Theories Beyond the Worldline

LEWIS, DANIEL

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From Modified Worldline Theories To String Phenomenology

Daniel J. Lewis

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Centre for Particle Theory
Department of Mathematical Sciences
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Abstract: Standard quantum field theories usually have high energy divergences which render them unacceptable as a complete theory of nature. Such divergences arise from the nature of point particle excitations. This thesis looks to go beyond such frameworks. We begin by asking the general question: what possible modifications of point particle theories are there which might give them desirable properties such as finite and well-behaved amplitudes at high energies? We use the worldline formalism to construct modifications to worldline theories which have a tower of internal states propagating along the worldline, mimicking the behaviour of string oscillator modes. We argue that string theory itself can be regarded as a special case. We show that this class of theories shares similar interesting properties with string theory, and can also be used to analyse aspects of string theory itself. We then move on to focus on the phenomenology of string theory. We study a particular compactification of type I string theory, with the aim of constructing a model with a small but positive cosmological constant which has supersymmetry broken at a high scale. Such models usually suffer stability problems, but we will show that there is a class in which all moduli are either stabilised or flat up to exponentially suppressed terms. These models could be useful starting points for constructing either truly stabilised de Sitter minima, or perhaps as providing a quintessence scenario.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. The results in chapters 4, 5 and Appendix B are based on the following works, in which the authors collaborated equally:

- Steven Abel, Emilian Dudas, Daniel Lewis and Hervé Partouche, 'Stability and vacuum energy in open string models with broken supersymmetry'. In: JHEP 10:226, 2019.
  https://arxiv.org/abs/1812.09714
- Steven Abel and Daniel Lewis, 'Worldline theories with Towers of Internal States'. In: JHEP 12:069, 2020
  https://arxiv.org/abs/2007.07242

No part of this thesis has been submitted elsewhere for any degree or qualification.

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Contents

Chapter 1

Introduction and Outline

This thesis is concerned with theories which are constructed in some sense 'beyond the worldline'. By this we mean that they are extensions and modifications to the usual first quantized theories designed to supply them with desirable properties that conventional quantum field theories cannot have.

One of the main goals of contemporary particle physics is to construct a complete theory of gravity which is valid at any energy scale. Most quantum field theories are not capable of achieving this. In particular, generic field theories have ultraviolet (UV) divergences, rendering them only acceptable as effective field theories. The standard model and effective theories of gravity fall into this class. To go beyond these problems, one is forced to understand why the UV divergences occur and what can be done to alleviate them. One way to do this is to return to the first quantized formalism in which such divergences can very clearly be analysed. This is the point of view taken in this thesis.

Another potential problem that standard local quantum field theory faces is that it is local. On the contrary, quantum gravity is widely believed to have some degree of non-locality. This provides another reason to move beyond local quantum field theories. Again, locality is much clearer when seen from a worldline perspective, in which one has more intuition into how one might construct less local models.

The most widely studied theory sharing this philosophy is string theory. Here one modifies a worldline to a worldsheet (or higher dimensional worldvolumes). Its success in removing UV divergences and containing gravitons, as well as being able to incorporate a large number of gauge groups and particle species, is phenomenal. The problem is now the other way round - one must search through its enormous numbers of vacua for one which contains the standard model embedded in a universe with a small cosmological constant.

Other modifications to worldline theories have not been studied nearly as much. This is usually because they often encounter unitarity or other deep problems, making them pale in comparison to string theory. Nevertheless it is interesting to wonder whether there are alternatives.

This thesis studies both options, first being more agnostic about different kinds of worldline modifications before focussing on the case of string theory. It is laid out in the following way. Chapters 2 and 3 are preliminary background reading which explain what worldline theories are and how they behave. They are reviews of literature and do not claim to have particularly original ideas, but are intended as conceptual background to the subsequent chapters. We do find, however, that good expositions of these subjects are not particularly easy to find, so that these chapters may be of some use. In particular,

- Chapter 2 is concerned with designing quantum theories whose state space forms a representation of some symmetry group. The aim of this chapter is to show how representations of the Poincaré group can be achieved on the worldline.
- Chapter 3 is a short review of the worldline formalism. We take a scalar particle for simplicity and describe its quantization and amplitudes. On the way, we highlight issues that will particularly concern us in later chapters.

The final two chapters are original and based on the papers [1, 2], in which the authors have contributed equally. We will follow both papers closely.

- In chapter 4, we carry out a study of a particular class of extensions to the worldline model, by adding towers of states with worldline massses. We will see how these theories, which mimic string theories, can sometimes be seen to have a stringy origin; whereas in other cases they do not seem to arise from a geometric picture. Either way, they exhibit good properties similar to string theory, and can also be used to study string theory itself in a slightly unusual manner. This chapter is based on the author's work [1] cowritten with Steven Abel.
- In chapter 5, we commit ourselves fully to string theory, with the aim of constructing certain vacua which have realistic properties such as a small positive cosmological constant. This chapter is based on the author's work [2], cowritten with Steven Abel, Emilian Dudas and Herve Partouche.

Finally there are two appendices:

- Appendix A is a lightning-quick review of type I string theory, intended as a kind of glossary for some of the ideas used in chapter 5.
- Appendix B contains original lengthy calculations which are used in chapter 5.

Chapter 2

Symmetries and Phase Spaces

2.1 Introduction

This introductory chapter is a review into how one can construct quantum mechanical systems exhibiting given symmetry groups. Our main goal is to give a recipe to construct quantum theories exhibiting irreducible representations of the Poincaré group and, in particular, to extract appropriate phase spaces on which to build Lagrangians realizing such symmetries.

This chapter has been included as conceptual background for the next. It is intended to be of some use to those who wish to build models of worldline theories. We begin with some rather general comments which may seem somewhat technical, but then proceed by giving several examples with symmetry groups of increasing complexity, before concluding with the Poincaré group itself.

On the way, we will see how Grassmann variables provide convenient classical coordinates for a phase space for the spin groups. We will also see how supersymmetry is naturally associated with certain representations of the Poincaré group. Our discussion has been influenced significantly by the book by Woodhouse [3] as well as [4, 5]. We have also utilised the connection between the Borel-Weil theorem and constrained phase spaces explained in [6].

2.2 Phase spaces of quantum mechanics systems: overview

Suppose we want to classify all quantum theories with a symmetry group G. In general, this is a difficult problem. We need to first find all unitary representations, whose underlying vector space in physics is the state space, and under which elements $g \in G$ are represented by unitary operators U_g, whose infinitesimal forms generate an algebra $\mathcal{A}$ of observables.

Then we need to find all consistent dynamics that the system allows. The easiest way to do this is when the system admits a Hamiltonian - the operator which generates time translation. Such a Hamiltonian may be built up from operators in $\mathcal{A}$, but its ultimate choice is up to model builders who might add constraints such as unitarity and locality.

The problem has a classical analogue which is more tractable. Here we aim to find a symplectic manifold on which G acts transitively, together with a consistent Hamiltonian. Such manifolds are often obtained using the so-called orbit method [4].

To go from the quantum to classical picture is fairly easy. We will refer to this process as 'dequantization'. It should generate a subspace $\mathcal{C}$ of the original Hilbert space, equipped with a symplectic form, such that not only operators on $\mathcal{H}$ should be mapped to functions on $\mathcal{C}$ but also their commutators should be mapped to the Poisson brackets of their associated functions. In other words, it is a Lie algebra homomorphism from the space of operators with commutator bracket to the space of functions on $\mathcal{C}$ with Poisson bracket.

To go back again is less straightforward. Here we aim to promote functions of the classical phase space $\mathcal{C}$ to operators acting on a certain Hilbert space $\mathcal{H}$ which we must ascertain. Moreovoer, we want the Poisson brackets of functions to map to commutators of their corresponding operators. The main problem is that $\mathcal{C}$ need not be topologically trivial and so the conventional old canonical quantization is not applicable - it is then difficult to decide on what space should the representation act on, and also what *polarization* (the separation of coordinates into conjugate positions and momenta) to use. Here quantization techniques such as geometric quantization become important [3]. This is a rather tedious business filled with many subtleties. However, we will find through our examples that in important cases, by working on a larger space than $\mathcal{C}$ which *does* admit an obvious quantization, we are then able to reduce to the desired space $\mathcal{C}$ by adding appropriate Lagrange multipliers to a Lagrangian or Hamiltonian. As usual, when we do this, the convenience of having a simpler phase space comes at the cost of introducing gauge symmetries.

This all will become relevant in the next chapter when we review how to construct worldline theories, whose states will be irreducible representations of the Poincaré group (or rather, its covering). Actually, one might be interested in worldline theories in which the states contain more symmetries (e.g. gauge symmetries from a spacetime perspective), in which case the general lessons of this chapter would be useful.

From quantum mechanics to classical. Let us suppose that we wish to find a classical analogue of a given quantum theory with symmetry group G. We could call this process 'dequantization'. More formally, we wish to obtain a symplectic space which admits a quantization to the original quantum system. Thus we will begin with a given quantum theory whose state space is a known complex projective linear or Hilbert space $P\mathcal{H}$ that forms a representation of the group G, where each element $g \in G$ is represented by the operator U_g. For simplicity, we will assume the representation is irreducible.

The corresponding classical space must have (at least) the following three properties: 1) it should be a symplectic (or at least Poisson) manifold; 2) it should have a polarization available - this is always the case if it has a complex structure; and 3) for an irreducible representation, the group G should act transitively on the classical space[1].

Actually, the complex Hilbert space $P\mathcal{H}$ already has properties (1) and (2) - it is in fact not only symplectic, but Kähler, as we shall see shortly. Moreover, we will show that it is possible to associate every quantum operator U_g with a function $\tilde{U}_g : P\mathcal{H} \to \mathbb{C}$ and also to introduce a Poisson bracket on the space of such functions which is compatible with the commutator of the corresponding operators. Thus $P\mathcal{H}$ already has the structure of a special kind of classical space with many of the desired properties. However, this space is rather too large to be the classical space we want - the group G may not act transitively on $P\mathcal{H}$ so that it fails property (3). Ideally we would like to reduce this symplectic space to a symplectic subspace for which G acts transitively.

This raises the idea of using single orbits $\mathcal{C}_\phi = \{U_g\phi \mid g \in G\}$ with $\phi \in P\mathcal{H}$, the so-called coherent states, as possible classical spaces. The action of G on these subspaces is at least transitive. The problem now is that this $\mathcal{C}_\phi$ is not in general a symplectic submanifold. However, it has been shown [7] that in the case that G is compact then $\mathcal{C}_\phi$ is a symplectic submanifold precisely when ϕ is a highest weight, and if this is so then $\mathcal{C}_\phi$ is diffeomorphic to the homogeneous manifold G/T where T is a maximal torus of G. The non-compact case is harder, and we will deal with it case by case as it arises, and assume in general that G is compact.

In this way we obtain a canonical symplectic manifold $\mathcal{C}$ which we can take as our classical phase space. It has a symplectic form ω which is induced from the symplectic form on $P\mathcal{H}$ and is what will 'remember' the original representation. We can then ask whether it is possible to reverse the process and quantize $(\mathcal{C}, \omega)$ to retrieve our original representation. This turns out to often be true, essentially due to the Borel-Weil theorem or its orbit method generalizations, which states that every irreducible representation of G is equivalent to a holomorphic line bundle over G/T, and we can use methods of geometric quantization to make this explicit [3].

Let us now give some details of the above constructions. The Kähler structure of a projective Hilbert space $P\mathcal{H}$ is often given as follows (see e.g. [8]). On its underlying real structure, $\mathcal{H}$ contains a complex structure J (a map $\mathcal{H} \to \mathcal{H}$ with $J^2 = -1$), whose ac-

[1] This is because during quantization, the coordinates of the classical space correspond to certain quantum states (coherent states) which linearly span the whole representation. Thus, if the action of G on the classical space is not transitive, its corresponding action on the quantized states will not be irreducible.

tion on $\mathcal{H}$ is multiplication by i. Meanwhile, $\mathcal{H}$ contains a complex hermitian form which decomposes as

$$\langle \phi, \psi \rangle = G(\phi, \psi) + i\Omega(\phi, \psi) \ . \tag{2.2.1}$$

It is easily seen that G is symmetric whilst Ω is antisymmetric, and therefore they give rise to a metric and symplectic form respectively when applied to tangent vectors. Finally, it is easy to show that G, Ω and J are compatible, i.e. $G(\phi, \psi) = \Omega(\phi, J\psi)$, turning $\mathcal{H}$ into a Kähler manifold, and with symplectic reduction $P\mathcal{H}$ is also turned into a Kähler manifold.

Next, projective representations are the same as unitary or antiunitary representations on $\mathcal{H}$. Given any operator $O \in \mathrm{End}(\mathcal{H})$, we can associate it with a map $\tilde{O} : P\mathcal{H} \to \mathbb{C}$ given by

$$\tilde{\mathcal{O}}(\phi) = \frac{\langle \phi, \mathcal{O}\phi \rangle}{\langle \phi, \phi \rangle} \ , \tag{2.2.2}$$

where ϕ is a representative in $P\mathcal{H}$.

Using the fact that G is a Lie group with Lie algebra $\mathfrak{g}$, there is also a natural action of $\mathfrak{g}$ on $\mathcal{H}$: $X \cdot \phi = \frac{d}{dt}\big|_{t=0} \exp(tX) \cdot \phi$. As an aside, these actions are often tied together in the form of a momentum map $\Phi : P\mathcal{H} \to \mathfrak{g}^*$ given by $\langle \Phi(\phi), X \rangle = \tilde{X}(\phi)$. Anyway, the unitary operators U_g give rise to vector fields X_g so that X_g at the point ϕ is the tangent vector $X_g \cdot \phi \in T_\phi \mathcal{H}$.

It is then possible to define a Poisson bracket on these functions induced from the commutator:

$$\{\tilde{U}_1, \tilde{U}_2\}(\phi) = -\frac{\Omega(X_1\phi, X_2\phi)}{\langle \phi, \phi \rangle} = \frac{\langle \phi, [X_1, X_2]\phi \rangle}{\langle \phi, \phi \rangle} \ . \tag{2.2.3}$$

Here X_i are as in the notation above the vector fields associated to $\tilde{U}_i$. This latter equality relies on the fact that the operators U_i are unitary and so the corresponding vector fields X_i are antihermitian[2]. An important special case is when we take the operators to be the Lie algebra representatives U_X, acting as $U_X(\phi) = X \cdot \phi$. In this case our tilde map is compatible with the commutator:

$$\{\tilde{U}_{X_1}, \tilde{U}_{X_2}\} = [\widetilde{U_{X_1}, U_{X_2}}] \ . \tag{2.2.4}$$

Now let us restrict the above structures to the single orbit $\mathcal{C}$ of G through a highest weight vector ϕ_0. Note that any orbit $\mathcal{C}_\phi$ is diffeomorphic to G/T where T is the subgroup fixing ϕ. Thus if ϕ_0 is a highest weight vector in $P\mathcal{H}$, T is a maximal torus and $\mathcal{C} \cong G/T$. This is a rather special kind of space. At each point, its tangent space is generated by real linear combinations of positive and negative roots, and this provides a complex structure. In fact, if $\mathfrak{g} = \mathrm{Lie}(G)$ has a basis $\{f_\alpha, e_\alpha, h_i\}$, with the h_is spanning the Cartan subalgebra

[2]This equation also holds, up to a sign, for antiunitary operators.

$\mathfrak{h} = \mathrm{Lie}(T)$, so that the positive and negative roots (over the complexified Lie algebra) are $a^\dagger_\alpha = e_\alpha + if_\alpha$ and $a_\alpha = e_\alpha - if_\alpha$, we see that every element of $\mathfrak{g}$ can be written uniquely as $z_\alpha a_\alpha + \theta_i h_i + \bar{z}_\alpha a^\dagger_\alpha$ for complex z_α and real θ_i. Thus over $\mathrm{Lie}(G/T) = \mathfrak{g}/\mathfrak{h}$ one can parametrise elements holomorphically as $X(z) = z_\alpha a^\dagger_\alpha + \bar{z}_\alpha a_\alpha$ and z becomes a local coordinate on G/T, giving it a complex structure. Notice how the fact that ϕ_0 is a highest weight vector is crucial here - we need all the non-root vectors to be projected out for this argument to work. See [7] for more details.

When we restrict our tilde map to $\mathcal{C}$, one finds in particular

$$\tilde{U}_X(U_g \cdot \phi_0) = \tilde{U}_{\mathrm{Ad}_g X}(\phi_0)\ . \tag{2.2.5}$$

We can view $\phi_0 \in \mathfrak{g}^*$ by writing $\phi_0(X) = \langle \phi_0, X\phi_0\rangle$. In this case we see that $g \cdot \phi(X) = (U_g \cdot \phi_0)(X) = \phi_0(\mathrm{Ad}_g X)$, so that the elements $\phi \in \mathcal{C}$ can be viewed as living on a coadjoint orbit through ϕ_0. We will have little more to say about this, but it does provide the starting point for the very general orbit method [4], which uses a correspondence between coadjoint orbits (with symplectic form) on a group G and representations of G.

The above essentially completes the classification of classical systems - the classical phase space $\mathcal{C}$ can be taken to be any of the following equivalent spaces:

- The set of coherent states $\{U_g \cdot \phi_0\}$ where ϕ_0 is a highest weight vector.
- The quotient G/T where T is a maximal torus.
- A coadjoint orbit of $\mathfrak{g}^*$ passing through ϕ_0.

This space is symplectic with symplectic form ω obtained from the reduction of $P\mathcal{H}$ to $\mathcal{C}$. It is also complex, which allows for a polarization. Furthermore, the map (2.2.4) ensures us that the Poisson bracket on $\mathcal{C}$ is compatible with the Lie bracket of observables on $P\mathcal{H}$ under our tilde map. It remains to be seen that we really can construct the original representation from $(\mathcal{C}, \omega)$.

From classical to quantum. As we have already noted, this is in general more complicated. The standard method proceeds by geometric quantization, which is essentially an exact reverse of the previous discussion. This is widely written about, see [3] for example, and as such we will not dwell on it here. Instead we will focus on examples of quantizations which avoid having to use the full machinery of geometric quantization by involving constraints on larger spaces which have a more obvious quantization.

2.3 Example 1: SU(2)

This compact semisimple Lie group provides us with an excellent illustration of the general points above. We will proceed in the same order as the previous subsection, first giving the irreducible representations, then the submanifold of coherent states, and finally the symplectic form on this submanifold together with the Poisson brackets of classical observables. This is tantamount to dequantizing the symmetry. Then we will say a few words about the quantization.

Dequantizing $SU(2)$. We begin with a quantum system exhibiting an irreducible $SU(2)$ symmetry. It is well known that the associated irreducible representations are realized as the $(2j)^{th}$ symmetric powers of the fundamental representation, where the conventional index j is a half-integer or integer. Such a Hilbert space is then conventionally given as

$$\mathcal{H}_j = \mathrm{Span}_{\mathbb{C}}\{|j, m, \sigma\rangle \ , | \ m = -j, -j+1, \ldots, j\} \ , \tag{2.3.1}$$

where the index m enumerates basis states and the label σ is included in order to remind us that their might be other dependencies. Having been reminded, we shall henceforth drop this label since it plays no immediate role. Thus we can assume that the Hilbert space is $\mathcal{H}_j = \mathbb{C}^{2j+1}$ and so states live in its projectivisation $P\mathcal{H}_j = \mathbb{C}P^{2j}$.

With the standard matrix representations, the vector $\phi_0 = [1 : 0 : \cdots : 0] \in P\mathcal{H}_j$ is a highest weight vector. It is easily shown that the orbit of $SU(2)$ through ϕ_0 is

$$\mathcal{C} = \{\phi_{\alpha,\beta} = [\alpha^{2j} : \alpha^{2j-1}\beta : \cdots : \beta^{2j}] \mid \alpha, \beta \in \mathbb{C}\} \ . \tag{2.3.2}$$

This is simply the Veronese embedding of $\mathbb{C}P^1$ into $\mathbb{C}P^{2j}$, so that this orbit is really just a copy of $S^2 \cong \mathbb{C}P^1$ sitting in $\mathbb{C}P^{2j}$. Indeed, from hereon we will speak of $[\alpha : \beta] \in \mathcal{C}$ using this identification. In line with the previous subsection, we say that $\mathcal{C}$ is a set of coherent states. It is well known that the 2-sphere is not only symplectic but Kähler, which gives us hope that $\mathcal{C}$ will provide a classical state space. In fact, the curve $\mathcal{C}$ bends enough so that it linearly spans[3] the entire $P\mathcal{H}_j$. The reader may for the sake of completeness be interested in the other orbits which do not go through ϕ_0. It turns out that these are generically three-dimensional lens spaces $SO(3)/\Gamma$, with Γ a discrete subgroup which are certainly not symplectic, or 2-spheres which are not simultaneously complex and symplectic submanifolds.

Before we go on, we should note that there are three further ways to view the submanifold $\mathcal{C}$:

[3] Technically, $P\mathcal{H}_j$ is the L^2 completion of $\mathcal{C}$.

- It is easily seen that the stabiliser subgroup which fixes ϕ_0 is the maximal torus T consisting of diagonal matrices in $SU(2)$. It follows that $\mathcal{C} \cong SU(2)/T$. Topologically this is the Hopf fibration $S^2 \cong S^3/S^1$.

- We can also view $\mathcal{C}$ as a coadjoint orbit. For this, first note how a Lie algebra element $A \in \mathfrak{g} = \mathfrak{su}(2)$ has a natural action on $\mathcal{H}$ as $A \cdot \phi = \frac{d}{dt}|_{t=0} e^{tA} \cdot \phi$. Using this and the hermitian form on $\mathcal{H}$, we can interpret $\phi \in \mathfrak{g}^*$ as having values $\phi(A) = \langle \phi, A \cdot \phi \rangle$. It follows that $g \cdot \phi_0(A) = \phi_0(\mathrm{Ad}_g A)$, so that $\mathcal{C} = \{g \cdot \phi_0 \mid g \in SU(2)\}$ is a coadjoint orbit through ϕ_0.

- The Hilbert space $\mathcal{H}$ can equivalently be described using a vector space of homogeneous polynomials, so that the state $|j, m\rangle$ corresponds with the basis element $z_1^{j+m} z_2^{j-m}$. For the projective Hilbert space $P\mathcal{H}$ the z_1 and z_2 should be taken as homogeneous coordinates. This equivalent representation is convenient because the Lie algebra acts as differential operators rather than large matrices. In particular, we see that the generators of the complexified Lie algebra, $a = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right)$, $a^\dagger = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and Cartan $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, act as

$$U_{a^\dagger} = z_2 \partial_{z_1} \ , \quad U_a = z_1 \partial_{z_2} \ , \quad U_h = i(z_1 \partial_{z_1} - z_2 \partial_{z_2}) \ . \tag{2.3.3}$$

  The coherent states in this picture are holomorphically embedded as $\sum_m z_1^{j-m} z_2^{j+m} = \frac{z_1^{2j} - z_2^{2j}}{z_1 - z_2}$ using the homogeneous coordinates.

Now let us see how operators on $\mathcal{H}_j$ induce functions on $\mathcal{C}$. The most interesting operators are the unitary representatives of group elements U_g and their corresponding Lie algebra operators. In the first case we have associated functions

$$\tilde{U}_g(\mathbf{z}) = \frac{\langle \phi_\mathbf{z}, U_g \phi_\mathbf{z} \rangle}{\langle \phi_\mathbf{z}, \phi_\mathbf{z} \rangle} = \left(\frac{\mathbf{z}^\dagger g \mathbf{z}}{\mathbf{z}^\dagger \mathbf{z}} \right)^{2j} \tag{2.3.4}$$

where $\mathbf{z} = [z_1, z_2]$ and $\phi_\mathbf{z}$ is defined above. In the second case, one has

$$\tilde{U}_X(\mathbf{z}) = \frac{2j \mathbf{z}^\dagger X \mathbf{z}}{\mathbf{z}^\dagger \mathbf{z}} \ . \tag{2.3.5}$$

For example, if we view $\mathcal{H}$ as the space of homogeneous functions of total degree $2j$, then one finds[4]

$$\begin{aligned} U_a^\dagger &= z_2\partial_{z_1} && \to \quad \tilde{a}^\dagger(z_1,z_2) = 2j\bar{z}_1 z_2/R^2 \\ U_a &= -z_1\partial_{z_2} && \to \quad \tilde{a}(z_1,z_2) = -2j\bar{z}_2 z_1/R^2 \\ U_h &= i(z_1\partial_{z_1} - z_2\partial_{z_2}) && \to \quad \tilde{h}(z_1,z_2) = 2j\cdot i(|z_1|^2-|z_2|^2)/R^2 \ , \end{aligned} \tag{2.3.7}$$

where $R^2 = |z_1|^2 + |z_2|^2$ and $a, a^\dagger, h$ are the representations of the corresponding Lie algebra elements living in the complexified Lie algebra $\mathfrak{su}(2)_{\mathbb{C}}$ (so that elements of $\mathfrak{su}(2)$ may be taken as real combinations $za^\dagger + \bar{z}a + xh$).

Similarly, the global operators induce functions

$$U_{g(\alpha,\beta)} = U_{\left(\begin{smallmatrix}\alpha & -\bar\beta \\ \beta & \bar\alpha\end{smallmatrix}\right)} \quad \to \quad \tilde{U}_{g(\alpha,\beta)}([z_1,z_2]) = \left((\bar{z}_1,\bar{z}_2)\left(\begin{smallmatrix}\alpha & -\bar\beta \\ \beta & \bar\alpha\end{smallmatrix}\right)\left(\begin{smallmatrix}z_1 \\ z_2\end{smallmatrix}\right)\right)^{2j}/R^{4j} \ . \tag{2.3.8}$$

The factors $1/R^2$ are crucial to making these well-defined functions on $\mathcal{C}$. If we omitted them, then they would only be sections of a holomorphic line bundle over $\mathcal{C}$.

Finally, one can check what the symplectic bracket on $\mathcal{C}$ looks like. One anticipates that it would be the standard Fubini-Study metric on $\mathbb{C}P^{2j}$ pulled back onto the set of coherent states. Indeed, parametrizing a local neighbourhood of G/T by

$$g(z) = \exp(za^\dagger + \bar{z}a) = \begin{pmatrix} \cos|z| & -\frac{z}{\bar{z}}\sin|z| \\ \frac{z}{\bar{z}}\sin|z| & \cos|z| \end{pmatrix} \ , \tag{2.3.9}$$

then $\mathcal{C}$ has a local coordinate system

$$p : z \mapsto \phi_z = U_{g(z)}\phi_0 \ . \tag{2.3.10}$$

Then vector fields on $\mathbb{C}$ push forward as

$$p_*\partial_z = U_{a^\dagger} \ , \quad p_*\bar\partial_z = U_a \ . \tag{2.3.11}$$

We then have, on the local patch

$$p^*\omega_z(\partial_z, \bar\partial_z) = \omega_{\phi_z}(\phi_z, [U_{a^\dagger}, U_a]\phi_z) \tag{2.3.12}$$

[4]These equations use homogeneous coordinates for $\mathbb{C}P^1$. They have analogues on, for example, the local coordinates parametrised by $[1, z]$:

$$\begin{aligned} a &= z\partial_z && \to \quad \tilde{a}^\dagger(z) = 2j\bar{z} \\ a^\dagger &= z^2\partial_z + 2j && \to \quad \tilde{a}(z) = 2jz \\ h &= i(2j - 2z\partial_z) && \to \quad \tilde{h}(z) = 2j\cdot i(1-|z|^2) \ . \end{aligned} \tag{2.3.6}$$

and a straightforward calculation reveals[5]

$$p^*\omega(z) = j\frac{\sin|z|}{|z|}\, dz \wedge d\bar{z}\ . \tag{2.3.13}$$

A similar calculation would give the symplectic form on the other patch. Notice that ω crucially involves j, so that it still 'remembers' our original representation.

To conclude, we have shown that the desired classical space is $\mathcal{C} = \mathbb{C}P^1$, with symplectic form as above. We have also shown which functions on it descend from the action of unitary operators in the original representation.

Quantization. Is it possible to go back? That is, if we are given the manifold $\mathcal{C}$ with some symplectic form ω, can we reconstruct the original representation? This is a prototypical case of geometric quantization, which has been described in detail in many references such as [3]. Rather than dwelling on it here, we would like to make a few salient observations. First, only certain symplectic forms which obey an integral condition appear in (2.3.13). So we expect some kind of Dirac quantization consistency condition to appear somewhere. Second, it is the symplectic form which must control which representation one finds - geometric quantization selects the representation by selecting the holomorphic line bundle over $\mathcal{C}$ with a connection whose curvature is ω. One can see this from our perspective, where the functions $\tilde{U}_g$ form an equivalent representation to what we started with via $g' \cdot U_g = \tilde{U}_{g'g}$, and if we ignore the normalization, they should be thought of as sections of line bundles over $\mathcal{C}$. The Borel-Weil theorem states that all the irreducible representations of $SU(2)$ can be realised on such sections of line bundles, with different line bundles correlating with different representations.

However, there is another route we can take [6]. This is to work on a larger space than $\mathcal{C}$ which admits a more obvious quantization. Then we use a gauge symmetry to project the physical states onto $\mathcal{C}$. In the case at hand, the space $\mathbb{C}^2$ provides an obvious example of such a larger space. At the end we would like to constrain $(z_1, z_2) \to [z_1, z_2] \in \mathbb{C}P^1$. Equation (2.3.7) is very suggestive. If the z_1 and z_2 are normalized to $R^2 = |z_1|^2 + |z_2|^2 = 1$, then the reverse map - the quantization - should simply turn z_i into an operator multiplying by z_i and $\bar{z}_i$ into ∂_{z_i} for $i = 1, 2$, with some kind of normal ordering scheme needed. Thus simultaneously quantizing $\mathbb{C}^2$, *along* with the Hamiltonian constraint:

$$H' = \mu(|z_1|^2 + |z_2|^2 - 1)\ , \tag{2.3.14}$$

where μ is a Lagrange multiplier, should recover our original representation. Notice that the constraint serves a double purpose - it restricts the phase space to the sphere S^3, but

[5]Often one uses the coordinates $z = \theta e^{i\phi}$ to put this into a more standard form.

being first class, also generates a $U(1)$ gauge symmetry which rotates (z_1, z_2) by a phase, further reducing the physical phase space to $\mathbb{C}P^1$.

The symplectic form on the larger space should then be taken to be $\omega = j \sum_i dz_i \wedge d\bar{z}_i$. If we normalize so that $\omega = \sum_i dz_i \wedge d\bar{z}_i$ so that $\{z_i, \bar{z}_i\} = 1$ for each i, then the constraint reads

$$H' = \mu(|z_1|^2 + |z_2|^2 - 2j)\ . \tag{2.3.15}$$

One can verify that this method will work. For instance, physical states must satisfy $H' f(z_1, z_2) = 0$ or, with a suitable normal ordering, $(z_1\partial_{z_1} + z_2\partial_{z_2})f = 2j$, which restricts the Hilbert space to homogeneous functions of total degree $2j$, just as we want. Meanwhile, the Lie algebra is generated by the functions in an exact reversal of the maps in (2.3.7).

Yet another perspective worth pointing out that is that H' is projecting elements of the universal enveloping algebra to those which have Casimmir $C_2 = j$. This can be seen be dequantizing the corresponding operator $\frac{1}{2}(aa^\dagger + a^\dagger a) + h^2$.

2.4 Example 2: SU(3)

Dequantization. Irreducible representations of $SU(3)$ are realized on $S^{j_1}(V) \otimes S^{j_2}(\Lambda^2 V)$ where $V = \mathbb{C}^3$ is the fundamental, so that the pair (j_1, j_2) labels the representation. In the standard matrix representations, the vector $\phi_0 = e_1^{j_1} \otimes e_{12}^{\otimes j_2}$ is a highest weight vector, where $\{e_i\}$ is a Cartesian basis for V and $e_{ij} = e_i \wedge e_j$.

It is useful to parametrise elements of $SU(3)$ by

$$g(z, w) = \begin{pmatrix} z_1 & w_1 & * \\ z_2 & w_2 & * \\ z_3 & w_3 & * \end{pmatrix} \tag{2.4.1}$$

where the third column is completely determined from the first two, and the first two columns must be orthonormal: $|\mathbf{z}|^2 = |\mathbf{w}|^2 = 1$ and $\mathbf{z}^\dagger \mathbf{w} = 0$.

The set of coherent states $\mathcal{C}$ consists of the orbit through this highest weight:

$$g(z, w) \cdot \phi_0 = (z_i e_i)^{\otimes j_1} \otimes (z_i w_j e_{ij})^{\otimes j_2}\ .$$

The right hand side can be thought of as tensor products of certain (1)-minors and (2)-minors of the matrix g. We know from the general discussion that, projectively, this space is $SU(3)/T$ where T is the maximal torus consisting of diagonal entries. By using a convenient basis of the Cartan subalgebra $h_1 = \mathrm{diag}(i, 0, -i)$ and $h_2 = \mathrm{diag}(0, i, -i)$, which have the property that $g(z, w)\exp(\theta h_1) = [e^{i\theta} z, w]$ and $g(z, w)\exp(\theta h_2) = [z, e^{i\theta} w]$, one sees that G/T is the subspace $\mathcal{F} \subset \mathbb{C}P^2 \times \mathbb{C}P^2$ such that if $([\mathbf{z}], [\mathbf{w}]) \in \mathcal{F}$ then

$\mathbf{z}^\dagger\mathbf{w} = 0$. This is an example of a flag manifold, identified with the nested strict inclusions $0 \subset \mathrm{Span}\{\mathbf{z}\} \subset \mathrm{Span}\{\mathbf{z}, \mathbf{w}\} \subset V$.

It is also possible to identify $\mathcal{C}$ with the space of polynomials $f(z)g(z,w)$ where f is homogeneous of degree j_1 and g is homogeneous of total degree j_1 and antisymmetric in z and w . In this case, the action of the Lie algebra on $\mathcal{C}$ can be very explicitly realized as differential operators

$$X \cdot f(z,w) = X_{ij}(z_i\partial_{z_j} + w_i\partial_{w_j})f(z,w) \; . \tag{2.4.2}$$

For example, the complexified Lie algebra element $a_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ acts as $-z_1\partial_{z_2} - w_1\partial_{w_1}$. The associated functions are found to be

$$\tilde{X}(z,w) = X_{ij}(z_i\bar{z}_j + z_i\bar{z}_j) \; . \tag{2.4.3}$$

Quantization of $\mathcal{C}$. Again, it would be rather frustrating to directly quantize the awkward flag manifold $\mathcal{C}$. Instead, it is simpler to begin with the space $\mathbb{C}^3 \times \mathbb{C}^3$ together with symplectic bracket $\omega = \sum_i j_1 dz_i \wedge d\bar{z}_i + j_2 dw_i \wedge d\bar{w}_i$ and then reduce [6]. For this, we need to constrain $|\mathbf{z}|^2 = |\mathbf{w}|^2 = 1$ and $\bar{\mathbf{z}} \cdot \mathbf{w}$. This leads to the addition of Lagrange multipliers in the Hamiltonian

$$H' = \mu(|\mathbf{z}|^2 - 1) + \lambda(|\mathbf{w}|^2 - 1) + \nu(\bar{\mathbf{z}} \cdot \mathbf{w} + \bar{\mathbf{w}} \cdot \mathbf{z}) \; . \tag{2.4.4}$$

The Lagrange multipliers not only constrain the phase space to $SU(3)$, but as before play a dual role in forcing the physical phase space to be $SU(3)/T$. That is, the first two constraints are first class and generate gauge symmetries multiplying $\mathbf{z}$ and $\mathbf{w}$ by separate phases, identifying $\mathbf{z} \sim \mu\mathbf{z}$ and similarly $\mathbf{w} \sim \lambda\mathbf{w}$. The final constraint is second class. We have made it real so that $e^{iH'}$ is unitary. It should be dealt with by Gupta-Blauer quantization, so that it vanishes on matrix elements rather than physical states. Alternatively, we can simply apply half the constraint $\bar{\mathbf{z}} \cdot \mathbf{w}\phi = 0$ to physical states ϕ. See [6] for more explicit details on this.

Quantization requires j_1 and j_2 to be non-negative integers, and in this case the Hilbert space becomes holomorphic homogeneous functions $f(z_i, w_i)$ of degree j_1 in z_i and degree j_2 in the w_i. The final term in $H(\mu, \lambda)$ corresponds to antisymmetrizing the final j_2 z_i and w_i variables. For example, with $(j_1, j_2) = (1,1)$ one finds a Hilbert space consisting of the span of functions $f(z_i, w_i) = z_i(z_k w_l - z_l w_k)$. It will be recognised that this really is reconstructing the representation $S^{j_1}(V) \otimes S^{j_2}(\Lambda^2 V)$ where $V = \mathbb{C}^3$ of $SU(3)$.

Between this example and the previous, it should now be clear how this approach extends to assigning phase spaces and Hamiltonians realizing any symmetry group $SU(N)$.

Anticommuting Formalism. By writing a basis $\{\theta_i\}$ of V, and thinking of these as odd coordinates or differentials, one can translate the polynomials above that generated $S^{j_1}(V) \otimes S^{j_2}(\Lambda^2 V)$ into products of mutually commuting sets of Grassmann variables θ_i^α with $\alpha = 1, \ldots, j_1 + j_2$ so that $\theta_i^\alpha \theta_j^\alpha = -\theta_j^\alpha \theta_i^\alpha$ for each α. As an example of this dictionary between the commuting and anticommuting variables:

$$z_1(z_1 w_2 - z_2 w_1) \leftrightarrow \theta_1^1 \theta_{12}^2 \tag{2.4.5}$$

where $\theta_{jk}^i = \theta_j^i \wedge \theta_k^i$. The corresponding Lie algebra has an action which can be deduced from $z_i \partial_{z_j} + w_i \partial_{w_j} \leftrightarrow \sum_{\alpha=1}^{j_1+j_2} \theta_i^\alpha \iota_{\theta_j^\alpha}$, where ι_θ is the interior product and θ acts as $\theta\wedge$. Applying this dictionary more generally allows us to reformulate our Hamiltonian constraints in terms of the θ_i^α variables. In fact a similar observation would apply to any $SU(2)$ as well, and $SU(N)$ more generally. Of course, this 'phase space' is dependent on the specific representation, unlike the phase space above. This observation is what underpins [9].

2.5 Example 3: The spin groups

Dequantization of the spinor representations. The spin group $\mathrm{Spin}(2n)$ has a spinorial representation that is often described in infinitesimal form via Clifford modules. Here we take a real vector space V of dimension n, complexify it and decompose it as $V_{\mathbb{C}} = W \oplus \bar{W}$ with respect to a complex structure. We then set $V_{\mathbb{C}}$ to act on the exterior algebra ΛW, where $w \in W$ acts as $w\wedge$, whilst $\bar{w} \in \bar{W}$ acts as interior multiplication ι_w. This forms a 2^n dimensional representation of the Clifford algebra $C(V)$ whose bracket is given by

$$\{w_i, \bar{w}_j\} = 2\delta_{ij} \ , \tag{2.5.1}$$

where $\{w_i\}$ forms a basis of W. The $\mathrm{spin}(n, \mathbb{C})$ Lie algebra consists of elements $v_1 v_2 - v_2 v_1 \in C(V)$ and the spin group can be locally obtained by exponentiating these. Because of its even grading, it acts reducibly, preserving $\Lambda W = \Lambda^{(\mathrm{even})} W \oplus \Lambda^{(\mathrm{odd})} W$ with each subspace giving the two spinorial representations. Let us focus on the subspace $\Lambda^{(\mathrm{even})} W$. The highest weight vector is projectively the line $[1] \in P\Lambda W$. Locally then, the set of coherent states can be identified as

$$\mathcal{C} = \{\phi_A = [e^{A_{ij} w_i w_j} \cdot 1]\} \ , \tag{2.5.2}$$

where the antisymmetric A_{ij} is arbitary and where we are using projective notation as usual. In other words, and ignoring the projectivisation, $\mathcal{C}$ is locally the exponentiation of 2-forms $A = A_{ij} w_i \wedge w_j$ on W. It can be shown [10] that the global form of $\mathcal{C}$ is really the space of complex structures on V, which has a local chart given precisely as above. This is completely in line with the Borel-Weil perspective, where representations of $\mathrm{Spin}(n)$ are built from line bundles (Pfaffian line bundles for the case of the spinor representations) over the space of complex structures on V.

