Gauge Theory as a String Theory in The Infinite and Zero Tension Limits

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Abstract

We investigate formulations of Yang-Mills theory as a string theory in zero and infinite tension limits. For the infinite tension case, a small Regge slope expansion of open bosonic string scattering amplitudes is performed. The leading order term is precisely the Yang-Mills amplitude plus string corrections. We explore monodromy relations among open string scattering amplitudes and their field theory counterparts. Diagrams regarding to these identities, namely Plahte diagrams, are studied. For the tensionless limit, the Yang-Mills theory is described by electric lines of force which can be interpreted as the theory of strings with contact interactions. Both theories agree with each other at the level of the expectation of Wilson loop. To address a non-Abelian theory, the string model is modified by introducing a scalar field into the worldsheet whose dynamics takes the form of the topological BF action. We further analyse an effective action of the BF theory and its connection to the two-dimensional Yang-Mills theory.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. Aspects of this thesis were carried out in collaboration with Prof. Paul Mansfield. This will be clearly referenced within the text by citation or other appropriate acknowledgement. Chapter 3 is based on [1] and Chapter 6 is based on [2].

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Chapter 1

Introduction

Yang-Mills theories serve as an important building block of the standard model which is probably one of the most successful theories of fundamental particles ever tested experimentally. They incorporate local internal symmetries called gauge symmetries at Lagrangian level. With the symmetry group $SU(3) \times SU(2) \times U(1)$, the model provides a unified description of three main forces of nature excluding gravity. In collaboration with the quantum field theory, it gives an extremely accurate result accounting for the measurement of the electron magnetic moment [3] with precision to more than 10 significant digits.

Connections between Yang-Mills theories and string theories have a long history. Originally, string theory was proposed to be a theory of strong interactions. This aspect was suggested in the context of scattering amplitudes for four-meson scattering. The S-matrix for this process, i.e. the renowned Veneziano amplitude [4], can be interpreted as the scattering amplitude for four scalar open strings. In the present day, perspective towards string theories has been shifted greatly from their original objective. They are often described as candidates for a unified theory of quantum gravity and gauge interactions.

There are several viewpoints suggesting possible connections between Yang-Mills and string theories. Dating back to the seventies, it was found that the string scattering amplitudes form vector fields reproduce those of Yang-Mills theories [5] in the limit of low energies. In this perspective, string theory is considered as an effective theory which reduces to Yang-Mills theory at low energy. Another connection has been noticed by 't Hooft [6] that in the large N limit, the $SU(N)$ gauge theory diagrams can be re-organized as a series in powers of $1/N$ which is equivalent to the perturbative expansion of string theory with the string coupling constant $1/N$. A more modern approach to relating the two theories is the AdS/CFT correspondence [7] where a string theory in Anti-de Sitter (AdS) space is dual to the conformal field theory on its spatial boundary.

Quite recently, it was shown that the Wilson loop for an Abelian Yang-Mills theory is equal to a partition function of a tensionless string with non-standard contact interactions [8,9]. In this viewpoint, the string represents an off-shell line of electric force stretching between the worldsheet boundary. However, a non-Abelian generalization of this model is not yet fully developed.

In this thesis, we are particularly interested in connections between Yang-Mills theory in two different limits, which are the infinite tension limit and zero tension limit. For the former case, the duality between the two theories is investigated through scattering amplitudes. As for applications, we will apply this point of view to investigate an interesting relation between string amplitudes called Plahte diagrams. For the latter case, we will explore a formulation of Yang-Mills theory as a string theory with non-standard interaction. A generalization of this model to non-Abelian gauge groups will be discussed in which it requires an introduction of new degrees of freedom into the worldsheet. Finally, a candidate for the dynamics of newly introduced worldsheet fields will be intensively analysed.

1.1 Yang-Mills theory

The Yang-Mills theory for a non-Abelian $SU(N)$ gauge group is described by the Lagrangian

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\text{tr}(F^{\mu\nu}F_{\mu\nu}) \tag{1.1.1}$$

where $F_{\mu\nu}$ is a field strength tensor of a gauge field A_μ defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + q[A_\mu, A_\nu]. \tag{1.1.2}$$

Both objects $F_{\mu\nu}$ and A_μ are the elements of the non-Abelian Lie algebra $SU(N)$ which can be written in terms of a set of generators $\{T^R\}$ as $F^{\mu\nu} = F_R^{\mu\nu}T^R$ and

$A = A_R T^R$ where the Lie index is an integer that runs from 1 to $N^2 - 1$. These generators are normalised as

$$\mathrm{tr}(T^A T^B) = \frac{1}{2}\eta^{AB}, \qquad [T^A, T^B] = i f^{AB}{}_C T^C \tag{1.1.3}$$

where f^{ABC} is the totally-antisymmetric structure constant.

The Lagrangian (1.1.1) is invariant under gauge transformation

$$A_\mu \to U A_\mu U^{-1} + \frac{1}{q} U(\partial_\mu U^{-1}) \equiv \frac{1}{q} U \mathcal{D}_\mu U^{-1} \tag{1.1.4}$$

where

$$U = \exp(-q\Lambda_R(x) T^R) \tag{1.1.5}$$

is an element of the gauge group G and the covariant derivative $\mathcal{D}_\mu$ is defined as

$$\mathcal{D}_\mu = \partial_\mu + q A_\mu. \tag{1.1.6}$$

Using the above expression, we can define the field strength tensor as the commutator of the covariant derivatives, i.e.

$$F_{\mu\nu} = \frac{1}{q}[\mathcal{D}_\mu, \mathcal{D}_\nu]. \tag{1.1.7}$$

It is not hard to see that the transformation (1.1.4) transforms the field strength tensor as

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1}. \tag{1.1.8}$$

The gauge invariant property is guaranteed by the trace in the Lagrangian.

However, as the theory has a very large symmetry group, a difficulty arises when promoting it to the quantum level. This is because the Yang-Mills action provides the same information towards any two physically equivalent gauge fields A which are different from each other by the gauge transformation (1.1.4). In path integral language, this makes the vacuum amplitude hugely divergent as there are an infinite number of equivalent configurations of the gauge field. To deal with this infinity, we can use the well-known Faddeev–Popov procedure [10] to impose a gauge condition which is particularly designed to pick one field from each gauge orbit. This results in introducing new Grassmannian fields called ghosts into the theory in which we will discuss in the next section.

1.1.1 Quantization of Yang-Mills theory

To quantize the theory, one need to define the path integral representing the partition function of the theory in Euclidean spacetime. The naive definition is to integrate over all configurations of the field A of an exponential of the Yang-Mills action, i.e.

$$Z = \int DA \exp(-S_{YM}[A]) \tag{1.1.9}$$

where

$$S_{YM}[A] = \int d^4x \mathcal{L}_{YM}(x). \tag{1.1.10}$$

However, as already stated, this functional integral is largely divergent as integrating out all configurations of the field A involves an infinite number of gauge equivalent configurations. Since the integrand is identical for any two gauge equivalent fields, we will obtain an infinite copy regarding to the integration over each gauge orbit.

To fix a gauge choice, we impose a gauge-fixing condition, $G[A] = 0$ where we will choose

$$G[A] = \partial^\mu A_\mu(x) - \omega(x). \tag{1.1.11}$$

Note that $G[A]$ is an element of the $SU(N)$ group. The above condition is imposed via an identity

$$1 = \int D\Lambda \delta(G[A^\Lambda])\text{det}\left(\frac{\delta G[A^\Lambda]}{\delta\Lambda}\right) \tag{1.1.12}$$

where A^Λ is a gauge transformed field expressed in (1.1.4). Since the theory should not depend on the specific gauge-fixing condition, we can take a weighted average over different choice of $\omega(x)$. As a result, it is natural to average the gauge condition using a Gaussian distribution by including the identity

$$N(\zeta) \int D\omega \exp\left[\int d^4x \frac{\omega^2(x)}{2\zeta}\right] = 1 \tag{1.1.13}$$

where $N(\zeta)$ is a normalisation factor and ζ is arbitrary real number.

It is not hard to see that the infinitesimal change of the gauge field is

$$\delta A^\Lambda_\mu = \partial_\mu\Lambda + q[A_\mu, \Lambda] = \mathcal{D}_\mu\Lambda. \tag{1.1.14}$$

Therefore, we can evaluate

$$\frac{\delta G[A^\Lambda]}{\delta\Lambda} = \partial^\mu\mathcal{D}_\mu. \tag{1.1.15}$$

According to the Faddeev-Popov procedure, the determinant is replaced by the functional integral over new anticommuting ghost fields c as

$$\det\left(\frac{\delta G[A^\Lambda]}{\delta\Lambda}\right) = \int D\bar{c}Dc\exp\left[2\int d^4x\ \mathrm{tr}(\bar{c}(\partial^\mu\mathcal{D}_\mu)c)\right]. \tag{1.1.16}$$

Despite having wrong spin statistics, we can still treat these new fields as additional particles which are involved in the computation of Feynman diagrams.

Substituting (1.1.12), (1.1.13) and (1.1.16) into (1.1.9), the Yang-Mills partition function now takes the form

$$\begin{aligned} Z =&\mathcal{N}(\zeta)\int D\Lambda DAD\omega D\bar{c}Dc\delta(G[A^\Lambda])\exp\left[-S_{\mathrm{YM}}[A]-S_{\mathrm{ghost}}[c,\bar{c},A]+\int d^4x\frac{\omega^2}{2\zeta}\right]\\ =&\mathcal{N}(\zeta)\int D\Lambda DAD\omega D\bar{c}Dc\delta(G[A])\exp\left[-S_{\mathrm{YM}}[A]-S_{\mathrm{ghost}}[c,\bar{c},A]+\int d^4x\frac{\omega^2}{2\zeta}\right]\\ =&\mathcal{N}(\zeta)\int D\Lambda D\bar{c}DcDA\exp\left[-S_{YM}-S_{\mathrm{ghost}}[c,\bar{c},A]+\int d^4x\frac{(\partial^\mu A_\mu)^2}{2\zeta}\right] \end{aligned} \tag{1.1.17}$$

where

$$S_{\mathrm{ghost}}[c,\bar{c},A] = -\int d^4x\bar{c}_A(\partial^\mu\mathcal{D}_\mu^{AB})c_B \tag{1.1.18}$$

To obtain the second line, we use the gauge-invariant property of the Yang-Mills action, i.e. $S_{\mathrm{YM}}[A]=S_{\mathrm{YM}}[A^\Lambda]$, together with a simple shift of variables from A to A^Λ then rename it back to A.

Consequently, The gauge-fixed Yang-Mills Lagrangian is written as

$$\mathcal{L}_{\mathrm{g.f.}}(x) = -\frac{1}{4}F^R_{\mu\nu}F^{\mu\nu}_R - \frac{(\partial^\mu A_\mu)^2}{2\zeta} - \bar{c}_A(\partial^\mu\mathcal{D}^{AB}_\mu)c_B. \tag{1.1.19}$$

The corresponding generating functional is

$$Z[J,\eta,\bar{\eta}] = \int DAD\bar{c}Dc\exp\left[-\int d^4x\left(\mathcal{L}_{\mathrm{g.f.}} - J^R_\mu A^\mu_R - \bar{\eta}_R c^R - \eta_R\bar{c}^R\right)\right]. \tag{1.1.20}$$

If we treat the cubic and quartic terms as perturbations to the free action, we can rewrite these terms in terms of functional differentiation as

$$\begin{aligned} S_I =&iq\int d^4x f^{ABC}\partial_\mu\left(\frac{\delta}{\delta J^A_\nu(x)}\right)\left(\frac{\delta}{\delta J^B_\mu(x)}\right)\left(\frac{\delta}{\delta J^{C\nu}(x)}\right)\\ &-\frac{q^2}{4}\int d^4x f^{ABC}f^{DEF}\eta_{AD}\left(\frac{\delta}{\delta J^B_\mu(x)}\right)\left(\frac{\delta}{\delta J^C_\nu(x)}\right)\left(\frac{\delta}{\delta J^{E\mu}(x)}\right)\left(\frac{\delta}{\delta J^{F\nu}(x)}\right), \end{aligned} \tag{1.1.21}$$

$$S_I^{\mathrm{ghost}} =iq\int d^4x f^{ABC}\partial_\mu\left(\frac{\delta}{\delta\eta^A(x)}\right)\left(\frac{\delta}{\delta J^B_\mu(x)}\right)\left(\frac{\delta}{\delta\bar{\eta}^C(x)}\right). \tag{1.1.22}$$

Consequently, the generating functional (1.1.20) becomes

$$Z[J,\eta,\bar{\eta}] = e^{-S_I - S_I^{\text{ghost}}} Z[J]\widetilde{Z}[\eta,\bar{\eta}] \tag{1.1.23}$$

where $Z[J]$ and $\widetilde{Z}[\eta,\bar{\eta}]$ are the gauge field part and the ghost part of the free theory generating functionals which take the forms

$$Z[J] = \int DA \exp\left[\int d^4x \; -\frac{1}{2}A_R^\mu\left(\partial^2\eta_{\mu\nu} - \left(1-\frac{1}{\zeta}\right)\partial_\mu\partial_\nu\right)\eta^{RS}A_S^\nu + J_\mu^R A_R^\mu\right] \tag{1.1.24}$$

and

$$\widetilde{Z}[\eta,\bar{\eta}] = \int D\bar{c}Dc \exp\left[\int d^4x \bar{c}_A\partial^2\eta^{AB}c_B + \bar{\eta}_R c^R + \eta_R\bar{c}^R\right] \tag{1.1.25}$$

The above integrals can be easily evaluated using the functional Gaussian integration formula. Thus, the partition functions $Z[J]$ and $\widetilde{Z}[\eta,\bar{\eta}]$ take the forms

$$Z[J] = (\det(M))^{-1/2} \exp\left(\frac{1}{2}\int d^4x d^4y J_\mu^A(x)(M^{-1})_{AB}^{\mu\nu}(x-y)J_\nu^B(y)\right) \tag{1.1.26}$$

and

$$\widetilde{Z}[\eta,\bar{\eta}] = (\det(N)) \exp\left(-\int d^4x d^4y \bar{\eta}^A(N^{-1})_{AB}(x-y)\eta^B(y)\right) \tag{1.1.27}$$

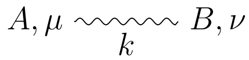
where $(M^{-1})_{AB}^{\mu\nu}(x-y)$ and $(N^{-1})_{AB}(x-y)$ are the propagators for gauge fields and ghosts respectively which can be expressed as

$$(M^{-1})_{AB}^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\eta_{AB}\left(\eta^{\mu\nu} - (1-\zeta)\frac{k^\mu k^\nu}{k^2}\right)\frac{e^{ik(x-y)}}{k^2} \tag{1.1.28}$$

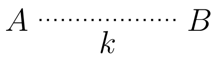
and

$$(N^{-1})_{AB}(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\eta_{AB}e^{ik(x-y)}. \tag{1.1.29}$$

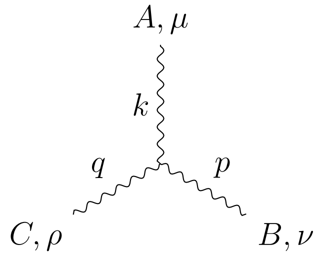
From the Lagrangian (1.1.19) we can obtain the Feynman rules presented in figure (1.1). The wavy lines denote the propagator of gauge particles and dotted lines represent that of ghosts.



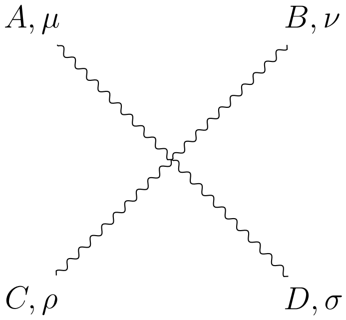
$$= \frac{1}{k^2}\left(\eta^{\mu\nu} - (1-\zeta)\frac{k^\mu k^\nu}{k^2}\right)\eta^{AB}$$



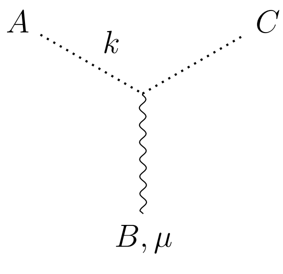
$$= \frac{1}{k^2}\eta^{AB}$$



$$= qf^{ABC}\big(\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu\big)$$



$$= q^2\big({f^{AB}}_E f^{CDE}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + {f^{AC}}_E f^{BDE}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) + {f^{AD}}_E f^{BCE}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\big)$$



$$= iqf^{ABC}k^\mu$$

Figure 1.1: Feynman rules of a non-Abelian gauge theory

1.1.2 Wilson loop

In gauge theory, a Wilson loop is a gauge-invariant observable which can be thought as a non-Abelian generalisation of the phase factor corresponding to the Aharonov–Bohm effect in quantum theory [11]. It is a very useful tool to understand quark confinement which was first pointed out by Wilson using lattice field theory [12]. The Wilson loop can be defined as the trace of the path-ordered exponential of a line integral of the gauge field A along a closed loop C,

$$W[C] = \mathrm{tr}\big(\mathcal{P}\big(e^{-q\oint_C A\cdot d\xi}\big)\big) \tag{1.1.30}$$

where $\mathcal{P}$ is a path-ordering operator which orders a product of operators along the path C as

$$\mathcal{P}(O_1(\xi_1)O_1(\xi_2)\dots O_N(\xi_N)) \equiv O_{P_1}(\xi_{P_1})O_{P_2}(\xi_{P_2})\dots O_{P_N}(\xi_{P_N}) \tag{1.1.31}$$

where on the right hand side the operators are ordered according to their positions along the path, i.e. $\xi_{P_1} \geq \xi_{P_2} \geq \dots \geq \xi_{P_N}$. The trace in (1.1.30) is computed over colour indices.

By Taylor expanding the exponential (1.1.30), the expectation value of the Wilson loop is

$$\langle W[C]\rangle = \mathrm{tr}\Bigg(\mathcal{P}\sum_{n=0}^{\infty}\frac{(-q)^n}{n!}\Big\langle\prod_{i=0}^{n}\oint_C d\xi_i^{\mu_i}A_{\mu_i}\Big\rangle\Bigg). \tag{1.1.32}$$

For a small coupling constant q, we can evaluate (1.1.32) perturbatively which requires the calculation of $\langle A^n\rangle$. The first non-trivial contribution to $\langle W\rangle$ is

$$\frac{q^2}{2}\mathrm{tr}\Bigg(\mathcal{P}\oint_C\oint_C d\xi_1^{\mu}d\xi_2^{\nu}\big\langle A_\mu(\xi_1)A_\nu(\xi_2)\big\rangle\Bigg). \tag{1.1.33}$$

According to the gauge-field propagator (1.1.28), the above term can be written as

$$\frac{q^2}{2}\mathrm{tr}\Bigg(\mathcal{P}\oint_C\oint_C d\xi_1^{\mu}d\xi_2^{\nu}\frac{d^4k}{(2\pi)^4}\eta_{AB}\bigg(\eta^{\mu\nu}-(1-\zeta)\frac{k^\mu k^\nu}{k^2}\bigg)\frac{e^{ik(x-y)}}{k^2}T^AT^B\Bigg). \tag{1.1.34}$$

If we ignore the self-interaction of the Yang-Mills theory, then the expectation value of the Wilson loop is evaluated as

$$\mathrm{tr}\mathcal{P}\exp\Bigg(\frac{q^2}{2}\oint_C\oint_C d\xi_1^{\mu}d\xi_2^{\nu}\frac{d^4k}{(2\pi)^4}\eta_{AB}\bigg(\eta^{\mu\nu}-(1-\zeta)\frac{k^\mu k^\nu}{k^2}\bigg)\frac{e^{ik(x-y)}}{k^2}T^AT^B\Bigg) \tag{1.1.35}$$

which only differs from the Abelian case by the path-ordering of the Lie generators.

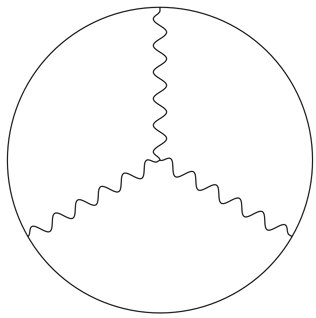


Figure 1.2: The simplest Wilson loop with the three gauge boson vertex

The main difference between the Abelian and non-Abelian theories is the existence of self-interactions. There are three and four gauge boson vertices which first appear in the expectation of the Wilson loop at q^4 and q^6 respectively. For the three-point vertex, we can calculate its contribution to the expectation of the Wilson loop at the lowest order by considering

$$\mathrm{tr}\mathcal{P}\left(-\frac{q^3}{3!}\langle(\oint d\xi^\mu A_\mu)^3\rangle\right) = -\frac{q^3}{3!}\frac{1}{Z[0]}\mathrm{tr}\mathcal{P}\prod_{i=1}^{3}\left(\oint d\xi_i^\mu T^{A_i}\frac{\delta}{\delta J^{\mu A_i}(\xi_i)}\right)Z[J,\eta,\bar{\eta}]\Big|_{J,\eta,\bar{\eta}=0}. \tag{1.1.36}$$

Keeping only the terms with only f^{ABC}, one obtains

$$\mathrm{tr}\mathcal{P}\Bigg[i\frac{q^4}{2}f^{ABC}T_AT_BT_C\prod_{i=1}^{3}\int\frac{d^4k_i}{(2\pi)^4}(2\pi)^4\delta^4(k_1+k_2+k_3)\oint\oint\oint\frac{1}{k_1^2k_2^2k_3^2}$$
$$\times\left(d\xi_1^\mu-\frac{k_1^\mu k_1\cdot d\xi_1}{k_1^2}\right)\left(d\xi_{2\mu}-\frac{k_{2\mu}k_2\cdot d\xi_2}{k_2^2}\right)ik_{1\nu}\left(d\xi_3^\nu-\frac{k_3^\nu k_3\cdot d\xi_3}{k_3^2}\right)e^{-i\sum_{i=1}^{3}k_i\cdot\xi_i}. \tag{1.1.37}$$

The expression (1.1.37) was computed in the Landau gauge ($\zeta=0$) and contributes to the Feynman diagram in figure 1.2.

1.1.3 Two-Dimensional Yang-Mills Theory

Although Yang-Mills theory in two dimensions is classically trivial and lacks propagating degrees of freedom, it possesses interesting properties. For example, when the theory is formulated on spacetimes of nontrivial topology, it serves as a tool for the study of the topology of the moduli spaces of flat connections on surfaces [13, 14]. Moreover, at large N, the theory is equivalent to a closed string theory or, to be precise, the large N expansion of the free energy W ($W = \ln Z$ where Z is the partition function) is equal to a string theory partition function with string coupling $g_s = 1/N$ and with string tension identified with $q^2 N$ where q is the Yang-Mills coupling. The coefficients of the expansions can be interpreted as sums over maps from the orientable worldsheet to the target space [15–18]. The idea was extended to describe $SO(N)$ and $Sp(N)$ gauge theories which include maps from non-orientable worldsheets [19,20]. The string description for finite N was also discussed in [21,22].

An action for two-dimensional Yang-Mills theory on an orientable 2D Riemannian manifold $\mathcal{M}$ with the gauge group G is

$$S_{\text{YM}} = -\frac{1}{2e^2} \int_{\mathcal{M}} d^2\xi \sqrt{g} \text{tr}(\mathcal{F}_{ij}\mathcal{F}^{ij}) \tag{1.1.38}$$

where e^2 is the gauge coupling. The partition function for (1.1.38) on the manifold $\mathcal{M}_h$ of genus h is given by a sum over all irreducible representations [23],

$$Z(A,h) = \sum_R (d_R)^{2-2h} \exp\big(-e_{\text{YM}}^2 A C_2(R)\big) \tag{1.1.39}$$

where A is an area of the sphere and R is an irreducible representation of $SU(N)$. d_R and $C_2(R)$ are the dimension and the quadratic Casimir of the representation R respectively. The Yang-Mills coupling constant e_{YM} is equal to $eq/2$.

Formulae for the vacuum expectation value of the Wilson loops have been calculated [13, 23–25]. For a contractible loop γ on $\mathcal{M}_h$, the expectation value of the Wilson loop takes the form

$$\langle \text{tr}_\mu \mathcal{P} e^{-q\oint_\gamma A\cdot d\xi}\rangle = \sum_{\lambda\in G}\sum_{\rho\in\lambda\otimes\mu} d_\lambda^{1-2h} d_\rho \exp\Big[-e_{\text{YM}}^2(C_2(\lambda)A(D) + C_2(\rho)A(D'))\Big] \tag{1.1.40}$$

where D is the surface enclosed by the loop γ and D' is its compliment on $\mathcal{M}_h$, i.e. the surface outside the region D. The Wilson loop is in the irreducible representation $\mu \in G'$, and $\lambda \in G'$ where G' is a span of representations of group G.

According to Witten [13, 14], Yang Mills theory in two dimensions can be formulated from a topological field theory called a BF theory. This theory is a diffeomorphism-invariant gauge theory. On a D-dimensional manifold $\mathcal{M}$ ($D \geq 2$) with structure group, a Lie group G, the classical action of the non-Abelian BF theory takes the form

$$S = 2\int_{\mathcal{M}} \mathrm{tr}(B \wedge F) \tag{1.1.41}$$

where B is a $(D-2)$-form in the fundamental representation of G. F is a curvature 2-form of a connection 1-form A defined by $F = dA + q[A, A]$. The trace implies a scalar product in the algebra. Notice that the action is topologically invariant because it is independent of the metric. The equations of motion with respect to B and A are

$$F = 0 \qquad \text{and} \qquad d_A B = 0 \tag{1.1.42}$$

where d_A is a covariant derivative defined as $d_A = d + q[A,\ \ ]$. The action is invariant under local gauge transformation with gauge parameter ω as

$$\delta A = d_A\omega \qquad \text{and} \qquad \delta B = [B, \omega] + d_A\eta. \tag{1.1.43}$$

The field η is a $(D-1)$-form corresponding to the non-Abelian symmetry of the B field, namely B symmetry which only appears when $D \geq 3$.

In the case $D = 2$, one can express (1.1.41) as

$$S = 2\int_{\mathcal{M}} d^2\xi\ \epsilon^{ij}\mathrm{tr}(\phi\mathcal{F}_{ij}) \tag{1.1.44}$$

or equivalently,

$$S[\phi, \mathcal{A}] = \int_{\mathcal{M}} d^2\xi \left(iq\mathcal{A}_{iA}\mathcal{A}_{jB}f^{ABC}\phi_C - 2\partial_i\phi_A\mathcal{A}_j^A \right)\epsilon^{ij} \tag{1.1.45}$$

where $\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i + q[\mathcal{A}_i, \mathcal{A}_j]$ is the field-strength of the gauge field $\mathcal{A}$. For clarity, we will use curly letters to to describe fields in two dimensions from now on. Both fields ϕ and $\mathcal{A}$ are elements of a non-Abelian group and so can be written in terms of a set of generators $\{T^R\}$ as $\phi = \phi_R T^R$ and $\mathcal{A} = \mathcal{A}_R T^R$.[1] To obtain (1.1.45), the boundary term, i.e. $\int d^2\xi\partial_i(\phi \cdot \mathcal{A}_j)\epsilon^{ij}$, is assumed to vanish. Notice that in two dimensions, the B field is a 0-form, thus, it is natural to replace it with the scalar field ϕ.

As mentioned earlier, the action (1.1.44) has a close connection to the Yang-Mills action in two dimensions as they are equivalent in the zero coupling constant

[1] $\mathrm{tr}(T^AT^B) = \frac{1}{2}\eta^{AB}$ and $[T^A, T^B] = if^{AB}{}_C T^C$

limit [13, 14]. This can be seen by adding a quadratic term with coupling constant e to the action and then integrating out the field ϕ in the path integral below using Gaussian integration

$$\begin{aligned}&\int D\mathcal{A}D\phi \exp\left(2\int_{\mathcal{M}} d^2\xi \left(\epsilon^{ij}\mathrm{tr}(\phi\mathcal{F}_{ij}) + e^2\sqrt{g}\mathrm{tr}(\phi\phi)\right)\right)\\ &= \int D\mathcal{A}\exp\left(\frac{1}{2e^2}\int_{\mathcal{M}} d^2\xi\sqrt{g}\mathrm{tr}(\mathcal{F}_{ij}\mathcal{F}^{ij})\right).\end{aligned} \tag{1.1.46}$$

In this way, the two-dimensional Yang-Mills theory is almost topological arising from the topological BF theory where the quadratic term can be seen as a deformation from the topological one.

1.2 String theory

Historically, string theory was proposed to be a theory of strong interactions. The theory was put under the spotlight when Nambu [26], Nielsen [27], and Susskind [28] independently suggest that the Veneziano model [4], which describes scattering amplitudes for four mesons, can be viewed as the scattering of extended one-dimensional objects or strings. However, the theory goes beyond what it was meant to be at the beginning. In fact, it is a candidate for a unified theory of quantum gravity and quantum field theory since gravitons naturally appear in the string spectrum when quantizing the theory. The shift in perspective is due to a striking discovery by Scherk and Schwarz [29] in which they showed the equality between the two models, i.e. string theory and quantum gravity, at low energy. More modern aspects of string theory includes the AdS/CFT correspondence [7] where string theories on a curved Anti-de Sitter (AdS) space are related to conformal field theories on the boundary of this space. This conjecture is a strong/weak coupling duality meaning that it can be used to study non-perturbative problems in quantum field theory using the weakly-coupled string theory context, see [30–32] for useful reviews.

In analogy to point particles where the action is described by an interval length of the particle worldline, one can define the string action by a proper area of the string worldsheet as

$$S_{\mathrm{NG}}[X] = -T \int_{\Sigma} d^2\xi \sqrt{-\det G_{ab}} \tag{1.2.47}$$

where G_{ab} is the induced metric on the worldsheet Σ embedded in the Minskowski spacetime defined as

$$G_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{1.2.48}$$

This action is known as Nambu-Goto action. The spacetime field X^μ is reparametrized by the worldsheet coordinates ξ^a. Note that we have used the notation $\partial_a = \frac{\partial}{\partial \xi^a}$ as a worldsheet derivative. T is called the string tension which relates to the Regge slope α' by

$$T = \frac{1}{2\pi\alpha'}. \tag{1.2.49}$$

However, the non-linearity of the Nambu-Goto action makes quantization rather difficult. To deal with this, one can introduce an auxiliary field, the intrinsic worldsheet metric g_{ab}, which allows one to obtain a classically equivalent action known as the Polyakov action

$$S_p[g,X] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\sqrt{g}g^{ab}\partial_a X^\mu(\xi)\partial_b X_\mu(\xi). \tag{1.2.50}$$

In fact, this action was first formulated by Brink, Di Vecchia and Howe [33] and then it was later quantized by Polyakov using the path integration approach [34]. Upon the elimination of g_{ab} through its equation of motion, one re-obtains the Nambu-Goto action.

The Polyakov action is invariant under local worldsheet reparametrisations or diffeomorphisms and also invariant under local rescalings of the metric, namely Weyl transformations. Under the infinitesimal changes of coordinates $\xi^a \to \tilde{\xi}^a = \xi^a - \epsilon^a$ and the metric rescaling $\delta g_{ab} = 2\Omega g_{ab}$, the combined effect of diffeomorphism and Weyl rescaling transforms the metric as

$$\delta g_{ab} = (P\cdot\epsilon)_{ab} + 2\tilde{\Omega}g_{ab} \tag{1.2.51}$$

where

$$(P\cdot\epsilon)_{ab} = \nabla_a\epsilon_b + \nabla_b\epsilon_a - \nabla_a\epsilon^a g_{ab} \tag{1.2.52}$$

$$\tilde{\Omega} = \Omega + \frac{1}{2}\nabla_a\epsilon^a \tag{1.2.53}$$

The operator P maps worldsheet vectors to symmetric traceless tensors. Apart from these worldsheet local symmetries, the action (1.2.50) obeys global spacetime isometries, i.e. Poincaré symmetries as well.

According to the worldsheet symmetries, one can locally fix the worldsheet metric to be simply the Minkowski metric, i.e. $g_{ab} = \eta_{ab}$, which can be referred to as conformal gauge. However, this argument is not valid globally as in higher genus worldsheets, there appears remaining parameters of the metric known as moduli which cannot be modded out by diffeomorphisms and Weyl transformations.

Although the conformal gauge is applied, it leaves large residual gauge symmetries known as conformal symmetries. They are simply the diffeomorphism which are generated by the conformal Killing vectors ϵ that fulfill the conformal Killing equation

$$(P\cdot\epsilon)_{ab} = 0. \tag{1.2.54}$$

Note that this effect on the metric can be undone by Weyl rescaling, thus leaving the action unchanged. Due to being locally conformally flat, we can choose convenient coordinates to be the exponential of complexified coordinates

$$z = e^{\xi^0 + i\xi^1}, \qquad \bar{z} = e^{\xi^0 - i\xi^1}. \tag{1.2.55}$$

This set of coordinates simplifies the equations of motion for the embedding coordinates X^μ to be the complex Laplace equation

$$\partial\bar{\partial}X^\mu = 0. \tag{1.2.56}$$

This implies that the field $X^\mu(z,\bar{z})$ can be decomposed into a sum of holomorphic and antiholomorphic parts, i.e. $X^\mu(z,\bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$.

Besides, this conformal gauge invariance is useful to compute string amplitudes. For a zero-genus worldsheet, this symmetry allows us to fix any three points of string state insertions in the amplitudes via Möbius transformations

$$z \to \frac{az+b}{cz+d} \tag{1.2.57}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. The group of Möbius transformations is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$ or $PSL(2,\mathbb{C})$.

Propagators for the fields X^μ can be obtained from the Dyson-Schwinger equations

$$\langle X^\mu(z,\bar{z})\frac{\delta S}{\delta X_\nu(w,\bar{w})}\rangle = \eta^{\mu\nu}\delta^2(z-w,\bar{z}-\bar{w}) \tag{1.2.58}$$

where $\langle\ldots\rangle$ represents expectation value which is defined by the functional integral

$$\langle\mathcal{F}[X]\rangle = \frac{1}{Z}\int DX\mathcal{F}[X]e^{-S_P[X]}. \tag{1.2.59}$$

$S_P[X]$ is the Euclidean Polyakov action with the conformally flat worldsheet metric. The partition function Z is to normalise the average so that $\langle 1\rangle = 1$. Following from (1.2.58), the two-point function is

$$\langle X^\mu(z,\bar{z})X^\nu(w,\bar{w})\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2. \tag{1.2.60}$$

Unlike the closed string where the entire worldsheet is mapped onto the full complex plane $\mathbb{C}$, the open string worldsheeet is described by upper-half complex plane $\mathcal{H}_+$

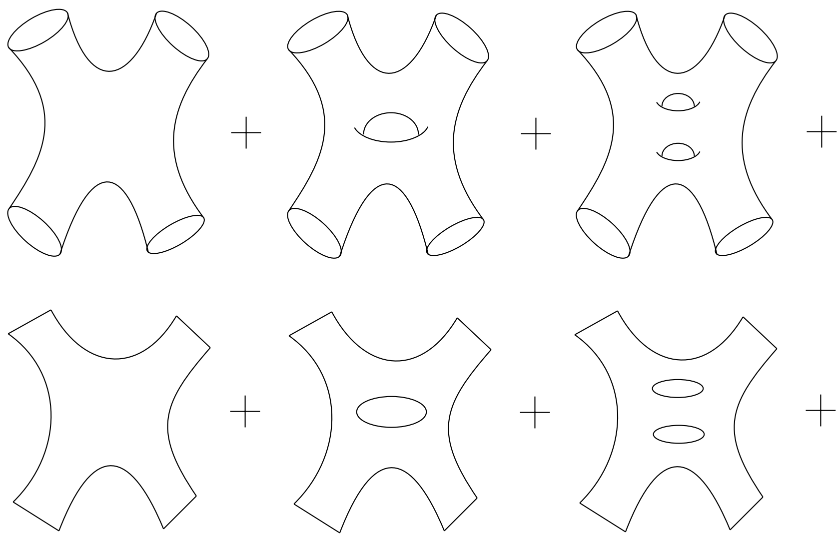


Figure 1.3: Diagrams contributing to the worldsheet expansion for four-point closed string (above) and open string (below) scattering

on which the boundary lies at the real axis. Correspondingly, The Green's function on the disk can be obtained from the method of images, thus, taking the form

$$\langle X^\mu(z)X^\nu(w)\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-w|^2 - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-\bar{w}|^2. \tag{1.2.61}$$

Note that the open string propagator obeys the Neumann condition. Other correlation functions can be calculated using Wick's theorem.

1.2.1 String Scattering Amplitudes

Served as a bridge between theories and experiments, scattering amplitudes are useful tools to make predictions for physical observables. In quantum theory, the transition amplitudes can be obtained by a sum over all histories connecting between the initial and final positions. Analogously, in string theory, one can represent the amplitudes as a sum over all worldsheets interpolating between the initial and final string configurations. In Euclidean signature, each worldsheet is weighed by a factor e^{-S_p} with the Polyakov action (1.2.50). The physical string states are inserted on the worldsheet as asymptotic incoming and outgoing states via the corresponding vertex operators. This correspondence between states and operators is a key feature in conformal field theory.

In string perspectives, scattering diagrams of N particles can be created by merging and splitting of N strings. The diagrams can be formed in many ways based on the worldsheet topologies. Therefore, to evaluate the amplitudes, we need to sum over all possible topologies. This sum gives the perturbative expansion of string theory in terms of loops. The figure 1.3 illustrates the contributions to a 4-point scattering for closed strings while the figure 1.3 displays the same scattering but for open strings. Using the Weyl invariance of the theory, we can deform these worldsheets into more familiar forms. For examples, the tree-level scatterings for closed and open strings can be conformally mapped into a sphere and a disk respectively. In the case of 1-loop diagrams, the worldsheets are reduced to a torus (for closed strings) and an annulus (for open strings). The external string states are mapped into punctures on the worldsheet for closed strings or along the boundaries for open strings. Nevertheless, in practice, it is convenient to conformally map the worldsheets onto the complex plane $\mathbb{C}$ (with certain identifications for those with loops). The entire complex plane with punctures on its bulk represents the worldsheet for closed string theory while the open string amplitudes are defined on the upper half-plane with punctures on the real axis.

