Complex Hyperbolic Triangle Groups

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Chapter 1

Introduction

Mostow [14] studied a group of complex hyperbolic isometries generated by three complex reflections. He assumed that each pair of them has the classical braiding relation, meaning they braid with length 3. Since the odd braiding relations imply that the reflections share the same order, the group was forced to have the restriction that all three reflections are of the same order. The group's fundamental domain was then constructed based on Dirichlet's fundamental domain. The paper mainly focused on arithmeticity.

There is a series of papers on complex hyperbolic isometry groups generated by three complex reflections afterwards: Livné [13] in 1980, then Deligne and Mostow [4],[5] in 1986-1993. Following Picard, Deligne and Mostow viewed the groups as monodromy groups of hypergeometric functions in several variables. They still focused on arithmeticity and did not consider fundamental domains.

Around 1988, Thurston [26] introduced an alternative way to view the groups through cone metrics on the sphere, although this was not published until later. This method has been used in many fundamental domain constructions afterwards.

After that, in 2005, Deraux-Falbel-Paupert [7] built some of the Deligne-Mostow groups' fundamental domains for the first time. Following suit, Parker ([17], [15]) used Thurston's construction to build fundamental domains for the Livné groups[13]. Then, there is a work of Boadi-Parker[3] that used Thurston's method to build fundamental domains for the Deligne-Mostow groups not considered by Deraux-Falbel-Paupert.

Some time afterwards, the results we will use to compare our groups with were published by Pasquinelli. She used Thurston's method to build fundamental domains for all the Deligne-Mostow groups. This was done in two papers, the first for the case of three-fold symmetry [21] and the second for the rest (two-fold symmetry) [22].

There tend to be restrictions on the braiding orders when constructing fundamental domains. Parker-Paupert [20] considered which possible braiding orders were allowed for groups generated by three complex reflections of the same order to generate proper fundamental domains. Then, Deraux-Parker-Paupert [8] experimentally worked out which of the Parker-Paupert groups were lattices using Dirichlet domains. (For more on Dirichlet domains, see Deraux [6], which can be compared to simpler fundamental domains in [10].)

Deraux-Parker-Paupert ([9],[10]) verified the experimental results in [8] by giving an algorithm to build fundamental domains for all the Deraux-Parker-Paupert groups (called DPP groups from now on) as well as the Deligne-Mostow groups (which are different from Pasquinelli's domains we will use as references). Lastly, Parker [18] showed that all the DPP and Deligne-Mostow groups, except some of the $\mathbf{E_2}$ groups, were monodromy groups of higher hypergeometric functions.

Our aim is to consider a group with more freedom, namely, a group generated by three complex reflections that braid with lengths $(2,4,4)$, called a $(2,4,4)$ group. With even braidings, there can be a variety of different generators' orders. We show that such groups in SU(2,1) (provided that they are non-elementary) also contain a fourth (conjugacy class of) complex reflections, giving rise to further $(2,4,4)$ groups after changing generators. This means that there are isomorphisms between $(2,4,4)$ groups with complex reflections of different orders. We show that the $(2,4,4)$ groups can be identified with the groups generated by Pasquinelli ([21]), and thus, are commensurable with Deligne-Mostow groups.

After we have the group structures, we consider a subgroup of the form (r,4,4;4) (Citing [10], a group of the form $(a,b,c;d)$ is a group generated by complex reflections R_1, R_2, R_3 where R_2 and R_3 braid with length a, R_3 and R_1 braid with length b, R_1 and R_2 braid with length c, R_1 and $R_3^{-1}R_2R_3$ braid with length d.) for the sake of geometric construction as we want to apply the algorithm in Deraux-Parker-Paupert [10] on this group to construct its fundamental domain.

We begin by building combinatorial fundamental domains for all the groups and partly geometrically realising these. There are difficulties from embedding results here so we will assume the embeddedness for now. Then, we can rely on the Poincare polyhedron theorem to get a presentation and find the orbifold Euler characteristic. Since Pasquinelli ([21],[22]) has technically summed up the lattices and their corresponding Euler characteristics in her works already, we compare ours to hers. The agreeing Euler characteristics suggest that the embeddedness assumption is valid. Consequently, we gain fundamental domains that are much simpler than Pasquinelli's yet yield the same result.

By comparing presentations, we can show that our groups are commensurable to Deligne-Mostow groups or DPP $\mathbf{E_2}$ groups. Recall the automorphisms used in symmetrizing our generators' orders mentioned earlier. This shows that all $\mathbf{E_2}$ groups are commensurable to Deligne-Mostow groups. Hence, we can show all the $\mathbf{E_2}$ groups are monodromy groups of higher hypergeometric functions as well as hypergeometric functions in several variables.

In this chapter, we give the introduction and necessary definitions. We start with complex hyperbolic reflections and their relations with braiding relations. Then, we state the Poincaré Polyhedron Theorem.

In chapter 2, we construct an arbitrary group generated by 3 complex reflections and defined by the braiding relations between the generators. In this case, we let it be a (2,4,4)-group. We then show that two of the generators, along with another reflection in the group are interchangable when we ignore their orders. After that, we identify the group's presentation with Pasquinelli's groups ([21],[22]), which has also been identified with Deligne-Mostow groups.

Then, in chapter 3, we consider a (r,4,4;4)- group, which we later show to be a subgroup of the (2,4,4)-group we created earlier.

In chapter 4, we use the group to construct a fundamental domain using the DPP algorithm. Then, we check all the necessary conditions for it to satisfy the Poincaré Polyhedron Theorem. We check the Euler characteristics of the polyhedron constructed and compare them with Deligne-Mostow's from [21] to convince that the embeddedness is indeed satisfied. After that, we give the table of all possible values for the orders of the group's generators along with the lattices associated to them from [21], concluding our work.

1.1 Reflections on Complex hyperbolic spaces

The space $\mathbb{C}^3$ is the 3-dimensional complex vector space. We equip it with a Hermitian form of signature $(2,1)$, resulting in a space called $\mathbb{C}^{2,1}$. This divides the space into three subspaces:

$$\begin{aligned} V_0 &= \{z \in \mathbb{C}^3 : \langle z, z\rangle = 0\}. \\ V_- &= \{z \in \mathbb{C}^3 : \langle z, z\rangle < 0\}. \\ V_+ &= \{z \in \mathbb{C}^3 : \langle z, z\rangle > 0\}. \end{aligned}$$

The complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ we are interested in is then the image of V_- under the canonical projection

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} z_1/z_3 \\ z_2/z_3 \\ 1 \end{pmatrix}$$

equipped with the Bergman metric, given by the distance function ρ defined by

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle z,w\rangle\langle w,z\rangle}{\langle z,z\rangle\langle w,w\rangle}.$$

A **complex reflection** is an isometry of complex hyperbolic space with a two dimensional eigenspace and a one dimensional eigenspace. When the two dimensional eigenspace contains negative vectors, it corresponds to a complex line in complex hyperbolic space, called the **mirror**, which is fixed pointwise by the transformation. (On the other hand, if the one dimensional eigenspace contains negative vectors, it corresponds to a unique fixed point in complex hyperbolic space.)

Consider a vector subspace U in $\mathbb{C}^{2,1}$ with (complex) dimension 2. Let $U^\perp$ be the vector subspace orthogonal to U with respect to the Hermitian form. That is, if $\underline{c} \in U^\perp$ then $\langle \underline{z}, \underline{c}\rangle = 0$ for all $\underline{z} \in U$. Now suppose that $U \cap V_-$ contains at least one non-trivial vector and hence an open subset of U. The projection of $U \cap V_-$ to $\mathbf{H}^2_{\mathbb{C}}$ is a **complex geodesic** C. Any non-trivial vector in $U^\perp$ is a **polar vector** to C.

A **bisector** is the locus of points in $\mathbf{H}^2_{\mathbb{C}}$ equidistant from two points. Suppose that p_1 and p_2 are points of $\mathbf{H}^2_{\mathbb{C}}$ and suppose that $\underline{p}_1$ and $\underline{p}_2$ are vectors in $\mathbb{C}^{2,1}$ whose images under the canonical projection are p_1 and p_2 respectively. We now show how to construct the bisector $\mathscr{B}(p_1,p_2)$ equidistant from p_1 and p_2. [11] Consider the vector subspace of $\mathbb{C}^{2,1}$ spanned by $\underline{p}_1$ and $\underline{p}_2$. By construction this is (complex) two-dimensional. Its image is a complex geodesic containing p_1 and p_2, called the **complex spine** of $\mathscr{B}(p_1,p_2)$ and is denoted Σ. As indicated above, this is a copy of the Poincaré disc. Let σ be the Poincaré geodesic in the complex spine Σ equidistant from p_1 and p_2. This is called the **real spine** of $\mathscr{B}(p_1,p_2)$. Consider a point $s \in \sigma$. Let C_s be the complex line through s orthogonal to Σ. This complex line is contained in the bisector $\mathscr{B}(p_1,p_2)$ and is called a **complex slice** of the bisector. The bisector $\mathscr{B}(p_1,p_2)$ is the union of all its slices; that is:

$$\mathscr{B}(p_1,p_2) = \bigcup_{s\in\sigma} C_s.$$

Moreover, the bisector $\mathscr{B}$ is foliated by totally geodesic subspaces in a second way. It is not hard to show that a Lagrangian plane R containing σ is also contained in $\mathscr{B}$. Moreover, $\mathscr{B}$ is the union of all Lagrangian planes containing σ.

To illustrate this definition, we give a simple example in the unit ball model $\mathbb{B}^2 = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$. Suppose that $0 < y < 1$ and that $p_1 = (iy, 0)$, $p_2 = (-iy, 0)$ in $\mathbb{C}^2$. Then Σ is the complex line with $z_2 = 0$ and the real spine σ is

$$\sigma = \{(s,0) \in \mathbb{C}^2 : -1 < s < 1\}.$$

For a fixed $s \in (-1,1)$ the slice C_s is given by

$$C_s = \{(s,z) \in \mathbb{C}^2 : |z|^2 < 1 - s^2\}.$$

For a given $\theta \in [0, 2\pi)$ the **meridian** R_θ is given by

$$R_\theta = \{(x, ye^{i\theta}) \in \mathbb{C}^2 \,:\, x^2 + y^2 < 1\}.$$

Pratoussevitch [23] gives a more explicit definition for complex reflections starting from a more common type of reflections, namely reflections of order 2. In this case, a complex reflection in a complex geodesic C is an isometry in $\mathrm{PU}(2,1)$ of order 2 with C as its set of fixed points. It is given by

$$\underline{z} \mapsto -\underline{z} + 2\frac{\langle \underline{z}, \underline{c} \rangle}{\langle \underline{c}, \underline{c} \rangle}\underline{c}$$

where $\underline{c}$ is a polar vector of C.

In a more general definition, a complex μ-reflection, for a unit complex number μ, is an element of $\mathrm{PU}(2,1)$ with a complex geodesic C as the fixed point set that rotates around this complex geodesic by the angle $\arg(\mu)$. It can be written as

$$\underline{z} \mapsto \underline{z} + (\mu - 1)\frac{\langle \underline{z}, \underline{c} \rangle}{\langle \underline{c}, \underline{c} \rangle}\underline{c}$$

where $\underline{c}$ is a polar vector of C. Note that a usual complex reflection mentioned earlier is a μ-reflection where $\mu = -1$.

With this, we write our reflections with angle $\frac{2\pi}{k}$ in complex hyperbolic spaces in the form

$$\begin{aligned}
R\left(\underline{z}\right) &= \underline{z} + \left(e^{\frac{2\pi i}{k}} - 1\right)\frac{\langle \underline{z}, \underline{c} \rangle}{\langle \underline{c}, \underline{c} \rangle}\underline{c} \\
&= \underline{z} + \left(e^{\frac{2\pi i}{k}} - 1\right)\frac{\underline{c}^* H \underline{z}}{\langle \underline{c}, \underline{c} \rangle}\underline{c} \\
&= \underline{z} + \left(e^{\frac{2\pi i}{k}} - 1\right)\underline{c}\frac{\underline{c}^* H \underline{z}}{\langle \underline{c}, \underline{c} \rangle} \\
&= \left(I + \left(e^{\frac{2\pi i}{k}} - 1\right)\underline{c}\frac{\underline{c}^* H}{\langle \underline{c}, \underline{c} \rangle}\right)\underline{z}
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the Hermitian form of signature (2,1) associated to a Hermitian matrix H and $\underline{c}$ is a vector polar to the mirror of the reflection. Consider R as the projective matrix $I + (e^{\frac{2\pi i}{k}} - 1)\underline{c}\frac{\underline{c}^* H}{\langle \underline{c}, \underline{c} \rangle}$. Then,

$$\begin{aligned}
R\underline{c} &= \left(I + \left(e^{\frac{2\pi i}{k}} - 1\right)\underline{c}\frac{\underline{c}^* H}{\langle \underline{c}, \underline{c} \rangle}\right)\underline{c} \\
&= \underline{c} + (e^{\frac{2\pi i}{k}} - 1)\underline{c} \\
&= e^{\frac{2\pi i}{k}}\underline{c}.
\end{aligned}$$

So, $\underline{c}$ is an eigenvector of R corresponding to the eigenvalue $e^{\frac{2\pi i}{k}}$. Moreover,

$$\begin{aligned} R\left(\underline{z}-\frac{\langle \underline{z},\underline{c}\rangle}{\langle \underline{c},\underline{c}\rangle}\underline{c}\right) &= \underline{z}-\frac{\langle \underline{z},\underline{c}\rangle}{\langle \underline{c},\underline{c}\rangle}\underline{c}+\left(e^{\frac{2\pi i}{k}}-1\right)\underline{c}\frac{\underline{c}^*H}{\langle \underline{c},\underline{c}\rangle}\left(\underline{z}-\frac{\langle \underline{z},\underline{c}\rangle}{\langle \underline{c},\underline{c}\rangle}\underline{c}\right) \\ &= \underline{z}-\frac{\langle \underline{z},\underline{c}\rangle}{\langle \underline{c},\underline{c}\rangle}\underline{c}. \end{aligned}$$

Thus, the vector $\underline{z}-\frac{\langle \underline{z},\underline{c}\rangle}{\langle \underline{c},\underline{c}\rangle}\underline{c}$ is an eigenvector of R corresponding to the eigenvalue 1. Since $\underline{z}$ is arbitrary, the eigenspace is of dimension 2 and the eigenvalue 1 is repeated. Hence, the reflections with angle $\frac{2\pi}{k}$ always have eigenvalues $e^{\frac{2\pi i}{k}}, 1, 1$.

We want to construct the reflections R_1, R_2, R_3 of order q, p, m associated to the polar vectors $n_1 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T, n_2 = \begin{pmatrix}0 & 1 & 0\end{pmatrix}^T, n_3 = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^T$, respectively. We also want the space to be $\mathbb{C}^{2,1}$, the complex vector space of dimension 3 with the Hermitian form of signature $(2,1)$. We firstly assume the form of the Hermitian matrix H associated with the Hermitian form as

$$H = \begin{pmatrix} a & -i\rho & i\bar{\tau} \\ i\bar{\rho} & b & -i\sigma \\ -i\tau & i\bar{\sigma} & c \end{pmatrix}$$

where a, b, c are real whilst $\rho, \tau, \sigma \in \mathbb{C}$. In the case of R_1, we have

$$\begin{aligned} R_1 &= I + (e^{\frac{2\pi i}{q}}-1)n_1\frac{n_1^*H}{\langle n_1, n_1\rangle} \\ &= I + (e^{\frac{2\pi i}{q}}-1)n_1\frac{1}{\langle n_1, n_1\rangle}\begin{pmatrix}1 & 0 & 0\end{pmatrix}H \\ &= I + (e^{\frac{2\pi i}{q}}-1)\frac{1}{\langle n_1, n_1\rangle}n_1\begin{pmatrix}a & -i\rho & i\bar{\tau}\end{pmatrix} \\ &= I + (e^{\frac{2\pi i}{q}}-1)\frac{1}{a}\begin{pmatrix} a & -i\rho & i\bar{\tau} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= I + \begin{pmatrix} e^{\frac{2\pi i}{q}}-1 & (e^{\frac{2\pi i}{q}}-1)\frac{-i\rho}{a} & (e^{\frac{2\pi i}{q}}-1)\frac{i\bar{\tau}}{a} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{2\pi i}{q}} & -i(e^{\frac{2\pi i}{q}}-1)\frac{\rho}{a} & i(e^{\frac{2\pi i}{q}}-1)\frac{\bar{\tau}}{a} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The matrix has determinant $e^{\frac{2\pi i}{q}}$. Since we want to work on the projective model and it will be more convenient to work with matrices with determinant 1, we let

$$R_1 = \begin{pmatrix} e^{\frac{4\pi i}{3q}} & -i(e^{\frac{4\pi i}{3q}} - e^{-\frac{2\pi i}{3q}})\frac{\rho}{a} & i(e^{\frac{4\pi i}{3q}} - e^{-\frac{2\pi i}{3q}})\frac{\bar{\tau}}{a} \\ 0 & e^{-\frac{2\pi i}{3q}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3q}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{4\pi i}{3q}} & -ie^{\frac{\pi i}{3q}}(2i\sin(\frac{\pi}{q}))\frac{\rho}{a} & ie^{\frac{\pi i}{3q}}(2i\sin(\frac{\pi}{q}))\frac{\bar{\tau}}{a} \\ 0 & e^{-\frac{2\pi i}{3q}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3q}} \end{pmatrix}.$$

Thus, it would be convenient to let a be $2\sin(\frac{\pi}{q})$ and R_1 would become

$$R_1 = \begin{pmatrix} e^{\frac{4\pi i}{3q}} & e^{\frac{\pi i}{3q}}\rho & -e^{\frac{\pi i}{3q}}\bar{\tau} \\ 0 & e^{-\frac{2\pi i}{3q}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3q}} \end{pmatrix}.$$

In the same way, let $b = 2\sin(\frac{\pi}{p})$ and $c = 2\sin(\frac{\pi}{m})$. Also, let $u = e^{\frac{\pi i}{3q}}, v = e^{\frac{\pi i}{3p}}$ and $w = e^{\frac{\pi i}{3m}}$. Then,

$$R_1 = \begin{pmatrix} u^4 & u\rho & -u\bar{\tau} \\ 0 & \bar{u}^2 & 0 \\ 0 & 0 & \bar{u}^2 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} \bar{v}^2 & 0 & 0 \\ -v\bar{\rho} & v^4 & v\sigma \\ 0 & 0 & \bar{v}^2 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} \bar{w}^2 & 0 & 0 \\ 0 & \bar{w}^2 & 0 \\ w\tau & -w\bar{\sigma} & w^4 \end{pmatrix}$$

whilst

$$H = \begin{pmatrix} 2\sin\left(\frac{\pi}{q}\right) & -i\rho & i\bar{\tau} \\ i\bar{\rho} & 2\sin\left(\frac{\pi}{p}\right) & -i\sigma \\ -i\tau & i\bar{\sigma} & 2\sin\left(\frac{\pi}{m}\right) \end{pmatrix}.$$

Lemma 1.1.1. The signature of H is $(2,1)$ if and only if $\det(H)$ is negative.

Proof. Since H is Hermitian, its eigenvalues are all real. Their summation is equal to the trace of H:

$$2\sin\left(\frac{\pi}{q}\right) + 2\sin\left(\frac{\pi}{p}\right) + 2\sin\left(\frac{\pi}{m}\right),$$

and their product is the determinant of H:

$$8\sin\left(\frac{\pi}{q}\right)\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{m}\right) - i\rho\tau\sigma + i\bar{\rho}\bar{\tau}\bar{\sigma} - 2|\sigma|^2\sin\left(\frac{\pi}{q}\right) - 2|\tau|^2\sin\left(\frac{\pi}{p}\right) - 2|\rho|^2\sin\left(\frac{\pi}{m}\right).$$

If the signature is indeed $(2,1)$, then the determinant is a product of two positive and one negative parameters, thus negative.

Since $tr\,(H)$ is obviously positive, the eigenvalues cannot be all negative and hence, $\det(H)$ being negative implies that the signature is $(2,1)$. $\square$

So, we now assume $\det(H)$ negative. This means that

$$\mathrm{Im}(\rho\tau\sigma) < |\sigma|^2 \sin\left(\frac{\pi}{q}\right) + |\tau|^2 \sin\left(\frac{\pi}{p}\right) + |\rho|^2 \sin\left(\frac{\pi}{m}\right) - 4\sin\left(\frac{\pi}{q}\right)\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{m}\right).$$

We can compute the traces of the transformations we are going to work with. The transformations that are words of length 3 or higher can be calculated using Theorem 7 from [23].

$$tr\,(R_1R_2) = uv[2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right) - |\rho|^2] + \bar{u}^2\bar{v}^2, \tag{1.1}$$

$$tr\,(R_2R_3) = vw[2\cos\left(\frac{\pi}{p} - \frac{\pi}{m}\right) - |\sigma|^2] + \bar{v}^2\bar{w}^2, \tag{1.2}$$

$$tr\,(R_1R_3) = uw[2\cos\left(\frac{\pi}{q} - \frac{\pi}{m}\right) - |\tau|^2] + \bar{u}^2\bar{w}^2, \tag{1.3}$$

$$tr\left(R_1R_3^{-1}R_2R_3\right) = uv[2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right) - |\sigma\tau - \bar{w}^3\bar{\rho}|^2] + \bar{u}^2\bar{v}^2, \tag{1.4}$$

$$tr\,(R_1R_2R_3) = uvw[u^3\bar{v}^3\bar{w}^3 + \bar{u}^3v^3\bar{w}^3 + \bar{u}^3\bar{v}^3w^3 - \bar{u}^3|\sigma|^2 - \bar{v}^3|\tau|^2 - \bar{w}^3|\rho|^2 + \rho\tau\sigma]. \tag{1.5}$$

Apart from the generators' orders, we will also define our groups based on the braiding relations between the generators. For an integer n, we say that two transformations A and B **braid** with length n if

$$(AB)^{\frac{n}{2}} = (BA)^{\frac{n}{2}}$$

when n is even, or

$$(AB)^{\frac{n-1}{2}}A = (BA)^{\frac{n-1}{2}}B$$

when n is odd. We say that $\mathrm{br}(A,B) = n$ in such a case.

For integers a, b, c, a (a,b,c)-group is a group generated by 3 complex reflections A, B, C with braiding relations $\mathrm{br}(B,C) = a$, $\mathrm{br}(A,C) = b$ and $\mathrm{br}(A,B) = c$.

Lemma 1.1.2. Let A, B be reflections with angles θ, ϕ, respectively. Then, A has 2-dimensional $e^{-\frac{i\theta}{3}}$ eigenspace and B has 2-dimensional $e^{-\frac{i\phi}{3}}$ eigenspace. Their eigenspaces intersect and AB has the eigenvalue $e^{-i\frac{\theta+\phi}{3}}$ associated to the common eigenspace.

Proof. As matrices acting on $\mathbb{C}^{2,1}$, both A and B have eigenspaces that are two dimensional with eigenvalues $e^{-i\frac{\theta}{3}}$ and $e^{-i\frac{\phi}{3}}$. Therefore, any non-trivial vector in the intersection of these two eigenspaces is both an $e^{-i\frac{\theta}{3}}$ eigenvector for A and an $e^{-i\frac{\phi}{3}}$ eigenvector for B. Hence it is an $e^{-i\frac{\theta+\phi}{3}}$ eigenvector for AB. □

Since there are 2-dimensional eigenspaces according to the eigenvalue $\bar{u}^2$ of R_1 and $\bar{v}^2$ of R_2. We have an eigenvalue $\bar{u}^2\bar{v}^2$ for R_1R_2 with 1-dimensional eigenspace.

Suppose that the mirrors M_1 and M_2 of R_1 and R_2 are ultraparallel. Then there is a complex line C_{12} orthogonal to M_1 and M_2. By construction, R_1 and R_2 send C_{12} to itself and they act on C_{12} as rotations of angles $\frac{2\pi}{q}$ and $\frac{2\pi}{p}$ with fixed points $M_1 \cap C_{12}$ and $M_2 \cap C_{12}$ respectively. Furthermore, Assume that $2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right) - |\rho|^2 = -2\cos\left(\frac{2\pi}{a}\right)$ for some integer a. Then, the set of eigenvalues $\bar{u}^2\bar{v}^2, -uve^{-\frac{2\pi i}{a}}, -uve^{\frac{2\pi i}{a}}$ satisfies both the trace and determinant of R_1R_2. We see that R_1R_2 acts on C_{12} as a rotation of angle $\frac{4\pi}{a}$. Hence $\langle R_1, R_2\rangle$ acts on C_{12} as a $(p, q, \frac{a}{2})$ triangle group. When a is even, we see that $(R_1R_2)^{\frac{a}{2}}$ fixes each point of C_{12}, but may itself rotate in the orthogonal direction. Since $(R_1R_2)^{\frac{a}{2}}$ acts trivially on C_{12} it must commute with both R_1 and R_2, and so it is in the centre of $\langle R_1, R_2\rangle$. Hence, $\langle R_1, R_2\rangle$ acts on $\mathbf{H}^2_{\mathbb{C}}$ as a central extension of a triangle group, with centre generated by $(R_1R_2)^{a/2}$. Thus, R_1 and R_2 braid with length a.

Theorem 1.1.3. Knapp's theorem, see $[12, pp.296-297]$ Let A and B be normalized elliptic matrices with distinct fixed points. The group generated by A and B is discrete if and only if the traces of AB, A and B satisfy one of the seven conditions:

(I) $|tr(AB)| < 2$ and AB has extreme negative trace,

(II) $|tr(AB)| \geq 2$,

(III) $tr(A) = tr(B)$ and $tr(AB) = 2\cos(\pi + \frac{2\pi}{n})$ with $n \geq 3$ and odd,

(IV) $tr(A) = 0, tr(B) = 2\cos(\pi - \frac{\pi}{n})$ and $tr(AB) = 2\cos(\pi + \frac{2\pi}{n})$ with $n \geq 3$ and odd (or the same thing with A and B interchanged),

(V) $tr(A) = 2\cos(\pi - \frac{\pi}{3}), tr(B) = 2\cos(\pi - \frac{\pi}{n})$ and $tr(AB) = 2\cos(\pi + \frac{3\pi}{n})$ with $n \geq 7$ and not divisible by 3 (or the same thing with A and B interchanged),

(VI) $tr(A) = tr(B) = 2\cos(\pi - \frac{\pi}{n})$ and $tr(AB) = 2\cos(\pi + \frac{4\pi}{n})$ with $n \geq 7$ and odd,

(VII) $tr(A) = 2\cos(\pi - \frac{\pi}{3}), tr(B) = 2\cos(\pi - \frac{\pi}{7})$ and $tr(AB) = 2\cos(\pi + \frac{2\pi}{7})$ (or the same thing with A and B interchanged).

In the case that a is odd, according to Knapp's theorem, it either falls into the case (III) and $p = q$, or the case (VII) and $q = 3, p = 7, a = 7$.

In the case that $p = q$, we have $(R_1R_2)^{\frac{a-1}{2}} R_1$ in the centre, so the braid length is still a.

This means that, if the braiding relation between R_1 and R_2 is $\mathrm{br}(R_1, R_2) = a$, then

$$2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right) - |\rho|^2 = -2\cos\left(\frac{2\pi}{a}\right). \tag{1.6}$$

In the same way, the braiding relations $\mathrm{br}(R_2, R_3) = b, \mathrm{br}(R_1, R_3) = c, \mathrm{br}(R_1, R_3^{-1}R_2R_3) = d$ hold if we have

$$2\cos\left(\frac{\pi}{p} - \frac{\pi}{m}\right) - |\sigma|^2 = -2\cos\left(\frac{2\pi}{b}\right), \tag{1.7}$$

$$2\cos\left(\frac{\pi}{q} - \frac{\pi}{m}\right) - |\tau|^2 = -2\cos\left(\frac{2\pi}{c}\right), \tag{1.8}$$

$$2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right) - |\sigma\tau - \bar{w}^3\bar{\rho}|^2 = -2\cos\left(\frac{2\pi}{d}\right), \tag{1.9}$$

respectively.

Lemma 1.1.4. If A and B are complex reflections satisfying an odd order braid relation, they are conjugate and share the same angle.

Proof. Let A and B braid with length $2k + 1$, then

$$(AB)^k A = B (AB)^k$$

and so,

$$(AB)^k A (AB)^{-k} = B,$$

that is, they are conjugate. □

Since we want our reflections to have different angles, the braiding relations between them should be even.

In a special case when $q = p = m$, which also means $u = v = w$, we can relate to [10] by substituting $u' = u^2 = v^2 = w^2$ and $\rho' = u\rho, \tau' = v\tau, \sigma' = w\sigma$. Then,

$$R_1 = \begin{pmatrix} u'^2 & \rho' & -u'\bar{\tau}' \\ 0 & \bar{u}' & 0 \\ 0 & 0 & \bar{u}' \end{pmatrix},$$

$$R_2 = \begin{pmatrix} \bar{u}' & 0 & 0 \\ -u'\bar{\rho}' & u'^2 & \sigma' \\ 0 & 0 & \bar{u}' \end{pmatrix},$$

$$R_3 = \begin{pmatrix} \bar{u}' & 0 & 0 \\ 0 & \bar{u}' & 0 \\ \tau' & -u'\bar{\sigma}' & u'^2 \end{pmatrix},$$

and the traces become

$$\begin{aligned} tr\,(R_1R_2) &= u'\left(2-|\rho'|^2\right)+\bar{u}'^2,\\ tr\,(R_2R_3) &= u'\left(2-|\sigma'|^2\right)+\bar{u}'^2,\\ tr\,(R_1R_3) &= u'\left(2-|\tau'|^2\right)+\bar{u}'^2,\\ tr\left(R_1R_3^{-1}R_2R_3\right) &= u'\left(2-|\tau'\sigma'-\bar{\rho}'|^2\right)+\bar{u}'^2,\\ tr\,(R_1R_2R_3) &= 3-|\rho'|^2-|\tau'|^2-|\sigma'|^2+\rho'\tau'\sigma', \end{aligned}$$

analogous to the calculation in [10]. In terms of braiding relations, if $\mathrm{br}_n(R_1,R_2)$, meaning $\mathrm{br}(R_1,R_2)=n$, and they share the same angle simplified as above, then

$$tr\,(R_1R_2) = -2u'\cos\left(\frac{2\pi}{n}\right)+\bar{u}'^2. \tag{1.10}$$

The Proposition 3.1 from [18] gives us a way to identify a set of complex reflections and its conjugates. We restate it in our terms:

Proposition 1.1.5. Let R_1,R_2,R_3 and R_1',R_2',R_3' be two sets of complex reflections with angles $\frac{2\pi}{p},\frac{2\pi}{q},\frac{2\pi}{r}$, respectively. Let ρ,σ,τ and ρ',σ',τ' be as defined above. The triples R_1,R_2,R_3 and R_1',R_2',R_3' are conjugate in $\mathrm{PGL}(3,\mathbb{C})$ if and only if one of the following is true:

1. If $\rho\sigma\tau\neq 0$ and $p,q,m\geq 3$, then
$$|\rho'|=|\rho|,|\sigma'|=|\sigma|,|\tau'|=|\tau|,\arg(\rho'\sigma'\tau')=\arg(\rho'\sigma'\tau).$$
2. If $\rho\sigma\tau\neq 0$ and $p=2,q=2,m=2$, then
$$|\rho'|=|\rho|,|\sigma'|=|\sigma|,|\tau'|=|\tau|,\arg(\rho'\sigma'\tau')=\pm\arg(\rho'\sigma'\tau).$$
3. if $\rho\sigma\tau=0$, then
$$|\rho'|=|\rho|,|\sigma'|=|\sigma|,|\tau'|=|\tau|.$$

1.2 Poincaré Polyhedron Theorem

A polyhedron E consists of facets of lower dimensions. We call the facets with codimension one, two, three and four as sides, ridges, edges and vertices, respectively and refer to the sets of facets as $F_i(E)$ where i is the facet's codimension.

Our aim is to ultimately construct fundamental polyhedra along with side pairing maps satisfying the Poincaré Polyhedron Theorem. There are many versions of statement for Poincaré polyhedron theorem. For the sake of consistent notation, we refer to the one stated in [9].

Next, we define side pairing maps. A **side pairing** for E is a map $\psi : F_1(E)\to \mathrm{Isom}(\mathbf{H}^n_{\mathbb{C}})$, whereas the map $P=\psi(s)$ is called a *side pairing map* associated with the side $s\in F_1(E)$, satisfying these conditions:

1. The side pairing map $P = \psi(s)$ associated with s maps s onto another side $s^- \in F_1(E)$ whilst preserving the cell structure. Moreover, $\psi(s^-) = P^{-1}$.
2. For $P = \psi(s)$, $P^{-1}(E) \cap E = s$ and $P^{-1}(E^o) \cap E^o = \emptyset$.
3. There is an open neighbourhood of each point in s^o that is contained in $E \cup P^{-1}(E)$.

If, moreover, there is a finite group $\Upsilon \leq \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$ of cell preserving automorphisms on E, we can add the side pairing maps to form a new group $\Gamma \leq \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$. In our case, it is enough to assume that Υ can be presented in terms of its generators and their relations. We put our facets into orbits under the action of Υ. We say that the side pairing ψ is **compatible with** Υ if $\psi(Ss) = S\psi(s)S^{-1}$ for every side $s \in F_1(E)$ and every $S \in \Upsilon$.

Now that we have side pairing, we continue to define ridge cycles and cycle relations, which are essential for the theorem. Consider a ridge $r_1 \in F_2(E)$. It lies on exactly two sides. We call them s_0^- and s_1, so $r_1 = s_0^- \cap s_1$. Since $P_1 = \psi(s_1)$ maps s_1 to another side s_1^- preserving the cell structure, it sends the ridge r_1 to another ridge r_2 in s_1^-. The ridge r_2 also lies in exactly two sides, say, s_1^- and s_2. Then, the map $P_2 = \psi(s_2)$ maps s_2 to a side s_2^- and the ridge r_2 to a ridge r_3 in s_2^-. By continuing this, we get a sequence of ridges $r_1, r_2, \ldots$ and a sequence of sides $s_1, s_2, \ldots$ along with the side pairing maps. The relation between them is that $r_i = s_i \cap s_{i-1}^-$ and $P_i = \psi(s_i)$.

Since the number of ridges is finite, there exists a ridge r_m with $m > 1$ where r_{m+1} is in the same Υ-orbit as r_1. In the case that r_2 is in the same orbit as r_1, we let m be 0. With this, we claim that there is a unique $S \in \Upsilon$ such that $Sr_{m+1} = r_1$. To see this, suppose that $Sr_{m+1} = r_1 = S'r_{m+1}$ with $S \neq S'$. Then, $S^{-1}S'$ is not the identity, but it fixes r_{m+1} pointwise along with the sides containing it, which is impossible.

