linking form in terms of the intersection pairing of B, as $H_2(B; \mathbb{Z})$ is zero.

If M bounds a connected, compact orientable four-manifold W such that the coefficient system on M extends over W then we have an exact sequence such that ∂ factors through the kernel of i which contains T^S

$$\begin{tikzcd} H_2(W;R^n) \arrow[r, "q"] & H_2(W,M;R^n) \arrow[r, "\partial"] \arrow[dr] & H_1(M;R^n) \arrow[r, "i"] & H_1(W;R^n). \\ & & \ker(i) \arrow[u] \arrow[ur, "0"] & \end{tikzcd}$$

Hence, we can identify T^S with a submodule of $H_2(W,M;R^n)/\operatorname{im}(q)$. We then prove the following:

Theorem 1.2.1. *Let R be a ring with involution and $S \subset R$ be a multiplicative subset which satisfies the left Ore condition and contains no zero divisors. Additionally, let M be a closed, connected, orientable, three-manifold and let W be a compact orientable four-manifold such that $\partial W = M$ and the following commutes*

$$\begin{tikzcd} & \pi & \\ \pi_1(M) \arrow[ur, two heads, "v"] \arrow[rr, "i_*"] & & \pi_1(W), \arrow[ul, two heads, "w"'] \end{tikzcd}$$

where i_* is the induced map on homotopy groups coming from the inclusion, and v and w are the group epimorphism, which correspond to the covers of M and W with group of deck transformation π. Then if $H_1(W;R^n)$ is S-torsion free then the Blanchfield form on M

$$\mathrm{Bl} : T^S \times T^S \to R_S/R$$

can be computed by

$$\mathrm{Bl}([x]\,,[y]) = -\frac{1}{r}\lambda(x_0,y_0)\frac{1}{\overline{s}}$$

where $r\,[x] = s\,[y] = 0$; $x_0, y_0 \in H_2(W;R^n)$ with $i(x_0) = rx$ and $i(y_0) = sy$ with $x, y \in H_2(W,M;R^n)$; and λ is the intersection form on W.

The proof method of this result is based on the work of Anthony Conway in [Con18] [1]. We show that his proof method works more generally and use this to prove

[1] In 2020, I informed Anthony Conway of an error in his published paper and he made some small changes to his paper on the Arxiv. The main difference was a small change to the definition of one of the pairings which resulted in small changes to some of the proofs.

Theorem 1.2.1.

Chapter 2

Algebraic and geometric intersections

2.1 Defining the intersection form

Let X be a CW complex and $\widetilde{X}$ be the regular cover of X, associated to the kernel of a surjective group homomorphism $\phi : \pi_1(X) \to \pi$. Let $Y \subset X$ and define $\widetilde{Y}$ to be the pre-image of Y under the covering map. Using deck transformations, $C_*(X,Y;\Lambda) \colon = C_*(\widetilde{X},\widetilde{Y})$ is given a left Λ-module structure, where $\Lambda := \mathbb{Z}\pi$; where Λ has a involution coming from extending the involution $\overline{g} := g^{-1}$ on π linearly.

Let N be a (R,Λ)-bimodule where R is a ring with involution. We define

$$C_*(X,Y;N) := N \otimes C_*\left(\widetilde{X},\widetilde{Y},\Lambda\right)$$

with corresponding homology denoted $H_*(X,Y;N)$. Both modules are left R-modules. We define the cochain complex by

$$C^*(X,Y;N) \colon = \mathrm{Hom}_{\mathrm{right}-\Lambda}\left(\overline{C_*(X,Y;\Lambda)},N\right),$$

where $\overline{C_*(X,Y;\Lambda)}$ is the involuted chain complex, with corresponding cohomology denoted $H^*(X, Y; N)$. This is a left R-module with the action of R defined by

$$(r \cdot f)(\sigma) \colon = rf(\sigma),$$

where $r \in R$, $f \in C^*(X, Y; N)$, and $\sigma \in \overline{C_*(X, Y; \Lambda)}$.

Set $N = R^n$, where the right action on R^n by Λ, comes from extending a representation $\rho : \pi \to U_n(R)$, where $U_n(R)$ is the group of unitary R matrices, linearly to get a ring representation $\Lambda \to M_{n\times n}(R)$. We also call this extension ρ. The right action of Λ on R^n is defined to be

$$v \cdot a := v\rho(a),$$

where $a \in \Lambda$ and $v \in R^n$ viewed as a row vector. Different choices of representation will result in different chain complexes and thus different homology and cohomology.

We now define a map $\langle -, - \rangle : R^n \times R^n \to R$ where

$$\langle v, w \rangle = v\overline{w}^T,$$

where $v, w \in R^n$.

The above map can easily be shown to be sesquilinear.

Definition 2.1.1. Define $\kappa : C^n(X, Y; R^n) \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(C_n(X, Y; R^n), R\right)}$ to be the map

$$f \mapsto \left((v \otimes \sigma) \mapsto v\overline{f(\sigma)}^T\right).$$

We denote the evaluation map by

$$p : H^n\left(\overline{\mathrm{Hom}_{\mathrm{left}-R}\left(C_*(X, Y; R^n), R\right)}\right) \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(H_n(X, Y; R^n), R\right)}$$

and define $\mathrm{ev} \colon := p \circ \kappa$. We often refer to ev as the evaluation map, however we will be specify which map if it is unclear.

Let W be a compact orientable connected four-manifold with ∂W which is equipped with Poincaré duality maps

$$PD : H^n(W; R^n) \to H_{4-n}(W, \partial W; R^n)$$

and

$$PD : H^n(W, \partial W; R^n) \to H_{4-n}(W; R^n).$$

Consider the following composition:

$$\Phi : H_2(W; R^n) \xrightarrow{q} H_2(W, \partial W; R^n) \xrightarrow{\mathrm{PD}^{-1}} H^2(W; R^n) \xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(H_2(W; R^n), R\right)}.$$

Definition 2.1.2. The *equivariant intersection form* on W is the map

$$\begin{aligned} H_2(W; R^n) \times H_2(W; R^n) &\to R \\ (a, b) &\mapsto \Phi(b)(a). \end{aligned}$$

From the definition, it is clear this form is sesquilinear and it is well known that the intersection form is hermitian [Ran02a].

2.2 Geometric Intersections

We define geometric intersections of 2-spheres in an oriented four-manifold M. This gives us a geometric method of computing the equivariant intersection form, when $R = \Lambda$, $N = \Lambda$, and $\rho = \mathrm{Id}$.

Fix an orientation of S^2 with a basepoint $x_0 \in S^2$. Let $(f, \gamma_f) : S^2 \to M$ be a based map of a sphere i.e a map paired with a path $\gamma_f : [0,1] \to M$ where $\gamma(0) = m_0$ and $\gamma(1) = f(x_0)$. We often suppress the γ_f in our notation and instead just write $f : S^2 \to M$ to be a based map, where it is implicit there exists a specific γ_f. Let $g : S^2 \to M$ be another based map. Assume f and g are generic transverse immersions and further assume the map $f \cup g$ is generic also. By compactness there exists a finite number of intersections between the images of f and g. Suppose x is an intersection of f and g where $p_1 \in f^{-1}(x)$ and $p_2 \in g^{-1}(x)$. Let $k_x : [0,1] \to S^2$ be a path such that $k_x(0) = x_0$ and $k_x(1) = p_1$. Similarly, define $l_x : [0,1] \to S^2$ with $l_x(0) = x_0$ and $l_x(1) = p_2$. We define

$$g_x = \gamma_f \cdot f(k_x) \cdot g\left(\bar{l}_x\right) \cdot \overline{\gamma}_g \in \pi_1(M)$$

Definition 2.2.1. We define the *geometric intersection* to be

$$\lambda_{\text{geo}}(f,g) = \sum_{x\in f(S^2)\cap g(S^2)} \varepsilon_x g_x \in \mathbb{Z}\pi,$$

where ε_x is the sign of the intersection at x.

The geometric intersection is independent of the choice of paths on each sphere as S^2 is simply connected. However, if we consider f with a different choice of basing path this may change the value of the geometric intersection. Hence, it is only well defined on based maps. The geometric intersection form agrees with the equivariant intersection form when $N = \Lambda$ and $\rho = \text{Id}$, since the basing paths and the maps f and g specify a homology classes in $H_2(M;\mathbb{Z}\pi)$ [Ran02b]. Hence, we will use λ to refer to both the geometric and the equivariant intersection forms.

Let $f^+\colon S^2 \to M$ be a normal push-off, using a section of the normal bundle of f which is transverse to the 0-section. To make sense of $\lambda_{geo}(f,f)$ we define:

$$\lambda_{\text{geo}}(f,f) := \lambda(f,f^+).$$

This is independent of the choice of normal push off.

We define a self-intersection number related to λ. Let I be the set of all double points of f and let $p \in I$. Let $p_1, p_2 \in f^{-1}(p)$ with $p_1 \neq p_2$ and let $\delta_1, \delta_2 : [0,1] \to S^2$ be paths from x_0 to p_1 and p_2 respectively. Set

$$g_p = \gamma_f \cdot f(\delta_1) \cdot f\left(\bar{\delta}_2\right) \cdot \bar{\gamma}_f \in \pi_1(M).$$

We could have swapped the roles of δ_1 and δ_2 and this would give us the loop g_p^{-1}. To resolve this we take values in the quotient $\mathbb{Z}\widetilde{\pi}\colon = \mathbb{Z}\pi/\{g \sim g^{-1}\}$. This is a group quotient where we view $\mathbb{Z}\pi$ as an abelian group.

Definition 2.2.2. The *self-intersection number* of a generic (based) immersion $f : S^2 \to M$ is defined to be

$$\mu(f)\colon = \sum_{p\in I} \varepsilon_p g_p \in \mathbb{Z}\widetilde{\pi},$$

where ε_p is the sign of the intersection at p.

We can relate the self-intersection number and geometric intersection number by the following.

Theorem 2.2.3. *Let $f : S^2 \to M$ be a generic immersion and let $\iota : \mathbb{Z} \to \mathbb{Z}\pi$ be the ring homomorphism where $n \mapsto n1$. Then the following equation holds*

$$\lambda\left(f, f\right) := \mu(f) + \overline{\mu(f)} + \iota\left(\chi(f)\right) \in \mathbb{Z}\pi, \tag{2.2.1}$$

where $\overline{\mu(f)}$ is given by taking the representatives we use to describe $\mu(f)$ and sending each group element to its inverse, and $\chi(f)$ is the Euler number of the normal bundle of f.

It is easily shown from this that μ is not a homotopy invariant. Cusp homotopies (defined in the next chapter) change the Euler number of the normal bundle by ± 2 but λ remains fixed as it is a homotopy invariant. Therefore must μ must change for λ to remain constant.

Chapter 3

Homotopy of surfaces and the Kirk invariant

3.1 The structure of link homotopy in dimension four

Definition 3.1.1. A map

$$f = f_1 \sqcup \ldots \sqcup f_n : \coprod_{i=1}^{n} S_i^2 \to S^4,$$

where $S_i^2 \cong S^2$, is called a *link map* if components disjoint in the domain are disjoint in the image, i.e $f_i(S_i^2) \cap f_j(S_j^2) = \phi$ for $i \neq j$. A *link homotopy* is a homotopy through link maps. We denote the set of n-component link maps up to link homotopy by $\mathrm{LM}^4_{\underbrace{2,\ldots,2}_{n}}$. If each of our spheres comes equipped with a basing path from the basepoint in S^4 to their surface we call it a *based link map*. The set of based link maps up to link homotopy is denoted $\mathrm{LM}^4_{\underbrace{2,\ldots,2}_{n}*}$.

We want to understand link maps up to link homotopy. Using results by Hirsch [Hir94] we can assume that each each link map is a generic, self-transverse immersion with only double points. We highlight three instances of homotopies between

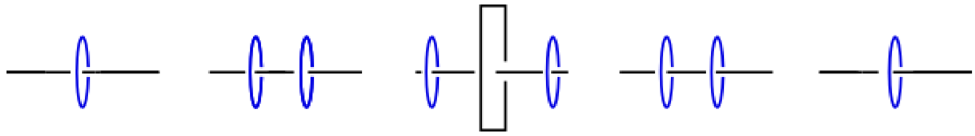

Figure 3.1: The local picture around the origin in $\mathbb{R}^4$ where the $x-y$ plane intersect the $z-t$ plane. The blue circles are the Clifford torus sitting around the origin.

generic immersions: finger moves, Whitney moves and cusp homotopies. This gives homotopy between link maps a rigid structure.

Let S be a generic immersed surface inside a four-manifold M, where S is compact, connected, and oriented. Let $p, q \in S$ be distinct points which are not intersections, and $\gamma : I \to M$ be an embedded arc which intersects p and q at the end points but does not intersect S at any other point. Take a small disc $D \subset S$ around p and push the disc along the path of γ past the end point q. This is a regular homotopy which creates two new intersections with opposing signs.

Definition 3.1.2. The homotopy described in the previous paragraph is called a *finger move*.

Finger moves can be used to simplify the complement of an immersed surface, at the expense of creating new intersections. For any double point of a generically immersed surface, there exists a homeomorphism of a neighbourhood of the double point to a ball B centred around the origin such that the double point is mapped to the origin, one sheet of the surface is mapped to $B \cap (\mathbb{R}^2 \times \{0\}) \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$, and the other sheet is mapped to $B \cap (\{0\} \times \mathbb{R}^2) \subset \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathbb{R}^4$.

Consider the torus $S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. This torus has one meridian travelling around the unit circle in the $z-t$ plane and the other meridian travelling around the unit circle in the $x-y$ plane. Such a torus is known as a *Clifford torus*. Hence, around each double point there exists a torus which has a meridian around one sheet and the other meridian around the other sheet. Consider the piece of the surface S, which for some choice of basepoint and path, the meridian is given by $\alpha \in \pi_1(M \setminus S)$. Doing a finger move on this piece along an element $\beta \in \pi_1(M \setminus S)$,

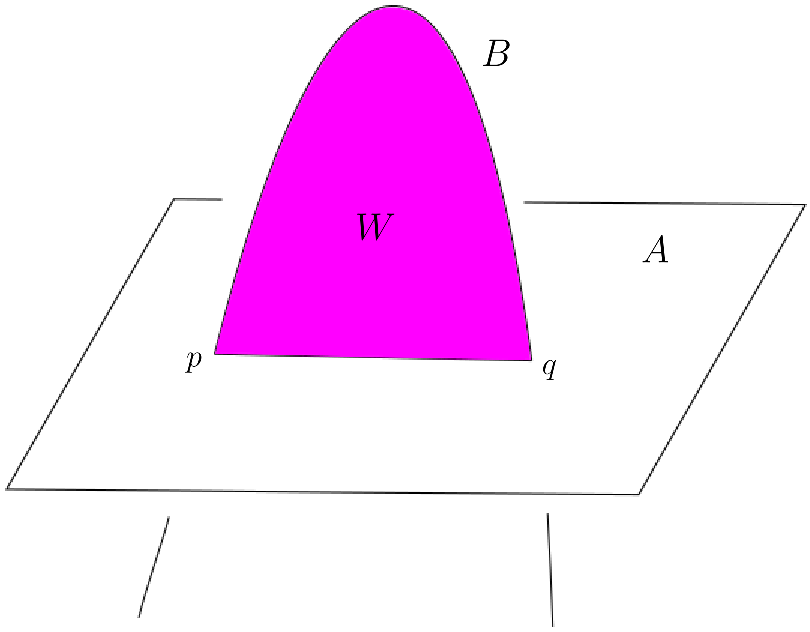

Figure 3.2: A local picture of a framed embedded Whitney disc W, pairing the intersections p and q, with sheet A lying in the present and sheet B having a arc lying in the present and the rest going into the past and future.

generates a new generically immersed surface $\widetilde{S}$. Since we have introduced Clifford tori into the complement of the surface, we have potentially introduced new relations into the fundamental group. A choice of basis curves on the tori are given by α and $\beta^{-1}\alpha\beta$ and since they lie on a torus both elements in the fundamental group of the complement of S must commute. Hence,

$$\pi_1(M \smallsetminus \widetilde{S}) = \pi_1(M \smallsetminus S) / \left\langle\left\langle \left[\alpha,\, \beta^{-1}\alpha\beta\right] \right\rangle\right\rangle .$$

A careful proof of this is given by [Cas86].

We now describe our second move. Let $p, q \in S$ be two oppositely signed double points of S. Assume there exist two embedded curves $\gamma, \gamma' : I \to S$, where γ is a path from p to q along one sheet (call this sheet A) and γ' travels from q to p along the other sheet (call this sheet B). Further, specify that the circle $\gamma \cup \gamma'$ bounds an embedded disc with interior in $M \smallsetminus S$. We call this disc W. Take a non-vanishing section of the normal bundle of ∂W, which is normal to A along γ and tangent to B along γ'. If this section can be extended to a non-vanishing section of the normal bundle of W, we say W is a framed Whitney disc. In the case where the Whitney

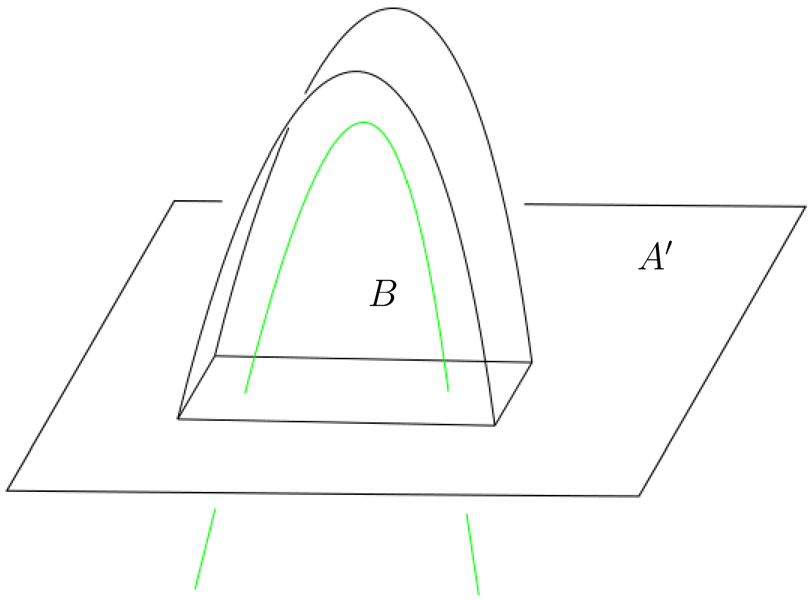

Figure 3.3: The result of doing a Whitney move on the Whitney disc W removing the intersections p and q, which changes the sheet A to A' and leaves sheet B unchanged.

disc is not framed we say W is a twisted Whitney disc. In the framed case we can push a neighbourhood of γ in A along the disc bundle of W. This is a regular homotopy. The result of this is to remove a neighbourhood of γ in A and glue in two parallel oppositely oriented copies of W and glue both them together with a strip. If W is embedded, this homotopy removes the intersections p and q with no new intersections on the resulting surface. If the Whitney disc is immersed, then the homotopy removes p and q but introduces four new intersections to the immersed surface S for each intersection in the interior of W.

Definition 3.1.3. We call the move described above a *Whitney move*.

In the description above there is nothing special about the roles of A and B. We could have chosen to extended a section normal to B and tangent to A on the boundary of the disc and pushed along that section.

Two intersections may have multiple Whitney discs pairing them up to isotopy, so removing intersections via a Whitney move requires a choice of Whitney disc.

Loosely speaking, finger moves and Whitney moves are inverse operations. If one performs a finger move then we create a pair of oppositely oriented intersection

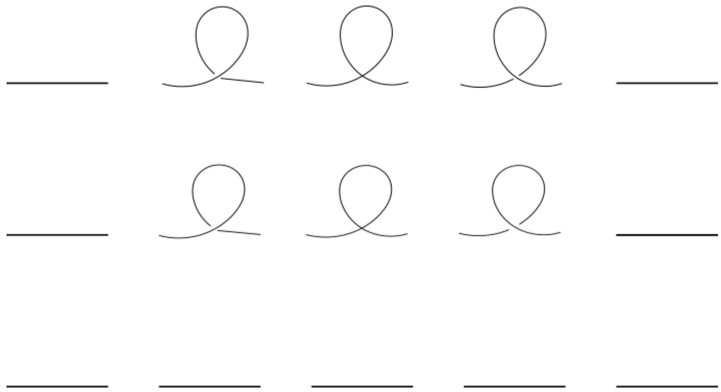

Figure 3.4: A description of a cusp homotopy, where the homotopy time goes from the bottom of the picture to the top.

which we can pair with a Whitney disc to undo the finger move.

The last move we consider is a cusp homotopy. Consider taking a map $I \to D^2$ with a single intersection, like the picture in the centre and top of Figure 3.4, and doing the homotopy like the one described which pulls out the "kink", which looks like going down the central column of Figure 3.4. Crossing with $D^1 \subset D^2$, this gives a homotopy $D^2 \times I \to D^4$, which goes from the standard embedding of $D^2 \subset D^4$ to a map $D^2 \to D^4$ with a line of self intersections. We then do a small homotopy sending part of the line into the future and the other into the past leaving only a single intersection. The bottom row of Figure 3.4 is the standard embedding and the top row is the result of a cusp homotopy

Definition 3.1.4. We call a homotopy of S like the one above, supported in a four-ball neighbourhood of a non-double point, a *cusp homotopy.*

Unlike finger moves and Whitney moves, the cusp homotopy is distinct since it is not a regular homotopy but a homotopy between immersions. The importance of these three moves in combination with one another is captured by the following proposition.

Proposition 3.1.5 ([FQ90])**.** *A generic (link-)homotopy between two generic immersions is (link-)homotopic to concatenations of isotopies, finger moves, Whitney moves, and cusp homotopies.*

Hence, we can think of a link homotopy between two link maps as a combination of isotopy, Whitney moves, finger moves and cusp homotopies.

3.2 Two-component link maps and the Kirk invariant

For this section all link maps will have two components unless stated otherwise. In 1988, Koschorke showed that, up to link homotopy, connect sum is well defined on two-component link maps [Kos88]. This turns $\mathrm{LM}^4_{2,2}$ an abelian monoid. In 1999 Bartels and Teichner showed that each element in $\mathrm{LM}^4_{2,2}$ has an inverse, establishing that $\mathrm{LM}^4_{2,2}$ is a group [BT99]. Two-component link maps have an invariant defined by Paul Kirk in 1988, which we now call the Kirk invariant. In 2018, Schneiderman and Teichner showed that the Kirk invariant is a complete invariant of $\mathrm{LM}^4_{2,2}$ [ST17]. We define this invariant since this is the inspiration for our three-component invariant.

Let $f = f_1 \sqcup f_2 : S^2_1 \coprod S^2_2 \to S^4$ be a link map and define

$$M_i = S^4 \smallsetminus \nu(f_j),$$

where $i \neq j$ and $\nu(f_j)$ is the regular neighbourhood of f_j.

We compute intersections $\lambda(f_1, f_1) \in \mathbb{Z}\pi_1(M_1)$. This is not a link homotopy invariant since the fundamental group of the complement of f_2 changes over a generic link homotopy. To fix this we use homology; since the homology of the complement of a generic sphere in S^4 is constant throughout a generic link homotopy. Define $\lambda(f_1, f_1) \in \mathbb{Z}H_1(M_1)$ to be $\sigma_1(f)$. Swapping the roles of f_1 and f_2 we get another invariant of f called $\sigma_2(f)$.

Definition 3.2.1. The *Kirk invariant* of a link map f is defined to be

$$\sigma(f) = (\sigma_1(f), \sigma_2(f)) \in \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}].$$

We can realise the image of the Kirk invariant [ST17] using a technique Fenn and

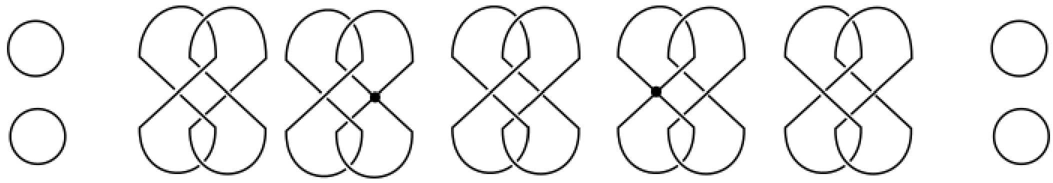

Figure 3.5: The JK-construction applied to the Whitehead link with time moving from left to right with a single intersection on each sphere.

Rolfsen created in their paper to construct the first example of a link homotopically non-trivial link map in dimension two [FR86].

Construction 3.2.2 (The Jin-Kirk construction)**.** Consider a two-component link, $L = L^1 \coprod L^2 \subset S^3 \times \left\{\frac{1}{2}\right\} \subset S^3 \times I$, where $I = [0,1]$. Suppose the linking number of L is zero; both components null-homotopic in the complement of the other and both components unknotted. Consider the trace of a homotopy inside $S^3 \times \left[\frac{1}{2}, 1\right]$ which fixes the second component and does a null homotopy of the first. Then take the trace of a homotopy running backwards from $S^3 \times \{\frac{1}{2}\}$ to $S^3 \times \{0\}$ with the first component fixed and a null-homotopy applied to the second component. Capping off both ends of the cylinder with a D^4 and both ends traces of the link with a D^2 gives a link map in S^4.