To determine string amplitudes, one can treat the series of loop expansion as a perturbation series meaning that the higher loop diagrams are less likely to contribute to the computation. This can be done by adjusting the string coupling to be preferably small. The string coupling can be added to the theory by adding a term to the Polyakov action as

$$S_{\text{string}} = S_P + \lambda\chi \tag{1.2.62}$$

where λ is a constant parameter and χ is the Gauss-Bonnet term defined as

$$\chi = \frac{1}{4\pi}\int_{\Sigma} d^2\xi\sqrt{g}R + \frac{1}{2\pi}\int_{\partial\Sigma} dsk \tag{1.2.63}$$

with R the Ricci scalar of the surface and k the geodesic curvature of the boundary. This quantity is topological invariant, thus, it preserves diffeomorphisms and Weyl transformations. In Euclidean space, it is equal to the Euler characteristic of that worldsheet given by

$$\chi = 2 - 2g - b \tag{1.2.64}$$

where g and b are the number of handles and boundaries of the worldsheet. This extra term keeps track of the topology in the path integral. We can define the string coupling $g_s = e^{\lambda}$. If $g_s \ll 1$, the higher loop terms are negligible as claimed.

Putting everything together, we can now tentatively write an expression for the amplitude describing the N-string scattering as

$$\mathcal{A}_n(k_1, k_1, \ldots, k_N) = \sum_{\text{Topologies}} \int \frac{DXDg}{\text{Vol}_{\text{Diff}\times\text{Weyl}}} e^{-S_P - \lambda\chi} \prod_{i=1}^{N} \mathcal{V}_i(k_i). \tag{1.2.65}$$

where $\mathcal{V}_i$ is the integrated vertex operator taking the form

$$\mathcal{V}_i(k_i) = g_s \int d^2 z V_i(z, \bar{z}, k_i) \tag{1.2.66}$$

for closed strings and

$$\mathcal{V}_i(k_i) = g_o \int dz V_i(z, k_i) \tag{1.2.67}$$

for open strings. g_o is the open-string coupling constant which relates to the coupling constant g_s by $g_o^2 = g_s$. In general we can write

$$V_i(z, \bar{z}, k_i) = f(z, \bar{z}) : e^{ik_i \cdot X(z,\bar{z})} : \quad \text{and} \quad V_i(z, k_i) = f(z) : e^{ik_i \cdot X(z)} : \tag{1.2.68}$$

where the polarizations of higher order string states are encoded in the function f. When $f = 1$, they correspond to tachyons. For massless string states,

$$f(z, \bar{z}) = \frac{2}{\alpha'} \xi_{\mu\nu} \partial X^{\mu} \bar{\partial} \bar{X}^{\nu}, \qquad f(z) = \frac{-i}{\sqrt{2\alpha'}} \xi_{\mu} \partial X^{\mu} \tag{1.2.69}$$

for closed and open string respectively. The normalisation constants used are followed from Polchinski [35].

To calculate (1.2.65), we need to deal with the overcounting of gauge equivalent configurations using the so-called Fadeev-Popov procedure. This is done by separating the gauge measure corresponding to the diffeomorphism and Weyl transformation. However, we need to be careful when turning the integral over all worldsheet metrics $\int Dg$ into the integral over all diffeomorphisms and Weyl rescalings, $\int d\zeta$, as there are two mismatches between them.

First, there exists a subset of diffeomorphisms and Weyl rescalings that leaves metric unchanged. Such diffeomorphisms are given by the conformal Killing vector satisfying (1.2.54). These affect the metric by a rescaling factor which can be undone by the Weyl transformation. Consequently, one must not integrate over this subset of transformations. In the presence of vertex operator insertions, these degrees of freedoms can be totally fixed by fixing the position of some of the vertex coordinates on the worldsheet. The number of fixing points depends on the global structure of the worldsheet.

Second, not all metrics can be reached from a reference metric by diffeomorphisms and Weyl transformations (1.2.51). One must include a variation of moduli space parametrized by a moduli vector t^α to a total change of the metric. As a result, the most general metric variation can then be generated by the gauge transformations and physical variations due to a change in the metric moduli as

$$\delta g^{\zeta}_{ab}(t) = (P\cdot\epsilon)_{ab} + 2\tilde{\Omega} g_{ab} + \delta t^\alpha \partial_{t^\alpha} g_{ab} \tag{1.2.70}$$

where ζ labels the gauge variables. α runs from 1 to μ where μ is the dimension of the moduli space.

The real dimension of the moduli space μ and the number of vertex coordinates one can fix κ are related by Riemann-Roch theorem as

$$\mu - \kappa = -3\chi \tag{1.2.71}$$

with χ the Euler characteristic (1.2.64). Furthermore,

$$\text{if } \chi > 0, \quad \kappa = 3\chi, \quad \mu = 0 \tag{1.2.72}$$

$$\text{if } \chi < 0, \quad \kappa = 0, \quad \mu = -3\chi. \tag{1.2.73}$$

For a sphere S^2, the Riemann-Roch theorem implies that $\mu = 0$ and $\kappa = 6$ meaning that we can fix three vertex insertions on the worldsheet. This is in agreement with the fact that the conformal group of S^2 is $PSL(2,\mathbb{C})$. In case of torus, the moduli space is parametrised by a complex number. Therefore, one can conclude that $\mu = \kappa = 2$ in this scenario.

To remove all gauge redundancies in the functional integral (1.2.65), we apply the gauge fixing condition $g_{ab} = \hat{g}^{\zeta}_{ab}(t)$ and fix κ vertex operator coordinates, $\xi^a_i = \hat{\xi}^a_i$, via the identity

$$1 = \Delta_{\mathrm{FP}}(g,\xi)\int d^\mu t \int D\zeta\, \delta(g - \hat{g}^{\zeta}_{ab}(t)) \prod_{(a,i)\in\mathfrak{f}} \delta(\xi^a_i - \hat{\xi}^a_i) \tag{1.2.74}$$

where Δ_{FP} is the Faddeev–Popov determinant. $\mathfrak{f}$ is the set of fixed vertex operator coordinates (a, i). Substitute (1.2.74) to (1.2.65), we obtain

$$\begin{aligned} \mathcal{A}_n(k_1, k_1, \ldots, k_N) =& g_s^{(N_c + \frac{1}{2}N_o)} \sum_{\text{Topologies}} \int d^\mu t \Delta_{\text{FP}}(\hat{g}, \hat{\xi}) \int DX e^{-S_P[\hat{g}(t)] - \lambda\chi} \\ & \times \prod_{(a,i)\notin \mathfrak{f}} \int d\xi_i^a \prod_{i=1}^N \left(\sqrt{\hat{g}(\xi_i)} V_i(\xi_i, k_i)\right) \end{aligned} \tag{1.2.75}$$

with N_c and N_o the number closed and open string external states. Using the Fourier decomposition of Dirac delta functions, the inverse of the Faddeev-Popov determinant becomes

$$\begin{aligned} \Delta_{\text{FP}}(\hat{g}, \hat{\xi})^{-1} = & \int d^\mu \delta t \int D\Omega D\epsilon^a \int D\beta_{ab} d^\kappa x \; e^{2\pi i(\beta | P\cdot\epsilon + 2\tilde{\Omega}\hat{g} + \delta t^\alpha \partial_{t^\alpha}\hat{g})} \\ & \times e^{2\pi i \sum_{(a,i)\in\mathfrak{f}} x_{ai}\epsilon^a(\hat{\xi}_i^a)} \end{aligned} \tag{1.2.76}$$

where the inner product of metrics $(h^{(1)}|h^{(2)})$ defined as

$$(h^{(1)}|h^{(2)}) = \int d^2\xi \sqrt{g} g^{ab} g^{cd} h_{ac}^{(1)} h_{bd}^{(2)}. \tag{1.2.77}$$

The integration over Ω constrains β_{ab} to be symmetric and traceless.

To obtain the inverse form of the integral (1.2.76), we replace all bosonic variables with Grassmann variables,

$$\epsilon^a \to c^a, \qquad \beta_{ab} \to b_{ab}, \qquad x_{ai} \to \eta_{ai}, \qquad t^\alpha \to \tau^\alpha. \tag{1.2.78}$$

Thus, the Fadeev-Popov determinant becomes

$$\begin{aligned} \Delta_{\text{FP}}(\hat{g}, \hat{\xi}) &= \int d^\mu \tau d^\kappa \eta \int DbDc e^{-\frac{1}{4\pi}(b|P\cdot c + \tau^\alpha \partial_{t^\alpha}\hat{g}) + \sum_{(a,i)\notin\mathfrak{f}} \eta_{ai} c^a(\hat{\xi}_i^a)} \\ &= \int DbDc \exp(-S_g) \prod_{\alpha=1}^{\mu} \frac{1}{4\pi}(b|\partial_{t^\alpha}\hat{g}) \prod_{(a,i)\in\mathfrak{f}} c^a(\hat{\xi}_i^a). \end{aligned} \tag{1.2.79}$$

In the last line we have performed the integration over the Grassmann parameters η and τ. The ghost action S_g is written as

$$\begin{aligned} S_g[b, c] &= \frac{1}{4\pi}(b|P\cdot c) = \frac{1}{4\pi}\int d^2\xi \sqrt{\hat{g}} b_{ab} \nabla^a c^b \\ &= \frac{1}{2\pi}\int d^2 z (b\bar{\partial}c + \bar{b}\partial\bar{c}) \end{aligned} \tag{1.2.80}$$

where we have used the abbreviations $b = b_{zz}, \bar{b} = b_{\bar{z}\bar{z}}, c = c^z$, and $\bar{c} = c^{\bar{z}}$.

All in all, after substituting (1.2.79) into (1.2.75), the gauge-fixed amplitude takes the form

$$\begin{aligned}\mathcal{A}_n(k_1, k_1, \ldots, k_N) =& g_s^{(N_c+\frac{1}{2}N_o)} \sum_{\text{Topologies}} \int d^\mu t \int DXDbDc\, e^{-S_P-S_g-\lambda\chi} \\ &\times \prod_{(a,i)\notin \mathfrak{f}} \int d\xi_i^a \prod_{\alpha=1}^{\mu} \frac{1}{4\pi}(b|\partial_{t^\alpha}\hat{g}) \prod_{(a,i)\in\mathfrak{f}} c^a(\hat{\xi}_i^a) \prod_{i=1}^{N} \big(\sqrt{\hat{g}(\xi_i)}V_i(\xi_i,k_i)\big).\end{aligned} \tag{1.2.81}$$

Note that at tree level no insertion of the anti-ghost field b is required as $\mu = 0$ at that level.

1.2.2 Tree-level Amplitudes

With all ingredients, we are now ready to compute scattering amplitudes at the lowest order. The tree-level amplitudes are described by the correlation functions evaluated on the Riemann surfaces. For closed strings, the states are inserted on the sphere S^2 which is conformally equivalent to the complex plane plus a point at infinity. In the case of open strings. they are evaluated on the disk or upper half-plane.

Let us first focus on the closed string amplitude. According to the Riemann-Roch theorem (1.2.71), neither insertion of anti-ghost b fields nor integration over the moduli parameters is required at the tree level. Accordingly, the closed string amplitude is factorised into two terms as

$$\begin{aligned}\mathcal{A}_n^{\text{cl}}(k_1, k_1, \ldots, k_N) =& g_s^{(N_c-2)} \langle c(z_1)\tilde{c}(\bar{z}_1)c(z_{N-1})\tilde{c}(\bar{z}_{N-1})c(z_N)\tilde{c}(\bar{z}_N)\rangle_{b,c} \\ &\langle \prod_{i=1,N-1,N} V_i(z_i,\bar{z}_i,k_i) \int_{\mathbb{C}^2} \prod_{j=2}^{N-2} d^2z_j V_j(z_j,\bar{z}_j,k_j)\rangle_X\end{aligned} \tag{1.2.82}$$

where we fix 3 positions of vertex operators to be at z_1, z_{N-1} and z_N. The vacuum expectation value of b, c fields, $\langle \ldots \rangle_{b,c}$, and that of X fields, $\langle \ldots \rangle_X$, are defined as

$$\langle \ldots \rangle_{b,c} = \int DbDc \ldots e^{-S_g[b,c]}, \qquad \langle \ldots \rangle_X = \int DX \ldots e^{-S_P[X]}. \tag{1.2.83}$$

Note that the open string amplitude can be retrieved from the expression above but without the anti-holomorphic parts as which we will discuss later.

To compute the correlation functions in the bc fields, we expand the ghost fields by suitable bases

$$c(z) = \sum_J \mathcal{c}_J \mathcal{C}_J(z), \qquad b(z) = \sum_J \mathcal{b}_J \mathcal{B}_J(z) \tag{1.2.84}$$

where $\mathcal{c}_J$ and $\mathcal{b}_J$ are Grassmann numbers. Both $\mathcal{C}_J$ and $\mathcal{B}_J$ independently form complete sets. Remember that the fields b and c are tensors with respect to world-sheet coordinate transformations, thus $\mathcal{C}_J$ and $\mathcal{B}_J$ are shorthand for $\mathcal{C}_J^z$ and $\mathcal{B}_{Jzz}$ respectively. They are defined as eigenfunctions of the eigenvalue equations

$$P^\intercal P \mathcal{C}_J = u_J^2 \mathcal{C}_J, \qquad P P^\intercal \mathcal{B}_J = v_J^2 \mathcal{B}_J \tag{1.2.85}$$

where the operator P maps a vector to symmetric traceless tensor defined in (1.2.52). The adjoint of P, on the other hand, maps any 2-tensor to a vector. They are normalised such that

$$\int d^2z \mathcal{C}_J \bar{\mathcal{C}}_{J'} = \delta_{JJ'}, \qquad \int d^2z \mathcal{B}_J \bar{\mathcal{B}}_{J'} = \frac{1}{8}\delta_{JJ'}. \tag{1.2.86}$$

According to (1.2.85), it is not hard to find that $P \cdot \mathcal{C}_J \propto \mathcal{B}_J$. Therefore, we can write

$$\mathcal{B}_J = \frac{1}{v_J}(P \cdot \mathcal{C}_J). \tag{1.2.87}$$

Substituting (1.2.84) into the ghost action (1.2.80), the action becomes

$$\begin{aligned} S_g[b,c] &= \frac{1}{4\pi}\int d^2z \big(b_{zz}(P\cdot c)^{zz} + b_{\bar z\bar z}(P\cdot c)^{\bar z\bar z}\big) \\ &= \frac{1}{4\pi}\int d^2z \sum_{J,K}\big(\mathcal{b}_J \mathcal{c}_K \mathcal{B}_J (P\cdot \mathcal{C}_K)^{zz} + \bar{\mathcal{b}}_J \bar{\mathcal{c}}_K \bar{\mathcal{B}}_J (P\cdot c)^{\bar z\bar z}\big) \\ &= \frac{1}{8\pi}\sum_J (v_J \mathcal{b}_J \mathcal{c}_J + \bar v_J \bar{\mathcal{b}}_J \bar{\mathcal{c}}_J) \end{aligned} \tag{1.2.88}$$

where the last line the equations (1.2.86) and (1.2.87) were applied.

Consequently, the vacuum expectation value with respect to the bc fields takes the form

$$\begin{aligned} \langle \prod_{i=1}^3 c(z_i)\tilde c(\bar z_i)\rangle_{b,c} &= \int \prod_{J\neq 0} D\mathcal{b}_J D\bar{\mathcal{b}}_J D\mathcal{c}_J D\bar{\mathcal{c}}_J e^{-S_g[\mathcal{b},\mathcal{c}]} \\ &\quad \times \int \prod_{i=1}^3 d\mathcal{c}_{0i} d\bar{\mathcal{c}}_{0i}\big(\sum_{j=1}^3 \mathcal{c}_{0j}\mathcal{C}_{0j}(z_i)\bar{\mathcal{c}}_{0j}\bar{\mathcal{C}}_{0j}(\bar z_i)\big). \\ &= \int \prod_{J\neq 0} D\mathcal{b}_J D\bar{\mathcal{b}}_J D\mathcal{c}_J D\bar{\mathcal{c}}_J e^{-S_g[\mathcal{b},\mathcal{c}]} \det \mathcal{C}_{0j}(z_i) \det \bar{\mathcal{C}}_{0j}(\bar z_i) \end{aligned} \tag{1.2.89}$$

Note that only zero modes of the ghost fields contribute to the insertions due to the Grassmannian properties. We denote the index $J = 0$ to refer to the zero modes of the bc fields whose eigenvalues equal to zeros according to (1.2.85). $\mathcal{C}_0$ is an eigenfunction with zero eigenvalue, thus satisfying the conformal Killing equation (1.2.54). On the sphere S^2, the conformal Killing vectors $\mathcal{C}_0(z)$ are basically $1, z, z^2$. Therefore, the determinant of $\mathcal{C}_{0j}(z_i)$ reads

$$\det \mathcal{C}_{0j}(z_i) = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_{N-1} & z_N \\ z_1^2 & z_{N-1}^2 & z_N^2 \end{vmatrix} = (z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N). \tag{1.2.90}$$

All in all, the expression (1.2.89) takes the form

$$\langle \prod_{i=1,N-1,N} c(z_i)\tilde{c}(\bar{z}_i)\rangle_{b,c} = C^g_{S^2}|z_1 - z_{N-1}|^2|z_1 - z_N|^2|z_{N-1} - z_N|^2 \tag{1.2.91}$$

where $C^g_{S^2}$ is the constant encoded the functional integral.

Now what we have left is to consider the expectation value with respect to the scalar field X. To do this, one can use Wick's theorem to calculate correlation functions by summing over all possible contractions with the propagator (1.2.61). It can be shown that the expectation value of product of vertex operators V_i (1.2.68) is proportional to the factor $\prod_{i<j}^N |z_j - z_i|^{\alpha' k_i\cdot k_j}$. As a result, we can write

$$\langle \prod_{i=1}^N V_i(z_i, \bar{z}_i, k_i)\rangle_X = iC^X_{S^2}(2\pi)^D\delta^D(\sum_i k_i)\prod_{i<j}^N |z_j - z_i|^{\alpha' k_i\cdot k_j}\mathcal{K}_N \tag{1.2.92}$$

where $\mathcal{K}_N$ is a collection of kinematic factors which depends on the external states of the amplitude. $\mathcal{K}_N = 1$ for tachyons and

$$\begin{aligned}\mathcal{K}_N = \exp\Bigg[2\alpha'\Bigg(\sum_{i>j}\frac{\xi_i\cdot\xi_j}{(z_i - z_j)^2} + \frac{\bar{\xi}_i\cdot\bar{\xi}_j}{(\bar{z}_i - \bar{z}_j)^2}\Bigg) \\ + \sqrt{2\alpha'}\sum_{i\neq j}\left(\frac{k_i\cdot\xi_j}{(z_i - z_j)} - \frac{k_i\cdot\bar{\xi}_j}{(\bar{z}_i - \bar{z}_j)}\right)\Bigg]\Bigg|_{\text{multilinear in }\xi,\bar{\xi}}\end{aligned} \tag{1.2.93}$$

for first excited string states such as gravitons. $\xi^{\mu\nu} = \xi^\mu\bar{\xi}^\nu$ is a polarization vector of the external states.

When substituting (1.2.91) and (1.2.92) into (1.2.82), the general expression for closed string amplitude at tree level takes the form

$$\begin{aligned}\mathcal{A}_n^{\text{cl}}(k_1, k_1, \ldots, k_n) =& ig_s^n C_{S^2} (2\pi)^D \delta^D(\sum_i k_i)|z_1 - z_{n-1}|^2 |z_1 - z_n|^2 |z_{n-1} - z_n|^2 \\ &\times \int_{\mathbb{C}^2} \prod_{i=2}^{n-2} d^2 z_i \prod_{j<k}^{n} |z_j - z_k|^{\alpha' k_j \cdot k_k} \mathcal{K}_n\end{aligned} \tag{1.2.94}$$

where $C_{S^2} = g_s^{-2} C_{S^2}^g C_{S^2}^X$. Note that the positions z_1, z_{n-1} and z_n are to be fixed to any distinct points. A traditional choice is setting $z_1 = 0$, $z_{n-1} = 1$ and $z_n = \infty$. This expression is only valid for a certain range of kinematics, i.e. $\alpha' k_i \cdot k_j > -1$. Beyond this regions, it requires analytic continuation which we will discuss in detail later in the next part.

The computation of open string amplitudes proceeds in a similar manner. By repeating the calculation presented earlier, it is not hard to find that the expression for an open string amplitude takes the same form as (1.2.94) but containing only the holomorphic parts, i.e.

$$\begin{aligned}\mathcal{A}_n^{\text{op}}(k_1, k_1, \ldots, k_n) =& ig_o^n C_{D_2} (2\pi)^D \delta^D(\sum_i k_i)|(x_1 - x_{n-1})(x_1 - x_n)(x_{n-1} - x_n)| \\ &\times \int \prod_{i=2}^{n-2} dx_i \prod_{j<k}^{n} (x_k - x_j)^{2\alpha' k_j \cdot k_k} \mathcal{K}_n.\end{aligned} \tag{1.2.95}$$

In this scenario, the conformal symmetry maps the interacting worldsheet to the disk D_2 with vertex operators inserted on its boundary. This is conformally equivalent to mapping the worldsheet onto the upper-half complex plane on which the vertex operators are aligned on the real axis. The points x_1, x_{n-1} and x_n are fixed due to the $SL(2, R)$ gauge symmetry. Note that there is a factor of 2 in the exponent which differs from the expression by closed string (1.2.94). This is because of the appearance of the image charge in the boundary propagator (1.2.61). All polarization vectors are encoded in the function $\mathcal{K}_n^{(l)}$ whose value depends on the external states of the amplitude we consider. $\mathcal{K}_n = 1$ for tachyons and

$$\mathcal{K}_n = \exp\left(\sum_{i>j} \frac{\xi_i \cdot \xi_j}{(x_i - x_j)^2} + \sqrt{2\alpha'} \sum_{i \neq j} \frac{k_i \cdot \xi_j}{(x_i - x_j)} \right)\Bigg|_{\text{multilinear in } \xi_i} \tag{1.2.96}$$

for an n-gauge field amplitude with n polarization vectors ξ_i.

Unlike closed strings, it is possible to introduce non-dynamical degrees of freedom at the ends of the strings which are the so-called Chan-Paton factors [36]. These factors carry colour degrees of freedom of gauge bosons. Accordingly, each open string state encodes two labels $i, j = 1, 2, \ldots, N$ in addition to those for the usual Fock space. This factor is presented in form of $N \times N$ matrices $(T^a)^i_j$. The reality condition implies that the T^a is Hermitian, thus, satisfying $U(N)$ gauge group. The matrices $(T^a)^i_j$ can be normalised to

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{1.2.97}$$

In consequence, the scattering amplitudes for open strings must include traces of the product of Chan-Paton factors. At tree level, this forms a single trace.

The introduction of Chan-Paton factors refers to the fact that each end point of the open string is confined on a $(p+1)$-dimensional hyperplane called a D_p-brane when the Dirichlet boundary conditions were applied to each endpoint in $(D-(p+1))$ spatial directions. In the case of N parallel branes, a Chan-Paton label, running from 1 to N, indicates which brane the string is attached to. In this way, a stack of N coincident branes gives a $U(N)$ gauge theory.

By summing over all possible ordering of these factors, the tree-level open string amplitudes can be written as

$$\begin{aligned}\mathcal{A}^{\mathrm{op}}_n(k_1, k_1, \ldots, k_n) = i\sqrt{2}^n g_o^n C_{D_2}(2\pi)^D \delta^D(\sum_i k_i) \sum_{(a_1,\ldots,a_n)\in S_n/\mathbf{Z}n} \mathrm{tr}(T^{a_1}T^{a_2}\ldots T^{a_n}) \\ \times \mathcal{A}^{\mathrm{op}}_n(1, 2, \ldots, n)\end{aligned} \tag{1.2.98}$$

where the summation is performed over the $(n-1)!$ non-cyclic permutations of the external legs. The partial amplitude $\mathcal{A}^{\mathrm{op}}_n(1, 2, \ldots, n)$ is color-ordered where vertex operators are inserted along the boundary at the specific ordering designated. The general expression for an n-point color-ordered open string amplitude is

$$\begin{aligned}\mathcal{A}^{\mathrm{op}}_n(1, 2, \ldots, n) =& |(x_1 - x_{n-1})(x_1 - x_n)(x_{n-1} - x_n)| \\ &\times \int \prod_{i=2}^{n-2} dx_i \prod_{i=1}^{n-1} \Theta(x_{i+1} - x_i) \prod_{j<k}^{n} (x_k - x_j)^{2\alpha' k_j \cdot k_k} \mathcal{K}^{(l)}_n\end{aligned} \tag{1.2.99}$$

where $\Theta(x_j - x_i)$ is the Heaviside step function forcing the ordering of external legs as $x_j > x_i$ since $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for otherwise.

In order to relate the string coupling constant g_s to the Yang-Mills and the gravitational coupling constants appearing in the low energy field theory, the precise normalization is required. The normalization factors C_{S^2} and C_{D_2} can be determined by requiring unitarity of the amplitudes. For instance, a 4-point tree-level amplitude which has simple poles due to the exchange of intermediate particles can factorise into two 3-point tree amplitudes at these poles. This allow us to fix its coefficient.

For simplicity, let consider the four-tachyon amplitude of the open bosonic string. According to (1.2.98) and (1.2.99), the amplitude takes the form

$$\begin{aligned}\mathcal{A}_4^{\text{tachyon}}(s,t,u) = {} & 4ig_o^4 C_{D_2}(2\pi)^D\delta^D(\sum_{i=1}^{4} k_i) \\ & \times \Big[\mathrm{tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3}) + \mathrm{tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})B(-\alpha's-1,-\alpha't-1) \\ & + \mathrm{tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4}) + \mathrm{tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})B(-\alpha't-1,-\alpha'u-1) \\ & + \mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \mathrm{tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})B(-\alpha's-1,-\alpha'u-1)\Big] \qquad (1.2.100)\end{aligned}$$

where the Euler beta function $B(a,b)$ defined as

$$B(a,b) = \int_0^1 dx\ x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \qquad (1.2.101)$$

with the Mandelstam variables

$$s = -(k_1+k_2)^2, \quad t = -(k_1+k_3)^2, \quad u = -(k_1+k_4)^2. \qquad (1.2.102)$$

The beta function $B(a,b)$ contains an infinite series of simple poles at $a=-n$ or $b=-n$ where n is a non-negative integer. Near the pole $s=-1/\alpha'$, the 4-point amplitude becomes

$$\begin{aligned}\mathcal{A}_4^{\text{tachyon}}(s,t,u) = & -4ig_o^4 C_{D_2}(2\pi)^D\delta^D(\sum_{i=1}^{4} k_i)\frac{1}{\alpha's+1}\mathrm{tr}(\{T^{a_1},T^{a_2}\}\{T^{a_3},T^{a_4}\}) \\ & + \text{terms analytic at } \alpha's=-1. \qquad (1.2.103)\end{aligned}$$

As a consequence of unitarity, we can factorise the amplitude (1.2.100) on the tachyonic pole, $s = -1/\alpha'$ into two on-shell 3-point amplitudes with the ansatz

$$i\int \frac{d^Dk}{(2\pi)^D}\frac{\mathcal{A}_3^{\text{tachyon}}(k_1,k_2,k)\mathcal{A}_3^{\text{tachyon}}(-k,k_3,k_4)}{-k^2+\alpha'^{-1}}$$
$$= -8i\alpha' g_o^6 C_{D_2}^2 (2\pi)^D \frac{1}{\alpha' s+1}\delta^D(\sum_{i=1}^{4}k_i)\sum_a \text{tr}(\{T^{a_1},T^{a_2}\}T^a)\text{tr}(T^a\{T^{a_3},T^{a_4}\})$$
$$= -4i\alpha' g_o^6 C_{D_2}^2 (2\pi)^D \frac{1}{\alpha' s+1}\delta^D(\sum_{i=1}^{4}k_i)\text{tr}(\{T^{a_1},T^{a_2}\}\{T^{a_3},T^{a_4}\}). \quad (1.2.104)$$

To obtain the last line, we have used the completeness relation of T^a_{ij} with the normalisation (1.2.97). Note that the expression for the 3-point tachyonic amplitude is

$$\mathcal{A}_3^{\text{tachyon}}(k_1,k_2,k_3) = \sqrt{2}^3 i g_o^3 C_{D_2}(2\pi)^D\delta^D(k_1+k_2+k_3)\text{tr}(\{T^{a_1},T^{a_2}\}T^{a_3}). \quad (1.2.105)$$

By comparing (1.2.103) and (1.2.104), it gives

$$C_{D_2} = \frac{1}{\alpha' g_o^2}. \quad (1.2.106)$$

In analogy to the open string, we can deduce the similar relation between the factor C_{S^2} and g_s by unitarity and that relation is

$$C_{S^2} = \frac{8\pi}{\alpha' g_s^2}. \quad (1.2.107)$$

1.2.3 Mixed Open and Closed String Amplitudes

Tree-level scattering amplitudes for processes that involve both open and closed strings have a world-sheet with the topology of a disc. This world-sheet can be conformally mapped to the upper complex half-plane $\mathcal{H}_+ = \{z \in \mathbb{C}|\text{Im}(z) \geq 0\}$. Open string vertices are inserted along the boundary of the world-sheet while closed string vertices are inserted in the bulk. According to Stieberger [37], tree-level string amplitudes involving N_o open and N_c closed strings can be expressed as linear combinations of string amplitudes of $N_0 + 2N_c$ open string scattering.

The generic expression for the mixed disk amplitude describing N_o open and N_c closed string scattering is

$$\mathcal{M}^{(N_o,N_c)} = V_{\text{CKG}}^{-1}\sum_{\sigma\in S_{N_o-1}}\int_{\mathcal{I}_\sigma}\prod_{i=1}^{N_o}dx_i\int_{\mathcal{H}_+}\prod_{j=1}^{N_c}d^2z_j\,\langle\prod_{i=1}^{N_o}V_o(x_i,k_i)\prod_{j=1}^{N_c}V_c(z_j,\bar{z}_j,k_j)\rangle_X \quad (1.2.108)$$

where V_o and V_c are the inserted vertex operators of open and closed string respectively. The factor V_{CKG} refers to the volume of the conformal Killing group which will be canceled out by fixing any three vertex positions. The integration along the boundary is subject to the integration region $\mathcal{I}_\sigma$ forcing the ordering of the open string variables. The integral is summed over all $(N_o - 1)!$ non-cyclic orderings σ.

In this thesis, we will only focus on the mixed amplitudes of $(N-2)$ open strings and a single closed string. The disk amplitudes in this case can be expressed by the integral,

$$\begin{aligned}\mathcal{M}^{(N-2,1)}(1,4,5,\ldots,N-2;p_1,p_2) &= V_{\text{CKG}}^{-1}\delta(\sum_{i\in\mathbb{N}_O,i=1}^{N-2} k_i + p_1 + p_2) \\ &\times \int_{\mathcal{I}_\sigma} \prod_{i\in\mathbb{N}_0,i=1}^{N-2} dx_i \prod_{r,s\in\mathbb{N}_0,r\neq s}^{N-2} |x_r - x_s|^{2\alpha' k_r\cdot k_s} \int_{\mathcal{H}_+} d^2z(z-\bar{z})^{2\alpha' p_1\cdot p_2} \\ &\times \prod_{i\in\mathbb{N}_0,i=1}^{N-2} (x_i - z)^{2\alpha' p_1\cdot k_i}(x_i - \bar{z})^{2\alpha' p_2\cdot k_i}\mathcal{F}_{N-2,1}(x_i,z,\bar{z}).\end{aligned} \tag{1.2.109}$$

The set $\mathbb{N}_O$ is $\{1,4,5,6,\ldots,N\}$ containing indices used for labeling the open strings. The polarization vectors are contained in the branch-free function $\mathcal{F}_{N-2,1}$. This function depends on the types of particles we consider. This expression is for the colour-ordered partial amplitude corresponding to a group factor $\text{Tr}(T_1T_4T_5\ldots T_{N-2})$ which gives rise to the integration region $\mathcal{I}_\sigma = \{x \in \mathbb{R}|x_1 < x_4 < x_5 < \ldots < x_{N-2}\}$. The closed string momenta p_1 and p_2 are assumed to be unrelated at first.

In order to integrate the closed string variables over all the complex upper half-plane, we rewrite $z = z_1 + iz_2$ where $z_1 \in (-\infty,\infty)$ and $z_2 \in [0,\infty)$. By analytically continuing the variable z_2 to the complex plane, we deform the z_2-contour line along the positive real axis to the pure imaginary axis with $\text{Im}(z_2) \geq 0$ by

$$z_2 \to ie^{-2i\epsilon}z_2 \simeq i(1-2i\epsilon)z_2. \tag{1.2.110}$$

The ϵ is present to make sure that the z_2-integral avoids all branch points located at $\pm i(x_i - z_1)$. Accordingly, this changes the expressions in the integrands of (1.2.109) as

$$
\begin{aligned}
(x_i - z)^a \to & (x_i - \xi - i\epsilon\delta)^a, \\
(x_i - \bar{z})^a \to & (x_i - \eta + i\epsilon\delta)^a, \\
(z - \bar{z})^a \to & (\xi - \eta + 2i\epsilon\delta)^a.
\end{aligned} \tag{1.2.111}
$$

where ξ, η are real variables defined as

$$
\xi = z_1 + z_2, \quad \eta = z_1 - z_2 \tag{1.2.112}
$$

whose values are subject to

$$
\eta - \xi \equiv \delta > 0. \tag{1.2.113}
$$

After changing variables $(z, \bar{z})$ to (ξ, η), the mixed string amplitude (1.2.109) takes the form

$$
\begin{aligned}
\mathcal{M}^{(N-2,1)}(1,4,5,\ldots,N-2;p_1,p_2) = & \frac{i}{2} V_{\mathrm{CKG}}^{-1} \delta(\sum_{i \in \mathbb{N}_O, i=1}^{N-2} k_i + p_1 + p_2) \\
\times \int_{\mathcal{I}_\sigma} \prod_{i \in \mathbb{N}_0, i=1}^{N-2} dx_i & \prod_{r,s \in \mathbb{N}_0, r \neq s}^{N-2} |x_r - x_s|^{2\alpha' k_r \cdot k_s} \int_{-\infty}^{\infty} d\xi \int_{\xi}^{\infty} d\eta (\xi - \eta + 2i\epsilon\delta)^{2\alpha' p_1 \cdot p_2} \\
\times \prod_{i \in \mathbb{N}_0, i=1}^{N-2} (x_i - \xi - i\epsilon\delta)^{2\alpha' p_1 \cdot k_i} & (x_i - \eta + i\epsilon\delta)^{2\alpha' p_2 \cdot k_i} \mathcal{F}_{N-2,1}(x_i, \xi, \eta).
\end{aligned} \tag{1.2.114}
$$

Note that the factor $\frac{i}{2}$ is due to the Jacobian when changing variables. At this point, the closed string variables turns into two open string ones ξ and η. To relate the integral expression (1.2.1092) to color-ordered open string amplitudes, we need to make sure that the integrand is in the right form as (1.2.99). By careful examination of branch cuts, we can form the following relations

$$
z^c = \begin{cases} e^{i\pi c}(-z)^c, & \mathrm{Im}(z) \geq 0 \\ e^{-i\pi c}(-z)^c, & \mathrm{Im}(z) < 0 \end{cases} \tag{1.2.115}
$$

when $\mathrm{Re}(z) < 0$.

Using the relations (1.2.115), one obtains

$$
\begin{aligned}
\mathcal{M}^{(N-2,1)}(1,4,5,\dots,N-2;p_1,p_2) = \frac{i}{2} V_{\text{CKG}}^{-1}\delta(\sum_{i=1}^{N} k_i)\int_{\mathcal{I}_\sigma}\prod_{i\in\mathbb{N}_0,i=1}^{N-2} dx_i \\
\times \prod_{r,s\in\mathbb{N}_0,r\neq s}^{N-2}|x_r-x_s|^{2\alpha' k_r\cdot k_s}\int_{-\infty}^{\infty}dx_2\int_{x_2}^{\infty}dx_3|x_3-x_2|^{2\alpha' k_2\cdot k_3}\Omega(x_2,x_3) \\
\times \prod_{i\in\mathbb{N}_0,i=1}^{N-2}|x_i-x_2|^{2\alpha' k_i\cdot k_2}|x_i-\bar{z}|^{2\alpha' k_i\cdot k_3}\mathcal{F}_{N-2,1}(x_i,z,\bar{z})\Lambda(x_i,x_2,x_3)
\end{aligned}
\tag{1.2.116}
$$

where we have redefined the closed string variables $x_2 \equiv \xi$ and $x_3 \equiv \eta$ and their corresponding momenta $k_2 = p_1$ and $k_3 = p_2$. The functions $\Omega(x_2, x_3)$ and $\Lambda(x_i, x_2, x_3)$ are the phase factors corresponding to the appropriate branch of the integrand. They are defined as follows:

$$
\begin{aligned}
\Omega(x_2,x_3) =& e^{2\pi i\alpha' k_2\cdot k_3\Theta(x_3-x_2)} \\
\Lambda(x_i,x_2,x_3) =& e^{-2\pi i\alpha' k_i\cdot k_2\Theta(x_2-x_i)}e^{2\pi i\alpha' k_i\cdot k_3\Theta(x_3-x_i)}
\end{aligned}
\tag{1.2.117}
$$

with $\Theta(x_j - x_i)$ being the Heaviside step function whose value equal to 1 for $x_j > x_i$ and 0 for otherwise. Therefore, we can write this partial mixed amplitude in terms of pure open string amplitudes as

$$
\begin{aligned}
\mathcal{M}^{(N-2,1)}(1,4,5,\dots,N-2;p_1,p_2) = \frac{i}{2}\sum_{m\in\mathbb{N}_O,m=1}^{N-2}\;\sum_{n\in\mathbb{N}_O,n=m+1}^{N-1} \\
\times \exp\left\{\pi i\alpha'\left(s_{23}-\sum_{i\in\mathbb{N}_O,i=1}^{m}s_{i,2}+\sum_{j\in\mathbb{N}_O,j=1}^{n}s_{j,3}\right)\right\} \\
\times \mathcal{A}_N(1,4,5,\dots,m,2,m+1,\dots,n,3,n+1,\dots,N) \\
+\frac{i}{2}\exp(\pi i\alpha' s_{23})\mathcal{A}_N(2,3,1,4,5,\dots,N).
\end{aligned}
\tag{1.2.118}
$$

Again, $s_{ij} = 2k_i \cdot k_j$. This result was presented in [38].