A **ridge cycle** of r_1 is the sequence $(r_1, r_2, \ldots, r_m)$. We also define the **cycle transformation** $T = T(r_1)$ of r_1 as the map $S \circ P_m \circ P_{m-1} \circ \cdots \circ P_2 \circ P_1$, which is a map fixing r_1 (but might not act as an identity on r_1). If T has finite order l, we call $T^l = id$, or simply T^l, **cycle relation** associated to r_1.

Lemma 1.2.1. For $i \geq 1$, $r_i = Sr_{m+i}$.

Proof. We are going to prove this using induction. Since ψ is compatible with Υ, we have $P_1 = \psi(s_1) = \psi(Ss_{m+1}) = SP_{m+1}S^{-1}$. We assume that if $P_k = SP_{m+k}S^{-1}$ and $r_k = Sr_{m+k}$, then

$$\begin{aligned} r_{k+1} &= P_k r_k \\ &= SP_{m+k}S^{-1}Sr_{m+k} \\ &= SP_{m+k}r_{m+k} \\ &= Sr_{m+k+1}, \end{aligned}$$

and

$$\begin{aligned} P_{k+1} &= \psi(r_{k+1}) \\ &= \psi(Sr_{m+k+1}) \\ &= S\psi(r_{m+k+1})S^{-1} \\ &= SP_{m+k+1}S^{-1}. \end{aligned}$$

Hence, the induction is fulfilled and we have $r_i = Sr_{m+i}$. $\square$

Note that, if we consider a ridge cycle of another ridge r_n in the ridge cycle of r_1, we end up with the ridge cycle $(r_n, r_{n+1}, \dots, r_m, S^{-1}r_1, S^{-1}r_2, \dots, S^{-1}r_{n-1})$ instead. This ridge cycle of r_n is just a shift of the ridge cycle of r_1 and the cycle transformation $T(r_n)$ is

$$\begin{aligned} &S \circ (S^{-1}P_{n-1}S) \circ (S^{-1}P_{n-2}S) \circ \dots \circ (S^{-1}P_1S) \circ P_m \circ P_{m-1} \circ \dots \circ P_n \\ &= P_{n-1} \circ P_{n-2} \circ \dots S \circ P_m \circ \dots \circ P_n, \end{aligned}$$

a cyclic permutation of T. If we switch the sides we picked at the first step, that is s_1 and s_0^-, then we end up with a ridge cycle with reversed order. Moreover, any action of Υ on the ridge r_1 will result in the same action on the ridge cycle as well and the cycle transformation becomes the conjugate of the original transformation T by that action. This just means that a single ridge in a ridge cycle is enough to determine the whole ridge cycle and to represent it.

We define $\mathfrak{C}(p_1)$ to be

$$\mathfrak{C}(r_1) = \{P_i \circ \dots \circ P_1 T^j : 0 \leq i \leq m-1, 0 \leq j \leq l-1\}.$$

If i is equal to 0, then that means there are no P_i terms and the same goes for when $m = 0$.

Lemma 1.2.2. For any element $C \in \mathfrak{C}(r_1)$, the ridge r_1 is contained in every image $C^{-1}(E)$ of E.

Proof. Since T is $S \circ P_m \circ \dots \circ P_1$,

$$T^{-1}(E) = P_1^{-1} \circ \dots \circ P_m^{-1} S^{-1}(E) = P_1^{-1} \circ \dots \circ P_m^{-1}(E).$$

By construction, we have that $r_i = s_i \cap s_{i-1}^-$ for $i \geq 1$. Since $P_i = \psi(s_i)$ and $P_{i-1}^{-1} = \psi(s_{i-1}^-)$, the side pairing condition says that $s_i = P_i^{-1}(E) \cap E$ and $s_{i-1}^- = P_{i-1}(E) \cap E$. Thus, $r_i \subseteq P_i^{-1}(E) \cap P_{i-1}(E) \cap E$.
For $i \geq 2$, we have $r_i = P_{i-1} \circ \dots \circ P_1(r_1)$. So,

$$\begin{aligned} r_1 &= P_1^{-1} \circ \dots \circ P_{i-1}^{-1}(r_i) \\ &\subseteq P_1^{-1} \circ \dots \circ P_{i-1}^{-1}(P_i^{-1}(E) \cap P_{i-1}(E) \cap E) \\ &\subseteq (P_1^{-1} \circ \dots \circ P_i^{-1}(E)) \cap (P_1^{-1} \circ \dots \circ P_{i-2}^{-1}(E)) \cap (P_1^{-1} \circ \dots \circ P_{i-1}^{-1}(E)). \end{aligned}$$

This means that r_1 is contained in $P_1^{-1} \circ \cdots \circ P_{i-1}^{-1}(E)$ for $i \geq 2$. Thus, it is in $T^{-1}(E)$ as well. Hence,

$$r_1 \subseteq \bigcap_{C \in \mathfrak{C}(r_1)} C^{-1}(E)$$

as we need. □

We say that E and Γ satisfy the **cycle condition** at r_1 if this intersection is precisely r_1 and all these copies of $\bigcup_{C \in \mathfrak{C}(r_1)} C^{-1}(E)$ tessellate around r. That is:

1. $r_1 = \bigcap_{C \in \mathfrak{C}(r_1)} C^{-1}(E)$.
2. If $C_1, C_2 \in \mathfrak{C}(r_1)$ with $C_1 \neq C_2$ then $C_1^{-1}(E^\circ) \cap C_2^{-1}(E^\circ) = \emptyset$.
3. For any element in r_1, there exists an open neighbourhood of its that is contained in $\bigcup_{C \in \mathfrak{C}(r_1)} C^{-1}(E)$.

Another thing to consider is the existence of cusps on the polyhedron. In such cases, it is necessary to consider its completeness when quotiented by the side pairing maps. If there are cusps on E, assume a **consistent system of horoballs** where each one is centred at a cusp and covers all the cusps. We can assume further that the horoballs are pairwise disjoint by choosing smaller ones. To show that such a system exists for a polyhedron, we can show that every cycle transformation that fixes a cusp is not loxodromic since, in that case, the horoballs based at the cusp are all preserved by the transformation.

We now state the complex hyperbolic Poincaré polyhedron theorem as used in [9](More on [14] and [19]).

Theorem 1.2.3. Suppose E is a smoothly embedded finite-sided polyhedron in $\mathbf{H}^2_{\mathbb{C}}$, together with a side pairing $\psi : F_1(E) \to \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$. Let $\Upsilon \leq \mathrm{Isom}(\mathbf{H}^2_{\mathbb{C}})$ be a group of automorphisms of E. Let Γ be the group generated by Υ and the side-pairing maps. Suppose the cycle condition is satisfied for each ridge in $F_2(E)$, and that there is a consistent system of horoballs at the cusps of E (if it has any). Then the images of E under the cosets of Υ in Γ tessellate $\mathbf{H}^2_{\mathbb{C}}$. That is

1. $\bigcup_{A \in \Gamma} A(E) = \mathbf{H}^2_{\mathbb{C}}$.
2. If $A \in \Gamma - \Upsilon$, then $E^\circ \cap A(E^\circ) = \emptyset$.

Moreover, Γ is discrete and a fundamental domain for its action on $\mathbf{H}^2_{\mathbb{C}}$ is obtained by intersecting E with a fundamental domain for Υ.

Finally, one obtains a presentation for Γ in terms of the generators given by the side pairing maps together with a generating set for Υ; the relations are given by the reflection relations, the cycle relations and the relations in a presentation for Υ.

Chapter 2

(2,4,4) and Deligne-Mostow groups

2.1 (2,4,4) groups

We explore groups of each type based on the braiding relations between the generators. Lemma 1.1.4 shows that odd braidings result in conjugate pairs of transformations, so we focus on the even braiding. The simplest ones are $(2,4,4)$-groups, i.e. groups generated by three complex reflections A, B, C with rotation angles $\frac{2\pi}{a}, \frac{2\pi}{b}, \frac{2\pi}{c}$, respectively, and braiding relations $\mathrm{br}(A,B) = \mathrm{br}(A,C) = 4, \mathrm{br}(B,C) = 2$. We assume A, B, C to have determinant 1 and write $\underline{n}_A, \underline{n}_B, \underline{n}_C$ as polar vectors to the mirrors of A, B, C, respectively. Furthermore, assume that all three of them do not share a common eigenvector.

Let A, B, C be in the same form as R_1, R_2, R_3 from earlier for the sake of defining their parameters. That is

$$A = \begin{pmatrix} e^{\frac{4\pi i}{3a}} & e^{\frac{\pi i}{3a}}\rho & -e^{\frac{\pi i}{3a}}\bar{\tau} \\ 0 & e^{\frac{-2\pi i}{3a}} & 0 \\ 0 & 0 & e^{\frac{-2\pi i}{3a}} \end{pmatrix},$$

$$B = \begin{pmatrix} e^{\frac{-2\pi i}{3b}} & 0 & 0 \\ -e^{\frac{\pi i}{3b}}\bar{\rho} & e^{\frac{4\pi i}{3b}} & e^{\frac{\pi i}{3b}}\sigma \\ 0 & 0 & e^{\frac{-2\pi i}{3b}} \end{pmatrix},$$

$$C = \begin{pmatrix} e^{\frac{-2\pi i}{3c}} & 0 & 0 \\ 0 & e^{\frac{-2\pi i}{3c}} & 0 \\ e^{\frac{\pi i}{3c}}\tau & -e^{\frac{\pi i}{3c}}\bar{\sigma} & e^{\frac{4\pi i}{3c}} \end{pmatrix}$$

According to the equation 1.6, the commutativity between B and C mean that the parameter linking the two (ρ in the equation 1.6) has to be zero. The third case in the Proposition 1.1.5 says that the moduli of parameters ρ, σ, τ determine the group up to conjugacy.

Lemma 2.1.1. The transformation $(AB)^2$ is a complex reflection whose order b' satisfies $\frac{1}{2} + \frac{1}{a} + \frac{1}{b} = \frac{k_b}{b'}$ for some integer k_b.

Proof. The 4-braiding of A and B implies that $(AB)^2$ commutes with both of them. The polar to mirror eigenvectors of theirs then generate a two-dimensional eigenspace for $(AB)^2$. Therefore, $(AB)^2$ is also a complex reflection.

Referring to 1.1, the trace of AB is $e^{-\frac{2\pi i}{3a}-\frac{2\pi i}{3b}}$. Since one of its eigenvalues is $e^{-\frac{2\pi i}{3a}-\frac{2\pi i}{3b}}$, the fact that it is also unimodular implies that its other eigenvalues are $ie^{\frac{\pi i}{3a}+\frac{\pi i}{3b}}$ and $-ie^{\frac{\pi i}{3a}+\frac{\pi i}{3b}}$. Thus, the eigenvalues of $(AB)^2$ are $e^{-\frac{4\pi i}{3a}-\frac{4\pi i}{3b}}$ and (repeated) $e^{\pi i+\frac{2\pi i}{3a}+\frac{2\pi i}{3b}}$. If b' is its order, then

$$\frac{1}{2} + \frac{1}{a} + \frac{1}{b} = \frac{k_b}{b'} \tag{2.1}$$

for some integer k_b. □

Symmetrically, the order of $(AC)^2$, c', satisfies

$$\frac{1}{2} + \frac{1}{a} + \frac{1}{c} = \frac{k_c}{c'} \tag{2.2}$$

for some integer k_c.

Lemma 2.1.2. The transformation $D = (BACA)^{-1}$ is a complex reflection of order d where $\frac{1}{2} + \frac{1}{a} = \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$.

Proof. We know that B has eigenvalues $e^{\frac{4\pi i}{3b}}$, associated to eigenvector n_B, and another eigenvalue $e^{\frac{-2\pi i}{3b}}$ while C has eigenvalues $e^{\frac{4\pi i}{3c}}$, associated to eigenvector n_C, and $e^{\frac{-2\pi i}{3c}}$. Since B and C commute, n_C has to be an eigenvector of B and thus, associates with the eigenvalue $e^{\frac{-2\pi i}{3b}}$, while n_B is the eigenvector associated to the eigenvalue $e^{\frac{-2\pi i}{3c}}$ of C.

The transformation $BACA$ can be written as $BC^{-1}(CACA)$. Since A braids with length 4 with C, $(CA)^2$ commutes with C and, by the same argument as above, n_C is also one of its eigenvectors and thus n_C is a common eigenvector for $B, C^{-1}, CACA$ associated with eigenvalues $e^{\frac{-2\pi i}{3b}}, e^{\frac{-4\pi i}{3c}}, e^{i\pi+\frac{2\pi i}{3a}+\frac{2\pi i}{3c}}$, respectively. Thus, n_C is an eigenvector of $BACA$ associated to eigenvalue $e^{i\pi+\frac{2\pi i}{3a}-\frac{2\pi i}{3b}-\frac{2\pi i}{3c}}$.

We can also write $BACA$ as $(BABA)(A^{-1}B^{-1}A)(A^{-1}CA)$. Again, we have that n_B is an eigenvector of $BABA$ and, since A commutes with $BABA$, so is $A^{-1}n_B$ (with the same eigenvalue as well). Thus, $A^{-1}n_B$ is the common eigenvector for $BABA, A^{-1}B^{-1}A$, $A^{-1}CA$ with eigenvalues $e^{i\pi+\frac{2\pi i}{3a}+\frac{2\pi i}{3b}}, e^{\frac{-4\pi i}{3b}}, e^{\frac{-2\pi i}{3c}}$, respectively. This means it is an eigenvector of $BACA$ with eigenvalue $e^{i\pi+\frac{2\pi i}{3a}-\frac{2\pi i}{3b}-\frac{2\pi i}{3c}}$.

Since n_C and $A^{-1}n_B$ share the same eigenvalue, in the case that they are linearly independent, we have ourselves a two-dimensional eigenspace and we have a one-dimensional eigenspace with eigenvalue

$$e^{\frac{-4\pi i}{3a}+\frac{4\pi i}{3b}+\frac{4\pi i}{3c}}.$$

Hence, $D = (BACA)^{-1}$ is a complex reflection.

Suppose otherwise that they are linearly dependent. We consider the reflection $(AB)^2$. It commutes with both A and B, and thus, with $A^{-1}BA$. This means that their polar to mirror eigenvectors $n_{(AB)^2}, A^{-1}n_B$ are orthogonal. Since n_C and $A^{-1}n_B$ are linearly dependent, n_C is also orthogonal to $n_{(AB)^2}$. Hence, $n_{(AB)^2}$ is a common eigenvector for A, B and C, a contradiction. □

We can see that, because

$$\begin{aligned}
(AD)^2(A^{-1}D^{-1})^2 &= (C^{-1}A^{-1}B^{-1})^2(A^{-1}BACA)^2\\
&= (C^{-1}A^{-1}B^{-1}C^{-1}A^{-1}B^{-1})(A^{-1}BACBACA)\\
&= C^{-1}A^{-1}B^{-1}C^{-1}(A^{-1}B^{-1}A^{-1}B)ACBACA\\
&= C^{-1}A^{-1}B^{-1}C^{-1}(BA^{-1}B^{-1}A^{-1})ACBACA\\
&= C^{-1}A^{-1}B^{-1}C^{-1}BA^{-1}B^{-1}CBACA\\
&= C^{-1}A^{-1}B^{-1}BC^{-1}A^{-1}CB^{-1}BACA\\
&= C^{-1}A^{-1}C^{-1}A^{-1}CACA\\
&= I,
\end{aligned}$$

the reflections A and D braid with length 4. Similarly to how we assume the order b' and c' for $(AB)^2$ and $(AC)^2$, respectively, the order d' of $(AD)^2$ satisfies

$$\frac{1}{2}+\frac{1}{a}+\frac{1}{d}=\frac{k_d}{d'} \tag{2.3}$$

for some integer k_d. Another relation we have is

$$\begin{aligned}
CD &= CA^{-1}C^{-1}A^{-1}B^{-1}\\
&= A^{-1}C^{-1}A^{-1}CB^{-1}\\
&= A^{-1}C^{-1}A^{-1}B^{-1}C\\
&= DC.
\end{aligned}$$

That is $\mathrm{br}(C, D) = 2$. This gives us the presentation

$$\Gamma_{2,4,4} = \left\langle A,B,C,D : \begin{array}{l} A^a \ = B^b \ = C^c \ = D^{\frac{2abc}{abc+2bc-2ab-2ac}} = BACAD \ = I, \\ (AB)^{\frac{4abk_b}{ab-2a-2b}} = (AC)^{\frac{4ack_c}{ac-2a-2c}} = (AD)^{\frac{4abck_d}{-2bc+ab+ac}} = I, \\ \mathrm{br}_4(A,B),\ \mathrm{br}_4(A,C),\ \mathrm{br}_4(A,D),\ \mathrm{br}_2(B,C), \mathrm{br}_2(C,D) \end{array} \right\rangle.$$

Another noteworthy transformation here is $C^{-1}AC$. It has the same order a as A. See that

$$\begin{aligned}\mathrm{br}(C^{-1}AC, B) &= \mathrm{br}(A, CBC^{-1})\\ &= \mathrm{br}(A, B)\\ &= 4,\end{aligned}$$

and

$$\begin{aligned}\mathrm{br}(C^{-1}AC, C) &= \mathrm{br}(A, C)\\ &= 4.\end{aligned}$$

We want to show that, by construction, the orders b, c, d of B, C, D are actually symmetric in the sense that there are groups of B, C, D's conjugates, in different orders, whose braiding relations and the relation $BACAD = I$ (when replaced by the conjugates in their respective sequential orders) still hold. Namely, groups that are almost homomorphic to $\Gamma_{2,4,4}$ but with permuted generators' orders, except for a.

To do that, it is enough to show that there are such groups whose generators' orders are (a, d, c, b) and (a, b, d, c). Let $\Gamma'_{2,4,4}$ be the presentation similar to $\Gamma_{2,4,4}$ without the order relations, that is

$$\Gamma'_{2,4,4} = \left\langle A, B, C, D : \begin{array}{l} BACAD \ = I, \\ \mathrm{br}_4(A, B),\ \mathrm{br}_4(A, C),\ \mathrm{br}_2(B, C) \end{array} \right\rangle.$$

As the relations $\mathrm{br}_4(A, D)$ and $\mathrm{br}_2(C, D)$ are inferred only by the braiding relations, they are true in the presentation, thus

$$\Gamma'_{2,4,4} = \left\langle A, B, C, D : \begin{array}{l} BACAD \ = I, \\ \mathrm{br}_4(A, B),\ \mathrm{br}_4(A, C),\ \mathrm{br}_4(A, D), \\ \mathrm{br}_2(B, C),\ \mathrm{br}_2(C, D) \end{array} \right\rangle.$$

Define the maps $\phi_{bd} : \Gamma'_{2,4,4} \to \langle A, D, C, A^{-1}C^{-1}A^{-1}BACA \rangle$,
$\phi_{cd} : \Gamma'_{2,4,4} \to \langle A, C^{-1}A^{-1}BAC, D, C \rangle$ so that

$$\begin{aligned}
\phi_{bd}(A) &= A, & \phi_{cd}(A) &= A,\\
\phi_{bd}(B) &= D, & \phi_{cd}(B) &= C^{-1}A^{-1}BAC,\\
\phi_{bd}(C) &= C, & \phi_{cd}(C) &= D,\\
\phi_{bd}(D) &= A^{-1}C^{-1}A^{-1}BACA, & \phi_{cd}(D) &= C.
\end{aligned}$$

Lemma 2.1.3. The map $\phi_{bd} : \Gamma'_{2,4,4} \to \langle A, D, C, A^{-1}C^{-1}A^{-1}BACA \rangle$ is an isomorphism.

Proof. Since A and C stay the same in both presentations, we have $\mathrm{br}(A, C) = 4$. We also assume $\mathrm{br}(A, D) = 4$ and $\mathrm{br}(C, D) = 2$ in both directions of the proof. Then,

$$\begin{aligned} \mathrm{br}(C, A^{-1}C^{-1}A^{-1}BACA) &= \mathrm{br}(ACACA^{-1}C^{-1}A^{-1}, B) \\ &= \mathrm{br}(C, B). \end{aligned}$$

Thus, $\mathrm{br}(C, A^{-1}C^{-1}A^{-1}BACA) = 2$ if and only if $\mathrm{br}(B, C) = 2$. Next,

$$\begin{aligned} \mathrm{br}(A, A^{-1}C^{-1}A^{-1}BACA) &= \mathrm{br}(ACAAA^{-1}C^{-1}A^{-1}, B) \\ &= \mathrm{br}(ACAC^{-1}A^{-1}, B) \\ &= \mathrm{br}(C^{-1}AC, B) \\ &= \mathrm{br}(A, CBC^{-1}) \\ &= \mathrm{br}(A, B). \end{aligned}$$

As the rest of the relations follow from these relations, the map is an isomorphism. □

Lemma 2.1.4. The map $\phi_{cd} : \Gamma'_{2,4,4} \to \langle A, C^{-1}A^{-1}BAC, D, C\rangle$ is an isomorphism.

Proof. To show that the presentations are equivalent, we identify the relations in each one. We already have $\mathrm{br}_4(A, D)$. Next, consider

$$\begin{aligned} \mathrm{br}(A, C^{-1}A^{-1}BAC) &= \mathrm{br}(CAC^{-1}, A^{-1}BA) \\ &= \mathrm{br}(ACAC^{-1}A^{-1}, B) \\ &= \mathrm{br}(C^{-1}AC, B) \\ &= 4. \end{aligned}$$

We also get

$$\begin{aligned} (C^{-1}A^{-1}BAC)^{-1}D(C^{-1}A^{-1}BAC)D^{-1} &= C^{-1}A^{-1}B^{-1}ACDC^{-1}A^{-1}BACD^{-1} \\ &= (C^{-1}A^{-1}B^{-1})A(CDC^{-1})A^{-1}(BAC)D^{-1} \\ &= (AD)ADA^{-1}(D^{-1}A^{-1})D^{-1} \\ &= ADADA^{-1}D^{-1}A^{-1}D^{-1} \\ &= I. \end{aligned}$$

Thus, $\mathrm{br}(C^{-1}A^{-1}BAC, D) = 2$. Lastly, similar to $BACAD = I$, we have

$$\begin{aligned} \left((C^{-1}AC)^{-1}B(C^{-1}AC)\right) ADAC &= C^{-1}A^{-1}CB(C^{-1}ACA)DAC \\ &= C^{-1}A^{-1}CB(ACAC^{-1})DAC \\ &= C^{-1}A^{-1}C(BACA)(C^{-1}D)AC \\ &= C^{-1}A^{-1}CD^{-1}(DC^{-1})AC \\ &= I. \end{aligned}$$

Hence, the two presentations are equivalent. □

Since the 3 presentations are presentations of the same group, albeit with different generators, the transformations ϕ_{bd} and ϕ_{cd} are automorphisms on $\Gamma'_{2,4,4}$.

The group $\Gamma_{2,4,4}$ is just the group $\Gamma'_{2,4,4}$ equipped with the generators' orders. The orders of generators (A, B, C, D) are (a, b, c, d), while $(A, D, C, A^{-1}C^{-1}A^{-1}BACA)$'s are (a, d, c, b) and $(A, C^{-1}A^{-1}BAC, D, C)$'s are (a, b, d, c). The rest of the orders follow from the previously assumed conditions. Thus, the parameters b, c and d are symmetric when the groups' generators are complex reflections without a common fixed point.

2.2 Deligne-Mostow groups

We will try to identify our groups with Deligne-Mostow groups, so we give some definitions on the terms used.

Definition 2.2.1. An N-tuple of real numbers $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ with $0 < \mu_n < 1$ for $n = 1, 2, \ldots, N$ is called a **ball N-tuple** if

$$\sum_{n=1}^{N} \mu_n = 2.$$

Definition 2.2.2. A ball N-tuple $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ is said to satisfy the ΣINT condition if there is a subset S of $\{1, 2, \ldots, N\}$ such that for each pair of $m, n \in \{1, 2, \ldots, N\}$ with $\mu_m + \mu_n < 1$, either

1. $1 - \mu_m - \mu_n = \frac{1}{\mu_{m,n}}$ where $\mu_{m,n} \in \mathbb{Z}$, or
2. $m, n \in S$ and $\frac{1}{2} - \mu_m = \frac{1}{2} - \mu_n = \frac{1}{\mu_{m,n}}$ where $\mu_{m,n} \in \mathbb{Z}$.

In [26], Thurston represented the lattices in terms of cone angles instead. This is done by considering a sphere with N cone singularities with angles $0 \leq \theta_i \leq 2\pi$ for $i = 1, 2, \ldots, N$, satisfying the discrete Gauss-Bonnet formula, i.e. the summation of the curvatures $2\pi - \theta_i$ at the cone point is 4π. The moduli space of cone metrics according to these angles with area 1 has a complex hyperbolic structure of dimension $N - 3$. Thurston got the condition for the cone points to lead to a lattice, which is equivalent to the ΣINT condition when $\theta_n = 2\pi(1 - \mu_n)$.

In [21], Pasquinelli gave the lattices based on the cone angles $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$. We will follow suit and refer to those lattices as references. Since they satisfy the discrete Gauss-Bonnet formula, the last term can be determined by the other four, and they can instead be represented by just 4 parameters. She let $\alpha, \beta, \theta, \phi$ be

$$\alpha = \frac{\theta_2}{2},\ \beta = \frac{\theta_3}{2},\ \theta = \frac{\theta_3}{2} + \frac{\theta_4}{2} - \pi,\ \phi = \frac{\theta_1}{2} + \frac{\theta_2}{2} - \pi.$$

Then, a cone metric corresponding to these angles is denoted by $(\alpha, \beta, \theta, \phi)$ and the sphere has the 5 cone singularities of angles

$$(2(\pi + \phi - \alpha), 2\alpha, 2\beta, 2(\pi + \theta - \beta), 2(\pi - \theta - \phi)).$$

In terms of the ball 5-tuples, μ_i is $1 - \frac{\theta_i}{2\pi}$. The maps (called moves) are also defined to move around these parameters:

$$R_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta}\frac{\sin\beta}{\sin(\beta-\theta)} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_2' = \begin{pmatrix} \sin\alpha\sin\theta e^{i(\alpha-\phi)} & \sin(\alpha-\phi)\sin\theta e^{i\alpha} & -\sin(\alpha-\phi)\sin\theta e^{i\alpha} \\ \sin(\beta-\theta)\sin\phi e^{i\beta} & \sin\phi\sin\beta e^{i(\beta-\theta)} & -\sin(\beta-\theta)\sin\phi e^{i\beta} \\ \sin(\theta+\phi)\sin\alpha e^{i\beta} & \sin(\theta+\phi)\sin\beta e^{i\alpha} & A \end{pmatrix},$$

$$J = \begin{pmatrix} \sin\alpha\sin\theta e^{i(\alpha+\phi)} & \sin(\alpha-\phi)\sin\theta e^{i\alpha} & -\sin(\alpha-\phi)\sin\theta e^{i\alpha} \\ \sin\alpha\sin\phi e^{i(\alpha+\theta+2\phi)} & \frac{\sin\phi\sin\alpha\sin\alpha}{\sin(\alpha-\theta)}e^{i\alpha} & -\sin\alpha\sin\phi e^{i(\alpha+\theta)} \\ \sin(\theta+\phi)\sin\alpha e^{i(\alpha+2\phi)} & \sin(\theta+\phi)\sin\alpha e^{i\alpha} & A \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} e^{2i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $A = \sin\theta\sin\phi - \sin(\theta+\phi)\sin\beta e^{i\alpha}$.

2.2.1 2-fold symmetry case

Pasquinelli is also interested in the case of 2-fold symmetry, namely when two of the cone points have the same cone angles. Since the cone angles are arbitrary, any pair of the angles would do. For convenience's sake, we assume $\alpha = \beta$, so that the cone angles are

$$(2(\pi + \phi - \alpha), 2\alpha, 2\alpha, 2(\pi + \theta - \alpha), 2(\pi - \theta - \phi)).$$

Also, let $R_0 = R_2' R_1' R_2'^{-1}$ and the parameters $p^*, p', k, k', l, l', d^*$ be

$$\begin{array}{llll} \frac{\pi}{p^*} = \theta, & \frac{\pi}{k} = \phi, & \frac{\pi}{l} = \alpha - \theta - \phi, & \frac{\pi}{d^*} = \pi - \alpha - \theta, \\ \frac{\pi}{p'} = \alpha - \frac{\pi}{2}, & \frac{\pi}{k'} = \pi + \theta + \phi - 2\alpha, & \frac{\pi}{l'} = \pi - \alpha - \phi. & \end{array}$$

So,

$$\frac{1}{l} = \frac{1}{2} - \frac{1}{p^*} - \frac{1}{k} + \frac{1}{p'}, \tag{2.4}$$

$$\frac{1}{l'} = \frac{1}{2} - \frac{1}{k} - \frac{1}{p'}, \tag{2.5}$$

$$\frac{1}{k'} = \frac{1}{p^*} + \frac{1}{k} - \frac{2}{p'}, \tag{2.6}$$

$$\frac{1}{d^*} = \frac{1}{2} - \frac{1}{p^*} - \frac{1}{p'}, \tag{2.7}$$

$$A_1^k = B_1^{p^*} = I. \tag{2.8}$$

The cone angles in terms of $p^*, p', k, k', l, l', d^*$ are then

$$\left(2\left(\frac{\pi}{2} + \frac{\pi}{k} - \frac{\pi}{p'}\right), \pi + \frac{2\pi}{p'}, \pi + \frac{2\pi}{p'}, 2\left(\frac{\pi}{2} + \frac{\pi}{p^*} - \frac{\pi}{p'}\right), 2\left(\pi - \frac{\pi}{p^*} - \frac{\pi}{k}\right)\right),$$

and the ball 5-tuple is

$$\left(\frac{1}{2} - \frac{1}{k} + \frac{1}{p'}, \frac{1}{2} - \frac{1}{p'}, \frac{1}{2} - \frac{1}{p'}, \frac{1}{2} - \frac{1}{p^*} + \frac{1}{p'}, \frac{1}{p^*} + \frac{1}{k}\right).$$

For this to satisfy the ΣINT condition, we want a subset S of $\{1,2,3,4,5\}$ with $\mu_m = \mu_n$ for all $m, n \in S$. Since we already have two equal parameters, μ_2 and μ_3, we let $S = \{2,3\}$.

Lemma 2.2.3. The ball 5-tuple $\left(\frac{1}{2} - \frac{1}{k} + \frac{1}{p'}, \frac{1}{2} - \frac{1}{p'}, \frac{1}{2} - \frac{1}{p'}, \frac{1}{2} - \frac{1}{p^*} + \frac{1}{p'}, \frac{1}{p^*} + \frac{1}{k}\right)$ satisfies the ΣINT condition with $S = \{2,3\}$. if and only if the paraneters $p^*, p', k, k', l, l', d^*$ are in the set $\mathbb{R}^- \cup \mathbb{Z}^+ \cup \{\infty\}$.

Proof. For $m = 2, n = 3$ we have

$$\mu_{2,3} = \frac{1}{\frac{1}{2} - (\frac{1}{2} - \frac{1}{p'})} = p'.$$

For other pairs, we have:

$$\begin{aligned}
\mu_{1,2} = \mu_{1,3} &= \frac{1}{1-(\frac{1}{2}-\frac{1}{k}+\frac{1}{p'})-(\frac{1}{2}-\frac{1}{p'})} &= k,\\
\mu_{1,4} &= \frac{1}{1-(\frac{1}{2}-\frac{1}{k}+\frac{1}{p'})-(\frac{1}{2}-\frac{1}{p^*}+\frac{1}{p'})} &= k',\\
\mu_{1,5} &= \frac{1}{1-(\frac{1}{2}-\frac{1}{k}+\frac{1}{p'})-(\frac{1}{p^*}+\frac{1}{k})} &= d^*,\\
\mu_{2,4} = \mu_{3,4} &= \frac{1}{1-(\frac{1}{2}-\frac{1}{p'})-(\frac{1}{2}-\frac{1}{p^*}+\frac{1}{p'})} &= p^*,\\
\mu_{2,5} = \mu_{3,5} &= \frac{1}{1-(\frac{1}{2}-\frac{1}{p'})-(\frac{1}{p^*}+\frac{1}{k})} &= l,\\
\mu_{4,5} &= \frac{1}{1-(\frac{1}{2}-\frac{1}{p^*}+\frac{1}{p'})-(\frac{1}{p^*}+\frac{1}{k})} &= l'.
\end{aligned}$$

In the case that a parameter is negative(or ∞), then there are $m, n \in \{1,2,3,4,5\}$ such that $\mu_{m,n}$ is negative(∞), that is, $\mu_m + \mu_n > 1 (= 1)$. Either case satisfies the ΣINT condition.

In the positive case, we have $\mu_{m,n} < 1$. The ΣINT condition is satisfied if and only if that parameter is also integer.

$\square$

As stated in [21], the presentation of the 2-fold symmetry case can be written as

$$\Gamma = \left\langle B_1,\, R_0,\, A_1 : \begin{array}{l} B_1^{p^*} = R_0^{p'} = (B_1R_0A_1)^{2k'} = A_1^k = (R_0B_1R_0A_1)^l = I, \\ (A_1R_0)^{2l'} = (B_1R_0)^{2d^*} = I, \\ \mathrm{br}_4(B_1,R_0),\ \mathrm{br}_2((B_1R_0A_1)^{-2},R_0),\ \mathrm{br}_2(A_1,B_1) \end{array} \right\rangle.$$

Proposition 2.2.4. The presentation

$$\Gamma_{2,4,4} = \left\langle A,B,C,D : \begin{array}{l} A^a = B^b = C^c = D^{\frac{2abc}{abc+2bc-2ab-2ac}} = BACAD = I, \\ (AB)^{\frac{4abk_b}{ab-2a-2b}} = (AC)^{\frac{4ack_c}{ac-2a-2c}} = (AD)^{\frac{4abck_d}{-2bc+ab+ac}} = I, \\ \mathrm{br}_4(A,B),\ \mathrm{br}_4(A,C),\ \mathrm{br}_2(B,C) \end{array} \right\rangle$$

is equivalent to

$$\Gamma = \left\langle B_1,\, R_0,\, A_1 : \begin{array}{l} B_1^{p^*} = R_0^{p'} = (B_1R_0A_1)^{2k'} = A_1^k = (R_0B_1R_0A_1)^l = I, \\ (A_1R_0)^{2l'} = (B_1R_0)^{2d^*} = I, \\ \mathrm{br}_4(B_1,R_0),\ \mathrm{br}_2((B_1R_0A_1)^{-2},R_0),\ \mathrm{br}_2(A_1,B_1) \end{array} \right\rangle.$$

Moreover, the parameters $p^*, p', k, k', l, l', d^*$ correspond to the parameters $b, a, c, \frac{2abck_d}{-2bc+ab+ac}$, d, $\frac{4ack_c}{ac-2a-2c}$, $\frac{4abk_b}{ab-2a-2b}$, respectively.