The construction of the link map above is, up to link homotopy, independent of the choice of null homotopy on each component as $\pi_2\left(S^3 \smallsetminus L^i\right) = 0$, where L^i is a component of the link we started the construction with. This gives a surjective map [ST17]

$$JK : \mathfrak{L} \twoheadrightarrow \mathrm{LM}^4_{2,2}\,,$$

where $\mathfrak{L}$ is the set of two-component links with each component null-homotopic in the complement of the other and both components unknotted.

Example 3.2.3. Let FR be the link map created from the JK construction described in Figure 3.5. Then

$$\sigma(FR) = \left(t + t^{-1} - 2, 2 - t - t^{-1}\right).$$

Schniederman and Teichner proved the following result which classified two-component link maps.

Theorem 3.2.4. *Let* $z = 2 - t - t^{-1}$ *and identify* $\mathbb{Z} = z \cdot \mathbb{Z}[z] / z^2 \cdot \mathbb{Z}[z]$. *Then the sequence*

$$0 \longrightarrow \mathrm{LM}^4_{2,2} \xrightarrow{\sigma} z \cdot \mathbb{Z}[z] \oplus z \cdot \mathbb{Z}[z] \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the right most map is addition, is exact.

Schniederman and Teichner show that $\mathrm{LM}^4_{2,2}$ has an interesting module structure.

Theorem 3.2.5. *Let* $R := \mathbb{Z}[z_1, z_2] / (z_1 z_2)$. *Then* $\mathrm{LM}^4_{2,2}$ *is a free* R*-module of rank* 1 *and as a* $\mathbb{Z}$*-module is free of infinite rank.*

Chapter 4

The Heisenberg group

In this section we develop the commutator calculus and group theory necessary for our three-component invariant. We discuss Milnor groups which will be necessary for generalising the Kirk invariant since the n-component invariant Kirk developed is based in homology which loses too much geometric information due to homology being abelian.

4.1 Milnor groups and commutator calculus

Definition 4.1.1. Let G be a group normally generated by elements $x_1, \ldots, x_n \in G$. Then we call the group

$$MG := G/\left\langle\left\langle\left[x_i, gx_ig^{-1}\right]\right\rangle\right\rangle \ 1 \leq i \leq n, \ \forall g \in G$$

the *Milnor group of* G. We call G a *Milnor group* if $G = MG$.

Remark 4.1.2. There is still a question about if the choice of normal generators effects the resulting quotient group. However, for any choice of normal generators we will choose $x_1, \ldots, x_n$ for a group G, any other set of normal generators we choose $x'_1, \ldots, x'_n$ will have the property $x'_i = gx_ig^{-1}$ for some $g \in G$. This will guarantee that this quotient is fixed for our purposes[1].

Milnor used such groups in the study of classical links up to link homotopy in [Mil54], where he was able to classify three-component links up to link homotopy. We will see that for three-components, one dimension up, we only care about about one particular Milnor group. Before studying this Milnor group we will first discuss commutator calculus and lower central series quotients.

Definition 4.1.3. Let G be a group and A and B be normal subgroups. Define the subgroup $[A, B]$ to be the normal subgroup generated by elements of the form $aba^{-1}b^{-1}$, where $a \in A$ and $b \in B$.

Swapping the roles of A and B in the above definition does not change the resulting subgroup.

Lemma 4.1.4. *The normal subgroups $[A, B]$ and $[B, A]$ are equal to each other.*

Proof. Consider the commutator

$$[a, b] = aba^{-1}b^{-1}$$

then we have

$$[b, a] = [a, b]^{-1} .$$

Thus, all the normal generators of $[A, B]$ have inverses which are normal generators of $[B, A]$ and so both subgroups are equal. $\square$

If we have a group G we can iteratively take commutators with G and this leads us to the following definition.

Definition 4.1.5. Let G be a group. Then the lower central series is the series

$$G = G_1 \trianglerighteq G_2 \trianglerighteq G_3 \trianglerighteq \cdots \trianglerighteq G_n \trianglerighteq \cdots ,$$

[1]The author tried to find a reference to show that the Milnor quotient independent of our choice of normal generators and was unable to find one. However, he did find that Freedman and Teichner in [FT95], define Milnor groups relative to a choice of normal generators, this suggest they are either possibly different or it is not known if all such quotients are equal.

where $G_{k+1} = [G, G_k]$, the normal subgroup generated by elements of the form $[a, b] := aba^{-1}b^{-1}$, where $a \in G$ and $b \in G_k$. We call G *nilpotent* if for some $s \in \mathbb{N}$, G_s is trivial. We call G *nilpotent of class* n if $n \in \mathbb{N}$ is the smallest number such that $G_{n+1} = \{1\}$

Before we discuss commutator calculus, let us prove the following.

Lemma 4.1.6. *If G is a group normally generated by k elements, then MG is nilpotent of class at most k.*

Proof. The proof is by induction and comes from [BT99]. In the $k = 1$ case the Milnor group of G is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/m$ for some $m \in \mathbb{N}$. Hence it is nilpotent of class 1. Assume the result is true for $k = n - 1$. Let G have normal generators x_i with $1 \leq i \leq n$ and let A_i be the normal closure of the cyclic subgroup created by x_i in MG. Let $x \in MG$ and $y \in (MG)_n$. For all i the commutator $y \in MG/A_i$ is trivial by the inductive hypothesis. We have that $y \in \cap_{i=1}^{n} A_i$. Hence, y is contained in the center of MG, thus $MG_{n+1} = \{1\}$. □

Proposition 4.1.7. *Let G be a group and $a, b, c \in G$. Then the following equalities hold:*

$$[ab, c] = [a, [b, c]]\,[b, c]\,[a, c]\,, \tag{4.1.1}$$

$$[a, bc] = [a, b]\,[b, [a, c]]\,[a, c] \tag{4.1.2}$$

$$\left[a^b, [c, b]\right][b^c, [a, c]]\,[c^a, [b, a]] = 1 \tag{4.1.3}$$

$$[a, [c, b]]\,[b, [a, c]]\,[c, [b, a]] = [c, b]^a\,[b, c]\,[a, c]^b\,[c, a]\,[b, a]^c\,[a, b] \tag{4.1.4}$$

Proof. The proofs consist of expanding and simplifying. □

Corollary 4.1.8. *Let G be a group and let $a \in G_k$, $b \in G_l$ and $c \in G_m$ then we have the following results:*

$$ab \equiv ba \mod G_{k+l}, \tag{4.1.5}$$

$$[ab, c] \equiv [a, c]\,[b, c] \mod G_{k+l+m}, \tag{4.1.6}$$

$$[a, bc] \equiv [a, b]\,[a, c] \mod G_{k+l+m}, \tag{4.1.7}$$

Proof. This proof was given in [MKS04]; we will recreate it here. Let us first prove Congruence (4.1.5). We know

$$ab = [a, b]\, ba,$$

so we need to prove that

$$[G_k, G_l] \subseteq G_{k+l}. \tag{4.1.8}$$

We prove this by induction. Take the case where $k = 1$, clearly we have $[G, G_l] = G_{l+1}$, by the definition of the lower central series. Assume that the result holds up to some $n \in \mathbb{N}$. Then we have

$$[G_{n+1}, G_l] = [[G, G_n], G_l] \subseteq [[G_n, G_l], G]\, [[G_l, G], G_n],$$

as a result of (4.1.3). By the inductive hypothesis we have $[[G_n, G_l], G] \subseteq G_{n+l+1}$. Furthermore, $[G_{l+1}, G_n] \subseteq G_{n+l+1}$. Hence, we have proven that for all $k, l \in \mathbb{N}$ we have $[G_k, G_l] \subseteq G_{k+l}$ and thus we have shown (4.1.5) holds. We now prove Congruence (4.1.6). From Equation (4.1.1),

$$[ab, c] = [a, [b, c]]\, [a, c]\, [b, c],$$

and by (4.1.8) we have $[G_k, [G_l, G_m]] \subseteq G_{k+l+m}$. Congruence (4.1.7) is proved in a similar fashion. □

Let $F(n)$ be the free group on n generators. Consider the free group on two generators $F := F(2)$, and its lower central series quotient F/F_3.

Lemma 4.1.9. *The group F/F_3 admits the presentation*

$$\langle x, y \mid [x, [x, y]], [y, [x, y]] \rangle.$$

Proof. Let N be the normal subgroup generated by $[x, [x, y]]$ and $[y, [x, y]]$. It is clear that $N \subseteq F_3$. We wish to show the relations defining N imply that $F_3 \subseteq N$. First, let us prove that for any $\omega \in F$ we have

$$[\omega, [x, y]] \equiv 1 \bmod N. \tag{4.1.9}$$

We do this by induction of the length of the word ω. It is clear in the cases where ω is equal to x or y. Checking their inverses we have

$$\left(x(\left[x^{-1},[x,y]\right]x^{-1}\right)^{-1}=[x,[x,y]]\,.$$

Hence

$$\left[x^{-1},[x,y]\right]\equiv 1 \text{ mod } N.$$

The case for y^{-1} is similar. Now assume that the result holds true for words of length $n-1$. Let ω be a word of length n. Now $\omega = g\omega'$ where g is equal to x, y x^{-1} or y^{-1} and ω' is a word of length $n-1$. Assume that $g=x$; the proofs of the other cases are similar. By Equation (4.1.1) and the inductive hypothesis we have

$$\begin{aligned}[x\omega',[x,y]] &= [x,[\omega',[x,y]]]\,[\omega',[x,y]]\,[x,[x,y]]\\ &\equiv 1 \text{ mod } N\end{aligned}$$

Every element of F_2 is a product of $[x,y]$, its inverse, and their conjugates. So any element can be expressed as a product of these commutators and as such they can be expressed as some minimum length as a product of commutators. We will prove that $[\omega,K]\in N$, where K is some product of commutators, by induction. The base case is done by checking that all commutators of the form $h\,[x,y]\,\bar{h}$, for some $h\in F$. Applying Congruence (4.1.9), we have

$$\left[\omega,h\,[x,y]\,h^{-1}\right]=h\left[h^{-1}\omega h,[x,y]\right]h^{-1}\equiv 1 \text{ mod } N.$$

Similarly,

$$\left[\omega,h\,[y,x]\,h^{-1}\right]=h\left[h^{-1}\omega h,[y,x]\right]h^{-1}\equiv 1 \text{ mod } N.$$

Assume this holds true for any element in F_2 that can be written as product of $n-1$ $[x,y]$, its inverse, and their conjugates. Let K be an element of F_2 which can be written n elements of such commutators. Then we have $K=LK'$ where L is equal to $h\,[x,y]\,h^{-1}$ or $h\,[y,x]\,h^{-1}$. Again assume $L=h\,[x,y]\,h^{-1}$; the proof of the other case is similar. Using Equation (4.1.2) and the inductive hypothesis we have

$$[\omega, LK] = [\omega, L]\,[L, [\omega, K']]\,[\omega, K'] \equiv 1 \bmod N.$$

Hence, $F_3 \subseteq N$ as required. □

Lemma 4.1.10. *The groups MF and F/F_3 are isomorphic.*

Proof. Let N be the normal subgroup we quotient F with to get MF. We have the following equivalence

$$\left(([x,[x,y]])^{[y,x]}\right)^{-1} = \left[x, yxy^{-1}\right].$$

Thus, adding the $[x,[x,y]] = 1$ relation is equivalent to adding the relation $[x, yxy^{-1}] = 1$. Similarly, adding the relation $[y,[y,x]] = 1$ is equivalent to $[y, xyx^{-1}] = 1$. Thus, we have $F_3 \subseteq N$. We need to show $[x, \omega x \omega^{-1}] = 1$ and $[y, \omega y \omega^{-1}] = 1$ hold for any reduced word ω when the relations for F/F_3 are added. We will do this by induction on the word length of ω where the base case is the relations given by F/F_3. Let k be one of the generators x, y or their inverses. Let $\omega = k\omega'$, where ω and ω' are freely reduced words of length k and $k-1$ respectively. Using Congruence (4.1.7) twice we have

$$\left[x, \omega x \omega^{-1}\right] = [x,k]\left[k, \left[x, \omega' x \omega'^{-1}\right]\right]\left[x, k^{-1}\right] \bmod F_3$$

By the inductive hypothesis the central term vanishes and we have

$$\begin{aligned}\left[x, \omega x \omega^{-1}\right] &\equiv [x,k]\left[x, k^{-1}\right] \bmod F_3\\ &\equiv [x,k]\,[k,x] \bmod F_3\\ &\equiv 1 \bmod F_3\end{aligned}$$

Since $F_3 \subseteq N$, these relations still hold mod N and thus $N = F_3$. □

4.2 The Heisenberg group and its homology

We show that F/F_3 is isomorphic to a group of integral matrices and use this to to compute the homology of the group. First, we define what we mean by the homology of a group.

Definition 4.2.1. Let G be a group. Then the group homology and cohomology of G are defined to be

$$H_*(G;\mathbb{Z}) := H_*(\mathbf{B}G;\mathbb{Z}), \qquad H^*(G;\mathbb{Z}) = H^*(\mathbf{B}G;\mathbb{Z}),$$

where $\mathbf{B}G$ is the classifying space.

We will work exclusively with discrete finitely presented groups. Their classifying spaces are Eilenberg-Maclane spaces, $K(G,1)$ i.e. connected CW complexes with fundamental group equal to G and all higher homotopy groups vanishing.

Definition 4.2.2. We call the group of matrices

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a,b,c \in \mathbb{Z} \right\} \subseteq \mathrm{ALM}(3,\mathbb{Z}),$$

the (integral) Heisenberg group.

We will show that both F/F_3 and H are isomorphic by making use of a normal form for a presentation of F/F_3.

Definition 4.2.3. Let $F(n)$ be the free group on n generators and G be a group such that there exist a surjective homomorphism $\pi : F(n) \to G$. Then a *normal form* is an injective map $s : G \to F(n)$ such that $\pi \circ s = I_G$.

We will first show the following result.

Lemma 4.2.4. *Let*

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \; Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; Z = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H.$$

Then

$$\begin{pmatrix} 1 & a & -c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = X^a Z^c Y^b.$$

Proof. This can be checked directly by matrix multiplication. □

Proposition 4.2.5. *The Heisenberg group and F/F_3 are isomorphic.*

Proof. Consider F/F_3 with the presentation

$$\langle x, y, z | \left[x, z\right], \left[y, z\right], \left[x, y\right] = z \rangle .$$

Let ω be a fully reduced word in x, y and z. Using the relations we can collect all of the z or z^{-1} to the back resulting in

$$\omega = \omega' z^t,$$

where ω' is some word in x and y. We can move each y and y^{-1} to the back by passing using the relations, introducing more z and z^{-1}. Eventually, we have the word

$$x^u z^w y^u = \omega \in F/F_3$$

for some $u, v, w \in \mathbb{Z}$. Define a homomorphism $\phi : F/F_3 \to H$ by specifying that $\phi(x) = X$, $\phi(y) = Y$ and $\phi(z) = Z$. This is well defined since

$$\phi\left(\left[x, z\right]\right) = \phi\left(\left[y, z\right]\right) = \phi\left(\left[x, y\right] z^{-1}\right) = 1$$

and is clearly surjective so all that remains to be proven is injectivity. Suppose the map failed to be injective then we would be able to find a,b, c, a', b' $c' \in \mathbb{Z}$ such that

$x^a z^c y^b \neq x^{a'} z^{c'} y^{b'}$ and

$$\phi\left(x^a z^c y^b\right) = \phi\left(x^{a'} z^{c'} y^{b'}\right),$$

and by Lemma 4.2.4 we have

$$\begin{pmatrix} 1 & a & -c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a' & -c' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence we have $a = a'$, $b = b'$ and $c = c'$. This contradicts $x^a z^c y^b \neq x^{a'} z^{c'} y^{b'}$ and completes the proof. $\square$

Consider the *continuous Heisenberg group*

$$H_{\mathbb{R}} = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}.$$

As a space, this is clearly homeomorphic to $\mathbb{R}^3$ and we can define a cocompact group action of H on $H_{\mathbb{R}}$ by matrix multiplication. We call the quotient space of this action P. Since we are multiplying by matrices with positive determinant, multiplication by elements of H is an orientation preserving diffeomorphism. Hence, the quotient space is a connected, closed, orientable three-manifold. Using the long exact sequence of the fibration, for

$$H \hookrightarrow \mathbb{R}^3 \to P,$$

we know that all homotopy groups vanish except the fundamental group which is isomorphic to the Heisenberg group. Hence, the quotient of the space is a model for $\mathbf{B}F/F_3$.

Proposition 4.2.6. *The homology of the Heisenberg group are given by*

$$H_n(F/F_3) = \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1, 2. \end{cases}$$

Proof. The $n = 0$ and $n = 3$ cases are obvious since $\mathbf{B}F/F_3$ is a connected, closed, orientable three-manifold. We have $H_1(F/F_3) = \mathbb{Z} \oplus \mathbb{Z}$ by the Hurewicz theorem. Consider the following sequence of isomorphisms

$$H_2\left(\mathbf{B}F/F_3;\mathbb{Z}\right) \longrightarrow H^1\left(\mathbf{B}F/F_3;\mathbb{Z}\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(H_1\left(\mathbf{B}F/F_3;\mathbb{Z}\right),\mathbb{Z}\right),$$

where the first arrow is given by Poincaré duality and the last arrow is the evaluation homomorphism. This implies $H_2\left(BF/F_3;\mathbb{Z}\right) \cong \mathbb{Z} \oplus \mathbb{Z}$. $\square$

Chapter 5

Algebraic topology of the complement of two-spheres

We want to use the intersection form on a component of a three-component link map in the complement of the other two spheres to get an element of $\mathbb{Z}\pi$ and then take the Milnor quotient and a choice of isomorphism to F/F_3 to get an element in $\mathbb{Z}[F/F_3]$. Doing this for each sphere gives an element of $(\mathbb{Z}[F/F_3])^3$. This section will establish that the Milnor group of the fundamental group of the complement of two generically immersed two-spheres in S^4 is isomorphic to F/F_3.

Lemma 5.0.1. *Let S be the image of generically immersed two-spheres given by a n-component link map, with $2d \in \mathbb{N}$ double points. Assume that the double points on* each *sphere algebraically cancel. Then*

$$H_i(S^4 \smallsetminus \nu(S)) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}^n & i = 1 \\ \mathbb{Z}^{2d} & i = 2 \\ \mathbb{Z}^{n-1} & i = 3 \textit{ and } n > 1 \\ 0 & \textit{otherwise.} \end{cases}$$

The second homology is generated by Clifford tori around each intersection.

Proof. First assume $n = 1$. Our goal is to use Mayer-Vietoris to compute the homology in such a way that the generators of the homology become clear. The tubular neighbourhood of the image of this immersion can be described as the result of a plumbing [FQ90]. The boundary of $S^4 \setminus \nu(S)$ is a plumbed three-manifold i.e the boundary of a plumbed four-manifold. Call this manifold Y. To compute the homology of Y we will construct from spaces we know and use Mayer-Vietoris. Consider

$$X := \left(S^2 \setminus \coprod_{i=1}^{4d} \mathring{D}^2 \right) \times S^1$$

with

$$\partial X = \coprod_{i=1}^{4d} S^1 \times S^1.$$

There are two discs removed from S^2 for each double point. We think of X as being a cobordism between $4d-1$ tori, call this part of the boundary A, and a single torus, call this part of the boundary B. We have $H_2(X) \cong \mathbb{Z}^{4d-1}$ and it is generated by the $2d-1$ tori in A. Let us call this basis $\{\beta_1, \dots, \beta_{4d-1}\}$. This basis satisfies the condition that

$$-\sum_{i=1}^{4d-1} \beta_i \in H_2(X)$$

is the element coming from inclusion of the boundary torus in B for some choice of basis of $H_2(B) \cong \mathbb{Z}$.

We have $H_1(X) \cong \mathbb{Z}^{4d}$, where the basis is given by a single meridian on each torus in A together with the single shared meridian they all have. Let $x_1, \dots, x_{4d-1}$ and y be these generators.

We pair up boundary components by gluing $\coprod_{i=1}^{d} S^1 \times S^1 \times I$, where one end is glued using the identity and the other is glued using either of the maps $f, g : S^1 \times S^1 \to S^1 \times S^1$ with

$$f(x, y) = (y, x) \text{ and } g(x, y) = (\bar{y}, \bar{x}).$$

Using Mayer-Vietoris gives:

$$0 \longrightarrow H_3(Y) \longrightarrow H_2\left(\coprod^{4d} S^1 \times S^1\right) \xrightarrow{\phi} H_2(X) \oplus H_2\left(\coprod^{2d} S^1 \times S^1 \times I\right) \longrightarrow H_2(Y) \longrightarrow \cdots$$

We have a basis $\{\alpha_1, \ldots, \alpha_{4d}\}$ of $H_2(\coprod_{i=1}^{4d} S^1 \times S^1)$, where α_i is represented by the ith torus. Let $\{\gamma_1, \ldots, \gamma_{2d}\}$ be a basis for $H_2\left(\coprod_{i=1}^{2d} S^1 \times S^1 \times I\right)$ where γ_i is a generator of $H_2\left(S^1 \times S^1 \times I\right)$ for the ith copy. These bases are chosen such that the map ϕ is given by

$$\phi(\alpha_i) = \begin{cases} \beta_i + (-1)^{1+i}\gamma_{\lceil \frac{i}{2} \rceil} & \text{if } 1 \leq i \leq 4d-1 \\ -\sum_{i=1}^{4d-1} \beta_i - \gamma_{2d} & \text{if } i = 4d, \end{cases}$$

and extended linearly. The kernel of ϕ is given by integer multiples of $\sum_{i=1}^{4d} \alpha_i$. Hence, $H_3(Y) \cong \mathbb{Z}$. From another part of the long exact sequence we have

$$\ldots \longrightarrow H_2\left(Y\right) \longrightarrow H_1\left(\coprod_{i=1}^{4d} S^1 \times S^1\right) \xrightarrow{\psi} H_1\left(X\right) \oplus H_1\left(\coprod_{i=1}^{2d} S^1 \times S^1 \times I\right) \longrightarrow \cdots$$

From this we can construct the short exact sequence

$$0 \longrightarrow \operatorname{coker}(\phi) \longrightarrow H_2(Y) \longrightarrow \ker(\psi) \longrightarrow 0.$$

since $\ker(\psi)$ is a free abelian group, this short exact sequence splits and

$$H_2(Y) \cong \ker(\psi) \oplus \operatorname{coker}(\phi).$$

As $\operatorname{coker}(\phi)$ is isomorphic to $\mathbb{Z}^{2d}$. We now describe the map ψ in order to compute $H_2\left(Y\right)$. We choose a basis of $H_1(\coprod_{i=1}^{4d} S^1 \times S^1)$, $\{v_1, w_1, v_2, w_2, \ldots, v_{4d}, w_{4d}\}$. Additionally, choose a basis $\{a_1, b_1, \ldots, a_{2d}, b_{2d}\}$ for $H_1(\coprod_{i=1}^{2d} S^1 \times S^1 \times I)$. These bases are chosen such that

$$\psi(v_i) = \begin{cases} x_i - a_{\lceil \frac{i}{2} \rceil} & \text{if } i \equiv 1 \mod 2 \\ x_i - b_{\lceil \frac{i}{2} \rceil} & \text{if } i \equiv 0 \mod 2 \text{ and } i \leq 2d \\ x_i + b_{\lceil \frac{i}{2} \rceil} & \text{if } i \equiv 0 \mod 2 \text{ and } 2d < i < 4d \\ -\sum_{i=1}^{4d-1} x_i + b_d & \text{if } i = 4d \end{cases}$$

and

$$\psi(w_i) = \begin{cases} y - b_{\lceil \frac{i}{2} \rceil} \text{ if } i \text{ is odd} \\ y - a_{\lceil \frac{i}{2} \rceil} \text{ if } i \equiv 0 \mod 2 \text{ and } i \leq 2d \\ y + a_{\lceil \frac{i}{2} \rceil} \text{ if } i \equiv 0 \mod 2 \text{ and } 2d < i \leq 4d. \end{cases} .$$

The kernel of the map ψ is given by multiples of

$$\sum_{i=1}^{4d} v_i - \sum_{i=1}^{2d} w_i + \sum_{i=2d+1}^{4d} w_i.$$

Hence, $H_2(Y) \cong \mathbb{Z}^{2d} \oplus \mathbb{Z}$.

Using the Mayer-Vietoris sequence we have the following isomorphism

$$0 \longrightarrow H_2(Y) \xrightarrow{\cong} H_2(S^4 \smallsetminus \nu(S)) \oplus H_2(\nu(S)) \longrightarrow 0. \tag{5.0.1}$$

The basis of coker(ϕ) is given by $2d$ Clifford tori represented by each $S^1 \times S^1 \times \left\{\frac{1}{2}\right\} \subset S^1 \times S^1 \times I$. Under the non-trivial map in the exact sequence (5.0.1) we see that each of these Clifford tori bound in $\nu(S)$ and thus each generator maps into $H_2(S^4 \smallsetminus \nu(S))$. The only other basis element in $H_2(Y)$ is given by the basis element of $\ker(\psi)$ however this must then map non-trivially to $H_2(\nu(S)) \cong H_2(S) \cong \mathbb{Z}$ and trivially into $H_2(S^4 \smallsetminus \nu(S))$. Hence the Clifford tori-generate the second homology of the complement of a generically immersed sphere in S^4.

