Applications of higher-form symmetries at strong and weak coupling

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Abstract: In this thesis we consider two distinct applications of higher-form symmetries in quantum field theory. First we explore the spontaneous breaking of higher-form symmetry in a holographic quantum field theory containing matter fields in the fundamental representation of the gauge group $U(N)$. At strong coupling, we numerically solve the bulk equations of motion to compute the current-current Green's function and demonstrate the existence of a goldstone mode. We then compare to direct analytic perturbative results obtained at weak coupling. In the second half of the thesis we work with a hydrodynamic effective field theory which possesses a higher-form symmetry. In particular, we consider a natural higher-derivative correction to force-free electrodynamics and compute a hydrodynamic transport coefficient from microscopics. Concretely, this is a perturbative QED calculation in a background magnetic field. Finally we compare our findings to astrophysical observations.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. The results in chapters 3 and 5 are based on the following collaborative works:

- Nabil Iqbal and Kieran Macfarlane. 'Higher-form symmetry breaking and holographic flavour' (July 2021).
  `arXiv:2107.00373 [hep-th]`
- Nabil Iqbal and Kieran Macfarlane. 'Microscopic computation of higher-derivative corrections to force-free electrodynamics' (In preparation)

No part of this thesis has been submitted elsewhere for any degree or qualification.

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Contents

Notation and conventions

Spacetime signature

We follow the conventions and notation of [3]. In particular, the metric signature is mostly plus, that is we have $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Lorentzian Green's functions are related to two-point functions by $iG_{12}(x,y) = \langle \mathcal{T}\phi_1(x)\phi_2(y)\rangle$ for fields ϕ_1 and ϕ_2 where $\mathcal{T}$ denotes time-ordering.

Indices

Unless otherwise stated, we normally use M, N to refer to 5d bulk indices, μ, ν to refer to 4d field theory indices, and i, j to refer to 3d spatial indices. In Chapter 3 our construction will involve embedding a $D7$-brane into a 10d spacetime. Here α, β will refer to $D7$-brane worldvolume coordinates and A, B will refer to 10d target space indices.

Differential forms

Our conventions for differential forms are those of [4], and we record some useful identities below:

$$d(\omega_p \wedge \eta_q) = d\omega_p \wedge \eta_q + (-1)^p \omega_p \wedge d\eta_q \tag{0.0.1}$$

$$\omega_p \wedge \eta_q = (-1)^{pq} \eta_q \wedge \omega_p \tag{0.0.2}$$

$$\omega_p \wedge \star\eta_p = \eta_p \wedge \star\omega_p \tag{0.0.3}$$

The square of the Hodge star acting on a p form in n dimensions on a metric with s minus signs in its eigenvalues is

$$\star^2 = (-1)^{s+p(n-p)} \ . \tag{0.0.4}$$

In particular, in Lorentzian signature in 4d acting on a 2-form, we have $\star_4^2 = -1$.

As in [5], we define

$$|A_p|^2 = \frac{1}{p!} A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p} \tag{0.0.5}$$

and we use the shorthand

$$A_p^2 \equiv A_p \wedge \star A_p \tag{0.0.6}$$

To translate between expressions involving forms and expressions involving components, we can use the identity

$$A_p^2 = |A_p|^2 \, \epsilon \tag{0.0.7}$$

where ϵ is the volume form associated with the metric determinant g

$$\epsilon = \star 1 = \sqrt{|g|} \, d^n x \tag{0.0.8}$$

The integral of an n-form Ω over an n-dimensional manifold $\mathcal{M}$ of signature s is defined by

$$\int_{\mathcal{M}} \Omega \equiv \int_{\mathbb{R}^n} (-1)^s (\star\Omega) \, \epsilon = \int_{\mathbb{R}^n} d^n x \, \sqrt{|g|} \, (-1)^s (\star\Omega) \tag{0.0.9}$$

So in particular, for a p-form we have

$$\int_{\mathcal{M}} A_p^2 = \int_{\mathbb{R}^n} d^n x \, \sqrt{|g|} \, |A_p|^2 \tag{0.0.10}$$

Fourier transforms

As is standard, we write a tilde to denote the Fourier transform of an object. So for example

$$\tilde{J}^{\mu\nu}(p) = \int d^4 x \, e^{-ip\cdot x} \, J^{\mu\nu}(x) \tag{0.0.11}$$

The inverse is then given by

$$J^{\mu\nu}(x) = \int \widetilde{dp} \, e^{ip\cdot x} \, \tilde{J}^{\mu\nu}(p) \tag{0.0.12}$$

Here the tilde over the measure $\widetilde{dp}$ is a shorthand for $\frac{d^4 p}{(2\pi)^4}$. Similarly $\widetilde{dp_\parallel} \equiv \frac{d^2 p_\parallel}{(2\pi)^2}$, that is we weight the denominator by the appropriate power of 2π.

For the Fourier transform of a Green's function G, we use the convention that

$$\tilde{G}_{12}(p) \equiv \int \widetilde{dp} \, e^{-ip\cdot x} \, G_{12}(x, 0) \tag{0.0.13}$$

and similarly in a 2-point function,

$$\langle \tilde{\phi}_1(p) \tilde{\phi}_2(-p) \rangle \equiv \int \widetilde{dp} \, e^{-ip\cdot x} \, \langle \mathcal{T} \phi_1(x) \phi_2(0) \rangle \tag{0.0.14}$$

Gamma matrices

Our metric sign convention fixes the Clifford algebra as

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \tag{0.0.15}$$

For concreteness, we use the Weyl (chiral) representation of the gamma matrices given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{0.0.16}$$

where $\sigma^\mu \equiv (\mathbb{1}_2, \sigma^i)$ with σ^i the Pauli matrices.

Chapter 1

Introduction

Symmetries are of crucial importance in theoretical physics in general and quantum field theory in particular. For example, symmetries provide aesthetically pleasing mathematical explanations for a wide range of natural phenomena, from classification of phases of matter to the Higgs mechanism. Symmetries also afford us a systematic way to solve many seemingly intractable problems by computing perturbations away from some idealised solvable system. Such calculations are abundant in perturbative quantum field theory and hydrodynamics.

This thesis focuses on applications of a novel type of symmetry in quantum field theory called *generalised global symmetries*, *higher-form symmetries* or *p-form symmetries*. These symmetries were first explored in [6] and, as the name suggests, generalise the notion of global symmetries in quantum field theories from point-like objects (particles) to extended objects with higher spatial dimensions (strings, branes, etc.). Helpfully, many of the features of ordinary global symmetries can be readily lifted up to the higher-form case. In particular, a higher-form symmetry is associated with a conserved symmetry current. Physically, the conservation of the symmetry current corresponds to the conservation of (the density of) extended objects such as strings and branes. Higher-form symmetries can be spontaneously broken (resulting in Goldstone modes [7, 8]), have anomalies (see e.g. [9] for an early example), and be used to build hydrodynamic theories [10, 11, 12].

The outline of the thesis is as follows. We begin in Chapter 2 with a very elementary treatment of symmetries in the familiar setting of Lagrangian mechanics. From here we bridge the gap to review generalised global symmetries in quantum field theory. In Chapter 3 we use the formalism of holography to explore the higher-form symmetry structure of a supersymmetric quantum field theory with matter fields in the fundamental representation of the gauge group. These results were first presented in [1]. We move on to consider a further application of higher-form symmetry in hydrodynamics, namely *force-free electrodynamics*. In Chapter 4 we review the essential aspects of hydrodynamics and force-free electrodynamics, then in Chapter 5 we compute a transport coefficient for force-free electrodynamics from a microscopic theory. These results form the basis of [2]. Finally we offer a conclusion and outlook in Chapter 6.

Chapter 2

Review of generalised global symmetries

We set the stage with a very simple example in the framework of classical Lagrangian mechanics. The discussion and notation readily generalise to Lagrangian field theory, and from there we can smoothly proceed to the quantum theory, clarifying various aspects along the way. Once we have discussed global symmetries in quantum theories, we can generalise further to higher-form symmetries. We end the chapter by considering higher-form symmetries of non-Abelian gauge theories.

2.1 Symmetries in classical Lagrangian mechanics

The material in this section is elementary undergraduate knowledge, brilliantly explained in e.g. [13]. Nonetheless, it provides an ideal starting point to clarify our perspective and notation, and should make the later generalisations more accessible.

2.1.1 Single point particle

Consider a classical point particle of mass m under the influence of a time-independent potential V in three-dimensional Euclidean space. We assume that the particle takes a path beginning at time t_0 and ending at time t_1. The Lagrangian is given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m|\dot{\mathbf{x}}|^2 - V(\mathbf{x}) \tag{2.1.1}$$

The action functional for the system is

$$S[\mathbf{x}; t_0, t_1] = \int_{t_0}^{t_1} dt\ L(\mathbf{x}, \dot{\mathbf{x}}) = \int_{t_0}^{t_1} dt\ \left(\frac{1}{2} m |\dot{\mathbf{x}}|^2 - V(\mathbf{x})\right) \tag{2.1.2}$$

Equations of motion

To find the equations of motion for the particle, we extremise the action functional S with respect to the path of the particle $\mathbf{x}(t)$. Consider a small variation of the path

$$x_i(t) \mapsto x_i(t) + \epsilon\ \delta x_i(t) \tag{2.1.3}$$

which vanishes on the endpoints

$$\delta x_i(t_0) = 0 \tag{2.1.4a}$$

$$\delta x_i(t_1) = 0 \tag{2.1.4b}$$

i.e. the point particle has a fixed initial position $\mathbf{x}_I = \mathbf{x}(t_0)$ and a fixed final position $\mathbf{x}_F = \mathbf{x}(t_1)$.[1]

Under this variation of the path, the Lagrangian deforms as

$$L \mapsto L + \epsilon\ \delta L + \mathcal{O}(\epsilon^2) \tag{2.1.5}$$

where

$$\begin{aligned} \delta L &= m\dot{\mathbf{x}} \cdot \delta\dot{\mathbf{x}} - \delta\mathbf{x} \cdot \nabla V \\ &= -\delta\mathbf{x} \cdot (m\ddot{\mathbf{x}} + \nabla V) + \frac{d}{dt}(m\ \delta\mathbf{x} \cdot \dot{\mathbf{x}}) \end{aligned}$$

So the change in the action is

$$S[\mathbf{x} + \epsilon\ \delta\mathbf{x}] - S[\mathbf{x}] = \epsilon\ \delta S[\mathbf{x}] + \mathcal{O}(\epsilon^2) \tag{2.1.6}$$

where

$$\delta S[\mathbf{x}] = \int_{t_0}^{t_1} dt\ \delta L = -\int_{t_0}^{t_1} dt\ \delta\mathbf{x} \cdot (m\ddot{\mathbf{x}} + \nabla V) + [m\ \delta\mathbf{x} \cdot \dot{\mathbf{x}}]_{t_0}^{t_1} \tag{2.1.7}$$

The second term on the right-hand side vanishes because, by definition, the variation $\delta\mathbf{x}$ vanishes at the endpoints. Along an extremising path $\mathbf{x}_C(t)$, δS must vanish for all $\delta\mathbf{x}(t)$, and hence $\mathbf{x}_C(t)$ satisfies the (classical) equation of motion or *Euler-Lagrange equation*

$$m\ddot{\mathbf{x}} + \nabla V = 0 \tag{2.1.8}$$

[1] Here i is understood to range from 1 to 3.

A path $\mathbf{x}(t)$ satisfying the classical equation of motion is said to be "on-shell".

We define the canonical conjugate momentum to be

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \tag{2.1.9}$$

Defining the force as $\mathbf{F} = -\nabla V$ in the usual way, we recover Newton's equation

$$\mathbf{F} = \dot{\mathbf{p}} \tag{2.1.10}$$

Time translation invariance and conservation of energy

What symmetries does this system have? Observe that if we make a small translation in time

$$t \mapsto t' = t - \epsilon \tag{2.1.11}$$

then this induces a transformation of the path

$$x_i(t) \mapsto \tilde{x}_i(t) = x_i(t+\epsilon) = x_i(t) + \epsilon\, \dot{x}_i(t) + \mathcal{O}(\epsilon^2) \tag{2.1.12}$$

i.e. $\delta x_i(t) = \dot{x}_i(t)$.[2]

Note that this induced transformation of the x_i says nothing about the endpoints, unlike the variation we took to find the equations of motion. Indeed, we made no such specification that $\dot{x}_i(t_0)$ should vanish. Confusingly, it is common to use the notation δx_i both for taking variations of the action as in (2.1.3) and for the change under an infinitesimal transformation such as (2.1.11).

The Lagrangian transforms as

$$\begin{aligned}
\delta L &= m\dot{\mathbf{x}} \cdot \delta\dot{\mathbf{x}} - \delta\mathbf{x} \cdot \nabla V \\
&= m\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} - \dot{\mathbf{x}} \cdot \nabla V \\
&= \frac{d}{dt}\left(\frac{1}{2}m|\dot{\mathbf{x}}|^2 - V(\mathbf{x})\right) \\
&= \frac{dL}{dt}
\end{aligned}$$

In particular, to leading order in ϵ the Lagrangian shifts by a total derivative. The transformed action is

$$S'[\mathbf{x}] = \int_{t_0-\epsilon}^{t_1-\epsilon} dt\; L(\tilde{\mathbf{x}}, \dot{\tilde{\mathbf{x}}})$$

[2]Note that the sign of ϵ appears to be opposite in the transformation of x_i compared to t, because $\tilde{x}_i(t') = x_i(t)$. This is called an active transformation - more on this later.

$$
\begin{aligned}
&= \int_{t_0-\epsilon}^{t_1-\epsilon} dt \left(L(\mathbf{x}, \dot{\mathbf{x}}) + \epsilon \, \frac{dL}{dt} \right) + \mathcal{O}(\epsilon^2) \\
&= \int_{t_0}^{t_1} dt \; L(\mathbf{x}, \dot{\mathbf{x}}) - \epsilon(L(t_1) - L(t_0)) + \epsilon \int_{t_0}^{t_1} dt \; \frac{dL}{dt} + \mathcal{O}(\epsilon^2) \\
&= S[\mathbf{x}] + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

where we used the Leibniz integral formula and the fundamental theorem of calculus.

To leading order in ϵ, the action is invariant under the transformation. Hence S' will yield the same equations of motion as the original action S. For this reason, we say that a transformation for which the action is invariant is a symmetry of the system, or a symmetry of the action. Note that in particular, we did not have to use the equations of motion to show that $S' = S$.

This particular transformation is time translation. So we say that the system is invariant under time translation, or that the system has a time translation symmetry.

Recall that in general the Lagrangian takes the form

$$L = T - V \tag{2.1.13}$$

where T is the kinetic energy of the system and V is the potential energy of the system.[3] The total energy of the system is given by

$$E = T + V = \frac{1}{2} m |\dot{\mathbf{x}}|^2 + V(\mathbf{x}) \tag{2.1.14}$$

Using the equations of motion, it is straightforward to show that the total energy is conserved, that is, it is constant in time. Indeed,

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt} \left(\frac{1}{2} m |\dot{\mathbf{x}}|^2 + V(\mathbf{x}) \right) \\
&= m \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \dot{\mathbf{x}} \cdot \nabla V \\
&= \dot{\mathbf{x}} \cdot (m \ddot{\mathbf{x}} + \nabla V) \\
&= 0
\end{aligned}
$$

At least superficially it may seem that these two facts, namely the existence of the symmetry and the conservation of the energy, are unrelated. In fact, the conservation of the total energy is a direct consequence of the time translation symmetry! One way to see this is to compare our particular transformation of the Lagrangian

$$L \mapsto L + \frac{d}{dt}(\epsilon \, L) + \mathcal{O}(\epsilon^2) \tag{2.1.15}$$

[3] At least for a conservative system.

with the transformation under a general variation:

$$L \mapsto L - \epsilon\, \delta\mathbf{x} \cdot (m\ddot{\mathbf{x}} + \nabla V) + \frac{d}{dt}\left(\epsilon\, m\delta\mathbf{x} \cdot \dot{\mathbf{x}}\right) + \mathcal{O}(\epsilon^2) \tag{2.1.16}$$

Equating the expressions to first order in ϵ and imposing the equations of motion we find that

$$\frac{d}{dt}(\epsilon\, L) = \frac{d}{dt}(\epsilon\, m\delta\mathbf{x} \cdot \dot{\mathbf{x}}) \tag{2.1.17}$$

Setting $\delta\mathbf{x} = \dot{\mathbf{x}}$ we recover $\dot{E} = 0$ as expected.

This is more than just an amusing observation. The conservation of energy of the particle allows us to study its dynamics even when the potential is more complicated, for example by considering stationary points of the potential and investigating their stability.

Rotational invariance and conservation of angular momentum

To consider another example, let's restrict to the case where the potential V is spherically symmetric, i.e. we have $V(\mathbf{x}) = V(|\mathbf{x}|)$. Consider rotating the path $\mathbf{x}(t)$ in a plane with unit normal $\mathbf{n}$ by an angle θ.

$$x_i(t) \mapsto M_{ij}(\mathbf{n}, \theta)\, x_j(t) \tag{2.1.18}$$

where $M \in SO(3)$. Using the defining property of $SO(3)$, we have

$$M_{ki}M_{kj} = \delta_{ij} \tag{2.1.19}$$

and so both $|\mathbf{x}|$ and $|\dot{\mathbf{x}}|$ are invariant under the rotation.

Hence $L \mapsto L$ and the rotation is a symmetry of the action.

What is the infinitesimal form of the rotation? If θ is small we can write

$$M_{ij} = \delta_{ij} + \theta A_{ij} + \mathcal{O}(\theta^2) \tag{2.1.20}$$

for some matrix A_{ij}.

Imposing $M_{ki}M_{kj}$ to leading order in θ gives

$$A_{ij} + A_{ji} = 0 \tag{2.1.21}$$

i.e. A is antisymmetric.

Imposing $\det M = 1$ to leading order in θ gives

$$\mathrm{Tr}\, A = 0 \tag{2.1.22}$$

i.e. A is traceless.

The infinitesimal form of the rotation is thus

$$x_i(t) \mapsto x_i(t) + \theta A_{ij} x_j(t) + \mathcal{O}(\theta^2) \tag{2.1.23}$$

i.e. we have $\delta x_i = A_{ij} x_j$, where A is antisymmetric and traceless. In fact,

$$A_{ij} = \epsilon_{ijk}\, n_k \tag{2.1.24}$$

Plugging this into the general transformation of the Lagrangian gives

$$\begin{aligned}
\delta L &= -\delta \mathbf{x} \cdot (m\ddot{\mathbf{x}} + \nabla V) + \frac{d}{dt}(m\, \delta\mathbf{x} \cdot \dot{\mathbf{x}}) \\
&= -\delta \mathbf{x} \cdot (m\ddot{\mathbf{x}} + \nabla V) + \frac{d}{dt}(m\, \epsilon_{ijk}\, n_k\, x_j \dot{x}_i) \\
&= -\delta \mathbf{x} \cdot (m\ddot{\mathbf{x}} + \nabla V) - \frac{d}{dt}(\mathbf{n} \cdot \mathbf{L})
\end{aligned}$$

where $\mathbf{L} = \mathbf{x} \times \mathbf{p} = \mathbf{x} \times (m\dot{\mathbf{x}})$ is the angular momentum.

We know that $\delta L = 0$ under the rotation. Hence when the equations of motion are satisfied, the component of the angular momentum parallel to the normal is conserved:

$$\frac{d}{dt}(\mathbf{n} \cdot \mathbf{L}) = 0 \tag{2.1.25}$$

However, this is true for every unit vector $\mathbf{n}$, and so we must have $\frac{d}{dt}\mathbf{L} = 0$.

To verify this statement we can use the equations of motion directly. As a result of the spherically symmetric potential we have that

$$\nabla V(|\mathbf{x}|) = V(|\mathbf{x}|) \nabla |\mathbf{x}| = \frac{V(|\mathbf{x}|)}{|\mathbf{x}|}\, \mathbf{x} \tag{2.1.26}$$

So

$$\begin{aligned}
\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}(\mathbf{x} \times \mathbf{p}) \\
&= m\dot{\mathbf{x}} \times \dot{\mathbf{x}} + \mathbf{x} \times \dot{\mathbf{p}} \\
&= \mathbf{x} \times (-\nabla V) \\
&= -\frac{V(|\mathbf{x}|)}{|\mathbf{x}|}\, (\mathbf{x} \times \mathbf{x}) \\
&= 0
\end{aligned}$$

as asserted above.

An important case is a Newtonian point particle of mass $m \ll M$ (say, a satellite) orbiting a fixed point particle of mass M (say, the Earth). In this case, with the larger mass at the origin and the smaller mass at a point $\mathbf{x}$, the potential is given by

$$V(|\mathbf{x}|) = -\frac{GM}{|\mathbf{x}|} \tag{2.1.27}$$

where G is Newton's gravitational constant. Physically, a satellite orbiting the Earth has a conserved angular momentum throughout its orbit.

2.1.2 General system

Now we generalise the discussion so far, while remaining in Lagrangian mechanics. We replace the coordinates of the particle $x_i(t)$ with n generalised coordinates $q_1(t), q_2(t), \cdots, q_n(t)$. The Lagrangian will be a function of $q_i(t), \dot{q}_i(t)$ and time t only. In particular, there will be no second-order or higher derivatives of q_i appearing in the action.

The action functional is

$$S[\mathbf{q}] = \int_{t_0}^{t_1} dt\ L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \tag{2.1.28}$$

Here we take the opportunity to clarify some universal abuses of notation that arise in this subject. The Lagrangian should be understood to be a function of $2n+1$ variables.

$$\begin{aligned} L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} &\to \mathbb{R} \\ (\mathbf{a}, \mathbf{b}, c) \qquad &\mapsto L(\mathbf{a}, \mathbf{b}, c) \end{aligned}$$

It just so happens that in practice, we always evaluate this function L at $\mathbf{a} = \mathbf{q}(t)$, $\mathbf{b} = \dot{\mathbf{q}}(t)$, and $c = t$. When we write an expression like $\frac{\partial L}{\partial \dot{q}_i}$, what we really mean is $\frac{\partial L}{\partial b_i}$ evaluated at $\mathbf{a} = \mathbf{q}(t)$, $\mathbf{b} = \dot{\mathbf{q}}(t)$ and $c = t$. With this in mind, it is perfectly legitimate to take derivatives with respect to $\mathbf{q}$ and $\dot{\mathbf{q}}$ independently.

Equations of motion

In order to determine the classical equation of motion of this system, we extremise the action functional $S[\mathbf{q}]$ with respect to the paths $q_i(t),\ i = 1, \ldots, n$ from t_0 to t_1 while keeping the endpoints fixed.

More rigorously, $\mathbf{q}$ is a function of a single variable defined on an interval

$$\begin{aligned} \mathbf{q} :\ [t_0, t_1] &\to \mathbf{R}^n \\ t \quad &\mapsto \mathbf{q}(t) \end{aligned}$$

satisfying $\mathbf{q}(t_0) = \mathbf{q}_I$ and $\mathbf{q}(t_1) = \mathbf{q}_F$. Let $\mathcal{Q}$ be the set of all such functions $\mathbf{q}$ which are also sufficiently smooth. We want to extremise the action functional

$$\begin{aligned} S : \mathcal{Q} &\to \mathbb{R} \\ \mathbf{q} &\mapsto S[\mathbf{q}] \end{aligned}$$

To do this we deform $\mathbf{q}$ by a small variation

$$q_i(t) \mapsto q_i(t) + \epsilon\, \delta q_i(t) \tag{2.1.29}$$

satisfying $\delta q_i(t_0) = \delta q_i(t_1) = 0$.

Evaluating the action after the deformation, we find

$$\begin{aligned} S[\mathbf{q} + \epsilon\, \delta\mathbf{q}] &= \int_{t_0}^{t_1} dt\; L(\mathbf{q} + \epsilon\, \delta\mathbf{q}, \dot{\mathbf{q}} + \epsilon\, \delta\dot{\mathbf{q}}, t) \\ &= \int_{t_0}^{t_1} dt\; L + \epsilon \int_{t_0}^{t_1} dt \left(\delta q_i(t) \frac{\partial L}{\partial q_i} + \delta \dot{q}_i(t) \frac{\partial L}{\partial \dot{q}_i} \right) + \mathcal{O}(\epsilon^2) \\ &= S[\mathbf{q}] \; + \epsilon \left\{ \int_{t_0}^{t_1} dt\; \delta q_i(t) \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) + \left[\delta q_i \frac{\partial L}{\partial \dot{q}_i} \right]_{t_0}^{t_1} \right\} + \mathcal{O}(\epsilon^2) \end{aligned}$$

Summation over the repeated index i is implied and we have integrated by parts. The first-order in ϵ term involving the endpoints vanishes because $\delta q_i = 0$ at the endpoints.

If we define the first variation of S to be

$$\delta S[\mathbf{q}] = \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} (S[\mathbf{q} + \epsilon\, \delta\mathbf{q}] - S[\mathbf{q}]) \right) \tag{2.1.30}$$

and the variational derivative of S by

$$\delta S[\mathbf{q}] = \int_{t_0}^{t_1} dt\; \delta q_i(t) \frac{\delta S}{\delta q_i(t)} \tag{2.1.31}$$

then we obtain

$$\frac{\delta S}{\delta q_i(t)} = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \tag{2.1.32}$$

For $\mathbf{q}_C \in \mathcal{Q}$ to be a stationary point of S, we require $\delta S[\mathbf{q}_C] = 0$ for all variations $\delta\mathbf{q}$, i.e. we require the variational derivative of S to vanish.

Hence $\mathbf{q}_C$ satisfies the Euler-Lagrange equations of motion for the classical path of the system

$$\frac{\delta S}{\delta q_i(t)} = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{2.1.33}$$

As before, a path $q_i(t)$ satisfying $\frac{\delta S}{\delta q_i(t)} = 0$ is said to be on-shell.

Transformations of the system

So far, we have seen two distinct transformations: time translation and spatial rotation. These two examples demonstrate the two essential classes of transformations which are possible in Lagrangian mechanics. With a view to generalising to field theory, we call time translation a *(space)time transformation*, and we call spatial rotation an *internal transformation*.

(Space)time transformations commonly cause much confusion, even in the straightforward setting of Lagrangian mechanics. At this point, we will make a useful definition and move on.[4] For us, an active (space)time transformation

$$t \mapsto T(t) \tag{2.1.34}$$

is defined to induce a transformation

$$q_i(t) \mapsto Q_i(t) = q_i(T^{-1}t) \tag{2.1.35}$$

Further, under such a transformation, the Lagrangian *is defined* to transform as

$$L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) \mapsto L(\mathbf{q}(T^{-1}t), \dot{\mathbf{q}}(T^{-1}t), t) \tag{2.1.36}$$

If instead we transform the generalised coordinates $\mathbf{q}$ themselves, we have

$$q_i(t) \mapsto Q_i(t) = \rho_i(\mathbf{q}(t), t) \tag{2.1.37}$$

for some functions ρ_i. The Lagrangian *is defined* to transform as

$$L \mapsto L' = L(Q_i(t), \dot{Q}_i(t), t) \tag{2.1.38}$$

If the action is left invariant by either type of transformation, i.e. $S[L'] = S[L]$, then the equations of motion are unchanged and we call such a transformation a symmetry. The reader may at this point why we bother distinguishing between the two types of transformation. After all, a (space)time transformation simply induces a transformation of the generalised coordinates and nothing more. We will soon see why. For us, a (space)time symmetry will always shift the Lagrangian by a total derivative, and is only a pseudosymmetry of the Lagrangian. An internal symmetry is a symmetry of the Lagrangian itself, that is it leaves the Lagrangian unchanged. This will have important implications shortly.

Of particular importance to us are continuous transformations connected to the identity. For such transformations we can write (for some infinitesimal real parameter

[4] The reader is welcome to disagree with our definition, but the formalism here will be self-consistent.

ϵ)

$$t \mapsto T(t) = t - \epsilon\, \delta t \tag{2.1.39}$$

for a (space)time transformation, and

$$q_i \mapsto \rho_i(q,t) = q_i(t) + \epsilon\, \delta q_i(t) \tag{2.1.40}$$

for an internal transformation.

For a (space)time transformation, the infinitesimal change in the Lagrangian is

$$\delta L = \left(\dot{q}_i \frac{\partial L}{\partial q_i} + \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) \delta t = \left(\frac{dL}{dt} - \frac{\partial L}{\partial t}\right) \delta t \tag{2.1.41}$$

For an internal transformation, the infinitesimal change in the Lagrangian is simply

$$\delta L = \delta q_i \frac{\partial L}{\partial q_i} + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right) + \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) \tag{2.1.42}$$

2.1.3 Aside: Active v passive transformations

Now we take a moment to clarify some confusing terminology and language associated with transformations in physics, namely the notion of "active" and "passive" transformations. The most general setting to formulate this distinction carefully is the framework of manifolds, diffeomorphisms and charts.

Let $\mathcal{M}$ be a manifold and let $A:\ \mathcal{M} \to \mathbb{R}^n$ be a chart for $\mathcal{M}$. Let $\phi:\ \mathcal{M} \to \mathbb{R}$ be a real scalar field on the manifold. Suppose we are given some smooth map $T:\ \mathbb{R}^n \to \mathbb{R}^n$.

In the passive picture, this represents a coordinate transformation $T:\ x \mapsto x'$, mapping the coordinates with respect to the chart A to coordinates with respect to another chart B. In other words, it is a transition map $T = B^{-1} \circ A$. The representation of the scalar field ϕ in the original coordinates is $\Phi_A = \phi \circ A^{-1} : \mathbb{R}^n \to \mathbb{R}$. Usually in field theory we refer to "$\phi(x)$" when we really mean $\Phi_A(x)$. For a passive transformation, the field itself does not transform, but its representative must be written in the new coordinates. Let $\Phi_B = \phi \circ B^{-1}$. We have $\Phi_A = \phi \circ A^{-1} = \phi \circ B^{-1} \circ B \circ A^{-1} = \Phi_B \circ T$. So $\Phi_A(x) = \Phi_B(Tx)$. In a common abuse of notation, we suppress the dependence on charts and simply write $\phi(x) \mapsto \phi(Tx)$. Physically, this is saying that nothing has actually moved - only the coordinates we give to points on the manifold.

On the other hand, in the active picture there is some diffeomorphism $\mathcal{T}:\ \mathcal{M} \to \mathcal{M}$. In this case, we interpret T as the representation of this diffeomorphism with respect to the chart A. More precisely, we have $T = A \circ \mathcal{T} \circ A^{-1}$. Note that there is only one chart involved in this picture - we are not doing a coordinate transformation, but instead moving the points around on the manifold itself. The diffeomorphism induces a pushforward on the scalar field ϕ which yields a new scalar field $\phi' = \mathcal{T}_*\phi = (\mathcal{T}^{-1})^*\phi = \phi \circ \mathcal{T}^{-1}$. The new scalar field has a representation Φ' in the coordinates of the chart given by $\Phi' = \phi' \circ A^{-1} = \phi \circ \mathcal{T}^{-1} \circ A^{-1} = \phi \circ A^{-1} \circ A \circ \mathcal{T}^{-1} \circ A^{-1} = \Phi \circ T^{-1}$. Hence the new scalar field is related to the old scalar field by $\Phi'(x) = \Phi(T^{-1}x)$. In a common abuse of notation, we simply write $\phi(x) \mapsto \phi(T^{-1}x)$. Physically, this is saying that points on the manifold have moved around, but we are using the same set of coordinates.

The point of all this is that simply writing "the transformation $x \mapsto x'$" by itself is insufficient to deduce the induced transformations on scalar fields and other objects. Usually active transformations can be assumed, however it is best to simply *define* clearly how a given transformation acts on everything to avoid ambiguity. From here onwards we will take this approach and clearly *define* all of our induced transformations.

2.1.4 Noether's theorem for Lagrangian mechanics

We conclude our recap of Lagrangian mechanics with the most important result.

Over a century ago the German Mathematician Emmy Noether proved in [14] that (in modern language) every continuous symmetry of a classical physical system gives rise to a conservation law. This statement is known as Noether's theorem.[5] The precise mathematical statement depends on the context. A more rigorous unification from an algebraic point of view is attempted in [15], but our treatment here is closer to the spirit of Noether's original work.

In Lagrangian mechanics, a continuous symmetry is a transformation such that the Lagrangian is deformed by (at most) a total derivative with respect to time:

$$\delta L = \frac{df}{dt} \tag{2.1.43}$$

In such a case, the action S is invariant, and the equations of motion are unchanged.

Here a conservation law means mathematically there is some function $K(t)$ such that along any on-shell path $q_i(t)$, we have

$$\frac{dK}{dt} = 0 \tag{2.1.44}$$

We usually call such a function a *conserved charge*. The upshot of Noether's theorem

[5] Not to be confused with Noether's theorems for rings - usually this is unambiguous for physicists.

is that given a non-trivial continuous symmetry of the system, we are guaranteed a conserved charge. Further, we get an algorithm to construct the charge using the symmetry transformation.

Precise statement

Given a Lagrangian $L(q_i(t), \dot{q}_i(t), t)$, and a transformation $q_i(t) \mapsto q_i(t) + \epsilon\, \delta q_i(t)$ such that $\delta L = \frac{df}{dt}$, there exists a non-trivial function $K(t)$ such that when the equations of motion are satisfied, we have $\frac{dK}{dt} = 0$.

Proof

From the above equations, we have that, under an arbitrary transformation,

$$\delta L = \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right) + \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) \tag{2.1.45}$$

In particular, when the transformation is a symmetry we have

$$\frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i} - f\right) = \delta q_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right) \tag{2.1.46}$$

When the Euler-Lagrange equations of motion are satisfied, the right-hand side vanishes and hence defining

$$K(t) = \delta q_i \frac{\partial L}{\partial \dot{q}_i} - f(t) \tag{2.1.47}$$

we have constructed a conserved charge as desired. □

(Space)time symmetry: energy conservation

From (2.1.41), we can deduce that a Lagrangian which does not depend explicitly on time, i.e. satisfies $\frac{\partial L}{\partial t} = 0$, admits a (space)time symmetry (with $\delta t = 1$, say). More explicitly, we have $\delta q_i = \dot{q}_i$ and the Lagrangian changes by a total derivative:

$$\delta L = \frac{dL}{dt} \tag{2.1.48}$$

Hence we can use the Noether recipe to construct a conserved charge:

$$E(t) = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \tag{2.1.49}$$

which is the generalisation of the conserved energy we obtained previously in (2.1.14). A more careful treatment of time transformations in classical mechanics and the corresponding Noether theorem is provided in [16].

Internal symmetry: momentum conservation

For internal symmetries, we assume that the Lagrangian itself is invariant. Hence the Noether recipe is even simpler and we obtain simply

$$K(t) = \delta q_i \frac{\partial L}{\partial \dot{q}_i} \tag{2.1.50}$$

In practice, this will be the jth component of linear momentum when $\delta q_i = \delta_{ij}$ or the angular momentum when $\delta q_i = A_{ij} q_j$ for some antisymmetric A.

2.2 Symmetries in classical Lagrangian field theory

Now we have clarified the fundamental points, we generalise from Lagrangian mechanics to Lagrangian field theory. Since the ultimate destination is quantum field theory, we will adopt relativistic notation from this point onwards. The material in this section is also covered in e.g. [3, 17, 18]

2.2.1 Free scalar field

The simplest example in Lagrangian field theory is a free scalar field. This is analagous to the single point particle considered in Lagrangian mechanics. For concreteness, we work in four-dimensional Minkowski spacetime with coordinates $x^\mu = (t, x^i)$. The scalar field is a function $\phi : \mathbb{R}^{1,3} \to \mathbb{R}$. Typically we write $\phi = \phi(x)$.

As before, we have a Lagrangian L which depends on ϕ and its first derivatives. We rule out explicit dependence on time, and write $L = L(\phi, \partial_\mu \phi)$. Similarly to before, the action is then $S[\phi] = \int_{t_0}^{t_1} dt \; L$.

However now that we have spatial dimensions to work with, we assume that the dynamics are local and we can write the Lagrangian as

$$L(\phi, \partial_\mu \phi) = \int_{\mathbb{R}} d^3x \; \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.2.1}$$

for some *Lagrangian density* $\mathcal{L}$. The action is then simply

$$S[\phi] = \int d^4x \; \mathcal{L}(\phi, \partial_\mu \phi) \tag{2.2.2}$$

In practice, we almost always work with the Lagrangian density and refer to it as "the Lagrangian".

For a free scalar, the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi - \frac{1}{2}\, m^2\phi^2 \tag{2.2.3}$$

where $g^{\mu\nu}$ is the Minkowski metric.

Equations of motion

To find the equations of motion for the scalar field, we extremise the action functional S with respect to the path of the field $\phi(x)$. Physically, in the Lagrangian mechanics setting we considered a particle living at a point $\mathbf{x}$ in space at each time t and tracing out some continuous path over time. In Lagrangian field theory, the field ϕ permeates all of space, but its value at a given point $\mathbf{x_0}$ can change over time.

Consider a small deformation of the field

$$\phi(x) \mapsto \phi(x) + \epsilon\, \delta\phi(x) \tag{2.2.4}$$

which vanishes at the temporal endpoints and decays to zero at spatial infinity

$$\delta\phi(t_0, x^i) = 0 \tag{2.2.5a}$$

$$\delta\phi(t_1, x^i) = 0 \tag{2.2.5b}$$

$$\delta\phi(t, x^i) \to 0 \quad \text{as } |\mathbf{x}| \to \infty \tag{2.2.5c}$$

i.e. the field has a fixed initial and final configurations, and is fixed at spatial infinity during its time evolution.

The Lagrangian deforms as

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon\, \delta\mathcal{L} + \mathcal{O}(\epsilon^2) \tag{2.2.6}$$

where

$$\delta\mathcal{L} = -g^{\mu\nu}\partial_\mu\phi\, \partial_\nu(\delta\phi) - m^2\phi\, \delta\phi \tag{2.2.7}$$

So the change in the action is

$$S[\phi + \epsilon\, \delta\phi] - S[\phi] = \epsilon\, \delta S[\phi] + \mathcal{O}(\epsilon^2) \tag{2.2.8}$$

where

$$\delta S[\phi] = \int d^4x\, \delta\mathcal{L} = \int d^4x\, \delta\phi\, \left(\Box - m^2\right)\phi - \int_{\partial\mathcal{M}} d^3y\, \delta\phi\, n^\mu\partial_\mu\phi \tag{2.2.9}$$

Here $\Box \equiv g^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator, $\partial\mathcal{M}$ schematically denotes the spatio-temporal boundary and n^μ is a normal vector to $\partial\mathcal{M}$. By definition, $\delta\phi$ vanishes on $\partial\mathcal{M}$ so the integral over y on the right-hand side vanishes.

For an extremising field $\phi_C(x)$, δS must vanish for all such $\delta\phi(x)$ and hence $\phi_C(x)$ satisfies the classical equation of motion

$$\left(\Box - m^2\right)\phi(x) = 0 \tag{2.2.10}$$

The equation of motion for a free scalar field is called the *Klein-Gordon equation.*

A field $\phi(x)$ satisfying the equation of motion is said to be "on-shell". We define the canonical conjugate momentum to be

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \dot{\phi}(x) \tag{2.2.11}$$

We can write the equation of motion more explicitly as in Lagrangian mechanics:

$$\dot{\pi} = \left(\nabla^2 - m^2\right)\phi \tag{2.2.12}$$

where $\nabla^2 \equiv \delta^{ij}\partial_i\partial_j$ is the Laplacian.

Spacetime translation invariance and conservation of stress tensor

Observe that if we make a small translation in spacetime

$$x^\mu \mapsto x^\mu - \epsilon\, a^\mu \tag{2.2.13}$$

then this induces a transformation of the field

$$\phi(x) \mapsto \hat{\phi}(x) = \phi(x) + \epsilon\, a^\mu\partial_\mu\phi(x) + \mathcal{O}(\epsilon^2) \tag{2.2.14}$$

i.e. $\delta\phi(x) = a^\mu\partial_\mu\phi(x)$.

Note that this induced transformation of ϕ says nothing about the boundary, unlike the variation we took to find the equations of motion. However in Lagrangian field theory this does not matter, because we assume that the Lagrangian density decays sufficiently quickly as we reach spatial infinity.[6]

The Lagrangian density transforms as

$$\begin{aligned}
\delta\mathcal{L} &= -\partial^\mu\phi\,\partial_\mu\left(a^\nu\partial_\nu\phi\right) - m^2\phi\, a^\nu\partial_\nu\phi \\
&= -\frac{1}{2}\, a^\nu\partial_\nu\left(\partial^\mu\phi\,\partial_\mu\phi + m^2\phi^2\right) \\
&= \partial_\nu\left(a^\nu\mathcal{L}\right)
\end{aligned}$$

[6] Otherwise the action would not be well-defined.

The transformed action is

$$\begin{aligned} S'[\phi] &= \int d^4x \ (\mathcal{L} + \epsilon\, \delta\mathcal{L}) + \mathcal{O}(\epsilon^2) \\ &= S[\phi] + \epsilon \int d^4x\, \partial_\mu \left(a^\mu \mathcal{L}\right) + \mathcal{O}(\epsilon^2) \\ &= S[\phi] + \epsilon \int_{\partial\mathcal{M}} d^3y\, n_\mu \left(a^\mu \mathcal{L}\right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

Since we assume the Lagrangian density vanishes on $\partial\mathcal{M}$, the term linear in ϵ vanishes. So the action is invariant, and hence the equations of motion are invariant. That is, spacetime translation is a *symmetry* of the theory.

Now it is appropriate to define the *stress tensor* or *energy-momentum tensor* of the theory. We write

$$T_{\mu\nu} = \partial_\mu\phi\, \partial_\nu\phi + g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\, \partial_\nu\phi - \frac{1}{2}\, g_{\mu\nu} \left(\partial^\rho\phi\, \partial_\rho\phi + m^2\phi^2\right) \tag{2.2.15}$$

Using the equations of motion, it is straightforward to show that the stress tensor is conserved. Indeed,

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu \left[\partial^\mu\phi\, \partial^\nu\phi - \frac{1}{2}\, g^{\mu\nu} \left(\partial^\rho\phi\, \partial_\rho\phi + m^2\phi^2\right)\right] \\ &= \left(\partial^\nu\phi\right) \left(\Box - m^2\right)\phi \\ &= 0 \end{aligned}$$

Again this was an inevitable consequence of Noether's theorem - invariance under spacetime translation implies conservation of the stress tensor. To see why, compare our particular transformation of the Lagrangian density with the transformation under a general variation. Imposing the equation of motion, we get

$$\partial_\mu \left(a^\mu \mathcal{L}\right) = -\partial_\mu \left(\delta\phi\, \partial^\mu\phi\right) \tag{2.2.16}$$

Setting $\delta\phi = a^\mu\partial_\mu\phi$ we recover $a_\nu \partial_\mu T^{\mu\nu} = 0$. But a_ν is an arbitrary constant, so conservation of the stress tensor follows immediately as expected.

There is an important but subtle point to be made here. In general, the stress tensor should be defined by coupling the Lagrangian to an arbitrary background metric $g_{\mu\nu}$ and taking a functional derivative of the action:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{2.2.17}$$

For the action to be invariant under an arbitrary diffeomorphism (smooth map between manifolds), the stress tensor must be conserved:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.2.18}$$

However, this equation should be understood in the spirit of "gauge invariance", i.e. the stress tensor is not a true conserved quantity, but rather a redundancy of our mathematical description.

The crucial extra ingredient required to construct a true conserved quantity is an *isometry* of spacetime. Generically, a vector field K generates an isometry if

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \tag{2.2.19}$$

and we also call K a *Killing field.* By contracting a Killing field K_μ with some parametrisation of a geodesic x^μ we can manufacture a Noether charge Q. In our trivial flat space example, translations in any direction are isometries, so we (loosely) say that $T^{\mu\nu}$ itself is a Nother current. A more precise statement is that Noether charges are conserved *along geodesics of the spacetime.* The details of symmetry and Noether's theorem associated with the stress tensor are very nicely explored in the notes [19].

The conserved stress tensor will play an important role in the quantised theory, particularly in the context of the computations in Chapter 5.

2.2.2 Complex scalar field

To consider another example of symmetry, we can work instead with a *complex* scalar field, i.e. a map $\phi : \mathbb{R}^{1,3} \to \mathbb{C}$. We denote the complex conjugate of ϕ as $\phi^\dagger$. The Lagrangian density is

$$\mathcal{L} = -g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi^\dagger - m^2\phi^\dagger\phi \tag{2.2.20}$$

which again yields the Klein-Gordon equation as the equation of motion.

Global $U(1)$ symmetry and conservation of charge

Observe that if we make the transformation (for constant $\alpha \in \mathbb{R}$)

$$\phi(x) \mapsto e^{i\alpha}\,\phi(x) \tag{2.2.21a}$$

$$\phi^\dagger(x) \mapsto e^{-i\alpha}\,\phi^\dagger(x) \tag{2.2.21b}$$

then the Lagrangian density, and thus the action, is invariant. We call this a *global* $U(1)$ symmetry, since the field $\phi(x)$ transforms the same way at every point x.

Infinitesimally, we can write

$$\phi(x) \mapsto \phi(x) + i\alpha\phi(x) + \mathcal{O}(\alpha^2) \tag{2.2.22}$$

Making contact with the notation from before, we have

$$\delta\phi(x) = i\phi(x) \tag{2.2.23}$$

and

$$\delta\mathcal{L} = 0 \tag{2.2.24}$$

If we define the current

$$j_\mu = i\left(\phi^\dagger\partial_\mu\phi - \phi\,\partial_\mu\phi^\dagger\right) \tag{2.2.25}$$

then we have

$$\begin{aligned}
\partial_\mu j^\mu &= ig^{\mu\nu}\partial_\mu\left(\phi^\dagger\partial_\nu\phi - \phi\,\partial_\nu\phi^\dagger\right) \\
&= i\left(\phi^\dagger\,\Box\phi - \phi\,\Box\phi^\dagger\right) \\
&= i\phi^\dagger\left(\Box - m^2\right)\phi - i\phi\left(\Box - m^2\right)\phi^\dagger \\
&= 0
\end{aligned}$$

when the equations of motion are satisfied. i.e. the current j^μ is *conserved*.

The conservation equation can be written more explicitly as

$$\frac{\partial j^0}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{2.2.26}$$

where $j^\mu = (j^0, \mathbf{j})$.

Integrating over some bounded spatial volume V with boundary surface $\partial V = S$ gives the integral form of the equation

$$\frac{dQ_V}{dt} + \mathcal{F}_V = 0 \tag{2.2.27}$$

where $Q_V = \int_V dV\ j^0$ is the charge inside the volume V and $\mathcal{F}_V = \int_S dS\ (\hat{\mathbf{n}}\cdot\mathbf{j})$ is the flux coming out of V. As is conventional, $\hat{\mathbf{n}}$ is the unit normal to S pointing outwards.

Indeed, using Stokes's theorem we have

$$\begin{aligned}
\frac{dQ_V}{dt} &= \frac{d}{dt}\int_V dV\ j^0 \\
&= \int_V dV\ \left(\frac{\partial j^0}{\partial t}\right) \\
&= -\int_V dV\ (\nabla\cdot\mathbf{j})
\end{aligned}$$

$$
\begin{aligned}
&= - \int_S dS \, (\hat{\mathbf{n}} \cdot \mathbf{j}) \\
&= -\mathcal{F}_V
\end{aligned}
$$

Making contact with classical electromagnetism, we call j^0 the *charge density* and $\mathbf{j}$ the *current density*. Physically the local charge Q_V inside some bounded region of space V changes at exactly the instantaneous rate to balance the current flowing across the boundary ∂V.

Once again, the conservation of the current is a consequence of the global symmetry.

2.2.3 General system

We can now generalise our discussion of classical field theory and work in d spacetime dimensions. Consider n general fields which we write schematically as $\Phi_A(x)$, $A = 1, 2, \ldots, n$. Note that the Φ_A are not necessarily scalars and can be in any representation of the Lorentz group. The most general Lagrangian density of interest is a function of the fields and their first derivatives. The final ingredient is the set of possible masses and interaction couplings which appear as the coefficients, which we write schematically as g_a.

The action functional is

$$
S[\Phi_A; g_a] = \int d^d x \, \mathcal{L} \left(\Phi_A(x), \partial_\mu \Phi_A(x); g_a \right) \tag{2.2.28}
$$

Equations of motion

Using our familiar technology we extremise the action functional by deforming the fields as

$$
\Phi_A \mapsto \Phi_A + \epsilon \, \delta\Phi_A \tag{2.2.29}
$$

The Lagrangian density deforms as

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \Phi_A} \, \delta\Phi_A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} \partial_\mu (\delta\Phi_A) \\
&= \delta\Phi_A \left(\frac{\partial \mathcal{L}}{\partial \Phi_A} - \partial_\mu \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} \right) + \partial_\mu \left(\delta\Phi_A \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} \right)
\end{aligned}
$$

where summation over A is implied.