Proof. Identify A, B, C with R_0, B_1, A_1, respectively. Then, their orders a, b, c correspond respectively with the orders p', p^*, k.

Assume the properties of A, B, C and D according to $\Gamma_{2,4,4}$. Then, we have

$$\begin{aligned} B_1R_0A_1 &\equiv BAC \\ &= (AD)^{-1}, \\ R_0B_1R_0A_1 &\equiv ABAC \\ &= AD^{-1}A^{-1}, \\ A_1R_0 &= CA, \\ B_1R_0 &= BA. \end{aligned}$$

This gives the relations for the rest of the orders. Now, we have $\mathrm{br}(B_1, R_0) = \mathrm{br}(A, B) = 4$ and $\mathrm{br}(A_1, B_1) = \mathrm{br}(B, C) = 2$. So, all that is left is to show $\mathrm{br}((B_1R_0A_1)^{-2}, R_0) = \mathrm{br}((BAC)^{-2}, A)$. See that

$$\begin{aligned} A^{-1}(BAC)^{-2}A(BAC)^2 &= A^{-1}C^{-1}A^{-1}(B^{-1}C^{-1})A^{-1}B^{-1}ABA(CB)AC \\ &= A^{-1}C^{-1}A^{-1}(C^{-1}B^{-1})A^{-1}B^{-1}ABA(BC)AC \\ &= A^{-1}C^{-1}A^{-1}C^{-1}B^{-1}A^{-1}B^{-1}(ABAB)CAC \\ &= A^{-1}C^{-1}A^{-1}C^{-1}B^{-1}A^{-1}B^{-1}(BABA)CAC \\ &= A^{-1}C^{-1}A^{-1}C^{-1}ACAC \end{aligned}$$

This means that $\mathrm{br}((BAC)^{-2}, A) = 2$ if and only if $\mathrm{br}(A, C) = 4$. Note that we only use the other two braiding relations in proving this. Thus, $\mathrm{br}((BAC)^{-2}, A) = \mathrm{br}((BAC)^2, A) = 2$ as required.

Conversely, assume the conditions for B_1, R_0, A_1 as in Γ. Since the existence of D and all the orders in $\Gamma_{2,4,4}$ result from the braiding relations of A, B, C, it is enough to prove that the braiding relations hold. We have $\mathrm{br}(A, B) = \mathrm{br}(B_1, R_0) = 4$ and $\mathrm{br}(B, C) = \mathrm{br}(A_1, B_1) = 2$ already. As mentioned before, these two braiding relations imply that the relation $\mathrm{br}(A_1, R_0) = 4$ is equivalent to $\mathrm{br}((B_1R_0A_1)^{-2}, R_0)$. Hence, the two presentations are equivalent. Because the orders of each transformation is either arbitrary or is a result of the braiding relations, they are naturally identified. $\square$

Since the orders of the two presentations correspond to each others, the relations between the orders in one presentation would also be the same as those in another. Thus, for the ΣINT condition to hold in the presentation $\Gamma_{2,4,4}$, according to the result we have from Γ, the integers k_b, k_c, k_d from 2.1,2.2,2.3 should all be 1. Hence,

$$b' = \frac{2ab}{ab - 2a - 2b},$$
$$c' = \frac{2ac}{ac - 2a - 2c},$$
$$d' = \frac{2ad}{ad - 2a - 2d}.$$

From this point on, we let b', c', d' be as such, and $d = \frac{2abc}{abc+2bc-2ab-2ac}$.

The ball 5-tuple we have in this case is

$$\left(\frac{1}{2} + \frac{1}{a} - \frac{1}{c}, \frac{1}{2} - \frac{1}{a}, \frac{1}{2} - \frac{1}{a}, \frac{1}{2} + \frac{1}{a} - \frac{1}{b}, \frac{1}{b} + \frac{1}{c}\right).$$

Basing on Lemma 2.2.3, the ΣINT condition is satisfied if the corresponding parameters in this presentation, a, b, c, d, b', c', d', are integers whenever they are positive. Also, note that the set of parameters (a, b, c, d) is the same as $(a, \phi(b), \phi(c), \phi(d))$ where ϕ is a permutation of b, c, d. Hence, the ball 5-tuple associated with them are also the same (after permuting some of the cone points).

2.2.2 3-fold Symmetry

To get a 3-fold symmetry, we want another element of the ball 5-tuple to be $\frac{1}{2} - \frac{1}{a}$. We have shown earlier that b, c, d are symmetric, meaning that it does not matter which element in the ball 5-tuple we choose. In this case, we choose $\frac{1}{2} + \frac{1}{a} - \frac{1}{b}$. Then, $b = \frac{a}{2}$.

Lemma 2.2.5. If $b = \frac{a}{2}$, then $d = c'$ and $c = d'$.

Proof. Consider the definitions for d and d', we have

$$\begin{aligned}\frac{1}{d} &= \frac{1}{2} + \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \\ &= \frac{1}{2} + \frac{1}{a} - \frac{2}{a} - \frac{1}{c} \\ &= \frac{1}{2} - \frac{1}{a} - \frac{1}{c} \\ &= \frac{1}{c'},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{d'} &= \frac{1}{2} - \frac{1}{a} - \frac{1}{d} \\ &= \frac{1}{2} - \frac{1}{a} - \frac{1}{c'} \\ &= \frac{1}{c}.\end{aligned}$$

$\square$

Lemma 2.2.6. The ball 5-tuple $\left(\frac{1}{2}+\frac{1}{a}-\frac{1}{c}, \frac{1}{2}-\frac{1}{a}, \frac{1}{2}-\frac{1}{a}, \frac{1}{2}-\frac{1}{a}, \frac{2}{a}+\frac{1}{c}\right)$ satisfies the ΣINT condition with $S=\{2,3,4\}$. if and only if the parameters a, c, d, b' are in the set $\mathbb{R}^- \cup \mathbb{Z}^+ \cup \{\infty\}$.

Proof. For the indices in $S=\{2,3,4\}$, we have

$$m_{2,3}=m_{2,4}=m_{3,4}=\frac{1}{\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{a}\right)}=a,$$

For other pairs, we have:

$$\begin{aligned} n_{1,2}=n_{1,3}=n_{1,4}=&\frac{1}{1-\left(\frac{1}{2}+\frac{1}{a}-\frac{1}{c}\right)-\left(\frac{1}{2}-\frac{1}{a}\right)} &=c,\\ n_{1,5}=&\frac{1}{1-\left(\frac{1}{2}+\frac{1}{a}-\frac{1}{c}\right)-\left(\frac{2}{a}+\frac{1}{c}\right)} &=b',\\ n_{2,5}=n_{3,5}=n_{4,5}=&\frac{1}{1-\left(\frac{1}{2}-\frac{1}{a}\right)-\left(\frac{2}{a}+\frac{1}{c}\right)} &=d. \end{aligned}$$

Similarly to Lemma 2.2.3, we can conclude that the ΣINT condition with $S=\{2,3,4\}$ is satisfied if and only if each parameter is an integer whenever it is positive.

□

We assume that the whole group does not have a common fix point. Moreover, let there be a map R whose square is B while sharing the same mirror as B and is conjugate to A (which always exists in case of complex reflections).

Lemma 2.2.7. Let A, R be non-commuting complex reflections in $\mathrm{SL}(2,1)$. Then, $\mathrm{br}_3(A,R)$ if and only if $\mathrm{br}_4(A,R^2)$ and A, R are conjugates.

Proof. See that the braiding relation $\mathrm{br}_3(A,R)$ means

$$\begin{aligned} AR^2AR^2A^{-1}R^{-2}A^{-1}R^{-2} &= AR(RAR)RA^{-1}R^{-1}(R^{-1}A^{-1}R^{-1})R^{-1}\\ &= AR(ARA)RA^{-1}R^{-1}(A^{-1}R^{-1}A^{-1})R^{-1}\\ &= (ARA)(RAR)A^{-1}R^{-1}A^{-1}R^{-1}A^{-1}R^{-1}\\ &= (RAR)(ARA)A^{-1}R^{-1}A^{-1}R^{-1}A^{-1}R^{-1}\\ &= I. \end{aligned}$$

Thus, $\mathrm{br}_3(A,R)$ implies $\mathrm{br}_4(A,R^2)$. Also, $A = RARA^{-1}R^{-1}$ is a conjugate of R.

Conversely, suppose that $\mathrm{br}_4(A,R^2)$ and that A, R are conjugates. Since they are conjugates, they share the same angle, say θ. Then, their eigenvalues are u^2, u^{-1}, u^{-1} where $u=e^{i\theta}$. The relation $\mathrm{br}_4(A,R^2)$ implies that AR^2AR^2 commutes with both A

and R^2 and, therefore, is a complex reflection.

Let n_{AR} be a polar vector of AR^2AR^2. It is then in the mirrors of A and R^2. Following what we did with R_1, R_2 in 1.1, we get that the trace of AR^2 is of the form

$$v^2 - 2v^{-1}\cos\left(\frac{2\pi}{4}\right) = v^2$$

where v^2 is its eigenvalue of the polar eigenvector. Now, consider the eigenvectors of R. There is n_{AR} lying in the mirror of R. Suppose that the eigenvector n_R is polar to the mirror of R and m_R is another eigenvector in its mirror. Then,

$$\begin{aligned} Rn_R &= u^2 n_R, \\ Rm_R &= u^{-1} m_R, \\ Rn_{AR} &= u^{-1} n_{AR}. \end{aligned}$$

We can write the matrix R with respect to this basis as

$$R = \begin{pmatrix} u^2 & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & u^{-1} \end{pmatrix}.$$

Now, since $An_{AR} = u^{-1}n_{AR}$, we have

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & u^{-1} \end{pmatrix}.$$

Since we know that the trace of A is $u^2 + u^{-1} + u^{-1}$, we have that $a + d = u^2 + u^{-1}$.

Consider that

$$\begin{aligned} AR^2 n_{AR} &= A(u^{-2} n_{AR}) \\ &= u^{-3} n_{AR}. \end{aligned}$$

Since n_{AR} is the polar vector of AR^2, we have $v^2 = u^{-3}$ and the trace of AR^2 is u^{-3}. Compare this to the trace we get from the product of A and R^2, we have

$$u^{-3} = au^4 + du^{-2} + u^{-3}.$$

Thus, $au^4 + du^{-2} = 0$. We can then solve for a and d as

$$a = \frac{u^{-1}}{1 - u^3}, d = \frac{-u^5}{1 - u^3}.$$

Now that we have these values, the trace of AR is

$$\begin{aligned} tr(AR) &= au^2 + du^{-1} + u^{-2} \\ &= \frac{u}{1-u^3} + \frac{-u^4}{1-u^3} + u^{-2} \\ &= u + u^{-2} \\ &= -2u\cos\left(\frac{2\pi}{3}\right) + u^{-2}. \end{aligned}$$

Hence, A and R braid with length 3. □

Since R shares B's mirror, it also commutes with C.

Lemma 2.2.8. The transformation $(AR)^3$ is the same as $(AB)^2$, and thus, is a complex reflection of order b'.

Proof. See that

$$\begin{aligned} (AR)^3 &= AR(ARA)R \\ &= AR(RAR)R \\ &= AR^2AR^2 \\ &= (AB)^2. \end{aligned}$$

Hence, it is a complex reflection of order b'. □

Lemma 2.2.9. The transformation $J = RAC$ is of order 3.

Proof. See that

$$\begin{aligned} JRJ^{-1} &= RACRC^{-1}A^{-1}R^{-1} \\ &= RA(CR)C^{-1}A^{-1}R^{-1} \\ &= RA(RC)C^{-1}A^{-1}R^{-1} \\ &= RARA^{-1}R^{-1} \\ &= A, \\ JAJ^{-1} &= RACAC^{-1}A^{-1}R^{-1} \\ &= R(ACAC^{-1}A^{-1})R^{-1} \\ &= R(C^{-1}AC)R^{-1} \\ &= C^{-1}(RAR^{-1})C \\ &= C^{-1}(A^{-1}RA)C, \\ J(C^{-1}A^{-1}RAC)J^{-1} &= RAC(C^{-1}A^{-1}RAC)C^{-1}A^{1}R^{-1} \\ &= R. \end{aligned}$$

Hence, J^3 commutes with $R, A, C^{-1}A^{-1}RAC$, and thus, with B. It also commutes with C because

$$\begin{aligned}
JCJ^{-1} &= RACCC^{-1}A^{-1}R^{-1} \\
&= RACA^{-1}R^{-1}, \\
J(RACA^{-1}R^{-1})J^{-1} &= RAC(RACA^{-1}R^{-1})C^{-1}A^{-1}R^{-1} \\
&= RA(CR)ACA^{-1}(R^{-1}C^{-1})A^{-1}R^{-1} \\
&= RA(RC)ACA^{-1}(C^{-1}R^{-1})A^{-1}R^{-1} \\
&= (RAR)(CACA^{-1}C^{-1})(R^{-1}A^{-1}R^{-1}) \\
&= (ARA)(A^{-1}CA)(A^{-1}R^{-1}A^{-1}) \\
&= ARCR^{-1}A^{-1} \\
&= ACA^{-1}, \\
J(ACA^{-1})J^{-1} &= RAC(ACA^{-1})C^{-1}A^{-1}R^{-1} \\
&= R(ACAC)A^{-1}C^{-1}A^{-1}R^{-1} \\
&= R(CACA)A^{-1}C^{-1}A^{-1}R^{-1} \\
&= RCR^{-1} \\
&= C.
\end{aligned}$$

Hence, J^3 fixes all the generators A, B, C. It is then the identity transformation. □

Since $J = RAC$, we have the transformation

$$\begin{aligned}
D &= (BACA)^{-1} \\
&= (R^2ACA)^{-1} \\
&= (R(RAC)A)^{-1} \\
&= (RJA)^{-1} \\
&= A^{-1}J^{-1}R^{-1}.
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
(AC)^2 &= (R^{-1}J)^2 \\
&= R^{-1}JR^{-1}J \\
&= R^{-1}JR^{-1}(J^{-1}J^{-1}) \\
&= R^{-1}(JR^{-1}J^{-1})J^{-1} \\
&= R^{-1}A^{-1}J^{-1},
\end{aligned}$$

meaning that $D = A^{-1}J^{-1}R^{-1} = R(AC)^2R^{-1}$ is a conjugate of $(AC)^2$. Thus, their orders, d and c', are the same, corresponding to our prior result. Another pair of transformations to consider are

$$\begin{aligned}(AD)^2 &= (J^{-1}R^{-1})^2\\ &= J^{-1}R^{-1}J^{-1}R^{-1}\\ &= J^2R^{-1}J^{-1}R^{-1}\\ &= J(JR^{-1}J^{-1})R^{-1}\\ &= JA^{-1}R^{-1},\end{aligned}$$

and $C = A^{-1}R^{-1}J = J^{-1}(AD)^2J$, which agree to the previous relation $c = d'$.

The resulting presentation is

$$\Gamma_J = \left\langle A, R, J : \begin{array}{l} A^a \ = R^a \ = J^3 \ = I, \\ (AR)^{\frac{6a}{a-6}} \ = (R^{-1}J)^{\frac{4ac}{ac-2a-2c}} \ = (A^{-1}R^{-1}J)^c \ = I, \\ \mathrm{br}_3(A,R),\ \mathrm{br}_4(A, A^{-1}R^{-1}J),\ \mathrm{br}_2(A^{-1}R^{-1}J, R) \end{array} \right\rangle.$$

In [22], Pasquinelli constructed the 3-fold case geometrically. But to give the subsequent group presentation, we only need to state the transformations and their algebraic relations. The group presentation given is

$$\Gamma' = \left\langle J', P, R_1', R_2' : \begin{array}{l} J'^3 \ = P^{3d^*} \ = R_1'^{p'} \ = R_2'^{p'} = (P^{-1}J')^k \ = (R_2R_1J')^l \ = I, \\ R_2' \ = PR_1'P^{-1} \ = J'R_1'J'^{-1}, P = R_1'R_2', \end{array} \right\rangle.$$

We claim that this presentation identifies with Γ_J by assuming $A \approx R_2', R \approx R_1'$ and $J \approx J'$.

Lemma 2.2.10. The presentation

$$\Gamma_J = \left\langle A, R, J : \begin{array}{l} A^a \ = R^a \ = J^3 \ = I, \\ (AR)^{\frac{6a}{a-6}} \ = (R^{-1}J)^{\frac{4ac}{ac-2a-2c}} \ = (A^{-1}R^{-1}J)^c \ = I, \\ \mathrm{br}_3(A,R),\ \mathrm{br}_4(A, A^{-1}R^{-1}J),\ \mathrm{br}_2(A^{-1}R^{-1}J, R) \end{array} \right\rangle$$

is isomorphic to the presentation

$$\Gamma' = \left\langle J', P, R_1', R_2' : \begin{array}{l} J'^3 \ = P^{3d^*} \ = R_1'^{p'} \ = R_2'^{p'} = (P^{-1}J')^k \ = (R_2'R_1'J')^l \ = I, \\ R_2' \ = PR_1'P^{-1} \ = J'R_1'J'^{-1}, P = R_1'R_2', \end{array} \right\rangle$$

given A, R, J are mapped to R_2', R_1', J, respectively.

Proof. Consider the map $\phi : \langle A, R, J\rangle \to \langle J', P, R_1', R_2'\rangle$ given by

$$\begin{aligned}\phi(A) &= R_2',\\ \phi(R) &= R_1',\\ \phi(J) &= J'.\end{aligned}$$

Since ϕ maps generators to generators, while P can also be written as $R_1'R_2'$, ϕ is a bijection. Then, for it to be homomorphic, $\phi(P) = \phi(R_1'R_2') = RA$. Now, we can identify some of the words in both presentations:

$$\begin{aligned}
\phi(AR) &= R_2'R_1' \\
&= (R_1'^{-1}P)R_1' \\
&= R_1'^{-1}PR_1', \\
\phi\left((R^{-1}J)^2\right) = \phi(R^{-1}A^{-1}J^{-1}) &= R_1'^{-1}R_2'^{-1}J'^{-1} \\
&= (J'R_2'R_1')^{-1} \\
&= \left(J'(R_2'R_1'J')J'^{-1}\right)^{-1}, \\
\phi(A^{-1}R^{-1}J) &= R_2'^{-1}R_1'^{-1}J' \\
&= P^{-1}J'.
\end{aligned}$$

This means that the orders of $A, R, J, AR, (R^{-1}J)^2, A^{-1}R^{-1}J$ are the same as those of $R_2', R_1', J', R_1'^{-1}PR_1', J'(R_2'R_1'J')J'^{-1}, P^{-1}J'$, respectively. Thus, the parameters

$$a, \frac{2a}{a-6}, \frac{2ac}{ac-2a-2c}, c$$

are equal to

$$p', d^*, l, k.$$

Now, consider the braiding relations, we have

$$\begin{aligned}
\mathrm{br}_3(A,R) &\iff ARA = RAR \\
&\iff \phi(ARA) = \phi(RAR) \\
&\iff R_2'R_1'R_2' = R_1'R_2'R_1' \\
&\iff R_2'P = PR_1' \\
&\iff R_2' = PR_1'P^{-1}, \\
\mathrm{br}_2(A^{-1}R^{-1}J, R) &\iff A^{-1}R^{-1}JR = RA^{-1}R^{-1}J \\
&\iff A^{-1}R^{-1}JR = A^{-1}R^{-1}AJ \\
&\iff JR = AJ \\
&\iff A = JRJ^{-1} \\
&\iff R_2' = J'R_1'J'^{-1}.
\end{aligned}$$

The relation $\mathrm{br}_4(A, A^{-1}R^{-1}J)$ is implied by $\mathrm{br}_2(A^{-1}R^{-1}J, R)$. Hence, the map ϕ is an isomorphism. $\square$

Lemma 2.2.11. Suppose that $b = \frac{a}{2}$ and a is even, the group

$$\Gamma_{2,4,4} = \left\langle A, B, C, D : \begin{array}{l} A^a \ = B^b \ = C^c \ = D^d = BACAD \ = I, \\ (AB)^{2b'} \ = (AC)^{2c'} \ = (AD)^{2d'} \ = I, \\ \mathrm{br}_4(A,B),\ \mathrm{br}_4(A,C),\ \mathrm{br}_2(B,C) \end{array} \right\rangle$$

is either the same group as or a subgroup of index 3 of

$$\Gamma_J = \left\langle A, R, J : \begin{array}{l} A^a \ = R^a \ = J^3 \ = I, \\ (AR)^{\frac{6a}{a-6}} \ = (R^{-1}J)^{\frac{4ac}{ac-2a-2c}} \ = (A^{-1}R^{-1}J)^c \ = I, \\ \mathrm{br}_3(A,R),\ \mathrm{br}_4(A, A^{-1}R^{-1}J),\ \mathrm{br}_2(A^{-1}R^{-1}J, R) \end{array} \right\rangle.$$

Proof. Let the elements A from both groups be the same, $R^2 = B$ and $J = RAC$. Then, $D = (BACA)^{-1} = (RJA)^{-1} = A^{-1}J^{-1}R^{-1}$. Since $b = \frac{a}{2}$ implies $c = d'$ and $d = c'$, all the relations on the generators' orders are the same. The braiding relation $\mathrm{br}_4(A,C)$ is already in both groups and the braiding relations $\mathrm{br}_3(A,R), \mathrm{br}_2(A^{-1}R^{-1}J,R)$ are equivalent to the braiding relations $\mathrm{br}_4(A,B), \mathrm{br}_2(B,C)$, respectively. Thus, A, R, J could generate all the generators A, B, C, D with all their properties intact.

In the case that $J \in \Gamma_{2,4,4}$, then so is $R = JC^{-1}A^{-1}$. Since we have $\mathrm{br}_2(C,R)$,

$$\begin{aligned} ARA &= A(JC^{-1}A^{-1})A \\ &= AJC^{-1} \\ &= JRC^{-1} \\ &= JC^{-1}R \\ &= (RAC)C^{-1}R \\ &= RAR. \end{aligned}$$

That is, $\mathrm{br}_3(A,R)$. Hence, the conditions in Γ_J hold. The two groups are the same.

Otherwise, when J is not contained in $\Gamma_{2,4,4}$, consider the relations it has with each generator of $\Gamma_{2,4,4}$. According to some relations of J in Lemma 2.2.9, we have

$$\begin{aligned}
JA &= C^{-1}A^{-1}RACJ \\
&= C^{-1}A^{-1}J^2 \\
&= C^{-1}A^{-1}J^{-1}, \\
J^{-1}A &= RJ^{-1} \\
&= RJJ \\
&= R(RAC)J \\
&= BACJ, \\
JB &= JR^2 \\
&= A^2J, \\
J^{-1}B &= J^{-1}R^2 \\
&= (C^{-1}A^{-1}RAC)^2J^{-1} \\
&= C^{-1}A^{-1}R^2ACJ^{-1} \\
&= C^{-1}A^{-1}BACJ^{-1}, \\
JC &= J(A^{-1}R^{-1}J) \\
&= (JA^{-1})R^{-1}J \\
&= (C^{-1}A^{-1}B^{-1}J^{-1})R^{-1}J^{-1} \\
&= C^{-1}A^{-1}B^{-1}(J^{-1}R^{-1})J^{-1} \\
&= C^{-1}A^{-1}B^{-1}(C^{-1}A^{-1}R^{-1}R^{-1})J^{-1} \\
&= C^{-1}A^{-1}B^{-1}(C^{-1}A^{-1}B^{-1})J^{-1} \\
&= (C^{-1}A^{-1}B^{-1})^2J^{-1}, \\
J^{-1}C &= J^{-1}A^{-1}R^{-1}J \\
&= (J^{-1}A^{-1})R^{-1}J \\
&= (R^{-1}J^{-1})R^{-1}J \\
&= (R^{-1}JJ)R^{-1}J \\
&= (ACJ)R^{-1}J \\
&= AC(JR^{-1})J \\
&= AC(A^{-1}J)J \\
&= ACA^{-1}J^{-1}.
\end{aligned}$$

Since $D = (BACA)^{-1}$, it can be excluded from the generators of $\Gamma_{2,4,4}$.

Clearly, any finite words in $J \cup \Gamma_{2,4,4}$ can be written in either the form $\gamma, \gamma J$ or γJ^{-1} where $\gamma \in \Gamma_{2,4,4}$. This is because, as we have shown above, we can shift J through any generators in $\Gamma_{2,4,4}$ to the right.

If any of the words $\gamma_1 J^j$ is the same as a word $\gamma_2 J^k$, then $J^{j-k} \in \Gamma_{2,4,4}$. Thus, $j \cong k($ mod 3) and $\gamma_1 = \gamma_2$. This also implies that $\Gamma_{2,4,4}, \Gamma_{2,4,4}J, \Gamma_{2,4,4}J^{-1}$ are all distinct. Thus, $\Gamma_{2,4,4}$ is a subgroup of index 3 of Γ_J. □

2.3 Thompson's $\mathbf{E_2}$ groups

In his thesis, Thompson [25] investigated groups generated by three complex reflection R_1, R_2, R_3 of order 2 so that the pairs $(R_i, R_{i+1}), (R_{i-1}R_iR_{i-1}^{-1}, R_{i+1})$ all braid to finite orders and so that $R_1R_2R_3$ also has finite order. Here the indices are taken mod 3. See also [10], where the order of the reflections is increased but the setting is the same. One of Thompson's families, called $\mathbf{E_2}$, is characterised by

$$\mathrm{br}_3(R_2, R_3), \mathrm{br}_4(R_1, R_2), \mathrm{br}_4(R_1, R_3), \mathrm{br}_4(R_1, R_2R_3R_2^{-1}), \mathrm{br}_6(R_1R_2R_1^{-1}, R_3).$$

When the R_j have order p=3,4,6,12, the group is a lattice. By construction, these groups correspond to the parameters $(a; b, c, d) = (p; p, 3, 6)$. Using the automorphisms given by permuting b, c, d we can identify them with some of the Deligne-Mostow groups.

- Case $p = 3$, the quadruple $(3; 3, 3, 6)$ corresponds to $\Gamma\left(\frac{1,1,3,3,4}{6}\right)$.
- Case $p = 4$, the quadruple $(4; 3, 4, 6)$ corresponds to $\Gamma\left(\frac{3,3,5,6,7}{12}\right)$.
- Case $p = 6$, the quadruple $(6; 3, 6, 6)$ corresponds to $\Gamma\left(\frac{2,2,2,3,3}{6}\right)$.
- Case $p = 12$, the quadruple $(12; 3, 6, 12)$ corresponds to $\Gamma\left(\frac{3,5,5,5,6}{12}\right)$.

Chapter 3

Geometry of (r,4,4;4) triangle groups

We are going to show that there is a subgroup of the group $\Gamma_{2,4,4}$ generated by complex reflections with braidings $(r,4,4;4)$, see Section 3.1 below. We will construct it according to the braiding relations provided under the assumption that the generators' mirrors are in general position, i.e. they do not all share a common fixed point.

We consider the case where the generators are the three reflections R_1, R_2, R_3 with arbitrary orders, which is allowed because of the even braidings. Hence, we choose the braidings with length 4 as 4 is the smallest length with some degree of freedom on the generators' orders excluding length 2 braidings as those are just commutativities.

3.1 (r,4,4;4) triangle groups

Let the braiding lengths $\mathrm{br}(R_1, R_2), \mathrm{br}(R_1, R_3)$ and $\mathrm{br}(R_1, R_3^{-1}R_2R_3)$ be 4 and $\mathrm{br}(R_2, R_3)$ be an arbitrary integer r. Since $\mathrm{br}(R_1, R_2) = 4$, their mirrors reflect each other and create a sequence of mirrors $R_1, R_2, R_2^{-1}R_1R_2, R_1R_2R_1^{-1}$ (The sequence with reversed direction is just a conjugate of this sequence). The same happens between R_1, R_3 and also between $R_1, R_3^{-1}R_2R_3$. In this sense, it is convenient to have a map that fixes R_1 and rotates the maps $R_2, R_3, R_3^{-1}R_2R_3$ around. Suppose that such map S_1 exists where $S_1R_1S_1^{-1} = R_1, S_1R_2S_1^{-1} = R_3$ and $S_1R_3S_1^{-1} = R_3^{-1}R_2R_3^{-1}$. Because $S_1R_2S_1^{-1} = R_3$, it means that R_2 and R_3 are conjugates and must have the same order. Now, since $\mathrm{br}(R_1, R_2) = \mathrm{br}(R_1, R_3) = \mathrm{br}(R_1, R_3^{-1}R_2R_3) = 4$, we also have the relations from

$(1.6), (1.8), (1.9)$,

$$|\rho|^2 = 2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right), \tag{3.1}$$

$$|\tau|^2 = 2\cos\left(\frac{\pi}{q} - \frac{\pi}{m}\right), \tag{3.2}$$

$$|\sigma\tau - \bar{w}^3\bar{\rho}|^2 = 2\cos\left(\frac{\pi}{q} - \frac{\pi}{p}\right). \tag{3.3}$$

Since $p = m$, the equations (3.1) and (3.2) imply that $|\rho|^2 = |\tau|^2$. Proposition 1.1.5 shows some ambiguity in the choices of parameters ρ, σ, τ. This means that we have some freedom in the choice of arguments for ρ, σ and τ, so we let $\rho = \bar{\tau}$. From the definition of S_1, we have

$$\begin{aligned} S_1R_2R_3R_2^{-1}S_1^{-1} &= S_1R_2S_1^{-1}S_1R_3S_1^{-1}S_1R_2^{-1}S_1^{-1} \\ &= R_3(R_3^{-1}R_2R_3)R_3^{-1} \\ &= R_2. \end{aligned}$$

and, from the braiding relation between R_1 and R_2, we can infer another relation:

$$\begin{aligned} R_1R_2R_3R_2^{-1}R_1R_2R_3R_2^{-1} &= R_1S_1^{-1}R_2S_1R_1S_1^{-1}R_2S_1 \\ &= S_1^{-1}R_1R_2R_1R_2S_1 \\ &= S_1^{-1}R_2R_1R_2R_1S_1 \\ &= R_2R_3R_2^{-1}R_1R_2R_3R_2^{-1}R_1, \end{aligned}$$

that is, $\mathrm{br}(R_1, R_2R_3R_2^{-1}) = 4$. Since the trace of $R_1R_2R_3R_2^{-1}$ is

$$tr\left(R_1R_2R_3R_2^{-1}\right) = uw[2\cos\left(\frac{\pi}{q} - \frac{\pi}{m}\right) - |\rho\sigma - \bar{v}^3\bar{\tau}|^2] + \bar{u}^2\bar{w}^2,$$

we have

$$|\rho\sigma - \bar{v}^3\bar{\tau}| = 2\cos\left(\frac{\pi}{q} - \frac{\pi}{m}\right).$$

Also, from (3.2), we get

$$|\tau|^2 = |\rho\sigma - \bar{v}^3\bar{\tau}|^2. \tag{3.4}$$

The equations (3.1) and (3.3) give

$$|\rho|^2 = |\sigma\tau - \bar{w}^2\bar{\rho}|^2.$$

Since $\rho = \bar{\tau}$ and $v = w$, we have

$$|\sigma - \bar{v}^3|^2 = 1.$$

This and (3.2) imply that $|\tau|^2 = 1$. Let y be such that $\sigma - \bar{v}^3 = \bar{v}^3y^6$, meaning that $|y| = 1$. We can now express σ in terms of v and y as

$$\sigma = \bar{v}^3y^3(y^3 + \bar{y}^3).$$

Proposition 3.1.1. Suppose that R_2 and R_3 are complex reflections with the same angle and that $\mathrm{br}(R_1, R_2) = \mathrm{br}(R_1, R_3) = \mathrm{br}(R_1, R_3^{-1}R_2R_3) = 4$. Then there exists a map S_1 satisfying $S_1^* H S_1 = H$ and

$$S_1R_1S_1^{-1} = R_1, \quad S_1R_2S_1^{-1} = R_3, \quad S_1R_3S_1^{-1} = R_3^{-1}R_2R_3.$$

Proof. Since R_2, R_3 have the same order, it is enough to show that there exists a map S_1 satisfying $S_1^*HS_1 = H$ and so that, up to scalar multiples. $S_1\mathbf{n}_1 = \mathbf{n}_1$, $S_1\mathbf{n}_2 = \mathbf{n}_3$, $S_1\mathbf{n}_3 = R_3^{-1}\mathbf{n}_2$.

Assume that the polar vector of R_j is the jth basis vector $\mathbf{n}_j$. Then $S_1\mathbf{n}_1 = \mathbf{n}_1$, $S_1\mathbf{n}_2 = \mathbf{n}_3$, $S_1\mathbf{n}_3 = R_3^{-1}\mathbf{n}_2$ imply that

$$S_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & f \\ 0 & h & f\bar{v}^3\bar{\sigma} \end{pmatrix}.$$

If $\tau = \bar{\rho}$

$$H = \begin{pmatrix} -iu^3 + i\bar{u}^3 & -i\rho & i\rho \\ i\bar{\rho} & -iv^3 + i\bar{v}^3 & -i\sigma \\ -i\bar{\rho} & i\bar{\sigma} & -iv^3 + i\bar{v}^3 \end{pmatrix}.$$

then

$$S_1^*HS_1 = \begin{pmatrix} |a|^2(-iu^3 + i\bar{u}^3) & \bar{a}hi\rho & -\bar{a}f(1 - \bar{v}^3\bar{\sigma})i\rho \\ -\bar{h}ai\bar{\rho} & |h|^2(-iv^3 + i\bar{v}^3) & \bar{h}fi\bar{v}^6\bar{\sigma} \\ \bar{f}a(1 - v^3\sigma)i\bar{\rho} & -\bar{f}hiv^6\sigma & |f|^2(-iv^3 + i\bar{v}^3) \end{pmatrix}.$$

From this we see that $|a| = |f| = |h| = 1$ and

$$1 = -\bar{a}h, \quad \sigma = -\bar{h}f\bar{v}^6\bar{\sigma}, \quad -1 = \bar{f}a(1 - v^3\sigma).$$

In particular, $h = -a$ and $v^3\sigma = 1 + \bar{a}f$. Without loss of generality, we suppose that $1 = \det(S_1) = -afh$, and so $f = \bar{a}^2$. Hence S_1 has the form

$$S_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & \bar{a}^2 \\ 0 & -a & a + \bar{a}^2 \end{pmatrix}.$$

Set $a = \bar{y}^2$ and $\sigma = \bar{v}^3(1 + y^6)$. Then, such S_1 that fulfils all of the conditions exists. $\square$

Hence, we have

$$S_1 = \begin{pmatrix} \bar{y}^2 & 0 & 0 \\ 0 & 0 & y^4 \\ 0 & -\bar{y}^2 & y^4 + \bar{y}^2 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} u^4 & u\rho & -u\rho \\ 0 & \bar{u}^2 & 0 \\ 0 & 0 & \bar{u}^2 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \bar{v}^2 & 0 & 0 \\ -v\bar{\rho} & v^4 & \bar{v}^2 y^3 (y^3 + \bar{y}^3) \\ 0 & 0 & \bar{v}^2 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} \bar{v}^2 & 0 & 0 \\ 0 & \bar{v}^2 & 0 \\ v\bar{\rho} & -v^4 \bar{y}^3 (y^3 + \bar{y}^3) & v^4 \end{pmatrix}.$$

The eigenvalues of S_1 are $y^4, \bar{y}^2, \bar{y}^2$. Seeing as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ is in an eigenspace of dimension 1 correspondig to the eigenvalue y^4 of S_1 which makes the eigenspace positive and $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}^T$ is in an eigenspace of dimension 2 corresponding to the eigenvalue $\bar{y}^2$ whose eigenspace is also positive, S_1 is indeed a complex reflection.