In the case, where $n > 1$ we have $H_3(Y) \cong \mathbb{Z}^n$ and from the long exact sequence we have

$$0 \longrightarrow H_4(S^4) \longrightarrow H_3(Y) \longrightarrow H_3(S^4 \smallsetminus \nu(S)) \oplus H_3(\nu(S)) \longrightarrow 0.$$

since $H_3(\nu(S))$ vanishes, $H_3(S^4 \smallsetminus \nu(S)) \cong \mathbb{Z}^{n-1}$. To compute $H_2(S^4 \smallsetminus \nu(S))$ we can do a similar calculation to the one-component case and show that the second homology is $\mathbb{Z}^d$. It is clear that for any $n \in \mathbb{N}$ we have $H_1(S^4 \smallsetminus S) = \mathbb{Z}^n$. □

Definition 5.0.2. We define a *grope* of class k (a k-grope) inductively. For $k = 1$ a grope is circle. A grope of class $k = 2$ is a compact oriented surface Σ of genus g, with one boundary component. A k-grope is a 2-complex is defined as follows. Let $\{\alpha_i, \beta_i, i = 1, \ldots, g\}$ be the standard symplectic basis for $H_1(\Sigma)$. For positive integers p_i, q_i with $p_i + q_i \geq k$ and $p_{i_0} + q_{i_0} = k$ for at least one index i_0, a k grope is given by gluing a p_i grope to each α_i and a q_i grope to each β_i. A *closed grope* is where a 2-cell has been glued in along the boundary of Σ.

Definition 5.0.3. Let X be a CW complex. Then $\phi_k(X) \subseteq H_2(X)$ is the subgroup of elements of $H_2(X)$ which can be represented by a map of a closed class k-grope into X. Alternatively, this subgroup can be defined as the kernel of the composition

$$H_2(X) \longrightarrow H_2\left(\pi_1\left(X\right)\right) \longrightarrow H_2\left(\pi_1\left(X\right)/\pi_1\left(X\right)_{k-1}\right)$$

Let $f : G \to K$ be a group homomorphism. A theorem by Stallings [Sta65] states that there is a relationship between the induced map on group homology and the induced maps between G and K's lower central series quotients. This theorem can also be used to relate statements about the topology of a space and the induced map on lower central series quotients of the fundamental group of a space. However, Stalling's theorem is insufficient for our purposes since the creation and elimination of self-intersections during a link homotopy changes the rank of the second homology of the complement. We require the following generalisation of Stalling's theorem.

Theorem 5.0.4 (Dwyer's Theorem [FT95])**.** *Let G and K be finitely generated groups and let $f : G \to K$ be a homomorphism which induces an isomorphism on the first group homology of these groups. Then for $k \geq 2$ the following are equivalent:*

1. *f induces an epimorphism on $H_2(G)/\phi_k(G) \to H_2(K)/\phi_k(K)$,*
2. *f induces an isomorphism $G/G_k \to K/K_k$,*
3. *f induces an isomorphism $H_2(G)/\phi_k(G) \to H_2(K)/\phi_k(K)$ and an injection $H_2(G)/\phi_{k+1}(G) \to H_2(K)/\phi_{k+1}(K)$.*

Like Stalling's theorem, Dwyer's theorem is stated in terms of groups but applies to spaces since we can always just attach cells of degree greater than three and this does not affect any of the statements of the theorem. Before we use Dwyer's theorem let us prove the following lemmas.

Lemma 5.0.5. *Let S be the image of a generic n-component link map and x_i be a meridian of the ith component. Then $x_1, \ldots x_n$ is a set of normal generators for the fundamental group of $S^4 \setminus S$.*

Proof. Define $X_S := S^4 \setminus \nu(S)$. To prove the result, we will give a Kirby diagram description of the normal bundle of S and turn the handle decomposition upside down and glue to the boundary of X_S to make S^4; from this it will be clear that we have a set of normal generators. Consider the normal neighbourhood of the ith sphere (a description of the normal bundle is given in [GS99, Section 6.1]). A Kirby diagram of a normal neighbourhood of an immersion is given by a single 0-handle, a single 2-handle, and k many 1-handles (one for each intersection on the immersed surface). Turning this upside down we have a single 2-handle, k many 3-handles, and a single 4-handle. We now attach the 2-handle to the ith boundary component, where attaching sphere of the dual two handle is a meridian of the ith sphere. To see the 2-handle attaches to the meridian of the sphere, notice that the 2-handle of the plumbing is the thickened 2-cell of the sphere and thus the dual handle from turning upside down the new attaching sphere is the meridian of the sphere. The fundamental group of this new space once we have attched the two handle is $\pi_1(S^4 \setminus \nu(S)) / \langle\langle x_i \rangle\rangle$. Doing this for each sphere we recover S^4 which has trivial fundamental group. Hence the meridians are a set of normal generators for the complement as required. $\square$

Lemma 5.0.6. *Let f be a link map and a generic immersion. We can do finger moves on f to get another link map $\tilde{f}$ such that*

$$\pi_1 \left(S^4 \setminus \nu(\tilde{f}) \right) \cong M\pi_1 \left(S^4 \setminus \nu(f) \right).$$

Proof. We can do self-finger moves on each component, using a finger which misses the other components, to introduce relations of the form $[\alpha, \beta\alpha\beta^{-1}]$, where $\alpha, \beta \in \pi_1(S^4 \setminus \nu(f))$. Since Milnor groups are nilpotent and normally generated by finitely many elements, by Theorem 5.0.5, they are finitely presented [BT99]. Hence, we only need to do finitely many to make the complement isomorphic to a Milnor group. $\square$

Proposition 5.0.7. *Let $f : S^2 \coprod S^2 \coprod S^2 \to S^4$ be a link map, with each component generically immersed. Then the Milnor group of the fundamental group of the complement of any two components is isomorphic to F/F_3.*

Proof. By Lemma 5.0.6 we can assume that the fundamental group of the complement of two components of the image is a Milnor group. Let K be the presentation complex for F/F_3, where F/F_3 is presented by

$$\langle x, y \mid [x,[x,y]], [y,[x,y]] \rangle .$$

There exists a map $g : K \to N$ where $N := S^4 \setminus \nu(f_i \cup f_j)$ with $i \neq j$. To calculate $H_*(K)$ we can compute the boundary map of the cellular chain complex using Fox derivatives and we see that the first and second boundary maps are zero. Hence

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1, 2 \end{cases},$$

where the second homology is generated by the relations given by the chosen presentation of F/F_3. Each relation can be represented by a closed class 3 grope inside K.

For N we showed that the second homology is generated by Clifford tori around each intersection in Lemma 5.0.1. Since f is a link map, the meridians of the torus are of the form α and $\beta\alpha\beta^{-1}$, where α is a meridian of the sphere which the intersection is on and $\beta \in \pi_1(N)$. Hence the $(1, -1)$ curve on each torus is trivial when viewed as an element of $H_1(N)$. Thus each Clifford torus generating $H_2(N)$ can be extended to a closed class 3-grope. This implies that g induces an isomorphism on $H_2(K)/\phi_3(K) \to H_2(N)/\phi_3(N)$, as both sides are equal to zero. By Dwyer's theorem this is equivalent to an isomorphism

$$g : F/F_3 \xrightarrow{\cong} \pi_1(N)/\pi_1(N)_3.$$

By Lemma 4.1.6, Milnor groups with two generators are nilpotent of at most degree two, hence we have an isomorphism between F/F_3 and the Milnor group of the complement of two spheres. □

Corollary 5.0.8. *Let* $f = f_1 \sqcup f_2 \sqcup f_3 : S^2 \coprod S^2 \coprod S^2 \to S^4$ *be a generic link map. Then we can apply finger moves to* f *to create a link map* $\widetilde{f}_1 \sqcup \widetilde{f}_2 \sqcup \widetilde{f}_3 : S^2 \coprod S^2 \coprod S^2 \to S^4$ *such that*

$$\pi_1(S^4 \smallsetminus \nu(\widetilde{f}_i \cup \widetilde{f}_j)) \cong F/F_3,$$

where $i \neq j$*.*

Proof. By Lemma 5.0.6, finger moves can be applied to f such that the complement of any two distinct components is a Milnor group. By Proposition 5.0.7, this Milnor group is isomorphic to F/F_3. □

Chapter 6

The three-component link map invariant

We now define our three-component invariant. We first do this in the case of based link maps and then explain how to make an invariant for unbased link maps.

6.1 Based link maps

Firstly, we choose the induced orientation on S^n from the standard orientation on $\mathbb{R}^{n+1}$. Fix a basepoint $s_0 \in S^4$ and consider a generically immersed, based, link map $f = f_x \sqcup f_y \sqcup f_z : S^2_x \coprod S^2_y \coprod S^2_z \to S^4$, where the basing paths for each sphere are given by γ^f_x, γ^f_y and γ^f_z respectively. Homotop γ^f_x, fixing the end points, such that on $[1-\varepsilon, 1]$, for some $\varepsilon > 0$, γ lies in the fibre of the normal bundle above the point $\gamma^f_x(1)$. Let U be an open subset of f_x, around $\gamma^f_x(1)$ which contains no intersections and admits a trivialisation of the normal bundle, $\phi : U \times D^2 \to \nu(f_x)|_U$. We define a meridian by the following concatenation of paths: travel along the basepoint γ^f_x from x_0 to $\gamma^f_x(1-\varepsilon)$, then travels along the generator of $\pi_1(U \times D^2 \smallsetminus U \times \{0\})$ which has linking number $+1$ with f_x, and then travels back along γ^f_x from $\gamma^f_x(1-\varepsilon)$ to the x_0. Call this meridian x. Define y and z to be similar meridians for the other respective components. These meridians are normal generators of $\pi_1(S^4 \smallsetminus \nu(f))$ and thus are generators of $M\pi_1\left(S^4 \smallsetminus \nu(f)\right)$. Let A_x, A_y and A_z be the normal closures of x, y and z respectively, inside $M\pi_1\left(S^4 \smallsetminus \nu(f)\right)$. By Proposition 5.0.7 we have

$$\begin{aligned}
M\pi_1\left(S^4 \smallsetminus \nu(f)\right)/A_x &\cong \left\langle y, z, s \mid [y,s], [z,s], [y,z]\, s^{-1} \right\rangle \\
M\pi_1\left(S^4 \smallsetminus \nu(f)\right)/A_y &\cong \left\langle z, x, t \mid [z,t], [x,t], [z,x]\, t^{-1} \right\rangle \\
M\pi_1\left(S^4 \smallsetminus \nu(f)\right)/A_z &\cong \left\langle x, y, u \mid [x,u], [y,u], [x,y]\, u^{-1} \right\rangle .
\end{aligned}$$

as $M\pi_1\left(S^4 \smallsetminus \nu(f)\right)/A_i = M\pi_1\left(S^4 \smallsetminus \left(\nu(f_j) \sqcup \nu(f_k)\right)\right)$, where $i, j, k \in \{x, y, z\}$ and are pairwise distinct. We define

$$\begin{aligned}
\Gamma_x &:= \left\langle y, z, s \mid [y,s], [z,s], [y,z]\, s^{-1} \right\rangle , \\
\Gamma_y &:= \left\langle z, x, t \mid [z,t], [x,t], [z,x]\, t^{-1} \right\rangle , \\
\Gamma_z &:= \left\langle x, y, u \mid [x,u], [y,u], [x,y]\, u^{-1} \right\rangle ,
\end{aligned}$$

and $K := \mathbb{Z}\Gamma_x \times \mathbb{Z}\Gamma_y \times \mathbb{Z}\Gamma_z$.

Remark 6.1.1. To clarify the above, we have taken a based link map and used the basing curves for each surface to define a meridian and then chosen isomorphisms to F/F_3 using the specified meridians.

Definition 6.1.2. Let f be a based three-component link map then

$$\sigma^3(f) = \left(\sigma_x(f), \sigma_y(f), \sigma_z(f)\right) := \left(M\left(\lambda(f_x, f_x)\right), M\left(\lambda(f_y, f_y)\right), M\left(\lambda(f_z, f_z)\right)\right) \in K.$$

Proposition 6.1.3. *The map σ^3 is a based link homotopy invariant.*

Proof. By Proposition 5.0.7, through a generic homotopy, the Milnor group of the fundamental group of the complement of two spheres, described by a link map, remains constant. We now check that under a generic homotopy the triple remains fixed.

Considering a link map f, we check that finger moves do not affect the intersection values. We first check that $\sigma_x(f) = M(\lambda(f_x, f_x))$ is not affected by any finger move on any component. The other components will follow similarly. It is clear that doing a finger move on the x component does not affect the intersection number since λ is a homotopy invariant.

We now check that a finger move on the y component does not affect $M\lambda(f_x, f_x)$. If we do a finger move on the y sphere we change the fundamental group of the complement of $\pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z)))$ to $\pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z))) \big/ \langle\langle [y, gyg^{-1}] \rangle\rangle$ for some $g \in \pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z)))$ [Cas86]. We thus have the following commutative diagram

$$\begin{tikzcd}
\pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z))) \arrow[r] \arrow[d] & M\pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z))) \arrow[d, "\mathrm{Id}"] \\
\pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z))) \big/ \langle\langle [y, gyg^{-1}] \rangle\rangle \arrow[r] & M\left(\pi_1(S^4 \setminus (\nu(f_y) \coprod (f_z))) \big/ \langle\langle [y, gyg^{-1}] \rangle\rangle\right).
\end{tikzcd}$$

The horizontal maps are the Milnor quotient maps. Hence, group elements are not changed by finger moves, since the finger move introduces relations already a consequence of the relations in $M\pi_1\left(S^4 \setminus (f_y \sqcup f_z)\right)$. Hence, $M\lambda(f_x, f_x)$ is invariant under finger moves on the y component. An analogous argument works for finger moves on the z component. Hence σ^3 is invariant under finger moves.

Let g be a generic immersed link map link homotopic to f. We can assume that that both maps have the same Euler number, since cusps do not change λ nor the fundamental group of the complement. Let $F : \left(S^2_x \coprod S^2_y \coprod S^2_z\right) \times I \to S^4$ be a generic homotopy such that $F(-, 0) = f$ and $F(-, 1) = g$. We may assume this is a regular homotopy since both Euler numbers are the same and by Proposition 3.1.5 we may assume F is a concatenations of finger moves, Whitney moves and isotopies. As finger moves are supported by an arc and Whitney moves are supported in the neighbourhood of a disc we may assume F does all the finger moves when $t \in \left(0, \frac{1}{2}\right)$ and Whitney moves when $t \in \left(\frac{1}{2}, 1\right)$. By the earlier argument, finger moves do not change the triple and thus the triple is equal on f and $F(-, \frac{1}{2})$. Running the homotopy backwards from g to $F(-, \frac{1}{2})$ is a sequence of finger moves, so the triple on g and $F(-, \frac{1}{2})$ is the same. By transitivity, σ^3 is a homotopy invariant. $\square$

6.1.1 Link maps (unbased)

Using the invariant for the based link maps we will develop an invariant for unbased link maps. This will come from taking a quotient of K which corresponds to changing the choice of basing paths for each sphere.

We first analyse the changes to σ^3 which occur when we change the basing of a single component. After this, we describe what happens when all the basings change. This will give us an orbit space for the action on K that correspond to changing the basing paths. This will be the correct quotient to describe an invariant of unbased link maps.

Lemma 6.1.4. *Let f and g be based link maps which are equal except that the basings of the x component are different. Then for some $h \in \Gamma_x$ we have*

$$M\lambda(g_x, g_x) = h\,(M\lambda(f_x, f_x))\,h^{-1}.$$

Proof. Let $\gamma_x^f : I \to S^4$ and $\gamma_x^g : I \to S^4$ be basing paths for the x component. To prove the result we will work with the self-intersection number μ and use Theorem 2.2.3 to calculate λ. Let $p \in f_x(S_x^2)$ be a self-intersection. Using the basing γ_x^f, the group element associated to the intersection is given by

$$\gamma_x^f \cdot f(\delta_1) \cdot f(\overline{\delta}_2) \cdot \overline{\gamma}_x^f.$$

Recall that δ_1 and δ_2 are paths on S^2 from $f^{-1}(\{\gamma_x^f(1)\})$ to the distinct elements $p_1, p_2 \in f_x^{-1}(\{p\})$ respectively. Using the basing path γ_x^g, the group element associated to the self-intersection becomes

$$\gamma_x^g \cdot f(\delta_1') \cdot f(\overline{\delta}_2') \cdot \overline{\gamma}_x^g,$$

where δ_1' and δ_2' are defined similarly to δ_1, δ_2. Let $\alpha : I \to \mathrm{im}(f_x)$ such that $\alpha(0) = \gamma_x^g(1)$ and $\alpha(1) = \gamma_x^f(1)$ such that α does not pass through an intersection at any point on the path. Also, define $h = \gamma_x^g \cdot \alpha \cdot \bar{\gamma}_x^f$. As S^2 is simply connected,

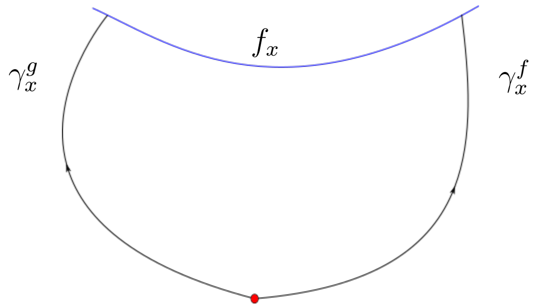

Figure 6.1: A schematic for the different choice of basing paths to the x component, for the based link maps f and g in Lemma 6.1.4

$\alpha \cdot f(\delta_1) \simeq f(\delta_1')$ and $f(\bar{\delta}_2) \cdot \bar{\alpha} \simeq f(\bar{\delta}_2')$. Using both of these homotopies we have

$$\begin{aligned}\gamma_x^g \cdot f(\delta_1') \cdot f(\bar{\delta}_2') \cdot \bar{\gamma}_x^g &\simeq \gamma_x^g \cdot \alpha \cdot f(\delta_1) \cdot f(\bar{\delta}_2) \cdot \bar{\alpha} \cdot \bar{\gamma}_x^g \\ &\simeq \gamma_x^g \cdot \alpha \cdot \bar{\gamma}_x^f \cdot \gamma_x^f \cdot f(\delta_1) \cdot f(\bar{\delta}_2) \cdot \bar{\gamma}_x^f \cdot \gamma_x^f \cdot \bar{\alpha} \cdot \bar{\gamma}_x^g \\ &= h\left(\gamma_x^f \cdot f(\delta_1) \cdot f(\bar{\delta}_2) \cdot \bar{\gamma}_x^f\right)\bar{h},\end{aligned}$$

where $h \in \Gamma_x$. Using Theorem 2.2.3 we have proven the result. □

Lemma 6.1.5. *Let f be a based link map such the triple $\sigma^3(f)$ is equal to*

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} x^i u^j y^k\right) \in K$$

Then for a based link map g, which is equal to f except that it differs by a basing path to the x component, the triple $\sigma^3(g)$ is equal to

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-ak+bi} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^{j+bk} x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} x^i u^{j-ai} y^k\right) \in K,$$

for some $a, b \in \mathbb{Z}$.

Proof. By the previous lemma we have

$$M\left(\lambda\left(g_x, g_x\right)\right) = hM\left(\lambda\left(f_x, f_x\right)\right)h^{-1},$$

for some $h \in \Gamma_x$. Since $h = y^{v_1} s^{q_x} z^{w_1}$ for $v_1, q_x, w_1 \in \mathbb{Z}$ then using the relations in

Γ_x we have

$$\begin{aligned}
M\left(\lambda\left(g_x, g_x\right)\right) &= y^{v_1} s^{q_x} z^{w_1} \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^j z^k \right) z^{-w_1} s^{-q_x} y^{-v_1} \\
&= y^{v_1} z^{w_1} \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^j z^k \right) z^{-w_1} y^{-v_1} \\
&= y^{v_1} \left(\sum_{(i,j,k)\in\mathbb{Z}^3} z^{w_1} y^i z^{-w_1} s^j z^k \right) y^{v_1} \\
&= y^{v_1} \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-w_1 i} z^k \right) y^{-v_1} \\
&= \sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-w_1 i} y^{v_1} z^k y^{-v_1} \\
&= \sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-w_1 i+v_1 k} z^k.
\end{aligned}$$

We now show how the other components transform. The details of this are slightly more subtle, as the group elements associated to the intersections do not change when considered as elements of $M\pi_1(S^4 \smallsetminus f_x \sqcup f_z)$ and $M\pi_1(S^4 \smallsetminus f_x \sqcup f_y)$. The reason why the values in Γ_y and Γ_z change is because our choice of isomorphisms from $M\pi_1(S^4 \smallsetminus f_x \sqcup f_z)$ and $M\pi_1(S^4 \smallsetminus f_x \sqcup f_y)$, to Γ_y and Γ_z respectively, change when we change the basing path. In order to compute what our invariant looks like on g we must describe the meridian of the first component of f in terms of the specified meridians of g.

Let x' be the meridian of the first component of g. Our goal is to find a loop $r \in \pi_1(S^4 \smallsetminus f)$ such $x = r^{-1} x' r$, where we consider r as in Γ_y and Γ_z and $r = h$ in Γ_x.

Let $\gamma^f_\varepsilon, \gamma^g_\varepsilon : I \to S^4$ be the basing paths of the x components f and g respectively which stop at $\gamma^f_x(1-\varepsilon)$ and $\gamma^f_x(1-\varepsilon)$. Recall that $\alpha : I \to f_x$ is a path between the two basing points of g and f starting at $\gamma^g_x(1)$ and ending at $\gamma^f_x(1)$ without going through any intersections. Let α' be a normal push-off of α from the surface with starting point $\gamma^g_x(1-\varepsilon)$ and with end point $\gamma^f_x(1-\varepsilon)$. Let p^g_x be the generator, based at $\gamma^g_x(1-\varepsilon)$, of $\pi_1(U \times D^2 \smallsetminus U \times \{0\})$ with positive linking number to f_x, where

U is an open subset around $\gamma^g_x(x)$ which trivialises the normal bundle. We have a homotopy $x \simeq \gamma^f_\varepsilon \cdot \overline{\alpha}' \cdot p^g_x \cdot \alpha \cdot \overline{\gamma}^f_\varepsilon$, which we can see by comparing Figure 6.2 and Figure 6.3. Hence,

$$\begin{aligned} x &\simeq \gamma^f_\varepsilon \cdot \overline{\alpha}' \cdot p^g_x \cdot \alpha \cdot \overline{\gamma}^f_\varepsilon \\ &\simeq \gamma^f_\varepsilon \cdot \overline{\alpha}' \cdot \overline{\gamma}^g_\varepsilon \cdot \gamma^g_\varepsilon \cdot p^g_x \cdot \overline{\gamma}^g_\varepsilon \cdot \gamma^g_\varepsilon \cdot \alpha \cdot \overline{\gamma}^f_\varepsilon \\ &= r^{-1} x' r, \end{aligned}$$

where $r = \gamma^g_\varepsilon \cdot \alpha \cdot \overline{\gamma}^f_\varepsilon$ and, by definition, $x' = \gamma^g_\varepsilon \cdot p^g_x \cdot \overline{\gamma}^g_\varepsilon$. Considering r as an element inside $M\pi_1(S^4 \smallsetminus f_y \sqcup f_z)$ we can see that it is homotopic to $h \in \Gamma_x$. Using this we have

$$\begin{aligned} M\left(\lambda(g_y, g_y)\right) &= \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^j x^k \\ &= \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^j (r^{-1}x'r)^k, \end{aligned}$$

where we consider $r \in \Gamma_y$. Similarly we have

$$M\left(\lambda(g_z, g_z)\right) = \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} (r^{-1}xr)^i u^j y^k,$$

where we consider $r \in \Gamma_z$. We consider r inside the different quotients, Γ_x, Γ_y and Γ_z. Inside these quotients we find that r is equal to $y^{v_1} s^{q_x} z^{w_1} \in \Gamma_x$, $z^{v_2} t^{q_y} x^{w_2} \in \Gamma_y$ and $x^{v_3} u^{q_z} y^{w_3}$, where $v_i, w_i \in \mathbb{Z}$ for $1 \leq i \leq 3$.

Claim 6.1.6. In the above we have $v_1 = w_3$, $w_1 = v_2$ and $w_2 = v_3$.

From this claim we set $v_1 = -a$ and $w_1 = -b$. We now have that the triple for g, substituting in the various quotients of r, and using Γ_i relations we have

$$\begin{aligned} &\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^{-a} z^{-b} y^i s^j z^k z^b y^a, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^j (z^b x' z^{-b})^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} (y^a x' y^{-a})^i u^j y^k \right) \\ &= \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-ak+bi} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^{j+bk} (x')^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} (x')^i u^{j-ai} y^k \right). \end{aligned}$$

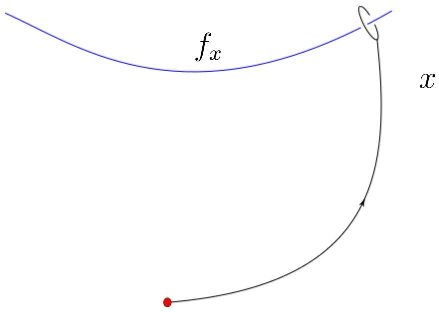

Figure 6.2: A schematic showing the meridian x associated to the surfaces basing path.