As usual, the Lie algebra elements have corresponding functions

$$\tilde{A}(\phi_B) = \langle \phi_B, A \cdot \phi_B \rangle \ , \tag{2.5.3}$$

where $\langle \cdot, \cdot \rangle$ is the hermitian form on ΛW [10]. Then the Poisson brackets, as before, are compatible with the commutators:

$$\{\tilde{A}, \tilde{B}\} = \widetilde{[A, B]} \ . \tag{2.5.4}$$

This completes our dequantization. The case for $\mathrm{Spin}(2n+1)$ is extremely similar, the main difference being that it acts irreducibly on ΛW.

Quantization. Direct quantization here is no better than in the $SU(3)$ case. There one had to deal with a rather awkward flag variety. Here we have to deal with the space of complex structures, which is also not particularly wieldly. In fact, even the functions $\tilde{A}(\phi_B)$ do not readily admit a simple explicit form to guide us.

Therefore, as in the previous examples, we look to bypass this difficulty by going to a larger space. Actually, the space $\{\phi(z) = e^{z_i w_i} \cdot 1\} \subset \Lambda W$ presents itself as a good candidate, since the action of $C(V)$ on it is just an odd coordinate version of the familiar multiplication and derivative operators that quantize $\mathbb{C}^n$.

Now, in the coordinates z_i, the quantization should promote z_i to the action of $z_i w_i$. But this means that the functions z_i^2 are promoted to zero! In other words, the phase space is most naturally identified with the algebra of Grassmann numbers ψ_i which anticommute with each other. This exposes the underlying supergeometry of this space, where the Clifford phase space is identified with $\mathbb{C}^{0|d}$.

Finally, one might be interested in getting the irreducible spinor representation rather than the entire Clifford algebra. This means restricting operators to consist of an even number of ψ_i and/or $\bar{\psi}_i$s and picking a ground state to be in $\Lambda^{(\mathrm{even})}W$ or $\Lambda^{(\mathrm{odd})}W$. This can be achieved by gauging a discrete $(-1)^F$ symmetry in the Hamiltonian, where $(-1)^F$ counts the number of fermionic oscillators. Such a gauging might not be possible due to an anomaly. In fact, for worldline models constructed on a space M this anomaly vanishes precisely if M admits a spin-structure [11]. Of course, if we are only interested in Dirac spinors, then this gauging is unnecessary.

The other representations. Once we have the spinor representation and its conjugate, all other representations can be obtained by appropriate projections of the possible tensor products. This becomes a little ad hoc - we will briefly return to it in the next section.

2.6 Example 4: The Poincaré group

In this section we are concerned with the group $\mathcal{P}^d = \mathbb{R}^d \rtimes \mathrm{Spin}(d)$ which covers the Poincaré group. Due to the discrete symmetries P and T it is disconnected, and we focus on the part connected to the identity.

Irreducible representations of $\mathcal{P}^d$ are induced from the representations of the abelian translation subgroup. The latter are just the characters $\chi_p = e^{ip\cdot x}$. The $\mathrm{Spin}(n)$ subgroup then acts on χ_p to give orbits labelled by $p^2 = -m^2$. The irreducible representations of P then act on sections of homogeneous hermitian line bundles over these orbits. The bundles are defined by choosing the fibre of a point p on the orbit to be the subgroup of $\mathrm{Spin}(d)$ which leaves p fixed - this is the little group.

Scalar Representations. For the scalar representation, $\mathrm{Spin}(d)$ acts trivially, and so we can take as a phase space $\mathbb{R}^d$, together with a projection in the Hamiltonian

$$H' = e(p^2 - m^2) \; , \tag{2.6.1}$$

which picks out a specific orbit as discussed above. Physically, this ensures that physical states are on-shell. We will discuss such models extensively in the next chapter.

Spinor Representation. We have noted above that the classical phase space associated with the spinor representation of $\mathrm{Spin}(d)$ can be taken to be $\mathbb{C}^{0|d}$ together with suitable constraints. It follows that the phase space for a spinorial representation of $\mathcal{P}^d$ is a subset of $\mathbb{C}^{d|0} \otimes \mathbb{C}^{0|d} \cong \mathbb{C}^{d|d}$. Yet again, quantization is easiest on this large space, in which case we must project to the irreducible representation, first by enforcing that p lies on the orbit $p^2 - m^2$ and then by restricting the Grassmann part of the phase space to be perpendicular to p. Hence we add in the Hamiltonian constraints

$$H' = e(p^2 - m^2) + \chi p \cdot \psi \; , \tag{2.6.2}$$

where now e and χ are acting as Lagrange multipliers. The term $Q = p\cdot\psi$ can be interpreted as a supercharge and can also be viewed as a Dirac operator. It generates supersymmetry transformations $\delta X \sim \epsilon\psi$ and $\delta\psi = \epsilon X$. We will see how e can be interpreted as a metric field whilst χ is a gravitino.

Other representations. It is possible to achieve more representations by including r copies of the fermion phase space so that the total phase space is $\mathbb{C}^{d|rd}$. By including more copies of the fermions one constructs the tensor product representations of the spinorial such as $S^{\otimes r}$. We can also include the conjugate representation $\bar{S}$ by including the dual representation with coordinates $\bar{\psi}$. Through further projections one can project onto all other irreducible representations of Spin(d) and hence $\mathcal{P}^d$. For example, if $S = \Lambda W$ is the Dirac representation, the decomposition $S \otimes \bar{S} = \sum_{p=0}^{d} \Lambda^p V$, where V is the vector representation, is obtained by viewing $\Lambda^p V$ as the subspace of $S \otimes \bar{S}$ created from the vacuum $(1, \psi_1\psi_2 \cdots \psi_d)$ by p copies of ψ_μ tensored with $d-p$ copies of its dual $\bar{\psi}_\mu$. The relevant projection operator onto the representation $\Lambda^p V$ is then $\psi \cdot \bar{\psi} + \bar{\psi} \cdot \psi - (d-2p)$ and so a good Hamiltonian is given by adding the constraints [12, 13]

$$H_1 = e(p^2 - m^2) + \chi p \cdot \psi + \bar{\chi} p \cdot \bar{\psi} + a(\bar{\psi}\psi - k) \ , \tag{2.6.3}$$

where the constant k could be chosen to be $d-2p$ depending exactly on how we normal order (if we choose symmetric normal ordering and $[\psi_\mu, \psi_\nu]_+ = \delta_{\mu\nu}$, then $k = d - 2p - 1$). These actions are also interesting because they exhibit an extended $N = 2$ supersymmetry (see for example [13]).

Chapter 3

Worldline Theories: A Review

In the previous chapter we have seen how to incorporate Poincaré degrees of freedom into a quantum mechanical system. We can now let these degrees of freedom propagate in time to achieve a theory of point particles. Such a framework often goes by the name of the "worldline formalism", with individual theories being termed "worldline theories". In this chapter we review these theories in a conceptual manner rather than computing many amplitudes. The most illustrative example of a worldline theory is that of a standard scalar particle - this is what we will spend most time on. In section 3.1 we introduce the classical action of a scalar particle propagating on some spacetime. In section 3.2 we discuss the quantum theory. Its first few subsections describe the path integral quantization and propagators of the theory. We will also briefly discuss its relation to the conventional spacetime field theory description in 3.2.3. In section 3.3 we add junctions, allowing worldlines to meet. This leads to the idea of 'worldgraphs' discussed in 3.4.

This chapter is intended as preparation for the next where we will describe novel modifications to these theories. It summarises known work and makes no claim to novel contributions. However, we hope that the exposition is useful, and are aware of no other review which treats worldline theories in such a conceptual way.

3.1 Introduction: A scalar particle with no interactions

This section introduces the most basic example of a worldline theory [14–21]. We will see how one can let the spin 0 representation of the Poincaré group become the state space of a quantum mechanical system consisting of a scalar field on a one-dimensional space. We begin by giving details of the classical theory before moving on to the quantum theory. We will then discuss how to add junctions to the theory. Finally, we will discuss amplitudes of the theory and their geometrical nature. We will see that the introduction of junctions requires a non-perturbative completion which is a quantum field theory associated to the scalar particle.

3.1.1 Classical theory

We begin with the following setup:

- Let M be a pseudo-Riemannian manifold with dimension D and metric $G_{\mu\nu}$ of signature $(1, D-1)$. We refer to M as *spacetime* or the *target space*.
- A *worldline* is an immersion of a one-dimensional manifold I into M. We allow I to have a boundary. The corresponding map $X : I \to M$, whose components are X^μ, can be interpreted as a collection of scalar fields on I.

Being an introductory example, we will assume that:

- The manifold I is either $I = [0,1]$ or S^1. Actually, we can assume $I = [0,1]$ in the latter case as long as we include a boundary condition $X(0) = X(1)$.
- The target space M is topologically trivial and can be covered by a single open patch.

With these assumptions, there are two topologically distinct 'elementary' cases[1] - closed and open worldlines, as defined in figure 3.1. There is clearly a procedure where more complicated worldlines, such as a figure of eight, can be decomposed into multiple copies of these elementary worldlines with suitable boundary conditions.

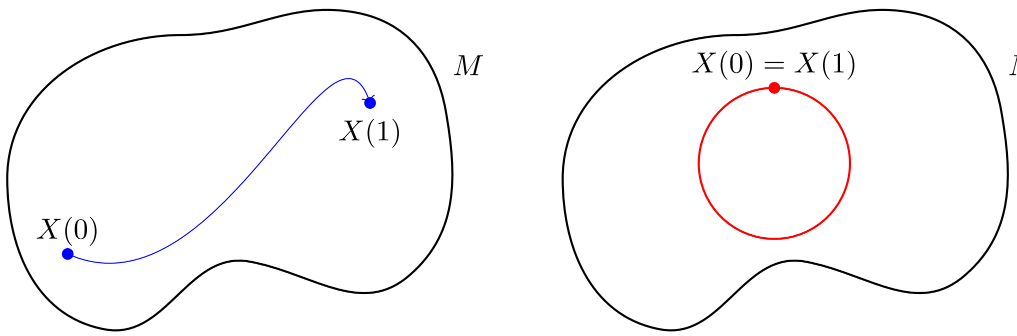

***Figure 3.1**: An open worldline (left) is where X does not meet itself. A closed worldline (right) is where X meets itself precisely once at $X(0) = X(1)$.*

[1]When considering several worldlines, this discussion is still valid for $D \geq 4$. For $D = 3$ an added complication exists - since the generic codimension of the immersion is one, the worldlines can knot. In physics these are known as anyons.

Actions and Hamiltonians. The next step is to introduce an action for this system. It follows from the previous chapter that such a quantum system exhibiting a scalar symmetry of the Poincaré group can be constructed from the Hamiltonian

$$H[X,P,e] = \int_I \left\{ P_\mu \dot{X}^\mu - \frac{1}{2} e(P^2 + m^2) \right\} d\tau \ , \tag{3.1.1}$$

where $e(\tau)$ is an auxiliary Lagrange multiplier field and $\dot{X}^\mu = dX^\mu/d\tau$. Defining $\dot{X}^2 = G_{\mu\nu}(X)\dot{X}^\mu \dot{X}^\nu$ and going over to the Lagrangian with $P_\mu = m\dot{X}_\mu/\sqrt{-\dot{X}^2}$ gives an action

$$S[X,e] = \frac{1}{2} \int_I \left\{ \frac{1}{e} \dot{X}^2 - em^2 \right\} d\tau \ . \tag{3.1.2}$$

The equation of motion of $e(\tau)$ yields a constraint $\dot{X}^2 + e^2 m^2 = 0$. If we assume the particle is not tachyonic, i.e. $m^2 \geq 0$, the only classical configurations are therefore time-like or null curves.

Equation (3.1.1) clearly identifies e as associated with the first class constraint $\phi = P^2 + m^2 = 0$. As we saw from the last chapter, this constraint projects onto the irreducible representation of the Poincaré group with Casimir $P^2 = -m^2$.

Symmetries. It is a general fact that a first class constraint ϕ enforces a gauge redundancy for a phase space element Q, infinitesimally acting as $\delta_\chi Q = \frac{1}{2}\chi\{Q,\phi\}$. Alternatively, we can note that the configuration space action (3.1.2) already has a symmetry: to make this most manifest, we define a tensor $g = g_{\tau\tau} d\tau^2$ where $g_{\tau\tau} = e^2$ and rewrite (3.1.2) as

$$S[X,g] = \frac{1}{2} \int_I \sqrt{g} \left\{ g^{\tau\tau} G_{\mu\nu}(X) \partial_\tau X^\mu \partial_\tau X^\nu - m^2 \right\} d\tau \ . \tag{3.1.3}$$

This has the form of one-dimensional quantum gravity with worldline metric g and cosmological constant $-m^2$. Indeed, the integrand is invariant under diffeomorphisms $f : I \to I$ which act as[2]

$$\tau \mapsto f(\tau) \ , \quad g_{\tau\tau} \mapsto \frac{1}{f'(\tau)^2} g_{\tau\tau} \ , \quad X^\mu(\tau) \mapsto X^\mu(\tau) \ . \tag{3.1.4}$$

Note that this means that the einbein transforms as $e(\tau) \mapsto e(\tau)/|f'(\tau)|$ where we have chosen the positive square root.

It is useful to write these transformations infinitesimally. With $f(\tau) = \tau + \xi$ one has, to first order in ξ,

$$\delta_\xi e = \partial_\tau(\xi e) \ , \quad \delta_\xi X^\mu = \xi \dot{X}^\mu \ . \tag{3.1.5}$$

[2]Note that if $f : I \to I$ is a diffeomorphism its derivative does not vanish.

The corresponding variation in the action vanishes except for a boundary term

$$\xi\left[\frac{\dot{X}^2}{e}+em^2\right]_{\partial I} . \tag{3.1.6}$$

For the open worldline, this generically requires $\xi(0)=\xi(1)=0$; that is, the diffeomorphism should fix the boundary of I. For the closed worldline, it is sufficient to have $\xi(0)=\xi(1)$ and the diffeomorphism group contains translations $\tau\mapsto\tau+c$ with c a constant.

For both kinds of worldline, we appear to have a choice of whether to gauge the entire diffeomorphism group or just its identity component (the other component reverses the orientation of I). This corresponds to unoriented and oriented worldlines respectively.

Gauge Fixing the Einbein. Let us conclude this section by analysing in more detail how the group of diffeomorphisms acts on the einbein. We will denote by $\mathrm{Met}(I)$ the space of all metrics on I. The previous discussion shows how the group $\mathrm{Diff}(I)$ of diffeomorphisms of I has a natural action on $\mathrm{Met}(I)$. Now, as measured by the einbein $e(\tau)$, the length of I given by

$$T=\int_I e(\tau)d\tau , \tag{3.1.7}$$

is clearly diffeomorphically invariant. Conversely, any two einbeins $e(\tau)$ and $e'(\tau')$ which yield the same such length can be related by the diffeomorphism $\tau\mapsto\tau'(\tau)$ given by

$$\int^{\tau'} e'(\sigma)d\sigma=\int^{\tau} e(\sigma)d\sigma . \tag{3.1.8}$$

Thus, if we restrict ourselves to positive einbeins $e(\tau)>0$, the space of orbits is $\mathrm{Met}(I)/\mathrm{Diff}(I)\cong\mathbb{R}_{>0}$. This is the *moduli space* of metrics on the worldline.

It follows that a third and final classical action can be found by using diffeomorphisms to partially gauge fix $e(\tau)=T$:

$$S[X,T]=\int_I\left\{\frac{1}{T}\dot{X}^2+m^2T\right\}d\tau , \tag{3.1.9}$$

where T is still a gauge parameter, but lives in a much smaller one-dimensional space - the upper half line.

Finally, throughout all of this, the reader will have noticed that there are ambiguities in the sign of the action which arise from the sign of $e(\tau)$. Being a Lagrange multiplier we can always use a convention that it is non-negative. However, a proper treatment of the quantum theory will force us to be more careful. We note that a negative sign of $e(\tau)$ is associated with the particle travelling backwards on the worldline.

Geometric Action. If we assume $m \neq 0$ then we can integrate out $e(\tau)$ in (3.1.2) leaving the geometric action[3]

$$S[X] = -m \int_I d\tau \sqrt{-G_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}\ . \tag{3.1.10}$$

In other words, the action is just the length in spacetime of the worldline. It is a matter of taste which we take to be more fundamental - equations (3.1.2) or (3.1.10) (assuming that their quantization is the same). However, from our philosophy of starting with irreducible representations of the Poincaré group, it is (3.1.2) to which we are naturally led. This has the added benefit of being directly applicable to the massless $m = 0$ case.

3.2 The quantum theory for a scalar particle with no interactions

In this section, we will quantize the classical theory of the previous section. A conceptually deep perspective using the idea of bordisms has been given by Atiyah and Segal [22] for generic quantum field theories. In this picture, correlation functions are naturally calculated by path integrals. We start by quantizing the X field whilst leaving $e(\tau)$ as a background field. Towards the end of this section, we will quantize the einbein as well.

In the vision of Atiyah and Segal (see for example [22]), a quantum one-dimensional theory has the following features:

- It assigns to every zero-dimensional manifold (i.e. a point), a Hilbert space $\mathcal{H}$.
- It assigns a partition function $\mathcal{Z}(C)$ for every closed one-dimensional manifold C. In our case, where the target space M is topologically trivial, this means C is the closed worldline.
- To every set of points $p_1, \ldots, p_n \in M$ which are bordant[4] via a one-dimensional manifold C, it assigns linear maps

$$\mathcal{Z}(C \backslash p_1 \cup \cdots \cup p_n) : \mathcal{H}_{p_1} \otimes \cdots \otimes \mathcal{H}_{p_n} \to \mathbb{C}\ , \tag{3.2.1}$$

  which are the *correlation functions* on C with local operators at $p_1, \ldots, p_n$.

Let us see, one-by-one, how these ideas translate into our theory:

- For every point, we assign the Hilbert space to be $\mathcal{H} = L^2(M, \mathbb{R})$, the space of square integrable functions on M. It is customary to enlarge this space to include

[3] Note how the sign ambiguity of $e(\tau)$ is transferred to the sign ambiguity of the square root in the geometric action.

[4] Two manifolds $M_1, M_2 \subset M$ are bordant if there exists a manifold X such that the boundary $\partial X = M_1 \coprod M_2$, the latter being the disjoint union.

distributions, in which case a basis is given by delta functions at $x \in M$ (in Dirac notation these are $|x\rangle$). We can define an operator $\hat{X}$ on $\mathcal{H}$ by $\hat{X}|x\rangle = x|x\rangle$ and extending linearly. This basis defines a dual space $\mathcal{H}^*$ with basis $\langle x|$ such that $\langle x|y\rangle = \delta(x-y)$.

- Every closed worldline C with modulus T is associated with a complex number via the partition function

$$Z(T) = \int \mathcal{D}X e^{-S_C[X,T]/\hbar}\,, \tag{3.2.2}$$

where we use the partially gauge fixed action (3.1.9).

- Given two points $x, y \in M$ which are bordant via a worldline $X(\tau)$ with modulus T, the map $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ is given by

$$|x\rangle \otimes |y\rangle \mapsto \langle x|y\rangle_T := \int_{X^\mu(0)=x^\mu}^{X^\mu(1)=y^\mu} \mathcal{D}X e^{-S_C[X,T]/\hbar}\,, \tag{3.2.3}$$

and extending linearly.

More generally, given x and y which are bordant via a worldline X, and n points lying inside X, say $z_i = X(\tau_i)$, we assign

$$|x\rangle \otimes |z_1\rangle \otimes \cdots \otimes |z_n\rangle \otimes |y\rangle \mapsto \int_{X(0)=x, X(\tau_i)=z_i}^{X(1)=y} \mathcal{D}X e^{-S[X,T]/\hbar}\,. \tag{3.2.4}$$

This should really be seen as a means to evaluate correlation functions

$$\langle x|\hat{\mathcal{O}}(\tau_1)\cdots\hat{\mathcal{O}}(\tau_n)|y\rangle = \int_{X(0)=x}^{X(1)=y} \mathcal{D}X\ \mathcal{O}(\tau_1)\cdots\mathcal{O}(\tau_n) e^{-S[X,T]/\hbar}\,, \tag{3.2.5}$$

where the operators $\mathcal{O}$ are built from $\hat{X}$ operators. The equivalence is seen by inserting complete states with $\hat{\mathcal{O}}(\tau_i)|z_i\rangle = \mathcal{O}(\tau_i)|z_i\rangle$ and using the sewing property of the path integral.

Of course, all of this implicitly requires us to be able to calculate the path integral. We will return to this shortly. But first, we must stop neglecting the gauge symmetry.

Integrating over the background metric. Up to now, we have kept the worldline metric as a background field. However, in the full theory, this must be integrated over. In fact, if the *spacetime* field X^μ has Minkowski signature, the previous path integrals over X won't actually converge, due to the wrong sign in the kinetic term of X^0. The gauge symmetry associated to e is crucial to removing the degrees of freedom that cause this difficulty.

Naively, one might want to define the quantum mechanical amplitudes using the path integral measures such as

$$\int \mathcal{D}X\mathcal{D}e \; e^{-S[X,e]} \; . \tag{3.2.6}$$

This is wrong because it ignores the gauge symmetry. Indeed, the total configuration space $\mathcal{E} = \{(e(\tau), X(\tau))\}$ has a huge redundancy - the same physical configuration is described by many points on this space. As we have noted, the gauge group $G = \mathrm{Diff}(I)$ of diffeomorphisms acts on this space as

$$f \cdot X = f^* X = X \circ f \; , \tag{3.2.7}$$

$$f \cdot e = f^* e = \frac{1}{|f'|} e \circ f \tag{3.2.8}$$

for $f \in G$, leaving the Lagrangian invariant. Thus, physically distinct field configurations are precisely the ones which are not diffeomorphically related to each other. The naive path integral overcounts by summing over all elements in $\mathcal{E}$, instead of representatives from gauge orbits.

Mathematically, we have a bundle $\mathcal{E} \to \mathcal{E}/G$, the base space being the physical configuration space, which can be visualised as the total configuration space where each orbit has been shrunk to a point. One can imagine that this might be geometrically very complicated.

Let us attempt to integrate over this base space, following Nakahara [23]. Take $\tilde{X}$ and e_T to be particular representatives in each gauge orbit. An obvious choice for the latter is $e_T = T$, the length of the worldline. Then any pair $(X, e) \in \mathcal{E}$ can be written as $X = f^* \tilde{X}$ and $e = f^* e_T$ for some diffeomorphism f. The choice of e_T characterises f only up to zero modes (constant translations), which form a subgroup H of G. The remaining ambiguity is fixed, however, by the choice of $\tilde{X}$. We now wish to change variables (e, X) to $(f, e_T, \tilde{X})$. Thus we have

$$\int \mathcal{D}e\mathcal{D}X e^{-S[X,e]} = \int \mathcal{D}f\mathcal{D}e_T\mathcal{D}\tilde{X} \; Je^{-S[f^*\tilde{X}, f^* e_T]} = \int \mathcal{D}f\mathcal{D}e_T\mathcal{D}\tilde{X} \; Je^{-S[\tilde{X}, e_T]} \; , \tag{3.2.9}$$

where J is an overall Jacobian and in the last equality we assume that the measure has no (one-dimensional gravitational) anomaly in the replacements $\tilde{X} \to f^* \tilde{X}$ and $e_T \to f^* e_T$. This clearly isolates the integration over the bundle $\mathcal{E}$ into vertical gauge fibres and horizontal physical fibres. Hence we *define* the path integral to be

$$\int \mathcal{D}e_T\mathcal{D}\tilde{X} \; Je^{-S} \; . \tag{3.2.10}$$

This process is colloquially called 'dividing out by the volume of G'.

There is, however, a problem due to the geometry of the base space: the space $\{e_T\}$ is very simple, whilst $\{\tilde{X}\}$ is not necessarily so. Indeed, the action of G on the space of metrics is so nice that there is a canonical choice $e(\tau) = T =: e_T$ for each orbit so that the space $\{e_T\}$ is simply the upper half line[5]. However, with this choice of e, the space $\tilde{X}$ is almost the space of maps $I \to M$, but we must identify functions related by zero modes: $\tilde{X}(\tau) \sim \tilde{X}'(\tau) := \tilde{X}(\tau + a)$ for a real constant a (this is only the case if $\mathrm{Diff}(I)$ contains constant translations, so that this only affects closed worldlines). Direct integration over such a space is not straightforward. Instead, it is simpler to reinstate the integration to sum over all functions $X(\tau)$ by separating the diffeomorphisms f into zero modes f_0 and the remainder f' and by using $\int \mathcal{D}X e^{-S[X,e_T]} = \int J_2 df_0 \int \mathcal{D}\tilde{X} e^{-S[\tilde{X},e_T]}$ where J_2 is some Jacobian. By doing this, one builds up Jacobians:

$$\begin{aligned}\int \mathcal{D}X\mathcal{D}e\, e^{-S[e,X]} &= \int \mathcal{D}X\mathcal{D}e\, e^{-S[e,X]} \\ &= \int df_0\mathcal{D}f'\mathcal{D}\tilde{X}\; de_T\; Je^{-S[e_T,\tilde{X}]} \\ &= \int df_0\mathcal{D}f'\mathcal{D}X\; de_T\; JJ_2^{-1}e^{-S[e_T,X]} \\ &= \int \mathcal{D}f\mathcal{D}X de_T\; V_H^{-1}JJ_2^{-1}e^{-S[e_T,X]}\end{aligned} \tag{3.2.11}$$

where V_H is the Jacobian associated with the change of variables decomposing $f \to (f_0, f')$ into its zero mode f_0 and non-zero modes f', whilst J is the Jacobian associated with the change of variables $(e, X) \to (e_T, \tilde{X}, f_0, f')$.

Let us now evaluate the necessary Jacobians. The tangent space at e_T and $\tilde{X}$ can be split into directions $\delta f', \delta f_0, \delta e_T$ and $\delta\tilde{X}$ and has an ultralocal diffeormorphism invariant inner product [24] defined by $(A, B) = \int d\tau e_T\; e_T^{-p+q} AB$ where A and B are such that their product is a (p, q)-form. With respect to this inner product, these directions are orthogonal. Now we can decompose arbitrary tangent vectors as $\delta e = \partial_\tau(e_T \delta f') + \delta e_T$ and $\delta X = \delta f' \partial_\tau \tilde{X} + \delta f_0 \partial_\tau \tilde{X} + \delta\tilde{X}$.

It then follows that the Jacobian $J = \det^{\frac{1}{2}}(P^\dagger, P)$ where

$$P = \frac{\delta(e, X)}{\delta(\delta f', e_T, \delta f_0, \tilde{X})} = \begin{pmatrix} D & 1 & 0 & 0 \\ \partial_\tau\tilde{X} & 0 & \partial_\tau\tilde{X} & 1\end{pmatrix} \tag{3.2.12}$$

In particular, J is the product of determinants J_1 and J_2 where $J_i = \det^{\frac{1}{2}}(P_i^\dagger, P_i)$ and P_1 is the top left block and P_2 is the bottom right block of P. The latter is precisely the same Jacobian J_2 coming from the change of variables $X \to \tilde{X} = f^*X$ for $f \in H$ (i.e. a zero mode) found in the third line of (3.2.11). Thus, in that equation we only need to find the Jacobian $JJ_2^{-1} = J_1$. The latter is easy to calculate by zeta function regularization or the

[5] Assuming we only consider positive $e(\tau)$.

equivalent path integral:

$$
\begin{aligned}
J_1 = \det{}'^{\,\frac{1}{2}}(P_1^\dagger, P_1) &= \int \mathcal{D}\delta f' d\delta e_T \mathcal{D}\delta\tilde{X} e^{-\int d\tau e_T^{-3}(e_T\partial_\tau \delta f')^2 + \int d\tau e_T\, e_T^{-2}\delta e_T^2} \\
&\propto \begin{cases} 1 & \text{for open boundary conditions,} \\ e_T^{1/2} & \text{for closed boundary conditions.} \end{cases}
\end{aligned}
\tag{3.2.13}
$$

In the above, note that for open boundary conditions the group H is trivial so that the integral over δe_T is empty and can be disregarded (equivalently the change of variables is simply $f = f'$). We have also normalized the path integral in $\delta f'$.

As for the other Jacobian V_H, when the worldline is closed so that H is non-trivial and contains the constant mode translations, it can be found from

$$
\int \mathcal{D}\delta f e^{-||\delta f||^2} = \int \mathcal{D}\delta f' d\delta f_0\, V_H e^{-\int d\tau e_T^3(\delta f_0^2 + \delta f'^2)} = e_T^{3/2} V_H^{-1} \int \mathcal{D}\delta f' e^{-||\delta f'||^2} \ , \tag{3.2.14}
$$

where we are using the diffeomorphism invariant inner product to normalize the path integrals. Of course, for the open worldline, H is trivial and so V_H is too. Thus

$$
V_H = \begin{cases} 1 & \text{(for an open worldline)} \ , \\ e_T^{3/2} & \text{(for a closed worldline)} \ . \end{cases} \tag{3.2.15}
$$

Combining these, one finds the gauge fixed path integral to be

$$
\int_{\text{Gauge fixed}} \mathcal{D}X\mathcal{D}e\, e^{-S[X,e]} = \begin{cases} \int \mathcal{D}X de_T e^{-S[X,e_T]} & \text{for open worldlines,} \\ \int \mathcal{D}X \frac{de_T}{e_T} e^{-S[X,e_T]} & \text{for closed worldlines.} \end{cases} \tag{3.2.16}
$$

3.2.1 Correlators on line segments

In this section, we give a few examples of the calculation of simple correlation functions. We use the theory with action (3.1.9) and where $I = [0, 1]$, but keep the worldline metric as a background field unless otherwise stated.

The Spacetime Propagator. We begin by evaluating the spacetime propagator

$$
K(x, y; T) = \langle x|y\rangle_T = \int_{X(0)=x}^{X(1)=y} \mathcal{D}X e^{-\int_0^1 d\tau \left\{\frac{1}{T}(\partial X)^2 + Tm^2\right\}/\hbar} \ , \tag{3.2.17}
$$

in a flat space M. This is a standard Gaussian path integral which can be evaluated by dividing the field X into a classical and fluctuating part as

$$
X(\tau) = X_c(\tau) + q(\tau) \ , \tag{3.2.18}
$$

such that X_c is a solution of the classical equation of motion together with boundary conditions $X_c(\tau_1) = x$ and $X_c(\tau_2) = y$, whilst the quantum piece $q(\tau)$ is any continuous path such that $q(0) = q(1) = 0$.

The classical trajectory is just the geodesic connecting x and y. In flat space then, the equation of motion and its solution are

$$\partial^2 X_c = 0 \ , \qquad X_c(\tau) = x + (y-x)\tau \ . \tag{3.2.19}$$

The quantum piece can be uniquely written as a Fourier series

$$q(\tau) = \sum_{n=1}^{\infty} a_n \sin(\pi n \tau) \ , \tag{3.2.20}$$

for real a_n. The path integral measure is then schematically $\prod_{n\geq 1,\mu} \int_{\infty}^{\infty} da_n^\mu$ up to some normalization. The importance of the quadratic action is that it splits as

$$S[X_c + q] = S[X_c] + S_q \ , \tag{3.2.21}$$

where the individual pieces are calculated to be

$$\begin{aligned} S_c[X_c] &= \frac{(x-y)^2}{T} + m^2 T \ , \\ S_q &= \frac{(\pi n a_n)^2}{T} \ , \end{aligned} \tag{3.2.22}$$

and cross terms between X_c and q vanish in the integral. Thus

$$\int \mathcal{D}X e^{-S/\hbar} = e^{-S[X_c]/\hbar} \int \prod_{n\neq 0,\mu} da_n^\mu e^{-S_q/\hbar} = e^{-S[X_c]/\hbar} \prod_{n=1}^{\infty} \left(\frac{T}{(\pi n)^2} \right)^{D/2} . \tag{3.2.23}$$

This product should be understood to be zeta function regularized, which can be considered as part of the definition of the path integral. (Physically, one could add counterterms to the action which carry out the same regularization). With this, one finally arrives at

$$K(x,y;T) = \frac{1}{T^{D/2}} e^{-\left\{\frac{(x-y)^2}{T} + m^2 T\right\}} \ . \tag{3.2.24}$$

Finally, in our quantum gravity example, the parameter T is itself gauged. Hence the only correlation function defined on any line segment is:

$$\langle x|y\rangle = \int_0^\infty dT \, \langle x|y\rangle_T = \int dT \frac{1}{T^{D/2}} e^{-\left\{\frac{(x-y)^2}{T} + m^2 T\right\}} \ . \tag{3.2.25}$$

This is just the Feynman propagator in Schwinger representation.

The Worldline Propagator: the Green Function. Let the worldline metric be a background field again and consider the following correlation function on a line segment with $I = [0,1]$:

$$G^{\mu\nu}(\tau_1, \tau_2; T) = \langle X^\mu(\tau_1) X^\nu(\tau_2)\rangle_T = \langle 0|X^\mu(\tau_1)X^\nu(\tau_2)|0\rangle_T \ . \tag{3.2.26}$$

There are several ways one can calculate this. Following our general prescription of calculating correlation functions, the worldline propagator is found by sewing appropriate path integrals together (see also figure 3.2). One then has

$$G^{\mu\nu}(\tau_1, \tau_2; T) = \frac{\int d^D x d^D y x^\mu y^\nu \, \langle 0|x\rangle_{T_1} \langle x|y\rangle_{T_2} \langle y|0\rangle_{T_3}}{\langle 0|0\rangle_T} \ , \tag{3.2.27}$$

where $T = T_1 + T_2 + T_3$ and, if $\tau_1 < \tau$ then $\tau_1 = T_1/T$ and $\tau_2 = (T_1 + T_2)/T$, otherwise $\tau_2 = T_1/T$ and $\tau_1 = (T_1 + T_2)/T$. This is clearly seen by the geometry of figure 3.2. Finally, the denominator in (3.2.27) is there to normalise the path integral correctly. After a little tedious algebra, one finds

$$G^{\mu\nu}(\tau_1, \tau_2; T) = \eta^{\mu\nu}\frac{T_1 T_3}{T} = \eta^{\mu\nu} T \Big\{ \Theta(\tau_2 - \tau_1)\tau_1(1-\tau_2) + \Theta(\tau_1 - \tau_2)\tau_2(1-\tau_1) \Big\} \ . \tag{3.2.28}$$

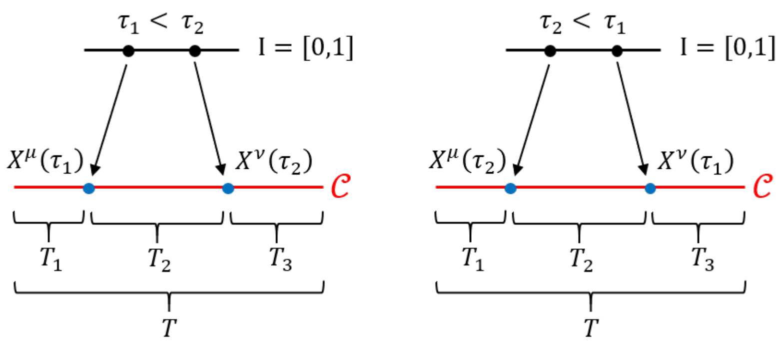

Figure 3.2: *A figure to illustrate the geometry of the worldline Green function. It can be found by sewing together the path integrals from $\tau = 0$ to τ_1, from τ_1 to τ_2 and τ_2 to τ_3, so that each individual worldline has modulus T_i. The overall modulus of the worldline is $T = T_1 + T_2 + T_3$.*

In passing, to see why G is a Green function for the worldline Laplacian, note that

$$0 = \int_{X(0)=0}^{X(1)=0} \mathcal{D}X \frac{\delta}{\delta X^\mu(\tau)} \left(e^{-S[X,T]} X^\nu(\tau') \right) \ . \tag{3.2.29}$$

Integrating the action by parts as $S = \int d\tau \sqrt{g}(X_\mu \partial_\tau (g^{\tau\tau}\partial_\tau X^\mu) + Tm^2)$ plus a boundary term, one instantly finds that $G(\tau, \tau'; T)$ must satisfy

$$\sqrt{g}\partial_\tau(g^{\tau\tau}\partial_\tau)G^{\mu\nu}(\tau, \tau'; T) = \eta^{\mu\nu}\delta(\tau) \ . \tag{3.2.30}$$

That is, the propagator G is the Green function corresponding to the worldsheet Laplacian. Using diffeomorphisms to fix $g_{\tau\tau} = T^2$ then rederives the solution (3.2.28) assuming boundary conditions $G(0, \tau; T) = G(\tau, 0; T) = 0$.

It might be of interest that the following function also acts as a Green function for the worldsheet Laplacian:

$$G(\tau, \tau'; T) = \frac{T}{2}|\tau - \tau'| \ , \tag{3.2.31}$$

which has the boundary condition that it vanishes as $\tau \to \tau'$.

3.2.2 Correlators on circles

Partition Function. We begin with the partition function on a circle, again treating the metric as a background field. This is the closed string correlator, given by the path integral

$$Z(T) = \int \mathcal{D}X e^{-S} \ . \tag{3.2.32}$$

This is accomplished, just as for the spacetime propagator, by expanding the field X into classical and fluctuating parts. Alternatively, we can simply use the sewing rules to sew a line segment into a circle. Then

$$Z(T) = \int d^D x \, \langle x|x\rangle_T = \frac{V_M}{T^{D/2}} e^{-m^2 T} \ , \tag{3.2.33}$$

where V_m is the volume of M - for flat space the resulting divergence is an infrared effect.

In the quantum gravity version, we must integrate over the moduli space $T > 0$. As usual, we can exchange the integral over small diffeomorphisms on I with the fixing of the metric to a constant $e(\tau) = T$. But now, there are also large diffeomorphisms $\tau \mapsto \tau + \tau_0$ with τ_0 constant which we must also account for. As we have already explained, this is done by dividing by an extra power of T. Thus the partition function is the number

$$Z = \int \frac{\mathcal{D}g}{V(\mathcal{G})} \int \mathcal{D}X e^{-S[X,g]} = \int_0^\infty \frac{dT}{T} Z(T) \ . \tag{3.2.34}$$

This expression can also be seen by sewing the open string correlator $\langle x_1|x_2\rangle$ setting $x_1 = x_2$. In this picture, the extra $1/T$ comes about because there are of order $\sim T$ different x_1 coordinates which give the same closed worldline.

It should be noted that if $Z(T)$ does not approach zero fast enough as $T \to 0$ - as in the above expression for $Z(T)$ - then this integral has a divergence. Because of the short length of the line segments, this is a UV divergence. Regularizing the theory, such as subtracting a term e^{-M^2T} from the partition function - a Pauli-Villars type regularization - will deal with this, but its justification can only be adequately met by a discussion of renormalization which would take us too far afield. The upshot is that something is going wrong as the closed worldline shrinks to zero. This problem will be addressed in the next chapter. We only note here that there are two essential ways of dealing with it - by using a more appropriate partition function such as from a supersymmetric theory which would force the partition function to vanish, or by being more radical and modifying the worldline formalism we have been discussing here.