Part I

Gauge Theory as Infinite-Tension String Theory

It is well known for a long time that in the limit of low energies (which refers to zero-slope or infinite string tension limit as $\alpha' = 1/(2\pi T) \to 0$) string theory reproduces the scattering amplitudes of certain field theories. When taking massless states into consideration, it is shown in [5] that this limit replicates tree diagrams of Yang–Mills theory. Similarly, if closed strings are concerned, one can reproduce amplitudes of quantum gravity [29, 39]. In this point of view, string theory leads to an effective field theory at low energy. This concept results in α' correction terms to the Lagrangian of usual field theories.

There are several approaches to derive the low energy effective action from string theory but the most straightforward one is to compute the S-matrix of the string amplitudes and then to construct a field theory action which reproduces them at each level of α'. Much research has been conducted on finding these α' correction terms based on both bosonic and superstring theories (see [40–44] as examples for open strings and [45–48] as those for closed strings). In the following section, we will review how to obtain the correction terms for Yang-Mills theory up to the order of α'^2. The calculation is mostly based on Tseytlin's paper [40].

This duality between string and gauge field theory presents many useful features. First of all, one can replace calculations of field theory amplitudes which could involve a large number of Feynman diagrams by those of string amplitudes which contains considerably fewer diagrams at each order. Secondly, there exist explicit expressions for scattering amplitudes at all loop levels [49]. Finally, understanding the structure of string amplitudes would provides a better insight into those of quantum field theories. In fact, many crucial field theory relations are closely tied to relations in string theory. Among them are the renowned BCJ relations of Bern, Carrasco and Johansson [50] and the Kleiss-Kujif (KK) relations [51] which relate to the string monodromy relations called Plahte identities [52].

The final feature expressed above is of particular interest to this thesis as it allows one to deduce field theory amplitude relations from the string theory at the vanishing string-slope limit. For example, Kawai, Lewellen and Tyle (KLT) relations [53] expressing closed string amplitudes as products of two open string amplitudes gives alternative descriptions of gravity as the square of gauge theory [54]. The squaring relations between gravity and the gauge theory were proven later at the quantum field theory level [55–57]. This relation beautifully allows us to compute troublesome gravitational amplitudes by looking at much simpler amplitudes in gauge theory. The connection seems miraculous as it connects together two theories which are distinct in structure and physical interpretation. This simplification is clear from the perspective of quantum field theory where the Feynman rules for gravity contain an infinite number of graviton interaction vertices whilst gauge theory contains only three and four-point interactions. We review the KLT relations and their derivation in the appendix A.1.

Another interesting relation in string amplitudes was discovered by Plahte in 1970, namely, the Plahte identities which are linear relations between color-ordered open string scattering amplitudes [52]. The relations are connected to the field theory BCJ relations in and the KK relations. By using these monodromy relations in string theory, the number of independent color-ordered open string amplitudes with n external legs is reduced from $(n-1)!$ as given by a cyclic property of the trace down to $(n-3)!$ [37, 58]. This is in congruence with the field theory of pure Yang-Mills amplitudes where one can use KK relations and BCJ relations to represent all color-ordered gauge amplitudes in terms of a basis $(n-3)!$ amplitudes.

The Plahte identities can be illustrated by geometric shapes in the complex plane [59]. Plahte identities for n-point open string amplitudes can be represented by n-sided polygons whose sides are proportional to the amplitudes and the angles are given by products of two corresponding momenta. An intriguing result is found in the specific case of the 4-point open tachyon amplitudes where the three amplitudes form a triangle. The area of this triangle is equal to the 4-point closed tachyon amplitude as a consequence of the KLT relations. However, this simple interpretation of the area as a closed string amplitudes is not easy to generalise directly to higher point amplitudes as it is not possible to contain all the relevant open string amplitudes in a single polygon.

The purpose of this part of the thesis is to explore aspects of string theory as gauge theory in the limit of infinite string-tension. We start by reviewing how to obtain the effective Lagrangian from the open bosonic string theory using the knowledge of string and quantum field theory amplitude in chapter 2. Then, the linear relations among color-ordered string amplitudes, namely Plahte identities, and their field theory limit are investigated in chapter 3.

Chapter 2

Yang-Mills Lagrangian and Its Corrections from String Theory

It has long been observed that field theory amplitudes can be recovered from string theory in the limit where the string tension becomes infinite. In this limit, the heavy string states becomes too massive and decouple leaving only the scalar and the massless string states. This idea was first investigated by Scherk [60] who showed that the Dual-Resonance model (a prehistoric name for string theory) reduces to a scalar field theory with cubic interactions when the scalar string modes are concerned. If the massless string states are selected, this limit reproduces a tree diagram of Yang-Mills theory for open strings [5] and that of Einstein's gravity for closed strings [29, 39].

As mentioned above, the vanishing Regge slope limit ($\alpha' \to 0$) gives massless Yang-Mills field theory. The proof is very straightforward by showing that the S-matrix computed from the Feynman rules of the Yang-Mills theory at tree level coincides with that of the string theory with massless external states when the zero-slope limit was applied. Let first consider the 3-gluon string amplitude at tree level. According to (1.2.98), (1.2.99), and (1.2.96), the gluonic amplitude takes the form

$$\begin{aligned}\mathcal{A}_3^{\text{gluon}}(k_1,k_2,k_3) =& 2i\frac{g_o}{\sqrt{\alpha'}}(2\pi)^D\delta^D(\sum_{i=1}^{3}k_i)\Big(\xi_1\cdot k_{23}\xi_2\cdot\xi_3+\xi_2\cdot k_{31}\xi_3\cdot\xi_1\\ &+\xi_3\cdot k_{12}\xi_1\cdot\xi_2+\frac{\alpha'}{2}\xi_1\cdot k_{23}\xi_2\cdot k_{31}\xi_3\cdot k_{12}\Big)\mathrm{tr}(T^{a_1}[T^{a_2},T^{a_3}])\end{aligned} \tag{2.0.1}$$

where $k_{ij} \equiv k_i - k_j$. The leading term is exactly the 3-point Yang-Mills amplitudes we found in the chapter one using the Feynman rules (See figure 1.1) with $q = g_o/\sqrt{2\alpha'}$. Apart from the Yang-Mills term, the expression (2.0.1) allows us to add a correction term to the gauge theory. One can find a suitable Lagrangian which reproduces the term at the α' order as

$$-\frac{\sqrt{4\alpha'}}{3} g_o \mathrm{tr}(F^{\mu}{}_{\nu} F^{\nu}{}_{\rho} F^{\rho}{}_{\mu}) \tag{2.0.2}$$

which was first discovered in [29]. The tensor $F_{\mu\nu}$ is defined as (1.1.2) with $q = g_o/\sqrt{2\alpha'}$. Hence, we can write the low energy effective Lagrangian as

$$\mathcal{L}_{\text{eff}} = -\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{4}{3}q(\sqrt{2}\alpha')\mathrm{tr}(F^{\mu}{}_{\nu} F^{\nu}{}_{\rho} F^{\rho}{}_{\mu}). \tag{2.0.3}$$

This method to obtain the effective Lagrangian is called S-matrix approach. It starts by writting down the most general gauge-invariant Lagrangian up to the desired order in α'. By doing so, the Bianchi identity

$$\mathcal{D}^{\mu}F^{\nu\rho} + \mathcal{D}^{\rho}F^{\mu\nu} + \mathcal{D}^{\nu}F^{\rho\mu} = 0 \tag{2.0.4}$$

was used to make sure that all coefficients in the Lagrangian are all independent. Then the unknown coefficients are fixed by comparing to the n-point on-shell string amplitudes. More specifically, to fix the coefficients of the α'^N order terms in the effective Lagrangian, one needs to know all terms at the order α'^k where $k = 1, 2, \ldots, N-1$ together with the expression for $(N+2)$-point gauge boson amplitude from the open string expanded at the order α'^N. For instance, to determine the coefficient of the $\alpha'\mathrm{tr}F^3$ term, it is required to know the 3-point amplitude. Likewise, to determine that of the $\alpha'^2\mathrm{tr}F^4$ and $\alpha'^2\mathrm{tr}\mathcal{D}^2F^2$, it is necessary to know the 4-point amplitude, and so forth.

To calculate the low energy effective Lagrangian up to α'^2 order, we write down the general expression for the effective Lagrangian as

$$\begin{aligned}\mathcal{L}_{\text{eff}} = -\frac{1}{2}\mathrm{tr}\Bigg[(F_{\mu\nu}F^{\mu\nu}) &+ q(\sqrt{2}\alpha')\Bigg(a_1(F^{\mu}{}_{\nu} F^{\nu}{}_{\rho} F^{\rho}{}_{\mu}) + a_2\mathcal{D}^{\lambda}F_{\lambda}{}^{\mu}\mathcal{D}^{\rho}F_{\rho\mu}\Bigg) \\ &+ q^2(\sqrt{2}\alpha')^2\Bigg(a_3F^{\mu\lambda}F^{\nu}{}_{\lambda}F_{\mu}{}^{\rho}F_{\nu\rho} + a_4F^{\mu}{}_{\lambda}F_{\nu}{}^{\lambda}F^{\nu\rho}F_{\mu\rho} + a_5F^{\mu\nu}F_{\mu\nu}F^{\lambda\rho}F_{\lambda\rho} \\ &+ a_6F^{\mu\nu}F^{\lambda\rho}F_{\mu\nu}F_{\lambda\rho} + a_7F^{\mu\nu}\mathcal{D}^{\lambda}F_{\mu\nu}\mathcal{D}^{\rho}F_{\rho\lambda} + a_8\mathcal{D}^{\lambda}F_{\lambda\rho}\mathcal{D}^{\rho}F_{\rho\nu}F^{\mu\nu} \\ &+ a_9\mathcal{D}^{\rho}\mathcal{D}^{\lambda}F_{\lambda}{}^{\mu}\mathcal{D}_{\rho}\mathcal{D}^{\nu}F\mu\nu\Bigg)\Bigg] + \mathcal{O}(\alpha'^3).\end{aligned} \tag{2.0.5}$$

By a careful examination of the possibility of field redefinition in [40], one can set the variables a_2, a_7, a_8, a_9 equal to zeroes as they are not sensitive to the S-matrix due to the equivalence theorem. Again, one can fix $a_1 = \frac{4}{3}$ by comparing 3-point string amplitude to that of the standard Yang-Mills theory. To fix $a_3, \dots, a_6$, we need to evaluate the 4-point gluonic string amplitudes. For simplicity, it is sufficient to consider only the $(\xi \cdot \xi)(\xi \cdot \xi)$-term in the four-point amplitude where ξ is a polarlization vector. As far as the color ordering with a group factor $\mathrm{tr}(T^1T^2T^3T^4)$ is concerned, the string amplitude (1.2.98) takes the form

$$8iq^2(2\pi)^4\delta(\sum_{i=1}^{4} k_i)\mathrm{tr}(T^1T^2T^3T^4)x_4^2 \int_0^1 dx x^{-\alpha' s}(1-x)^{-\alpha' t}\mathcal{K}_4 \tag{2.0.6}$$

where the Mandelstam variables and the function $\mathcal{K}_4$ were defined in (1.2.102) and (1.2.96) respectively. Note that we have fixed the vertex positions x_1, x_3 and x_4 to be $0, 1$ and ∞ respectively. The factor x_4^2 will get cancelled out when expanding $\mathcal{K}_4$ to obtain the $(\xi \cdot \xi)(\xi \cdot \xi)$-term. Therefore, the amplitude (2.0.6) becomes

$$\begin{aligned}
&8iq^2(2\pi)^4\delta(\sum_{i=1}^{4} k_i)\mathrm{tr}(T^1T^2T^3T^4) \\
&\times \Big[B(-\alpha' s - 1, 1 - \alpha' t)\xi_{1234} + B(1-\alpha' s, -\alpha' t - 1)\xi_{1423} + B(1 - \alpha' s, 1 - \alpha' t)\xi_{1324}\Big] \\
&= 8iq^2(2\pi)^4\delta(\sum_{i=1}^{4} k_i)\mathrm{tr}(T^1T^2T^3T^4) \\
&\times \alpha'^2 \Gamma \begin{bmatrix} \alpha' s, \alpha' t \\ \alpha' u \end{bmatrix} \left(\frac{tu}{1+\alpha' s}\xi_{1234} + \frac{su}{1+\alpha' t}\xi_{1423} + \frac{st}{1+\alpha' u}\xi_{1324}\right)
\end{aligned} \tag{2.0.7}$$

where

$$\Gamma \begin{bmatrix} x, y \\ z \end{bmatrix} \equiv \frac{\Gamma(-x)\Gamma(-y)}{\Gamma(1+z)}. \tag{2.0.8}$$

and $\xi_{1234} \equiv (\xi_1 \cdot \xi_2)(\xi_3 \cdot \xi_4)$. The beta function was expressed in (1.2.101). Using the approximation

$$\Gamma(1+\epsilon) = 1 - \gamma\epsilon + \frac{1}{2}(\gamma^2 + \frac{1}{6}\pi^2)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{2.0.9}$$

where γ is the Euler constant, one obtains

$$\Gamma \begin{bmatrix} \alpha' s, \alpha' t \\ \alpha' u \end{bmatrix} = \frac{1}{\alpha'^2 st} - \frac{1}{6}\pi^2 + \mathcal{O}(\alpha'). \tag{2.0.10}$$

After expanding (2.0.7) using (2.0.10), at the order α'^2, (2.0.7) becomes

$$8iq^2\alpha'^2(2\pi)^4\delta(\sum_{i=1}^{4}k_i)\mathrm{tr}(T^1T^2T^3T^4)\bigg(-\frac{1}{6}\pi^2\Big[tu\xi_{1234}+su\xi_{1423}+st\xi_{1324}\Big] +su\xi_{1234}+tu\xi_{1423}+u^2\xi_{1324}\bigg). \tag{2.0.11}$$

To obtain the above expression, the expansion of $\frac{1}{1+x}$ was used. By comparing (2.0.11) with the amplitude produced by the field theory expression (2.0.5), one can find the values of $a_3,\ldots,a_6$ as

$$a_3=2a_4=\frac{1}{3}\pi^2. \qquad a_5=-\frac{1}{12}\pi^2-\frac{1}{2}, \qquad a_6=-\frac{1}{24}\pi^2+\frac{1}{2}. \tag{2.0.12}$$

Note that to make the comparison, we substitute $A^R_\mu = T^R\xi_\mu e^{ik\cdot x}$ to the effective Lagrangian (2.0.5) then keep only the contribution with $\mathrm{tr}(T^1T^2T^3T^4)$ and the $(\xi\cdot\xi)(\xi\cdot\xi)$-term. More details can be found in [40].

The fact that string theory reproduces field theories at low energy proves useful for understanding the structure of amplitude relations for both string theory and field theory. In this thesis, we are particularly interested in the geometric structure of Plahte identities (string monodromy relations). Consequently, in the next chapter, we will investigate aspects of geometric diagrams based on the identities called Plahte diagrams as well as discuss some possible applications and related issues.

Chapter 3

Plahte Diagrams for String Scattering Amplitudes

We devote this chapter to discuss the linear relations between tree-level open string amplitudes called Plahte identities and investigate the geometric structure of diagrams representing them, namely Plahte diagrams. Since we may encounter open string amplitude $\mathcal{A}_n^{\text{op}}$ several times during the chapter, it is more convenient to drop the superscript on the string amplitude symbol. From this point forward, we use $\mathcal{A}_n$ and $\mathscr{A}_n$ to represent open and closed string amplitude respectively.

3.1 Plahte Identities

We will here briefly review the derivation of the Plahte identities. According to (1.2.99), the general expression for an n-point color-ordered open string amplitude is

$$\begin{aligned}\mathcal{A}_n(1,2,\ldots,n) = \int \prod_{i=1}^{n} dz_i \frac{|z_{ab}z_{bc}z_{ac}|}{dz_a dz_b dz_c} \prod_{i=1}^{n-1} \Theta(x_{i+1}-x_i)\\ \times \prod_{1\leq i<j\leq n} |z_{ij}|^{2\alpha' k_i\cdot k_j} \mathcal{K}_n.\end{aligned} \tag{3.1.1}$$

This expression is valid for both bosonic and supersymmetric string [61]. For the bosonic string, $dz_i = dx_i$ while for the superstring case $dz_i = dx_i d\theta_i$ and $z_{ij} = x_i - x_j + \theta_i\theta_j$. Invariance under Möbius transformations allows us to set any three

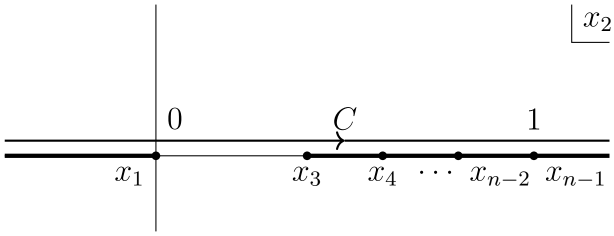


Figure 3.1: Contour in upper half-plane and branch cuts along the real axis.

arbitrary integration variables, i.e. z_a, z_b and z_c equal to any fixed distinct values. A conventional choice is $x_1 = 0$, $x_{n-1} = 1$ and $x_n = +\infty$ for the bosonic string as well as $\theta_{n-1} = \theta_n = 0$ for the supersymmetric case.

The function $\mathcal{K}_n$ is a branch free function which comes from the operator product expansion of vertex operators. $\mathcal{K}_n = 1$ for tachyons and it equals (1.2.96) for massless vectors. In addition, $\mathcal{K}_n = \int \prod_{i=1}^{n} d\eta_i \times \exp\Big[\sum_{i\neq j}\Big(\frac{\sqrt{\alpha'}\eta_i(\theta_i-\theta_j)(\xi_i\cdot k_j)-\eta_i\eta_j(\xi_i\cot\xi_j)}{(x_i-x_j+\theta_i\theta_j)}\Big)\Big]$ for the superstring amplitude where η_i are Grassmann variables.

Consider the complex integral

$$\int_{-\infty}^{\infty} dx_2 \int \prod_{i=3}^{n-2} dx_i \Bigg(\Theta(x_3 - x_1)\prod_{i=3}^{n-1}\Theta(x_{i+1} - x_i) \\ \times \prod_{1\leq i<j\leq n}(x_j - x_i)^{2\alpha' k_i\cdot k_j}\mathcal{K}_n\Bigg) \tag{3.1.2}$$

where we choose $x_1 = 0$, $x_{n-1} = 1$ and $x_n = +\infty$. The ordering of variables x_i is $x_1 < x_3 < x_4 < \ldots < x_{n-1} < x_n$ as a result of the Heaviside step functions. The integrand in (3.1.2) contains $n-2$ branch points with respect to x_2 where all branch points are situated along the real axis. The integration with respect to the variable x_2 can be performed slightly above the real axis to avoid the branch points and can then be closed in the upper half plane as illustrated in the figure 3.1. The integral vanishes due to the absence of singularities. To relate the term $(x_j - x_i)^{2\alpha' k_i\cdot k_j}$ to the $|z_{ij}|^{2\alpha' k_i\cdot k_j}$ in (3.1.1) the relations (1.2.115) are useful. As a result, we can obtain the Plahte identity for bosonic string as

$$e^{2\pi i\alpha' k_1\cdot k_2}\mathcal{A}_n(2,1,3,\ldots,n) + \mathcal{A}_n(1,2,3,\ldots,n) + e^{-2\pi i\alpha' k_2\cdot k_3}\mathcal{A}_n(1,3,2,\ldots,n) \\ + \ldots + e^{-2\pi i\alpha' k_2\cdot(k_3+k_4+\ldots+k_{n-1})}\mathcal{A}_n(1,3,\ldots,n-1,2,n) = 0. \tag{3.1.3}$$

The other Plahte identities can be found by a similar approach but using different orderings and integration variables. Note that all the amplitudes appearing in a Plahte identity involve the same states and polarizations.

The complex conjugate relation can be obtained using a similar contour in the lower half-plane. The combination of these two identities leads to the following relations:

$$\begin{aligned}0 =&\mathcal{A}_n(2,1,3,\ldots,n)+\mathcal{C}_{k_1,k_2}\mathcal{A}_n(1,2,3,\ldots,n)\\&+\mathcal{C}_{k_2,k_1+k_3}\mathcal{A}_n(1,3,2,\ldots,n)\\&+\ldots+\mathcal{C}_{k_2,k_1+k_3+\ldots+k_{n-1}}\mathcal{A}_n(1,3,4,\ldots,n-1,2,n)\end{aligned}\tag{3.1.4}$$

and

$$\begin{aligned}0 =&\mathcal{S}_{k_1,k_2}\mathcal{A}_n(1,2,3,\ldots,n)+\mathcal{S}_{k_2,k_1+k_3}\mathcal{A}_n(1,3,2,\ldots,n)\\&+\ldots+\mathcal{S}_{k_2,k_1+k_3+\ldots+k_{n-1}}\mathcal{A}_n(1,3,4,\ldots,n-1,2,n)\end{aligned}\tag{3.1.5}$$

where we use the notation $\mathcal{S}_{k_i,k_j}\equiv\sin(2\pi\alpha' k_i\cdot k_j)$ and $\mathcal{C}_{k_i,k_j}\equiv\cos(2\pi\alpha' k_i\cdot k_j)$. These are analytic relations between the amplitudes and so although they are derived from the integral expression (3.1.1) which only converges when all the momenta satisfy $2\alpha' k_i\cdot k_j>-1$. They will continue to hold for the analytic continuations of the amplitudes away from this restricted kinematic region. In the limit $\alpha'\to 0$, The relation (3.1.4) becomes

$$A_n(2,1,3,\ldots,n)=(-1)\sum_\sigma A_n(1,\sigma,n)\tag{3.1.6}$$

where $\sigma\in OP(\{2\}\cup\{3,4,\ldots,n-1\})$ which is a set of ordered permutation preserving ordering within both sets, i.e. $\{2\}$ and $\{3,4,\ldots,n-1\}$. The above equation expresses the Kleiss-Kujif relations in field theory [51].

Besides, when applying the same limit to the equation (3.1.5), we obtain

$$\begin{aligned}0 =&s_{12}A_n(1,2,3,\ldots,n)+(s_{12}+s_{23})A_n(1,3,2,\ldots,n)\\&+(s_{12}+s_{23}+s_{24})A_n(1,3,4,2,\ldots,n)\\&+\ldots+(s_{12}+s_{23}+\ldots+s_{1(n-1)})A_n(1,3,4,\ldots,n-1,2,n)\end{aligned}\tag{3.1.7}$$

where $s_{ij}\equiv(k_i+k_j)^2=2k_i\cdot k_j$. This equation is exactly the BCJ relation [50]. This means the Plahte identities can be viewed as a string generalisation of the field theory relations, i.e. Kleiss-Kujif and BCJ relations.

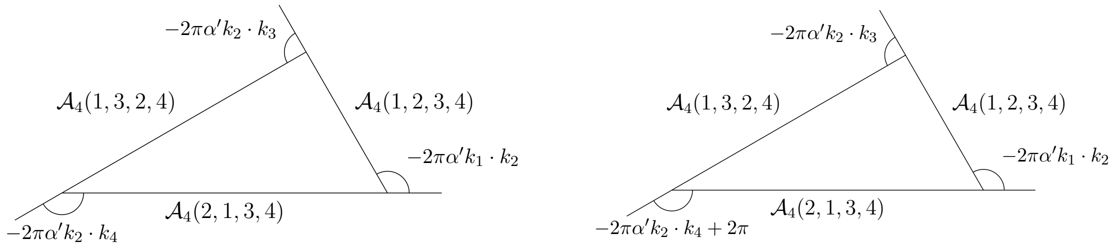


Figure 3.2: Triangle representing the 4-point tachyon amplitudes (left) and gauge amplitudes (right) from the Plahte identities.

Generally, the Plahte identities expressed in (3.1.3) are valid for N-point amplitudes in both bosonic and supersymmetric string theory as transforming the integrand in the expression (3.1.1) for the superstring theory from $z_{ij}^{2\alpha' k_i \cdot k_j}$ to $z_{ji}^{2\alpha' k_i \cdot k_j}$ encounters the same phase correction (1.2.115).

3.2 Plahte Diagrams

An intriguing feature of Plahte identities is that they can be depicted geometrically. Let us first explore the simplest example for open string amplitudes, i.e. the scattering of four tachyons. According to (3.1.3), the Plahte identity for this process is

$$\mathcal{A}_4(2,1,3,4) + e^{-2\pi i\alpha' k_1 \cdot k_2}\mathcal{A}_4(1,2,3,4) + e^{-2\pi i\alpha' k_2 \cdot (k_1+k_3)}\mathcal{A}_4(1,3,2,4) = 0 \quad (3.2.8)$$

Combining (3.2.8) with its complex conjugate relation along with the mass-shell condition, $k^2 = 1/\alpha'$, yields

$$\frac{\mathcal{A}_4(1,2,3,4)}{\sin(2\pi\alpha' k_2 \cdot k_4)} = \frac{\mathcal{A}_4(2,1,3,4)}{\sin(2\pi\alpha' k_2 \cdot k_3)} = \frac{\mathcal{A}_4(1,3,2,4)}{\sin(2\pi\alpha' k_1 \cdot k_2)}. \quad (3.2.9)$$

The above relation can be easily pictured as the triangle in fig 3.2 (left) whose sides refer to the open string amplitudes and angles determined by the product of corresponding momenta. The sum of the external angles is equal to 2π is guaranteed by the conservation of momentum and the kinematic relation for tachyons $k^2 = 1/\alpha'$. Note that the Plahte diagrams are constructed in the kinematic region where all partial amplitudes are positive. The area of this triangle Δ is quadratic in open string amplitudes and in [59] the KLT relations were used to show that this is proportional to the four-point tachyonic closed string amplitude $\mathscr{A}_4$ i.e.

$$\mathscr{A}_4 = -8i\frac{\kappa^2}{\pi\alpha'}\sin(2\pi\alpha' k_1 \cdot k_2)\mathcal{A}_4(1,2,3,4)\widetilde{\mathcal{A}}_4(2,1,3,4) = -16i\frac{\kappa^2}{\pi\alpha'}\Delta \qquad (3.2.10)$$

where the un-tilded and tilded expressions represent the left-moving and right-moving modes of open string amplitudes respectively. For tachyonic scattering, there is no difference between these modes as there is no involvement of polarization vectors. See appendix A.1 for a description of the KLT relations.

Unlike the tachyonic case, the kinematics for gauge particles, i.e. $k^2 = 0$, makes the external angle sum zero. To deal with this problem, the factor 2π is added to one of the angles to assure that all angles sum up into full circle. Note that this 2π phase shift is allowed as it does not change the form of the Plahte identities. The Plahte diagram for four gauge particle scattering is that of figure 3.2 (right). The connection between the area of the Plahte diagram and the gauge closed string amplitude is slightly trickier as the area of a diagram for particles with polarisations only refers to the closed string amplitude with polarisations corresponding to those of the open string amplitudes.

According to the figure 3.2 (below), the area of the diagram is

$$\frac{1}{2}\sin(2\pi\alpha' k_1 \cdot k_2)\xi_{\mu_1}\dots\xi_{\mu_4}\cdot\xi_{\nu_1}\dots\xi_{\nu_4}\mathcal{A}_4^{\mu_1\dots\mu_4}(1,2,3,4)\mathcal{A}_4^{\nu_1\dots\nu_4}(2,1,3,4), \qquad (3.2.11)$$

where $\mathcal{A}_4(\sigma) = \xi_{\mu_1}\dots\xi_{\mu_4}\mathcal{A}_4^{\mu_1\dots\mu_4}(\sigma)$ is the open string amplitude containing the polarization vectors ξ_i. One can see that this area corresponds to the gauge closed string amplitude $\mathscr{A}_4 = \xi_{\mu_1\nu_1}\dots\xi_{\mu_4\nu_4}\mathscr{A}_4^{\mu_1\nu_1\dots\mu_4\nu_4}$ only when the corresponding polarization vector is $\xi_{\mu\nu} = \xi_\mu\xi_\nu$. However, in general we can regain the KLT relation for four-point gauge amplitudes by considering the tensor component of the equation as

$$\mathscr{A}_4^{\mu_1\nu_1\dots\mu_4\nu_4} = -8i\frac{\kappa^2}{\pi\alpha'}\sin(2\pi\alpha' k_1\cdot k_2)\mathcal{A}_4^{\mu_1\dots\mu_4}(1,2,3,4)\mathcal{A}_4^{\nu_1\dots\nu_4}(2,1,3,4). \qquad (3.2.12)$$

The above relation is independent of any polarization vectors which means we can always contract the relation with any polarization vectors we want to consider.

In general, the Plahte identities (3.1.3) for n particle scattering can be depicted by n-sided polygons in the complex plane whose sides are given by colour-ordered

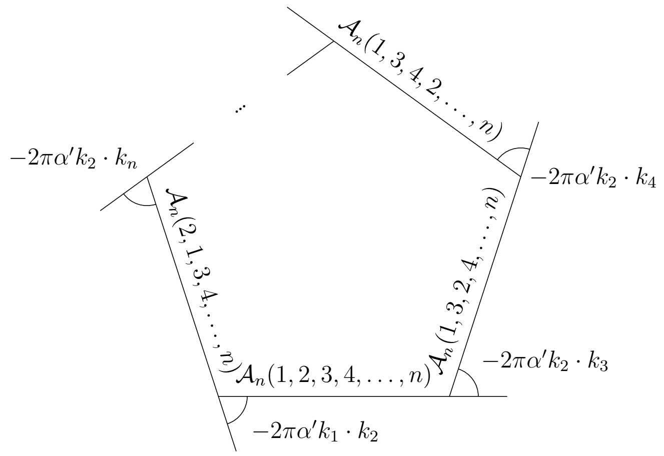


Figure 3.3: Plahte diagram for N-point open tachyon string amplitudes

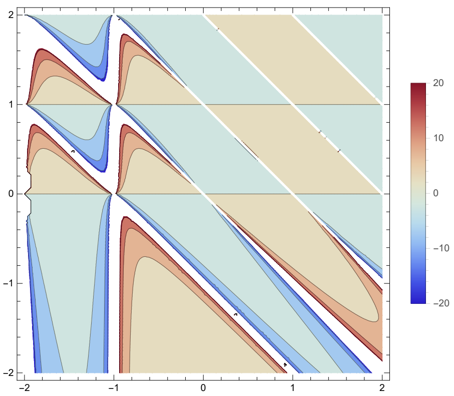
open string amplitudes and its angles correspond to products of two momenta. For tachyon scattering, the Plahte diagram is shown in figure 3.3. To obtain the diagram for gauge particles, one of the external angles need to be added by 2π as discussed.

However, we need to emphasise that in all the Plahte diagrams we constructed in this section we took the partial amplitudes to be real, positive and finite. This is not true for general kinematics as the amplitudes have to be defined by analytic continuation and then it is possible for open string amplitudes to be negative or even divergent. Therefore, we need to take an extra care to construct Plahte diagrams with those features. We will discuss more of these aspects in the next section.

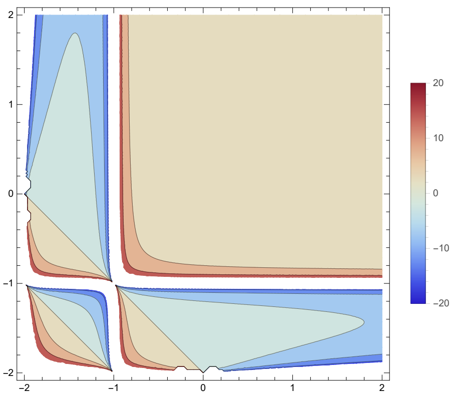
3.3 Plahte Diagrams with Negative Amplitudes and Their Dynamics

So far, in drawing the diagrams in the complex plane, we have taken all amplitudes involved in the diagram to be positive, (as given by the integral expression), but in general this is not the case. The most basic example is the four-point tachyon open string amplitudes. Although the integral expression for the amplitude obtained by Koba and Nielsen [62], i.e.

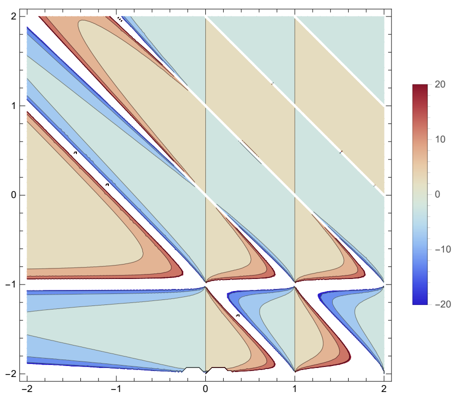
$$\mathcal{A}_4(1,2,3,4) = \int_0^1 dx\ x^{2\alpha' k_1 \cdot k_2}(1-x)^{2\alpha' k_2 \cdot k_3}, \tag{3.3.13}$$



$\mathcal{A}_4$(2,1,3,4)



$\mathcal{A}_4$(1,2,3,4)



$\mathcal{A}_4$(1,3,2,4)

Figure 3.4: Contour plots for three partial open tachyon amplitudes with $2\alpha' k_1 \cdot k_2$ and $2\alpha' k_2 \cdot k_3$ being X-axis and Y-axis

seems to be positive, it is ill-defined outside the regime where $2\alpha' k_1 \cdot k_2 > -1$ and $2\alpha' k_2 \cdot k_3 > -1$. To obtain the amplitude outside this region, the integral need to be defined by analytic continuation. In this example, it is not hard to see that the expression for this integral is

$$\mathcal{A}_4(1,2,3,4) = \frac{\Gamma(1+s)\Gamma(1+t)}{\Gamma(2+s+t)} \tag{3.3.14}$$

which provides the analytic continuation to the entire complex plane. We use the notation that $s \equiv 2\alpha' k_1 \cdot k_2$ and $t \equiv 2\alpha' k_2 \cdot k_3$.

The figure (3.4) shows value of all three partial amplitudes for four-point tachyon scattering based on the expression (3.3.14). The graph was plotted in the kinematic space where $2\alpha' k_1 \cdot k_2$ and $2\alpha' k_2 \cdot k_3$ are X-axis and Y-axis respectively. It is obvious from the figure that some amplitudes become negative. This occurs in regions depicted in blue while the reddish regions show the amplitude being positive. The white areas indicate lines of divergences where the amplitudes are infinitely large.

Alternatively, one can use the Plahte relation in (3.2.9) to provide an analytic continuation of the amplitude beyond the region where the integrals converge. For example, $\mathcal{A}_4(1,2,3,4)$ can be defined with in $t > -1$ and $s+t < -1$ via $\mathcal{A}_4(1,3,2,4)$ using

$$\mathcal{A}_4(1,2,3,4) = -\frac{\sin \pi(s+t)}{\sin \pi s}\mathcal{A}_4(1,3,2,4). \tag{3.3.15}$$

It is clear from the relation that the amplitude blows up and vanishes when s and $s+t$ is a negative integer respectively. Obviously, it is the same behaviour as would be obtained from the equation (3.3.14). Similarly, we can evaluate $\mathcal{A}_4(1,2,3,4)$ using $\mathcal{A}_4(1,2,3,4) = -(\sin \pi(s+t)/\sin \pi t)\mathcal{A}_4(2,1,3,4)$ within the region $s > -1$ and $s+t < -1$ in which $\mathcal{A}_4(2,1,3,4)$ is well-defined.

The remaining region in kinematic space can be determined by considering a map between $\mathcal{A}_4(s-1, t-1)$ and $\mathcal{A}_4(-s,-t)$ where $\mathcal{A}_4(a,b)$ is defined as the integral $\int_0^1 dx x^a(1-x)^b$. Consider a product $\mathcal{A}_4(s-1,t-1)\mathcal{A}_4(-s,-t)$

$$\begin{aligned} &= \int_0^1 dx \int_0^1 dy\ x^{s-1}(1-x)^{t-1}y^{-s}(1-y)^{-t} \\ &= \int_0^\infty dx \int_0^\infty dy\ (x+1)^{-s-t}x^{t-1}(y+1)^{s+t+2}y^{-t}. \end{aligned} \tag{3.3.16}$$

A change of variables for x and y as $(x,y) \to (\frac{1}{x}-1, \frac{1}{y}-1)$ is applied to obtain the last line. The integral can then be performed in plane polar coordinates (r,θ) along with the change of variable, $R = (r\cos\theta + 1)/(r\sin\theta + 1)$, which yields

$$\begin{aligned} &\int_0^{\pi/2} d\theta \int_1^{\cot\theta} dR\ R^{-s-t}\frac{\cot\theta^t \sec\theta}{(\cos\theta - \sin\theta)} \\ &= \int_0^{\pi/2} d\theta\ \frac{\tan\theta^{s-1} - \tan\theta^{-t}}{(1-s-t)(\cos\theta - \sin\theta)}\sec\theta \\ &= \int_0^\infty dp \frac{p^{s-1} - p^{-t}}{(1-s-t)(1-p)} \end{aligned} \tag{3.3.17}$$

where we have substituted $p = \tan\theta$ in the last line. By splitting the integral into two pieces which are the integral from 0 to 1 and that from 1 to ∞, along with a change of variable $p \to 1/p$ applied to the latter part, the integral takes the form

$$\frac{1}{1-s-t}\int_0^1 dp \frac{1}{(1-p)}\left[(p^{s-1} - p^{-s}) + (p^{t-1} - p^{-t})\right] \tag{3.3.18}$$

The term $(1-p)^{-1}$ can be Taylor expanded as $\sum_{n=0}^{\infty} p^n$ allowing the integration to be done to yield the result

$$\frac{1}{1-s-t}\sum_{n=0}^{\infty}\left[\left(\frac{1}{s+n}-\frac{1}{n-s+1}\right)+\left(\frac{1}{t+n}-\frac{1}{n-t+1}\right)\right]. \tag{3.3.19}$$

It can be shown numerically that the above expression is equal to $\pi(\cot(\pi s)+\cot(\pi t))$. Consequently, the relation between $\mathcal{A}_4(s-1,t-1)$ and $\mathcal{A}_4(-s,-t)$ is

$$\mathcal{A}_4(s-1,t-1)\mathcal{A}_4(-s,-t)=\frac{\pi(\cot(\pi s)+\cot(\pi t))}{1-s-t}. \tag{3.3.20}$$

Note that this is evaluated within $0<s<1$ and $0<t<1$. However, we can still use equation (3.3.20) as the analytic continuation to determine a value of the amplitude of undetermined points in kinematic space. It is not hard to see that the equation (3.3.20) can be directly derived from the expression (3.3.14) as well.