We now replace x' with x to identify the x meridian given by g to get

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-ak+bi} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^{j+bk} x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} x^i u^{j-ai} y^k \right)$$

To complete the proof, we need to account for the different possible choice of r resulting from the choice of push off of α. The different choice of α does not effect a and b as these are linking numbers of the path with the y and z component and upon the inclusion of the x component all the different paths become homotopic and thus these values are unchanged by our choice. The only value which is changed is the power of x in Γ_y and Γ_z but this does not affect the end results as the triple is independent of p'. This completes the proof modulo the proof of Claim 6.1.6. $\square$

Remark 6.1.7. It is clear from the proof of Lemma 6.1.5 that we only need to consider the loop r as an element of Γ_x to know how to change the triple for the new basing path and, since elements in the center do not matter, we only need to look at the curve as an element of $\Gamma_x/\,[\gamma_x, \gamma_x] \cong H_1(S^4 \setminus f_y \sqcup f_z)$. As v_1 is the linking of the loop with the S^2_y sphere and w_1 is the linking number of the loop S^2_z, it is clear we only need to know the linking number of the loop with the other components to calculate how the triple transforms.

We now prove the claim that was in the previous proof.

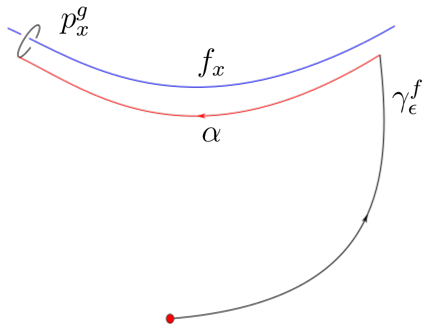

Figure 6.3: A schematic showing the result of a homotopy of the x meridian, which used the push-off of the surface α.

Proof of Claim 6.1.6. We will prove that the following diagram commutes

$$\begin{tikzcd} M\pi_1(S^4 \smallsetminus f) \arrow[r, "\mathrm{Ab}"] \arrow[d, "q"] & \mathbb{Z}^3 \arrow[d] \\ \Gamma_x \times \Gamma_y \times \Gamma_z \arrow[r, "\mathrm{Ab}\times\mathrm{Ab}\times\mathrm{Ab}"] & \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2, \end{tikzcd}$$

where Ab is the quotient map to the abelianisation for the various groups involved. The map q takes a group and sends it to its various representatives in each factor. Notice that $(\mathrm{Ab}\times\mathrm{Ab}\times\mathrm{Ab})\,(\Gamma_x \times \Gamma_y \times \Gamma_z) = H_1(S^4 \smallsetminus f_y \sqcup f_z) \times H_1(S^4 \smallsetminus f_z \sqcup f_x) \times H_1(S^4 \smallsetminus f_x \sqcup f_y) \cong \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2$ and $H_1(S^4 \smallsetminus f) \cong \mathbb{Z}^3$. The map $\mathbb{Z}^3 \to \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2$ is given by

$$x^\alpha y^\beta z^\gamma \mapsto ((\beta,\gamma),(\alpha,\gamma),(\alpha,\beta))$$

Proving the above diagram commutes is equivalent to showing that

$$\begin{tikzcd} M\pi_1(S^4 \smallsetminus f) \arrow[r, "i"] \arrow[d, "\mathrm{Ab}"] & \Gamma_x \arrow[d, "\mathrm{Ab}"] \\ H_1(S^4 \smallsetminus f) \arrow[r, "H(i)"] & H_1(S^4 \smallsetminus f_y \sqcup f_z) \end{tikzcd}$$

commutes, as the roles of x, y and z are symmetric. We can write $r = x^\alpha y^\beta z^\gamma \eta$ where η is a product of commutators. Then $\mathrm{Ab}(i(r)) = \mathrm{Ab}(y^\beta z^\gamma s^p) = y^\beta z^\gamma$. Computing the other path, we have $H(i)\,(\mathrm{Ab}(r)) = H(i)(x^\alpha y^\beta z^\gamma) = y^\beta z^\gamma$. This proves the claim. □

We now show how the triple changes when all three components change their basing paths

Corollary 6.1.8. *Let g_1 be a based link map with triple $\sigma^3(g_1)$ equal to*

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk}y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk}z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk}x^i s^j y^k\right) \in K.$$

Let g_2 be the same link map as g_1 with potentially different basings then

$$\sigma^3(g_2) = \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk}y^i s^{j+(b-c)i+(f-a)k} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk}z^i t^{j+(d-e)i+(b-c)k} x^k,\right.$$
$$\left.\sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk}x^i s^{j+(f-a)i+(d-e)k} y^k\right) \in K,$$

for some $a, b, c, d, e, f \in \mathbb{Z}$.

Proof. Let $r_x = y^a z^b s^\alpha \in \Gamma_x$, $r_y = z^c x^d t^\beta \in \Gamma_y$ and $r_z = x^e y^d u^\gamma \in \Gamma_z$ be the loops we construct from the two basings on each component. We now repeat the proof of Lemma 6.1.5 on each sphere: changing the x-basing we have

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk}y^i s^{j-ak+bi} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk}z^i t^{j+bk} x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk}x^i u^{j-ai} y^k\right);$$

and changing the y-basing

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk}y^i s^{j-ak+(b-c)i} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk}z^i t^{j+(b-c)k+di} x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk}x^i u^{j-ai+dk} y^k\right).$$

Changing the z-basing we arrive at the desired result,

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk}y^i s^{j+(b-c)i+(f-a)k} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk}z^i t^{j+(b-c)k+(d-e)i} x^k,\right.$$
$$\left.\sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk}x^i u^{j+(f-a)i+(d-e)k} y^k\right).$$

One can check that the above does not depend on the order in which we changed the basing paths. However, it is clear geometrically that this is the case. $\square$

We wish to show that we can achieve all values of $a, b, c, d, e, f \in \mathbb{Z}$ by changing basing paths. Let g be a link map and define B_g to be the set consisting of triples of basing paths to the x, y and z spheres. We define a map $\zeta : B_g \times B_g \to \mathbb{Z}^6$. Let $Q, Q' \in B_g$. Let γ_x, γ_y and γ_z be the basing paths for each sphere given by Q and let γ'_x, γ'_y and γ'_z be the basing paths in Q'. Define a loop

$$h_x := \gamma_x \cdot \delta \cdot \bar{\gamma}'_x,$$

where δ is the unique path up to homotopy between the points $\gamma_x(1)$ and $\gamma'_x(1)$ which is the image of a path in the pre-images. We define h_y and h_z similarly. Thus we define

$$\begin{aligned}\zeta\left(Q, Q'\right) := \Big(\operatorname{lk}\left(h_x, g(S^2_y)\right), \operatorname{lk}\left(h_x, g(S^2_z)\right), \operatorname{lk}\left(h_y, g(S^2_z)\right),& \\ \operatorname{lk}\left(h_y, g(S^2_x)\right), \operatorname{lk}\left(h_z, g(S^2_x)\right), \operatorname{lk}\left(h_z, g(S^2_y)\right) \Big).&\end{aligned}$$

Lemma 6.1.9. *The map, $\zeta : B_g \times B_g \to \mathbb{Z}^6$, is surjective.*

Proof. Let m_x, m_y and m_z be meridians defined using the basing paths of Q and let $a, b, c, d, e, f \in \mathbb{Z}$. Let Q_m be the basing paths consisting of the paths $m_y^a \cdot m_z^b \cdot \gamma_x$, $m_z^c \cdot m_x^d \cdot \gamma_y$ and $m_x^e \cdot m_y^f \cdot \gamma_z$. Hence,

$$\zeta\left(Q_m, Q\right) = (a, b, c, d, e, f).$$

□

Definition 6.1.10. Let $(a, b, c, d, e, f) \in \mathbb{Z}^6$ and

$$\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} x^i u^j y^k \right) \in K.$$

Then we define

$$\begin{aligned}(a, b, c, d, e, f) \cdot &\left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} x^i u^j y^k \right) \\ = &\Bigg(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^{j-ci+fk-ak+bi} z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b_{ijk} z^i t^{j-ei+bk-ck+di} x^k, \\ &\qquad \sum_{(i,j,k)\in\mathbb{Z}^3} c_{ijk} x^i u^{j-ai+ck-ek+fi} y^k \Bigg).\end{aligned}$$

Lemma 6.1.11. *The operation described in Definition 6.1.10 is a group action on K by $\mathbb{Z}^6$.*

Proof. One must check that for all $g \in K$

$$(0,0,0,0,0,0) \cdot g = g$$

and

$$(a,b,c,d,e,f) \cdot ((a',b',c',d',e',f') \cdot g) = (a+a', b+b', c+c', d+d', e+e', f+f') \cdot g.$$

This is clear from the algebra. □

Let $\widetilde{K}$ be the orbit space of this action.

Definition 6.1.12. Let f be a three-component link map. Then we define a map $\widetilde{\sigma}^3 : \mathrm{LM}^4_{2,2,2} \to \widetilde{K}$ by

$$\widetilde{\sigma}^3(f) := [\sigma^3(f)] \in \widetilde{K}.$$

Proposition 6.1.13. *The map $\widetilde{\sigma}^3$ is a well defined map on* $\mathrm{LM}^4_{2,2,2}$.

Proof. We showed that in the based case under link homotopy the triple remains fixed. It is clear from Lemma 6.1.5 and the definition of $\widetilde{K}$ that the map is well defined. □

The invariant $\widetilde{\sigma}^3$ contains all the data of the Kirk invariant on its two-component sublink maps.

Lemma 6.1.14. *Let $i : \mathrm{LM}^4_{2,2,2} \to \mathrm{LM}^4_{2,2}$ be the map which forgets about the ith sphere. Additionally, let $p_i : K \to (\mathbb{Z}[\mathbb{Z}])^2$ be the map given by projecting onto the factors which are not i and setting the ith meridian equal to* 1. *Then the following diagram commutes*

$$\begin{array}{ccc} \mathrm{LM}^4_{2,2,2} & \xrightarrow{i} & \mathrm{LM}^4_{2,2} \\ \big\downarrow{\scriptstyle \widetilde{\sigma}^3} & & \big\downarrow{\scriptstyle \sigma} \\ K & \xrightarrow{p_i} & (\mathbb{Z}[\mathbb{Z}])^2, \end{array} \tag{6.1.1}$$

where i is the map that forgets the ith component, and p_i is the map where you project onto two factors which are not i and sets ith component's meridian equal to 1.

6.2 Examples of three-component linking behaviour

Our three-component invariant, $\tilde{\sigma}^3$, can differentiate between the trivial link map and link maps which are trivial when any component is removed, and thus can detect linking behaviour which only occurs when there are at least three-components.

Theorem 6.2.1. *Let L be the link given in the centre of Figure 6.4. Then we can apply a null homotopy of components of the link to construct a representative of $f \in \mathrm{LM}^4_{2,2,2}$ with some choice of basing path such that*

$$\tilde{\sigma}^3(f) = \left(z\,(s-1) + z^{-1}\left(s^{-1}-1\right),\, 0,\, x\,(1-u) + x^{-1}\left(1-u^{-1}\right)\right) \neq 0.$$

Hence, F is link homotopically non-trivial. However, the removal of any component gives a trivial component two-component link map.

Proof. Using Figure 6.4, we have a description of a three-component link map inside S^4 which we call f. To compute the first component of σ^3, consider the null-homotopy from the central time slice to the end. The sign of the self-intersections in the corresponding link map is labelled on Figure 6.4. The corresponding element of γ_x at the first intersection on this region is z (we could have chosen $\overline{z}$ here). The corresponding element of Γ_x of the next self-intersection in this region is given by $zyz^{-1}y^{-1}z^{-1} \in \Gamma_x$. Hence, the first component of $\sigma^3(f)$ is given by

$$z\,([y,\, z]-1) + z^{-1}\,([z,\, y]-1) \in \mathbb{Z}\Gamma_x.$$

We now compute the third component of σ^3. Using the Wirtinger presentation one can show that the meridian of the first component, below the second component, is given by $y^{-1}xy$. We now study the homotopy from the start to the central time slice. The group element associated to the first double point in this region is $x \in \Gamma_z$. The group element associated to the next intersection in this region is $y^{-1}xy \in \gamma_z$

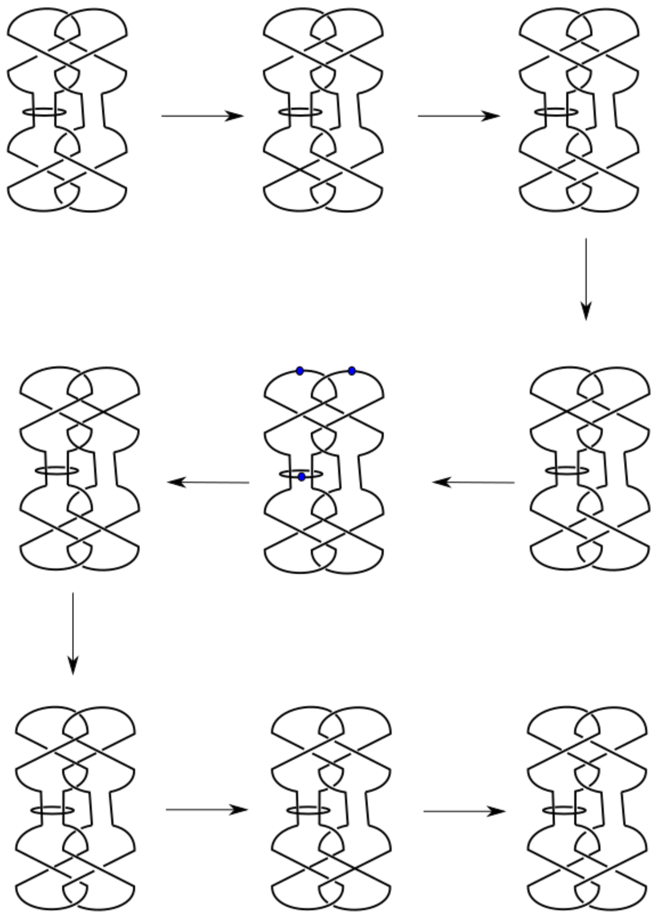

Figure 6.4: This is a sequence of times slices in $D^3 \times I$ inside S^4 describing a link map where we have specified a basing paths and meridians on the central time slice by blue points on each link component. The x component is the component at the top on the left, the second components is the circle round the band and the third component is the right most component. The orientation of each components travels through the basepoint to the left.

this is equivalent to the group element $x\,[x,\,y]$. Hence the third component is

$$x\,(1-[x,y])+x^{-1}\,(1-[y,\,x])\in\mathbb{Z}\Gamma_z.$$

The second component is embedded and thus the second component of σ^3 vanishes.

If we remove a sphere the link becomes trivial. To see this, we use repeated applications of Lemma 6.1.14. Notice that $p_i(\widetilde{\sigma}^3(f))=0$ for each i. Using that the Kirk invariant is injective [ST17] this implies that $i(f)$ is a trivial link map for all i which proves the result. □

Paul Kirk had a version of his invariant for more than two components [Kir88]. If we consider the link map constructed using Figure 6.4, his invariant fails to be able to distinguish between this non-trivial link map and a trivial link map. We will give a name for link maps which a trivial once a component is removed.

Definition 6.2.2. We call a link map *Brunnian* if the removal of any single component results in a trivial link map up to link homotopy. We will denote the set of three-component Brunnian link maps by $\mathrm{Bl}^4_{2,2,2}$.

Example 6.2.3. Let f be the link map from Theorem 6.2.1 with the same basing paths and orientations. Let $g=f\circ k$ where $k: S^2_x\amalg S^2_y\amalg S^2_z\to S^2_x\amalg S^2_y\amalg S^2_z$ which is a reflection on S^2_y and the identity elsewhere. This gives us

$$\begin{aligned}\sigma^3(f)&=\left(z\,(s-1)+z^{-1}\left(s^{-1}-1\right),\,0,\,x\,(1-u)+x^{-1}\left(1-u^{-1}\right)\right)\\ \sigma^3(g)&=\left(z\left(s^{-1}-1\right)+z^{-1}\,(s-1)\,,\,0,\,x\left(1-u^{-1}\right)+x^{-1}\,(1-u)\right).\end{aligned}$$

If we consider the based verison of the invariant it is clear that these are two different based link maps. However, we must check that the unbased case gives two distinct equivalence classes. Let us fix the representative of $\sigma^3(g)$ as above. The equivalence class for $\sigma^3(f)$ is described by

$$\left(z\left(s^{1-a}-s^{-a}\right)+z^{-1}\left(s^{-1+a}-s^{a}\right),0,x\left(u^{-a}-u^{1-a}\right)+x^{-1}\left(u^{a}-u^{-1+a}\right)\right),$$

where $a \in \mathbb{Z}$. Taking the difference and looking at the first component we get

$$z(s^{1-a} - s^{-a} - s^{-1} + 1) + z^{-1}(s^{-1+a} - s^{a} - s + 1).$$

A necessary condition for both link maps to be equal is to have an $a \in \mathbb{Z}$ such that

$$s^{1-a} - s^{-a} - s^{-1} + 1 = 0.$$

Clearly, such an a must have $|a| < 2$. Checking each value of a remaining, it is evident we can never solve the above equation. Hence, as unbased link maps they are not equal.

This establishes the following.

Theorem 6.2.4. *For each $n \geq 3$ there exists link maps $f = f_1 \sqcup \cdots \sqcup f_n$ and $f' = f'_1 \sqcup \cdots \sqcup f'_n$ such that for i, $f_i(S^2) = f'_i(S^2)$, but f and f' are not link homotopic*

6.3 New invariants

Frequently, it is difficult to differentiate two elements of $\mathrm{LM}^4_{2,2,2}$ using $\tilde{\sigma}^3$ since it can be difficult to tell if two representatives of elements of $\widetilde{K}$ are in the same equivalence class. This section will focus on extracting new invariants from $\tilde{\sigma}^3$ which also are independent of our choice of basing path for each component.

Let $f \in \mathrm{LM}^4_{2,2,2}$ and write

$$\tilde{\sigma}^3(f) = \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a^f_{ijk} y^i s^j z^k, \sum_{(i,j,k)\in\mathbb{Z}^3} b^f_{ijk} z^i t^j x^k, \sum_{(i,j,k)\in\mathbb{Z}^3} c^f_{ijk} x^i u^j y^k \right)$$

for some choice of basing paths for each sphere, where $a^f_{ijk}, b^f_{ijk}, c^f_{ijk} \in \mathbb{Z}$. Let, A_f be the unordered n-tuple of non-zero a^f_{ijk}. Define B_f and C_f similarly. We have the following

Proposition 6.3.1. *Let f, g be link homotopic link maps. Then $A_f = A_g$, $B_f = B_g$ and $C_f = C_g$.*

Proof. The proposition follows from immediately from the definition of $\widetilde{\sigma}^3$. $\square$

Proposition 6.3.2. *Let $f \in \mathrm{LM}^4_{2,2,2}$ with some choice of basing path. Suppose g is a link map link homotopic to f. Then for any choice of basing path there exist an $l \in \mathbb{Z}$ such that $a^g_{ilk} = a_{ijk}$ and*

$$j \equiv l \mod \gcd(i,k).$$

Proof. It is clear that the exponent of s does not change if we do a link homotopy of a based link map, so we must show that this is independent of our choice of basing path. Notice that a change of basing paths does the following transformation:

$$y^i s^j z^k \mapsto y^i s^{j+(b-c)i+(f-a)k} z^k$$

Hence, taking j modulo $\gcd(i,k)$ we have a number independent of our choice of basing path, which proves the proposition. $\square$

Similar invariants can be derived by looking at the power of the t and u terms in σ_y and σ_z respectively. These invariants are similar to Milnor's triple linking number introduced in [Mil54], with the only significant difference being that there is no obvious sense in which we have symmetry relations. However, this invariant requires us to throw away information of the other intersections. We will now construct a similar invariant but which keeps track of all changes to the intersections.

Let n_1, n_2 and n_3 non-negative integers with $n = 2n_1 + 2n_2 + 2n_3$. Let $\mathcal{S}_{n_k}$ be the symmetric group on n_k variables when n_k is non-zero. We denote $\mathcal{S}_{n_1} \times \mathcal{S}_{n_2} \times \mathcal{S}_{n_3}$ by $\mathcal{S}_{(n_1,n_2,n_3)}$. This group has an action on $\mathbb{Z}^n = (\mathbb{Z}^2)^{n_1} \times (\mathbb{Z}^2)^{n_2} \times (\mathbb{Z}^2)^{n_3}$ given by permuting components.

We can think of the action of $\mathcal{S}_{(n_1,n_2,n_3)}$ as being represented by a group of matrices $P_{(n_1,n_2,n_3)}$.

We define

$$A_k(\mathbb{Z}^n) := \{x + N \mid x \in \mathbb{Z}^n \text{ and } N \text{ is a submodule of } \mathbb{Z}^n \text{ of rank } k\}.$$

Definition 6.3.3. Take the quotient space

$$\mathcal{A}_{(n_1,n_2,n_3)} := \coprod_{k=0}^{3} A_k\left(\mathbb{Z}^n\right)/P_{(n_1,n_2,n_3)},$$

and define

$$\mathcal{A} := \coprod_{(n_1,n_2,n_3)\in\mathbb{Z}^3_{\geq 0}} \mathcal{A}_{(n_1,n_2,n_3)}.$$

We now wish to construct a map

$$\mu : K \to \mathcal{A}$$

which descends to a well-defined map

$$\overline{\mu} : \widetilde{K} \to \mathcal{A}.$$

Let

$$v = \left(\sum_{(i,j,k)\in\mathbb{Z}^3} a_{ijk} y^i s^j z^k, \sum_{(i',j',k')\in\mathbb{Z}^3} b_{i'j'k'} z^{i'} t^{j'} x^{k'}, \sum_{(i'',j'',k'')\in\mathbb{Z}^3} c_{i''j''k''} x^{i''} u^{j''} y^{k''} \right)$$

and $n_1 = |A_v|$, $n_2 = |B_v|$ and $n_3 = |C_v|$. Additionally, place an ordering on the elements of A_v, B_v, and C_v. We write the lth element of A_v, where $1 \leq l \leq n_1$, as $a_{i_l j_l k_l}$ corresponding to the group element $y^{i_l} s^{j_l} z^{k_l}$. We write the lth element of B_v and lth element of C_v similarly. We now define $\mu(v)$ to be the set of vectors that satisfy

$$\begin{aligned}\sum_{l=1}^{n_1}(j_l + q_1 k_l + q_2 i_l)e_{2l-1} + \sum_{l=1}^{n_1}(a_{i_l j_l k_l})e_{2l} \\ + \sum_{l=1}^{n_2}\left(j'_l + q_2 k'_l + q_3 i'_l\right) e_{n_1+2l-1} + \sum_{l=1}^{n_2}(b_{i'_l j'_l k'_l})e_{2l} \\ + \sum_{l=1}^{n_3}\left(j''_l + q_1 i''_l + q_3 k''_l\right) e_{n_1+n_2+2l-1} + \sum_{l=1}^{n_3}(c_{i''_l j''_l k''_l})e_{2l} \in \mathcal{A}.\end{aligned}$$

where $q_1, q_2, q_3 \in \mathbb{Z}$ and e_l to the vector which has all zeros except for a 1 in the lth position.

Proposition 6.3.4. *The map $\mu : K \to \mathcal{A}$ is well defined and descends to a map on the quotient $\overline{\mu} : \widetilde{K} \to \mathcal{A}$.*

Proof. Indeterminacy arises from our choice of ordering on the intersections. However, this is accounted for by the choice of quotients used to construct $\mathcal{A}$.

The submodule associated to the affine space defined by μ is dependent on the powers of the non-commutator generators of each group element which is unchanged by the action of $\mathbb{Z}^6$, which defines $\widetilde{K}$. Furthermore, the vector given by the powers of the commutator terms all lie in the same affine space so when considering different representatives we assign the same affine space. □

This invariant is similar to the total Milnor quotient in [DNOP20] by Davis, Nagel, Powell and Orson. They made use of exterior algebra since Milnor's $\bar{\mu}$ invariants obey some relations by permuting elements. We do not have these symmetry relations, as our objects that are "triple linking" are a closed loop and two immersed two-spheres.

The map $\overline{\mu}$ is clearly stronger than our previous invariant which considered the power of the commutator term of the intersection which is shown in the following example.

Example 6.3.5. Let f be the link map from Theorem 6.2.1 and g be a similar link map which can be represented by a similar movie but the band travels around the second component twice. We thus have

$$\widetilde{\sigma}^3(f) = \Big(z(s-1) + z^{-1}(s^{-1}-1),\, 0,\, x(1-u) + x^{-1}(1-u^{-1})\Big)$$

and

$$\widetilde{\sigma}^3(g) = \Big(z(s^2-1) + z^{-1}(s^{-2}-1),\, 0,\, x(1-u^2) + x^{-1}(1-u^{-2})\Big).$$

Computing $\overline{\mu}$ for each we get

$$\overline{\mu}\left(\sigma^3(f)\right) = \Big(1+q_1,\, 1,\, q_1,\, -1,\, -1-q_1,\, 1,\, -q_1,\, -1,$$
$$1+q_1,\, -1,\, q_1,\, 1,\, -1-q_1,\, -1,\, -q_1,\, 1\Big)$$

and

$$\overline{\mu}\left(\sigma^3(f)\right) = \Big(2+q_1,\, 1,\, q_1,\, -1,\, -2-q_1,\, 1,\, -q_1,\, -1,$$
$$2+q_1,\, -1,\, q_1,\, 1,\, -2-q_1,\, -1,\, -q_1,\, 1\Big),$$

for some choice of ordering of the intersections.