Imposing as usual that the field deformations vanish on the spacetime boundary, the total derivative term vanishes under the integral sign, and hence the variational derivative of the action with respect to Φ_A is

$$\frac{\delta S}{\delta \Phi_A} = \frac{\partial \mathcal{L}}{\partial \Phi_A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} \tag{2.2.30}$$

At extremal S, this must vanish for arbitrary deformations of the fields (as long as the deformations vanish on the spacetime boundary). Hence the Euler-Lagrange equations of motion are

$$\frac{\partial \mathcal{L}}{\partial \Phi_A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} = 0; \quad A = 1, 2, \ldots, n \tag{2.2.31}$$

Transformations of the system

We can again split out transformations into two classes.

A spacetime transformation is a map

$$x^\mu \mapsto X^\mu(x) \tag{2.2.32}$$

which induces fields transformations

$$\Phi_A(x) \mapsto \hat{\Phi}_A(x) = \Phi_A\left(X^{-1}x\right) \tag{2.2.33}$$

The Lagrangian density transforms as

$$\mathcal{L}\left(\Phi_A(x), \partial_\mu \Phi_A(x); g_a\right) \mapsto \mathcal{L}\left(\Phi_A(X^{-1}x), \partial_\mu \Phi_A(X^{-1}x); g_a\right) \tag{2.2.34}$$

If instead we transform the fields directly, we have

$$\Phi_A(x) \mapsto \hat{\Phi}_A(x) \tag{2.2.35}$$

and the Lagrangian density transforms as

$$\mathcal{L} \mapsto \mathcal{L}' = \mathcal{L}\left(\hat{\Phi}_A(x), \partial_\mu \hat{\Phi}_A(x)\right) \tag{2.2.36}$$

This is called an *internal* transformation.

If the action is invariant under either type of transformation then we call the transformation a symmetry. A spacetime symmetry will shift the Lagrangian density by a total spacetime derivative, while an internal symmetry is a symmetry of the Lagrangian density itself (and not only the action).

Of particular interest are continuous transformations connected to the identity. For a spacetime transformation we have

$$x^\mu \mapsto X^\mu(x) = x^\mu - \epsilon\, \delta x^\mu \tag{2.2.37}$$

and for an internal transformation we have

$$\Phi_A(x) \mapsto \Phi_A(x) + \epsilon\, \delta\Phi_A(x) \tag{2.2.38}$$

For clarity, we emphasise that the parameter δx^μ is constant but that generically $\delta\Phi_A$ can be a function of spacetime x^μ. This is analagous to the earlier (simpler) case of Lagrangian mechanics, where the parameter δt in (2.1.39) was constant but δq_i in (2.1.40) could be a function of t.

For a spacetime transformation, the infinitesimal change in the Lagrangian density is then simply given by

$$\delta\mathcal{L} = \delta x^\mu\, \partial_\mu \mathcal{L} \tag{2.2.39}$$

For an internal transformation, the infinitesimal change in the Lagrangian density is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_A}\,\delta\Phi_A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\partial_\mu(\delta\Phi_A) = \delta\Phi_A\left(\frac{\partial\mathcal{L}}{\partial\Phi_A} - \partial_\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\right) + \partial_\mu\left(\delta\Phi_A\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)}\right) \tag{2.2.40}$$

Note that in a sense the spacetime transformation is a special case of the internal transformation when the Lagrangian has no explicit dependence on x^μ. It is straightforward to check the consistency of these two variations by setting $\delta\Phi_A(x) = \delta x^\nu\, \partial_\nu\Phi_A(x)$.

2.2.4 Noether's theorem for Lagrangian field theory

As the reader will anticipate, we can lift our derivation of Noether's theorem to Lagrangian field theory with appropriate generalisations.

In Lagrangian field theory, a continuous symmetry is a transformation such that the Lagrangian density is deformed by (at most) a total derivative with respect to spacetime:

$$\delta\mathcal{L} = \partial_\mu f^\mu \tag{2.2.41}$$

In such a case, the action S is invariant, and the equations of motion are unchanged.

The suitable generalisation of a conserved charge is a conserved *current*. Mathematically, there is some vector j^μ such that on-shell, we have

$$\partial_\mu j^\mu = 0 \tag{2.2.42}$$

Precise statement

Given a Lagrangian density $\mathcal{L}(\Phi_A, \partial_\mu \Phi_A)$ and a transformation $\Phi_A(x) \mapsto \Phi_A + \epsilon\ \delta\Phi_A(x)$ such that $\delta\mathcal{L} = \partial_\mu f^\mu$, there exists a non-trivial current j^μ such that on-shell, we have $\partial_\mu j^\mu = 0$.

Proof

Under an arbitrary transformation, we have

$$\delta\mathcal{L} = \partial_\mu \left(\delta\Phi_A \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} \right) + \delta\Phi_A \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} - \frac{\partial \mathcal{L}}{\partial \Phi_A} \right) \tag{2.2.43}$$

In particular, when the transformation is a symmetry, we have

$$\partial_\mu \left(\delta\Phi_A \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} - f^\mu \right) = \delta\Phi_A \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} - \frac{\partial \mathcal{L}}{\partial \Phi_A} \right) \tag{2.2.44}$$

When the fields are on-shell, the right-hand side vanishes and hence defining

$$j^\mu = \delta\Phi_A \, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_A)} - f^\mu \tag{2.2.45}$$

we have constructed a conserved current as desired.

Spacetime symmetry

In Minkowski space, the spacetime symmetry group consists of translations

$$\delta x^\mu = a^\mu \tag{2.2.46}$$

and Lorentz transformations

$$\delta x^\mu = \Lambda^\mu_\nu \, x^\nu \tag{2.2.47}$$

where $\Lambda_{\mu\nu}$ is constrained to be antisymmetric.

Together these transformations form the Poincaré group. To see why the Poincaré transformations are symmetries, it is straightforward to check using (2.2.39) that we have

$$\delta\mathcal{L} = \partial_\mu(\delta x^\mu \mathcal{L}) \tag{2.2.48}$$

i.e. the Lagrangian density deforms by a total derivative. The associated conserved quantities are the stress tensor (for translations) and the angular momentum (for Lorentz transformations).

Internal symmetry

For internal symmetries, the Lagrangian is invariant and we obtain a conserved current

$$j^\mu = \delta\Phi_A \,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_A)} \tag{2.2.49}$$

Physically this can correspond to conservation of e.g. electric charge or baryon number. When the fields live in representations of non-Abelian gauge groups, the internal symmetries can be more complicated, but we will not discuss that here.

2.2.5 Aside: Differential forms

So far we have worked with Lagrangians and fields explicitly in components. While very concrete, the equations have a tendency to become littered with indices and can get quite complicated. An elegant approach to deal with this problem is to work instead with *differential forms*. As well as being succinct, using forms instead of components also allows us to generalise more easily to curved spacetimes. This will be particularly useful to us in Chapter 3 where we work extensively in bulk AdS spacetime. Some work is required upfront to build intuition for the formalism of differential forms and the associated operations, but this investment will more than pay off in the later chapters.

In what follows we use the conventions of [4]. Another useful resource is [19].

Physicist's definition

We define a *p-form* or *differential form* to be an antisymmetric $(0, p)$ tensor field. Geometerically, given a smooth spacetime manifold $\mathcal{M}$ with dual tangent bundle $\mathcal{TM}^*$, a $(0, p)$ tensor field is a map

$$\omega : (\mathcal{TM}^*)^p \to \mathbb{R} \tag{2.2.50}$$

For such ω to be a p-form, we further require that with respect to any basis, the components of ω are antisymmetric.

$$\omega_{\mu_1\mu_2\ldots\mu_p} = \omega_{[\mu_1\mu_2\ldots\mu_p]} \tag{2.2.51}$$

In our work we will exclusively use the dual coordinate basis, and write the components of a p-form as

$$\omega_p = \frac{1}{p!}\,\omega_{\mu_1\mu_2\ldots\mu_p}\,dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p} \tag{2.2.52}$$

Here the subscript p on the left-hand side reminds us that ω_p is a p-form.

New forms from old

The *wedge product* denoted by $\wedge$ can be defined component-wise for a p-form ω_p and a q-form η_q by

$$(\omega \wedge \eta)_{\mu_1\mu_2\ldots\mu_{p+q}} = \frac{(p+q)!}{p!\, q!}\, \omega_{[\mu_1\mu_2\ldots\mu_p}\eta_{\mu_{p+1}\mu_{p+2}\ldots\mu_{p+q}]} \tag{2.2.53}$$

$\omega_p \wedge \eta_q$ is thus a $(p+q)$-form.

The *exterior derivative* denoted by d can be defined component-wise for a p-form ω_p by

$$(d\omega)_{\mu_1\mu_2\ldots\mu_{p+1}} = (p+1)\, \partial_{[\mu_1}\omega_{\mu_2\ldots\mu_{p+1}]} \tag{2.2.54}$$

$d\omega_p$ is thus a $(p+1)$-form.

The *Hodge star* denoted by $\star$ can be defined component-wise for a p-form ω_p by

$$(\star\omega)_{\mu_1,\ldots\mu_{n-p}} = \frac{1}{(n-p)!}\, \epsilon^{\nu_1\ldots\nu_p}{}_{\mu_1\ldots\mu_{n-p}}\, \omega_{\nu_1\ldots\nu_p} \tag{2.2.55}$$

where n is the number of spacetime dimensions and

$$\epsilon_{\mu_1\mu_2\ldots\mu_n} = \sqrt{|g|}\, \mathrm{sgn}\left((\mu_1\mu_2\ldots\mu_n)\right) \tag{2.2.56}$$

is the Levi-Civita tensor density. Here g is the metric determinant and $(\mu_1\mu_2\ldots\mu_n)$ denotes a cycle. The sign of the cycle takes the value 0 if it is not a permutation, $+1$ if it is an even permutation and -1 if it is an odd permutation.

$\star\omega_p$ is thus an $(n-p)$-form.

Finally, we can use the Hodge star to integrate n-forms as follows. Define the *volume form* by

$$\epsilon = \star 1 = \sqrt{|g|}\, d^n x \tag{2.2.57}$$

where we identified $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = d^n x$.

For an n-form Ω_n, we have

$$\Omega_n = \omega(x)\, \epsilon \tag{2.2.58}$$

for some function $\omega(x)$. In fact, it turns out that $\omega = (-1)^s\, (\star\Omega_n)$, where s is the signature of the metric on the manifold.

Hence we can define the integral of Ω_n over the manifold $\mathcal{M}$ by

$$\int_{\mathcal{M}} \Omega_n = (-1)^s \int_{\mathbb{R}^n} d^n x\, \sqrt{|g|}\, (\star\Omega_n) \tag{2.2.59}$$

Utility of differential forms

Now we can generalise our earlier work in Lagrangian field theory. Given an n-dimensional spacetime manifold $\mathcal{M}$, we can instead make the Lagrangian density $\mathcal{L}$ an n-form. Then the action is simply given by

$$S = \int_{\mathcal{M}} \mathcal{L} \tag{2.2.60}$$

Note that the manifold is not necessarily flat Minkowski spacetime.

Returning to our earlier example, consider a complex scalar field $\phi(x)$. The action is simply

$$S[\phi] = -\int \left(d\phi \wedge \star(d\phi)^\dagger + m^2 \phi \wedge (\star\phi)^\dagger \right) \tag{2.2.61}$$

Taking a variation of ϕ (leaving $\phi^\dagger$ fixed), we have

$$\begin{aligned} \delta S &= -\int \left(d(\delta\phi) \wedge \star(d\phi)^\dagger + m^2 \delta\phi \wedge (\star\phi)^\dagger \right) \\ &= \int \delta\phi \wedge \left(d \star (d\phi)^\dagger - m^2 (\star\phi)^\dagger \right) - \int_{\partial\mathcal{M}} \delta\phi \wedge \star(d\phi)^\dagger \end{aligned}$$

where we used Stoke's theorem for differential forms. Neglecting the boundary term and acting with $\star$, we extract the equation of motion

$$\left(\star d \star d - m^2 \star^2 \right) \phi = 0 \tag{2.2.62}$$

If we assume that the metric is flat with signature $s = -1$, it is straightforward to show that $\star d \star d\phi = -\Box\phi$ and $\star^2 \phi = -\phi$, so we recover the usual Klein-Gordon equation.

Finally, we turn to the conserved Noether current j^μ satisfying $\partial_\mu j^\mu = 0$. If we interpret j instead as a *1-form*, we can write the elegant conservation equation

$$d \star j = 0 \tag{2.2.63}$$

This generalises nicely to higher-form symmetries as we will discuss later.

2.3 Symmetries in quantum field theory

2.3.1 Quantisation and the path integral

The most accessible route to defining a quantum field theory is to quantise a known Lagrangian field theory. In the quantum theory, our classical fields $\Phi(x)$ are promoted to *quantum fields* or *operator-valued fields*, i.e. at each point x in spacetime, $\Phi(x)$ is an operator. Note that the quantum field theory may consist of multiple fundamental fields $\Phi_1(x), \Phi_2(x), \cdots$, but for brevity in abstract discussion we will schematically denote all the field content as $\Phi(x)$. In particular, the fields may not be in the scalar (trivial) representation of the Lorentz group.

Given some Lagrangian density $\mathcal{L}$ and an action $S[\Phi] = \int d^d x \ \mathcal{L}\left(\Phi, \partial_\mu \Phi; g_a\right)$, we define the *path integral* $\mathcal{Z}_0[g_a]$ by

$$\mathcal{Z}_0 = \int [D\Phi] \ \exp\left(iS[\Phi; g_a]\right) \tag{2.3.1}$$

where $[D\Phi]$ is the path integral measure.

Roughly speaking, rather than only considering the on-shell field configurations, we integrate over *all* configurations for the fields, and weight them by (the exponential of) their corresponding actions. By construction, the classical field configurations minimise the action, and so (after Wick-rotating to Euclidean space), their contributions to the path integral dominate because they are the least exponentially suppressed.

Given a path integral, we can define *correlation functions* or *n-point functions* of local operators. For operators $\mathcal{O}_A(x)$, we define

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \frac{1}{\mathcal{Z}_0}\int [D\Phi] \ \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n) \ \exp\left(iS[\Phi; g_a]\right) \tag{2.3.2}$$

These are the fundamental objects of study in quantum field theory.

Note that the $\mathcal{O}_A$ are not necessarily fundamental fields appearing in the Lagrangian. They may be composite operators e.g. $\mathcal{O}(x) = \Phi_1(x)\Phi_2(x)$ or something entirely more exotic.

In practice, we usually introduce a sourcing field J for each fundamental field in the action and consider a *generating functional* $\mathcal{Z}[J]$ defined by

$$\mathcal{Z}[J] = \int [D\Phi] \ \exp\left(iS[\Phi; g_a] + i\int d^d x \ J(x)\cdot\Phi(x)\right) \tag{2.3.3}$$

where $J \cdot \Phi$ schematically denotes possible contractions.

The generating functional provides a systematic mechanism to compute perturbative quantum corrections arising from interactions between fields, order by order in the coupling strengths g_a. These computations are codified by *Feynamn rules* which are used to draw and calculate individual *Feynman diagrams*. Usually we refer to the generating functional as simply "the path integral", noting that $\mathcal{Z}[0] = \mathcal{Z}_0$.

2.3.2 Noether's theorem for quantum field theory

Quantum mechanically, the relevant object to discuss is the generating functional, rather than the action. Observe that in the path integral, we integrate over all field configurations, and so (from the point of view of the path integral) the fields are essentially dummy variables. For arbitrary transformations of the fields $\Phi \mapsto \Phi'$ we can write

$$\mathcal{Z}[J] = \int [D\Phi'] \, \exp\left(iS[\Phi'; g_a] + i\int d^dx \; J(x) \cdot \Phi'(x)\right) \tag{2.3.4}$$

If $S[\Phi'] = S[\Phi]$, we say that the transformation is a symmetry. In classical field theory, Noether's theorem guaranteed the existence of a conserved current. However, in quantum field theory we are faced with the additional complication of the path integral measure. If the path integral measure is invariant under the transformation: $[D\Phi'] = [D\Phi]$, then we have a quantum mechanical symmetry. On the other hand, if the path integral measure is not invariant, then the quantum theory does not inherit the classical symmetry and we call it *anomalous*.

Following [18], we can use a nice trick to generalise Noether's theorem to quantum field theory. This is also well-explained in [20]. Given an infinitesimal transformation $\Phi \mapsto \Phi' = \Phi + \epsilon \; \delta\Phi$, we can consider a more general class of transformations by promoting the infinitesimal parameter ϵ to a spacetime-dependent field $\epsilon(x)$. Sometimes we will drop $\mathcal{O}(\epsilon^2)$ terms without explicitly stating so, but from the context it will be clear.

Under the general continuous transformation

$$\Phi \mapsto \Phi' = \Phi + \epsilon(x) \; \delta\Phi \tag{2.3.5}$$

the Lagrangian transforms as

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon(x) \; \delta\Phi \; \frac{\partial \mathcal{L}}{\partial \Phi} + (\partial_\mu \epsilon(x) \; \delta\Phi) \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \tag{2.3.6}$$

In particular, when the transformation with ϵ constant is a global symmetry of the action, the infinitesimal change in the Lagrangian must be a total derivative, i.e.

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon \; \partial_\mu f^\mu \tag{2.3.7}$$

Equating the order ϵ terms when epsilon is constant, we find that

$$\delta\Phi \; \frac{\partial \mathcal{L}}{\partial \Phi} + \partial_\mu \left(\delta\Phi\right) \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial_\mu f^\mu \tag{2.3.8}$$

But this equation is independent of $\epsilon(x)$, and so we can substitute back into the general transformation of the Lagrangian to obtain

$$\mathcal{L} \mapsto \mathcal{L} + \epsilon(x)\, \partial_\mu f^\mu + \partial_\mu \left(\epsilon(x)\right)\, \delta\Phi\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \tag{2.3.9}$$

The action must then transform as follows

$$S[\Phi + \epsilon\, \delta\Phi] - S[\Phi] = \int_{\partial\mathcal{M}} d^{d-1}y\, n_\mu \left(\epsilon(y)\, \delta\Phi(y)\, \frac{\partial(\partial_\mu \mathcal{L})}{\partial \Phi}\right) - \int_{\mathcal{M}} d^d x\, \epsilon(x)\, \partial_\mu j^\mu(x) \tag{2.3.10}$$

where j^μ is the usual Noether current.

Discarding the boundary integral over y, we see immediately that if $S[\Phi + \epsilon] = S[\Phi]$ and ϵ is constant, then j^μ is conserved as expected. This is nothing more than the classical Noether theorem.

We can push this further in the quantum theory. For now let's assume that we have a global symmetry transformation under which the path integral measure is invariant. Using the above technology, we can write the path integral as

$$\begin{aligned}
\mathcal{Z}[K] &= \int [D\Phi] \exp\left(iS[\Phi] - i\int d^d x\, \epsilon(x)\, \partial_\mu j^\mu(x)\right) \exp\left[i\int d^d y\, K(y)\cdot(\Phi(y) + \epsilon(y)\, \delta\Phi(y))\right] \\
&= \int [D\Phi]\, e^{iS[\Phi] + i\int d^d z\, K(z)\cdot\Phi(z)} \left(1 - i\int d^d x\, \epsilon(x)\, \partial_\mu j^\mu(x)\right)\left(1 + i\int d^d y\, K(y)\cdot \epsilon(y)\, \delta\Phi(y)\right) \\
&= \mathcal{Z}[K] + i\int [D\Phi]\, e^{iS[\Phi] + i\int d^d z\, K(z)\cdot\Phi(z)} \int d^d x\, \epsilon(x) \left(K(x)\cdot\ \delta\Phi(x) - \partial_\mu j^\mu(x)\right)
\end{aligned}$$

where we wrote K for the source field instead of J, to avoid confusion with the symmetry current j^μ.

Now we see the utility of promoting ϵ to a spacetime-dependent field. To satisfy the above equation, we must have for all $\epsilon(x)$ that

$$\int [D\Phi]\, e^{iS[\Phi] + i\int d^d z\, K(z)\cdot\Phi(z)} \int d^d x\, \epsilon(x) \left(K(x)\cdot\ \delta\Phi(x) - \partial_\mu j^\mu(x)\right) = 0 \tag{2.3.11}$$

In particular, we can choose $\epsilon(x) = K\cdot\delta\Phi - \partial_\mu j^\mu$ so that the integrand inside the functional integral is strictly positive (after Wick-rotating). Hence we must have that

$$\int [D\Phi] \exp\left(iS[\Phi] + i\int d^d z\, K(z)\cdot\Phi(z)\right)\left(K(x)\, \delta\Phi(x) - \partial_\mu j^\mu(x)\right) = 0 \tag{2.3.12}$$

In a sense, this is the generalisation of Noether's theorem to the quantum theory.

Observe that we are still free to choose the source $K(x)$ and so we have considerable scope to derive further identities. For example, we can evaluate the expression with K identically vanishing to immediately get

$$\partial_\mu \left\langle j^\mu(x)\right\rangle = 0 \tag{2.3.13}$$

In other words, the one-point function of the Noether current operator is conserved.

Instead, we could first take a functional derivative with respect to $K(y)$ to obtain

$$\int [D\Phi]\ e^{iS[\Phi]+i\int d^dz\ K(z)\Phi(z)} \Big[i\Phi(y)(K(x)\ \delta\Phi(x) - \partial_\mu j^\mu(x)) + \delta\Phi(x)\ \delta^{(4)}(x-y) \Big] = 0 \tag{2.3.14}$$

Setting $K = 0$ in this expression then yields

$$\partial_\mu^{(x)} \langle \Phi(y) j^\mu(x) \rangle = \frac{1}{i}\ \delta^{(4)}(x-y) \langle \delta\Phi(x) \rangle \tag{2.3.15}$$

This is known as a *Ward identity* for the theory. The term involving a delta-function on the right-hand side is called a *contact term*.

By successively taking n functional derivatives and evaluating at $K = 0$, we can obtain a more general identity:

$$i\partial_\mu^{(x)} \langle \Phi(y_1)\Phi(y_2)\cdots\Phi(y_n) j^\mu(x) \rangle = \sum_{i=1}^{n} \delta^{(4)}(x-y_i) \langle \delta\Phi(x) \prod_{\substack{j=1 \\ j\neq i}}^{n} \Phi(y_j) \rangle \tag{2.3.16}$$

In particular, when $\delta\Phi = iq\Phi$, i.e. a usual $U(1)$ symmetry, we have

$$\partial_\mu^{(x)} \langle \Phi(y) j^\mu(x) \rangle = q\ \delta^{(4)}(x-y) \langle \Phi(x) \rangle \tag{2.3.17}$$

Roughly speaking, correlation functions containing the Noether current are conserved up to contact terms. In other words, the charged operators under the global symmetry current are the local operators $\Phi(x)$. The charged excitations of these operators are physically interpreted as point particles, e.g. electrons. In more modern language, we refer to "ordinary" global symmetries as 0-form symmetries. The conserved vector current j can be thought of as a 1-form.

In quantum theories, it is a common abuse of notation to write operator-valued equations involving j and Φ without specifying that they only hold *inside the path integral.*

The quantum analogue of Noetherian local charge conservation is the existence of quantum mechanical operators whose eigenvalues are "good" quantum numbers, in the sense that the operators commute with the Hamiltonian. For example, the conserved $U(1)$ current of a complex scalar field theory guarantees the existence of a number operator given by an integral over an arbitrary timeslice by

$$N = \int \star j = \int d^3x\ j^0 = i\int d^3x\ \left(\phi\ \dot{\phi}^\dagger - \dot{\phi}\ \phi^\dagger\right) \tag{2.3.18}$$

Roughly speaking, the number operator acts on a state to count the net number of particles (number of particles minus number of antiparticles). In the quantum theory, the presence of the $U(1)$ global symmetry is equivalent to the number operator being indepedent of the chosen timeslice; i.e. the net number of particles in a given state of the Fock space is conserved as time evolves. In fact, we did not have to define N on a timeslice at all; an arbitrary codimension-1 manifold will do.[7] This notion will generalise later when we consider higher-form symmetries in quantum field theory.

2.3.3 Anomalies

General discussion

Now we consider an important new possibility in the quantum theory compared to the classical theory, namely *anomalies*. Anomalies are not the main focus of this thesis, but to provide a complete overview of symmetries in quantum field theory it is important to explain briefly what they are and to provide a simple example.

As alluded to already, a *quantum anomaly* arises when the path integral measure $[D\Phi]$ transforms non-trivially under a (classical) symmetry transformation. Confusingly, such transformations are sometimes referred to as "anomalous symmetries", but this is misleading, since in the quantum theory such transformations are *not* symmetries at all. It is perhaps more accurate to describe these transformations as classical symmetries which are not present in the full quantum theory.

We can generalise our treatment of Ward identities from above. Suppose that we have a classical symmetry transformation of the action so that

$$S[\Phi + \alpha\ \delta\Phi] - S[\Phi] = -\int \alpha \wedge (d \star j) \tag{2.3.19}$$

where j is the Noether current, and we wrote $\alpha(x)$ instead of ϵ to avoid confusion with the volume form.

Now if the path integral measure $D[\Phi]$ picks up a phase under the transformation, we have

$$[D\Phi] \mapsto [D\Phi]\ \exp\left(i\int \alpha \wedge \star\mathcal{A}\right) = [D\Phi]\left(1 + i\int \alpha \wedge \star\mathcal{A}\right) \tag{2.3.20}$$

for some function $\mathcal{A}(x)$.

The generating functional can then be written as

$$\begin{aligned}\mathcal{Z}[K] &= \int [D\Phi]\left(1 + i\int \alpha\wedge\star\mathcal{A}\right) e^{iS[\Phi]}\left(1 - i\int \alpha\wedge(d\star j)\right) e^{i\int K\wedge\star\Phi}\left(1 + i\int \alpha\ K\wedge\star\delta\Phi\right)\\ &= \int [D\Phi]\ e^{iS[\Phi]}\ e^{i\int K\wedge\star\Phi}\left[1 + i\int \alpha\wedge(\star\mathcal{A} - d\star j + K\wedge\star\delta\Phi)\right]\end{aligned}$$

[7]In n ambient spacetime dimensions, a codimension of k simply means a dimension of $n-k$.

$$= \mathcal{Z}[K] + i\int [D\Phi] \int \alpha \wedge (\star\mathcal{A} - d\star j + K \wedge \star\delta\Phi)$$

Since $\alpha(x)$ is arbitrary, we obtain

$$\left\langle \exp\left(i\int K \wedge \star\Phi\right)(\mathcal{A} + \star d\star j + K\,\delta\Phi)\right\rangle = 0 \tag{2.3.21}$$

In particular, with the source K switched off, we have

$$\langle \mathcal{A} + \star d \star j\rangle = 0 \tag{2.3.22}$$

which in components in flat space is simply

$$\langle \partial_\mu j^\mu(x)\rangle = \langle \mathcal{A}(x)\rangle \tag{2.3.23}$$

Roughly speaking, the function $\mathcal{A}(x)$ quantifies the extent to which the current is no longer conserved.

Example: ABJ anomaly of free Dirac fermion

Let's consider the most well-known example of a classical 0-form global symmetry which is anomalous in the full quantum theory. This is the famous *ABJ anomaly* first studied in [21, 22, 23]. The setting is the theory of a free massless Dirac fermion.

First we consider the classical action and demonstrate the existence of a 0-form global symmetry. Suppose we have a massless free Dirac fermion ψ with Lagrangian density given by

$$\mathcal{L} = i\gamma^\mu\bar{\psi}\partial_\mu\psi \tag{2.3.24}$$

We can make the $U(1)$ transformation

$$\psi \to \psi' = e^{i\alpha}\psi; \quad \alpha \in \mathbb{R} \tag{2.3.25}$$

so that $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$ and $\mathcal{L} \to \mathcal{L}$. The action is invariant so this transformation is a symmetry transformation. By Noether's theorem, there must therefore be an associated conserved 1-form current j.

Taking a variation of the action and neglecting boundary terms, we find that

$$\delta S = i\int d^4x\,(\delta\bar{\psi}\,\not{\partial}\psi - \bar{\psi}\overleftarrow{\not{\partial}}\,\delta\psi) \tag{2.3.26}$$

where the left arrow indicates that the derivative is acting to the left.

Hence imposing $\delta S = 0$ gives the classical equation of motion

$$i\,\not{\partial}\psi = 0 \tag{2.3.27}$$

If instead we follow the usual trick by "gauging" the $U(1)$ parameter $\alpha \to \alpha(x)$ and perform the infinitesimal transformation $\delta\psi = i\alpha(x)\psi$, we get

$$\delta S = -\int d^4x \, (\partial_\mu \alpha) \, (\bar{\psi}\gamma^\mu\psi) = \int d^4x \, \alpha \, \partial_\mu(\bar{\psi}\gamma^\mu\psi) \tag{2.3.28}$$

When α is constant, this transformation is a symmetry, and so we must have $\delta S = 0$ for all α. This allows us to identify the conserved *vector current*

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi \tag{2.3.29}$$

This is a classical statement. Quantum mechanically, we can derive a Ward identity from the path integral to show that the charged objects under the vector symmetry are the local operators ψ and $\bar{\psi}$. The charged excitations under the symmetry (that which the charge operator counts) are electrons and positrons.

Interestingly, there is a further classical symmetry associated with the Dirac field: the *axial symmetry*. Consider coupling the Dirac fermion to a background gauge field A_1, by promoting the partial derivative ∂ to a covariant derivative D given by

$$D = \partial - igA \tag{2.3.30}$$

for some coupling strength g.

Defining the gamma matrix γ_5 by

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.3.31}$$

we can deform the Dirac field by

$$\psi \mapsto e^{i\alpha\gamma_5}\psi; \quad \alpha \in \mathbb{R} \tag{2.3.32}$$

This induces a transformation $\bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}$, so under this transformation, the Lagrangian density is invariant. Hence by Noether's theorem there is an associated conserved 1-form current. In this case, the conserved current is the *axial current* defined by

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \tag{2.3.33}$$

However, it turns out that in the full quantum theory, there is an anomaly and so the axial current is no longer conserved in the usual sense of the Ward identities. Comparing to our earlier notation, the anomaly function $\mathcal{A}(x)$ is given by

$$\mathcal{A}(x) = \frac{g^2}{4\pi^2} \star (F \wedge F) = -\frac{g^2}{16\pi^2} \, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{2.3.34}$$

where $F = dA$.

Hence quantum mechanically we have

$$d \star \langle j_A \rangle = \frac{g^2}{4\pi^2} \star \langle F \wedge F \rangle \tag{2.3.35}$$

This anomaly was originally discovered perturbatively at one-loop using so-called "triangle diagrams", see [21, 22, 23]. Concretely, there does not exist a regularisation scheme that preserves axial symmetry, and so after renormalising at one-loop, the quantum corrections forbid the conservation of the axial current.

Later in [24] the anomaly was computed by calculating the Jacobian factor in the path integral with functional determinants. The calculations are lengthy and involved; more readable treatments are given in the usual textbooks [3, 18] and notes [20].

The ABJ anomaly quantitatively affects the decay width of neutral pions to pair of photons, and so can be observed experimentally.

2.3.4 Spontaneous symmetry breaking

In this part we revisit the complex scalar field to cover the final aspect of conventional (0-form) global symmetries, namely *spontaneous symmetry breaking.*

We introduce a quartic coupling to the complex scalar field theory. The Lagrangian density is now

$$\mathcal{L} = -(\partial_\mu \phi)^\dagger (\partial^\mu \phi) - V(\phi) \tag{2.3.36}$$

where

$$V(\phi) = m^2 \, |\phi|^2 + \frac{\lambda}{4} \, |\phi|^4 \tag{2.3.37}$$

and $\lambda > 0$.

As before, we have a $U(1)$ global symmetry transformation $g(\alpha)$ parametrised by α:

$$g(\alpha) : \phi(x) \mapsto e^{i\alpha} \, \phi(x), \quad \alpha \in \mathbb{R} \tag{2.3.38}$$

If $m^2 > 0$ then the vacuum expectation value of ϕ is $\langle \phi \rangle = 0$ because this is the unique minimum of the classical potential. Thus $\mathcal{L}$ describes a massive complex scalar of mass m with a quartic self-interaction governed by the coupling λ.

However if $m^2 < 0$, we write $\mu^2 = -m^2 > 0$ and then

$$V(\phi) = \frac{\lambda}{4} \left(|\phi|^2 - \frac{v^2}{2} \right)^2 \tag{2.3.39}$$

where $v = \frac{2\mu}{\sqrt{\lambda}}$ and we added a constant μ^4/λ to the potential which doesn't affect the physics. This is called the "Wine Bottle" potential because its plot looks like the bottom of a wine bottle - see Figure 2.1.

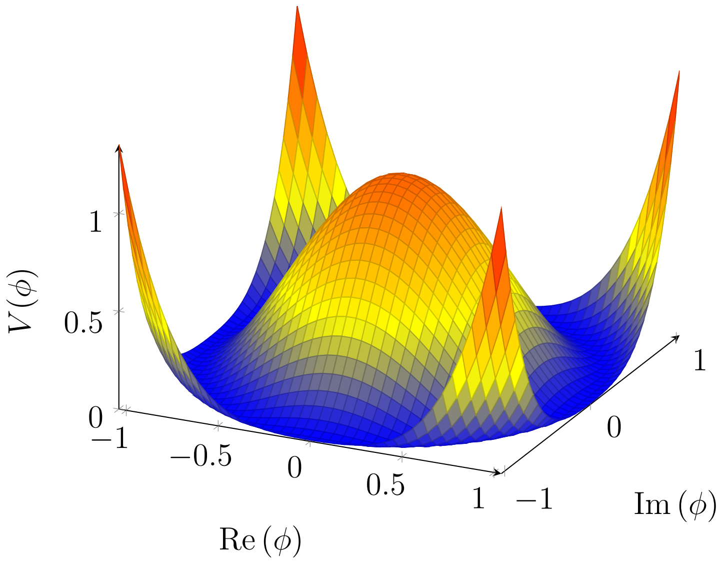

Figure 2.1: Plot of the "Wine Bottle" potential for $\lambda = 4$ and $v^2 = 2$

Clearly $V(\phi) \geq 0$ for all configurations $\phi(x)$. Consider the manifold of minima of the potential given by

$$\mathcal{V} \equiv \{\phi(x) \mid V(\phi) = 0\} = \left\{ \frac{v}{\sqrt{2}}\, e^{i\theta} \,\middle|\, \theta \in \mathbb{R} \right\} \cong S^1 \tag{2.3.40}$$

$\mathcal{V}$ is diffeomorphic to S^1 so we can parametrise the vacuum states as

$$\phi_0(\theta) \equiv \frac{v}{\sqrt{2}}\, e^{i\theta} \tag{2.3.41}$$

The $U(1)$ transformation $g(\alpha)$ then acts as

$$\begin{aligned} g(\alpha):\quad & \mathcal{V} \to \mathcal{V} \\ & \phi_0(\theta) \mapsto \phi_0(\theta + \alpha) \end{aligned}$$

So although g maps $\mathcal{V}$ into itself, the map g has no fixed points $\phi_0(\eta)$. We say that the $U(1)$ global symmetry is *spontaneously broken* or *non-linearly realised.*

We can expand ϕ about an arbitrary state $\phi_0(\eta) \in \mathcal{V}$ and write [8]

$$\phi(x) = \frac{v + \rho(x)}{\sqrt{2}} \, \exp\left(i\eta + \frac{i\theta(x)}{v} \right) \tag{2.3.42}$$

[8] The factor of v in the exponent is chosen so that $\theta(x)$ will be canonically normalised in the Lagrangian.

where $\rho(x)$ and $\theta(x)$ are real degrees of freedom. We say that the field ϕ has obtained a non-zero vacuum expectation value given by $\langle\phi\rangle = \phi_0(\eta)$.

Then we can write the action in terms of the new degrees of freedom via

$$|\phi(x)|^2 - \frac{v^2}{2} = \frac{1}{2}\,\rho^2 + v\rho \tag{2.3.43a}$$

$$(\partial_\mu\phi)^\dagger(\partial^\mu\phi) = \frac{1}{2}\,|d\rho|^2 + \frac{1}{2}\left(1+\frac{\rho}{v}\right)^2 |d\theta|^2 \tag{2.3.43b}$$

So the Lagrangian density can be written as

$$\mathcal{L} = -\frac{1}{2}\,|d\rho|^2 - \frac{1}{2}\left(1+\frac{\rho}{v}\right)^2 |d\theta|^2 - \frac{1}{2}\,m_\rho^2\,\rho^2\left(1+\frac{\rho}{2v}\right)^2 \tag{2.3.44}$$

where the mass for the gapped mode $\rho(x)$ is given by

$$m_\rho^2 = \frac{1}{2}\,\lambda v^2 \tag{2.3.45}$$

As expected, there is no dependence on the choice of vacuum state parametrised by η. This is a consequence of the existence of a global symmetry: we get the same result independent of which vacuum we expand around. However, the fact that we had to make an arbitrary choice of *some* vacuum is crucial!

Hence the Lagrangian describes a theory of a massive real scalar $\rho(x)$ with mass m_ρ and a massless real scalar $\theta(x)$. The gapless mode $\theta(x)$ is called a *Goldstone boson* or simply *Goldstone mode*.[9] ρ has cubic and quartic self-interactions, and there are also $\theta\theta\rho\rho$ and $\theta\theta\rho$ interactions.

Intuitively, it makes sense that "moving along the rim of the wine bottle" between the degenerate vacua costs little energy, and thus corresponds to the gapless mode $\theta(x)$, whereas moving in a radial direction "uphill" to a higher potential costs much more energy, and thus corresponds to the gapped mode $\rho(x)$.

Recall we have a conserved symmetry current j, exactly as when the symmetry is not spontaneously broken. This current can be written in terms of the new degrees of freedom in this case as

$$j = -\left(1+\frac{\rho}{v}\right)^2 d\theta \tag{2.3.46}$$

There is a massive field $\rho(x)$ and a massless field $\theta(x)$, so if we consider low energies $E \ll m_\rho$, we can ignore the massive excitations. The low energy effective action is then simply

$$S_{\text{eff}} = -\int \left(\frac{1}{2}\,(d\theta)^2\right) \tag{2.3.47}$$

[9]In general when a continuous global symmetry group G is spontaneously broken to a continuous global symmetry group H, there arise $\dim(G) - \dim(H)$ Goldstone bosons. In this case $G = U(1)$ and $H = \{\mathbb{1}\}$, the trivial group, so there is just one Goldstone boson.

so the theory describes a free gapless scalar field θ.

However, consider the symmetry transformation inherited by θ. We now have

$$\theta \mapsto \theta + \alpha v \tag{2.3.48}$$

i.e. $\delta\theta = v$. We usually call this a *shift symmetry*, and this is the non-linear realisation of the $U(1)$ global symmetry of the complex scalar field. The associated conserved symmetry current is

$$j = -d\theta \tag{2.3.49}$$

In fact, we can choose to characterise spontaneous symmetry breaking by its effect on the Noether current. When spontaneous symmetry breaking occurs, the current j becomes *exact*, i.e. we have $j = d\beta$ for some 0-form β. Interestingly, this automatically guarantees the existence of a second conserved current. Indeed, defining $k(x) = \star j(x)$, we have

$$d \star k = d\left(\star^2 j\right) = -dj = -d^2\beta = 0 \tag{2.3.50}$$

Later in Section 2.4.3 we will make the connection between co-closed forms[10] and conserved quantities more explicit. They play a key role in the construction of topological operators.

In quantum field theories, the distinction between spontaneously broken symmetries and anomalies is very subtle, as explored in [25].

2.4 Higher-form symmetries

2.4.1 Free Maxwell field

The free Maxwell theory, which classically is nothing more than relativistic electrodynamics in vacuum, provides an excellent concrete introduction to so-called 1-form symmetries. We will begin with the familiar textbook formalism, before introducing more sophisticated and recently developed concepts.

Consider a gauge field A_1 with field strength $F_2 = dA_1$. As a shorthand we write $F^2 \equiv F \wedge \star F$. For now we will work in four-dimensional Minkowski space. The action is given by

$$S[A] = -\int d^4x \left(\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}\right) = -\int \left(\frac{1}{2g^2} F^2\right) \tag{2.4.1}$$

[10] Co-closed forms are forms whose Hodge duals are closed.

where g^2 is a coupling constant.

Equations of motion

Consider a small deformation to the gauge field that vanishes on the spacetime boundary:

$$A \mapsto A + \epsilon\, \delta A \tag{2.4.2}$$

The induced transformation on the Lagrangian density is

$$\mathcal{L} \mapsto \mathcal{L} - \epsilon\, d(\delta A) \wedge \star F \tag{2.4.3}$$

Hence the action transforms as

$$\delta S = -\frac{1}{g^2}\int d(\delta A) \wedge \star F = -\frac{1}{g^2}\int_{\partial\mathcal{M}} (\delta A \wedge \star F) - \frac{1}{g^2}\int_{\mathcal{M}} \delta A \wedge (d \star F) \tag{2.4.4}$$

Since δA vanishes on $\partial\mathcal{M}$, we can read off the equation of motion

$$d \star F = 0 \tag{2.4.5}$$

which in components is the familiar free Maxwell equation

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.4.6}$$

Electric 1-form symmetry

Observe that if we consider the specific transformation given by

$$\delta A = \Lambda \tag{2.4.7}$$

with Λ a *closed* 1-form, i.e. $d\Lambda = 0$, then

$$\delta S = 0 \tag{2.4.8}$$

So shifting the gauge field by a closed 1-form is a symmetry of the action. Every symmetry we considered so far was generated simply by a *function*, i.e. a 0-form. This new type of symmetry is called a 1-*form symmetry* because it is generated by a 1-form. This is a particular case of a *higher-form symmetry* or *generalised global symmetry*. In free Maxwell theory, this is usually called the *electric* 1-*form symmetry*.

Straightforwardly from the equations of motion, we obtain a conserved 2-form current J_e defined by

$$J_e^{\mu\nu} = \frac{1}{g^2}F^{\mu\nu} \tag{2.4.9}$$

satisfying $d \star J_e = 0$. We emphasise that this is a *classical* statement. The quantum analogue follows shortly.

Ward identity

To compare with the 0-form example, we define the *Wilson loop* for a closed curve C by

$$W[C] = \exp\left(i \int_C dx^\mu A_\mu\right) \tag{2.4.10}$$

Neglecting the boundary term (e.g. by insisting that Λ vanishes there), we can relax our condition that Λ is closed to obtain

$$\delta S = -\int \Lambda \wedge (d \star J_e) \tag{2.4.11}$$

The Wilson loop deforms as

$$\delta W[C] = \left(i \int_C \Lambda\right) W[C] \tag{2.4.12}$$

The path integral for the quantum theory is

$$\mathcal{Z} = \int [DA] \, \exp\left(iS[A]\right) \tag{2.4.13}$$

The path integral measure $[dA]$ is invariant under the 1-form transformation, and so similarly to the 0-form symmetry case, we obtain

$$\begin{aligned}
\mathcal{Z} &= \int [DA] \, \exp\left(iS[A] - i\epsilon \int \Lambda \wedge (d \star J_e)\right) \\
&= \int [DA] \, \exp(iS[A]) \left(1 - i\epsilon \int \Lambda \wedge (d \star J_e) + \mathcal{O}(\epsilon^2)\right) \\
&= \mathcal{Z} - i\epsilon \left\langle \int \Lambda \wedge (d \star J_e) \right\rangle + \mathcal{O}(\epsilon^2)
\end{aligned}$$

so we have

$$\left\langle \int \Lambda \wedge (d \star J_e) \right\rangle = 0 \tag{2.4.14}$$

But this must hold for all $\Lambda(x)$, and hence we have

$$d \star \langle J_e \rangle = 0 \tag{2.4.15}$$

Now we consider insertions of the Wilson loop in the path integral. Define a generating functional

$$\mathcal{Z}_1 = \int [DA] \, \exp\left(iS[A]\right) W[C] \tag{2.4.16}$$

The 1-form transformation deforms both the action and the Wilson loop, so we have

$$\begin{aligned}
\mathcal{Z} &= \int [DA] \; \exp\left(iS[A] - i\epsilon \int \Lambda \wedge (d \star J_e)\right) W[C] \left(1 + i\epsilon \int_C \Lambda\right) \\
&= \int [DA] \; \exp(iS[A]) W[C] \left[1 + i\epsilon \left(\int_C \Lambda - \int \Lambda \wedge (d \star J_e)\right) + \mathcal{O}(\epsilon^2)\right] \\
&= \mathcal{Z}_1 + i\epsilon \left\langle \left(\int_C \Lambda - \int \Lambda \wedge (d \star J_e)\right) W[C] \right\rangle + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Hence for all Λ the order ϵ term vanishes and we have

$$d \star \langle J_e \, W[C] \rangle = \delta_C \, \langle W[C] \rangle \tag{2.4.17}$$

where we introduced the 3-form δ_C defined by

$$\int_{\mathcal{M}} \omega_1 \wedge \delta_C = \int_C \omega_1 \tag{2.4.18}$$

for any 1-form ω_1. The notation is deliberately suggestive: δ_C is a differential form generalisation of the Dirac delta function. More rigorous treatment of the underlying theory of integration of forms on manifolds is given in [26].

(2.4.17) is the Ward identity for the $U(1)$ 1-form global symmetry. In other words, the Wilson loops are the *charged operators* under the 1-form symmetry. The associated charged excitations are *electric field lines*. This construction is entirely analagous to the conventional Ward identity given by (2.3.17).

Note that Wilson loops $W[C]$ are *line operators* depending on a curve C, as opposed to *local operators* $\Phi(x)$ depending only on a spacetime point x. Our constructions hint at the possibility that the gauge field A_1, although a local operator itself, is not necessarily the correct choice of fundamental field from which to build the theory. Therefore it would be interesting to construct the theory regarding the Wilson loops as fundamental. We could image sourcing the Wilson loops in the path integral by a line operator source and taking a formal derivative with respect to this source. This would allow us to derive Ward identities with multiple Wilson loops and contact terms. In fact, this is the approach adopted in [27], taking inspiration from earlier work in [28].

Gauge invariance

There is a subtle distinction to be made with a *local gauge transformation*. A local gauge transformation is defined by

$$A \to A + d\lambda \tag{2.4.19}$$

and leaves the Lagrangian density invariant. However this is not regarded as a global symmetry of the system, but rather a redundancy in our mathematical description of the physics. This redundancy is referred to as *local gauge invariance* or, confusingly, *gauge symmetry*.

We should emphasise that the symmetries of interest in this thesis are *global symmetries*, rather than gauge "symmetries". Gauge invariance is an important feature of the equations which we can exploit to make computations simpler, but it does not have a physical meaning, as opposed to global symmetries, whose physical reality manifests in conserved currents. Again, this hints that gauge invariance could be an indication that the gauge fields are not the correct degrees of freedom from which to build the theory.

It is often said that the photon is massless as an inevitable result of gauge invariance. However, a more modern perspective should view the massless photon as a *Goldstone mode* of a spontaneously broken 1-form electric symmetry! This idea was originally proposed in [6], with further discussion in [7, 8]. Note that this is distinct from the *superconducting phase* in which the photon acquires an effective mass and the 1-form electric symmetry is *unbroken*, see e.g. [29].

2.4.2 Electromagnetic duality

Now we discuss a useful method for deepening our understanding of theories with p-form gauge fields. Given a purely kinetic action that depends on a p-form field A_p only through its field strength $F_{p+1} = dA_p$, we can *dualise* by adding a term to the Lagrangian and integrating out F_{p+1}. If we work in n Lorentzian spacetime dimensions, the electromagnetic dual field is an $(n-p-2)$-form $\tilde{A}_{n-p-2}$.

This algorithm is described in e.g. [5] and we reproduce it below for completeness.

The initial classical action is

$$S[A_p] = -\int \left(\frac{1}{2}F_{p+1}^2\right) \tag{2.4.20}$$

The associated path integral in the quantum field theory is

$$\mathcal{Z} = \int [DA] \exp(iS[A_p]) \tag{2.4.21}$$

We can make a change of variables in the path integral so that we integrate instead over field strengths $F_{p+1} = dA_p$. However, we must insist on the constraint that F_{p+1} is closed, i.e. $dF_{p+1} = 0$. Hence we can introduce a Lagrange multiplier term into the action, and then integrate out F_{p+1} from the path integral. This gaussian field integral will contribute an overall multiplicative constant to the path integral.

Let Λ_{n-p-2} be such a Lagrange multiplier. The action for F and Λ is given by

$$S[F_{p+1}, \Lambda_{n-p-2}] = \int \left(-\frac{1}{2}\, F_{p+1}^2 + \Lambda_{n-p-2} \wedge dF_{p+1} \right) \tag{2.4.22}$$

The equation of motion for Λ is algebraic as there is no kinetic term for Λ.

We can use Stokes's theorem for differential forms and neglect the boundary integral to obtain

$$S[F_{p+1}, \Lambda_{n-p-2}] = \int \left(-\frac{1}{2}\, F_{p+1}^2 - (-1)^{p(n-p)}\, F_{p+1} \wedge d\Lambda_{n-p-2} \right) \tag{2.4.23}$$

Deriving the equation of motion for F in the usual way, we find that

$$\star\, F_{p+1} + (-1)^{p(n-p)}\, d\Lambda_{n-p-2} = 0 \tag{2.4.24}$$

We can substitute this into the action to obtain an effective action for Λ. We usually rename Λ to $\tilde{A}$ to remind us of its origin. This yields simply

$$S_{\text{eff}}[\tilde{A}_{n-p-2}] = -\int \left(\frac{1}{2} d\tilde{A}_{n-p-2}^2 \right) \tag{2.4.25}$$

Observe that the initial action had a symmetry generated by the transformation

$$\delta A_p = \Xi_p \tag{2.4.26}$$

for a closed p-form Ξ_p.