Due to the analytic continuation, it is clear that the four-point tachyon amplitudes become negative in some regions. The question is how do negative amplitudes affect the Plahte diagram. Since we can always write any amplitude as $\mathcal{A}=(-)\mathcal{A}e^{\pm i\pi}$, the sign of the amplitude can be absorbed by shifting the phase angles next to the amplitude by $+\pi$ or $-\pi$. Each adjacent angle needs to be shifted either by $+\pi$ or $-\pi$ differently in order to keep the sum of the external angles unchanged at 2π.

To put it into a clearer perspective, let us give the example of a four-point tachyonic Plahte diagram. We will investigate how the Plahte diagram behaves as the kinematic variables flow from point A to E along the blue line in the figure (3.5)(top left). Θ_{ij} is a shorthand for $2\pi\alpha' k_i\cdot k_j$. We start our examination at the point A in which all three color-ordered amplitudes are positive. When it approaches the point B, the amplitudes $\mathcal{A}_4(1,3,2,4)$ and $\mathcal{A}_4(2,1,3,4)$ diverge, thus, close to the left of the point B, the diagram becomes a pair of infinite parallel lines with $\mathcal{A}_4(1,2,3,4)$ as a finite bridge between those lines shown in figure 3.5(B^-)

As the kinematic variables flow past the point B, the amplitude $\mathcal{A}_4(1,3,2,4)$ and $\mathcal{A}_4(2,1,3,4)$ become negative. This leads to a shift for $-\Theta_{12}$ and $-\Theta_{23}$ by $-\pi$ and π respectively. This is a clear example of how the changing sign of an amplitude ends up shifting the phase angles by $\pm\pi$. Moving to the point C, the Plahte diagram is

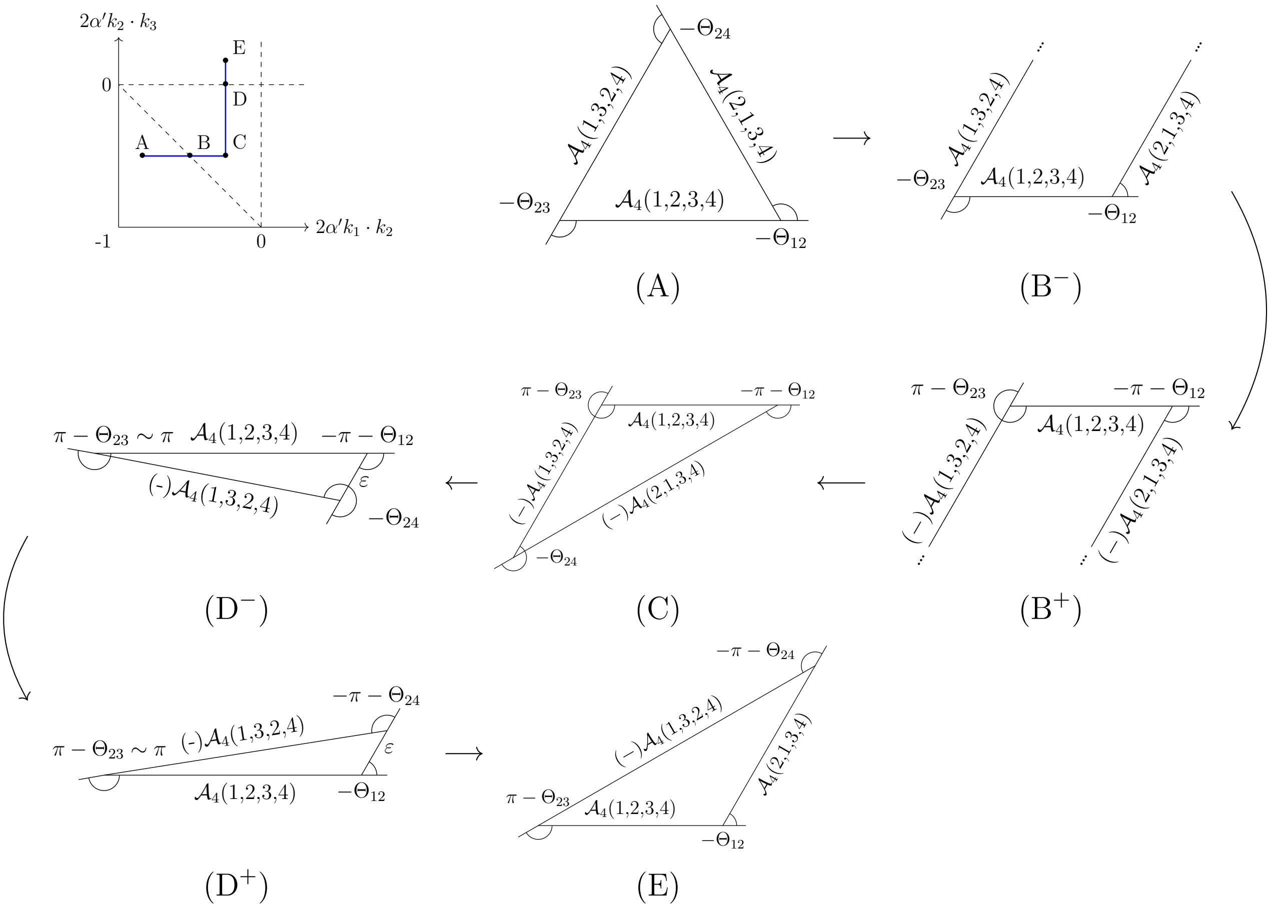


Figure 3.5: Dynamics of Plahte diagram for four tachyon scattering with the kinematic variables flowing from (A) to (E)

now a triangle illustrated in figure 3.5(C). Remember that the shifts we made in the angles do not alter the sum of the external angles as can be easily checked.

Moving towards the point D, $\mathcal{A}_4(2,1,3,4)$ becomes smaller and vanishes at the point D. The diagram is now a line with finite length at this point. When it crosses the point D, the amplitudes $\mathcal{A}_4(1,2,3,4)$ and $(-)\mathcal{A}_4(1,3,2,4)$ are flipped with each other shifting the angles $-\pi-\Theta_{12}$ and $-\Theta_{24}$ to $-\Theta_{12}$ and $-\pi-\Theta_{24}$ respectively. This shift reflects the fact that in this region $\mathcal{A}_4(2,1,3,4)$ is negative and to compensate this the adjacent angles need to be shifted by π and $-\pi$.

According to this example, the amplitudes change their signs when their values pass through zero or infinity which is similar to what happens at the points D and B respectively in the figure 3.5. This corresponds to the Plahte diagram becoming a line with finite length or a pair of infinitely long parallel lines.

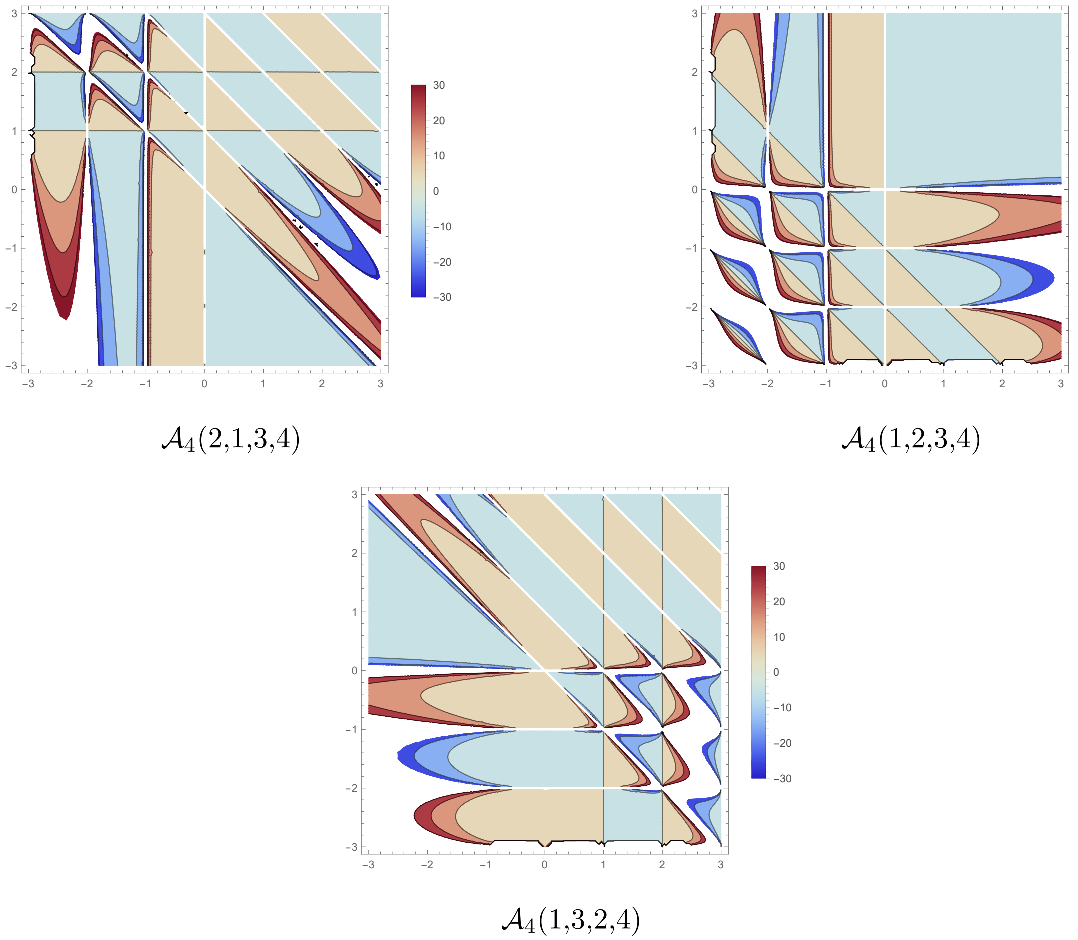


Figure 3.6: Contour plots for three partial gluon amplitudes with $2\alpha' k_1 \cdot k_2$ and $2\alpha' k_2 \cdot k_3$ being X-axis and Y-axis

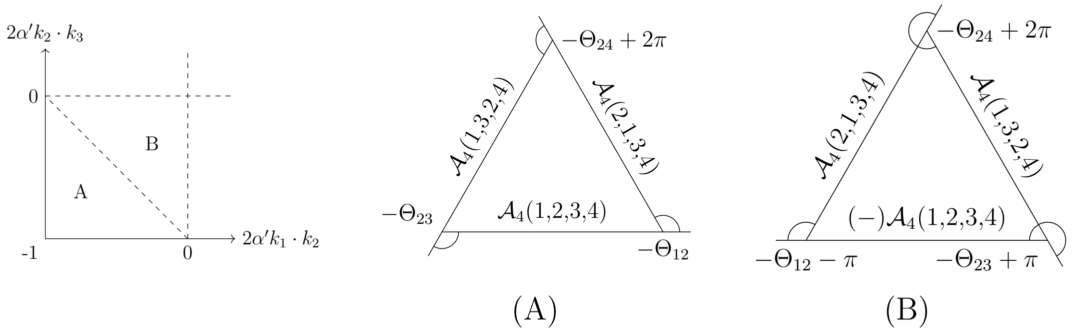


Figure 3.7: Plahte diagrams for four gluon scattering in the kinematic regions (A) and (B).

Lets move to another case of interest, Plahte diagrams for gauge bosons. Like the tachyonic case, an open gauge string amplitude can be negative in certain kinematic regions, for instance the region around the origin. This can be seen easily using the

BCJ relation,

$$\frac{A_4(1,2,3,4)}{s_{24}} = \frac{A_4(2,1,3,4)}{s_{23}} = \frac{A_4(1,3,2,4)}{s_{12}}, \tag{3.3.21}$$

which is basically the field theory version of (3.2.9). It is unavoidable that at least one partial amplitude must gain a different sign from others as $s_{12} + s_{23} + s_{24} = 0$.

The amplitude for multi-gluon scattering can be obtained from superstring theory which directly relates to the Yang-Mills amplitude in the infinite tension limit. The amplitude for four-point gluon scattering is well-known [61],

$$\mathcal{A}_4^{\mathrm{SUSY}}(1,2,3,4) = \frac{\Gamma(1+\alpha' s_{12})\Gamma(1+\alpha' s_{23})}{\Gamma(1+\alpha' s_{12}+\alpha' s_{23})} A_4^{\mathrm{YM}}. \tag{3.3.22}$$

Let us consider the Plahte diagram for four gluon scattering assuming that the particle 1 and 2 have negative helicity while the two remaining particles have positive helicity. In this scenario, the 4-point Yang-Mills amplitude A_4^{YM} can be obtained from the Parke-Taylor formula [63],

$$A_4^{\mathrm{YM}}(1^-,2^-,3^+,4^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.3.23}$$

The above equation is expressed in the spinor-helicity formalism. See [64] for more details. We can retrieve the expression for the amplitudes in terms of kinematic momenta by considering the absolute square of the amplitude as

$$\left|A_4^{\mathrm{YM}}(1^-,2^-,3^+,4^+)\right|^2 = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[12]^4}{[12][23][34][41]} = \left(\frac{s_{12}}{s_{23}}\right)^2. \tag{3.3.24}$$

Therefore, $A_4^{\mathrm{YM}}(1^-,2^-,3^+,4^+)$ is basically a square root of (3.3.24) up to a certain phase factor. With this calculation, it is not hard to see that all three partial Yang-Mills amplitudes are

$$\begin{aligned} A_4^{\mathrm{YM}}(1^-,2^-,3^+,4^+) &= \frac{s_{12}}{s_{23}}e^{i\phi_1}, \qquad A_4^{\mathrm{YM}}(2^-,1^-,3^+,4^+) = \frac{s_{12}}{s_{13}}e^{i\phi_2}, \\ \text{and} \quad A_4^{\mathrm{YM}}(1^-,3^+,2^-,4^+) &= \frac{(s_{12})^2}{s_{13}s_{23}}e^{i\phi_3} \end{aligned} \tag{3.3.25}$$

where ϕ_i is a phase argument corresponding to each amplitude. These phase terms can be determined by BCJ and Kleiss-Kujif relations in equation (3.1.6) and (3.1.7). This allow us to constraint $\phi_1 = \phi_2 = \phi_3 = \phi$.

Figure 3.6 shows the contour plots of all three partial amplitudes for four gluon scattering according to the equation (3.3.22) with $\phi = \pi$. The horizontal and vertical axes of this plot are $2\alpha' k_1 \cdot k_2$ and $2\alpha' k_2 \cdot k_3$ respectively. It is clear from the figure that at one partial amplitude must have a different sign to the others around the origin. Similar to the tachyonic Plahte diagram, the negative amplitudes can be compensated by shifting their adjacent angles by π and $-\pi$ differently which can be directly seen in figure 3.7.

In figure 3.7, the gluonic Plahte diagrams are constructed for the two different regions depicted in the leftmost figure. All partial amplitudes are positive in region A while the amplitude $\mathcal{A}_4(1,2,3,4)$ is negative in region B. Consequently, the angles next to $\mathcal{A}_4(1,2,3,4)$ are shifted from $-\Theta_{12}$ and $-\Theta_{23}$ to $-\Theta_{12}+\pi$ and $-\Theta_{23}-\pi$ respectively as claimed earlier.

To sum up, in order to draw a Plahte diagram with negative amplitudes, the external angles next to those amplitudes need to be shifted by π and $-\pi$ to absorb their negative sign. In general, when a momentum product $2\pi\alpha' k_i \cdot k_j$ is equal to $n\pi$ where n is an integer, it implies a condition where at least one amplitude is about to change its sign. This can be seen by the Plahte identities in (3.1.4).

For four-point scattering, when $2\pi\alpha' k_i \cdot k_j$ is equal to $n\pi$ with an integer n, Plahte diagram becomes either a line with finite length or an infinite parallel lines as discussed earlier. The former occurs when the integer $n > l$ while the latter occurs for otherwise. We use the letter l as an identifying integer whose value refers to different types of particles. The values of $l = -1$ and 0 correspond to tachyons and gauge bosons respectively.

3.4 Plahte Diagrams for 5-point Amplitudes

In this section, we will explore the Plahte diagrams for 5-point scattering amplitudes. For simplicity, we will first focus on the scattering of tachyons. The Plahte diagrams for five tachyons is illustrated by quadrilaterals which are directly derived from Plahte identities. Let consider the integral

$$\int_{-\infty}^{\infty} dx_2 \int_0^1 dx_3\; x_2^{2\alpha' k_1\cdot k_2} x_3^{2\alpha' k_1\cdot k_3} (1-x_2)^{2\alpha' k_2\cdot k_4} (1-x_3)^{2\alpha' k_3\cdot k_4} (x_3-x_2)^{2\alpha' k_2\cdot k_3} \tag{3.4.26}$$

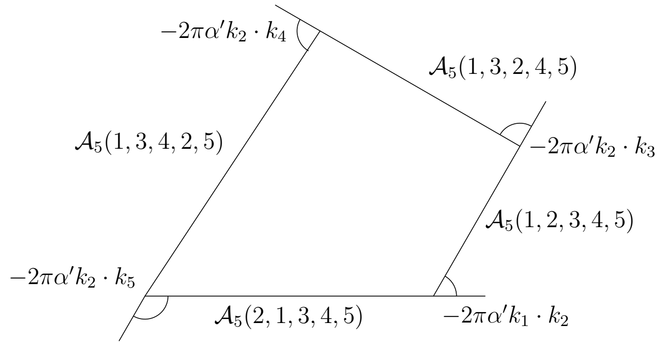


Figure 3.8: An example of Plahte diagrams for 5-point tachyonic scattering.

where x_1, x_4 and x_5 are fixed at $x_1 = 0, x_4 = 1$ and $x_5 = \infty$. This integral corresponds to the Plahte diagram illustrated in figure (3.8). The integration needs to be implemented with careful examination of branch cuts and the use of equation (1.2.115) to provide the correct definition of the partial open string amplitudes.

There are thirty Plahte diagrams in total for five-particle scattering. These can be obtained from similar integrals but with different choices of gauge-fixed and integration variables. The number of diagrams can be halved with the help of reflection symmetry.

Although the Plahte diagrams in figure 3.8 are deduced for tachyons, they generalise to other particle states with a suitable phase shift as discussed in the previous section. Polarization vectors are also included in the amplitudes for excited particles. Again, if an amplitude is negative, the diagram needs to be adjusted as discussed previously.

It is unavoidable for Plahte diagrams to share some common sides as there are twelve possible orderings for partial amplitudes and only fifteen diagrams in total [1]. Therefore we are able to build a bigger picture by combining diagrams together.

We start by putting the quadrilateral □ABCD in the middle and then attach other quadrilaterals around it. The combined Plahte diagram is shown in figure (3.9). Obviously, this combined diagram comprises of five different quadrilaterals. Different combined diagrams can be obtained in a similar manner but starting with

[1] Actually they consist of half of all possible ordering as the other amplitudes are related by reflection symmetry.

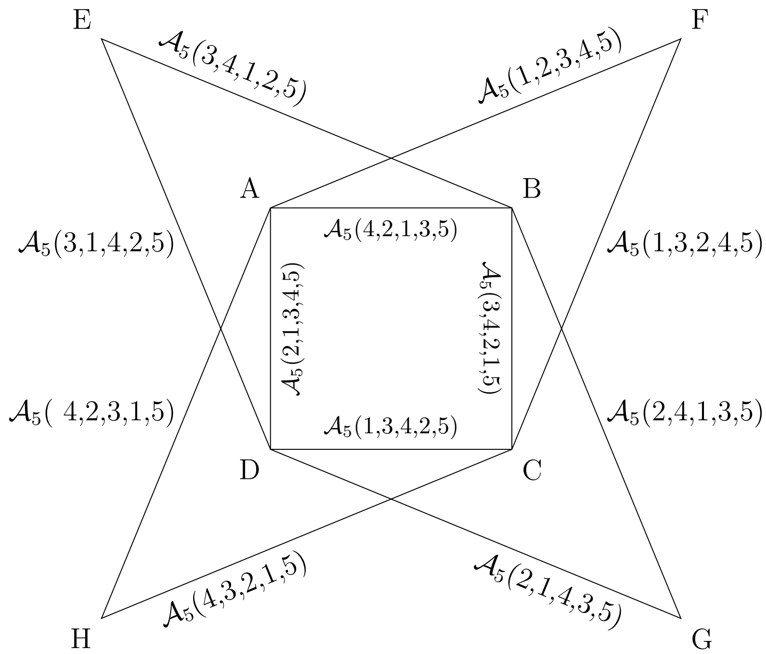


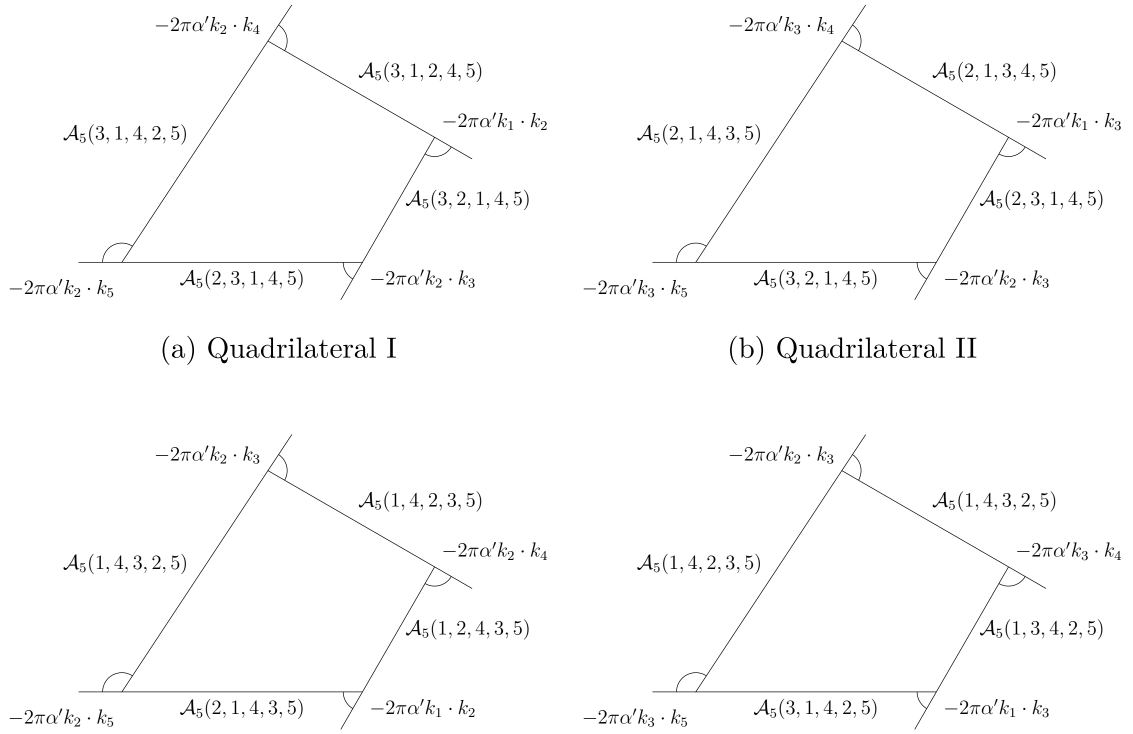
Figure 3.9: Combined Plahte diagram of 5-point tachyon scattering

a different quadrilateral at the centre. We could extend our combined diagrams still further by considering the peripheral quadrilaterals as new centres. Then we attach another four diagrams surrounding each, however, this would make the resulting diagram quite complicated and hard to analyse. For that reason, we will content ourselves with the diagram as it is presented in figure (3.9).

Unlike the 4-particle cases, for 5-particles the area of each Plahte diagram is no longer simply proportional to a closed string amplitude. However, this does not mean there is no geometric relation between the closed string amplitudes and the Plahte diagrams. According to the KLT relations for 5-point tachyon string amplitudes,

$$\begin{aligned}\mathscr{A}_5 =& -8i\frac{\kappa^3}{(\pi\alpha)^2}\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(2,1,4,3,5)\\ &+\text{exchange of }(2\leftrightarrow 3).\end{aligned} \tag{3.4.27}$$

The terms on the right-hand-side of this correspond to areas of parts of the combined



(a) Quadrilateral I

(b) Quadrilateral II

(c) Quadrilateral III

(d) Quadrilateral IV

Figure 3.10: Plahte diagrams for 5-point tachyonic scattering

Plahte diagram of figure 3.9:

$$\mathscr{A}_5 = -16i\frac{\kappa^3}{(\pi\alpha)^2}\left(\frac{\langle\triangle\mathrm{EBC}\rangle\langle\triangle\mathrm{BCH}\rangle}{(BC)^2} + \frac{\langle\triangle\mathrm{FCD}\rangle\langle\triangle\mathrm{CDE}\rangle}{(CD)^2}\right.$$
$$\left. + \frac{\langle\triangle\mathrm{GDA}\rangle\langle\triangle\mathrm{DAF}\rangle}{(AD)^2} + \frac{\langle\triangle\mathrm{HAB}\rangle\langle\triangle\mathrm{ABG}\rangle}{(AB)^2}\right) \tag{3.4.28}$$

where the angle bracket $\langle \quad \rangle$ denotes the area of the object inside. The elements entering this relation are triangles and squares built on the sides of the central quadrilateral. Note that the above argument is generally correct for all types of particles not only for tachyons. Moreover, we can further reshape the equation (3.4.27) into a new form using elementary Euclidean geometry.

Let us consider the quadrilaterals I and II in figure 3.10. We can extract the Plahte identities for each diagram as

$$\mathcal{S}_{k_2,k_4}\mathcal{A}_5(3,1,4,2,5) = \mathcal{S}_{k_1,k_2}\mathcal{A}_5(3,2,1,4,5) + \mathcal{S}_{k_2,k_1+k_3}\mathcal{A}_5(2,3,1,4,5) \tag{3.4.29a}$$

$$\mathcal{S}_{k_3,k_4}\mathcal{A}_5(2,1,4,3,5) = \mathcal{S}_{k_1,k_3}\mathcal{A}_5(2,3,1,4,5) + \mathcal{S}_{k_3,k_1+k_2}\mathcal{A}_5(3,2,1,4,5). \tag{3.4.29b}$$

From the above relations we can rewrite (3.4.27) to obtain the KLT relation in the form

$$\begin{aligned}\mathscr{A}_5 =& -8i\frac{\kappa^3}{(\pi\alpha)^2}\Big[\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_1,k_3}\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(2,3,1,4,5)\\ &+\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_1+k_2}\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(3,2,1,4,5)\Big]\\ &+\text{exchange of }(2\leftrightarrow 3).\end{aligned} \tag{3.4.30}$$

Similarly, we can perform the same trick but this time we consider the quadrilaterals III and IV instead. This yields the KLT relation in the form

$$\begin{aligned}\mathscr{A}_5 =& -8i\frac{\kappa^3}{(\pi\alpha)^2}\Big[\mathcal{S}_{k_2,k_4}\mathcal{S}_{k_3,k_4}\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(1,4,2,3,5)\\ &+\mathcal{S}_{k_3,k_4}\mathcal{S}_{k_2,k_3+k_4}\mathcal{A}_5(1,2,3,4,5)\mathcal{A}_5(1,4,3,2,5)\Big]\\ &+\text{exchange of }(2\leftrightarrow 3).\end{aligned} \tag{3.4.31}$$

This is not a surprising result since these equivalent forms of the KLT relations were presented in the original paper [53]. However, Plahte diagrams give us a simple geometrical way of obtaining them.

Although we deduced the geometrical expression for the KLT relations from the specific combined Plahte diagram for figure 3.9, the relation (3.4.28) is generally valid for any of the combined pictures. In other words, we can find the KLT relations from any combined diagram not just the one presented in figure 3.9, using the relation (3.4.28). The newly obtained KLT relations will be of the same form as equation (3.4.27) but with particles' labels interchanged.

In deriving the KLT relations, we are free to fix the positions of any three vertices positions in the integral representation of the closed string amplitude. Then, the integral is factorised into products of open string amplitudes. In the original paper [53], the fixing choice, $z_1 = 0$, $z_4 = 1$ and $z_5 = \infty$ was utilised for five particle scattering. Choosing different vertex positions to be fixed would result in exactly the KLT relations in equation (3.4.27) but with different particles' labels interchanged.

As there are fifteen different Plahte diagrams for 5-point amplitudes, it suggests that fifteen different KLT relations can be obtained from reordering the particles 1 to 5 as well. Obviously, there are 5! ways to rearrange five objects. However, not all of them corresponds to distinct KLT relations as some permutations do not change the form of relations. It is not hard to see that these following permutations keep the KLT relation in (3.4.27) invariant: First, swapping particle 1 and 4, second, swapping particle 2 and 3, and third, swapping particle 1 and 2 together with particle 3 and 4 simultaneously. As a result, the total number of different KLT relations are $5!/(2^3) = 15$ as claimed.

Let us explore some more aspects of the combined diagrams in figure (3.9). One may notice that there are only eight partial amplitudes (excluding their reflections) taking part in the diagram. In order to include all twelve color-ordered amplitudes we need to enlarge the diagram. The extended version of the combined diagram in figure (3.9) is shown below in figure (3.11). The central quadrilaterals in figure (3.9) are disassembled into a cross-like structure in our new picture. Notice that quadrilaterals □BCDE, □AFCD, □ABGD, and □ABCH in figure (3.9) are the same quadrilaterals □OBGC, □OAFB, □OAED, and □OCHD in figure (3.11) respectively.

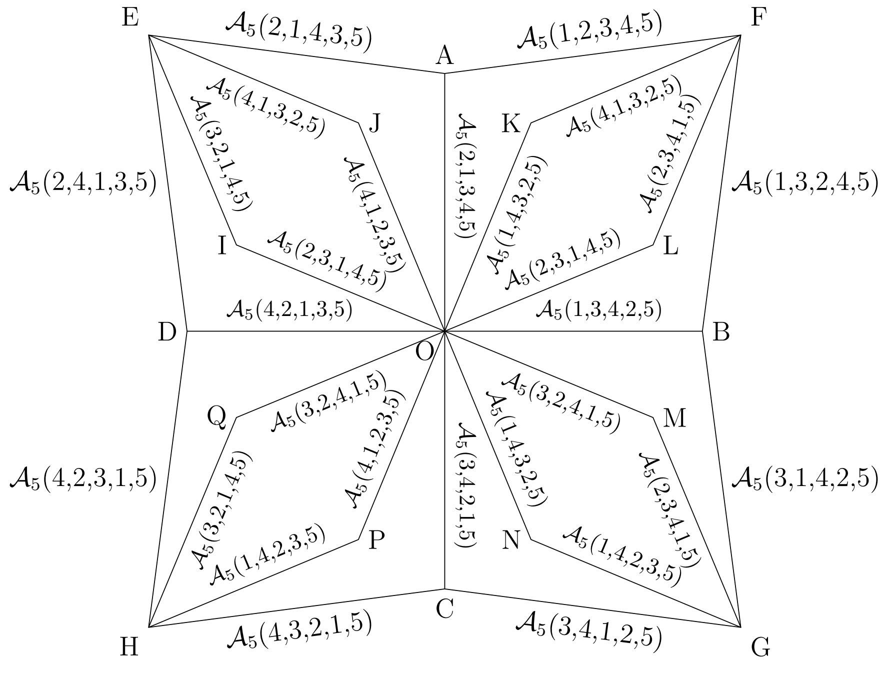


Figure 3.11: An extended version of combined Plahte diagram containing all possible color-ordered scattering amplitudes

As we are able to combine all diagrams into one figure in the complex plane, it is natural to say that we can express all 5-point amplitudes in terms of any two amplitudes by implementing simple Euclidean geometry. Our choice is $\mathcal{A}_5(1,2,3,4,5)$ and $\mathcal{A}_5(1,3,2,4,5)$ and for convenience we call them $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let start by considering the quadrilateral □OAFB. The vector sum of the displacements along the sides vanishes, $\mathbf{OA}+\mathbf{AF}+\mathbf{FB}+\mathbf{BO}=\mathbf{0}$. Taking the vector product of this relation with **OA** and then with **BO** leads to

$$\begin{aligned}
\mathcal{S}_{k_2,k_5}\mathcal{A}_5(1,3,4,2,5) &= \mathcal{S}_{k_1,k_2}\mathcal{A}_1 + \mathcal{S}_{k_2,k_1+k_3}\mathcal{A}_2 \\
\mathcal{S}_{k_2,k_5}\mathcal{A}_5(2,1,3,4,5) &= \mathcal{S}_{k_2,k_3+k_4}\mathcal{A}_1 + \mathcal{S}_{k_2,k_4}\mathcal{A}_2.
\end{aligned}$$

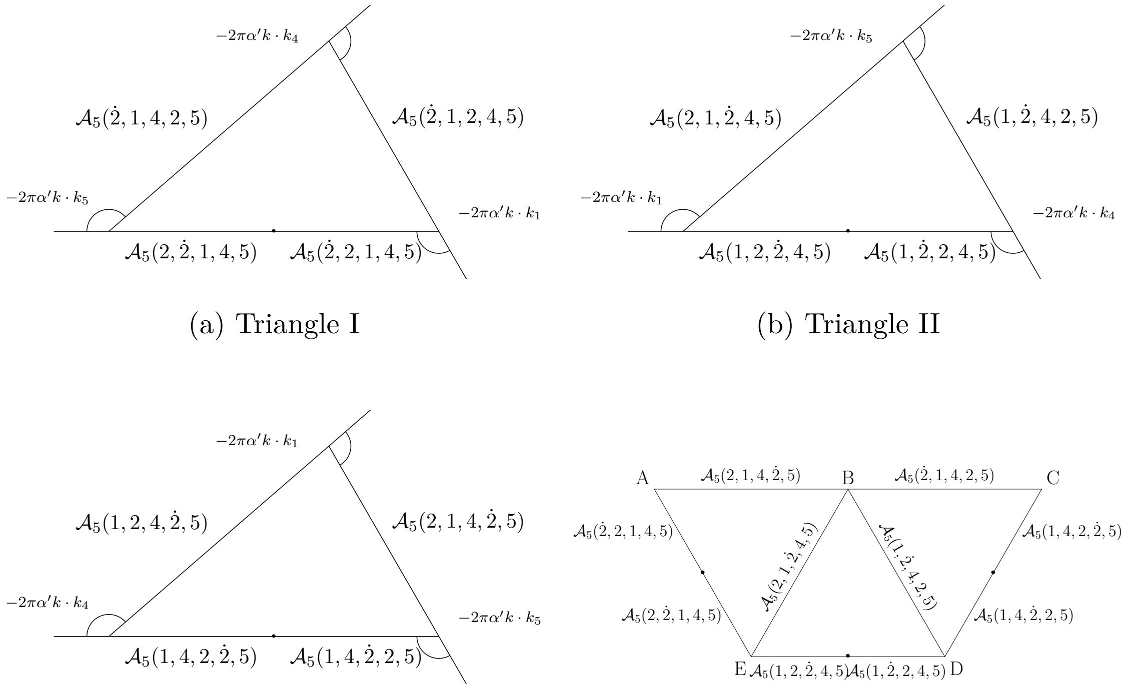
The remaining relations can be obtained by implementing a similar approach to the different quadrilaterals. This yields

$$\begin{aligned}
\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_1,k_4}\mathcal{A}(2,3,1,4,5) &= (-1)^{l+1}\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\mathcal{A}_1 - \mathcal{S}_{k_2,k_4}\mathcal{S}_{k_1,k_3+k_4}\mathcal{A}_2 \\
\mathcal{S}_{k_3,k_5}\mathcal{S}_{k_1,k_4}\mathcal{A}(1,4,2,3,5) &= (-1)^{l+1}\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4}\mathcal{A}_1 - \mathcal{S}_{k_1,k_3}\mathcal{S}_{k_4,k_1+k_2}\mathcal{A}_2 \\
\mathcal{S}_{k_1,k_4}\mathcal{S}_{k_2,k_5}\mathcal{S}_{k_3,k_5}\mathcal{A}(2,1,4,3,5) &= (-1)^{l}\mathcal{S}_{k_1,k_3}\mathcal{S}_{k_2,k_4}\mathcal{S}_{k_5,k_2+k_3}\mathcal{A}_2 \\
&\quad + (\mathcal{S}_{k_2,k_3+k_4}\mathcal{S}_{k_3,k_1+k_2}\mathcal{S}_{k_1,k_4} + (-1)^{l+1}\mathcal{S}_{k_2,k_3}\mathcal{S}_{k_1,k_2}\mathcal{S}_{k_3,k_4})\mathcal{A}_1 \qquad (3.4.32)
\end{aligned}$$

where the remaining five amplitudes are obtained by exchanging labels $2 \leftrightarrow 3$. The integer l is the identifying number defined previously. This result agrees with [37,58] which explicitly computes the expansion of color-ordered 5-point gauge amplitudes in terms of a minimal choice of two amplitudes, i.e. $\mathcal{A}_1$ and $\mathcal{A}_2$.