Each of these affine lines has a unique real affine line in $\mathbb{R}^8$ which intersects every point in the affine subspace. Computing the distance of these "completed" affine lines from the origin we get 10 and 16 respectively. The actions of $P_{4,0,4}$ are orthogonal so both distances are unchanged by the action of $P_{4,0,4}$.

If we post compose σ^3 with $\overline{\mu}$ we have a map from $\mathrm{LM}^4_{2,2,2} \to \mathcal{A}$, we will often refer to this map $\overline{\mu}$ as it will be clear from context which domain we are considering.

Proposition 6.3.6. *Let $f \in \mathrm{LM}^4_{2,2,2}$ such that the representative affine space $\overline{\mu}(f)$ is a single vector. Then $f \in \mathrm{Bl}^4_{2,2,2}$.*

Proof. If $\overline{\mu}(f)$ is represented by a single vector then, in $\sigma^3(f)$, each component is given by a Laurent polynomial in the commutator terms, s, t and u. Remove a component to get a two-component link map. Using Proposition 6.1.14 three times to compute the Kirk invariant and using the injectivity of the Kirk invariant proves the result. □

For our final invariant on $\widetilde{K}$ we seek to place a notion of size of elements of our invariant.

Definition 6.3.7. Let $A \subset \mathbb{Z}^3$ be finite and $d : \mathbb{Z}^3 \times \mathbb{Z}^3 \to \mathbb{R}$ be a metric. We call

$$D_d(A) := \max_{(a,b)\in A\times A} d(a,b),$$

the *diameter* of A.

Take a representative of $\sigma^3(f)$, we define the following subset of $\mathbb{Z}^3$

$$A'_f := \left\{(i,j,k) \in \mathbb{Z}^3 | a_{ijk} \neq 0\right\},$$

We can define the sets B'_f and C'_f analogously. The sets A'_f and B'_f and C'_f are contained within equivalence classes of subsets of $\mathbb{Z}^3$, which we will denote by $\left[A'_f\right]$, $\left[B'_f\right]$ and $\left[C'_f\right]$, where the equivalences classes are given by the action of $\mathbb{Z}^6$ on $\mathbb{Z}\Gamma_i$.

Definition 6.3.8. Let $f \in K$ and $d : \mathbb{Z}^3 \times \mathbb{Z}^3 \to \mathbb{R}$ be a metric. We define

$$W\left(f\right) := \min_{f \in [f]} \left(D_d\left(A'_f\right) + D_d\left(B'_f\right) + D_d\left(C'_f\right)\right),$$

to be the *width of f with respect to d.*

Chapter 7

Constructing new link maps

7.1 Annular link maps

Our inspiration for Theorem 6.2.1 was to imagine taking the connect sum of two-component link maps but around one of the tubes place an unknotted sphere such that the tube links this sphere. Our goal is to formalise this idea and provide a formula for $\widetilde{\sigma}^3$ for three-component link maps constructed via this method. We now introduce some formalism.

Let $D_1, \ldots D_n$ be disjoint embedded discs in B^3 with boundary C_i such that $C_i \cap \partial B^3 = \emptyset$ for all i. Additionally have all C_i lie inside $\mathbb{R}^2 \times \{0\} \cap B^3$ and are not nested in the plane.

Definition 7.1.1. A *n-component* 2*-string link* is a smooth/topologically flat proper embedding of

$$\coprod_{i=1}^{n} S^1 \times [0,1] \to B^3 \times [0,1] \,,$$

such that the image of each annulus is bounded by $C_i \times \{0\}$ and $C_i \times \{1\}$, with compatible orientation. An *Annular link map* is defined similarly but we allow this map to be an immersion with self-intersections, on the same component, in the interior of each annulus.

We consider annular link maps up to link homotopy i.e. a homotopy through annular link map. Denote the set of three-component annular up to link homotopy by $\mathrm{ALM}^4_{2,2,2}$. Let $\mathrm{EASL}^4_{2,2,2} \subset \mathrm{ALM}^4_{2,2,2}$ be the subgroup of three-component annular link maps which are link homotopic to a topologically flat embedded annular link map. An embedded annular link map which maps each component as $(p,t) \mapsto (f_t(p), t)$, where each f_t is an embedding for all $t \in I$, is called a *pure braid* .

We construct a link map from an annular link map by taking $B^3 \times I$ and gluing along another $B^3 \times I$ along $S^2 \times I$ giving $S^3 \times I = B^3 \times I \cup_{S^2 \times I} B^3 \times I$. We now cap off both ends of $S^3 \times I$ with D^4 and cap off each end of the annulus with a slice disc which is link homotopic to the collection of disjoint D_i, giving rise to a map $\mathrm{ALM}^4_{2,2,2} \to \mathrm{LM}^4_{2,2,2}$. If $X \in \mathrm{ALM}^4_{2,2,2}$ denote its *link map closure* by $f_X \in \mathrm{LM}^4_{2,2,2}$.

The set of annular link maps can be equipped with a multiplication. Suppose we have two annular link maps $X, X' : \coprod_{i=1}^n S^1 \times [0,1] \to B^3 \times [0,1]$. We define

$$(X \cdot X')\,(p,t) = \begin{cases} X(p, 2t) & 0 \leq t \leq \frac{1}{2} \\ X'(p, 2t-1) & \frac{1}{2} < t \leq 1. \end{cases}$$

If we consider the set of n-component annular link maps up to link homotopy then $\mathrm{ALM}^4_{2,2}$ becomes a group with multiplication given by

$$[X] \cdot [X'] := [X \cdot X']\,.$$

The inverse is given by considering the involution $r : B^3 \times [0,1] \to B^3 \times [0,1]$, with

$$r(p,t) = (p, 1-t)$$

and also let $Y : \coprod_{i=1}^n S^1 \times [0,1] \to B^3 \times [0,1]$ with

$$Y(p,t) = X(p, 1-t).$$

We then have a annular link map $r \circ Y : \coprod_{i=1}^n S^1 \times [0,1] \to B^3 \times [0,1]$ for which

$$[r \circ Y] \cdot [X] = [X] \cdot [r \circ Y] = [O]\,,$$

Figure 7.1: An example of an embedded annular link map.

where O is the trivial annular link map. The proof that this gives an inverse can be found in [MY21]. We can show that this multiplication is non-abelian using the σ^3 invariant. We first define the following. For a annular link map X we will denote its inverse by $\overline{X}$.

Definition 7.1.2. Let B be a three-ball contained inside $\mathring{B}^3$. Let X be a three-component annular link map with one of the components contained in a $B \times I \subset B^3 \times I$ where the other components do not intersect $B \times I$ and the image of this component is a $C_i \times I$ for some $i \in \{1, 2, 3\}$. The other two components are the image of a JK construction which has not been capped off. Then we call X a *three-component JK construction*.

Proposition 7.1.3 ([MY21])**.** *The group of annular link maps up to link homotopy is non-abelian if $n \geq 3$.*

Proof. Consider the three-component JK construction inside $B^3 \times I$ based on the JK construction using the Whitehead link, where the second component is trivial. Denote this by X, and similarly denote its inverse in $\mathrm{ALM}^4_{2,2,2}$ by $\overline{X}$. Let J be the annular link map described by Figure 7.1, inside the $B^3 \times I$. Consider the stacking $XJ\overline{X}$, with link map closure $f_{XJ\overline{X}}$. We have

$$\sigma^3(f_{XJ\overline{X}}) = (z(s-1) + \bar{z}(\bar{s}-1),\, 0,\, x(1-u) + \bar{x}(1-\bar{u}))\,,$$

as it is the same link map as in Theorem 6.2.1. Consider the link map closure $f_{X\overline{X}J}$. Notice that we can retract the tubes given by J so that this is the same link map given by the closure of $X\overline{X}$. This gives a trivial link map [MY21] and thus $XJ\overline{X}$ and $X\bar{X}J$ are not equivalent. $\square$

Definition 7.1.4. Let $L \subset B^3$ be an oriented n-component link of circles. The *group of singular concordances of* L is the set of flat topological immersions

$$\coprod_{i=1}^{n} S^1 \times I \to B^3 \times I,$$

which is a link map such that image of the boundary each annulus is $L \times \{0\}$ and $L \times \{1\}$ with compatible orientation and the only self-intersections are in the interior of each component, up to link homotopy. We denote this group by $C\,(L)$ and the group operation is given by stacking the concordances.

To prove this has a group structure is similar to showing that the group of annular link maps also has a group structure.

Proposition 7.1.5. *Let L and L' be concordant links in B^3. Then the groups $C(L)$ and $C(L')$ are isomorphic.*

Proof. A similar proof is given in [MY21]. Let Y be a concordance between L' and L and let $X \in C\,(L)$ and $X' \in C\,(L')$. We have two homomorphisms $\xi : C(L) \to C(L')$, $\xi' : C(L') \to C(L)$ defined by

$$\xi(X) = YX\overline{Y}$$

and

$$\xi'(X') = \overline{Y}X'Y.$$

We have

$$\xi' \circ \xi(X) = \overline{Y}YX\overline{Y}Y \sim_{lh} X.$$

Hence, $\xi' \circ \xi = \mathrm{Id}_{C(L)}$. Similarly, $\xi \circ \xi' = \mathrm{Id}_{C(L')}$. □

Corollary 7.1.6 ([MY21])**.** *Let L be a n-component slice link. The group of singular concordances of L up to link homotopy is non-abelian for $n \geq 3$.*

Proof. By the previous proposition, the group of (singular) concordances of a slice link up to link homotopy is isomorphic to the group of annular link map. Hence, we only have to show that the group of annular link maps isn't abelian for $n \geq 3$. However, this is shown in Proposition 7.1.3. □

7.1.1 A three-component annular link map invariant

Both Proposition 7.1.3 and Corollary 7.1.6 were proven in [MY21] by showing that the subgroup of embedded annular link maps is non-abelian, using the longitudes of each component; whereas we studied the self-intersections of annular link maps.

We now create an invariant of annular link maps which incorporates the longitude information but also records self-intersection information similar to our invariant for link maps.

First we specify a choice of basing path for each annular link map with three components. Take the basepoint to be $(s_0, 0) \in B^3 \times I$ and let p_i be a point in a tubular neighbourhood of the ith component of the unlink O_3 inside $B^3 \times \{0\}$. We define a meridian for the ith component by taking a path from $(s_0, 0)$ to p_i with the curve lying only inside $B^3 \times \{0\}$ and disjoint from the discs D_i. Finally, travel along a small loop around the ith component, such that the linking number is positive, and return along the path which took you to p_i. We call the meridians x, y and z for the first second and third components respectively. The basing curves for the ith component is given by taking the same path to p_i for the meridians and then travelling radially down the fibre of the normal bundle to the ith component.

Definition 7.1.7. We call such a collection of meridians and basing curves a *basing system* for a annular link map.

We defined basing systems for annular link map above in terms of the unlink in the $B^3 \times \{0\}$ time slice. However, if $B^3 \times \{t\} \cap X = O_3 \times \{t\}$ at $t \in I$ we will say there is a basing system there too, where we take the basepoint to be (s_0, t) for these other basing systems. Hence, we will often talk about a *basing system at* $t \in I$.

Annular link maps specify an automorphism of the lower central series quotients of the fundamental group of the complement of the unlink. To prove this we will first prove the following Lemma.

Lemma 7.1.8. *Let X be the image of a generic n-component annular link map inside $B^3 \times I$. Then $M\pi_1(B^3 \times I \setminus X)$ is isomorphic to the Milnor free group on n-generators.*

Proof. We can apply finger moves to the annular link map such that $\pi_1(B^3 \times I \setminus X) = M\pi_1(B^3 \times I \setminus X)$, which are normally generated by meridians of the unlink in $B^3 \times \{1\}$. Using Seifert-Van Kampen we have

$$\pi_1\left(S^3 \times I \setminus X\right) = \pi_1\left(B^3 \times I \setminus X\right) * \pi_1\left(B^3 \times I\right) = \pi_1\left(B^3 \times I \setminus X\right),$$

where $S^3 \times I \setminus X = B^3 \times I \setminus X \cup_{S^2 \times I} B^3 \times I$. Capping off $S^3 \times \{1\}$ with D^4 and choosing slice discs for each component which are link homotopic to the standard embedding of discs bounded by the n-component unlink call these discs Δ.

This gives a collection of singular but disjoint slice discs for the unlink in D^4. We call this collection Δ'. By Seifert-Van Kampen, we have

$$\pi_1\left(D^4 \setminus \Delta'\right) = \pi_1\left(S^3 \times I \setminus X\right) *_{F(n)} \pi_1(D^4 \setminus \Delta),$$

Since $\pi_1(D^4 \setminus \Delta) \cong F(n)$ where the generators of $F(n)$ is the choice of meridians of the unlink we made previously. However, the amalgamation identifies the generators of $\pi_1(S^3 \times I \setminus X)$ and $\pi_1(D^4 \setminus \Delta)$, Hence

$$\pi_1\left(B^3 \times I \setminus X\right) \cong \pi_1\left(D^4 \setminus \Delta'\right)$$

By a result in [FT95], we have that $M\pi_1(D^4 \setminus \Delta') \cong MF(n)$. This proves the result. □

Lemma 7.1.9. *Let X be a Annular link map. Then the inclusion maps*

$$j_i : \left(B^3 \times \{i\} \setminus O_3, (s_0, i)\right) \to \left(B^3 \times I \setminus X, (s_0, i)\right),$$

where $i = 0, 1$, induces an isomorphism on the Milnor groups of the fundamental groups of each space.

Proof. By Lemma 7.1.8, we have that the Milnor group of the complement of X

inside $B^3 \times I$ is the Milnor free group on three-generators. It is clear that since we are sending generators to generators and both are the free Milnor object of the same number of generators we have an isomorphism. $\square$

We define

$$\begin{aligned}
\pi_{O_3} &:= \pi_1\left(B^3 \times \{0\} \smallsetminus O_3, (s_0, 0)\right), \\
\pi_{O_3'} &:= \pi_1\left(B^3 \times \{1\} \smallsetminus O_3, (s_0, 1)\right), \\
\pi_X^i &:= \pi_1\left(B^3 \times I \smallsetminus X, (s_0, i)\right).
\end{aligned}$$

By Lemma 7.1.9

$$\begin{aligned}
(j_0)_* &: M\pi_{O_3} \to M\pi_C^0 \\
(j_1)_* &: M\pi_{O_3'} \to M\pi_C^1
\end{aligned}$$

are isomorphisms. Let $\alpha : [0,1] \to (B^3 \times I \smallsetminus X)$ be given by

$$\alpha(t) = (s_0, t).$$

Let $\psi_\alpha : \pi_X^0 \to \pi_X^1$ be the isomorphism defined by

$$\gamma \mapsto \bar{\alpha} \cdot \gamma \cdot \alpha.$$

Hence, we have two isomorphisms

$$\begin{aligned}
(j_1)_*^{-1} \circ \psi_\alpha \circ (j_0)_* &: M\pi_{O_3} \to M\pi_{O_3'}, \\
(j_0)_*^{-1} \circ \psi_\alpha^{-1} \circ (j_1)_* &: M\pi_{O_3'} \to M\pi_{O_3}.
\end{aligned}$$

A more geometric description of the content of these isomorphisms is to consider the basing system for the annular link map lying in the $t = 0$ and another basing system lying in the $t = 1$ slice. Let the m_i and the m_i' be meridians for the ith component at each time slice $t = 0$ and $t = 1$ respectively. We can map

$$m_i' \mapsto (j_0)_*^{-1} \circ \psi_\alpha^{-1} \circ (j_1)_*(m_i') = g_i^{-1} m_i g_i \in M\pi_{O_3}$$

for some $g_i \in M\pi_{0_3}$. This g_i is represented by a longitude of the ith component of the annular link map which we can decompose in terms of the meridians of the basing system at $t = 0$.

Definition 7.1.10. Let X be an annular link map with a basing system at both $t = 0$ and $t = 1$. Then the *ith longitude* is a loop based at $(s_0, 0)$ and is defined by concatenating the following paths:

- use the basing system at $t = 0$ travel to from $(s_0, 0)$ to $(p_i, 0)$.

- Next, take a path in a regular neighbourhood of the ith component, which at no point lies in the fibre above a double point, and travels between $(p_i, 0)$ to $(p_i, 1)$.

- Use the basing system at $t = 1$ to travel from $(p_i, 1)$ to $(s_0, 1)$.

- Finally, travel along the path $\overline{\alpha} : [0, 1] \to B^3 \times I$ defined by $\overline{\alpha}(t) = (s_0, 1 - t)$.

Once we have fixed a choice of basing path and meridians at either end of the cylinder there are, up to homotopy, $\mathbb{Z}^2$ many choices of longitude of the ith component, determined by the linking number of the longitude with the ith component and how much "wraps around" the annulus. To deal with this indeterminacy we always choose longitudes with this linking number equal to zero. The indeterminacy given by wrapping round the cylinder does not affect the end result up to homotopy, as we have an unlink in the boundary.

Remark 7.1.11. For three-component JK constructions we will always assume that the longitudes of the ith component is trivial in Γ_i.

Let $\mathrm{Aut}(MF(3))$ be the group of automorphisms of the free Milnor group on three generators, with the usual group structure given by composition of functions. Recall that $K = \mathbb{Z}\Gamma_x \times \mathbb{Z}\Gamma_y \times \mathbb{Z}\Gamma_z$. We will define a map $\Phi : \mathrm{Aut}(MF(3)) \to \mathrm{Aut}(K)$ where $\mathrm{Aut}(K)$ is the group of $\mathbb{Z}$ linear ring automorphisms of K. Notice that for $\psi \in \mathrm{Aut}(MF(3))$ there exists a map $\psi_i : \Gamma_i \to \Gamma_i$ such that

$$\begin{array}{ccc} MF(3) & \xrightarrow{\psi} & MF(3) \\ \big\downarrow & & \big\downarrow \\ \Gamma_i & \xrightarrow{\psi_i} & \Gamma_i \end{array}$$

commutes for each $i \in \{x, y, z\}$. We use the same notation for these maps to denote extension to the corresponding group ring.

Evaluating ψ on each of the generators we have

$$\psi(x) = \tau_x^{-1} x \tau_x,$$
$$\psi(y) = \tau_y^{-1} y \tau_y, \text{ and}$$
$$\psi(z) = \tau_z^{-1} z \tau_z.$$

Denote the image of τ_i in the quotient group by $[\tau_i] \in \Gamma_i$.

Given $(a, b, c) \in K$, we define $\Phi : \mathrm{Aut}(MF(3)) \to \mathrm{Aut}(K)$ to be

$$\Phi(\psi)(a, b, c) = \left([\tau_x] \, \psi_x(a) \, [\tau_x]^{-1}, [\tau_y] \, \psi_y(b) \, [\tau_y]^{-1}, [\tau_z] \, \psi_z(c) \, [\tau_z]^{-1} \right).$$

Lemma 7.1.12. *The map $\Phi : \mathrm{Aut}(MF(3)) \to \mathrm{Aut}(K)$ is a group homomorphism.*

Proof. Let $\alpha, \beta \in \mathrm{Aut}\, MF(3)$. We wish to show that that $\Phi(\beta \circ \alpha) = \Phi(\beta) \circ \Phi(\alpha)$. We write

$$\Phi(\alpha)(a, b, c) = \left([\tau_x] \, \alpha_x(a) \, [\tau_x]^{-1}, [\tau_y] \, \alpha_y(b) \, [\tau_y]^{-1}, [\tau_z] \, \alpha_z(c) \, [\tau_z]^{-1} \right)$$

and

$$\Phi(\beta)(a, b, c) = \left([\upsilon_x] \, \beta_x(a) \, [\upsilon_x]^{-1}, [\upsilon_y] \, \beta_y(b) \, [\upsilon_y]^{-1}, [\upsilon_z] \, \beta_z(c) \, [\upsilon_z]^{-1} \right).$$

We calculate

$$\begin{aligned} \beta \circ \alpha(x) &= \beta\left(\tau_x^{-1} x \tau_x \right) \\ &= \beta(\tau_x^{-1}) \beta(x) \beta(\tau_x) \end{aligned}$$

$$= \beta(\tau_x^{-1})\upsilon_x^{-1} x \upsilon_x \beta(\tau_x),$$

Defining similarly on y and z, we have

$$\Phi(\beta\circ\alpha)(a,b,c) = \Big([\upsilon_x]\,\beta_x([\tau_x])\,(\beta\circ\alpha)_x\,(a)\,\beta_x([\tau_x]^{-1})\,[\upsilon_x]^{-1},$$
$$[\upsilon_y]\,\beta_y([\tau_y])\,(\beta\circ\alpha)_y\,(b)\,\beta_y([\tau_y]^{-1})\,[\upsilon_y]^{-1},\ [\upsilon_z]\,\beta_z([\tau_z])\,(\beta\circ\alpha)_z\,(a)\,\beta_z([\tau_z]^{-1})\,[\upsilon_z]^{-1}\Big)$$
$$= \Big([\upsilon_x]\,\beta_x([\tau_x])\beta_x\left(\alpha_x\,(a)\right)\beta_x([\tau_x]^{-1})\,[\upsilon_x]^{-1},$$
$$[\upsilon_y]\,\beta_y([\tau_y])\beta_y\left(\alpha_y\,(a)\right)\beta_y([\tau_y]^{-1})\,[\upsilon_y]^{-1},\ [\upsilon_z]\,\beta_z([\tau_z])\beta_z\left(\alpha_z\,(a)\right)\beta_z([\tau_z]^{-1})\,[\upsilon_z]^{-1}\Big)$$

As $(\beta\circ\alpha)_i = \beta_i\circ\alpha_i$ we have

$$\Phi(\beta\circ\alpha)(a,b,c) = \Phi(\beta)\left(\left([\tau_x]\,\alpha_x(a)\,[\tau_x]^{-1},[\tau_y]\,\alpha_y(b)\,[\tau_y]^{-1},[\tau_z]\,\alpha_z\,(c)\,[\tau_z]^{-1}\right)\right)$$
$$= \Phi(\beta)\circ\Phi(\alpha)\left((a,b,c)\right),$$

as required. $\square$

Define

$$L := K \rtimes \mathrm{Aut}(MF(3))$$

where we treat K an abelian group given by addition. The group multiplication in L is defined to be

$$(x_1,y_1)\,(x_2,y_2) := (x_1 + \Phi(y_1)\,(x_2)\,,y_1\circ y_2)\,,$$

a semi-product, where Φ determines the action of $MF(3)$ on K.

A link homotopy invariant for annular link maps is given by the following: let X be a annular link map in $B^3\times I$, let X_x, X_y and X_z be the x, y and z components respectively. Define

$$\Theta_x(X) = M\lambda\,(X_x,X_x)\in\mathbb{Z}\Gamma_x$$

where the normal push-off which is used keeps the boundary of X_x inside $\mathring{B}^3\times\{0,1\}$, and the interior of X_x inside $\mathring{B}^3\times\mathring{I}$, and we use the basing system at $B^3\times\{0\}$ to compute $\lambda(X_x, X_y)$ and to specify an isomorphism to $\mathbb{Z}\Gamma_x$. We define $\Theta_y(X)$ and $\Theta_z(X)$ similarly. Let $\eta(X)$ be the element of $\mathrm{Aut}(MF(3))$ given by considering the isomorphisms $M\pi_1(B^3 \times \{1\} \smallsetminus O_3, (s_0, 1)) \to M\pi_1(B^3 \times \{0\} \smallsetminus O_3, (s_0, 0))$ given by X.

Definition 7.1.13. Let X be an annular link map. We define

$$\Theta(X) := ((\Theta_x(X), \Theta_y(X), \Theta_z(X)), \eta(X)) \in L.$$

Proposition 7.1.14. *Let X and X' be link homotopic annular link maps. Then*

$$\Theta(X) = \Theta(X').$$

Proof. The proof for the factors involving Θ_x, Θ_y and Θ_z are the same as for the case of three-component link maps. We must show that the longitudes are unaffected by link homotopy. However, this is clear because the link homotopy is a concatenations of isotopy, cusps, Whitney moves and finger moves. These moves do not affect the longitude. This completes the proof. □

Theorem 7.1.15. *The map*

$$\Theta : \mathrm{ALM}^4_{2,2,2} \to L,$$

is a group homomorphism.

Proof. We must show that

$$\Theta(X \cdot X') = \Big((\Theta_x(X), \Theta_y(X), \Theta_z(X)) + \Phi(\eta(X)) (\Theta_x(X'), \Theta_y(X'), \Theta_z(X')), \\ \eta(X) \circ \eta(X') \Big).$$

We will first show that the stacking operation on annular link maps corresponds to composition in $\mathrm{Aut}(MF(3))$. Let $g \in \pi_1(B^3 \times \{1\} \smallsetminus O_3) \cong MF(3)$ and let $\alpha_0, \alpha_{\frac{1}{2}} : I \to B^3 \times I \smallsetminus (X \cdot X')$ where

$$\alpha_0(t) = \left(s_0, \frac{t}{2}\right)$$

and

$$\alpha_{\frac{1}{2}}(t) = \left(s_0, \frac{1+t}{2}\right).$$

Rebase g to get $\alpha_{\frac{1}{2}} \cdot g \cdot \overline{\alpha}_{\frac{1}{2}}$ and think of it as an element of $\pi_1\left(B^3 \times \left[\frac{1}{2}, 1\right] \smallsetminus X', \left(s_0, \frac{1}{2}\right)\right)$. We then pull this back using the induced map on the Milnor group of the fundamental groups given by $B^3 \times \{\frac{1}{2}\} \smallsetminus O_3 \hookrightarrow B^3 \times I \smallsetminus X$. This gives us an element of $MF(3) \cong M\pi_1\left(B^3 \times \{\frac{1}{2}\}, \left(s_0, \frac{1}{2}\right)\right)$ and equal to $\eta(X')(g)$. We follow a similar argument rebasing using α_0 so that the induced map from the stacking is $\eta(X) \circ \eta(X')$, as required.