Green function on a circle. We now calculate the correlator

$$G_\circ(\tau_1,\tau_2;T) = \langle 0|X^\mu(\tau_1)X^\nu(\tau_2)|0\rangle_T \tag{3.2.35}$$

on a circle with modular parameter T. Using methods as above, one sees that $G_\circ$ is a Green function for the circle Laplacian with a zero mode removed:

$$\Delta G_\circ(\tau_1,\tau_2;T) = \delta(\tau_1-\tau_2) - 1\ . \tag{3.2.36}$$

A solution with appropriate boundary conditions is

$$G_\circ(\tau_1,\tau_2;T) = 2\pi T\left(\tau_{12}^2 - |\tau_{12}|\right)\ , \tag{3.2.37}$$

where $\tau_{12} = \tau_1 - \tau_2$.

3.2.3 Many worldlines and field theories in spacetime

The next logical step is to define the theory for a composite system of several disjoint[6] worldlines, as depicted in figure 3.3. We can regard this theory as one where the total one-dimensional universe $\mathcal{C}$ is a union of disjoint universes $\mathcal{C}_i$ (the worldlines).

We assume that each worldline is given a chart defined on the same interval $I = [0,1]$ and with coordinate functions $X_i^\mu(\tau)$. Furthermore, each has its own einbein $e_i(\tau)$. Then at time τ, the general state of n disconnected worldlines is the tensor product $|x_1,\ldots,x_n\rangle = S(|x_1\rangle\otimes\cdots\otimes|x_n\rangle)$. For indistinguishable particles, S should have the effect of symmetrizing or antisymmetrizing since the square modulus of the amplitudes should remain invariant under any permutation.

[6] In the next section we discuss the case of interacting particles.

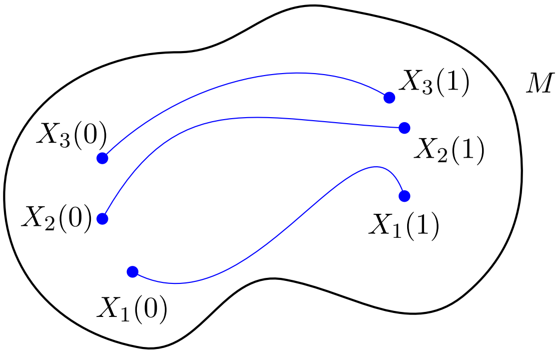

Figure 3.3: *The trajectory of multiple worldlines $X_i(\tau)$ in M.*

The action for N worldlines is naturally the sum

$$S[X_1, \dots, X_N, e_1, \dots, e_N] = \sum_{i=1}^{N} S[X_i, e_i] \;, \tag{3.2.38}$$

and it is straightforward to calculate the corresponding correlation functions. Of course, for a large number of particles this will become increasingly complex, and the path integral measure

$$\int \mathcal{D}X_1 \cdots \mathcal{D}X_N \tag{3.2.39}$$

will become undefined - at least rigorously - in the limit $N \to \infty$. In the next section, we want to allow the particle number to vary. This is done by allowing worldlines to meet at junctions, and summing over the resulting worldgraphs will force us to take seriously such a limit. In the limit $N \to \infty$, the states $|x_1, \dots, x_n\rangle$ (with n arbitrary) span a Fock space. But for the moment, N is to be kept fixed and the worldlines disjoint.

We will now describe the reformulation of this theory as a spacetime field theory - that is, one which is defined on M itself. We introduce creation operators $a^\dagger(p)$ which act on the Fock space in the natural manner:

$$a^\dagger(p)\,|0\rangle = |p\rangle \;, \qquad a^\dagger(p_1)a^\dagger(p_2)\,|0\rangle = |p_1, p_2\rangle \;, \tag{3.2.40}$$

and so on, with the physical states $|p\rangle$ obeying the on-shell condition $p^2 = -m^2$ due to the Lagrange constraint in (3.1.2). Next, we define the conjugates $a(p)$ to annihilate the vacuum and to have the commutation relation $[a(p_1), a^\dagger(p_2)] = \delta^{(D)}(p_1 - p_2)$. Then the operator-valued functional

$$\varphi(X) = \int dp\, \langle X|p\rangle\, a(p) + h.c. \tag{3.2.41}$$

has the property that

$$\varphi(X)\,|0\rangle = |X\rangle \ . \tag{3.2.42}$$

We can understand this as a collection of all the positions the particle occupies as it moves through spacetime. In other words, $\varphi(X)$ creates the entire trajectory $X(\tau)$ of the particle.

Now let us see how we can construct a single operator that acts as the Hamiltonian on the N-particle state (see [25] for this approach). One has, on a generic state $|p\rangle$, the single particle Hamiltonian acting as

$$H\,|p\rangle = (p^2 + m^2)\,|p\rangle \tag{3.2.43}$$

and so the N particle Hamiltonian acts as

$$\begin{aligned} H^{(N)}\,|p_1, \ldots, p_N\rangle &= \sum H\,|p\rangle_i \otimes |p_1, \ldots, \hat{p}_i, \ldots, p_N\rangle \\ &= \sum (p_i^2 + m^2)\,|p_1, \ldots, p_N\rangle \\ &= \left(\int d^d p (p^2 + m^2) a_p^\dagger a_p\right) |p_1, \ldots, p_N\rangle \ , \end{aligned} \tag{3.2.44}$$

where H is the single particle Hamiltonian of the previous sections and $\hat{p}_i$ means the omission of p_i. In the above, depending on how the Fock space is symmetrised, there could be a sign in the first line, but this would disappear again in the second. If we Fourier transform the above using the definition of $\varphi(X)$ above, then the upshot is that the N-particle Hamiltonian has an equivalent form as an operator

$$H^{(N)} = \int d^d x (-(\partial\varphi(X))^2 + m^2\varphi^2) \ , \tag{3.2.45}$$

acting on the Fock space. This procedure is called second quantization and is familiar from statistical physics[7]. Because it is independent of N, one could conceivably take a limit $N \to \infty$.

On the other hand, the second quantized formalism is not completely without subtlety. One can intuitively see that if we use the corresponding Lagrangian to do path integrals, then what was before a path integral over all possible trajectories $X_i(\tau)$ must become a path integral over all possible $\varphi(X)$. This kind of path integral is mathematically less well defined.

[7] It is sometimes said that the name ‘second quantization’ is wrong because we are not actually quantizing anything - merely changing formalism. We agree with this, but the terminology is too widespread to avoid.

$$|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \qquad \mapsto \qquad \lambda \int dz$$

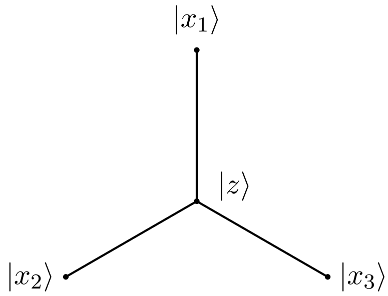

***Figure 3.4**: A trivalent junction of worldlines.*

$$\langle x_1|V_p(\tau)|x_2\rangle \qquad =$$

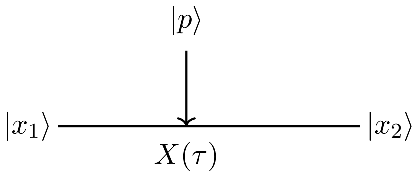

***Figure 3.5**: A state of definite momentum p interacting with a worldline.*

3.3 Junctions and vertex operators

In this section, we relate our worldline theory to perturbative scalar quantum field theory with interactions. We begin by allowing the worldlines to join each other - we will call these sites 'junctions'. It is simplest to restrict to the case of trivalent junctions only, as illustrated in figure 3.4. Such a junction must then be associated[8] with the correlation function

$$Z_3 : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \to \mathbb{C}\ , \qquad |x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \mapsto \lambda \int dy\, \langle x_1|y\rangle\, \langle y|x_2\rangle\, \langle y|x_3\rangle\ , \tag{3.3.1}$$

where λ is some real number. Since any trivalent worldgraph can be sewn from line segments, circles and these trivalent junctions, this uniquely defines the theory on all trivalent worldgraphs.

A basic example of this is when we want an external state of definite momentum p to interact with an open worldline. In this case, we have an associated amplitude

$$A(x_1, x_2; p) = \lambda \int dy\, \langle x_1|y\rangle\, \langle y|p\rangle\, \langle y|x_2\rangle = \int_I d\tau\, \langle x_1|\lambda e^{ip\cdot X(\tau)}|x_2\rangle\ . \tag{3.3.2}$$

In other words, the insertion into correlators of the vertex operator

$$V_p(\tau) = \lambda e^{-ip\cdot X(\tau)} \tag{3.3.3}$$

[8]This form of the correlation function is more-or-less unique if we want the interaction to be truly local. One could instead attempt to be more creative by allowing non-local vertices but these generically suffer from unitarity problems.

has the effect of an external state with definite momentum p interacting at the point τ in the system (see figure 3.5). For an insertion into a closed loop, one should instead use the integrated vertex operator

$$V_p(\tau) = \lambda T e^{ip\cdot X(\tau)} , \tag{3.3.4}$$

which accounts for the degeneracy in the number of positions on the loop that the external state could be injected in.

As an example to illustrate all of this, let us calculate the one-loop two-point amplitude with trivalent vertices:

$$A(p_1, p_2) = \int_0^\infty \frac{dT}{T} \int_{[0,1]^2} dv_1 dv_2 T^2 \left\langle 0 | V_{p_1}(v_1) V_{p_2}(v_2) | 0 \right\rangle_{S^1(T)} . \tag{3.3.5}$$

Recall that the extra $1/T$ factor accounts for the large diffeomorphisms, corresponding to rotations of the circle. Alternatively, one can drop that extra factor as long as we fix the position of the first vertex operator to $v_1 = 0$, so that the diffeomorphism group no longer includes constant translations. Thus:

$$A(p_1, p_2) = \int_0^\infty dT \int_0^1 dv T^2 \left\langle 0 | V_{p_1}(0) V_{p_2}(v) | 0 \right\rangle_{S^1(T)} . \tag{3.3.6}$$

This is just

$$\begin{aligned} A(p_1, p_2) &= \int_0^\infty dT\, T^2 \int_0^1 dv \left\langle 0 | e^{ip_1\cdot X(0)} e^{ip_2\cdot X(v)} | 0 \right\rangle_{S^1(T)} \\ &= \int_0^\infty dT \mathcal{Z}(T)\, T^2 \int_0^1 dv e^{-p_{1\mu} p_{2\nu} \langle X^\mu(0) X^\nu(v) \rangle_T} \\ &= \int_0^\infty \frac{dT}{T^{d/2-2}} \int_0^1 dv e^{-p_1\cdot p_2 T v(1-v) - m^2 T} , \end{aligned} \tag{3.3.7}$$

using the Green function in equation (3.2.37) and the partition function $\mathcal{Z}(T)$ of the circle. This is the the same as the familiar two-point function in φ^3 theory. In a famous series of papers by Bern and Kosower (in particular [16]), it was pointed out that these kinds of worldline techniques can result in a faster approach to calculate Feynman diagrams than more traditional methods.

3.4 Worldgraphs

We conclude our discussion of the one-dimensional scalar theory by describing the full quantum gravity in which worldgraphs are summed over. What junctions are allowed is regarded as an input to the theory[9]. For simplicity, we will only consider trivalent junctions.

[9] This is precisely analagous to the way that interaction terms in a spacetime theory are chosen by hand - e.g. φ^3 or φ^4 terms in the potential.

The quantum gravitational path integral must now sum over all possible graphs. Over each graph, it integrates over all posssible trajectories having that topology. As before, this reduces to an integral over all the moduli T_i of the graph, which we can interpret as the proper length of each leg. Figure 3.6 illustrates this.

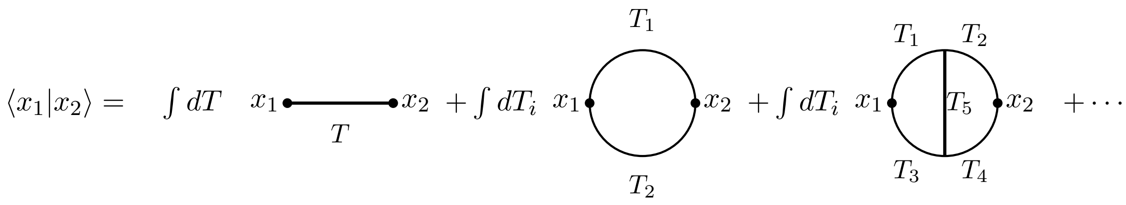

***Figure 3.6**: The quantum gravitational path integral as an expansion of trivalent worldgraphs.*

Here all the moduli T_i should be integrated over for each graph. There is an interesting mathematical interpretation of this [26, 27] as a sum over integrals of cell complexes M_g, where each complex M_g consists of g-loop diagrams, such that the cell boundary consists of the limits where one modulus shrinks to zero. This is reminiscent of the loop expansion of string theory.

By ordering the expansion by the number of vertices, the amplitudes can be written as an infinite sum

$$A = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots \tag{3.4.1}$$

where A_n corresponds to the amplitude of the graphs with n vertices. This is where we meet the limits of the worldline formalism, for generically this series eventually diverges. Even though one might be able to resum the series, the theory can only really be thought of as a perturbative approximation of something more fundamental. In our case, that non-perturbative completion is scalar spacetime quantum field theory, which has precisely this perturbative expansion of its amplitudes.

If we are conservative, we can regard the worldline formalism simply as a perturbative definition of certain quantum field theories useful for practical perturbative calculations. However, one can take a more radical approach by attempting to modify the worldline theory and seeing whether a non-perturbative completion is now needed. Even if this is the case, if such a modified worldline theory can be constructed with UV safe amplitudes, it still provides a very attractive proposal for what a UV complete theory could look like. This is the philosophy of perturbative string theory, but we can apply it more generally. This is the subject of the next chapter.

Chapter 4

Worldline Theories with Towers of States

4.1 Introduction

There are several reasons to go beyond local quantum field theories (LQFTs). In general, QFTs - including the standard model - suffer from UV divergences which render them unacceptable as truly fundamental theories. We can identify two philosophies that one could adopt in order to design better behaved theories. More conservatively, one can stay within the framework of LQFT and try to impose extra mechanisms or symmetries (e.g. conformal or supersymmetries) in order to tame UV divergences. The second approach is more radical - to replace standard quantum field theory altogether by some new UV completion. The advantage of the worldline description we have discussed in the previous chapter is that it very clearly exposes how UV divergences arise from short distance singularities. Its perspective gives us more intuition into how such divergences could be remedied by modifying the theory in some way. This lends itself to more creative types of model building. String theory is the prime example of this approach, successfully removing short distance singularities by replacing worldlines with higher dimensional worldvolumes. But in this chapter, we want to be more general and ask whether other modifications can be chosen which also have viable UV completions, or at least with favourable perturbative UV properties.

Moreover, in the context of local quantum field theories, it appears that gravity can at best be regarded as an effective field theory. This raises another reason why one might like to go beyond the LQFT regime - there is much reason to think that gravity requires some degree of non-locality in its quantum description. It is, therefore, of interest to modify the worldline theory so that it resembles a non-local field theory.

Our proposed modification in this chapter is to build a tower of internal states on top of a standard scalar particle theory. The resulting theories will consist of a propagating particle which has a possibly infinite number of harmonic oscillator degrees of freedom. The most extreme version is to have an infinite tower of states on the worldline which mimic the internal modes of a string. In fact, we will explicitly carry out a dimensional reduction of the string to obtain such a theory. Ultimately, however, we aim to be agnostic about the origins of the tower, and simply adopt a model builder's attitude of exploring what kinds of towers lead to interesting physics. In fact, we will *in some cases* find geometric origins; for example, from the modes of a string in a curved background, or from a higher dimensional membrane, but in others, no geometric origins seem to present themselves.

The theories we will construct resemble standard string physics, yielding UV finiteness and even modular invariance. They will rely on the use of Borcherds products to construct modular invariant partition functions. We will also take some time to explore theories with a truncated tower consisting of a finite number of internal states, which are less well-behaved but still have interesting properties.

An alternative way to view this chapter is to consider it as a more conceptual study into how string theory 'works' from the particle point of view. Our modified theories can act as a bridge between point particle theories and string theory - increasing the number of states in the tower adds corrections which bring us successively closer to a theory of strings. We can use this bridge to measure the number of ingredients that are needed to be added to a worldline theory in order for it to share a certain property with string theory. For example, one can analyse what makes string theory UV complete, or ask how the soft amplitudes that string theory enjoys arise, from our perspective. We will see that our worldline theories can be argued to be UV complete based purely on their spectrum. We will also find that, even if we truncate the tower to just one extra state, one obtains a saddle point in certain amplitudes, which enables them to mimic the Gross-Mende saddle points in string theory.

We can also use these theories to analyse how non-locality plays a role in string theory. This issue has been extensively studied [28–33] and it is revealed that, in certain regimes, string theory behaves locally whilst in others it does not. A simple way to see this ambiguity is as follows. The spacetime propagator of string theory can be written as a sum over individual string excitations, Kaluza-Klein and winding modes, so that one can obtain a spectral function resembling a local field theory (with an infinite number of states) [34]. On the other hand, measuring these states individually is impossible - resonances from increasingly high energy modes become increasingly broad as the number of decay channels to lower energy modes increases. This means they overlap with each other and at very high energies we do not see the individual peaks at all. Furthermore, in certain kinematic regimes, string amplitudes behave in a way impossible for local quantum field theories - violating the Froissart bound - in a way reminiscent of non-local field theories [29]. Our approach allows some insight into how this works from a particle perspective.

Of particular interest to us are the Green functions associated with loops. This is for two reasons. First, the Green function is what measures how the theory behaves when probed at small distances, by which we mean when vertices of a diagram are brought close together compared to the diagram's size itself. A normal scalar particle's Green function is not entirely satisfactory because it vanishes as vertices collide, reducing amplitudes to behave like the divergent partition function. Second, if non-locality is present in the theory, then this should be reflected in the behaviour of Green functions as vertices separate. For example, the one-loop Green function of string theory makes a diagram, whose vertex separations z are small compared with the loop size τ, equivalent to a diagram with vertex separations z/τ on a loop of size $1/\tau$. This has the consequence that the theory looks like one with a minimal length, where vertex collisions on large loops of size τ can be interpreted as non-local interactions on the order $1/\tau$. This non-locality however arises *only* at one-loop. At tree level, propagators really do look like a local field theory (with many states). This 'tree-level plus nonlocality-at-one-loop' has inspired the "infinite-derivative-field theories" of [35–38]. It is interesting for us, therefore, to study the Green functions of our theories in order to see whether the spacetime theory can be similarly interpreted to resemble a non-local field theory.

This chapter is based on the author's work [1] and closely follows the plan of that paper. In section 4.2, we give a worldline description of string theory by compactifying the worldsheet action to a worldline, thus obtaining our first theory of a particle with an infinite tower of internal states. We will use this as a model theory for the next subsections in which we analyse its physical implications. After this, we make a brief diversion to note how our theory can be expressed as a non-local *worldline* theory. In 4.2.2 and 4.2.3, we reconstruct the modular invariant partition function from the worldline techniques of the previous chapter, before finishing the section with a discussion of the one-loop Green functions. In section 4.3, we describe the truncation of this model theory to a finite number of internal states and explore the consequences. In section 4.4, we discuss the hard-scattering limit of the amplitudes of the theory and show that one obtains Gross-Mende-like behaviour even with a truncated tower. Finally in 4.5, we take a step back and search for other theories with different worldline tower spectra, which share similar properties such as modular invariant partition functions, and discuss whether they have geometrical origins. We also will include a brief description on the addition of fermions into this framework.

4.2 A worldline perspective on string theory

We begin by investigating the relationship between non-local particle theories and string theories. To do this, we construct a theory on the worldline which secretly encodes the modes of a string. This enables us to see how string behaviour emerges from the addition of heavier modes onto the worldline. From this bottom-up perspective of string theory, we will then move on to studying the behaviour of the loop amplitudes and their Green functions, enabling us to picture how such a worldline theory can be seen to be non-local in spacetime.

4.2.1 String theory as Kaluza-Klein on a cylinder

We have seen in the previous chapter that the traditional worldline formalism treats a quantum field theory (QFT) perturbatively as a one-dimensional gravitational theory, whose universes correspond to legs of a corresponding spacetime Feynman diagram. In the previous chapter we saw how the simplest such theory, which describes the propagation of space-time scalars, is

$$S[X_m, g] = \frac{1}{4\pi\ell_s^2}\int_M d\tau\sqrt{g}\left\{g^{\tau\tau}G_{\mu\nu}\partial_\tau X_0^\mu \partial_\tau X_0^\nu\right\}\,, \tag{4.2.1}$$

where the worldline scalar field is now denoted by X_0^μ for reasons to become clear; also, we have introduced a length scale ℓ_s, and set $G_{\mu\nu}$ to be the background metric. Now, however, we will add a tower of internal states with worldline masses, so that the action becomes

$$S[X_m, g] \;=\; \frac{1}{2\pi\ell_s^2}\int_0^1 d\tau\sqrt{g}\left\{\frac{1}{2}g^{\tau\tau}|\partial_\tau X_0|^2 + \sum_{m=1}^{N}\left(g^{\tau\tau}|\partial_\tau X_m|^2 + (2\pi)^2 f_m|X_m|^2\right)\right\}\,, \tag{4.2.2}$$

where the X_m are complex worldline scalar fields, N is either a large integer or formally infinite (as it is in string theory), and the worldline mass-squared f_m is some function of the integers. As we will shortly see, for certain functions $f_m = m^2$, the modes X_m can be interpreted as the Kaluza-Klein modes of a propagating string whose worldsheet is a cylinder. The central idea is to see how much of string physics is captured by the addition of some or all of these modes.

To derive such an action from string theory, we will carry out a straightforward dimensional reduction of the Polyakov action for closed strings propagating with a cylindrical worldsheet. We start with the simplest Polyakov action:

$$S[X, g] \;=\; \frac{1}{4\pi\ell_s^2}\int_\Sigma d^2\sigma\sqrt{g}\left\{g^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right\}\,. \tag{4.2.3}$$

As usual, we could add a cosmological constant and an antisymmetric $B_{\mu\nu}$ field, but for the sake of simplicity we will not. Let us summarise the other assumptions and conventions before proceeding: throughout, we will assume the target space metric $G_{\mu\nu}$ is flat and can be Wick rotated to Euclidean signature. The path integral involves summing over all Riemann surfaces Σ and this topological dependence has been notationally suppressed in $S[g, X]$. We extend this sum to include Riemann surfaces with boundaries - so that, in a moment, we will be taking Σ to be a cylinder. The manifold Σ is parametrised, at least locally, by σ_a with $a = 1, 2$, and has a Euclidean metric (so that the Minkowski time-like coordinate is $\sigma_0 = i\sigma_2$). Assuming that there are $D = 26$ X^μ fields, the theory has no conformal anomaly and so the conformal symmetry of the action is a true gauge symmetry. In the following, we often drop the μ subscripts and take just one X field, but this is merely for notational simplicity. For the previous statements to hold without including ghost fields, we assume light-cone gauge when necessary.

Now let us make a dimensional reduction of this theory on a cylindrical worldsheet. This is simply the propagator in string theory, or in other words, the amplitude for a string to move between two given circles in space-time. To parametrise it we let $\sigma_2 \in [0, 1]$ and take a periodic $\sigma_1 \in [0, 1]$. Diffeomorphisms and Weyl invariance can be used to completely gauge fix the cylinder metric to $ds^2 = d\sigma_1^2 + T^2 d\sigma_2^2$, where the parameter T is the one Teichmüller parameter of the cylinder, analogous to the Schwinger parameter for a line segment. It will be more convenient, however, to retain some gauge freedom. Indeed, working on the covering space $\mathbb{C}$ parametrised by $z = \sigma_1 + iT\sigma_2$, the diffeomorphism $z \mapsto z + z_2 A$ twists one of the boundary circles in $\partial\Sigma$ by an amount A. Clearly we should identify $A \sim A + 1$ and regard A as being a one-dimensional $U(1)$ gauge field. The corresponding metric is $g = d\sigma_1^2 + 2A d\sigma_1 \otimes d\sigma_2 + (T^2 + A^2) d\sigma_2^2$. We will say that the cylinder is untwisted when $A = 0$. Of course, this metric also descends to that of a torus (when we also identify $z \sim z + iT$ in the covering space), in which case A is precisely the real part of the modular parameter, usually denoted τ_1.

We then carry out a Kaluza-Klein compactification on the cylindrical worldsheet, with the ansatz

$$X(\sigma_1, \sigma_2) \;=\; \sum_{m\in\mathbb{Z}} X_m(\sigma_2) e^{2\pi i m \sigma_1} \;, \tag{4.2.4}$$

together with a reality condition $X_{-m} = X_m^\dagger$, and with metric[1] (which depends only on σ_2)

$$g_{ab} \;=\; \begin{pmatrix} 1 & A_\tau \\ A_\tau & g_{\tau\tau} + A_\tau^2 \end{pmatrix} \;. \tag{4.2.5}$$

[1]In principle, we could consider the more general ansatz where the written metric is multiplied by $e^{2\phi}$. This conformal factor is cancelled in the action and so is irrelevant. The field ϕ does turn up in the dimensional reduction of the Ricci scalar, where it has the interpretation of being an emergent spatial coordinate, belonging to a dimensionally reduced Liouville action.

In the above, we are anticipating the notation $\tau := \sigma_2$ as the Euclidean worldline coordinate, in which case the gauge field is $A = A_\tau d\tau$. We have also left $g_{\tau\tau}$ as a general one-dimensional metric (as opposed to fixing it to T^2 as in the previous paragraph). Obviously, the X_m are then simply related to the string oscillator modes. Upon dimensional reduction and retaining all the fields, the resulting worldline action is

$$S[X_m, A, g] \;=\; \frac{1}{2\pi\ell_s^2}\int_0^1 d\tau\sqrt{g_{\tau\tau}}\left\{\frac{1}{2}g^{\tau\tau}|D_\tau X_0|^2 + \sum_{m=1}^{\infty}\left(g^{\tau\tau}|D_\tau X_m|^2 + (2\pi m)^2|X_m|^2\right)\right\}, \tag{4.2.6}$$

where the covariant derivative is $D_\tau X_m = (\partial_\tau - 2\pi i m A_\tau)X_m$. As discussed, because it is pure gauge, we can locally set $A = 0$ and enforce the associated Gauss constraint $\sum_m \mathrm{Im}(X_m^\dagger \partial X_m) = 0$, which is equivalent to the level-matching conditions. The gauge fixed action with $A = 0$ and $g_{\tau\tau} = T^2$ becomes

$$S[X_m, T] \;=\; \frac{1}{2\pi\ell_s^2}\int_0^1 d\tau\left\{\frac{1}{2T}(\partial_\tau X_0)^2 + \sum_{m=1}^{\infty}\left(\frac{1}{T}|\partial_\tau X_m|^2 + T(2\pi m)^2|X_m|^2\right)\right\}. \tag{4.2.7}$$

Note that the $U(1)$ gauge symmetry described by A acts on the X_m fields *simultaneously*: under $A \mapsto A + d\alpha$ we have $X_m \mapsto e^{2\pi i m\alpha}X_m$ for all m, so it applies a single constraint to the spectrum. Equivalently, since each X_m has a standard global $U(1)$ symmetry, the gauge symmetry is taken to be the diagonal of the product "$U(1)^\infty$" global symmetry. Although locally A can be gauged away, it will become important when the worldline topology is non-trivial.

As anticipated, this dimensional reduction gives us a concrete way to view the string as an infinite tower of complex one dimensional fields with worldline masses. Note that the field X_0, the centre of mass of the string, is to be interpreted as the worldline coordinate field, whereas the massive fields are harmonics corresponding to fluctuations of the extended string around it. As we have seen in chapters 1 and 2, internal degrees of freedom in the worldline formalism are always realised by adding extra worldline fields, and constraints on the spectrum as worldline gauge symmetries. The main difference between the theories of this chapter and the previous one is that we are now adding an infinite number of states (so that they really describe a continuous object) which have worldline masses (making the theory non-local in spacetime).

Relation to theories with worldline non-locality

The previous dimensionally reduced action, say with $A = 0$, can be rewritten as a worldline theory that has non-locality on the worldline via an infinite-derivative kinetic term. More precisely, one can rewrite the action as

$$S = \frac{1}{2\pi\ell_s^2}\int d\tau\sqrt{g}\, G_{\mu\nu}\, g^{\tau\tau}\partial_\tau X^\mu(\tau) F(\sqrt{\Box})\partial_\tau X^\nu(\tau)\ , \tag{4.2.8}$$

where we define[2] $\sqrt{\Box} = \sqrt{g^{\tau\tau}\partial_\tau^2}$, which, in terms of the einbein $e(\tau)$, can be written $\sqrt{\Box} = e^{-1}\partial_\tau$. Gauge fixing $g_{\tau\tau} = T^2$ and making the particular choice

$$F(\sqrt{\Box}) = \frac{\tan(\frac{1}{2}\sqrt{\Box})}{\sqrt{\Box}}\ , \tag{4.2.9}$$

gives the worldline theory that was considered in [39]. (Further aspects[3] of this non-local action have been explored in [31, 39, 40].) This theory can be seen to be equivalent to (4.2.7) through the identity

$$\frac{1}{2z\tan(z/2)} = \frac{1}{z^2} + 2\sum_{n=1}^{\infty}\frac{1}{z^2-(2\pi n)^2}\ . \tag{4.2.10}$$

That is, the formal inverse of the Green function of $\tan(\frac{1}{2T}\partial_\tau)\cdot\frac{2}{T}\partial_\tau$ is the infinite sum of individual Green functions $G_m = 1/(-\partial_\tau^2 + (2\pi mT)^2)$. It is then reasonable to suppose that the nonlocal action in (4.2.8) is equivalent to an infinite number of fields having action involving a kinetic term $-\partial_\tau^2 + (2\pi mT)^2$, with the two being related to each other by integration by parts.

An alternative approach would be to start by considering all nonlocal worldline actions, rather than theories augmented by towers of massive worldline states. Our approach is of more generality, for any infinite-derivative theory can be recast as one with an infinite tower of states, whilst in our formulation, we also allow for the tower to be finite. Moreover, the amplitudes of our worldline theory are easily computed using standard techniques for local field theories, making them more practical.

4.2.2 Strings from particles: propagators

From hereon, we will take the action (4.2.7) and ignore its origins. We would like to discuss how such an action is able to reproduce the stringy behaviour from where it has its origins. As in the previous chapter, a good place to start is by calculating the propagators.

In the last chapter, we saw how the propagator of a point particle travelling from x_0 to y_0 could be expressed as an integral over a kernel $K(x_0, y_0; T)$, where T is the Schwinger

[2] We are content to use the expression for the Laplacian assuming constant metric so that $\Box = g^{\tau\tau}\partial_\tau^2$. This is because we can always gauge fix the metric to be constant. More generally, $\Box$ would be the usual Laplacian.

[3] Note that we are working in Euclidean signature on the worldline, whereas these papers work in Minkowski.

parameter (which for us is always in string units). In our current theory with a tower of internal states, each field travels across the same line segment with Schwinger parameter T (always to be measured in string units). The action for a single massive field X_m, gauge fixing $A = 0$, is

$$S_m[X_m, T] = \frac{1}{2\pi\ell_s^2}\int_0^1 d\tau \left\{ \frac{1}{T}|\partial_\tau X_m|^2 + T(2\pi m)^2|X_m|^2 \right\} . \tag{4.2.11}$$

The amplitude for a single massive state X_m to go from an initial boundary condition x_m to final y_m is given by the Mehler kernel:

$$\begin{aligned} K_m(x_m, y_m; T) &= \int_{X_m(0)=x_m}^{X_m(1)=y_m} \mathcal{D}X_m e^{-S[X_m,T]} \\ &= \frac{\pi}{2}\cdot\frac{m}{\sinh(2\pi mT)} e^{-\frac{m((|x_m|^2+|y_m|^2)\cosh(2\pi mT)-2\mathrm{Re}(x_m\cdot y_m))}{\ell_s^2\sinh(2\pi mT)}} , \end{aligned} \tag{4.2.12}$$

which can be found by the usual heat kernel methods. The expression in the exponent is the on-shell action, whilst the prefactor arises from a regularization of a functional determinant. It is important to note that this regularization is physically meaningful in that it gives rise to a counterterm in the vacuum energy. For D identical fields, the determinant (and therefore prefactor) gains a power D. The $m = 0$ case is the same as we have studied in the last chapter, but can also be reproduced from (4.2.12) using the limit $m \to 0$ and noting that the X_0 field is scalar rather than complex, removing half the degrees of freedom and hence square-rooting the functional determinant. For convenience, we rewrite it here:

$$K_0(x_0, y_0; T) = \frac{1}{2T^{1/2}} e^{-\frac{(x_0-y_0)^2}{4\pi\ell_s^2 T}} . \tag{4.2.13}$$

Again, with D such fields, the denominator's exponent becomes $D/2$.

Putting everything together, in D target-space dimensions and hence with $D-2$ independent fields (assuming light-cone gauge), the total propagator becomes a product over all (infinitely many) fields, each with boundary conditions x_m^μ and y_m^μ. We can write this product as a regularised limit[4]:

$$K(\{x_m^\mu\},\{y_m^\mu\};T) = \lim_{N\to\infty} \frac{e^{-\frac{(x_0-y_0)^2}{4\pi\ell_s^2 T}}}{(2T)^{(D-2)/2}} \left(\prod_{m=1}^{N} \frac{m}{\sinh(2\pi m T)}\right)^{D-2} \times\ e^{-\sum_{m=1}^{N} \frac{m((|x_m|^2+|y_m|^2)\cosh(2\pi mT)-2\mathrm{Re}(x_m\cdot y_m))}{\ell_s^2\sinh(2\pi mT)}}, \quad (4.2.14)$$

$$= \frac{e^{-\frac{(x_0-y_0)^2}{4\pi\ell_s^2 T}}}{(2T)^{(D-2)/2}} \frac{1}{\eta(2iT)^{2(D-2)}}\ e^{-\sum_{m=1}^{\infty} \frac{m((|x_m|^2+|y_m|^2)\cosh(2\pi mT)-2\mathrm{Re}(x_m\, y_m))}{\ell_s^2\sinh(2\pi mT)}}.$$

Note the appearance at this stage (already at tree-level) of the Dedekind eta function $\eta(\tau)$, which has arisen from zeta function regularisation[5]: $\prod_{m=1}^{\infty}\sin(2\pi m\tau) \stackrel{\zeta}{=} \sqrt{-2i}\eta(2\tau)$ for $\tau\in\mathbb{C}$.

As a simple example, when the initial and final states are point-like, all the higher harmonics vanish and the propagator becomes

$$K(x_0, x_m=0, y_0, y_m=0;T) = \frac{1}{(2T)^{(D-2)/2}|\eta(2iT)|^{2(D-2)}}\ e^{-\frac{(x_0-y_0)^2}{4\pi\ell_s^2 T}}. \quad (4.2.15)$$

This expression is equivalent to that of the sum of all the standard QFT propagators over the string spectrum. It is in this sense that we say (as in the introduction) that the propagator is that of a local field theory. This provides a very explicit version of the expressions given in [34] but is also to be expected from open/closed string duality.

4.2.3 Sewing the cylinder into a torus

Having studied the tree level propagator, the next step is to look at loop diagrams. For this, we should start with the partition function. From the string perspective, we want to sew the cylinder into a torus. We will see how, under this sewing, the gauge parameter A becomes globally important and, together with the Schwinger parameter T, the modular parameter $A+iT$ arises in the amplitude.

The reason why we cannot take $A=0$ any more is that one must now consider the twisted boundary conditions as states go round the loop. These are $x_m \sim x_m e^{2\pi i m A}$ for all m with arbitrary $A\in(-1/2,1/2]$. We can think of the different gauge transformations A as

[4]In this expression we drop constant prefactors which are irrelevant. Note that the Dedekind eta function comes about as explained in the next footnote. One also has the regularised product $\prod_{m=1}^{\infty} m = \sqrt{2\pi}$, so that this is absorbed into some overall constant.

[5]Indeed, writing $q=e^{2\pi i\tau}$ one has $\prod_{m=1}^{\infty}\sin(2\pi m\tau) = \prod_{m=1}^{\infty}\frac{-1}{2i}q^{-m}(1-q^{2m})$. A careful regularisation allows us to split the divergent product into three, two of which require regularisation: $\prod_{m=1}^{\infty} -1/2i$ and $\prod_{m=1}^{\infty} q^{-m}$. Both of these have standard zeta function regularisations, the former regularised to $\sqrt{-2i}$ and the latter to $q^{1/12}$. Then one can use the definition of the Dedekind eta function $\eta(\tau)=q^{1/24}\prod_{m=1}^{\infty}(1-q^m)$ to match the expression in the text.

defining different sectors of the theory, with the path integral having to go over all sectors. Keeping A and T as background fields, in the sector defined by A, the partition function after integrating over all possible regularised kernels becomes:

$$\begin{aligned}
\mathcal{Z}(A,T) &= \lim_{N\to\infty} \prod_{m=1}^{N} \int_{-\infty}^{\infty} dx^{\mu}_{m} K(\{x^{\mu}_{m}\},\{x^{\mu}_{m} e^{2\pi i m A}\};T) \ , \\
&= \lim_{N\to\infty} \prod_{m=1}^{N} \int_{-\infty}^{\infty} dx^{\mu}_{m} \frac{1}{T^{(D-2)/2}} \left(\frac{m}{\sinh(2\pi m T)}\right)^{D-2} e^{-\frac{m|\sin(\pi m(A+iT))|^2 |x_m|^2}{\sinh(2\pi m T)}} \ , \\
&= \lim_{N\to\infty} \frac{1}{T^{(D-2)/2}} \prod_{m=1}^{N} \frac{1}{|\sin(\pi m(A+iT))|^{2(D-2)}} \ . \qquad (4.2.16)
\end{aligned}$$

Upon zeta function regularisation of the product, one recovers the Dedekind eta function of the full one-loop partition function:

$$\mathcal{Z}(A,T) = \frac{1}{T^{(D-2)/2}|\eta(A+iT)|^{2(D-2)}} \ . \qquad (4.2.17)$$

Hence we naturally obtain the string torus partition function with $A+iT$ being the Teichmüller parameter[6].

There are two remaining integrations to be done: over $T \in (0,\infty)$ and $A \in (-1/2, 1/2)$. However, there is a large degeneracy in the sewing procedure, which we would like to see from the worldline perspective. First, as usual, there is a $1/T$ coming from the $\sim 2\pi T$ ways of sewing the line segment together to form the same circle. Second, because we have assumed lightcone gauge, we should include another $1/T$ factor coming from the two remaining zero modes belonging to the lightcone coordinates $X^{\pm}$ (alternatively this factor would be seen in a proper treatment of the Polyakov action involving ghost fields - these ghosts would then also appear in the dimensionally reduced worldline theory).

Next, the partition function is now exhibiting an unasked-for modular $SL(2,\mathbb{Z})$ symmetry. The real part of this symmetry, i.e. $A \to A+1$, is put in by hand for the gauge symmetry to be compact. Meanwhile, a $T \to 1/T$ symmetry is already a symmetry in the prefactor of the kernel in equation (4.2.15) when the ends are coincident, and becomes manifest in the loop diagram. Therefore, one should not attempt to integrate over more than one representative T when we sew the propagator into a loop. This restricts the integration of $A+iT$ to the fundamental domain of $SL(2,\mathbb{Z})$. This argument is purely from the worldline approach - we have not made reference to a worldsheet. Of course, the worldline theory 'grows' a stringy interpretation, but from our perspective such an interpretation is not needed.

[6]For this calculation, note that factor of 2 in the argument of η compared to (4.2.15) is effectively the result of a double-angle identity.