Before finishing this section, let us explore another case of interest. We will now consider a special case of 5-point scattering amplitudes where the momenta of two particles are equal, says $k_2 = k_3 = k$. For tachyon scattering, it causes some Plahte diagrams to become degenerate as the particles 2 and 3 are now indistinguishable. In this scenario, all component quadrilaterals in the combined diagram (3.9) can be decomposed into tree triangles which are illustrated in the figure (3.12). Note that we use $\dot{2}$ instead of the number 3 in the amplitudes to signify this indistinguishability.

Again, combining all three triangles gives us a combined diagram for this special case which is presented in the figure (3.12). It is not hard to find that the connection



(a) Triangle I (b) Triangle II

(c) Triangle III (d) Combined diagram

Figure 3.12: Plahte diagrams for 5-point tachyonic scattering with $k_2 = k_3 = k$.

between the KLT relations and the Plahte diagrams now takes form,

$$\mathcal{A}_5\Big|_{k_2=k_3} = -\,32i\frac{\kappa^3}{(\pi\alpha)^2}\left(\frac{\langle\triangle\text{ABE}\rangle\langle\triangle\text{BDE}\rangle}{(BE)^2}\right) = -32i\frac{\kappa^3}{(\pi\alpha)^2}\left(\frac{\langle\triangle\text{BDE}\rangle\langle\triangle\text{BCD}\rangle}{(BD)^2}\right). \tag{3.4.33}$$

One can notice that the partial amplitude in the denominator is actually a side shared between two triangles.

The equation (3.4.33) does not hold for any excited state particles in general as it involves polarization vectors. Therefore, switching the order of particles 2 and 3 no longer keeps the amplitude invariant. However, if we consider the special case where the polarization vectors of particle 2 and 3 form a rank-2 symmetric traceless tensor, $\xi_{\mu\nu}$, the KLT expression in the equation (3.4.33) holds. This special case of a 5-point amplitude will be useful when discussing the mixed open and closed string amplitudes in the next section.

3.5 Applications to Mixed Disk Amplitudes

Unlike pure closed string amplitudes in which there is no interaction between left- and right- moving world-sheet fields, in mixed scattering amplitudes the interaction between these modes prevents us from expanding the amplitude into a sum of products of color-ordered open string amplitudes. Instead, the amplitude involving N_0 open strings and N_C closed strings can be mapped into a sum of color-ordered $(N_0 + 2N_C)$-point open string amplitudes [37].

The simplest example of a mixed disk amplitude is $\mathcal{M}^{(3,1)}$ with three open strings and one closed string. In this scenario, we label the three open strings states by the number $1, 4$ and 5. Each carries the corresponding momentum k_1, k_4, and k_5 respectively. The closed string in the mixed disk amplitude will be replaced by a pair of collinear open strings which both carry half of the closed string momentum. If we assign both open string momenta by $k_2 = k_3 = k$ satisfying $k^2 = -l/\alpha'$ where l is the identifying number we defined earlier, the closed string momentum is now $2k$.. According to the formula (1.2.118), the mixed amplitude can be written as

$$\begin{aligned}\mathcal{M}^{(3,1)}(1,4,5;k,k) =& \frac{i}{2}\bigg(\mathcal{A}_5(2,\dot{2},1,4,5) + \mathcal{A}_5(1,2,\dot{2},4,5) + \mathcal{A}_5(1,4,2,\dot{2},5) \\ &+ e^{2\pi i\alpha' k\cdot k_1}\mathcal{A}_5(2,1,\dot{2},4,5) + e^{2\pi i\alpha' k\cdot(k_1+k_4)}\mathcal{A}_5(2,1,4,\dot{2},5) \\ &+ e^{2\pi i\alpha' k\cdot k_4}\mathcal{A}_5(1,2,4,\dot{2},5)\bigg). \end{aligned} \tag{3.5.34}$$

Due to interchangeability between particles 2 and 3, we rewrite $\dot{2}$ instead of 3 in this case. Taking the real part of the above equation into consideration, this yields

$$\begin{aligned}\mathcal{M}^{(3,1)}(1,4,5;k,k) =& -\frac{1}{2}\bigg(\mathcal{S}_{k,k_1}\mathcal{A}_5(2,1,\dot{2},4,5) - \mathcal{S}_{k,k_5}\mathcal{A}_5(2,1,4,\dot{2},5) \\ &+ \mathcal{S}_{k,k_4}\mathcal{A}_5(1,2,4,\dot{2},5)\bigg). \end{aligned} \tag{3.5.35}$$

The open string amplitudes that result from the mixed closed/open string amplitude are actually the open string amplitudes we previously considered in the special case where $k_2 = k_3$. Therefore, the equation (3.5.35) can be further simplified using the

Plahte diagrams in figure (3.12). These (from triangle I to III) directly give us:

$$\text{I)} \qquad \mathcal{S}_{k,k_5}\mathcal{A}_5(\dot{2},1,4,2,5) = \mathcal{S}_{k,k_1}\mathcal{A}_5(\dot{2},1,2,4,5) \tag{3.5.36a}$$

$$\text{II)} \qquad \mathcal{S}_{k,k_1}\mathcal{A}_5(2,1,\dot{2},4,5) = \mathcal{S}_{k,k_4}\mathcal{A}_5(1,\dot{2},4,2,5) \tag{3.5.36b}$$

$$\text{III)} \qquad \mathcal{S}_{k,k_4}\mathcal{A}_5(1,2,4,\dot{2},5) = \mathcal{S}_{k,k_5}\mathcal{A}_5(2,1,4,\dot{2},5). \tag{3.5.36c}$$

The above relations simplify the disk amplitude in (3.5.35) to

$$\begin{aligned}\mathcal{M}^{(3,1)}(1,4,5;k,k) &= -\frac{1}{2}\mathcal{S}_{k,k_1}\mathcal{A}_5(2,1,\dot{2},4,5) = -\frac{1}{2}\mathcal{S}_{k,k_5}\mathcal{A}_5(2,1,4,\dot{2},5)\\ &= -\frac{1}{2}\mathcal{S}_{k,k_4}\mathcal{A}_5(1,2,4,\dot{2},5).\end{aligned} \tag{3.5.37}$$

Geometrically, the right-hand sides of the above equations are nothing but the height of each Plahte diagram in figure 3.12.

The relations (3.5.37) also provide a description for mixed graviton and gauge boson amplitudes. The graviton in the mixed disk amplitude can be split into pairs of collinear gauge vectors. In the field theory limit, the left-hand side term in (3.5.37) is described by Einstein-Yang-Mills theory which express the decay of a graviton into three gauge bosons [65].

The collinear limit for Yang-Mills amplitudes may seem troublesome for our mixed disk amplitudes. It is known that the partial amplitudes with adjacent gauge bosons contain collinear divergence [66]. Fortunately, these singularities are absent from the partial amplitudes in the expression (3.5.37) as the collinear pair are not adjacent.

Furthermore, one can also make a connection between closed string amplitudes and mixed disk amplitudes. According to the equation (3.4.33) and (3.5.37), it is not hard to obtain

$$\mathscr{A}_5\Big|_{k_2=k_3=k} = -32i\frac{\kappa^3}{(\pi\alpha')^2}\big(\mathcal{M}^{(3,1)}(1,4,5;k,k)\big)^2. \tag{3.5.38}$$

This expresses the 5-point closed tachyon string amplitudes with any two momenta being equal as a quadratic in the disk amplitudes describing the scattering of three open string and one closed string tachyon.

More interestingly, this allows us to compute the specific case of the five-point graviton scattering amplitude as a product of the scattering amplitudes of three massless gauge bosons and a graviton. The relation is presented in tensor form as

$$\mathcal{A}_5{}^{\mu_1\nu_1\ldots\mu_5\nu_5}\Big|_{k_2=k_3=k} = \\ -32i\frac{\kappa^3}{(\pi\alpha')^2}\mathcal{M}^{(3,1)\mu_1\mu_2\ldots\mu_5}(1,4,5;k,k)\mathcal{M}^{(3,1)\nu_1\nu_2\ldots\nu_5}(1,4,5;k,k). \tag{3.5.39}$$

The symmetric traceless polarization vectors $\xi_{\mu\nu}$ are to be contracted with both sides to obtain the scattering amplitude.

3.6 Comments on the Connection between Plahte Diagrams and BCFW Recursion Relations

In the first decade of this century the study of scattering amplitudes benefitted considerably from the discovery of the Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion relations [67, 68]. The relations allow us to express tree-level amplitudes as products of other tree-level amplitudes with fewer particles. The key idea for deriving the on-shell recursion relations is based on the fact that any tree-level scattering amplitude is a rational function of the external momenta, thus, one can turn an amplitude A_n into a complex meromorphic function $A_n(z)$ by deforming the external momenta through introducing a complex variable z. The deformed momenta are required to be on-shell and satisfy momentum conservation. For a scattering process involving n particles, we can choose an arbitrary pair of particle momenta to be shifted. Our choice is given by

$$k_1 \to \hat{k}_1(z) = k_1 - qz \tag{3.6.40a}$$

$$k_n \to \hat{k}_n(z) = k_n + qz \tag{3.6.40b}$$

where q is a reference momentum which obeys $q \cdot q = k_1 \cdot q = k_n \cdot q = 0$.

The unshifted amplitude $A_n(Z = 0)$ can be obtained from a contour integration in which the contour is large enough to enclose all finite poles. According to Cauchy's theorem,

$$A_n(0) = \oint dz \frac{A_n(z)}{z} - \sum_{\text{poles}} \text{Res}_{z=z_{\text{poles}}}. \tag{3.6.41}$$

If the amplitude is well-behaved at large z (which is the case for most theories), then the amplitude at $z = 0$ is equal to the sum of the residues over the finite poles. For Yang-Mills theory, the residue at a finite pole is the product of amplitudes with at least two fewer particles and one leg for an exchanged particle. In Yang-Mills a sum over the helicities of the intermediate gauge boson and in general theories a sum over all allowed intermediate particle states must also be done. In the general case, the BCFW recursion relation is

$$A_n(0) = \sum_{\substack{\text{poles} \\ \alpha}} \sum_{\substack{\text{physical} \\ \text{states}}} A_L(\ldots, P(z_\alpha)) \frac{2}{P^2 + M^2} A_R(-P(z_\alpha), \ldots) \tag{3.6.42}$$

with P being the momentum of the exchanged particle with mass M.

The validity of equation (3.6.41) requires the absence of a pole at infinity. In the case that there exists such a pole, one must include the residue at infinity. However, the residue at this pole does not have a similar physical interpretation to the residues at finite poles. A detailed discussion can be found in [69].

The idea of deforming scattering amplitudes can be applied to string theory as well. Despite the infinite number of physical states of intermediate particles, many works have successfully addressed the string theory versions of BCFW on-shell recursion relations [70–73].

There are links between Plahte diagrams and the BCFW on-shell recursion relations. We have noticed that when we collapse any two adjacent sides of a 5-point gluonic Plahte diagram to the diagonal line, the diagonal line along with two remaining partial amplitudes forms a triangle. It turns out that the corresponding Plahte identities for the triangle coincide with the BCJ relations derived from the BCFW recursion relations of the five gluon scattering amplitudes.

As an explicit example, consider the triangle in the figure 3.13. The external angles next to the diagonal line of a triangle are parametrized by ϕ. The parameter ϕ can be evaluated using BCFW on-shell recursion relations in which we will see later that it corresponds to the shifted momenta in (3.6.40). Without much effort,

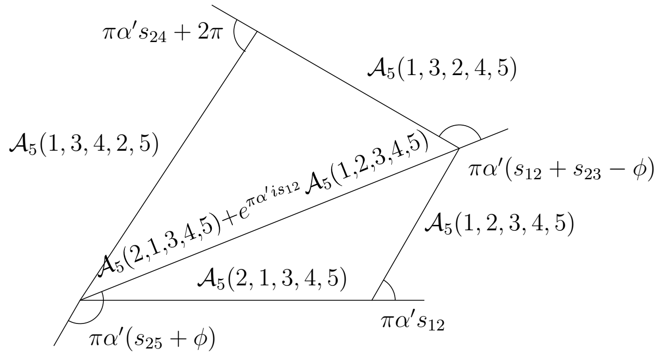


Figure 3.13: An triangle made from the diagonal line of the five-point gluonic Plahte diagram.

one can find the Plahte identities for the triangle as

$$\begin{aligned} \frac{|\mathcal{A}_5(1,3,2,4,5)|}{\sin(\pi\alpha'(s_{25}+\phi))} &= \frac{|\mathcal{A}_5(2,1,3,4,5)+e^{\pi\alpha' i s_{12}}\mathcal{A}_5(1,2,3,4,5)|}{\sin(\pi\alpha' s_{24})} \\ &= \frac{|\mathcal{A}_5(1,3,4,2,5)|}{\sin(\pi\alpha'(s_{12}+s_{23}-\phi))}. \end{aligned} \tag{3.6.43}$$

For convenience, let us make a specific choice of polarizations, say negative helicity for particles one and five and positive helicity for those remaining. Let us calculate the on-shell recursion relations for $A_5(1^-,2^+,3^+,4^+,5^-)$ based on the $[5,1\rangle$-shift (the shifted momentum $q=|5\rangle[1|$). According to the equation (3.6.42), the amplitude breaks down into two terms as

$$\begin{aligned} A_5(1^-,2^+,3^+,4^+,5^-) &= \hat{A}_3(\hat{1}^-,2^+,-\hat{P}_{12}^+)\frac{1}{s_{12}}\hat{A}_4(\hat{P}_{12}^-,3^+,4^+,\hat{5}^-) \\ &\quad +\hat{A}_4(\hat{1}^-,2^+,3^+,\hat{P}_{45}^-)\frac{1}{s_{45}}\hat{A}_3(-\hat{P}_{45}^+,4^+,\hat{5}^-). \end{aligned} \tag{3.6.44}$$

The notation P_{ij} means k_i+k_j. Note that all hatted terms are evaluated at the residue value such that $\hat{s}_{ij}=0$ or $z=z_{ij}=-P_{ij}^2/2q\cdot P_{ij}$. Now let us take a closer look at the subamplitude $\hat{A}_3(-\hat{P}_{45}^+,4^+,\hat{5}^-)$. According to the Parke-Taylor formula,

$$\hat{A}_3(-\hat{P}_{45}^+,4^+,\hat{5}^-) = \frac{[\hat{P}4]^3}{[4\hat{5}][\hat{5}\hat{P}]}. \tag{3.6.45}$$

It turns out that all spinor products in above expression are zeroes. More detailed analysis can be found in [64]. As there are three powers in the numerator compared with the two in the denominator, the subamplitude $\hat{A}_3(-\hat{P}_{45}^+,4^+,\hat{5}^-)$ vanishes. Consequently, only the first term from (3.6.44) contributes.

Using a similar approach, we can then find the remaining partial amplitudes as

$$
\begin{aligned}
A_5(1^-,3^+,2^+,4^+,5^-) =& \hat{A}_3(\hat{1}^-,3^+,-\hat{P}_{13}^+)\frac{1}{s_{13}}\hat{A}_4(\hat{P}_{13}^-,2^+,4^+,\hat{5}^-) \\
A_5(1^-,3^+,4^+,2^+,5^-) =& \hat{A}_3(\hat{1}^-,3^+,-\hat{P}_{13}^+)\frac{1}{s_{13}}\hat{A}_4(\hat{P}_{13}^-,4^+,2^+,\hat{5}^-) \\
A_5(2^+,1^-,3^+,4^+,5^-) =& -A_5(1^-,2^+,3^+,4^+,5^-) \\
& + \hat{A}_3(\hat{1}^-,3^+,-\hat{P}_{13}^+)\frac{1}{s_{13}}\hat{A}_4(\hat{P}_{13}^-,4^+,\hat{5}^-,2^+). \qquad (3.6.46)
\end{aligned}
$$

Notice that these colour-ordered amplitudes can now be related to each other if we exploit the BCJ relations for the subamplitude $\hat{A}_4(\hat{P}_{13}^-,2^+,4^+,\hat{5}^-)$:

$$
\begin{aligned}
\frac{|A_5(1,3,2,4,5)|}{s_{25}+2z_{13}q\cdot k_2} &= \frac{|A_5(2,1,3,4,5)+A_5(1,2,3,4,5)|}{s_{24}} \\
&= \frac{|A_5(1,3,4,2,5)|}{(s_{12}+s_{23})-2z_{13}q\cdot k_2} \qquad (3.6.47)
\end{aligned}
$$

where $z_{13} = -P_{13}^2/2q\cdot P_{13}$.

Clearly, The BCJ relations (3.6.47) resemble the field theory version of the Plahte identities in the equation (3.6.43) with $\phi = 2z_{13}q\cdot k_2$. The parameter ϕ which refers to the amount angles are shifted by is now related to the shifted momentum $zq\cdot k_i$ from BCFW recursion relations as claimed. For other sets of polarisations, the shifted angles can be obtained in a similar manner but with different choices of shifted momenta.

3.7 Plahte Diagrams with Complex Momenta

String scattering amplitudes considered as mathematical objects have been widely studied in past few decades. For example, they provide a close connection to local zeta functions especially in the framework of p-adic string theory [74–77]. Recently, the work of Bocardo-Gaspar, Veys and Zúñiga-Galindo [78] established in a rigorous mathematical way that the integral expressions for open string amplitudes (3.1.1) are bona fide integrals which admit meromorphic continuations as complex functions in the kinematic parameters.

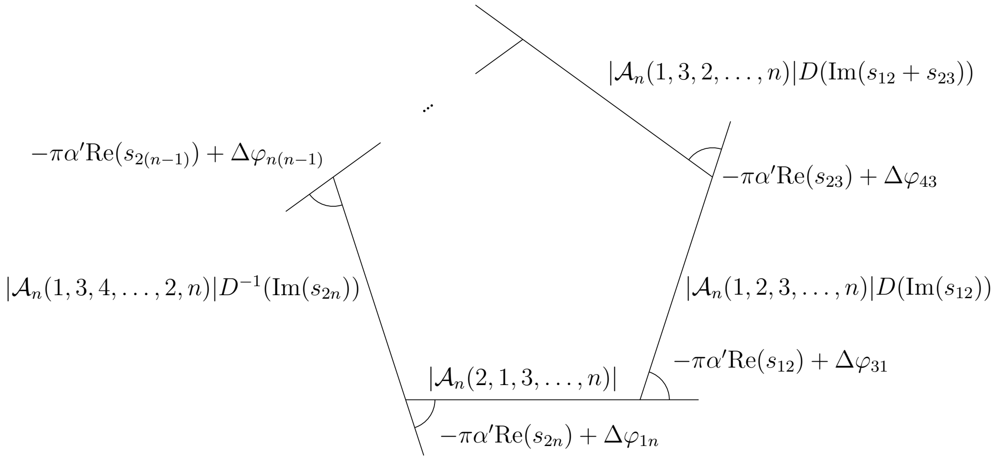


Figure 3.14: Plahte diagram for N-point open tachyon string amplitudes with complex momenta corresponding to the Plahte identity (3.7.48)

When the momenta k_i are taken to be complex the Plahte diagram is deformed. External angles between the sides representing amplitudes are shifted by the differences of the internal phases of the corresponding amplitudes. Besides, the partial amplitudes themselves are re-scaled due to the presence of the imaginary component of the kinematic variables $k_i \cdot k_j$.

Let us consider the generalization of the Plahte identity for n particles scattering. As the momenta are allowed to become complex, the amplitudes also become complex so we write them in Euler's form as $\mathcal{A}_n(\sigma) = |\mathcal{A}_n(\sigma)|e^{i\varphi_\sigma}$ where σ is a certain ordering of open string vertices. In this scenario, the Plahte identity in equation (3.1.3) becomes

$$
\begin{aligned}
&|\mathcal{A}_n(2,1,3,\ldots,n)|e^{i\varphi_1} + |\mathcal{A}_n(1,2,3,\ldots,n)|e^{-\pi i\alpha' s_{12}+i\varphi_3} \\
&\quad + |\mathcal{A}_n(1,3,2,\ldots,n)|e^{-\pi i\alpha'(s_{12}+s_{23})+i\varphi_4} \\
&\quad + \ldots + |\mathcal{A}_n(1,3,\ldots,n-1,2,n)|e^{-\pi i\alpha'(s_{12}+s_{23}+\ldots+s_{2(n-1)})+i\varphi_n} = 0 \qquad (3.7.48)
\end{aligned}
$$

where $s_{ij} = 2k_i \cdot k_j$. The internal phase φ_i is labelled by the particle ordering with particle i next to the particle 2 to its right. Note that the complex momenta k_i are constrained by $\sum_{i=1}^{n} k_i = 0$ and $k_i \cdot k_i = -l/\alpha'$ where $l = -1$ and 0 for tachyons and gauge bosons respectively.

According to the Plahte identity (3.7.48), it is not hard to see that the internal phases φ_i alter the external angles in the Plahte diagram and also that the imaginary components of kinematic variables $\mathrm{Im}(s_{ij})$ lead to a re-scaling of the sides of the diagram.

The Plahte diagram corresponding to the Plahte identity (3.7.48) is presented in figure 3.14. A scaling factor $D(x)$ and an internal phase difference Δ_{ij} are defined as

$$D(x) \equiv e^{\pi\alpha' x} \tag{3.7.49}$$

$$\Delta\varphi_{ij} \equiv \varphi_i - \varphi_j \tag{3.7.50}$$

where x is real. The Plahte diagram presented is a generalisation of that of figure 3.3.

Recall that in order to get the identity (3.7.48), the integral (3.1.2) was performed along the contour which is closed in the upper-half plane. Another identity can be found using the same integral but with the contour closed in the lower-half plane instead. This yields

$$\begin{aligned} &|\mathcal{A}_n(2,1,3,\ldots,n)|e^{i\varphi_1} + |\mathcal{A}_n(1,2,3,\ldots,n)|e^{\pi i\alpha' s_{12}+i\varphi_3} \\ &\quad + |\mathcal{A}_n(1,3,2,\ldots,n)|e^{\pi i\alpha'(s_{12}+s_{23})+i\varphi_4} \\ &\quad + \ldots + |\mathcal{A}_n(1,3,\ldots,n-1,2,n)|e^{\pi i\alpha'(s_{12}+s_{23}+\ldots+s_{2(n-1)})+i\varphi_n} = 0. \end{aligned} \tag{3.7.51}$$

Unlike the identities with real kinematic variables, closing the contour in the upper-half or lower-half plane generates distinct Plahte identities. It can be seen from the relations (3.7.48) and (3.7.51) that both identities provide different information. As a result, they create different Plahte diagrams. The diagram corresponding to the identity (3.7.51) is illustrated in figure 3.15.

It is clear from the figures that the external angles and sides are shifted and rescaled differently in both diagrams. However, when all imaginary parts of kinematic variables s_{ij} are tuned off, both diagrams become identical.

We now give an explicit example. For simplicity, we phrase the discussion only for four-point scattering. By combining equations (3.7.48) and (3.7.51), we can find relations among the complex amplitudes as

$$\frac{|\mathcal{A}_4(1,2,3,4)|e^{i\varphi_3}}{\sin(\pi\alpha' s_{24})} = \frac{|\mathcal{A}_4(2,1,3,4)|e^{i\varphi_1}}{\sin(\pi\alpha' s_{23})} = \frac{|\mathcal{A}_4(1,3,2,4)|e^{i\varphi_4}}{\sin(\pi\alpha' s_{12})} \tag{3.7.52}$$

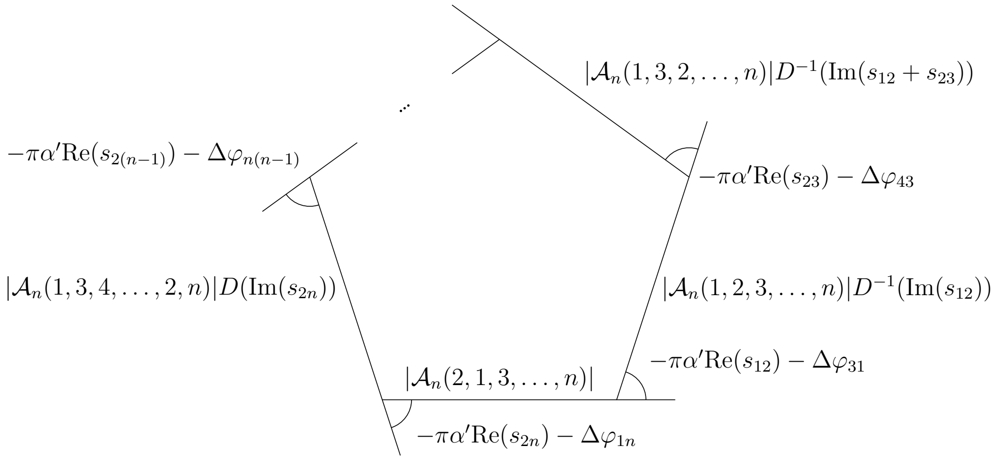


Figure 3.15: Plahte diagram for N-point open tachyon string amplitudes with complex momenta corresponding to the Plahte identity (3.7.51)

which is a complex continuation of equation (3.2.9). It relates all partial amplitudes to one amplitude. The relation above also allows us to find connections between internal phases φ_i as linear relations. By dividing the equation (3.7.52) with its own conjugation, one can obtain

$$\begin{aligned} &\varphi_1 = \varphi_3 - \frac{i}{2}\ln\left(\frac{\sin(\pi\alpha'\bar{s}_{24})\sin(\pi\alpha' s_{23})}{\sin(\pi\alpha' s_{24})\sin(\pi\alpha'\bar{s}_{23})}\right), \\ \text{and} \qquad &\varphi_4 = \varphi_3 - \frac{i}{2}\ln\left(\frac{\sin(\pi\alpha'\bar{s}_{24})\sin(\pi\alpha' s_{12})}{\sin(\pi\alpha' s_{24})\sin(\pi\alpha'\bar{s}_{12})}\right). \end{aligned} \tag{3.7.53}$$

Equivalently, The relations (3.7.53) can also be expressed as

$$\begin{aligned} &\varphi_1 = \varphi_3 - \frac{1}{2}\arctan(\mathcal{K}(\pi\alpha' s_{24})) + \frac{1}{2}\arctan(\mathcal{K}(\pi\alpha' s_{23})), \\ \text{and} \qquad &\varphi_4 = \varphi_3 - \frac{1}{2}\arctan(\mathcal{K}(\pi\alpha' s_{24})) + \frac{1}{2}\arctan(\mathcal{K}(\pi\alpha' s_{12})), \end{aligned} \tag{3.7.54}$$

where $\mathcal{K}(z)$ is defined as

$$\frac{2\sin(\mathrm{Re}(z))\cos(\mathrm{Re}(z))\sinh(\mathrm{Im}(z))\cosh(\mathrm{Im}(z))}{\sin^2(\mathrm{Re}(z))\cosh^2(\mathrm{Im}(z)) - \cos^2(\mathrm{Re}(z))\sinh^2(\mathrm{Im}(z))}. \tag{3.7.55}$$

The above relations are valid for all types of particles.

Part II

Gauge Theory as Tensionless String Theory

In this part we will discuss formulations of non-Abelian Yang-Mills theory as a tensionless string with contact interactions. This provides a new way to relate non-Abelian Yang-Mills theories to string theories. The idea of this formalism is initiated by Mansfield [79] which expresses an electromagnetic field strength tensor of two moving charges as a string-like object supported on a worldsheet bounded by particle worldlines. The formulation of quantum electrodynamics (QED) as the tensionless limit of a spinning string with contact interaction was formulated in [8] and [9]. However, the extension to that of non-Abelian case is still not yet fully-developed.

The main obstacle that impedes the non-Abelian generalisation is incorporating Lie algebras into the theory. This can be done by simply introducing Lie algebra-valued fields into the worldsheet. However, one needs to find a suitable dynamics to describe them. These new degrees of freedom have to generate the interaction vertices of Yang-Mills theory and, when considering the Wilson loop, path-ordering of Lie algebra generators on the world-sheet boundary.

Chapter 4

QED as String with Contact Interaction

Throughout scientific history, Maxwell's theory and its quantum version, QED, prove themselves some of the most successful theories to confront experiment. These theories were developed using fields as the physical fundamental objects. The string theory on the other hand suggests that 1-dimensional extended objects or strings are the fundamental building blocks. The arrival of string theory raises the question whether it possible to reinterpret the strings as the fundamental objects for QED.

The idea of considering strings as fundamental and dynamical objects of electromagnetism can be traced back to Faraday. However, his view toward lines of force as the physical substances dropped out when Maxwell introduced the perspective of fields for electromagnetism described mathematically by vector calculus. The advent of string theory provided the mathematics to bring Faraday's idea of lines of force back to life. Classical electromagnetism was reformulated as the statistical mechanics of lines of electric flux with dynamics described by the string action in four dimensions [79]. This perspective was also applied to the quantum counterpart. In [8] and [9], QED emerges in the tensionless limit of string theory with contact interaction. The equality between the two models was tested through a computation of Wilson loops.

In this chapter, we aim to review how QED can be described by the tensionless limit of string with contact interactions which mostly based on [79], [8] and [9].

4.1 Electrostatics

Before we consider time-dependent electromagnetic fields, we consider the simplest case of electric fields produced by two stationary opposite charged particles. These two charges are located with position vector $\mathbf{a}$ and $\mathbf{b}$ with charges $+q$ and $-q$ respectively. The electric field can be gained from the Gauss' law

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{q}{\epsilon_0}(\delta^3(\mathbf{x}-\mathbf{a}) - \delta^3(\mathbf{x}-\mathbf{b})). \tag{4.1.1}$$

By inspection, one of the possible solutions allowed from the Gauss' law is in the form of string-like solution

$$\mathbf{E}_C(\mathbf{x}) = \frac{q}{\epsilon_0}\int_C \delta^3(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{4.1.2}$$

The solution is an electric line of force extending along the curve C which is an arbitrary curve joining the points $\mathbf{a}$ and $\mathbf{b}$. The direction of the electric field is tangent to the curve. One can easily show that $\mathbf{E}_C(\mathbf{x})$ satisfies the Gauss' law as

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot \mathbf{E}_C(\mathbf{x}) &= \frac{q}{\epsilon_0}\int_C \nabla_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y})d\mathbf{y} = -\frac{q}{\epsilon_0}\int_C \nabla_{\mathbf{y}}\delta^3(\mathbf{x}-\mathbf{y})d\mathbf{y}\\ &= \frac{q}{\epsilon_0}(\delta^3(\mathbf{x}-\mathbf{a}) - \delta^3(\mathbf{x}-\mathbf{b})). \end{aligned}\tag{4.1.3}$$

The expression (4.1.2) has the same mathematical form as the Dirac string which was introduced to describe the magnetic field of a monopole [80].

However, when taking the Faraday's law, $\nabla \times \mathbf{E} = 0$, into consideration, the expression of $\mathbf{E}_C$ fails at the classical level. This implies that the string $\mathbf{E}_C$ is not a physical object. Instead, to regain the classical theory, we will make the assumption that the theory is stochastic in a manner that the classical electric field is reproduced when the electric strings are to be averaged over with a suitable Boltzmann weight, βH [79]. The average of any observable Ω is given by the functional integral

$$\langle \Omega \rangle = \frac{1}{Z}\int \mathcal{D}\mathbf{y}\Omega e^{-\beta H} \tag{4.1.4}$$

where Z is a normalisation so that $\langle 1 \rangle = 1$.

To perform an average calculation, we split the macroscopic charge q into microscopic charge of magnitude q_0. Each pair of them are accounted for the terminus of a line of force in which each electric string is given by (4.1.2) with q replaced by q_0. There are q/q_0 lines in total and each adds up to give the classical electric field. Since we know that the classical solution which satisfies both Gauss' law and Faraday's law is

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3} - \frac{\mathbf{x}-\mathbf{b}}{|\mathbf{x}-\mathbf{b}|^3}\right), \tag{4.1.5}$$

we need to find an appropriate Boltzmann factor such that

$$\frac{1}{4\pi}\left(\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^3} - \frac{\mathbf{x}-\mathbf{b}}{|\mathbf{x}-\mathbf{b}|^3}\right) = \frac{1}{Z}\int \mathcal{D}\mathbf{y}\int_C \delta^3(\mathbf{x}-\mathbf{y})d\mathbf{y}e^{-\beta H}. \tag{4.1.6}$$

A natural choice for the Boltzmann factor arises from the heat-kernel associated with diffusion and Brownian motion which takes the form

$$\langle \mathbf{b}|e^{-\hat{H}_0 T}|\mathbf{a}\rangle = \int \mathcal{D}\mathbf{y}e^{-\int_0^T dt\frac{\dot{y}^2}{2}} = \frac{e^{-\frac{|\mathbf{a}-\mathbf{b}|^2}{2T}}}{(2\pi T)^{3/2}} \tag{4.1.7}$$

where $\hat{H}_0 = \hat{\mathbf{p}}^2/2$ is the Hamiltonian for a free scalar bosonic particle. The curve is parametrised by the parameter t from 0 to T so that $\mathbf{y}(0) = \mathbf{a}$ and $\mathbf{y}(T) = \mathbf{b}$. Via a Wick rotation, the above term is related to the propagator of a free scalar boson traveling from $\mathbf{a}$ to $\mathbf{b}$. To obtain the Dirac-delta function from the partition function, we introduce a source term $\mathbf{A}$ to get

$$\begin{aligned}\langle \mathbf{E}_C(\mathbf{x})\rangle &= \frac{1}{Z}\int \mathcal{D}\mathbf{y}\left(\frac{q}{\epsilon_0}\int_C \delta^3(\mathbf{x}-\mathbf{y})d\mathbf{y}\right)e^{-\int_0^T dt\frac{\dot{y}^2}{2}} \\ &= \frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\int \mathcal{D}\mathbf{y}e^{-\int_0^T dt\frac{\dot{y}^2}{2}+\int_\mathbf{a}^\mathbf{b}\mathbf{A}(\mathbf{y})d\mathbf{y}}\bigg|_{\mathbf{A}=0}.\end{aligned} \tag{4.1.8}$$

The exponent of the equation (4.1.8) is equivalent to the action of a point particle coupled to a gauge field $\mathbf{A}$. Therefore, in the heat-kernel language, the Hamiltonian $\hat{H}_0$ is modified to $\hat{H} = (\hat{\mathbf{p}}^2 + i\frac{q}{\epsilon_0}\mathbf{A}(\hat{\mathbf{q}}))^2/2$. Consequently, we can rewrite the equation (4.1.8) as

$$\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\langle \mathbf{b}|e^{-\hat{H}T}|\mathbf{a}\rangle\bigg|_{\mathbf{A}=0} = -\int_0^T dt\langle \mathbf{b}|e^{-\hat{H}_0(T-t)}\frac{\delta\hat{H}}{\delta\mathbf{A}}\bigg|_{\mathbf{A}=0}e^{-\hat{H}_0 t}|\mathbf{a}\rangle. \tag{4.1.9}$$

It is not hard to compute that

$$2\frac{\delta\hat{H}}{\delta\mathbf{A}(\mathbf{x})}\bigg|_{\mathbf{A}=0} = \frac{q}{\epsilon_0}\big(i\hat{\mathbf{p}}\delta^3(\hat{\mathbf{q}}-\mathbf{x}) + \delta^3(\hat{\mathbf{q}}-\mathbf{x})i\hat{\mathbf{p}}\big). \tag{4.1.10}$$

We then apply the completeness relation of the position states, $\mathbb{1} = \int d^3\mathbf{c}|\mathbf{c}\rangle\langle\mathbf{c}|$, and use the notations, $\langle\mathbf{x}|\hat{\mathbf{p}} = -i\nabla_{\mathbf{x}}\langle\mathbf{x}|$ and $\hat{\mathbf{p}}|\mathbf{x}\rangle = i\nabla_{\mathbf{x}}|\mathbf{x}\rangle$. This yields

$$\begin{aligned}
2\frac{\delta\hat{H}}{\delta\mathbf{A}(\mathbf{x})}\bigg|_{\mathbf{A}=0} &= \frac{q}{\epsilon_0}\int d^3\mathbf{c}\Big(i\hat{\mathbf{p}}|\mathbf{c}\rangle\langle\mathbf{c}|\delta^3(\hat{\mathbf{q}}-\mathbf{x}) + \delta^3(\hat{\mathbf{q}}-\mathbf{x})|\mathbf{c}\rangle\langle\mathbf{c}|i\hat{\mathbf{p}}\Big) \\
&= \frac{q}{\epsilon_0}\int d^3\mathbf{c}\Big(-(\nabla_{\mathbf{c}}|\mathbf{c}\rangle)\langle\mathbf{c}|\delta^3(\hat{\mathbf{q}}-\mathbf{x}) + \delta^3(\hat{\mathbf{q}}-\mathbf{x})|\mathbf{c}\rangle(\nabla_{\mathbf{c}}\langle\mathbf{c}|)\Big) \\
&= \frac{q}{\epsilon_0}|\mathbf{x}\rangle\overleftrightarrow{\nabla}\langle\mathbf{x}|
\end{aligned} \tag{4.1.11}$$

where $|\mathbf{x}\rangle\overleftrightarrow{\nabla}\langle\mathbf{x}| = |\mathbf{x}\rangle\nabla\langle\mathbf{x}| - \nabla|\mathbf{x}\rangle\langle\mathbf{x}|$. Therefore, the equation (4.1.9) becomes

$$\frac{\delta}{\delta\mathbf{A}(\mathbf{x})}\langle\mathbf{b}|e^{-\hat{H}T}|\mathbf{a}\rangle\bigg|_{\mathbf{A}=0} = -\frac{q}{2\epsilon_0}\int_0^T dt\langle\mathbf{b}|e^{-\hat{H}_0(T-t)}|\mathbf{x}\rangle\overleftrightarrow{\nabla}\langle\mathbf{x}|e^{-\hat{H}_0 t}|\mathbf{a}\rangle. \tag{4.1.12}$$

The partition terms above are recognised as the heat kernels from (4.1.7). Thus, the average of the electric string becomes

$$\langle\mathbf{E}_C(\mathbf{x})\rangle = -\frac{q}{2\epsilon_0}\frac{(2\pi T)^{3/2}}{e^{-|\mathbf{a}-\mathbf{b}|^2/2T}}\int_0^T dt\frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2(T-t)}}}{(2\pi(T-t))^{3/2}}\overleftrightarrow{\nabla}\frac{e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2t}}}{(2\pi t)^{3/2}}. \tag{4.1.13}$$

However, the parameter T is a dimensionful quantity which does not appear in the classical observable (4.1.5). To get rid of this we can set a value to T to be exceptionally large. For the large T, the integral is negligible everywhere except the small regions near the boundaries where $t \approx 0$ and $t \approx T$.