This is an example of a *p-form symmetry*; the general definition will be given later.

There is a corresponding conserved $(p+1)$-form current

$$J_{p+1} = F_{p+1} \tag{2.4.27}$$

satisfying $d \star J = 0$.

After taking the electromagnetic dual, the new action has a symmetry generated by the transformation

$$\delta \tilde{A}_{n-p-2} = \tilde{\Xi}_{n-p-2} \tag{2.4.28}$$

for a closed $(n-p-2)$-form $\tilde{\Xi}_{n-p-2}$.

This is a $(n-p-2)$-form symmetry. The corresponding conserved current is

$$\tilde{J}_{n-p-1} = d\tilde{A}_{n-p-2} = \pm \star F_{p+1} \tag{2.4.29}$$

In the above, we presented the case of a single free p-form field. However, we emphasise that it is also possible to extend electromagnetic duality to interacting theories as long as the Lagrangian density depends on the field of interest only through its field strength. We will exploit this fact extensively in Chapter 3.

Magnetic 1-form symmetry

Now we consider the canonical example of electromagnetic duality from which it historically obtains its name. We work in $n = 4$ spacetime dimensions with $p = 1$, i.e. a free Maxwell field A_1. We already know that there exists an electric 1-form symmetry whose charged operators are Wilson loops and whose charged quantum excitations are electric field lines.

By considering the electromagnetic dual of the Maxwell field, we obtain the *magnetic gauge field* $\tilde{A}_1$, whose field strength $\tilde{F}_2 = d\tilde{A}_1$ is given by

$$\tilde{F}_2 = \star F_2 \tag{2.4.30}$$

The free Maxwell theory can thus be presented in terms of the magnetic gauge field. Immediately we obtain the magnetic 1-form symmetry whose associated conserved current is

$$J_m = \star F_2 \tag{2.4.31}$$

The charged operators under the magnetic 1-form symmetry are 't Hooft lines $\tilde{W}[C]$ which can be constructed from the magnetic gauge field as

$$\tilde{W}[C] = \exp\left(i \int_C \tilde{A}_1\right) \tag{2.4.32}$$

The associated charged quantum excitations under the symmetry are magnetic field lines.

Particle-vortex duality

For a more exotic example, we can work in $n = 3$ spacetime dimensions and again consider $p = 1$, i.e. a free ordinary gauge field A_1. Taking the electromagnetic dual, we obtain a 0-form θ. The inherited 0-form symmetry is *precisely* the shift symmetry we studied in our earlier discussion of spontaneous symmetry breaking. This is an example of *particle-vortex duality*, and raises the following interesting question: which degree of freedom provides the "correct" or "fundamental" presentation of the theory? The answer is simply that they are equivalent viewpoints of the same quantum field theory; with these symmetries, neither is more "fundamental" than the other.

2.4.3 Abstract discussion

Amazingly, higher-form symmetries were only very recently systematically studied in [6]. We give a brief overview of the salient points presented therein using our present notation. In general, a p-form global symmetry is associated with a $(p+1)$-form current J which obeys a conservation equation

$$d \star J = 0 \tag{2.4.33}$$

The conventional case is $p = 0$; in this case we have a 1-form current that counts a density of particles. In this thesis we will focus on the case $p = 1$; in this case we have a conserved 2-form current satisfying $\partial_\mu J^{\mu\nu} = 0$. For 1-form global symmetries, the charged excitations are e.g. strings and field lines. For $p > 1$, the charged excitations are e.g. membranes and defects.

Topological operators

Given a p-form symmetry with a conserved $(p+1)$-form current J, we can define a *conserved charge* on a codimension-$(p+1)$ manifold $\mathcal{M}$ by

$$Q(\mathcal{M}) = \int_{\mathcal{M}} \star J \tag{2.4.34}$$

For a 0-form symmetry on a timeslice $\mathcal{M}$ this is precisely the usually defined Noetherian charge.

Taking the exponential of this charge, we obtain a *topological operator*

$$U_g(\mathcal{M}) = \exp\left(g \int_{\mathcal{M}} \star J\right) \tag{2.4.35}$$

for g an element of the symmetry group.

For the free Maxwell theory we have extensively discussed, there are topological operators U^E associated with the electric 1-form symmetry and U^M associated with the magnetic 1-form symmetry. These are sometimes referred to as *Gukov-Witten operators*. See the original paper [6] for a more detailed discussion.

Interestingly, such topological operators can also be constructed for *discrete* higher-form symmetries, for which there is no conserved current J.

2.4.4 Higher-form symmetries in non-Abelian gauge theory

In Chapter 3 we will study some aspects of the realisation of higher-form symmetries in quantum field theories with holographic duals. We will focus on the study of gauge theory coupled to probe matter in the fundamental representation. As we will review below, in the case where the gauge group is $SU(N)$, this theory has no 1-form symmetries, though it does have a 0-form symmetry associated with baryon number; we will clarify some aspects of how the holographic representation of this baryon number intertwines with the (explicitly broken) putative center symmetry of the pure gauge theory. In the case where the gauge group is $U(N)$ however, the theory has an unbroken 1-form symmetry associated with the conservation of magnetic flux of the "U(1) factor" in the gauge group. We will study the realisation of this symmetry, identifying the charged line operators and studying the correlation function of its currents. As we will elaborate on below, we note that this is perhaps the simplest holographic model in which such a continuous 1-form symmetry can be spontaneously broken, motivating our study.

We now review how higher-form symmetries are realised in various types of non-Abelian gauge theory with and without matter couplings.

$SU(N)$ **gauge theory:** Let us begin our study by reviewing the higher-form symmetry structure of $SU(N)$ gauge theory with only adjoint matter. If we have access to a Lagrangian description of the theory, the action is

$$S = \int d^4x \left(-\frac{1}{g_{YM}^2} \operatorname{tr} |F|^2 + \cdots \right) \tag{2.4.36}$$

where F is the non-Abelian field strength, and the $\cdots$ refers to possible supersymmetrisations or other terms in the action. This action depends only on the Lie algebra of the group $su(N)$. As it turns out, though the Lie algebra specifies the action, it does not actually *fully* define the theory itself. This is because the full theory contains line operators, and the spectrum of line operators depends on the global form of the gauge group [30]. Let us first consider the case where the global form of the gauge group is $SU(N)$; we then have the usual Wilson lines in the fundamental representation of the gauge group in the theory:

$$W(C) \equiv \operatorname{Tr} \mathrm{P} \exp \left(\oint_C A \right) \tag{2.4.37}$$

In a modern understanding, these Wilson lines are charged under 1-form symmetries. To be more precise, there is a $\mathbb{Z}_N$ valued surface operator $U_q(\mathcal{M}_2)$ that is defined on closed 2-manifolds and is *topological* in that it is invariant under small modifications of the 2-manifold $\mathcal{M}_2$. This surface operator has a non-trivial braiding algebra with the Wilson line, i.e. we have inside the path integral,

$$U(\mathcal{M}_2)\, W(C) = \exp\left(\frac{2\pi i q}{N}\right) W(C) \qquad q \in \{0, \ldots, N-1\} \tag{2.4.38}$$

if $\mathcal{M}_2$ wraps the curve C. This 1-form symmetry is a refinement of the usual "center" symmetry of non-Abelian gauge theory, under which adjoint matter fields Φ^a_b are invariant:

$$\Phi^a_b \to \exp\left(\frac{2\pi i q}{N}\right) \delta^a_c\ \Phi^c_d\ \exp\left(-\frac{2\pi i q}{N}\right) \delta^d_b = \Phi^a_b \tag{2.4.39}$$

(More explicitly, the insertion of a surface operator $U(\mathcal{M}_2)$ induces a gauge transformation that is not single-valued as one winds around $\mathcal{M}_2$; instead this gauge transformation returns to itself only up to an element of the center of the gauge group [31]. However if all fields transform in the adjoint, this operation is non-singular from the point of view of the gauge fields).

Let us now consider what happens if we instead couple this theory to N_f flavours of bosons and fermions charged in the fundamental under $SU(N)$, i.e. if the action is taken to be:

$$\begin{aligned} S' = \int d^4x \Bigg[& -\frac{1}{g^2_{YM}} \operatorname{tr} |F|^2 + \\ & + \sum_{i=1}^{N_f} \Big(-|\partial\phi_i - iA\phi_i|^2 - m^2_\phi\, |\phi_i|^2 + \bar{\psi}_i \left(i\gamma^\mu \left(\partial_\mu - iA_\mu\right) - m_\psi\right) \psi_i \Big) + \cdots \Bigg] \end{aligned} \tag{2.4.40}$$

The 1-form symmetry above is now explicitly broken – though the line operator $W(C)$ can still be defined, the operator U_q is no longer topological and thus there is no longer a 1-form symmetry.[11]

However there is a new 0-form symmetry: the baryon number current, which acts as a diagonal phase rotation on both ϕ_i and ψ_i. The associated conserved current is defined in the usual way as:

$$j^\mu_B = \sum_{i=1}^{N_f} \left(\bar{\psi}_i \gamma^\mu \psi_i + 2\ \mathrm{Im}\, \phi^\dagger_i D^\mu \phi_i \right) \tag{2.4.41}$$

It is worth noting that the local gauge-invariant operators that are charged under this baryon number symmetry are fully antisymmetrized products of N_c fundamental fields, and so will have charge N_c in the appropriate units.

Symmetry structure of $U(N)$ gauge theory: Let us now change the theory under consideration by studying instead the $U(N)$ gauge theory with only adjoint

[11] In familiar 0-form language, a fundamental matter field transforms as $\phi^a \to \exp\left(\frac{2\pi i q}{N}\right) \delta^a_b\ \phi^b = \exp\left(\frac{2\pi i q}{N}\right) \phi^a \neq \phi^a$.

matter. It is convenient to write the gauge group as

$$U(N) = \frac{U(1) \times SU(N)}{\mathbb{Z}_N} \tag{2.4.42}$$

If we have access to a Lagrangian description of the $U(N)$ gauge theory, it is straightforward to see that the gauge field corresponding to the $U(1)$ factor separates off, and the action (2.4.36) can now be written as

$$S = \int d^4x \left(-\frac{1}{2g_1^2}|f|^2 - \frac{1}{g_{YM}^2} \operatorname{tr} |F|^2 + \cdots \right) \tag{2.4.43}$$

where f corresponds to the field strength of the new $U(1)$ gauge field $f = da$. As all matter is in the adjoint, nothing couples to the $U(1)$ gauge field, which has a free Maxwell action. There is thus a precisely marginal $U(1)$ gauge coupling which we have named g_1.

Unlike above, where we had only a single discrete $\mathbb{Z}_N$ 1-form symmetry, this theory has two continuous $U(1)$ 1-form symmetries corresponding to the simultaneous conservation of electric $U(1)_e$ and magnetic $U(1)_b$ flux. Their respective conserved currents $J^{\mu\nu}_{e,b}$ are:

$$J^{\mu\nu}_e = \frac{1}{g_1^2} f^{\mu\nu} \qquad J^{\mu\nu}_b = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma} \tag{2.4.44}$$

We can generalise our earlier discussion of 1-form symmetries (in Section 2.4.1) from Maxwell theory to non-Abelian gauge theory as follows. In the phase described by a free $U(1)$ gauge theory action both of these symmetries are spontaneously broken, and the usual 4d photon is the Goldstone mode of this breaking. The line operator that is charged under the electric 1-form symmetry is the usual $U(1)$ Wilson line, and that charged under the magnetic 1-form symmetry is the t'Hooft line. The diagnosis of this symmetry breaking in terms of these line operators is discussed in [7, 8].

Following along the lines of the discussion above, we now add the same flavour degrees of freedom to the $U(N)$ gauge theory. The $U(1)$ gauge field a now couples minimally to both ϕ and ψ:

$$\begin{aligned} S' = \int d^4x \Bigg[&- \frac{1}{2g_1^2}|f|^2 - \frac{1}{g_{YM}^2} \operatorname{tr} |F|^2 + \\ &\sum_{i=1}^{N_f} \Big(-|\partial\phi_i - ia\phi_i - iA\phi_i|^2 - m_\phi^2 \, |\phi_i|^2 + \bar{\psi}_i \left(i\gamma^\mu \left(\partial_\mu - ia_\mu - iA_\mu \right) - m_\psi \right) \psi_i \Big) + \cdots \Bigg] \end{aligned} \tag{2.4.45}$$

where a and A are the $U(1)$ and $SU(N)$ gauge potentials respectively. This changes the dynamics of the $U(1)$ 1-form symmetries described above. Importantly, the symmetry corresponding to conservation of electric flux is now generically *explicitly broken* by the presence of the electrically charged matter; the simplest way to see this is to note that the conserved current identified in (2.4.44) is now no longer conserved; indeed the $U(1)$ Maxwell equations are simply:

$$\partial_\nu J_e^{\mu\nu} = j_B^\mu[\phi, \psi], \tag{2.4.46}$$

where $j_B^\mu[\phi, \psi]$ is precisely the baryon number current (2.4.41). Physically, this simply captures the idea that electric field lines can now end on the electric charges that are carried by ϕ and ψ.

The magnetic flux current $J_b^{\mu\nu}$ is still conserved, as is clear from its definition:

$$\partial_\nu J_b^{\mu\nu} = 0 \tag{2.4.47}$$

Thus, this theory has a single $U(1)$ 1-form symmetry.

However, the realisation of this symmetry now depends on the dynamics of the ϕ, ψ fields. Let us now consider how the energy scale of interest E compares to the masses m_ϕ, m_ψ:

1. $E \ll m_{\phi,\psi}$: In this case the matter fields are gapped and can essentially be ignored. We are then in the same situation as when there were no flavour fields at all; $j^\mu[\phi, \psi]$ is effectively zero, and both the electric and magnetic flux currents are conserved. The associated symmetries are again both *spontaneously* broken, as described around (2.4.44). In particular, the relevant line operators should display a perimeter law in this phase.

2. $E \gg m_{\phi,\psi}$: In this case we probe electric charge fluctuations in the vacuum, and the electric flux symmetry is explicitly broken. The magnetic flux symmetry is now realised differently; in particular, it is no longer spontaneously broken. Relatedly, in this regime the $U(1)$ gauge coupling g_1 runs logarithmically with the energy scale E.

Summary: In the following chapter, we will study the manifestation of the higher-form symmetry structures described above in a strongly coupled model, given by the holographic realisation of maximally supersymmetric $\mathcal{N} = 4$ Super-Yang-Mills coupled to matter in the fundamental. We will primarily focus on the case of the $U(N)$ gauge theory where we have a continuous 1-form symmetry, but along the way we will clarify some aspects of the $SU(N)$ case as we proceed.

Chapter 3

Application 1: Holographic flavour

In this chapter we study the higher-form symmetry structure of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with added matter fields in the fundamental representation of the gauge group. This quantum field theory has a holographic dual; the added matter fields correspond to probe $D7$-branes in the bulk.

The structure is as follows. In Section 3.1 we introduce the (well-known) holographic bulk action and discuss its symmetry structure, also discussing some lower-dimensional examples to build some intuition for the extensive dualisations that follow. In Sections 3.2 and 3.3 we discuss the bulk dynamics and appropriate charged operators in the duality frames that are appropriate for the $SU(N)$ and $U(N)$ gauge theories respectively. Finally in Section 3.4 we numerically compute the spectral function for the 2-form current in the $U(N)$ theory and compare with expectations at weak coupling.

3.1 Symmetries of holographic flavour

In this section we describe the holographic dual of the system described at the end of Chapter 2; in particular we study the maximal supersymmetrisation of the gauge theory, i.e. $\mathcal{N} = 4$ SYM with holographic flavour added. In most discussions of holography it is implicitly assumed that the gauge group is $SU(N)$; for us however the precise distinction between the $U(N)$ and $SU(N)$ gauge theories will be of considerable importance. This issue has been clarified recently (see [32] for a perspective from higher-form symmetry) and we briefly review it here, taking special care with the issues that will be relevant for our construction.

Some preliminary remarks on holography are in order. Holographic duality is the statement that a quantum field theory in d spacetime dimensions is equivalent to a gravity theory in $(d+1)$ spacetime dimensions (its *dual theory* or *bulk dual*). Roughly speaking, the extra spacetime dimension accounts for the running of the couplings with energy scale, i.e. the renormalisation group flow.

A crucial aspect of holography is that strongly-coupled quantum field theories are dual to weakly-coupled gravity theories. This makes holography very useful for studying quantum field theories in regimes which are perturbatively inaccessible with conventional techniques such as Feynman diagrams. This aspect of holography may be referred to as *strong-weak duality*.

There is an extensive "dictionary" that can be used to translate between objects in the field theory and objects in the gravity theory. For example, a conserved current of a 0-form symmetry in the field theory is dual to a 1-form gauge field in the bulk. More generally, a conserved current of a p-form global symmetry in the field theory is dual to a $(p+1)$-form gauge field in the bulk. The stress tensor of the field theory is dual to the metric of the gravity theory.

The boundary conditions of the bulk degrees of freedom are dual to the global structure of the field theory gauge group. In this chapter we will determine the appropriate such boundary conditions (corresponding to each of $SU(N)$ and $U(N)$) for these bulk fields by studying the symmetry properties of the dual field theory.

Many examples of holography have been constructed and detailed reviews can be found in e.g. [33] and [34].

3.1.1 Bulk holographic action

The holographic dual of the above is Type IIB string theory on $\mathrm{AdS}_5 \times S^5$, giving rise to kinetic terms for the NS-NS 2-form B_2 and the R-R 2-form C_2, as well as a Chern-Simons term. After compactifying on the S^5 we obtain an action which is an integral over all of AdS_5. To add fundamental matter, we wrap $N_f \ll N_c$ probe $D7$-branes around the S^5 [35]. See e.g. [33] for a review of holographic flavour.

The final form of the dimensionally-reduced action on AdS_5 is

$$\begin{aligned} S_{\text{bulk}} &= S_{\text{kin}} + S_{\text{CS}} + N_f\, S_{\text{DBI}} \\ &= \frac{N_c^2}{8\pi^2 R^3} \int \left[-\frac{1}{2} H_3^2 - \frac{1}{2} \left(\frac{\lambda}{4\pi N_c} G_3 \right)^2 + \kappa B_2 \wedge \left(\frac{\lambda}{4\pi N_c} G_3 \right) - \frac{1}{2} \kappa^2 \mu f(z) \left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2 \right)^2 \right] \end{aligned} \tag{3.1.1}$$

Note our conventions for writing differential forms. For a p-form Ω_p in n-dimensions we can define a corresponding n-form by

$$\Omega_p^2 \equiv \Omega_p \wedge \star \Omega_p \tag{3.1.2}$$

Sometimes it is preferable to work in components, in which case we borrow from [5] and write

$$|\Omega_p|^2 \equiv \frac{1}{p!}\,\Omega_{\mu_1\mu_2\ldots\mu_p}\Omega^{\mu_1\mu_2\ldots\mu_p} \tag{3.1.3}$$

It is straightforward to translate between these descriptions using the identity

$$\Omega_p^2 = |\Omega_p|^2\sqrt{|g|}\;d^n x \tag{3.1.4}$$

The forms appearing in the action are the field strengths $H_3 = dB_2$, $G_3 = dC_2$ and $F_2 = dA_1$. Note that we have used the unusual name G_3 for the field strength of the R-R form to avoid confusion with the field strength of the $D7$-brane Maxwell field. Our zoo of higher-form fields is extensive – and will become even more so as we dualise fields in the bulk – so we have provided an index in Appendix 3.C. The constants in the action are given by

$$\kappa = \frac{4}{R} \tag{3.1.5a}$$

$$\mu = \frac{N_f}{N_c}\frac{\lambda}{32\pi^2} \tag{3.1.5b}$$

μ denotes the relative dynamical importance of the flavour and colour degrees of freedom. The function f is given by

$$f(z) = \begin{cases} 1-(z/z_c)^2 & z \le z_c \\ 0 & z > z_c \end{cases} \tag{3.1.6}$$

We will work with AdS_5 in Poincaré coordinates:

$$ds^2 = \frac{R^2}{z^2}\left(dx^\mu dx_\mu + dz^2\right) \tag{3.1.7}$$

where R is the AdS radius.

Some words about the probe limit are in order here. Usually one considers the limit where $\mu \to 0$, and thus where the backreaction of the flavour degrees of freedom on the colour degrees of freedom can be ignored. We will work with the simple quadratic action above, but we will allow μ to take on finite values. This corresponds to studying some subset of the interplay between flavour and colour degrees of freedom, in particular those associated with the realisation of the symmetries. As explained below, this results in novel effects associated with the Higgsing of the 2-form B field by the DBI gauge field A_1; these effects qualitatively affect the physics but are invisible in the strict probe limit. Strictly speaking, however, we are not studying all aspects of this interplay, because we neglect the gravitational backreaction of the flavour branes; thus the approximation we take should be considered as an illustrative one that is designed to highlight the physics of interest.

Let us now note the gauge symmetry of the action above. We have two independent 1-form gauge transformations shifting B_2 and C_2 respectively; we also have a 0-form gauge transformation shifting the DBI worldvolume field. The full transformation of the fields is

$$\delta B_2 = d\Xi_1^{(B)} \tag{3.1.8a}$$

$$\delta C_2 = d\Xi_1^{(C)} \tag{3.1.8b}$$

$$\delta A_1 = -\frac{\sqrt{\lambda}}{2\pi R^2}\left(\Xi_1^{(B)} + d\xi^{(A)}\right) \tag{3.1.8c}$$

Note the simultaneous transformation of A_1 and B_2 under a shift by $\Xi_1^{(B)}$; this encodes the fact that string worldsheets can end on the D-brane, and will be of considerable importance to us.

We turn now to the Chern-Simons term $B_2 \wedge G_3$; this well-known term [36, 37, 38] is closely related to the physics of higher-form symmetry in holography [32] and will play a key role in our analysis. Obtaining the precise prefactor can be somewhat subtle; it can naively be thought of as arising from a dimensional reduction of a 10d Chern-Simons term $B_2 \wedge dc_4 \wedge G_3$ involving the R-R 4-form c_4. Integrating over the S^5 we pick up a factor of the flux $dc_4 \sim N_c$, giving a term with qualitatively the correct form, as first noted in [38].

However, this is not quite consistent: in fact, a covariant action for the (self-dual) RR 4-form does not actually exist, and pursuing the above route results in an inconsistent normalisation for the Chern-Simons term, as noted in [39]. In this work we take a different approach. Consistency of the theory in the presence of magnetic charges actually requires the coefficient of this term to be quantized; we review this Dirac quantisation condition in Appendix 3.A.2 and identify the integer coefficient of the term with N_c, as we expect on symmetry grounds.

Finally, the last term arises from the dimensional reduction of the DBI action. We can embed N_f probe $D7$-branes into the target space by means of the DBI action:

$$N_f\, S_{\text{DBI}} = -N_f\, \tau_7 \int d^8\xi\, \sqrt{-\det\left(g_{\alpha\beta} + B_{\alpha\beta} + 2\pi l_s^2\, F_{\alpha\beta}\right)} \tag{3.1.9}$$

where ξ^α are the brane worldvolume coordinates, $g_{\alpha\beta}$ are the components of the induced worldvolume metric on the $D7$-brane, $B_{\alpha\beta}$ are the components of B_2 and $F_{\alpha\beta}$ are the components of $F_2 = dA_1$, the Maxwell field strength living on the brane worldvolume. τ_p is the effective Dp-brane tension after absorbing the effect of the dilaton $e^\Phi = g_s$ and is given by $\tau_7 = \frac{1}{g_s}\frac{1}{(2\pi)^7 l_s^8}$.

The S^5 factor in the metric may be written as

$$d\Omega_5^2 = d\theta^2 + \cos^2\theta \; d\psi^2 + \sin^2\theta \; d\Omega_3^2, \tag{3.1.10}$$

where we have chosen coordinates that make manifest an $S^3 \subset S^5$. As usual for a D3/D7 embedding, the desired brane configuration fills all of AdS and wraps the 3-sphere around the S^5. Thus the embedding is parametrized by the transverse coordinates ψ, θ; ψ is a Killing direction and may be taken to be constant, and the appropriate solution for $\theta(z)$ corresponding to massive holographic flavour is [35]:

$$\theta(z) = \theta_c \equiv \begin{cases} \arccos(z/z_c) & z \leq z_c \\ 0 & z > z_c \end{cases} \tag{3.1.11}$$

A careful matching to the field theory shows that

$$z_c = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{m_F} \tag{3.1.12}$$

where m_F is the mass of the flavour degrees of freedom [40, 41].

Geometrically, this embedding means that the S^3 wrapping the S^5 is of maximal size ($\theta = \pi/2$) at $z = 0$ on the boundary, and the $D7$-brane vanishes ($\theta = 0$) at the critical value $z = z_c$. For $z > z_c$ the $D7$-brane has no effect; this is dual to the fact that at energies smaller than the mass gap the flavour degrees of freedom can no longer be excited. If we set the mass to zero θ is constant and the theory is conformal.

If we now substitute the on-shell angle $\theta = \theta_c$ back into the DBI action and expand to quadratic order in B_2 and F_2 we obtain the following quadratic action for the fluctuations of the DBI gauge field:

$$N_f \; S_{\mathrm{DBI}} = -\frac{N_c^2}{8\pi^2 R^3}\int \frac{1}{2}\kappa^2 \mu f(z)\left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}}F_2\right)^2 \tag{3.1.13}$$

Some details about this computation are given in Appendix 3.A.3.

3.1.2 Examples in lower dimensions

The action (3.1.1) has several interesting features, arising from the interplay of the higher-form symmetry with the baryon number symmetry. To the best of our knowledge these have not yet been fully explained in the literature, and we will unpack these below. It is first helpful to orient ourselves with some more familiar examples in lower dimension.

Let us begin with the following action for a Goldstone mode in three dimensions:

$$S_1 = -\int d^3x\, \frac{v^2}{2}|d\theta|^2 \tag{3.1.14}$$

Clearly this has a single degree of freedom, which is gapless. The situation is however very different if we consider gauging this scalar Goldstone with a 1-form gauge field a_μ, resulting in the following action

$$S_2 = \int d^3x \left(-\frac{1}{2g^2}\,|da|^2 - \frac{v^2}{2}\,|a - d\theta|^2\right) \tag{3.1.15}$$

This theory is now massive; the Goldstone mode is eaten by the photon, resulting in a gapped theory with mass gap g^2v^2. As turning on a small gauge coupling g results in a mass gap, the weak coupling limit and the infrared limit don't commute.

Higgsing a gauge field is one way to obtain a mass gap. Another way to give a gauge field a mass is through a Chern-Simons term [42]. Let us thus imagine removing the Goldstone mode and adding a second gauge field b to obtain the following theory:

$$S_3 = \int d^3x \left(-\frac{1}{2g^2}\,|da|^2 - \frac{1}{2v^2}\,|db|^2 + a\wedge db\right) \tag{3.1.16}$$

What is the spectrum of this theory? An illuminating way to understand this is to dualise the gauge field b to a scalar ψ. Following the standard algorithm, we find

$$S_3' = \int d^3x \left(-\frac{1}{2g^2}\,|da|^2 - \frac{v^2}{2}\,|d\psi - a|^2\right) \tag{3.1.17}$$

where in terms of the original degree of freedom $db = g^2\star(d\psi - a)$. This is essentially the same as the Higgs-ed theory studied above in (3.1.15), and is also gapped. Thus we see that adding a Chern-Simons term and Higgsing a gauge field are the same mechanism, just written in different duality frames.

Finally, let us imagine *both* adding a Chern-Simons term and Higgsing, i.e. we study the following action:

$$S_4 = \int d^3x \left(-\frac{1}{2g^2}|da|^2 - \frac{1}{2v^2}|db|^2 + a\wedge db + \frac{v^2}{2}|a - d\theta|^2\right) \tag{3.1.18}$$

What is the spectrum now? It is again helpful to dualise the b field, after which we obtain:

$$S_4' = \int d^3x \left(-\frac{1}{2g^2}|da|^2 - \frac{v^2}{2}|d\psi - a|^2 - \frac{v^2}{2}|d\theta - a|^2\right) \tag{3.1.19}$$

i.e. after a duality this is like "Higgsing twice". But we only have one photon to eat a putative Goldstone; thus there should remain one Goldstone left uneaten, which can be seen by rewriting the action to be:

$$S_4'' = \int d^3x \left(-\frac{1}{2g^2}|da|^2 - \frac{v^2}{4}|d\psi + d\theta - 2a|^2 - \frac{v^2}{4}|d\psi - d\theta|^2 \right) \tag{3.1.20}$$

Thus the gauge-charged combination $\psi + \theta$ is eaten, and forms part of a massive photon; however the gauge-invariant combination $\psi - \theta$ remains uneaten, and is a gapless mode in the spectrum. The general lesson is that if we try to give a gauge field a mass *twice*, both by Higgsing and by adding a Chern-Simons term, then we find that a gapless mode survives. We could also have chosen to dualise the scalar θ in (3.1.18) into a 1-form gauge field; in this case the gapless mode would have appeared to be a gauge field and not a scalar, but this gauge field would be related to $\psi - \theta$ by the usual Abelian duality.

What does all of this have to do with AdS/CFT? In most discussions of holographic flavour in AdS/CFT, one works in the probe limit, considering the limit $\mu \sim \frac{N_f}{N_c}$ to 0 and studying the fluctuations of the DBI worldvolume gauge field A_1, which then decouples from the other fields, and whose action takes the form $S \sim \int (dA_1)^2$. This is the action of a massless photon and is analogous to a higher-form version of (3.1.14). The field theory dual of this massless gauge field is the baryon number current.

However, if we consider the complete action (3.1.1), we see that this is actually somewhat dangerous; in fact the gauge field does not appear by itself but rather in the gauge-invariant combination $B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2$. At first glance, this appears somewhat problematic, as the action contains the following terms:

$$\frac{N_c^2}{8\pi^2 R^3} \int \left[-\frac{1}{2}(dB_2)^2 - \frac{1}{2}\kappa^2 \mu \left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2 \right)^2 + \cdots \right] \tag{3.1.21}$$

(where for simplicity we consider a case where $f(z)$ is constant). Compare this to (3.1.15); it actually now appears that the field A_1 has been *eaten* by the higher-form gauge field B_2, in a higher-form analogue of the Higgs mechanism. This suggests that the theory should have only massive modes, with the mass scaling like $\kappa^2 \mu$. In a sense, the infrared limit no longer commutes with the probe limit. However, this is clearly a nonsensical result: the dual field theory should still have a baryon number current, which should be dual to a bulk gauge field that is exactly massless to all orders in $\frac{N_f}{N_c}$.

The resolution comes in studying the full action, which contains an extra field C_2 which has its own kinetic term as well as a mixed Chern-Simons term coupling it to B_2. The action now appears more like a higher-form version of (3.1.18), which indeed *does* support a gapless mode, though it is not apparent at first glance. In this work we will unpack the analogous mechanism in the AdS/CFT context; we will indeed find that the bulk action always supports a gapless mode. For a certain set of boundary conditions (those which are dual to the $SU(N)$ gauge theory), this mode is dual to the baryon number current. For a different set of boundary conditions (those which are dual to the $U(N)$ gauge theory), this mode is dual to the 2-form current for magnetic flux.

The fact that the existence of this gapless mode depends crucially on the interplay between the Chern-Simons and DBI terms is dual to the fact that in the field theory the baryon number current is intertwined in some sense with the 1-form center $\mathbb{Z}_N$ symmetry of the pure gauge theory.

3.2 $SU(N)$ gauge theory

The bulk may be understood in various duality frames. We begin by studying it in a frame which is useful when the dual field theory is the $SU(N)$ gauge theory coupled to fundamental flavour.

3.2.1 Bulk action

To begin, it is helpful to Poincaré dualise the R-R form C_2 to a 1-form $\tilde{C}_1$ in the usual way. The procedure is explained in for example Appendix B.4 of [5]; applying the algorithm we find:

$$\tilde{G}_2 = \frac{\lambda}{4\pi N_c} \star G_3 - \kappa B_2 \tag{3.2.1}$$

where $\tilde{G}_2 = d\tilde{C}_1$.

Substituting into the action now gives a different presentation of the same theory.

$$S = -\frac{N_c^2}{8\pi^2 R^3} \int \left[\frac{1}{2} H_3^2 + \frac{1}{2}\kappa^2 \left(\left(B_2 + \frac{1}{\kappa}\tilde{G}_2\right)^2 + \mu f(z) \left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2\right)^2\right)\right] \tag{3.2.2}$$

Observe now that from the duality relation (3.2.1), $\tilde{C}_1$ inherits the gauge-transformation under the 1-form gauge symmetry:

$$\delta\tilde{C}_1 = -\kappa \left(\Xi_1^{(B)} + d\xi^{(C)}\right) \tag{3.2.3}$$

which ensures that the action is indeed still gauge-invariant. Here $\Xi^{(B)}$ is the original 1-form gauge transformation of B_2, whereas $\xi^{(C)}$ is an emergent 0-form gauge transformation that exists only in this duality frame. In a sense it is the magnetic dual of the 1-form gauge transformation of C_2, which has been dualised away.

We can diagonalise the action in this duality frame to better understand its spectrum. After diagonalising, it will also be easier to fix a gauge and solve the equations of motion. To this end, it is convenient to define the function $h(z)$ by

$$h(z) = \frac{1}{1 + \mu f(z)} \tag{3.2.4}$$

so that we can define the new 1-form fields

$$\eta_1 = \frac{1}{\kappa}\,\tilde{C}_1 - \frac{2\pi R^2}{\sqrt{\lambda}}\,A_1 \tag{3.2.5a}$$

$$\tau_1 = \frac{2\pi R^2}{\sqrt{\lambda}}\,A_1 + h\,\eta_1 \tag{3.2.5b}$$

and their respective field strengths

$$Y_2 = d\eta_1 \tag{3.2.6a}$$

$$T_2 = d\tau_1 \tag{3.2.6b}$$

These linear combinations inherit the following gauge-transformations

$$\delta\eta_1 = d\xi \tag{3.2.7a}$$

$$\delta\tau_1 = -\Xi_1^{(B)} + h\,d\xi \tag{3.2.7b}$$

where $\xi = \xi^{(A)} - \xi^{(C)}$. Observe that η_1 has the same gauge transformation as an ordinary free Maxwell field. When the field theory is gapless, η_1 is precisely a massless gauge field in the bulk, so is the holographic dual of a 0-form symmetry in the field theory. This 0-form symmetry corresponds to baryon number conservation.

The diagonalised action can now be written cleanly as

$$S = -\frac{N_c^2}{8\pi^2 R^3}\int\left\{\frac{1}{2}H_3^2 + \frac{1}{2}\kappa^2\left[(1-h)Y_2^2 + h^{-1}\left(B_2 + T_2 + \eta_1\wedge dh\right)^2\right]\right\} \tag{3.2.8}$$

Gauge-invariance is easy to check: H_3, Y_2 and $B_2 + T_2 + \eta_1 \wedge dh$ are each individually gauge-invariant quantities.

This action is somewhat complicated, as it is dual to an RG flow captured holographically by the profile of the function $f(z)$. To understand the symmetry structure, it is helpful to consider the limit of zero fermion mass $m_F \to 0$. We now have $z_c \to \infty$, and so $f(z) = 1$ for all z. This gives $h(z) = (1+\mu)^{-1}$.

We then find:

$$S \to -\frac{N_c^2}{8\pi^2 R^3}\int \left[\frac{1}{2}H_3^2 + \frac{1}{2}\kappa^2(1+\mu)(B_2 + d\tau_1)^2 + \frac{1}{2}\kappa^2\left(\frac{\mu}{1+\mu}Y_2^2\right)\right] \tag{3.2.9}$$

We can partially gauge-fix to an analogue of unitary gauge in which we set $T_2 = 0$. This describes a 2-form gauge field B_2 which has been Higgs-ed by the 1-form τ_1; the resulting dynamical bulk field is massive. It also has a precisely massless 1-form gauge field η_1, as anticipated above. This is dual to the baryon number current. Note that this structure arose out of the interplay between the Chern-Simons term and the flavour terms; this is dual to the interplay between the $U(1)$ baryon number current and the $\mathbb{Z}_N$ center symmetry of the field theory. In the remainder of this section we further describe some universal aspects of this interplay.

Up to boundary terms, the variation of the action is

$$\begin{aligned}\delta S = -\frac{N_c^2}{8\pi^2 R^3}\int \Big\{&\delta B_2 \wedge \star\left[\star d \star dB_2 + \kappa^2 h^{-1}(B_2 + T_2 + \eta_1\wedge dh)\right] \\ &- \kappa^2\, \delta\eta_1 \wedge \star\left[\star d((1-h)\star d\eta_1) - h^{-1}\star(dh\wedge\star(B_2+T_2+\eta_1\wedge dh))\right] \\ &+ \kappa^2\, \delta\tau_1\wedge d\left[h^{-1}\star(B_2+T_2+\eta_1\wedge dh)\right]\Big\}\end{aligned} \tag{3.2.10}$$

Hence the equations of motion are[1]

$$\star\, d\star dB_2 + \kappa^2 h^{-1}(B_2 + T_2 + h'\eta_1\wedge dz) = 0 \tag{3.2.11a}$$

$$(1-h)\star d\star d\eta_1 + h^{-1}h'\star[dz\wedge\star(-h\, d\eta_1 + B_2 + T_2 + h'\eta_1\wedge dz)] = 0 \tag{3.2.11b}$$

where $h' \equiv \frac{dh}{dz}$.

The spectrum of fields thus consists of a massive 2-form gauge field B_2, a massless 1-form gauge field η_1 which is dual to the baryon number current, and a 1-form gauge field τ_1 which appears only through its field strength T_2. T_2 is of less physical importance since it can be easily gauged away; in a sense it simply provides the longitudinal degrees of freedom of the massive tensor B_2.

If we are studying the $SU(N)$ gauge theory coupled to fundamental flavour, it is important to note that the appropriate boundary conditions at the AdS boundary are those where we hold fixed the boundary value of η_1; this guarantees that we obtain a conserved 0-form baryon number current j_b^μ in the boundary theory. As we will see, this will be different when we study the $U(N)$ gauge theory.

[1]The τ_1 equation is redundant since it follows by taking the exterior derivative of the B_2 equation.

3.2.2 Boundary Conditions

We briefly take a moment to unravel the dualities to uncover the appropriate boundary conditions on the original form fields, namely B_2, A_1 and C_1.

At stated above, the appropriate boundary condition on η is to hold it fixed at the UV boundary, i.e.

$$\partial_\mu \lim_{z\to 0} \eta = 0 \tag{3.2.12}$$

This boundary condition implies that the field strength Y_2 satisfies

$$Y_{\mu\nu} = 0, \qquad z \to 0 \tag{3.2.13}$$

From the duality (3.2.1) and the definition of η in (3.2.5a) we find that the field strengths are related by

$$\kappa Y_2 = \frac{\lambda}{4\pi N_c} \star G_3 - \kappa\left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2\right) \tag{3.2.14a}$$

$$T_2 = \frac{2\pi R^2}{\sqrt{\lambda}} F_2 + hY_2 + h' dz \wedge \eta_1 \tag{3.2.14b}$$

Putting the boundary condition (3.2.13) into (3.2.14a) we find that

$$\frac{\lambda}{4\pi N_c}(\star G)_{\mu\nu} = \kappa\left(B_{\mu\nu} + \frac{2\pi R^2}{\sqrt{\lambda}} F_{\mu\nu}\right), \qquad z \to 0 \tag{3.2.15}$$

If we further fix the gauge so that $T_2 = 0$ and note that $h'(0) = 0$ then we have simply

$$F_{\mu\nu} = 0, \qquad z \to 0 \tag{3.2.16a}$$

$$\frac{\lambda}{4\pi N_c}(\star G)_{\mu\nu} = \kappa B_{\mu\nu}, \qquad z \to 0 \tag{3.2.16b}$$

3.2.3 Charged operators

We now describe the bulk object that is charged under the baryon number symmetry.

Baryon vertices in pure $SU(N)$ gauge theory

Let us first review the conventional case with no flavour branes [38]. There we set $N_f \to 0$, and we find simply:

$$S = -\frac{N_c^2}{8\pi^2 R^3} \int \left[\frac{1}{2} H_3^2 + \frac{1}{2}\kappa^2 (B_2 + T_2)^2\right] \tag{3.2.17}$$

With no flavour branes the DBI gauge field A_1 does not exist, and from (3.2.4) and (3.2.5b) we see that when $h = 1$ we have simply $\tau_1 = \kappa^{-1}\tilde{C}_1$, i.e. τ_1 is directly the magnetic dual of the RR 2-form.

We will now revisit this action from the point of view of symmetry. Note that under the 1-form gauge transformation of B_2, τ_1 must also transform:

$$B_2 \to B_2 + d\Xi_1 \qquad \tau_1 \to \tau_1 - \Xi_1 \tag{3.2.18}$$

We now study the bulk objects that are charged under these gauge symmetries. We have fundamental string worldsheets, which couple minimally to B_2 as

$$\frac{1}{2\pi l_s^2}\int_{\mathcal{M}} B_2 \tag{3.2.19}$$

We turn now to τ_1; as τ_1 is a usual 1-form gauge field in the bulk, it couples naturally to one-dimensional particle worldlines in AdS$_5$. What are these objects?

From the definition of the duality relationship (3.2.1), we can see that these objects couple magnetically to the RR 2-form C_2. In the ten-dimensional picture, these are thus $D5$-branes. Of their six dimensional worldvolume, five of them are wrapped on the S^5, and the remaining one dimension traces a worldline on AdS$_5$. By using the normalisations in Appendix 3.A.2 one can verify that a single such $D5$-brane couples to τ_1 as

$$\frac{N_c}{2\pi l_s^2}\int_L \tau_1 \tag{3.2.20}$$

Note however that this coupling alone is not invariant under the 1-form gauge transformation (3.2.18): indeed we see that the the one-dimensional worldline L can exist only as the *boundary* of N_c string worldsheets, i.e. the combined coupling

$$N_c\left(\frac{1}{2\pi l_s^2}\int_{\mathcal{M}} B_2\right) + \frac{N_c}{2\pi l_s^2}\int_{\partial\mathcal{M}} \tau_1 \tag{3.2.21}$$

is invariant. This fact – that the wrapped $D5$-brane is the endpoint of N_c fundamental strings and thus acts as a baryon vertex in the dual field theory – can also be understood directly from the original Chern-Simons coupling [38, 36, 37, 43]. Here we simply restate it in an alternative (bulk) duality frame.

It is now instructive to imagine the bulk worldline intersecting the AdS boundary at a point. Each of the N_c string worldsheets will also intersect the boundary as a series of curves ending at the same point. Holographically, the combined object is a non-dynamical baryon vertex serving as the endpoint of N_c Wilson lines in the fundamental representation. It is clear that the baryon vertex, being tied to N_c Wilson lines, is not a local operator in the field theory; the bulk dual of this statement is that it does not correspond to a free particle worldline but rather only as the boundary of N_c F-strings.

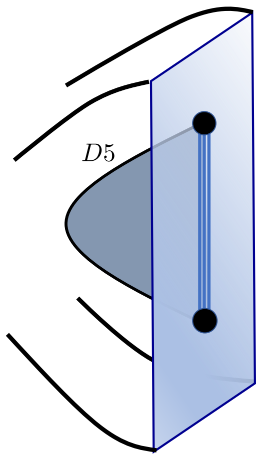

Figure 3.1: The hanging $D5$-brane forms 1-dimensional worldline in the bulk; it is the boundary of N_c F-string worldsheets which also intersect the AdS boundary. At their intersection with the boundary they define the insertion of fundamental Wilson lines in the dual CFT.

Baryon operator in theory with dynamical flavour

We now restore the flavour branes, i.e. we return to (3.2.9). In the bulk, we now have a new massless field η_1, which we understand is dual to the baryon number current in the field theory $U(1)_B$. In the dual field theory, we now expect the existence of local baryon operators that carry charge N_c (in units of the baryon charge of the fundamental gauge-charged fields).

What is the bulk dual of this operator? Consider a general particle-like object in the bulk that couples to both η_1 and τ_1, i.e.

$$\int_L (q_\eta \eta_1 + q_\tau \tau_1) \tag{3.2.22}$$

As argued above, any coupling to τ_1 will necessarily mean that the particle has strings attached, in order to ensure gauge-invariance. Let us consider an object which has $q_\tau = 0$. As η_1 does not transform under the 1-form gauge transformation (3.2.18), this coupling is entirely gauge-invariant on its own. Thus a particle in the bulk that couples in this way is dual to a local boundary operator that carries baryon charge. From field theory considerations, we understand that this object should be related to a bound state of a $D5$-brane and F-strings in some manner.

The presence of the new field η_1 lets the $D5$-brane exist as an independent object that is untethered to any strings. To see this more explicitly, we can express this coupling in terms of the original fields $\tilde{C}_1$ and A_1; we find that the unit quantized $D5$-brane couples as

$$q_\eta \int_L \eta_1 = \frac{N_c}{2\pi l_s^2} \int_L \left(\frac{2\pi R^2}{\sqrt{\lambda}} A_1 - \kappa\, \tilde{C}_1 \right) \tag{3.2.23}$$

where we have used the quantized coupling to $\tilde{C}_1$ worked out above, and where the coupling to A_1 is correlated with that of $\tilde{C}_1$ by the condition that $q_\tau = 0$. This can be compared to the coupling of a single F-string ending on the $D7$-brane:

$$\frac{1}{2\pi l_s^2} \left(\int_{ws} B_2 + \int_{\partial ws} \frac{2\pi R_2}{\sqrt{\lambda}} A_1 \right) \tag{3.2.24}$$

In other words, the coupling to A_1 is as N_c F-strings. Microscopically one can actually imagine that the $D5$-brane is connected by very small strings to the flavour $D7$-brane, where the string charge is now carried by the A_1 field living on the brane. The resulting composite object is the particle-like excitation that we describe above. Related work in different holographic models to directly construct bulk objects carrying baryon number can be found in [44, 45, 46]; see in particular [46]. We stress that our construction makes no real statements about the dynamics of the internal structure, and simply shows how their charges are captured in the low-energy description.

3.3 $U(N)$ gauge theory

We would now like to understand the theory with the boundary conditions that are appropriate to having a $U(N)$ gauge theory dual. We now expect to obtain a 2-form conserved current on the boundary; it is thus appropriate to study the bulk in a different duality frame.

3.3.1 Bulk action

We can Poincaré dualise the field τ_1 by integrating out its field strength T_2 in the usual way. This yields a 2-form $\mathcal{A}_2$ with field strength $\mathcal{F}_3 = d\mathcal{A}_2$ given by:

$$\mathcal{F}_3 = \kappa^2 h^{-1} \star (B_2 + T_2 + \eta_1 \wedge dh) \tag{3.3.1}$$

Substituting into the action and integrating by parts gives

$$S = -\frac{N_c^2}{8\pi^2 R^3}\int\left[\frac{1}{2}H_3^2 + \frac{1}{2}\kappa^2(1-h)Y_2^2 + \frac{1}{2}\kappa^{-2}h\,\mathcal{F}_3^2 + B_2\wedge\mathcal{F}_3 - Y_2\wedge\mathcal{A}_2\wedge dh\right] \tag{3.3.2}$$

Note the last term in the action where $\mathcal{A}_2$ appears explicitly; this arises from an integration by parts so that the action depends on $Y_2 = d\eta_1$ and not η_1 explicitly. As a result we can now dualise η_1 using exactly the same procedure to give a 2-form $\mathcal{P}_2$ whose field strength $\mathcal{Q}_3 = d\mathcal{P}_2$ is given by

$$\mathcal{Q}_3 + \mathcal{A}_2\wedge dh = \kappa^2(1-h)\star Y_2 \tag{3.3.3}$$

(Note that $\mathcal{P}_2$ can be thought of as – modulo mixing with other fields – the electric-magnetic dual of η_1, i.e. the bulk field dual to the baryon number current). Substituting this back into the action then gives:

$$S = -\frac{N_c^2}{8\pi^2 R^3}\int\left[\frac{1}{2}H_3^2 + B_2\wedge\mathcal{F}_3 + \frac{1}{2}\kappa^{-2}\left((1-h)^{-1}(\mathcal{Q}_3+\mathcal{A}_2\wedge dh)^2 + h\mathcal{F}_3^2\right)\right] \tag{3.3.4}$$

Note that we have a gauge freedom given by

$$\delta\mathcal{A}_2 = d\Xi_1 \tag{3.3.5a}$$

$$\delta\mathcal{P}_2 = -\Xi_1\wedge dh + d\Lambda_1 \tag{3.3.5b}$$

under which the action is invariant, where Λ_1 is a new free 1-form gauge parameter. From the perspective of the 2-form picture, two of the equations of motion in the 1-form picture are simply the Bianchi identities $d\mathcal{F}_3 = 0$ and $d\mathcal{Q}_3 = 0$.

The spectrum is easiest to understand in the case where the flavour mass is zero so that $dh = 0$. We then have two coupled 2-forms B_2 and $\mathcal{A}_2$ which constitute a massive degree of freedom. We also have a single massless 2-form $\mathcal{P}_2$ whose dependence is only through its field strength $\mathcal{Q}_3$; this massless bulk is dual to the only conserved 2-form current $J^{\mu\nu} = J_b^{\mu\nu}$, identified in (2.4.44).