4.2.4 Green functions and vertex operators

Having calculated the one-loop partition function, we now like to discuss amplitudes. To do this, we will need to analyse the Green functions and also the possible vertex operators. We restrict ourselves to line segments and circles, although more generally, one can sew these together to make worldgraphs. It is clear from the action in (4.2.6) that, with fixed background T and A, one has

$$\langle X_m^\dagger(v_1) X_n(v_2)\rangle_{T,A} \;=\; 2\pi\ell_s^2\,\delta_{m,n} G_m(v_{12}; A, T)\;, \tag{4.2.18}$$

where $G_m(v_{12}; A, T)$ is the Green function corresponding to the kinetic operator

$$L_m \;=\; -T^{-1}(\partial_v - 2\pi i m A)^2 + T m^2\;. \tag{4.2.19}$$

Over the real line, the Green function satisfying $LG_m(v_{12}) = \delta(v_{12})$ is easily found to be

$$G_m(v_1, v_2; A, T) \;=\; \begin{cases} \frac{1}{4\pi|m|} e^{2\pi i m A v_{12}} e^{-2\pi|m||v_{12}|T} & \text{if } m \neq 0 \\ \frac{1}{2}T|v_{12}| & \text{if } m = 0\;. \end{cases} \tag{4.2.20}$$

Over a circle parametrised by $v \in [0, 1]$, one instead finds

$$G_m^\circ(v_1, v_2; A, T) \;=\; \begin{cases} \frac{1}{4\pi|m|} \sum_{k\in\mathbb{Z}} e^{2\pi i m A(k - v_{12})} e^{-2\pi|m||k-v_{12}|T} & \text{if } m \neq 0 \\ \frac{T}{2}(v_{12}^2 - |v_{12}|) & \text{if } m = 0\;, \end{cases} \tag{4.2.21}$$

where the latter satisfies $LG_0(v12) = \delta(v_{12}) - 1$, subtracting the zero mode piece. The latter is easily obtained from the former via the method of images.

Vertex operators

The next ingredient for constructing general graphs is the worldline vertex operator. The relevant vertex operators for the field X_m are of the form $V_{m;0} = e^{ip\cdot X_m}$. However, this is not gauge invariant: under a gauge transformation with gauge parameter $g = e^{2\pi i u}$, it transforms as $e^{ip\cdot X_m} \mapsto e^{ip\cdot X_m e^{2\pi i m u}}$. Thus, there is a class of gauge-equivalent vertex operators $V_{u;m} = e^{ip\cdot e^{2\pi i m u} X_m}$ with $u \in [0, 1]$, and a gauge invariant vertex operator

$$V \;=\; \int du \;\;\cdot\;\; \prod_m e^{ip\cdot e^{2\pi i m u} X_m}\;. \tag{4.2.22}$$

Here we are using the sewing rule that all states are to be emitted at the same position on the worldline (i.e. every leg of the worldline universe has a whole tower on it) and the only conserved momentum belongs to the zero mode. Higher level states have vertex operators induced from this. For example, one can also define vertex operators

$$V_g \;=\; \int du \quad \cdot \quad \sum_n \partial X_n \prod_m e^{ip\cdot e^{2\pi imu}X_m} \;. \tag{4.2.23}$$

We illustrate this with the two-point loop amplitude. Here we place two ordinary vertex operators on the circle and then integrate over all sectors to obtain

$$\begin{aligned}\mathcal{A}_2 \;&=\; \prod_{m_1,m_2\in\mathbb{Z}} \int d\mu \int_0^1 dv \langle V^\dagger_{u_1;m_1}(p,0)V_{u_2;m_2}(q,v)\rangle_A \\ &=\; \int d\mu \int_0^1 dv e^{-p\cdot q\sum_{m\in\mathbb{Z}} e^{2\pi im(u_1-u_2)}2\pi\ell_s^2 G_m(v;A,T)}\,\mathcal{Z}(A,T)\;,\end{aligned} \tag{4.2.24}$$

where we use (4.2.18) and where $\mathcal{Z}(A,T)$ is the partition function calculated above in the sector A, and the integral $\int d\mu$ denotes integration over all $T+iA\in\mathcal{F}$ with $\mathcal{F}$ the fundamental domain of $SL(2,\mathbb{Z})$, as well as $u_1,u_2\in[0,1]$.

Comparison with string theory

Of course, the usual string loop amplitudes have been studied long ago and we can compare them to the above. We will expand the usual string expressions into modes in a way similar to [16]. Schematically, string theory one-loop amplitudes are of the form

$$\mathcal{A}_n \;=\; \int_{\mathcal{F}} d^2\tau\;\mathcal{Z}(\tau)\int d^2 z_i \prod_{\substack{i=1\\ j<i}}^{n} e^{-k_i\cdot k_j 2\pi\ell_s^2 G(z_{ij};\tau)}\;, \tag{4.2.25}$$

where now we integrate over the Teichmüller parameters of the torus $\tau=\tau_1+i\tau_2$, and the vertex operator positions are two-dimensional (complex) numbers z_i, integrated over the torus $T(\tau)$ with modular parameter τ. The function $G(z_{ij};\tau)$ is the Green function on the torus (see equation (4.2.27)). There can also be complex prefactors to the exponential which depend on the precise form of the vertex operators - we ignore these since they are irrelevant for our discussion. For a torus with modular parameter $\tau=\tau_1+i\tau_2$, the Green function presented in its usual form is

$$2\pi\ell_s^2 G(z;\tau) \;=\; \langle X(z)X(0)\rangle_{T^2(\tau)} \;=\; -\frac{\ell_s^2}{2}\left(\ln\left|\frac{\vartheta_1(z)}{\vartheta_1'(0)}\right|^2 - \frac{2\pi z_2^2}{\tau_2}\right)\;, \tag{4.2.26}$$

where $z=z_1+iz_2=u+\tau v$ is a coordinate on the torus and $\vartheta_1(z)$ is a Jacobi theta function. The field theory limit involves taking $\alpha'=\ell_s^2\to 0$, keeping $T=\pi\alpha'\tau_2$ fixed, which plays the role of the Schwinger parameter. In other words, it arises as the large τ_2

limit. A Fourier expansion of $G(z)$ gives [41]:

$$
\begin{aligned}
G(z,\tau) \;&=\; \frac{\tau_2}{2}(v^2-|v|) \;+\; \sum_{\substack{m\neq 0\\ k\in\mathbb{Z}}} \frac{1}{4\pi|m|} e^{2\pi i m(u+\tau_1(k+v))} e^{-2\pi\tau_2|m||k-v|} \\
&\qquad\qquad +\; 2\ln 2\pi \;+\; 2\sum_{\substack{m\neq 0\\ k\geq 1}} \frac{1}{4\pi|m|} e^{2\pi i k m\tau_1} e^{-2\pi k|m|\tau_2}\;.
\end{aligned}
\tag{4.2.27}
$$

As indicated above, the torus Green function separates into a term $v^2-|v|$ familiar from the scalar particle X_0^μ and discussed in the last chapter, as well as an infinite tower of stringy corrections, which become increasingly important when α' becomes large. In the next section, these corrections are sufficient for two and four-point amplitudes to have soft string-like behaviour in certain kinematical limits. The final line of (4.2.27) is just a zero mode which is irrelevant for our purposes.

Comparing to the the circle Green function in (4.2.21), we find that they match when all the massive worldline modes X_m are summed:

$$
2\pi\ell_s^2 G(z=u_{12}+\tau v_{12};\tau) = \sum_{m\in\mathbb{Z}} \langle \left(e^{-2\pi i m u_1} X_m^\dagger(v_1)\right)\left(e^{2\pi i m u_2} X_m(v_2)\right)\rangle|_{T=\tau_2, A=\tau_1} + \text{zero mode}\;.
\tag{4.2.28}
$$

4.3 Truncation: “mock” string theory

The reader may argue that the compactification of bosonic closed string theory onto a worldline, as considered above, is not strictly a one-dimensional worldline theory, since the modes are simply being the degrees of freedom of a string. To make a genuine particle theory, we will now truncate the spectrum so that the number of massive states is N. As we will see, this ‘pruning’ of the infinite-tower theory retains certain physical properties of its stringy cousin. However, it will also suffer certain setbacks. The problem is that, just like in normal Kaluza-Klein theory, the limit $N\to\infty$ is not really continuous - no finite approximation ever really resembles the higher dimensional theory.

Truncating to N massive fields leaves the action

$$
S[X_m,A,g] \;=\; \frac{1}{2\pi\ell_s^2}\int_0^1 d\tau\sqrt{g_{\tau\tau}}\left\{\frac{1}{2}g^{\tau\tau}|D_\tau X_0|^2 + \sum_{m=1}^{N}\left(g^{\tau\tau}|D_\tau X_m|^2 + (2\pi m)^2|X_m|^2\right)\right\}\,,
\tag{4.3.1}
$$

which is now well-defined on the worldline. Such a truncation is reminiscent of matrix models. The corresponding one-loop diagram is as in (4.2.16), but again we do not take the $N \to \infty$ limit:

$$\mathcal{Z}(A,T) = \frac{1}{T^{d/2}} \prod_{m=1}^{N} 4^d |q^{-m/2}|^{-2d} |1-q^m|^{-2d} \,, \tag{4.3.2}$$

where $q = e^{2\pi i(A+iT)}$ and $d = D-2$. Of course, the Green functions of such a theory remain the same as in the previous section. With the $U(1)$ gauge symmetry, the gauge invariant vertex operators are inherited from $\int du e^{ip\cdot \sum_m e^{2\pi i m u} X_m}$ as before.

From the spacetime perspective, this theory, just like its infinite-tower cousin, does not directly give us a spacetime Lagrangian. However, since in second quantization the field X^μ_m would be replaced with a state $|X^\mu_m\rangle$, one still finds an infinite number of higher spin states whose mass can be read off from a q-expansion of the partition function (4.3.2):

$$\mathcal{Z}(A,T) \;=\; \frac{1}{T^{d/2}} \cdot 4^{Nd} |q|^{N(N+1)d/2} \cdot \prod_{m=1}^{N} \sum_{r,s\geq 0} c_{r,s} q^{mrd} \bar{q}^{msd} \,, \tag{4.3.3}$$

for appropriate d-dependent degeneracies $c_{r,s}$. The imaginary part involves the unlevel matched states with $r \neq s$. These are projected out upon level matching, in which case we are left with an infinite tower of states with integer mass levels starting at $N(N+1)d$. In this one-dimensional theory, there is freedom to add a cosmological constant to the action which would decrease or increase the physical mass. For example, by introducing a negative cosmological constant on the worldline one can tune the above expression so that the tower begins with massless scalars[7].

Although these theories are interesting in the sense that they are non-local, and we will show that their (regulated) amplitudes also have interesting properties, they do suffer the major setback that they are no longer UV finite. This is because the truncation breaks the modular invariance of the kernel (4.2.15) so that our argument to restrict integration of $T + iA$ to the fundamental domain of $SL(2,\mathbb{Z})$ is no longer valid. In principle, one could expand each $X_m(\tau)$ into modes, but truncate their number so that the theory still retains a $T \to 1/T$ symmetry. This would provide us with a modular invariant regularization of the theory.

4.4 Softening of amplitudes

One remarkable feature about string theory is that it exhibits extremely soft behaviour in the high energy, fixed-angle regime; this was originally studied by Gross and Mende

[7]For the transition to string theory in the formal limit $N \to \infty$, counterterms to the cosmological constant should be generated which regulate the sum "$1+2+3+\cdots$" to $-1/12$, thereby introducing the tachyon.

[42]. In particular, in the kinematical region $s \gg 1$ (in string units) with $|t|/s$ fixed, individual perturbative amplitudes are exponentially suppressed with order $\exp(-sf(\theta))$. (When resummed, however, one finds a less severe $\exp(-\sqrt{s}f(\theta))$ suppression, which is now above the Cerulus-Martin bound [43]). In this section, we will study this suppression from the worldline point of view. We will show how it originates from the addition of exponential terms into the one-loop Green function so that, even with a single internal state with worldline mass, a similar saddle point occurs. Increasing the number of states in the tower brings us successively closer to the result of Gross and Mende. Let us examine this in more detail.

4.4.1 Gross-Mende softening of amplitudes in string theory

From the last chapter, it follows that an ordinary scalar QFT, whose spacetime field has been Wick rotated to Euclidean signature, has a two point function

$$\mathcal{A}_2 \;=\; \int_0^\infty \frac{dT}{T}\mathcal{Z}(T)\int dx_1 dx_2 T^{-1} e^{-Tx_{12}(1-x_{12})p^2}\,. \tag{4.4.1}$$

where $x_{12} = x_2 - x_1$ is the distance between the two vertex operator insertions, T is the radius of the circle, and $\mathcal{Z}(T)$ is the circle partition function[8]. The function $Tx_{12}(1-x_{12})$ is, of course, precisely the Green function we found earlier. On a fixed T-subspace, a saddle point argument in x_{12} shows that the largest contributions occur when the vertex operators are as far away as possible - they repel each other. However, there is no saddle point in T, which contributes increasingly in the UV region at $T \to 0$. Thus, one does not expect an exponential softening of amplitudes.

This discussion changes in string theory. To be concrete, we will take closed bosonic string theory as before. Then 1) the one-loop amplitude is finite since the integration of the moduli avoids the dangerous UV region $T \to 0$, and relatedly 2) due to the modular transformation of the Green function, not only do the vertex insertions repel each other, but the Green function for loop moduli τ and $-1/\tau$ are related. This leads to a saddle point in the modular parameter τ as well, as shown in [42].

It will be convenient to review the four-point one-loop amplitude in some more detail. Taking equation (4.2.25), we use the parametrisation $z_i = u_i + \tau v_i$, where u_i and v_i are in $[0, 1]$ for each i. One can fix $z_4 = 0$ using the one conformal Killing vector of the torus. It is useful to define $s_{ij} := -(k_i + k_j)^2$ and we will use the convention that $s := s_{12}$, $t := s_{14}$ and $u := s_{13}$.

Famously, in the fixed angle limit where t and u scale with s, the exponent in (4.2.25) has a saddle when $\hat{z}_1 = \frac{1}{2}$, $\hat{z}_2 = \frac{\tau}{2}$ and $\hat{z}_3 = \frac{1}{2}+\frac{\tau}{2}$ (assuming we have fixed $z_4 = 0$), which is where

[8]For example, with just one scalar field $\mathcal{Z}(T) = T^{-D/2}e^{-m^2T}$.

one would intuitively expect the vertex operators to lie using the electrostatic analogy [42]. There are other saddles, but these are subdominant. It is then a straightforward exercise to find a saddle point for τ_2; this turns out to be

$$\begin{aligned} \hat{\tau}_2 &= i\frac{K(-u/s)}{K(-t/s)} \,, \\ &\simeq -\frac{1}{\pi}\log\left(-\frac{t}{16s}\right) \,, \end{aligned} \tag{4.4.2}$$

where $K(z)$ is the elliptic integral of the first kind, and the approximation is in the $|t|/s \ll 1$ limit. The softness of string amplitudes in this kinematic limit is entirely due to the existence of this saddle point.

This saddle point should disappear in the field theory limit, where we take $\alpha' \to 0$ with fixed $T = \pi\alpha'\tau_2$. To check this, first we break the string amplitude into three pieces

$$\begin{aligned} \mathcal{A}_4(s,t,u) &= 2\mathcal{A}_4(s,t) + 2\mathcal{A}_4(t,u) + 2\mathcal{A}_4(u,s) \\ &= 2\int_{R_s} d^2z_i I(z_i) + 2\int_{R_t} d^2z_i I(z_i) + 2\int_{R_u} d^2z_i I(z_i) \,, \end{aligned} \tag{4.4.3}$$

with R_s, R_t and R_u disjoint, corresponding to the s, t and u channels in the field theory. On a tangential note, it is interesting to remark that cycling (stu), which maps the individual integrals into each other, maps the saddle point $\hat{\tau}_2$ to points as depicted in figure 4.1. Thus, if one is only interested in saddle points, one can choose a single integral in the above and integrate τ over the fundamental domain of the congruence subgroup $\Gamma_0(2)$, also depicted in figure 4.1.

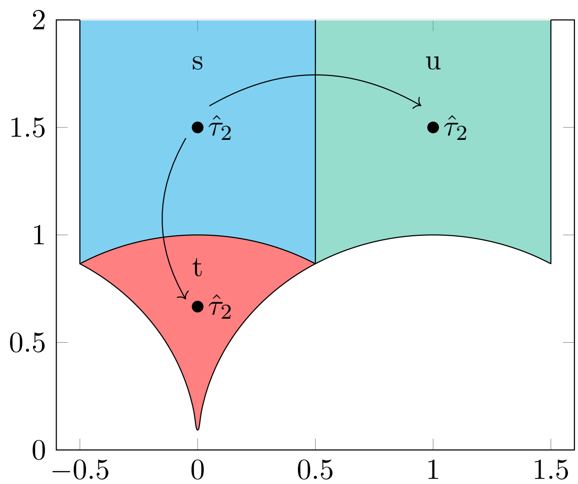

***Figure 4.1**: Congruence subgroup $\Gamma_0(2)$, a subset of the complex plane. The Gross-Mende saddle is mapped under cycling s, t and u to copies of the $SL(2,\mathbb{Z})$ fundamental domain as shown.*

Using Feynman parameters α_i and excising vertex operator collisions, the plane wave factor in the R_s region becomes the standard field theory expression

$$\prod_{i<j} \exp\left\{ -k_i \cdot k_j G(z_{ij}|\tau_2)\right\} \longrightarrow \exp\left\{ -T(s\alpha_1\alpha_3 + t\alpha_2\alpha_4)\right\} \qquad \text{as } \alpha' \to 0\,, \tag{4.4.4}$$

in the field theory limit. The other regions are obtained by cycling (stu). In each region, the saddle point behaviour is destroyed. Indeed, whilst on the Feynman parameter α-subspace there are still saddle points at $\alpha_1 = \alpha_3 = -\frac{t}{2u}$ and $\alpha_2 = \alpha_4 = -\frac{s}{2u}$ (which of course correspond to aforementioned Gross-Mende extremal points at $v_1 = -\frac{t}{2u}$, $v_2 = \frac{1}{2}$ and $v_3 = \frac{1}{2} - \frac{t}{2u}$ and $v_4 = 0$), the Schwinger parameter T has no extrema, as stated in the previous subsection.

4.4.2 Softening in the truncated theory

We claim that our truncated theory, in certain kinematical regimes, also has stringy saddle points. As an example, let us take the theory truncated to $N = 1$. Inserting four of the gauge invariant integrated vertex operators as discussed in the previous section, the s-channel amplitude of the mock string theory contains the plane wave factor:

$$\begin{aligned}\prod_{i<j} \exp\left\{ -k_i \cdot k_j G(z_{ij} = i\hat{v}_{ij}|\tau = A + iT)\right\} \;=\; & \exp\left\{ \frac{\alpha'}{2}\Big(sv_1(v_3 - v_2) + tv(v_2 - v_1)(1 - v_3)\right. \\ & \left. + \sum_{i<j}\sum_{k=0,1} s_{ij}\cos(2\pi(u_{ij} + kAv_{ij}))e^{-2\pi T|v_{ij}|}\Big)\right\}. \end{aligned} \tag{4.4.5}$$

Recall that the u_i parameters are here associated with the gauge invariant vertex operators, and are to be integrated over $[0, 1]$.

The v_i-subspace contains the same saddle points $\hat{v}_i$ as in section 4.4.1, so that

$$\begin{aligned}\prod_{i<j} \exp\left\{ -k_i \cdot k_j G(i\hat{v}_{ij}|A + iT)\right\} \;=\; & \exp\left\{ \frac{\alpha'}{2}\Big(-\pi\tau_2 \frac{st}{2u}\right. \\ & \left. + \sum_{i<j}\sum_{k=0,1} s_{ij}\cos(2\pi(u_{ij} + kA\hat{v}_{ij}))e^{-2\pi T|\hat{v}_{ij}|}\Big)\right\}. \end{aligned} \tag{4.4.6}$$

First note that even with this $N = 1$ truncation, one obtains an exponentially good approximation to the torus Green function at large τ_2. Now, thanks to the exponential corrections which crucially come with varying signs, the above expression has saddles. To be explicit, let us work in the limit $t \to 0$ and $u \sim -s$. Then, at leading order, one finds a deep extremum at $u_1 = u_2 = 1/2$, $u_3 = 0$ and $A = 0$. At these values, the above becomes

$$\prod_{i<j} \exp\left\{ - k_i \cdot k_j G(i\hat{v}_{ij}|A + iT)\right\} \;\simeq\; \exp\left\{\frac{\alpha'}{2}\left(\pi T \frac{t}{2} + 8se^{-\pi T}\right)\right] , \tag{4.4.7}$$

so there is an extremum at $T = \hat{T}$ given by

$$\exp(-\pi\hat{T}) = \frac{t}{16s} , \tag{4.4.8}$$

matching the Gross-Mende first approximation in (4.4.2).

Of course, as we include more states in the truncated theory, one will obtain increasingly better approximations to the real saddle point. However, we would like to stress that just a single state, coming with a correction of the type $\exp(-|v_{ij}|T)$, is needed to ensure exponential string-like suppression of amplitudes in the fixed-angle regime.

4.4.3 Two point amplitude

To complete this section, we will briefly discuss the two-point one loop amplitude between the lightest scalar vertex operators. As a reference, in string theory this is

$$\mathcal{A}_2(s) \;=\; \int_{\mathcal{F}} d^2\tau \int_{T(\tau)} d^2z \langle V(0)V(z)\rangle_{T(\tau)} \;=\; \int_{\mathcal{F}} d^2\tau \int_{T(\tau)} d^2z \mathcal{Z}(\tau) e^{-2\pi\ell_s^2 k_1\cdot k_2 G(z;\tau)} , \tag{4.4.9}$$

where k_1 and k_2 are the incoming momenta which must be on-shell for this to be valid, whilst $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$, $T(\tau)$ is the torus with modulus τ, and $G(z;\tau)$ is given in (4.2.26). Momentum conservation then implies $k_1 + k_2 = 0$, which completely fixes $s = k_1 \cdot k_2$. By using the formal trick of breaking Lorentz invariance, however, we are able to vary s.

Now consider the mock string theory with an $N = 1$ truncation. It has a one-loop two-point amplitude

$$\mathcal{A}_2(s) \;=\; \int_0^\infty dT \int_{-1/2}^{1/2} dA \int_{[0,1]^2} dvdu \mathcal{Z}(T,A) e^{-2\pi\ell_s^2 k_1\cdot k_2 (G_0^\circ(v,A,T) + e^{2\pi i u} G_1^\circ(v,A,T))} \tag{4.4.10}$$

with G_m° as in (4.2.21). We will simply write $s = k_1 \cdot k_2$ here, so that when on-shell and with incoming momentum k, one has $s = -k^2$. Consider large s, in which we would like to use a saddle point approximation. The Green function G_0° contributes a saddle in the v-subspace at $\hat{v} = 1/2$ (this is just like what happens in the usual field theory)[9]. Again, whilst the Schwinger parameter T of the field theory amplitude does not have any extrema, here the addition of exponential terms to the Green function is sufficient for there to be a saddle

[9]There is a similar saddle point in the v-subspace when more Green functions G_m are added by symmetry.

even for T. Indeed, a small calculation shows that the saddle lies at $A = u = 1/2$ and constant positive T. It therefore follows that this amplitude is exponentially suppressed at large s.

We would like to conclude this section with a brief account of how these kinds of amplitudes impact on renormalisation. The exponential suppression in our theories begins at energies $s \gg 1$, a region we call the deep UV, where one really is exciting the higher internal mass modes. It is an interesting question whether one can make sense of renormalisation at these kinds of scales, which are well above any conventional effective field theory cutoff. The conventional wisdom is to take a Wilsonian viewpoint, where one integrates out UV degrees of freedom to leave behind an IR effective field theory. In these theories, as in string theory, this is not so straightforward. To be consistent, one needs a modular invariant cutoff, which would appear to simultaneously remove IR degrees of freedom along with those in the UV. There is, however, an alternative. This is to use the amplitudes themselves to define couplings and their runnings. For example, the two-point string function can be understood as defining a coupling constant in the effective field theory.

In detail, a one-loop correction to a spacetime field theory, say QED for concreteness, should be of the form

$$\delta\mathcal{L} = C_1(p^2, \tilde{\mu}^2) F_{\mu\nu} F^{\mu\nu} \ . \tag{4.4.11}$$

Here $\tilde{\mu}$ is the renormalization scale which should disappear from all amplitudes. One can choose an on-shell renormalisation scheme in which one fixes an energy scale μ_* and demands that $C_1(p^2 = \mu_*^2, \mu_*^2) = 0$. This eliminates $\tilde{\mu}$ and keeps logarithmic corrections small when the (Euclidean) energy is close to μ_*^2. In this case, the coupling constant can be thought of as a function of $\mu^2 = p^2$, so that a beta function is defined by

$$\beta_g(\mu) = \frac{g}{2} \frac{\partial C_1(p^2 = \mu^2, \mu_*^2)}{\partial \log \mu} \ . \tag{4.4.12}$$

In other words, we are using the two-point amplitude at energy scale s to renormalise the theory at a scale $\mu^2 = s$. But crucially, here we have a UV completion, so one might as well replace C_1 with the actual amplitude $\mathcal{A}_2(s)$, which will track the beta function to arbitrarily high energy scales. On doing this, the beta function should, just like the two-point amplitude, become exponentially suppressed at large s, essentially vanishing in the deep UV. In this sense, one can regard the theory as exhibiting a form of asymptotic safety.

4.5 Generalisations

In the previous sections, we considered worldline theories which had towers of particles corresponding to those of worldsheet Kaluza-Klein vibrations of the string (or its truncations), and as such they had integer masses. We have seen that they enjoy string-like properties without needing to refer to a worldsheet. Therefore, it is interesting to take a step back and consider a more general class of worldline theories parametrised by different spectra of worldline masses. We want to be agnostic about the origins of such towers - maybe they come from some geometrical object of higher dimension; maybe they are more arithmetical with no geometric interpretation. The idea is to search this class of theories for those which share properties previously discussed - such as modular invariant partition functions which would render them UV finite. We will proceed sequentially, first generalising the worldline theories to include arbitrary masses of the particle towers, before discussing certain special spectra which render their subclass UV finite. We will then include a brief discussion on the addition of multiple towers which could represent higher dimensional membranes. Finally, we will discuss the addition of worldline fermions into the picture which, as we know from chapter 1, are necessary to produce spacetime fermions [17, 44–47].

Generalised towers: Our first examples consist of deforming the mass of the field X_m to be some function $f_m = f(m)$ rather than m. This leads to a natural generalization of equation (4.2.6):

$$S \;=\; \int d\tau\sqrt{g}\sum_{m=-N}^{N}\left(g^{\tau\tau}|D_\tau X_m|^2+(2\pi)^2 f_m^2|X_m|^2\right)\Bigg\} \qquad \text{with } X_{-m}=X_m^\dagger\,. \tag{4.5.1}$$

By choosing the field X_0 to remain massless (i.e. set $f_0 = 0$), it can still be interpreted as a coordinate field that embeds the worldline into a Poincaré-invariant spacetime. There is then great freedom to choose values for the remaining masses f_m such that at low energies the theory automatically reverts to the standard worldline theory.

We can repeat the story outlined in section 4.2 for these more general towers. Focussing on the partition function[10], the calculations proceed exactly as before, but with f_m in place of m. In particular, the partition function in one dimension now has the form

$$\mathcal{Z}(A,T) \;=\; \frac{1}{T^{1/2}}\prod_{m=1}^{N}\frac{1}{|\sin(\pi f_m(A+iT))|^2}\,. \tag{4.5.2}$$

Taking the formal limit $N\to\infty$ will be of particular interest to us. As before, one should regularise this product. We illustrate this with two simple examples:

[10]Note that the theory's Green functions will also change relative to before.

1. Let $f_m = m + \alpha$ for $m \geq 1$ and some $\alpha \geq 0$. Then the regularised partition function is

$$\mathcal{Z}(\tau = A + iT) \;=\; \frac{1}{2T^{1/2}} \left| q^{\frac{1}{4}(\alpha^2 - \alpha + \frac{1}{6})} \prod_{m \geq 1} \left(1 - e^{2\pi i m \alpha} q^m\right) \right|^{-2} ,$$

where $q = e^{2\pi i \tau}$ and the polynomial comes about from regularising using the Hurwitz zeta function $\zeta(s, \alpha)$, which has $\zeta(-1, \alpha) = -\frac{1}{2} B_2(\alpha) = -\frac{1}{2}(\alpha^2 - \alpha + \frac{1}{6})$.

2. Let $f_m = \sqrt{m^2 + \alpha^2}$ for $m \geq 1$ and some $\alpha \geq 0$. Then the regularised partition function is

$$\mathcal{Z}(\tau = A + iT) \;=\; \frac{1}{2T^{1/2}} \left| q^{c_\alpha} \prod_{m \geq 1} \left(1 - q^{\sqrt{m^2 + \alpha^2}}\right) \right|^{-2} ,$$

with $c_\alpha = \frac{1}{(2\pi)^2} \sum_{k=1}^{\infty} \int_0^{\infty} dx e^{-k^2 x - \frac{\pi^2 \alpha^2}{x}}$.

The first of these examples corresponds in string theory to the string modes being shifted to $\alpha_{m+\alpha}$, and so is reminiscent of what happens in an orbifold theory. The second example is reminiscent of a string in a plane wave background, see [48, 49].

By combining different towers, one can begin to *design* partition functions. For example, we can formally construct the half-integral Jacobi theta functions ϑ_1 and ϑ_2 by combining two worldline towers, with masses $f_m = (m - 1 + \alpha)/2$ and $f_m = (m - 1 - \alpha)/2$ for $m \geq 1$. This has associated partition function

$$\begin{aligned} \mathcal{Z}(\tau) \;&=\; \frac{1}{2\sqrt{T}} \left| q^{-\zeta(-1,-\alpha)/2} \prod_{m=0}^{\infty} (1 - q^{m-\alpha}) \right|^{-2} \cdot \frac{1}{2\sqrt{T}} \left| q^{-\zeta(-1,\alpha)/2} \prod_{m=0}^{\infty} (1 - q^{m+\alpha}) \right|^{-2} \\ &=\; \frac{1}{4T} \left| \frac{\vartheta \begin{bmatrix} \alpha - \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}}{\eta} \right|^{-2} . \end{aligned} \tag{4.5.3}$$

With our bosonic fields, it is not possible to construct the integral Jacobi theta functions ϑ_3 and ϑ_4 with these scalar Lagrangians - one would need anti-periodic boundary conditions, and towers of worldline fermions. We will discuss this later.

Mass degeneracies: By writing the partition function as a regularised product of sinh functions leads to a connection with the Bocherds' product formula [50]. To make this precise, let us first refine our notation to include a degeneracy of masses, so that the Lagrangian is[11]

$$S \;=\; \frac{1}{2} \int d\tau \sqrt{g} \left\{ g^{\tau\tau} \sum_{k=1}^{c(0)} |D_\tau X_{0,k}|^2 + 2 g^{\tau\tau} \sum_{m=1}^{N} \sum_{k=1}^{c(m)} \left(|D_\tau X_{m,k}|^2 + (2\pi)^2 f_m^2 |X_{m,k}|^2 \right) \right\} , \tag{4.5.4}$$

[11] In these equations, the worldline modulus τ should not be confused with the Teichmüller parameter τ.

where the fields $X_{m,1}, \ldots, X_{m,c(m)}$ all have the same mass f_m. The associated partition function is now

$$\mathcal{Z}(\tau := A + iT) \;=\; \frac{1}{T^{c(0)/2}} \left| q^{-C} \prod_{m=1}^{\infty} (1 - q^{f_m})^{c(m)} \right|^{-2} , \tag{4.5.5}$$

where C is the regularised sum $\frac{1}{2}\sum_{m=1}^{\infty} f_m c(m)$. We will assume that we can tune the constant C by including a cosmological term $\int \sqrt{g}\Lambda$ in the action. Then C, f_m and $c(m)$ essentially become free parameters. There is a rich theory expressing automorphic forms as such products. For example, according to Borcherds in [50], given a weakly holomorphic modular form $f(\tau) = \sum c(m) q^m$ of weight $1/2$ for $\Gamma_0(4)$ whose integer coefficients vanish unless $n = 0$ or 1 mod 4, the function

$$\Psi(\tau) \;=\; q^{-h} \prod_{m \geq 1} (1 - q^m)^{c(m^2)} , \tag{4.5.6}$$

is a meromorphic modular function for some character of $SL_2(\mathbb{Z})$ of integral weight. Here, h is the constant term of $f(\tau) \sum H(n) q^n$ where $H(n)$ is the Hurwitz class function, with $H(0) = -1/12$. This correspondence between modular functions and products is an example of a so-called theta lift [51]. It gives us a way to express certain modular forms as infinite products, from which we can reverse-engineer a partition function with nice modular properties by choosing appropriate masses f_m and degeneracies $c(m)$[12].

In fact we have effectively encountered this already in section 4.2, because the case $f(\tau) = \vartheta_3(\tau) = 1 + 2q + 2q^4 + 2q^9 + \cdots$, lifts to $\Psi = q^{1/12} \prod_{m \geq 1} (1 - q^m)^2 = \eta(\tau)^2$. For a less trivial example, it is possible to lift a weight $1/2$ modular form to the Eisenstein series $E_6(\tau)$, which furnishes a product expansion of the form

$$E_6(\tau) \;=\; (1 - q)^{504} (1 - q^2)^{143388} (1 - q^3)^{51180024} \cdots \tag{4.5.7}$$

where the exponents are all known and positive. We refer to section 15 of [50] for more details. In this way, we can design a partition function which looks like

$$\mathcal{Z}(\tau) \;=\; \frac{1}{T^6 |E_6(\tau)|^2} . \tag{4.5.8}$$

Multiple indices: Although formally covered by the previous cases, it is interesting to rearrange our notation in order to allow for worldline theories whose indices span a lattice.

[12] A limitation so far is that the degeneracies $c(m)$ of our bosonic worldline theory must necessarily be positive. However, this constraint can change if we add worldline fermions.

For example, consider the following action

$$S[g, X_m] = \frac{1}{2}\int d\tau\sqrt{g}\Big\{g^{\tau\tau}(\partial_\tau X_0)^2+2\sum_{(m,n)\neq(0,0)}^{N}\Big(g^{\tau\tau}|D_\tau X_{m,n}|^2+(2\pi)^2(m^2+n^2)|X_{m,n}|^2\Big)\Big\}\ , \tag{4.5.9}$$

(with $X_{-m,-n} = X^\dagger_{m,n}$) where the worldline fields now have two indices $X_{m,n}$ which span the lattice $\mathbb{Z}^2$. For example, such an action could come from an object with topology $S^1 \times S^1$. In this case, what has been called the "closed bucatini" partition function, is

$$\mathcal{Z}(\tau) \ = \ \frac{1}{T^{1/2}}q^{-E^*(i,-1)}\prod_{(m,n)\neq(0,0)}(1-q^{\sqrt{m^2+n^2}})\ , \tag{4.5.10}$$

where $E^*(\tau, s) = \frac{1}{2}\sum_{(c,d)\neq(0,0)}\frac{\tau_2^s}{|c\tau+d|^{2s}}$ is an Eisenstein series, to be evaluated at $\tau = i$.

Extension to fermions: As we know from chapter 1, in order to describe spacetime fermions, it is necessary to include worldline fermions into our formalism - these will typically exhibit worldline supersymmetry. Alternatively, one could simply be interested in adding worldline fermions without worrying about the spacetime picture. Either way, it is fairly straightforward to generalize the results of the previous sections to include fermions. We will make just a few brief comments on their significance to our modified worldline theory model building.

First, suppose that the worldline fermions are supersymmetric partners to the worldline bosons of (4.2.6), with ψ_m a tower of two-component Grassmann fields. We will consider their propagation on a line with Teichmüller parameter T and the following action:

$$S_\psi \ = \ \frac{i}{8\pi\ell_s^2}\sum_{m\in\mathbb{Z}+\nu}\int_0^1 d\tau\,\psi_m^\dagger\begin{pmatrix} D+2\pi mT & 0 \\ 0 & D-2\pi mT\end{pmatrix}\psi_m\ , \tag{4.5.11}$$

where $D\psi_m = (\partial_\tau - 2\pi i mT)\psi_m$ is the same covariant derivative as before, and the above is to be supplemented with $\psi_{-m} = \psi_m^\dagger$. Due to the latter condition, we must restrict the parameter ν to either $\nu = 0$ or $\nu = 1/2$. Not surprisingly from a string perspective, the above action would arise naturally from dimensionally reducing the kinetic term of a worldsheet fermion, in which case the choices $\nu = 0$ or $1/2$ correspond to an R or NS fermion.

Following the same route as in section 4.2, the kernel corresponding to ψ_m propagating on the line segment can easily be calculated. Actually, since the on-shell action vanishes, this kernel is also essentially equal to the one-loop partition function. We let $\mu = 0$ or $1/2$ depending on whether ψ_m is to be a Grassmann boson or fermion respectively. The final result is[13]

$$K_m(\tau := A + iT) \;=\; |\mu - \tau m|^2 \prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left|1 - \frac{\tau m}{n+\mu}\right|^2 \;=\; \left|\frac{\mu \sin(\pi(m\tau - \mu))}{\sin(\pi\mu)}\right|^2 , \qquad (4.5.12)$$

up to constant factors. In the string theory, the entire tower is present, and one finds, again using zeta function regularisation, an effective kernel for the tower being

$$K^{\mu,\nu}(\tau) = \left|\frac{\vartheta[{}^{\mu}_{\nu}](\tau)}{\eta(\tau)}\right|^2 . \qquad (4.5.13)$$

From the string perspective this is all extremely standard. It is interesting to note, however, that the vanishing of the ϑ_1 can be seen to arise purely from the zero-mode, ψ_0, in the $(\mu,\nu) = (0,0)$ sector.

One may now adjust the fermion tower analagously to the previous discussion, so that there is no a priori stringy origin for it. Again, we can use Borcherds products to reverse engineer towers with interesting properties. The fermions are useful because they allow the numbers $c(m)$ in (4.5.6) to be negative. By combining bosonic and fermionic towers, one is then free to construct even more modular invariant partition functions. For example, with a boson and fermion tower with $(\mu,\nu) = (0,0)$, and omitting the ψ_0 term which would cause the partition function to vanish, one can construct partition functions such as the modular invariant

$$\frac{1}{T^4 |E_4(\tau)|^2} , \qquad (4.5.14)$$

where the Eisenstein series $E_4(\tau)$ has a product expansion with even and odd powers of $(1-q^n)$ [50].

4.6 Conclusions

In this chapter, we have shown how a worldline theory with a spectrum of worldline masses can be viewed as a non-local theory in spacetime with special properties. We have seen how higher dimensional theories such as string theory can be encoded in such a formalism, where the worldline masses become particular excitations of modes on a string. But our framework is quite general and in fact we have seen that there are other theories with interesting arithmetical properties but which do not appear to have such a geometric interpretation. We have found a method, using Borcherd products, to construct a subclass of these theories which exhibit modular invariant partition functions in common with string theory.

We have also seen how string theory itself can be analysed from this point of view. Some

[13] Note that we are consistently using τ to be $A + iT$ whenever we refer to the partition function.

aspects of string theory, such as its modular invariance, are encoded in the precise spectrum of worldline masses, so that, through Borcherd products, they combine to produce something modular invariant. On the other hand, other aspects, such as stringy high-energy fixed-angle behaviour, become realizable even with a truncated worldline spectrum - it is the Green functions that are responsible for this behaviour, and correcting a particle's one-loop Green function only by a single exponential term (as long as the worldline gauge symmetry is present) is sufficient to give the amplitudes saddle points.