In the infinite T limit, the exponential outside the integral turns to identity. The integral approximated at $t \approx 0$ is then

$$-\frac{q}{2\epsilon_0}\int_0^\infty \nabla\frac{e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2t}}}{(2\pi t)^{3/2}}dt. \tag{4.1.14}$$

Similarly at $t \approx T$ in the limit of large T, the integral (4.1.13) becomes

$$\frac{q}{2\epsilon_0}\int_0^\infty \nabla\frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2(T-t)}}}{(2\pi(T-t))^{3/2}}dt = \frac{q}{2\epsilon_0}\int_0^\infty \nabla\frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2t}}}{(2\pi t)^{3/2}}dt \tag{4.1.15}$$

where the change of integrating variable, $(T-t) \to t$, was applied at the end.

Substituting (4.1.14) and (4.1.15) to (4.1.13), the integral takes the form

$$\begin{aligned}
\langle\mathbf{E}_C(\mathbf{x})\rangle &= -\frac{q}{2\epsilon_0}\nabla\int_0^\infty dt\left(\frac{e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2t}}}{(2\pi t)^{3/2}} - \frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2t}}}{(2\pi t)^{3/2}}\right) \\
&= -\frac{q}{4\pi\epsilon_0}\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{a}|} - \frac{1}{|\mathbf{x}-\mathbf{b}|}\right)
\end{aligned} \tag{4.1.16}$$

The result turns out to be exactly the classical electric field from (4.1.5). This implies that the classical electrostatics can be interpreted as the statistical theory of Dirac electric strings. We will see in the next example that this perspective can be applied to a dynamical system of electromagnetism as well.

4.2 Time-dependent Electromagnetism

We move to a more interesting example of two moving particles with opposite charges. This scenario leads us to formulate the string interpretation for a field strength tensor. Unlike the electrostatic case, these two moving charges generate time-dependent lines of electric flux which form a worldsheet Σ on which the field strength tensor is supported. Again, to regain the classical field strength tensor which satisfies all the set of Maxwell's equations, the functional integration over all possible worldsheet is implemented with a suitable Boltzmann factor.

Consider the situation where there are two charges moving with the position vectors a_μ and b_μ with charges q and $-q$ respectively. The electromagnetic current density for this system is

$$J^\mu(x) = q\int_{-\infty}^{\infty} dt\Big(\delta^4(x-a)\dot{a}^\mu - \delta^4(x-b)\dot{b}^\mu\Big). \tag{4.2.17}$$

We propose the solution for the antisymmetric field strength tensor which satisfies the Maxwell's equation, $\partial^\mu F_{\mu\nu} = J_\nu$, with the above four-current as

$$F_{\mu\nu}(x) = -q\int_\Sigma \delta^4(x-y)d\Sigma_{\mu\nu}(y) \tag{4.2.18}$$

for any surface Σ which is bounded by the two particle worldlines. This solution describes that the field strength tensor $F_{\mu\nu}$ is supported on the worldsheet Σ. We parametrise the worldsheet by two worldsheet coordinates (ξ^1, ξ^2). $d\Sigma_{\mu\nu}$ is an infinitesimal area element on the surface defined as

$$d\Sigma_{\mu\nu}(y) = \epsilon^{ab}\partial_a y_\mu \partial_b y_\nu d^2\xi \tag{4.2.19}$$

where $\partial_a = \frac{\partial}{\partial \xi^a}$ is a worldsheet derivative.

The proof of the solution (4.2.18) is very straight forward by taking a partial derivative to (4.2.18) to get

$$\begin{aligned}\partial_x^\mu F_{\mu\nu}(x) &= -q\int_\Sigma \partial_x^\mu \delta^4(x-y)d\Sigma_{\mu\nu}(y) = q\int_\Sigma \partial_y^\mu \delta^4(x-y)d\Sigma_{\mu\nu}(y)\\ &= q\int_\Sigma d^2\xi \partial_y^\mu \delta^4(x-y)(\partial_1 y_\mu \partial_2 y_\nu - \partial_2 y_\mu \partial_1 y_\nu)\\ &= q\int_\Sigma d^2\xi(\partial_1 \delta^4(x-y)\partial_2 y_\nu - \partial_2\delta^4(x-y)\partial_1 y_\nu)\end{aligned} \tag{4.2.20}$$

where we applied the chain rule to get the last line. To further the calculation, the Green's theorem,

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = \int_{\partial D} Pdx + Qdy, \tag{4.2.21}$$

is used where the functions P and Q are both functions of x and y. Therefore, this gives

$$\begin{aligned}\partial_x^\mu F_{\mu\nu}(x) &= q\int_{\partial\Sigma} \delta^4(x-y)(\partial_1 y_\nu d\xi^1 + \partial_2 y_\nu d\xi^2)\\ &= q\int_{\partial\Sigma} \delta^4(x-y)dy_\nu.\end{aligned} \tag{4.2.22}$$

When taking the boundary values into account which are evaluated at the two worldlines. This verifies that $F_{\mu\nu}$ defined in (4.2.18) satisfies the Gauss' law.

Similar to the electric string in electrostatics, the worldsheet solution of the field strength tensor is not a classical object as it fails to satisfy the remaining Maxwell equation

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0. \tag{4.2.23}$$

Indeed, we can re-obtain the classical tensor $F_{\mu\nu}$ associated with the two charged particles by averaging it over all surfaces, where each surface is weighted by the Polyakov action. To verify the argument given, we define the worldsheet average of a quantity Ω over all surfaces Σ spanning $\partial\Sigma$ as

$$\langle\Omega\rangle_\Sigma = \frac{1}{Z_P}\int \mathcal{D}g\mathcal{D}Y\Omega e^{-S_p[g,Y]} \tag{4.2.24}$$

where $S_p[g,Y]$ is the Polyakov action (1.2.50). g_{ab} is the intrinsic metric of the worldsheet Σ.

Now let us compute the worldsheet average of the field strength tensor (4.2.18):

$$\langle F_{\mu\nu}(x)\rangle_\Sigma = \frac{1}{Z}\int \mathcal{D}g\mathcal{D}Y\Big(-q\int_\Sigma \delta^4(x-Y)d\Sigma_{\mu\nu}(Y)\Big)e^{-S_p[g,Y]} \tag{4.2.25}$$

with a boundary made by particle-antiparticle loop.

To perform the above functional integration, we will first refrain from integrating over g_{ab} by giving a fixed value for it and we will eventually find out that the averaging does not depend on the value we pick. We can reshape (4.2.25) by using a Fourier decomposition of the Dirac delta function together with introducing a source term j^a_μ for the spacetime coordinate Y to obtain

$$\begin{aligned}\langle F_{\mu\nu}(x)\rangle_\Sigma &= -\frac{q}{Z}\int \frac{d^4k}{(2\pi)^4}\int \mathcal{D}Y\, d\Sigma_{\mu\nu}(Y)\Big)e^{-ik\cdot(x-Y)-S_p[g,Y]}\\ &= -\frac{q}{Z}\int \frac{d^4k}{64\pi^4\alpha'^2}d^2\xi\epsilon^{ab}\frac{\partial}{\partial j^{\mu a}}\frac{\partial}{\partial j^{\nu b}}\int \mathcal{D}Y\, e^{-S'}\Big|_{j=0}\end{aligned} \tag{4.2.26}$$

where

$$2\pi\alpha' S' = \int_\Sigma \Bigg(\sqrt{g}g^{ab}\frac{1}{2}\frac{\partial Y}{\partial\tilde{\xi}^a}\cdot\frac{\partial Y}{\partial\tilde{\xi}^b} + \Big[ik\cdot(x-Y) + Y\cdot j^a\frac{\partial}{\partial\tilde{\xi}^a}\Big]\delta^2(\tilde{\xi}-\xi)\Bigg)d^2\tilde{\xi} \tag{4.2.27}$$

with $\frac{\partial}{\partial\xi^a}j^a_\mu = 0$. Note that on the last line we relabel k by $k/(2\pi\alpha')$.

We then write the field Y as a sum of the classical path and a quantum fluctuation, i.e. $Y = Y_{\rm cl} + \overline{Y}$. The classical path $Y_{\rm cl}$ is a solution to the Euler-Lagrange equation for S':

$$-\frac{\partial}{\partial\tilde{\xi}^a}\Big(\sqrt{g}g^{ab}\frac{\partial Y^\mu_{\rm cl}}{\partial\tilde{\xi}^b}\Big) = \Big[ik^\mu - j^\mu_a\frac{\partial}{\partial\tilde{\xi}_a}\Big]\delta^2(\tilde{\xi}-\xi). \tag{4.2.28}$$

One can write $Y_{\rm cl}$ using the Dirichlet Green function for the Laplacian, $G(\xi,\tilde{\xi})$ whose value satisfies

$$-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\tilde{\xi}^a}\Big(\sqrt{g}g^{ab}\frac{\partial}{\partial\tilde{\xi}^b}\Big)G(\xi,\tilde{\xi}) \equiv -\triangle G(\xi,\tilde{\xi}) = \frac{1}{\sqrt{g}}\delta^2(\tilde{\xi}-\xi), \tag{4.2.29}$$

as

$$\begin{aligned}Y_{\rm cl}(\xi) &= \int_\Sigma d^2\tilde{\xi}G(\xi,\tilde{\xi})\Big(ik^\mu - j^\mu_a\frac{\partial}{\partial\tilde{\xi}_a}\Big)\delta^2(\tilde{\xi}-\xi)\\ &\quad + \oint_{\partial\Sigma}\sqrt{g}g^{ab}\epsilon_{bc}\frac{\partial G(\xi,\tilde{\xi})}{\partial\tilde{\xi}^a}y(\tilde{\xi})\ d\tilde{\xi}^c\end{aligned} \tag{4.2.30}$$

where $y(\xi)$ is the boundary value of $Y_{\rm cl}$.

After expanding, $Y = Y_{\rm cl} + \overline{Y}$, (4.2.25) takes the form

$$\langle F_{\mu\nu}(x)\rangle_\Sigma = -\, q\frac{e^{-S_P[g,Y_{\rm cl}]}}{Z}\int \frac{d^4k}{64\pi^4\alpha'^2}d^2\xi\ \epsilon^{ab}\frac{\partial}{\partial j^a_\mu}\frac{\partial}{\partial j^b_\nu}\int \mathcal{D}\overline{Y}\ e^{-S''}\Bigg|_{j=0} \tag{4.2.31}$$

where

$$\begin{aligned}2\pi\alpha' S'' =& \int_\Sigma d^2\tilde{\xi}\Bigg(-\frac{1}{2}\overline{Y}\frac{\partial}{\partial\tilde{\xi}^a}\sqrt{g}g^{ab}\frac{\partial}{\partial\tilde{\xi}^b}\overline{Y} + (ik\cdot\overline{Y} - \overline{Y}\cdot j^a\frac{\partial}{\partial\tilde{\xi}^a})\Bigg)\delta^2(\xi-\tilde{\xi})\\ &-\Bigg(ik_\mu + j^a_\mu\frac{\partial}{\partial\xi^a}\Bigg)Y^\mu_{\rm cl}(\xi) + ik\cdot x(\xi).\end{aligned} \tag{4.2.32}$$

Note that the boundary term which includes $\overline{Y}$ vanishes as the quantum fluctuation is zero at boundary. We then carry out the Gaussian integral in $\overline{Y}$ with the Laplacian operator in the quadratic term. The equation (4.2.31) becomes

$$\langle F_{\mu\nu}(x)\rangle_\Sigma = -\, q\int \frac{d^4k}{64\pi^4\alpha'^2}d^2\xi\ \epsilon^{ab}\frac{\partial}{\partial j^{\mu a}}\frac{\partial}{\partial j^{\nu b}}e^{-\tilde{S}}\Bigg|_{j=0} \tag{4.2.33}$$

with

$$\begin{aligned}2\pi\alpha'\tilde{S} =& -\frac{1}{2}(ik + j^a\frac{\partial}{\partial\xi^a})\cdot(ik + j^b\frac{\partial}{\partial\tilde{\xi}^b})G(\xi,\tilde{\xi})\Bigg|_{\xi=\tilde{\xi}}\\ &-\Bigg(ik_\mu + j^a_\mu\frac{\partial}{\partial\xi^a}\Bigg)Y^\mu_{\rm cl}(\xi) + ik\cdot x(\xi).\end{aligned} \tag{4.2.34}$$

The determinant factor as well as the source-independent terms were cancelled out by Z in the denominator.

Since the Green function at co-incident points, $G(\xi,\xi)$, is divergent, it should be regulated with a short-distance cut-off, ϵ. The Green function can be constructed from the heat kernel $\mathcal{G}$ as

$$G_\epsilon(\xi,\tilde{\xi}) = \int_\epsilon^\infty \mathcal{G}(\xi,\tilde{\xi};\tau)d\tau. \tag{4.2.35}$$

satisfying

$$\frac{\partial}{\partial\tau}\mathcal{G} = -\triangle\mathcal{G}, \qquad \mathcal{G}(\xi,\tilde{\xi};0) = \frac{1}{\sqrt{g}}\delta^2(\xi-\tilde{\xi}). \tag{4.2.36}$$

If u_n is an eigenfunction of the Laplacian with the eigenvalue λ_n vanishing on the boundary $\partial\Sigma$, the heat kernel can be expressed in the spectral representation as

$$\mathcal{G}(\xi,\tilde{\xi};\tau) = \sum_n u_n(\xi)u_n(\tilde{\xi})e^{-\tau\lambda_n}. \tag{4.2.37}$$

When substituting (4.2.37) into (4.2.35), the regulated Green function takes the form

$$G_\epsilon(\xi,\tilde{\xi}) = \sum_n u_n(\xi)u_n(\tilde{\xi})\frac{e^{-\epsilon\lambda_n}}{\lambda_n}. \tag{4.2.38}$$

Let $\psi(\xi)$ denote the value of Green function at co-incident points, i.e.

$$\psi(\xi) = G_\epsilon(\xi,\xi) = \int_\epsilon^\infty \mathcal{G}(\xi,\xi;\tau). \tag{4.2.39}$$

This function vanishes on the boundary and is non-negative elsewhere due to the properties of the eigenfunction u_n as well as the fact that λ_n is always grater than zero. Furthermore, we can utilise (4.2.38) to write the derivative of Green function at co-incident points in terms of ψ as

$$\frac{\partial}{\partial\xi}G(\xi,\tilde{\xi})\bigg|_{\xi=\tilde{\xi}} = \frac{1}{2}\frac{\partial}{\partial\xi}\psi(\xi). \tag{4.2.40}$$

This allows us to rewrite (4.2.34) as

$$\begin{aligned}2\pi\alpha'\tilde{S} =& \frac{1}{2}k^2\psi(\xi) - \frac{i}{2}k\cdot j^a\frac{\partial}{\partial\xi^a}\psi(\xi) - \frac{1}{2}j^a\cdot j^b\frac{\partial^2}{\partial\xi^a\partial\xi^b}G(\xi,\tilde{\xi})\bigg|_{\xi=\tilde{\xi}}\\ &-\left(ik_\mu + j^a_\mu\frac{\partial}{\partial\xi^a}\right)Y^\mu_{\text{cl}}(\xi) + ik\cdot x(\xi).\end{aligned} \tag{4.2.41}$$

Turning back to (4.2.33), the calculation is proceeded by differentiating (4.2.33) with respect to j and then set j to zero to get

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_\Sigma =& -q\int\frac{d^4k}{64\pi^4\alpha'^2}d^2\xi\ \frac{\epsilon^{ab}}{(2\pi\alpha')^2}\left(\frac{i}{2}k^\mu\frac{\partial\psi(\xi)}{\partial\xi^a}+\frac{\partial Y^\mu_{\text{cl}}}{\partial\xi^a}\right)\left(\frac{i}{2}k^\nu\frac{\partial\psi(\xi)}{\partial\xi^b}+\frac{\partial Y^\nu_{\text{cl}}}{\partial\xi^b}\right)\\ &\times\exp\left(\frac{-1}{2\pi\alpha'}\left(\frac{1}{2}k^2\psi(\xi) - ik\cdot(Y_{\text{cl}}-x)\right)\right)\\ =& -q\int\frac{d^4k}{4(2\pi)^6\alpha'^4}d^2\xi\epsilon^{ab}\Big(ik^{[\mu}\partial_a\psi(\xi)\partial_bY^{\nu]}_{\text{cl}} + 2\partial_aY^{[\mu}_{\text{cl}}\partial_bY^{\nu]}_{\text{cl}}\Big)\\ &\times\exp\left(\frac{-1}{2\pi\alpha'}\left(\frac{1}{2}k^2\psi(\xi) - ik\cdot(Y_{\text{cl}}-x)\right)\right).\end{aligned} \tag{4.2.42}$$

Notice that we have raised the indices of the field strength tensor as it is more convenient to utilise the notation of antisymmetric square brackets. There is no contribution from the second derivative of the point-coinciding Green function as it vanishes when producted with ϵ^{ab}.

It is straightforward to integrate (4.2.42) over k using the Gaussian integration formula. Consequently, (4.2.42) takes the form

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_\Sigma =& q\int \frac{d^2\xi}{4(2\pi)^3\alpha'^2\psi^3}\epsilon^{ab}\Big[(Y_{\rm cl}-x)^{[\mu}\partial_a\psi\partial_b Y_{\rm cl}^{\nu]}-2\psi\partial_a Y_{\rm cl}^{[\mu}\partial_b Y_{\rm cl}^{\nu]}\Big]\\ &\times\exp\Big(-|Y_{\rm cl}-x|^2/(4\pi\alpha'\psi)\Big).\end{aligned}\tag{4.2.43}$$

This splits into two integrals.

Now, we need to investigate the value of $\psi(\xi)$. As mentioned earlier, we can evaluate the Green function through the heat kernel via (4.2.35). The general form of the heat kernel can be written using the Seeley-DeWitt expansion [81] which can be modified to a manifold with boundary [82] [83]. If $\sigma_r(\xi,\tilde{\xi})$ is twice the square of the length of the geodesic path connecting between ξ and $\tilde{\xi}$ with r reflections at the boundary, then the heat kernel is obtained by writing

$$\mathcal{G}(\xi,\tilde{\xi};\tau)=\frac{1}{4\pi\tau}\sum_r\exp\Big(-\frac{\sigma_r(\xi,\tilde{\xi})}{2\tau}\Big)\Omega_r(\xi,\tilde{\xi};\tau).\tag{4.2.44}$$

The function Ω_r can be expanded as a power series of τ which is

$$\Omega_r(\xi,\tilde{\xi};\tau)=\sum_n^\infty a_n^r(\xi,\tilde{\xi})\tau^n\tag{4.2.45}$$

where $a_n^r(\xi,\tilde{\xi})$ are called the Seeley-DeWitt coefficients. According to [82], for $\xi=\tilde{\xi}$, the coefficients of the first few orders are evaluated as

$$a_0^0(\xi,\xi)=1,\quad a_1^0(\xi,\xi)=\frac{1}{6}R(\xi),\quad a_0^1(\xi,\xi)=-1,\quad a_1^1(\xi,\xi)=-\frac{1}{6}R(\xi).\tag{4.2.46}$$

The divergence of $\psi(\xi)$ is associated with the short-time behaviour of the heat kernel. In this limit as $\tau\to 0$ and for $\xi\approx\tilde{\xi}$, it is sufficient to obtain the asymptotic version of (4.2.44) by including only zero and one reflection terms as

$$\mathcal{G}(\xi,\tilde{\xi};\tau)=\frac{1}{4\pi\tau}\Big[\exp\Big(-\frac{\sigma_0}{2\tau}\Big)-\exp\Big(-\frac{\sigma_1}{2\tau}\Big)\Big].\tag{4.2.47}$$

At the co-incident points, $\sigma_0=0$. Therefore, $\psi(\xi)$ reads

$$\begin{aligned}\psi(\xi)&\approx\int_\epsilon^\infty d\tau\,\frac{1}{4\pi\tau}\Big(1-\exp\Big(-\frac{\sigma_1}{2\tau}\Big)\Big)\\ &=\begin{cases}\frac{\sigma_1}{(8\pi\epsilon)}, & \sigma_1\ll\epsilon\\ \frac{1}{4\pi}\ln\Big(\frac{\sigma_1}{\epsilon}\Big), & \sigma_1\gg\epsilon.\end{cases}\end{aligned}\tag{4.2.48}$$

The function ψ ranges from zero to a large positive value as ξ moves away from the boundary. This implies that the integrand (4.2.43) is suppressed outside a very tiny strip bordering the boundary $\partial\Sigma$ when the regulator ϵ is removed.

We then reparametrize the worldsheet coordinates (ξ^1, ξ^2) into $(\vartheta, \eta = 4\pi\alpha'\psi)$ where η is constant at the boundary. Now, the worldsheet coordinates ϑ and η can be seen as an angular and radial-like coordinates.

According to the new coordinates, let consider the second integral (up to a constant) in (4.2.43),

$$\int_\Sigma \frac{d\vartheta d\eta}{\eta^2}\frac{\partial y_{\rm cl}^{[\mu}}{\partial\vartheta}\frac{\partial y_{\rm cl}^{\nu]}}{\partial\eta}\ e^{-|y_{\rm cl}-x|^2/\eta} \tag{4.2.49}$$

where $Y_{\rm cl}(\xi)$ is replaced by its value evaluated near the boundary $y_{\rm cl}(\vartheta, \eta)$ due to the suppression of ψ inside the worldsheet. Since the leading contributions of the integrand are located inside a tiny strip near the boundary, we can set the cut-off limit for η by the width of the strip Λ. The integral (4.2.49) can be approximated as

$$\int d\vartheta \frac{\partial y_{\rm cl}^{[\mu}}{\partial\vartheta}\frac{\partial y_{\rm cl}^{\nu]}}{\partial\sigma_1}\int_0^\Lambda \frac{d\eta}{\eta^2}\frac{\partial\sigma_1}{\partial\eta}\ e^{-|y_{\rm cl}-x|^2/\eta}. \tag{4.2.50}$$

Using (4.2.48),

$$\int_0^\Lambda \frac{d\eta}{\eta^2}\frac{\partial\sigma_1}{\partial\eta}\ e^{-|y_{\rm cl}-x|^2/\eta} \sim \frac{2\epsilon}{\alpha'}\int_0^\Lambda \frac{d\eta}{\eta^2}\ e^{-|y_{\rm cl}-x|^2/\eta}. \tag{4.2.51}$$

However, we know for sure that the above integral is positive and less than

$$\frac{2\epsilon}{\alpha'}\int_0^\infty \frac{d\eta}{\eta^2}\ e^{-|y_{\rm cl}-x|^2/\eta} = \frac{2\epsilon}{\alpha'}\frac{1}{|y_{\rm cl}-x|^2}. \tag{4.2.52}$$

As a result, this term is negligible.

What remains is the first term in the integral (4.2.43) which is

$$\langle F^{\mu\nu}(x)\rangle_\Sigma = 2q\alpha' \int \frac{d\vartheta d\eta}{\eta^3}(y_{\rm cl}-x)^{[\mu}\partial_\eta\psi\partial_\vartheta y_{\rm cl}^{\nu]}\ e^{-|y_{\rm cl}-x|^2/\eta} \tag{4.2.53}$$

where $\partial_\vartheta = \frac{\partial}{\partial\vartheta}$ and $\partial_\eta = \frac{\partial}{\partial\eta}$. This integral is written in the new coordinates (ϑ, η) and the derivative of $y_{\rm cl}$ with respect to η was ignored. Again, we replaced the classical path $Y_{\rm cl}$ by its boundary value $y_{\rm cl}$. The integration over η is executed to get

$$\begin{aligned}\langle F^{\mu\nu}(x)\rangle_\Sigma &= \frac{q}{2\pi}\int d\vartheta \frac{1}{|y_{\mathrm{cl}}-x|^4}(y_{\mathrm{cl}}-x)^{[\mu}\partial_\vartheta y_{\mathrm{cl}}^{\nu]} \\ &= \frac{q}{2\pi}\oint \frac{1}{|y_{\mathrm{cl}}-x|^4}(y_{\mathrm{cl}}-x)^{[\mu}dy_{\mathrm{cl}}^{\nu]} \\ &= \frac{q}{4\pi}\left(\partial^\mu \oint \frac{1}{|y_{\mathrm{cl}}-x|^2}dy^\nu_{\mathrm{cl}} - \partial^\nu \oint \frac{1}{|y_{\mathrm{cl}}-x|^2}dy^\mu_{\mathrm{cl}}\right). \end{aligned} \tag{4.2.54}$$

This solution $\langle F^{\mu\nu}(x)\rangle_\Sigma$ satisfies the remaining Maxwell equation (4.2.23) as claimed. As a result, the classical solution of the field strength tensor can be obtained by carrying out a statistical average of the electromagnetic string worldsheet (4.2.18) over string configurations in Polyakov action.

4.3 The Abelian Yang-Mills Action and its relation to string theory

In the previous section, the string formulation of electromagnetic field strength tensor gives a hint for reformulating the theory of electromagnetism as the stochastic theory of electric lines of force. Using the field strength defined in (4.2.18), we can formulate a string theory with non-standard interaction directly from an Abelian gauge theory.

Consider the Lagrangian for pure electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4.3.55}$$

which is an Abelian version of (1.1.1). We then simply insert the line of force solution (4.2.18) into the action giving

$$S_{\mathrm{EM}} = \int d^4x\mathcal{L}(x) = \frac{q^2}{4}\int_\Sigma d\Sigma_{\mu\nu}(X(\xi))\delta^4(X(\xi)-X(\tilde{\xi}))d\Sigma^{\mu\nu}(X(\tilde{\xi})) \tag{4.3.56}$$

where the area element $d\Sigma$ is defined in (4.2.19). Due to the appearance of the delta function, the action is non-vanishing when the worldsheet coordinates coincide, $\xi = \tilde{\xi}$, or when any two points on the worldsheet contact to each other, $X(\xi) = X(\tilde{\xi})$. This splits (4.3.56) into two pieces as

$$S_{\mathrm{EM}} = \frac{q^2}{4}\delta^2(0)\mathrm{Area}(\Sigma) + \frac{q^2}{4}\int_\Sigma d\Sigma_{\mu\nu}(X(\xi))\delta^4(X(\xi)-X(\tilde{\xi}))d\Sigma^{\mu\nu}(X(\tilde{\xi}))\Big|_{\xi\neq\tilde{\xi}}. \tag{4.3.57}$$

The first piece contains the area of the worldsheet corresponding to the Nambu-Goto action of string theory, albeit with a divergent coefficient. See appendix A.2 for the illustration of how the Nambu-Goto action arises from the first piece of the action. The latter piece provides more interesting interpretation. It implies a contact interaction which occurs when the worldsheet intersects with itself. This is unusual from string perspectives in which the standard interactions in string theory are caused by joining and splitting worldsheets. Similar interactions have previously been discussed by Kalb and Ramond [84]. From now on we denote the second term of (4.3.57) by $S_I[X]$.

As the Nambu-Goto action is quite difficult to work with, we may replace it with the classically equivalent Polyakov action. In [85], it was shown perturbatively that the partition function of a tensionless four-dimensional string with the contact interaction S_I whose worldsheet Σ spans the closed loop $\partial\Sigma$ is similar to the Wilson loop for Abelian gauge theory associated with the closed curve $\partial\Sigma$ in flat Euclidean space at the first leading order. This suggests that the expectation value of the Wilson loop could be expressed as the worldsheet average of exponential of S_I. However, a difficulty arises as divergences appear when exponentiating S_I which potentially spoils the suppression.

To illustrate this, let determine the partition function of the action (4.3.57) in which the Nambu-Goto action is replaced by the Polyakov action $S_P[X,g]$, i.e.

$$Z = \frac{1}{Z_{\mathrm{P}}} \int \mathcal{D}X\mathcal{D}g \; e^{-S_P[X,g]-S_I[X]} \tag{4.3.58}$$

where Z_p is a normalisation such that the above quantity reduces to 1 when the coupling constant q is tuned off. The partition function can be written as a power series of the expectation value of S_I which is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_I^n \rangle_\Sigma. \tag{4.3.59}$$

When treating the coupling constant q to be a small parameter, we can neglect all insignificant terms of higher orders leaving only the first order interaction, $\langle S_I \rangle_\Sigma$, to consider. Remember that the worldsheet average $\langle \Omega \rangle_\Sigma$ was defined as (4.2.24).

Let look closely to the expression of the interaction term (4.3.57). We then apply the Fourier decomposition to the delta function to obtain

$$S_I = \frac{q^2}{4} \int \frac{d^4k}{(2\pi)^4} d^2\xi d^2\xi' \; V_k^{\mu\nu}(\xi) V_{-k\ \mu\nu}(\xi') \tag{4.3.60}$$

where $V_k^{\mu\nu}(\xi)$ is the vertex operator defined as

$$V_k^{\mu\nu}(\xi) = \epsilon^{ab}\partial_a X^\mu(\xi)\partial_b X^\nu(\xi) e^{ik\cdot X(\xi)}. \tag{4.3.61}$$

The expression (4.3.60) can be separated into two terms by introducing a projection operator $\mathbb{P}_k$ which is defined as

$$\mathbb{P}_k(X)^\mu = X^\mu - k^\mu \frac{k\cdot X}{k^2}. \tag{4.3.62}$$

The operator projects any 4-vectors onto their transverse directions comparing to the vector k. Therefore, the vertex operator now takes the form

$$\begin{aligned} V_k^{\mu\nu}(\xi) =& \epsilon^{ab}\partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu e^{ik\cdot X} + 2\epsilon^{ab}\partial_a(k\cdot X) k^{[\mu}\partial_b \mathbb{P}_k(X)^{\nu]}\frac{e^{ik\cdot X}}{k^2} \\ =& \epsilon^{ab}\partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu e^{ik\cdot X} - \partial_a\left(2i\epsilon^{ab}k^{[\mu}\partial_b \mathbb{P}_k(X)^{\nu]}\frac{e^{ik\cdot X}}{k^2}\right). \\ \equiv& \tilde{V}_k^{\mu\nu}(\xi) - \partial_a\left(2i\epsilon^{ab}k^{[\mu}\partial_b \mathbb{P}_k(X)^{\nu]}\frac{e^{ik\cdot X}}{k^2}\right) \end{aligned} \tag{4.3.63}$$

where $\tilde{V}_k^{\mu\nu}(\xi)$ is a projected vertex operator defined as

$$\tilde{V}_k^{\mu\nu} = \epsilon^{ab}\partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu e^{ik\cdot X}. \tag{4.3.64}$$

Note that as $k_\mu \mathbb{P}_k(X)^\mu = 0$, the vertex operator must satisfy $k_\mu V_k^{\mu\nu} = k_\mu \tilde{V}_k^{\mu\nu} = 0$. Consequently, when inserting (4.3.63) into (4.3.60), we find

$$\begin{aligned} S_I =& \frac{q^2}{4} \int \frac{d^4k}{(2\pi)^4} d^2\xi d^2\xi' \; \tilde{V}_k^{\mu\nu}(\xi) \tilde{V}_{-k\ \mu\nu}(\xi') \\ &+ \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \oint_{\partial\Sigma}\oint_{\partial\Sigma} d\mathbb{P}_k(X)^\mu(\xi) \frac{e^{ik\cdot(X(\xi)-X(\xi'))}}{k^2} d\mathbb{P}_k(X)_\mu(\xi'). \end{aligned} \tag{4.3.65}$$

To obtain the above expression, Stoke's theorem was used.

It turns out that averaging over the worldsheet using the standard string action will suppress the first term. To see this, we use Wick's theorem to evaluate the expectation of products of fields. According to Wick's theorem for the bosonic string,

$$X^\mu(\xi)X^\nu(\xi') =: X^\mu(\xi)X^\nu(\xi') : +\alpha'\delta^{\mu\nu}G(\xi,\xi') \tag{4.3.66}$$

The colons indicate normal ordering which means no further self-contractions are to be carried out between the fields contained within. $G(\xi,\xi')$ is the Green function for the worldsheet Laplacian. Since the field X can be expanded around the classical field X_c, so the expectation value of the normal ordered part is

$$\langle : X^\mu(\xi) X^\nu(\xi') : \rangle_\Sigma = X_c^\mu X_c^\nu. \tag{4.3.67}$$

It is not hard to find that the expression for the projected vertex operator is

$$\tilde{V}_k^{\mu\nu} =: \tilde{V}_k^{\mu\nu} : e^{-\alpha' \pi k^2 G(\xi,\xi)}. \tag{4.3.68}$$

To do so, the expression for an exponential of the field X

$$e^{ik\cdot X} =: e^{ik\cdot X} : e^{-\alpha' \pi k^2 G(\xi,\xi)} \tag{4.3.69}$$

together with the relation

$$\epsilon^{ab} \partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu =: \epsilon^{ab} \partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu : \tag{4.3.70}$$

were used. Note that the Wick contraction between $\partial_a \mathbb{P}_k(X)^\mu$ and $e^{ik\cdot X}$ vanishes which can be seen by the following steps:

$$\begin{aligned}
\langle \partial_a \mathbb{P}_k(X)^\mu e^{ik\cdot X} \rangle_\Sigma &= \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} \langle \partial_a \mathbb{P}_k(X)^\mu X \rangle : X^{n-1} : \\
&= ik_\nu \langle \partial_a \mathbb{P}_k(X)^\mu X^\nu \rangle : e^{ik\cdot X} : \\
&= ik_\nu \left(\alpha' \delta^{\mu\nu} \partial_a G(\xi,\xi) - \alpha' \frac{k^\mu k^\nu}{k^2} \partial_a G(\xi,\xi) \right) : e^{ik\cdot X} := 0.
\end{aligned} \tag{4.3.71}$$

The Green's function at co-incident points $G(\xi,\xi)$ is zero on the worldsheet boundary and diverges as ξ moves away from the boundary into the interior of the worldsheet as previously expressed in (4.2.48). Since the theory was in the Euclidean signature, $k^2 > 0$, the projected vertex operator (4.3.68) is suppressed inside the worldsheet for which $\alpha' k^2$ is finite. This suppression gets further amplified when taking the tensionless limit $\alpha' \to \infty$ into consideration. What remains in the expectation of S_I is that of the second term in (4.3.65)

$$\langle S_I \rangle_\Sigma = \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \oint_{\partial\Sigma} \oint_{\partial\Sigma} d\mathbb{P}_k(X)^\mu(\xi) \frac{e^{ik\cdot(X(\xi)-X(\xi'))}}{k^2} d\mathbb{P}_k(X)_\mu(\xi') \tag{4.3.72}$$

For short, we will write $X' = X(\xi')$. With (4.3.62), it is obvious that

$$d\mathbb{P}_k(X)_\mu d\mathbb{P}_k(X')^\mu = dX_\mu d\mathbb{P}_k(X')^\mu \tag{4.3.73}$$

Therefore, (4.3.72) becomes

$$\begin{aligned}\langle S_I \rangle_\Sigma &= \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \oint_{\partial\Sigma} \oint_{\partial\Sigma} dX_\mu \Big(dX'^\mu - \frac{k^\mu}{k^2}(k \cdot dX') \Big) \frac{e^{ik\cdot(X-X')}}{k^2} \\ &= \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \oint_{\partial\Sigma} \oint_{\partial\Sigma} dX_\mu dX'_\nu \Big(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \Big) \frac{e^{ik\cdot(X-X')}}{k^2}. \end{aligned} \tag{4.3.74}$$

This is exactly the first sub-leading term of the expectation value of the Wilson loop (1.1.34) evaluated in the Landau gauge ($\zeta = 0$). This suggests that the expectation value of the Wilson loop could be expressed as the worldsheet average of exponential of S_I. However, a difficulty arises as divergences appear when exponentiating S_I which potentially spoil the suppression. Fortunately, no such terms are produced in the supersymmetric generalisation of the model. It appears that the expectation value of super Wilson loop for (non-supersymmetric) Abelian gauge theory can be expressed as the worldsheet average of the spinning string with a contact interaction [8], [9].

Chapter 5

Non-Abelian Yang-Mills Theory as Tensionless String with Contact Interactions

We have learned from the previous chapter that the Wilson loop for four-dimensional Abelian gauge theories can be obtained from a string theory with non-standard contact interactions. The purpose of this section is to investigate if we can generalise the string model to reproduce the expectation of the non-Abelian Wilson loop in the Yang-Mills theory. This requires an introduction of Lie algebra-valued worldsheet degrees of freedom to try to reproduce the Lie algebra structure of Yang-Mills propagators. The additional fields are expected to reduce to path-ordered generators on the boundary.

We will now present two possible modifications of the string model describing non-Abelian Yang-Mills theory. The two models are based on papers by Curry and Mansfield [85, 86]. Although both models can produce non-interaction parts of the Wilson loop correctly, they still lack structures to generate the self-interaction contributions. Thus, they cannot be considered as a non-Abelian generalization of [8, 9]. However, we will present a possible solution towards the non-Abelian generalization of the string model at the end of this chapter.