We need to check the effect of stacking on the self-intersection information. We may assume that the intersections on X and X' have cancelling sign. The intersection on the X region is not affected since those intersections will use the same meridians and basing paths to decompose the group elements and thus their contributions are not affected. We must check that the values of the intersections in the X' region. First notice that the $\Theta_x(X'), \Theta_y(X'), \Theta_z(X')$ are equal to the sum of the intersections on the X' region using the basing system at $B^3 \times \{\frac{1}{2}\}$. We now must rebase and describe the group elements using the basing system at $B^3 \times \{0\}$. To calculate $\Theta_x(X')$ intersections using the meridians of the basing system at $B^3 \times \{0\}$ we apply the automorphism on Γ_x induced by $\eta(X)$. We then must take into account our choice of basing path. However, this is just conjugation by the longitude of the x component of X as required. The proofs for the y and z components are analogous. □

From the construction of Θ we have the following result.

Theorem 7.1.16. *The following diagram commutes*

$$\begin{tikzcd}
\mathrm{ALM}^4_{2,2,2} \arrow[r, two heads] \arrow[d, "\Theta"] & \mathrm{LM}^4_{2,2,2} \arrow[r, two heads, "i"] \arrow[d, "\widetilde{\sigma}^3"] & \mathrm{LM}^4_{2,2} \arrow[d, "\sigma"] \\
L \arrow[r] & \widetilde{K} \arrow[r, "p_i"] & (\mathbb{Z}[\mathbb{Z}])^2 .
\end{tikzcd} \tag{7.1.1}$$

Proposition 7.1.17. *Let X be an embedded three-component annular link map such that $\Theta(X) = (a, \phi)$ where $a \neq 0$. Then X is not link homotopic to a topologically flat embedding.*

7.1.2 Describing link maps

We now wish to construct link maps by giving a description of a annular link maps and then calculating Θ to determine the values of σ_3 and $\tilde{\sigma}_3$ after taking the closure.

Definition 7.1.18. We call an embedded annular link a *ribbon braid* if it is the trace of an isotopy in B^3, where for each time slice we have an unlink with each component lying in a $B^3 \cap \mathbb{R}^2 \times \{q\}$ for some $q \in \mathbb{R}^2$.

Remark 7.1.19. This is different from how ribbon braids are defined in [ABMW14], where the components lying in a plane parallel to each other condition is loosened.

Let $X := X_1 J_1 X_2 \ldots J_{n-1} X_n$ be a stacking of annular link maps where the X_i are three-component JK constructions and the J_i are ribbon braids. We wish to describe X in such a way which makes it easier to calculate the $\widetilde{\sigma}^3$ value of the link map closure f_X. We will draw on the work of [ABMW14] and [BH08] to do this.

Let $p_1, \ldots, p_n$ be pairwise distinct elements in $\mathring{I}$. Let $f : \coprod_{i=1}^{n} I_i \hookrightarrow I \times I$ with $I_i = [0,1]$ and $f(j) = \{p_i\} \times \{j\}$ with $j \in \partial I_i$ be a topologically flat, transverse immersion such that $f(i) = (x, i)$ for some $x \in I$. For all double points p, place a partial ordering on each elements in the pre-image. If we can say that one element in the pre-image is less than or greater than the other - using the partial ordering - then we delete a small neighbourhood of the lower point, if we cannot this is called a *welded crossing* and we place a circle around this crossing. We call this a *virtual braid diagram.*

Definition 7.1.20. We call the set of virtual braid diagrams, up to standard Reidmeister moves; virtual Reidemeister moves described in Figures 7.3, 7.4, 7.5, 7.6; and the over crossing move shown in Figure 7.7, *welded braids.*

We can stack welded braids together to produce new welded braids. This gives a group structure on the set of welded braids. Denote this group $\mathfrak{W}$. There is a well defined map from $\mathfrak{W}$ to ribbon braids called the tube map which is discussed

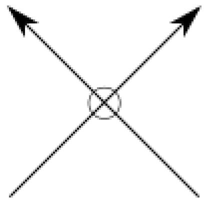

Figure 7.2: A welded crossing

Figure 7.3: The first virtual Reidemeister move.

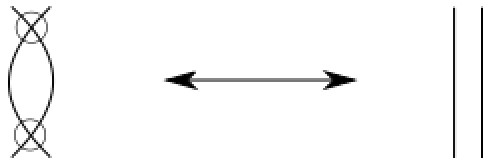

Figure 7.4: The second virtual Reidemeister move.

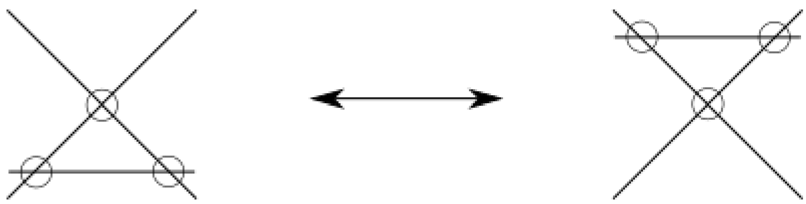

Figure 7.5: The third virtual Reidemeister move.

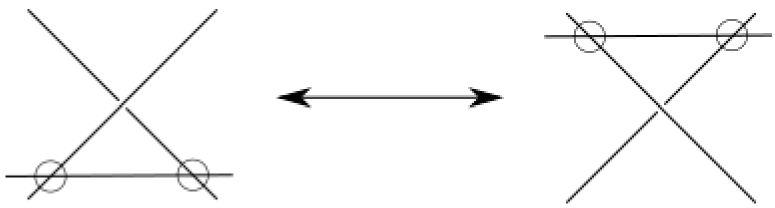

Figure 7.6: The fourth virtual Reidemeister move.

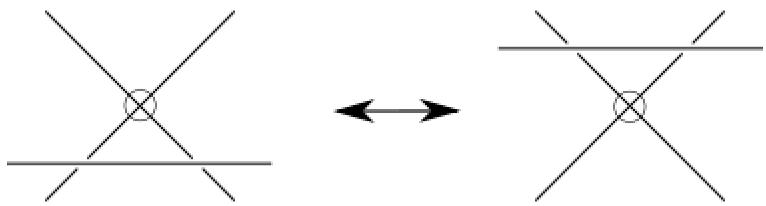

Figure 7.7: Overcrossing on welded braids.

in [ABMW14] and due to the results of Brendle and Hatcher [BH08] this is an isomorphism. Hence we can represent ribbon braids by a welded braid diagram.

Definition 7.1.21. Let L be a welded braid, where each strand is oriented from $I \times \{0\}$ to $I \times \{1\}$, an *overstrand* is a piece of arc with each endpoint being either the boundary or the underpass of a classical crossing, with no classical crossing in the interior of this piece of the arc. Let $O(L)$ be the set of overstrands of L and let $C(L)$ be the set of classical crossings of L. For $c \in C(L)$ we define the following:

- ϵ_c is the sign of the crossing at c
- s_c^0 is the overstrand containing the highest pre-image at the crossing,
- s_c^- is the overstrand whose boundary element is the lowest preimage of c and the orientation of the strand point into the crossing,
- s_c^+ is the overstrand whose boundary element is the lowest preimage of c and the orientation of the strand point out of the crossing.

The fundamental group of L is defined to be

$$\pi_1(L) := \left\langle O(L) \mid s_c^+ = \left(s_c^-\right)^{\left(s_c^0\right)^{\epsilon_c}} \forall c \in C(L) \right\rangle.$$

Brendle and Hatcher, in [BH08], showed that the tube map provides the following isomorphism

$$\pi_1(L) \cong \pi_1\left(B^3 \times I \setminus \mathrm{Tube}(L)\right).$$

In fact, there is a one-to-one correspondence between the generators and relations, coming from classical crossing of the welded braid and Wirtinger presentation given by a broken surface diagram for the image of L under the tube map [1]. Thus, we can use welded braids to compute longitudes of elements and compute the isomorphisms of $\mathrm{Aut}(MF(3)$ induced by a ribbon braid, which makes constructing examples easier.

[1]A comprehensive account of broken surface diagrams can be found in [ABMW14].

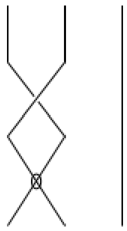

Figure 7.8: A welded braid description of the ribbon braid described in 7.1, where the x-component is starts and ends on the left, the y-component starts and ends in the centre, and the z-component is on the right.

It follows that we can use these diagrams to specify any automorphisms of $MF(3)$ using results from [ABMW14].

Example 7.1.22. Consider the welded braid as in Figure 7.8. Then the longitude of the x-component is overstrand of the y component. Hence the element $\psi \in \operatorname{Aut}(MF(3))$ described by this welded braid is given by

$$\psi(x) = y^{-1}xy,$$

where $\psi(y) = y$ and $\psi(z) = z$.

Definition 7.1.23. Let $f \in \mathrm{LM}^4_{2,2}$ and $g \in F/F_3$ then we define $\sigma_i^g(f)$ to be the Laurent polynomial $\sigma_i(f)$ evaluated on g.

Definition 7.1.24. Let X be a three-component annular link map and $i \in \{1, 2, 3\}$. Then X^i is a two-component annular link map which is X with the ith component removed.

Let $X := X_1 J_1 X_2 \dots J_{n-1} X_n$ be a stacking of JK constructions. Furthermore, we define:

- Let k_i and k_i' be the difference of the number of positive and negative classical crossing of a longitude of the x, with the y and z components respectively, for the Welded Braid which is mapped to the substack $J_1 \dots J_{i-1}$ under the tube map, when $i > 1$. We set $k_1 = k_1' = 0$.

- Let l_i and l'_i be the difference of the number of positive and negative classical crossing of a longitude of the y component, with the z and x components respectively, for the welded braid which is mapped to the substack $J_1 \dots J_{i-1}$ under the Tube map, when $i > 1$. We set $l_1 = l'_1 = 0$.

- Let m_i and m'_i be the difference of the number of positive and negative classical crossing of a longitude of the z component, with the x and y components respectively, for the welded braid which is mapped to the substack $J_1 \dots J_{i-1}$ under the tube map, when $i > 1$. We set $m_1 = m'_1 = 0$.

Theorem 7.1.25. *Let f_X be a link map arising from the stacking X. Then, for some choice of basing path, we have*

$$\begin{aligned}
\sigma_x(f_X) &= \sum_{i=1}^{n} z^{k'_i - l_i} \sigma_1^y\left(f_{X_i^3}\right) z^{-k'_i + l_i} + y^{k_i - m'_i} \sigma_1^z\left(f_{X_i^2}\right) y^{-k_i + m'_i}\\
\sigma_y(f_X) &= \sum_{i=1}^{n} z^{l_i - k'_i} \sigma_2^x\left(f_{X_i^3}\right) z^{-l_i + k'_i} + x^{l'_i - m_i} \sigma_1^z\left(f_{X_i^1}\right) x^{-l'_i + m_i}\\
\sigma_z(f_X) &= \sum_{i=1}^{n} y^{m'_i - k_i} \sigma_2^x\left(f_{X_i^2}\right) y^{-m'_i + k_i} + x^{m_i - l'_i} \sigma_2^y\left(f_{X_i^1}\right) x^{-m_i + l'_i}.
\end{aligned}$$

Proof. In this proof, we will use μ for our calculation instead of λ. Hence, we will assume that the sum of the signed intersections are zero.

Recall that $\eta(X_i) = \mathrm{Id}$ as X_i is a three-component JK construction and for each J_i we have $\theta_p(J_i) = 0$ for $p \in \{x, y, z\}$. We first calculate Θ and then using commutative diagram (7.1.1) we will map to the image of the link map closure under σ^3. As Θ is a homomorphism

$$\begin{aligned}
\Theta(X) = \Bigg(&(\theta_x(X_1), \theta_y(X_1), \theta_z(X_1))\\
&+ \sum_{i=2}^{n} \Phi\left(\eta\left(\Pi_{j=1}^{i-1} J_j\right)\right)(\theta_x(X_i), \theta_y(X_i), \theta_z(X_i)), \eta(J_1 \dots J_{n-1})\Bigg).
\end{aligned}$$

We now calculate $\Theta_x(X_i)$, the calculations for $\theta_y(X_i)$ and $\theta_z(X_i)$ are similar. As X_i is a three-component JK construction there are three cases depending on which component is embedded. In the case where the x component is embedded we know that $\theta_x(X_i) = 0$ and thus $\sigma_1\left(f_{X_i^3}\right) = 0$ and $\sigma_1\left(f_{X_i^2}\right) = 0$. Hence, the claim is true in this first case. The next case to consider is to suppose x has self-intersections and y is the embedded component, split from the other two components. For each intersection on the x component we associate an element of Γ_x. Since the y component is split from the rest of the components, the group elements of the intersections of the x components are powers of the z meridian coming from the basing system at $B^3 \times \{\frac{1}{2^{1+2n-2i}}\}$. Taking the sum over all the intersections of the x component and considering their values inside Γ_x/A_y and we have $\sigma_1(f_{X_i^2})$. Thus $\theta_x(X_i) = \sigma_1^z(f_{X_i^2})$. Since $\sigma_1^y(f_{X_i^3}) = 0$ we have

$$\Theta_x(X_i) = \sigma_1^z(f_{X_i^2}) + \sigma_1^y(f_{X_i^3}).$$

A similar argument shows that this equation holds in the remaining case. We now show that

$$\begin{aligned}
&\Phi(\eta(J_1 \dots J_{i-1}))\left((\theta_x(X_i), \theta_y\left(X_i\right), \theta_z(X_i))\right) \\
&\qquad = \Big(z^{k_i'-l_i}\sigma_1^y\left(f_{X_i^3}\right)z^{-k_i'+l_i} + y^{k_i-m_i'}\sigma_1^z\left(f_{X_i^2}\right)y^{-k_i+m_i'}, \\
&\qquad\quad z^{l_i-k_i'}\sigma_2^x\left(f_{X_i^3}\right)z^{-l_i+k_i'} + x^{l_i'-m_i}\sigma_1^z\left(f_{X_i^1}\right)x^{-l_i'+m_i}, \\
&\qquad\qquad y^{m_i'-k_i}\sigma_2^x\left(f_{X_i^2}\right)y^{-m_i'+k_i} + x^{m_i-l_i'}\sigma_2^y\left(f_{X_i^1}\right)x^{-m_i+l_i'}\Big).
\end{aligned}$$

Since $\Phi\left(\eta(J_1 \dots J_{i-1})\right)$ can be determined by $\tau_x^i \in \Gamma_x$, $\tau_y^i \in \Gamma_y$, $\tau_z^i \in \Gamma_z$, the longitudes of the x,y and z longitudes of $J_1 \dots J_{i-1}$. We know that as the longitude of a component welded braid which represents the $J_1 \dots J_{i-1}$ is equivalent to the longitude of the corresponding component of $J_1 \dots J_{i-1}$. Given $\tau_x = y^{k_i}z^{k_i'}s^{p_i^x}$, $\tau_y = z^{l_i}x^{l_i'}t^{p_i^y}$ and $\tau_z = x^{m_i}y^{m_i'}u^{p_i^z}$, where $p_i^x, p_i^y, p_i^z \in \mathbb{Z}$. Thus evaluating $\Phi\left(\eta(J_1 \dots J_n)\right)$ on $\theta_x(X_i)$ gives

$$\begin{aligned}
&\tau_x\sigma_1^{z^{-l_i}yz^{l_i}}(f_{X_i^3})\tau_x^{-1} + \tau_x\sigma_1^{z^{-m_i'}yz^{m_i'}}\left(f_{X_i^2}\right)\tau_x^{-1} \\
&\qquad = z^{k_i'}\sigma_1^{z^{-l_i}yz^{l_i}}(f_{X_i^3})z^{-k_i'} + y^{k_i}\sigma_1^{z^{-m_i'}yz^{m_i'}}\left(f_{X_i^2}\right)y^{-k_i} \\
&\qquad\qquad = z^{k_i'-l_i}\sigma_1^y\left(f_{X_i^3}\right)z^{-k_i'+l_i} + y^{k_i-m_i'}\sigma_1^y\left(f_{X_i^3}\right)z^{-k_i+m_i'}
\end{aligned}$$

A similar result shows the required effect on $\theta_y(X_i)$ and $\theta_z(X_i)$. Using commutative

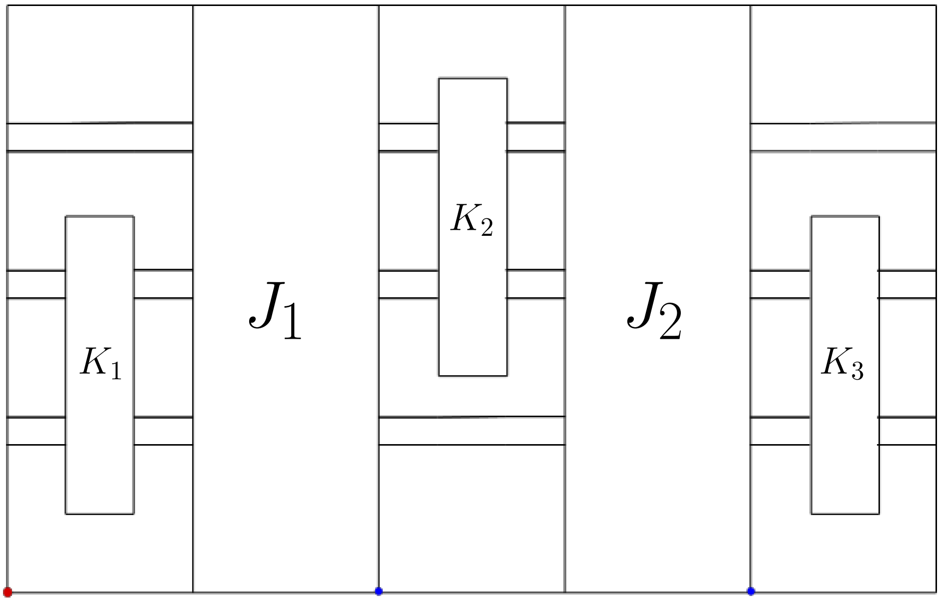

Figure 7.9: A schematic for the proof of Theorem 7.1.25 where the red point represents the basepoint we want to make our calculations from and the blue represent the basepoints we use to make our initial calculations of the intersections of K_i before we then rebase and use the isomorphism induced by the stack of singular pure braids to the left of the basepoint.

diagram (7.1.1), we have the result.

□

This result gives the image of $\widetilde{\sigma}^3$ for three-component link maps which are the result of connect sum of link maps which have two components which non-trivially link and another component split from those two which is trivial. One may wonder if all values of $k_i, k_i', l_i, l_i', m_i, m_i'$ are possible. All values can be achieved because the welded crossing introduce no relations and you can always move strands into position using welded crossings, which do not add any extra information into the longitude in Γ_i.

It follows from Theorem 7.1.25 that connect sum does not give a well defined group structure like in the two-component case for link maps.

Corollary 7.1.26. *There exists different choices of tubings for connect sum giving different resulting link homotopy classes in* $\mathrm{LM}^4_{2,2,2}$.

Proof. Let J be the braid described by Figure 7.1 and let X be the three-component JK construction which keeps the y component trivial and let $\overline{X}$ be the reversed mirror image of this JK construction. Applying σ^3 to the link maps $f_{XJ\overline{X}}$ and $f_{XJJ\overline{X}}$. From Theorem 7.1.25 we have

$$\sigma^3\left(f_{XJ\overline{X}}\right) = \left(z\left(s-1\right)+\overline{z}\left(s-1\right),0,x\left(1-u\right)+\overline{x}\left(1-\overline{u}\right)\right)$$
$$\sigma^3\left(f_{XJJ\overline{X}}\right) = \left(z\left(s^2-1\right)+\overline{z}\left(\overline{s}^2-1\right),0,x\left(1-u^2\right)+\overline{x}\left(1-\overline{u}^2\right)\right).$$

Using Example 6.3.5, we can see that these link maps are distinct from one another. □

It is worth noting that Theorem 7.1.25 and the proof of Proposition 7.1.3 show that the group of embedded annular link map up to link homotopy is not a normal subgroup of the group of annular link maps. This is because if we take $J \in \mathrm{EASL}^4_{2,2,2}$ such that

$$\Theta(J) = ((0,0,0),\eta)$$

with $eta \neq \mathrm{Id}$. Let $X \in SL^4_{2,2,2}$ such that

$$\Theta(X) = ((\theta_x(X),\theta_y(X),\theta_z(X)),\mathrm{Id})$$

with $(\theta_x(X),\theta_y(X),\theta_z(X)) \neq (0,0,0)$. Then we can arrange that

$$\Theta(XJ\overline{X}) = (a,\eta) \in L$$

such that $a \neq 0$. However the $\mathrm{EASL}^4_{2,2,2}$ is contained in the following normal subgroup.

Proposition 7.1.27. *Let N be the set of annular link maps such that their link map closures are Brunnian. Then N is a normal subgroup of the group of singular link concordances.*

Proof. We first show N is a subgroup and then prove that it is normal. Let Y_1 and Y_2 be annular link maps whose link map closures are Brunnian. We will show that

$f_{Y_1Y_2}$ is a Brunnian link map. As Θ is a homomorphism

$$\Theta\left(Y_1Y_2\right) = \Big((\theta_x(Y_1), \theta_y(Y_1), \theta_z(Y_1)) + \Phi(\eta(Y_1))\left((\theta_x(Y_2), \theta_y(Y_2), \theta_z(Y_2))\right), \\ \eta(Y_1) \circ \eta(Y_2)\Big).$$

Taking the link map closure we know that for some choice of basing paths we have

$$\widetilde{\sigma}^3\left(f_{Y_1Y_2}\right) = (\theta_x(Y_1), \theta_y(Y_1), \theta_z(Y_1)) + \Phi(\eta(Y_1))\left((\theta_x(Y_2), \theta_y(Y_2), \theta_z(Y_2))\right)$$

where we have added using the basings provided by the annular link map. Recall the map $p_i : \widetilde{K} \to \left(\mathbb{Z}\left[\mathbb{Z}\right]\right)^2$ where we project onto the factors which aren't i and in each component set the ith meridian equal to 1. We will do the case for $i = 3$ as the other cases are similar We have that

$$p_3\left(\widetilde{\sigma}^3\left(f_{Y_1Y_2}\right)\right) = \left(\sigma_1(f_{Y_1^3}) + \sigma_1(f_{Y_2^3}), \sigma_2(f_{Y_1^3}) + \sigma_2(f_{Y_2^3})\right)$$

Since the link map closures of each annular link map is trivial the terms in each sum is zero and by the injectivity of the Kirk invariant Y_1Y_2 is in N.

Suppose that $Y \in N$, we will show that $\overline{Y} \in N$. Note that

$$\widetilde{\sigma}^3\left(f_{\overline{Y}}\right) = \Phi\left(\eta(Y)^{-1}\right)\left(-\theta_x(Y), -\theta_y(Y), -\Theta_z(Y)\right)$$

Applying the map p_i for each i and using the injectivity of the Kirk invariant we have $\overline{Y} \in N$.

As the trivial annular link map is clearly in N we have shown that N is a subgroup.

We now show that N is normal. Let $X \in \mathrm{ALM}^4_{2,2,2}$ and $Y \in N$. We will just focus on the intersection information of $XY\overline{X}$ since the automorphism data is not relevant when we map to $\mathrm{LM}^4_{2,2}$. The intersection information is given by

$$(\theta_x(X), \theta_y(X), \theta_z(X)) + \Phi\left(\eta(X)\right)\left(\theta_x(Y), \theta_y(Y), \theta_z(Y)\right) \\ + \Phi\left(\eta(XY)\right)\left(\theta_x(\overline{X}), \theta_y(\overline{X})), \theta_z(\overline{X})\right).$$

Since

$$\begin{aligned}\left(\theta_x(\overline{X}),\theta_y(\overline{X}),\theta_z(\overline{X})\right) &= -\Phi(\eta(X)^{-1})\left((\theta_x(X),\theta_y(X),\theta_z(X))\right)\\ &= -\Phi(\eta(\overline{X}))\left((\theta_x(X),\theta_y(X),\theta_z(X))\right),\end{aligned}$$

the intersection information becomes

$$\begin{aligned}&(\theta_x(X),\theta_y(X),\theta_z(X)) + \Phi\left(\eta(X)\right)(\theta_x(Y),\theta_y(Y),\theta_z(Y))\\ &\qquad - \Phi\left(\eta(XY\overline{X})\right)(\theta_x(X),\theta_y(X)),\theta_z(X))\end{aligned}$$

We now write

$$p_3\left(\widetilde{\sigma}^3\left(f_{XY\overline{X}}\right)\right) = \left(\sigma_1(X^3)-\sigma_1(X^3),\sigma_2(X^3)-\sigma_2(X^3)\right) = (0,0).$$

By injectivity of the Kirk invariant removing the third component gives a trivial link map. A similar result can be shown for removing the first and second component and this proves the result. □

Hence, the normal closure of the set of three-component embedded link maps is contained within N.