Proposition 3.1.2. The map S_1 braids with length 4 to both R_2 and R_3. In fact, $S_1R_2S_1R_2 = R_2S_1R_2S_1 = S_1R_3S_1R_3 = R_3S_1R_3S_1$.

Proof. By the settings of S_1, we also have that

$$S_1R_2R_3 = R_3S_1R_3 = R_3R_3^{-1}R_2R_3S_1 = R_2R_3S_1,$$

that is, it commutes with R_2R_3. Moreover,

$$\begin{aligned} S_1R_2S_1R_2 &= R_3S_1R_3S_1 = R_3R_3^{-1}R_2R_3S_1S_1 = R_2R_3S_1^2, \\ S_1R_3S_1R_3 &= S_1R_3R_3^{-1}R_2R_3S_1 = S_1R_2R_3S_1 = R_2R_3S_1^2, \\ R_2S_1R_2S_1 &= R_2R_3S_1^2, \\ R_3S_1R_3S_1 &= R_3R_3^{-1}R_2R_3S_1^2 = R_2R_3S_1^2. \end{aligned}$$

Thus, $S_1R_2S_1R_2 = R_2S_1R_2S_1 = S_1R_3S_1R_3 = R_3S_1R_3S_1$. □

We now have a system between R_1, R_2, R_3, S_1. We also see that any mirror in the sequence of mirrors $R_2, R_3, R_3^{-1}R_2R_3, R_3^{-1}R_2^{-1}R_3R_2R_3, ...$ is just a conjugate of the previous mirror in the sequence by S_1 and can be written as $S_1^kR_2S_1^{-k}$. From this, together with the fact that $S_1R_1S_1^{-1} = R_1$, we have that any map in the sequence braids with length 4 with R_1.

Lemma 3.1.3. Let k be a positive integer, the word $S_1^kR_2S_1^{-k}$ is

1. $(R_3^{-1}R_2^{-1})^{\frac{k-1}{2}}R_3(R_2R_3)^{\frac{k-1}{2}}$ when k is odd, and
2. $(R_3^{-1}R_2^{-1})^{\frac{k}{2}}R_2(R_2R_3)^{\frac{k}{2}}$ when k is even.

Proof. Using induction on k starting from $k = 0$, we have

$$S_1^0 R_2 S_1^0 = R_2 = (R_3^{-1}R_2^{-1})^0 R_2 (R_2R_3)^0.$$

Now, suppose that k is odd, then

$$\begin{aligned}
S_1^k R_2 S_1^{-k} &= (R_3^{-1}R_2^{-1})^{\frac{k-1}{2}}R_3(R_2R_3)^{\frac{k-1}{2}},\\
S_1^{k+1} R_2 S_1^{-k-1} &= S_1(R_3^{-1}R_2^{-1})^{\frac{k-1}{2}}R_3(R_2R_3)^{\frac{k-1}{2}}S_1^{-1}\\
&= (S_1R_3^{-1}R_2^{-1}S_1^{-1})^{\frac{k-1}{2}}S_1R_3S_1^{-1}(S_1R_2R_3S_1^{-1})^{\frac{k-1}{2}}\\
&= (R_3^{-1}R_2^{-1})^{\frac{k-1}{2}}R_3^{-1}R_2R_3(R_2R_3)^{\frac{k-1}{2}}\\
&= (R_3^{-1}R_2^{-1})^{\frac{k-1}{2}}(R_3^{-1}R_2^{-1})R_2(R_2R_3)(R_2R_3)^{\frac{k-1}{2}}\\
&= (R_3^{-1}R_2^{-1})^{\frac{k+1}{2}}R_2(R_2R_3)^{\frac{k+1}{2}},
\end{aligned}$$

and when k is even, we have

$$\begin{aligned}
S_1^k R_2 S_1^{-k} &= (R_3^{-1}R_2^{-1})^{\frac{k}{2}}R_2(R_2R_3)^{\frac{k}{2}},\\
S_1^{k+1} R_2 S_1^{-k-1} &= S_1(R_3^{-1}R_2^{-1})^{\frac{k}{2}}R_2(R_2R_3)^{\frac{k}{2}}S_1^{-1}\\
&= (S_1R_3^{-1}R_2^{-1}S_1^{-1})^{\frac{k}{2}}S_1R_2S_1^{-1}(S_1R_2R_3S_1^{-1})^{\frac{k}{2}}\\
&= (R_3^{-1}R_2^{-1})^{\frac{k}{2}}R_3(R_2R_3)^{\frac{k}{2}}\\
&= (R_3^{-1}R_2^{-1})^{\frac{(k+1)-1}{2}}R_3(R_2R_3)^{\frac{(k+1)-1}{2}}
\end{aligned}$$

as we required. □

If we assume that S_1 is of finite order n, then

$$S_1^n R_2 S_1^{-n} = R_2.$$

However, the above lemma stated that either

$$S_1^n R_2 S_1^{-n} = (R_3^{-1}R_2^{-1})^{\frac{n-1}{2}}R_3(R_2R_3)^{\frac{n-1}{2}},$$

or

$$S_1^n R_2 S_1^{-n} = (R_3^{-1}R_2^{-1})^{\frac{n}{2}}R_2(R_2R_3)^{\frac{n}{2}}.$$

In the former case, we have

$$\begin{aligned}
R_2 &= (R_3^{-1}R_2^{-1})^{\frac{n-1}{2}}R_3(R_2R_3)^{\frac{n-1}{2}},\\
(R_2R_3)^{\frac{n-1}{2}}R_2 &= R_3(R_2R_3)^{\frac{n-1}{2}},
\end{aligned}$$

while the latter gives us

$$\begin{aligned}
R_2 &= (R_3^{-1}R_2^{-1})^{\frac{n}{2}} R_2 (R_2R_3)^{\frac{n}{2}},\\
(R_2R_3)^{\frac{n}{2}} R_2 &= R_2(R_2R_3)^{\frac{n}{2}},\\
R_2(R_3R_2)^{\frac{n}{2}} &= R_2(R_2R_3)^{\frac{n}{2}},\\
(R_3R_2)^{\frac{n}{2}} &= (R_2R_3)^{\frac{n}{2}}.
\end{aligned}$$

In either case, we have that n is divisible by $r = \mathrm{br}(R_2, R_3)$. To see that it is indeed n, suppose otherwise. According to the above computation,

$$\begin{aligned}
S_1^r R_2 S_1^{-r} &= (R_3^{-1}R_2^{-1})^{\frac{r-1}{2}} R_3 (R_2R_3)^{\frac{r-1}{2}}\\
&= R_2,
\end{aligned}$$

or

$$\begin{aligned}
S_1^r R_2 S_1^{-r} &= (R_3^{-1}R_2^{-1})^{\frac{r}{2}} R_2 (R_2R_3)^{\frac{r}{2}}\\
&= R_2.
\end{aligned}$$

Thus, S_1^r commutes with R_2. Since $R_3 = S_1R_2S_1^{-1}$, it also commutes with S_1^r. Recall that R_1 commutes with S_1, and thus, with S_1^r. Ultimately, S_1^r commutes with R_1, R_2, R_3, hence, it is an identity. The order of S_1 is r.

We now have a group generated by R_1, R_2, R_3, S_1 with defined orders. With the same arguments made in Lemma 2.1.1, we also have the orders q', r' of $(R_1R_3)^2, (R_3S_1)^2$ satisfying

$$\begin{aligned}
\frac{1}{2} - \frac{1}{p} - \frac{1}{q} &= \frac{1}{q'},\\
\frac{1}{2} - \frac{1}{p} - \frac{1}{r} &= \frac{1}{r'}.
\end{aligned}$$

The group generated by them is

$$\left\langle R_1, R_2, R_3, S_1 : \begin{array}{l} R_1^q = R_3^p = S_1^r = I, \\ (R_1R_3)^{\frac{4pq}{pq-2p-2q}} = (R_3S_1)^{\frac{4pr}{pr-2p-2r}} = I, \\ \mathrm{br}_2(R_1, S_1),\ \mathrm{br}_4(R_1, R_3),\ \mathrm{br}_4(R_3, S_1), \\ S_1R_2 = R_3S_1 \end{array} \right\rangle.$$

Any relations involving R_2 can be inferred from $S_1R_2 = R_3S_1$, so we just left them out. Considering just the transformations R_1, R_3, S_1, we have braiding relations

$$\begin{aligned}
&\mathrm{br}_4(R_1, R_3),\\
&\mathrm{br}_4(R_3, S_1),\\
&\mathrm{br}_2(R_1, S_1).
\end{aligned}$$

These relations are the same as those of A, B, C in $\Gamma_{2,4,4}$. Moreover, their orders and the orders of their products relate in the same ways as those in $\Gamma_{2,4,4}$ as well. Similarly to Lemma 2.1.2, we have a complex reflection $S_2 = (R_1R_3S_1R_3)^{-1}$ of order s where $\frac{1}{2} + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$.

Similarly to the braiding relations of D in $\Gamma_{2,4,4}$ where $\mathrm{br}_4(A, D)$ and $\mathrm{br}_2(C, D)$, we have $\mathrm{br}_4(R_3, S_2)$ and $\mathrm{br}_2(S_1, S_2)$. With this, we have the exact same group $\langle R_1, R_3, S_1, S_2\rangle$ as $\langle A, B, C, D\rangle$.

We want to create a new system containing S_2 and the other two reflections $R_1R_2R_1^{-1}$ and $R_2^{-1}R_1R_2$ left from the sequence of R_1 and R_2. These two act in a similar way to R_1 and R_2 as well. First of all, they braid with length 4 since

$$\begin{aligned}
\mathrm{br}(R_1R_2R_1^{-1}, R_2^{-1}R_1R_2) &= \mathrm{br}(R_1^{-1}R_1R_2R_1^{-1}R_1, R_1^{-1}R_2^{-1}R_1R_2R_1) \\
&= \mathrm{br}(R_2, R_2R_1R_2^{-1}) \\
&= \mathrm{br}(R_2, R_1).
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{br}(R_2^{-1}R_1R_2, R_3) &= \mathrm{br}(R_2R_2^{-1}R_1R_2R_2^{-1}, R_2R_3R_2^{-1}) \\
&= \mathrm{br}(R_1, R_2R_3R_2^{-1}) \\
&= \mathrm{br}(S_1R_1S_1^{-1}, S_1R_2R_3R_2^{-1}S_1^{-1}) \\
&= \mathrm{br}(R_1, R_2).
\end{aligned}$$

That is, $\mathrm{br}(R_2^{-1}R_1R_2, R_3) = 4$. So, we want $R_2^{-1}R_1R_2$ to act as R_1 and $R_1R_2R_1^{-1}$ to act as R_2 for this system. However, $R_1R_2R_1^{-1}$ and R_3 need not braid with length r. See that S_2 acts as S_1 for the system in the sense that

$$\begin{aligned}
S_2R_2^{-1}R_1R_2S_2^{-1} &= S_2(S_1^{-1}R_3^{-1}S_1)R_1(S_1^{-1}R_3S_1)S_2^{-1} \\
&= (S_2S_1^{-1})R_3^{-1}(S_1R_1S_1^{-1})R_3(S_1S_2^{-1}) \\
&= (S_1^{-1}S_2)R_3^{-1}(R_1)R_3(S_2^{-1}S_1) \\
S_2R_1R_2R_1^{-1}S_2^{-1} &= R_3, \\
S_2R_3S_2^{-1} &= R_3^{-1}R_2^{-1}R_1R_2R_3.
\end{aligned}$$

Since

$$\begin{aligned}
R_1 &= (R_1R_2R_1^{-1})^{-1}(R_2^{-1}R_1R_2)(R_1R_2R_1^{-1}), \\
R_2 &= (R_2^{-1}R_1R_2)(R_1R_2R_1^{-1})(R_2^{-1}R_1R_2)^{-1},
\end{aligned}$$

we have

$$\begin{aligned}
S_2R_1S_2^{-1} &= R_3^{-1}R_2^{-1}R_1R_2R_3^{-1}, \\
S_2R_2S_2^{-1} &= R_2^{-1}R_1R_2R_3R_2^{-1}R_1^{-1}R_2.
\end{aligned}$$

For the sake of labeling, we name the maps $R_1R_2R_1^{-1}$ as R_4 and $R_2^{-1}R_1R_2$ as R_5. Then S_2 interacts with R_5, R_4, R_3 in the same way S_1 does with R_1, R_2, R_3.

Proposition 3.1.4. The map S_2 commutes with R_5 and braids with length 4 with both R_3 and R_4. Moreover, $S_2R_4S_2^{-1} = R_3$ and $S_2R_3S_2^{-1} = R_3^{-1}R_4R_3$. In the sense of polar unit vectors, the new ones n_4 and n_5 corresponding to R_4 and R_5 can be written as $n_4 = R_1n_2$ and $n_5 = R_2^{-1}n_1$. The transformation S_2 maps n_4 to the direction of n_3 and preserves the direction of n_5.

The proof is done in the same way we handled the maps S_1, R_1, R_2, R_3 since there is a symmetry between those maps with the maps S_2, R_5, R_4, R_3.

Now that we have the presentation of the group and S_2 is defined, we have finally realized our aim of constructing a subgroup of $\Gamma_{2,4,4}$.

Lemma 3.1.5. The group

$$\Gamma'_{r,4,4;4} = \left\langle R_3, S_1, S_2 : \begin{array}{l} R_3^p = S_1^r = S_2^s = (R_3S_1R_3S_2)^q = I, \\ (S_1R_3S_2)^{\frac{4pq}{pq-2p-2q}} = (R_3S_1)^{\frac{4pr}{pr-2p-2r}} = (R_3S_2)^{\frac{4ps}{ps-2p-2s}} = I, \\ \mathrm{br}_2(S_1,S_2),\ \mathrm{br}_4(R_3,S_1),\ \mathrm{br}_4(R_3,S_2) \end{array} \right\rangle,$$

whose (r,4,4;4)-subgroup is

$$\Gamma_{r,4,4;4} = \left\langle R_1, R_2, R_3 : \begin{array}{l} R_1^q = R_2^p = R_3^p = I, \\ (R_1R_2)^{\frac{4pq}{pq-2p-2q}} = (R_1R_3)^{\frac{4pq}{pq-2p-2q}} = I, \\ \mathrm{br}_4(R_1,R_2),\ \mathrm{br}_4(R_1,R_3),\ \mathrm{br}_4(R_1,R_3^{-1}R_2R_3),\ \mathrm{br}_r(R_2,R_3) \end{array} \right\rangle,$$

is isomorphic to the group

$$\Gamma_{2,4,4} = \left\langle A, B, C, D : \begin{array}{l} A^a = B^b = C^c = D^{\frac{2abc}{abc+2bc-2ab-2ac}} = BACAD = I, \\ (AB)^{\frac{4ab}{ab-2a-2b}} = (AC)^{\frac{4ac}{ac-2a-2c}} = (AD)^{\frac{4abc}{-2bc+ab+ac}} = I, \\ \mathrm{br}_4(A,B),\ \mathrm{br}_4(A,C),\ \mathrm{br}_2(B,C) \end{array} \right\rangle$$

when $p = a, q = b, r = c$ and $\frac{1}{2} + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$.

Proof. We will first show that $\Gamma'_{r,4,4;4}$ contains R_1 and R_2, including all their relations. This can be done by letting R_1 be $(R_3S_1R_3S_2)^{-1}$ and R_2 be $S_1^{-1}R_3S_1$. Assume all the relations in $\Gamma'_{r,4,4;4}$. Then, R_1 is of the same order q as $R_1^{-1} = R_3S_1R_3S_2$, and since R_2 is a conjugate of R_3, the order of R_2 is also p. Moreover,

$$\begin{aligned} R_1R_3 &= (S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1})R_3 \\ &= S_2^{-1}R_3^{-1}S_1^{-1}. \end{aligned}$$

Hence, its order is as required. We also have that

$$\begin{aligned} R_1R_2 &= R_1(S_1^{-1}R_3S_1) \\ &= (R_1S_1^{-1}R_3S_1) \\ &= (S_1^{-1}R_1)R_3S_1 \\ &= S_1^{-1}R_1R_3S_1, \end{aligned}$$

a conjugate of R_1R_3. Thus, they share the same order. Now, for the braiding relations, we start with the braiding relations between R_1 and R_3. See that

$$\begin{aligned}
R_3R_1R_3R_1 &= R_3(S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1})R_3(S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1}) \\
&= R_3(S_2^{-1}R_3^{-1}S_1^{-1})(S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1}) \\
&= R_3S_2^{-1}R_3^{-1}(S_1^{-1}S_2^{-1})R_3^{-1}S_1^{-1}R_3^{-1} \\
&= R_3S_2^{-1}R_3^{-1}(S_2^{-1}S_1^{-1})R_3^{-1}S_1^{-1}R_3^{-1} \\
&= (R_3S_2^{-1}R_3^{-1}S_2^{-1})(S_1^{-1}R_3^{-1}S_1^{-1}R_3^{-1}) \\
&= (S_2^{-1}R_3^{-1}S_2^{-1}R_3)(R_3^{-1}S_1^{-1}R_3^{-1}S_1^{-1}) \\
&= S_2^{-1}R_3^{-1}S_2^{-1}(R_3R_3^{-1})S_1^{-1}R_3^{-1}S_1^{-1} \\
&= S_2^{-1}R_3^{-1}S_2^{-1}S_1^{-1}R_3^{-1}S_1^{-1} \\
&= S_2^{-1}R_3^{-1}(S_2^{-1}S_1^{-1})R_3^{-1}S_1^{-1} \\
&= S_2^{-1}R_3^{-1}(S_1^{-1}S_2^{-1})R_3^{-1}S_1^{-1} \\
&= (S_2^{-1}R_3^{-1}S_1^{-1})(S_2^{-1}R_3^{-1}S_1^{-1}) \\
&= (R_1R_3)(R_1R_3) \\
&= R_1R_3R_1R_3.
\end{aligned}$$

Thus, we have $\mathrm{br}_4(R_1, R_3)$. Next, since

$$\begin{aligned}
R_1S_1 &= (S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1})S_1 \\
&= S_2^{-1}(R_3^{-1}S_1^{-1}R_3^{-1}S_1) \\
&= S_2^{-1}(S_1R_3^{-1}S_1^{-1}R_3^{-1}) \\
&= (S_2^{-1}S_1)R_3^{-1}S_1^{-1}R_3^{-1} \\
&= (S_1S_2^{-1})R_3^{-1}S_1^{-1}R_3^{-1} \\
&= S_1R_1,
\end{aligned}$$

we have that R_1 and S_1 commute. This leads to $\mathrm{br}_4(R_1, R_2)$ because

$$\begin{aligned}
R_2R_1R_2R_1 &= (S_1^{-1}R_3S_1)R_1(S_1^{-1}R_3S_1)R_1 \\
&= S_1^{-1}R_3(S_1R_1S_1^{-1})R_3(S_1R_1) \\
&= S_1^{-1}R_3(R_1)R_3(R_1S_1) \\
&= S_1^{-1}(R_3R_1R_3R_1)S_1 \\
&= S_1^{-1}(R_1R_3R_1R_3)S_1 \\
&= (S_1^{-1}R_1)R_3R_1R_3S_1 \\
&= (R_1S_1^{-1})R_3(S_1S_1^{-1})R_1R_3S_1 \\
&= R_1(S_1^{-1}R_3S_1)(S_1^{-1}R_1)R_3S_1 \\
&= R_1(R_2)(R_1S_1^{-1})R_3S_1 \\
&= R_1R_2R_1R_2.
\end{aligned}$$

The third braiding relation can be obtained in the same way using the fact that

$$\begin{aligned} R_3^{-1}R_2R_3 &= R_3^{-1}(S_1^{-1}R_3S_1)R_3 \\ &= R_3^{-1}R_3S_1R_3S_1^{-1} \\ &= S_1R_3S_1^{-1}. \end{aligned}$$

This is just the case of braiding between R_1 and $R_2 = S_1^{-1}R_3S_1$ where we switch S_1 and S_1^{-1}. Thus, we claim $\mathrm{br}_4(R_1, R_3^{-1}R_2R_3)$. The last braiding relation $\mathrm{br}_r(R_2, R_3)$ is because of the fact that $R_2 = S_1^{-1}R_3S_1$. As shown earlier, Lemma 3.1.3 implies that the braid length between R_2 and R_3 is the same as the order of S_1.

Next, we show the isometry between $\Gamma_{2,4,4}$ and $\Gamma'_{r,4,4;4}$. Let $\phi : \Gamma'_{r,4,4;4} \to \Gamma_{2,4,4}$ be a homomorphism defined by

$$\begin{aligned} \phi(R_3) &= A, \\ \phi(S_1) &= C, \\ \phi(S_2) &= D. \end{aligned}$$

Then, the orders match up well. The order d of D satisfies

$$\begin{aligned} \frac{1}{d} &= \frac{1}{2} + \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \\ &= \frac{1}{2} + \frac{1}{p} - \frac{1}{q} - \frac{1}{r} \\ &= \frac{1}{s}. \end{aligned}$$

The generator $B = D^{-1}A^{-1}C^{-1}A^{-1}$ is, by homomorphism, defined to be $S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1}$. In other words, it is R_1 we defined above. Thus, it works like R_1 and satisfies the rest of the order relations. The braiding relations $\mathrm{br}_4(A, B), \mathrm{br}_4(A, C), \mathrm{br}_2(B, C)$ are equivalent to the braiding relations $\mathrm{br}_4(R_1, R_3), \mathrm{br}_4(R_3, S_1)$ and $\mathrm{br}_2(R_1, S_1)$ we already had. Conversely, the relations $\mathrm{br}_2(S_1, S_2), \mathrm{br}_4(R_3, S_2)$ are equivalent to $\mathrm{br}_2(C, D), \mathrm{br}_4(A, D)$ which were already known relations in $\Gamma_{2,4,4}$. Hence, they are isomorphic as required. $\square$

Chapter 4

Deraux-Parker-Paupert's Algorithm

In [10], the group generated by three reflections of the same order but with distinct mirrors was studied. It was then used to construct fundamental polyhedra for the group. Since the group Γ generated in the previous chapter is just a generalization of the group, we want to apply the algorithm to it as well.

4.1 DPP-Fundamental Domain Construction

We will construct our fundamental domain in the same manner as in [10], so we will roughly state the algorithm used there. Since the notations a, b, c are used in [10], from this point on, they will be used as such and must not be confused with the ones used in $\Gamma_{2,4,4}$. We start off with combinatorial construction and make a geometric realization of it later. In [10], the fundamental domain is set to be bounded by the spherical shell surrounding the fixed point of $R_1R_2R_3$.

Being bounded by a spherical shell means that the domain is chosen to be on the 'inside' of a copy of S^3, which is the boundary of a 4-ball. To make sure of that, there are some rules from [10] we want to mention and some notation we need to define.

The polyhedron was built by connecting pyramids in bisectors. Each pyramid, sequentially, could be determined by an ordered triangle that is one of its faces.

Given an ordered complex hyperbolic triangle, let the edges be associated to complex reflections a, b, c and call the respective edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$. There is also the assumption that a, b and c have the same order p. The edge $\mathbf{a}$ is called the **base**, while the other two edges $\mathbf{b}$ and $\mathbf{c}$ intersect at the **apex** of the triangle. For convenience's sake and to keep the reference consistent, we will denote the inverses with bars.

The reflections b, c both act on the projective line of complex line through the intersection of b and c as rotations of angle $\frac{2\pi}{p}$. Their product bc is also a rotation on the projective line. An example given in [10] is the case where their braid length is 5. The result is as seen in Figure 4.1.

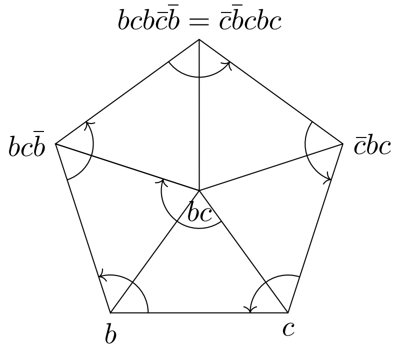

Figure 4.1: Triangle group on the projective line through the intersection of b and c

Any counter-clockwise consecutive rotations have a product of bc. This can help us keep track of our orientation. The 5-braiding between b and c gives a pentagon similarly to what we have with R_2, R_3 and S_1 earlier when $r = 5$.

This also applies to any braiding lenth n, which will hold the property that the products between any two counter-clockwise consecutive rotations are bc. When considering the pyramid with the mirror of a as the base and b, c as lateral edges, the result is the same as when we start with a and any consecutive pair in the sequence as the lateral edges in the same orientation, such as $\bar{c}bc$ and $\bar{c}\bar{b}cbc$. The pyramid in the 5-braiding case is as in Figure 4.2.

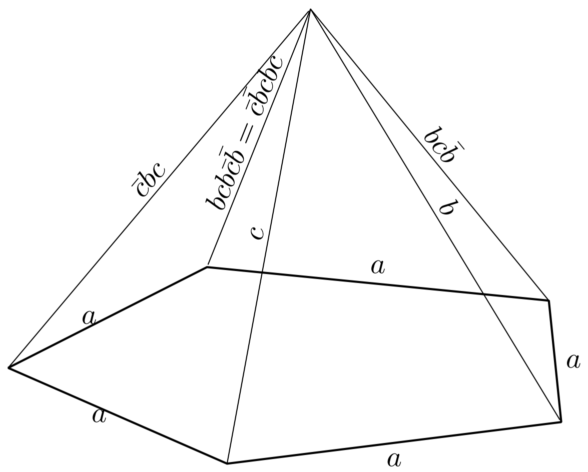

Figure 4.2: Pyramid corresponding to the ordered triangle a, b, c

By shifting through the triangle edges a, b, c, either to b, c, a or c, a, b, the result pyramid might be different due to the braiding relations between them and will be discussed later. Note that, this pyramid with a pentagon as its base is just an example in the case of $\mathrm{br}_5(b, c)$. In the generalized case where $\mathrm{br}_r(b, c)$, as in our case where we think of R_1, R_2, R_3 as a, b, c, respectively, this results in a similar pyramid with an r-gon as its base. This, too, will be explored later.

4.1.1 Side pairing

The aim of using the polyhedron as fundamental domain will require it to satisfy the Poincaré Polyhedron theorem. In order to do that, it is necessary to define the side pairing. The sides in the space are actually the 3-faces.

In this case, the pyramids were intended to be the sides. For each pyramid, there is a base and there is naturally another pyramid paired with it, that is, the image pyramid under the reflection whose mirror is at the base, or under its inverse. For instance, the previous pyramid associated to the triangle $a; b, c$ has the mirror of a as the base. The side pairing map associated to the pyramid is either a or a^{-1}.

Consider the triangle $a; b, c$, since the reflection a fixes the edge associated with it, there are two possible images of the triangle: $a; ab\bar{a}, ac\bar{a}$ and $a; \bar{a}ba, \bar{a}ca$ depending on whether it is under a or a^{-1}. That leads to a rule frequently used in [10], and consequently here, to keep tracks of the choices, the **123-rule**.

The name 123-rule corresponds to the fact that we want to keep building around the fixed point of $R_1R_2R_3$, and also to the algorithm of the rule itself. The rule states that, we will choose a pyramid associated to a triangle $a; b, c$ if and only if either $abc = 123$

or $bca = 123$ where 123 is $R_1R_2R_3$ and the products abc and bca are done in the triangle-group sense where we take the braiding relations into account. An example is the choices we left unfinished earlier, between the pyramids associated to $a; ab\bar{a}, ac\bar{a}$ and $a; \bar{a}ba, \bar{a}ca$. In this case, since we started with $a; b, c$, we let $abc = 123$ and so $a(ab\bar{a})(ac\bar{a}) = 1(12\bar{1})(13\bar{1}) = 123$, while $a(\bar{a}ba)(\bar{a}ca) = 1(\bar{1}21)(\bar{1}31) = 231 \neq 123$. Thus, the former pyramid is selected as the mirrored side.

There are many restrictions here for the sake of selection to satisfy nice properties. First of all, the selection should be done so that the block is invariant under the stabiliser, which is P in the symmetry case, and Q in the non-symmetry case. In our case, the stabiliser is a bit different and will be shown later.

Next , we want nice ridge cycles. To be precise, for each ridge(2-face, in this case) on a side, it should be contained in exactly two sides. According to the side pairing, the paired sides contain a common ridge, its base. It is naturally selected to be contained in exactly those two sides. For the lateral sides, the choices are not as obvious. Take the triangle $a; b, c$, for example. This triangle can be shifted to $b; c, a$ and $c; a, b$, both associated to their own pyramids. According to the 123-rule, either $abc = 123$ or $bca = 123$. If $abc = 123$, then the triangle $c; a, b$ satisfies the rule, while $b; c, a$ does not. Otherwise, when $bca = 123$, the rule applies to $b; c, a$ but not $c; a, b$. Similarly to the pyramid associated to $a; b, c$, for these pyramids to be well defined, the braiding relations between a and b(a and c in the latter case) should be finite.

4.1.2 Geometric Realization

Realizing the block geometrically was done dimension by dimension, starting from the 0-faces, the vertices. There is some notation for specific terms in [10]. We will continue to use it in this section, although we will later use our own labels for our case instead.

Vertex realization. Let the pyramid be $a; b_1, b_2$, a pyramid with the mirror of a as its base and one of its lateral ridges is the triangle with edges $\mathbf{a}$,$\mathbf{b_1}$,$\mathbf{b_2}$. The rest of the lateral edges are labelled (in order) as $\mathbf{b_3}$,$\mathbf{b_4}$,...,$\mathbf{b_n}$, respectively. An additional assumption was made so that the mirrors of the edges of any triangle are pairwise distinct, and that they do not all intersect at the same point. Complex lines may not intersect in $\mathbf{H}^2_{\mathbb{C}}$, but they always do in $P^2_{\mathbb{C}}$.

Lines $\mathbf{v}$ and $\mathbf{w}$ with distinct polar vectors v and w, respectively, intersect at a unique point u in $P^2_{\mathbb{C}}$. We denote this by $u = v \boxtimes w$, additionally to the usual line-intersection notation (See [23] for how this is properly defined). The point u lies in $\mathbf{H}^2_{\mathbb{C}}$ if and only if $\langle u, u\rangle < 0$. When $\langle u, u\rangle > 0$, the two lines have a common perpendicular complex line $\mathbf{u}$.

Now, we begin the realization for each vertex. There are three types of vertices: the top ,the base and the mid vertices.

The **top vertices** are the vertices resulted from intersection of two lateral edges. In fact, if a top vertex lies in $\mathbf{H}^2_{\mathbb{C}}$ or its boundary, then every lateral ridge shares the same top vertex. Otherwise, complex lines containing lateral edges intersect in $P^2_{\mathbb{C}}$ outside of $\mathbf{H}^2_{\mathbb{C}}$, so there will be a complex line, which will be called $\mathbf{d}$, polar to the intersection point. This line is also common among all the ridges and will intersect the n lateral lines, resulting in n vertices. We will also call them top vertices in this case, and we say that the vertex $b_1 \boxtimes b_2$ is *truncated* by the line $\mathbf{d}$.

A **base vertex** is, as the name suggested, the vertice at the intersection point of the base a and another line. For any k, the intersection point between $\mathbf{a}$ and $\mathbf{b_k}$ is called d_k. If this point lies in the closure of $\mathbf{H}^2_{\mathbb{C}}$, then it is called a base vertex. If not, there is a line $\mathbf{d_k}$ polar to d_k and perpendicular to $\mathbf{a}$,$\mathbf{b_k}$. Then, the line $\mathbf{d_k}$ intersects $\mathbf{a}$ at a point $a \boxtimes d_k$, which we will also call a base vertex, and intersects $\mathbf{b_k}$ at a point $b_k \boxtimes d_k$ called a mid vertex.

Edge realization. From the vertices obtained, we can link two of them up by a geodesic arc to get an edge. There are different types of edges depending on the pair of vertices they connect.

If the apex of the pyramid is truncated, then there is a common complex line $\mathbf{d}$ polar to the apex and a corresponding n top vertices, $\mathbf{d} \cap \mathbf{b_1}, \mathbf{d} \cap \mathbf{b_2}, \ldots, \mathbf{d} \cap \mathbf{b_n}$. The top edges are the ones joining $\mathbf{d} \cap \mathbf{b_k}$ to $\mathbf{d} \cap \mathbf{b_{k+1}}$ (including the one between $\mathbf{d} \cap \mathbf{b_n}$ and $\mathbf{d} \cap \mathbf{b_1}$).

Now, consider the base. Recall that d_k is the intersection point between $\mathbf{a}$ and $\mathbf{b_k}$. However, like the apex, it may not be in $\mathbf{H}^2_{\mathbb{C}}$. In that case, there is a mid vertex $\mathbf{d_k} \cap \mathbf{b_k}$. When linked to the top vertex $\mathbf{b_k} \cap \mathbf{d_{k+1}}$($\mathbf{d} \cap \mathbf{b_k}$ if the apex is truncated), we get an edge joining a top vertex and a mid vertex.

Moreover, in the case that there is a mid vertex, we also get an edge joining the mid vertex $\mathbf{d_k} \cap \mathbf{b_k}$ to the base vertex $\mathbf{d_k} \cap \mathbf{a}$ as well.

In the case that there is no mid vertex on $\mathbf{b_k}$, then the top vertex (either $\mathbf{b_k} \cap \mathbf{d_{k+1}}$ or $\mathbf{d} \cap \mathbf{b_k}$) can be directly joined to the base vertex $\mathbf{b_k} \cap \mathbf{a}$ by an edge.

The last kind of edges are the bottom edges joining base vertices to other base vertices. These base vertices can be either in the form $\mathbf{b_k} \cap \mathbf{a}$ or $\mathbf{d_k} \cap \mathbf{a}$ depending on if there is a truncation there (and the forms of both base vertices are also independent to each other).