5.1 The First Model

The first model is a generalisation for the boundary fields ψ, whose correlation function generates path-ordering on the boundary, into the interior of the worldsheet. This was inspired by looking at the expression for bosonic non-Abelian Wilson loop (1.1.35) assuming that the self-interactions are turned off. Using [87, 88], one can replace path-ordered operator by a functional integral over an anti-commuting field ψ defined on the worldsheet boundary $\partial\Sigma$ as

$$\int D\psi^\dagger D\psi \psi^\dagger(1)\psi(0) \exp\Big(\int d\tau \psi^\dagger \dot{\psi} \\ + \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \oint_{\partial\Sigma} \oint_{\partial\Sigma} (\psi^\dagger T^A \psi d\mathbb{P}_k(X)^\mu)\Big|_\xi \frac{e^{ik\cdot(X(\xi)-X(\xi'))}}{k^2} (\psi^\dagger T^B \psi d\mathbb{P}_k(X)_\mu)\Big|_{\xi'} \eta_{AB} \Big). \tag{5.1.1}$$

Remember that the above expression was considered in Landau gauge ($\zeta = 0$). This leads to the non-Abelian modification to the contact interaction S_I as

$$S_I^\phi = \frac{q^2}{4} \int_\Sigma d\Sigma_{\mu\nu}(\xi) \phi^R(\xi) \delta^4(X(\xi) - X(\tilde{\xi})) d\Sigma^{\mu\nu}(\tilde{\xi}) \phi_R(\tilde{\xi}). \tag{5.1.2}$$

where

$$\phi^R = \psi^\dagger T^R \psi. \tag{5.1.3}$$

As a consequence of the δ-function, this interaction is gauge invariant under the spacetime gauge transformation $\phi(\xi) \to U(X(\xi))\phi(\xi)U^{-1}(X(\xi))$ where $U(\xi)$ is defined in (1.1.5). At the boundary, the dynamics of the Grassmanian fields are described by

$$S_\psi = \int_0^1 \psi^\dagger \dot{\psi} d\tau \tag{5.1.4}$$

which leads to the boundary propagator [89]

$$\langle \psi_a^\dagger(\tau_1)\psi_b(\tau_2)\rangle_\psi = \frac{1}{2}\delta_{ab}\mathrm{sign}(\tau_1 - \tau_2) \tag{5.1.5}$$

where the anti-periodic boundary conditions were applied. Consequently, the correlation function of ϕ^R at boundary is

$$\langle \psi^\dagger(1)\phi^R(\xi)\phi^S(\xi')\psi(0)\rangle_\psi = \mathcal{P}(T^R T^S). \tag{5.1.6}$$

To extend the fields to live inside the worldsheet, the generalisation for (5.1.5) is required.

It is straightforward to show that using the modified contact interaction (5.1.2), one can recreate the expression (1.1.35). To demonstrate this, Let first rewriting S_I^ϕ in terms of vertex operators $V_k^{\mu\nu}(\xi)$ as

$$S_I^\phi = \frac{q^2}{4}\int \frac{d^4k}{(2\pi)^4}\int_\Sigma\int_\Sigma d^2\xi d^2\xi' \ \phi^R(\xi)V_k^{\mu\nu}(\xi)V_{-k\ \mu\nu}(\xi')\phi_R(\xi') \tag{5.1.7}$$

where $V_k^{\mu\nu}(\xi)$ was defined in (4.3.61). Similar to the previous chapter, the projection operator $\mathbb{P}_k$ in (4.3.62) was used to write

$$\begin{aligned}\phi^R(\xi)V_k^{\mu\nu}(\xi) =&\phi^R\tilde{V}_k^{\mu\nu}(\xi) + (\partial_a\phi^R)\left(2i\epsilon^{ab}k^{[\mu}\partial_b\mathbb{P}_k(X)^{\nu]}\frac{e^{ik\cdot X}}{k^2}\right)\\ &-\partial_a\left(2i\phi^R\epsilon^{ab}k^{[\mu}\partial_b\mathbb{P}_k(X)^{\nu]}\frac{e^{ik\cdot X}}{k^2}\right)\end{aligned} \tag{5.1.8}$$

where the projected vertex operator $\tilde{V}_k^{\mu\nu}(\xi)$ was defined as (4.3.64). Notice the second term on the right-hand side provides the difference from the Abelian result. Thus, the contact interaction becomes

$$\begin{aligned}S_I^\phi =&\frac{q^2}{4}\Bigg[\int\frac{d^4k}{(2\pi)^4}\int_\Sigma\int_\Sigma d^2\xi d^2\xi' \ \phi^R(\xi)\tilde{V}_k^{\mu\nu}(\xi)\tilde{V}_{-k\ \mu\nu}(\xi')\phi_R(\xi')\\ &+2\int_\Sigma\int_\Sigma d^2\xi d^2\xi' \ \big(\partial_a\phi^R\epsilon^{ab}\partial_b\mathbb{P}_k(X)^\mu\big)(\xi)\frac{e^{ik\cdot(X(\xi)-X(\xi'))}}{k^2}\big(\partial_r\phi_R\epsilon^{rs}\partial_s\mathbb{P}_k(X)_\mu\big)(\xi')\\ &+4\int_\Sigma d^2\xi\oint_{\partial\Sigma}\big(\partial_a\phi^R\epsilon^{ab}\partial_b\mathbb{P}_k(X)^\mu\big)(\xi)\frac{e^{ik\cdot(X(\xi)-X(\xi'))}}{k^2}(d\mathbb{P}_k(X)_\mu\phi_R)(\xi')\\ &+2\int\frac{d^4k}{(2\pi)^4}\oint_{\partial\Sigma}\oint_{\partial\Sigma}(\phi^R d\mathbb{P}_k(X)^\mu)(\xi)\frac{e^{ik\cdot(X(\xi)-X(\xi'))}}{k^2}(d\mathbb{P}_k(X)_\mu\phi_R)(\xi')\Bigg].\end{aligned} \tag{5.1.9}$$

The terms in the second and the third line are not present in the Abelian model. When averaging (5.1.9), only the last term survives due to an appearance of $e^{\pm ikX}$ in the interior of the worldsheet which will be suppressed via the Wick theorem (4.3.69) in the tensionless limit. However, to neglect the second and third line, we assume that the expectations $\langle\phi^R\partial_a\phi^S\rangle_\psi$ and $\langle\partial_a\phi^R\partial_b\phi^S\rangle_\psi$ do not generate terms that spoil the suppression.

Consequently, using (5.1.6), the expectation of S_I^ϕ takes the form

$$\langle S_I^\phi\rangle_{\psi,\Sigma} = \langle S_I\rangle_\Sigma\ \mathcal{P}(T^RT_R) \tag{5.1.10}$$

where $\langle S_I \rangle_\Sigma$ was expressed in (4.3.74). As far as the boundary terms are concerned, if we exponentiate (5.1.10), we would obtain the exponential (5.1.1) possibly with the help of worldsheet supersymmetry to eliminate divergences [8, 9]. Remember that this equality was evaluated by neglecting the contributions of the bulk terms which may lead to the self interactions in the Wilson loop via contractions of the derivatives of ϕ.

According to [85], it was suggested that the functional $n[C_1, C_2]$, which is defined as

$$n[C_1, C_2] = \int_{C_1} \int_{C_2} \delta^2(x_1 - x_2)\epsilon_{ab} dx_1^a dx_2^b = -n[C_2, C_1] \tag{5.1.11}$$

which counts the number of times the two curves intersect (in oriented way), can be used to generalise the path-ordering along a boundary to define inside the worldsheet. It was found that when averaging (5.1.11) over the curve C_1 and C_2, the result coincides with the boundary propagator of the field ψ in (5.1.5) which is

$$\langle n[C_1, C_2] \rangle_{C_1, C_2} = k(b_1 - b_2)/|b_1 - b_2| \tag{5.1.12}$$

where k is a constant and b_i is the end point of the curves C_i on the boundary. The expectation over the curve C is defined as the functional integral

$$\langle \Omega \rangle_C = \int_\Sigma d^2 a \sqrt{h(a)} \left(\frac{1}{Z} \int Dx \; \Omega \; e^{-S[x]} \right) \tag{5.1.13}$$

with

$$S[x] = \frac{1}{2} \int_0^\infty dt h_{rs}(x) \dot{x}^r \dot{x}^s \tag{5.1.14}$$

where h_{rs} is an induced metric along the curve. The curve C is parametrised by t, $t \in [0, \infty)$, with the end points a and b (which is located on the boundary).

By moving the end points b_1 and b_2 inside the worldsheet, $\langle n[C_1, C_2] \rangle_{C_1, C_2}$ can be considered as a continuation of (5.1.5). Unfortunately, this model does not contain the correct structure to obtain the three-gluon vertex of Yang-Mills theory. This argument was explicitly shown in [90] and in which the authors also provide a suggestion on how to use the model obtain that self-interaction vertex.

5.2 The Second Model

As mentioned above, although the first model can incorporate the path-ordering into the worldsheet interior, it does not provide a correct structure to retrieve the self-interaction of the Wilson loop. Therefore, we continue seeking for a better model to describe non-Abelian Wilson loop as a tensionless string with contact interactions.

In order to introduce the Lie algebra-valued fields ϕ^R onto the worldsheet, we need a Lagrangian to describe the dynamics of the new degrees of freedom ϕ^R. This has to be gauge-invariant to preserve the spacetime gauge invariance of the contact interaction (5.1.2) and Weyl invariant to satisfy the usual organising principle of string theories. It also has to generate the extra interactions of non-Abelian gauge theories which are absent from Abelian ones. A candidate for that action is the BF action we introduced earlier in (1.1.44), i.e.

$$S_{\mathrm{BF}}[\phi,\mathcal{A}] = 2\int_{\Sigma} d^2\xi\ \epsilon^{ij}\mathrm{tr}(\phi\mathcal{F}_{ij})$$

with the field strength tensor $\mathcal{F}_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i + q[\mathcal{A}_i,\mathcal{A}_j]$. Remember that both fields ϕ and $\mathcal{A}$ are Lie algebra-valued fields. The worldsheet 1-form $\mathcal{A}$ is intrinsic to the worldsheet and it differs from the actual spacetime gauge field A in the gauge theory whose dynamics we wish to reformulate.

We then define the partition function corresponding to the BF action as

$$Z = \frac{1}{\mathrm{Vol}}\int D\phi D\mathcal{A} e^{-S_{\mathrm{BF}}[\phi,\mathcal{A}]}\mathrm{tr}\Big(\mathcal{P}\Big(e^{-q\oint_C \mathcal{A}\cdot d\xi}\Big)\Big). \tag{5.2.15}$$

where we insert a Wilson loop along the boundary of Σ. To remove all the gauge redundancy, we apply the axial gauge-fixing condition via the insertion

$$1 = \int D\Lambda\delta(\mathbf{n}\cdot\mathcal{A}^{\Lambda})\mathrm{det}\left(\frac{\delta\mathbf{n}\cdot\mathcal{A}^{\Lambda}}{\delta\Lambda}\right) \tag{5.2.16}$$

with a fixed vector $\mathbf{n}$. Therefore, the partition function (5.2.15) takes the form

$$\begin{aligned} Z &= \frac{1}{\mathrm{Vol}}\int D\Lambda D\phi D\mathcal{A}\delta(\mathbf{n}\cdot\mathcal{A}^{\Lambda})\mathrm{det}(\mathbf{n}\cdot\mathcal{D})e^{-S_{\mathrm{BF}}[\phi,\mathcal{A}]}\mathrm{tr}\Big(\mathcal{P}\Big(e^{-q\oint_C \mathcal{A}\cdot d\xi}\Big)\Big) \\ &= \mathcal{N}\int D\phi D\mathcal{A}\delta(\mathbf{n}\cdot\mathcal{A})\mathrm{det}(\mathbf{n}\cdot\mathcal{D})e^{-S_{\mathrm{BF}}[\phi,\mathcal{A}]}\mathrm{tr}\Big(\mathcal{P}\Big(e^{-q\oint_C \mathcal{A}\cdot d\xi}\Big)\Big). \end{aligned} \tag{5.2.17}$$

To obtain the last line, we used the fact that the integrand and the measures are gauge invariant which can then be renamed from $(\phi^{\Lambda},\mathcal{A}^{\Lambda})$ to $(\phi,\mathcal{A})$.

By introducing a source term for the scalar field, one can construct a generating functional as

$$Z[J] = \mathcal{N} \int D\phi D\mathcal{A}\delta(\mathbf{n}\cdot\mathcal{A})\det(\mathbf{n}\cdot\mathcal{D})e^{-S_{\mathrm{BF}}[\phi,\mathcal{A}]+2\int d^2\xi \mathrm{tr}(J\phi)}\mathrm{tr}\big(\mathcal{P}\big(e^{-q\oint_C \mathcal{A}\cdot d\xi}\big)\big). \tag{5.2.18}$$

Integrating out the field ϕ generates the constraint via δ-function as

$$Z[J] = \widetilde{\mathcal{N}} \int D\phi D\mathcal{A}\delta(\mathbf{n}\cdot\mathcal{A})\det(\mathbf{n}\cdot\mathcal{D})\delta(\frac{1}{2}(\epsilon^{ij}\mathcal{F}_{ij} - J))\mathrm{tr}\big(\mathcal{P}\big(e^{-q\oint_C \mathcal{A}\cdot d\xi}\big)\big). \tag{5.2.19}$$

For simplicity, we will content ourselves to consider the worldsheet with the topology of a disk D^2 which is topologically equivalent to the upper half-plane. Accordingly, it is convenient to work in Cartesian coordinates, hence, (5.2.19) taking the form

$$Z[J] = \widetilde{\mathcal{N}} \int D\mathcal{A}\delta(\mathcal{A}_y)\det(\mathcal{D}_y)\delta(\mathcal{F} - \frac{1}{2}J)\mathrm{tr}\big(\mathcal{P}\big(e^{-q\int \mathcal{A}_x dx}\big)\big). \tag{5.2.20}$$

where $\mathcal{F} = \partial_x\mathcal{A}_y - \partial_y\mathcal{A}_x + q[\mathcal{A}_x, \mathcal{A}_y]$ and we chose the reference vector $\mathbf{n}$ to be a unit vector pointing in y-direction.

After integrating out $\mathcal{A}_y$, (5.2.20) becomes

$$Z[J] = \widetilde{\mathcal{N}} \int D\mathcal{A}_x\det(\partial_y)\delta(\partial_y\mathcal{A}_x + \frac{1}{2}J)\mathrm{tr}\big(\mathcal{P}\big(e^{-q\int \mathcal{A}_x dx}\big)\big). \tag{5.2.21}$$

This requires us to solve the constraint, $\partial_y\mathcal{A}_x = -\frac{1}{2}J$. By setting

$$J(\mathbf{x}) = \int d^2\mathbf{x}'\lambda(\mathbf{x}')\delta^2(\mathbf{x} - \mathbf{x}'), \tag{5.2.22}$$

one can obtain the solution as

$$\mathcal{A}_x = \frac{1}{2}\int d^2\mathbf{x}'\lambda(\mathbf{x}')\theta(y' - y)\delta(x' - x). \tag{5.2.23}$$

As a result, the generating function can be written as

$$Z[J] = \widetilde{\mathcal{N}}\mathrm{tr}\Big(\mathcal{P}\Big(\exp\Big[-\frac{q}{2}\int d^2\mathbf{x}'\lambda(\mathbf{x}')\Big]\Big)\Big), \tag{5.2.24}$$

Remember that the Wilson loop is evaluated on the worldsheet boundary, thus $\theta(y' - y) = 1$ as $y = 0$.

Now we will consider expectation value of a product of ϕ on the boundary of the form

$$\langle \phi^{R_1}(x_1)\phi^{R_2}(x_2)\dots\phi^{R_n}(x_n)\rangle_{\phi,\mathcal{A}} = \frac{1}{Z[0]}\frac{\delta^n Z[J]}{\delta\lambda_{R_1}(x_1)\delta\lambda_{R_2}(x_2)\dots\lambda_{R_n}(x_n)}\Bigg|_{J=0}$$
$$= \left(-\frac{q}{2}\right)^n \mathcal{P}\,\mathrm{tr}(T^{R_1}T^{R_2}\dots T^{R_n}). \qquad (5.2.25)$$

The expression (5.2.25) is the exact prescription to reformulate the expectation value of the non-Abelian Wilson loop without self-interactions presented in (1.1.35). This can be seen by evaluating the expectation value of $e^{-\tilde{S}_I^\phi}$

$$\sum_{n=0}^{\infty}\frac{\langle(-\tilde{S}_I^\phi)^n\rangle_{\phi,\mathcal{A},\Sigma}}{n!} =$$
$$\mathcal{P}\mathrm{tr}\sum_{n=0}^{\infty}\frac{q^{2n}}{2^n n!}\prod_{i=1}^{n}\bigg(\int\frac{d^4k_i}{(2\pi)^4}\oint\oint d\mathbb{P}_{k_i}(X)^\mu(\xi)d\mathbb{P}_{k_i}(X)_\mu(\xi')\frac{e^{ik_i\cdot(X(\xi)-X(\xi'))}}{k_i^2}T^{R_i}T^{R_i}\bigg).$$
$$(5.2.26)$$

where $\tilde{S}_I^\phi$ is the re-scaled S_I^ϕ as

$$S_I^\phi = \frac{q^2}{4}\tilde{S}_I^\phi, \qquad (5.2.27)$$

i.e.

$$\tilde{S}_I^\phi = \int_\Sigma d\Sigma_{\mu\nu}(\xi)\phi^R(\xi)\delta^4(X(\xi)-X(\tilde{\xi}))d\Sigma^{\mu\nu}(\tilde{\xi})\phi_R(\tilde{\xi}). \qquad (5.2.28)$$

Note that this calculation was done neglecting the effect of bulk terms in the contact interaction. At the order q^{2n}, the expression (5.2.26) describes a Wilson loop with n pairs of gauge propagators which freely propagate between the boundary.

As a consequence, in this string model, we expect that the expectation value of exponential of the rescaling contact interaction $\tilde{S}_I^\phi$ written as

$$\langle e^{-\tilde{S}_I^\phi}\rangle_{\Sigma,\phi,\mathcal{A}} = \frac{1}{\widetilde{Z}}\int D\phi D\mathcal{A}DXDg\exp\bigg(-S_{\mathrm{P}}[X,g]-\tilde{S}_I^\phi[X,\phi]-S_{\mathrm{BF}}[\phi,\mathcal{A}]\bigg)$$
$$\times\,\mathrm{tr}\Big(\mathcal{P}\Big(e^{-q\oint_C\mathcal{A}\cdot d\xi}\Big)\Big), \qquad (5.2.29)$$

where $\widetilde{Z}$ is a normalisation constant which makes $\langle 1\rangle_{\Sigma,\phi,\mathcal{A}} = 1$, would reproduce the expectation of the non-Abelian Wilson loop.

However, in order to verify if the proposed string model provides a valid description of the non-Abelian Yang-Mills theory or not, we need to find out whether the model is able to reproduce self-interactions of the gauge fields. According to

(5.2.26), it is obvious that no such terms appear when evaluating solely on the boundary. Thus, we need to include the bulk terms into consideration.

Since a contribution of the three gluon vertex in the Wilson loop is first observed at $\mathcal{O}(q^4)$, we need to investigate the expectation of $(\tilde{S}_I^\phi)^2$ in our string model. To do this, we start by rewriting the expression for $\tilde{S}_I^\phi$ in (5.2.28) as

$$\begin{aligned}\tilde{S}_I^\phi = \int \frac{d^4k}{(2\pi)^4}\bigg[&\int_\Sigma\int_\Sigma d^2\xi d^2\xi' \ \phi^R(\xi)\tilde{V}_k^{\mu\nu}(\xi)\tilde{V}_{-k\ \mu\nu}(\xi')\phi_R(\xi') \\ &+ 2\frac{\eta_{RS}}{k^2}(C+B)_k^{R\mu}(C+B)^S_{\mu\,-k}\bigg] \end{aligned} \tag{5.2.30}$$

where we defined the bulk integral $C_k^{R\mu}$ and the boundary integral $B_k^{R\mu}$ as

$$C_k^{R\mu} = \int_\Sigma d^2\xi\big(\partial_i\phi^R\epsilon^{ij}\partial_j\mathbb{P}_k(X)^\mu\big)(\xi)e^{ik\cdot X(\xi)} \tag{5.2.31}$$

and

$$B_k^{R\mu} = \oint_{\partial\Sigma}\big(\phi^R d\mathbb{P}_k(X)^\mu\big)(\xi)e^{ik\cdot X(\xi)} \tag{5.2.32}$$

with the projection operator $\mathbb{P}_k(X)^\mu$ defined in (4.3.62). Omitting the terms containing the projected vertex $\tilde{V}_k^{\mu\nu}$ whose expectation values of such terms vanish due to an appearance of $e^{\pm ik\cdot X}$, $(\tilde{S}_I^\phi)^2$ reads

$$\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\frac{\eta_{PQ}}{k^2}(C+B)_k^{P\mu}(C+B)^Q_{\mu\,-k}\frac{\eta_{RS}}{k'^2}(C+B)_{k'}^{R\nu}(C+B)^S_{\nu\,-k'}. \tag{5.2.33}$$

It is not hard to see that the expectation of the quartic term of the boundary integral, i.e. B^4, corresponds to the Wilson loop with a pair of non-interacting gauge propagators joining two pairs of vertices on the boundary.

To reproduce the Wilson loop with three-point vertex in figure 1.2, we expect the expectation of the expression (5.2.33) to contain three boundary vertices. Therefore, the candidates which potentially provide those vertices are the terms $B\cdot \overbracket{CC}\cdot B$ and $B\cdot\overbracket{CB}\cdot B$ where $A\cdot B \equiv \eta_{RS}A_k^{\mu R}B^S_{\mu\,-k}$ and the linking lines denote Wick's contraction. If we are lucky, the contractions between two integrals would reproduce the third boundary vertex we wish for.

Let us consider the first candidate, i.e. $B\cdot \overset{\sqcap}{CC}\cdot B$ which takes the form

$$
\begin{aligned}
&4\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\frac{\eta_{PQ}}{k^2}\frac{\eta_{RS}}{k'^2}\\
&\times\oint_{\partial\Sigma}\big(\phi^P d\mathbb{P}_k(X)^\mu\big)(\xi_1)e^{ik\cdot X_1}\int_\Sigma d^2\xi_2\big(\partial_i\phi^Q\epsilon^{ij}\partial_j\mathbb{P}_{-k}(X)_\mu\big)(\xi)e^{-ik\cdot X_2}\\
&\times\int_\Sigma d^2\xi_3\big(\partial_k\phi^R\epsilon^{kl}\partial_l\mathbb{P}_{k'}(X)_\nu\big)(\xi)e^{ik'\cdot X_3}\oint_{\partial\Sigma}\big(\phi^S d\mathbb{P}_{-k'}(X)^\nu\big)(\xi_4)e^{-ik'\cdot X_4}.
\end{aligned}\tag{5.2.34}
$$

According to [90], To produce exactly the three point vertex, we would require

$$\langle\partial_i\phi^A(\xi_1)\partial_j\phi^B(\xi_2)\rangle_{\mathcal{A},\phi}=\epsilon_{ij}f^{ABC}\delta^{(2)}(\xi_1-\xi_2)\langle\phi_C\rangle_{\mathcal{A},\phi}. \tag{5.2.35}$$

With this assumption, the expectation of (5.2.34) is written as

$$
\begin{aligned}
\Big\langle &4\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\frac{\eta_{PQ}}{k^2}\frac{\eta_{RS}}{k'^2}\oint_{\partial\Sigma}\big(\phi^P d\mathbb{P}_k(X)^\mu\big)(\xi_1)e^{ik\cdot X_1}\\
&\times\int_\Sigma d^2\xi f^{QRT}\phi_T(\xi)\epsilon^{ij}\partial_i\mathbb{P}_{-k}(X)_\mu\partial_j\mathbb{P}_{k'}(X)_\nu e^{i(k'-k)\cdot X}\\
&\qquad\times\oint_{\partial\Sigma}\big(\phi^S d\mathbb{P}_{-k'}(X)^\nu\big)(\xi_4)e^{-ik'\cdot X_4}\Big\rangle_{\mathcal{A},\phi,\Sigma}.
\end{aligned}\tag{5.2.36}
$$

As $k\cdot\mathbb{P}_k(X)=0$, the integrand in the second line becomes

$$f^{QRT}\int_\Sigma d^2\xi\phi_T(\xi)\epsilon^{ij}\partial_iX_\mu\partial_jX_\nu e^{i(k'-k)\cdot X} \tag{5.2.37}$$

which can be seen as a new vertex operator. By expanding (5.2.37) using a projection of X along $(k'-k)$, there exists the term

$$\frac{-i}{2}f^{QRT}\int_\Sigma d^2\xi\phi_T(\xi)\epsilon^{ij}\frac{1}{(k'-k)^2}(k'-k)_{[\mu}\partial_ie^{i(k'-k)\cdot X}\partial_j\mathbb{P}_{(k'-k)}(X)_{\nu]} \tag{5.2.38}$$

in which we can turn it into an integral along the boundary using an integration by parts which yields

$$\frac{i}{2}f^{QRT}\oint_{\partial\Sigma}d^2\xi\phi_T(\xi)\frac{1}{(k'-k)^2}(k'-k)_{[\mu}d\mathbb{P}_{(k'-k)}(X)_{\nu]}e^{i(k'-k)\cdot X}. \tag{5.2.39}$$

Substituting (5.2.39) to (5.2.36) and relabeling the dummy indices on the momenta, we reproduce the Wilson loop with three gluon vertex (1.1.37). Note that we utilised (5.2.25) to obtain the trace of path-ordered product of the Lie generators.

Unfortunately, the assumption (5.2.35) cannot hold as there is no solution for a two-point function satisfying (5.2.35). This can be seen by the following argument.

Assuming that the product $\phi^A(\xi_1)\phi^B(\xi_2)$ can be expanded in terms of functions which depend only on the separation between the two spatial points when the points are near each other. Therefore to the leading order, the partial derivative at the point ξ_1 is equal to minus the partial derivative at the point ξ_2 of the two-point function as

$$\partial_{\xi_1}\langle\phi^A(\xi_1)\phi^B(\xi_2)\rangle\Big|_{\xi_1\sim\xi_2} = -\partial_{\xi_2}\langle\phi^A(\xi_1)\phi^B(\xi_2)\rangle\Big|_{\xi_1\sim\xi_2} \tag{5.2.40}$$

As a consequence, we can write

$$\partial_{\xi_1^i}\partial_{\xi_2^j}\langle\phi^A(\xi_1)\phi^B(\xi_2)\rangle\Big|_{\xi_1\sim\xi_2} = -\partial_{\xi_1^i}\partial_{\xi_1^j}\langle\phi^A(\xi_1)\phi^B(\xi_2)\rangle\Big|_{\xi_1\sim\xi_2}. \tag{5.2.41}$$

Clearly, the indices i and j on the right hand side is symmetric which contradicts (5.2.35) which is antisymmetric.

The first candidate fails leaving the remaining one to investigate. To evaluate the term $B\cdot \overset{\sqcap}{C}B\cdot B$, it requires to calculate $\langle\phi^R(\xi_1)\partial_i\phi^S(\xi_2)\rangle_{\mathcal{A},\phi}$. We proceed by considering a variation of the expectation of the field ϕ as

$$\begin{aligned}
0 =&\delta_{\mathcal{A}}\langle\phi\rangle_{\mathcal{A},\phi} = -\langle\phi\delta_{\mathcal{A}}S_{\mathrm{BF}}\rangle_{\mathcal{A},\phi} + \langle\phi\delta_{\mathcal{A}}\mathrm{tr}\mathcal{P}(-q\oint_{\partial\Sigma}\mathcal{A}\cdot d\xi)\rangle_{\mathcal{A},\phi}\\
=&-2\langle\phi\int_\Sigma d^2\xi\epsilon^{ij}\mathrm{tr}(\phi\mathcal{D}_i\delta\mathcal{A}_j)\rangle_{\mathcal{A},\phi} - q\langle\phi\mathrm{tr}\mathcal{P}\oint_{\partial\Sigma}\delta\mathcal{A}\cdot d\xi)\rangle_{\mathcal{A},\phi}\\
=&2\langle\phi\int_\Sigma d^2\xi\epsilon^{ij}\mathrm{tr}(\mathcal{D}_i\phi\delta\mathcal{A}_j)\rangle_{\mathcal{A},\phi} + 2\langle\phi\oint_{\partial\Sigma}\mathrm{tr}\phi\delta\mathcal{A}\cdot d\xi\rangle_{\mathcal{A},\phi}\\
&-q\langle\phi\mathrm{tr}\mathcal{P}\oint_{\partial\Sigma}\delta\mathcal{A}\cdot d\xi)\rangle_{\mathcal{A},\phi}.
\end{aligned} \tag{5.2.42}$$

We can consider the variation of the gauge field along the boundary and inside the worldsheet separately. The relation of the expectation value of the boundary terms is nothing but a reproduction of (5.2.25) we found earlier. By functionally differentiating the bulk term with respect to the gauge field, we obtain

$$\langle\phi^A(\xi_1)\mathcal{D}_i\phi^B(\xi_2)\rangle_{\mathcal{A},\phi} = 0 \tag{5.2.43}$$

which implies the relation

$$\langle\phi^A(\xi_1)\partial_i\phi^B(\xi_2)\rangle_{\mathcal{A},\phi} = -qf^{BCD}\langle\phi^A(\xi_1)\mathcal{A}_i^C(\xi_2)\phi^D(\xi_2)\rangle_{\mathcal{A},\phi}. \tag{5.2.44}$$

We can then evaluate the right-hand side of (5.2.44) by introducing the source J as

$$\langle\phi^A(\xi_1)\mathcal{A}_i^C(\xi_2)\phi^D(\xi_2)\rangle_{\mathcal{A},\phi} = \frac{\delta}{\delta J(\xi_1)_A}\frac{\delta}{\delta J(\xi_2)_D}\langle\mathcal{A}_i^C(\xi_2)\rangle_{\mathcal{A},\phi}\Big|_{J=0} \tag{5.2.45}$$

where the partition function with the source J was stated in (5.2.18).

The expectation of the gauge field with the source J is written as

$$\langle \mathcal{A}_i^R \rangle_{\mathcal{A},\phi} = \mathcal{N} \int D\phi D\mathcal{A}\delta(\mathbf{n}\cdot\mathcal{A})\det(\mathbf{n}\cdot\mathcal{D})A_i^R e^{-S_{\mathrm{BF}}[\phi,\mathcal{A}]+2\int d^2\xi \mathrm{tr}(J\phi)}\mathrm{tr}\big(\mathcal{P}\big(e^{-q\oint_C \mathcal{A}\cdot d\xi}\big)\big). \tag{5.2.46}$$

For convenience, we will implement the calculation in the upper half-plane together with setting the reference vector $\mathbf{n}$ to be pointed in y direction. Therefore, (5.2.46) becomes

$$\begin{aligned}\langle \mathcal{A}_i^R \rangle_{\mathcal{A},\phi} &= \mathcal{N} \int D\phi D\mathcal{A}_x \det(\partial_y)\delta_i^x \mathcal{A}_x^R e^{-S_{\mathrm{BF}}[\phi,\mathcal{A}]+2\int d^2\xi \mathrm{tr}(J\phi)}\mathrm{tr}\big(\mathcal{P}\big(e^{-q\int \mathcal{A}_x dx}\big)\big)\\ &= \mathcal{N} \int D\mathcal{A}_x \det(\partial_y)\delta_i^x \mathcal{A}_x^R \delta(\mathcal{F} - \frac{1}{2}J)\mathrm{tr}\big(\mathcal{P}\big(e^{-q\int \mathcal{A}_x dx}\big)\big)\end{aligned} \tag{5.2.47}$$

where $\mathcal{F} = \partial_x\mathcal{A}_y - \partial_y\mathcal{A}_x + q[\mathcal{A}_x, \mathcal{A}_y]$. Again, if the source J takes the same form as (5.2.22), we then have the solution for $\mathcal{A}_x$ as (5.2.23). Accordingly, the expectation of the gauge field takes the form

$$\langle \mathcal{A}_i^R(\mathbf{x}) \rangle_{\mathcal{A},\phi} = \frac{1}{2}\delta_i^x \int d^2\mathbf{x}'\lambda^R(\mathbf{x}')\theta(y'-y)\delta(x'-x)\,\mathrm{tr}\mathcal{P}e^{-\frac{q}{2}\int d^2\mathbf{x}''\lambda(\mathbf{x}'')}. \tag{5.2.48}$$

Taking a double functional derivative with respect to λ, we obtain

$$\begin{aligned}\frac{\delta}{\delta\lambda(\mathbf{x}_1)_A}\frac{\delta}{\delta\lambda(\mathbf{x}_2)_D}\langle \mathcal{A}_i^C(\mathbf{x}_2) \rangle_{\mathcal{A},\phi} &= \frac{1}{2}\delta_i^x \frac{\delta}{\delta\lambda(\mathbf{x}_1)_A}\bigg(\delta^{CD}\delta(0)\mathrm{tr}\mathcal{P}e^{-\frac{q}{2}\int d^2\mathbf{x}''\lambda(\mathbf{x}'')}\\ &-\frac{q}{2}\int d^2\mathbf{x}'\lambda^C(\mathbf{x}')\theta(y'-y_2)\delta(x'-x_2)\mathrm{tr}\mathcal{P}T^D e^{-\frac{q}{2}\int d^2\mathbf{x}''\lambda(\mathbf{x}'')}\bigg).\end{aligned} \tag{5.2.49}$$

However, when contracting the above equation with the structure constant f^{BCD} as (5.2.44), the first term on the right-hand side vanishes. Thus, omitting the first term, the expression for (5.2.49) at $\lambda = 0$ becomes

$$-\frac{q}{4}\delta_i^x\delta^{AC}\theta(y_1-y_2)\delta(x_1-x_2)\,\langle\phi^D(\mathbf{x}_2)\rangle_{\mathcal{A},\phi}. \tag{5.2.50}$$

The above contraction allows the functional integration of X to be implemented inside the bulk, thus, it gets suppressed by the Wick's contraction of $e^{\pm ik\cdot X}$. Consequently, the term $B\cdot CB\cdot B$ does not contribute to the theory.

5.3 Potential Modification to the Second Model

As a result, the proposed string model lacks the correct structure to produce the self-interaction vertices, thus, this cannot be considered as a true non-Abelian generalisation of the string model with contact interactions. However, there is always light at the end of the tunnel. We noticed a hint on how the self-interaction contributions might arise in the theory by introducing the gauge field $\mathcal{A}$ into the interior of the worldsheet. To see this, we introduce a new bulk vertex as

$$\widetilde{C}^R_{\mu\, k} = \int_\Sigma d^2\xi \mathcal{A}^R_i \epsilon^{ij} \partial_j X_\mu e^{ik\cdot X} \tag{5.3.51}$$

and we will find that the Wilson loop with three-point vertex in (1.2) can be obtained from the expectation

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \frac{\eta_{PQ}}{k^2} \frac{\eta_{RS}}{k'^2} \langle B^{P\mu}_k \widetilde{C}^Q_{\mu\, -k} \widetilde{C}^R_{\nu\, k'} B^{S\nu}_{-k'} \rangle_{\mathcal{A},\phi,\Sigma}. \tag{5.3.52}$$

Consider the product of the new bulk vertices in (5.3.52) which is

$$\widetilde{C}^Q_{\mu\, -k} \widetilde{C}^R_{\nu\, k'} = \int \int_\Sigma d^2\xi d^2\xi' \mathcal{A}^Q_i(\xi) \mathcal{A}^R_k(\xi') \epsilon^{ij} \epsilon^{kl} \partial_j X_\mu(\xi) \partial_l X_\nu(\xi') e^{-ik\cdot X(\xi)+ik'\cdot X(\xi')}. \tag{5.3.53}$$

If the two-point function of the worldsheet gauge field is

$$\langle \mathcal{A}^A_i(\xi_1) \mathcal{A}^B_j(\xi_2) \rangle_{\mathcal{A}} \sim f^{ABC} \phi_C \epsilon_{ij} \delta^{(2)}(\xi_1 - \xi_2) \tag{5.3.54}$$

where $\sim$ denotes that this is satisfied upto some factors plus other irrelevant terms, the expectation with respect to the gauge field of (5.3.53) becomes

$$\langle \widetilde{C}^Q_{\mu\, -k} \widetilde{C}^R_{\nu\, k'} \rangle_{\mathcal{A}} \sim f^{QRT} \int_\Sigma d^2\xi \phi_T(\xi) \epsilon^{ij} \partial_i X_\mu(\xi) \partial_j X_\nu(\xi) e^{i(k'-k)\cdot X(\xi)} \tag{5.3.55}$$

which is similar to (5.2.37). Therefore we can repeat the same calculation as previously discussed to re-express (5.3.52) in the form of (1.1.37).

In fact, we will see later in the next chapter that the relation (5.3.54) holds for the general Lie algebras. To be more precise, the expectation of the worldsheet gauge propagator is

$$\langle \mathcal{A}^A_i(\xi_1) \mathcal{A}^B_j(\xi_2) \rangle_{\mathcal{A}} \sim \Theta^{AB} \epsilon_{ij} \delta^{(2)}(\xi_1 - \xi_2) \tag{5.3.56}$$

where $f^{ABC}\phi_C$ is a part of the function Θ^{AB} in which we will define and discuss in the next chapter.

With this hint, we propose a further modification to the contact interaction (5.2.28) by adding the worldsheet gauge field $\mathcal{A}$ into the worldsheet as

$$\tilde{S}_I^{\phi,\mathcal{A}}[X,\phi,\mathcal{A}] = \int_\Sigma d\Sigma_{\mu\nu}(\xi)d\Sigma^{\mu\nu}(\tilde{\xi})\mathrm{tr}\Big(\phi(\xi)\langle\mathcal{W}_{C_1}(\xi,\tilde{\xi})\rangle_{C_1}\delta^4(X(\xi)-X(\tilde{\xi})) \times \phi(\tilde{\xi})\langle\mathcal{W}_{C_2}(\tilde{\xi},\xi)\rangle_{C_2}\Big) \tag{5.3.57}$$

where

$$\mathcal{W}_C(\xi_1,\xi_2) = \mathcal{P}\exp(-q\int_C \mathcal{A}\cdot d\omega) \tag{5.3.58}$$

is a Wilson line along an arbitrary curve C whose end points are ξ_1 and ξ_2. To avoid picking an arbitrary path, we will average over all possible paths using the approach [85]. The expectation value over all the paths C is given by

$$\langle\Omega\rangle_C = \mathcal{N}\int dx\Omega e^{-S[x]} \tag{5.3.59}$$

with

$$S[x] = \frac{1}{2}\int_0^\infty dt h_{rs}(x)\dot{x}^r\dot{x}^s. \tag{5.3.60}$$

Note that the above expression is similar to the functional integral (5.1.13) except for the fact that the two end points are fixed.