7.1.3 Other constructions

Schneiderman and Teichner showed how to turn $\mathrm{LM}^4_{2,2}$ into a module over the ring $\mathbb{Z}\left[z_1,z_2\right]/\left(z_1z_2\right)$ [ST17]. Since we cannot turn $\mathrm{LM}^4_{2,2,2}$ into an abelian group using connect sum, we cannot give a module structure for the three-component case. However, we can use their ideas to construct new link maps.

Suppose we have a three-component link map f where we have done enough finger moves so that the complement of any two of the spheres is F/F_3. Take an oppositely oriented normal push off of the x component. Then tube both copies of the x component along a commutator of the meridians of the y and z components. This gives a new three-component link map which we will denote by $\mu_{[y,z]}\cdot f$.

Lemma 7.1.28. *The map $\mu_{[y,z]} \cdot f$ is a well defined element of* $\mathrm{LM}^4_{2,2,2}$.

Proof. This proof is a similar argument used in [ST17] to establish a similar operation on two-component link maps. □

We can also define similarly constructions on the y and z components. Denote these link maps by $\mu_{[z,x]} \cdot f$ and $\mu_{[x,y]} \cdot f$ respectively. We can iterate and combine these operations on any given link map.

There is a subset of $\mathrm{LM}^4_{2,2,2}$ on which we will calculate some values of σ^3 where we have applied these operations.

Proposition 7.1.29. *Let $f \in \mathrm{Bl}^4_{2,2,2}$. Then for any choice of basing curves of the components and meridians and $n \geq 2$ we have:*

1. $\widetilde{\sigma}^3(\mu^n_{[y,z]} \cdot f) = ((2 - s - \bar{s})^n \, \sigma_x(f),\ 0,\ 0)$,

2. $\widetilde{\sigma}^3(\mu^n_{[z,x]} \cdot f) = \Big(0,\ (2 - t - \bar{t})^n \, \sigma_y(f),\ 0\Big)$,

3. $\widetilde{\sigma}^3(\mu^n_{[x,y]} \cdot f) = (0,\ 0,\ (2 - u - \bar{u})^n \, \sigma_z(f))$.

Proof. Assume that the complement of any two components of f is already F/F_3. We will only prove the first item on the list since the rest are automatically proven by symmetry. Make a choice of basing path and meridian and for each component. Notice that for all $j \in \mathbb{N}$ we have

$$\begin{aligned}
\sigma_x\left(\mu^j_{[y,z]} \cdot f\right) &= \lambda\left((1-s)\,\mu^{j-1}_{[y,z]} \cdot f, (1-s)\,\mu^{j-1}_{[y,z]} \cdot f\right) \\
&= (2 - s - \bar{s})\,\lambda\left(\mu^{j-1}_{[y,z]} \cdot f,\ \mu^{j-1}_{[y,z]} \cdot f\right) \\
&= (2 - s - \bar{s})^j \, \sigma_x(f).
\end{aligned}$$

For computing the other components first consider what happens when $j = 1$. This operation changes the group elements corresponding to the intersections by replacing the meridians of x with some power the commutators t or u. We now have

$$\sigma_y(\mu_{[y,z]} \cdot f) = \sum_{i=-\infty}^{\infty} a_i z^i$$

and

$$\sigma_z(\mu_{[y,z]} \cdot f) = \sum_{i=-\infty}^{\infty} b_i y^i,$$

where $a_i \in \mathbb{Z}[t, t^{-1}]$ and $b_i \in \mathbb{Z}[u, u^{-1}]$ where $\bar{a}_i = a_{-i}$ and $\bar{b}_i = b_{-i}$. The above sums are finite since there is finitely many intersections on each sphere. For all $i \in \mathbb{Z}$ we have $a_i(1) = b_i(1) = 0$ since f is a Brunnian link map. When we apply $\mu_{[y,z]}$ again, the σ_y and σ_z terms vanish as the only dependence on x is contained in the commutator terms a_i and b_i. So setting x equal to a commutator makes $a_i(1) = b_i(1) = 0$ and thus

$$\sigma_y(\mu^2_{[y,z]} \cdot f) = 0$$

and

$$\sigma_z(\mu^2_{[y,z]} \cdot f) = 0.$$

These terms remain trivial under repeated application of $\mu_{[y,z]}$. This proves the result, as we can realise the homotopies of each disc as a link homotopy of the sphere. $\square$

Chapter 8

n-component link maps

We now define an n-component link map homotopy invariant. We will mostly sketch out the process and not prove things explicitly since the proofs will be analogous to the three-component case. We first need to prove the following proposition.

Proposition 8.0.1. *The fundamental group complement of k generically immersed spheres given by a link map is isomorphic to the free Milnor group on k generators, where a meridian of each sphere is a generator.*

Proof. Let f be an k-component link map and apply finger moves to arrange that the complement is a Milnor group. Using compactness we can apply a link homotopy to f which moves it it below into the southern hemisphere of S^4. For each component of f, do a link homotopy which takes a finger from each sphere and brings up to the northern hemisphere and the intersection with the equatorial S^3 is a single connected component, resulting in n-component unlink at the equator.

We then cut along the equator and this decomposes S^4 into two D^4 both with the same n-component unlink in their boundary S^3, where each component bounds an immersed disc in the four-ball and does not intersect any other component. On both discs we do finger moves to make the fundamental group of the complement a Milnor group. Using a result from [FT95] we know that the fundamental group of each of these complements is Milnor free group on k generators. Using Seifert-Van Kampen this proves the result. □

Let $f = f_1 \sqcup \dots f_n : S_1^2 \coprod \dots \coprod S_n^2 \to S^4$ be a based link map. For each basing path we associate a meridian of the component , similar to the three-component case. Call these meridians $x_1, \dots x_n$. Let

$$\tau = (n, n-1, \dots, 2, 1) \in \mathcal{S}_n,$$

recall $\mathcal{S}_n$ is the symmetric group on n variables. Then define

$$\Gamma_i := \left\langle x_{\tau^{i-1}(1)}, x_{\tau^{i-1}(2)}, \dots, x_{\tau^{i-1}(n)} \mid x_{\tau^{i-1}(1)},\, r_1,\, r_2,\, r_3, \dots,\, r_s \right\rangle$$

where $r_1, \dots, r_s$ are the minimal relations for the free Milnor group on n generators. Clearly each Γ_i is the Milnor group on $n-1$ generators

Definition 8.0.2. Let $f = f_1 \sqcup \dots \sqcup f_n : S_1^2 \coprod \dots \coprod S_n^2 \to S^4$ be a based link map then we define

$$\sigma^n(f) = (M(\lambda(f_1, f_1)), \dots, M(\lambda(f_i, f_i)), \dots, M(\lambda(f_n f_n))) \in \prod_{i=1}^{n} \mathbb{Z}\Gamma_i.$$

Proposition 8.0.3. *Let f and g be link homotopic based link maps. Then*

$$\sigma^n(f) = \sigma^n(g).$$

Proof. The proof of this result is analogous to the proof of Proposition 6.1.3. □

We now find the correct quotient to consider which removes the dependence on basing and gives a unbased link map invariant.

Choose another set of basing paths for f, similarly to the three component case, this pair of basing curves specifies elements of an n-tuple of $(g_1, g_2, \dots, g_n) \in \prod_{i=1}^n \Gamma_i$. Let $\tilde{g}_i \in \Gamma_j$ where $i \neq j$ where $\tilde{g}_i$ is defined to be the image of g_i under the sequence of maps

$$\Gamma_i \twoheadrightarrow \Gamma_i / \langle\langle x_j \rangle\rangle \hookrightarrow \Gamma_j,$$

where the left most arrow is the quotient map and the right most map is the natural inclusion map. The notation omits which generator we have omitted, however it will be clear from the context. We define the following map $\psi_{g_i} : \mathbb{Z}\Gamma_i \to \mathbb{Z}\Gamma_i$ where $\psi_{g_i}(r) = g_i r g_i^{-1}$. Furthermore, let $\phi^i_{g_j} : \mathbb{Z}\Gamma_i \to \mathbb{Z}\Gamma_i$ with

$$\phi^i_{g_j}(x_k) = \begin{cases} \tilde{g}_j^{-1} x_k \tilde{g}_j & k = i \\ x_k & \text{otherwise} \end{cases}$$

extended linearly. Define $w_i : \mathbb{Z}\Gamma_i \to \mathbb{Z}\Gamma_i$ where

$$w_i := \psi_{g_{\tau^{i-1}(1)}} \circ \phi^i_{g_{\tau^{i-1}(2)}} \circ \phi^i_{g_{\tau^{i-1}(3)}} \circ \ldots \circ \phi^i_{g_{\tau^{i-1}(n)}}.$$

Define an action by

$$(g_1, \ldots g_n) \cdot (r_1, \ldots, r_n) = (w_1(r_1), w_2(r_2), \ldots, w_n(r_n)).$$

This action corresponds to the changing basing paths similarly to the action defined in Lemma 6.1.11 in the three-component case. We define our n-component invariant.

Definition 8.0.4. Let f be an n-component link map. Choose a collection of basing paths for each sphere. Define

$$\sigma_i(f) := M(\lambda(f_i, f_i)).$$

Define

$$\widetilde{\sigma}^n(f) = (\sigma_1(f), \ldots \sigma_n(f)) \in \prod_{i=1}^{n} \mathbb{Z}\Gamma^i / \sim.$$

Let $1 \le i_1 < \cdots < i_k \le n$ let us define the map

$$(i_1, \ldots, i_k) : \mathrm{LM}^4_{\underbrace{2,\ldots,2}_{n}} \to \mathrm{LM}^4_{\underbrace{2,\ldots,2}_{n-k}}$$

where $(i_1, \ldots, i_k)(f)$ is the link map f where we forget about each i_j sphere. Define the $p_{(i_1,\ldots,i_k)} : \prod_{i=1}^{n} \mathbb{Z}\Gamma^i / \sim \to \prod_{i=1}^{n-k} \mathbb{Z}\Gamma^i / \sim$ to be the projection onto the factors which are not one of the i_j and set all $x_{i_j} = 1$ and relabel.

Proposition 8.0.5. *The following diagram commutes*

$$\begin{tikzcd}
\mathrm{LM}^4_{2,\ldots,2} \arrow[r, "\underline{i}"] \arrow[d, "\widetilde{\sigma}^n"] & \mathrm{LM}^4_{2,\ldots,2} \arrow[d, "\widetilde{\sigma}^{n-k}"] \\
\prod_{i=1}^{n} \mathbb{Z}\Gamma^i / \sim \arrow[r] & \prod_{i=1}^{n-k} \mathbb{Z}\Gamma^i / \sim \ ,
\end{tikzcd}$$

where $\underline{i} := (i_1, \ldots, i_k)$.

The proof is omitted.

Chapter 9

Noncommutative Blanchfield forms and the intersection form

We now shift our focus away from link homotopy and turn our attention to how we can use the intersection form on a four-manifold with a single boundary component to recover linking information on the boundary. Explicitly, we show we can compute the Blanchfield pairing on a three-manifold in terms of intersections forms of four-manifolds for relatively weak assumptions.

9.1 Organisation of Chapter 9

In Section 9.2, we recall the relevant algebra to make sense of localisation with non-commutative rings; define the twisted Blanchfield pairing. In Section 9.3, we proceed carefully through Conway's proof method checking his method extends to the non-commutative case with the twisted Blanchfield pairing and then prove the result.

9.2 Algebraic and topological preliminaries

9.2.1 Algebra

We revise the basics of the Ore-condition.

Definition 9.2.1. Let R be a ring. We call $S \subset R$ a multiplicative subset if $0 \notin S$, $1 \in S$ and if whenever $a, b \in S$ then $ab \in S$.

We restrict ourselves to the following multiplicative subsets.

Definition 9.2.2. Let R be a ring with involution and $S \subset R$ a multiplicative subset. We say that S satisfies the left Ore-condition if for all $a \in R$ and for all $s \in S$ we have

- $Sa \cap Rs \neq \emptyset$,

- if $as = 0$ then there exists a $u \in S$ such that $ua = 0$,

We further specify that S is closed under involution and does not contain any zero-divisors.

Definition 9.2.3. Let R be a ring and $S \subset R$ be a multiplicative subset which satisfies the left Ore-condition. We define the localisation

$$R_S := R \times S / \sim$$

where $(a, s) \sim (b, t)$ if $\exists c, d \in R$ such that

1. $ca = db$

2. $cs = dt \in S$.

As with localisation of commutative rings, we denote the equivalence class (a, s) by $\frac{a}{s}$. The following proposition allows us to turn R_S into a ring,. We omit the proof but one can be found in [Ore31].

Proposition 9.2.4. *Let R be a ring and $S \subset R$ satisfy the left Ore-condition then R_S can be turned into a ring by defining the addition to be*

$$\frac{a}{s} + \frac{b}{t} = \frac{ca + db}{u}$$

where $c, d \in R$ such that $cs = dt = u \in S$. We define the multiplication to be

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ka}{lt}$$

where $k \in R$ and $l \in S$ such that $ks = lb$.

Remark 9.2.5. As we restrict ourselves to multiplicative subsets which do not contain zero divisors, the inclusion $R \to R_S$ is an injective ring homomorphism. It is known that R_S as a left-R module is flat [Ste75].

Definition 9.2.6. Let N be a left R-module. We call the kernel of the canonical map $N \to R_S \otimes N$ the *S-torsion submodule*, denoted by N^S.

By a Corollary 3.3 in [Ste75] we have that

$$N^S = \{x \in N \mid \exists s \in S \text{ s.t } sx = 0\}.$$

9.2.2 The Blanchfield form

Recall from Chapter 2 that we defined a map $\langle -, - \rangle : R^n \times R^n \to R$, we now provide the following generalisation. Let $P = R, R_S$ or R_S/R and define the map $\langle -, - \rangle : R^n \times P \otimes_R R^n \to P$ where

$$\langle v, p \otimes w \rangle = v\overline{w}^T\overline{p}.$$

One can easily show that this pairing is well defined and sesquilinear with respect to R.

Definition 9.2.7. We call a multiplicative subset of $S \subset R$ *sensible* if the above bilinear map is non-singular for all $P \in \{R, R_S, R_S/R\}$.

One can check that the map

$$\begin{aligned}\kappa : \mathrm{Hom}_{\mathrm{right}\,-\mathbb{Z}\pi}\left(\overline{C_*(X,Y;\Lambda)}, P\otimes_R R^n\right) \to \quad & \overline{\mathrm{Hom}_{\mathrm{left}\,-R}\left(C_*(X,Y;R^n),P\right)}\\ f \mapsto \quad & ((v\otimes\sigma)\mapsto\langle v, f(\sigma)\rangle)\end{aligned}$$

is an isomorphism of left R-modules if S is sensible.

$$H^i\left(X,Y;P\otimes R^n\right) \to H^i\left(\overline{\mathrm{Hom}_{\mathrm{left}\,-R}\left(C_*(X,Y;R^n),P\right)}\right).$$

We use this to define an evaluation map

$$p : H^i\left(\overline{\mathrm{Hom}_{\mathrm{left}\,-R}\left(C_*(X,Y;R^n),P\right)}\right) \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(H_i\left(X,Y;R^n\right),P\right)}.$$

we define $\mathrm{ev} := p\circ\kappa$.

Let M be a connected, closed, orientable three-manifold. Consider the sequence of maps

$$\begin{aligned}T^S &\xrightarrow{PD^{-1}} H^2(M;R^n)^S\\ &\to \ker\left(H^2(M;R^n)\to H^2(M;R_S\otimes_R R^n)\right)\\ &\xrightarrow{\beta^{-1}} H^1(M;R_S/R\otimes_R R^n)/\ker(\beta)\\ &\xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_{\mathrm{left}\,-R}\left(T^S, R_S/R\right)},\end{aligned} \tag{9.2.1}$$

where T^S is the S-torsion submodule of the first map is the inverse of the Poincaré duality map; the second map is the inclusion map; the third map is the "inverse" of the Bockstein map from the long exact sequence induced by the short exact sequence,

$$0\to C^*\left(M;R^n\right)\to C^*\left(M;R_S\otimes_R R^n\right)\to C^*\left(M;R_S/R\otimes_R R^n\right)\to 0,$$

once we have descended to the quotient using the first isomorphism theorem; and the fourth arrow is the evaluation map. We must show this final map is well-defined.

Lemma 9.2.8. *The map*

$$\mathrm{ev} : H^1\left(X, Y, R_S/R \otimes R^n\right) \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(H_1\left(X, Y; R^n\right), R_S/R\right)}$$

descends to a map

$$H^1\left(M, R_S/R \otimes R^n\right) / \ker(\beta) \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(T^S; R_S/R\right)}.$$

Proof. Let $\phi \in \ker(\beta)$ and $q_* : H^1\left(M; R_S \otimes R^n\right) \to H^1\left(M; R_S/R \otimes R^n\right)$ be the induced map from the short exact sequence. By the long exact sequence there exists $\psi \in H^1\left(M; R_S \otimes R^n\right)$ such that $q_* \circ \psi = \phi$. We want to show that

$$\mathrm{ev}(q_* \circ \psi)(x) = 0$$

for all $x \in T^S$.

Consider the diagram

$$\begin{array}{ccc} H^2(X, Y; R_S \otimes R^n) & \xrightarrow{\quad q_* \quad} & H^2(X, Y; R_S/R \otimes R^n) \\ \Big\downarrow \mathrm{ev} & & \Big\downarrow \mathrm{ev} \\ \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(H_1\left(X, Y; R^n\right), R_S\right)} & \xrightarrow{\widetilde{q}_*} & \overline{\mathrm{Hom}_{\mathrm{left}-R}\left(H_1\left(X, Y; R^n\right), R_S/R\right)}, \end{array} \tag{9.2.2}$$

we will show that this diagram commutes. We have

$$\mathrm{ev}(q_* \circ \psi)\left(\sum_{i=1}^k x_i \otimes \sigma_i\right) = \sum_{i=1}^k x_i \overline{q\left(f(\sigma_i)\right)}$$

Since the quotient map commutes with the involution we have

$$\mathrm{ev}(q_* \circ \psi)\left(\sum_{i=1}^k x_i \otimes \sigma_i\right) = \sum_{i=1}^k x_i q\left(\overline{\psi(\sigma_i)}\right).$$

Taking the other path we have

$$\begin{aligned}(\widetilde{q}_* \circ \mathrm{ev})\left(\sum_{i=1}^k x_i \otimes \sigma_i\right) &= q\left(\sum_{i=1}^k x_i \overline{\psi(\sigma_i)}\right) \\ &= \sum_{i=1}^k x_i q\left(\overline{\psi(\sigma_i)}\right),\end{aligned}$$

thus the diagram commutes. If $x \in T^S$ then there exists an $s \in S$ such that $sx = 0$.

Hence

$$\mathrm{ev}(\psi)(x) = s^{-1}s\,\mathrm{ev}(\psi)(sx) = s^{-1}\,\mathrm{ev}(\psi)(0) = 0.$$

By the commutativity of (9.2.2) we have the result. □

Define

$$\Omega : T^S \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\,(T^S; R_S/R)}.$$

to be the map coming from the sequence of maps in (9.2.1).

Definition 9.2.9. The *Blanchfield pairing* is defined by

$$\begin{aligned}\mathrm{Bl} : T^S \times T^S &\to R_S/R\\ (a,b) &\mapsto \Omega(b)(a).\end{aligned}$$

Proposition 9.2.10. *Let S be sensible. Then the blanchfield pairing is non-singular if and only if the evaluation map*

$$p : H^i\left(\overline{\mathrm{Hom}_{\mathrm{left}-R}\,(C_*(X,Y;R^n),P)}\right) \to \overline{\mathrm{Hom}_{\mathrm{left}-R}\,(H_*(X,Y;R^n),P)}$$

is an isomorphism.

Proof. The adjoint of the blanchfield form is $\Omega = p \circ \kappa \circ \beta^{-1} \circ PD^{-1}$. If S is sensible then Ω is an isomorphism if and only if p is an isomorphism, as all other functions in the composition are isomorphisms, and the Blanchfield form is non-singular if and only if Ω is an isomorphism. □

9.3 Establishing the main result

9.3.1 Outline of the proof

We outline our proof method inspired by Conway. Let W be a orientable, connected, compact four-manifold with boundary M. We want to show how one can compute Blanchfield form using the intersection form of W. We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^*(W,M;R^n) \arrow[r, "i^{(W,M)}_{R,R_S}"] \arrow[d, "i^{(W,M),W)}_{R}"] & C^*(W,M;R_S \otimes R^n) \arrow[r, "i^{(W,M)}_{R_S,R_S/R}"] \arrow[d, "i^{(W,M),W)}_{R_S}"] & C^*(W,M;R_S/R \otimes R^n) \arrow[r] \arrow[d, "i^{(W,M),W)}_{R_S/R}"] & 0 \\
0 \arrow[r] & C^*(W;R^n) \arrow[r, "i^{W}_{R,R_S}"] \arrow[d, "i^{W,M}_{R}"] & C^*(W;R_S \otimes R^n) \arrow[r, "i^{W}_{R_S,R_S/R}"] \arrow[d, "i^{W,M}_{R_S}"] & C^*(W;R_S/R \otimes R^n) \arrow[r] \arrow[d, "i^{W,M}_{R_S/R}"] & 0 \\
0 \arrow[r] & C^*(M;R^n) \arrow[r, "i^{M}_{R,R_S}"] \arrow[d] & C^*(M;R_S \otimes R^n) \arrow[r, "i^{M}_{R_S,R_S/R}"] \arrow[d] & C^*(M;R_S/R \otimes R^n) \arrow[r] \arrow[d] & 0, \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{9.3.1}$$

where each column and row is exact. When we descend to cohomology we use the same notation for each map.

First, we establish that there exists a pairing

$$\theta : \partial^{-1}\left(T^S\right) \times \partial^{-1}\left(T^S\right) \to R_S,$$

where $\partial : H_2(W,M;R^n) \to H_1(M;R^n)$ is the boundary map in the long exact sequence of the pair for (W,M) and the form fits into a commutative diagram

$$\begin{tikzcd}
\partial^{-1}\left(T^S\right) \times \partial^{-1}\left(T^S\right) \arrow[r, "-\theta"] \arrow[d, "\partial\times\partial"] & R_S \arrow[d] \\
T^S \times T^S \arrow[r, "\Omega"] & R_S/R
\end{tikzcd}$$

We then show there is another map

$$\psi : \partial^{-1}\left(T^S\right) \times \partial^{-1}\left(T^S\right) \to R_S/R$$

which is defined by considering taking elements $x, y \in \partial^{-1}\left(T^S\right)$ then $r, s \in S$ such that $r\partial(x) = s\partial(y) = 0$. Using the long exact sequence of the pair

$$\cdots \longrightarrow H_2(W;R^n) \xrightarrow{q} H_2(W,M;R^n) \xrightarrow{\partial} H_1(M;R^n) \longrightarrow \cdots \tag{9.3.2}$$

we have that there are $x_0, y_0 \in H_2(W;R^n)$ such that $q(x_0) = rx$ and $q(y_0) = sy$. We then define

$$\psi(x,y) = \frac{1}{r}\lambda(x_0,y_0)\frac{1}{\bar{s}}.$$

We then show that $\theta = \psi$ and prove the result.

9.3.2 Covering the Blanchfield pairing

The next series of lemmas will establish the existence of θ which is a "lift" of the Blanchfield form. Let us define

$$\left(i_{R_S}^{(W,M),W}\right)^{-1} : i_{R,R_S}^{W}\left(\ker\left(i_{R_S}^{W,M}\circ i_{R,R_S}^{W}\right)\right) \to \frac{H^2(W,M;R_S\otimes R^n)}{\ker\left(i_{R_S}^{(W,M),W}\right)}.$$

which is defined by the following. Consider $x \in \ker\left(i_{R_S}^{W,M}\circ i_{R,R_S}^{W}\right)$, by definition, $i_{R_S}^{W,M}\left(i_{R,R_S}^{W}(x)\right) = 0$. By exactness, there exists $y \in H^2(W,M;R_S\otimes R^n)$ such that $i_{R,R_S}^{W}(x) = i_{R_S}^{(W,M),W}(y)$. We thus define $\left(i_{R_S}^{(W,M),W}\right)^{-1}(i_{R,R_S}^{W}(x)) = y$ where y is thought of as an element of $\frac{H^2(W;R_S\otimes R^n)}{\ker\left(i_{R_S}^{(W,M),W}\right)}$. The quotient considered makes this map well defined by the first isomorphism theorem.

Definition 9.3.1. We say a diagram of left R-modules of the form

$$\begin{tikzcd}
A_0 \arrow[r, "\alpha_1"] \arrow[d, "\beta_1"] & A_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{n-1}"] & A_{n-1} \arrow[r, "\alpha_n"] & A_n \arrow[d, "\alpha_{n+1}"] \\
B_1 \arrow[d, "\beta_2"] & & & & A_{n+1} \arrow[d, "\alpha_{n+2}"] \\
\vdots \arrow[d, "\beta_{p-1}"] & & & & \vdots \arrow[d, "\alpha_{n+m-1}"] \\
B_{p-1} \arrow[d, "\beta_p"] & & & & A_{n+m-1} \arrow[d, "\alpha_{n+m}"] \\
B_p \arrow[r, "\beta_{p+1}"] & B_{p+1} \arrow[r, "\beta_{p+2}"] & \cdots \arrow[r, "\beta_{p+q-1}"] & B_{p+q-1} \arrow[r, "\beta_{p+1}"] & A_{n+m} = B_{p+q}
\end{tikzcd}$$

anticommutes if $\alpha_{n+m}\circ\cdots\circ\alpha_1 = -\beta_{p+q}\circ\cdots\circ\beta_1$.