The equations of motion in the 2-form picture are

$$d\star\left[(1-h)^{-1}(\mathcal{Q}_3+\mathcal{A}_2\wedge dh)\right] = 0 \tag{3.3.6a}$$

$$d\star H_3 - \mathcal{F}_3 = 0 \tag{3.3.6b}$$

$$d\star(h\mathcal{F}_3) + \kappa^2 H_3 - (1-h)^{-1}dh\wedge\star(\mathcal{Q}_3+\mathcal{A}_2\wedge dh) = 0 \tag{3.3.6c}$$

From the perspective of the 1-form picture, two of these equations give the Bianchi identities $dY_2 = 0$ and $dT_2 = 0$. The third equation is the same in both pictures.

Thus, to obtain the bulk dual to the $U(N)$ gauge theory coupled to flavour, we should use AdS/CFT boundary conditions where we hold fixed the boundary value of the 2-form field $\mathcal{P}$. The usual rules of AdS/CFT will then guarantee that in the dual field theory, we will obtain a 2-form conserved current J, as expected.

We note that the $U(N)$ theory seems to contains one extra parameter as compared to the $SU(N)$ theory; as explained around (2.4.45), the coupling constant g_1 associated to the "$U(1)$ factor" seems to be an extra parameter that can be tuned. In a universal sense this can be understood as a double-trace coupling associated to the 2-form current J. When there are flavor degrees of freedom present this coupling is expected to run logarithmically, becoming strong in the UV. Thus, due to dimensional transmutation the extra data that needs to be provided is not a dimensionless coupling but rather the energy scale for the Landau pole at which this coupling becomes strong. As explained in [32, 47], the boundary conditions for a massless 2-form field such as $\mathcal{P}$ in AdS_5 indeed require one to specify such a scale. We will see this explicitly when solving the bulk equations of motion in later sections.

We provide a few more details; as usual, J is obtained by taking a functional derivative of the bulk on-shell action with respect to the boundary value of $\mathcal{P}_2$. If we set

$$\lim_{z\to 0}\mathcal{P}_2 = p_2 \tag{3.3.7}$$

and use the equation of motion then we obtain

$$\begin{aligned} J^{\mu\nu} = \frac{\delta S_{\text{on-shell}}}{\delta p_{\mu\nu}} &= \frac{\mathcal{N}}{2\kappa^2}\frac{\delta}{\delta p_{\mu\nu}}\int d\left[(1-h)^{-1}\mathcal{P}_2\wedge\star(\mathcal{Q}_3+\mathcal{A}_2\wedge dh)\right] \\ &= \frac{\mathcal{N}}{2}\frac{\delta}{\delta p_{\mu\nu}}\int d(\mathcal{P}_2\wedge Y_2) \end{aligned}$$

From here we can conclude that the 2-form symmetry current is:

$$J^{\mu\nu} = \lim_{z\to 0}\frac{\mathcal{N}}{2}\left(\star_4 Y_2\right)^{z\mu\nu} \tag{3.3.8}$$

where the normalisation is given by $\mathcal{N} = \frac{N_c^2}{8\pi^2 R^3}$.

3.3.2 Boundary Conditions

As we did for the $SU(N)$ gauge theory previously, we now explicitly spell out the boundary conditions for the original form fields which are appropriate for the $U(N)$ gauge theory.

We hold $\mathcal{P}_2$ fixed at the UV boundary, or in other words the field strength $\mathcal{Q}_3$ satisfies

$$\mathcal{Q}_{\mu\nu\rho} = 0, \qquad z\to 0 \tag{3.3.9}$$

Using (3.3.3) and again the fact that $h'(0) = 0$, we obtain simply

$$(\star Y)_{\mu\nu\rho} = 0, \qquad z \to 0 \tag{3.3.10}$$

Taking the Hodge star of (3.2.14a) and (3.2.14b) we obtain the boundary conditions

$$(\star F)_{\mu\nu\rho} = 0, \qquad\qquad z \to 0 \tag{3.3.11a}$$

$$\frac{\lambda}{4\pi N_c} G_{\mu\nu\rho} = -\kappa(\star B)_{\mu\nu\rho}, \qquad z \to 0 \tag{3.3.11b}$$

In a sense these boundary conditions are the electromagnetic dual of the boundary conditions we previously obtained for the $SU(N)$ gauge theory. This is expected because at the UV boundary the field strength of η_1 is the electromagnetic dual of the field strength of $\mathcal{P}_2$.

The non-vanishing components of the field strengths in each case (namely $Y_{z\mu}$ for $SU(N)$ and $\mathcal{Q}_{z\mu\nu}$ for $U(N)$) encode the respective conserved symmetry current. For $SU(N)$ this is the baryon number current and for $U(N)$ this is the magnetic flux current.

3.3.3 Charged line operator

We would now like to understand the bulk operators that are charged under the 2-form gauge field $\mathcal{P}_2$. In the field theory, these are dual to line operators that are charged under the corresponding 1-form symmetry. In this subsection only we will work only to first order in μ to simplify the formulas. We begin by tracing back through the chain of dualities; from (3.2.5a) to (3.2.5b), in the small μ limit we find:

$$\tau_1 = \frac{1}{\kappa}\tilde{C}_1 + 2\pi\ell_s^2\mu A_1 \qquad \eta_1 = \frac{1}{\kappa}\tilde{C}_1 - 2\pi\ell_s^2 A_1 \tag{3.3.12}$$

Furthermore, in the same limit we find

$$d\mathcal{P}_2 = \kappa^2\mu \star d\eta_1 \qquad d\mathcal{A}_2 = \kappa^2\left(B_2 + d\tau_1\right) \tag{3.3.13}$$

An object which couples minimally to $\mathcal{P}_2$ is one that appears on the right hand side of the equation of motion $d \star d\mathcal{P}_2 = 0$; we thus need to find bulk objects that couple *magnetically* to the fields $\tilde{C}_1$ and A_1. As $\tilde{C}_1$ is the magnetic dual of the RR 2-form C_2, the object coupling magnetically to it is simply a D1-string. In the Appendix we work out the normalisation of this coupling in our conventions to show that

$$\frac{1}{\kappa}\int_{S^2} d\tilde{C}_1 = \frac{(2\pi\ell_s)^2}{N_c} \tag{3.3.14}$$

where here the S_2 wraps a $D1$-string that is hanging down into AdS$_5$.

The object which couples magnetically to the DBI worldvolume gauge field A_1 is somewhat more interesting. We will call this object the DBI monopole. In this section we will work in the case where $N_f = 1$; the situation for generic N_f is more interesting still and we will touch on it briefly later. A similar problem was discussed in [48] in a lower dimensional construction, and we may take over the same ideas. The desired magnetically charged object turns out to be a wrapped $D5$-brane that *ends* on the $D7$ flavour brane. To be more precise, recall from the earlier sections that the $D7$ flavour brane wraps an $S^3 \subset S^5$:

$$d\Omega_5^2 = d\theta^2 + \cos^2\theta \, d\psi^2 + \sin^2\theta \, d\Omega_3^2 \tag{3.3.15}$$

where the S^3 is spanned by the coordinates Ω_3. The $D7$-brane does not extend in θ: more precisely, for each value of the radial coordinate z, the $D7$-brane sits at a single $\theta(z)$. In the conformal case, it lives at $\theta_{D7} = \frac{\pi}{2}$ for all z, whereas in the non-conformal case θ_{D7} interpolates from $\frac{\pi}{2}$ at the UV boundary to 0 in the interior.

In contrast, consider a $D5$-brane that wraps this S^3 and *ends* on the $D7$-brane. The $D5$-brane extends in θ from $\theta = 0$ to the θ_{D7} coordinate of the $D7$-brane, as shown in Figure 3.2. It sits at a particular value of ψ; as ψ is a Killing direction this choice is arbitrary.

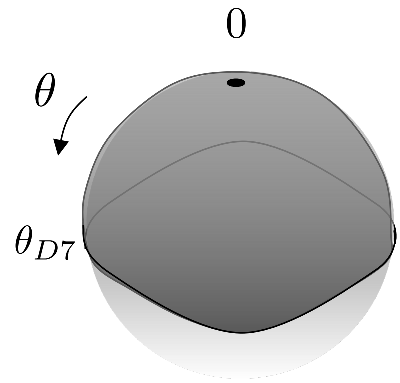

Figure 3.2: The $D5$-brane wrapping half of the $S^4 \subset S^5$ formed by (θ, Ω_3), ending on the $D7$-brane which lives at $\theta = \theta_{D7}$. The remaining 2 coordinates of the $D5$-brane worldvolume form a two-dimensional string worldsheet in the bulk.

The boundary of the $D5$-brane is a five-dimensional manifold; three of these dimensions are compact and form the S^3, and the remaining two dimensions define a two dimensional manifold $\mathcal{M}_2$ which extends into the bulk of AdS$_5$. As is well known [49], the boundary of this $D5$-brane appears magnetically charged to the DBI gauge field A_1 living on the $D7$-brane worldvolume. Hence the wrapped $D5$-brane is the DBI monopole that we seek.

In the Appendix, we work out the coupling of this brane and show that the coupling to one such wrapped brane is

$$2\pi\ell_s^2 \int_{S^2} F_2 = (2\pi\ell_s)^2 \tag{3.3.16}$$

where $F_2 = dA_1$ and the S^2 surrounds $\mathcal{M}_2$ in AdS$_5$. By comparing this to (3.3.14) and (3.3.12), we see that the $D1$-brane couples to $\mathcal{P}$ with $1/N_c$ the charge of the DBI monopole. We may write an effective coupling to the $\mathcal{P}$ field for both of these objects:

$$S = q_{D1} \int_{D1} \mathcal{P} + q_{D5} \int_{D5} \mathcal{P} \tag{3.3.17}$$

The overall normalisation of q_{D1} and q_{D5} depends on the (arbitrary) convention chosen to normalize $\mathcal{P}$ in our action, but their ratio is fixed on topological grounds to be N_c^{-1}.

$$\frac{q_{D1}}{q_{D5}} = \frac{1}{N_c} \tag{3.3.18}$$

Let us now turn to an understanding of this from the dual field theory. The intersection of the $D1$ string with the AdS boundary defines a t'Hooft line in the $SU(N)$ gauge theory sector. Similarly the wrapped $D5$-brane defines a t'Hooft line for the $U(1)$ gauge theory sector; the simplest way to see this is to note that when evaluated at the boundary, (3.3.16) is precisely the definition of a t'Hooft line. It has been previously noted (see e.g. Appendix C of [32]) that from the point of view of the $U(1)$ magnetic 1-form current, the charge of a non-Abelian t'Hooft line has $U(1)$ charge of $1/N_c$-th the Dirac monopole, consistent with (3.3.18).

Let us now understand the dynamics of symmetry breaking. Consider the wrapped DBI monopole such that it intersects the AdS boundary on a 1d curve C. This defines the insertion of a line operator into the field theory $\langle W(C)\rangle$, and as usual from the rules of AdS/CFT we should compute:

$$\langle W(C)\rangle \sim \exp\left(-S_{D5}[C]\right) \tag{3.3.19}$$

with $S_{D5}[C]$ the on-shell action of the wrapped $D5$-brane. We now seek to understand the dependence of this answer on the curve C; if it depends only locally on the data of the curve C (e.g. as a perimeter law) then the symmetry is spontaneously broken. If it depends non-locally – e.g. as an area law, or more generally in any way that cannot be locally determined by the curve, then the symmetry is unbroken.

The precise holographic arguments are a higher-form analogue of the arguments presented in [48]. Consider first the case where the mass of the flavour degrees of freedom is zero, i.e. $z_c \to \infty$. In this case the S^3 factor of the $D7$-brane remains the same size everywhere in the bulk, i.e. it is independent of z. As the brane always hangs down into the bulk, this defines a minimal area problem, essentially the same as in the usual studies of Wilson lines from holography [50]. It is clear from the geometry that the on-shell action will always depend more strongly on the curve itself than its perimeter. Thus by the previous paragraph, the symmetry is *unbroken*. See Figure 3.3 for a visualisation of this geometry.

Let us now consider the case where the mass of the flavour degrees of freedom is nonzero. Then there is a value of z_c at which the $D7$-brane caps off. At this value of z_c the wrapped $D5$-brane also pinches off and is allowed to smoothly end. There are now two disconnected possibilities for the topology of the hanging DBI monopole; it can form topologically a disc, or it can be topologically a cylinder which hangs straight down and ends where the brane caps off. For sufficiently large sizes of the curve, the cylinder solution will dominate. Such topologically non-trivial phase transitions are common in holography [51, 52, 53]. As the surface now hangs straight down, the action will depend only on the perimeter of the curve (multiplied by a constant distance in the holographic direction), and in this phase the $U(1)$ symmetry is spontaneously broken, as expected.

Finally, one could attempt to generalize the construction of defect operators to the case $N_F > 1$; in this case there is presumably an extra quantum number associated with which of the N_F D7 branes the D5 brane ends on. It seems that a careful study of the braiding algebra of bulk operators should allow the holographic identification of the mixed symmetry of rank $\gcd(N, N_F)$ identified by [54]. We leave this for later study.

3.4 Fluctuation spectrum

In the remainder of this chapter we study only the $U(N)$ theory, with its associated 1-form symmetry associated with magnetic flux. We have argued above that if the flavour degrees of freedom are gapped, then the 1-form symmetry is spontaneously broken, as can be seen from the fact that the charged line operator exhibits a perimeter law. On general grounds, we thus expect that there exists a gapless Goldstone mode in the spectrum [7, 8]. In this section we will explicitly solve the equations of motion to show the existence of this Goldstone mode. We first digress slightly to explain precisely what a Goldstone mode means in this context.

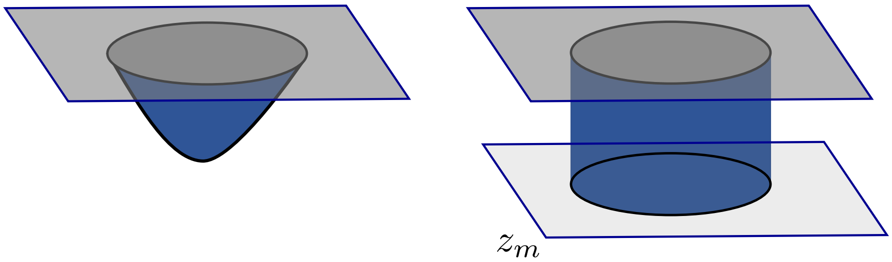

Figure 3.3: Two distinct topologies that are possible for the DBI monopole. On the left is the situation when the flavour sector is gapless; the $D7$-brane then has no boundary, and the DBI monopoles hangs down into the bulk with a disc topology, whose action depends non-locally on the data describing the boundary curve. On the right, when the flavour brane ends, the $D5$-brane is also allowed to end, permitting a cylinder topology. The action of this configuration depends only on the perimeter of the boundary curve.

Consider a completely general Lorentz-invariant four-dimensional quantum field theory with a conserved 2-form current $J^{\mu\nu}$. For simplicity, let us study the theory in Euclidean signature; as explained in (e.g.) [32], the two-point function of the current in momentum space then takes the general form

$$\begin{aligned}&\langle J^{\mu\nu}(k)J^{\rho\sigma}(-k)\rangle \\ &= \left(-\frac{1}{k^2}\left(k^\mu k^\rho g^{\nu\sigma} - k^\nu k^\rho g^{\mu\sigma} - k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\sigma g^{\mu\rho}\right) + \left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)\right) f_{JJ}\left(\frac{|k|}{m}\right)\end{aligned} \tag{3.4.1}$$

where here f_{JJ} is a dimensionless function and m is a scale. The correlation is completely determined by the function f. In this context, spontaneous breaking of the symmetry means that $f_{JJ}\left(\frac{|k|}{\Lambda}\right)$ approaches a constant as $k \to 0$; the $k \to 0$ limit then results in a gapless mode from the inverse factors of k^{-2} arising from the tensor structure.

A simple example is given by pure 4d electrodynamics; here the 1-form symmetry is broken, and the correlator of the magnetic flux $J = \star F$ takes precisely this form with

$$f_{JJ}(k) = \frac{1}{g^2} \tag{3.4.2}$$

where g^2 is the electromagnetic coupling.

An example where the symmetry is *not* spontaneously broken is given by the holographic example studied in [32]. Here the theory in question was a simple bottom-up holographic realisation of a 1-form symmetry, and the function f_{JJ} was given by

$$f_{JJ}(k) = \frac{1}{g^2 \log\left(\frac{|k|}{\Lambda}\right)} \tag{3.4.3}$$

where Λ is a Landau pole, i.e. a UV scale where the theory breaks down, as described in [32]. We note here that f_{JJ} vanishes at $k \to 0$, and the symmetry is not spontaneously broken. A similar result is found whenever there are electrically charged degrees of freedom present that are massless.

In this section we will explicitly solve the bulk equations of motion and compute the function f_{JJ} in our theory, showing that the low-frequency limit does not vanish. We will then compare it to expectations at weak coupling.

3.4.1 Goldstone modes and numerics

We will proceed by computing the correlation function of spatial components of J^{ij} with the (Euclidean) momentum purely in the time direction. Although the Green's function of interest is better extracted in the "2-form" duality frame with the fields $\mathcal{P}_2$ and $\mathcal{A}_2$, it is easier to solve the equations of motion in the "1-form" duality frame consisting of the fields η_1 and τ_1. Our strategy will be to solve the bulk equations of motion in the 1-form frame and then exploit a simple correspondence between the frames at the UV boundary to extract the Green's function.

For numerical convenience, we will set $\mu = 1$. As explained below (3.1.7), we are working in an illustrative approximation where we capture some aspect of the backreaction of the flavor degrees of freedom on the color dynamics, while neglecting gravitational backreaction. The results below do not depend qualitatively on this choice of μ, but this $\mathcal{O}(1)$ choice allows us to conveniently find numerical solutions to the equations of motion.

1-form

We solve the equations of motion given in (3.2.11) by partially fixing the gauge so that $T_2 = 0$. Next we Fourier transform the fields in the field theory directions and exploit Lorentz invariance to choose the momentum $k^\mu = (\omega, 0)$. This allows us to expand some expressions involving differential forms in terms of their components as

$$\star d \star d\eta_1 = \frac{z}{R^2}\left\{\left(z\eta_i'' - \eta_i' + z\omega^2\eta_i\right) dx^i + i\omega z(\eta_t' - i\omega\eta_z)\, dz + \left[z(\eta_t'' - i\omega\eta_z') - \eta_t' + i\omega\eta_z\right]\, dt\right\} \tag{3.4.4}$$

and

$$H_3 = \frac{1}{2} B'_{ij}\, dx^i \wedge dx^j \wedge dz + \left(B'_{it} + \frac{1}{2} i\omega B_{ij} dx^i \wedge dx^j \wedge dt - i\omega B_{iz} \right) dx^i \wedge dt \wedge dz \quad (3.4.5)$$

We next note that for a general 2-form Ω_2, we have

$$\star\left(dz \wedge \star\Omega_2\right) = -\frac{z^2}{R^2} (\Omega_{iz}\, dx^i + \Omega_{tz}\, dt) \quad (3.4.6)$$

Similarly, for a general 3-form Ω_3 we have

$$\star d \star \Omega_3 = -\frac{z}{R^2} \left\{ \frac{1}{2} \left((z\Omega_{ijz})' + i\omega z \Omega_{ijt} \right) dx^i \wedge dx^j + (z\Omega_{itz})' dx^i \wedge dt + i\omega z \Omega_{itz} dx^i \wedge dz \right\} \quad (3.4.7)$$

In pure AdS we can also show that

$$\partial_S\, \epsilon^{MNP}{}_{QR} = \frac{1}{z}\, \delta^z_S\, \epsilon^{MNP}{}_{QR} \quad (3.4.8)$$

which allows us to write

$$\star d \star H_3 = -\frac{z}{R^2} \left\{ \frac{1}{2} ((zB'_{ij})' - \omega^2 z B_{ij}) dx^i \wedge dx^j + (zH_{zit})' dx^i \wedge dt + i\omega z H_{zit} dx^i \wedge dz \right\} \quad (3.4.9)$$

where

$$H_{zit} = B'_{it} - i\omega B_{iz} \quad (3.4.10)$$

Now we can write the equations of motion more explicitly in components. The B_2 equation of motion is

$$\begin{aligned} & zh \left[\frac{1}{2} \left(zB'_{ij} \right)' dx^i \wedge dx^j + (zH_{zit})' dx^i \wedge dt + i\omega z H_{zit} dx^i \wedge dz \right] \\ & = 16 \left[(B_{iz} + h'\eta_i)\, dx^i \wedge dz + (B_{tz} + h'\eta_t) dt \wedge dz + \frac{1}{2} B_{ij}\, dx^i \wedge dx^j + B_{it}\, dx^i \wedge dt \right] \end{aligned} \quad (3.4.11)$$

We are interested in the vector channel, namely the components with a single spatial index; after imposing the duality relation at the boundary this contains the information of the transverse channel of the J^{ij} correlation function.

$$zh(zH_{zit})' = 16 B_{it} \quad (3.4.12a)$$

$$zh(i\omega z H_{zit}) = 16(B_{iz} + h'\eta_i) \quad (3.4.12b)$$

These can be combined to give

$$zh \left[\frac{z(B'_{it} + i\omega h'\eta_i)}{16 - \omega^2 z^2 h} \right]' - B_{it} = 0 \quad (3.4.13)$$

where we eliminated B_{iz} using

$$B_{iz} = \frac{i\omega z^2 h B'_{it} - 16h'\eta_i}{16 - \omega^2 z^2 h} \tag{3.4.14}$$

We also have the η_1 equation

$$(1-h)\left[\left(z\eta''_i - \eta'_i + z\omega^2\eta_i\right)dx^i + i\omega z(\eta'_t - i\omega\eta_z)dz + (z(\eta''_t - i\omega\eta'_z) - \eta'_t + i\omega\eta_z)dt\right] \tag{3.4.15}$$

$$- zh^{-1}h'[(h\eta'_i + h'\eta_i + B_{iz})dx^i + (h\eta'_t - i\omega h\eta_z + B_{tz} + h'\eta_t)dt] = 0 \tag{3.4.16}$$

We are most interested in the vector channel equation

$$(1-h)\left(\eta''_i - \frac{1}{z}\eta'_i + \omega^2\eta_i\right) - h^{-1}h'\left(h\eta'_i + h'\eta_i + \frac{i\omega z^2 h B'_{it} - 16h'\eta_i}{16 - \omega^2 z^2 h}\right) = 0 \tag{3.4.17}$$

Observe that equation 3.4.17 contains no information for $z > z_c$, since $h(z) = 1$ and $h'(z) = 0$ in that region. Tracing this back to Eq 3.2.11, we conclude that η_1 simply does not exist for $z > z_c$. In this interpretation, the field B_2 starts out life in the deep interior of the bulk and evolves continuously through the D-brane cap until it reaches the UV boundary. However, the field η_1 does not exist on the IR side of the D-brane cap - it begins its life at $z = z_c$ and evolves to the UV boundary. As η_1 started its life as the DBI worldvolume gauge field (which was then mixed together with other bulk fields to obtain the physical spectrum), it makes sense that it only exists where the D-brane is present.

However, we now need to understand how to evolve the existing fields through the transition at $z = z_c$. Imposing continuity of the η_1 equation of motion yields a useful boundary condition. As $z \to z_c$ from below we have $h(z) \to 1$ and $h'(z) \to 2\mu/z_c$, so we obtain

$$dz \wedge \star\left(-d\eta_1 + B_2 + T_2 + \frac{2\mu}{z_c}\eta_1 \wedge dz\right) = 0$$

i.e.

$$B_{\mu z} + T_{\mu z} - \partial_\mu\eta_z + \left(\partial_z + \frac{2\mu}{z_c}\right)\eta_\mu = 0 \tag{3.4.18}$$

With our gauge choice, the relevant boundary condition at the cap is given by

$$\eta'_i = -\left(\frac{i\omega B'_{it} - 2\omega^2\mu\eta_i}{16 - \omega^2}\right); \quad z = z_c\ . \tag{3.4.19}$$

Finally, if we expand the dynamical equations of motion in the UV, from the asymptotic behavior of the fields we can read off the dual conformal dimensions (using e.g. [33])

$$\Delta_\eta = 3 \tag{3.4.20a}$$

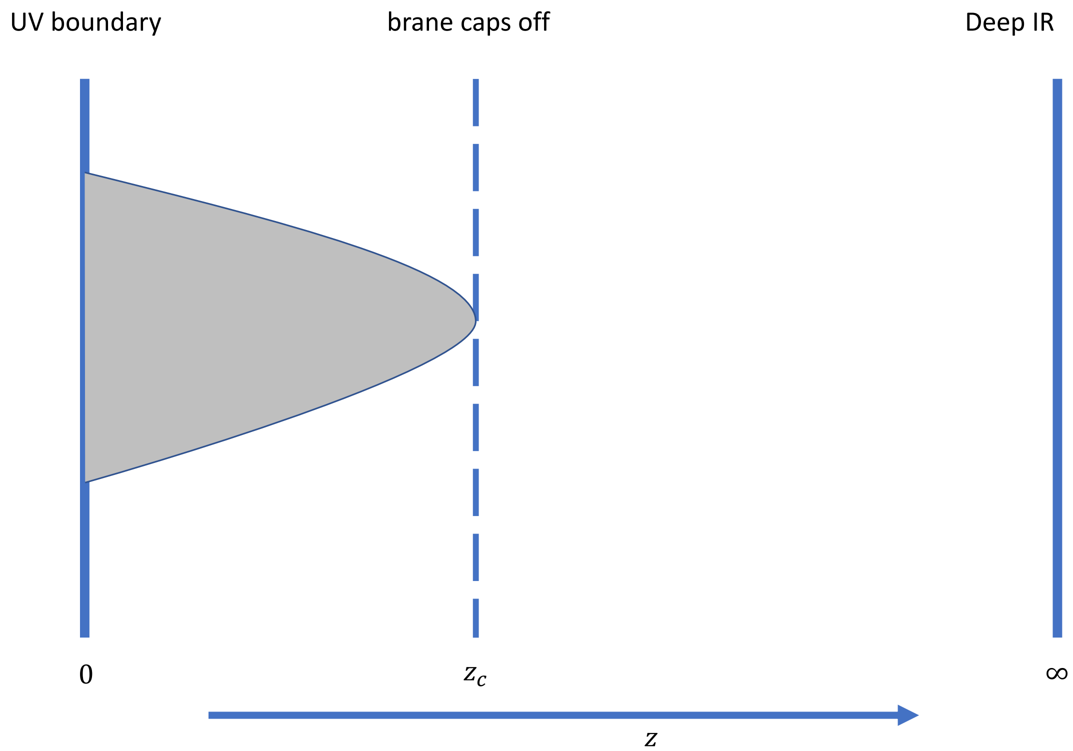

Figure 3.4: Brane caps off at $z = z_c$

$$\Delta_b = 2 + 4\sqrt{1+\mu} \tag{3.4.20b}$$

As η is dual to the conserved baryon current, its dimension is fixed at 3 as expected; B is dual to a massive tensor mode that does not have a simple universal interpretation.

To solve the equations of motion, we now Wick-rotate to Euclidean signature by setting $\omega = i\tilde{\omega}$. The equations of motion become

$$(1-h)\left(\eta_i'' - \frac{1}{z}\eta_i' - \tilde{\omega}^2\eta_i\right) + h^{-1}h'\left(-h\eta_i' - h'\eta_i + \frac{\tilde{\omega}z^2hB_{it}' + 16h'\eta_i}{16+\tilde{\omega}^2z^2h}\right) = 0 \tag{3.4.21a}$$

$$zh\left[\frac{z(B_{it}' - \tilde{\omega}h'\eta_i)}{16+\tilde{\omega}^2z^2h}\right]' - B_{it} = 0 \tag{3.4.21b}$$

We can also rewrite the equations in terms of a dimensionless holographic radial coordinate and frequency by defining

$$\zeta = z/z_c \tag{3.4.22a}$$

$$w = \tilde{\omega}\, z_c \tag{3.4.22b}$$

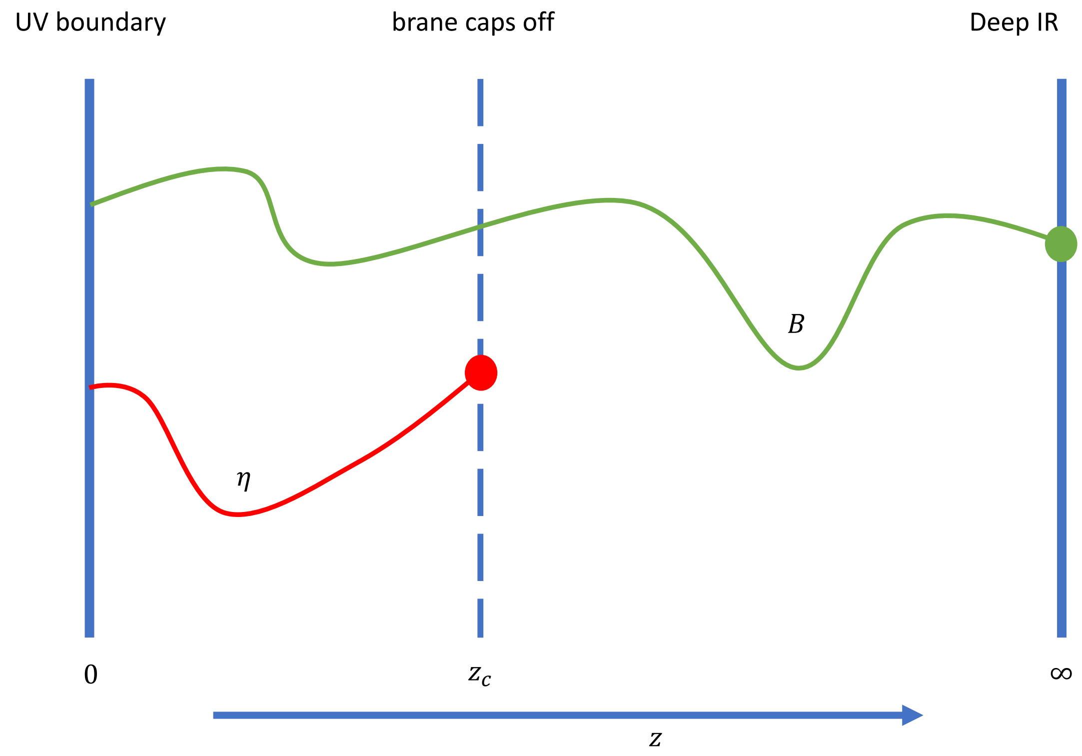

Figure 3.5: B_2 (in green) evolves continuously from the IR to the UV boundary; η_1 (in red) is “born” at the brane cap and evolves to the UV

Dropping the subscripts i, t and exploiting the fact that $z\partial_z = \zeta\partial_\zeta$, we have

$$(1-h)\left(\frac{d^2\eta}{d\zeta^2} - \frac{1}{\zeta}\frac{d\eta}{d\zeta} - w^2\eta\right) + h^{-1}\frac{dh}{d\zeta}\left(-h\frac{d\eta}{d\zeta} - \frac{dh}{d\zeta}\eta + \frac{z_c w\zeta^2 h\frac{dB}{d\zeta} + 16\frac{dh}{d\zeta}\eta}{16 + w^2\zeta^2 h}\right) = 0 \tag{3.4.23a}$$

$$\zeta h\frac{d}{d\zeta}\left[\frac{\zeta}{16 + w^2\zeta^2 h}\left(z_c\frac{dB}{d\zeta} - w\frac{dh}{d\zeta}\eta\right)\right] - z_c B = 0 \tag{3.4.23b}$$

Note that instances of z_c remain - this is to be expected since it is precisely the mass scale $m_{\text{meson}} = z_c^{-1}$ which breaks conformal invariance of the dual field theory. However, the factors of z_c appear only when multiplied by B. Hence we can define $b = z_c B$, so that

$$(1-h)\left(\frac{d^2\eta}{d\zeta^2} - \frac{1}{\zeta}\frac{d\eta}{d\zeta} - w^2\eta\right) + h^{-1}\frac{dh}{d\zeta}\left(-h\frac{d\eta}{d\zeta} - \frac{dh}{d\zeta}\eta + \frac{w\zeta^2 h\frac{db}{d\zeta} + 16\frac{dh}{d\zeta}\eta}{16 + w^2\zeta^2 h}\right) = 0 \tag{3.4.24a}$$

$$\zeta h \frac{d}{d\zeta}\left[\frac{\zeta}{16+w^2\zeta^2 h}\left(\frac{db}{d\zeta}-w\frac{dh}{d\zeta}\eta\right)\right]-b \qquad = 0 \tag{3.4.24b}$$

The boundary condition at the cap is given in Euclidean signature by

$$\frac{d\eta}{d\zeta}=\frac{w\frac{db}{d\zeta}-2\mu w^2\eta}{16+w^2} \tag{3.4.25}$$

2-form

The above set of equations is a closed system that can be conveniently numerically solved. However we are ultimately interested in studying the behavior of the system in the $U(N)$ duality frame, in which the physics is encoded in the fields $\mathcal{P}_2$ and $\mathcal{A}_2$ rather than η_1 and τ_1. To relate them, we note that in the UV ($z\to 0,\ dh=0$), we can match the fields using (3.3.3) to get $d\mathcal{P}_2=\frac{\kappa^2\mu}{1+\mu}\star d\eta_1$. After a Wick rotation we can fix some UV scale z_Λ to get

$$\frac{w}{z_c}\,\mathcal{P}_{12}(z_\Lambda)=\alpha\,\frac{\eta_3'(z_\Lambda)}{z_\Lambda} \tag{3.4.26a}$$

$$z\,\mathcal{P}_{12}'(z_\Lambda)=\alpha\,\frac{w}{z_c}\,\eta_3(z_\Lambda) \tag{3.4.26b}$$

where $\alpha=\frac{16\mu}{(1+\mu)R}$.

As in [32], in the UV we have

$$\mathcal{P}_{jk}\sim p_{jk}+J_{jk}\log z,\quad z\to 0 \tag{3.4.27}$$

We may also directly verify that the leading order asymptotic behaviour of $\eta_3(z)$ is given by

$$\eta_3(z)\sim\eta_0+\eta_2 z^2+\bar{\eta}_2 z^2\log z,\quad z\to 0 \tag{3.4.28}$$

Hence matching these components at the cutoff we find that

$$J_{12}=\alpha\,\frac{w}{z_c}\,\eta_0 \tag{3.4.29a}$$

$$\frac{w}{z_c}(p_{12}+J_{12}\log z_\Lambda)=\alpha\,(2\eta_2+\bar{\eta}_2+2\bar{\eta}_2\log z_\Lambda) \tag{3.4.29b}$$

Consistency of the unambiguous coefficients of $\log z_\Lambda$ fixes the coefficient $\bar{\eta}_2$ to be

$$\bar{\eta}_2=\frac{1}{2}\left(\frac{w}{z_c}\right)^2\eta_0 \tag{3.4.30}$$

We now turn to the interpretation of the logarithm in Eq (3.4.27). As explained in detail in [32], this logarithm arises from the fact that the double-trace coupling J^2

is marginally irrelevant. This marginal irrelevance breaks conformality, and more information must be given to specify the theory. (Indeed, the only conformal theory with a continuous 1-form symmetry in four dimensions is free Maxwell electrodynamics [7, 55]). This information can be given in the form of the value of the double-trace coupling $\frac{1}{\theta}J^2$ at a particular scale. (Note that in this strongly coupled model one can now identify θ with the gauge coupling of the $U(1)$ sector g_1 in (2.4.45)).

Following the algorithm in [32], we can now determine the source p_{12} by

$$p_{12} = \mathcal{P}_{12}(z_\Lambda) - \frac{J_{12}}{\theta} = 2\alpha\,\frac{z_c}{w}\,\eta_2 + J_{12}\log\bar{z}_* \tag{3.4.31}$$

Here the scale $\bar{z}_*$ is given by

$$\bar{z}_* \equiv e^{1/2}\,z_* \equiv e^{1/2}\,z_\Lambda\,e^{-1/\theta} \tag{3.4.32}$$

The value of this scale should be understood as the Landau pole where the theory breaks down; as $\theta > 0$, we note that it is an extremely small scale, much smaller than the cutoff. Concretely, we can numerically extract the 2-point function content $f_{JJ}(\omega)$ by solving the equations of motion for B_{3t} and η_3. See Appendix 3.B for further details of this method.

Results

Here we present a plot of the numerically calculated Green's function as a function of $w = \omega z_c$ for various values of the dimensionless number $\gamma \equiv z_c/\bar{z}_*$, i.e. the meson mass in units of the Landau pole scale.

Note that at weak coupling the mass gap is given by the bare flavor mass m_F. However at strong coupling the mass gap is the mass of the lightest meson which is given by $\frac{1}{z_c} = \frac{1}{2\pi}\frac{\sqrt{\lambda}}{m_F}$, where here m_F should be understood as the coefficient of the relevant mass deformation in the UV. We have thus chosen to plot the result in units of the physical meson mass z_c. We observe that the asymptotic behaviour is as expected: for small w the leading order contribution is a constant which depends on γ. For large w we expect logarithmic behaviour, but this is difficult to see explicitly because we cannot numerically access the solution for an exponential range of values of w.

3.4.2 Comparison to weak coupling

Here we will try to compare the functional form of the results above to a weak-coupling computation. By weak-coupling, we mean that we will take the non-Abelian

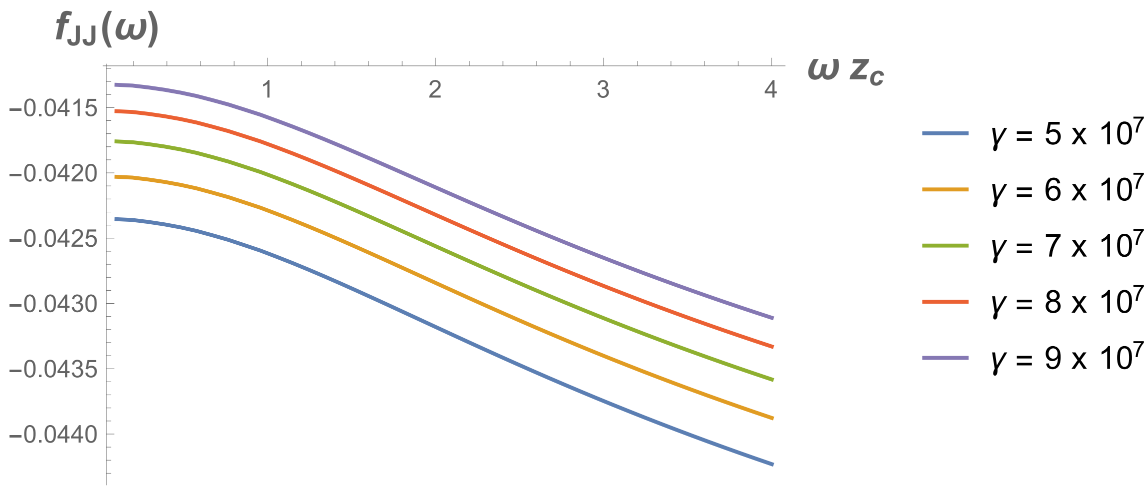

Figure 3.6: The symmetry current correlator at strong coupling as a function of $w = \omega z_c$ for various masses, computed numerically using holography. For this plot we set $\bar{z}_* = 10^{-8}$.

t'Hooft coupling λ to zero; however we will keep fixed the Landau pole associated with the $U(1)$ factor. Note that in the λ to zero limit, the $U(1)$ sector of the field theory is effectively super QED with N_f flavours, i.e. a $U(1)$ gauge field a_1 coupled to N_f Dirac fermions and N_f complex scalars of mass m with coupling constant g_1. Up to a normalisation, the current associated with the 1-form global symmetry is $J_b = \star f$.

The current-current correlator can be shown to be

$$\langle \tilde{J}_b^{\mu\nu}(k)\tilde{J}_b^{\rho\sigma}(-k)\rangle = \epsilon^{\mu\nu\alpha\beta}\epsilon^{\rho\sigma\gamma\delta}k_\alpha k_\gamma\, \tilde{\Delta}_{\beta\delta}(k) \tag{3.4.33}$$

where $\tilde{\Delta}_{\mu\nu}(k)$ is the loop-corrected photon propagator in momentum space. We are interested in the purely spatial components $\langle \tilde{J}_b^{xy}(k)\tilde{J}_b^{xy}(-k)\rangle$.

The contributions to the photon propagator $\Delta_{\mu\nu}$ arise from resumming scalar and fermion loops as in Figure 3.7 and Figure 3.8:

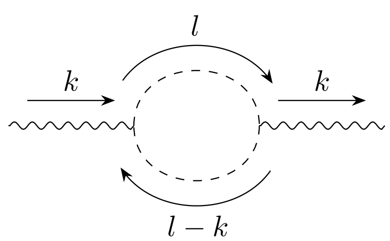

Figure 3.7: Scalar loop diagram contributing to correction of photon propagator

This is a textbook calculation - see e.g. [3] for a reference which matches our conventions. We use dimensional regularisation in the $\overline{MS}$ renormalisation scheme

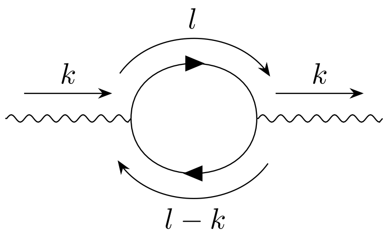

Figure 3.8: Fermion loop diagram contributing to correction of photon propagator

and put momentum purely in the time direction. This allows us to write

$$\langle \tilde{J}_b^{xy}(\omega)\tilde{J}_b^{xy}(-\omega)\rangle = \left\{1 - \frac{N_f\ g_1^2(\mu)}{4\pi^2}\int_0^{1/2} dy\ (1-2y^2)\log\left[\frac{1+(1/4-y^2)\ \hat{\omega}^2}{(\mu/m)^2}\right] + \mathcal{O}(g_1^4)\right\}^{-1} \tag{3.4.34}$$

where μ is an arbitrary mass scale and we define a dimensionless number by $\hat{\omega} \equiv \omega/m$. The coupling g_1 runs logarithmically with the energy scale. Let's fix the coupling g_1 at some UV scale μ_Λ to be g_R. Then the Landau pole scale μ^* at which the renormalized coupling g_1 diverges is related to μ_Λ by

$$\mu^* = \mu_\Lambda\ e^{1/\chi} \tag{3.4.35}$$

where here χ is given by

$$\chi = \frac{5N_f\ g_R^2}{24\pi^2} \tag{3.4.36}$$

Note that we have combined the fermion contribution of $N_f\ g_R^2/(6\pi^2)$ with the scalar contribution of $N_f\ g_R^2/(24\pi^2)$. Here μ^* is the physical scale which we should identify with the holographic Landau pole z^* when comparing the two theories.

This gives an expression for the current-current correlator in terms of the Landau pole scale and the double-trace coupling as

$$\left\langle \tilde{J}_b^{xy}(\omega)\tilde{J}_b^{xy}(-\omega)\right\rangle^{-1} = \frac{6}{5}\chi\int_{-1/2}^{1/2} dy\ (1-2y^2)\ \frac{1}{2}\log\left[\frac{1+(1/4-y^2)\ \hat{\omega}^2}{(\mu^*/m)^2}\right] \tag{3.4.37}$$

See Figure 3.9 for a plot of the correlator at weak coupling. As we can see, the weak-coupling and strong-coupling plots are extremely similar: they approach a constant for small ω in relation to the relevant mass scale (as dictated by the spontaneous symmetry breaking), and diverge logarithmically for large ω (as dictated by the running of the coupling at large frequencies). It is curious to note that these two properties appear to be sufficient to control the correlator at all scales, giving the same dynamical behavior from strongly coupled gravity and from weakly coupled Feynman diagrams.

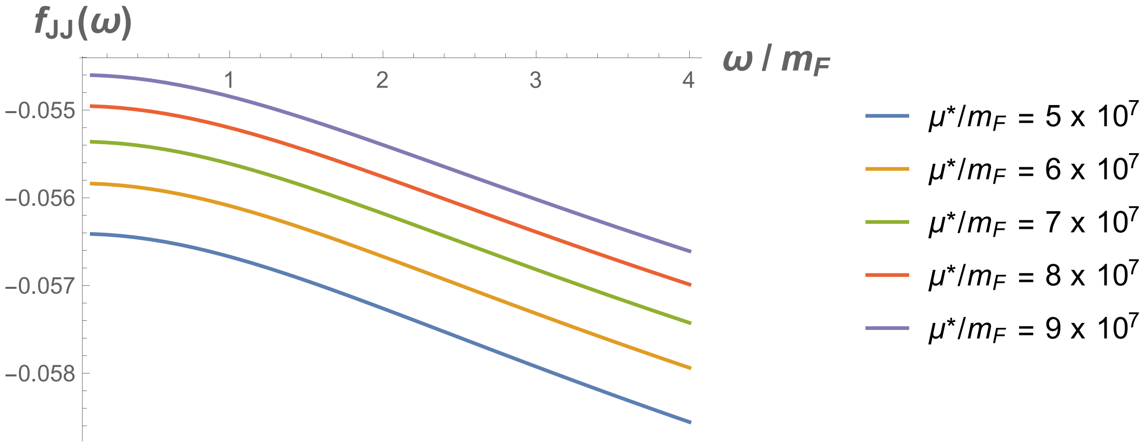

Figure 3.9: The symmetry current correlator at weak coupling as a function of $\hat{\omega} = \omega/m_F$, computed analytically in perturbation theory

Conclusion: In this chapter, we have studied the realisation of 1-form symmetries in perhaps the simplest holographic model in which such a symmetry could be spontaneously broken; along the way we have clarified some aspects of the interplay between 0-form baryon number symmetry and the 1-form $\mathbb{Z}_N$ symmetry in $SU(N)$ gauge theory. We identified the charged line operator and verified the expected behavior of the current-current correlation function, demonstrating the existence of the expected Goldstone mode. We can identify various directions for future research. It would be very interesting to extend this study to finite temperatures, where we could expect to make contact with recent symmetry-based formulations of magnetohydrodynamics [10]. In a more formal direction, it would also be very interesting to understand the bulk holographic dual of the colour-flavour-center symmetry identified in [54].

3.A Normalisations

To translate between field theory quantities and bulk quantities we use the holographic dictionary [34]

$$\frac{R^4}{l_s^4} = \lambda \equiv g_{YM}^2 N_c = 4\pi g_s N_c \tag{3.A.1}$$

3.A.1 Kinetic Terms

The kinetic terms for B_2 and C_2 are of the form

$$S_{\text{kin}} = -\int_{\text{AdS}_5} \left(\frac{1}{2}\mathcal{N}_B^2\, H_3^2 + \frac{1}{2}\mathcal{N}_C^2\, G_3^2 \right) \tag{3.A.2}$$

and our task is to find the factors $\mathcal{N}_B$ and $\mathcal{N}_C$.

Consider the type IIB low energy supergravity action in the NS-NS sector[2] (see e.g. [5]):

$$S_{\text{NS}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G}\; e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}|H_3|^2 \right) \tag{3.A.3}$$

where G is the 10-dimensional metric in the string frame, Φ is the dilaton field, R is the Ricci scalar and κ_{10} is the gravitational coupling in 10 spacetime dimensions given by $2\kappa_{10}^2 = (2\pi)^7 l_s^8$.

If we choose the dilaton to be constant with $e^\Phi = g_s$ and choose the string frame metric to be the usual metric on $\text{AdS}_5 \times S^5$, the relevant term is

$$S_{\text{NS}} = \frac{1}{(2\pi)^7 l_s^8 g_s^2} \int_{\text{AdS}_5 \times S^5} \left(-\frac{1}{2} H_3^2 \right) \tag{3.A.4}$$

We dimensionally reduce on the S^5 which yields a factor of $V_5 = \pi^3 R^5$

$$S_{\text{eff}} = -\frac{R^5}{128\pi^4 l_s^8 g_s^2} \int_{\text{AdS}_5} \left(\frac{1}{2} H_3^2 \right) \tag{3.A.5}$$

where now the integral is taken only over the AdS_5 directions. We thus conclude that

$$\mathcal{N}_B^2 = \frac{R^5}{128\pi^4 g_s^2 l_s^8} = \frac{N_c^2}{8\pi^2 R^3} \tag{3.A.6}$$

The analysis for the R-R kinetc term is similar. The supergravity action in the R sector is

$$S_{\text{R}} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x\sqrt{-G} \left(|F_1|^2 + |\hat{G}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right) \tag{3.A.7}$$

[2]To match the conventions of [5] with ours we have $\int d^{10}x\sqrt{-G}\, |F_{p+1}|^2 = \int F_{p+1}^2$.

where the relevant quantity for us is $\hat{G}_3 \equiv G_3 - C_0 \wedge H_3$ and $G_3 = dC_2$ is the R-R field strength. Setting $C_0 = 0$ we have the term

$$S_{\mathrm{R}} = -\frac{R^5}{128\pi^4 l_s^8} \int_{AdS_5} \left(\frac{1}{2} G_3^2\right) \tag{3.A.8}$$

after compactifying on the S^5. By comparing with the original action we can identify

$$\mathcal{N}_C^2 = \frac{R^5}{128\pi^4 l_s^8} = \frac{\lambda^2}{128\pi^4 R^3} \tag{3.A.9}$$

as promised.