The latter behaviour is extremely difficult to accomplish for theories constructed in spacetime. For example, a large class of (non-local) spacetime propagators can be written in Schwinger form as $\pi(p^2) = \int_0^\infty dt \exp(-p^2 F(t) - m^2 t)$, such that $\pi(p^2)$ reduces to the traditional Feynman propagator at low energies [52, 53]. This kind of parametrisation makes it easy to build models of non-local theories also having this property. A non-trivial example is $\pi(p^2) = e^{-p^2}/(p^2+m^2)$ [35–38] which corresponds to $F(t) = t+1$. Even ignoring physical constraints such as unitarity, it is easy to show that a one-loop four point diagram with such propagators will not have a saddle point. Instead, we saw that the saddle point associated to our worldline theory came about in a rather delicate way, involving the worldline gauge field.

In our view, our framework opens a door to a wide class of theories written in the worldline formalism which have been unexplored. Interesting other examples can be constructed from dimensionally reducing other kinds of geometric objects, or by considering strings on non-trivial backgrounds. It would be interesting to see if there is always a geometric interpretation for all the modular invariant worldline theories. Or relatedly, is there a geometric interpretation of Borcherds products? Either way, we have only scratched the surface of the space of the theories considered in this chapter.

There is a an obstacle to developing a full perturbative theory in our framework. This is the question of how to define interactions. At the one-loop level, we can get away with vertex operators, but at two-loops and beyond, we will have to introduce a consistent way of joining these worldline theories together to form worldgraphs (with internal states propagating over each leg). A straightforward sewing-together of multiple worldlines at a point, as discussed in the previous chapter, will not work here. We can see this by looking at the equivalent picture in string theory. Here, a closed string dividing into two would, with a choice of time-slices, produce a single worldline which discontinuously breaks into two separate worldlines - not at a local trivalent junction. This is the same kind of problem as encountered in string field theory. Its solution requires an introduction of a non-local sewing rule for worldlines, a problem we leave open.

Chapter 5

Stability in Models of Open Strings

This work is based on [2], cowritten by the author. We follow the plan of that paper closely.

5.1 Introduction

In this chapter we shift our focus exclusively to string theory. We will attempt to construct string models which solve practical problems in string phenomenology. Principally, these are:

- The cosmological problem, which asks why a low energy effective field theory (such as the standard model) should have a very small vacuum energy - this requires a remarkable cancellation between UV and IR degrees of freedom.
- Constructing a string model with a stable de Sitter vacuum. Although claimed models do exist in the literature (e.g. [54]), recently they have become rather controversial. In fact, there are now swampland conjectures stating that no stable de Sitter string vacuum should exist [55].

The string theories in ten dimensions that have been studied most have unbroken supersymmetry, so that their ten dimensional cosmological constant vanishes. Much of the history of string phenomenology has involved carrying out compactifications of such theories which are both analytically tractable and give desirable properties. This includes the study of compactifications on Calabi-Yau manifolds [56], orbifold models [57], fermionic constructions [58–60], or Gepner models [61, 62]. In most of these studies, the compactifications do not break supersymmetry and so the four dimensional cosmological constant also vanishes. Sooner or later, however, one is bound to have to break supersymmetry for realistic phenomenology to take place.

It is therefore natural to wonder why one can't break supersymmetry before compactification. Indeed, there are already tachyon-free non-supersymmetric ten-dimensional string models [63, 64]. They can be regarded as having spontaneous supersymmetry breaking at the string scale M_s so that, on compactification to D dimensions, one expects a quantum effective potential of the order M_s^D [65]. In general, however, research into these theories has been neglected in comparison with their supersymmetric cousins, mainly because the non-supersymmetric higher dimensional theories often suffer from instabilities, usually having runaway potentials in the metric moduli, potentially to large AdS-like vacua [66].

But in fact there are large classes of non-supersymmetric models whose leading order (or even higher order) terms in the effective potential vanish - see for example [67–73]. These theories often arise from interpolations betwen supersymmetric theories, which allow them to be analytically tractable [74–76]. In recent years, they have had something of a resurgence in interest due to the property that some have exponentially suppressed one-loop effective potentials [77–88]. These models often rely on a massless boson-fermion degeneracy at tree level.

In this chapter, we study these kinds of models in the context of type I string theory. We will use a Scherk-Schwarz mechanism [56, 89–93] to break supersymmetry at the compactification scale. The resulting lower dimensional theories are truly non-supersymmetric, with a supersymmetry breaking scale

$$\mathcal{M} = \text{the Scherk-Schwarz supersymmetry breaking scale},$$

given by the mass of the gravitino. Without any extra ingredients, these models have a large negative cosmological constant. To remedy this, we will add Wilson lines which wrap internal cycles. At special points in moduli space, one finds configurations with equal numbers of massless fermions and bosons, so that the effective potential becomes exponentially small. The core of this work will involve analysing the stability of these models. Using the ideas of frozen Wilson lines, we will show that there exists a subclass of configurations that completely stabilise the Wilson lines at one loop and have exponentially suppressed effective potential.

Our work will be extremely geometrical in nature. We will use T-duality to map between Wilson lines and D-brane coordinates, so that the Wilson line moduli space maps to brane configurations. This gives us a rather elegant intuition into how the theory should behave as we move around in the moduli space, without actually having to do much non-trivial mathematics. Nevertheless, we will encounter some subtleties on the way, and everything will be checked against proper string calculations.

One may still worry about the cosmology that the exponentially suppressed terms lead to - can we really take the modulus $\mathcal{M}$ to be flat? The lack of stabilisation of $\mathcal{M}$ is really what prevents us from calling the effective potential a true cosmological constant. We will adopt the view that, since its presence only occurs in exponentially suppressed terms, one can produce parametrically large 'table-top' potentials on which the modulus is effectively stable. The related scenario would be to formulate a theory of quintessence on these models. Actually, the potential of the theories discussed here falls too steeply to account for an accelerating universe. Nevertheless, one can hope that a more sophisticated model could be developed, based on the ideas raised here, which would have a moduli space with sufficiently flat directions to formulate a physically reasonable quintessence. Another potential strategy would be to stabilize $\mathcal{M}$ by adding competing perturbative terms to keep it at values much less than M_s. In this chapter we will simply take $\mathcal{M}$ to be a background field and discuss its stability towards the end.

In section 5.2 we explain the main ideas which we will use to construct our models, before going on to study the warm-up case of a simple circle compactification in 5.3. We will use a combination of simple counting methods, along with some group theory considerations, to estimate the stability within the Wilson line moduli space, before checking against a full string calculation. In section 5.4 we discuss models compactified on a torus T^D, again giving a heuristic picture before displaying the full string calculation. In all of these sections we work with the perturbative $O(32)$ gauge group of type I string. In section 5.5 we discuss the lifting of these gauge groups to the non-perturbative theory with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$. Finally, we make our conclusions in section 5.6. This includes some brief comments on the relation of our work to the swampland program. There are two appendices which complement this chapter. Appendix A is a quick review of some of the concepts used here, whilst Appendix B contains the details of the full string one-loop calculations of the effective potential which are quoted in this chapter.

5.2 The basic idea

In this section, we want to clearly explain the main ideas which allow for non-supersymmetric models with massless boson-fermion degeneracy. We begin by returning to the two bullet points that we began the previous section with, starting with a discussion of the vacuum energy of string models.

From the effective field theory point of view, a first quantized formalism, with given background moduli fields, reveals that the quantum effective potential in D dimensions is

$$\mathcal{V} \;=\; \int_{1/\Lambda^2}^{\infty} \frac{dt}{t^{1+D/2}}\, \mathrm{Str}\, e^{-M^2 t}\,, \tag{5.2.1}$$

where M is the mass operator, and the supertrace is the sum over all states, accounting for spin-statistics. The mass scale Λ acts as a UV cutoff and the sum is over states of mass less than this. The IR divergences at $t \to \infty$ are not a problem and can be regularized by (for example) giving each particle a massive partner with opposite statistics. Assuming a finite number of states with mass $m \ll \Lambda$, one can expand the exponential to find

$$\mathcal{V} \;=\; \sum_{0 \le n < D/2} b_n \cdot \Lambda^{D-2n} \,\mathrm{Str}\, M^{2n} \;+\; b_{D/2} \cdot \mathrm{Str}\, M^D \,\log(M^2/\Lambda^2) \;+\; (\text{UV finite}) \tag{5.2.2}$$

where $b_n = 1/(n!(D/2-n))$ and $b_{D/2} = 1/(D/2)!$ for even dimensions or vanishes for odd dimensions. If the supersymmetry is spontaneously broken, Str $1 = 0$, and occasionally Str $M^2 = 0$. The final divergent term, Str $M^D \log \Lambda^2$, can be removed with a renormalization procedure. The cosmological constant problem still occurs, however, due to the presence of the term Str $M^D \log M^2$, which in principle requires an unrealistically precise cancellation from the UV completion. If all the moduli are stabilised, then the effective potential above really can be said to be the (one-loop) cosmological constant.

One can refine the above to include the presence of a Kaluza-Klein (KK) spectrum. In this case, the partition function is dressed with sums $\sum_{m\in\mathbb{Z}} \exp(-m^2 M_1^2 t)$ where M_1 is the mass scale of the first KK mode. Let us assume M_1 is the lowest non-zero mass scale, so that one has an effective potential

$$\mathcal{V} = \sum_{m\in\mathbb{Z}} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t^{1+D/2}} \,\mathrm{Str}\, e^{-M^2 t - m^2 M_1^2 t} \,, \tag{5.2.3}$$

where M is either zero or greater than M_1. In such circumstances, it is best to Poisson resum the light KK tower so that

$$\mathcal{V} = \frac{\sqrt{\pi}}{M_1} \sum_{\ell\in\mathbb{Z}} \int_{1/\Lambda^2}^{\infty} \frac{dt}{t^{1+\frac{D+1}{2}}} \,\mathrm{Str}\, e^{-M^2 t - \frac{\ell^2}{M_1^2 t}} \,. \tag{5.2.4}$$

The point of doing this is that now all terms with $M \neq 0$ are exponentially suppressed, as can be seen from a saddle point approximation. Note that when either M or ℓ is non-zero, one can remove the UV parameter $\Lambda \to \infty$. Doing this for the $M \neq 0$ piece gives

$$\mathcal{V}|_{M\neq 0} \sim \sqrt{\pi}\; M_1^D \,\Gamma\!\left(\frac{D+1}{2}\right) \sum_{\ell\in\mathbb{Z}} \frac{1}{\ell^{D+1}} \,\mathcal{H}_{\frac{D+1}{2}}\!\left(\frac{M\ell}{M_1}\right) , \tag{5.2.5}$$

where $\mathcal{H}_\nu(z)$ is defined in (B.1.14) and whose asymptotics, given in (B.1.15), show that these heavy terms are absorbed into errors of order[1]:

$$\mathcal{V}|_{M\neq 0} \ \sim \ \mathcal{O}\Big((MM_1)^{D/2} e^{-M/M_1} \Big) \ . \tag{5.2.6}$$

This leaves the modes with $M = 0$ which contribute

$$\mathcal{V}|_{M=0} \ \sim \ \frac{2}{D+1}\Lambda^{D+1}/M_1 + M_1^D \ \Gamma\Big(\frac{D+1}{2}\Big) \sum_{\ell\neq 0} \frac{1}{\ell^{D+1}} \ . \tag{5.2.7}$$

Actually, we will find that in the Scherk-Schwarz models, there is a cancellation which removes the even ℓ terms, including the first $\ell = 0$ term, so that one finds

$$\mathcal{V}_{KK} \ \sim \ (n_F^{(0)} - n_B^{(0)}) M_1^D + \mathcal{O}((MM_1)^{D/2} e^{-M^2/M_1^2}) \ . \tag{5.2.8}$$

The situation is quite analogous in string theory when compactified to D dimensions. In this case, one finds a cut-off proportional to the string scale M_s. For closed strings, the one-loop effective potential is

$$\mathcal{V} = M_1^D \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{\tau_2^{1+D/2}} \ \mathrm{Tr}\ (-1)^F q^{M_L^2} \bar{q}^{M_R^2} \ , \tag{5.2.9}$$

(which really is the vacuum energy if all moduli have been stabilised), where $q = e^{2\pi i \tau}$ and M_L (M_R) are the mass operators in the left (right) moving spectra, incorporating the KK and winding modes originating from lattice sums. The physical states are those which are level matched ($M_L = M_R$), and for much of the τ integral (the region $\tau_2 > 1$ below the string scale) the τ_1 integral projects onto these physical states. Let us be extremely schematic by taking a circle compactification so that level matched states have masses $((mR)^2 + (n/R)^2 + k)M_s^2$. There are now three scales - the string scale M_s, the KK scale M_s/R and the winding mode scale $M_s R$. Let M_1 be the lightest scale of these background metric fields. By Poisson resumming if necessary, one can ensure that all other towers have mass scales at least at the string scale M_s. One therefore expects

$$\mathcal{V} = M_1^D (n_F^{(0)} - n_B^{(0)}) + \mathcal{O}((M_s M_1)^{D/2} e^{-M_s^2/M_1^2}) \ , \tag{5.2.10}$$

where M_1 is now the lightest mass scale of the background metric fields.

Breaking supersymmetry before or during compactification will usually lift the massless boson-fermion degeneracy: one expects $n_F^{(0)} - n_B^{(0)} \neq 0$, so that a large cosmological constant is produced. Actually, if the metric fields are not stabilised, then the coupling to M_1 will introduce a large tadpole in a metric modulus - a catastrophic disaster.

[1] Even three orders of magnitude of difference between M_1 and M, which is not particularly constraining, suppresses massive states by a factor of more than 10^{-120}, below what is required experimentally.

Because of the above discussion, we will be interested primarily in models which contain a massless boson-fermion degeneracy in the compactified theory. The idea is that if $n_F^{(0)} - n_B^{(0)} = 0$ then not only are tadpoles exponentially small so that the associated moduli are essentially flat, but also, as a consequence, one really should have a very small cosmological constant. We will also be interested in the positive potential case with $n_F^{(0)} > n_B^{(0)}$, for such models could be used as a starting point for quintessence[2].

Now let us come to the method we will use to break supersymmetry. The compactification we want to consider is straightforward toroidal compactification, but with a Scherk-Schwarz mechanism which shifts the masses of fermionic KK towers by a half-integral shift, completely removing supersymmetry. Such a compactification can be thought of as a freely acting orbifold $(-1)^F\delta$, where δ rotates a compactified circle $X \to X + \pi R$ for normalized coordinates $X \in [0, 2\pi R]$ and where $R_{SS} = R/2$ is the Scherk-Schwarz radius (i.e. the radius of the physical compactified circle), from hereon referred to as R. It has the effect of shifting the KK towers coming from compactification in the Scherk-Schwarz circle

$$m_{n,k}^2 = \left(\frac{n + F/2}{R}\right)^2 + k^2 M_s^2 \ , \tag{5.2.11}$$

where F is the spacetime fermion number $0, 1$ and the k contributions come from the string oscillator states (we are neglecting vacuum contributions for simplicity). The supersymmetry breaking scale is identified with the mass of the gravitino, so that it is of order $\mathcal{M} = 1/2R$. The advantage of this kind of compactification is that string amplitudes can be fully calculated without resorting to approximations, so that analysing stringy physics is within our control.

At this point, however, we face the central problem: this type of compactification results in many more massless bosons than fermions. One expects AdS vacua. The central idea of this chapter is to remedy this by including non-trivial Wilson lines. On the same circle as before, a KK tower charged under a Wilson line a^α with charge vector Q_α (with α indexing the sixteen Cartan $U(1)$ factors of $SO(32)$) has shifted masses

$$m_n^2 = \left(\frac{n + F/2 + Q_\alpha a^\alpha}{R}\right)^2 . \tag{5.2.12}$$

Note that an open string has ends charged under precisely two of the Cartan $U(1)$s, with charges ± 1. More generally, in a compactification on a torus T^{10-D}, a KK tower of open strings coming from the Scherk-Schwarz circle, with charges Q_α with respect to the Wilson

[2]As discussed in the introduction, the simple models discussed here have too steep a potential for quintessence to be viable. But it is conceivable that there are more elaborate variations on our idea in which one could flatten the modulus.

line in the I^{th} direction, a_I^α with $I = D, D+1, \ldots, 9$, has a mass spectrum:

$$m_{n,k}^2 = G^{IJ} P_I P_J + k M_s^2 \ , \qquad \text{where } P_I = n_I + F/2 + Q_\alpha a_I^\alpha \ . \tag{5.2.13}$$

One can see that, through an interplay between the Scherk-Schwarz twist and the Wilson lines, certain configurations can increase the number of massless fermions relative to bosons.

The above gives us a mechanism to construct configurations with both Scherk-Schwarz supersymmetry breaking *and* with a possibly small cosmological constant, barring the exponentially small instabilities in the metric moduli. We must now also ensure that the Wilson line moduli are stabilised. This will impose extra constraints on the set of possible configurations. At the effective field theory level, one requires the quantum potential to satisfy

$$\frac{\partial \mathcal{V}}{\partial a_I^\alpha} = 0 \ , \tag{5.2.14}$$

for a possible vacuum and

$$\frac{\partial^2 \mathcal{V}}{\partial a_I^\alpha \partial a_I^\beta} = 0 \ , \tag{5.2.15}$$

for a stable vacuum, for each Wilson line. When we vary Wilson lines, $n_F^{(0)} - n_B^{(0)}$ likewise varies. One expects the above equations to be valid when there is no masss scale between 0 and $\mathcal{M}$. This can be seen by a simple argument. Starting with a stable point in moduli space, and turning on a Wilson line, a tower of fermions becomes massive with a mass scale between zero and that of the Scherk-Schwarz supersymmetry breaking scale $\mathcal{M}$. As the Wilson line increases, eventually the fermion mass becomes equal to $\mathcal{M}$, in which case its bosonic superpartner now becomes massless, restoring criticality and enhancing the symmetry group.

The second stability condition can be given a field theory interpretation as follows. Expanding the effective potential 5.2.1 for small Wilson lines, and assuming stability has been achieved (so that (5.2.14) is satisfied), there is a mass term in $\mathcal{V}$ proportional to

$$\left(\sum_{Q \in R_F} - \sum_{Q \in R_B} \right) Q_\alpha Q_\beta G^{IJ} a_I^\alpha a_J^\beta + \cdots \tag{5.2.16}$$

The sum over weights Q can be understood as follows. The spectrum arranges itself into families of representations R_F for fermions and R_B for bosons, which have charges Q_α for the corresponding Cartan $U(1)$. The sum over states then becomes a signed sum over weights of these representations. (In the next section, we will give an example of this in action). Let us assume that the metric is block diagonal for simplicity[3], with $G^{99} \gg G^{ij}$ (where $i, j = D, D+1, \ldots, 8$). Restricting to a single representation R, the mass matrix is

[3] This is not necessary for this argument to work, but is useful in comparing with the later sections.

approximately diagonal:

$$\frac{\partial^2 \mathcal{V}}{\partial a_I^\alpha \partial a_I^\beta} \propto T_R\big(G^{99} a_9^\alpha a_9^\beta + G^{ij} a_i^\alpha a_j^\beta\big)\delta_{\alpha\beta} \ , \tag{5.2.17}$$

where we have used $\sum_{Q\in R} Q_\alpha Q_\beta = T_R \delta_{\alpha,\beta}$, with T_R the Dynkin index of the corresponding representation and the sum being over all weights. The right hand matrix is positive definite. More generally, by decomposing the representations into representations R_i over the simple groups G_i of the gauge group, the Wilson lines corresponding to the simple group G_i have masses proportional to

$$m^2 \propto N_{Bi} T_{Bi} - N_{Fi} T_{Fi} \ , \tag{5.2.18}$$

where T_{Ri} is the Dynkin index of the representation R_i and N_B, N_F are the number of bosonic/fermionic families in such a representation.

It follows that, as we increase the number of massless fermions in order to raise the effective potential, one might sacrifice the stability of a Wilson line they are charged under. This shows how non-trivial constructing our intended sorts of models could be.

In the next sections, we will discuss these models extensively. We will use T-duality to transform to a highly geometric picture where a combination of simple geometric reasoning and effective field theory ideas teaches us most of the essential features.

5.3 Compactification to nine dimensions

In this section we combine the ingredients above and apply a Scherk-Schwarz compactification on a circle to type I string theory, also adding Wilson lines. This warm-up section will illustrate many important features of the more general commpactifications we study next. We set R to be the radius of this compact dimension, measured in string units. The radius is of course a field in its own right but we treat it as a background field for now. To avoid a Hagedorn instability, we assume $R \gg R_H = \sqrt{2}\ell_s$ (the Hagedorn radius).

In the previous section we described our overall method. In the case at hand, we begin by applying the Scherk-Schwarz method, shifting the KK fermion masses by a half-integer:

$$m_n^2 = \left(\frac{n + F/2}{R}\right)^2 M_s^2 \ . \tag{5.3.1}$$

This gives us a supersymmetry breaking scale of

$$\mathcal{M} = \frac{M_s}{2R} \ . \tag{5.3.2}$$

Next, we turn on an $SO(32)$ Wilson line on the compact dimension:

$$\begin{aligned}\mathcal{W} &= \mathrm{diag}(e^{2\pi i a_\alpha}, e^{-2\pi i a_\alpha} \; ; \; \alpha = 1, \ldots, 16) \\ &= \mathrm{diag}(e^{2\pi i a_1}, e^{-2\pi i a_1}, \ldots, e^{2\pi i a_{16}}, e^{-2\pi i a_{16}}) \; .\end{aligned} \tag{5.3.3}$$

In this case, the open strings with Chan-Paton factor $|\alpha\beta\rangle$ get a further mass shift of

$$m_n^2 = \left(\frac{n + F/2 + a_\alpha - a_\beta}{R}\right) M_s^2 \; . \tag{5.3.4}$$

As discussed in Appendix A, the T-dual theory is an orientifold of type IIA theory, whose compact dimension is a line segment with two O8 planes located at the ends $X^9 = 0$ and $\pi\tilde{R}$, with $\tilde{R} = 1/R$. To avoid an RR tadpole coming from the non-zero RR charges of the O-planes, one must include 16 D-branes. The D-brane positions are dual to the Wilson lines, so that a Wilson line a_α is associated with a D-brane at position $\pi\tilde{R}a_\alpha$. Thus the moduli space associated with the Wilson lines is precisely the same space as D-brane positions.

It is more convenient usually to double the degrees of freedom by viewing the line segment as a circle in which any object at coordinate X^9 should have a mirror object at coordinate $-X^9$. In this picture, the two O-planes lie at antipodal points on the circle and 32 half-branes can move around, such that away from the O-planes they are always accompanied by an image half-brane. This is illustrated in figure 5.1.

The Wilson lines spontaneously break the $SO(32)$ gauge symmetry, generically to $U(1)^{16}$. Using the T-dual picture, it is far easier to explain how the gauge groups arise:

- A single D-brane situated away from an O-plane is associated with a $U(1)$ symmetry.

  In the $\frac{1}{2}$-brane picture: the $\frac{1}{2}$-brane and its image collectively contribute a $U(1)$.

- A stack of q D-branes situated away from the O-plane enhances this to $U(q)$.

  In the $\frac{1}{2}$-brane picture: a stack of q $\frac{1}{2}$-branes, accompanied by their q image branes, contribute a gauge group $U(q)$.

- At the O-planes: the gauge symmetry from a stack of q D-branes enhances even more to $SO(2q)$. Thus the full $SO(32)$ symmetry happens when all the D-branes sit at a single O-plane.

  In the $\frac{1}{2}$-brane picture: the q D-branes fractionate into $2q$ $\frac{1}{2}$-branes generating an $SO(2q)$ symmetry.

There is also a gauge symmetry coming from the compactification of the metric G and RR two-form C_2, each giving a single $U(1)$ gauge field. Hence, in the T-dual picture, if we have p_1 half branes at $X^9 = 0$, p_2 half-branes at $X^9 = \pi\tilde{R}$ and r_σ half-branes at points $X^9 = X_\sigma$, the total gauge symmetry group is

$$U(1)^2_{G,C} \times SO(p_1) \times SO(p_2) \times \prod_\sigma U(r_\sigma)\,, \tag{5.3.5}$$

where the RR tadpole condition requires $p_1 + p_2 + \sum_\sigma 2r_\sigma = 32$.

One can see from (5.3.4) that the open strings stretching between two branes become massless if 1) they are bosons and they end on the same stack of branes, or 2) they are fermions with length half of the circle length. Let us illustrate this. If all the D-branes sit at one orientifold plane, the only massless open strings are the bosonic strings which start and end on that stack. These are unoriented and there are $16 \cdot (16-1)/2$ of them, so that they fill out the adjoint of $SO(32)$. Moving a pair of $\frac{1}{2}$-branes away leaves a gauge group $SO(30) \times U(1)$, where now there are $15 \cdot (15-1)/2$ unoriented strings at the orientifold filling out the adjoint of $SO(30)$, and also 2 oriented strings with endpoints on the two $\frac{1}{2}$-branes which provide the adjoint of $U(1)$. As the pair of $\frac{1}{2}$-branes make their way to the O-plane on the other side, the bosonic strings with endpoints on the $U(1)$ half-branes become unoriented and enhance the symmetry to $SO(2)$. Meanwhile the fermions that stretch between it and the $SO(30)$ stack become massless. There are $30 \cdot 2$ of them, filling out the bifundamental of $SO(30) \times SO(2)$.

We would now like to find what are the stable points in the Wilson line moduli space. That is, one wishes to let the branes freely move and see where they settle. As we have already noted, such stable points are expected to be where there is gauge symmetry enhancement. Since there is symmetry enhancement when branes reach the O-planes, one expects the

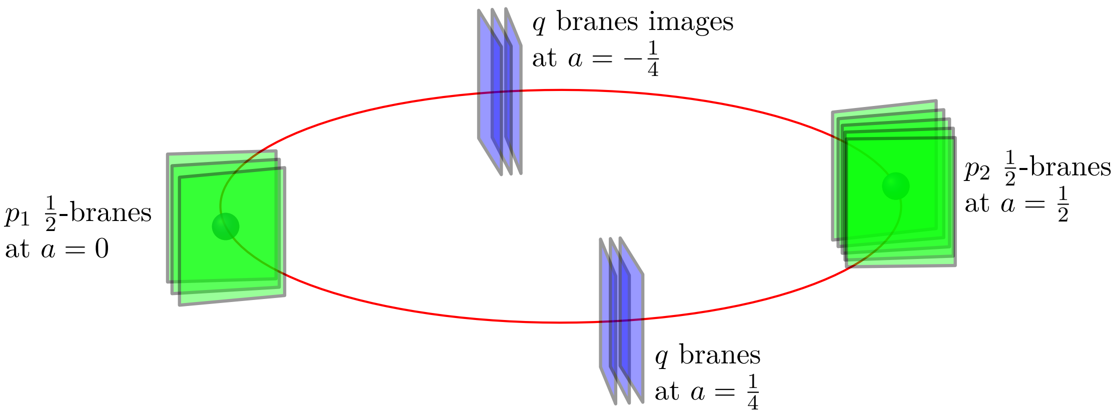

Figure 5.1: *A D9-brane configuration in the T-dual picture, in which the Wilson lines become the positions of D8-branes along $\tilde{X}^9$. In the example depicted, the gauge group is $U(1)^2_{G,C} \times SO(p_1) \times SO(p_2) \times U(q)$.*

O-planes to be attractors to the D-branes.

However, there is an interesting point at $a_i = \frac{1}{4}$, illustrated in figure 5.1. At this point fermions stretching across the q $\frac{1}{2}$-branes and their images also become massless. If there are also p D-branes at one of the O-planes, then there are equal number of bosonic and fermionic strings between $a = 0$ and $a = \frac{1}{4}$ of masses $\mathcal{M}/2$ - these exhibit a kind of 'fake supersymmetry'. Thus it is worth taking this point seriously in our analysis.

5.3.1 Counting massless bosons and fermions

As we have just seen, the number of massless fermions and bosons can be inferred by simple geometric reasoning. Let us do this more generally, placing the branes at the interesting symmetric points. Suppose we place p_1 half-branes at $a_i = 0$, p_2 half-branes at $a_i = 1/2$, q branes at $a_i = 1/4$ and also their images at $a_i = 1/4$. One sees that the number of massless bosons and fermions is

$$\begin{aligned} n_B^{(0)} &= 8 \cdot \big(8 + \frac{p_1(p_1-1)}{2} + \frac{p_2(p_2-1)}{2} + q\bar{q}\big) \\ n_F^{(0)} &= 8 \cdot \big(p_1 p_2 + \frac{q(q-1)}{2} + \frac{\bar{q}(\bar{q}-1)}{2}\big) \ , \end{aligned} \tag{5.3.6}$$

where $\bar{q}$ is numerically equal to q but is useful to remind us that it represents strings stretching to an image brane. The counting is as follows. The first $8 \cdot 8$ comes from the NS-NS sector. The type I orientifold projects out the Kalb-Ramond B_2 field, as well as the RR C_0 and C_4 forms. This leaves the dilaton, metric and C_2 form, which give $1 + 35 + 28 = 64$ states. The other terms in (5.3.6) simply count the number of massless open strings stretching between the branes, as we did in the example in the last subsection, taking into account the fact that the massless fermions must stretch across half of circle, and massless bosons must end on the same brane stack. These terms group together to give the correct dimensions of the appropriate representations of $SO(p_1) \times SO(p_2) \times U(q)$. We see that the 64 closed string states are not charged (as is expected), whilst the remaining bosonic gauge bosons group to form the adjoints of $SO(p_1)$, $SO(p_2)$ and $U(q)$. We also see that the fermions stretching between the two orientifolds form the bifundamental (p_1, p_2) representation of $SO(p_1) \times SO(p_2)$, and the remaining fermion terms come from the antisymmetric and $\overline{\text{antisymmetric}}$ representations of $U(q)$.

Combining these, we find

$$n_F^{(0)} - n_B^{(0)} = 4\big(-(p_1 - p_2)^2 + 2(p_1 + p_2) - 48\big) \ , \tag{5.3.7}$$

where we have used the tadpole cancellation $p_1 + p_2 + 2q = 32$ to eliminate q. At the most symmetric point with the 32 $\frac{1}{2}$-branes at one of the orientifolds, one finds the minimum value $n_F^{(0)} - n_B^{(0)} = -4032$. This configuration should therefore stabilise the Wilson line moduli, although we cannot quite call it a vacuum since $\mathcal{M} = 1/R$ is not yet stabilised (see [86–88] for cosmological solutions).

Perturbatively, the gauge group of type I string theory is $O(32)$. In this group, there is a second disconnected part of the moduli space which can be interpreted as having one half-brane stuck at an orientifold. The corresponding Wilson lines are

$$\mathcal{W} = \mathrm{diag}(e^{2\pi i a_1}, e^{-2\pi i a_1}, \ldots, e^{2\pi i a_{15}}, e^{-2\pi i a_{15}}, 1, -1) \ . \tag{5.3.8}$$

Note that there are now only 15 dynamical Wilson line parameters. Over this part of the moduli space, (5.3.7) is still valid, and the minimum is found when 31 half-branes are located at one of the orientifolds, with the remaining half-brane stuck at the other. In this case, $n_F^{(0)} - n_B^{(0)} = -3356$. This is where we make our crucial observation: freezing a half-brane increases the value of $n_F^{(0)} - n_B^{(0)}$ (although it is still negative). This is of importance in the remainder of this chapter.

5.3.2 The effective potential in 9 dimensions

The actual string computation of the one-loop amplitude is given in Appendix B.2. One finds

$$\mathcal{V} = \frac{\Gamma(5)}{\pi^{14}} \mathcal{M}^9 \sum_n \frac{\mathcal{N}_n(\mathcal{W})}{(2n+1)^{10}} + \mathcal{O}\big((M_s \mathcal{M})^{9/2} e^{-\pi M_s/\mathcal{M}}\big) \ , \tag{5.3.9}$$

valid when $\mathcal{M}$ is small compared to the string scale. For the above we have defined

$$\begin{aligned}\mathcal{N}_n(\mathcal{W}) &= 4\big(-16 - \mathrm{tr}\mathcal{W}^{2n} + \mathrm{tr}\big(\mathcal{W}\big)^{2(2n+1)}\big) \\ &= -16\bigg(\sum_{\substack{\alpha,\beta=1 \\ r\neq s}}^{N} \cos(2\pi(2n+1)a_\alpha)\cos(2\pi(2n+1)a_\beta) + N - 4\bigg)\end{aligned} \tag{5.3.10}$$

where N is the total number of dynamical Wilson lines (i.e. $N = 16$ or 15).

If we place all Wilson lines at the symmetric points $a = 0, \frac{1}{2}$ or $\pm\frac{1}{4}$, the above simplifies considerably. In fact, $\mathcal{N}_n$ becomes independent of n and one recovers the result of the previous section:

$$\mathcal{V} = \xi_9 \mathcal{M}^9 (n_F^{(0)} - n_B^{(0)}) + \mathcal{O}\big((M_s\mathcal{M})^{9/2} e^{-\pi M_s/\mathcal{M}}\big) \ , \tag{5.3.11}$$

where ξ_9 is an unimportant positive constant given in the appendix and where the first term is given by (5.3.7).

We can then proceed to study stability at these points. All we need to do is calculate the derivatives, which are[4]

$$\frac{\partial \mathcal{V}}{\partial a_\alpha} \cong \begin{cases} 0 \\ (p_1 - p_2)\xi_9' \mathcal{M}^9 & \text{for } a_\alpha = \frac{1}{4} \end{cases} , \tag{5.3.12}$$

where ξ_9' is another positive constant. Hence critical points require either $q = 0$ or $p_1 = p_2$.

Finally, we can calculate the second derivative to establish stability. It is block diagonal:

$$\left(\frac{\partial^2 \mathcal{V}}{\partial a_r \partial a_s} \right) = \xi_9'' \mathcal{M}^9 \text{diag}\Big((\frac{p_1 - p_2}{2} - 1) I_{\lfloor p_1/2 \rfloor}, (\frac{p_2 - p_1}{2} - 1) I_{\lfloor p_2/2 \rfloor}, A \Big) , \tag{5.3.13}$$

where I_d is the $d \times d$ identity matrix, $\lfloor x \rfloor$ is the integer part of x, A is the $q \times q$ matrix $A_{\alpha\beta} = \delta_{\alpha\beta} - 1$, and ξ_9'' is another positive constant.

The matrix A has $(q-1)$ eigenvalues $+1$ and a single eigenvalue $-(q-1)$. This renders the $U(1)$ of the $U(q)$ unstable. Hence stable configurations must have $q \leq 1$. It also follows that stable configurations must simultaneously satisfy $p_1 - p_2 - 2 \geq 0$ (if $p_1 \geq 2$) and $p_2 - p_1 - 2 \geq 0$ (if $p_2 \geq 2$). This can only happen if one of p_1 or p_2 is less than 2. Putting everything together one finds that the only stable configurations are $(p_1, p_2, q) = (32, 0, 0)$ and $(31, 1, 0)$ up to labelling[5].

We would like to note the solutions where $n_F^{(0)} - n_B^{(0)} = 0$ correspond to the gauge groups $SO(18) \times SO(14)$, $SO(12)^2 \times U(4)$ and $SO(14) \times SO(12) \times U(3)$. However, the above discussion shows that these are not even inflection points in the potential. In fact, the third group even contains tadpoles.

5.3.3 Stability conditions From field theory

According to the discussion in section 5.2, one should be able to arrive at a similar conclusion using a field theory approximation. To adapt it for the case at hand, we want to expand the nine-dimensional field theory 1-loop effective potential written in the first quantized form:

$$\mathcal{V} = -\frac{\mathcal{M}^9}{2(2\pi)^9} \int_0^\infty \frac{d\tau_2}{\tau_2^{1+9/2}} \text{Str} e^{-\pi\tau_2 M^2} , \tag{5.3.14}$$

where

$$M = \frac{m + F/2 + Q_\alpha a^\alpha}{R} . \tag{5.3.15}$$

[4] Of course, this equation is valid only up to $\mathcal{O}((M_s\mathcal{M})^{9/2} \exp(-\pi M_s/\mathcal{M})$ terms.

[5] The $q = 1$ case is discounted, due to its stable configuration requiring $p_1 = p_2$ due to (5.3.12), in which case there is no solution satisfying the tadpole condition.

Expanding the exponential to quadratic form in the Wilson lines and taking the second derivative, one finds

$$\frac{\partial^2 \mathcal{V}}{\partial a_\alpha^2}|_{a_i=0} = 4\xi_9'' \mathcal{M}^9 (T_B - T_F) \ , \tag{5.3.16}$$

where T_B and T_F are the Dynkin indices whose values for the relevant representations of $SO(n)$ and $U(n)$ are listed in table 5.1.

G	Representation R	$\dim R$	T_R
$SO(p)$, $p \geq 2$	fundamental	p	1
	adjoint	$\frac{p(p-1)}{2}$	$p-2$
$SU(q)$, $q \geq 2$	fundamental	q	1
	adjoint	q^2-1	$2q$
	antisymmetric	$\frac{q(q-1)}{2}$	$q-2$
	antisymmetric	$\frac{q(q-1)}{2}$	$q-2$

Table 5.1*: A list of representations of simple Lie groups with their dimensions and Dynkin indices, using the convention that the Dynkin index is normalized to 1 on the fundamental.*

A limitation of this is that it strictly only applies for $a_i = 0$, or by symmetry at $a_i = \frac{1}{2}$. However, even at the points with $p_1 = p_2$ and q nonzero, one expects it should still hold, for at these points the fake supersymmetry coming from equal numbers of bosons and fermions of mass $\mathcal{M}/2$ renders these states invisible to the supertrace.

Once we know the Dynkin indices, we just need to check how many massless states are charged under which representations of the gauge group. Indeed,

- At the $SO(p_1)$ stack: there are 8 bosons charged under each Cartan of $SO(p_1)$ forming the adjoint representation, $8p_2$ fermions in the fundamental (coming from the bifundamental of $SO(p_1) \times SO(p_2)$).
- At the $SO(p_2)$ stack: there are 8 bosons forming the adjoint, and $8p_2$ fermions in the fundamental.
- At the $U(q)$ stack: there are 8 bosons in the adjoint, 8 fermions in the antisymmetric and 8 fermions in the conjugate $\overline{\text{antisymmetric}}$.

Combining these and summing the relevant Dynkin indices, matches the calculation before:

$$\frac{\partial^2 \mathcal{V}}{\partial a_\alpha^2} \propto \mathcal{M}^9 \begin{cases} p_1 - p_2 - 2 & \text{for the } SO(p_1) \text{ Wilson lines} \\ p_2 - p_1 - 2 & \text{for the } SO(p_2) \text{ Wilson lines} \\ 2 & \text{for the } U(q) \text{ Wilson lines} \end{cases} \tag{5.3.17}$$

which completely matches the previous section.

5.4 Compactifications to D dimensions

We have just seen that in nine dimensions the disconnected branch of the $O(32)$ Wilson moduli space contains a stable configuration whose effective potential is a little higher than the $SO(32)$ model. However, it was still negative. In this section, we will compactify more dimensions so that one has more orientifold planes. In this case, one can freeze more half-branes, and in this way come up with a systematic method to increase the effective potential still further. The hope is that we will eventually find stable models (with respect to the Wilson lines at least) which have $n_F^{(0)} - n_B^{(0)} \geq 0$. What follows is very similar to the previous section, but now with this more general toroidal compactification. We focus on the configurations where the all half-branes are localised at the orientifold planes, since the previous section has illustrated that free branes usually move towards these fixed points anyway.