Ridge realization. For the sake of realizing the bottom ridge, we restate an assumption from [10].

Assumption. The (ordered) polygon obtained by taking the bottom edges joining the bottom vertices $\mathbf{a} \cap \mathbf{b_k}$ or $\mathbf{a} \cap \mathbf{d_k}$ is an embedded (piecewise smooth) topological circle in the (closure in $\mathbf{H}^2_{\mathbb{C}}$ of the) complex line $\mathbf{a}$, equivalently this polygon bounds a disk in that (closed) complex line.

Thus, the bottom ridge is defined. There may be a similar ridge at the top of the pyramid in the case that $\mathbf{d} = \mathbf{b_1} \cap \mathbf{b_2}$ lies outside of $\mathbf{H}^2_{\mathbb{C}}$. The n top edges create an n-gon referred to as **top ridge**. The embeddedness is also assumed for it to be a ridge.

For the lateral ridges, there are many possibilities depending on the existence of truncations. The figure 4.3 show every types of possible lateral ridge containing the edges $\mathbf{b_k}$ and $\mathbf{b_{k+1}}$.

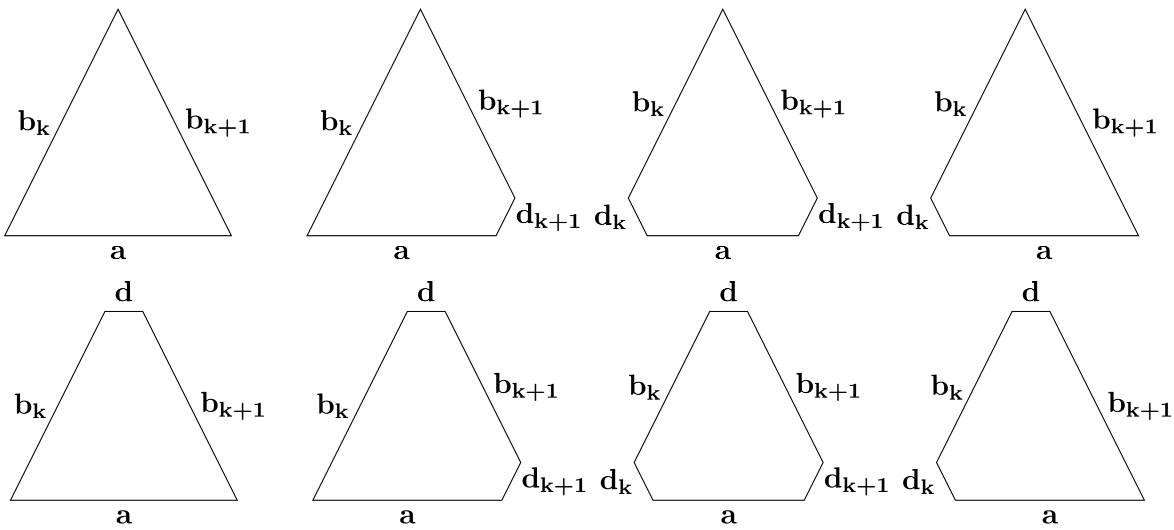

Figure 4.3: Possible lateral ridges.

There will be a need to discuss triangles with vertices outside of the closure of $\mathbf{H}^2_{\mathbb{C}}$. Thus, it is necessary to define some notation on these triangles. A triple of pairwise distinct complex lines $\mathbf{e_1}$,$\mathbf{e_2}$,$\mathbf{e_3}$ with polar vectors e_1, e_2, e_3, respectively, encompasses a triangle. The triangle's *vertices* are the intersections of any pair of the lines, denoted by $v_i = e_j \boxtimes e_k$ where i, j, k are pairwise distinct. The vertex v_i is called the vertex opposite to the edge $\mathbf{e_i}$.

We define a height similarly to a triangle's height for this complex triangle. A *complex height* of the triangle through v_i is a complex geodesic in $\mathbf{H}^2_{\mathbb{C}}$ orthogonal to $\mathbf{e_i}$ with projective extension containing the opposite vertex v_i. If such height exists through v_i, then it is also unique. It is, in fact, the complex line polar to $v_i \boxtimes e_i$.

The intersection point between the height (if it exists) and the edge $\mathbf{e_i}$ is called the *foot* of the complex height. The foot is given by $f_i = v_i - \frac{\langle v_i, e_i \rangle}{\langle e_i, e_i \rangle} e_i$. The height through

v_i exists if and only if the foot f_i is negative.

A proper definition of complex hyperbolic triangles given in [10] is:

Definition 4.1.1. *(Definition 4.3 from [10])* A *complex hyperbolic triangle* is a triple of pairwise distinct complex lines that admits three complex heights.

From here on, we let all the triangles be complex hyperbolic triangles. Hence, each one has three pairwise distinct edges with three different well-defined complex heights. Here are some useful results of the assumption obtained in [10]:

Proposition 4.1.2. (*Proposition 4.4 from [10]*) Given a complex hyperbolic triangle $\mathbf{a}$, $\mathbf{b}$,$\mathbf{c}$ in complex hyperbolic space, there is a unique bisector $\mathbf{B_a}$ such that

1. $\mathbf{a}$ is a complex slice of $\mathbf{B_a}$ and,
2. the extended real spine of $\mathbf{B_a}$ contains $\mathbf{b} \boxtimes \mathbf{c}$.

Lemma 4.1.3. (*Lemma 4.6 from [10]*) Let L be a complex line orthogonal to a complex slice of a bisector $\mathbf{B}$. Then $L \cap \mathbf{B}$ is a geodesic, contained in a meridian of $\mathbf{B}$.

Proposition 4.1.4. (*Proposition 4.5 from [10]*) The bisector $\mathbf{B_a}$ from Proposition 4.1.2 contains the 1-skeleton of the geometric realization for $a; b, c$.

Since each ridge is contained in exactly two sides, there should also be another side constructed by choosing $\mathbf{b}$ or $\mathbf{c}$, according to the 123-rule, as the base instead of $\mathbf{a}$.

The bisectors $\mathbf{B_a}$, $\mathbf{B_b}$ and $\mathbf{B_c}$ are called the natural bisectors associated to the triangle $a; b, c$. The triangle is said to be real if it is the complexification of a triangle in a copy of $\mathbf{H}^2_{\mathbb{R}}$. This is equivalent to saying that their polar vectors can be scaled so that their pairwise inner products are real.

Finally, we have

Proposition 4.1.5. (*Proposition 4.7 from [10]*) Let $a; b, c$ be a non-real complex hyperbolic triangle. The natural bisectors satisfy the following properties.

1. $\mathbf{B_a} \cap \mathbf{B_b} = \mathbf{B_b} \cap \mathbf{B_c} = \mathbf{B_a} \cap \mathbf{B_c}$.
2. The above intersections have at most two connected components, and each component is a proper smooth disk, called a Giraud disk, in $\mathbf{H}^2_{\mathbb{C}}$.
3. The 1-skeleton of the corresponding ridge is contained in (the closure of) only one of the connected components.

These assumptions make sure that each pyramid is contained in a unique bisector, and that the base and top ridges are in complex lines, while lateral ridges are in Giraud disks.

4.2 Γ-Fundamental Domain Construction

We will now start the construction on the group Γ we had, mimicking the algorithm from the previous section. We declare the shell for the domain to be the spherical shell surrounding the fixed point of S_1S_2, i.e. the intersection of the mirrors of S_1 and S_2. The transformation S_1S_2 is equal to $(R_1R_2R_3)^{-1}$, thus, having the same notation as the fixed point used in [10].

In the equilateral case of [10], the constructed polyhedron was stabilised by the transformation P, where $P^3 = R_1R_2R_3$. In our case, the polyhedron will be stabilised by the group $\langle S_1, S_2\rangle \cong C_r \times C_s$.

The domain we are about to construct consists of hyperbolic pyramids. We will start from the triangles which are ridges on these pyramids. Citing the settings of [10], for any ordered triangle in complex hyperbolic space, we can denote the triangle by its edges.

In our case, there are three pyramids associated to three triangles that we will use to build our polyhedron. The shell consists of these three pyramids and all their images under the elements of $\langle S_1, S_2\rangle$. The first ordered triangle's edges are R_1, R_2, R_3, with R_1 being the base. Consider the reflections R_2, R_3. These two mirrors intersect at the mirror of $(R_2R_3)^r$ and create a triangle with one edge being the shell. While R_2, R_3 act as rotation on the triangle by angle $\frac{2\pi}{p}$, their product R_2R_3 also rotates the triangle, although with different angle, as illustrated in Figure 4.4.

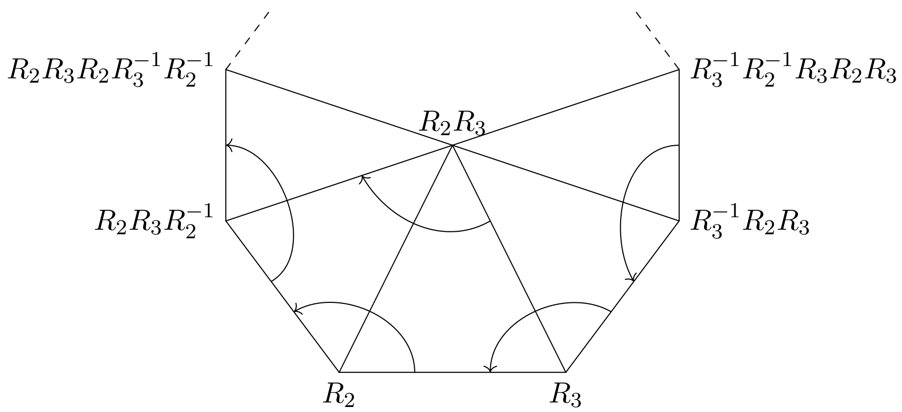

Figure 4.4: Triangle group for $R_1; R_2, R_3$

The products between any two consecutive rotations in counter-clockwise direction are always R_2R_3. Since R_2 braids with R_3 with length r, the figure 4.4 is an r-gon.

We also apply the 123-rule here, where R_1, R_2, R_3 are denoted as $1, 2, 3$, respectively. This means that the triangles we will consider are, for example, $R_1; R_2, R_3$ and $R_3; R_2, R_1$. The former is the one we discussed before and will be paired with

$R_1; R_1R_3R_1^{-1}, R_1, R_1R_2R_1^{-1}$. The latter, $R_3; R_2, R_1$, is chosen instead of $R_2; R_1R_3$ because of the 123-rule. The side pairing map chosen for it is, according to the algorithm, R_3^{-1} and it is paired with $R_3; R_3^{-1}R_1R_3, R_3^{-1}R_2R_3$. Since R_1 and R_2 braid with length 4, there are four triangles generated by them.

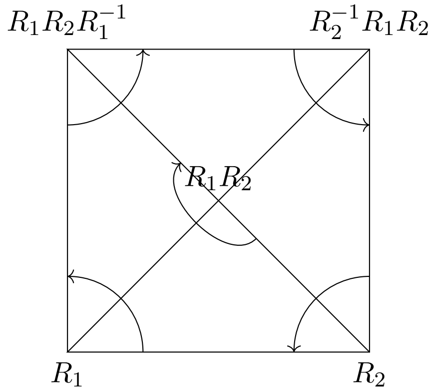

Figure 4.5: Triangle group $R_3; R_3^{-1}R_1R_3, R_3^{-1}R_2R_3$

Hence, we get the two pyramids as sides inside the shell, both containing the ridge R_1, R_2, R_3.

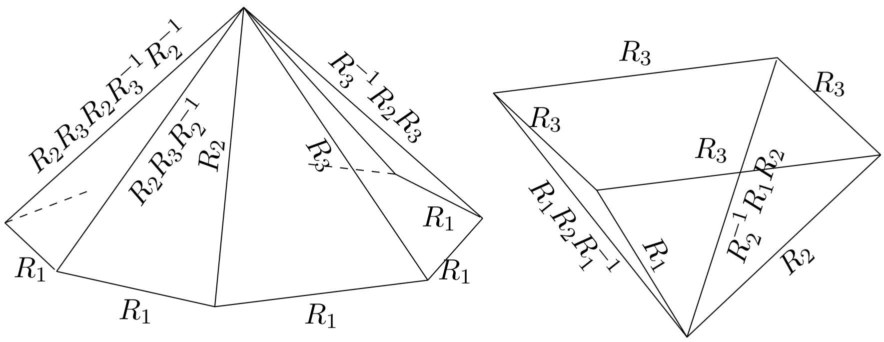

Figure 4.6: (1) Pyramid corresponding to the ordered triangle $R_1; R_2, R_3$.
(2) Pyramid corresponding to the ordered triangle $R_3; R_1, R_2$.

The reason that the pyramid associated to $R_3; R_1, R_2$ is shown upside down in figure 4.6 is so that the side-pairing map R_3^{-1} will map it to a pyramid in the original orientation as opposed to what the side pairing map R_1 does to the $R_1; R_2, R_3$-pyramid.

Another side we will consider is associated to the triangle $R_2^{-1}R_1R_2; R_1R_2R_1^{-1}, R_3$, or just, in short, $R_5; R_4, R_3$. This satisfies the 123-rule. The reason we chose this triangle and not the other two ($R_1R_2R_1^{-1}; R_1, R_3$ and $R_2; R_2^{-1}R_1R_2, R_3$) is because they will later be seen in orbits of the ones we are working on now.

We have similar relations between R_3, R_4 and S_2 as we have with R_2, R_3, S_1. Here, since we assume the order s of S_2, we have $\mathrm{br}_s(R_3, R_4)$. Since they interact with each other in the same way, the pyramid associated to $R_5; R_4, R_3$ will be just like the one associated to $R_1; R_2, R_3$, albeit with s-gon as its base instead of an r-gon.

4.3 Geometric realization for Γ

With the setting we have established from the previous section, we can geometrically realize the group Γ(and, in turn, $\Gamma_{2,4,4}$) as a polyhedron in the projective space. We label the vertex associated to each R_i with the integer i and, for the sake of labelling, we think of S_1 and S_2 as R_6 and R_7. We make slight changes to the notation of vertices from the previous section.

Vertex realization Considering the triangle $R_1; R_2, R_3$, we have r copies of this triangle generated by S_1. Every triangle has the mirror of R_1 as its base and words generated by R_2 and R_3 as lateral edges. The edges are either in the form $R_2R_3R_2 \dots R_2^{-1}R_3^{-1}R_2^{-1}$ or $R_3^{-1}R_2^{-1}R_3^{-1} \dots R_3R_2R_3$. We will depict them in the vertices connecting to them using only the indices of R put consecutively into words, while the inverses are noted by bars. For example, we use $23\bar{2}$ for $R_2R_3R_2^{-1}$ and $\bar{3}\bar{2}323$ for $R_3^{-1}R_2^{-1}R_3R_2R_3$. Moreover, we use $z_{w_1w_2} = w_1 \boxtimes w_2$ where w_1, w_2 are words consisting of elements in $\{1, 2, 3, 4, 5, 6, 7\}$. Recall that $R_4 = R_1R_2R_1^{-1}, R_5 = R_2^{-1}R_1R_2, R_6 = S_1$ and $R_7 = S_2$. Also, note that $R_5 \boxtimes R_4$ is actually the vertex $z_{12} = R_1 \boxtimes R_2$ and we will denote it as such instead of z_{45}.

As for when $z_{w_1w_2}$ lies outside $\mathbf{H}^2_{\mathbb{C}}$ ($\langle z_{w_1w_2}, z_{w_1w_2}\rangle > 0$), we get a truncation and there are vertices $z_{w_1w_2} \boxtimes z_{w_1}$ and $z_{w_1w_2} \boxtimes z_{w_2}$ in the shell, instead. This notation also works for $z_{12} \boxtimes z_4$ as $54 = \bar{2}1212\bar{1} = 12$. We denote $z_{w_1w_2} \boxtimes z_{w_1}$ by $z^{w_1}_{w_1w_2}$.

In this way, we can realize the vertices as done in section 4.1.2 with a different system of notation. Note that the pyramid associated to $R_1; R_2, R_3$ is made up of r tetrahedra in the same S_1 orbit. Hence, we will only pay attention to one of the tetrahedra, namely, the one with $R_1; R_2, R_3$ as a lateral ridge, from now on. We consider only the tetrahedron with $R_5; R_4, R_3$ for the pyramid associated to it.

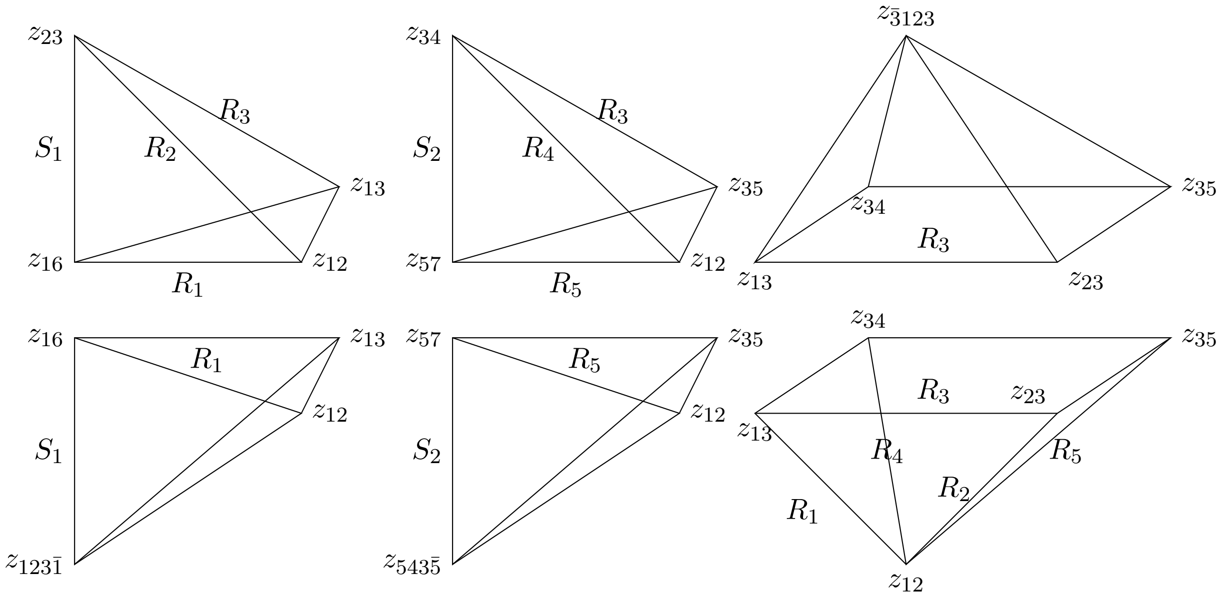

Figure 4.7: Sides of the polyhedron

Edge and ridge realizations Edges and ridges are also realized like in section 4.1.2. We denote the edge joining two vertices by an ordered pair containing them in any order (thus, the orders of the pairs are meaningless and is more akin to a set) and denote the ridges containing n vertices by an n-tuple, also in any order.

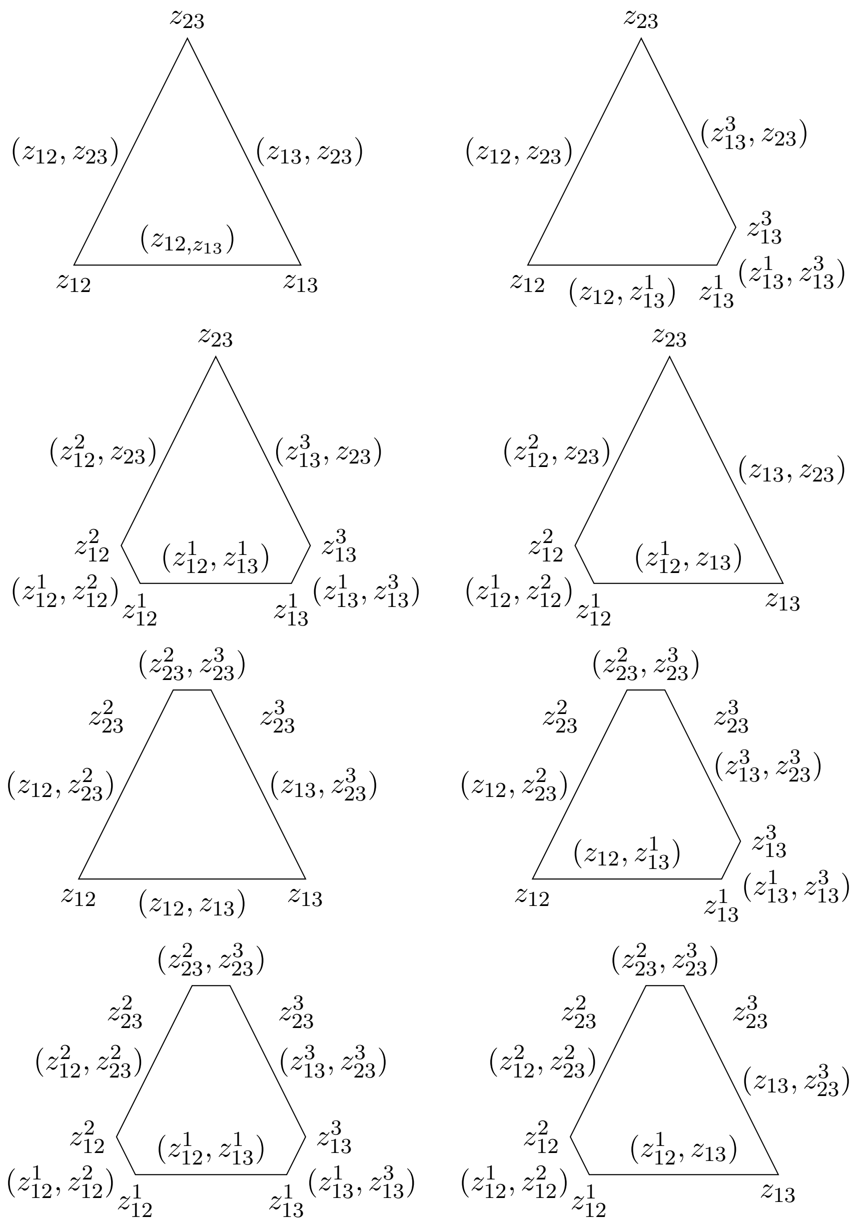

Figure 4.8: Possible ridges on $R_1; R_2, R_3$.

4.3.1 Side pairing

We define the polyhedron by its sides, which are shown in figure 4.7 along with their images under the side pairing maps.

We call the side containing $z_{23}, z_{16}, z_{13}, z_{12}$ as s_1, the one containing $z_{34}, z_{57}, z_{35}, z_{12}$ as s_2 and the one containing $z_{12}, z_{23}, z_{13}, z_{34}, z_{35}$ as s_3. The side s_1 shares the ridge

(z_{23}, z_{13}, z_{12}) with the side s_3, while s_2 shares the ridge (z_{34}, z_{35}, z_{12}) with s_3.

Our Υ is the group $\langle S_1, S_2 : S_1^r = S_2^s = I, S_1S_2 = S_2S_1\rangle$. Now, consider the actions of S_1 and S_2 on our sides.

Starting with the action of S_1 on the side s_1, it shifts the ridge (z_{23}, z_{16}, z_{12}) to the ridge (z_{23}, z_{16}, z_{13}) which is another ridge on this side and also shifts it to another ridge $(z_{23}, z_{16}, z_{1\bar{3}23})$. Thus, S_1 just maps the side s_1 to another side sharing the ridge (z_{23}, z_{16}, z_{13}) with s_1. Continuing this and we end up with a looped sequence of sides with r distinct sides.

The transformation S_2 does the same to the side s_2 where it shifts s_2 to a side sharing the ridge z_{34}, z_{57}, z_{35} with s_2 and continues to shift the sides around until it goes back to s_2 again with a total of s sides in the sequence.

The side pairing ψ is defined by $\psi(s_1) = R_1$ and $R_1(s_1) = s_1'$, $\psi(s_2) = R_5$ and $R_5(s_2) = s_2'$, $\psi(s_3) = R_3$ and $R_3(s_3) = s_3'$ where s_1', s_2', s_3' are the mirrored sides of their respective sides in figure 4.7. Moreover, the side pairing maps $\psi(s_1'), \psi(s_2'), \psi(s_3')$ are just the inverses of their counterparts.

For a side Ss_i where $S \in \Upsilon$ and $i = 1, 2$ or 3, the side pairing map assigned to $\psi(Ss_i)$ is $S\psi(s_i)S^{-1}$ so that the side pairing ψ is compatible with Υ.

The map ψ we have defined now satisfies the first condition of a side pairing along with the compatibility with Υ. Since all the side pairing maps are cell-preserving maps and the images of sides under them only coincide in one side each, the condition $R^{-1}(E)\cap E = s$ is fulfilled for each side pairing maps R associated to the side s.

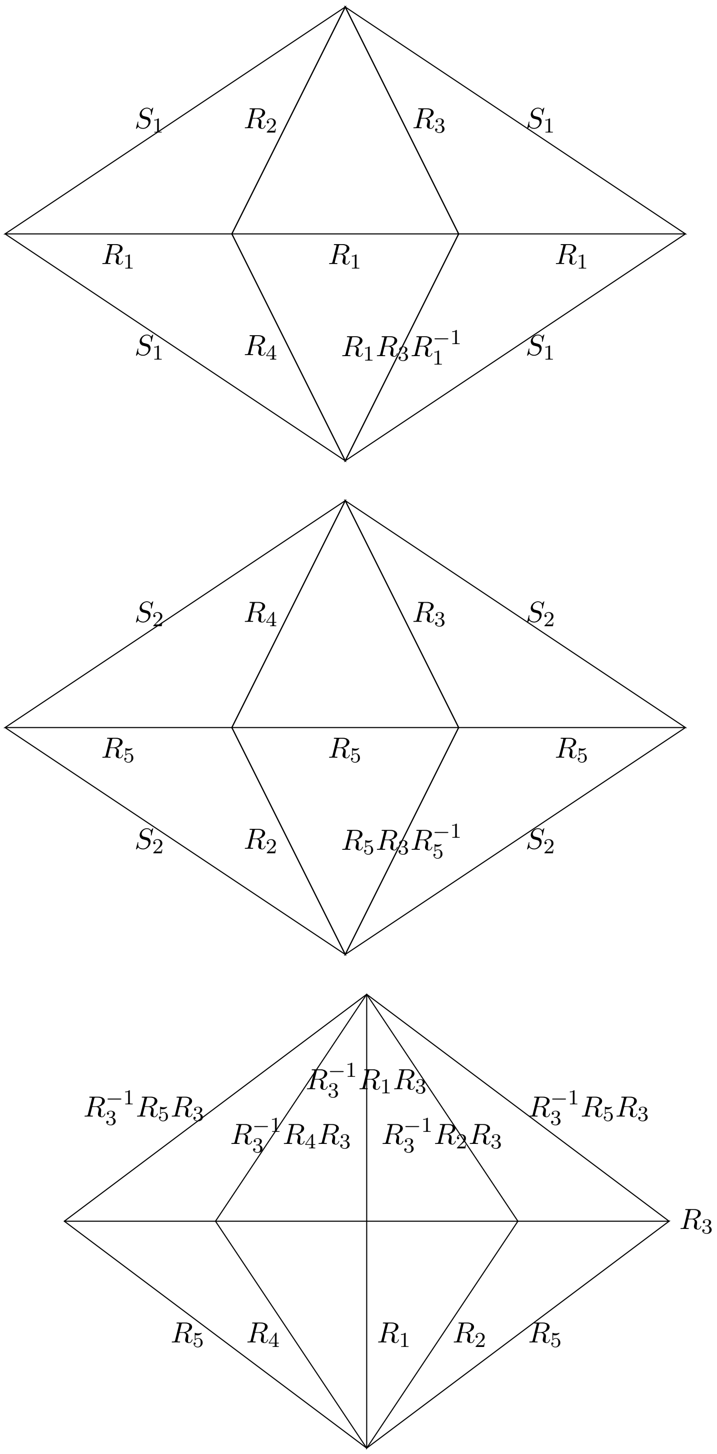

Figure 4.9: Some ridges and their images under the side pairing maps

There are also the ridges which lie on the mirrors of R_1, R_5 and R_3 on the sides

s_1, s_2, s_3, respectively, as well. These are preserved by their respective side pairing maps.

4.4 Ridge cycles

Each ridge lies in two sides. For example, $R_1; R_2, R_3$ and $R_3; R_2, R_1$ are the same ridge seen in s_1 and s_3, respectively. In this section, we construct ridge cycles starting from a ridge, passing through sides it is contained in and applying side pairing maps until it goes back to the original one.

We list the ridges, the sides they are contained in, the maps associated with the sides and the resulting cycles in table 4.1 first.

Note that, for the last three cycles in the table, the orders q', r', s' are the orders of R_1R_3, S_1R_3, S_2R_3, respectively. The words shown in the cycle column are the ridge cycles to the power of their orders so that they are all actually the identity.

Ridge	Sides	Map	Cycle
(z_{16}, z_{12}, z_{13})	$s_1 \cap s_1'$	R_1	
(z_{16}, z_{12}, z_{13})	$s_1 \cap s_1'$		R_1^q
(z_{57}, z_{45}, z_{35})	$s_2 \cap s_2'$	R_5	
(z_{57}, z_{45}, z_{35})	$s_2 \cap s_2'$		R_5^q
$(z_{13}, z_{23}, z_{53}, z_{43})$	$s_3 \cap s_3'$	R_3	
$(z_{13}, z_{23}, z_{53,z_{43}})$	$s_3 \cap s_3'$		R_3^p
(z_{23}, z_{16}, z_{12})	$s_1 \cap S_1^{-1}s_1$	R_1	
$(z_{123\bar{1}}, z_{16}, z_{12})$	$s_1' \cap S_1^{-1}s_1'$	$S_1^{-1}R_1^{-1}S_1$	
(z_{23}, z_{16}, z_{12})	$s_1 \cap S_1^{-1}s_1$		$S_1^{-1}R_1^{-1}S_1R_1$
(z_{34}, z_{57}, z_{12})	$s_2 \cap S_2^{-1}s_2$	R_5	
$(z_{543\bar{5}}, z_{57}, z_{12})$	$s_2' \cap S_2^{-1}s_2'$	$S_2^{-1}R_5^{-1}S_2$	
(z_{34}, z_{57}, z_{12})	$s_2 \cap S_2^{-1}s_2$		$S_2^{-1}R_5^{-1}S_2R_5$
(z_{23}, z_{12}, z_{13})	$s_1 \cap s_3$	R_1	
$(z_{123\bar{1}}, z_{12}, z_{13})$	$s_1' \cap S_2^{-1}s_3$	$S_2^{-1}R_3S_2$	
$(z_{123\bar{1}}, z_{12}, z_{1231\bar{2}\bar{1}})$	$S_2^{-1}R_3S_2R_1s_3\cap$	$S_2^{-1}S_1^{-1}R_3S_1S_2$	
	$S_2^{-1}S_1^{-1}s_3$		
$(z_{123\bar{1}}, z_{12312\bar{3}\bar{2}\bar{1}}, z_{1231\bar{2}\bar{1}})$	$R_1R_2R_3R_2^{-1}R_1^{-1}S_2^{-1}S_1^{-1}s_3\cap$	S_1S_2	
	$S_2^{-1}S_1^{-1}s_1$		
(z_{23}, z_{12}, z_{13})	$s_1 \cap s_3$		$R_3S_1R_3S_2R_1$
(z_{34}, z_{45}, z_{35})	$s_2 \cap s_3$	R_5	
$(z_{543\bar{5}}, z_{45}, z_{35})$	$s_2' \cap S_1^{-1}s_3$	$S_1^{-1}R_3S_1$	
$(z_{543\bar{5}}, z_{45}, z_{5435\bar{4}\bar{5}})$	$S_1^{-1}R_3S_1R_5s_3\cap$	$S_1^{-1}S_2^{-1}R_3S_2S_1$	
	$S_1^{-1}S_2^{-1}s_3$		
$(z_{543\bar{5}}, z_{54354\bar{3}\bar{4}\bar{5}}, z_{5435\bar{4}\bar{5}})$	$R_5R_4R_3R_4^{-1}R_5^{-1}S_1^{-1}S_2^{-1}s_3\cap$	S_2S_1	
	$S_1^{-1}S_2^{-1}s_2$		
$(z_{34}, z_{45}, (z_{35})$	$s_2 \cap s_3$		$R_3S_2R_3S_1R_5$
$(z_{12}^1, z_{12}^2, z_{12}^5, z_{12}^4)$	s_3	R_3^{-1}	
$(z_{\bar{3}123}^{\bar{3}13}, z_{\bar{3}123}^{\bar{3}23}, z_{\bar{3}123}^{\bar{3}53}, z_{\bar{3}123}^{\bar{3}43})$	s_3'	$S_1^{-1}S_2^{-1}$	
$(z_{12}^5, z_{12}^4, z_{12}^1, z_{12}^2)$	s_3		$(S_1^{-1}S_2^{-1}R_3^{-1})^{2q'}$
$(z_{23}^2, z_{23}^3, z_{23}^6)$	s_1	R_1	
$(z_{123\bar{1}}^{12\bar{1}}, z_{123\bar{1}}^{13\bar{1}}, z_{123\bar{1}}^6)$	s_1'	$S_1^{-1}S_2$	
$(z_{23}^2, z_{23}^3, z_{23}^6)$	s_1		$(S_1^{-1}S_2R_1)^{r'}$
$(z_{34}^4, z_{34}^3, z_{34}^7)$	s_2	R_5	
$(z_{543\bar{5}}^{54\bar{5}}, z_{543\bar{5}}^{53\bar{5}}, z_{543\bar{5}}^7)$	s_2'	$S_2^{-1}S_1$	
$(z_{34}^4, z_{34}^3, z_{34}^7)$	s_2		$(S_2^{-1}S_1R_5)^{s'}$

Table 4.1: Ridges and their corresponding sides, side pairing maps and ridge cycles

To see how this table came to be, we trace the actions of side pairing maps and the transformation on the ridges we have as done in section 1.2. Note that we can just add $S_1S_1^{-1}, S_1^{-1}S_1, S_2S_2^{-1}$ or $S_2^{-1}S_2$ in between the word to group maps in the word so that it is in the form AB where A is a word consisting of only elements from Υ and B is a word consisting of only side pairing maps. Thus, we can just find a sequence of maps for each ridge that preserve it and get a ridge cycle from that. For example, see the ridge cycles for the ridges (z_{23}, z_{13}, z_{12}) and (z_{34}, z_{35}, z_{45}) in figures 4.10 and 4.11.