3.A.2 Chern-Simons Term

Suppose we have a Chern-Simons term in the action of the form

$$S_{\mathrm{CS}} = \frac{k}{2\pi} \int B_2 \wedge G_3 = \kappa\, \mathcal{N}_B\, \mathcal{N}_C \int B_2 \wedge G_3 \tag{3.A.10}$$

The coupling to $D1$-branes and $F1$-strings respectively is $S_{\mathrm{D1}} = \mu_1 \int C_2$ and $S_{\mathrm{F1}} = \frac{1}{2\pi l_s^2} \int B_2$, where $\mu_1^{-1} = 2\pi l_s^2$ is the tension of a $D1$-brane. A higher-form Dirac quantisation condition gives

$$\frac{\mu_1}{2\pi} \int_{S_3} F_3 \in \mathbb{Z} \tag{3.A.11}$$

For a magnetic monopole of unit charge we have (in $d = 10$)

$$dG_3 + \frac{2\pi}{\mu_1} \delta_4(W) = 0 \tag{3.A.12}$$

where $W = S^5 \times L$ is the 6d worldvolume of the $D5$-brane sourcing the monopole and L is the worldline of the monopole, i.e.a timelike curve in AdS$_5$.

By taking the wedge with $d\Omega_5 \wedge \Xi_1$ and integrating over all 10 dimensions we obtain

$$\int_{\mathrm{AdS}} dG_3 \wedge \Xi_1 + \frac{2\pi}{\mu_1} \int_L \Xi_1 = 0 \tag{3.A.13}$$

which allows us to write the gauge variation of the Chern-Simons term as

$$\delta S_{\mathrm{CS}} = \frac{k}{\mu_1} \int_L \Xi_1 \tag{3.A.14}$$

This must be cancelled by the gauge variation of M F_1 strings which end on the worldline L,

$$M \delta S_{F1} = \frac{M}{2\pi l_s^2} \int_{F1} \delta B_2 = \frac{M}{2\pi l_s^2} \int_L \Xi_1 \tag{3.A.15}$$

We identify the integer M with the number of colours in the field theory N_c, as in [38].

Hence

$$k = \frac{\mu_1 N_c}{2\pi l_s^2} = \frac{N_c}{4\pi^2 l_s^4} = \frac{N_c \lambda}{4\pi^2 R^4} \tag{3.A.16}$$

This gives

$$\kappa = \frac{k}{2\pi \mathcal{N}_B \mathcal{N}_C} = \frac{4}{R} \tag{3.A.17}$$

3.A.3 DBI term

Here we will describe the basic setup to add a $D7$-brane to $\mathrm{AdS}_5 \times S^5$ by wrapping an S^3 around the S^5. We will follow a similar approach to [40]. The final result will be a contribution to the action of

$$-\mathcal{N}_B^2 \int \left[\frac{1}{2}\kappa^2 \mu f(z) \left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2\right)^2\right] \tag{3.A.18}$$

where the factor μ and the function $f(z)$ will be determined.

The $10d$ string frame metric G_{AB} is given by

$$ds^2 = G_{AB} dX^A dX^B = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx^i dx^j \delta_{ij}) + R^2 d\Omega_5^2 \tag{3.A.19}$$

Here $i, j \in \{1, 2, 3\}$ are spatial indices and A, B index the coordinates z, t, x^i and all the angles of S^5. R is the AdS radius and we parametrise the 5-sphere as

$$d\Omega_5^2 = d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\Omega_3^2, \tag{3.A.20}$$

where $d\Omega_3^2$ is the standard metric for a 3-sphere, the angle $\psi \in [0, 2\pi]$ is azimuthal and the angle θ takes values in $[0, \frac{\pi}{2}]$. This coordinate choice is analogous to the so-called Hopf coordinates on S^3. For our purposes, these coordinates provide a simpler way to embed a 3-sphere inside a 5-sphere than the usual hyperspherical coordinates.

We can embed a probe $D7$-brane into the target space by means of the DBI action:

$$S_{\mathrm{DBI}} = -\tau_7 \int d^8\xi \sqrt{-\det\left(g_{\alpha\beta} + B_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta}\right)} \tag{3.A.21}$$

where ξ^α are the brane worldvolume coordinates, $g_{\alpha\beta}$ is the induced worldvolume metric on the $D7$-brane, $B_{\alpha\beta}$ are the components of the NS-NS 2-form and $F_2 = dA_1$ is the Maxwell field strength living on the brane.

τ_p is the effective Dp-brane tension after absorbing the effect of the dilaton $e^{\Phi} = g_s$ and is given by equation (13.3.23) of [5] as

$$\tau_7 = \frac{1}{g_s} \frac{1}{(2\pi)^7 l_s^8} \tag{3.A.22}$$

The desired brane configurations fills all of AdS and wraps a 3-sphere around the S^5, so the transverse fluctuations will be the θ and ψ angles. Since G_{AB} is independent of ψ, i.e.∂_ψ is a Killing vector of the target space metric, we will take ψ to be a constant. Crucially, the θ angle will be a function of the AdS radial coordinate: $\theta = \theta(z)$.

The worldvolume metric is the pull-back of the target space metric onto the worldvolume

$$g_{\alpha\beta} = \frac{\partial X^A}{\partial \xi^\alpha} \frac{\partial X^B}{\partial \xi^\beta} G_{AB} \tag{3.A.23}$$

If we choose static gauge $\xi^\alpha = X^\alpha$ then we simply have

$$g_{zz} = G_{zz} + \left(\frac{d\theta}{dz}\right)^2 G_{\theta\theta} = G_{zz}(1 + (z\theta')^2) \tag{3.A.24a}$$

$$g_{\alpha\beta} = G_{\alpha\beta}, \qquad \alpha \neq z \tag{3.A.24b}$$

Let's write $B_{\alpha\beta} + 2\pi l_s^2 F_{\alpha\beta} \equiv \tilde{B}_{\alpha\beta}$. We can expand the determinant in the DBI Lagrangian as

$$\begin{aligned}
\det(g_{\alpha\beta} + \tilde{B}_{\alpha\beta}) &= \det(g_{\alpha\gamma}\delta^\gamma_\beta + g_{\alpha\gamma} g_{\beta\delta} \tilde{B}^{\gamma\delta}) \\
&= \det(g_{\alpha\gamma}) \det(\delta^\gamma_\beta + g_{\beta\delta}\tilde{B}^{\gamma\delta}) \\
&= \det(G_{MN})(1 + (z\theta')^2)(R^2 \sin^2\theta)^3 \det(\delta^\gamma_\beta + g_{\beta\delta}\tilde{B}^{\gamma\delta})
\end{aligned}$$

For a traceless matrix A we have $\det(\mathbb{1} + A) = 1 - \frac{1}{2}\operatorname{Tr}(A^2) + \mathcal{O}(A^3)$.

Since $g_{\beta\delta}\tilde{B}^{\gamma\delta}$ is traceless, the leading order behaviour of $\det(\delta^\gamma_\beta + g_{\beta\delta}\tilde{B}^{\gamma\delta})$ is given by

$$\det(\delta^\gamma_\beta + g_{\beta\delta}\tilde{B}^{\gamma\delta}) = 1 + \frac{1}{2}\tilde{B}^{\alpha\beta}\tilde{B}_{\alpha\beta} + \cdots \tag{3.A.25}$$

Putting this into the DBI action and performing the integral over the unit 3-sphere to obtain a factor of $2\pi^2$ gives

$$S_{\text{DBI}} = -\frac{1}{g_s}\frac{2\pi^2 R^3}{(2\pi)^7 l_s^8} \int \sqrt{-G_{MN}}\, d^5x \left[\sqrt{1 + (z\theta')^2}\ \sin^3\theta \left(1 + \frac{1}{4}\tilde{B}^{\alpha\beta}\tilde{B}_{\alpha\beta}\right)\right] \tag{3.A.26}$$

to quadratic order in $\tilde{B}_{\alpha\beta}$.

Let's turn off the Kalb-Ramond field and the Maxwell field. Now we have an effective action of the form

$$S = \mathcal{N} \int dz L[\theta(z), \theta'(z)] \tag{3.A.27}$$

where

$$L = \frac{\sin^3 \theta}{z^5} \sqrt{1 + (z\theta')^2} \tag{3.A.28}$$

and we absorbed the integral over field theory directions into the overall normalisation factor $\mathcal{N}$.

Solving the Euler-Lagrange equation gives the single-valued on-shell angle

$$\theta(z) = \theta_c \equiv \begin{cases} \arccos(z/z_c) & z \leq z_c \\ 0 & z > z_c \end{cases} \tag{3.A.29}$$

Geometrically, this means that the S^3 wrapping the S^5 is of maximal size ($\theta = \pi/2$) at $z = 0$ on the boundary, and the $D7$-brane vanishes ($\theta = 0$) at the critical value $z = z_c$. For $z > z_c$ the $D7$-brane has no effect.

It is straightforward to show that

$$\sqrt{1 + (z\theta_c')^2}\ \sin^3 \theta_c = f(z) \tag{3.A.30}$$

where f is the dimensionless scalar function given by

$$f(z) = \begin{cases} 1 - (z/z_c)^2 & z \leq z_c \\ 0 & z > z_c \end{cases} \tag{3.A.31}$$

Using the holographic dictionary we find that $2\pi l_s^2 = \frac{2\pi R^2}{\sqrt{\lambda}}$. In general to add N_f flavours we simply add N_f probe $D7$-branes. After holographic renormalisation this gives an overall contribution to the action of

$$N_f S_{\mathrm{DBI}} = -\frac{N_c^2}{8\pi^2 R^3} \int \left[\frac{1}{2} \left(\frac{4}{R} \right)^2 \frac{N_f}{N_c} \frac{\lambda}{32\pi^2} f(z) \left(B_2 + \frac{2\pi R^2}{\sqrt{\lambda}} F_2 \right)^2 \right] \tag{3.A.32}$$

from which we can read off that

$$\mu = \frac{N_f}{N_c} \frac{\lambda}{32\pi^2} \tag{3.A.33}$$

as expected.

3.A.4 Couplings of other branes

Here we work out the couplings to various other bulk objects in our normalisation. We will sometimes make use of the form delta function $\delta_{\mathcal{M}_p}(x)$. This is a delta function that is nonzero only if x is on the submanifold $\mathcal{M}_p$; more precisely it is a $d-p$-form such that $\delta_{\mathcal{M}_p}(x) = 0$ if $x \notin \mathcal{M}_p$ and the integral over any p-form C_p satisfies:

$$\int_{\mathbb{R}^d} \delta_{\mathcal{M}_p} \wedge C_p = \int_{\mathcal{M}_p} C_p \ . \tag{3.A.34}$$

Wrapped $D5$-brane: baryon vertex

We recall the action:

$$S = -\frac{N_c^2}{8\pi^2 R^3} \int \left[\frac{1}{2} H_3^2 + \frac{1}{2}\kappa^2 (B_2 + T_2)^2\right] \tag{3.A.35}$$

where $T_2 = d\tau_1$. Recall from (3.2.1) also that τ_1 is related to the original RR and NS 2-forms as

$$d\tau_1 = \frac{\lambda}{4\pi N_c \kappa} \star dC_2 - B_2 \tag{3.A.36}$$

We now start with a $D5$-brane wrapped on the S^5, and would like to determine its coupling to the field τ_1; in other words we add to the action a term

$$q_\tau \int_L \tau_1 \tag{3.A.37}$$

and would like to determine the coefficient q_τ. For simplicity, study a configuration with $B_2 = 0$; varying the action with respect to τ_1 we have

$$\frac{N_c^2 \kappa^2}{8\pi^2 R^3} d \star d\tau_1 = q_\tau \delta_L(x) \tag{3.A.38}$$

Integrating both sides of this over a ball with boundary S^3 that intersects the worldline L, we have

$$\frac{N_c^2 \kappa^2}{8\pi^2 R^3} \int_{S^3} \star d\tau_1 = q_\tau \tag{3.A.39}$$

Now use $G_3 = dC_2$ and insert (3.A.36) to find that for a minimally charged $D5$-brane as in (3.A.11) we have

$$q_\tau = \frac{N_c}{2\pi l_s^2} \tag{3.A.40}$$

This is N_c times the "unit charge" of a single F-string in the appropriate units, as we discuss in the bulk of the text.

Wrapped $D5$-brane with boundary: DBI monopole

We now discuss a different bulk object, though also arising from a wrapped $D5$-brane, here wrapping a half S^4 and ending on the D7 flavour branes. The geometry is described around (3.3.15). Here we work out the precise charges; the computation outlined here is a higher-dimensional analogue of the calculations in [48]. The relevant parts of the bulk action are

$$N_F T_7 \int_{D7} 2\pi l_s^2\, F_2 \wedge C_6 + T_5 \int_{D5} C_6 + \cdots \tag{3.A.41}$$

We study the case with $N_f = 1$. We study the configuration where the $D5$-brane has a boundary $\partial D5$ ending on the $D7$-brane. This boundary means that the coupling to C_6 alone is no longer gauge invariant; indeed if we now do a 5-form gauge transformation of the RR 6-form C_6, $C_6 \to C_6 + d\Lambda_5$, we find that gauge-invariance requires that

$$T_7\; 2\pi l_s^2\; dF_2 = -T_5\; \delta_{\partial D5}(x) \tag{3.A.42}$$

and thus that if we consider an S^2 that surrounds $\partial D5$ on the $D7$-brane worldvolume, we have that

$$\int_{S^2} F_2 = 2\pi \tag{3.A.43}$$

where we have used that $\frac{T_5}{T_7(2\pi l_s^2)} = 2\pi$. Thus the edge of the wrapped brane couples magnetically to the DBI worldvolume field. As expected, this is the magnetic flux that saturates the Dirac quantisation condition, where the conjugate electric charge is viewed as the endpoint of an F-string ending on the $D7$-brane.

$D1$ string

Here we work out the coupling of the $D1$ string to $\tilde{C}_1$, which is the magnetic dual of the RR 2-form C_1. We begin with the relevant part of the effective 5d kinetic term for C_2 from (3.1.1), which is

$$S = -\frac{\lambda^2}{128\pi^3 R^3} \int_{AdS_5} \frac{1}{2}(dC_2)^2 + \cdots \tag{3.A.44}$$

From here we and the coupling to a $D1$ string used in 3.A.2 we find that the equation of motion in the presence of a $D1$ source is

$$\int_{S^2} \star dC_2 = \frac{64\pi^3}{l_s^2}\frac{R^3}{\lambda^2} \tag{3.A.45}$$

where the integral is taken over an S^2 that surrounds the $D1$ brane. Next, using the relation between C_2 and $\tilde{C}_1$ in (3.2.1) in a configuration where $B = 0$, we find that

$$\int_{S^2} dC_1 = \frac{16\pi^2 l_s^2}{N_C R} \tag{3.A.46}$$

Restoring the factor of κ^{-1} this reduces to (3.3.14) quoted in the main text.

3.B Numerical Solution

Here we give a brief explanation of the numerical procedure used to obtain the spectral function in Figure 3.6.

An overview: the equations of motion for B_2 and η_1 are numerically solved twice, each time with different boundary conditions. The solution for η_1 is dualised to a solution for $\mathcal{P}_2$ in the UV. The solutions for B_2 and $\mathcal{P}_2$ are then used to construct the Green's function f_{JJ}. For concreteness, we work with the components B_{3t} and η_3.

3.B.1 Equations of motion

The boundary conditions used are the values of the fields B and η at the $D7$-brane cap $\zeta = 1$. This then fixes the derivatives of the fields as follows. The derivative of B is determined by solving the equation of motion in the range $1 < \zeta < \infty$ and the derivative of η is determined by imposing continuity as $\zeta \to 1$ from below.

In the region $1 < \zeta < \infty$, the equation of motion for B is considerably simpler. In fact, it can be reduced to a first-order nonlinear differential equation for the new field Σ defined by

$$\Sigma(\zeta) \equiv \frac{1}{B(\zeta)} \frac{dB}{d\zeta} \tag{3.B.1}$$

The appropriate asymptotic boundary condition in the IR is constrained by regularity to be

$$\Sigma(\zeta) \sim -w; \quad \zeta \to \infty \tag{3.B.2}$$

This is the consequence of the asymptotics of B itself:

$$B(\zeta) \sim e^{-w\zeta} \quad \zeta \to \infty \tag{3.B.3}$$

After solving for Σ we can read off the value of the derivative of B at the cap as

$$\frac{dB}{d\zeta}(1) = \Sigma(1)B(1) \tag{3.B.4}$$

The coupled equations of motion are solved up to some UV-cutoff scale (a minimum value for z_c), at which η_1 can be straightforwardly dualised to $\mathcal{P}_2$. $\mathcal{P}_2$ corresponds to a 1-form global symmetry, so its asymptotic form in the UV is well-understood.

3.B.2 Asymptotic analysis

To extract the data needed for the Green's functions, we need a careful understanding of the asymptotic falloffs of various fields. In the 1-form picture, one finds the following form for the fields:

$$B(z \to 0) \sim z^{-\nu}\left(b_{0,-} + z^2 b_{2,-} + \cdots\right) + z^{\nu}\left(b_{0,+} + z^2 b_{2,+} + \cdots\right) \tag{3.B.5}$$

Here, by the usual rules of AdS/CFT, $b_{0,-}$ is the source and $b_{0,+}$ is the response. Similarly, we may expand the field $\eta(z)$ at infinity: we find

$$\eta(z \to 0) \sim \eta_0 + \eta_2 z^2 + \bar{\eta}_2 z^2 \log z + \cdots + z^{4-\nu}\left(\eta_{-,0} + z^2 \eta_{-,2} + \cdots\right) \tag{3.B.6}$$

Here a somewhat unfamiliar role is played by the terms in $z^{4-\nu}$; these arise from the mixing between the two bulk fields. The coefficients $\eta_{-,0}, \eta_{-,2}$ are all proportional to $b_{0,-}$ and may be explicitly calculated from the asymptotic analysis of the equations of motion.

Numerically it is more practical to fit the solutions of the equations of motion to the known asymptotic form using linear regression. This "numerical holographic renormalisation" allows us to pick out the coefficients we need. Crucially however, we implemented the numerics using the ζ coordinate defined by $\zeta = \frac{z}{z_c}$. We can write the above asymptotic expansions in this coordinate system as

$$\begin{aligned} B(\zeta \to 0) &\sim z_c^{-\nu}\,\zeta^{-\nu}\left(b_{0,-} + z_c^2\,\zeta^2\,b_{2,-} + \cdots\right) + z_c^{\nu}\,\zeta^{\nu}\left(b_{0,+} + z_c^2\,\zeta^2\,b_{2,+} + \cdots\right) \\ &= \zeta^{-\nu}\left(\hat{b}_{0,-} + \cdots\right) + \zeta^{\nu}\left(\hat{b}_{0,+} + \cdots\right) \end{aligned} \tag{3.B.7}$$

A linear regression in the ζ coordinate system will thus fit the coefficients

$$\hat{b}_{0,-} \equiv z_c^{-\nu} b_{0,-} \tag{3.B.8a}$$

$$\hat{b}_{0,+} \equiv z_c^{\nu}\, b_{0,+} \tag{3.B.8b}$$

This scaling can be accounted for, but will anyway cancel out at the end of our calculation.

However for η the presence of the logarithmic term in the asymptotic expansion produces a more subtle transformation. We have (after holographic renormalisation)

$$\begin{aligned} \eta(\zeta \to 0) &\sim \eta_0 + \eta_2 z_c^2 \zeta^2 + \bar{\eta}_2 z_c^2 \zeta^2 (\log z_c + \log \zeta) + \cdots \\ &= \eta_0 + z_c^2\left(\eta_2 + \bar{\eta}_2 \log z_c\right)\zeta^2 + \bar{\eta}_2\, z_c^2\, \zeta^2 \log \zeta + \cdots \\ &= \hat{\eta}_0 + \hat{\eta}_2 \zeta^2 + \hat{\bar{\eta}}_2 \zeta^2 \log \zeta + \cdots \end{aligned} \tag{3.B.9}$$

Hence for η a naive linear regression in the ζ coordinates will fit the coefficients

$$\hat{\eta}_0 \equiv \eta_0 \tag{3.B.10a}$$

$$\hat{\eta}_2 \equiv z_c^2 \left(\eta_2 + \bar{\eta}_2 \log z_c\right) \tag{3.B.10b}$$

$\bar{\eta}_2$ is given in terms of η_0 by consistency, so we can invert these transformations to obtain η_0 and η_2, i.e.the expansion coefficients in the z coordinate system. These are the physically useful constituents for computing the Green's function.

We finally map these coefficients to the source and response in the 2-form picture via

$$p = 2\alpha \, \frac{z_c}{w} \, \eta_2 + J_{12} \log \bar{z}_* \tag{3.B.11a}$$

$$J = \alpha \, \frac{w}{z_c} \, \eta_0 \tag{3.B.11b}$$

3.B.3 Source-response method

To construct the Green's function we refer to the source-response picture, in which the Green's function is understood as acting on the source to produce a response. Labelling the fields as I, J, the sources as S_I and the responses as R_I, the components G_{IJ} of the Green's function are thus given by

$$G_{IJ} S_J = R_I \tag{3.B.12}$$

Hence for example,

$$G_{PB} S_B + G_{PP} S_P = R_P \tag{3.B.13}$$

To extract G_{PP}, we need to obtain two sets of the source and response data, which we label as $S_I^{(1)}$ and $S_I^{(2)}$, etc. We thus obtain a straightforward matrix equation

$$\begin{pmatrix} S_B^{(1)} & S_P^{(1)} \\ S_B^{(2)} & S_P^{(2)} \end{pmatrix} \begin{pmatrix} G_{PB} \\ G_{PP} \end{pmatrix} = \begin{pmatrix} R_P^{(1)} \\ R_P^{(2)} \end{pmatrix} \tag{3.B.14}$$

which we can trivially invert to find

$$G_{PP} = \frac{S_B^{(1)} R_P^{(2)} - S_B^{(2)} R_P^{(1)}}{S_B^{(1)} S_P^{(2)} - S_B^{(2)} S_P^{(1)}} \tag{3.B.15}$$

In our earlier notation, we have $S_B = b_{0,-}$, $R_B = b_{0,+}$, $S_P = p$, $R_P = J$ and $f_{JJ} = G_{PP}$. Hence running the numerical algorithm twice provides all the data we need to input into equation 3.B.15 to compute the Green's function of interest.

3.C Index of symbols

For the convenience of the reader, here we present a roughly alphabetical list of the symbols in this chapter, a brief description, and where it is first defined. As a rule, the subscript on a form indicates the degree of the form.

1. A_1: the usual DBI worldvolume gauge field living on the flavour brane. First appears in (3.1.1), where $F_2 = dA_1$.

2. $\mathcal{A}_2$: the magnetic dual of the 1-form field τ_1.

3. B_2: the NS-NS 2-form. First appears in (3.1.1).

4. C_2: the R-R 2-form. First appears in (3.1.1).

5. $\tilde{C}_1$: the magnetic dual of the R-R 2-form C_2. Defined in (3.2.1).

6. $f(z)$: the function describing how the brane caps off in the bulk. Defined in (3.1.6).

7. $f_{JJ}(w)$: the scalar function capturing the dependence of the symmetry current two-point function on w. Defined in (3.4.1).

8. F_2: the field strength of A_1. First appears in (3.1.1).

9. $\mathcal{F}_3$: the field strength of $\mathcal{A}_2$. Defined in (3.3.1).

10. G_3: the field strength of C_2. First appears in (3.1.1).

11. $\tilde{G}_2$: the field strength of $\tilde{C}_1$. Defined in (3.2.1).

12. $h(z)$: a function of f first defined in (3.2.4)

13. H_3: the field strength of B_2. First appears in (3.1.1).

14. m_F: the fermion mass, i.e.the mass gap at weak coupling. First appears in (3.1.12).

15. m_{meson}: the lightest meson mass, z_c^{-1}, i.e.the mass gap at strong coupling.

16. N_c: the number of colours of the gauge group.

17. N_f: the number of flavours of fundamental matter; the number of $D7$-branes in the bulk.

18. $\mathcal{P}_2$: the magnetic dual of η_1. Defined in (3.3.3).

19. $\mathcal{Q}_3$: the field strength of $\mathcal{P}_2$. Defined in (3.3.3).

20. R: the AdS radius. First appears in (3.1.7).

21. T_2: the field strength of τ_1. Defined in (3.2.6b).

22. w: the dimensionless number ωz_c. Defined in (3.4.22b).

23. Y_2: the field strength of η_1. First defined in (3.2.6a).

24. z: the radial AdS coordinate.

25. z_c: the value of z where the brane caps off. First appears in (3.1.6).

26. η_1: a combination of A_1 and $\tilde{C}_1$. Defined in (3.2.5a).

27. κ: Factor appearing in (3.1.1) equal to $\frac{4}{R}$. Defined in (3.1.5a).

28. μ: the ratio of mass contributions from A_1 and $\tilde{C}_1$. Defined in (3.1.5b).

29. τ_1: a combination of A_1 and $\tilde{C}_1$. Defined in (3.2.5b).

30. ζ: the dimensionless number z/z_c. Defined in (3.4.22a).

Chapter 4

Review of relativistic hydrodynamics

In this short chapter we provide a brief review of relativistic hydrodynamics and contrast it with the recent symmetry-based reformulations of magnetohydrodynamics and force-free electrodynamics. This will lay the groundwork for the following chapter in which we compute a transport coefficient in this framework from microscopics.

4.1 Relativistic hydrodynamics

An excellent review of relativistic hydrodynamics is given in [56]. The key points are as follows. As discussed in Chapter 2, relativistic theories in a flat spacetime have a translational symmetry, and hence a conserved stress tensor $T^{\mu\nu}$ satisfying

$$\partial_\mu T^{\mu\nu} = 0 \tag{4.1.1}$$

We can also add a further ingredient, namely a $U(1)$ ordinary global symmetry. By Noether's theorem, this guarantees a classically conserved 0-form current j satisfying

$$\partial_\mu j^\mu = 0 \tag{4.1.2}$$

In hydrodynamics, the relevant degrees of freedom are no longer the stress tensor $T^{\mu\nu}$ and the symmetry current j^μ, but rather the *hydrodynamic variables* of fluid velocity u^μ, inverse temperature β and chemical potential μ. We normalise the fluid velocity by $u^\mu u_\mu = -1$. The equations expressing the relationship between the microscopic $T^{\mu\nu}$ and j^μ in terms of the hydrodynamic variables are called *constitutive relations*.

In the *thermal equilibrium* state of the system, u^μ, β and μ are constant functions of the spacetime. Hydrodynamics is concerned with small fluctuations of the system around the thermal equilibrium, and hence we consider *derivative expansions* of the hydrodynamic variables. That is, we write the constitutive relations order-by-order in derivatives of u^μ, β and μ.

Physically, when no deviations from thermal equilibrium are allowed, we are in the regime of *ideal hydrodynamics* or *non-dissipative hydrodynamics*. This system can be solved in certain cases, but for general boundary conditions even guaranteeing the *existence* of a physical solution remains a deep open problem, see [57]. Here the constitutive relations are given simply by

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) \tag{4.1.3a}$$

$$j^\mu = n u^\mu \tag{4.1.3b}$$

where ϵ is the equilibrium energy density, p is the equilibrium pressure and n is the equilibrium charge density.

At non-trivial orders in derivatives, there are dissipative corrections to the constitutive relations. Hydrodynamics thus provides a systematic way to organise such corrections. From an effective field theory point of view, the coefficients of such corrections, called *transport coefficients*, must be determined separately from the underlying microscopic theory. At first order in conventional hydrodynamics these transport coefficients are viscosities.

In quantum field theory we can use a *Kubo formula* to compute transport coefficients from correlation functions involving j^μ and $T^{\mu\nu}$. This is the approach adopted in Chapter 5. The Kubo formulas themselves are universal in the sense that they are independent of the specific microscopics. The transport coefficient is only determined when the precise microscopics are specified.

4.2 Magnetohydrodynamics

In conventional magnetohydrodynamics (MHD), there is an additional ingredient provided by a dynamical electromagnetic field. The equations of magnetohydrodynamics are used extensively to understand astrophysical plasmas. See [58] for a discussion of relativistic MHD.

From the gauge field strength and the Levi-Civita tensor, we can construct a 2-form current $J^{\mu\nu}$.

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{4.2.1}$$

From a symmetries perspective, there is in fact no longer an ordinary $U(1)$ global symmetry, but instead a 1-form $U(1)$ global symmetry, providing a conserved 2-form current $J^{\mu\nu}$ satisfying

$$\partial_\mu J^{\mu\nu} = 0 \tag{4.2.2}$$

Physically, the 1-form symmetry is related to the conservation of magnetic flux lines. Strongly-coupled MHD can be constructed as a hydrodynamic theory from first principles, as in [10]. As before, we have a conserved stress tensor $T^{\mu\nu}$.

The constitutive relations at ideal order are

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} - \mu\rho\, h^\mu h^\nu \tag{4.2.3a}$$

$$J^{\mu\nu} = \rho(u^\mu h^\nu - u^\nu h^\mu) \tag{4.2.3b}$$

where h^μ is a spacelike vector pointing in the direction of the magnetic field lines. We normalise $h^\mu h_\mu = 1$.

Crucially, the constitutive relations break Lorentz invariance by fixing some preferred choice of direction h^μ along the field lines. This distinguishes MHD from force-free electrodynamics which we discuss below.

4.3 Force-free electrodynamics

4.3.1 FFE in astrophysics

In astrophysics, *force-free electrodynamics* (FFE) is conventionally thought of as a description of a particular regime of a strongly magnetised plasma, one where free electric charges are sufficiently plentiful to screen the electric field to zero, but sufficiently dilute that their stress-energy may be ignored in comparison to the energy stored in the electromagnetic field. As conventionally formulated, FFE comprises a self-consistent set of equations describing the non-linear dynamics of the magnetic field itself [59, 60] (see [61] for a review). One can imagine that the FFE equations of motion are a sort of Navier-Stokes equations for the magnetic field.

In components, the FFE equations of motion (in a curved spacetime) are given by

$$\nabla_{[\mu} F_{\rho\sigma]} = 0 \tag{4.3.1a}$$

$$F_{\sigma\nu} \nabla_\mu F^{\mu\nu} = 0 \tag{4.3.1b}$$

This theory is conventionally used for a coarse-grained description of the magnetospheres of astrophysical objects. It is important to note however that the usual formulation of FFE imposes the degeneracy condition

$$F \wedge F = 0 \tag{4.3.2}$$

or equivalently in components

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \epsilon_{ijk}F_{0i}F_{jk} = \vec{E}\cdot\vec{B} = 0 \tag{4.3.3}$$

There is thus never an electric field *along* a magnetic field line. We now note that the Lorentz force means that stray charged particles are essentially confined to move along magnetic field lines. This means that there is no possibility for an accelerating electric field, i.e. an electric field that accelerates charged particles to high energies.

A typical application of FFE pertains to the study of neutron stars, in particular to pulsars. Pulsars exhibit a rich phenomenology, including coherent emission of radio waves and the acceleration of particle winds. From the above, it appears that FFE alone does accommodate a natural mechanism to accelerate particles into jets or cosmic rays. Further, it appears that FFE has no built-in length scales; these two features mean that FFE does not seem to account in a simple manner for the full plethora of observed phenomena associated with pulsars. [62, 63, 64]. We will discuss these consequences in further detail in Chapter 5.

As a theory then, (ideal) FFE can be considered incomplete. Theoretically, the conventional formulation of FFE does not make clear in precisely what sense it is an approximation, or how one might systematically improve on the approximation in some small parameter.

4.3.2 Generalised FFE

From the point of view of higher-form symmetries, FFE (in the astrophysics literautre) can be considered as the ideal (no dissipation) limit of a hydrodynamic theory, which we will call *generalised FFE* or confusingly, FFE. Recently, FFE was reformulated from the point of view of effective field theory in [65]. This reformulation was made possible by the identification of a higher-form symmetry that is associated with the conservation of magnetic flux. Importantly, this reformulation makes clear that FFE is an expansion in powers of derivatives, essentially placing it on the same footing as conventional hydrodynamic theories [56]. It also makes it possible to systematically incorporate higher-derivative corrections.

We first present an extremely brief overview of the formalism, assuming that the reader is already somewhat familiar with [65]. We work in terms of $J^{\mu\nu}$, which is related to the conventional electromagnetic field strength as

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{4.3.4}$$

As explained in detail in [6], this is the conserved current for the 1-form symmetry associated with magnetic flux conservation. It was shown in [10] that this symmetry principle is useful for organising magnetohydrodynamics. In [65] it was further shown that a formulation of FFE exists where the magnetic field and stress tensor can be conveniently expressed in terms of a 2-form $u^{\rho\sigma}$ which can be thought of as the volume form of the dynamical magnetic field. To leading order in derivatives, one finds

$$J^{\mu\nu} = Bu^{\mu\nu} \tag{4.3.5a}$$

$$T^{\mu\nu} = B^2\left(\frac{1}{2}g^{\mu\nu} + \Omega^{\mu\nu}\right) \tag{4.3.5b}$$

where here $\Omega_{\mu\nu} = u_{\mu\rho}u^{\rho}_{\nu}$ is a projector onto the worldsheet of the magnetic field lines, B is the magnitude of the magnetic field, and we have made a choice of an "equation of state". (Note that in this formalism B should be viewed as a kind of "thermodynamic variable", as explained in [65]). $u^{\mu\nu}$ is not an arbitrary 2-form; it has a fixed magnitude and satisfies a particular degeneracy relation $u \wedge u = 0$.

Conservation of the 2-form current and the stress tensor is equivalent to the field equations of motion for FFE given in (4.3.1). This is the usual expression for FFE; however within the effective field theory framework it is only the first term in an infinite series of higher-derivative corrections.

One interesting comparison between the hydrodynamic theories discussed so far emerges from the contrast between the various corresponding constitutive relations (4.1.3), (4.2.3) and (4.3.5).

4.3.3 Higher-derivative corrections

In this thesis, we focus on a particular higher-derivative correction, one which was shown in [66] to produce a correction with $\vec{E} \cdot \vec{B} \neq 0$, and hence a non-trivial accelerating electric field, in a typical (toy model of a) pulsar geometry. We present the first microscopic computation of such a higher derivative correction in Chapter 5 by computing a one-loop diagram in QED.

The possible set of leading higher-derivative corrections was classified in [66]. In the following chapter we focus on one particular correction β_2 from that analysis, which leads to a correction of the form:

$$J^{\mu\nu} = Bu^{\mu\nu} - 3\nabla_\sigma\left(\beta_2(B)\nabla_{[\alpha}u_{\beta\gamma]}\Omega^{\alpha\sigma}\Pi^{\beta\mu}\Pi^{\nu\gamma}\right) + \cdots \tag{4.3.6}$$

where here $\Pi^{\mu\nu} = g^{\mu\nu} - \Omega^{\mu\nu}$ is an orthogonal projector. The addition of this term is conceptually similar to adding e.g. viscosity to ideal hydrodynamics. We will thus call β_2 a *transport coefficient*, where we borrow the term from hydrodynamics. (As we will see, the Lorentz-invariance of the FFE effective theory gives it a rather different character to more familiar transport coefficients such as viscosity).

The key point here is that once such a term is included, it is no longer generically true that $J \wedge J \sim \vec{E} \cdot \vec{B} = 0$. The generic existence of accelerating electric fields now leads to the tremendously exciting possibility of observational consequences. In fact, it was shown in [66] that the term shown above is generically active on a toy model of a pulsar geomtery. In the following chapter, we compute $\beta_2(B)$ from first principles.

Chapter 5

Application 2: Force-free electrodynamics

In this chapter, we compute the transport coefficient associated with a particular higher-derivative correction to force-free electrodynamics. Concretely, this is a perturbative QED calculation involving fermions in a background magnetic field.

The structure is as follows. In Section 5.1 we give an overview of the perturbative computation including the relevant Feynman diagrams. In Section 5.2 we compute the transport coefficient for a toy model consisting of a complex scalar. Then in Section 5.3 we use a similar but more complicated method to compute the transport coefficient for the physical model which consists of a Dirac fermion. Finally in Section 5.4 we compare the results of the calculation for the physical model with appropriate astrophysical observations and discuss the implications. Many further details of the computations, including our choice of conventions, can be found in the appendix to this chapter.

5.1 Overview of calculation

As usual, to calculate a transport coefficient from a micrsocopic description we should use a Kubo formula (see [56] for a review of the formalism in conventional hydrodynamics). For FFE the relevant Kubo formulas were recently derived in [66]; that for β_2 is given by

$$\beta_2 = -\left[\frac{\partial^2}{\partial p_1^2}\ \tilde{G}^{02,23}_{JT}(p_1)\right]\Bigg|_{p_1=0} = i\left[\frac{\partial^2}{\partial p_1^2}\ \langle \tilde{J}^{02}(p^1)\tilde{T}^{23}(-p^1)\rangle\right]\Bigg|_{p_1=0} \tag{5.1.1}$$

This is written in a non-covariant manner; the Kubo formula was derived from fluctuations about a homogenous magnetic field pointing in the 3 direction, i.e. in equilibrium we have $J^{03} = B$, where B is the background magnetic field. p_1 is a transverse spatial momentum. With our conventions, the Green's function is related to the two-point function by

$$i\, G_{JT}^{\mu\nu,\rho\sigma}(x,y) \equiv \langle J^{\mu\nu}(x) T^{\rho\sigma}(y)\rangle \tag{5.1.2}$$

We note that J is odd under charge conjugation symmetry, which also acts on the background magnetic field as $B \to -B$; thus this correlator clearly vanishes in the Lorentz-invariant vacuum where $B = 0$. The final answer for β_2 will thus be odd under $B \to -B$, as anticipated in [65]. While we will verify this explicitly in what follows.

We will calculate this correlation function at one-loop order for both a complex scalar and a Dirac fermion. We note that in general it is difficult to compute transport coefficients from perturbative quantum field theory, as the hydrodynamic limit typically does not commute with the weak-coupling limit, which manifests itself in the need to sum infinitely many Feynman diagrams. However in this case this particular correlation function is a *static* correlator, evaluated at $\omega = 0$. In some ways, this perhaps more akin to a correction to thermodynamics than a dynamical transport coefficient. We will nevertheless continue to call it a transport coefficient in this current work, and we will initiate its study by computing it to leading order in perturbation theory, and in the conclusion we will discuss the validity of this calculation.

The general method is as follows. First we calculate the Fourier transform of the correlation function $\langle \mathcal{T} J^{02}(x) T^{23}(y)\rangle$ with momentum purely in a spatial direction transverse to the magnetic field, and then take partial derivatives with respect to this momentum. As T is a bilinear in the fundamental fields, this correlation function itself can be constructed by calculating a three-point function of fundamental fields and applying appropriate spatial derivatives and integrals. For the complex scalar we will use the three-point function $\langle \mathcal{T} A_\mu(x)\phi^\dagger(y)\phi(z)\rangle$ and for the Dirac fermion we will use the three-point function $\langle \mathcal{T} A_\mu(x)\bar{\psi}(y)\gamma_\nu\psi(z)\rangle$.

To calculate $\langle \mathcal{T} J^{02}(x) T^{23}(y)\rangle$ we take appropriate spatial derivatives of the relevant three-point function and then sew together the ends of the diagram to make a loop, bringing y and z to a single point. This sewing process is illustrated by Figures 5.1 and 5.2.

By building the 2-form current J and the stress tensor T out of the fundamental fields and using a modified form of the propagator for the matter fields, we can perform a one-loop calculation to obtain the transport coefficient. For the scalar field, the only non-vanishing contribution comes from the so-called Schwinger phase of the

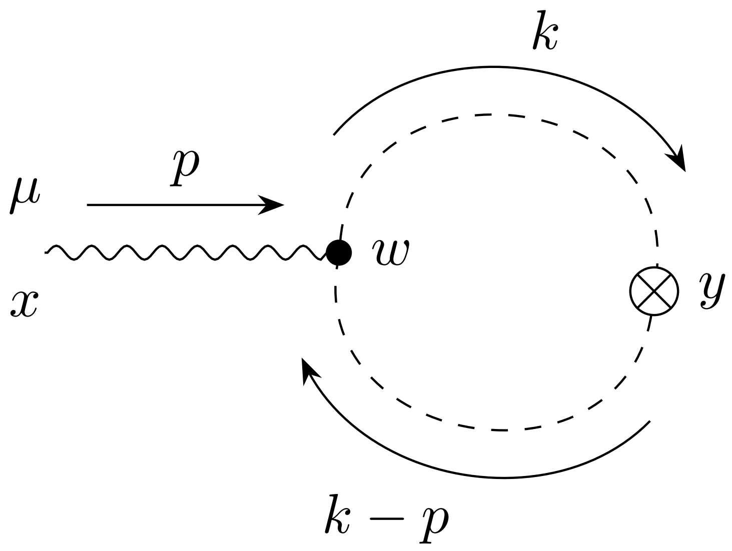

Figure 5.1: Feynman diagram for $\langle \mathcal{T} J^{02}(x) T^{23}_{(S)}(y) \rangle$ formed by sewing together the ends of the three-point function $\langle \mathcal{T} A_\mu(x) \phi^\dagger(y) \phi(z) \rangle$

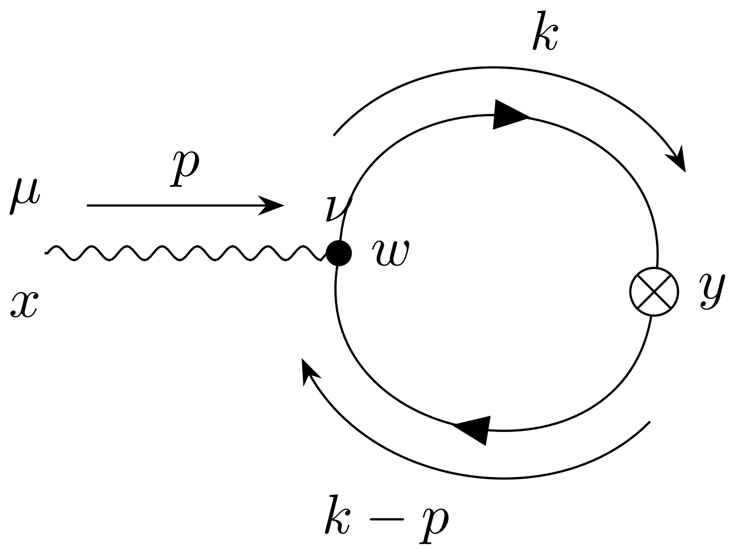

Figure 5.2: Feynman diagram for $\langle \mathcal{T} J^{02}(x) T^{23}_{(F)}(y) \rangle$ formed by sewing together the ends of the three-point function $\langle \mathcal{T} A_\mu(x) \bar{\psi}(y) \gamma_\nu \psi(z) \rangle$

modified scalar propagator. The Schwinger phase breaks the transverse translational symmetry of the propagator. In the fermion case, as well as the Schwinger phase we also have contributions to the transport coefficient which can be physically understood to emerge from the intrinsic spin of the electron.

A key component of the calculation is the gauge-invariant regularisation procedure. We will regularise a 2-dimensional loop integral over the longitudinal momenta using (a method similar to) the Pauli-Villars procedure.

The computation yields a fully analytic formula for the transport coefficient in terms of the dimensionless number $b \equiv B/B_{cr}$, where $B_{cr} = m^2/e$ is the critical magnetic field strength of the particle (complex scalar or Dirac fermion). The transport coefficient is an odd function of b, with an asymptotic expansion for small b given by

$$\beta_2^{(S)} = \frac{e}{120\pi^2} \frac{B}{B_{cr}^{(S)}} + \mathcal{O}\left[\left(\frac{B}{B_{cr}^{(S)}}\right)^3\right] \tag{5.1.3}$$

for the complex scalar, and

$$\beta_2^{(F)} = -\frac{e}{240\pi^2} \frac{B}{B_{cr}^{(F)}} + \mathcal{O}\left[\left(\frac{B}{B_{cr}^{(F)}}\right)^3\right] \tag{5.1.4}$$

for the Dirac fermion.

We will adopt a notation where i, j range over $1, 2$ - the *transverse directions* - and a, b range over $0, 3$ - the *longitudinal directions*. Alternatively we may write the components in pairs as $x_\perp \equiv (x^1, x^2) \equiv x^i$ and $x_\parallel = (x^0, x^3) \equiv x^a$.

5.2 Scalar field

An outline of the calculation follows. First we will introduce a modified version of the scalar propagator in the presence of a background magnetic field. Next we will explain how to construct the current stress tensor Green's function $\tilde{G}^{02,23}_{JT}(p_1)$ from a loop diagram involving these propagators. This provides the input to the Kubo formula for the transport coefficient $\beta_2^{(S)}$. Finally we will regularise the loop integral in a gauge-invariant manner to analytically evaluate the transport coefficient and present the result. Supplementary details and derivations can be found in Appendix 5.B.

5.2.1 Modified scalar propagator

Following the conventions of [3], we define the free Feynman propagator for the scalar in a vacuum background to be

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon} \tag{5.2.1}$$

This is a Green's function for the Klein-Gordon equation, ie the classical equation of motion for a free scalar field:

$$(-\Box^{(x)} + m^2)\, \Delta(x-y) = \delta^{(4)}(x-y) \tag{5.2.2}$$

where $\Box \equiv \partial_\mu \partial_\nu g^{\mu\nu}$ and the superscript (x) emphasises that derivatives are taken with respect to the position x.

In ordinary scalar QED calculations with a vacuum background, this is the propagator used to evalute Feynman diagrams. However, in this work we impose a fixed background magnetic field of strength $F_{12} = B_3 = B$ oriented along the positive 3-direction. In the calculation that follows, we will assume that $B > 0$, while simultaneously noting which aspects of the final answer depend on the sign of B. Adapting a calculation in the appendix of [67], we can find a modified scalar propagator in the presence of such a background magnetic field. As is conventional, we fix the background field to be in Landau gauge given by $A_2 = Bx^1$. Writing $D = d - ieA$ for the covariant derivative associated with the background gauge field, the Green's equation for the modified propagator is

$$\left(-g^{\mu\nu} D_\mu^{(x)} D_\nu^{(x)} + m^2\right) G^{(\phi)}(x, y) = \delta^{(4)}(x - y) \tag{5.2.3}$$

where the superscript (ϕ) emphasises that we are working with a scalar. The presence of the magnetic field together with the charge of the scalar introduces a magnetic length scale l defined by

$$l = (eB)^{-1/2} \tag{5.2.4}$$

It is convenient to Fourier transform the longitudinal directions $x_\parallel$ and leave the transverse direction $x_\perp$ alone as follows

$$G^{(\phi)}(p_\parallel; x_\perp, y_\perp) = \int d^2 x_\parallel \; e^{-ip_\parallel \cdot (x_\parallel - y_\parallel)} \; G^{(\phi)}(x, y) \tag{5.2.5}$$

We can then solve the Green's equation to obtain

$$G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) = \frac{1}{2\pi} e^{i\Phi(x_\perp, y_\perp)} e^{-\frac{1}{2}\xi} \sum_{n=0}^{\infty} \frac{L_n(\xi)}{l^2(m^2 + p_\parallel^2) + 2n + 1} \tag{5.2.6}$$

where Φ is the so-called Schwinger phase defined by

$$\Phi(x_\perp, y_\perp) = \frac{1}{2l^2}(x^1 + y^1)(x^2 - y^2) \tag{5.2.7}$$

and the L_n are orthogonal Laguerre polynomials. The variable ξ is defined by

$$\xi = \frac{1}{2l^2} |x_\perp - y_\perp|^2 \tag{5.2.8}$$

For a detailed derivation of this propagator see Appendix 5.B.1.

We interpret the sum over n as a sum over Landau levels.

If we flip the orientation of the magnetic field $B \mapsto -B$, then the Green's function simply maps to its complex conjugate. In particular, the only change is the change of sign of the Schwinger phase $\Phi(x_\perp, y_\perp) \mapsto -\Phi(x_\perp, y_\perp) = \Phi(y_\perp, x_\perp)$. As anticipated around (5.1.2), our final answer should be odd under $B \to -B$; thus we expect that in the scalar field case, the only non-vanishing contribution to $\beta_2^{(S)}$ arises from the Schwinger phase, which is the only ingredient that is sensitive to the sign of B. We will see this explicitly from the calculation below.

Further, it is often helpful to Fourier transform the translationary-invariant part of the propagator, i.e. the propagator without the Schwinger phase included:

$$\tilde{G}^{(\phi)}(p) \equiv \int d^2x_\perp \, e^{-ip_\perp \cdot (x_\perp - y_\perp)} \, e^{-i\Phi} \, G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) = 2l^2 e^{-p_\perp^2 l^2} \sum_{n=0}^{\infty} \frac{(-1)^n L_n(2p_\perp^2 l^2)}{l^2(m^2 + p_\parallel^2) + 2n + 1} \tag{5.2.9}$$

See Appendix 5.D.2 for the relevant details concerning Fourier transforms of Laguerre polynomials.

5.2.2 Computation of Feynman diagram

Construction of current stress tensor correlator

Now that we have a modified form of the scalar propagator, we need to derive an expression for the current stress tensor correlator using the usual Feynman diagram formalism.