5.4.1 Counting bosons and fermions

We compactify type I string theory on a torus T^{10-D} with internal metric G_{IJ}, $I, J = D, \ldots, 9$. We will keep the Scherk-Schwarz acting in the 9^{th} direction only and include Wilson lines in all compact dimensions. The Scherk-Schwarz mechanism gives us a supersymmetry breaking scale of the order

$$\mathcal{M} = \frac{\sqrt{G^{99}}}{2}\,. \tag{5.4.1}$$

Again, by T-dualizing one obtains a more geometric representation of the Wilson lines. It is best to T-dualize all internal coordinates, so that the resulting compact space is now a box with 2^D corners, at which sit 2^D O$(D-1)$-planes. The metric on this T-dual box is $\tilde{G}_{IJ} = G^{IJ}$. Again, to saturate the RR charge, one must introduce 16 branes, which can be thought to dissolve into pairs of $\frac{1}{2}$ D-branes at the O-planes.

One can now introduce a Wilson line a_I for each compact cycle. The dictionary between these and the dual D-brane positions is as follows. One can collate the Wilson lines into rows of a matrix (a_I^α). The columns of this matrix are essentially the dual D-brane coordinates normalized to lie in $[0, 1]$, or in other words, the coordinate[6] $2\pi a_I^\alpha \sqrt{\tilde{G}_{II}}$. Figure 5.2 illustrates this T-dual picture, where we have placed all branes at the orientifolds.

Some useful notation. It is convenient to index the O-planes with the label $A = 1, \ldots, 2^D$. Moreover, it is useful to choose an ordering of the A, such that the O-planes $\{2A-1\}$ all lie on the same Scherk-Schwarzed circle coordinate, with O-planes $2A-1$ and $2A$ facing each other in purely the Scherk-Schwarz direction. It is also convenient to define

[6]For this, note that the T-dual box has side lengths $\pi\sqrt{\tilde{G}_{II}}$.

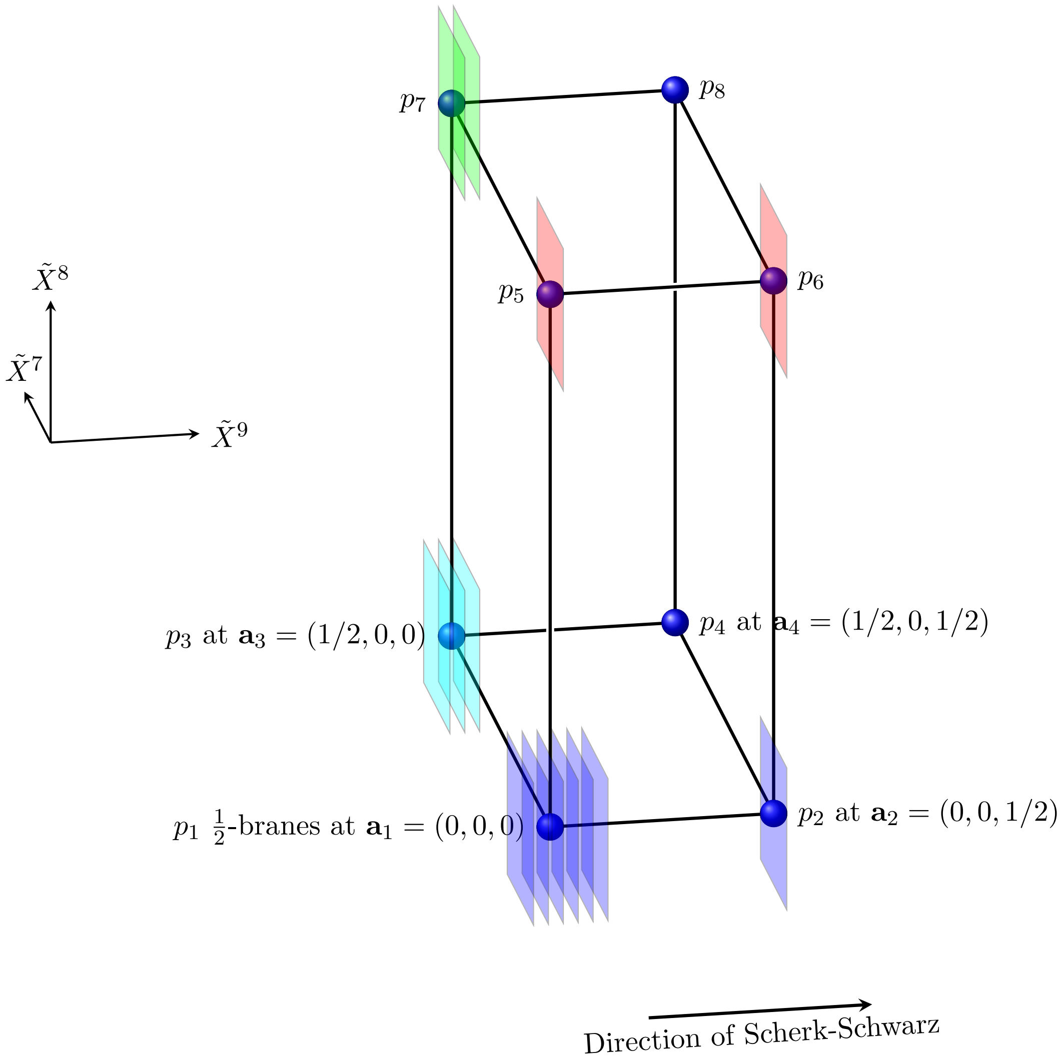

***Figure 5.2**: Configuration of D7-branes and O7-planes in type IIB orientifold, in $D = 7$ dimensions. At each corner of the internal "3-box", there is an orientifold plane coincident with p_A $\frac{1}{2}$-branes, $A = 1, \ldots, 8$. The stacks of p_{2A-1} and p_{2A} $\frac{1}{2}$-branes, $A = 1 \ldots, 4$, are separated in direction $\tilde{X}^9$, along which the Scherk Schwarz mechanism is implemented. In reality, there is a total of 32 $\frac{1}{2}$-branes.*

a vector $\mathbf{a}_S$ which points in the Scherk-Schwarz direction, so that only its $I = 9$ coordinate is nonvanishing: $\mathbf{a}_S = (\mathbf{0}', \frac{1}{2})$. We also define p_A to be the number of $\frac{1}{2}$-D-branes on the A^{th} O-plane. It is sometimes useful to select the coordinates of the internal space which are perpendicular to the Scherk-Schwarz direction. For this, we use indices $i = D, \ldots, 8$ and primed vectors like $\mathbf{n}' = (n_D, \ldots, n_8)$.

Counting massless states. With this rather clunky notation dealt with, we now want to calculate the number of massless states in the models where we place all $\frac{1}{2}$ D-branes at the orientifolds. We can do this as in the last section by simply looking at the geometry.

In particular, the only massless bosons are those which start and end on the same brane stack, as before. The massless fermions can only come from strings stretching in purely the Scherk-Schwarz direction. Indeed, if they stretch in other directions, the combination of the Wilson line shifts with the Scherk-Schwarz shift will never be integral or half-integral. Thus all we need to do is place p_A branes on each O-plane A and count the number of ways a string can stretch across in the Scherk-Schwarz direction. It is then clear that:

$$n_B^{(0)} = 8\left(8 + \sum_{A=1}^{2^{10-D}} \frac{p_A(p_A-1)}{2}\right), \qquad n_F^{(0)} = 8 \sum_{A=1}^{2^{10-D}/2} p_{2A-1}p_{2A} \,. \tag{5.4.2}$$

This can be understood as follows. There are $p_{2A-1}p_{2A}$ possible strings generating the bifundamental $SO(p_{2A-1}) \times SO(p_{2A})$. Meanwhile, there are $p_A(p_A - 1)$ massless bosonic strings whose endpoints lie on the same stack of branes at the A^{th} O-plane, forming the adjoint of $SO(p_A)$. Finally, there are the usual chargeless 64 closed strings coming from metric and RR compactification.

It is convenient sometimes to denote as L the set of pairs (A, B) such that there are branes located at p_A and p_{A+1}. In this case, the right hand side of the above is $\sum_{(A,B)\in L} p_A p_B$.

Combining these expressions with the closed string sector and using the tadpole condition, $\sum_{A=1}^{10-D} p_A = 32$, one finds

$$n_F^{(0)} - n_B^{(0)} = 8\left(8 - \frac{1}{2}\sum_{A=1}^{2^{10-D}/2} (p_{2A-1} - p_{2A})^2\right) . \tag{5.4.3}$$

5.4.2 Stability from the field theory perspective

Before blindly looking for solutions to $n_F^{(0)} - n_B^{(0)} = 0$, it is better to check what the constraints from stability of these models look like. In this section, we adopt our supertrace approximation, while in the next section we will do the full string calculation. To make things simpler for us, we want to first consider a block diagonal metric $G_{IJ} = \left(\begin{smallmatrix} G_{99} & 0 \\ 0 & I \end{smallmatrix}\right)$ with the assumption $G_{99} \gg 1$ (equivalent to $\mathcal{M}/M_s \ll 1$). This ensures that $\mathcal{M}$ is the lowest mass scale in the theory.

As before, we can expand the supertrace form of the quantum effective potential for small

Wilson lines as[7]

$$\mathcal{V} = \mathcal{M}^D \Bigg(\sum_{\text{weights } Q \in R_B} - \sum_{\text{weights } Q \in R_F} \Bigg) \Bigg[-\xi_D + c_2 Q_\alpha Q_\beta \Bigg(\sum_{i,j=D}^{8} G^{ij} a_i^\alpha a_j^\beta + G^{99} a_I^9 a_I^9 \Bigg) + \cdots \Bigg] \tag{5.4.4}$$

where a_I^α are the Wilson lines, $c_2 > 0$ and R_B and R_F are the reducible representations of all the massless bosons and fermions at the point a_α. We are now familiar with this, and so we can immediately claim that stability is, at least for small a_I^α, just a signed sum of Dynkin indices.

Let us take the configurations we are interested in, with p_{2A-1} $\frac{1}{2}$-branes at the O-plane $2A-1$ and p_{2A} at the O-plane $2A$. At the p_{2A-1} stack, there are, up to an overall normalization: 1 bosonic state in the adjoint representation charged under the Cartan $U(1)$ forming the adjoint, and p_{2A} charged fermions transforming in the fundamental. Continuing in this way and using table 5.1 again, one finds that the Wilson line at a_α^I has a positive mass if:

$$\begin{cases} p_{2A-1} - 2 - p_{2A} \geq 0 & \text{for the Wilson lines at } 2A-1 \text{ (if } p_{2A-1} \geq 2) \\ p_{2A} - 2 - p_{2A-1} \geq 0 & \text{for the Wilson lines at } 2A \qquad \text{(if } p_{2A} \geq 2), \end{cases} \tag{5.4.5}$$

for all A for the configuration to be stable. These are reasonably severe constraints, forcing either $(p_{2A-1}, p_{2A}) = (n, 0)$ for $n \geq 0$ or $(n, 1)$ for $n \geq 3$ or $n = 1$ (up to ordering). With all $\frac{1}{2}$-branes confined to the O-planes we can then summarise all of our constraints.

$$\begin{aligned} &\text{The tadpole constraint:} && \sum_A p_A = 32 \\ &\text{Boson-Fermion degeneracy:} && \sum_A (p_{2A-1} - p_{2A})^2 = 16 \\ &\text{The stability condition:} && (p_{2A-1}, p_{2A}) = \begin{cases} (n,0), (0,n) \text{ for } n \geq 0 \\ (n,1), (1,n) \text{ for } n \geq 3 \text{ or } n = 1 \end{cases} \end{aligned} \tag{5.4.6}$$

There are still many $O(32)$ models satisfying these conditions. Table 5.2 gives a list of models with $n_F^{(0)} - n_B^{(0)} = 0$, listing their open string gauge group.

[7]Note that the normalized Wilson lines y_α^I are given by $a_I^\alpha = y_I^\alpha \mathcal{M}$. Because we will only be interested in stability, we will not need to use y_I^α.

Open string gauge group	D
$\big[SO(5) \times SO(1)\big] \times \big[SO(1) \times SO(1)\big]^{13}$	5
$SO(4) \times \big[SO(1) \times SO(1)\big]^{14}$	5
$\big[SO(4) \times SO(1)\big] \times \big[SO(3) \times SO(1)\big] \times SO(1)^3 \times \big[SO(1) \times SO(1)\big]^{10}$	5
$\big[SO(4) \times SO(1)\big] \times SO(2) \times SO(1)^3 \times \big[SO(1) \times SO(1)\big]^{11}$	5
$\big[SO(4) \times SO(1)\big] \times SO(1)^7 \times \big[SO(1) \times SO(1)\big]^{10}$	4
$SO(3) \times \big[SO(3) \times SO(1)\big] \times SO(1)^3 \times \big[SO(1) \times SO(1)\big]^{11}$	5
$SO(3) \times SO(2) \times SO(1)^3 \times \big[SO(1) \times SO(1)\big]^{12}$	4
$SO(3) \times SO(1)^7 \times \big[SO(1) \times SO(1)\big]^{11}$	4
$SO(2)^u \times \big[SO(3) \times SO(1)\big]^v \times SO(1)^{16-4(u+v)} \times \big[SO(1) \times SO(1)\big]^{8+u}, u+v \leq 4$	

***Table 5.2**: A list of models in five dimensions. On the left is the gauge group coming from the open sector. The bracketed notation $[SO(p_1) \times SO(p_2)]$ means that the gauge group is originating from stacks of branes separated purely in the Scherk-Schwarz direction. Gauge groups without brackets come from stacks of branes which do not face other branes in the Scherk-Schwarz direction. The factors $SO(1)$ are trivial, but useful to keep track of what the brane configuration is. On the right hand side is the maximum spacetime dimension that these models live in. If a model works for $D = 5$, it wil also embed into $D = 4$ or lower. Note that each gauge group in the table corresponds to several inequivalent brane configurations.*

Let us conclude this section with a few comments about these models.

- The models require a minimum number (5) of internal dimensions to exist.

- It is useful to notate the gauge groups by $\prod_A [SO(p_{2A-1}) \times SO(p_{2A})]$, where the bracket reminds us that the pair comes from branes separated in purely the Scherk-Schwarz direction.

- Factors $SO(2)$ and $[SO(3) \times SO(1)]$ only have marginal stability and should be investigated further.

- If we allow for models with a greater number of massless fermions than bosons, then they face more severe constraints. The highest value of the effective potential, which has trivially stable Wilson lines (at least perturbatively) is the configuration $[SO(1) \times SO(1)]^{16}$, valid in dimensions $D \leq 5$. This totally reduces the rank of $O(32)$ to 0, although the closed string sector will still contribute its own gauge group associated to the compactification.

- If we expand our search to include models with a negative value of $n_F^{(0)} - n_B^{(0)}$, then there are many more stable models. In particular, the $SO(32)$ model lies in this part of moduli space, with a large negative effective potential.

5.4.3 The effective potential in D dimensions

In this section, we will review the full string calculation of the one-loop effective potential in D dimensions. See Appendix B.3 for details. For now, we will consider a certain regime when the metric G_{IJ} has G_{99} dominant, so that it satisfies

$$G_{99} \gg |G_{ij}| \gg G^{99} , \quad G_{99} \gg |G_{9j}|^2 . \tag{5.4.7}$$

(In the next section we will consider the larger region of moduli space when all metric moduli G_{II} can become large). As in the previous section, we will set $G_{99} \gg 1$ to avoid a Hagedorn instability. Figure 5.2 illustrates this geometry by making the dual $\tilde{X}^9$ dimension smaller than the others.

The calculation of the one-loop effective potential is a little tedious, so we have relegated it to Appendix B.3. Expanding the Wilson lines $a_I^\alpha = \langle a_I^\alpha \rangle + \varepsilon_I^\alpha$, where the background is 0 or 1/2 whilst ε_I^α is a deviation, one finds

$$\mathcal{V} = \xi_D \mathcal{M}^D \sum_{l_9} \frac{\mathcal{N}_{2l_9+1}(\epsilon, G)}{|2l_9+1|^{D+1}} + \mathcal{O}((M_s\mathcal{M})^{\frac{D}{2}} e^{-2\pi c M_s/\mathcal{M}}) \tag{5.4.8}$$

where ξ_D is a constant, c is of order $\mathcal{O}(1)$ and $\mathcal{N}_{2l_9+1}$ is

$$\begin{aligned}\mathcal{N}_{2l_9+1}(\varepsilon, G) \; &= 4\Bigg\{ -16 - 0 - \sum_{(\alpha,\beta)\in L} (-1)^F \cos\Big[2\pi(2l_9+1)\big(\varepsilon_\alpha^9 - \varepsilon_\beta^9 + \frac{G^{9i}}{G^{99}}(\varepsilon_\alpha^i - \varepsilon_\beta^i)\big)\Big] \\ &\qquad \times \mathcal{H}_{\frac{D+1}{2}}\left(\pi|2l_9+1|\frac{\big[(\varepsilon_\alpha^i - \varepsilon_\beta^i)\hat{G}^{ij}(\varepsilon_\alpha^j - \varepsilon_\beta^j)\big]^{\frac{1}{2}}}{\sqrt{G^{99}}}\right) \\ &\quad + \sum_\alpha \cos\Big[4\pi(2l_9+1)\big(\varepsilon_\alpha^9 + \frac{G^{9i}}{G^{99}}\varepsilon_\alpha^i\big)\Big]\mathcal{H}_{\frac{D+1}{2}}\left(2\pi|2l_9+1|\frac{\big[\varepsilon_\alpha^i\,\hat{G}^{ij}\,\varepsilon_\alpha^j\big]^{\frac{1}{2}}}{\sqrt{G^{99}}}\right)\Bigg\} .\end{aligned}$$

We have kept this equation in the form produced by the string calculation, so that the four terms correspond to the torus, Klein bottle, annulus and Möbius strip amplitudes respectively. The function $\mathcal{H}_\nu(x)$, which is defined in (B.1.14), has the behaviour that it approximates to 1 for $x \ll 1$, whilst it is exponentially suppressed for $x \gg 1$ (see (B.1.15)). As a reminder for the reader, the set L is the set of pairs of brane stacks separated purely by the Scherk-Schwarz direction, so that the sum is really over all pairs of branes with $\langle a_I^\alpha \rangle - \langle a_I^\beta \rangle = a_{S,I}$. Finally, the hatted metric defined by

$$\hat{G}^{ij} = G^{ij} - \frac{G^{i9}}{G^{99}} G^{99} \frac{G^{9j}}{G^{99}} = G^{ij} + \mathcal{O}(1/G^{99}) , \tag{5.4.9}$$

can be understood to be an internal metric on the transverse subspace to the Scherk-Schwarz direction.

Although it looks a little unwieldly, equation (5.4.8) has quite simple consequences. First, if we move all the Wilson lines to the orientifold planes, which corresponds to setting all $\varepsilon^I_\alpha = 0$, then one recovers the previous results of subsections 5.4.1 and 5.4.2. Indeed, at these points it is straightforward to recover (5.4.3):

$$\mathcal{N}(\varepsilon^I_\alpha = 0, G) = 4(-16 + (p_1 - p_2)^2) = n_F^{(0)} - n_B^{(0)} . \tag{5.4.10}$$

It follows that the potential has leading terms

$$\mathcal{V} = (n_F^{(0)} - n_B^{(0)})\xi_D \mathcal{M}^D + \mathcal{O}((M_s\mathcal{M})^2 e^{-2\pi c M_s/\mathcal{M}}) , \tag{5.4.11}$$

as we have claimed in the introduction.

Similarly, one can form the Hessian of the effective potential with respect to the Wilson lines in order to establish stability of the configurations. Using Appendix B.3, we find

$$\mathcal{V} = (n_F^{(0)} - n_B^{(0)})\xi_D \mathcal{M}^D + \frac{1}{2}\xi''_D \mathcal{M} \sum_\alpha \left(\frac{p_{A(\alpha)} - p_{\hat{A}(\alpha)}}{2} - 1 \right) \varepsilon^I_\alpha \Delta_{IJ} \varepsilon^J_\alpha + \mathcal{O}(\varepsilon^4) \tag{5.4.12}$$

plus the usual exponentially suppressed terms. Here α is a sum over the dynamical Wilson line degrees of freedom, $A(\alpha)$ is the O-plane on which the dynamical brane at a^I_α is situated and $\tilde{A}(\alpha)$ is the partner O-plane along the Scherk Schwarz direction. Finally, the mass matrix is given by

$$\Delta_{IJ} = \begin{pmatrix} \frac{G^{ij}}{(D-1)G^{99}} + \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & 1 \end{pmatrix} . \tag{5.4.13}$$

In particular, Δ has positive eigenvalues: $9 - D$ of them are large of order $\mathcal{O}(G_{99})$ and the last one is $\mathcal{O}(1)$. This is precisely the same result as we found in the previous subsection, although the advantage of the string computation is that it explicitly shows that the corrections are exponentially small. Before we move on, it is interesting to note that the configurations with $[SO(2)]$ and $[SO(3) \times SO(1)]$ that we found in the last subsection, are indeed marginal at one-loop. It is now time to expand the range of validity of our models by considering the case when all torus moduli can become large.

5.4.4 Extension to more general tori

The previous section made progress by assuming the G_{99} component of the torus metric was dominant. This is quite a narrow part of the metric moduli space. In this section, we would like to generalise this to the much larger moduli space satisfying

$$G_{II} \gg 1 , \tag{5.4.14}$$

for each $I = D, \ldots, 9$. In particular, we are now including the case where $G_{ii} \gg G_{99}$. Forcing our models to be stable on this larger region of moduli space comes at the cost of further restrictions.

A similar calculation to the previous section (see appendix B.3) now shows that

$$\mathcal{V} = \xi_D M_s^D \sqrt{\det G} \sum_{\mathbf{l}} \frac{\mathcal{N}_{\mathbf{l}}(\mathcal{W})}{(\tilde{l}_I G_{IJ} \tilde{l}_J)^5} + \mathcal{O}(M_s^D \sqrt{\det G} G_{99}^{-11/4} e^{-2\pi\sqrt{G_{99}}}) \,. \tag{5.4.15}$$

In this equation, the sum is over integer vectors ℓ_I where $\tilde{\ell} = (l_D, \ldots, l_8, 2l_9 + 1)$ and we have

$$\mathcal{N}_l(\mathcal{W}) = 4(-16 - 0 - \mathrm{tr}(\mathcal{W}_D^{l_D}, \ldots, \mathcal{W}_9^{l_9})) + \mathrm{tr}(\mathcal{W}_D^{2l_D}, \ldots, \mathcal{W}_9^{2l_9})) \,, \tag{5.4.16}$$

where the four contributions come from the torus, Klein bottle, annulus and Möbius strip amplitudes respectively. The factors

$$\mathcal{W}_I = \mathrm{diag}(e^{i a_I^\alpha} \ ; \ \alpha = 1, \ldots, 32) \tag{5.4.17}$$

are the Wilson lines along the I^{th} cycle of the torus. Recall that, although in this parametrisation it looks as if there are 32 Wilson lines, not all the a_I^α should be treated independently (they correspond to half-branes which must be paired with their images when they are situated away from O-planes).

Setting all $\frac{1}{2}$-branes to lie at the O-planes so that there are p_A at the A^{th} O-plane, this gives

$$\mathcal{N}_l(\mathcal{W}) = 8\bigg(-8 - \frac{1}{2} \sum_{A,B=1}^{2^{10-D}/2} (p_{2A-1} - p_{2A})(p_{2B-1} - p_{2B})(-1)^{2\mathbf{l}' \cdot (\mathbf{a}'_A - \mathbf{a}'_B)} + \frac{1}{2} \sum_{A=1}^{2^{10-D}} p_A \bigg) \,. \tag{5.4.18}$$

Here, the primed vectors denote just the $I = D, \ldots, 8$ components.

This expression is quite interesting. For the leading term in the effective potential to vanish (so that there is massless boson-fermion degeneracy), the quantity $\mathcal{N}_l$ must vanish, which implies that it is independent of $\mathbf{l}'$. This can only happen if all $p_{2A-1} - p_{2A} = 0$ except for a single pair of O-planes, say $A = 1, 2$. This seems to reduce considerably the number of possible models with suppressed effective potential.

The reader may wonder why the results of the previous sections do not appear to show this behaviour. It comes down to the fact that previously we had been assuming that $\mathcal{M} \sim \sqrt{G^{99}} M_s$ was the lowest mass scale of the theory. But in the region we are currently discussing, an even lower mass scale $\sqrt{G^{ii}}$ can develop as the i^{th} cycle of the type I torus becomes dominantly large. As this happens, the KK tower associated to the i^{th} cycle becomes very light. If this tower feels any Scherk-Schwarz supersymmetry breaking, then it will start contributing to $n_F^{(0)} - n_B^{(0)}$ non-trivially. A quick geometric argument should convince the reader that the brane configurations under which such towers do not get a Scherk-Schwarz breaking are precisely those that have at most a single stack of branes separated from the rest in the Scherk-Schwarz direction. The other case is when $p_{2A-1} = p_{2A}$ for multiple A. For these kinds of brane configurations, the Scherk-Schwarz breaking in the i^{th} direction is boson-fermion degenerate, so that one still expects a stable minimum. Of course, one can combine these two possibilities to obtain the solutions in the previous paragraph.

Let us then find all the remaining stable models. The above discussion shows that we might as well have p_1 $\frac{1}{2}$-branes lying at $A = 1$, p_2 $\frac{1}{2}$-branes at $A = 2$, and then the remainder must satisfy $p_{2A'-1} = p_{2A'}$ for all other A'. Writing the Wilson lines so that ε_I^α is the deviation from their background positions, one finds from (5.4.16) that

$$\mathcal{N}_{\tilde{\ell}} = -16\Bigg(\sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{\frac{1}{2}(p_1-p_2)} \cos(2\pi\tilde{\ell}\cdot\varepsilon^\alpha)\cos(2\pi\tilde{\ell}\cdot\varepsilon^\beta) + \frac{p_1-p_2}{2} - 4\Bigg) . \tag{5.4.19}$$

In particular, one recovers

$$n_F^{(0)} - n_B^{(0)} = 8\big(8 - \frac{1}{2}(p_1-p_2)^2\big) . \tag{5.4.20}$$

Taylor expanding $\mathcal{N}_{\tilde{\ell}}$ gives a mass term for the ϵ_α^I which looks like

$$\delta\mathcal{V} = \xi_D''\sqrt{\det G}(p_1-p_2-2)\sum_{r=1}^{\frac{1}{2}(p_1-p_2)} \varepsilon_\alpha^I \Delta_{IJ} \varepsilon_\alpha^J + \cdots , \tag{5.4.21}$$

where there are terms of higher order in the Wilson line moduli, as well as the usual exponentially suppressed terms. The matrix Δ_{IJ} in the above is given by

$$\Delta_{IJ} = \sum_{\mathbf{l}} \frac{\tilde{l}_I\tilde{l}_J}{(\tilde{\ell}_K G_{KL}\tilde{\ell}_L)^5} , \tag{5.4.22}$$

which is clearly positive definite (since G_{IJ} is). We conclude that we must take $p_1-2-p_2 \geq 0$ for stability, but there do not appear to be any further constraints. It is worth pointing out that the $SO(2)$ and $[SO(3)\times SO(1)]$ configurations are still flat in this larger G_{IJ} regime.

To summarise, we need our models to satisfy:

$$\begin{aligned}
&\text{The tadpole condition:} && \sum p_A = 0 \\
&\text{Massless fermion-boson degeneracy:} && (p_1 - p_2)^2 = 16\ ,\ p_{2A'-1} = p_{2A'} \ \text{ for } A' \geq 3, \\
&\text{Stability constraint:} && p_1 - p_2 - 2 \geq 0 \text{ (up to ordering) .}
\end{aligned} \tag{5.4.23}$$

There are just two gauge groups (although multiple configurations) which satisfy these:

$$SO(4) \times [SO(1) \times SO(1)]^{14} \quad \text{and} \quad [SO(5) \times SO(1)] \times [SO(1) \times SO(1)]^{13}\ . \tag{5.4.24}$$

These occur for $D \leq 5$. These models are now fairly robust - the moduli G_{IJ} (including $\mathcal{M} \sim \sqrt{G^{99}}$) are only contained in the exponentially suppressed terms and can be taken to be essentially flat, whilst the potential is independent of the RR moduli from C_2. Their effective potential is nearly vanishing and all Wilson lines are stabilized. This is about as good as we can hope to get from such a simple theory.

5.5 Nonperturbative models

In the previous section, we worked with the perturbative symmetry group $O(32)$. However, the real non-perturbative gauge group of type I string theory is $\mathrm{Spin}(32)/\mathbb{Z}_2$, and it is important to check which of our models lift to this group, so that the Wilson lines provide consistent non-perturbative backgrounds.

The difficulty of lifting an $O(N)$ bundle to a $\mathrm{Spin}/\mathbb{Z}_2$ bundle can be seen in two steps. First, we would like the gauge fields to lie in $SO(N)$. Second, we need them to lift to the full gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$. The first condition is not particularly stringent. It merely implies

$$\det \mathcal{W}_I = 1\ , \tag{5.5.1}$$

for all of the Wilson lines. For many of the gauge groups we have constructed, this can be made valid. In particular, one simply needs

$$\sum_\alpha a^I_\alpha \in \mathbb{Z}\ , \tag{5.5.2}$$

for each I, in the notation of the previous section.

The second stage is to promote the $SO(32)$ Wilson lines to $\mathrm{Spin}(32)$ Wilson lines (i.e. express them as even products of gamma matrices). Non-perturbative particles like the type I D-string, which transforms spinorially under the gauge group, must also have consistent holonomies as they traverses internal cycles. Therefore, the lifted Wilson lines must commute in their spinorial representation [94]. Whilst the Wilson lines $\mathcal{W}_I$ trivially commute in the vector representation, there may not be a consistent lift in which they commute in Spin(32).

The first and second Stiefel-Whitney classes are what measure the failure of the two steps of this lifting [95]. The first Stiefel-Whitney coefficient measures the failure of an $O(32)$ bundle to lift to $SO(32)$, whilst the second measures its failure to have a spin structure (and therefore a lift to Spin(32)). Actually, one only needs a modified Stiefel-Whitney class since the full non-perturbative group is a $\mathbb{Z}_2$ quotient of Spin(32) [96], but bundles which only lift to this quotient group rather than the full Spin(32) do not have 'vector structure', meaning that their topology makes it impossible for vector representations to arise[8] [97].

The simplest example of this in the context of our theories is the nine-dimensional compactification we discussed in section 5.3. There, the Wilson line associated with gauge group $SO(p_1) \times SO(p_2)$ has Wilson line $W = \mathrm{diag}(1^{p_1}, (-1)^{p_2})$. This has determinant $(-1)^{p_2}$, so that only if p_2 is even does the Wilson line belong to $SO(32)$.

From the Stiefel-Whitney perspective, one works over the cohomology $H^*(S^1, \mathbb{Z}_2)$ of the circle, which has θ as its generator of H^1. Given a Wilson line a_α, let $D(a_\alpha) = 0$ if $a_\alpha = 0$ and $D(a_\alpha) = 1$ if $a_\alpha = \frac{1}{2}$. Then the total Stiefel-Whitney form is [95]

$$w = \prod_\alpha (1 + 2a_\alpha \theta)^{D(a_\alpha)} \ . \tag{5.5.3}$$

For our $SO(p_1) \times SO(p_2)$ gauge group, this reduces to $w = 1 + p_2\theta$. The first Stiefel-Whitney class is the one-form component, which vanishes in $\mathbb{Z}_2$ only if p_2 is even.

Now let us compactify two dimensions on a torus. We will place all branes at the O-planes, so that the resulting gauge group is $[SO(p_1) \times SO(p_2)] \times [SO(p_3), SO(p_4)]$. The corresponding Wilson lines are $\mathcal{W}_9 = \mathrm{diag}(I_{p_1}, -I_{p_2}, I_{p_3}, -I_{p_4})$ and $\mathcal{W}_8 = \mathrm{diag}(I_{p_1}, I_{p_2}, -I_{p_3}, -I_{p_4})$. For these both to lie in $SO(N)$ one requires $1 = \det \mathcal{W}_9 = (-1)^{p_2+p_4}$ and $1 = \det \mathcal{W}_9 = (-1)^{p_3+p_4}$. This prevents gauge groups like $[SO(30) \times SO(1)] \times SO(1)$ coming from $SO(32)$. Note that the gauge group $[SO(29) \times SO(1)] \times [SO(1) \times SO(1)]$ does indeed lie in $SO(32)$.

Again, let us look at this using Stiefel-Whitney classes. Now we have two generators of the torus θ_1 and θ_2. One now defines $D(a_I^\alpha) = 0$ if $a_I^\alpha = 0$, and $D(a_I^\alpha) = 1$ if $a_I^\alpha = \frac{1}{2}$. Then the Stiefel-Whitney form is

$$w = \prod_\alpha (1 + 2a_1^\alpha \theta_1 + 2a_I^\alpha \theta_2)^{D(a_I^\alpha)} \ . \tag{5.5.4}$$

[8] Note that for the type I string, and its dual heterotic string, only particles in the adjoint and spinorial representations occur, which is why the gauge group can be considered to be a quotient of Spin(32) by a central $\mathbb{Z}_2$ subgroup.

With p_i as above, one finds

$$\begin{aligned} w &= (1+\theta_1)^{p_2}(1+\theta_2)^{p_3}(1+\theta_1+\theta_2)^{p_4} \\ &= 1 + (p_2+p_4)\theta_1 + (p_3+p_4)\theta_2 + (p_2p_3+p_2p_4+p_3p_4)\theta_1\theta_2 \ . \end{aligned} \tag{5.5.5}$$

We see that the vanishing of w_1 precisely reproduces the constraints illustrated above for the gauge group to descend from $SO(32)$. However, the vanishing of the second cohomology term w_2 gives us a new constraint - the combination $p_2p_3+p_2p_4+p_3p_4$ must be even. This prevents the aforementioned group $[SO(29)\times SO(1)]\times[SO(1)\times SO(1)]$, which does lie in $SO(32)$, being lifted to Spin(32). We note that there is an elegant inductive argument using 'reduction on a circle' to study the Stiefel-Whitney classes for higher dimensional compactifications [95].

We will not check further how the vanishing of w_2 constrains our models. It appears, although we have not formally checked, that most of them become inconsistent when this constraint is taken into account (although there is always an interesting $SO(1)^{32}$ model [98] with positive potential). However, one should not despair - the models in this paper are supposed to be 'proof-of-concept'. Their gauge groups are too small to be phenomenologically useful, so it is less relevant whether these particular configurations satisfy non-perturbative conditions. However, it is possible, as we will discuss in the conclusion, to make many variations on these kinds of theories, such as orbifolding them in different ways. Such a procedure could potentially make these kinds of models more viable; in which case, one would certainly need to check the non-perturbative constraints fully.

5.6 Conclusions

In this chapter, we have analysed the perturbative stability of type I compactifications with Scherk-Schwarz-type supersymmetry breaking and non-trivial Wilson lines over internal cycles. The advance over previous works is that we have constructed models which have an exponentially suppressed effective potential, exponentially flat metric and RR moduli, and stable or at least marginally stable Wilson lines. These theories are then an attractive starting point for designing theories with an exponentially small cosmological constant.

We have found that the geometrical nature of the type I orientifolds makes them extremely intuitive string theories to analyse. They have a T-dual geometric realization which makes understanding these models very simple - especially when compared with their heterotic counterparts. The Wilson lines are mapped under T-duality to the positions of D-branes, so that the moduli space can be explored by moving branes around. We have seen how one can then use simple geometric arguments to find the leading order of the effective potential, whilst relatively trivial group theoretical arguments can reliably estimate the stability of the Wilson lines.

Our 'trick' to increase the effective potential from the deep AdS values expected from 'vanilla' Scherk-Schwarz compactifications is to freeze fractional D-branes to orientifold planes. Perturbatively, such configurations lie in disconnected pieces of the moduli space of $O(32)$ flat connections. By freezing them, we have allowed for strings to stretch non-trivially across the internal space. Normally, these would be massive, but together with the Scherk-Schwarz twist, the fermions can become massless. Thus, freezing of branes gives us a way to increase the number of massless fermions until one reaches boson-fermion degeneracy, at which point the leading order of the effective potential vanishes and the remainder is exponentially small.

All of this assumes that the metric moduli are kept very large compared with the string scale. Future work could be done to construct models which stabilize these moduli. Alternatively, it would be interesting to construct theories of quintessence based on the ideas raised here.

In the introduction to this chapter, we referred to the swampland programme (see [99] for a review). Let us see how our work intertwines with this. First, a major swampland conjecture is that an effective potential $\mathcal{V}$ coming from a string theory should satisfy $|\mathcal{V}'| > c \cdot \mathcal{V}$ for some order one constant c [55]. Our models have exponentially small effective potentials of the type $\mathcal{V} \sim e^{-R}$ where R is the radius of the Scherk-Schwarz circle. These potentials satisfy the conjecture in that $|\mathcal{V}'|/\mathcal{V} \sim R$ which can become large. A related conjecture is that string theory requires some sort of quintessence in order to explain dark energy [100]. Our models could potentially provide a good starting point for accomplishing this, for they have exponentially suppressed moduli. Such exponential potentials also remove the 'double fine-tuning' criticism that is sometimes levelled against quintessence - that one needs two tunings for both $\mathcal{V}$ and $\mathcal{V}'$ to be very small. However, a naive application of this won't work here - one needs to find a way to increase the exponent of the effective potential to beyond the critical value (which is $\sqrt{2}$ in string units in four dimensions). This would be very interesting future work.

A potential problem with the models considered here is that their gauge groups are too small to accommodate the standard model or its extensions, with $SO(5)$ the largest simple group that can be constructed. However, by including extra ingredients such as orbifolding the theory, one should be able to increase the size of gauge groups. This is because the orbifold introduces new $O5$ planes which must come with new $D5$ branes in order to avoid an RR tadpole, which increases the size of the starting gauge group. In this scenario, one would try to construct the standard model on a D5 brane, with the $D9$ 'hidden sector' playing the role of dark energy. In [101], such a study has been carried out, confirming that there are models with much larger gauge groups. However, these models still require a proper consideration of the non-perturbative constraints that are required for the $O(32)$ gauge bundles to lift to the actual non-perturbative gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$.

Finally, it would be interesting to do a study of the cosmology of these theories. Cosmologies of similar theories have been considered in [86–88]. In particular, the time evolution of the Scherk-Schwarz mass scale $\mathcal{M}$ varies in general and one must find a way of keeping it large. This is an open problem which would be the next logical step to address.

Chapter 6

Conclusion

This thesis explores modifications to standard world-line theories. The underlying theme is to investigate possible UV completions of standard quantum field theories by using a particle point of view. It has shown how the first quantized, or worldline, formalism can act as a useful starting point for the creative design of other new theories. The advantage of this approach is that it provides an insight into how one might remove the high energy divergences that often plague quantum field theories, rendering them only as effective field theories of something more fundamental. We have studied more creative modifications of worldline theories, as well as focusing on perhaps the most successful example, string theory.

The bulk of the original research in this thesis is contained in chapters 4 and 5. In chapter 4, we have seen that there is a broad class of theories written in the worldline formalism with extra towers of internal states which mimic the oscillator modes of a string. It has been argued that these theories are more general than string theory and share various useful properties. For example, some of them have modular invariant partition functions as well as Gross-Mende-like behaviour at large energies. This research also provides a tool to study string theory itself from an unusual point of view. We have only scratched the surface of these modified worldline theories, and it would be very interesting to know whether there is a better unifying framework for them. Can they, for example, always be thought of as arising from strings or higher dimensional membranes on different backgrounds? This is a possible future direction of research.