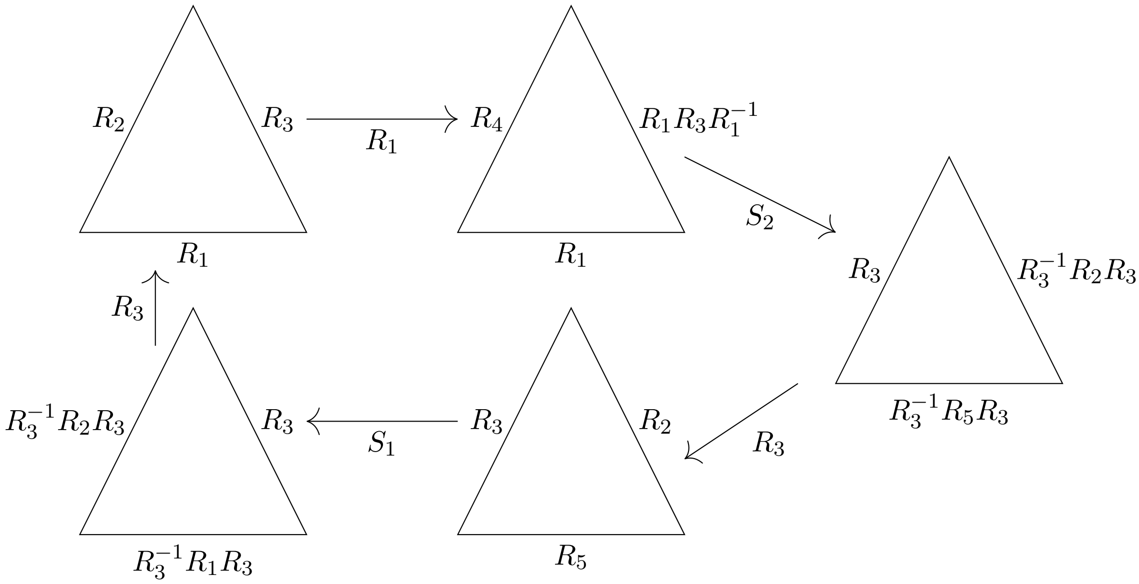

Figure 4.10: Ridge cycle of (z_{23}, z_{12}, z_{13})

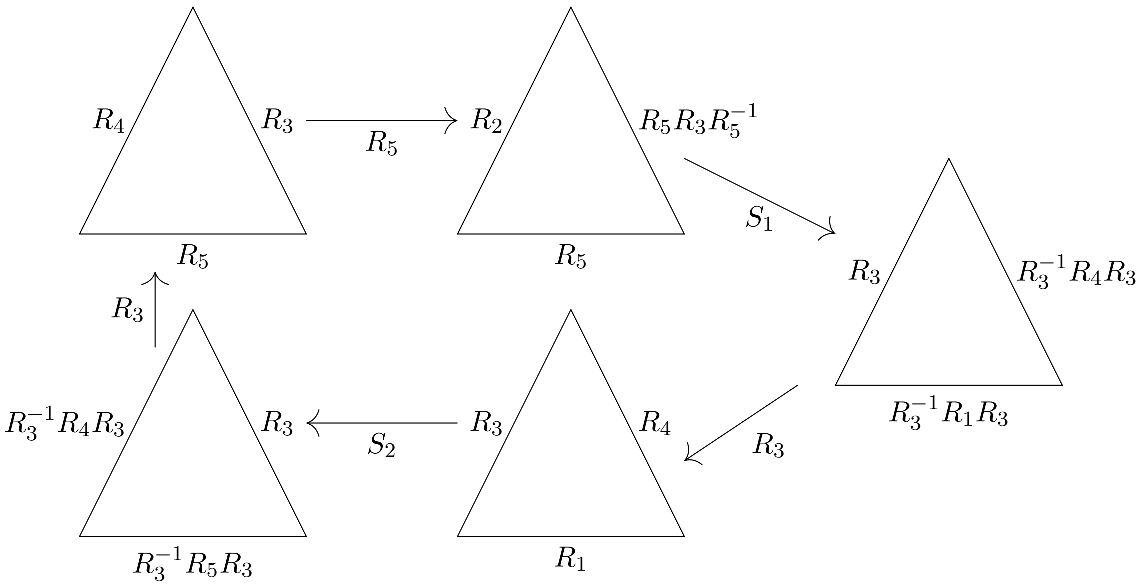

Figure 4.11: Ridge cycle of (z_{34}, z_{35}, z_{45})

The cycle we get for (z_{23}, z_{12}, z_{13}) is

$$\begin{aligned} R_3S_1R_3S_2R_1 &= R_3S_1(S_2S_2^{-1})R_3S_2R_1 \\ &= R_3S_1S_2(S_2^{-1}R_3S_2)R_1 \\ &= (S_1S_2(S_1S_2)^{-1})R_3(S_1S_2)(S_2^{-1}R_3S_2)R_1 \\ &= (S_1S_2)((S_1S_2)^{-1}R_3(S_1S_2))(S_2^{-1}R_3S_2)R_1. \end{aligned}$$

This corresponds to the method of tracing along the side pairing maps associated to the side a ridge is on since (z_{23}, z_{12}, z_{13}) is on the side s_1 with $\phi(s_1) = R_1$. Then the ridge is sent to $(z_{123\bar{1}}, z_{12}, z_{13}) = (z_{123\bar{1}}, z_{12}, z_{13})$ on the side s_1' by R_1. It also lies on the side $S_2^{-1}s_3$ with $\phi(S_2^{-1}s_3) = S_2^{-1}R_3S_2 = R_4$. Then, the ridge is sent by R_4 to the ridge $(z_{123\bar{1}}, z_{12}, z_{1231\bar{2}\bar{1}})$ This ridge lies on the sides $R_4S_2^{-1}s_3$ and $S_2^{-1}S_1^{-1}s_3$. The latter is associated with the side pairing map $\phi(S_2^{-1}S_1^{-1}s_3) = S_2^{-1}S_1^{-1}R_3S_1S_2$. The ridge is sent to $(z_{123\bar{1}}, z_{12312\bar{3}\bar{2}\bar{1}}, z_{1231\bar{2}\bar{1}})$ which is on the sides $R_1R_2R_3R_2^{-1}R_1^{-1}S_2^{-1}S_1^{-1}s_3$ and $S_2^{-1}S_1^{-1}s_1$. Thus, the map S_1S_2 sends it back to where it started.

Since the ridge (z_{34}, z_{35}, z_{45}) is symmetric to the previous ridge, it acts in the same way. This ridge is the intersection of s_2 and s_3 and is sent by R_5 to $(z_{543\bar{5}}, z_{45}, z_{35})$ on the sides s_2' and $S_1^{-1}s_3$. The corresponding transformation $\phi(S_1^{-1}s_3) = S_1^{-1}R_3S_1$ sends it to $(z_{543\bar{5}}, z_{45}, z_{5435\bar{4}\bar{5}})$ on the sides $S_1^{-1}R_3S_1R_5s_3$ and $S_1^{-1}S_2^{-1}s_3$. The latter side is associated with the side pairing map $S_1^{-1}S_2^{-1}R_3S_2S_1$ which sends the ridge to $(z_{543\bar{5}}, z_{54354\bar{3}\bar{4}\bar{5}}, z_{5435\bar{4}\bar{5}})$ on the sides $R_5R_4R_3R_4^{-1}R_5^{-1}S_1^{-1}S_2^{-1}s_3$ and $S_1^{-1}S_2^{-1}s_2$. Finally, S_1S_2 sends it back where we started.

These cycles show that the maps $R_3S_1R_3S_2R_1$ and $R_3S_2R_3S_1R_5$ are in the stabilizers for their respective ridges and, since these ridges are only preserved by the identity map, are exactly the identity map. This gives us the relation $S_2 = R_3^{-1}S_1^{-1}R_3^{-1}R_1^{-1} = R_3^{-1}R_2^{-1}R_1^{-1}S_1^{-1}$. Since

$$S_1R_1 = \begin{pmatrix} u^4\bar{y}^2 & u\bar{y}^2\rho & -u\bar{y}^2\rho \\ 0 & 0 & \bar{u}^2y^4 \\ 0 & -\bar{u}^2\bar{y}^2 & \bar{u}^2\bar{y}^2 + \bar{u}^2y^4 \end{pmatrix}$$

and

$$\begin{aligned} R_2R_3 &= \begin{pmatrix} \bar{v}^2 & 0 & 0 \\ -v\bar{\rho} & v^4 & \bar{v}^2y^3(y^3+\bar{y}^3) \\ 0 & 0 & \bar{v}^2 \end{pmatrix} \begin{pmatrix} \bar{v}^2 & 0 & 0 \\ 0 & \bar{v}^2 & 0 \\ v\bar{\rho} & -v^4\bar{y}^3(y^3+\bar{y}^3) & v^4 \end{pmatrix} \\ &= \begin{pmatrix} \bar{v}^4 & 0 & 0 \\ \bar{v}y^6\bar{\rho} & -v^2 - v^2y^6 - v^2\bar{y}^6 & v^2y^3(y^3+\bar{y}^3) \\ \bar{v}\bar{\rho} & -v^2\bar{y}^3(y^3+\bar{y}^3) & v^2 \end{pmatrix}, \end{aligned}$$

we have

$$S_2 = (S_1 R_1 R_2 R_3)^{-1} = \begin{pmatrix} u^4\bar{v}^4y^4 + \bar{u}^2v^2y^4 - \bar{u}^2v^2\bar{y}^2 & -uv^2y^4\rho & uv^2y^4\rho \\ \bar{u}^2\bar{v}y^4\bar{\rho} & -\bar{u}^2v^2y^4 - \bar{u}^2v^2\bar{y}^2 & \bar{u}^2v^2y^4 \\ \bar{u}^2\bar{v}\bar{y}^2\bar{\rho} & -\bar{u}^2v^2\bar{y}^2 & 0 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \bar{u}^4v^4y^2 & -\bar{u}v^4y^2\rho & \bar{u}v^4y^2\rho \\ \bar{u}^4vy^2\bar{\rho} & -u^2\bar{v}^2y^2 - \bar{u}^4v^4y^2 & \bar{u}^4v^4y^2 \\ \bar{u}^4v\bar{y}^4\bar{\rho} & -\bar{u}^4v^4\bar{y}^4 & -u^2\bar{v}^2y^2 + \bar{u}^4v^4\bar{y}^4 \end{pmatrix}.$$

By seeing all the ridges these cycles cover, we can see that all the ridges are in a ridge cycle as required. However, this is only for the non-truncation cases. If a truncation exists, there are some more ridges to consider.

4.4.1 Truncated Polyhedron

In the case that a vertex $\underline{v}$ of the polyhedron lies outside the space, i.e. $\langle \underline{v}, \underline{v} \rangle > 0$, the polyhedron is cut and gives birth to a new ridge. The ridge could be either a triangle or a rectangle depending on which vertex is cut. For instance, if the vertex z_{13} is outside the boundary, the vertex z_{12} which is in the same orbit also lies outside the boundary and this results in two more new ridges.

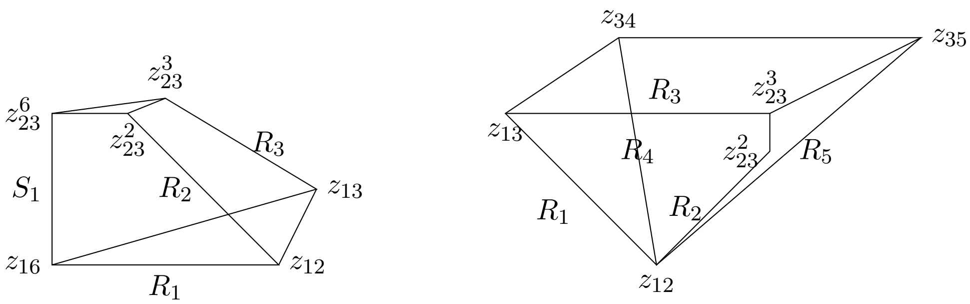

Figure 4.12: Polyhedron truncated at z_{23}

Symmetrically, in the case that the polyhedron is truncated at z_{34}, we have two new ridges identical to the previous case.

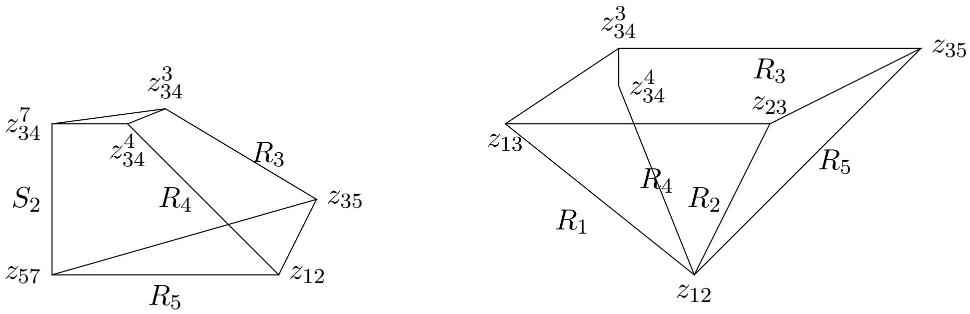

Figure 4.13: Polyhedron truncated at z_{34}

Another vertex which could be truncated at is z_{12}, which is actually named z_{12} and z_{45} as well. This angle is in the same orbit as z_{13} and z_{35} under S_1 and S_2, respectively, so the polyhedron is truncated through these two vertices as well.

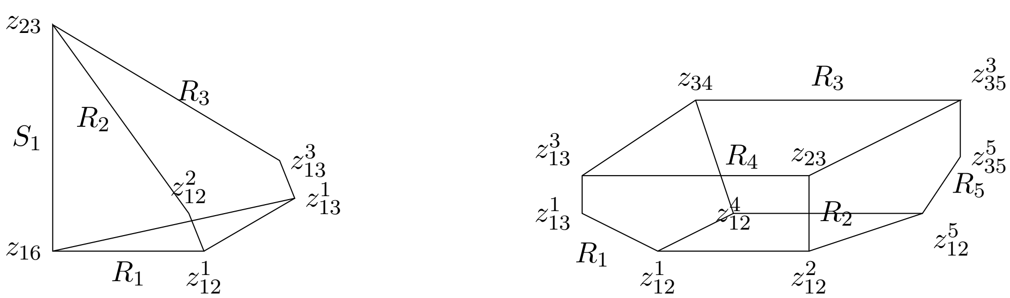

Figure 4.14: Polyhedron truncated at z_{12}

These truncations lead to new ridges at where the polyhedon is truncated at, we have a triangle with vertices on the mirrors of S_1, R_2, R_3, another triangle with S_2, R_4, R_3 and a rectangle with vertices on R_1, R_2, R_4, R_5, respectively.

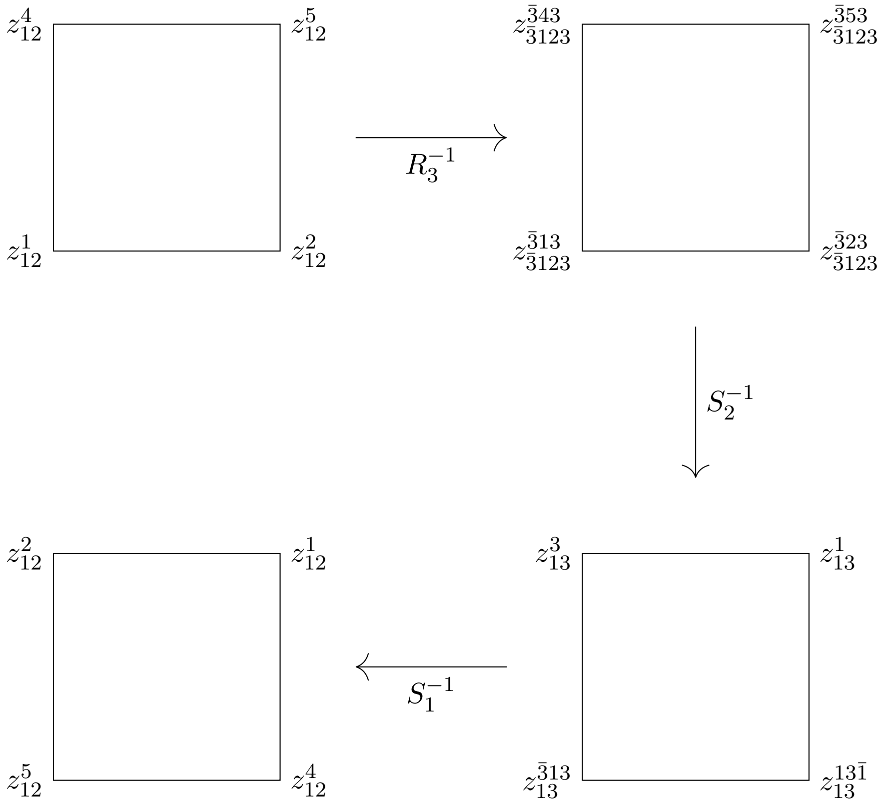

Figure 4.15: Half the ridge cycle of the ridge bounded by R_1, R_2, R_4, R_5

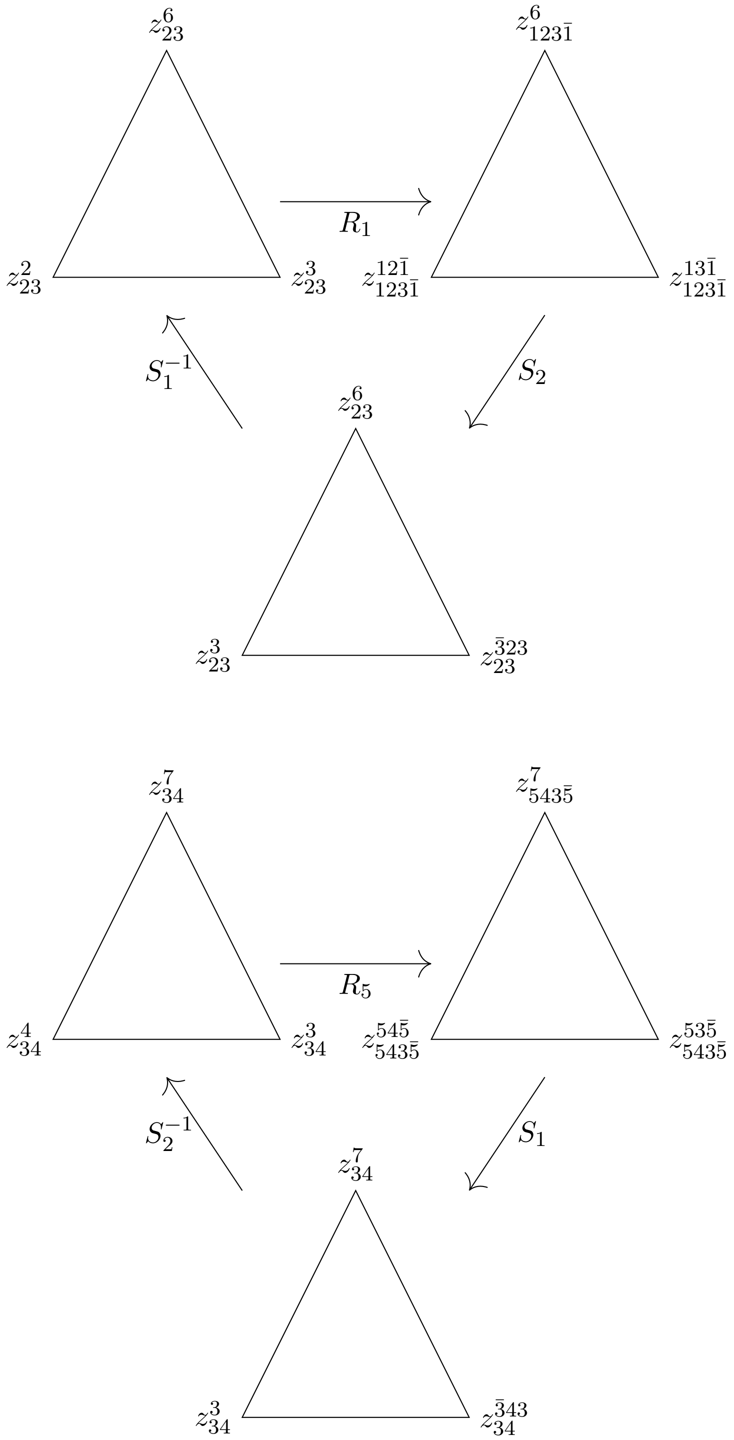

Figure 4.16: Ridge cycles for the ridges bounded by S_1, R_2, R_3 and by S_2, R_4, R_3

These cycles are exactly the cycle relations we need for each truncation. There are ridge cycle $(S_1^{-1}S_2^{-1}R_3^{-1})^2$ for the ridge $(z_{12}^1, z_{12}^2, z_{12}^5, z_{12}^4)$, ridge cycle $S_1^{-1}S_2^{-1}R_1$ for $(z_{23}^2, z_{23}^3, z_{23}^6)$ and ridge cycle $S_2^{-1}S_1R_5$ for $(z_{34}^3, z_{34}^4, z_{34}^7)$. These cycles need not be identity and we will consider their orders later in section 4.6.

4.5 Consistent system of horoballs

Another aspect to consider for our polyhedra is the existence of a consistent system of horoballs in the case where cusps exist. As stated earlier in section 1.2, the existence depends on whether the generators of the cusps' stabilisers are non-loxodromic. To see that, we consider the stabilisers for each vertex that could possibly be a cusp in the polyhedron.

The cusps exist when any of our vertices lies on the space boundary, namely, when q', r' or s' is ∞. Each case of ∞ is independent to another and two or more can exist at the same time like in the case of truncations.

We will now identify the stabilisers for each of our vertices. This can be done by taking into account all the cycles that preserve a vertex from the section 4.3. For a start, consider the vertex z_{12}. It lies on the sides s_1, s_2, s_3. The vertex is on the mirror of R_1 on the side s_1. On the side s_3, it is on the mirror of $R_2^{-1}R_1R_2$. This means that these two maps are in its stabiliser.

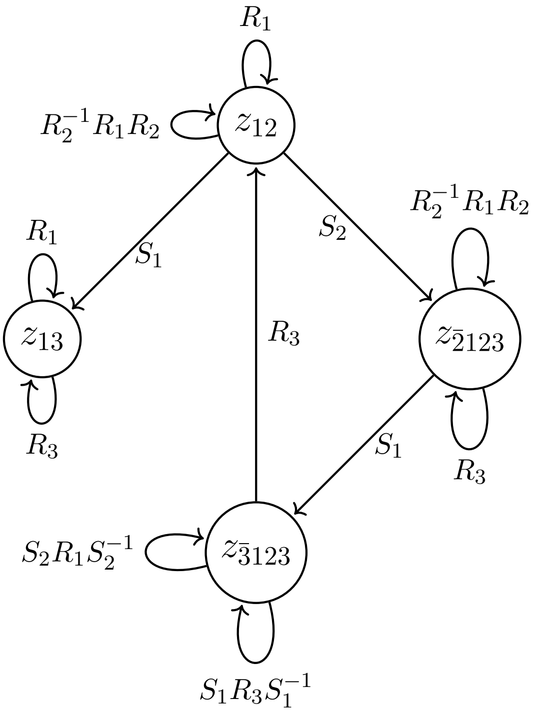

Figure 4.17: Cycles around the vertex z_{12}

By tracing through all of the cycles of z_{12} in our fundamental polyhedron, we obtained the figure 4.17. According to it, any cycles that preserve the vertex z_{12} would be a combination of words in the form

$$R_3 Z_{\bar{3}123} S_1 Z_{\bar{2}123} S_2 Z_{12},$$

where Z_{12} is a transformation in $\langle R_1, R_2^{-1}R_1R_2\rangle$, $Z_{\bar{2}123}$ is a transformation in $\langle R_3, R_2^{-1}R_1R_2\rangle$ and $Z_{\bar{3}123}$ is a transformation in $\langle S_2R_1S_2^{-1}, S_1R_3S_1^{-1}\rangle = \langle R_3^{-1}R_2^{-1}R_1R_2R_3, R_3^{-1}R_2R_3\rangle$. However, seeing as

$$\begin{aligned}
R_3S_2 &= S_2R_1R_2R_1^{-1},\\
R_2^{-1}R_1R_2S_2 &= S_2R_2^{-1}R_1R_2,\\
S_2R_1S_2^{-1}S_1S_2 &= S_1S_2R_1,\\
S_1R_3S_1^{-1}S_1S_2 &= S_1S_2R_1R_2R_1^{-1},\\
R_3S_2S_1 &= R_2^{-1}R_1^{-1},
\end{aligned}$$

any words generated from these would end up being generated by $R_1, R_2^{-1}R_1R_2, R_1R_2R_1^{-1}$, or simply just by R_1, R_2. Thus, the stabilizer for z_{12} is $\langle R_1, R_2\rangle$.

Proposition 4.5.1 (Proposition 4.6 of [16])**.** When $q' > 0$, the group $\langle R_1, R_2\rangle$ is the stabilizer of z_{12} and is a central extension of the orientation preserving subgroup of a $(2, p, q)$ triangle group by a group of order q', thus its order is $2q'^2$.

Proof. The eigenvalues of R_1 are $e^{\frac{4\pi i}{3q}}, e^{-\frac{2\pi i}{3q}}, e^{-\frac{2\pi i}{3q}}$. The eigenvalues of R_2 are $e^{\frac{4\pi i}{3p}}, e^{-\frac{2\pi i}{3p}}, e^{-\frac{2\pi i}{3p}}$. The eigenvalues of R_1R_2 are $ie^{\frac{\pi i}{3p}+\frac{\pi i}{3q}}, -ie^{\frac{\pi i}{3p}+\frac{\pi i}{3q}}, e^{-\frac{2\pi i}{3p}-\frac{2\pi i}{3q}}$. So, $\langle R_1R_2\rangle$ acts as the orientation preserving subgroup of a $(2, p, q)$ triangle group. We also have that $(R_1R_2)^2$ generates the centre of $\langle R_1, R_2\rangle$ with order q'. □

In the case where $q' \leq 0$, however, the truncation does not happen and there are no new ridges, hence, no new ridge cycles.

Lemma 4.5.2. If $q' < 0$, it is an integer.

Proof. From the cases of possible values for p and q, we have that q is either $2, 3, 4$ or 5. In the case that it is 2, $q' = -p$ which is an integer. When $q = 3$, negativity occurs only when p is $3, 4$ or 5 resulting in q' being $-6, -12$ and -30, respectively. When $q = 4$, p can only be 3 and $q' = -12$. The last case we have $(p, q, q') = (3, 5, -30)$. Hence, q' is an integer when it is negative. □

In the case of z_{23} and its orbit, we have the following figure:

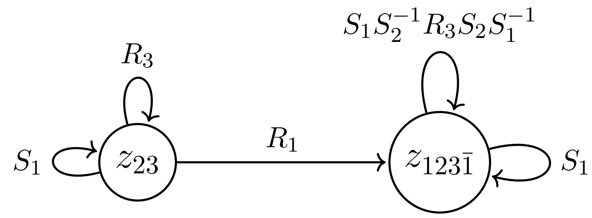

Figure 4.18: Cycles around the vertex z_{23}

As there are only the transformations R_3 and S_1 and no other cycles exist that preserve the vertex z_{23}, the stabiliser is $\langle R_3, S_1\rangle$. And, in the same way, we have for $z_{34} = z_{312\bar{1}}$ the figure:

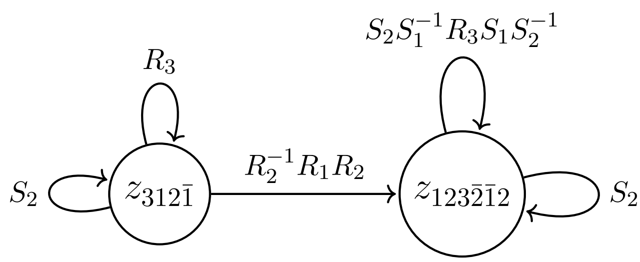

Figure 4.19: Cycles around the vertex z_{34}

This means its stabiliser is $\langle R_3, S_2 \rangle$.

By applying the same logic as we did to z_{12}, we get the following propositions.

Proposition 4.5.3. When $r' > 0$, the group $\langle S_1, R_2 \rangle$ is the stabilizer of z_{23} and is a central extension of the orientation preserving subgroup of a $(2, p, r)$ triangle group by a group of order r', thus its order is $2r'^2$.

Proposition 4.5.4. When $s' > 0$, the group $\langle S_2, R_3 \rangle$ is the stabilizer of z_{34} and is a central extension of the orientation preserving subgroup of a $(2, p, s)$ triangle group by a group of order s', thus its order is $2s'^2$.

Since there are automorphisms to interchange between q, r and s, we can use them to interchange q', r' and s', respectively. Thus, these propositions are derived from Proposition 4.5.1.

When $q' = \infty$, the vertex z_{12} is a cusp. As shown in Proposition 4.5.1, its stabiliser is $\langle R_1, R_2 \rangle$. Since the generators are both complex reflections, they are non-loxodromic. This means they preseve the horoballs and so do any of their combinations. Thus, we have a consistent system of horoballs on this cusp.

Similarly, the case when $r' = \infty$, the vertex z_{23} is a cusp and the stabiliser is $\langle S_1, R_3 \rangle$ while when $s' = \infty$, we get the cusp at z_{34} with stabiliser $\langle S_2, R_3 \rangle$. Since S_1, S_2, R_3 are all complex reflections, the same reasoning implies that all of the stabilisers preserve the horoballs as required.

In the case when there are truncations, there are extra vertices whose stabilisers might be irrelevant to the consistent system of horoballs, but are still useful when calculating for Euler characteristics.

Proposition 4.5.5. The stabiliser of z_{12}^1, when it exists, is $\langle R_1, (R_1 R_2)^2 \rangle$.

Proof. The figure 4.20 shows the actions of transformations on z_{12}^1.

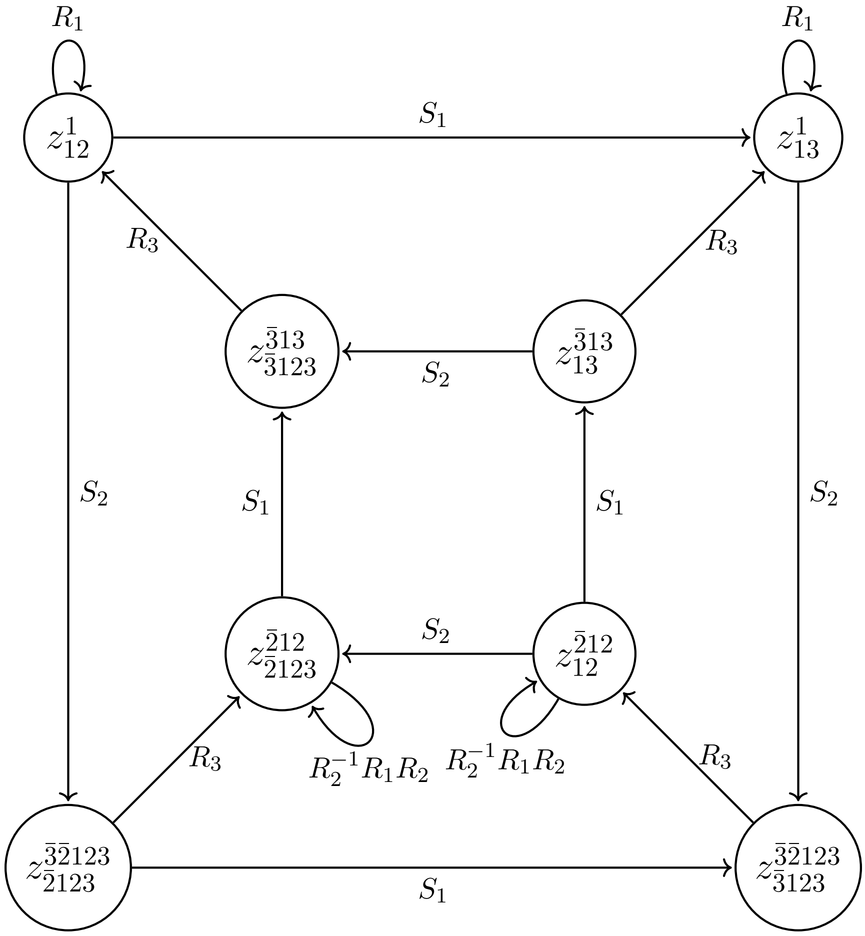

Figure 4.20: Cycles around the vertex z_{12}^{1}

We pick a spanning tree of the graph, which in practice can be any of them. The one we pick is as in 4.21. We want to represent each edge in the original graph with a circuit starting from z_{12}^{1} through the shortest path in the spanning tree to one end of the desired edge and return from the other end through another shortest path in the tree back to z_{12}^{1}. Note that, edges on the tree are represented by the identity map I.