The 2-form current associated with the higher-form symmetry is simply the Poincaré dual of the Maxwell field strength

$$J = \star F \tag{5.2.10}$$

and so in components we have

$$J^{02}(x) = F_{13}(x) = \partial_1^{(x)} A_3(x) - \partial_3^{(x)} A_1(x) \tag{5.2.11}$$

The relevant component of the scalar stress tensor can be readily constructed in the canonical way as

$$T^{23}_{(S)}(y) = \partial_2^{(y)} \phi^\dagger(y) \, \partial_3^{(y)} \phi(y) + \partial_3^{(y)} \phi^\dagger(y) \, \partial_2^{(y)} \phi(y) \tag{5.2.12}$$

Hence we can write the time-ordered current stress tensor correlator[1] in terms of three-point functions of the fundamental fields, where we sew two of the ends together

[1] Note that Kubo formulas in conventional hydrodynamics are usually formulated in terms of the retarded correlator; however as this particular Kubo formula is evaluated at $\omega = 0$, the time-ordered and retarded correlators coincide.

to form a loop as in Figure 5.1.

$$\begin{aligned}&\langle \mathcal{T} J^{02}(x) T^{23}_{(S)}(y)\rangle \\ &= \left[\left(\partial_2^{(y)}\partial_3^{(z)} + \partial_3^{(y)}\partial_2^{(z)}\right)\left(\partial_1^{(x)}\langle \mathcal{T} A_3(x)\phi^\dagger(y)\phi(z)\rangle - \partial_3^{(x)}\langle \mathcal{T} A_1(x)\phi^\dagger(y)\phi(z)\rangle\right)\right]_{z=y}\end{aligned} \tag{5.2.13}$$

The Feynman diagram associated with the three-point function can be found in Figure 5.3.

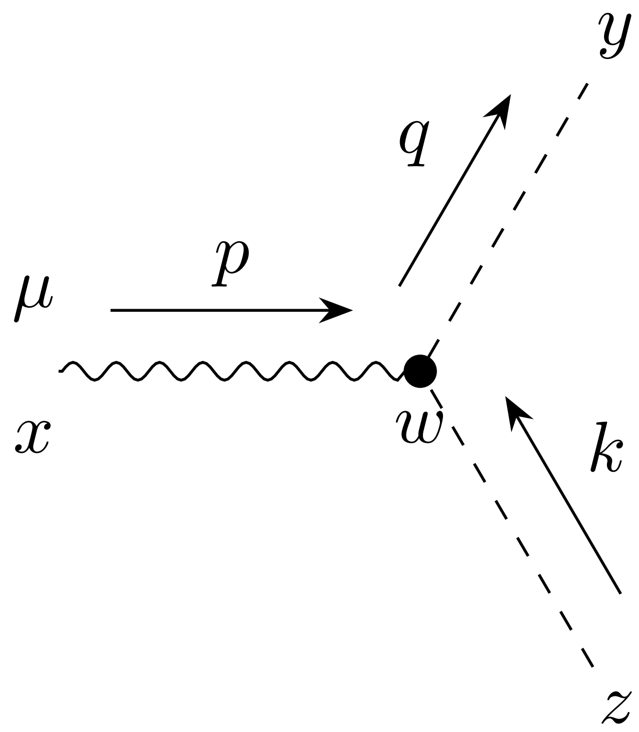

Figure 5.3: Feynman diagram for $\langle \mathcal{T} A_\mu(x)\phi^\dagger(y)\phi(z)\rangle$ as used in the complex scalar calculation

From the Feynman rules, the three-point function $\langle \mathcal{T} A_\mu(x)\phi^\dagger(y)\phi(z)\rangle$ is given by

$$\begin{aligned}&\langle \mathcal{T} A_\nu(x)\phi^\dagger(y)\phi(z)\rangle \\ &= -ie\, g^{\rho\sigma}\int d^4w\, G^{(\gamma)}_{\nu\rho}(x,w)\left[G^{(\phi)}(z,w)\partial_\sigma^{(w)}G^{(\phi)}(w,y) - G^{(\phi)}(w,y)\partial_\sigma^{(w)}G^{(\phi)}(z,w)\right]\end{aligned} \tag{5.2.14}$$

since each propagator appears with a factor of $-i$ and the vertex appears with a factor of $-e$. Here $G^{(\gamma)}_{\mu\nu}(x,w)$ denotes the ordinary photon propagator in position space. To lowest order in the coupling this photon propagator is not modified by the magnetic field from its familiar vacuum value; in the conclusion we discuss how one could systematically improve on this result.

For ease of calculation and writing it is useful to separate the photon propagator from the scalar propagators by defining

$$V_\sigma(w,y,z) \equiv G^{(\phi)}(z,w)\ \partial_\sigma^{(w)} G^{(\phi)}(w,y) - G^{(\phi)}(w,y)\ \partial_\sigma^{(w)} G^{(\phi)}(z,w) \tag{5.2.15}$$

so that the three-point function has a neat form.

$$\langle \mathcal{T} A_\nu(x) \phi^\dagger(y) \phi(z) \rangle = -ie\ g^{\rho\sigma} \int d^4w\ G^{(\gamma)}_{\nu\rho}(x,w)\ V_\sigma(w,y,z) \tag{5.2.16}$$

We can further define

$$U_\sigma(w,y,z) \equiv \left(\partial_2^{(y)} \partial_3^{(z)} + \partial_3^{(y)} \partial_2^{(z)}\right) V_\sigma(w,y,z) \tag{5.2.17}$$

and thus succinctly construct the gauge-field stress tensor correlator.

$$\langle \mathcal{T} A_\nu(x) T^{23}_{(S)}(y) \rangle = -ie\ g^{\rho\sigma} \int d^4w\ G^{(\gamma)}_{\nu\rho}(x,w)\ U_\sigma(w,y,y) \tag{5.2.18}$$

Next we differentiate with respect to x^μ, take a Fourier transform and restrict the momentum to the 1-direction. The resulting expression is indepdendent of the gauge choice for the photon propagator, so for simplicity we can choose the Feynman gauge where we have

$$\tilde{G}_{\mu\nu}(p^1) = \frac{g_{\mu\nu}}{(p^1)^2 - i\epsilon} \tag{5.2.19}$$

We can now write the following compact expression for the Fourier-transformed current stress tensor correlator.

$$\langle \tilde{J}^{02}(p^1) \tilde{T}^{23}_{(S)}(-p^1) \rangle = \frac{e}{p^1} \int d^4w\ e^{-ip_1 w^1}\ U_3(w,0,0) \tag{5.2.20}$$

A short manipulation in momentum space shows that U_3 receives contributions from spatial derivatives of the Schwinger phase as well as plane waves.

$$U_3(w,0,0) = i \int \widetilde{dq}\ \widetilde{dk}\ \tilde{G}^{(\phi)}(q)\ \tilde{G}^{(\phi)}(k)\ e^{iw\cdot(q-k)}\ (q_3+k_3) \left(q_2 k_3 + k_2 q_3 + \frac{w^1(q_3+k_3)}{2l^2} \right) \tag{5.2.21}$$

Hence after integrating out the longitudinal position $w_\parallel$ and the longitudinal momentum $k_\parallel$, the Green's function relating the current and the stress tensor is

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{2e}{l^2 p^1} \int d^2w_\perp\ \widetilde{dq}\ \widetilde{dk_\perp}\ e^{-ip_1 w^1} e^{iw_\perp\cdot(q_\perp - k_\perp)}\ G^{(\phi)}(q_\perp, q_\parallel)\ G^{(\phi)}(k_\perp, q_\parallel)\ q_3^2\ w^1 \tag{5.2.22}$$

Substituting the expression for the Fourier-transform of the translationally-invariant part of the scalar propagator from equation 5.2.9 we obtain a longitudinal loop integral for the Green's function.

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{e}{2\pi^2 l^2 p^1}\int \widetilde{dq_\parallel}\; q_3^2 \int d^2w_\perp\; w^1\; e^{-ip_1 w^1} e^{-w_\perp^2/2l^2} \sum_{n,n'=0}^{\infty} \frac{L_n(w_\perp^2/2l^2)L_{n'}(w_\perp^2/2l^2)}{\lambda_n(q_\parallel)\lambda_{n'}(q_\parallel)} \tag{5.2.23}$$

where the denominators inside the series are given by

$$\lambda_n(q_\parallel) = l^2(m^2 + q_\parallel^2) + 2n + 1 \tag{5.2.24}$$

Note that the loop is given in a mixed representation; due to the lack of translational invariance from the Schwinger phase, we perform the integral over the transverse directions in position space, though we do the integral over the longitudinal direction in momentum space.

Evaluation of loop integral

The Kubo formula for the transport coefficient $\beta_2^{(S)}$ contains derivatives of $\tilde{G}^{02,23}_{JT}(p_1)$ with respect to p_1. Here we use the Kubo formula to write $\beta_2^{(S)}$ in terms of a longitudinal loop integral. If we change variables to plane polar coordinates for $w_\perp$ then the dependence on p_1 takes a simple form.

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{e}{\pi l^2 p^1}\int \widetilde{dq_\parallel}\; q_3^2 \int_0^\infty dr\; r^2 e^{-\frac{r^2}{2l^2}} \sum_{n,n'=0}^{\infty} \frac{L_n(\frac{r^2}{2l^2})L_{n'}(\frac{r^2}{2l^2})}{\lambda_n(q_\parallel)\lambda_{n'}(q_\parallel)} \int_0^{2\pi}\frac{d\theta}{2\pi}\; \cos\theta\; e^{-ip_1 r\cos\theta} \tag{5.2.25}$$

Note that the azimuthal integral over θ is a standard representation of the Bessel function of the first kind.

$$\int_0^{2\pi}\frac{d\theta}{2\pi}\;\cos\theta\; e^{-ip_1 r\cos\theta} = -i\; J_1(p_1 r) \tag{5.2.26}$$

The Bessel function admits the Maclaurin expansion

$$\frac{J_1(p_1 r)}{p^1} = \frac{r}{2} - \frac{r^3 p_1^2}{16} + \mathcal{O}(p_1^4) \tag{5.2.27}$$

Hence applying the Kubo formula in Eq. 5.1.1 to the current stress tensor Green's function and changing variables to $x = r^2/2l^2$ gives

$$\beta_2^{(S)} = -\frac{iel^4}{2\pi}\int \widetilde{dq_\parallel}\; q_3^2 \int_0^\infty dx\; x^2\; e^{-x} \sum_{n,n'=0}^{\infty} \frac{L_n(x)L_{n'}(x)}{\lambda_n(q_\parallel)\lambda_{n'}(q_\parallel)} \tag{5.2.28}$$

The integral over x can be related to the inner-product structure of generalised Laguerre polynomials. If we define

$$I_{n,n'} \equiv \int_0^\infty dx\; x^2\; e^{-x} L_n(x)\; L_{n'}(x) \tag{5.2.29}$$

then we can write $I_{n,n'}$ in terms of quadratic polynomials in n as follows.

$$I_{n,n} = f_0(n) \tag{5.2.30a}$$

$$I_{n,n+1} = f_1(n) \tag{5.2.30b}$$

$$I_{n,n+2} = f_2(n) \tag{5.2.30c}$$

$$I_{n,n'} = 0, \quad |n-n'| > 2 \tag{5.2.30d}$$

(A limited number of off-diagonal entries in this inner product are nonzero due to the factor of x^2; see Appendix 5.D.1 for further details on this derivation and Appendix 5.D.1 for quick reference of explicit expressions for the polynomials). This structure allows us to perform one of the sums over n', and gives

$$\beta_2^{(S)} = -\frac{iel^4}{2\pi}\int \widetilde{dq_\parallel}\, q_3^2 \sum_{n=0}^{\infty}\left[\frac{f_0(n)}{\lambda_n(q_\parallel)^2} + \frac{2f_1(n)}{\lambda_n(q_\parallel)\lambda_{n+1}(q_\parallel)} + \frac{2f_2(n)}{\lambda_n(q_\parallel)\lambda_{n+2}(q_\parallel)}\right] \tag{5.2.31}$$

After Wick-rotating and rescaling the longitudinal momentum integral by l, we can finally write

$$\beta_2^{(S)} = \frac{e}{16\pi^2}\int_0^\infty dx\; x\; S^{(\phi)}(x;\mu) \tag{5.2.32}$$

where $\mu = (ml)^2$ is a dimensionless number and $S^{(\phi)}$ is a series given by

$$S^{(\phi)}(x;\mu) = \sum_{n=0}^{\infty}\frac{1}{\mu+x+2n+1}\left[\frac{f_0(n)}{\mu+x+2n+1} + \frac{2f_1(n)}{\mu+x+2n+3} + \frac{2f_2(n)}{\mu+x+2n+5}\right] \tag{5.2.33}$$

Regularisation

We can now naively explicitly perform the sum above to obtain

$$S^{(\phi)}(x;\mu) = \frac{1}{8}\left\{\left[3\left(\mu+x\right)^2+1\right]\psi^{(1)}\left(\frac{1}{2}\left(\mu+x+1\right)\right) - 2\left(3\mu+3x+2\right)\right\} \tag{5.2.34}$$

where $\psi^{(1)}$ is the polygamma function. However, when this is inserted into the expression for $\beta_2^{(S)}$ we obtain a quartic (in momentum) UV divergence; the integral over x (i.e. the squared Euclidean norm of the Wick-rotated longitudinal loop momentum) is not well-defined. Hence our previous expressions for β_2 only make sense if it is understood that we need to regulate the divergence.

This divergence arises from our regularisation procedure; a hard cutoff on the longitudinal loop momentum is not gauge-invariant. We need to regularise the loop integral using a scheme which preserves gauge-invariance. A straightforward choice is the Pauli Villars procedure. To review, we introduce a heavy fictitious ghost particle of mass $\Lambda \gg m$ by replacing the scalar propagator $G^{(\phi)}(x, y; m)$.

$$G^{(\phi)}(x, y; m) \to G^{(\phi)}(x, y; m) - G^{(\phi)}(x, y; \Lambda) \tag{5.2.35}$$

For finite Λ the integral is thus finite by construction; as Λ is removed we return to the original problem, and thus Λ is a gauge-invariant regulator.

In principle we would have to repeat all of the previous steps of the loop integral calculation. However it is straightforward to see that this is equivalent to making the following replacement in the expression for the transport coefficient.

$$\frac{1}{\lambda_n(q_\parallel; m)} \to \frac{l^2(\Lambda^2 - m^2)}{\lambda_n(q_\parallel; m)\ \lambda_n(q_\parallel; \Lambda)} \tag{5.2.36}$$

That is, for each Landau level we suppress the propagator by $l^2\left(q_\parallel^2 + \Lambda^2\right) + 2n + 1$ in exchange for a heavy mass factor in the numerator. This allows us to regularise the longitudinal momentum integrand as

$$\begin{aligned}
&\hat{S}^{(\phi)}(x; \mu, \Lambda) \\
&= \sum_{n=0}^{\infty} \left\{ \frac{(l^2\Lambda^2 - \mu)^2}{(\mu + x + 2n + 1)(l^2\Lambda^2 + x + 2n + 1)} \left[\frac{2 + 6n + 6n^2}{(\mu + x + 2n + 1)(l^2\Lambda^2 + x + 2n + 1)} \right. \right. \\
&\quad \left. \left. + \frac{2(-4 - 8n - 4n^2)}{(\mu + x + 2n + 3)(l^2\Lambda^2 + x + 2n + 3)} + \frac{2(2 + 3n + n^2)}{(\mu + x + 2n + 5)(l^2\Lambda^2 + x + 2n + 5)} \right] \right\}
\end{aligned} \tag{5.2.37}$$

Here we have written a hat over S to emphasise the regularisation. This sum can be explicitly evaluated, but we omit the full expression here for brevity. After performing the sum over n, we now take the limit $\Lambda \to \infty$ (for fixed $l > 0$) to remove the regulator and recover the corrected series $\hat{S}$.

$$\hat{S}^{(\phi)}(x; \mu) = \frac{1}{8} \left\{ \left[3\,(\mu + x)^2 + 1 \right] \psi^{(1)}\left(\frac{1}{2}\,(\mu + x + 1) \right) - 6\,(\mu + x) \right\} \tag{5.2.38}$$

We now obtain the transport coefficient $\beta_2^{(S)}$ by performing the remaining integral:

$$\beta_2^{(S)} = \frac{e}{16\pi^2} \int_0^\infty dx\ x\ \hat{S}^{(\phi)}(x; \mu) \tag{5.2.39}$$

which – unlike for the hard cutoff used earlier – is UV-finite.

5.2.3 Results

So far we have worked with the dimensionless quantity $\mu = m^2 l^2 = m^2/(eB)$. When presenting the results, it is convenient to introduce the *critical magnetic field* B_{cr}

which is defined by

$$B_{cr} = \frac{m^2}{e} \tag{5.2.40}$$

Physically this corresponds to the magnetic field strength at which the energy gap between adjacent Landau levels is the same order as the particle's rest energy [64]. This allows us to work with a different dimensionless quantity $b \equiv B/B_{cr} = \mu^{-1}$.

After removing the Pauli-Villars regulator, the regularised transport coefficient for the scalar field is given in terms of b by the lengthy but fully analytic formula

$$\beta_2^{(S)} = \frac{e}{128\pi^4}\Bigg[-\frac{\pi^2}{b^3}\Big(b^2(144b\log(A) + 48\log(A) + 5b\log(16) + 4(5b+3)\log(\pi) + 3 + \log(4096)) \\ - 4\left(b^2+3\right) b \log\Gamma\left(\frac{1}{2}\left(1+\frac{1}{b}\right)\right) + 96b^2\left(\psi^{(-2)}\left(\frac{b+1}{2b}\right) - 3b\psi^{(-3)}\left(\frac{b+1}{2b}\right)\right) + 1\Big) - 36\zeta(3)\Bigg] \tag{5.2.41}$$

where $A \approx 1.28$ is the Glaisher-Kinkelin constant and we have analytically continued the index of the polygamma function.

The transport coefficient is plotted against b in Figure 5.4.

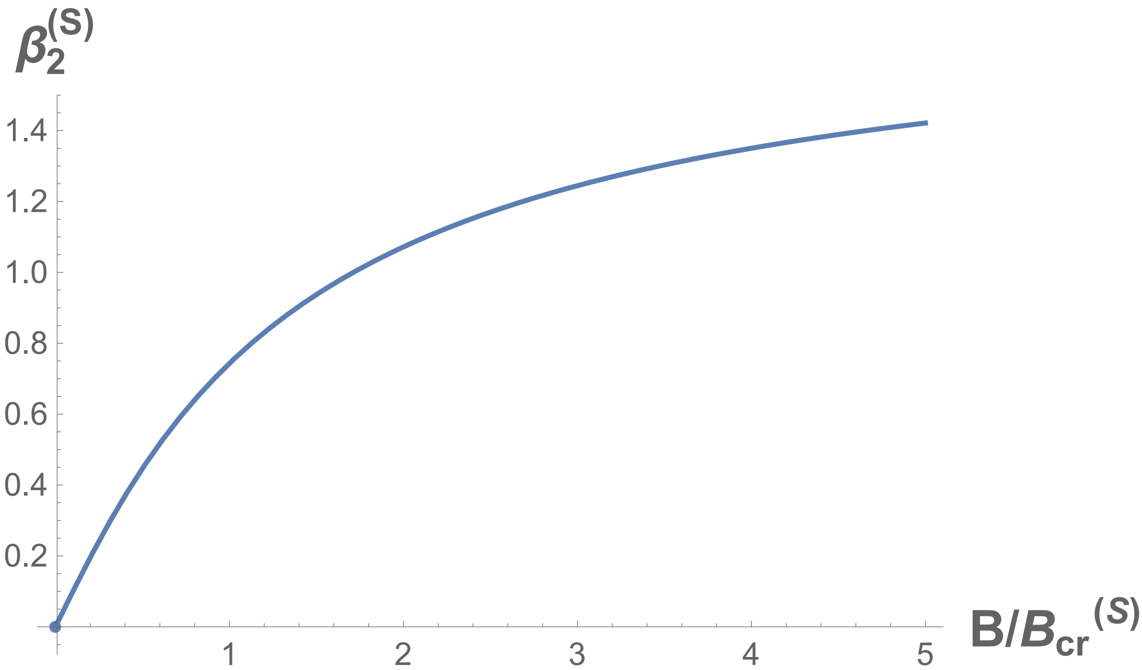

Figure 5.4: Plot of $\beta_2^{(S)}$ (in units of $\frac{e}{120\pi^2}$) for the complex scalar as a function of $B/B_{cr}^{(S)}$

It grows as B grows larger, and vanishes for small field. Within our perturbative calculation, this arises from the one-loop nature of the computation; a larger magnetic field means that there is more energy to pull forth scalar charge particles from the vacuum. For small b, we have, as promised in the Introduction:

$$\beta_2^{(S)} = \frac{eb}{120\pi^2} + \mathcal{O}\left(b^2\right) \tag{5.2.42}$$

5.3 Massive Dirac fermion

The above calculation has laid out the methodology for calculating such transport coefficients using the toy model of a complex scalar field. With an eye towards eventual astrophysical calculations, we now perform the same calculation for a Dirac fermion that (in principle) denotes the electron of our own universe. We will use very similar steps as in the complex scalar computation, with a few new complications arising from the mathematical structure of the electron spin.

The sequence of this section will be as follows. First we will introduce a modified version of the fermion propagator in the presence of a background magnetic field. Next we will explain how to construct the current stress tensor Green's function $\tilde{G}^{02,23}_{JT}(p_1)$ from a loop diagram involving these propagators. This provides the input to the Kubo formula for the transport coefficient $\beta_2^{(F)}$. Finally we will regularise the loop integral in a gauge-invariant manner to analytically evaluate the transport coefficient and present the result. Supplementary details and derivations can be found in Appendix 5.C.

5.3.1 Modified fermion propagator

As with the complex scalar, we follow the conventions of [3] by defining the free Feynman propagator for the Dirac fermion in a vacuum background to be

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{(-\not{k}+m)}{k^2+m^2-i\epsilon} \tag{5.3.1}$$

This is a Green's function for the Dirac equation, ie the classical equation of motion for a free Dirac field:

$$\left(m - i\not{\partial}^{(x)}\right) S_F(x-y) = \delta^{(4)}(x-y) \tag{5.3.2}$$

where $\not{\partial} \equiv \gamma^\mu \partial_\mu$ and the superscript (x) emphasises that derivatives are taken with respect to the coordinate x.

In ordinary QED calculations with a vacuum background, this is the propagator used to evalute Feynman diagrams. However, in this work we impose a fixed background magnetic field of strength $F_{12} = B_3 = B > 0$ oriented along the positive 3-direction.

Adapting a calculation in the Appendix of [67], we can find a modified fermion propagator in the presence of such a background magnetic field. Our conventions are as before; we write the magnetic field in Landau gauge $A_2 = Bx^1$. Writing $D = d - ieA$ for the covariant derivative associated with the background gauge field, the Green's equation for the modified propagator is

$$\left(m - i\not{D}^{(x)}\right) G^{(\psi)}(x,y) = \delta^{(4)}(x-y) \tag{5.3.3}$$

where the superscript (ψ) emphasises that we are working with a Dirac fermion, and as before the magnetic length is $l = (eB)^{-1/2}$.

It is convenient to Fourier transform the longitudinal directions $x_\parallel$ and leave the transverse direction $x_\perp$ alone as follows

$$G^{(\psi)}(p_\parallel; x_\perp, y_\perp) = \int d^2x_\parallel \; e^{-ip_\parallel\cdot(x_\parallel - y_\parallel)} \; G^{(\psi)}(x,y) \tag{5.3.4}$$

We can then solve the Green's equation to obtain

$$G^{(\psi)}(p_\parallel; x_\perp, y_\perp) = \frac{1}{2\pi} e^{i\Phi} e^{-\frac{1}{2}\xi} \sum_{n=0}^{\infty} \frac{(m - \gamma^a p_a)(L_n(\xi)P_+ + L_{n-1}(\xi)P_-) - \frac{i}{l^2}\gamma^j(x_j - y_j)L^{(1)}_{n-1}(\xi)}{l^2(m^2 + p_\parallel^2) + 2n} \tag{5.3.5}$$

where Φ is the so-called Schwinger phase defined by

$$\Phi(x_\perp, y_\perp) = \frac{1}{2l^2}(x^1 + y^1)(x^2 - y^2) \tag{5.3.6}$$

and the $L_n^{(\alpha)}$ are generalised Laguerre polynomials. The variable ξ is defined by

$$\xi = \frac{1}{2l^2}|x_\perp - y_\perp|^2 \tag{5.3.7}$$

A new ingredient for fermion is the degrees of freedom associated with the spin; for this we introduce the spin projectors $P_\pm$ defined by

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm i\gamma^1\gamma^2) \tag{5.3.8}$$

See Appendix 5.C.1 for a detailed derivation of the results above.

We interpret the sum over n as a sum over Landau levels.

Note that above we assumed $B > 0$. As in the scalar case, the final answer should be odd under charge conjugation, which acts as $B \to -B$; thus, as for the scalar, we expect a contribution to the answer from the presence of the Schwinger phase. However, for the case of the fermion, there are also new contributions arising from the coupling to the fermion spin. Indeed if we allow $B < 0$, then then the combination of Dirac matrices appearing in (5.3.8) is modified to be $(1 \pm i\ \mathrm{sign}(eB)\gamma^1\gamma^2)$, as discussed in more detail in Appendix 5.C.1. This non-analyticity in B allows for novel contributions to $\beta^{(F)}$. Physically, the familiar Schwinger phase represents a contribution from the orbital motion of the particles, and these new terms will represent a contribution from the intrinsic spin degrees of freedom. We will explicitly discuss these contributions as we proceed and verify that the final answer is odd in B, as required.

To obtain a more useful expression for our purposes, we can Fourier transform the translationary-invariant part of the propagator.

$$\tilde{G}^{(\psi)}(p) = 2l^2 e^{-p_\perp^2 l^2} \sum_{n=0}^{\infty} \frac{(-1)^n D_n(p)}{l^2(m^2+p_\parallel^2)+2n} \tag{5.3.9}$$

where the numerator is given by

$$D_n(p) = (m - \gamma^a p_a)\left(L_n\left(2p_\perp^2 l^2\right)P_+ - L_{n-1}\left(2p_\perp^2 l^2\right)P_-\right) + 2\gamma^j p_j L_{n-1}^{(1)}(2p_\perp^2 l^2) \tag{5.3.10}$$

See Appendix 5.D.2 for the relevant details concerning Fourier transforms of Laguerre polynomials.

5.3.2 Computation of Feynman diagram

Construction of current stress tensor correlator

As in the scalar calculation, we have $J^{02}(x) = F_{13}(x)$. However, for fields with spin the construction of the stress tensor is more subtle. More precisely, we need the Belifante stress tensor $T^{\mu\nu}_{(F)}$, which turns out to be the symmetrisation of the canonical stress tensor $T^{\mu\nu}_{(0)}$. We can construct the canonical stress tensor via the usual Noether construction.

$$T^{(0)}_{\mu\nu} = \frac{i}{2}\left[\bar{\psi}\gamma_\mu\partial_\nu\psi - (\partial_\nu\bar{\psi})\gamma_\mu\psi\right] \tag{5.3.11}$$

The Belifante stress tensor is then

$$T^{(F)}_{\mu\nu} = T^{(0)}_{(\mu\nu)} = \frac{i}{4}\left[\bar{\psi}\gamma_\mu\partial_\nu\psi - (\partial_\nu\bar{\psi})\gamma_\mu\psi + \bar{\psi}\gamma_\nu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma_\nu\psi\right] \tag{5.3.12}$$

More details on the relationship between the fermion stress tensors can be found in e.g. [68]. It is more helpful to write this as

$$T^{\mu\nu}_{(F)}(y) = \frac{i}{4}\left[(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\left(\partial_\rho^{(z)} - \partial_\rho^{(y)}\right)\bar{\psi}(y)\gamma_\sigma\psi(z)\right]\Bigg|_{z=y} \tag{5.3.13}$$

The relevant component is

$$T^{23}_{(F)}(y) = \frac{i}{4}\left[\left(\partial_2^{(z)} - \partial_2^{(y)}\right)\bar{\psi}(y)\gamma_3\psi(z) + \left(\partial_3^{(z)} - \partial_3^{(y)}\right)\bar{\psi}(y)\gamma_2\psi(z)\right]\Bigg|_{z=y} \tag{5.3.14}$$

A helpful mnemonic when relating this expression to the scalar stress tensor is that we have exchanged a partial derivative ∂_μ for a gamma matrix $i\gamma_\mu$.

Hence we can write the time-ordered current stress tensor correlator in terms of three-point functions, where we sew two of the ends together as in Figure 5.2.

$$\begin{aligned}\langle \mathcal{T} J^{02}(x) T^{23}_{(F)}(y)\rangle = \frac{i}{4}\Bigg[&\left(\partial_2^{(z)} - \partial_2^{(y)}\right)\left(\partial_1^{(x)}\langle A_3(x)\bar{\psi}(y)\gamma_3\psi(z)\rangle - \partial_3^{(x)}\langle A_1(x)\bar{\psi}(y)\gamma_3\psi(z)\rangle\right)\\ &+\left(\partial_3^{(z)} - \partial_3^{(y)}\right)\left(\partial_1^{(x)}\langle A_3(x)\bar{\psi}(y)\gamma_2\psi(z)\rangle - \partial_3^{(x)}\langle A_1(x)\bar{\psi}(y)\gamma_2\psi(z)\rangle\right)\Bigg]\Bigg|_{z=y}\end{aligned} \tag{5.3.15}$$

The Feynman diagram for the three-point function is shown in Figure 5.5.

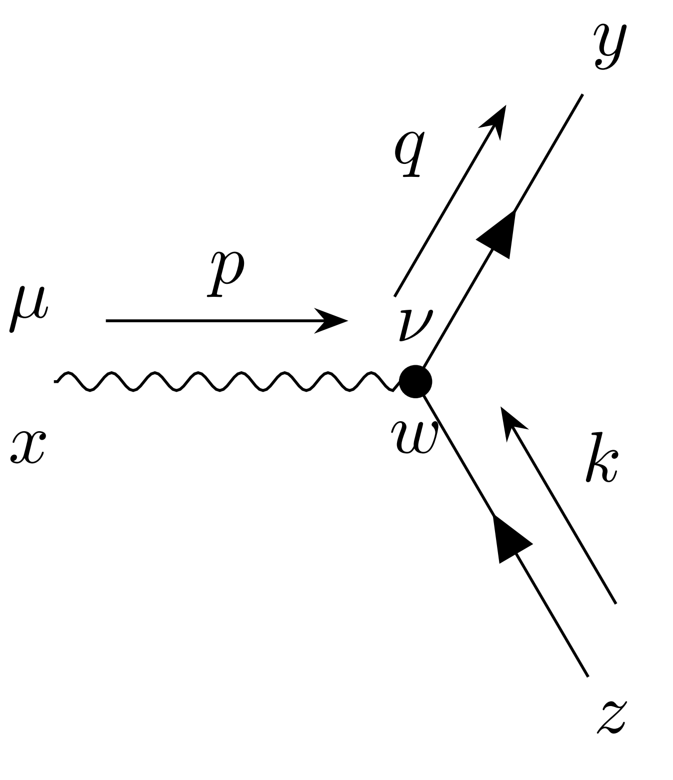

Figure 5.5: Feynman diagram for $\langle \mathcal{T} A_\mu(x)\bar{\psi}(y)\gamma_\nu\psi(z)\rangle$ as used in the Dirac fermion calculation

From the Feynman rules, the relevant (suitably time-ordered) three-point function is

$$\langle \mathcal{T} A_\mu(x)\bar{\psi}(y)\gamma_\nu\psi(z)\rangle = e\ g^{\rho\sigma}\int d^4w\ G^{(\gamma)}_{\mu\rho}(x,w)\,\mathrm{Tr}[\gamma_\nu G^{(\psi)}(z,w)\gamma_\sigma G^{(\psi)}(w,y)] \tag{5.3.16}$$

We now introduce the following notation

$$V_{\mu\nu}(w,y,z) \equiv \mathrm{Tr}\left[\gamma_\mu G^{(\psi)}(z,w)\gamma_\nu G^{(\psi)}(w,y)\right] \tag{5.3.17}$$

so that the three-point function is simply

$$\langle \mathcal{T} A_\mu(x)\bar{\psi}(y)\gamma_\nu\psi(z)\rangle = e\ g^{\rho\sigma}\int d^4w\ G^{(\gamma)}_{\mu\rho}(x,w)V_{\nu\sigma}(w,y,z) \tag{5.3.18}$$

It will also be useful to define

$$U_{\mu\nu\rho}(w,y,z) \equiv \left(\partial^{(z)}_\rho - \partial^{(y)}_\rho\right) V_{\mu\nu}(w,y,z) \tag{5.3.19}$$

so that we can compactly write the gauge-field stress tensor correlator as

$$\langle \mathcal{T} A_\mu(x)T^{23}_{(F)}(y)\rangle = \frac{ie}{4}\ g^{\rho\sigma}\int d^4w\ G^{(\gamma)}_{\mu\rho}(x,w)\left[U_{23\sigma}(w,y,y) + U_{32\sigma}(w,y,y)\right] \tag{5.3.20}$$

As before, differentiate and antisymmetrise, then Fourier transform and restrict the momentum to the 1-direction. The resulting Fourier-transformed current stress tensor correlator is

$$\langle \tilde{J}^{02}(p^1)\tilde{T}^{23}_{(F)}(-p^1)\rangle = -\frac{e}{4p^1}\int d^4w\ e^{-ip_1w^1}\ \left[U_{233}(w,0,0) + U_{323}(w,0,0)\right] \tag{5.3.21}$$

Working in momentum space, we have

$$U_{233}(w,0,0) = i\int \widetilde{dq}\ \widetilde{dk}\ \left(q_2 + k_2 + \frac{w^1}{l^2}\right)\ e^{iw\cdot(q-k)}\,\mathrm{Tr}\left[\gamma_3\ \tilde{G}^{(\psi)}(q)\ \gamma_3\ \tilde{G}^{(\psi)}(k)\right] \tag{5.3.22a}$$

$$U_{323}(w,0,0) = i\int \widetilde{dq}\ \widetilde{dk}\ (q_3 + k_3)\ e^{iw\cdot(q-k)}\,\mathrm{Tr}\left[\gamma_2\ \tilde{G}^{(\psi)}(k)\ \gamma_3\ \tilde{G}^{(\psi)}(q)\right] \tag{5.3.22b}$$

Hence the Green's function relating the current and the stress tensor is

$$\begin{aligned}&\tilde{G}^{02,23}_{JT}(p_1)\\ &= \frac{-e}{4p^1}\int d^4w\ \widetilde{dq}\ \widetilde{dk}\ e^{-ip_1w^1}e^{iw\cdot(q-k)}\\ &\qquad \left\{\left(q_2 + k_2 + \frac{w^1}{l^2}\right)\ \mathrm{Tr}\left[\gamma_3\ \tilde{G}^{(\psi)}(q)\ \gamma_3\ \tilde{G}^{(\psi)}(k)\right] + (q_3 + k_3)\ \mathrm{Tr}\left[\gamma_2\ \tilde{G}^{(\psi)}(k)\ \gamma_3\ \tilde{G}^{(\psi)}(q)\right]\right\}\end{aligned} \tag{5.3.23}$$

Now we can expand the Green's function as sums over Landau levels by substituting the expression for the Fourier-transformed fermion propagator. This gives

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{-el^4}{p^1} \int d^4w\, \widetilde{dq}\, \widetilde{dk}\, e^{-ip_1 w^1} e^{iw\cdot(q-k)} e^{-l^2 q_\perp^2}\, e^{-l^2 k_\perp^2} \sum_{n,n'=0}^{\infty} \frac{(-1)^{n+n'}}{\sigma_n(q_\parallel;m)\sigma_{n'}(k_\parallel;m)} \left[\left(q_2 + k_2 + \frac{w^1}{l^2}\right) \mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k)) + (q_3+k_3)\,\mathrm{Tr}(\gamma^3 D_n(q)\gamma^2 D_{n'}(k))\right] \tag{5.3.24}$$

where the denominator factors are given by

$$\sigma_n(q_\parallel;m) = l^2(m^2 + q_\parallel^2) + 2n \tag{5.3.25}$$

Next we can expand the traces using

$$\mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k)) = -2m^2[L_n(2q_\perp^2 l^2)L_{n'}(2k_\perp^2 l^2) + L_{n-1}(2q_\perp^2 l^2)L_{n'-1}(2k_\perp^2 l^2)] - 16q^i k_i L^{(1)}_{n-1}(2q_\perp^2 l^2) L^{(1)}_{n'-1}(2k_\perp^2 l^2) \tag{5.3.26a}$$

$$\mathrm{Tr}(\gamma^3 D_n(q)\gamma^2 D_{n'}(k)) = 4[L_{n-1}(2q_\perp^2 l^2)L^{(1)}_{n'-1}(2k_\perp^2 l^2)q_3(k_2+ik_1) - L^{(1)}_{n-1}(2q_\perp^2 l^2)L_{n'}(2k_\perp^2 l^2)k_3(q_2+iq_1) + L_{n'-1}(2k_\perp^2 l^2)L^{(1)}_{n-1}(2q_\perp^2 l^2)k_3(q_2-iq_1) - L^{(1)}_{n'-1}(2k_\perp^2 l^2)L_n(2q_\perp^2 l^2)q_3(k_2-ik_1)] \tag{5.3.26b}$$

For further details of the trace structure see Appendix 5.C.3. We can perform the integral over the longitudinal position $w_\parallel$ and $k_\parallel$, and invert the Fourier transforms of the generalised Laguerre polynomials, using

$$(-1)^n\, l^2 e^{-p_\perp^2 l^2} L_n\left(2p_\perp^2 l^2\right) = \frac{1}{4\pi}\int d^2x_\perp\, e^{-ip_\perp\cdot x_\perp} e^{-x_\perp^2/4l^2} L_n\left(x_\perp^2/2l^2\right) \tag{5.3.27a}$$

$$(-1)^n\, l^2 p_j\, e^{-p_\perp^2 l^2} L^{(1)}_{n-1}\left(2p_\perp^2 l^2\right) = -\frac{i}{8\pi l^2}\int d^2x_\perp\, x_j\, e^{-ip_\perp\cdot x_\perp} e^{-x_\perp^2/4l^2} L^{(1)}_{n-1}\left(x_\perp^2/2l^2\right) \tag{5.3.27b}$$

The resulting longitudinal loop integral for the Green's function is

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{2e}{(4\pi)^2 p^1 l^2}\int \widetilde{dq_\parallel}\, d^2w_\perp\, e^{-ip_1 w^1} e^{-w_\perp^2/2l^2} \sum_{n,n'=0}^{\infty} \frac{1}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)} \Bigg\{ w^1\, m^2 \left(L_n\left(w_\perp^2/2l^2\right) L_{n'}\left(w_\perp^2/2l^2\right) + L_{n-1}\left(w_\perp^2/2l^2\right) L_{n'-1}\left(w_\perp^2/2l^2\right)\right) - \frac{2w_1 w_\perp^2}{l^4} L^{(1)}_{n-1}\left(w_\perp^2/2l^2\right) L^{(1)}_{n'-1}\left(w_\perp^2/2l^2\right) + 4\, q_3^2\, w_1\, L^{(1)}_{n'-1}\left(w_\perp^2/2l^2\right)\left[L_n\left(w_\perp^2/2l^2\right) - L_{n-1}\left(w_\perp^2/2l^2\right)\right]\Bigg\} \tag{5.3.28}$$

Further details of this lengthy calculation are given in Appendix 5.C.2. Physically, we can understand the terms proportional to m^2 as arising from the Schwinger phase structure, since these take the same form as in the scalar calculation. The other new terms which involve generalised Laguerre polynomials can thus be understood to emerge from the spin components of the electron itself. As before, we have written the one-loop calculation in a mixed representation, going to momentum space in the longitudinal direction but staying in position space for the transverse directions.

Evaluation of loop integral

Next we want to perform the integral over $w_\perp$. We can use plane polar coordinates exactly as we did for the scalar. The azimuthal integral will yield a Bessel function of the first kind, which we can input into the Kubo formula. Repeating the same steps as in Section 5.2.2 gives a complicated but tractable integral:

$$\begin{aligned}\beta_2^{(F)} & \\ = -\frac{iel^4}{8\pi}\int \widetilde{dq_\parallel}\sum_{n,n'=0}^{\infty}\frac{1}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)}\int_0^\infty dx\; x^2\; e^{-x}\Big\{ & m^2\left(L_n(x)L_{n'}(x)+L_{n-1}(x)L_{n'-1}(x)\right) \\ & -\frac{4x}{l^2}L^{(1)}_{n-1}(x)L^{(1)}_{n'-1}(x)+4\;q_3^2\;L^{(1)}_{n'-1}(x)\left[L_n(x)-L_{n-1}(x)\right]\Big\}\end{aligned} \tag{5.3.29}$$

The integral over x can be related to the inner-product structure of generalised Laguerre polynomials. The terms here are more complicated than in the scalar case and we will need to define the following more general integral.

$$I^{(a,b,c)}_{n,n'} \equiv \int_0^\infty dx\; x^{2+c}\; e^{-x}L^{(a)}_n(x)L^{(b)}_{n'}(x) \tag{5.3.30}$$

Then $I^{(0,0,0)}_{n,n'} = I_{n,n'}$, the integral we defined in the scalar calculation. We can now write the expression for the transport coefficient more compactly as

$$\begin{aligned}\beta_2^{(F)} = -\frac{iel^2}{8\pi}\int \widetilde{dq_\parallel}\sum_{n,n'=0}^{\infty} & \frac{1}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)} \\ & \left[\mu\left(I^{(0,0,0)}_{n,n'}+I^{(0,0,0)}_{n-1,n'-1}\right)-4\;I^{(1,1,1)}_{n-1,n'-1}+4\;(lq_3)^2\left(I^{(0,1,0)}_{n,n'-1}-I^{(0,1,0)}_{n-1,n'-1}\right)\right]\end{aligned} \tag{5.3.31}$$

where we defined the dimensionless number $\mu \equiv (ml)^2$. It should be understood that any term in the series with a negative subscript vanishes. It is sensible to group the terms as follows.

$$S_1^{(\psi)} = \sum_{n,n'=0}^{\infty} \frac{I_{n,n'}^{(0,0,0)} + I_{n-1,n'-1}^{(0,0,0)}}{\sigma_n(q_\parallel; m)\sigma_{n'}(q_\parallel; m)} \tag{5.3.32a}$$

$$S_2^{(\psi)} = \sum_{n,n'=0}^{\infty} \frac{I_{n-1,n'-1}^{(1,1,1)}}{\sigma_n(q_\parallel; m)\sigma_{n'}(q_\parallel; m)} \tag{5.3.32b}$$

$$S_3^{(\psi)} = \sum_{n,n'=0}^{\infty} \frac{I_{n,n'-1}^{(0,1,0)} - I_{n-1,n'-1}^{(0,1,0)}}{\sigma_n(q_\parallel; m)\sigma_{n'}(q_\parallel; m)} \tag{5.3.32c}$$

We emphasise again that S_1 is associated with the Schwinger phase while S_2 and S_3 are associated with the intrinsic electron spin. We can evaluate the integrals as polynomials in n, then separate out the lowest Landau levels and rearrange to write the series as

$$S_1^{(\psi)} = \frac{2}{\sigma_0^2} - \frac{8}{\sigma_0\sigma_1} + \frac{4}{\sigma_0\sigma_2} + \sum_{n=1}^{\infty} \frac{1}{\sigma_n} \left(\frac{g_0(n)}{\sigma_n} + \frac{2\, g_1(n)}{\sigma_{n+1}} + \frac{2\, g_2(n)}{\sigma_{n+2}} \right) \tag{5.3.33a}$$

$$S_2^{(\psi)} = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} \left(\frac{h_0(n-1)}{\sigma_n} + \frac{2\, h_1(n-1)}{\sigma_{n+1}} + \frac{2\, h_2(n-1)}{\sigma_{n+2}} \right) \tag{5.3.33b}$$

$$S_3^{(\psi)} = \frac{2}{\sigma_0} \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) + \sum_{n=1}^{\infty} \frac{1}{\sigma_n} \left(\frac{p_0(n)}{\sigma_n} + \frac{p_1(n)}{\sigma_{n+1}} + \frac{p_2(n)}{\sigma_{n+2}} \right) \tag{5.3.33c}$$

Here $g(n), h(n), p(n)$ are all polynomials in n of at most cubic order. Their explicit forms are given in Appendix 5.D.1, along with thorough computational detatils in Appendix 5.D.1.

After Wick-rotating and rescaling the longitudinal momentum integral by l, we have

$$\beta_2^{(F)} = \frac{e}{32\pi^2} \int_0^{\infty} dx \, \left[\mu \, S_1^{(\psi)}(x; \mu) - 4 \, S_2^{(\psi)}(x; \mu) + 2x \, S_3^{(\psi)}(x; \mu) \right] \tag{5.3.34}$$

where $\mu = ml$ is dimensionless and we have

$$\begin{aligned} S_1^{(\psi)}(x; \mu) = & \frac{2}{(\mu + x)^2} - \frac{8}{(\mu + x)(\mu + x + 2)} + \frac{4}{(\mu + x)(\mu + x + 4)} \\ & + \sum_{n=1}^{\infty} \frac{1}{\mu + x + 2n} \left[\frac{g_0(n)}{\mu + x + 2n} + \frac{2\, g_1(n)}{\mu + x + 2n + 2} + \frac{2\, g_2(n)}{\mu + x + 2n + 4} \right] \end{aligned} \tag{5.3.35a}$$

$$S_2^{(\psi)}(x; \mu) = \sum_{n=1}^{\infty} \frac{1}{\mu + x + 2n} \left[\frac{h_0(n-1)}{\mu + x + 2n} + \frac{2\, h_1(n-1)}{\mu + x + 2n + 2} + \frac{2\, h_2(n-1)}{\mu + x + 2n + 4} \right] \tag{5.3.35b}$$

$$S_3^{(\psi)}(x; \mu) = \frac{2}{(\mu + x)(\mu + x + 2)} - \frac{2}{(\mu + x)(\mu + x + 4)}$$

$$+\sum_{n=1}^{\infty}\frac{1}{\mu+x+2n}\left[\frac{p_0(n)}{\mu+x+2n}+\frac{p_1(n)}{\mu+x+2n+2}+\frac{p_2(n)}{\mu+x+2n+4}\right] \tag{5.3.35c}$$

Regularisation

It is possible to explicitly perform the sums over n to obtain expressions for the $S_j^{(\psi)}(x;\mu)$ in terms of polygamma functions. However as we would expect, we again obtain a UV divergence when we try to perform the integral in (5.3.34) to obtain $\beta_2^{(F)}$. Again we need to regularise the loop integral in a gauge-invariant manner.

Inspired by the scalar calculation, a gauge-invariant regularisation scheme is to make the replacement

$$\frac{1}{\sigma_n(q_\parallel;m)}\to\frac{1}{\sigma_n(q_\parallel;m)}-\frac{1}{\sigma_n(q_\parallel;\Lambda)}=\frac{l^2(\Lambda^2-m^2)}{\sigma_n(q_\parallel;m)\,\sigma_n(q_\parallel;\Lambda)} \tag{5.3.36}$$

in the equations for $S_j^{(\psi)}$. This is similar to the Pauli-Villars procedure, in that the integral is finite for finite Λ, and as we take $\Lambda\to\infty$ we obtain the original problem. However it is subtly different to the usual replacement $G^{(\psi)}(x,y;m)\to G^{(\psi)}(x,y;m)-G^{(\psi)}(x,y;\Lambda)$ due to the presence of the mass m in the numerator of the fermion propagator.

We should pause for a moment to consider a physical picture of this regularisation procedure. One way to understand this procedure is that rather than regulating the original 4d fermion, we are instead regulating each effective two-dimensional degree of freedom traveling along the magnetic field lines. The transverse and longitudinal directions are thus now explicitly on a different footing. But really the symmetry between transverse and longitudinal directions was broken from the very beginning when we introduced a magnetic field along the 3-direction. Thus the regulator breaks no further symmetries, and it is computationally far more straightforward than a regulator of the original 4d fermion.

The rest of the computation proceeds as for the scalar. Once we have regularised expressions $\hat{S}_j^{(\psi)}(x;\mu,\Lambda)$ we can remove the regulator by taking the limit $\Lambda\to\infty$ to obtain new functions

$$\hat{S}_j^{(\psi)}(x;\mu)=\lim_{\Lambda\to\infty}S_j^{(\psi)}(x;\mu,\Lambda) \tag{5.3.37}$$

The transport coefficient is then given by the finite integral

$$\beta_2^{(F)}=\frac{e}{32\pi^2}\int_0^\infty dx\;\left[\mu\,\hat{S}_1^{(\psi)}(x;\mu)-4\,\hat{S}_2^{(\psi)}(x;\mu)+2x\,\hat{S}_3^{(\psi)}(x;\mu)\right] \tag{5.3.38}$$

5.3.3 Results

We have a fullly analytic expression for the transport coefficient as a function of $b = B/B_{cr} = \mu^{-1}$.

$$\beta_2^{(F)} = \frac{e}{64\pi^2 b^3}\left[8b^2 \log\left(\frac{\pi^3 A^6}{b}\right) - 48b^2\psi^{(-2)}\left(1+\frac{1}{2b}\right) - 2\left(2b^2+3\right)\psi^{(0)}\left(1+\frac{1}{2b}\right)\right.$$
$$\left. + 24b \log\Gamma\left(\frac{1}{2}\left(2+\frac{1}{b}\right)\right) + 2b(b(2b-3+\log(256))-3)+3\right] \tag{5.3.39}$$

Here $A \approx 1.28$ the Glaisher-Kinkelin constant and $\psi^{(n)}$ the analytically continued polygamma function.