The second major part of this thesis, chapter 5, has focused on the phenomenology of string theory. Our aim has been to produce stable vacua with an almost vanishing but positive cosmological constant. More precisely, we have studied compactifications of type I string theory which break supersymmetry at a high scale. Usually these compactifications are fraught with instabilities so that one does not find any true vacua. However, through a combination of the specific type of supersymmetry breaking we have used and the addition of Wilson lines, we have found models which have an effective potential which vanishes up to exponentially suppressed terms. In these models, all moduli are stabilised, marginal, or flat up to exponential terms. The latter qualification is the only thing which prevents us saying that the small effective potential really is a small cosmological constant. Thus, by attempting to stabilise these moduli, or looking at their cosmology, these models provide a starting point to constructing theories with very small but positive cosmological constants. The latter should provide an interesting direction for future work.

Appendix A

Basic Features of Type I String Theory

This appendix is a brief introduction to type I string theories, intended as a light companion to chapter 4. We do not aim to be particularly complete, only focusing on aspects of the theory which were used in that chapter. There are several excellent reviews already [102, 103] and books such as [4, 104, 105] provide good references.

A.1 A brief review of orientifolds and type I string theory

Type I string theory is the only consistent supersymmetric string theory in ten dimensions which includes open strings. It is best thought of as a quotient of type IIB string theory. We will begin with a brief discussion of the latter.

Type IIB string theory consists of one left-moving and one right-moving two-dimensional superconformal field theory, of the same chirality, tensored together. The degrees of freedom in either sector are essentially 8 bosonic $X(z)$ and 8 fermionic $\psi(z)$ fields which can be expanded into an infinite number of string modes. Out of these one can build an infinite spectrum of massive states. They group together in conformal families which form two species: descendents from a bosonic Neveu-Schwarz vacuum $|0\rangle$ and a fermionic Ramond vacuum $|S\rangle$ of positive chirality.

This is not quite the full definition of IIB string theory. Indeed, such a theory is inconsistent on its own, and requires a gauged $(-1)^f$ symmetry, where $(-1)^f$ counts the total (left and right) number of worldsheet fermionic oscillators[1,2]. There are then a priori four sectors on each side of the theory, coming from primary states enacted on by an even number of fermionic oscillators and an arbitrary number of bosonic oscillators. These transform in the trivial (O), vector (V), chiral spinor (S), or antichiral spinor (C) representations of $SO(8)$. The discrete gauge symmetry is chosen to project out the tachyonic O and also the C family, so that, on each side of the theory, states live in either V or S. Due to the boundary conditions of the fields, IIB theory must be interpreted as a theory of closed strings only.

Type IIB string theory exhibits a discrete worldsheet parity symmetry Ω exchanging the left and right hand sectors. This only works because the ground (primary) states are the same on each side of the theory. It is thus possible to gauge this symmetry, which results in what is called type I theory. The gauge symmmetry introduces new 'twisted' states, under which mirrored sides of the closed string are identified, so they are really unoriented open strings[3].

The endpoints of an open string X are able to couple to a gauge field through the inclusion of a boundary term to the worldsheet action:

$$q \int_{\partial\Sigma} d\tau A_\mu \partial_\tau X^\mu \ . \tag{A.1.1}$$

Integrating this, assuming constant A, shows that either end of the string is charged under A with charges $\pm q$. This extends to any semisimple Lie group G - in this case the open strings have a basis of Chan-Paton states:

$$|\phi\rangle \otimes |ij\rangle \ , \tag{A.1.2}$$

where $|\phi\rangle$ is a state made from the vacuum by applying the usual string oscillators, whilst the Chan-Paton factors $|ij\rangle$ are the states which have charge 1 with respect to the i^{th} $U(1)$ Cartan of G, and -1 with respect to the j^{th}. By considering, for example, the consistency of string interactions, it can be shown [107] that an unoriented string can only be charged under either $SO(32)$ or USp(32), depending on the level of the state. The open string states in type I string theory transform in the vector representation of $SO(32)$.

[1] One also needs to specify how f acts on the Ramond-Ramond ground states. One convention is to let it be $+1$ for $|S\rangle$ and -1 for $|C\rangle$ on each side of the theory.

[2] A $(-1)^f$ gauged theory is required by modular invariance and is equivalent to summing over spin structures [106]. Actually, to achieve this, the type II theories gauge $(-1)^{f_L}$ and $(-1)^{f_R}$ independently, where f_L and f_R are the left and right worldsheet fermion numbers - this is the non-chiral GSO projection. By choosing the convention that f_L acts on the left vacuum as $+1$ on S and -1 on C, one finds that there are two possible theories, IIA and IIB, depending on how f_R acts on the right spinorial vacua. There are two other chiral ways to gauge the $(-1)^f$ symmetry - these give the type 0 theories, which are non-supersymmetric and contain a tachyon.

[3] The terminology of twisted states is debatable here, see [104].

Non-perturbatively, however, the gauge group is $\mathrm{Spin}(32)/\mathbb{Z}_2$. One can see this by using a duality with heterotic string theory, which suggests there are D1 strings in type I theory which are spinorially charged under the gauge group.

Type IIB theory contains in its NS-NS sector a graviton G, antisymmetric tensor B and dilaton ϕ. In its RR-RR sector, it contains, up to duality, the fields C_0, C_2 and C_4 where C_p are p-form fields. A close examination of the action of Ω on the spectrum shows that it projects out B_2, C_0 and C_4. Meanwhile, states in the NS-R sector and R-NS sector simply become identified under Ω.

We also note that the action of Ω on the open string oscillators is

$$\Omega \alpha_k^\mu \Omega = (-1)^k \alpha_k^\mu \, . \tag{A.1.3}$$

Worldline supersymmetry enforces a similar action on the fermionic oscillators.

Quotients of string theories by products of Ω with other symmetries are called orientifold theories, or simply orientifolds. They usually admit a description in terms of a certain number of non-perturbative non-dynamical O-planes and non-perturbative dynamical D-branes. In particular, type I string theory has open strings which can be viewed as ending on 16 spacetime filling D-branes, coincident with a single spacetime filling O-plane [105].

A.2 Type I one-loop worldsheets

The one-loop diagram in type IIB string theory has the topology of a torus, with modular parameter τ which is integrated over the moduli space of $SL(2,\mathbb{Z})$. For example, the partition function is

$$\mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6} \, \mathrm{Str}\, q^{L_0 - 1/2} \bar{q}^{\bar{L}_0 - 1/2} \, , \tag{A.2.1}$$

written in the operator formalism, where $q = e^{2\pi i \tau}$ and $\mathcal{F}$ is the fundamental domain for $SL(2,\mathbb{Z})$. One can also think of this as an integral over the strip $\mathcal{S} = \{-\frac{1}{2} < \tau_1 < \frac{1}{2} \, , \; \tau_2 > 0\}$ where the torus worldsheet introduces a gauge symmetry relating $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$, thus restricting the integral to the fundamental domain.

In order to gauge worldsheet parity Ω, a projection $\frac{1}{2}(1+\Omega)$ should be introduced in the supertrace. The resulting amplitude can therefore be separated into two contributions $\mathcal{T} + \mathcal{K}$, where one finds

$$\mathcal{T} = \frac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6} \, \mathrm{Str}\, q^{L_0 - 1/2} \bar{q}^{\bar{L}_0 - 1/2} \, , \qquad \mathcal{K} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \, \mathrm{Str}\, \Omega q^{L_0 - 1/2} \bar{q}^{\bar{L}_0 - 1/2} \, . \tag{A.2.2}$$

The right hand term is called a Klein bottle amplitude because the Ω projection enforces a sewing rule where the closed string must meet itself with a change of orientation, so that the topology is a Klein bottle. Such an operator insertion is only consistent when the underlying torus modulus is purely imaginary, $\tau = i\tau_2$, and then there is no worldsheet symmetry to prevent us from integrating over all moduli $\tau_2 > 0$. The effect of inserting Ω is to interchange left and right moving modes, so that each contributes equally:

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\, \mathrm{Str}\, (q\bar{q})^{L_0-1} \,. \tag{A.2.3}$$

Hence the characters and Dedekind eta functions for the Klein bottle should be evaluated at $2i\tau_2$.

The open string sector has a similar projection. One now considers an open string one-loop amplitude, again placing a $\frac{1}{2}(1+\Omega)$ projection inside it, so that there are two pieces:

$$\mathcal{A} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\, \mathrm{Str}\, q^{\frac{1}{2}(L_0-1/2)} \,, \qquad \mathcal{M} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^6}\, \mathrm{Str}\, \Omega q^{\frac{1}{2}L_0-1/2} \,. \tag{A.2.4}$$

The left hand side has the topology of a cylinder whose modular parameter is $i\tau_2/2$. The factor of $\frac{1}{2}$ is intuitive from the open string being half the length of the closed string. The right hand side has the topology of the Möbius strip, thanks to the insertion of Ω. Since Ω acts on states with a sign depending on the state's level, the right hand side is equivalent to evaluating the trace at $\tau = \frac{1+i\tau_2}{2}$.

These supertraces break into sums over Verma modules. It is convenient to introduce the characters for these Verma modules by

$$\chi_R = \mathrm{Tr}_R\, q^{L_0-c/24} = q^{h-c/24}\sum_{n\geq 0} a_n q^n \,, \tag{A.2.5}$$

where c is the central charge, h is the weight of the primary and a_n are the degeneracies for each level. In particular, one can refine these if there is a chirality projection [108]:

$$\begin{aligned} O_8 &= \mathrm{Tr}_{NS}(1+(-1)^f)q^{L_0-c/24} \,, \\ V_8 &= \mathrm{Tr}_{NS}(1-(-1)^f)q^{L_0-c/24} \,, \\ S_8 &= \mathrm{Tr}_{R}(1+(-1)^f)q^{L_0-c/24} \,, \\ C_8 &= \mathrm{Tr}_{R}(1-(-1)^f)q^{L_0-c/24} \,, \end{aligned} \tag{A.2.6}$$

where f is the worldsheet fermion number. In the next appendix, we will make use of explicit forms of these.

The Möbius strip is a little unusual in that the Ω projection results in a sign change depending on the parity of the string oscillator level. Naively, this can be accommodated in the characters by evaluating them at $\chi(i\tau_2 + 1/2)$. But this introduces an overall sign, which we must remove. This is why, for Möbius amplitudes, it is convenient to introduce hatted characters which differ from their unhatted cousins precisely by this sign:

$$\hat{\chi}_R(\frac{1}{2}+\frac{i\tau_2}{2}) = e^{-i\pi(h-c/24)}\chi_R(\frac{1}{2}+\frac{i\tau_2}{2}) \;, \tag{A.2.7}$$

where as indicated, one should evaluate them at $\frac{1}{2}+\frac{i\tau_2}{2}$.

The open string one-loop diagrams also have closed channel tree level interpretations. The annulus becomes a closed string cylindrical worldsheet, the Klein bottle becomes a closed string that propagates between two cross-caps and the Möbius strip becomes a closed string propagating into a single cross-cap. The associated moduli can be read by looking at the corresponding worldsheet diagrams 'sideways' and rescaling. One finds the natural moduli (i.e. the 'length' of the cylinders) are

$$\ell = \begin{cases} 1/2\tau_2 & \text{(Klein bottle, Möbius strip)} \;, \\ 2/\tau_2 & \text{(annulus)} \;. \end{cases} \tag{A.2.8}$$

The equivalence to closed string propagation can be seen by changing variables from τ_2 to ℓ and using a modular transform to change terms involving $1/\ell$ to terms involving ℓ.

Finally, we note that the amplitudes $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ are individually divergent. The tadpole conditions arise from their sum being convergent. Technically, there are two tadpoles: a cancellation across the NS-NS sector and another cancellation across the R-R sector. In type I string theory these are related by supersymmetry. The vanishing of the RR tadpole is essential, being associated with the cancelling of non-zero RR-flux. In the D-brane picture, it comes about from cancelling the RR-charge of the O-plane with a certain number of D-branes. The NS-NS tadpole can in principle not need to vanish - it is associated with a non-trivial dilaton contribution to the vacuum energy.

A.3 The Scherk-Schwarz mechanism

A.3.1 Scherk-Schwarz in spacetime field theories

The Scherk-Schwarz mechanism was originally proposed as a 'generalized dimensional reduction' for field theories formulated in spacetime [109, 110]. The idea is that if the theory includes some discrete global symmetry G, then one can consider new boundary conditions such that fields that wrap cycles return to themselves up to a non-trivial element of G.

To illustrate this, suppose ϕ is some field of our D dimensional theory and we proceed to compactify the D^{th} dimension to a circle of radius R whose coordinate is $y \in [0, 2\pi R]$. In the most basic dimensional reduction, one imposes the boundary condition

$$\phi(x, y + 2\pi R) = \phi(x, y) \ . \tag{A.3.1}$$

This means that there is an expansion $\phi(x, y) = \sum_{n\in\mathbb{Z}} \phi_n(x)e^{iny/R}$ where the ϕ_n are the usual Kaluza-Klein modes with quantized momenta $p_D = n/R$ and have masses $m_n^2 = (n/R)^2$ in the lower dimensional theory (assuming ϕ is massless for simplicity).

But now suppose that ϕ is charged under a discrete global symmetry with generator Q (ϕ may thus have components). Then, as ϕ goes round the circle, it is allowed to transform more generally as

$$\phi(x, y + 2\pi R) = e^{iQ} \cdot \phi(x, y) \ . \tag{A.3.2}$$

It then follows that the field $\phi'(x, y) = e^{-iyQ/2\pi R}\phi(x, y)$ satisfies the boundary condition (A.3.1) and we can expand it in the corresponding Kaluza-Klein modes. In this way one finds that

$$\phi(x, y) = \sum_{n\in\mathbb{Z}} e^{iQy/2\pi R} \cdot \phi'_n(x)e^{iny/R} \ . \tag{A.3.3}$$

(Comparing to the usual Kaluza-Klein expansion, one could say that the Kaluza-Klein modes now have a y-dependence.) Assume now that Q acts diagonally for simplicity and that the charge of ϕ is q. Then we see that the compact momenta are now quantized as

$$p_D = \frac{n + q}{2\pi R} \ , \tag{A.3.4}$$

so that one obtains effective masses which are shifted:

$$m_n^2 = \left(\frac{n + q}{2\pi R}\right)^2 \ . \tag{A.3.5}$$

This gives us a mechanism to control and raise the masses of certain particles in a given theory. Note that this Scherk-Schwarz mechanism can be regarded as a spontaneous breaking of the discrete symmetry group, where in the decompactification limit $R \to \infty$ the symmetry is restored.

There is a simple application of this to fermions compactified on the circle - indeed the following is what is often meant by the Scherk-Schwarz mechanism. Suppose $\psi(x, y)$ is, for simplicity, a massless fermion in D dimensions. It is charged under a $(-1)^F$ symmetry (where F is the spacetime fermion number) so that $\psi(x, y)$ can return to itself up to a sign as it traverses the circle. We can then impose the following boundary conditions on a generic bosonic or fermionic state:

$$\psi(x, y + 2\pi R) = e^{\pi i F}\psi(x, y) \ . \tag{A.3.6}$$

This yields masses

$$m_n^2 = \frac{n + F/2}{R} \, . \tag{A.3.7}$$

In particular, the Kaluza-Klein tower for the fermions is shifted relative to the bosons. This gives a way to break supersymmetry at the scale of the compactification. Concretely, if we begin with a supersymmetric theory containing a massless gravitino, then compactifing á la Scherk-Schwarz will give it an effective $D-1$ dimensional mass of order $M \sim 1/2R$.

A.3.2 Scherk-Schwarz in string theory

The Scherk-Schwarz mechanism was first discussed in the context of string theory in [90] (see also [91, 93, 111]). In the worldsheet formulation of string theory, we can use the state-operator correspondence to now insist that the vertex operators $V(X)$ must transform by the symmetry generator as X traverses a compact cycle (in worldsheet time). The crucial difference to the field theory version described above is that now one must also include a twisted sector, where the vertex operators also transform as the closed string X winds round a compact circle (at fixed worldsheet time). This is imposed by modular invariance, the residual part of diffeomorphism invariance of the worldsheet, which effectively allows worldsheet time and space to be interchanged.

In the case of the fermion example, this is all nicely summed up by saying that the Scherk-Schwarz theory is an orbifold of a compact circle of radius $2R$, with $\mathbb{Z}_2$ symmetry generator $g = \delta(-1)^F$, where F is the total spacetime fermion number and δ is a half-shift of the circle. The resulting theory has a physical compact circle of radius R, with antiperiodic boundary conditions for the fermions along the circle[4].

This orbifold is a little subtle due to the twisted sector. For simplicity, let us briefly consider the case where we orbifold a type II string theory by $(-1)^F$ only, before going on to the Scherk-Schwarz orbifold. The untwisted sector is easy to describe - it consists of states invariant under $(-1)^F$, i.e. the bosons, so that the fermionic NS-R and R-NS sectors are projected out and only the NS-NS and R-R sectors of the theory are retained. In the twisted sector, one considers new boundary conditions where the string closes only up to a factor $(-1)^F$. A priori, this only reverses periodicity and antiperiodicity for the fermions and, since we are summing over spin structures anyway, might not seem to give anything new. The subtlety here is that the GSO projection is reversed - the path integral includes factors $1-(-1)^{f_L}$ and $1 \mp (-1)^{f_R}$ for the type IIA/B theories [90, 113]. There

[4] A modern approach to orbifolds as described in [112] views an orbifold as a theory with a discrete 1-form gauge symmetry, where one sums over topological line operators placed into the worldsheet. The ordinary path integral corresponds to an operator trace $\mathrm{tr} q^H \bar{q}^{\bar{H}}$. If we now include in the path integral a topological line operator of charge g 'horizontally', i.e. at fixed worldsheet time, this is equivalent to inserting g into the trace. Alternatively, placing it 'vertically', so that it is more properly termed a 'topological defect', modifies the Hamiltonian and generates the twisted sector. Since it is a gauge symmetry, one sums over all possible insertions of the line operator, so that the path integral gets four contributions in total.

are several ways to see this. The simplest is to insist that all amplitude integrands are modular invariant. This is because modular invariance is a residual part of the gauged diffeomorphism invariance of the worldsheet and so must be enforced. One can show that only this reversed GSO projection in the twisted sector gives rise to modular invariant amplitudes when the untwisted and twisted sectors are combined [114]. Alternatively, a twist field enforcing the boundary conditions can be inserted to generate the twisted sector [115]. Such a twist field has a left and right worldsheet fermion number of 1, so that the twisted sector vacuum is shifted by a single worldsheet fermion. For example, if the untwisted sector consists of states in V and S in the left hand moving sector, then the corresponding twisted sector consists of states in O and C.

Let us note in passing that these $(-1)^F$ orbifold theories turn out to be precisely the type 0A and 0B string theories referred to earlier. They are non-supersymmetric and include a tachyon, arising from the twisted sector. In fact, one can freely go between type 0A/B and IIA/B by orbifolding with a combination of worldsheet $(-1)^{f_{L/R}}$ and spacetime $(-1)^{F_{L/R}}$ operators (see e.g. [116]).

The generalization to the Scherk-Schwarz orbifold $\delta(-1)^F$ is rather trivial now. Due to the addition of δ, the untwisted sector corresponds to states with even winding number (relative to the physical circle of radius R) and the usual GSO projection. But in this case, the presence of δ means that the spacetime fermions are not projected out - it instead enforces Scherk-Schwarz boundary conditions on them, so that the states have quantized momentum

$$p = \frac{n + F/2}{R} \; , \tag{A.3.8}$$

and consequently a half-integer shift in the fermionic KK tower's masses.

Meanwhile, the twisted sector corresponds to odd winding modes which experience a $(-1)^F$ twist as the string closes onto itself. As just discussed, the latter requires the opposite GSO projection. As in the twisted sector, fermions have their KK numbers shifted by a half-integer. As in the field theory, the gravitino has a raised mass and supersymmetry is spontaneously broken at the compactification scale.

A.4 Wilson lines in string theory

We will now describe Wilson lines and their occurrence in string theory, taking the type I string for concreteness.

Wilson lines are configurations of gauge fields which are topological in nature. They occur when a constant gauge field wraps an internal cycle. Suppose we compactify type I theory on a circle of radius R and coordinate $y \in [0, 2\pi R]$. Then an $SO(32)$ connection A must have a boundary condition so that it is single-valued up to a gauge transformation:

$$A(x, y + 2\pi R) = gA(x, y)g^{-1} + g^{-1}dg \tag{A.4.1}$$

for some element $g \in SO(32)$. Let us assume the background is flat so that $dA = 0$. Then A must be at most constant. In flat space, such a field can be gauged away - apply the gauge transformation $g(x) = \exp(-x\cdot A)$ where x are the spacetime coordinates. But on the circle, one can only diagonalize A with a gauge field - the group element $g(x) = \exp(-x \cdot A)$ is now not well-defined on the compact circle. Thus, gauge-inequivalent solutions are classified by a moduli space $\{\theta_i \in [0, 2\pi] \mid i = 1, \ldots, 16\}$, such that representative gauge fields are

$$A = \mathrm{diag}(i\theta_1/2\pi R, \ldots, i\theta_{16}/2\pi R) \ . \tag{A.4.2}$$

By choosing such a solution, one generically breaks the gauge group $SO(32) \to U(1)^{16}$. One can show that if p Wilson lines θ_i are equal to each other, but not to 0 or πR, then the subgoup leaving them invariant is $U(p)$. In the exceptional cases, this is enhanced to $SO(2p)$.

The open strings are charged under this Cartan subalgebra, so that the state $|ij\rangle$ has charge $+1$ under the action of the i^{th} $U(1)$ and charge -1 under the action of the j^{th} $U(1)$.

The quantity $\exp(\int dyA_D)$ of this solution is called a Wilson line. Charged states with Chan-Paton factors $|ij\rangle$ pick up the phase $e^{i(\theta_i-\theta_j)}$ as they traverse the circle. In line with the comments of the previous subsection, the Kaluza-Klein modes get their compact momenta shifted:

$$p_D = \frac{2\pi n + \theta_i - \theta_j}{2\pi R} \ . \tag{A.4.3}$$

The masses are shifted accordingly.

T-duality and a geometric version of Wilson lines. T-duality interchanges type IIA and IIB string theory when they are compactified on a circle. Following [104, 105], it acts as follows. Starting with the IIB theory whose 9^{th} dimension is compactified to a circle of radius R, T-duality applies a spacetime parity operation to only the right hand compactified sector of the theory. That is, it maps bosonic $X^9_R(z) \mapsto -X^9_R(z)$, the fermions as $\psi^9_R(z) \mapsto -\psi^9_R(z)$. Meanwhile, on the Ramond sector, it interchanges the chirality of the vacua as $|S\rangle \mapsto |C\rangle$ where S and C are the opposite-chirality spinor vacua[5]. It is well known that this procedure gives a complete equivalence of type IIB string theory on a circle of radius R with type IIA on a circle of radius $\tilde{R} = 1/R$ (using string units).

Since type I theory is IIB/Ω, it follows that it has a dual orientifold theory of IIA. This

[5]If the reader prefers to think of the Ramond sector as gamma matrices then heuristically T-duality acts as $\Gamma^\mu \mapsto \Gamma^D\Gamma^\mu$.

dual is often called type I' string theory [117].

Under T-duality, Ω is mapped to an operator Ω' defined on type IIA theory which interchanges $X^9_L \leftrightarrow -X^9_R$ and $X^\mu_L \leftrightarrow X^\mu_R$ for the non-compact coordinates X^μ whilst also interchanging the compact coordinates as $\psi^9_L \leftrightarrow -\psi^9_R$ and $\psi^\mu_L \leftrightarrow \psi^\mu_R$. Finally, Ω' interchanges the left and right chiralities of the Ramond-Ramond ground state $|S\rangle \otimes |C\rangle \leftrightarrow |C\rangle \otimes |S\rangle$ and similarly swaps the NS-R with R-NS sectors.

The theory IIA/Ω' is thus an orientifold of IIA where one gauges the worldsheet parity together with the spacetime parity of the compactified coordinate. Thus we can think of the compactified dimension as being a line segment $y \in [0, \pi\tilde{R}]$ with two fixed ends (where $\tilde{R} = 1/R$). These are the two orientifold planes (O-planes) where the strings are unoriented. These planes are non-dynamical non-perturbative objects. It is often helpful, both for intuition and also for calculational purposes, to instead view the compact line segment as a circle with two O-planes antipodal to each other, making sure any object with coordinate $\tilde{X}^9$ lying between the two O-planes has an appropriate image at coordinate $-\tilde{X}^9$ (see figure 5.1 for an illustration).

More generally, it is straightforward to apply this to type I theory compactified on a flat torus T^D with radii R_I $(I = d, d+1, \ldots, 9)$. The T-dual type I' orientifold theory is an orientifold of IIA, where the internal space is T^D quotiented by parity. In other words, the internal space is a D-dimensional box with 2^D corners - again, these are the O-planes. See figure 5.2 for an illustration with $D = 3$.

Wilson lines and D-branes. We have just seen that the type I' T-dual of type I compactified on a box torus T^D has 2^D O-planes. It can be shown that they collectively carry -16 units of Ramond-Ramond charge, and since the internal space is compact, we require 16 D-branes to provide consistency.

There is also a very nice argument in chapter 8 of [104] which shows how the Wilson lines of type I theory are dual to the positions of these 16 D-branes in the type I' theory. In this correspondence, the variables θ_i correspond to Wilson lines on the type I side, but correspond to the positions of D-branes on the type I' side. This correspondence gives us a completely geometric picture for model building in type I toroidal compactifications with Wilson lines - we simply have to move the D-branes around. Moreovoer, the type I - heterotic correspondence relates the properties of our type I' theory to far less geometric properties on the heterotic side.

Half branes. As we mentioned, it is often convenient to think of type I theory on a circle as type I' where we retain the whole compact circle (with radius $R' = 1/R$), but add in appropriate images. In this picture, one has 32 'half D-branes', half of which are images of the other.

These half-branes are not completely calculational artifacts however. As we have discussed in the main text, there are perfectly consistent configurations where one half-brane gets stuck on an O-plane by itself. These configurations form a disconnected part of the Wilson line moduli space, for the half-brane can only leave the orientifold if it is accompanied by its image partner.

For example, for the circle compactification, one might have 31 half D-branes on one of the orientifolds, whilst the remaining D-brane lives on the other. The corresponding gauge group could be called $SO(31) \otimes SO(1)$, where of course the $SO(1)$ factor is trivial, but is convenient to remind us that there is a single half-brane stuck somewhere. This is an example of rank reduction. This example is not perfect because such a Wilson line does not actually live in the non-perturbative gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$. But, as discussed in 5.5, there are similar configurations in lower dimensional compactifications which do have consistent half-branes.

Appendix B

Calculation of the Effective Potential in Chapter 5

In this appendix, we present the full calculations of the string loop amplitudes that accompany chapter 5. In section B.1, we will list our definitions and conventions. In section B.2, we calculate the one-loop effective potential for Scherk-Schwarz compactifications on a circle with Wilson lines. Finally, in section B.3, we calculate the one-loop effective potential in D dimensions. These calculations are done with no field theory approximations.

B.1 Definitions and conventions

From the previous appendix, we know that the type I string one-loop partition function is a sum

$$\mathcal{V} = -\frac{M_s^D}{2(2\pi)^D}(\mathcal{T}+\mathcal{K}+\mathcal{A}+\mathcal{M})\,, \tag{B.1.1}$$

in D dimensions, where the individual terms are given in the operator formalism as

$$\begin{aligned} \mathcal{T} &= \int_{\mathcal{F}} \frac{d\tau_1 d\tau_2}{\tau_2^{1+D/2}} \,\mathrm{Str}\, q^{L_0-\frac{1}{2}}\bar{q}^{\bar{L}_0-\frac{1}{2}}\,, \quad \mathcal{K} = \int_0^\infty \frac{d\tau_2}{\tau_2^{1+D/2}}\,\mathrm{Str}\,\Omega q^{L_0-\frac{1}{2}}\bar{q}^{\bar{L}_0-\frac{1}{2}}\,, \\ \mathcal{A} &= \int_0^\infty \frac{d\tau_2}{\tau_2^{1+D/2}}\,\mathrm{Str}\, q^{\frac{1}{2}(L_0-\frac{1}{2})}\,, \qquad \mathcal{M} = \int_0^\infty \frac{d\tau_2}{\tau_2^{1+D/2}}\,\mathrm{Str}\,\Omega q^{\frac{1}{2}(L_0-\frac{1}{2})}\,. \end{aligned} \tag{B.1.2}$$

Here, $\mathcal{F}$ is the fundamental domain of $SL(2,\mathbb{Z})$, $q = e^{2\pi i\tau}$ where τ is the Teichmüller parameter, and L_0 are the zero-order Virasoro operators, playing a similar role to the mass operator in the analagous calculations in field theory.

Compactification on a torus with metric G_{IJ} gives rise to a sum over winding modes and Kaluza Klein (KK) modes. Disregarding powers of τ_2, these can be summarised by the lattice sum

$$\Lambda_{\mathbf{m},\mathbf{n}}(\tau) = q^{\frac{1}{4}P_L^2}\bar{q}^{\frac{1}{4}P_R^2} \; , \tag{B.1.3}$$

where the exponents are

$$\begin{aligned} P_L^2 &= P_I^L G^{IJ} P_J^L \quad \text{where} \quad P_I^L = m_I + G_{IJ}n^J \; , \\ P_R^2 &= P_I^R G^{IJ} P_J^R \quad \text{where} \quad P_I^R = m_I - G_{IJ}n^J \; , \end{aligned} \tag{B.1.4}$$

for $I = D, \ldots, 9$. Here, m_I and n^I are to be interpreted as KK and winding numbers along the compact direction X^I, and G^{IJ} are the components of the inverse metric.

The lattice sums associated with the open string amplitudes do not directly contain winding mode sums, for the open string cannot wind. However, it is possible to Poisson resum the KK modes so as to make them look like winding modes. This will be useful when we want to go between open and closed channel forms of the amplitudes. Hence, we define two sets of open string lattice sums:

$$\begin{aligned} P_{\mathbf{m}}(i\tau_2) &= \Lambda_{\mathbf{m},\mathbf{0}}(\tau) = e^{-2\pi\tau_2 m_I G^{IJ} m_J} \; , \\ W_{\mathbf{n}}(i\ell) &= \Lambda_{\mathbf{0},\mathbf{n}}(\tau) = e^{-\frac{\pi}{2}\ell n^I G_{IJ} n^J} \; . \end{aligned} \tag{B.1.5}$$

As remarked, these are related by Poisson resummation:

$$\sum_{\mathbf{m}\in\mathbb{Z}^d} P_{\mathbf{m}+\mathbf{a}}(i\tau_2) = \begin{cases} (2\ell)^{\frac{10-D}{2}}\sqrt{\det G}\sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i\mathbf{n}\cdot\mathbf{a}} W_{2\mathbf{n}}(i\ell) \; , & \text{where } \ell = \frac{1}{2\tau_2} \text{ for } \mathcal{K},\mathcal{M} \; , \\ (\frac{\ell}{2})^{\frac{10-D}{2}}\sqrt{\det G}\sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i\mathbf{n}\cdot\mathbf{a}} W_{\mathbf{n}}(i\ell) \; , & \text{where } \ell = \frac{2}{\tau_2} \text{ for } \mathcal{A} \; , \end{cases} \tag{B.1.6}$$

where we have written the same expression twice, but shuffled factors of 2 around. The only reason for doing this is because the annulus has a closed string channel modulus more naturally written as $2i/\tau_2$, whilst the others have closed string channel moduli $i/2\tau_2$ or $\frac{1}{2} + i/2\tau_2$ in the case of the Klein bottle and Möbius strip respectively. This is explained in Appendix A.

These lattice sums are closely related to the Jacobi forms and Dedekind functions defined as:

$$\vartheta_{\alpha,\beta}(z,\tau) = \sum_{m\in\mathbb{Z}} q^{\frac{1}{2}(m-\alpha/2)^2} e^{2\pi i(z-\beta/2)(m-\alpha/2)} \; , \quad \eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \; . \tag{B.1.7}$$

There are four special cases: $\vartheta_1 = \vartheta_{1,1}$, $\vartheta_2 = \vartheta_{1,0}$, $\vartheta_3 = \vartheta_{0,1}$ and $\vartheta_4 = \vartheta_{0,0}$. Note that ϑ_1 vanishes identically.

Out of the theta functions, one can define $SO(8)$ characters (also explained in the previous appendix):

$$\begin{aligned} O_8 &= \frac{\theta_3^4 + \theta_4^4}{2\eta^4} \ , \quad V_8 = \frac{\theta_3^4 - \theta_4^4}{2\eta^4} \ , \\ S_8 &= \frac{\theta_2^4 + \theta_1^4}{2\eta^4} \ , \quad C_8 = \frac{\theta_2^4 - \theta_1^4}{2\eta^4} \ . \end{aligned} \tag{B.1.8}$$

Numerically, $V_8 = S_8 = C_8$, but since these sums arise from different conformal families, it is a wise idea to keep them abstract so that we can identify from where in the spectrum certain contributions originate.

The characters behave as a vector-valued modular function under modular transformations:

$$\begin{pmatrix} O_8 \\ V_8 \\ S_8 \\ C_8 \end{pmatrix} (\tau) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} O_8 \\ V_8 \\ S_8 \\ C_8 \end{pmatrix} (-1/\tau) \ . \tag{B.1.9}$$

The Dedekind eta function transforms as a weight $\frac{1}{2}$ modular form:

$$\eta(\tau) = \frac{1}{\sqrt{-i\tau}} \eta(-1/\tau) \ . \tag{B.1.10}$$

For the Möbius transformation, it is useful to adopt a hatted character $\hat{\chi}_R$ defined by

$$\hat{\chi}_R(\frac{1}{2} + i\tau_2) = e^{-i\pi(h - c/24)} \chi_R(\frac{1}{2} + i\tau_2) \ , \tag{B.1.11}$$

where h is the weight of the associated primary state of the Verma module R and c is the central charge. In the case of the Möbius strip, the transformation from the open string channel to closed is called the P-transformation, under which

$$\begin{pmatrix} \hat{O}_8 \\ \hat{V}_8 \\ \hat{S}_8 \\ \hat{C}_8 \end{pmatrix} \left(\frac{1 + i\tau_2}{2} \right) = \mathrm{diag}(-1, 1, 1, 1) \begin{pmatrix} \hat{O}_8 \\ \hat{V}_8 \\ \hat{S}_8 \\ \hat{C}_8 \end{pmatrix} (\frac{1}{2} + i\ell) \ , \tag{B.1.12}$$

where $\ell = 1/2\tau_2$. For the bosonic oscillators, one requires

$$\hat{\eta}(\frac{1}{2} + i\frac{\tau_2}{2}) = \sqrt{2\ell}\, \hat{\eta}\big(\frac{1}{2} + i\ell\big) \ . \tag{B.1.13}$$

Finally, we will frequently encounter Bessel-function-like integrals of the kind:

$$\mathcal{H}_\nu(z) := \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{dx}{x^{1+\nu}} e^{-\frac{1}{x} - z^2 x} = \frac{2}{\Gamma(\nu)} z^\nu K_\nu(2z) \ , \tag{B.1.14}$$

where K_ν is a modified Bessel function of the second kind. It has asymptotic behaviour:

$$\mathcal{H}_\nu(z) \sim \begin{cases} \frac{\sqrt{\pi}}{\Gamma(\nu)} z^{\nu-\frac{1}{2}} e^{-2z} & \text{for } |z| \gg 1 \, , \\ 1 - \frac{z^2}{\nu-1} + \mathcal{O}(z^4) & \text{for } |z| \ll 1 \, . \end{cases} \tag{B.1.15}$$

B.2 The calculation of the effective potential in 9 dimensions

The torus amplitude for type IIB. Because the IIB theory is a tensor product of right- and left-moving conformal field theories, its torus amplitude in ten dimensions can be factored as

$$\mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \sum_{i,j} \bar{\chi}_i(\bar{q}) X_{ij} \chi_j(q) \, , \tag{B.2.1}$$

where we sum over the possible conformal families, both on the right and on the left moving side. In the above, the integer-valued matrix X_{ij} counts the multiplicity of these families and corrects for spin-statistics. For the IIB theory itself, since it is built from the Verma modules with primaries in the vector V and spinor S representations, one has

$$\mathcal{T} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} (V_8\bar{V}_8 - V_8\bar{S}_8 - S_8\bar{V}_8 + S_8\bar{S}_8) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} |V_8 - S_8|^2 \, . \tag{B.2.2}$$

Numerically this vanishes, as it should for a supersymmetric theory.

When we compactify the IIB theory on, for example a circle, then extra lattice sums coming from Kaluza-Klein and winding states will be generated. In the simplest cases, these lattice sums factor completely outside of the characters, with each internal dimension contributing a $\sqrt{\tau_2}$. For example, compactification on a circle without a Scherk-Schwarz twist is

$$\mathcal{T}' = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{3/2}} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} |V_8 - S_8|^2 \Lambda_{m,n} \, , \tag{B.2.3}$$

where it should be understood that $\Lambda_{m,n}$ should mean the summing over all integers[1].

When we compactify with the Scherk-Schwarz boundary conditions however, the KK tower spectra becomes dependent on the spin statistics and so the lattice sums intertwine non-trivially with the characters. As explained in A.3, the resulting theory has an untwisted sector consisting of the type IIB spectra with even winding modes and shifted masses for the fermions, and a twisted sector consisting of odd winding modes and opposite GSO projection, again with half-integer shifted fermion masses. Hence we have in total the

[1]There are two reasons this is a useful convention. First, it stops clumsy summations occuring in every formula. Second, there are times when the lattice KK modes become shifted, so that one sums over all half-integers m. This is clearly notated by writing $\Lambda_{m+\frac{1}{2},n}$.

amplitude

$$\mathcal{T} = \frac{1}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{3/2}} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} \Big\{ (V_8\bar{V}_8 + S_8\bar{S}_8)\Lambda_{m,2n} - (V_8\bar{S}_8 + S_8\bar{V}_8)\Lambda_{m+\frac{1}{2},2n} \\ + (O_8\bar{O}_8 + C_8\bar{C}_8)\Lambda_{m,2n+1} - (O_8\bar{C}_8 + C_8\bar{O}_8)\Lambda_{m+\frac{1}{2},2n+1} \Big\} , \tag{B.2.4}$$

where we have now included a factor $\frac{1}{2}$ which comes from the orientifold projection so that this can be regarded as a type I string amplitude. Note that the lattices $\Lambda_{m,n}$ depend on $G^{99} = 1/R_9^2$ where R_9 is the radius of the compact circle.

The Klein bottle amplitude. The Klein bottle amplitude can read from (**??**), which includes all propagating states. This is because the Klein bottle worldsheet symmetrises the NS-NS sector, antisymmetrises the RR sector, and projects out winding modes. It then follows that[2]

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{11/2}} \cdot \frac{1}{\eta^8} \sum_m (V_8 - S_8) P_m \ , \tag{B.2.5}$$

where characters are evaluated at $2i\tau_2$.