Lemma 9.3.2. *The following diagram anticommutes.*

$$\begin{tikzcd}
\ker(H^2(W;R^n)\to H^2(W;R_S\otimes R^n) \arrow[rr, "i_R^{W,M}"] \arrow[d, "i^W_{R,R_S}"] & & i_R^{W,M}\ker(H^2(W;R^n)\to H^2(M;R_S\otimes R^n) \arrow[d, "\beta^{-1}"] \\
i^W_{R,R_S}\ker(H^2(W;R^n)\to H^2(M;R_S\otimes R^n) \arrow[d, "\left(i_{R_S}^{(W,M),W}\right)^{-1}"] & & \frac{H^1(M;R_S/R\otimes R^n)}{\ker\beta} \arrow[d, "\delta_{R_S/R}"] \\
\frac{H^2(W,M;R_S\otimes R^n)}{\ker\left(i_{R_S}^{(W,M),M}\right)} \arrow[rr, "i^{(W,M)}_{R_S,R_S/R}"] & & \frac{H^2(W,M;R_S/R\otimes R^n)}{\mathrm{im}(\delta_{R_S/R}\circ i^M_{R_S,R_S/R})}
\end{tikzcd} \tag{9.3.3}$$

Proof. Conway shows that if we have nine cochain complexes as in (9.3.1) we can establish the result. While Conway's uses cohomology with commutative ring for his proof, the method works for non-commutative rings. This is because Conway makes no use of commutativity of the coefficients and uses a mixture of diagram chasing and exactness arguments. The proof is in the Appendix of [Con18]. ☐

Lemma 9.3.3. *Let W be a four-manifold such that $H_1(W;R^n)$ is S-torsion free. Then the following statements hold:*

1. *Poincaré duality restricts to a well-defined map*

$$\partial^{-1}\left(T^S\right)\to\ker\left(H^2(W;R^n)\to H^2(M;R_S\otimes_R R^n)\right)$$

2. *Poincaré duality restricts to a map*

$$T^S\to i_R^{W,M}\left(\ker\left(H^2(W;R^n)\to H^2(M;R_S\otimes R^n)\right)\right).$$

Proof. To prove the first item we consider the following diagram:

$$\begin{tikzcd}
H_2(W,M;R^n) \arrow[r, "\mathrm{PD}^{-1}"] \arrow[d, "\partial"] & H^2(W;R^n) \arrow[r, "i^W_{R,R_S}"] \arrow[d, "i_R^{W,M}"] & H^2(W;R_S\otimes R^n) \arrow[d, "i_{R_S}^{W,M}"] \\
H_1(M;R^n) \arrow[r, "\mathrm{PD}^{-1}"] & H^2(M;R^n) \arrow[r, "i^M_{R,R_S}"] & H^2(M;R_S\otimes R^n).
\end{tikzcd} \tag{9.3.4}$$

Let $x\in\partial^{-1}\left(T^S\right)$. We must show that $\left(i^W_{R,R_S}\circ i_R^{W,M}\circ\mathrm{PD}\right)(x)=0$. Choose $s\in S$

such that $s\partial(x) = \partial(sx) = 0$. Hence,

$$\begin{aligned}
\left(i^{W}_{R,R_S} \circ i^{W,M}_{R} \circ \mathrm{PD}^{-1}\right)(x) &= s^{-1}s\left(i^{W}_{R,R_S} \circ i^{W,M}_{R} \circ \mathrm{PD}^{-1}\right)(x)\\
&= s^{-1}\left(i^{W}_{R,R_S} \circ i^{W,M}_{R} \circ \mathrm{PD}^{-1}\right)(sx)\\
&= s^{-1}\left(i^{W}_{R,R_S} \circ \mathrm{PD}^{-1} \circ \partial\right)(sx)\\
&= s^{-1}\left(i^{W}_{R,R_S} \circ \mathrm{PD}^{-1}\right)(s\partial(x))\\
&= 0.
\end{aligned}$$

We now prove the second item. Let $a \in T^S$. As $H_1(W;R^n)$ is S-torision free there exist an $x \in H_2(W,M;R^n)$ such that $\partial(x) = a$. Since a is a S-torsion element we know $\mathrm{PD}^{-1}(x) \in \ker\left(H^2(W;R^n) \to H^2(M;R_S \otimes R^n)\right)$. Taking $i^{W,M}_{R}\left(\mathrm{PD}^{-1}(x)\right)$ we see by commutativity of (9.3.4) that $PD(a) = i^{W,M}_{R}(PD(x))$ as required. □

Lemma 9.3.4. *For the four-manifold W with $\partial W = M$ the following hold*

1. *The evaluation map on $H^2(W,M;R_S)$ induces a well defined map*

$$ev : \frac{H^2(W,M;R_S \otimes R^n)}{\ker\left(i^{(W,M),W}_{R_S}\right)} \to \overline{\mathrm{Hom}_R\left(\partial^{-1}\left(T^S\right), R_S\right)}.$$

2. *The evaluation map on $H^2(W,M;R_S/R \otimes R^n)$ induces a well defined map*

$$ev : \frac{H^2(W,M;R_S/R \otimes R^n)}{\mathrm{im}\left(\delta_{R_S/R} \circ i^{M}_{R_S,R_S/R}\right)} \to \overline{\mathrm{Hom}_\Lambda\left(\partial^{-1}\left(T^S\right), R_S/R\right)}.$$

Proof. Let $\phi \in \ker\left(H^2(W,M;R_S \otimes R^n) \to H^2(W;R_S \otimes R^n)\right)$. We must show that after applying ev and restricting to $\partial^{-1}(T^S)$ we have the zero map. By the long exact sequence of the pair

$$\phi \in \mathrm{im}\left(H^1(M;R_S \otimes R^n) \xrightarrow{\delta_{R_S}} H^2(W,M;R_S \otimes R^n)\right).$$

Then, for some $\psi \in H^1(M;R_S \otimes R^n)$, we have $\delta_{R_S}\psi = \phi$. For $x \in \partial^{-1}(T^S)$ there exists an $s \in S$ such that $s\partial(x) = 0$ and

$$\mathrm{ev}(\delta_{R_S}\psi)(x) = \mathrm{ev}(\psi)(\partial(x)) = s^{-1}s\,\mathrm{ev}(\psi)(\partial(x)) = s^{-1}\,\mathrm{ev}(\psi)(s\partial(x)) = 0.$$

We now prove the second item. Let ϕ be an element of $H^1(M; R_S \otimes R^n)$. Similarly we must show that $\delta_{R_S/R} \circ i^M_{R_S,R_S/R} \circ \phi$ maps to zero under the evaluation map, when restricted to $\partial^{-1}(T^S)$. Applying $i^M_{R_S,R_S/R}$ and $\delta_{R_S/R}$ and evaluating we have

$$\mathrm{ev}\left(\delta_{R_S/R} \circ i^M_{R_S,R_S/R} \circ \phi\right)(x) = \mathrm{ev}(i^M_{R_S,R_S/R} \circ \phi)(\partial(x)),$$

Since ϕ takes values in R_S it vanishes on S-torsion elements. This proves the result.

□

Lemma 9.3.5. *Let W be a compact, connected, orientable, four-manifold with boundary $\partial W = M$ and $H_1(W; R^n)$ S-torsion free. In the following diagram the triangle and squares commute and the pentagon anticommutes.*

$$\begin{tikzcd}
\partial^{-1}\left(T^S\right) \arrow[r, "\partial"] \arrow[dd, "\left(i^{(W,M),W}_{R_S}\right)^{-1} \circ i^W_{R,R_S} \circ PD"'] & T^S \arrow[d, "\beta^{-1} \circ PD"] \arrow[ddr, bend left, "\mathrm{ev} \circ \beta^{-1} \circ \mathrm{PD}"] & \\
 & \frac{H^1(M;R_S/R \otimes R^n)}{\ker(\beta)} \arrow[d, "\delta_{R_S/R}"] \arrow[dr, "\mathrm{ev}"] & \\
\frac{H^2(W,M;R_S \otimes R^n)}{\ker(i^{(W,M),W}_{R_S})} \arrow[r, "i^{(W,M)}_{R_S,R_S/R}"] \arrow[d, "\mathrm{ev}"] & \frac{H^2(W,M;R_S/R \otimes R^n)}{im\left(\delta_{R_S/R} \circ i^M_{R_S,R_S/R}\right)} \arrow[dr, "\mathrm{ev}"] & \overline{\mathrm{Hom}_R\left(T^S, R_S/R\right)} \arrow[d, "\partial^*"] \\
\overline{\mathrm{Hom}_R\left(\partial^{-1}\left(T^S\right), R_S\right)} \arrow[rr] & & \overline{\mathrm{Hom}_R\left(\partial^{-1}\left(T^S\right), R_S/R\right)}
\end{tikzcd} \tag{9.3.5}$$

Proof. By Lemmas 9.3.2 and 9.3.3 the pentagon containing $\partial^{-1}(T^S)$ commutes. The lowest rectangle commutes by Lemma 9.3.4. The rectangle on the right, containing ∂^*, clearly commutes. The triangle on the upper right clearly commutes by the definition of the Blanchfield form. □

From the left most column of (9.3.5) we can define a pairing on $\partial^{-1}\left(T^S\right)$. Consider the composition

$$\begin{aligned}
\Theta : \partial^{-1}\left(T^S\right) &\xrightarrow{\mathrm{PD}} \ker\left(H^2\left(W; R^n\right) \to H^2\left(\partial W; R_S \otimes R^n\right)\right) \\
&\xrightarrow{i^W_{R,R_S}} \ker\left(H^2\left(W; R^n\right) \to H^2\left(M; R_S \otimes R^n\right)\right)
\end{aligned}$$

$$\rightarrow \frac{H^2\left(W, M; R_S \otimes R^n\right)}{\ker\left(i_{R_S}^{(W,M),W}\right)}$$

$$\xrightarrow{\text{ev}} \overline{\operatorname{Hom}_\Lambda\left(\partial^{-1}\left(T^S\right), R_S\right)}.$$

We define a pairing on the $\partial^{-1}\left(T^S\right)$ by

$$\theta(x,y) := \Theta(y)(x).$$

By the previous lemma the following diagram commutes

$$\begin{tikzcd}
\partial^{-1}\left(T^S\right) \times \partial^{-1}\left(T^S\right) \arrow[r, "-\theta"] \arrow[d, "\partial\times\partial"] & R_S \arrow[d, "q"] \\
T^S \times T^S \arrow[r, "\mathrm{Bl}"] & R_S/R
\end{tikzcd}$$

as $\operatorname{ev}\circ\beta^{-1}\circ \mathrm{PD}^{-1}$ is the adjoint of the Blanchfield form.

9.3.3 Showing the forms are equal

We recall the definition of ψ. Suppose $x, y \in \partial^{-1}(T^S)$. Then there exists an $r, s \in S$ such that $r\partial(x) = s\partial(y) = 0$. From the long exact sequence of the pair, there exists $x_0, y_0 \in H_2(W; R^n)$ which map to rx and sy. We now define the pairing by

$$\psi(x,y) = \frac{1}{r}\lambda\left(x_0, y_0\right)\frac{1}{\overline{s}}.$$

Lemma 9.3.6. *The pairing ψ is well-defined.*

Proof. Let $x_0' \in H_2(W; R^n)$ and a non-zero $r' \in S$ be such that $i\left(x_0'\right) = r'x$. Then we have

$$\frac{1}{r}\lambda\left(x_0, y_0\right)\frac{1}{\overline{s}} - \frac{1}{r'}\lambda\left(x', y_0\right)\frac{1}{\overline{s}} = \left(\frac{\lambda\left(x_0, y_0\right)}{r} - \frac{\lambda\left(x_0', y_0\right)}{r'}\right)\frac{1}{\overline{s}}.$$

From the definition of addition there exist $p, p' \in R$ such that $pr = p'r' \in S$. Hence,

$$\left(\frac{\lambda\left(x_0, y_0\right)}{r} - \frac{\lambda\left(x_0', y_0\right)}{r'}\right)\frac{1}{\overline{s}} = \left(\frac{p\lambda\left(x_0, y_0\right)}{pr} - \frac{p'\lambda\left(x_0', y_0\right)}{p'r'}\right)\frac{1}{\overline{s}}$$

$$= \left(\frac{\lambda \left(p x_0 - p' x_0', y_0 \right)}{pr} \right) \frac{1}{s}.$$

As $q\left(px_0 - p'rx_0'\right) = \left(pr - p'r'\right)x = 0$ we have

$$\lambda\left(px_0 - p'x_0', y_0\right) = 0.$$

Using that the intersection form is hermitian and a similar argument on the second component shows the pairing is well-defined. $\square$

We now show that θ and ψ are equal. First we define some notation which will be helpful in the proof. We define $j : \partial^{-1}\left(T^S\right) \to R_S \otimes \mathrm{im}\,(q)$ as

$$j(x) = \frac{1}{r} \otimes q\left(x_0\right),$$

for some $x_0 \in H_2(W; R^n)$ such that $q(x_0) = rx$. This map is well-defined as $\frac{1}{r} \otimes q\left(x_0\right) = \frac{1}{r} \otimes rx = 1 \otimes x$. Let

$$K \colon = \ker\left(H^2\left(W; R^n\right) \xrightarrow{i_R^{W,M}} H^2\left(M; R^n\right) \xrightarrow{i_{R,R_S}^M} H^2\left(M; R_S \otimes R^n\right) \right).$$

From Lemma 9.3.2 we have a map $(i_{R_S}^{(W,M),W})^{-1}$ which we will relabel to

$$k^* : i_{R,R_S}^W\left(K\right) \to \frac{H^2\left(W, M; R_S \otimes R^n\right)}{\ker\left(i_{R_S}^{(W,M),W}\right)}$$

Let $V := H_2\left(W; R^n\right)$ and consider the following diagram

$$\begin{tikzcd}
R_S \otimes \frac{V}{\ker(q)} \arrow[r, "\mathrm{Id}_{R_S} \otimes q", "\cong"'] \arrow[dd, "\widetilde{\Phi}"] & R_S \otimes \mathrm{im}\,(q) \arrow[d, "\mathrm{Id}_{R_S} \otimes PD"] & \partial^{-1}\left(T^S\right) \arrow[l, "j"'] \arrow[d, "PD"] \\
 & R_S \otimes i_R^{(W,M),W} \circ PD\,(V) \arrow[d, "\mathrm{Id}_{R_S} \otimes i_{R,R_S}^W"] & K \arrow[d, "i_{R,R_S}^W"] \\
\overline{\mathrm{Hom}_\Lambda\left(R_S \otimes \frac{V}{\ker(q)}, R_S\right)} \arrow[d, "j^*\left(\mathrm{Id}_{R_S} \otimes \left(q^{-1}\right)^*\right)"] & R_S \otimes i_{R,R_S}^W \circ i_\Lambda^{(W,M),W} \circ PD\,(V) \arrow[l, "\widetilde{\mathrm{ev}}"'] \arrow[r, "\mathrm{mult}"] \arrow[d, "\mathrm{Id}_{R_S} \otimes k^*"] & i_{R,R_S}^W\,(K) \arrow[d, "k^*"] \\
\overline{\mathrm{Hom}_\Lambda\left(\partial^{-1}\left(T^S\right), R_S\right)} & R_S \otimes k^* \circ i_{R,R_S}^W \circ i_R^{(W,M),W} \circ PD\,(V) \arrow[l, "\widetilde{\mathrm{ev}}"'] \arrow[r, "\mathrm{mult}"] & \frac{H^2(W,M;R_S \otimes R^n)}{\ker\left(i_{R_S}^{(W,M),W}\right)} \arrow[ll, bend left, "\mathrm{ev}"]
\end{tikzcd} \tag{9.3.6}$$

Proving that ψ and θ are equal will be a consequence of proving that this diagram commutes, but first we will define some of the maps in the diagram above. The map $\widetilde{\Phi} : R_S \otimes \frac{V}{\ker(q)} \to \overline{\operatorname{Hom}\left(R_S \otimes \frac{V}{\ker(q)}, R_S\right)}$ is defined by

$$\widetilde{\Phi}\left(b \otimes [y_0]\right)\left(a \otimes [x_0]\right) = a\Phi\left([y_0]\right)\left([x_0]\right)\bar{b},$$

where the map Φ is the adjoint of the intersection form. The mult map is defined by

$$\operatorname{mult}\left(a \otimes \phi\right) = a \cdot \phi.$$

The map $\tilde{\text{ev}} : R_S \otimes k^* \circ i^W_{R,R_S} \circ i^{(W,M),W}_R \circ \mathrm{PD}^{-1}\left(V\right) \to \overline{\operatorname{Hom}_R\left(R_S \otimes \frac{V}{\ker(q)}, R_S\right)}$ is defined by

$$\widetilde{\text{ev}}\left(b \otimes \phi\right)\left(x\right) = \text{ev}(\phi)(x)\bar{b}.$$

We define $\widetilde{\text{ev}} : R_S \otimes i^W_{R,R_S} \circ i^{(W,M),W}_R \circ \mathrm{PD}^{-1}\left(V\right) \to \overline{\operatorname{Hom}_R\left(R_S \otimes \frac{V}{\ker(q)}, R_S\right)}$ by

$$\widetilde{\text{ev}}\left(b \otimes \phi\right)\left(a \otimes x\right) = a\,\text{ev}(\phi)(x)\bar{b}.$$

We must show all the maps in the diagram are also well defined. The right most column is well defined by Lemma 9.3.3. The central Poincaré duality map is well defined as $\mathrm{PD}^{-1} \circ q = i^{(W,M),W}_R \circ PD$. We must show that $i^{(W,M),W}_R \circ \mathrm{PD}^{-1}\left(V\right) \subset K$ to show the mult maps are well-defined. However, this is true by exactness, as $i^{W,M}_R \circ i^{(W,M),W}_R = 0$. It follows that mult is well-defined. Finally, we show the $\widetilde{\text{ev}}$ maps are well-defined but this follows from our observations of $i^{(W,M),W}_R \circ \mathrm{PD}^{-1}\left(V\right) = \mathrm{PD}^{-1} \circ q(V)$.

Proposition 9.3.7. *The pairings θ and ψ are equal.*

Proof. Showing that the diagram (9.3.6) commutes will prove the result, as $\text{ev} \circ k^* \circ i^W_{R,R_S} \circ PD^{-1}$ is the adjoint of θ and $j^*\left(\mathrm{Id}_{R_S} \otimes \left(q^{-1}\right)^*\right) \circ \widetilde{\Phi} \circ \left(\mathrm{Id}_{R_S} \otimes q\right) \circ j$ is the adjoint of ψ. By the definition of Φ, we have that the upper left hand square commutes. The bottom right square and lower triangle also commute. We now prove that the upper right rectangle commutes. For $x \in \partial^{-1}\left(T^S\right)$, mapping under j we have

$$j(x) = \frac{1}{r} \otimes q\left(x_0\right).$$

Applying $\left(\mathrm{Id}_{R_S} \otimes i^W_{R,R_S}\right) \circ \left(\mathrm{Id}_{R_S} \otimes \mathrm{PD}^{-1}\right)$, we have $\frac{1}{r} \otimes i^W_{R,R_S} \circ \mathrm{PD}^{-1} \circ (q(x_0))$. Applying the multiplication map we have

$$\mathrm{mult}\left(\frac{1}{r} \otimes i^W_{R,R_S} \circ \mathrm{PD}^{-1} \circ (q(x_0))\right) = \mathrm{mult}\left(\frac{1}{r} \otimes r i^W_{R,R_S} \circ \mathrm{PD}^{-1}(x)\right) = i^W_{R,R_S} \circ \mathrm{PD}^{-1}(x),$$

as required. We finally prove that the lower left square commutes. Let $\phi \in H^2(W, M; R^n)$, $b \in R_S$ and $x \in \partial^{-1}\left(T^S\right)$. By the definition of j, $\left((\mathrm{Id}_{R_S} \otimes q^{-1}) \circ j\right)(x) = \frac{1}{r} \otimes [x_0]$ where $x_0 \in H_2(W)$ such that $q(x_0) = rx$ where $r \in R$. Thus

$$\begin{aligned}
\widetilde{\mathrm{ev}}&\left(b \otimes \left(i^W_{R,R_S} \circ i^{(W,M),W}_R\right)(\phi)\right)\left(\left(\left(id_{R_S} \otimes q^{-1}\right) \circ j\right)(x)\right) \\
&= \widetilde{\mathrm{ev}}\left(b \otimes \left(i^W_{R,R_S} \circ i^{(W,M),W}_R\right)(\phi)\right)\left(\frac{1}{r} \otimes [x_0]\right) \\
&= \frac{1}{r}\left(\mathrm{ev}\left(\left(i^W_{R,R_S} \circ i^{(W,M),W}_R\right)(\phi)\right), [x_0]\right)\bar{b} \\
&= \frac{1}{r}\left(\mathrm{ev}(\phi)\left(q(x_0)\right)\right)\bar{b} \\
&= \mathrm{ev}(\phi)(x)\bar{b}.
\end{aligned}$$

Taking the other path to the left hand corner we have

$$\widetilde{\mathrm{ev}}\left(\left(b \otimes k^* \circ i^W_{R,R_S} \circ i^{(W,M),W}_R\right)(\phi)\right)(x) = \mathrm{ev}(i^{(W,M)}_{R,R_S}(\phi))(x)\,\bar{b} = \mathrm{ev}(\phi)(x)\bar{b}. \quad (9.3.7)$$

We will clarify the first equivalence in (9.3.7). Consider the following commutative diagram:

$$\begin{array}{ccc}
H^2(W, M; R^n) & \xrightarrow{i^{(W,M)}_{R,R_S}} & H^2(W, M; R_S \otimes R^n) \\
\Big\downarrow{\scriptstyle i^{(W,M),W}_R} & & \Big\downarrow{\scriptstyle i^{(W,M),W}_{R_S}} \\
H^2(W; R^n) & \xrightarrow{i^{(W,M)}_{R,R_S}} & H^2(W; R_S \otimes R^n) \\
& & \Big\downarrow{\scriptstyle i^{W,M}_{R_S}} \\
& & H^2(M; R_S \otimes R^n).
\end{array} \quad (9.3.8)$$

Suppose an element of K is equal to $i^{(W,M),M}_R(\eta)$ for some $\eta \in H^2(W, M; R)$. From commutativity of the diagram, we know that $i^{(W,M)}_{R,R_S}(\eta) = k^* \circ i^W_{R,R_S} \circ i^{(W,M),W}_R(\eta)$ in $\frac{H^2(W,M;R_S \otimes R^n)}{\ker\left(i^{(W,M)}_{R,R_S}\right)}$. Now that we have established (9.3.7) we have completed the proof. $\square$

Proof of Theorem 1.2.1. Recall we can identify

$$\ker\left(H_1(M;R^n) \to H_1(W;R^n)\right) = H_2(W,M;R^n)/\operatorname{im}(q).$$

It is clear that $T^S \subset \ker\left(H_1(M;R^n) \to H_1(W;R^n)\right)$ as $H_1(W;R^n)$ is S-torsion free. Suppose that $[x], [y] \in T^S$. Then for some $r, s \in S$, $r[x] = s[y] = 0$. Choose lifts $x, y \in H_2(W,M;R^n)$. It follows that

$$\mathrm{Bl}\left([x],[y]\right) = -\theta(x,y) = -\psi(x,y),$$

by Lemma 9.3.5 and Proposition 9.3.7. We must show that the choices of lifts do not affect the end result. Suppose x' is a lift of $[x]$. Then by the long exact sequence there exists some $v \in H_2(W;R^n)$ such that $q(v) = x - x'$. Since $r[x] = r[x'] = 0$, for some $r \in S$ we have $q(rv) = rq(v) = r(x-x')$. For some $s \in S$ we have $s[y] = 0$ and

$$\begin{aligned}\psi(x-x',y) = \frac{1}{r}\lambda(rv,y_0)\frac{1}{\overline{s}} = \lambda(v,y_0)(1,\overline{s}) &= \mathrm{ev}(\mathrm{PD}^{-1}(q(y_0)))(v)\frac{1}{\overline{s}}\\ &= \mathrm{ev}(\mathrm{PD}^{-1}(sy))(v)\frac{1}{\overline{s}}\\ &= \mathrm{ev}(\mathrm{PD}^{-1}(y))(v)\end{aligned}$$

as this takes values in R we have proved the result. Using that λ is hermitian we can similarly show that a different choice of lift for $[y]$ does not change the result. □

Corollary 9.3.8. *If M is the boundary of a four-manifold W such that the conditions of Theorem 1.2.1 are satisfied, then the Blanchfield form is hermitian.*

Proof. By Theorem 1.2.1, for some choice of $x_0, y_0 \in H_2(W;R^n)$

$$\begin{aligned}\mathrm{Bl}([x],[y]) = \frac{1}{r}\lambda(x_0,y_0)\frac{1}{\overline{s}} &= \frac{1}{r}\overline{\lambda(y_0,x_0)}\frac{1}{\overline{s}}\\ &= \overline{\frac{1}{s}\lambda(y_0,x_0)\frac{1}{\overline{r}}}\\ &= \mathrm{Bl}([y],[x])\end{aligned}$$

□

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