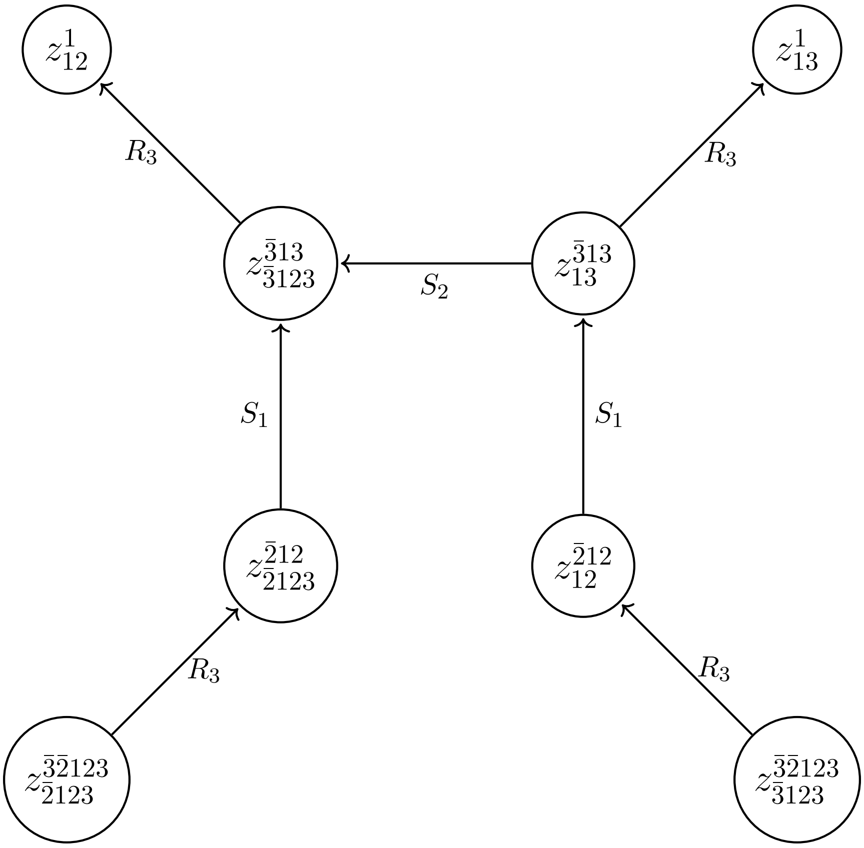

Figure 4.21: A spanning tree for the graph in figure 4.20

The resulting circuits are as in the following table:

Edge	Original transformation	Circuit
(z_{12}^{1}, z_{13}^{1})	S_1	$R_3S_2R_3^{-1}S_1$
		$= R_2^{-1}R_1^{-1}R_2^{-1}$
$(z_{13}^{1}, z_{3123}^{\bar{3}\bar{2}123})$	S_2	$R_3S_2S_1R_3S_2R_3S_2^{-1}R_3^{-1}$
		$= R_1^{-1}$
$(z_{3123}^{\bar{3}\bar{2}123}, z_{\bar{2}123}^{\bar{3}\bar{2}123})$	S_1^{-1}	$R_3S_1R_3S_1^{-1}R_3^{-1}S_1^{-1}S_2^{-1}R_3^{-1}$
		$= R_2R_1R_2$
$(z_{\bar{2}123}^{\bar{3}\bar{2}123}, z_{12}^{1})$	S_2^{-1}	$S_2^{-1}R_3^{-1}S_1^{-1}R_3^{-1}$
		$= R_1$
$(z_{12}^{\bar{2}12}, z_{\bar{2}123}^{\bar{2}12})$	S_2	$R_3S_1S_2S_1^{-1}S_2^{-1}R_3^{-1}$
		$= I$
(z_{13}^{1}, z_{13}^{1})	R_1	$R_3S_2R_3^{-1}R_1R_3S_2^{-1}R_3^{-1}$
		$= R_1$
$(z_{12}^{\bar{2}12}, z_{12}^{\bar{2}12})$	$R_2^{-1}R_1R_2$	$R_3S_2S_1R_2^{-1}R_1R_2S_1^{-1}S_2^{-1}R_3^{-1}$
		$= R_1$
$(z_{\bar{2}123}^{\bar{2}12}, z_{\bar{2}123}^{\bar{2}12})$	$R_2^{-1}R_1R_2$	$R_3S_1R_2^{-1}R_1R_2S_1^{-1}R_3^{-1}$
		$= R_1$

In this way, we can represent any circuit starting at z_{12}^{1} by a combination of these circuits through the tree. As all the circuits can be generated by $R_1, (R_1R_2)^2$, and since both of them are indeed in the stabiliser, the stabiliser of z_{12}^{1} is $\langle R_1, (R_1R_2)^2\rangle$. $\square$

Following the proposition, we can find the stabiliser for $z_{12}^{\bar{2}12}$ in the same way.

Proposition 4.5.6. The stabiliser of $z_{12}^{\bar{2}12}$, when it exists, is $\langle R_2^{-1}R_1R_2, (R_1R_2)^2\rangle$.

Proof. The graph around $z_{12}^{\bar{2}12}$ is the same as that of z_{12}^{1} from 4.5.5. We also chose the same spanning tree. Since there is a path $R_3S_2S_1$ from $z_{12}^{\bar{2}12}$ to z_{12}^{1}, if we extend a circuit C starting from z_{12}^{1} into a circuit $S_1^{-1}S_2^{-1}R_3^{-1}CR_3S_2S_1$, we get a circuit starting from $z_{12}^{\bar{2}12}$, instead. In fact, we get every of its circuits this way. Since R_1 and $(R_1R_2)^2$ generate the stabiliser for z_{12}^{1}, their equivalents $S_1^{-1}S_2^{-1}R_3^{-1}R_1R_3S_2S_1 = R_2^{-1}R_1R_2$ and $S_1^{-1}S_2^{-1}R_3^{-1}R_1R_2R_1R_2R_3S_2S_1 = (R_1R_2)^2$ generate the stabiliser for $z_{12}^{\bar{2}12}$. $\square$

Now, for another orbit of vertices $z_{12}^{2}, z_{12}^{12\bar{1}}$, we have a similar figure of circuits around them.

Proposition 4.5.7. The stabiliser of z_{12}^{2}, when it exists, is $\langle R_2, (R_1R_2)^2\rangle$.

Proof. As we have done in 4.5.5, we have the graph:

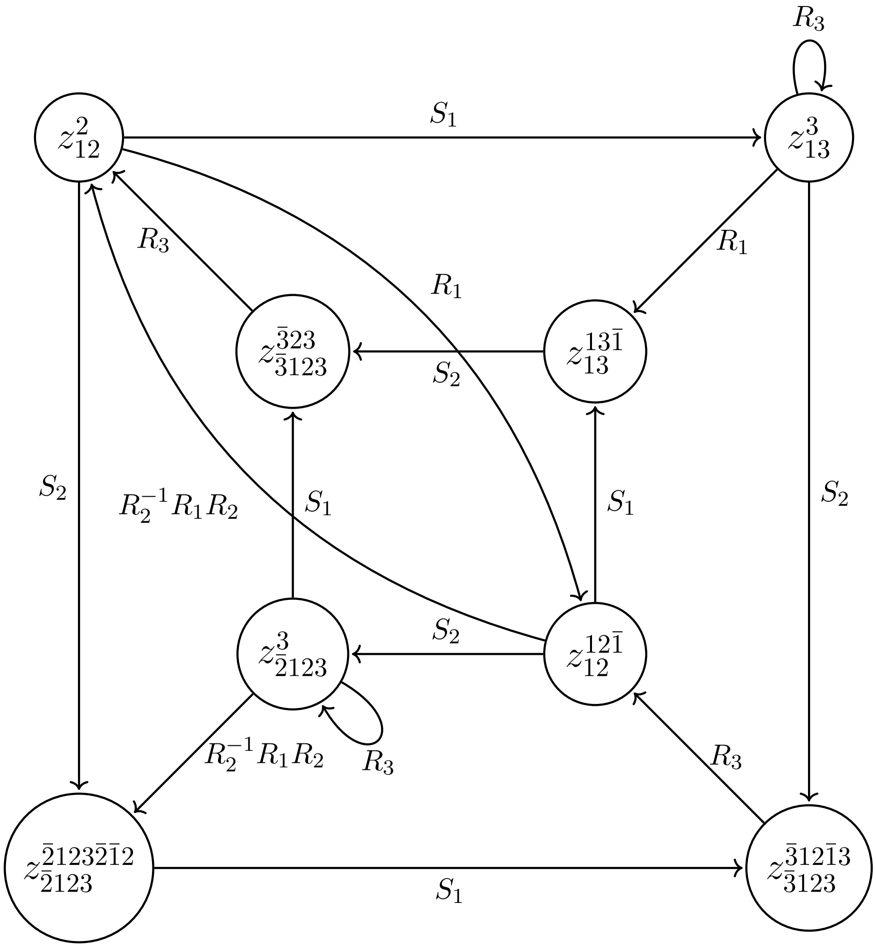

Figure 4.22: Cycles around the vertex z^2_{12}

The spanning tree we choose is:

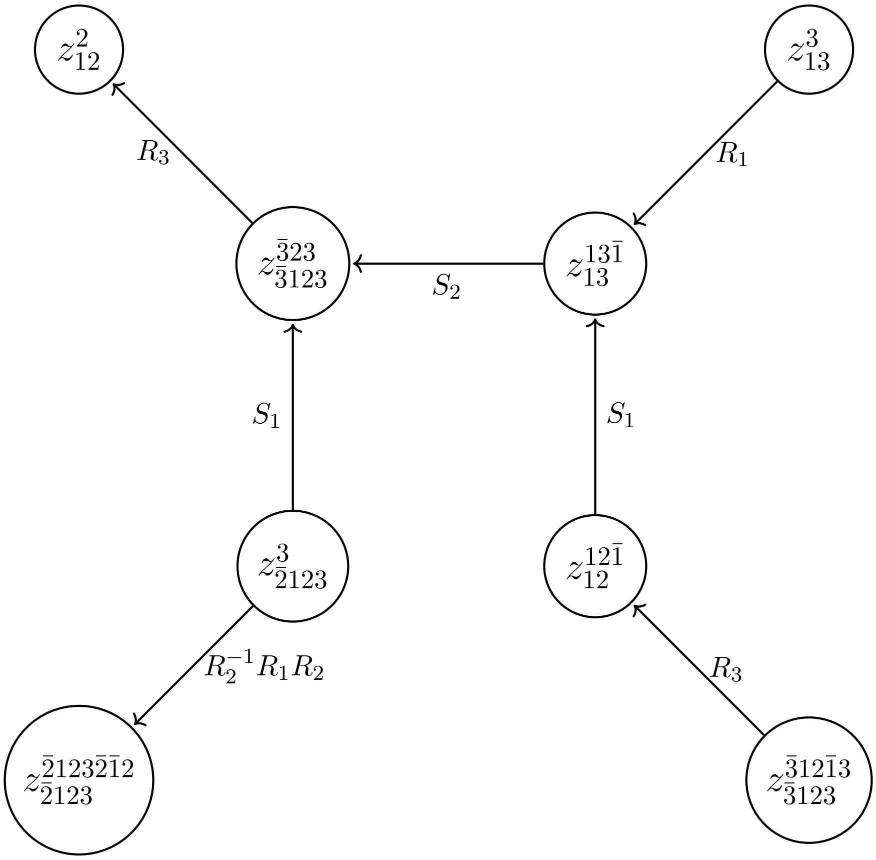

Figure 4.23: A spanning tree for the graph in figure 4.20

The table we will get for this one is:

Edge	Original transformation	Circuit
(z^2_{12}, z^3_{13})	S_1	$R_3S_2R_1S_1$ $= R_2^{-1}$
$(z^3_{13}, z^{\bar{3}12\bar{1}3}_{\bar{3}123})$	S_2	$R_3S_2S_1R_3S_2R_1^{-1}S_2^{-1}R_3^{-1}$ $= R_1^{-1}R_2^{-1}R_1^{-1}$
$(z^{\bar{3}12\bar{1}3}_{\bar{3}123}, z^{\bar{2}123\bar{2}\bar{1}2}_{\bar{2}123})$	S_1^{-1}	$R_3S_1R_2^{-1}R_1^{-1}R_2S_1^{-1}R_3^{-1}S_1^{-1}S_2^{-1}R_3^{-1}$ $= R_2$
$(z^{\bar{2}123\bar{2}\bar{1}2}_{\bar{2}123}, z^2_{12})$	S_2^{-1}	$S_2^{-1}R_2^{-1}R_1R_2S_1^{-1}R_3^{-1}$ $= R_1R_2R_1$
$(z^{12\bar{1}}_{12}, z^3_{\bar{2}123})$	S_2	$R_3S_1S_2S_1^{-1}S_2^{-1}R_3^{-1}$ $= I$
$(z^2_{12}, z^{12\bar{1}}_{12})$	R_1	$R_3S_2S_1R_1$ $= R_2^{-1}$
$(z^{12\bar{1}}_{12}, z^2_{12})$	$R_2^{-1}R_1R_2$	$R_2^{-1}R_1R_2S_1^{-1}S_2^{-1}R_3^{-1}$ $= R_1R_2R_1$
(z^3_{13}, z^3_{13})	R_3	$R_3S_2R_1R_3R_1^{-1}S_2^{-1}R_3^{-1}$ $= R_2$
$(z^3_{\bar{2}123}, z^3_{\bar{2}123})$	R_3	$R_3S_1R_3S_1^{-1}R_3^{-1}$ $= R_2$

Hence, the stabiliser of z^2_{12} is $\langle R_2, (R_1R_2)^2\rangle$. □

Proposition 4.5.8. The stabiliser of $z^{12\bar{1}}_{12}$, when it exists, is $\langle R_1R_2R_1^{-1}, (R_1R_2)^2\rangle$.

Proof. There is an edge R_1 going from z^2_{12} to $z^{12\bar{1}}_{12}$. By the same argument as in 4.5.6, the circuits starting at $z^{12\bar{1}}_{12}$ are exactly the circuits in the form $R_1CR_1^{-1}$ where C is a circuit starting at z^2_{12}. According to 4.5.7, the stabiliser of z^2_{12} is $\langle R_2, (R_1R_2)^2\rangle$. Thus, the stabiliser of $z^{12\bar{1}}_{12}$ is $R_1\langle R_2, (R_1R_2)^2\rangle R_1^{-1} = \langle R_1R_2R_1^{-1}, (R_1R_2)^2\rangle$. □

Proposition 4.5.9. The stabiliser of z^2_{23}, when it exists, is $\langle R_2, (S_1R_2)^2\rangle$.

Proof. Following 4.5.5, the graph for circuits around z^2_{23} and the chosen spanning tree is:

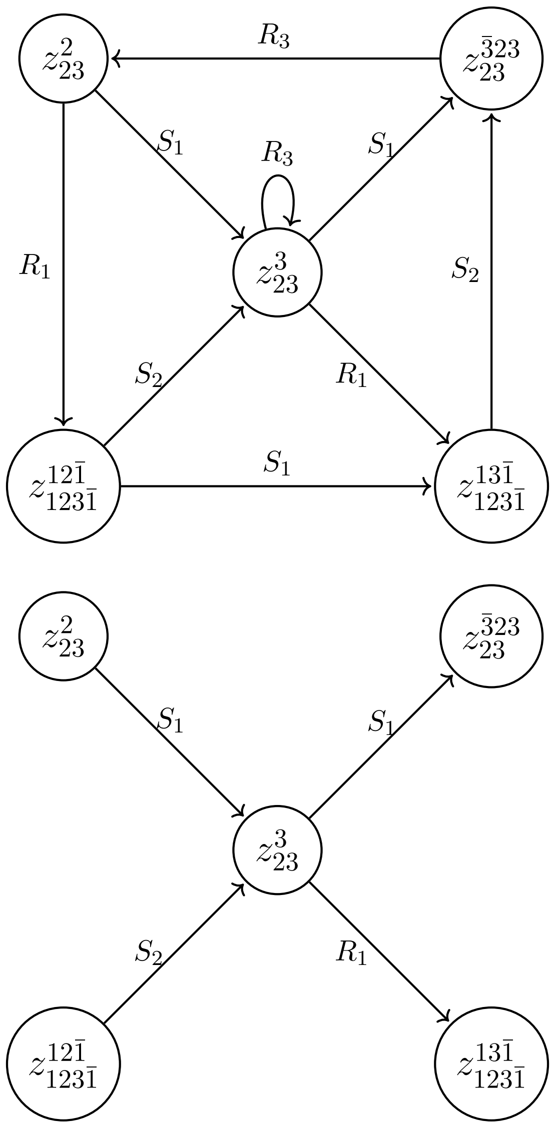

Figure 4.24: Circuits around the vertex z_{23}^{2} and its spanning tree

Each edge is represented by the circuits as in the following table:

Edge	Original transformation	Circuit
$(z_{23}^{2}, z_{23}^{\bar{3}23})$	R_3^{-1}	$S_1^{-2}R_3^{-1}$
		$=S_1^{-1}R_2^{-1}S_1^{-1}$
$(z_{23}^{\bar{3}23}, z_{123\bar{1}}^{13\bar{1}})$	S_2^{-1}	$S_1^{-1}R_1^{-1}S_2^{-1}S_1^{2}$
		$=R_2S_1R_2S_1$
$(z_{123\bar{1}}^{13\bar{1}}, z_{123\bar{1}}^{12\bar{1}})$	S_1^{-1}	$S_1^{-1}S_2S_1^{-1}R_1S_1$
		$=S_1^{-1}R_2^{-1}S_1^{-1}R_2^{-1}$
$(z_{123\bar{1}}^{12\bar{1}}, z_{23}^{2})$	R_1^{-1}	$R_1^{-1}S_2^{-1}S_1$
		$=S_1R_2S_1R_2$
(z_{23}^{3}, z_{23}^{3})	R_3	$S_1^{-1}R_3S_1$
		$=R_2$

All the circuits are generated exactly by R_2 and $(S_1R_2)^2$. Thus, the stabiliser of z_{23}^{2} is $\langle R_2, (S_1R_2)^2\rangle$ □

Proposition 4.5.10. The stabiliser of z_{23}^{3}, when it exists, is $\langle R_3, (S_1R_3)^2\rangle$.

Proof. According to 4.24, the circuits starting at z_{23}^{3} are as in the table:

Edge	Original transformation	Circuit
$(z_{23}^{2}, z_{23}^{\bar{3}23})$	R_3^{-1}	$S_1^{-1}R_3^{-1}S_1^{-1}$
$(z_{23}^{\bar{3}23}, z_{123\bar{1}}^{13\bar{1}})$	S_2^{-1}	$R_1^{-1}S_2^{-1}S_1$
		$=R_3S_1R_3S_1$
$(z_{123\bar{1}}^{13\bar{1}}, z_{123\bar{1}}^{12\bar{1}})$	S_1^{-1}	$S_2S_1^{-1}R_1$
		$=S_1^{-1}R_3^{-1}S_1^{-1}R_3^{-1}$
$(z_{123\bar{1}}^{12\bar{1}}, z_{23}^{2})$	R_1^{-1}	$S_1R_1^{-1}S_2^{-1}$
		$=S_1R_3S_1R_3$

The circuits are generated by R_3 and $S_1R_3S_1R_3$. Hence, the stabiliser of z_{23}^{3} is $\langle R_3, (S_1R_3)^2\rangle$ □

Proposition 4.5.11. The stabiliser of $z_{12\bar{1}3}^{12\bar{1}}$, when it exists, is $\langle R_1R_2R_1^{-1}, (S_2R_1R_2R_1^{-1})^2\rangle$.

Proof. Symmetrically to 4.5.9, we can draw the graph around $z_{12\bar{1}3}^{12\bar{1}}$ as:

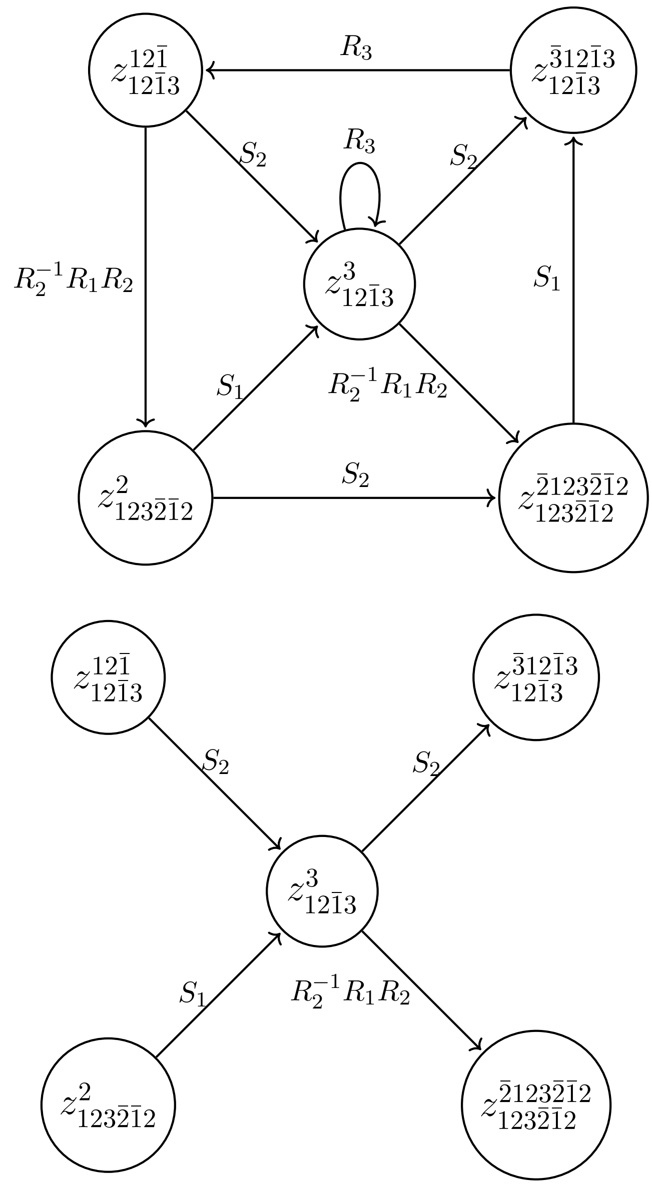

Figure 4.25: Circuits around the vertex $z^{12\bar{1}}_{12\bar{1}3}$ and its spanning tree

The circuits representation of each edge is then:

Edge	Original transformation	Circuit
$(z_{12\bar{1}3}^{12\bar{1}}, z_{12\bar{1}3}^{\bar{3}12\bar{1}3})$	R_3^{-1}	$S_2^{-2}R_3^{-1}$
		$=S_2^{-1}R_1R_2^{-1}R_1^{-1}S_2^{-1}$
$(z_{12\bar{1}3}^{\bar{3}12\bar{1}3}, z_{123\bar{2}\bar{1}2}^{\bar{2}123\bar{2}\bar{1}2})$	S_1^{-1}	$S_2^{-1}R_2^{-1}R_1^{-1}R_2S_1^{-1}S_2^2$
		$=S_2R_1R_2R_1^{-1}S_2R_1R_2R_1^{-1}$
$(z_{123\bar{2}\bar{1}2}^{\bar{2}123\bar{2}\bar{1}2}, z_{123\bar{2}\bar{1}2}^{2})$	S_2^{-1}	$S_2^{-1}S_1S_2^{-1}R_2^{-1}R_1R_2S_2$
		$=S_2^{-1}R_1R_2^{-1}R_1^{-1}S_2^{-1}R_1R_2^{-1}R_1^{-1}$
$(z_{123\bar{2}\bar{1}2}^{2}, z_{12\bar{1}3}^{12\bar{1}})$	$R_2^{-1}R_1^{-1}R_2$	$R_2^{-1}R_1^{-1}R_2S_1^{-1}S_2$
		$=S_2R_1R_2R_1^{-1}S_2R_1R_2R_1^{-1}$
$(z_{12\bar{1}3}^{3}, z_{12\bar{1}3}^{3})$	R_3	$S_2^{-1}R_3S_2$
		$=R_1R_2R_1^{-1}$

The generators for all these words are $R_1R_2R_1^{-1}$ and $(S_2R_1R_2R_1^{-1})^2$. Thus, the stabiliser is $\langle R_1R_2R_1^{-1}, (S_2R_1R_2R_1^{-1})^2\rangle$. $\square$

Proposition 4.5.12. The stabiliser of $z_{12\bar{1}3}^{3}$, when it exists, is $\langle R_3, (S_2R_3)^2\rangle$.

Proof. From the graph 4.25, we can get the circuits starting at $z_{12\bar{1}3}^{3}$ as:

Edge	Original transformation	Circuit
$(z_{12\bar{1}3}^{12\bar{1}}, z_{12\bar{1}3}^{\bar{3}12\bar{1}3})$	R_3^{-1}	$S_2^{-1}R_3^{-1}S_2^{-1}$
$(z_{12\bar{1}3}^{\bar{3}12\bar{1}3}, z_{123\bar{2}\bar{1}2}^{\bar{2}123\bar{2}\bar{1}2})$	S_1^{-1}	$R_2^{-1}R_1^{-1}R_2S_1^{-1}S_2$
		$=R_3S_2R_3S_2$
$(z_{123\bar{2}\bar{1}2}^{\bar{2}123\bar{2}\bar{1}2}, z_{123\bar{2}\bar{1}2}^{2})$	S_2^{-1}	$S_1S_2^{-1}R_2^{-1}R_1R_2$
		$=S_2^{-1}R_3S_2^{-1}R_3$
$(z_{123\bar{2}\bar{1}2}^{2}, z_{12\bar{1}3}^{12\bar{1}})$	$R_2^{-1}R_1^{-1}R_2$	$S_2R_2^{-1}R_1^{-1}R_2S_1^{-1}$
		$=R_3S_2R_3S_2$

The generators are $R_3, (S_2R_3)^2$ and so, the stabiliser is $\langle R_3, (S_2R_3)^2\rangle$. $\square$

4.6 Angles of central elements

In the section 4.3, we have obtained some words which preserve ridges and thus, are contained in some stabilisers. Moreover, we claim that these words are indeed central elements of the stabilisers as well.

Lemma 4.6.1. The words $(S_1^{-1}S_2^{-1}R_3^{-1})^2, S_1^{-1}S_2R_1$ and $S_2^{-1}S_1R_5$ are central elements in the groups $\langle R_1, R_2\rangle, \langle S_1, R_2\rangle$ and $\langle S_2, R_3\rangle$, respectively.

Proof. Using the fact that $S_1S_2R_1R_2R_3 = I$, we have

$$\begin{aligned}(S_1^{-1}S_2^{-1}R_3^{-1})^2 &= (S_2^{-1}S_1^{-1}R_3^{-1})^2 \\ &= (R_1R_2)^2.\end{aligned}$$

Also,

$$\begin{aligned}S_1^{-1}S_2R_1 &= S_1^{-1}S_1^{-1}R_3^{-1}R_2^{-1} \\ &= (S_1^{-1}R_2^{-1})^2.\end{aligned}$$

Now, because $R_3S_2R_3S_1R_5 = I$, we have

$$S_2^{-1}S_1R_5 = (S_2^{-1}R_3^{-1})^2.$$

So, the three words become simpler words $(R_1R_2)^2, (S_1^{-1}R_2^{-1})^2, (S_2^{-1}R_3^{-1})^2$. Since $\mathrm{br}(R_1, R_2) = \mathrm{br}(S_1, R_2) = \mathrm{br}(S_2, R_3) = 4$, they commute with the generators of, and thus are central elements of, the groups $\langle R_1, R_2\rangle, \langle S_1, R_2\rangle$ and $\langle S_2, R_3\rangle$, respectively. □

The orders of R_1 and R_5 are both q and the orders of R_2, R_3 and R_4 are p. The order of S_1 is the same as the braid length between R_2 and R_3 which is r. Let the order of S_2 be s. Note that, by the symmetry between (S_1, R_1, R_2, R_3) and (S_2, R_5, R_4, R_3), the braid length between R_4 and R_3 is s, the order of S_2.
Now, since

$$S_2 = \begin{pmatrix} \bar{u}^4v^4y^2 & -\bar{u}v^4y^2\rho & \bar{u}v^4y^2\rho \\ \bar{u}^4vy^2\bar{\rho} & -u^2\bar{v}^2y^2 - \bar{u}^4v^4y^2 & \bar{u}^4v^4y^2 \\ \bar{u}^4v\bar{y}^4\bar{\rho} & -\bar{u}^4v^4\bar{y}^4 & -u^2\bar{v}^2y^2 + \bar{u}^4v^4\bar{y}^4 \end{pmatrix},$$

its eigenvalues are $\bar{u}^4v^4\bar{y}^4, -u^2\bar{v}^2y^2, -u^2\bar{v}^2y^2$. This means that $\frac{1}{2} + \frac{1}{p} - \frac{1}{q} - \frac{1}{r}$ is an integer multiple of $\frac{1}{s}$ and we assume that it is indeed $\frac{1}{s}$. Thus, $\frac{1}{2} + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$.

To make things simpler, we try to see the symmetry between the maps that we have by assuming an automorphism from $(R_1, R_2, R_3, S_1, S_2)$ to $(R_5, R_4, R_3, S_2, S_1)$. These two presentations are indeed equivalent as $\mathrm{br}(R_1, R_2) = \mathrm{br}(R_1, R_3) = \mathrm{br}(R_1, R_3^{-1}R_2R_3) = \mathrm{br}(R_3, R_5) = \mathrm{br}(R_4, R_5) = \mathrm{br}(R_3^{-1}R_4R_3, R_5) = 4$ and $\mathrm{br}(R_2, R_3) = r = |S_1|$ whilst $\mathrm{br}(R_4, R_3) = s = |S_2|$.
There is also an automorphism from $(R_1, R_2, R_3, S_1, S_2)$ to $(S_1, R_2, R_4, R_1, R_3S_2R_3^{-1})$ as the braiding relations $\mathrm{br}(S_1, R_2) = \mathrm{br}(S_1, R_4) = \mathrm{br}(S_1, R_4^{-1}R_2R_4) = 4$ as well and we would want $\mathrm{br}(R_2, R_4)$ to be the same as the order of R_1, which is q.

Lemma 4.6.2. The maps R_2 and R_4 braid with length q.

Proof. Consider that, for an integer k, $(R_2R_4)^k = (R_2R_1R_2)^k(R_1)^{-k}$ and $(R_4R_2)^k = R_1R_2(R_2R_1R_2)^{k-1}R_1^{-k}R_2$. Then,

$$\begin{aligned}(R_2R_4)^k(R_4R_2)^{-k} &= (R_1)^{-k}(R_2R_1R_2)^kR_2^{-1}(R_2R_1R_2)^{1-k}R_1^kR_2^{-1}R_1^{-1}\\ &= (R_1)^{-k}(R_2R_1R_2)^{k-1}R_2(R_2R_1R_2)^{1-k}R_1^{k+1}R_2^{-1}R_1^{-1}\\ &= (R_1)^{-k-1}(R_2R_1R_2)^{k-2}R_2(R_2R_1R_2)^{2-k}R_1^{k+2}R_2^{-1}R_1^{-1}\\ &\vdots\\ &= (R_1)^{1-2k}R_2(R_1)^{2k}R_2^{-1}R_1^{-1}.\end{aligned}$$

When $2k = q$, we have that $(R_2R_4)^k = (R_4R_2)^k$.
In the case of $(R_2R_4)^kR_2$ and $(R_4R_2)^kR_4$, we have $(R_2R_4)^kR_2 = (R_2R_1R_2)^k(R_1)^{-k}R_2$ and $(R_4R_2)^kR_4 = R_1R_2(R_2R_1R_2)^kR_1^{-k-1}$. Then,

$$\begin{aligned}(R_2R_4)^kR_2R_4^{-1}(R_4R_2)^-k &= (R_1)^{-k}(R_2R_1R_2)^kR_2(R_2R_1R_2)^{-k}R_1^{k+1}R_2^{-1}R_1^{-1}\\ &= (R_1)^{-k-1}(R_2R_1R_2)^{k-1}R_2(R_2R_1R_2)^{1-k}R_1^{k+2}R_2^{-1}R_1^{-1}\\ &= (R_1)^{-k-2}(R_2R_1R_2)^{k-2}R_2(R_2R_1R_2)^{2-k}R_1^{k+3}R_2^{-1}R_1^{-1}\\ &\vdots\\ &= (R_1)^{-2k}R_2(R_1)^{2k+1}R_2^{-1}R_1^{-1}.\end{aligned}$$

When $2k+1 = q$, we have that $(R_2R_4)^kR_2 = (R_4R_2)^kR_4$.
The group $\langle R_1, R_2\rangle$ is a central extension of a triangle group $(2, p, q)$ with only the relations of the forms $R_1^q, R_2^p, (R_1R_2)^2$. If there is a power j such that $(R_2R_4)^j = (R_4R_2)^j$, then we have in the triangle group

$$(R_2R_4)^j = (R_2R_1R_2)^jR_1^{-j} = R_1^{-2j}$$

and

$$(R_4R_2)^j = R_2^{-1}(R_2R_1R_2)^jR_1^{-j}R_2 = R_2^{-1}R_1^{-2j}R_2 = (R_2^{-1}R_1R_2)^{-2j}.$$

In another case when $(R_2R_4)^jR_2 = (R_4R_2)^jR_4$, we have

$$(R_2R_4)^jR_2 = (R_2R_1R_2)^jR_1^{-j}R_2 = R_1^{-2j}R_2$$

and

$$(R_4R_2)^jR_4 = R_2^{-1}(R_2R_1R_2)^jR_1^{-j-1} = R_1R_2R_1^{-2j-1}.$$

In any case, for the two to be equal, we need $R_1^{-2j} = (R_2^{-1}R_1R_2)^{-2j}$ or $R_1^{-2j-1} = (R_2^{-1}R_1R_2)^{-2j-1}$. As both sides are powers of elliptic maps with distinct fixed points, they are equal if and only if they are identity, meaning that the power is a multiple of q. Hence, R_2 and R_4 braid with length q. □

This means that groups with parameters $(p; q, r, s)$ and $(p; q, s, r)$ are isomorphic and the groups with parameters $(p; q, r, s)$ and $(p; s, r, q)$ are isomorphic. So the permutation among (q, r, s) are allowed and we can assume $q \le r \le s$ when calculating for possible values of these parameteres.

4.7 Possible parameters

Let $q' = \frac{2pq}{pq-2p-2q}, r' = \frac{2pr}{pr-2p-2r}, s' = \frac{2ps}{ps-2p-2s}$. We note that $\frac{1}{q'}, \frac{1}{r'}, \frac{1}{s'}$ are less than a half so q', r', s' are greater than 2 when they are positive.
Since $\frac{3}{q} > \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = \frac{1}{p} + \frac{1}{2} > \frac{1}{2}$, $2 < q < 6$. Moreover, from the relations $\frac{1}{2} - \frac{1}{p} - \frac{1}{r} \leq \frac{1}{3}$ and $\frac{1}{2} - \frac{1}{p} - \frac{1}{s} \leq \frac{1}{3}$, $\frac{1}{3} - \frac{1}{r} - \frac{1}{s} \leq \frac{2}{p}$. But since $\frac{1}{p} + \frac{1}{2} = \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$, we have

$$\frac{1}{q} - \frac{1}{6} \leq \frac{3}{p}. \tag{4.1}$$

We will start from the case where $q = 2$. In this case, $\frac{1}{n} = \frac{1}{2} - \frac{1}{p} - \frac{1}{q} = -\frac{1}{p}$ always holds. Also, from (4.1), $p \leq 9$. So,

p	r	s
3	4	12
3	6	6
4	5	20
4	6	12
4	8	8
5	10	10
6	12	12
9	18	18

are the possible parameters generality. Next is when $q = 3$ in which case $p \leq 18$. Also, $\frac{1}{n} = \frac{1}{2} - \frac{1}{p} - \frac{1}{q} = \frac{1}{6} - \frac{1}{p}$, so $3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 18$ are the possible values for p. Thus,

p	r	s
3	3	6
3	4	4
4	3	12
4	4	6
6	4	12
6	6	6
8	4	24
10	5	15
12	6	12
18	9	9

are the possible parameters. For $q = 4$, $p \leq 36$ and $\frac{1}{4} - \frac{1}{p} = \frac{1}{n}$ means that the possible values of p are $3, 4, 5, 6, 8, 12, 20$. So the possible parameters are

p	r	s
4	4	4
5	4	5
6	4	6
8	4	8
12	4	12
12	6	6

And when $q = 5$, $\frac{3}{10} + \frac{1}{p} = \frac{1}{r} + \frac{1}{s} \leq \frac{2}{q} = \frac{2}{5}$, which means $p \geq 10$. Also, $p \leq 90$ and $\frac{3}{10} - \frac{1}{p} = \frac{1}{n}$. The possible values of p are 10 and 20. We only have $p = 10, r = s = 5$ as plausible parameters.