The open string amplitudes. The open string sector depends on the Wilson lines. A generic configuration in nine dimensions has p_1 half-branes located at $a = 0$, p_2 half-branes located at $a = \frac{1}{2}$, q half-branes located at $a = \frac{1}{4}$ and their mirrors at $a = -\frac{1}{4}$, and finally sets of r_σ half-branes located at $a = r_\sigma$ together with their mirrors at $a = -r_\sigma$. We then have the annulus amplitude as

[2]One could also take a more methodical alternative and compute the amplitude from first principles.

$$
\begin{aligned}
\mathcal{A} \;=\; & \frac{1}{2}\int_0^{\infty}\frac{d\tau_2}{\tau_2^{\frac{11}{2}}}\frac{1}{\eta^8}\sum_{m_9}\Big\{\big(p_1^2+p_2^2+2q\bar{q}+2\sum_\sigma r_\sigma\bar{r}_\sigma\big)\big(V_8P_{m_9}-S_8P_{m_9+\frac{1}{2}}\big)\\
&+2p_1p_2\big(V_8P_{m_9+\frac{1}{2}}-S_8P_{m_9}\big)+q^2\big(V_8P_{m_9+\frac{1}{2}}-S_8P_{m_9}\big)+\bar{q}^2(V_8P_{m_9-\frac{1}{2}}-S_8P_{m_9})\\
&+\sum_\sigma r_\sigma^2\big(V_8P_{m_9+2a_\sigma}-S_8P_{m_9+\frac{1}{2}+2a_\sigma}\big)+\sum_\sigma \bar{r}_\sigma^2\big(V_8P_{m_9-2a_\sigma}-S_8P_{m_9+\frac{1}{2}-2a_\sigma}\big)\\
&+2p_1q\big(V_8P_{m_9+\frac{1}{4}}-S_8P_{m_9-\frac{1}{4}}\big)+2p_1\bar{q}\big(V_8P_{m_9-\frac{1}{4}}-S_8P_{m_9+\frac{1}{4}}\big)\\
&+2p_2q\big(V_8P_{m_9-\frac{1}{4}}-S_8P_{m_9+\frac{1}{4}}\big)+2p_2\bar{q}\big(V_8P_{m_9+\frac{1}{4}}-S_8P_{m_9-\frac{1}{4}}\big)\\
&+2\sum_\sigma p_1r_\sigma\big(V_8P_{m_9+a_\sigma}-S_8P_{m_9+\frac{1}{2}+a_\sigma}\big)+2\sum_\sigma p_1\bar{r}_\sigma\big(V_8P_{m_9-a_\sigma}-S_8P_{m_9+\frac{1}{2}-a_\sigma}\big)\\
&+2\sum_\sigma p_2r_\sigma\big(V_8P_{m_9+\frac{1}{2}+a_\sigma}-S_8P_{m_9+a_\sigma}\big)+2\sum_\sigma p_2\bar{r}_\sigma\big(V_8P_{m_9+\frac{1}{2}-a_\sigma}-S_8P_{m_9-a_\sigma}\big)\\
&+2\sum_\sigma qr_\sigma\big(V_8P_{m_9+\frac{1}{4}+a_\sigma}-S_8P_{m_9-\frac{1}{4}+a_\sigma}\big)+2\sum_\sigma q\bar{r}_\sigma\big(V_8P_{m_9+\frac{1}{4}-a_\sigma}-S_8P_{m_9-\frac{1}{4}-a_\sigma}\big)\\
&+2\sum_\sigma \bar{q}r_\sigma\big(V_8P_{m_9-\frac{1}{4}+a_\sigma}-S_8P_{m_9+\frac{1}{4}+a_\sigma}\big)+2\sum_\sigma \bar{q}\bar{r}_\sigma\big(V_8P_{m_9-\frac{1}{4}-a_\sigma}-S_8P_{m_9+\frac{1}{4}-a_\sigma}\big)\\
&+\sum_{\sigma\neq\tau} r_\sigma r_\tau\big(V_8P_{m_9+a_\sigma+a_\tau}-S_8P_{m_9+\frac{1}{2}+a_\sigma+a_\tau}\big)+\sum_{\sigma\neq\tau}\bar{r}_\sigma\bar{r}_\tau\big(V_8P_{m_9-a_\sigma-a_\tau}-S_8P_{m_9+\frac{1}{2}-a_\sigma-a_\tau}\big)\\
&+2\sum_{\sigma\neq\tau} r_\sigma\bar{r}_\tau\big(V_8P_{m_9+a_\sigma-a_\tau}-S_8P_{m_9+\frac{1}{2}+a_\sigma-a_\tau}\big)\Big\}\,, \qquad \text{(B.2.6)}
\end{aligned}
$$

where the characters are now evaluated at $i\tau_2/2$.

Finally, the Möbius amplitude for this configuration is

$$
\begin{aligned}
\mathcal{M} \;=\; & -\frac{1}{2}\int_0^{+\infty}\frac{d\tau_2}{\tau_2^{11/2}}\frac{1}{\hat{\eta}^8}\sum_{m_9}\Big\{(p_1+p_2)\big(\hat{V}_8P_{m_9}-\hat{S}_8P_{m_9+\frac{1}{2}}\big)\\
&+\;q\big(\hat{V}_8P_{m_9+\frac{1}{2}}-\hat{S}_8P_{m_9}\big)+\bar{q}\big(\hat{V}_8P_{m_9-\frac{1}{2}}-\hat{S}_8P_{m_9}\big) \qquad \text{(B.2.7)}\\
&+\;\sum_\sigma r_\sigma\big(\hat{V}_8P_{m_9+2a_\sigma}-\hat{S}_8P_{m_9+\frac{1}{2}+2a_\sigma}\big)+\sum_\sigma\bar{r}_\sigma\big(\hat{V}_8P_{m_9-2a_\sigma}-\hat{S}_8P_{m_9+\frac{1}{2}-2a_\sigma}\big)\Big\}\,,
\end{aligned}
$$

where the characters are evaluated at $\frac{1}{2}+i\tau_2/2$. In each of the amplitudes $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$, there are UV divergences. That their summation should be finite yields the RR tadpole condition which must be satisfied for consistent string amplitudes. The sign of $\mathcal{M}$ is important for (and can be determined by) such tadpole cancellations.

We can use these expressions to determine the number of massless bosonic and fermionic states. From the bosonic sector this is easy - the Klein bottle amplitude numerically vanishes, and in the torus ampltitude, the only massless states can come from the $V_8\bar{V}_8 +$

$S_8\bar{S}_8$ (the other terms cannot be massless due to the lattice sums). By expanding $V_8/\eta^8 = S_8/\eta^8 = 8 + \mathcal{O}(q)$, one finds $\frac{1}{2} \cdot (8^2 + 8^2) = 64$ states.

Similarly, scouring through the annulus and Möbius strip amplitudes, looking for P_m contributions which allow $m = 0$ terms, and then expanding the characters gives us massless states

$$\begin{aligned} n_B^{(0)} &= 8\left(8 \;+\; \frac{p_1(p_1-1)}{2} + \frac{p_2(p_2-1)}{2} \;+\; q\bar{q} \;+\; \sum_\sigma r_\sigma \bar{r}_\sigma\right), \\ n_F^{(0)} &= 8\left(p_1 p_2 \;+\; \frac{q(q-1)+\bar{q}(\bar{q}-1)}{2} \;+\; \sum_{\substack{\sigma<\tau \\ a_\sigma + a_\tau = \frac{1}{2}}} (r_\sigma r_\tau + \bar{r}_\sigma \bar{r}_\tau)\right) . \end{aligned} \tag{B.2.8}$$

We can explicitly check this by ensuring that these fill out the correct dimensions of the corresponding representations. Indeed, the 64 string states do seem to be uncharged, whilst the p_1, p_2, q and r_σ gauge bosons are filling out the adjoints of $SO(p_1)$, $SO(p_2)$, $U(q)$ and $U(r_\sigma)$ respectively. Meanwhile, the fermionic side consists of bifundamentals of $SO(p_1) \times SO(p_2)$, the antisymmetric and $\overline{\text{antisymmetric}}$ of $U(q)$, and the bifundamentals of $U(r_\sigma) \times U(r_\tau)$ whenever the brane stacks are separated by a distance of $1/2$.

We can now proceed to calculate the effective potential. To do this, it is useful to double the degrees of freedom in the Wilson line and parametrise it as

$$\mathcal{W} = \mathrm{diag}(e^{2\pi i a_\alpha}\ ,\ \alpha = 1, \ldots, 32)\ . \tag{B.2.9}$$

This is purely a helpful trick. We must remember that not all the a_α are independent - we have seen from chapter 4 that they correspond to the position of half-branes, and dynamical half-branes must be accompanied by a mirror partner so that one really has a full brane in the dual theory. In other words, if $a_\alpha \neq 0$ or $1/2$, then it should have a partner a_β with $a_\beta = -a_\beta$.

The main use of parametrising Wilson lines by (B.2.9) is that it allows for the far more elegant form of annulus and Möbius amplitudes:

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{\frac{11}{2}}}\ \frac{1}{\eta^8} \sum_{m_9} \sum_{\alpha,\beta} \left(V_8 P_{m_9 + a_\alpha - a_\beta} - S_8 P_{m_9 + \frac{1}{2} + a_\alpha - a_\beta}\right)\ , \\ \mathcal{M} &= -\frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{\frac{11}{2}}}\ \frac{1}{\hat{\eta}^8} \sum_{m_9} \sum_{\alpha} \left(\hat{V}_8 P_{m_9 + 2a_\alpha} - \hat{S}_8 P_{m_9 + \frac{1}{2} + 2a_\alpha}\right)\ . \end{aligned} \tag{B.2.10}$$

By using Poisson resummation (B.1.6), and the transformations (B.1.5) and (B.1.12), these can be converted into the dual closed string form:

$$
\begin{aligned}
\mathcal{A} &= \frac{2^{-5}}{2} R_9 \int_0^\infty \frac{d\ell}{\eta^8} \sum_{n_9} \Big(\left(\mathrm{tr}\, \mathcal{W}^{2n_9}\right)^2 (V_8 - S_8) W_{2n_9} + \left(\mathrm{tr}\, \mathcal{W}^{2n_9+1}\right)^2 (O_8 - C_8) W_{2n_9+1} \Big) \ , \\
\mathcal{M} &= -R_9 \int_0^\infty \frac{d\ell}{\hat{\eta}^8} \sum_{n_9} \left(\mathrm{tr}\, \mathcal{W}^{2n_9}\right) \left(\hat{V}_8 - (-1)^{n_9} \hat{S}_8\right) W_{2n_9} \ ,
\end{aligned}
$$

where $\ell = 2/\tau_2$ and $\ell = 1/2\tau_2$ for the annulus and Möbius strip amplitudes, and the characters are evaluated at $i\ell$ and $\frac{1}{2} + i\ell$ respectively. One can also write $\mathcal{K}$ in the transverse channel

$$
\mathcal{K} = \frac{2^5}{2} R_9 \int_0^\infty \frac{d\ell}{\eta^8} \sum_{n_9} (V_8 - S_8) W_{2n} \ , \tag{B.2.11}
$$

where $\ell = 1/2\tau_2$ and the characters are evaluated at $i\ell$. There is a tadpole that should force the leading divergent terms in $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ to vanish when summed over. It can be checked that this happens precisely when

$$
p_1 + p_2 + 2q + 2 \sum_\sigma r_\sigma = 32 \ . \tag{B.2.12}
$$

In the limit that R_9 is large, the winding and oscillator modes become subdominant to the light KK states. In this approximation, one can calculate the remaining approximate integral using (B.1.14) and (B.1.15) to give

$$
\mathcal{T} = \frac{\Gamma(5)}{\pi^5} \frac{8}{R_9^9} \sum_{n_9} \frac{16}{(2n+1)^{10}} + \mathcal{O}\left(R_9^{-9/2} e^{-4\pi R_9}\right) \ . \tag{B.2.13}
$$

One similarly obtains

$$
\mathcal{A} + \mathcal{M} = \frac{\Gamma(5)}{\pi^5} \frac{8}{R_9^9} \sum_{n_9} \frac{(\mathrm{tr}(\mathcal{W}^{2n_9+1}))^2 - \mathrm{tr}\mathcal{W}^{2(2n_9-1)}}{(2n_9+1)^{10}} + \mathcal{O}\left(R_9^{-9/2} e^{-4\pi R_9}\right) \ . \tag{B.2.14}
$$

Hence the total potential is

$$
\mathcal{V} = \frac{\Gamma(5)}{\pi^{14}} \frac{M_s^9}{(2R_9)^9} 4 \sum_{n_9} \frac{-16 - \left(\mathrm{tr}\ \mathcal{W}^{2n_9+1}\right)^2 + \mathrm{tr}\ \mathcal{W}^{2(2n_9+1)}}{(2n_9+1)^{10}} + \mathcal{O}\left(R_9^{-9/2} e^{-4\pi R_9}\right) \ , \tag{B.2.15}
$$

which is what is claimed in (5.3.11).

B.3 The string calculation in D dimensions

This section essentially copies the previous section, but now compactifies type I string theory on a torus T^D with internal metric G_{IJ} ($I, J = 1, \ldots, D$) and with the Scherk-

Schwarz twist still only in the X^9 direction. The torus loop amplitude is now

$$\mathcal{T} = \frac{1}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\eta^8\bar{\eta}^8} \sum_{\mathbf{m},\mathbf{n}} \Big\{ \big(V_8\bar{V}_8 + S_8\bar{S}_8\big)\Lambda_{\mathbf{m},(\mathbf{n}',2n_9)} - \big(V_8\bar{S}_8 + S_8\bar{V}_8\big)\Lambda_{\mathbf{m}+\mathbf{a}_S,(\mathbf{n}',2n_9)} + \big(O_8\bar{O}_8 + C_8\bar{C}_8\big)\Lambda_{\mathbf{m},(\mathbf{n}',2n_9+1)} - \big(O_8\bar{C}_8 + C_8\bar{O}_8\big)\Lambda_{\mathbf{m}+\mathbf{a}_S,(\mathbf{n}',2n_9+1)} \Big\} , \tag{B.3.1}$$

where we assign the Scherk-Schwarz shift vector $\mathbf{a}_S = (\mathbf{0}', \frac{1}{2})$ to point only in the 9-direction, and also split $\mathbf{n} = (\mathbf{n}', n_9)$.

The Klein bottle contribution is rather trivial, since the closed string states are not charged under the Wilson lines:

$$\mathcal{K} = \frac{1}{2}\int_0^{+\infty} \frac{d\tau_2}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\eta^8} \sum_{\mathbf{m}} (V_8 - S_8) P_{\mathbf{m}} \ . \tag{B.3.2}$$

As before, this numerically vanishes.

The open string channel depends on the exact Wilson lines present. As explained in the main text, it is convenient to move over to the IIA orientifold picture by T-dualizing all compact dimensions so that the D Wilson lines a^I_α are dual to the coordinates of the branes (or half-branes if we double the degrees of freedom as before). As explained in the main text, we are mainly interested in the cases where we place p_A half-branes at the A^{th} O-plane. We use a labelling so that the $(2A-1)^{th}$ O-planes sits at $\mathbf{a}_{2A-1} = (\mathbf{a}'_{2A-1}, 0)$ for some $\mathbf{a}'_{2A-1}$, whilst the $(2A)^{th}$ O-plane sits at $\mathbf{a}_{2A} = \mathbf{a}'_{2A-1} + \mathbf{a}_S = (\mathbf{a}'_{2A-1}, \frac{1}{2})$. Or in other words, the pairs of O-planes $(2A-1, 2A)$ face each other across the Scherk-Schwarz direction. In this case,

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\int_0^{+\infty} \frac{d\tau_2}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\eta^8} \sum_{\mathbf{m}} \sum_{A,B=1}^{2^{10-D}} p_A p_B (V_8 P_{\mathbf{m}+\mathbf{a}_A-\mathbf{a}_B} - S_8 P_{\mathbf{m}+\mathbf{a}_S+\mathbf{a}_A-\mathbf{a}_B}) \ , \\ \mathcal{M} &= -\frac{1}{2}\int_0^{+\infty} \frac{d\tau_2}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\hat{\eta}^8} \sum_{\mathbf{m}} \sum_{A=1}^{2^{10-D}} p_A (\hat{V}_8 P_{\mathbf{m}} - \hat{S}_8 P_{\mathbf{m}+\mathbf{a}_S}) \ . \end{aligned} \tag{B.3.3}$$

As before, massless states can be read off from the coefficients of lattice sums which allow massless states, and by expanding the characters in q. One finds

$$\begin{aligned} n_B^{(0)} &= 8\left(8 + \sum_{A=1}^{2^{10-D}} \frac{p_A(p_A-1)}{2}\right) , \\ n_F^{(0)} &= 8 \sum_{A=1}^{2^{10-D}/2} \frac{p_{2A-1}p_{2A} + p_{2A}p_{2A-1}}{2} \ . \end{aligned} \tag{B.3.4}$$

We explained in the main text that this counting is to be expected from pure geometrical reasoning - the numbers of states have organised themselves into the correct dimensions of the representations of the gauge group that the massless open strings transform in. It now follows by using the tadpole condition $\sum p_A = 32$ that

$$n_F^{(0)} - n_B^{(0)} = 8\left(8 - \frac{1}{2}\sum_{A=1}^{2^{10-D}/2} (p_{2A-1} - p_{2A})^2\right) . \tag{B.3.5}$$

We can now move on to study the effective potential. We will do this in two regimes as we did in the main text.

The first regime is when we assume that the Scherk-Schwarz supersymmetry breaking scale $\mathcal{M}^2 \sim G^{99}$ is smaller than any of the other scales present (KK scales for other cycles, winding modes and string modes). To ensure this, we assume

$$G^{99} \ll |G_{ij}| \ll G_{99} \ , \qquad |G_{9j}| \ll \sqrt{G_{99}} \ , \tag{B.3.6}$$

for $i, j = D, \ldots, 8$. One also wants $G_{99} \gg 1$ to avoid a Hagedorn instability.

Meanwhile, each Wilson line $\mathcal{W}_I$ has a diagonal representative:

$$\mathcal{W}_I \ = \ \text{diag}\left(e^{2i\pi a_I^\alpha}; \alpha = 1, \ldots, 32\right) , \quad I = D, \ldots, 9 \ . \tag{B.3.7}$$

In this, we are again doubling the degrees of freedom, and will restrict pairs of Wilson lines to be related to each other later. This helps simplify some of the formulae.

The annulus. With the notation above, one has

$$\mathcal{A} \ = \ \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\eta^8} \sum_{\mathbf{m}} \sum_{\alpha,\beta} (V_8 P_{\mathbf{m}+\mathbf{a}_\alpha - \mathbf{a}_\beta} - S_8 P_{\mathbf{m}+\mathbf{a}_S+\mathbf{a}_\alpha-\mathbf{a}_\beta}) \ . \tag{B.3.8}$$

Expanding $V_8/\eta^8 = S_8/\eta^8 = 8\sum_{k\geq 0} c_k e^{-\pi k \tau_2}$, where $c_0 = 1$, and Poisson resumming over m_9, we obtain

$$\begin{aligned}\mathcal{A} \ = \ & (G^{99})^{\frac{D}{2}} \frac{\Gamma\left(\frac{D+1}{2}\right)}{\pi^{\frac{D+1}{2}}} 8 \sum_{k\geq 0} c_k \sum_{\alpha,\beta} \sum_{\mathbf{m}'} \sum_{l_9} \frac{1}{|2l_9+1|^{D+1}} \\ & \cdot \ \cos\Big[2\pi(2l_9+1)\Big(a_9^\alpha - a_9^\beta + \frac{G^{9i}}{G^{99}}(m_i + a_i^\alpha - a_i^\beta)\Big)\Big] \, \mathcal{H}_{\frac{D+1}{2}}\left(\pi|2l_9+1|\frac{M_{\mathcal{A}}}{\sqrt{G^{99}}}\right) ,\end{aligned} \tag{B.3.9}$$

where the function $\mathcal{H}_\nu$ is given in (B.1.14). The quantity $M_{\mathcal{A}}$ is the mass

$$M_{\mathcal{A}}^2 = (m_i + a_i^\alpha - a_i^\beta)\hat{G}^{ij}(m_j + a_j^\alpha - a_j^\beta) + k \ , \tag{B.3.10}$$

where the hatted metric is

$$\hat{G}^{ij} \;=\; G^{ij} - \frac{G^{i9}}{G^{99}} G^{99} \frac{G^{9j}}{G^{99}} = G^{ij} + \mathcal{O}\Big(\frac{1}{G_{99}}\Big), \quad i,j = D,\ldots,8 \,. \tag{B.3.11}$$

This can be regarded as the effective inverse metric of the internal space transverse to the Scherk-Schwarz direction.

We would now like to examine this amplitude when we place the half-branes at the O-planes. For this, it is more useful to divide the Wilson line into its background value and a dynamical variation:

$$a_I^\alpha \;=\; \langle a_I^\alpha \rangle + \varepsilon_I^\alpha \,, \tag{B.3.12}$$

where the background will be $\langle a_\alpha^I \rangle = 0$ or $1/2$ for each α and I. If we stare at (B.3.9) we see that most states have $M_{\mathcal{A}} \sim \mathcal{O}(1)$ and they will be highly exponentially suppressed. The only dominant contributions occur when $M_{\mathcal{A}} = 0$. For this one needs both $k = 0$ and $m_i = 0$ for each i, but also $a_i^\alpha - a_i^\beta = 0$. The latter restricts us to the case where the pair of branes at a_I^α and a_J^α should either be coincident or face each other across the Scherk-Schwarz direction. Later we will find it useful to call the set of all such pairs L. Then the dominant contribution is a sum over such pairs. Expanding, using the asymptotics of $\mathcal{H}_\nu$ in (B.1.15), one finds

$$\begin{aligned}\mathcal{A} \;=\; & \big(\sqrt{G^{99}}\big)^D \frac{\Gamma\big(\frac{D+1}{2}\big)}{\pi^{\frac{D+1}{2}}} 8 \sum_{(\alpha,\beta)\in L} (-1)^{2(\langle a_9^\alpha\rangle - \langle a_9^\beta\rangle)} \sum_{l_9} \frac{\cos\Big[2\pi(2l_9+1)\big(\varepsilon_9^\alpha - \varepsilon_9^\beta + \frac{G^{9i}}{G^{99}}(\varepsilon_i^\alpha - \varepsilon_i^\beta)\big)\Big]}{|2l_9+1|^{D+1}} \\ & \cdot \mathcal{H}_{\frac{D+1}{2}}\left(\pi|2l_9+1|\frac{\big[(\varepsilon_i^\alpha - \varepsilon_i^\beta)\hat{G}^{ij}(\varepsilon_j^\alpha - \varepsilon_j^\beta)\big]^{\frac{1}{2}}}{\sqrt{G^{99}}}\right) \;+\; \mathcal{O}\Big(\big(\sqrt{G^{99}}\big)^{\frac{D}{2}} e^{-\frac{2\pi c}{\sqrt{G^{99}}}}\Big) \,,\end{aligned} \tag{B.3.13}$$

where $c > 0$ is the next lowest value of $M_{\mathcal{A}}$ so that it is of order $\mathcal{O}(1)$.

The Möbius strip. The Möbius strip amplitude is

$$\mathcal{M} = \frac{1}{2}\int_0^\infty \frac{d\tau_2}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\hat{\eta}^8} \sum_{\mathbf{m}} \sum_{\alpha} (\hat{V}_8 P_{\mathbf{m}+2\mathbf{a}_\alpha} - \hat{S}_8 P_{\mathbf{m}+\mathbf{a}_S+2\mathbf{a}_\alpha}) \,. \tag{B.3.14}$$

Again, the m_9 sum can be Poisson resummed to one over l_9 and we expand out all the string characters as a q series. This time, one finds

$$\begin{aligned}\mathcal{M} = & -\big(G^{99}\big)^{\frac{D}{2}} \frac{\Gamma\big(\frac{D+1}{2}\big)}{\pi^{\frac{D+1}{2}}} 8 \sum_{k\geq 0} (-1)^k c_k \sum_{\alpha}\sum_{\mathbf{m}'}\sum_{l_9} \frac{1}{|2l_9+1|^{D+1}} \\ & \cos\Big[2\pi(2l_9+1)\Big(2a_9^\alpha + \frac{G^{9i}}{G^{99}}(m_i + 2a_i^\alpha)\Big)\Big]\, \mathcal{H}_{\frac{D+1}{2}}\Big(\pi|2l_9+1|\frac{M_{\mathcal{M}}}{\sqrt{G^{99}}}\Big) \,,\end{aligned} \tag{B.3.15}$$

where we have now introduced a Möbius analogue to (B.3.10):

$$M_{\mathcal{M}}^2 \;=\; (m_i + 2a_i^\alpha)\hat{G}^{ij}(m_j + 2a_j^\alpha) + k \;. \tag{B.3.16}$$

Similarly to before, the states in which $M_{\mathcal{M}}$ vanishes are dominant - the rest just get thrown away as exponentially suppressed terms. For $M_{\mathcal{M}} = 0$, one requires that $m_i + 2\langle a_i^\alpha\rangle = 0$. Hence either $m_i = 0$ or -1, with each giving the expression for the potential. It soon follows that

$$\begin{aligned}\mathcal{M} = -\left(\sqrt{G^{99}}\right)^D \frac{\Gamma\left(\frac{D+1}{2}\right)}{\pi^{\frac{D+1}{2}}} 8\sum_\alpha\sum_{l_9} \frac{\cos\left[4\pi(2l_9+1)\left(\varepsilon_9^\alpha + \frac{G^{9i}}{G^{99}}\,\varepsilon_i^\alpha\right)\right]}{|2l_9+1|^{D+1}} \\ \times \mathcal{H}_{\frac{D+1}{2}}\left(2\pi|2l_9+1|\frac{\left[\varepsilon_i^\alpha\,\hat{G}^{ij}\,\varepsilon_j^\alpha\right]^{\frac{1}{2}}}{\sqrt{G^{99}}}\right) + \mathcal{O}\left(\left(\sqrt{G^{99}}\right)^{\frac{D}{2}} e^{-\frac{2\pi c}{\sqrt{G^{99}}}}\right) \;.\end{aligned} \tag{B.3.17}$$

The Klein bottle. To provide us with a light breather, the Klein bottle amplitude vanishes precisely so we do not need to worry about it.

The torus. Although not strictly needed for the potential of the Wilson lines, the torus amplitude does play a role in the overall effective potential, so it is still good to calculate it. However, this is a more tricky business than before due to the integration over the two dimensional $SL(2,\mathbb{Z})$ moduli space. Using Poisson resummation over all the internal KK modes allows $\mathcal{T}$ to be written in Lagrangian form:

$$\begin{aligned}\mathcal{T} \;=\; \frac{1}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D+2}{2}}}\,\frac{1}{\eta^8\bar{\eta}^8}\,\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\,\frac{\theta\left[{a \atop b}\right]^4}{\eta^4}\,\frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}}\,\frac{\bar{\theta}\left[{\tilde{a} \atop \tilde{b}}\right]^4}{\bar{\eta}^4} \\ \cdot\,\frac{\sqrt{\det G}}{\tau_2^{\frac{10-D}{2}}}\sum_{\mathbf{l},\mathbf{n}} e^{-\frac{\pi}{\tau_2}(l^I+n^I\bar{\tau})G_{IJ}(l^J+n^J\tau)}(-1)^{l_9(a+\tilde{a})+n_9(b+\tilde{b})} \;.\end{aligned} \tag{B.3.18}$$

Notice how the sum over spin structures (a,b) and $(\tilde{a},\tilde{b})$ are coupling to the KK and winding numbers in the Scherk-Schwarz direction: $n_9, \tilde{l}_9$. This is what is responsible for the breaking of supersymmetry.

Unfolding the integral. An element $\gamma \in SL(2,\mathbb{Z})$ is given uniquely by two coprime integers p and q so that

$$\gamma = \begin{pmatrix} p & \star \\ q & \star \end{pmatrix} \tag{B.3.19}$$

where the asterisked entries are determined by Euclid's algorithm. Meanwhile, the element γ acts on the modular parameter τ so that $\gamma\cdot\tau_2 = \frac{\tau_2}{|p+q\tau|^2}$. Hence, we can write the following sum as

$$\sum_{(n,m)\in\mathbb{Z}^2\backslash\{(0,0)\}}^{\infty} e^{-|n+m\tau|^2/\tau_2} = \sum_{k=1}^{\infty}\sum_{(p,q)=1}^{\infty} e^{-k^2|p+q\tau|^2/\tau_2} = \sum_{k=1}^{\infty}\sum_{\gamma\in SL(2,\mathbb{Z})} \gamma\cdot e^{-k^2/\tau_2} \ . \quad \text{(B.3.20)}$$

Due to the exponential convergence, one can exchange the sums. Given a modular invariant function $f(\tau)$, this allows us to unfold certain kinds of integrals against the fundamental domain as

$$\begin{aligned}\sum_{(n,m)\in\mathbb{Z}^2\backslash\{(0,0)\}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} e^{-|n+m\tau|^2/\tau_2} f(\tau) &= \sum_{\gamma\in SL(2,\mathbb{Z})} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\gamma\cdot\sum_{k=1}^{\infty} e^{-k^2/\tau_2} f(\tau) \\ &= \sum_{k=1}^{\infty}\int_S \frac{d^2\tau}{\tau_2^2} e^{-k^2/\tau_2} f(\tau) \ ,\end{aligned} \quad \text{(B.3.21)}$$

where S is the strip $\{|\tau_1| < \frac{1}{2} \ , \ \tau_2 > 0\}$, which arises from changing variables for each γ[3].

We can apply this to the case at hand. Let us focus on the (l_9, n_9) sum. First note that when $l_9 = n_9 = 0$, the coupling to the spin structure vanishes, and the term vanishes by supersymmetry. This is to be expected - these states have no Scherk-Schwarz shift. This leaves the $(l_9, n_9) \neq (0,0)$ terms. In the approximation that G_{99} is much larger than the other metric moduli, the lattice sum approximately splits into a single exponential $\exp(-\pi G_{99}|l_9 + n_9\tau|^2/\tau_2)$ multiplied by what must be a modular invariant function. Using the unfolding technique then gives an integral over the strip of the same function but now with $n_9 = 0$ and $l_9 \neq 0$. Actually, due to supersymmetry, the l_9 can be taken to be over all the integers. We Poisson resum on all remaining n_I and l_I to find

$$\mathcal{T} = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\eta^8\bar{\eta}^8}\sum_{\mathbf{m},\mathbf{n}'}\Big\{\big(V_8\bar{V}_8 + S_8\bar{S}_8\big)\Lambda_{\mathbf{m},(\mathbf{n}',0)} - \big(V_8\bar{S}_8 + S_8\bar{V}_8\big)\Lambda_{\mathbf{m}+\mathbf{a}_S,(\mathbf{n}',0)}\Big\} \ . \quad \text{(B.3.22)}$$

This is now much more tractable. Integrating over τ_1, which implements the level matching condition, and doing the usual trick of Poisson resumming on m_9 and expanding all the characters, one finds

$$\begin{aligned}\mathcal{T} = (G^{99})^{\frac{D}{2}} \frac{\Gamma\big(\frac{D+1}{2}\big)}{\pi^{\frac{D+1}{2}}}\, 2\cdot 8^2 \sum_{k,\tilde{k}\geq 0} c_k c_{\tilde{k}} \sum_{\mathbf{m}',\mathbf{n}'} \delta_{\mathbf{m}'\cdot\mathbf{n}'+k-\tilde{k},0} \\ \sum_{l_9} \frac{\cos\Big[2\pi(2l_9+1)\frac{G^{9i}}{G^{99}}m_i\Big]}{|2l_9+1|^{D+1}} \mathcal{H}_{\frac{D+1}{2}}\Big(\pi|2l_9+1|\frac{M_{\mathcal{T}}}{\sqrt{G^{99}}}\Big) \ ,\end{aligned} \quad \text{(B.3.23)}$$

[3] The point is that $SL(2,\mathbb{Z})\backslash\mathcal{F} = S$.

where we have defined

$$M_{\mathcal{T}}^2 \;=\; P_i^L \hat{G}^{ij} P_j^L + k \;=\; P_i^R \hat{G}^{ij} P_j^R + \tilde{k}\;. \tag{B.3.24}$$

Clearly, the dominant terms that contribute have non-vanishing $M_{\mathcal{T}}^2$, which requires $k=0$ and $\mathbf{m}'=\mathbf{n}'=0$. These terms then contribute

$$\mathcal{T} \;=\; (G^{99})^{\frac{D}{2}}\,\frac{\Gamma\left(\frac{D+1}{2}\right)}{\pi^{\frac{D+1}{2}}}\,8\sum_{l_9}\frac{16}{|2l_9+1|^{D+1}} \;+\; \mathcal{O}\Big(\big(\sqrt{G^{99}}\big)^{\frac{D}{2}}\,e^{-\frac{2\pi c}{\sqrt{G^{99}}}}\Big)\;. \tag{B.3.25}$$

Summing all four amplitudes $\mathcal{T}$, $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ recovers the expression (5.4.8), with

$$\xi_D = \frac{\Gamma((D+1)/2)}{\pi^{(3D+1)/2}} \tag{B.3.26}$$

in that equation.

B.3.1 The effective potential for more general metric moduli

In this section we want to calculate the effective potential, but now want to lose the assumption that $\sqrt{G^{99}}$ is the minimal KK mass scale. We will consider the space of metric moduli with

$$G_{II} \gg 1\;, \tag{B.3.27}$$

for each I. The idea here is to force all winding modes to be heavier than the string scale so that there will be a clear heirarchy of scales $M_{KK} \ll M_{\text{string}} \ll M_{\text{winding}}$. The lowest mass scale is then $M_1^2 \sim \inf G_{II}$, which could be lower than the Scherk-Schwarz scale $\mathcal{M}^2 \sim G^{99}$. In this regime, one wants to Poisson resum *all* the internal modes. The calculations are so analagous to before that we will simply state the results. One finds

$$\begin{aligned}
\mathcal{A} \;&=\; \frac{8\Gamma(5)}{\pi^5}\sqrt{\det G}\sum_{k\geq 0} c_k \sum_{\alpha,\beta}\sum_{\mathbf{l}}\frac{e^{2i\pi\tilde{l}\cdot(\mathbf{a}_\alpha-\mathbf{a}_\beta)}}{(\tilde{l}_I G_{IJ}\tilde{l}_J)^5}\,\mathcal{H}_5\Big(\pi\sqrt{k\,\tilde{l}_I G_{IJ}\tilde{l}_J}\Big),\\
\mathcal{M} \;&=\; -\frac{8\Gamma(5)}{\pi^5}\sqrt{\det G}\sum_{k\geq 0}(-1)^k c_k\sum_{\alpha}\sum_{\mathbf{l}}\frac{e^{4i\pi\tilde{l}\cdot\mathbf{a}_\alpha}}{(\tilde{l}_I G_{IJ}\tilde{l}_J)^5}\,\mathcal{H}_5\Big(\pi\sqrt{k\,\tilde{l}_I G_{IJ}\tilde{l}_J}\Big)\;,
\end{aligned} \tag{B.3.28}$$

where $\tilde{l}_i = l_i$ but $\tilde{l}_9 = (2l_9+1)$. Just as before, only the terms in which the argument of $\mathcal{H}$ vanishes dominate - these require $k=0$ for both terms. The sum of the annulus and Möbius amplitudes can be conveniently arranged to give a generalization of (B.2.14):

$$\begin{aligned}
\mathcal{A}+\mathcal{M} = \frac{8\Gamma(5)}{\pi^5}\sqrt{\det G}\sum_{\mathbf{l}}&\frac{\left(\mathrm{tr}\,(\mathcal{W}_D^{\tilde{l}_D}\cdots\mathcal{W}_9^{\tilde{l}_9})\right)^2-\mathrm{tr}\,(\mathcal{W}_D^{2\tilde{l}_D}\cdots\mathcal{W}_9^{2\tilde{l}_9})}{(\tilde{l}_I G_{IJ}\tilde{l}_J)^5}\\
&+\;\mathcal{O}\big(\sqrt{\det G}\,G_{99}^{-\frac{11}{4}}\,e^{-2\pi\sqrt{G_{99}}}\big)\;,
\end{aligned} \tag{B.3.29}$$

where the Wilson lines $\mathcal{W}_I$ are defined by (B.3.7).

Klein bottle and torus. As usual the Klein bottle amplitude vanishes, so that we only have to compute the torus partition function.

We could unfold the torus integral in the same way as we did previously. However, it is clear that one can essentially discount the non-trivial winding modes already - in the end these massive states will be heavily exponentially suppressed, more than the leading exponential suppression terms. Using this shortcut, one can neglect the second line of (B.3.1) and Poisson resum the first line into Lagrangian form:

$$\begin{aligned}\mathcal{T} \;=\; & \frac{1}{2}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^{\frac{D+2}{2}}} \frac{1}{\eta^8\bar{\eta}^8}\, \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\frac{\theta\left[{a \atop b}\right]^4}{\eta^4}\, \frac{1}{2}\sum_{\tilde{a},\tilde{b}=0}^{1}(-1)^{\tilde{a}+\tilde{b}+\tilde{a}\tilde{b}}\frac{\bar{\theta}\left[{\tilde{a} \atop \tilde{b}}\right]^4}{\bar{\eta}^4} \\ & \cdot \frac{\sqrt{\det G}}{\tau_2^{\frac{10-D}{2}}}\sum_{\mathbf{l}} e^{-\frac{\pi}{\tau_2}l_I G_{IJ} l_J}(-1)^{l_9(a+\tilde{a})} \;+\; \mathcal{O}(e^{-c\inf G_{II}})\;,\end{aligned} \tag{B.3.30}$$

where $c = \mathcal{O}(1)$ is positive. Note that, for l_9 even, the coupling to the Wilson lines vanishes, so that the entire term also vanishes due to supersymmetry. Thus the sum over $\mathbf{l}$ can be assumed to be over $(l_i, 2l_9+1) = \tilde{l}$ in the notation of chapter 5. Finally, the error in extending the integration over the strip is exponentially small, of the order $\mathcal{O}(\exp(-c' G_{99}))$ for some order 1 constant c'. Eventually, one finds

$$\begin{aligned}\mathcal{T} \;&=\; \frac{\sqrt{\det G}}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^{+\infty}\frac{d\tau_2}{\tau_2^{1+5}}\,\frac{\theta_2^4}{\eta^{12}}\,\frac{\bar{\theta}_2^4}{\bar{\eta}^{12}}\sum_{\mathbf{l}} e^{-\frac{\pi}{\tau_2}\tilde{l}_I G_{IJ}\tilde{l}_J} \;+\; \mathcal{O}(e^{-c\inf G_{II}}) \\ &=\; \frac{\Gamma(5)}{\pi^5}\sqrt{\det G}\, 8\sum_{k\geq 0} c_k^2 \sum_{\mathbf{l}} \frac{16}{(\tilde{l}_I G_{IJ}\tilde{l}_J)^5}\,\mathcal{H}_5\Big(2\pi\sqrt{k\,\tilde{l}_I G_{IJ}\tilde{l}_J}\Big) \;+\; \mathcal{O}(e^{-c\inf G_{II}}) \\ &=\; \frac{\Gamma(5)}{\pi^5}\sqrt{\det G}\, 8\sum_{\mathbf{l}}\frac{16}{(\tilde{l}_I G_{IJ}\tilde{l}_J)^5} \;+\; \mathcal{O}\big(\sqrt{\det G}\, G_{99}^{-\frac{11}{4}} e^{-4\pi\sqrt{G_{99}}}\big)\;.\end{aligned} \tag{B.3.31}$$

In total, the 1-loop effective potential then takes the final form

$$\begin{aligned}\mathcal{V} \;=\; & \frac{\Gamma(5)}{\pi^{D+5}}\frac{M_s^D}{2^D}\sqrt{\det G}\; 4\sum_{\mathbf{l}}\frac{-16 - \big(\mathrm{tr}\,(\mathcal{W}_D^{\tilde{l}_D}\cdots\mathcal{W}_9^{\tilde{l}_9})\big)^2 + \mathrm{tr}\,(\mathcal{W}_D^{2\tilde{l}_D}\cdots\mathcal{W}_9^{2\tilde{l}_9})}{(\tilde{l}_I G_{IJ}\tilde{l}_J)^5} \\ & + \mathcal{O}\big(M_s^D\sqrt{\det G}\, G_{99}^{-\frac{11}{4}} e^{-2\pi\sqrt{G_{99}}}\big)\;,\end{aligned} \tag{B.3.32}$$

as stated in (5.4.15).

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