The transport coefficient is plotted against b in Figure 5.6. The asymptotic behaviour for small b is

$$\beta_2^{(F)} = -\frac{eb}{240\pi^2} + \mathcal{O}\left(b^3\right) \tag{5.3.40}$$

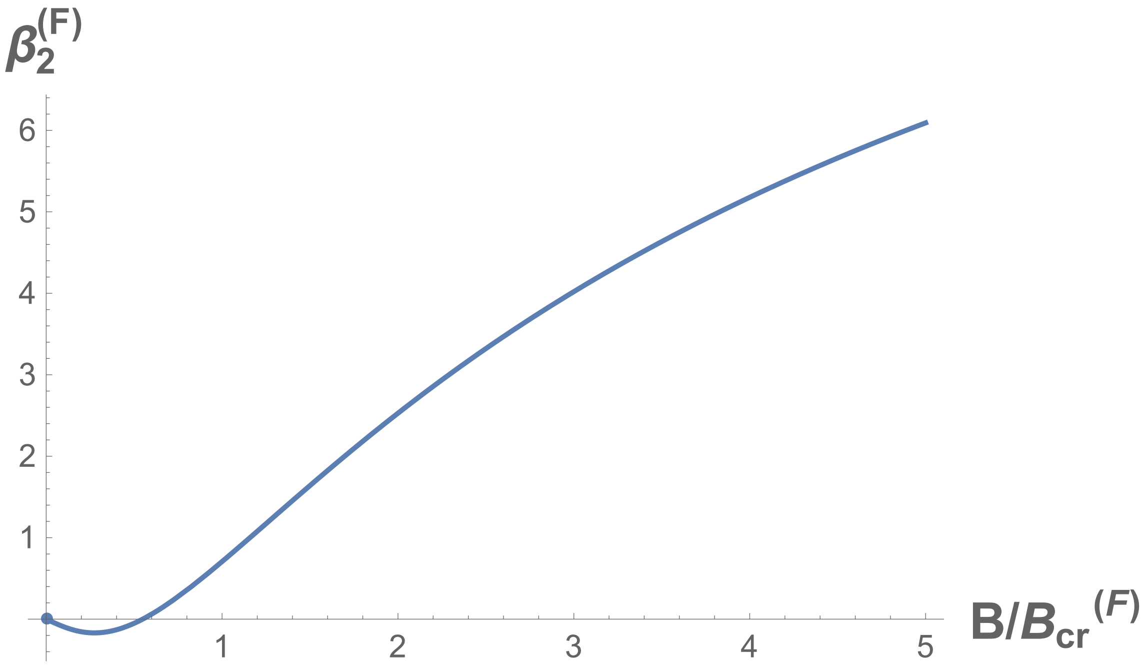

Figure 5.6: Plot of the transport coefficient in units in units of $e/240\pi^2$ for the Dirac fermion as a function of B/B_{cr}

Note that the sign changes from positive to negative at $b \approx 0.548$. Physically, this can be understood in terms of the two competing contributions to the transport coefficient, namely that from the Schwinger phase (which is positive) and that from the intrinsic spin (which is negative). Our computation demonstrates that for high Landau levels (large n), the dominant contribution to $\beta_2^{(F)}$ arises from the intrinsic spin of the electron, since each summand of the series is cubic in n, while the

Schwinger phase contribution is quadratic. A heuristic argument for this follows: for weak magnetic fields ($b \ll 1$) it is energetically possible for the electrons to occupy high Landau levels, hence for $b \ll 1$, the spin contribution dominates and $\beta_2^{(F)}$ has a negative sign. When the magnetic field is very strong ($b \gg 1$), the energy considerations force most of the electrons into the lowest Landau level, where the dominant contribution comes from the Schwinger phase and the spin structure is unimportant. This further explains why for asymptotically large b, the scalar and fermion plots have the same behaviour.

5.4 Conclusion

For small values of $b = B/B_{cr}$, the transport coefficient $\beta_2^{(F)}$ arising from a one-loop calculation for a massive Dirac fermion in QED has an expansion in odd powers of b given by

$$\beta_2^{(F)} = \frac{eB}{240\pi^2 B_{cr}} \left[-1 + \frac{46}{7} \left(\frac{B}{B_{cr}} \right)^2 + \frac{224}{11} \left(\frac{B}{B_{cr}} \right)^4 + \mathcal{O}\left((B/B_{cr})^6 \right) \right] \tag{5.4.1}$$

where as usual $eB_{cr} = m^2$, with m the mass of the fermion. This expression – and the corresponding result for the scalar – are the main results of this work.

We now discuss the *validity* of this calculation. It is well-known that calculating transport coefficients from perturbative quantum field theory is notoriously difficult; at its core this has to do with the fact that the hydrodynamic limit of an interacting quantum field theory (where one works on scales longer than the mean free path) is in tension with the weakly coupled limit of a quantum field theory (where the mean free path generally diverges as the coupling is taken to zero) [69, 70]. This conceptual tension manifests itself in the need to resum infinitely many Feynman diagrams to correctly account for IR divergences when calculating finite temperature transport coefficients in perturbative quantum field theory [71, 72]; indeed, if this is not done, then quantum field theory results do not agree with those from elementary kinetic theory.

The situation in our case, where we are studying a zero temperature – yet still coarse-grained and thus "hydrodynamic" – effective theory is somewhat less clear. There is less understanding in the literature on precisely what the domain of validity of FFE should be, i.e. which scale precisely plays the role of the mean free path (and why). Furthermore, it seems to us that the static correlator that we compute above is not afflicted by the precise IR divergences reported on in [72], which are intimately related to the analytic structure of *finite-temperature* quantum field theory (or, physically, by the presence of a thermal bath, which is absent in our calculation above). Thus we do not see an obvious pathology with the above calculation. This situation clearly deserves further study. It would be very interesting to study the same transport coefficient holographically in an intrinsically strongly coupled study (see e.g. [73] for some work on the holographic description of FFE). Another (diametrically opposite approach) would be to build a kinetic theory framework (using a density of particles in an external magnetic field) and calculate the same transport coefficient.

In the remainder of this conclusion, we optimistically take the result above at face value and turn to possible applications. We began this work by discussing the fact that there exist observed phenomena associated with pulsars – for example coherent radiation emission, or the presence of particle winds – that resist a simple theoretical explanation in terms of conventional FFE alone. As described in the introduction, a nonzero β_2 will now generically result in a non-trivial value for an accelerating electric field such that $\vec{E} \cdot \vec{B} \neq 0$. It is very interesting to ask whether our result for β_2 could potentially have observable consequences. We now speculate on this.

Here we will confine our attention to only one potential application; the creation of pulsar winds. We remind the reader that a pulsar is a rapidly rotating compact object with a large magnetic field. Despite considerable effort, the precise mechanism by which particle winds are generated by a pulsar remains theoretically challenging (see e.g. [74] for a review). It is natural to ask if our computation of β_2 – and concommitant production of an accelerating electric field can shed light on this.

Following [65], we will study the effects of this term on the *Michel monopole*, which is a caricature of a pulsar that models it as a rotating magnetic monopole with magnetic charge q and frequency Ω [75]. (For a more realistic model, one would *split* the monopole, i.e. flip the sign of the magnetic charge at $\theta = \frac{\pi}{2}$; this introduces a current sheet along the equator, and is now thought to be a reasonable facsimile of an actual pulsar). The Michel monopole field-strength is given by

$$F = q \sin\theta d\theta \wedge (d\phi - \Omega d(t-r)) \tag{5.4.2}$$

This is a solution to conventional FFE. In [65] a general expansion for β_2 was considered in powers of B as:

$$\beta_2 = BL^2 + \mathcal{O}(B^2) + \cdots \tag{5.4.3}$$

with L an arbitrary scale. Inserting this into the FFE constitutive relations arising from (4.3.6), it was shown that on the Michel magnetic monopole an accelerating electric field was created whose magnitude is

$$E_0 = -\frac{4qL^2}{r^3}\Omega\cos\theta \qquad E_0 \equiv \frac{\mathbf{E}\cdot\mathbf{B}}{\sqrt{\mathbf{B}^2 - \mathbf{E}^2}} \tag{5.4.4}$$

It was emphasised in [65] that this results in the acceleration of charged particles away from the pulsar, where the acceleration can now be calculated using elementary physics in terms of the scale L. Can this contribute to a reasonable pulsar wind? It is often estimated that one requires a Lorentz factor of these accelerated particles of $\gamma \sim 10^3$ [74]. Inserting typical numbers for pulsars (i.e. a magnetic field $B \sim 10^{12}$ Gauss, a stellar radius of ten kilometers, and a pulsar period on the order of seconds), [65] argued that one could obtain the desired acceleration if the scale L was on the order of metres. (Tantalisingly, this is also the same scale at which coherent pulsar *radiation* happens).

The new contribution of our work is that we now have an explicit calculation for L. Comparing the phenomenological expansion (5.4.3) with the microscopic calculation (5.4.1), we see that we have

$$L^2 = \frac{e}{240\pi^2 B_{cr}} = \frac{1}{240\pi^2}\frac{1}{m^2} \tag{5.4.5}$$

In other words, somewhat predictably L is set by the only scale in the problem; the electron mass. As a length scale $L = \frac{\bar{\lambda}_e}{\sqrt{240\pi^2}} \sim 10^{-14}$ metres, of the order of the Compton wavelength of the electron, which is many orders of magnitude too small to account for the precise application suggested in [65].

However, we stress that the given formulae are valid for a broader range of b and could potentially find a fruitful use in another setting. In particular, one might imagine that the stronger the magnetic field the more relevant our calculation; we note the existence of *magnetars* where the magnetic field can be up to two orders of magnitude greater than the critical field strength [76]. It would be extremely interesting to solve the equations of generalised FFE with a higher-derivative correction in a more complicated pulsar geometry using our $\beta_2^{(F)}$; one might entertain hope that this new effective field theory approach could shed light on long-standing problems in pulsar physics.

5.A Conventions

Transverse and longitudinal directions

In this chapter we have a fixed background magnetic field in the 3-direction. This effectively splits the whole 4-dimensional spacetime into *longitudinal directions* $0, 3$ and *transverse directions* $1, 2$. It is thus useful to devise an appropriate notation. We use i, j to range over $1, 2$ and a, b range over $0, 3$. Alternatively we may write the components in pairs as $x_\perp \equiv (x^1, x^2) \equiv x^i$ and $x_\parallel = (x^0, x^3) \equiv x^a$.

Gauge-fixing

We work with the background gauge field in the Landau gauge defined by

$$A_\mu = (0, 0, Bx^1, 0) \tag{5.A.1}$$

which fixes the magnetic field to be $F_{12} = B_3 = B$. For the one-loop calculation there is no gauge-fixing required since we restrict the photon momentum to be purely in the 1-direction.

5.B Details of scalar calculation

5.B.1 Derivation of scalar propagator

In this appendix we provide a self-contained derivation of the scalar propagator in a background magnetic field. Our approach follows [67] but is more detailed and uses different conventions. The strategy will be to find a Green's function for the modified equation of motion for a free complex scalar. We will take the Lagrangian to be (see [3])

$$\mathcal{L} = -(D^\mu\phi)^\dagger(D_\mu\phi) - m^2\phi^\dagger\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{5.B.1}$$

where $F = dA$ and the covariant derivative is defined by $D \equiv d - ieA$.[2] This yields the following equation of motion for ϕ

$$\left(-g^{\mu\nu}D_\mu D_\nu + m^2\right)\phi(x) = 0 \tag{5.B.2}$$

Making contact with the above vacuum background case, the Green's equation for the scalar field is defined by

$$\left(-g^{\mu\nu}D_\mu^{(x)}D_\nu^{(x)} + m^2\right)G^{(\phi)}(x, y) = \delta^{(4)}(x - y) \tag{5.B.3}$$

Following [67], we will work with a constant background magnetic field oriented along the positive z-direction:

$$F_{12} = B_3 = B > 0 \tag{5.B.4}$$

It is convenient to work in Landau gauge given by

$$A_2 = Bx^1 \tag{5.B.5}$$

[2] That is, we choose the electric charge of the "scalar electron" to be $q = -e < 0$.

We thus adopt a notation where i, j range over $1, 2$ - the *transverse directions* - and a, b range over $0, 3$ - the *longitudinal directions*. Alternatively we may write the components in pairs as $x_\perp \equiv (x^1, x^2) \equiv x^i$ and $x_\parallel = (x^0, x^3) \equiv x^a$. Then the Green's equation becomes

$$\left(-g^{ij} D_i^{(x)} D_j^{(x)} - g^{ab} \partial_a^{(x)} \partial_b^{(x)} + m^2\right) G^{(\phi)}(x, y) = \delta^{(4)}(x - y) \tag{5.B.6}$$

since the gauge field vanishes in the longitudinal directions with our gauge choice.

Now we Fourier transform in the longitudinal components:

$$G^{(\phi)}(p_\parallel; x_\perp, y_\perp) = \int d^2 x_\parallel \; e^{-ip_\parallel \cdot (x_\parallel - y_\parallel)} \; G^{(\phi)}(x, y) \tag{5.B.7}$$

This gives the Green's equation

$$\left(-\pi_\perp^2 + p_\parallel^2 + m^2\right) G^{(\phi)}(p_\parallel; x_\perp, y_\perp) = \delta^{(2)}(x_\perp - y_\perp) \tag{5.B.8}$$

where

$$\pi_1 = D_1^{(x)} = \partial_1^{(x)} \tag{5.B.9a}$$

$$\pi_2 = D_2^{(x)} = \partial_2^{(x)} - i(x^1/l^2) \tag{5.B.9b}$$

$$\pi_\perp^2 = \pi_1^2 + \pi_2^2 \tag{5.B.9c}$$

$l = (eB)^{-1/2}$ is the magnetic length, and $p_\parallel^2 = p_a p_b g^{ab} = -(p^0)^2 + (p^3)^2$.

Observe that if we now change the orientation of the magnetic field by flipping $B \mapsto -B$, this is implemented by replacing $l \mapsto -il$, and so we would replace $\pi_\perp^2$ by $(\pi_\perp^2)^*$. Clearly this is equivalent to changing the sign of the "scalar electron" charge $q \to -q$ since $(D_\mu)* = \partial_\mu + i(-q)A_\mu$.

At this point we introduce the wavefunctions $\Psi_{n,q_2}(x_\perp)$ defined for $n \in \mathbb{Z}_{\geq 0}$ and $q_2 \in \mathbb{R}$ by

$$\Psi_{n,q_2}(x_\perp) = \frac{1}{\sqrt{2\pi l}} e^{-iq_2 x^2} \psi_n\left(\frac{x^1}{l} + q_2\, l\right) \tag{5.B.10}$$

where $\psi_n(\eta)$ are so-called Hermite functions.[3] They are defined by

$$\psi_n(\eta) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\eta^2/2} H_n(\eta) \tag{5.B.11}$$

where H_n are Hermite polynomials. The ψ_n are scaled so that they are orthonormal:

$$\int_{-\infty}^{\infty} d\eta \; \psi_n(\eta) \psi_m(\eta) = \delta_{nm} \tag{5.B.12}$$

[3]Hermite functions $\psi_n(\eta)$ are energy eigenfunctions of the quantum harmonic oscillator in the usual position space representation, with $\eta = \sqrt{\frac{m\omega}{\hbar}}x$.

Hermite functions satisfy the recurrence relations

$$\psi_n'(\eta) = \sqrt{\frac{n}{2}}\psi_{n-1}(\eta) - \sqrt{\frac{n+1}{2}}\psi_{n+1}(\eta) \tag{5.B.13a}$$

$$\eta\psi_n(\eta) = \sqrt{\frac{n}{2}}\psi_{n-1}(\eta) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(\eta) \tag{5.B.13b}$$

for all $n \in \mathbb{Z}_{\geq 0}$ if we set $\psi_{-1} = 0$.

If we let $\eta = \frac{x^1}{l} + q_2 l$ we have

$$l\pi_1\Psi_{n,q_2}(x_\perp) = \frac{d}{d\eta}\Psi_{n,q_2}(x_\perp) \tag{5.B.14a}$$

$$l\pi_2\Psi_{n,q_2}(x_\perp) = -i\eta\Psi_{n,q_2}(x_\perp) \tag{5.B.14b}$$

Hence using the recurrence relations for the Hermite functions, we can show that Ψ_{n,q_2} satisfies the eigenvalue equation

$$\pi_\perp^2\Psi_{n,q_2} = -\frac{1}{l^2}\left(2n+1\right)\Psi_{n,q_2} \tag{5.B.15}$$

The Ψ_{n,q_2} satisfy completeness and normalisability conditions

$$\delta^{(2)}(x_\perp - y_\perp) = \int_{-\infty}^{\infty} dq_2 \, \left[\sum_{n=0}^{\infty}\Psi_{n,q_2}(x_\perp)\Psi^*_{n,q_2}(y_\perp)\right] \tag{5.B.16a}$$

$$\delta_{nn'}\,\delta(q_2 - q_2') = \int d^2x_\perp\;\Psi^*_{n,q_2}(x_\perp)\Psi_{n',q_2'}(x_\perp) \tag{5.B.16b}$$

We can now write the left-hand side of the Green's equation as

$$\begin{aligned}
&\left(-\pi_\perp^2 + p_\parallel^2 + m^2\right) G^{(\phi)}(p_\parallel; x_\perp, y_\perp) \\
&= \left(-\pi_\perp^2 + p_\parallel^2 + m^2\right)\int d^2z_\perp\;\delta^{(2)}(x_\perp - z_\perp)G^{(\phi)}(p_\parallel; z_\perp, y_\perp) \\
&= \int d^2z_\perp\;G^{(\phi)}(p_\parallel; z_\perp, y_\perp)\int_0^\infty dq_2\sum_{n=0}^{\infty}\left[\Psi^*_{n,q_2}(z_\perp)\left(-\pi_\perp^2 + p_\parallel^2 + m^2\right)\Psi_{n,q_2}(x_\perp)\right] \\
&= \frac{1}{l^2}\int d^2z_\perp\;G^{(\phi)}(p_\parallel; z_\perp, y_\perp)\int_{-\infty}^{\infty} dq_2\sum_{n=0}^{\infty}\left[\Psi^*_{n,q_2}(z_\perp)\lambda_n(p_\parallel; m)\Psi_{n,q_2}(x_\perp)\right]
\end{aligned}$$

where

$$\lambda_n(p_\parallel; m) = l^2(m^2 + p_\parallel^2) + 2n + 1 \tag{5.B.17}$$

Next multiply both sides of the Green's equation by $\Psi^*_{n',q_2'}(x_\perp)$ and integrate with respect to $x_\perp$. Using the completeness of the wavefunctions we obtain (after relabelling)

$$\frac{l^2}{\lambda_n(p_\parallel; m)}\Psi^*_{n,q_2}(y_\perp) = \int d^2z_\perp\;G^{(\phi)}(p_\parallel, m; z_\perp, y_\perp)\Psi^*_{n,q_2}(z_\perp) \tag{5.B.18}$$

Finally, multiply by $\Psi_{n,q_2}(x_\perp)$, sum over n and integrate over q_2 to obtain

$$G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) = l^2 \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \left[\frac{1}{\lambda_n(p_\parallel; m)} \Psi_{n,q}(x_\perp) \Psi^*_{n,q}(y_\perp) \right] \tag{5.B.19}$$

Now we can substitute the expressions for the wavefunctions and complete the square of the resulting exponent

$$\begin{aligned}
&G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) \\
&= l^2 \int dq_2 \left[\sum_n \frac{1}{\lambda_n} \Psi_{n,q_2}(x_\perp) \Psi^*_{n,q_2}(y_\perp) \right] \\
&= \frac{l^2}{2\pi l} \int dq_2\, e^{-iq_2(x^2-y^2)} \left[\sum_n \frac{1}{\lambda_n} \psi_n \left(\frac{x^1}{l} + q_2\, l \right) \psi_n \left(\frac{y^1}{l} + q_2\, l \right) \right] \\
&= \frac{l}{2\pi} \int dq_2\, e^{\left(-l^2 \tilde{q_2}^2 + i\Phi(x_\perp, y_\perp) - \frac{|x_\perp - y_\perp|^2}{4l^2} \right)} \left[\sum_n \frac{1}{2^n n! \sqrt{\pi}} \frac{1}{\lambda_n} H_n \left(\frac{x^1}{l} + q_2\, l \right) H_n \left(\frac{y^1}{l} + q_2\, l \right) \right]
\end{aligned}$$

where we defined a shifted complexified momentum

$$\tilde{q_2} = q_2 + \frac{1}{2l^2} [(x^1 + y^1) + i(x^2 - y^2)] \tag{5.B.20}$$

and a Schwinger phase

$$\Phi(x_\perp, y_\perp) = \frac{1}{2l^2} (x^1 + y^1)(x^2 - y^2) \tag{5.B.21}$$

The Green's function becomes

$$G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) = \frac{l}{2\pi} e^{i\Phi} e^{-\frac{1}{2}\xi} \int dq_2\, e^{-l^2 \tilde{q_2}^2} \left[\sum_n \frac{1}{2^n n! \sqrt{\pi}} \frac{1}{\lambda_n} H_n(l\tilde{q_2} - \alpha) H_n(l\tilde{q_2} + \alpha^*) \right] \tag{5.B.22}$$

where

$$\alpha = \frac{1}{2l} [(x^1 - y^1) + i(x^2 - y^2)] \tag{5.B.23}$$

and

$$\xi = \frac{1}{2l^2} |x_\perp - y_\perp|^2 = 2|\alpha|^2 \tag{5.B.24}$$

We want to perform a change of variable from q_2 to $\tilde{q_2}$ in order to do the integral. This is equivalent to shifting the contour to a horizontal line parallel to the real axis. This will give the same result as integrating along the real axis so we don't worry about it here. We also need to rescale $\tilde{q_2}$ via $u = \tilde{q_2} l$. To perform the resulting u integral, we will use the following integral identity

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} H_m(x + y) H_n(x + z) = \sqrt{\pi}\, 2^n\, m!\, z^{n-m}\, L_m^{(n-m)}(-2yz) \tag{5.B.25}$$

which is valid for $m < n$, where $L_m^{(\alpha)}(x)$ are the generalised Laguerre polynomials.

$L_m^{(0)}$ are ordinary Laguerre polynomials of order m. This identity is straightforward to prove by Taylor expanding the Hermite polynomials, using the orthogonality relation, and then comparing the resulting finite sum to the closed form of the generalised Laguerre polynomial. This allows us to write the Green's function as

$$\begin{aligned}G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) &= \frac{1}{2\pi} e^{i\Phi} e^{-\frac{1}{2}\xi} \int du\, e^{-u^2} \left[\sum_n \frac{1}{2^n n! \sqrt{\pi}} \frac{1}{\lambda_n} H_n(u-\alpha) H_n(u+\alpha^*)\right] \\ &= \frac{1}{2\pi} e^{i\Phi} e^{-\frac{1}{2}\xi} \sum_n \frac{1}{\lambda_n} L_n(2|\alpha|^2) \\ &= \frac{1}{2\pi} e^{i\Phi(x_\perp, y_\perp)} e^{-\frac{1}{2}\xi} \sum_{n=0}^{\infty} \frac{L_n(\xi)}{\lambda_n(p_\parallel; m)}\end{aligned}$$

The final expression for the mixed representation Green's function is thus

$$G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) = \frac{1}{2\pi} e^{i\Phi(x_\perp, y_\perp)} e^{-\frac{1}{2}\xi} \sum_{n=0}^{\infty} \frac{L_n(\xi)}{l^2(m^2 + p_\parallel^2) + 2n + 1} \tag{5.B.26}$$

We interpret the sum over n as a sum over Landau levels. If we flip the orientation of the magnetic field $B \mapsto -B$, then the Green's function simply maps to its complex conjugate. In particular, the only change is the change of sign of the Schwinger phase $\Phi(x_\perp, y_\perp) \mapsto -\Phi(x_\perp, y_\perp) = \Phi(y_\perp, x_\perp)$.

Further, we can Fourier transform the translationary invariant part of the propagator:

$$\tilde{G}^{(\phi)}(p) \equiv \int d^2x_\perp\, e^{-ip_\perp \cdot (x_\perp - y_\perp)}\, e^{-i\Phi}\, G^{(\phi)}(p_\parallel, m; x_\perp, y_\perp) = 2l^2 e^{-p_\perp^2 l^2} \sum_{n=0}^{\infty} \frac{(-1)^n L_n(2p_\perp^2 l^2)}{l^2(m^2 + p_\parallel^2) + 2n + 1} \tag{5.B.27}$$

See Appendix 5.D.2 for the relevant details concerning Fourier transforms of Laguerre polynomials.

5.B.2 Computation of Feynman diagram

Here we provide further details of the Feynman diagram computation for the complex scalar.

Fourier transform of current stress tensor correlator

We are interested in the momentum-space correlator $\langle \tilde{J}^{02}(p^1) \tilde{T}^{23}(-p^1) \rangle$. Starting from the 3-point function, we have

$$\langle \mathcal{T} A_\nu(x) \phi^\dagger(y) \phi(z) \rangle = -ie\, g^{\rho\sigma} \int d^4w\, G^{(\gamma)}_{\nu\rho}(x, w)\, V_\sigma(w, y, z) \tag{5.B.28}$$

Act on both sides with $\left(\partial_2^{(y)}\partial_3^{(z)} + \partial_3^{(y)}\partial_2^{(z)}\right)$ and evaluate at $z = y$ to get

$$\langle \mathcal{T} A_\nu(x) T^{23}(y)\rangle = -ie\, g^{\rho\sigma} \int d^4w\, G^{(\gamma)}_{\nu\rho}(x,w)\, U_\sigma(w,y,y) \tag{5.B.29}$$

Now we can differentiate with respect to x^μ and antisymmetrise over the indices μ and ν to get

$$\langle \mathcal{T} F_{\mu\nu}(x) T^{23}(y)\rangle = -ie\, g^{\rho\sigma} \int d^4w\, \left[\partial_\mu^{(x)} G^{(\gamma)}_{\nu\rho}(x,w) - \partial_\nu^{(x)} G^{(\gamma)}_{\mu\rho}(x,w)\right] U_\sigma(w,y,y) \tag{5.B.30}$$

The photon propagator is given by

$$G^{(\gamma)}_{\mu\rho}(x,w) = \int \widetilde{dp}\, e^{ip\cdot(x-w)} \tilde{G}^{(\gamma)}_{\mu\rho}(p) = \int \widetilde{dp}\, e^{ip\cdot(x-w)} \frac{1}{p^2 - i\epsilon}\left(g_{\mu\rho} - (1-\zeta)\,\frac{p_\mu p_\rho}{p^2}\right) \tag{5.B.31}$$

for some gauge-fixing parameter ζ.

Hence we have

$$\langle \mathcal{T} F_{\mu\nu}(x) T^{23}(0)\rangle = e\, g^{\rho\sigma} \int d^4w\, \widetilde{dp}\, e^{ip\cdot(x-w)} \left[p_\mu \tilde{G}^{(\gamma)}_{\nu\rho}(p) - p_\nu \tilde{G}^{(\gamma)}_{\mu\rho}(p)\right] U_\sigma(w,0,0) \tag{5.B.32}$$

Taking the Fourier transform we simply have

$$\langle \tilde{F}_{\mu\nu}(p) \tilde{T}^{23}(-p)\rangle = e\, g^{\rho\sigma} \left[p_\mu \tilde{G}^{(\gamma)}_{\nu\rho}(p) - p_\nu \tilde{G}^{(\gamma)}_{\mu\rho}(p)\right] \int d^4w\, e^{-ip\cdot w}\, U_\sigma(w,0,0) \tag{5.B.33}$$

Setting $\mu = 1, \nu = 3$ and $p^\mu = (0, p^1, 0, 0)$, we identify $F_{13} = J^{02}$ to get

$$\langle \tilde{J}^{02}(p^1) \tilde{T}^{23}(-p^1)\rangle = e\, p_1\, \tilde{G}^{(\gamma)}_{3\rho}(p^1)\, g^{\rho\sigma} \int d^4w\, e^{-ip_1 w^1}\, U_\sigma(w,0,0) \tag{5.B.34}$$

Observe that when the momentum is purely in the 1-direction, the expression for the photon propagator simplifies to

$$\tilde{G}^{(\gamma)}_{3\rho}(p^1) = \frac{g_{3\rho}}{(p^1)^2} \tag{5.B.35}$$

which allows us to write

$$\langle \tilde{J}^{02}(p^1) \tilde{T}^{23}(-p^1)\rangle = \frac{e}{p^1} \int d^4w\, e^{-ip_1 w^1}\, U_3(w,0,0) \tag{5.B.36}$$

Now to calculate $U_3(w,0,0)$. Start with $V_3(w,y,z)$ and use the momentum-space representation of the scalar propagator.

$$G^{(\phi)}(z,w) = \int \widetilde{dk}\, e^{ik\cdot(z-w)} e^{i\Phi(z_\perp, w_\perp)}\, \tilde{G}^{(\phi)}(k) \tag{5.B.37}$$

This allows us to write V_3 as

$$V_3(w,y,z) = i\int \widetilde{dq}\,\widetilde{dk}\;(q_3+k_3)\;e^{iq\cdot(w-y)}e^{ik\cdot(z-w)}e^{i\Phi(z_\perp,w_\perp)}e^{-i\Phi(y_\perp,w_\perp)}\;\tilde{G}^{(\phi)}(q)\tilde{G}^{(\phi)}(k) \tag{5.B.38}$$

Hence differentiating gives

$$\begin{aligned} U_3(w,y,z) = i\int \widetilde{dq}\,\widetilde{dk}\;e^{iq\cdot(w-y)}e^{ik\cdot(z-w)}e^{i\Phi(z_\perp,w_\perp)}e^{-i\Phi(y_\perp,w_\perp)}\tilde{G}^{(\phi)}(q)\tilde{G}^{(\phi)}(k) \\ (q_3+k_3)\left\{k_3\left[q_2+\partial_2^{(y)}\Phi(y_\perp,w_\perp)\right]+q_3\left[k_2+\partial_2^{(z)}\Phi(z_\perp,w_\perp)\right]\right\} \end{aligned}$$

When we evaluate at $z=y$, the Schwinger phases cancel, and we obtain a simpler expression.

$$\begin{aligned} &U_3(w,y,y) \\ &= i\int \widetilde{dq}\,\widetilde{dk}\;e^{i(q-k)\cdot(w-y)}\tilde{G}^{(\phi)}(q)\tilde{G}^{(\phi)}(k)(q_3+k_3)\left[k_3q_2+q_3k_2+(q_3+k_3)\;\partial_2^{(y)}\Phi(y_\perp,w_\perp)\right] \end{aligned} \tag{5.B.39}$$

The Schwinger phase is given by

$$\Phi(y_\perp,w_\perp) = \frac{1}{2l^2}(y^1+w^1)(y^2-w^2) \tag{5.B.40}$$

so the relevant derivative is

$$\partial_2^{(y)}\Phi(y_\perp,w_\perp) = \frac{w^1+y^1}{2l^2} \tag{5.B.41}$$

Hence evaluating at $y=0$, the expression we need is

$$U_3(w,0,0) = i\int \widetilde{dq}\,\widetilde{dk}\;\tilde{G}^{(\phi)}(q)\;\tilde{G}^{(\phi)}(k)\;e^{iw\cdot(q-k)}\;(q_3+k_3)\left(q_2k_3+k_2q_3+\frac{w^1(q_3+k_3)}{2l^2}\right) \tag{5.B.42}$$

Plugging this into the expression for the correlator, we can immediately do the integral over the longitudinal directions $w_\parallel$ to get a factor of $\delta^{(2)}(q_\parallel - k_\parallel)$. Then we can integrate over (say) $k_\parallel$. This gives

$$\langle \tilde{J}^{02}(p^1)\tilde{T}^{23}(-p^1)\rangle = \frac{2ie}{p^1}\int d^2w_\perp\;\widetilde{dq}\;\widetilde{dk_\perp}\;e^{-ip_1w^1}e^{iw_\perp\cdot(q_\perp-k_\perp)}\;G^{(\phi)}(q_\perp,q_\parallel)\;G^{(\phi)}(k_\perp,q_\parallel)q_3^2\left(q_2+k_2+\frac{w^1}{l^2}\right) \tag{5.B.43}$$

With our conventions the Green's function is related to the correlator by

$$i\;\tilde{G}^{02,23}_{JT}(p_1) = \langle \tilde{J}^{02}(p^1)\tilde{T}^{23}(-p^1)\rangle \tag{5.B.44}$$

Hence we can write

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{2e}{p^1} \int d^2w_\perp \, \widetilde{dq} \, \widetilde{dk_\perp} \, e^{-ip_1 w^1} e^{iw_\perp \cdot (q_\perp - k_\perp)} \, G^{(\phi)}(q_\perp, q_\parallel) \, G^{(\phi)}(k_\perp, q_\parallel) \, q_3^2 \left(q_2 + k_2 + \frac{w^1}{l^2} \right) \tag{5.B.45}$$

Similarly to before, we write $\widetilde{dk_\perp}$ as a shorthand for $\frac{d^2k_\perp}{(2\pi)^2}$. We will show that only the term in the integrand with a factor of w^1/l^2 has a non-vanishing contribution to the correlator. For the remaining $(q_2 + k_2)$ piece we can do the integral over w^2 and k_2 then expand the propagator expressions to get

$$\begin{aligned}
&\int dw_2 \, dq_2 \, dk_2 \, e^{iw^2(q_2 - k_2)} \, G^{(\phi)}(q_\perp, q_\parallel) \, G^{(\phi)}(k_\perp, q_\parallel) \; (q_2 + k_2) \\
&= \int dq_2 \, dk_2 \, \delta(q_2 - k_2) \, G^{(\phi)}(q_\perp, q_\parallel) \, G^{(\phi)}(k_\perp, q_\parallel) \; (q_2 + k_2) \\
&= 2 \int dq_2 \, G^{(\phi)}(q_\perp, q_\parallel) \, G^{(\phi)}(k_1, q_2, q_\parallel) \, q_2 \\
&= 8l^4 \int dq_2 \, q_2 \, e^{-l^2(k_1^2 + q_1^2 + 2q_2^2)} \sum_{n,n'=0}^{\infty} \left[\frac{(-1)^n L_n(2l^2(q_1^2 + q_2^2))}{l^2 \left(q_\parallel^2 + m^2 \right) + 2n + 1} \right] \left[\frac{(-1)^{n'} L_{n'}(2l^2(k_1^2 + q_2^2))}{l^2 \left(q_\parallel^2 + m^2 \right) + 2n' + 1} \right]
\end{aligned}$$

This explicitly shows that the integrand is odd in q_2, so since we integrate q_2 from $-\infty$ to ∞ the integral over q_2 vanishes as claimed. Hence we have

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{2e}{l^2 p^1} \int d^2w_\perp \, \widetilde{dq} \, \widetilde{dk_\perp} \, e^{-ip_1 w^1} e^{iw_\perp \cdot (q_\perp - k_\perp)} \, G^{(\phi)}(q_\perp, q_\parallel) \, G^{(\phi)}(k_\perp, q_\parallel) \, q_3^2 \, w^1 \tag{5.B.46}$$

Now we go full circle back to the mixed representation of the scalar propagator, using

$$G^{(\phi)}(k_\perp, q_\parallel) = \frac{1}{2\pi} \int d^2x_\perp \, e^{-ik_\perp \cdot x_\perp} e^{-x_\perp^2/4l^2} \sum_{n=0}^{\infty} \frac{L_n(x_\perp^2/2l^2)}{\lambda_n(q_\parallel)} \tag{5.B.47}$$

This allows us to invert the transverse direction Fourier transforms to get

$$\int \widetilde{dq_\perp} \, \widetilde{dk_\perp} \, e^{iw_\perp \cdot (q_\perp - k_\perp)} \, G(q_\perp, q_\parallel) \, G(k_\perp, q_\parallel) = \frac{1}{(2\pi)^2} \, e^{-w_\perp^2/2l^2} \sum_{n,n'=0}^{\infty} \frac{L_n(w_\perp^2/2l^2) L_{n'}(w_\perp^2/2l^2)}{\lambda_n(q_\parallel) \lambda_{n'}(q_\parallel)} \tag{5.B.48}$$

So substituting back into the Green's function gives

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{e}{2\pi^2 l^2 p^1} \int d^2w_\perp \, \widetilde{dq_\parallel} \, q_3^2 \, w^1 \, e^{-ip_1 w^1} e^{-w_\perp^2/2l^2} \sum_{n,n'=0}^{\infty} \frac{L_n(w_\perp^2/2l^2) L_{n'}(w_\perp^2/2l^2)}{\lambda_n(q_\parallel) \lambda_{n'}(q_\parallel)} \tag{5.B.49}$$

Wick rotation and rescaling of longitudinal integral

Here we fill in the details of the Wick rotation of the longitudinal loop momenta. We have [4]

$$\beta_2^{(S)} = -\frac{iel^4}{2\pi}\int \widetilde{dq_\parallel}\; q_3^2 \sum_{n,n'=0}^{\infty} \frac{I_{n,n'}}{\lambda_n(q_\parallel)\lambda_{n'}(q_\parallel)} \tag{5.B.50}$$

where $I_{n,n'}$ is a Laguerre-type integral. We can Wick rotate the longitudinal momentum $q_\parallel$ by setting $q^0 = i\Omega$, $q^3 = k$. Then $d^2 q_\parallel = i\, d\Omega\, dk$. We get

$$\beta_2^{(S)} = \frac{el^4}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\, k^2 \sum_{n,n'=0}^{\infty}\frac{I_{n,n'}}{[l^2(m^2+\Omega^2+k^2)+2n+1][l^2(m^2+\Omega^2+k^2)+2n'+1]} \tag{5.B.51}$$

Now the denominator is symmetric in Ω and k so we can write

$$\beta_2^{(S)} = \frac{el^4}{4\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi} \sum_{n,n'=0}^{\infty}\frac{(\Omega^2+k^2)I_{n,n'}}{[l^2(m^2+\Omega^2+k^2)+2n+1][l^2(m^2+\Omega^2+k^2)+2n'+1]} \tag{5.B.52}$$

Rescale and transform to polar coordinates with the following change of variables.

$$l\,\Omega = \rho\cos\varphi \tag{5.B.53a}$$

$$l\,k = \rho\sin\varphi \tag{5.B.53b}$$

Then $l^2\, d\Omega\, dk = \rho\, d\rho\, d\varphi$ and the expression becomes

$$\beta_2^{(S)} = \frac{e}{8\pi^2}\int_0^{2\pi}\frac{d\varphi}{2\pi}\int_0^{\infty} d\rho\,\rho^3 \sum_{n,n'=0}^{\infty}\frac{I_{n,n'}}{(\mu+\rho^2+2n+1)(\mu+\rho^2+2n'+1)} \tag{5.B.54}$$

where $\mu = ml$ is a dimensionless number. Clearly the azimuthal integral integrates to 1, and as before we can make this even simpler by substituting $x = \rho^2$ to get

$$\beta_2^{(S)} = \frac{e}{16\pi^2}\int_0^{\infty} dx\; x\; S^{(\phi)}(x;\mu) \tag{5.B.55}$$

where $S^{(\phi)}$ is the series given by

$$S^{(\phi)}(x;\mu) = \sum_{n,n'=0}^{\infty}\frac{I_{n,n'}}{(\mu+x+2n+1)(\mu+x+2n'+1)} \tag{5.B.56}$$

[4] We assume throughout that the series and all integrals commute with each other.

5.C Details of fermion calculation

5.C.1 Derivation of fermion propagator

Much of the work carries over from the scalar case. Beginning with the QED Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \tag{5.C.1}$$

We obtain the equation of motion for the Dirac spinor

$$(i\not{D} - m)\psi = 0 \tag{5.C.2}$$

So we have the Green's equation

$$\left(m - i\not{D}^{(x)}\right) G^{(\psi)}(x,y) = \delta^{(4)}(x-y) \tag{5.C.3}$$

We can Fourier transform in the longitudinal directions to get

$$\left(m + \gamma^a p_a - i\gamma^i \pi_i\right) G^{(\psi)}(p_\parallel; x_\perp, y_\perp) = \delta^{(2)}(x_\perp - y_\perp) \tag{5.C.4}$$

and then act on each side with $(m - \gamma^a p_a + i\gamma^i \pi_i)$ to get

$$\left(m^2 + p_\parallel^2 - \pi_\perp^2 - \frac{i}{l^2}\gamma^1\gamma^2\right) G^{(\psi)}(p_\parallel; x_\perp, y_\perp) = (m - \gamma^a p_a + i\gamma^i \pi_i)\, \delta^{(2)}(x_\perp - y_\perp) \tag{5.C.5}$$

Using the exact same wavefunctions $\Psi_{n,q_2}(x_\perp)$ as we did for the scalar calculation, we have

$$l\gamma^i \pi_i \Psi_{n,q_2}(x_\perp) = \left(\gamma^1 \frac{d}{d\eta} - i\gamma^2 \eta\right) \Psi_{n,q_2} \tag{5.C.6}$$

Using the representation of the Ψ_{n,q_2} in terms of Hermite polynomials $\psi_n(\eta)$ in (5.B.10), we have:

$$\begin{aligned}
l\gamma^i \pi_i \Psi_{n,q_2}(x_\perp) &= \frac{e^{-iqx^2}}{\sqrt{2\pi l}} \gamma^1 \left[P_+ \left(\frac{d}{d\eta} + \eta\right) + P_- \left(\frac{d}{d\eta} - \eta\right)\right] \psi_n(\eta) \\
&= \frac{e^{-iqx^2}}{\sqrt{2\pi l}} \gamma^1 \left[P_+ \sqrt{2n}\, \psi_{n-1}(\eta) - P_- \sqrt{2(n+1)}\, \psi_{n+1}(\eta)\right] \\
&= \gamma^1 \left[\sqrt{2n}\, \Psi_{n-1,q_2}\, P_+ - \sqrt{2(n+1)}\, \Psi_{n+1,q_2}\, P_-\right]
\end{aligned}$$

where the projectors $P_\pm$ are defined by

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm i\gamma^1\gamma^2) \tag{5.C.7}$$

Above, we assumed $B > 0$; however it will later be useful to keep track of which terms depend on the sign of B, as these tell us about the transformation of the propagator under charge conjugation. If we take $B \to -B$, this flips the sign of the Schwinger phase, and thus the sign of the exponent in Ψ_{n,q_2}. Tracing through, this also flips the sign of the term in $-i\gamma_2\eta$ term in (5.C.6) and thus interchanges P_+ and P_i in subsequent expressions; thus to obtain an expression valid for all B, we should also write:

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm i \,\mathrm{sign}(eB)\gamma^1\gamma^2) \tag{5.C.8}$$

Though we will mostly work with $B > 0$ and use the simpler expression (5.C.7), it is helpful to keep this non-analyticity in B in mind.

This expression is valid for all n if we understand Ψ_{n-1} to be 0. We can write the left-hand side of the Green's equation as

$$\begin{aligned}&\left(m^2 + p_\parallel^2 - \pi_\perp^2 - \frac{i}{l^2}\gamma^1\gamma^2\right) G^{(\psi)}(p_\parallel; x_\perp, y_\perp)\\ &= \frac{1}{l^2}\int d^2 z_\perp \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \left[\Psi^*_{n,q_2}(z_\perp)(\lambda_n(p_\parallel; m) - i\gamma^1\gamma^2)\Psi_{n,q_2}(x_\perp) G^{(\psi)}(p_\parallel; z_\perp, y_\perp)\right]\end{aligned} \tag{5.C.9}$$

and the right-hand side as

$$\begin{aligned}&(m - \gamma^a p_a + i\gamma^i\pi_i)\, \delta^{(2)}(x_\perp - y_\perp)\\ &= \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \Psi^*_{n,q_2}(y_\perp)\Big\{(m - \gamma^a p_a)\Psi_{n,q}(x_\perp)\\ &\qquad + \frac{i\gamma^1}{l}\left[\sqrt{2n}\; \Psi_{n-1,q_2}(x_\perp)\; P_+ - \sqrt{2(n+1)}\; \Psi_{n+1,q_2}(x_\perp)\; P_-\right]\Big\}\end{aligned} \tag{5.C.10}$$

Multiplying the Green's equation by $\Psi^*_{n',q_2'}(x_\perp)$, integrating with respect to $x_\perp$ and relabelling allows us to invert to get

$$\begin{aligned}&\frac{1}{l^2}\int d^2 z_\perp \left[\Psi^*_{n,q_2}(z_\perp)(\lambda_n(p_\parallel; m) - i\gamma^1\gamma^2) G^{(\psi)}(p_\parallel; z_\perp, y_\perp)\right]\\ &= (m - \gamma^a p_a)\Psi^*_{n,q_2}(y_\perp) + \frac{i\gamma^1}{l}\left[\sqrt{2(n+1)}\; \Psi^*_{n+1,q_2}(y_\perp)\; P_+ - \sqrt{2n}\; \Psi^*_{n-1,q_2}(y_\perp)\; P_-\right]\end{aligned} \tag{5.C.11}$$

Observe that

$$\lambda_n - i\gamma^1\gamma^2 = (\lambda_n - 1)P_+ + (\lambda_n + 1)P_- \tag{5.C.12}$$

so we can act on both sides with its inverse to get

$$\int d^2 z_\perp \left(\Psi^*_{n,q_2}(z_\perp) G^{(\psi)}(p_\parallel; z_\perp, y_\perp)\right)$$

$$
\begin{aligned}
&= l^2 \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_n + 1}\right) \Bigg\{ (m - \gamma^a p_a) \Psi^*_{n,q_2}(y_\perp) \\
&\qquad\qquad + \frac{i\gamma^1}{l} \left[\sqrt{2(n+1)}\, \Psi^*_{n+1,q_2}(y_\perp)\, P_+ - \sqrt{2n}\, \Psi^*_{n-1,q_2}(y_\perp)\, P_- \right] \Bigg\} \\
&= l^2 \Bigg[(m - \gamma^a p_a) \Psi^*_{n,q_2}(y_\perp) \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_n + 1}\right) \\
&\qquad\qquad + \frac{i\gamma^1}{l} \left(\frac{\sqrt{2(n+1)}}{\lambda_n + 1} \Psi^*_{n+1,q_2}(y_\perp) P_+ - \frac{\sqrt{2n}}{\lambda_n - 1}\, \Psi^*_{n-1,q_2}(y_\perp) P_- \right) \Bigg]
\end{aligned}
$$

Then as we did for the scalar, multiply by $\Psi_{n,q_2}(x_\perp)$, sum over n and integrate with respect to q_2. On the left-hand side this extracts the Green's function. On the right-hand side we run the same method as for the scalar: expand the wavefunction expressions, complete the square of the exponent, then shift the contour so we can do the momentum integral using the Hermite polynomial integral identity.

$$
\begin{aligned}
&(2\pi)\, e^{-i\Phi} e^{\frac{1}{2}\xi}\, G^{(\psi)}(p_\parallel; x_\perp, y_\perp) \\
&= (2\pi)\, e^{-i\Phi} e^{\frac{1}{2}\xi}\, l^2 \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \Psi_{n,q_2}(x_\perp) \Bigg[(m - \gamma^a p_a) \Psi^*_{n,q_2}(y_\perp) \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_n + 1}\right) \\
&\qquad\qquad + \frac{i\gamma^1}{l} \left(\frac{\sqrt{2(n+1)}}{\lambda_n + 1} \Psi^*_{n+1,q_2}(y_\perp) P_+ - \frac{\sqrt{2n}}{\lambda_n - 1}\, \Psi^*_{n-1,q_2}(y_\perp) P_- \right) \Bigg] \\
&= e^{-i\Phi} e^{\frac{1}{2}\xi}\, l \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \psi_n\left(\frac{x^1}{l} + q_2 l\right) \Bigg[(m - \gamma^a p_a) \psi_n\left(\frac{y^1}{l} + q_2 l\right) \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_n + 1}\right) \\
&\qquad\qquad + \frac{i\gamma^1}{l} \left(\frac{\sqrt{2(n+1)}}{\lambda_n + 1} \psi_{n+1}\left(\frac{y^1}{l} + q_2 l\right) P_+ - \frac{\sqrt{2n}}{\lambda_n - 1}\, \psi_{n-1}\left(\frac{y^1}{l} + q_2 l\right) P_- \right) \Bigg] \\
&= \int du\, e^{-u^2} \sum_{n=0}^{\infty} \frac{H_n(u + \alpha^*)}{2^n n! \sqrt{\pi}} \Bigg[(m - \gamma^a p_a) H_n(u - \alpha) \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_n + 1}\right) \\
&\qquad\qquad + \frac{i\gamma^1}{l} \left(\frac{H_{n+1}(u - \alpha)}{\lambda_n + 1}\, P_+ - \frac{2n\, H_{n-1}(u - \alpha)}{\lambda_n - 1}\, P_- \right) \Bigg] \\
&= \sum_{n=0}^{\infty} \left[(m - \gamma^a p_a) L_n(\xi) \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_n + 1}\right) + \frac{i\gamma^1}{l} \left(\frac{(-2\alpha) L_n^{(1)}(\xi)}{\lambda_n + 1}\, P_+ - \frac{2\alpha^* L_{n-1}^{(1)}(\xi)}{\lambda_n - 1}\, P_- \right) \right] \\
&= \sum_{n=0}^{\infty} \left[(m - \gamma^a p_a) L_n(\xi) \left(\frac{P_+}{\lambda_n - 1} + \frac{P_-}{\lambda_{n+1} - 1}\right) - \frac{2i\gamma^1}{l} \left(\frac{\alpha L_n^{(1)}(\xi)}{\lambda_{n+1} - 1}\, P_+ + \frac{\alpha^* L_{n-1}^{(1)}(\xi)}{\lambda_n - 1}\, P_- \right) \right] \\
&= \sum_{n=0}^{\infty} \frac{F_n(p_\parallel; x_\perp, y_\perp)}{\lambda_n(p_\parallel; m) - 1}
\end{aligned}
$$

where Φ is the Schwinger phase and $2l^2\xi = |x_\perp - y_\perp|^2$, exactly as before.