4.8 Euler Characteristic

Euler characteristics can be calculated from fundamental domains which, in our case, is the polyhedron we constructed. We then form orbits of facets of the polyhedron and find the stabiliser for each orbit. The Euler characteristic is the reciprocal sum of the orders of odd-dimension facets' stabilisers subtracted by those of the even-dimension.
Recall that p is the order of $R_2, R_3, R_1 R_2 R_1^{-1} = R_4$,
q is the order of $R_1, R_2^{-1} R_1 R_2 = R_5$,
r is the order of $S_1 = R_6$,
s is the order of $S_2 = R_7$,
q', r', s' is $\frac{2pq}{pq-2p-2q}, \frac{2pr}{pr-2p-2r}, \frac{2ps}{ps-2p-2s}$, respectively.
Without lost of generality, we write down the Euler characteristic for each case of (p, q, r, s) under the assumption that $q \leq r \leq s$ in the following table:

p	q	r	s	q'	r'	s'	χ
3	2	4	12	-3	-12	12	$\frac{7}{96}$
3	2	6	6	-3	∞	∞	$\frac{1}{12}$
3	3	3	6	-6	-6	∞	$\frac{1}{12}$
3	3	4	4	-6	-12	-12	$\frac{1}{12}$
4	2	5	20	-4	20	5	$\frac{99}{800}$
4	2	6	12	-4	12	6	$\frac{13}{96}$
4	2	8	8	-4	8	8	$\frac{9}{64}$
4	3	3	12	-12	-12	6	$\frac{7}{48}$
4	3	4	6	-12	∞	12	$\frac{17}{96}$
4	4	4	4	∞	∞	∞	$\frac{3}{16}$
5	2	10	10	-5	5	5	$\frac{3}{20}$
5	4	4	5	20	20	10	$\frac{99}{400}$
6	2	12	12	-6	4	4	$\frac{7}{48}$
6	3	4	12	∞	12	4	$\frac{11}{48}$
6	3	6	6	∞	6	6	$\frac{1}{4}$
6	4	4	6	12	12	6	$\frac{13}{48}$
8	3	4	24	24	8	3	$\frac{11}{48}$
8	4	4	8	8	8	4	$\frac{9}{32}$
9	2	18	18	-9	3	3	$\frac{13}{108}$
10	3	5	15	15	5	3	$\frac{37}{150}$
10	5	5	5	5	5	5	$\frac{3}{10}$
12	3	6	12	12	4	3	$\frac{1}{4}$
12	4	4	12	6	6	3	$\frac{13}{48}$
12	4	6	6	6	4	4	$\frac{7}{24}$
18	3	9	9	9	3	3	$\frac{13}{54}$

To verify this, we will calculate the Euler characteristic for each set of parameters. Note that the polyhedron looks different depending on if there is a truncation or not. So, we can find as a formulae the Euler characteristic for each case starting from the non-truncation case. Then, in the truncation case we can just find the differences the truncations make on the Euler characteristic. In the cases with several truncations, those truncations are independent and we can compute the Euler characteristic by adding all the changes from each truncation. The following lemma will be needed when figuring out the stabiliser of the edge (z_{12}, z_{13}) and can be applied to many other edges.

Lemma 4.8.1. The orientation of the edge joining z_{12} and z_{13} cannot be reversed by the provided side pairing maps.

Proof. We claim that the generators of all the side pairing maps do not reverse the orientation of the edge. To see that, we portray some of the facets as a graph:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & z_{2\bar{3}\bar{2}123} & \rightarrow & z_{\bar{2}123} & \rightarrow & z_{\bar{3}123} & \rightarrow & z_{\bar{3}\bar{2}3123} & \rightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & z_{123\bar{2}} & \rightarrow & z_{12} & \rightarrow & z_{13} & \rightarrow & z_{1\bar{3}23} & \rightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & z_{12312\bar{3}\bar{2}\bar{1}} & \rightarrow & z_{1231\bar{2}\bar{1}} & \rightarrow & z_{1231\bar{3}\bar{1}} & \rightarrow & z_{1231\bar{3}\bar{2}3\bar{1}} & \rightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the indices refer to two of the mirrors that the vertices lie on. The edges are displayed in arrows where the directions of the arrows represent the orientations. The map S_1 shifts the arrows once to the right whilst S_2 shift them up once. The map R_3 fixes $z_{\bar{2}123}, z_{13}$ and maps $z_{\bar{3}123}$ to z_{12}, thus, preserves the orientations. Since the edge lies on the mirror of R_1, the map fixes the arrows. Hence, these maps preserve the orientations of the arrows. As they generate the side pairing group, the rest in the group also do. Thus, none of them reverse the orientation of the edge. □

1. In the non-truncation case, the parameters q', r', s' are all negative.
   - **Vertex orbits** There are a total of five distinct orbits.
     - There are two orbits with one element: $\{z_{16}\}$ and $\{z_{57}\}$. The stabilisers for these orbits are $\langle R_1, S_1\rangle$ and $\langle R_5, S_2\rangle$, respectively. As R_1 commutes with S_1 and R_5 commutes with S_2, the orders of the stabilisers are the products of their orders, which are qr and qs.
     - ($\triangle$) The orbit denoted by triangle marks in 4.26 is $\{z_{23}, z_{123\bar{1}}\}$ with stabiliser generated by R_3 and S_1, Proposition 4.5.3 shows that its order is $2r'^2$.

- ($\square$) Symmetrically, the orbit denoted by rectangle marks is $\{z_{34}, z_{543\bar{5}}\}$ whose stabiliser is generated by R_3, S_2 and according to Proposition 4.5.4, the order is $2s'^2$.
- ($\circ$) The last orbit denoted by circles is $\{z_{13}, z_{12}, z_{\bar{3}123}, z_{35}\}$. The stabiliser is $\langle R_1, R_2\rangle$ and the order, according to Proposition 4.5.1, is $2q'^2$.

- **Edge orbits** Any pair of edges with end points from the same pair of vertices orbits are in the same orbit. Therefore, we can denote these orbits by a pair of marks at the end points. There are a total of seven such orbits.
  - ($\triangle, z_{16}$) We start with the orbit $\{(z_{23}, z_{16}), (z_{123\bar{1}}, z_{16})\}$. Its stabiliser is $\langle S_1\rangle$ of order r.
  - ($\circ, z_{16}$) The other orbit containing z_{16} is $\{(z_{13}, z_{16}), (z_{12}, z_{16})\}$. Its stabiliser is $\langle R_1\rangle$ of order q.
  - ($\square, z_{57}$) Symmetrically, the orbit $\{(z_{34}, z_{57}), (z_{543\bar{5}}, z_{57})\}$ has stabiliser $\langle S_2\rangle$ of order s.
  - ($\circ, z_{57}$) The orbit $\{(z_{12}, z_{57}), (z_{35}, z_{57}\}$ has stabiliser $\langle R_5\rangle$ of order q.
  - ($\triangle, \circ$) The next orbit is $(z_{23}, z_{13}), (z_{23}, z_{12}), (z_{23}, z_{\bar{3}123}), (z_{123\bar{1}}, z_{13}), (z_{123\bar{1}}, z_{12})$ with stabiliser $\langle R_3\rangle$ of order p.
  - ($\square, \circ$) The orbit $\{(z_{34}, z_{12}), (z_{34}, z_{\bar{3}123}), (z_{34}, z_{35}), (z_{543\bar{5}}, z_{12}), (z_{543\bar{5}}, z_{35})\}$ also has stabliser $\langle R_3\rangle$ of order p.
  - ($\circ, \circ$) The last orbit is the one containing points from the same orbit: $\{(z_{13}, z_{12}), (z_{13}, z_{\bar{3}123}), (z_{12}, z_{35}), (z_{\bar{3}123}, z_{35})\}$. In cases like this we have to be mindful of the existence of a map that preserves the orbit but changes the orientation as well. In this case, there is no such map according to Lemma 4.8.1 and the stabiliser is $\langle R_1\rangle$ of order q.

- **Ridge orbits** We can also see the orbits of the ridges in a similar way: through the orbits of the vertices they contain. There are also seven ridge orbits. Some of them lie on the Giraud disk and, hence, have trivial stabilliser of order 1. The only orbit of vertices that can appear twice in the same ridge is the one containing z_{12}. The Lemma 4.8.1 shows that such ridges cannot be flipped by a side pairing map.
  - ($\triangle, \circ, z_{16}$) The orbit $\{(z_{23}, z_{13}, z_{16}), (z_{23}, z_{12}, z_{16}), (z_{123\bar{1}}, z_{13}, z_{16}), (z_{123\bar{1}}, z_{12}, z_{16})\}$ lie on the Giraud disk.
  - ($\square, \circ, z_{57}$) The orbit isomorphic to that one is $\{(z_{34}, z_{35}, z_{57}), (z_{34}, z_{12}, z_{57}), (z_{543\bar{5}}, z_{35}, z_{57}), (z_{543\bar{5}}, z_{12}, z_{57})\}$, so it also lies on the Giraud disk.
  - ($\triangle, \circ, \circ$) Another orbit on the Giraud disk is $\{(z_{23}, z_{13}, z_{12}), (z_{23}, z_{13}, z_{\bar{3}123}), (z_{23}, z_{12}, z_{35}), (z_{23}, z_{\bar{3}123}, z_{35}), (z_{123\bar{1}}, z_{13}, z_{12})\}$.
  - ($\square, \circ, \circ$) Again, since the orbit $\{(z_{34}, z_{12}, z_{35}), (z_{34}, z_{\bar{3}123}, z_{35}), (z_{34}, z_{13}, z_{12}), (z_{34}, z_{13}, z_{\bar{3}123}), (z_{543\bar{5}}, z_{12}, z_{35})\}$ is isomorphic to the previous orbit, it has trivial stabiliser.

- ($z_{16}, \circ, \circ$) The orbit $\{(z_{16}, z_{13}, z_{12})\}$ lies on the R_1 mirror plane and so it has stabiliser $\langle R_1 \rangle$ of order q.
- ($z_{57}, \circ, \circ$) Similarly, the orbit $\{(z_{57}, z_{12}, z_{35})\}$ lies on the mirror plane of R_5 and has stabiliser $\langle R_5 \rangle$ of order q.
- ($\triangle, \circ, \square, \circ$) The last orbit $\{(z_{23}, z_{13}, z_{34}, z_{35})\}$ lies on the R_3 mirror plane and so, R_3 is in the stabiliser. We claim that a map that fixes z_{23} and z_{34} but switches z_{13} with z_{35} does not exist for if it did, it must commute with R_3 and is contained in the stabilisers of z_{23} and z_{34}, namely $\langle R_3, S_1 \rangle$ and $\langle R_3, S_2 \rangle$. This means that it can be written as a product of some power of $R_3S_1R_3S_1$ and some power of R_3 whilst can also be written as a product of some power of $R_3S_2R_3S_2$ and some power of R_3. In either case, since $R_3S_1R_3S_1$, $R_3S_2R_3S_2$ and R_3 either fix z_{13} and z_{35} or shift them around without switching them, the claim is proven. Thus, the ridge has stabiliser $\langle R_3 \rangle$ of order p.

- **Side orbits** There are only three orbits of sides: $\{S_1, S_1'\}, \{S_2, S_2'\}, \{S_3, S_3'\}$ all with trivial stabilisers.

- **The polyhedron** The polyhedron D has stabiiser $\langle S_1, S_2 \rangle$ of order rs.

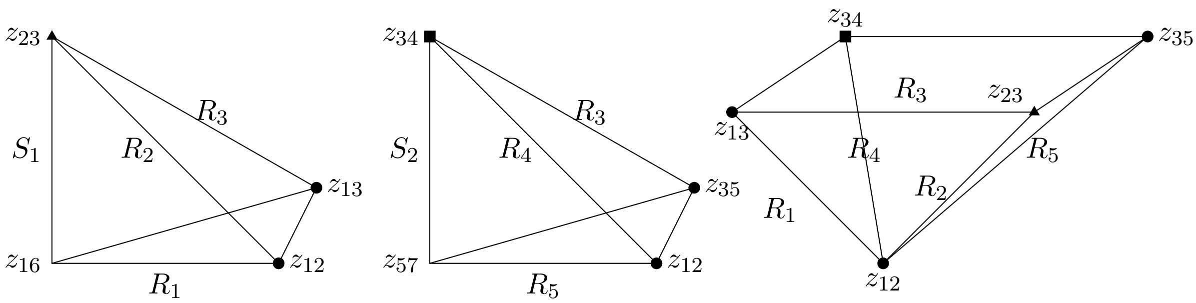

Figure 4.26: Orbits in the polyhedron

Orbit	Stabiliser	Order
$z_{23}, z_{123\bar{1}}$	$\langle R_3, S_1\rangle$	$2r'^2$
$z_{34}, z_{543\bar{5}}$	$\langle R_3, S_2\rangle$	$2s'^2$
z_{16}	$\langle R_1, S_1\rangle$	qr
z_{57}	$\langle R_5, S_2\rangle$	qs
z_{13}, z_{12}, z_{35}	$\langle R_1, R_2\rangle$	$2q'^2$
$(z_{23}, z_{16}), (z_{123\bar{1}}, z_{16})$	$\langle S_1\rangle$	r
$(z_{23}, z_{12}), (z_{23}, z_{\bar{3}123}), (z_{23}, z_{13}), (z_{23}, z_{35}), (z_{123\bar{1}}, z_{12}), (z_{123\bar{1}}, z_{13}))$	$\langle R_3\rangle$	p
$(z_{16}, z_{12}), (z_{16}, z_{13})$	$\langle R_1\rangle$	q
$(z_{12}, z_{13}), (z_{12}, z_{35}), (z_{\bar{3}123}, z_{13}), (z_{\bar{3}123}, z_{35})$	R_1	q
$(z_{34}, z_{57}), (z_{543\bar{5}}, z_{57})$	$\langle S_2\rangle$	s
$(z_{34}, z_{35}), (z_{34}, z_{12}), (z_{34}, z_{\bar{3}123}), (z_{34}, z_{13}), (z_{543\bar{5}}, z_{35}), (z_{543\bar{5}}, z_{12})$	$\langle R_3\rangle$	p
$(z_{57}, z_{35}), (z_{57}, z_{12})$	$\langle R_5\rangle$	q
$(z_{23}, z_{16}, z_{12}), (z_{23}, z_{16}, z_{13}), (z_{123\bar{1}}, z_{16}, z_{12}), (z_{123\bar{1}}, z_{16}, z_{13})$	1	1
$(z_{23}, z_{12}, z_{13}), (z_{23}, z_{12}, z'_{13}), (z_{23}, z_{35}, z_{12}), (z_{23}, z_{35}, z_{\bar{3}123}), (z_{123\bar{1}}, z_{12}, z_{13})$	1	1
(z_{16}, z_{12}, z_{13})	$\langle R_1\rangle$	q
$(z_{34}, z_{57}, z_{35}), (z_{34}, z_{57}, z_{12}), (z_{543\bar{5}}, z_{57}, z_{35}), (z_{543\bar{5}}, z_{57}, z_{12})$	1	1
$(z_{34}, z_{35}, z_{12}), (z_{34}, z_{13}, z_{12}), (z_{34}, z_{35}, z_{\bar{3}123}), (z_{34}, z_{13}, z_{\bar{3}123}), (z_{543\bar{5}}, z_{35}, z_{12})$	1	1
(z_{57}, z_{35}, z_{12})	$\langle R_5\rangle$	q
$(z_{13}, z_{23}, z_{34}, z_{35})$	R_3	p
S_1, S'_1	1	1
S_2, S'_2	1	1
S_3, S'_3	1	1
D	$\langle S_1, S_2\rangle$	rs

The Euler characteristic of the polyhedron is then

$$\begin{aligned}
\chi(\mathbf{H}^2_{\mathbb{C}}/\Gamma) &= \frac{1}{r'} + \frac{1}{qr} + \frac{1}{q'} + \frac{1}{s'} + \frac{1}{qs} - \frac{2}{p} - \frac{3}{q} - \frac{1}{r} - \frac{1}{s} + 4 + \frac{1}{p} + \frac{2}{q} - 3 + \frac{1}{rs} \\
&= \frac{(2p+2q-pq)^2}{8p^2q^2} + \frac{(2p+2r-pr)^2}{8p^2r^2} + \frac{(2p+2s-ps)^2}{8p^2s^2} + \frac{1}{qr} + \frac{1}{qs} + \frac{1}{rs} - \frac{2}{p} + \frac{1}{2} \\
&= \frac{3}{2p^2} - \frac{3}{2p} + \frac{1}{2}\left(\frac{1}{q^2} + \frac{1}{r^2} + \frac{1}{s^2} + \frac{2}{qr} + \frac{2}{qs} + \frac{2}{rs}\right) + \frac{1}{p}\left(\frac{1}{p} + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{p} + \frac{1}{2}\right) - \frac{2}{p} + \frac{7}{8} \\
&= \frac{3}{2p^2} - \frac{3}{2p} + \frac{1}{2}\left(\frac{1}{p^2} + \frac{1}{p} + \frac{1}{4}\right) + \frac{1}{p^2} - \frac{2}{p} + \frac{5}{8} \\
&= \frac{3}{p^2} - \frac{3}{p} + \frac{3}{4} = 3\left(\frac{1}{2} - \frac{1}{p}\right)^2.
\end{aligned}$$

If the constructed polyhedron got truncated at one of its vertices, there will be changes to vertices and edges which will affect the overall Euler characteristic as well. In our case, there are three possible vertices to be truncated, z_{12}, z_{23} and z_{34}.

2. In the case where the polyhedron is truncated at z_{12}, this means $q' > 0$.

- **Vertex Orbits** There are eight new vertices $z^1_{12}, z^2_{12}, z^5_{12}, z^4_{12}$, along with their reflections across the R_3-mirror. This also creates more vertices through the orbits of the edges as shown in the figure 4.14 all of which can be put into two distinct orbits: $\{z^1_{13}, z^1_{12}, z^1_{\bar{3}123}, z^5_{12}, z^5_{\bar{3}123}, z^5_{35}\}$ and $\{z^3_{13}, z^2_{12}, z^2_{\bar{3}123}, z^4_{12}, z^4_{\bar{3}123}, z^3_{35}\}$. The original orbit lies on the mirrors of R_1 and R_3 whilst these two new orbits lie on one of them and the mirror of the new ridge $(R_1R_2)^2$ on the boundary. So, their stabilisers are $\langle R_1, (R_1R_2)^2\rangle$ and $\langle R_3, (R_1R_2)^2\rangle$ of orders $\frac{1}{qq'}$ and $\frac{1}{pq'}$, respectively.
- **Edge orbits** As for the edge orbits, we gain one new orbit $\{(z^1_{12}, z^2_{12}), (z^1_{12}, z^4_{12}), (z^2_{12}, z^5_{12}), (z^4_{12}, z^5_{12}), (z^1_{13}, z^3_{13}), (z^3_{35}, z^5_{35}), (z^1_{\bar{3}123}, z^2_{\bar{3}123}), (z^1_{\bar{3}123}, z^4_{\bar{3}123}), (z^2_{\bar{3}123}, z^5_{\bar{3}123}), (z^4_{\bar{3}123}, z^5_{\bar{3}123})\}$ with stabiliser $\langle (R_1R_2)^2\rangle$ of order q' and another orbit $\{(z^1_{12}, z^1_{13}), (z^5_{12}, z^5_{35}), (z^1_{\bar{3}123}, z^1_{13}), (z^5_{\bar{3}123}, z^5_{35})\}$ with the same stabiliser as the original edge.
As for the edge orbits, we gain one new orbit $\{(z^1_{12}, z^2_{12}), (z^1_{12}, z^4_{12}), (z^2_{12}, z^5_{12}), (z^4_{12}, z^5_{12}), (z^1_{13}, z^3_{13}), (z^3_{35}, z^5_{35}), (z^1_{\bar{3}123}, z^2_{\bar{3}123}), (z^1_{\bar{3}123}, z^4_{\bar{3}123}), (z^2_{\bar{3}123}, z^5_{\bar{3}123}), (z^4_{\bar{3}123}, z^5_{\bar{3}123})\}$ with stabiliser $\langle (R_1R_2)^2\rangle$ of order q' and another orbit $\{(z^1_{12}, z^1_{13}), (z^5_{12}, z^5_{35}), (z^1_{\bar{3}123}, z^1_{13}), (z^5_{\bar{3}123}, z^5_{35})\}$ with the same stabiliser as the original edge.
- **Ridge orbits** There is also one new orbit of ridges $\{(z^1_{12}, z^2_{12}, z^5_{12}, z^4_{12}), (z^1_{\bar{3}123}, z^2_{\bar{3}123}, z^5_{\bar{3}123}, z^4_{\bar{3}123})\}$. This one has stabiliser $\langle R_1R_2\rangle$ of order $2q'$.

This means that, the change in Euler characteristic in the case that the truncation exist is

$$-\frac{1}{2q'^2} + \frac{1}{qq'} + \frac{1}{pq'} - \frac{1}{q'} + \frac{1}{2q'} = -\frac{3}{2}\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}\right)^2.$$

3. When the vertex z_{23} lies beyond the boundary, the parameter r' is positive.

- **Vertex orbits** There are two new orbits of vertices: $\{z^6_{23}, z^6_{123\bar{1}}\}$ with stabiliser $\langle S_1, (R_3S_1)^2\rangle$ of order rr' and $\{z^2_{23}, z^3_{23}, z^2_{123\bar{1}}, z^3_{123\bar{1}}\}$ with stabiliser $\langle R_3, (S_1R_3)^2\rangle$ of order pr'.

- **Edge orbits** There are two orbits of edges which are just the original ones truncated at one end. There are also two new orbits: $\{(z^6_{23}, z^2_{23}), (z^6_{23}, z^3_{23}), (z^6_{123\bar{1}}, z^2_{123\bar{1}}), (z^6_{123\bar{1}}, z^3_{123\bar{1}})\}$ with stabiliser $\langle (R_3S_1)^2\rangle$ of order r' and $\{(z^2_{23}, z^3_{23}), (z^2_{123\bar{1}}, z^3_{123\bar{1}})\}$ with stabiliser $\langle R_3S_1\rangle$ of order $2r'$.
- **Ridge orbits** The new orbit of ridges $\{(z^6_{23}, z^2_{23}, z^3_{23}), (z^6_{123\bar{1}}, z^2_{123\bar{1}}, z^3_{123\bar{1}})\}$ has stabiliser $(R_3S_1)^2$ of order r'.

Thus, the change in Euler characteristic is

$$-\frac{1}{2r'^2} + \frac{1}{rr'} + \frac{1}{pr'} - \frac{1}{r'} - \frac{1}{2r'} + \frac{1}{r'} = -\frac{3}{2}\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{r}\right)^2.$$

4. In the case of truncation at z_{34}, as the vertex is symmetric to the former, we can get the resulting change in Euler characteristic by replacing r, r' with s, s'. Hence, the change is $-\frac{3}{2}\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{s}\right)^2$.

Summarizing all of these results, we get our main theorem.

Theorem 4.8.2. The fundamental domain of a non-elementary group generated by three complex reflections

$$\Gamma'_{r,4,4;4} = \left\langle R_3, S_1, S_2 : \begin{array}{l} R_3^p = S_1^r = S_2^s = (R_3S_1R_3S_2)^q = I, \\ (S_1R_3S_2)^{\frac{4pq}{pq-2p-2q}} = (R_3S_1)^{\frac{4pr}{pr-2p-2r}} = (R_3S_2)^{\frac{4ps}{ps-2p-2s}} = I, \\ \mathrm{br}_2(S_1,S_2),\ \mathrm{br}_4(R_3,S_1),\ \mathrm{br}_4(R_3,S_2) \end{array} \right\rangle$$

where $\frac{1}{2}+\frac{1}{p}=\frac{1}{q}+\frac{1}{r}+\frac{1}{s}$, has its Euler characteristic determined by the parameters (p,q,r,s), given by

$$\chi(\mathbf{H}^2_{\mathbb{C}}/\Gamma_{r,4,4;4}) = 3\left(\frac{1}{2}-\frac{1}{p}\right)^2 - \frac{3}{2}\xi_1\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{q}\right)^2 - \frac{3}{2}\xi_2\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{r}\right)^2 - \frac{3}{2}\xi_3\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{s}\right)^2$$

where

$$\xi_1 = \begin{cases} 1 \text{ when } \frac{1}{2}-\frac{1}{p}-\frac{1}{q} > 0, \\ 0 \text{ when } \frac{1}{2}-\frac{1}{p}-\frac{1}{q} \le 0, \end{cases}$$

$$\xi_2 = \begin{cases} 1 \text{ when } \frac{1}{2}-\frac{1}{p}-\frac{1}{r} > 0, \\ 0 \text{ when } \frac{1}{2}-\frac{1}{p}-\frac{1}{r} \le 0, \end{cases}$$

$$\xi_3 = \begin{cases} 1 \text{ when } \frac{1}{2}-\frac{1}{p}-\frac{1}{s} > 0, \\ 0 \text{ when } \frac{1}{2}-\frac{1}{p}-\frac{1}{s} \le 0. \end{cases}$$

The theorem agrees with many existing results. For example,

- Theorem 5.3 of [24] has the Euler characteristic formulae

$$\begin{aligned} \chi &= \frac{p^2+12p-60}{16p^2} - \left(\frac{1}{4}-\frac{1}{2p}-\frac{1}{r}\right)^2 \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)^2 - \frac{1}{2}\left(\frac{1}{2}-\frac{3}{p}\right)^2 - \frac{1}{2r^2} - \frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}-\frac{1}{p}\right)^2, \end{aligned}$$

  which is one-third of what we have in Theorem 4.8.2 in the 3-fold maximally truncated case because our groups are subgroups of index 3 and thus, have three times the Euler characteristics. Also, recall that the 3-fold symmetry implies $q=\frac{p}{2}$ and $\frac{1}{2}=\frac{1}{p}+\frac{1}{r}+\frac{1}{s}$.

- Mostow's second kind group (Theorem 5.1 in [24]) has the Euler characteristic formulae

$$\begin{aligned}\chi &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)^2+\frac{1}{r}\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{r}\right)\\ &= \left(\frac{1}{2}-\frac{1}{p}\right)^2-\frac{1}{2r^2}-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r}-\frac{1}{p}\right)^2.\end{aligned}$$

  This is our 3 fold case where $\xi_1=0, \xi_2=\xi_3=1$.

There is also a special case when $q<0$, which we do not list here. This will collapse the ridges on the mirrors of R_1 and R_5. This will, in opposition to what a truncation does, add to the value of the Euler characteristic by $\frac{3}{q^2}$. Some examples for this case are Livné groups $(p;-p,2,\frac{p}{2})$ and Mostow's first kind groups $(p,q,\frac{p}{2},s)$, both of which have $q<0$.

4.9 Lattices

Now that we have the sets of parameters and their Euler characteristics from Theorem 4.8.2, we want to identify each of the configurations (p,q,r,s) with a Deligne-Mostow lattice.

By comparing our parameters to Pasquinelli's through $\Gamma_{2,4,4}$ (See Section 2.2), the cone angles

$$\left(\frac{1}{2}+\frac{1}{a}-\frac{1}{c},\frac{1}{2}-\frac{1}{a},\frac{1}{2}-\frac{1}{a},\frac{1}{2}+\frac{1}{a}-\frac{1}{b},\frac{1}{b}+\frac{1}{c}\right)$$

become

$$\left(\frac{1}{2}+\frac{1}{p}-\frac{1}{r},\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{1}{2}+\frac{1}{p}-\frac{1}{q},\frac{1}{q}+\frac{1}{r}\right).$$

in the 2 fold case. In the 3 fold case, we have the quintuple

$$\left(\frac{1}{2}+\frac{1}{a}-\frac{1}{c},\frac{1}{2}-\frac{1}{a},\frac{1}{2}-\frac{1}{a},\frac{1}{2}-\frac{1}{a},\frac{2}{a}+\frac{1}{c}\right),$$

which is

$$\left(\frac{1}{2}+\frac{1}{p}-\frac{1}{r},\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{2}{p}+\frac{1}{r}\right).$$

Since $\frac{1}{2}+\frac{1}{p}=\frac{1}{q}+\frac{1}{r}+\frac{1}{s}$, they can be written as

$$\left(\frac{1}{q}+\frac{1}{s},1-\frac{1}{q}-\frac{1}{r}-\frac{1}{s},1-\frac{1}{q}-\frac{1}{r}-\frac{1}{s},\frac{1}{r}+\frac{1}{s},\frac{1}{q}+\frac{1}{r}\right)$$

in both cases. This complies with the fact that q,r and s are symmetric. Also, note that the quintuples can be written in terms of just three parameters from (p,q,r,s). Paquinelli expresses the lattices in the form of (p^*, k, p') which is (q, r, p) in our case (although, as we tend to list our parameters increasingly, taking advantage of the symmetry between q, r, s, the corresponding parameters may not be in this order, as seen in table 4.2).

(p, q, r, s)	χ	Associated lattice in [21]	Associated volume	Deligne-Mostow lattice
$(3, 2, 4, 12)$	$\frac{7}{96}$	$(2, 4, 3)$	$\frac{7}{96}$	$\Gamma(\frac{2,2,4,7,9}{12})$
$(3, 2, 6, 6)$	$\frac{1}{12}$	$(6, 6, 3)$	$\frac{1}{12}$	$\Gamma(\frac{1,1,2,4,4}{6})$
$(3, 3, 3, 6)$	$\frac{1}{12}$	$(3, 3, 3)$	$\frac{1}{12}$	$\Gamma(\frac{1,1,3,3,4}{6})$
$(3, 3, 4, 4)$	$\frac{1}{12}$	$(4, 4, 3)$	$\frac{1}{12}$	$\Gamma(\frac{2,2,6,7,7}{12})$
$(4, 2, 5, 20)$	$\frac{99}{800}$	$(4, 5)$	$\frac{33}{800}$	$\Gamma(\frac{5,5,5,11,14}{20})$
$(4, 2, 6, 12)$	$\frac{13}{96}$	$(4, 6)$	$\frac{13}{288}$	$\Gamma(\frac{3,3,3,7,8}{12})$
$(4, 2, 8, 8)$	$\frac{9}{64}$			$\Gamma(\frac{2,2,2,5,5}{8})$
$(4, 3, 3, 12)$	$\frac{7}{48}$	$(3, 3, 4)$	$\frac{7}{48}$	$\Gamma(\frac{3,3,5,5,8}{12})$
$(4, 3, 4, 6)$	$\frac{17}{96}$	$(3, 4, 4)$	$\frac{17}{96}$	$\Gamma(\frac{3,3,5,6,7}{12})$
$(4, 4, 4, 4)$	$\frac{3}{16}$			$\Gamma(\frac{1,1,2,2,2}{4})$
$(5, 2, 10, 10)$	$\frac{3}{20}$	$(10, 10, 5)$	$\frac{3}{20}$	$\Gamma(\frac{2,3,3,6,6}{10})$
$(5, 4, 4, 5)$	$\frac{99}{400}$	$(4, 4, 5)$	$\frac{99}{400}$	$\Gamma(\frac{6,6,9,9,10}{20})$
$(6, 2, 12, 12)$	$\frac{7}{48}$	$(12, 12, 6)$	$\frac{7}{48}$	$\Gamma(\frac{2,4,4,7,7}{12})$
$(6, 3, 4, 12)$	$\frac{11}{48}$			$\Gamma(\frac{4,4,4,5,7}{12})$
$(6, 3, 6, 6)$	$\frac{1}{4}$	$(6, 6)$	$\frac{1}{12}$	$\Gamma(\frac{2,2,2,3,3}{6})$
$(6, 4, 4, 6)$	$\frac{13}{48}$	$(4, 4, 6)$	$\frac{13}{48}$	$\Gamma(\frac{4,4,5,5,6}{12})$
$(8, 3, 4, 24)$	$\frac{11}{48}$			$\Gamma(\frac{7,9,9,9,14}{24})$
$(8, 4, 4, 8)$	$\frac{9}{32}$			$\Gamma(\frac{3,3,3,3,4}{8})$
$(9, 2, 18, 18)$	$\frac{13}{108}$	$(18, 18, 9)$	$\frac{13}{108}$	$\Gamma(\frac{2,7,7,10,10}{18})$
$(10, 3, 5, 15)$	$\frac{37}{150}$			$\Gamma(\frac{4,6,6,6,8}{15})$
$(10, 5, 5, 5)$	$\frac{3}{10}$			$\Gamma(\frac{2,2,2,2,2}{5})$
$(12, 3, 6, 12)$	$\frac{1}{4}$			$\Gamma(\frac{3,5,5,5,6}{12})$
$(12, 4, 4, 12)$	$\frac{13}{48}$			$\Gamma(\frac{4,4,5,5,6}{12})$
$(12, 4, 6, 6)$	$\frac{7}{24}$			$\Gamma(\frac{4,5,5,5,5}{12})$
$(18, 3, 9, 9)$	$\frac{13}{54}$			$\Gamma(\frac{2,4,4,4,4}{9})$

Table 4.2: Table of possible values for (p, q, r, s) and their corresponding lattices.

4.10 Conclusion

We considered groups $\langle A, B, C\rangle$ generated by complex reflections that satisfy the braid relations $\mathrm{br}_2(B, C)$, $\mathrm{br}_4(A, B)$ and $\mathrm{br}_4(A, C)$. We showed that these groups all contain a further complex reflection D defined by $BACAD = I$ and so that $\mathrm{br}_4(A, D)$ and $\mathrm{br}_2(C, D)$ are satisfied. Such structure lets us identify the group presentation (when D is included) to Pasquinelli's groups ([21],[22]), showing that each such group is isomorphic to a group in the family containing Deligne-Mostow groups. We also showed that B, C, D can be permuted and the group will still retain the same structure except the orders of the elements, effectively proving that all the Thompson groups of type $\mathbf{E}_2$ are subgroups of a group of this type. We used this to show that all Thompson groups of this type are monodromy groups of higher hypergeometric functions.

We then considered a subgroup $\langle R_1, R_2, R_3\rangle$ of $\langle A, B, C\rangle$ generated by complex reflections satisfying the braid relations $\mathrm{br}_r(R_2, R_3)$, $\mathrm{br}_4(R_1, R_2)$, $\mathrm{br}_4(R_1, R_3)$ and $\mathrm{br}_4(R_1, R_3^{-1}R_2R_3)$. We applied the Deraux-Parker-Paupert algrorithm [10] to this group to produce a combinatorial fundamental domain. We were not able to prove that this combinatorial fundamental domain can be made geometric, but we are able to produce strong evidence that this may be done. For example, we can compute the orbifold Euler characteristic of the polyhedron. The condition that this fundamental domain satisfies the Poincaré polyhedron theorem is equivalent to the Deligne-Mostow group indicated above satisfying ΣINT and its orbifold Euler characteristic matches ours.

Chapter 5

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