In the lowest Landau level, we have

$$\frac{F_0(p_\parallel;x_\perp,y_\perp)}{\lambda_0-1}=\frac{(m-\gamma^a p_a)P_+}{l^2(m^2+p_\parallel^2)} \tag{5.C.13}$$

In every excited Landau level, we have

$$F_n(p_\parallel;x_\perp,y_\perp)=(m-\gamma^a p_a)(L_n(\xi)P_+ + L_{n-1}(\xi)P_-)-\frac{2i\gamma^1}{l}(\alpha P_+ + \alpha^* P_-)L^{(1)}_{n-1}(\xi) \tag{5.C.14}$$

We can easily show that

$$2l\gamma^1(\alpha P_+ + \alpha^* P_-)=\gamma^j z_j \tag{5.C.15}$$

where $z_\perp \equiv x_\perp - y_\perp$. Hence our final expression for the numerator is

$$F_n(p_\parallel;x_\perp,y_\perp)=(m-\gamma^a p_a)(L_n(\xi)P_+ + L_{n-1}(\xi)P_-)-\frac{i}{l^2}\gamma^j z_j L^{(1)}_{n-1}(\xi) \tag{5.C.16}$$

This is valid for $n=0$ also if we understand that $L^{(\alpha)}_{-1}=0$, so we can write

$$G^{(\psi)}(p_\parallel;x_\perp,y_\perp)=\frac{1}{2\pi}e^{i\Phi}e^{-\frac{1}{2}\xi}\sum_{n=0}^{\infty}\frac{(m-\gamma^a p_a)(L_n(\xi)P_+ + L_{n-1}(\xi)P_-)-\frac{i}{l^2}\gamma^j z_j L^{(1)}_{n-1}(\xi)}{l^2(m^2+p_\parallel^2)+2n} \tag{5.C.17}$$

Further, we can Fourier transform the translationary invariant part of the propagator:

$$\tilde{G}^{(\psi)}(p)=2l^2 e^{-p_\perp^2 l^2}\sum_{n=0}^{\infty}\frac{(-1)^n D_n(p)}{l^2(m^2+p_\parallel^2)+2n} \tag{5.C.18}$$

where the numerator is given by

$$D_n(p)=(m-\gamma^a p_a)\left(L_n\left(2p_\perp^2 l^2\right)P_+ - L_{n-1}\left(2p_\perp^2 l^2\right)P_-\right)+2\gamma^j p_j L^{(1)}_{n-1}(2p_\perp^2 l^2) \tag{5.C.19}$$

See Appendix 5.D.2 for the relevant details concerning Fourier transforms of Laguerre polynomials.

5.C.2 Computation of Feynman diagram

For completeness, we provide some further details of the computation of the current stress tensor Green's function for the fermion.

We have the three-point function

$$\langle \mathcal{T} A_\mu(x)\bar{\psi}(y)\gamma_\nu\psi(z)\rangle = e\, g^{\rho\sigma}\int d^4w\; G^{(\gamma)}_{\mu\rho}(x,w)V_{\nu\sigma}(w,y,z) \tag{5.C.20}$$

Differentiating, we get

$$\left(\partial_\lambda^{(z)} - \partial_\lambda^{(y)}\right) \langle \mathcal{T} A_\mu(x) \bar{\psi}(y) \gamma_\nu \psi(z) \rangle = e\, g^{\rho\sigma} \int d^4w\; G^{(\gamma)}_{\mu\rho}(x,w) U_{\lambda\nu\sigma}(w,y,z) \quad (5.C.21)$$

Now add the components with $\lambda = 2, \nu = 3$ and $\lambda = 3, \nu = 2$ then evaluate at $z = y$.

$$\langle \mathcal{T} A_\mu(x) T^{23}_{(F)}(y) \rangle = \frac{ie}{4}\; g^{\rho\sigma} \int d^4w\; G^{(\gamma)}_{\mu\rho}(x,w) \left[U_{23\sigma}(w,y,y) + U_{32\sigma}(w,y,y)\right] \quad (5.C.22)$$

Differentiating with respect to x^λ and antisymmetrising, we have

$$\begin{aligned} &\langle \mathcal{T} F_{\lambda\mu}(x) T^{23}_{(F)}(y) \rangle \\ &= -\frac{e}{4}\; g^{\rho\sigma} \int d^4w\; \widetilde{dp}\; e^{ip\cdot(x-w)} \left[p_\lambda \tilde{G}^{(\gamma)}_{\mu\rho}(p) - p_\mu \tilde{G}^{(\gamma)}_{\lambda\rho}(p)\right] \left[U_{23\sigma}(w,y,y) + U_{32\sigma}(w,y,y)\right] \end{aligned} \quad (5.C.23)$$

Now take the Fourier transform with momentum purely in the 1-direction.

$$\langle \tilde{F}_{13}(p^1) \tilde{T}^{23}_{(F)}(-p^1) \rangle = -\frac{e}{4}\; p_1\; g^{\rho\sigma}\; \tilde{G}^{(\gamma)}_{3\rho}(p^1) \int d^4w\; e^{-ip\cdot w}\; \left[U_{23\sigma}(w,0,0) + U_{32\sigma}(w,0,0)\right] \quad (5.C.24)$$

Hence after substituting the Fourier transform of the photon propagator and identifying the 2-form current, we have

$$\langle \tilde{J}_{02}(p^1) \tilde{T}^{23}_{(F)}(-p^1) \rangle = -\frac{e}{4p^1} \int d^4w\; e^{-ip\cdot w}\; \left[U_{233}(w,0,0) + U_{323}(w,0,0)\right] \quad (5.C.25)$$

Now to write the relevant components of $U_{\mu\nu\rho}(w,0,0)$ in terms of the propagators $\tilde{G}^{(\psi)}(q)$.

$$\begin{aligned} &U_{233}(w,y,z) \\ &= \left(\partial_2^{(z)} - \partial_2^{(y)}\right) V_{33}(w,y,z) \\ &= \left(\partial_2^{(z)} - \partial_2^{(y)}\right) \mathrm{Tr}\left[\gamma_3\; G^{(\psi)}(z,w)\; \gamma_3\; G^{(\psi)}(w,y)\right] \\ &= \left(\partial_2^{(z)} - \partial_2^{(y)}\right) \int \widetilde{dq}\; \widetilde{dk}\; e^{i\Phi(z_\perp,w_\perp)} e^{iq\cdot(w-y)} e^{i\Phi(w_\perp,y_\perp)} e^{ik\cdot(z-w)}\; \mathrm{Tr}\left[\gamma_3\; \tilde{G}^{(\psi)}(q)\; \gamma_3\; \tilde{G}^{(\psi)}(k)\right] \\ &= i \int \widetilde{dq}\; \widetilde{dk}\; (q_2 + k_2 + \partial_2^{(z)}\Phi(z_\perp,w_\perp) + \partial_2^{(y)}\Phi(y_\perp,w_\perp)) e^{i\Phi(z_\perp,w_\perp)} e^{iq\cdot(w-y)} e^{i\Phi(w_\perp,y_\perp)} e^{ik\cdot(z-w)} \\ &\qquad \mathrm{Tr}\left[\gamma_3\; \tilde{G}^{(\psi)}(q)\; \gamma_3\; \tilde{G}^{(\psi)}(k)\right] \end{aligned}$$

If we evaluate this expression at $z = y$ then the Schwinger phases cancel and the expression simplifies to

$$\begin{aligned} &U_{233}(w,y,y) \\ &= i \int \widetilde{dq}\; \widetilde{dk}\; (q_2 + k_2 + 2\; \partial_2^{(y)}\Phi(y_\perp,w_\perp))\; e^{i(q-k)\cdot(w-y)}\; \mathrm{Tr}\left[\gamma_3\; \tilde{G}^{(\psi)}(q)\; \gamma_3\; \tilde{G}^{(\psi)}(k)\right] \end{aligned}$$

$$= i \int \widetilde{dq}\, \widetilde{dk} \, \left(q_2 + k_2 + \frac{w^1 + y^1}{l^2} \right) \, e^{i(q-k)\cdot(w-y)} \, \mathrm{Tr} \left[\gamma_3 \, \tilde{G}^{(\psi)}(q) \, \gamma_3 \, \tilde{G}^{(\psi)}(k) \right]$$

where we used

$$2 \, \partial_2^{(y)} \Phi(y_\perp, w_\perp) = \frac{y^1 + w^1}{l^2} \tag{5.C.26}$$

Finally evaluating at $y = 0$ gives

$$U_{233}(w, 0, 0) = i \int \widetilde{dq}\, \widetilde{dk} \, \left(q_2 + k_2 + \frac{w^1}{l^2} \right) \, e^{iw\cdot(q-k)} \, \mathrm{Tr} \left[\gamma_3 \, \tilde{G}^{(\psi)}(q) \, \gamma_3 \, \tilde{G}^{(\psi)}(k) \right] \tag{5.C.27}$$

Similarly we also have

$$\begin{aligned}
U_{323}(w, y, z) &= \left(\partial_3^{(z)} - \partial_3^{(y)} \right) V_{23}(w, y, z) \\
&= \left(\partial_3^{(z)} - \partial_3^{(y)} \right) \mathrm{Tr} \left[\gamma_2 \, G^{(\psi)}(z, w) \, \gamma_3 \, G^{(\psi)}(w, y) \right] \\
&= \left(\partial_3^{(z)} - \partial_3^{(y)} \right) \int \widetilde{dq}\, \widetilde{dk} \; e^{i\Phi(z_\perp, w_\perp)} e^{iq\cdot(w-y)} e^{i\Phi(w_\perp, y_\perp)} e^{ik\cdot(z-w)} \, \mathrm{Tr} \left[\gamma_2 \, \tilde{G}^{(\psi)}(k) \, \gamma_3 \, \tilde{G}^{(\psi)}(q) \right] \\
&= i \int \widetilde{dq}\, \widetilde{dk} \; (q_3 + k_3) e^{i\Phi(z_\perp, w_\perp)} e^{iq\cdot(w-y)} e^{i\Phi(w_\perp, y_\perp)} e^{ik\cdot(z-w)} \; \mathrm{Tr} \left[\gamma_2 \, \tilde{G}^{(\psi)}(k) \, \gamma_3 \, \tilde{G}^{(\psi)}(q) \right]
\end{aligned}$$

Again evaluating at $z = y$:

$$U_{323}(w, y, y) = i \int \widetilde{dq}\, \widetilde{dk} \; (q_3 + k_3) \; e^{i(q-k)\cdot(w-y)} \, \mathrm{Tr} \left[\gamma_2 \, \tilde{G}^{(\psi)}(k) \, \gamma_3 \, \tilde{G}^{(\psi)}(q) \right]$$

Finally evaluating at $y = 0$ gives

$$U_{323}(w, 0, 0) = i \int \widetilde{dq}\, \widetilde{dk} \; (q_3 + k_3) \; e^{iw\cdot(q-k)} \, \mathrm{Tr} \left[\gamma_2 \, \tilde{G}^{(\psi)}(k) \, \gamma_3 \, \tilde{G}^{(\psi)}(q) \right] \tag{5.C.28}$$

After substituting these expressions into the current stress tensor Green's function, we obtain

$$\begin{aligned}
&\tilde{G}^{02,23}_{JT}(p_1) \\
&= \frac{-e}{4p^1} \int d^4 w \; \widetilde{dq}\, \widetilde{dk} \; e^{-ip_1 w^1} e^{iw\cdot(q-k)} \\
&\qquad \left\{ \left(q_2 + k_2 + \frac{w^1}{l^2} \right) \, \mathrm{Tr} \left[\gamma_3 \, \tilde{G}^{(\psi)}(q) \, \gamma_3 \, \tilde{G}^{(\psi)}(k) \right] + (q_3 + k_3) \; \mathrm{Tr} \left[\gamma_2 \, \tilde{G}^{(\psi)}(k) \, \gamma_3 \, \tilde{G}^{(\psi)}(q) \right] \right\}
\end{aligned} \tag{5.C.29}$$

Now we can expand the Green's function as sums over Landau levels by substituting the expression for the Fourier-transformed fermion propagator. This gives

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{-el^4}{p^1} \int d^4w\, \widetilde{dq}\, \widetilde{dk}\, e^{-ip_1w^1} e^{iw\cdot(q-k)} e^{-l^2q_\perp^2}\, e^{-l^2k_\perp^2} \sum_{n,n'=0}^{\infty} \frac{(-1)^{n+n'}}{\sigma_n(q_\|;m)\sigma_{n'}(k_\|;m)}$$
$$\left[\left(q_2 + k_2 + \frac{w^1}{l^2}\right) \mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k)) + (q_3+k_3)\,\mathrm{Tr}(\gamma^3 D_n(q)\gamma^2 D_{n'}(k))\right] \tag{5.C.30}$$

As in the scalar case, we will demonstrate that the terms linear in q_2 and k_2 must vanish. Indeed,

$$\begin{aligned}
&\int d^4w\, \widetilde{dq}\, \widetilde{dk}\, q_2\, e^{-ip_1w^1} e^{iw\cdot(q-k)} e^{-l^2q_\perp^2}\, e^{-l^2k_\perp^2} \left[\frac{\mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k))}{\sigma_n(q_\|;m)\sigma_{n'}(k_\|;m)}\right] \\
&= \int dw_1\, \widetilde{dq}\, \frac{dk_1}{2\pi}\, q_2\, e^{-iw^1(p_1-q_1+k_1)} e^{-2l^2q_2^2}\, e^{-l^2(q_1^2+k_1^2)} \left[\frac{\mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k_1,q_2,q_\|))}{\sigma_n(q_\|;m)\sigma_{n'}(q_\|;m)}\right] \\
&= 0
\end{aligned}$$

since the integrand is odd in q_2 and we integrate q_2 over all of $\mathbb{R}$. Similarly we have

$$\int d^4w\, \widetilde{dq}\, \widetilde{dk}\, k_2\, e^{-ip_1w^1} e^{iw\cdot(q-k)} e^{-l^2q_\perp^2}\, e^{-l^2k_\perp^2} \left[\frac{\mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k))}{\sigma_n(q_\|;m)\sigma_{n'}(k_\|;m)}\right] = 0$$

So the expression simplifies to

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{-el^4}{p^1} \int d^4w\, \widetilde{dq}\, \widetilde{dk}\, e^{-ip_1w^1} e^{iw\cdot(q-k)} e^{-l^2q_\perp^2}\, e^{-l^2k_\perp^2} \sum_{n,n'=0}^{\infty} \frac{(-1)^{n+n'}}{\sigma_n(q_\|;m)\sigma_{n'}(k_\|;m)}$$
$$\left[\frac{w^1}{l^2} \mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k)) + (q_3+k_3)\,\mathrm{Tr}(\gamma^3 D_n(q)\gamma^2 D_{n'}(k))\right] \tag{5.C.31}$$

Now we can expand the traces using

$$\begin{aligned}
&\mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k)) \\
&= -2m^2[L_n(2q_\perp^2l^2)L_{n'}(2k_\perp^2l^2) + L_{n-1}(2q_\perp^2l^2)L_{n'-1}(2k_\perp^2l^2)] - 16q^ik_iL^{(1)}_{n-1}(2q_\perp^2l^2)L^{(1)}_{n'-1}(2k_\perp^2l^2)
\end{aligned} \tag{5.C.32a}$$

$$\begin{aligned}
&\mathrm{Tr}(\gamma^3 D_n(q)\gamma^2 D_{n'}(k)) \\
&= 4[L_{n-1}(2q_\perp^2l^2)L^{(1)}_{n'-1}(2k_\perp^2l^2)q_3(k_2+ik_1) - L^{(1)}_{n-1}(2q_\perp^2l^2)L_{n'}(2k_\perp^2l^2)k_3(q_2+iq_1) \\
&\qquad + L_{n'-1}(2k_\perp^2l^2)L^{(1)}_{n-1}(2q_\perp^2l^2)k_3(q_2-iq_1) - L^{(1)}_{n'-1}(2k_\perp^2l^2)L_n(2q_\perp^2l^2)q_3(k_2-ik_1)]
\end{aligned} \tag{5.C.32b}$$

Once again we can neglect the terms which are odd in q_2 and k_2 since they vanish under the integral sign. Hence

$$\tilde{G}^{02,23}_{JT}(p_1) = -\frac{el^4}{p^1}\int d^4w\, \widetilde{dq}\, \widetilde{dk}\, e^{-ip_1w^1}e^{iw\cdot(q-k)}e^{-l^2q_\perp^2}\, e^{-l^2k_\perp^2}\sum_{n,n'=0}^{\infty}\frac{(-1)^{n+n'}}{\sigma_n(q_\parallel;m)\sigma_{n'}(k_\parallel;m)}$$
$$\Bigg\{\frac{-2w^1}{l^2}\Big(m^2[L_n(2q_\perp^2l^2)L_{n'}(2k_\perp^2l^2)+L_{n-1}(2q_\perp^2l^2)L_{n'-1}(2k_\perp^2l^2)]$$
$$+8q^ik_iL^{(1)}_{n-1}(2q_\perp^2l^2)L^{(1)}_{n'-1}(2k_\perp^2l^2)\Big)$$
$$+4i(q_3+k_3)\Big(L_{n-1}(2q_\perp^2l^2)L^{(1)}_{n'-1}(2k_\perp^2l^2)q_3k_1-L^{(1)}_{n-1}(2q_\perp^2l^2)L_{n'}(2k_\perp^2l^2)k_3q_1$$
$$-L_{n'-1}(2k_\perp^2l^2)L^{(1)}_{n-1}(2q_\perp^2l^2)k_3q_1+L^{(1)}_{n'-1}(2k_\perp^2l^2)L_n(2q_\perp^2l^2)q_3k_1\Big)\Bigg\} \tag{5.C.33}$$

An obvious and straightforward step is to perform the longitudinal position integral over $w_\parallel$. This will yield a delta function so that we can do the $k_\parallel$ integral. This simplifies the expression to

$$\tilde{G}^{02,23}_{JT}(p_1) = -\frac{el^4}{p^1}\int d^2w_\perp\, \widetilde{dq}\, \widetilde{dk_\perp}\, e^{-ip_1w^1}e^{iw_\perp\cdot(q_\perp-k_\perp)}e^{-l^2q_\perp^2}\, e^{-l^2k_\perp^2}\sum_{n,n'=0}^{\infty}\frac{(-1)^{n+n'}}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)}$$
$$\Bigg\{\frac{-2w^1}{l^2}\Big(m^2[L_n(2q_\perp^2l^2)L_{n'}(2k_\perp^2l^2)+L_{n-1}(2q_\perp^2l^2)L_{n'-1}(2k_\perp^2l^2)]$$
$$+8q^ik_iL^{(1)}_{n-1}(2q_\perp^2l^2)L^{(1)}_{n'-1}(2k_\perp^2l^2)\Big)$$
$$+8iq_3^2\Big(L_{n-1}(2q_\perp^2l^2)L^{(1)}_{n'-1}(2k_\perp^2l^2)k_1-L^{(1)}_{n-1}(2q_\perp^2l^2)L_{n'}(2k_\perp^2l^2)q_1$$
$$-L_{n'-1}(2k_\perp^2l^2)L^{(1)}_{n-1}(2q_\perp^2l^2)q_1+L^{(1)}_{n'-1}(2k_\perp^2l^2)L_n(2q_\perp^2l^2)k_1\Big)\Bigg\} \tag{5.C.34}$$

To save writing, we can suppress the arguments of the (generalised) Laguerre polynomials - the argument being implied by the subscript (e.g. n for $2q_\perp^2l^2$).

Then we have

$$\tilde{G}^{02,23}_{JT}(p_1) = -\frac{2el^4}{p^1}\int d^2w_\perp\,\widetilde{dq}\,\widetilde{dk_\perp}\;e^{-ip_1w^1}e^{iw_\perp\cdot(q_\perp-k_\perp)}e^{-l^2q_\perp^2}\;e^{-l^2k_\perp^2}\sum_{n,n'=0}^{\infty}\frac{(-1)^{n+n'}}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)}$$
$$\left\{\frac{-w^1}{l^2}\left[m^2(L_nL_{n'}+L_{n-1}L_{n'-1})+8q^ik_iL^{(1)}_{n-1}L^{(1)}_{n'-1}\right]\right.$$
$$\left.+\,4iq_3^2\left[k_1L^{(1)}_{n'-1}\left(L_{n-1}+L_n\right)-q_1L^{(1)}_{n-1}\left(L_{n'-1}+L_{n'}\right)\right]\right\} \tag{5.C.35}$$

Now we can invert the transverse Fourier transforms, using

$$(-1)^n\;l^2e^{-p_\perp^2l^2}L_n\left(2p_\perp^2l^2\right) = \frac{1}{4\pi}\int d^2x_\perp\;e^{-ip_\perp\cdot x_\perp}e^{-x_\perp^2/4l^2}L_n\left(x_\perp^2/2l^2\right) \tag{5.C.36a}$$
$$(-1)^n\;l^2p_j\;e^{-p_\perp^2l^2}L^{(1)}_{n-1}\left(2p_\perp^2l^2\right) = -\frac{i}{8\pi l^2}\int d^2x_\perp\;x_j\;e^{-ip_\perp\cdot x_\perp}e^{-x_\perp^2/4l^2}L^{(1)}_{n-1}\left(x_\perp^2/2l^2\right) \tag{5.C.36b}$$

Hence we have the lengthy expression

$$\tilde{G}^{02,23}_{JT}(p_1)$$
$$= -\frac{2e}{(4\pi)^2p^1}\int d^2w_\perp\,\widetilde{dq_\parallel}\,\widetilde{dq_\perp}\,\widetilde{dk_\perp}\;d^2y_\perp\;d^2z_\perp\;e^{-ip_1w^1}e^{iq_\perp\cdot(w_\perp-y_\perp)}e^{-ik_\perp\cdot(w_\perp+z_\perp)}e^{-y_\perp^2/4l^2}e^{-z_\perp^2/4l^2}$$
$$\sum_{n,n'=0}^{\infty}\frac{1}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)}\left\{\frac{-w^1}{l^2}\left[m^2\Bigg(L_n\left(y_\perp^2/2l^2\right)L_{n'}\left(z_\perp^2/2l^2\right)\right.\right.$$
$$\left.+\,L_{n-1}\left(y_\perp^2/2l^2\right)L_{n'-1}\left(z_\perp^2/2l^2\right)\Bigg)-\frac{2y^jz_j}{l^4}L^{(1)}_{n-1}\left(y_\perp^2/2l^2\right)L^{(1)}_{n'-1}\left(z_\perp^2/2l^2\right)\right]$$
$$+\,\frac{2q_3^2}{l^2}\left[z_1L^{(1)}_{n'-1}\left(z_\perp^2/2l^2\right)\left(L_n\left(y_\perp^2/2l^2\right)-L_{n-1}\left(y_\perp^2/2l^2\right)\right)-\right.$$
$$\left.\left.y_1L^{(1)}_{n-1}\left(y_\perp^2/2l^2\right)\left(L_{n'}\left(z_\perp^2/2l^2\right)-L_{n'-1}\left(z_\perp^2/2l^2\right)\right)\right]\right\}$$

We can straightforwardly do the transverse momentum integrals, yielding delta functions. Then after trivial integrals over $y_\perp$ and $z_\perp$ we have

$$\tilde{G}^{02,23}_{JT}(p_1) = \frac{2e}{(4\pi)^2p^1l^2}\int d^2w_\perp\,\widetilde{dq_\parallel}\;e^{-ip_1w^1}e^{-w_\perp^2/2l^2}\sum_{n,n'=0}^{\infty}\frac{1}{\sigma_n(q_\parallel;m)\sigma_{n'}(q_\parallel;m)}$$
$$\left\{w^1\;m^2\left(L_n\left(w_\perp^2/2l^2\right)L_{n'}\left(w_\perp^2/2l^2\right)+L_{n-1}\left(w_\perp^2/2l^2\right)L_{n'-1}\left(w_\perp^2/2l^2\right)\right)\right.$$
$$-\,\frac{2w_1w_\perp^2}{l^4}L^{(1)}_{n-1}\left(w_\perp^2/2l^2\right)L^{(1)}_{n'-1}\left(w_\perp^2/2l^2\right)$$

$$+ 4\, q_3^2\, w_1\, L^{(1)}_{n'-1}\left(w_\perp^2/2l^2\right)\left[L_n\left(w_\perp^2/2l^2\right) - L_{n-1}\left(w_\perp^2/2l^2\right)\right]\Bigg\}$$

5.C.3 Trace identities

From the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} \tag{5.C.37}$$

we can derive that

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = -4g^{\mu\nu} \tag{5.C.38a}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{5.C.38b}$$

$$\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}) = 0 \tag{5.C.38c}$$

Together with the projector properties,

$$P_\pm P_\pm = P_\pm \tag{5.C.39a}$$

$$P_\pm P_\mp = 0 \tag{5.C.39b}$$

and the relations

$$[P_\pm, \gamma^a] = 0 \tag{5.C.40a}$$

$$P_\pm\gamma^i = \gamma^i P_\mp \tag{5.C.40b}$$

we find that

$$\mathrm{Tr}(\gamma^a P_\pm \gamma^j) = 0 \tag{5.C.41}$$

and

$$\mathrm{Tr}(\gamma^3\gamma^i\gamma^2\gamma^a P_\pm) = 2g^{3a}(g^{i2} + ig^{i1}) \tag{5.C.42}$$

Hence we can show that

$$\begin{aligned}
&\mathrm{Tr}(\gamma^3 D_n(q)\gamma^3 D_{n'}(k)) \\
&= -2m^2[L_n(2q_\perp^2 l^2)L_{n'}(2k_\perp^2 l^2) + L_{n-1}(2q_\perp^2 l^2)L_{n'-1}(2k_\perp^2 l^2)] \\
&\quad - 16q^i k_i L^{(1)}_{n-1}(2q_\perp^2 l^2)L^{(1)}_{n'-1}(2k_\perp^2 l^2)
\end{aligned} \tag{5.C.43}$$

and

$$\begin{aligned}
&\mathrm{Tr}(\gamma^3 D_n(q)\gamma^2 D_{n'}(k)) \\
&= 2(q_a k_i L_{n-1}L^{(1)}_{n'-1} - k_a q_i L^{(1)}_{n-1}L_{n'})\,\mathrm{Tr}(\gamma^3\gamma^i\gamma^2\gamma^a P_+) \\
&\qquad + 2(k_a q_i L_{n'-1}L^{(1)}_{n-1} - q_a k_i L^{(1)}_{n'-1}L_n)\,\mathrm{Tr}(\gamma^3\gamma^i\gamma^2\gamma^a P_-) \\
&= 4[L_{n-1}(2q_\perp^2 l^2)L^{(1)}_{n'-1}(2k_\perp^2 l^2)q_3(k_2 + ik_1) - L^{(1)}_{n-1}(2q_\perp^2 l^2)L_{n'}(2k_\perp^2 l^2)k_3(q_2 + iq_1)
\end{aligned}$$

$$+ L_{n'-1}(2k_\perp^2 l^2) L_{n-1}^{(1)}(2q_\perp^2 l^2) k_3(q_2 - iq_1) - L_{n'-1}^{(1)}(2k_\perp^2 l^2) L_n(2q_\perp^2 l^2) q_3(k_2 - ik_1)]$$

5.D Laguerre polynomials

5.D.1 Laguerre polynomial structure

This appendix provides the details of the orthogonality structure of generalised Laguerre polynomials. Using this, we thoroughly demonstrate a method to evaluate position-space integrals which arise in the calculation of the current stress tensor Green's function.

Scalar part

First define the following integral.

$$I_{n,n'} \equiv \int_0^\infty dx\; x^2\; e^{-x} L_n(x)\; L_{n'}(x) \tag{5.D.1}$$

This looks very similar to the inner product of generalised Laguerre polynomials:

$$\langle n|n'\rangle \equiv \int_0^\infty dx\; x^2\; e^{-x}\; L_n^{(2)}(x)\; L_{n'}^{(2)}(x) = (n+2)(n+1)\; \delta_{n,n'} \tag{5.D.2}$$

The generalised Laguerre polynomials with superscript 0 are the usual Laguerre polynomials, and the generalised Laguerre polynomials of order 0 are just the constant 1 function:

$$L_n^{(0)} = L_n \tag{5.D.3a}$$
$$L_0^{(\alpha)} = 1 \tag{5.D.3b}$$

so we have $I_{0,0} = \langle 0|0\rangle = 2$. For $n > 0$ we can use the "three-point-rule" of generalised Laguerre polynomials

$$L_n^{(\alpha)} = L_n^{(\alpha+1)} - L_{n-1}^{(\alpha+1)} \tag{5.D.4}$$

to write $I_{n,n'}$ in terms of $\langle n|n'\rangle$. For example we have

$$L_1^{(0)} = L_1^{(2)} - 2L_0^{(2)} \tag{5.D.5}$$

which gives $I_{1,0} = \langle 1|0\rangle - 2\,\langle 0|0\rangle = -4$. We summarise the first few values below, obtained similarly.[5]

$$
\begin{aligned}
I_{0,0} &= \langle 0|0\rangle &&= 2 && \text{(5.D.6a)}\\
I_{0,1} &= \langle 1|0\rangle - 2\,\langle 0|0\rangle &&= -4 && \text{(5.D.6b)}\\
I_{1,1} &= \langle 1|1\rangle + 4\,\langle 0|0\rangle &&= 14 && \text{(5.D.6c)}\\
I_{0,2} &= \langle 2|0\rangle - 2\,\langle 1|0\rangle + \langle 0|0\rangle &&= 2 && \text{(5.D.6d)}\\
I_{1,2} &= \langle 2|1\rangle - 2\,\langle 2|0\rangle - 2\,\langle 1|1\rangle + 5\,\langle 1|0\rangle - 2\,\langle 0|0\rangle &&= -16 && \text{(5.D.6e)}
\end{aligned}
$$

In general for $n \geq 2$ we have

$$
\begin{aligned}
I_{0,n} &= \langle 0|n-2\rangle - 2\,\langle 0|n-1\rangle + \langle 0|n\rangle && \text{(5.D.7a)}\\
I_{1,n} &= \langle 1|n-2\rangle - 2\,\langle 1|n-1\rangle + \langle 1|n\rangle - 2\,I_{0,n} && \text{(5.D.7b)}
\end{aligned}
$$

When $n, n' \geq 2$, we can apply the three-point rule twice to each Laguerre polynomial to obtain (inductively)

$$
\begin{aligned}
I_{n,n'} = \langle n|n'\rangle \;+\; 4\,\langle n-1|n'-1\rangle \;+\; \langle n-2|n'-2\rangle \\
- \left[2\,\langle n|n'-1\rangle - \langle n|n'-2\rangle + 2\,\langle n-1|n'-2\rangle + (n \leftrightarrow n')\right]
\end{aligned}
\tag{5.D.8}
$$

From the above we deduce that $I_{n,n'} = 0$ whenever $|n - n'| > 2$. For $n \geq 2$ we have

$$
\begin{aligned}
I_{n,n} &= 2 + 6n + 6n^2 &&\equiv f_0(n) && \text{(5.D.9a)}\\
I_{n,n+1} &= -4 - 8n - 4n^2 &&\equiv f_1(n) && \text{(5.D.9b)}\\
I_{n,n+2} &= 2 + 3n + n^2 &&\equiv f_2(n) && \text{(5.D.9c)}
\end{aligned}
$$

In fact, we can check that $f_0(0) = 2 = I_{0,0}$ and $f_0(1) = 14 = I_{1,1}$. Similarly we can evaluate f_1 and f_2 at $n = 0, 1$ and compare with $I_{0,1}$ and so on. We conclude that the quadratics f_0, f_1 and f_2 give the respective values of $I_{n,n}, I_{n,n+1}$ and $I_{n,n+2}$ for all $n \geq 0$. This is sufficient for the scalar loop calculation.

Fermion part

To compute the Dirac fermion transport coefficient, there is further work to do. We will need to slightly generalise the above knowledge of the Laguerre polynomial integrals. Define the integral

$$
I^{(a,b,c)}_{n,n'} \equiv \int_0^\infty dx\; x^{2+c}\, e^{-x} L^{(a)}_n(x) L^{(b)}_{n'}(x) \tag{5.D.10}
$$

[5] $I_{n,n'}$ is trivially symmetric under the exchange of n and n' so we can assume $n \leq n'$.

so that $I^{(0,0,0)}_{n,n'} = I_{n,n'}$, the integral we defined in the scalar part. For the fermion, in addition to $I^{(0,0,0)}_{n,n'}$ we will need to compute $I^{(1,1,1)}_{n,n'}$ and $I^{(0,1,0)}_{n,n'}$.

To calculate $I^{(1,1,1)}_{n,n'}$ we can apply the exact method we used for $I^{(0,0,0)}_{n,n'}$ since the level of the generalised Laguerre polynomials in the integrand is once again 2 less than the power of x in the integrand. Hence we can apply the 3-point rule twice and use the orthogonality relation of generalised Laguerre polynomials. We immediately deduce that $I^{(1,1,1)}_{n,n'} = 0$ whenever $|n - n'| > 2$ and that $I^{(1,1,1)}_{n,n'} = I^{(1,1,1)}_{n',n}$.

Recall the 3-point rule

$$L^{(\alpha)}_n = L^{(\alpha+1)}_n - L^{(\alpha+1)}_{n-1} \tag{5.D.11}$$

We can generalise the inner product defined earlier to

$$\langle n|n'\rangle^{(\alpha)} = \int_0^\infty dx\, x^\alpha\, e^{-x}\, L^{(\alpha)}_n(x)\, L^{(\alpha)}_{n'}(x) = \frac{\Gamma(\alpha+n+1)}{n!}\, \delta_{n,n'} \tag{5.D.12}$$

and identify our earlier product $\langle n|n'\rangle \equiv \langle n|n'\rangle^{(2)}$. Here the relevant inner product is with $\alpha = 3$:

$$\langle n|n'\rangle^{(3)} = \int_0^\infty dx\, x^3\, e^{-x}\, L^{(3)}_n(x)\, L^{(3)}_{n'}(x) = (n+3)(n+2)(n+1)\, \delta_{n,n'} \tag{5.D.13}$$

Using this technology we have $I^{(1,1,1)}_{0,0} = 3! = 6$ and $I^{(1,1,1)}_{1,1} = 48$. For $n \geq 2$ we can apply the 3-point rule twice to write

$$L^{(1)}_n = L^{(3)}_n - 2L^{(3)}_{n-1} + L^{(3)}_{n-2} \tag{5.D.14}$$

So that

$$\begin{aligned} I^{(1,1,1)}_{n,n} &= \langle n|n\rangle^{(3)} + 4\,\langle n-1|n-1\rangle^{(3)} + \langle n-2|n-2\rangle^{(3)} \\ &= 6\,(1+n)^3 \\ &\equiv h_0(n) \end{aligned}$$

where we defined the polynomial $h_0(n)$. Note that $h_0(0) = 6 = I^{(1,1,1)}_{0,0}$ and $h_0(1) = 48 = I^{(1,1,1)}_{1,1}$, so in fact we have $I^{(1,1,1)}_{n,n} = h_0(n)$ for all $n \geq 0$.

We can do a similar calculation to show that

$$\begin{aligned} I^{(1,1,1)}_{n,n+1} &= -2\left(\langle n|n\rangle^{(3)} + \langle n-1|n-1\rangle^{(3)}\right) \\ &= -2(n+1)(n+2)(2n+3) \\ &\equiv h_1(n) \end{aligned}$$

for $n \geq 2$, and checking the edge cases by hand shows that $I^{(1,1,1)}_{n,n+1} = h_1(n)$ for all $n \geq 0$.

Finally we have

$$\begin{aligned} I^{(1,1,1)}_{n,n+2} &= \langle n|n\rangle^{(3)} \\ &= (n+1)(n+2)(n+3) \\ &\equiv h_2(n) \end{aligned}$$

and similar checks show that $I^{(1,1,1)}_{n,n+2} = h_2(n)$ for all $n \geq 0$.

In summary,

$$I^{(1,1,1)}_{n,n} = 6\,(1+n)^3 \qquad \equiv h_0(n) \tag{5.D.15a}$$
$$I^{(1,1,1)}_{n,n+1} = -2(n+1)(n+2)(2n+3) \equiv h_1(n) \tag{5.D.15b}$$
$$I^{(1,1,1)}_{n,n+2} = (n+1)(n+2)(n+3) \qquad \equiv h_2(n) \tag{5.D.15c}$$

Using the now familiar method, the final integral we need is

$$I^{(0,1,0)}_{n,n'} \equiv \int_0^\infty dx\; x^2\; e^{-x} L_n(x) L^{(1)}_{n'}(x) \tag{5.D.16}$$

Assume first that $n \geq 3$ and $n' \geq 1$ to avoid any edge issues. Using the notation and identities mentioned previously, we can write

$$I^{(0,1,0)}_{n,n'} = \left(\langle n|^{(2)} - 2\,\langle n-1|^{(2)} + \langle n-2|^{(2)}\right)\left(|n'\rangle^{(2)} - |n'-1\rangle^{(2)}\right) \tag{5.D.17}$$

We can then exhaustively consider all cases:

$$I^{(0,1,0)}_{n,n-2} = \langle n-2|n-2\rangle^{(2)} \qquad = n(n-1) \tag{5.D.18a}$$
$$I^{(0,1,0)}_{n,n-1} = -2\,\langle n-1|n-1\rangle^{(2)} - \langle n-2|n-2\rangle^{(2)} = -n(1+3n) \tag{5.D.18b}$$
$$I^{(0,1,0)}_{n,n} = \langle n|n\rangle^{(2)} + 2\,\langle n-1|n-1\rangle^{(2)} \qquad = (1+n)(2+3n) \tag{5.D.18c}$$
$$I^{(0,1,0)}_{n,n+1} = -\,\langle n|n\rangle^{(2)} \qquad = -(n+1)(n+2) \tag{5.D.18d}$$

with $I^{(0,1,0)}_{n,n'} = 0$ if $n' \geq n+2$ or $n' \leq n-3$. We can then check all edge cases by hand - it turns out that $I_{n,n'}$ agrees with the above formulae for every $n, n' \geq 0$, as expected from our earlier results.

Summary

For completeness and easy reference, we provide a full list of the polynomials in n appearing in the series.

The scalar calculation uses the following quadratics.

$$f_0(n) = 2 + 6n + 6n^2 \tag{5.D.19a}$$

$$f_1(n) = -4 - 8n - 4n^2 \tag{5.D.19b}$$

$$f_2(n) = 2 + 3n + n^2 \tag{5.D.19c}$$

The fermion calculation uses the following quadratics and cubics.

$$g_0(n) = 4 + 12n^2 \tag{5.D.20a}$$

$$g_1(n) = -4 - 8n - 8n^2 \tag{5.D.20b}$$

$$g_2(n) = 2 + 4n + 2n^2 \tag{5.D.20c}$$

$$h_0(n) = 6\,(1+n)^3 \tag{5.D.20d}$$

$$h_1(n) = -2(n+1)(n+2)(2n+3) \tag{5.D.20e}$$

$$h_2(n) = (n+1)(n+2)(n+3) \tag{5.D.20f}$$

$$p_0(n) = -6n^2 \tag{5.D.20g}$$

$$p_1(n) = 2(1+2n)^2 \tag{5.D.20h}$$

$$p_2(n) = -2(1+n)^2 \tag{5.D.20i}$$

5.D.2 Fourier transform of Laguerre polynomials

To work fully in momentum space, we need to take the Fourier transform of a product of a Laguerre polynomial and a gaussian. This is straightforward using the generating function for Laguerre polynomials and the result that gaussians map to gaussians under the Fourier transform.

It is easy to derive the following expression for the generating function.

$$G_L(t,x) \equiv \sum_n t^n L_n(x) = \frac{e^{-tx/(1-t)}}{1-t} \tag{5.D.21}$$

We know that the Fourier transform of a multidimensional gaussian with diagonal covariances σ^2 is given by

$$\int d^n x\; e^{-ip\cdot x} e^{-x^2/2\sigma^2} = (2\pi\sigma^2)^{n/2} e^{-\sigma^2 p^2/2} \tag{5.D.22}$$

and this can be checked easily by completing the square and shifting the contour.

Combining these two facts we find the following Fourier transform for the generating function.

$$\int d^2x\; e^{-ip\cdot x} e^{-x^2/4l^2} G_L(t, x^2/2l^2) = 4\pi l^2 e^{-p^2 l^2} G_L(-t, 2p^2 l^2) \tag{5.D.23}$$

Extracting the relevant polynomial from the generating function by successive differentiation with respect to t gives the result we need for the scalar.

$$\int d^2x\ e^{-ip\cdot x}e^{-x^2/4l^2}L_n(x^2/2l^2) = 4\pi l^2 e^{-p^2l^2}(-1)^n L_n(2p^2l^2) \tag{5.D.24}$$

This is enough to immediately derive the Fourier-transform of the translationally-invariant part of the scalar propagator.

$$\tilde{G}^{(\phi)}(p) \equiv \int d^2x_\perp\ e^{-ip_\perp\cdot(x_\perp - y_\perp)}\ \frac{e^{-\frac{1}{2}\xi}}{2\pi}\sum_{n=0}^{\infty}\frac{L_n(\xi)}{\lambda_n(p_\parallel;m)} = 2l^2 e^{-p_\perp^2 l^2}\sum_{n=0}^{\infty}\frac{(-1)^n L_n(2p_\perp^2 l^2)}{l^2(m^2+p_\parallel^2)+2n+1} \tag{5.D.25}$$

For the fermion, we will also need the following integral.

$$\int d^2x\ x_j\ e^{-ip\cdot x}e^{-x^2/4l^2}L_{n-1}^{(1)}(x^2/2l^2) = i(4\pi l^2)(2p_j l^2)\ e^{-p^2l^2}(-1)^n L_{n-1}^{(1)}(2p^2l^2) \tag{5.D.26}$$

This can be computed using a recurrence relation for Laguerre polynomials. Indeed, let $v = 2p^2l^2$. Then we have

$$\begin{aligned}
\int d^2x\ x_j\ e^{-ip\cdot x}e^{-x^2/4l^2}L_{n-1}^{(1)}(x^2/2l^2) &= i\frac{\partial}{\partial p_j}\int d^2x\ e^{-ip\cdot x}e^{-x^2/4l^2}L_{n-1}^{(1)}(x^2/2l^2)\\
&= i\frac{\partial}{\partial p_j}\sum_{k=0}^{n-1}\int d^2x\ e^{-ip\cdot x}e^{-x^2/4l^2}L_k(x^2/2l^2)\\
&= i(4\pi l^2)\frac{\partial}{\partial p_j}\sum_{k=0}^{n-1}e^{-p^2l^2}(-1)^k L_k(2p^2l^2)\\
&= i(4\pi l^2)\frac{\partial v}{\partial p_j}\frac{d}{dv}\sum_{k=0}^{n-1}e^{-v/2}(-1)^k L_k(v)\\
&= i(4\pi l^2)(4p_j l^2)\sum_{k=0}^{n-1}e^{-v/2}(-1)^k\left(-\frac{1}{2}L_k(v)+L_k'(v)\right)\\
&= i(4\pi l^2)(2p_j l^2)\ e^{-p^2l^2}\left[-1+\sum_{k=1}^{n-1}(-1)^{k+1}\left(L_k(v)+2L_{k-1}^{(1)}(v)\right)\right]\\
&= i(4\pi l^2)(2p_j l^2)\ e^{-p^2l^2}\left[-1+\sum_{k=1}^{n-1}(-1)^{k+1}\left(L_k^{(1)}(v)+L_{k-1}^{(1)}(v)\right)\right]\\
&= i(4\pi l^2)(2p_j l^2)\ e^{-p^2l^2}(-1)^n L_{n-1}^{(1)}(2p^2l^2)
\end{aligned}$$

Note that in the last equality we exploited a telescoping sum. Using these integral formulas, we can take the Fourier transform of the (transeversely) translationary-invariant part of the fermion propagator. That is, we can Fourier transform $e^{-i\Phi}G^{(\psi)}(p_\parallel;x_\perp,y_\perp)$, since only the Schwinger phase breaks transverse translational symmetry. Write $z_\perp = x_\perp - y_\perp$. Then we have

$$\int d^2x_\perp\ e^{-ip_\perp\cdot(x_\perp-y_\perp)}\ e^{-i\Phi}\ G^{(\psi)}(p_\parallel;x_\perp,y_\perp)$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int d^2 z_\perp \; e^{-ip_\perp\cdot z_\perp} e^{-z_\perp^2/4l^2} \sum_{n=0}^{\infty} \frac{F_n(p_\parallel; x_\perp, y_\perp)}{\lambda_n - 1} \\
&= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{1}{\lambda_n - 1}\int d^2 z_\perp \; e^{-ip_\perp\cdot z_\perp} e^{-z_\perp^2/4l^2}\Bigg[(m-\gamma^a p_a)\left(L_n\left(\frac{z^2}{2l^2}\right)P_+ + L_{n-1}\left(\frac{z^2}{2l^2}\right)P_-\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \frac{i}{l^2}\gamma^j z_j L_{n-1}^{(1)}\left(\frac{z^2}{2l^2}\right)\Bigg] \\
&= \frac{4\pi l^2 \; e^{-p^2 l^2}}{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n - 1}\left[(m-\gamma^a p_a)\left(L_n\left(2p_\perp^2 l^2\right)P_+ - L_{n-1}\left(2p_\perp^2 l^2\right)P_-\right) + 2\gamma^j p_j L_{n-1}^{(1)}(2p_\perp^2 l^2)\right] \\
&= 2l^2 \; e^{-p^2 l^2}\sum_{n=0}^{\infty}\frac{(-1)^n D_n(p)}{l^2(m^2+p_\parallel^2)+2n}
\end{aligned}
$$

where

$$
D_n(p) = (m-\gamma^a p_a)\left(L_n\left(2p_\perp^2 l^2\right)P_+ - L_{n-1}\left(2p_\perp^2 l^2\right)P_-\right) + 2\gamma^j p_j L_{n-1}^{(1)}(2p_\perp^2 l^2) \tag{5.D.27}
$$

Chapter 6

Conclusion

In this thesis we reviewed higher-form symmetries in quantum field theory and demonstrated their utility in two separate applications.

The first application was a supersymmetric quantum field theory with a holographic dual. This quantum field theory had matter fields in the fundamental representation of the gauge group. When the gauge group was $SU(N)$, the $U(1)$ 1-form symmetry was explicitly broken, but when the gauge group was $U(N)$, the $U(1)$ 1-form symmetry was spontaneously broken. We demonstrated this by numerically computing the two-point function of the symmetry current to show the existence of a Goldstone mode.

In the same way that ordinary global symmetries have been so successful as a tool for classifying phases of matter in quantum field theory, higher-form symmetries provide a pleasing way to classify the phases of more exotic systems, most notably the reinterpretation of the (massless) photon as a Goldstone mode of a spontaneously broken 1-form symmetry. The application given in Chapter 3 can be extended further to include e.g. finite temperature quantum field theories, implemented holographically by the presence of a black hole.

The second application was to a hydrodynamic effective field theory. This hydrodynamic theory was conceived to generalise the theory of force-free electrodynamics by providing a systematic way to compute higher-derivative corrections. Using a Kubo formula, we computed the transport coefficient of one such correction perturbatively from microscopics, namely QED in a background magnetic field. This particular correction provided a mechanism to generate non-zero $\mathbf{E} \cdot \mathbf{B}$ and hence accelerate radiation away from pulsars or other compact astrophysical objects.

Many more complicated astrophysical geometries can be considered, with potentially interesting phenomenological implications. This is analagous to the rich behaviour of the non-dissipative Navier-Stokes equations when situated in various regimes according to the fluid geometry and values of the hydrodynamic transport coefficients.

By now it will be clear to the reader that higher-form symmetries have a wide variety of applications in quantum field theory. The examples presented in this thesis are just a starting point of the exploration of this new exciting framework, and there are many promising directions for further fruitful computations to deepen our understanding of existing knowledge and shine a light on open problems.

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