One can see that, at the leading order, the newly proposed interaction (5.3.57) reduces to (5.1.2) when expanding $\mathcal{W}_C$ perturbatively. Unlike (5.1.2), the modified model (5.3.57) enjoys the gauge symmetry at the worldsheet level. However, whether or not the new model provides a true description for non-Abelian Yang-Mills theory is not yet known which requires further investigations.

Chapter 6

Effective Lagrangian for Non-Abelian Two-Dimensional Topological Field Theory

Over the past few decades, the study of topological field theories has been important for both mathematics and physics. The key feature of the theories is that observables depend only on the global structure of the space where the theories are defined. Topological field theories can be broadly categorized into two classes, namely Schwarz-type and Witten-type. Well-known examples of the Schwarz-type theories are the three-dimensional Chern-Simons model [91] as well as BF theories [92, 93]. A representative for the latter class is topological Yang-Mills theory [13, 94].

BF theories are the only known topological Schwarz-type theories which can be extended to any arbitrary dimension of spacetime. They can be considered as a generalization of Chern-Simons theory. The theories contribute greatly in many areas in physics such as theory of gravity [95–101] and quite recently, in condensed matter physics [102–108]. Moreover, in our work's perspective, the BF action (1.1.41) also provides a candidate to describe the dynamics of the additional quantities ϕ on the string worldsheet. Although it is not certain whether our string model (5.2.29) is the true non-abelian generalisation of [8] as there appears issues relating to reproduction of self-interactions, it still worth exploring aspects towards this theory.

We would like to devote this chapter to investigate an effective theory for $2D$

non-Abelian topological BF theory. By doing so, the non-Abelian gauge fields are integrated out from the BF action (1.1.45) to obtain an effective theory containing solely scalar fields. Expressions for the $SU(2)$ and $SU(3)$ effective actions will be explicitly stated. Before continuing through the following sections, it is useful for the reader to have a review on the general background of $2D$ non-Abelian BF theory in the section 1.1.3.

6.1 $SU(2)$ Effective BF Theory

We begin our calculation with the simplest model for the non-Abelian two-dimensional topological field theory, i.e. the BF theory for $SU(2)$. The partition function for this theory is defined as

$$Z = \frac{1}{\text{Vol}} \int D\phi D\mathcal{A}\; e^{-S[\phi,\mathcal{A}]} \tag{6.1.1}$$

where $S[\phi, \mathcal{A}]$ is expressed in (1.1.45). The functional integral is divided by the volume of the gauge symmetry which is denoted by Vol.

To obtain an effective theory for the scalar field ϕ, the gauge field $\mathcal{A}$ needs to be integrated out. For that purpose, we express all fields in terms of a set of orthonormal bases in Lie vector space, i.e. $\hat{\phi}$, $\hat{E}_+$, and $\hat{E}_-$, as

$$\phi^A = \varphi\hat{\phi}^A \qquad \text{and} \qquad \mathcal{A}_i^A = \chi_i\hat{\phi}^A + a_i^+\hat{E}_+^A + a_i^-\hat{E}_-^A. \tag{6.1.2}$$

Note that these bases are ξ-dependent. They are defined throughout the manifold point by point. Obviously, we have chosen a unit vector $\hat{\phi}$ to align in the direction of ϕ at each point. In terms of the usual cross products, the ξ-dependent bases give the following relations:

$$\hat{\phi} \times \hat{E}_+ = \hat{E}_+, \quad \hat{E}_+ \times \hat{E}_- = \hat{\phi}, \quad \hat{E}_- \times \hat{\phi} = \hat{E}_-. \tag{6.1.3}$$

Remember that the cross product is implemented in the Lie vector space which relates to the usual commutation relations.

Substituting (6.1.2) into (1.1.45), the action takes the form

$$S[\phi, \mathcal{A}] = \int_{\mathcal{M}} d^2\xi \left(2iq\varphi\, a_i^+ a_j^- - 2\partial_i\phi_A\chi_j\hat{\phi}^A - 2\partial_i\phi_A a_j^+\hat{E}_+^A - 2\partial_i\phi_A a_j^-\hat{E}_-^A\right)\epsilon^{ij}. \tag{6.1.4}$$

To obtain the first term, the relations (6.1.3) were utilised. Note that the structure constant f^{ABC} is equal to ϵ^{ABC} for $SU(2)$.

Rewriting all the fields using (6.1.2), the measure $D\mathcal{A}$ now turns into $D\chi Da^+ Da^-$. Integrating out χ would generate a constraint via the Dirac delta function as

$$\begin{aligned}\int D\chi_i \exp\left(\int_{\mathcal{M}} d^2\xi\ 2\hat{\phi}^A \partial_i \phi_A \chi_j \epsilon^{ij}\right) &= \int D\chi_i \exp\left(\int_{\mathcal{M}} d^2\xi\ \frac{1}{\varphi}\partial_i \varphi^2 \chi_j \epsilon^{ij}\right) \\ &= \mathcal{N} \prod_{\forall \xi \in \mathcal{M}} \varphi^2 \delta^{(2)}(\underline{\partial}\varphi^2) = \mathcal{N} \prod_{\forall \xi \in \mathcal{M}} \delta^{(2)}(\underline{\partial}\varphi).\end{aligned} \tag{6.1.5}$$

To obtain the first line, the relation (6.1.2) was used. This constraint means φ^2 (equivalently $|\phi|^2$) is constant throughout the space $\mathcal{M}$.

To proceed with the path integration with respect to the field a_i^α with $\alpha = \pm$, it is better to change the spacetime coordinates ξ^1 and ξ^2 into complex coordinates which are defined by

$$z = \xi^1 + i\xi^2 \qquad \text{and} \qquad \bar{z} = \xi^1 - i\xi^2. \tag{6.1.6}$$

In these new coordinates, the field a_i^α becomes complex fields b^α where

$$b^\alpha = \frac{1}{2}(a_1^\alpha - i a_2^\alpha) \qquad \text{and} \qquad \bar{b}^\alpha = \frac{1}{2}(a_1^\alpha + i a_2^\alpha). \tag{6.1.7}$$

Therefore, the path integral (6.1.1) takes the form

$$Z = \frac{1}{\text{Vol}} \int D\phi D b D\bar{b} \prod_{\forall \xi \in \mathcal{M}} \delta^{(2)}(\underline{\partial}\varphi)\ e^{-S[\phi, b, \bar{b}]} \tag{6.1.8}$$

where

$$S[\phi, b, \bar{b}] = \int_{\mathcal{M}} d^2 z \left(-\bar{b}^\alpha 2iq\varphi \epsilon_{\alpha\beta} b^\beta + 2\bar{\partial}\phi_A \hat{E}^A_\alpha b^\alpha - 2\partial \phi_A \hat{E}^A_\alpha \bar{b}^\alpha \right). \tag{6.1.9}$$

We can then use the Gaussian integration formula to integrate out the complex field b,

$$\int D b D\bar{b}\ e^{-\int d^2 z(-\bar{b}^\alpha M_{\alpha\beta} b^\beta + \bar{J}_\alpha b^\alpha + J_\alpha \bar{b}^\alpha)} = \mathcal{N}_0 \frac{e^{-\int d^2 z (\bar{J}_\alpha (M^{-1})^{\alpha\beta} J_\beta}}{\prod_{\forall \xi} \det(M)}. \tag{6.1.10}$$

According to (6.1.9), it is not hard to see that

$$M_{\alpha\beta} = 2iq\varphi\epsilon_{\alpha\beta}, \quad J_\alpha = -2\partial\phi_A \hat{E}^A_\alpha, \quad \text{and} \quad \bar{J}_\alpha = 2\bar{\partial}\phi_A \hat{E}^A_\alpha. \tag{6.1.11}$$

Using the fact that $\epsilon^{ij}\epsilon_{ij} = 2$, the inverse and the determinant of the matrix M are

$$(M^{-1})^{\alpha\beta} = \frac{-i}{4q\varphi}\epsilon^{\alpha\beta}, \qquad \text{and} \qquad \det(M) = -4q^2\varphi^2. \tag{6.1.12}$$

Consequently, we can express the path integral as

$$Z \sim \int D\phi \prod_{\forall\xi\in\mathcal{M}} \frac{-i}{(q\varphi)^2}\delta^{(2)}(\underline{\partial}\varphi)\ \exp\left[-\int_{\mathcal{M}} d^2z \frac{i}{q\varphi}\bar{\partial}\phi_A\partial\phi_B(\hat{E}^A_\alpha\epsilon^{\alpha\beta}\hat{E}^B_\beta)\right]. \tag{6.1.13}$$

We can rewrite the term $\hat{E}^A_\alpha\epsilon^{\alpha\beta}\hat{E}^B_\beta$ as

$$\hat{E}^A_\alpha\epsilon^{\alpha\beta}\hat{E}^B_\beta = \hat{E}^A_+\hat{E}^B_- - \hat{E}^B_+\hat{E}^A_- = (\hat{E}_+ \times \hat{E}_-)_C\epsilon^{ABC} \tag{6.1.14}$$

which can be evaluated using (6.1.3). As a result, the cross product on the right-hand side is simply the unit vector $\hat{\phi}$. Thus, the effective action for two-dimensional SU(2) BF theory can be written as

$$\int_{\mathcal{M}} d^2z \frac{i}{q|\phi|^2}\bar{\partial}\phi_A\partial\phi_B\phi_C\epsilon^{ABC} \tag{6.1.15}$$

or equivalently in the (ξ^1, ξ^2) coordinates as

$$\int_{\mathcal{M}} d^2\xi \frac{i}{2q|\phi|^2}\partial_i\phi_A\partial_j\phi_B\epsilon^{ij}\phi_C\epsilon^{ABC}. \tag{6.1.16}$$

Now, let us give an interpretation of the effective action (6.1.16). The effective action can be seen as a winding number (up to a constant). To see this, it needs to be noted that the unit vector $\hat{\phi}(\xi)$ maps a point on the manifold $\mathcal{M}$ into a point on S^2, i.e. $\hat{\phi} : \mathcal{M} \to S^2$. Furthermore, the integrand of the action (6.1.16),

$$\frac{1}{2}\partial_i\hat{\phi}_A\partial_j\hat{\phi}_B\epsilon^{ij}\hat{\phi}_C\epsilon^{ABC}, \tag{6.1.17}$$

is the area element on the target space S^2. This can be seen as follows: the variations of the manifold coordinates $\delta\xi^1$ and $\delta\xi^2$ correspond to two infinitesimal tangent vectors $\delta\xi^1\partial_1\hat{\phi}$ and $\delta\xi^2\partial_2\hat{\phi}$ on S^2. The cross product of these two vectors has direction $\hat{\phi}$ and magnitude δA. Consequently, the triple product, $\delta\xi^1\delta\xi^2(\partial_1\hat{\phi} \times \partial_2\hat{\phi}) \cdot \hat{\phi}$, is basically an infinitesimal area on the target space S^2 as claimed.

The integration over all manifold coordinates ξ of the integrand (6.1.17) yields the total area of the unit sphere times an integer corresponding to the winding number n as

$$\frac{1}{2}\int_{S^2} d^2\xi\partial_i\hat{\phi}_A\partial_j\hat{\phi}_B\epsilon^{ij}\hat{\phi}_C\epsilon^{ABC} = 4\pi n. \tag{6.1.18}$$

Note that the above term is proportional to the effective action (6.1.16) as the magnitude of the field ϕ, $|\phi|$, is constant due to the constraint (6.1.5).

6.2 Partition Function for $SU(2)$ Yang-Mills Theory on Sphere

On a sphere, the partition function for $SU(N)$ Yang-Mills theory is

$$Z_{\mathrm{YM}}(A) = \sum_R (d_R)^2 \exp\big(- e_{\mathrm{YM}}^2 A C_2(R)\big) \tag{6.2.19}$$

which is directly from (1.1.39). A is an area of the sphere and R is an irreducible representation of $SU(N)$. d_R and $C_2(R)$ are the dimension and the quadratic Casimir of the representation R respectively. For $SU(2)$, the representation R is characterized by a positive half-integer l. This yields

$$d_R = 2l+1 \qquad \text{and} \qquad C_2(R) = l(l+1). \tag{6.2.20}$$

Therefore, the partition function takes the form

$$Z_{\mathrm{YM}}(A) = \sum_{m=0}^{\infty} (m+1)^2 \exp\Big(- \frac{e_{\mathrm{YM}}^2}{4} A((m+1)^2 - 1)\Big) \tag{6.2.21}$$

where $l = m/2$.

Our purpose in this section is to re-obtain the partition function (6.2.21) by using the effective $SU(2)$ BF theory found in the previous section. To do this, we need to be more careful in integrating out the complex b field in (6.1.8) as one may notice that φ^2 in the determinant (6.1.12) will apparently get cancelled out by the Jacobian of the measure $D\phi = \varphi^2 d\varphi d\Omega$ with Ω denoting a direction of the scalar field. If the previous statement were true, we would not get the prefactor in the formula (6.2.21). This implies that the cancellation needs to be partial. It is due to the difference in the degrees of freedom between the scalar field and the vector field.

To put it into clearer perspective, let us evaluate the $SU(2)$ partition function, i.e.

$$Z = \frac{1}{\mathrm{Vol}} \int D\phi D\chi D\bar{\chi} Db D\bar{b} \exp\big(- S[\phi,\chi,\bar{\chi}] - S[\phi,b,\bar{b}]\big) \tag{6.2.22}$$

where

$$S[\phi,\chi,\bar{\chi}] = 2\int_{\mathcal{M}} d^2 z \hat{\phi}_A (\partial \phi^A \bar{\chi} - \bar{\partial}\phi^A \chi) \tag{6.2.23}$$

and $S[\phi, b, \bar{b}]$ is expressed as (6.1.9). We then expand all the fields in terms of eigenfunctions of the scalar Laplacian,

$$\nabla^2 u_\lambda = \lambda u_\lambda \tag{6.2.24}$$

where λ is an eigenvalue of the eigenfunction u_λ. Therefore, the expression for the real scalar field φ is

$$\varphi = \sum_{\lambda \neq 0} c_\lambda u_\lambda + \varphi_0 \tag{6.2.25}$$

where the zero mode term $\varphi_0 = c_0 u_0$ and those for the complex vector fields are

$$b^\alpha = \sum_{\lambda \neq 0} e^\alpha_\lambda \partial u_\lambda, \qquad \bar{b}^\alpha = \sum_{\lambda \neq 0} \bar{e}^\alpha_\lambda \bar{\partial} u_\lambda \tag{6.2.26}$$

$$\chi = \sum_{\lambda \neq 0} f_\lambda \partial u_\lambda, \qquad \bar{\chi} = \sum_{\lambda \neq 0} \bar{f}_\lambda \bar{\partial} u_\lambda. \tag{6.2.27}$$

Note that there is no zero mode expansion for the vector fields as $\partial u_0 = 0$ and u_λ forms a complete set of orthonormal basis satisfying

$$\int d^2\xi \sqrt{g} u_\lambda(\xi) u_{\lambda'}(\xi) = \delta_{\lambda\lambda'} \qquad \text{and} \qquad \sqrt{g} \sum_\lambda u_\lambda(\xi) u_\lambda(\xi') = \delta^{(2)}(\xi - \xi'). \tag{6.2.28}$$

Now, let first take a look at the integral

$$\int D\chi D\bar{\chi} \exp\big(-S[\phi, \chi, \bar{\chi}]\big). \tag{6.2.29}$$

By using the basis expansions, the integral (6.2.29) takes the form

$$\int |J_1| \prod_\lambda df_\lambda d\bar{f}_\lambda \exp\left(2 \int d^2z \sum_{\lambda,\lambda'} c_\lambda (\partial u_\lambda \bar{\partial} u_{\lambda'} \bar{f}_{\lambda'} - \bar{\partial} u_\lambda \partial u_{\lambda'} f_{\lambda'})\right) \tag{6.2.30}$$

where J_1 is the Jacobian determinant when changing variables from χ and $\bar{\chi}$ to f_λ and $\bar{f}_\lambda$. Therefore, it can be computed by

$$J_1 = \det\left(\frac{\delta(\chi, \bar{\chi})}{\delta(f_\lambda, \bar{f}_\lambda)}\right) \equiv \det(M) \tag{6.2.31}$$

The determinant of the matrix can be evaluated from the relation

$$\det(M) = \sqrt{\det(M^\dagger M)}. \tag{6.2.32}$$

According to (6.2.27), $\frac{\delta\chi(z)}{\delta f_\lambda} = \partial u_\lambda(z)$ and $\frac{\delta\bar{\chi}(z)}{\delta f_\lambda} = \bar{\partial} u_\lambda(z)$. Therefore,

$$J_1 = \sqrt{\det\begin{pmatrix} \int d^2z \bar{\partial} u_\lambda \partial u_{\lambda'} & 0 \\ 0 & \int d^2z \bar{\partial} u_\lambda \partial u_{\lambda'} \end{pmatrix}} = \prod_\lambda \lambda, \tag{6.2.33}$$

where (6.2.28) was utilised to obtain the last expression and the product is over the non-zero eigenvalues.

When applying the completeness relation (6.2.28) to the exponent of (6.2.30), it is not hard to see that the integral becomes

$$\begin{aligned}
\int \prod_{\lambda} (-2i\lambda) d(\mathrm{Re}(f_\lambda)) d(\mathrm{Im}(f_\lambda)) \exp\big(4ic_\lambda \lambda \mathrm{Im}(f_\lambda)\big) \\
= \int \prod_{\lambda} d(\mathrm{Re}(f_\lambda))(-4\pi i\lambda)\delta(4c_\lambda\lambda) \\
= \prod_{\lambda} \mathrm{Vol}(\mathrm{Re}(f_\lambda))(-\pi i\delta(c_\lambda)).
\end{aligned} \tag{6.2.34}$$

Similar to the expression (6.1.5), the above term provides a constraint on the theory via the Dirac delta function $\delta(c_\lambda)$ requiring the modulus of the scalar field φ to be constant, i.e. $\varphi = \varphi_0$, throughout the space.

The volume of the real number, $\mathrm{Vol}(\mathrm{Re}(f_\lambda))$, can be cancelled with the volume of the gauge symmetry in (6.2.22). To see this, let us apply a particular choice of gauge-fixing to our calculation. We consider a gauge condition that makes the direction of the scalar field, $\hat{\phi}$, constant everywhere except for an infinitesimal region. After this gauge has been applied, there is left the residual gauge symmetry which does not alter the direction $\hat{\phi}$.

Expanding an infinitesimal gauge transformation parameter ω as

$$\omega = \omega^\phi \hat{\phi} + \omega^+ \hat{E}_+ + \omega^- \hat{E}_- \tag{6.2.35}$$

where all components are real, the gauge transformation of the scalar field (1.1.5) implies that the residual symmetry has $\omega^\pm = 0$. Now, let us investigate the effect of this residual symmetry on the gauge field $\mathcal{A}$ where $\mathcal{A}$ takes the form

$$\mathcal{A} = \chi\hat{\phi} + b^+ \hat{E}_+ + b^- \hat{E}_-. \tag{6.2.36}$$

According to (1.1.5), a variation of the gauge field with respect to the residual gauge transformation is

$$\begin{aligned}
\delta_\omega \mathcal{A} &= \partial\omega + q[\mathcal{A}, \omega] \\
&= -i\partial\omega^\phi \hat{R} - i\omega^\phi \partial\hat{R} + q\omega^\phi \left(\frac{1}{\sqrt{2}}(b^+ - b^-)\hat{B}_1 + \frac{i}{\sqrt{2}}(b^+ + b^-)\hat{B}_2 \right)
\end{aligned} \tag{6.2.37}$$

where we have re-defined the bases to be

$$\hat{R} = i\hat{\phi}, \qquad \hat{B}_1 = \frac{1}{\sqrt{2}}(\hat{E}_+ + \hat{E}_-), \qquad \hat{B}_2 = \frac{-i}{\sqrt{2}}(\hat{E}_+ - \hat{E}_-). \tag{6.2.38}$$

Note that these bases resemble a set of ordinary unit vectors in three-dimensional sphere in which they obey the following algebras;

$$[\hat{R}, \hat{B}_1] = \hat{B}_2, \qquad [\hat{B}_2, \hat{R}] = \hat{B}_1, \qquad [\hat{B}_1, \hat{B}_2] = \hat{R}. \tag{6.2.39}$$

As a result, if the sphere is characterised by the usual polar angle α and azimuthal angle θ, the variation (6.2.37) becomes

$$\delta_\omega \mathcal{A} = -i\partial\omega^\phi \hat{R} + \omega^\phi \left(i\frac{\partial \alpha}{\partial z}\sin\theta + \frac{q}{\sqrt{2}}(b^+ - b^-) \right)\hat{B}_1 - i\omega^\phi \left(\frac{\partial \theta}{\partial z} - \frac{q}{\sqrt{2}}(b^+ + b^-) \right)\hat{B}_2. \tag{6.2.40}$$

Comparing the result to the actual variation of the gauge field (6.2.36), it implies that a variation of the field χ is in the residual gauge orbit when it is real. Remember that the variation of the field χ is equivalent to that of the function f_λ according to (6.2.27). Consequently, $\mathrm{Vol}(\mathrm{Re}(f_\lambda))$ is the residual gauge volume as claimed.

Moving on to the next integral to consider, the Gaussian functional integral of the vector fields b^α in the partition function (6.2.22) can be written in terms of scalar functions e_λ and $\bar{e}_\lambda$ according to the Laplacian eigenfunction expansion (6.2.26) as

$$|J_2| \int \prod_\lambda de_\lambda d\bar{e}_\lambda e^{-S[e,\bar{e}]} \tag{6.2.41}$$

where J_2 is the Jacobian determinant resulted from the change of variables from b and $\bar{b}$ into e and $\bar{e}$. The action $S[e, \bar{e}]$ is defined as

$$\begin{aligned} S[e,\bar{e}] = \int_{\mathcal{M}} d^2z \Bigg(&- 2iq\varphi_0 \sum_{\lambda,\lambda'} \bar{e}^\alpha_\lambda \epsilon_{\alpha\beta} e^\beta_{\lambda'} \bar{\partial} u_\lambda \partial u_{\lambda'} \\ &+ 2\varphi_0 \bar{\partial}\hat{\phi}_A \hat{E}^A_\alpha \sum_\lambda e^\alpha_\lambda \partial u_\lambda - 2\varphi_0 \partial\hat{\phi}_A \hat{E}^A_\alpha \sum_\lambda \bar{e}^\alpha_\lambda \bar{\partial} u_\lambda \Bigg). \end{aligned} \tag{6.2.42}$$

To obtain the above action, the constraint (6.2.34) is applied making the length of ϕ constant. The first term of the action (6.2.42) vanishes when $\lambda \neq \lambda'$ due to the completeness relation (6.2.28) which yields

$$\begin{aligned} S[e,\bar{e}] =& 2iq\varphi_0 \sum_\lambda \lambda \bar{e}^\alpha_\lambda \epsilon_{\alpha\beta} e^\beta_\lambda \\ &+ 2\varphi_0 \int d^2z \Big(\bar{\partial}\hat{\phi}_A \hat{E}^A_\alpha \sum_\lambda e^\alpha_\lambda \partial u_\lambda - \partial\hat{\phi}_A \hat{E}^A_\alpha \sum_\lambda \bar{e}^\alpha_\lambda \bar{\partial} u_\lambda \Big). \end{aligned} \tag{6.2.43}$$

The Jacobian determinant J_2 is

$$J_2 = \det\left(\frac{\delta(b(z), \bar{b}(z))}{\delta(e_\lambda, \bar{e}_\lambda)}\right). \tag{6.2.44}$$

By using the relation (6.2.32) and fact that $\frac{\delta b^\alpha(z)}{\delta e^\beta_\lambda} = \delta^\alpha_\beta \partial u_\lambda(z)$ and $\frac{\delta \bar{b}^\alpha(z)}{\delta \bar{e}^\beta_\lambda} = \delta^\alpha_\beta \bar{\partial} u_\lambda(z)$, one can obtain

$$J_2 = \sqrt{\det\begin{pmatrix} \int d^2z \bar{\partial} u_\lambda \partial u_{\lambda'} \delta^{\alpha\beta} & 0 \\ 0 & \int d^2z \bar{\partial} u_\lambda \partial u_{\lambda'} \delta^{\alpha\beta} \end{pmatrix}} = \prod_\lambda \lambda^2. \tag{6.2.45}$$

where (6.2.28) was utilised and again the product is over non-zero eigenvalues.

We can then calculate the Gaussian integral (6.2.41) over the complex field e_λ and $\bar{e}_\lambda$ using (6.1.10). It becomes

$$\int \prod_\lambda \lambda^2 de_\lambda d\bar{e}_\lambda e^{-S[e,\bar{e}]} = \frac{\exp(-S_{\text{eff}}[\varphi_0.n])}{\prod_\lambda -(2q\varphi_0)^2} \tag{6.2.46}$$

where

$$S_{\text{eff}}[\varphi_0, n] = i\frac{\varphi_0}{q} \int d^2z \bar{\partial} \hat{\phi}_A \partial \hat{\phi}_B \hat{\phi}_C \epsilon^{ABC} = 4\pi n i \frac{\varphi_0}{q}. \tag{6.2.47}$$

Note that the effective action is related to the winding number n as shown in (6.1.18).

The last element to consider is the decomposition of the measure $D\phi$. This can be obtained by considering a small variation of the field ϕ as

$$\delta\phi = \delta\varphi\hat{\phi} + \varphi\delta\hat{\phi} \tag{6.2.48}$$

with

$$\delta\hat{\phi} = \delta\omega^+ \hat{E}_+ + \delta\omega^- \hat{E}_- \tag{6.2.49}$$

where $\delta\omega^\pm$ are small variations in the tangent directions. The variations $\delta\varphi$ and $\delta\omega^\pm$ can be expanded in terms of the eigenfunction u_λ as

$$\delta\varphi(\xi) = \sum_m \delta c_m u_m(\xi) \tag{6.2.50}$$

$$\delta\omega^\pm(\xi) = \sum_m \delta\mu^\pm_m u_m(\xi) \tag{6.2.51}$$

Consequently, we can rewrite the measure as

$$D\phi = |J_3| \prod_m dc_m d\mu^+_m d\mu^-_m \equiv |J_3| \prod_m dc_m d\Omega \tag{6.2.52}$$

where the Jacobian determinant can be computed by

$$J_3 = \det\left(\frac{\delta\phi^A(\xi)}{\delta(c_m, \mu_p^+, \mu_q^-)}\right) \equiv \det(M_{IJ}). \tag{6.2.53}$$

M_{IJ} is the Jacobian matrix where the row index $I \equiv A, \xi$ and the column index $J \equiv m, p, q$. Again, the relation (6.2.32) is used to determine the Jacobian determinant.

As $\frac{\delta\phi^A(\xi)}{\delta c_m} = \hat{\phi}^A(\xi)u_m(\xi)$ and $\frac{\delta\phi^A(\xi)}{\delta\mu_m^\pm} = \hat{E}_\pm^A(\xi)\varphi u_m(\xi)$, It is not hard to see that $M^\dagger M$ is

$$\begin{pmatrix} \left(\int_\xi u_m(\xi)u_{m'}(\xi)\right)_{mm'} & 0 & 0 \\ 0 & 0 & \left(\int_\xi \varphi^2 u_m(\xi)u_{m'}(\xi)\right)_{mm'} \\ 0 & \left(\int_\xi \varphi^2 u_m(\xi)u_{m'}(\xi)\right)_{mm'} & 0 \end{pmatrix} \tag{6.2.54}$$

where $\int_\xi$ is a shorthand for $\int \sqrt{g}d^2\xi$. Note that the objects in the parentheses are the matrix elements in row m and column m'. We can then utilise the fact that the value of φ is the constant φ_0 throughout the space due to the constraint (6.2.34). This allows us to obtain the absolute value of the Jacobian determinant as

$$|J_3| = \prod_m (\varphi_0)^2. \tag{6.2.55}$$

It is clear that the product of $(\varphi_0)^2$ in (6.2.55) cannot be completely cancelled by the one in (6.2.46) as mentioned. The cancellation leaves a single factor of $(\varphi_0)^2$ behind. This remaining factor accounts for the pre-factor of the partition function as we shall see later.

In consequence, when substituting (6.2.34), (6.2.46), (6.2.52), and (6.2.55) into (6.1.8), the gauge-fixed partition function takes the form

$$Z = \mathcal{N}\int dc_0\left(\prod_\lambda dc_\lambda \delta(c_\lambda)\right)\varphi_0^2 \sum_{n=-\infty}^{\infty} \exp(-S_{\text{eff}}[\varphi_0, n]) \tag{6.2.56}$$

where $S_{\text{eff}}[\varphi_0, n]$ is expressed in (6.2.47).

According to (1.1.46), we can relate the BF theory to two-dimensional Yang-Mills theory by adding a quadratic term in the scalar field. Consequently, the partition function for two-dimensional Yang-Mills is

$$
\begin{aligned}
Z &= \widetilde{\mathcal{N}} \int_0^\infty \varphi_0^2 d\varphi_0 \sum_{n=-\infty}^{\infty} \exp(-S_{\text{eff}}[\varphi_0, n] - e^2 \int_{S^2} d^2\xi \sqrt{g}\varphi^2) \\
&= \widetilde{\mathcal{N}} \int_0^\infty \varphi_0^2 d\varphi_0 \sum_{n=-\infty}^{\infty} \exp\big(-\frac{4\pi i}{q} n\varphi_0 - e^2\varphi_0^2 A\big).
\end{aligned}
\tag{6.2.57}
$$

where A is the area of the sphere. The infinite sum of the Euler exponential provides a Dirac delta function. This discretises the possible values of φ_0 in the theory as

$$
\sum_{n=-\infty}^{\infty} \exp\big(-\frac{4\pi i}{q} n\varphi_0\big) = \frac{q}{2}\delta\big(\varphi_0 \text{mod} \frac{q}{2}\big). \tag{6.2.58}
$$

Therefore, it is not hard to see that the expression (6.2.57) turns into

$$
Z \sim \sum_{m=1}^{\infty} m^2 \exp\big(-\frac{(eq)^2}{4} A m^2\big). \tag{6.2.59}
$$

The result (6.2.59) is in agreement with the expression (6.2.21). They differ by the factor -1 in the exponent which can be adjusted by a local counter term.

6.3 Generalization to an Arbitrary Lie Algebra

In this section, we would like to generalise the approach we used in section 6.1 to an arbitrary Lie algebra. As seen in the earlier section, one of the key elements in our calculation is to expand the fields in terms of a set of suitable Lie bases. For a general Lie algebra, we will work in the Cartan-Weyl basis.

We will denote the Cartan generator H^a and Weyl generator E^α where $a = 1, \dots, N-1$ and α is a root of eigenvalue equation, $\text{ad}_{H^a}(E^\alpha) = \alpha^a E^\alpha$. The roots α forms a vector space Φ. The generators H^a and E^α satisfy the following algebra:

$$
\begin{aligned}
&[H^a, H^b] = 0, \qquad [H^a, E^\alpha] = \alpha^{(a)} E^\alpha, \\
\text{and} \qquad &[E^\alpha, E^\beta] = \begin{cases} N^{\alpha\beta} E^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ H^\alpha & \text{if } \alpha + \beta = 0 \end{cases}
\end{aligned}
\tag{6.3.60}
$$

where H^α is defined as $H^\alpha = \alpha_a H^a$. The Cartan generators H^a are diagonal traceless matrices in the adjoint representation.

Again, we start the calculation with the action (1.1.45) with the path integral defined by (6.1.1). The calculation proceeds by expanding the fields ϕ and $\mathcal{A}_i$ in the Cartan-Weyl basis as

$$\phi = \phi_a H^a \qquad \text{and} \qquad \mathcal{A}_i = \chi_{ia} H^a + a_{i\alpha} E^\alpha. \tag{6.3.61}$$

Similar to the SU(2) case, these bases are ξ-dependent. The Cartan generators were chosen such that the field ϕ lies within their subalgebra.

To relate Lie indices A with the Cartan and Weyl indices a and α, we introduce unit vectors $\hat{H}^a_A$ and $\hat{E}^\alpha_A$ in Lie vector space which are defined as δ^a_A and δ^α_A respectively. As a result, the inner products among the vectors are

$$\hat{H}^a_A \hat{H}^{Ab} = \eta^{ab}, \qquad \hat{E}^\alpha_A \hat{E}^{A\beta} = \eta^{\alpha\beta}, \qquad \hat{H}^a_A \hat{E}^{A\alpha} = 0 \tag{6.3.62}$$

where η^{ab} and $\eta^{\alpha\beta}$ are Killing metrics of Cartan and Weyl generators respectively. The completeness relation is

$$\hat{H}^A_a \hat{H}^a_B + \hat{E}^A_\alpha \hat{E}^\alpha_B = \delta^A_B. \tag{6.3.63}$$

It is not hard to write the field ϕ and $\mathcal{A}_i$ in terms of the unit vectors as

$$\phi^A = \phi^a \hat{H}^A_a \qquad \text{and} \qquad \mathcal{A}^A_i = \chi^a_i \hat{H}^A_a + a^\alpha_i \hat{E}^A_\alpha. \tag{6.3.64}$$

Using the relations (6.3.64), one can find the topological field theory action (1.1.45) as

$$S[\phi, \chi, a] = \int_{\mathcal{M}} d^2\xi \left(iqf^{ABC} \phi_C a^\alpha_i a^\beta_j \hat{E}_{\alpha A} \hat{E}_{\beta B} - 2(\partial_i \phi_A) a^\alpha_j \hat{E}^A_\alpha - 2(\partial_i \phi_A) \chi^a_j \hat{H}^A_a \right) \epsilon^{ij}. \tag{6.3.65}$$

Notice that there is no contribution from diagonal components of $\mathcal{A}^A_i$ to the first term as the Cartan subalgebra is commutative.

To obtain the effective Lagrangian of the field ϕ, we need to integrate out the variables χ^a_i and a^α_i. According to the action (6.3.65), integrating out χ^a_i would provide a constraint via the Dirac-delta function as

$$\begin{aligned} \int D\chi_{ja} \exp(2 \int d^2\xi (\partial_i \phi^A) \chi_{ja} \hat{H}^a_A \epsilon^{ij}) &= \mathcal{N} \prod_{a=1}^{N-1} \delta^{(2)}((\underline{\partial} \phi^A) \hat{H}^a_A) \\ &= \mathcal{N} \prod_{a=1}^{N-1} \delta^{(2)}(2\mathrm{tr}((\underline{\partial} \phi) H^a)). \end{aligned} \tag{6.3.66}$$

This implies that the derivative of the field ϕ, i.e. $\partial_i\phi$, has no H^a component. This provides a constraint on the theory as $\mathrm{tr}(\phi\partial_i\phi) = 0$.

This constraint (6.3.66) also implies that the square of the field ϕ, i.e. $\phi^A\phi_A \equiv |\phi|^2$, is constant throughout the space which is similar to what we found earlier in the $SU(2)$ theory. Apart from that, it also implies the existence of the new invariant quantity,

$$d^{ABC}\phi_A\phi_B\phi_C \tag{6.3.67}$$

where d^{ABC} is a totally symmetric third rank tensor defined by

$$d^{ABC} = 2\mathrm{tr}(\{T^A, T^B\}T^C). \tag{6.3.68}$$

This can be seen as follows. As the field ϕ lies only in the Cartan directions, the Lie indices in (6.3.67) are summed over the Cartan indices. However, $\partial_i\phi$ has no Cartan components due to the constraint (6.3.66) which makes $\partial_i(d^{ABC}\phi_A\phi_B\phi_C) = 0$. This gives (6.3.67) constant as claimed.

Up to this point we have ignored a boundary in (1.1.45). We will now consider the effect of including this term $2\int d^2\xi\partial_i(\phi_A\mathcal{A}^A{}_j)\epsilon^{ij}$. It affects the constraints. To see this, let consider the case when the manifold $\mathcal{M}$ has the topology of a disk. This manifold can be mapped to the upper-half plane parameterised by Cartesian coordinates. Therefore, the boundary term takes the form

$$-2\int d^2\mathbf{x}\delta(y)\phi_A\mathcal{A}^A{}_x. \tag{6.3.69}$$

By expanding the gauge field $\mathcal{A}$ as (6.3.64), this turns the theory constraints (6.3.66) into

$$\prod_a \delta(2\mathrm{tr}(\partial_x\phi)H^a)\delta(2\mathrm{tr}(\partial_y\phi - \delta(y)\phi)H^a). \tag{6.3.70}$$

This implies that the squared of the field ϕ is no longer constant throughout the manifold $\mathcal{M}$. There is a discontinuity of $|\phi|^2$ at the boundary in the y direction as

$$|\phi|^2(x,\epsilon) = 3|\phi|^2(x,0). \tag{6.3.71}$$

To perform the path integration with respect to the field a_i^α, we apply the same trick we used in the previous section. We change the spacetime coordinates ξ^1 and ξ^2 into the complex coordinates z and $\bar{z}$ which were previously defined in (6.1.6). Of course, this coordinate transformation modifies the field a_i^α into the complex field b^α as stated in (6.1.7).

As a result, the partition function now takes form

$$Z = \frac{1}{\text{Vol}} \int D\phi_A Db^\alpha D\bar{b}^\alpha \prod_{a=1}^{N-1} \delta^{(2)}(2\text{tr}((\underline{\partial}\phi)H^a))\exp(-S[\phi, b, \bar{b}]) \tag{6.3.72}$$

where the action is expressed in the complex coordinates as

$$S[\phi, b, \bar{b}] = 2\int_D d^2z \left(iqf^{ABC}\phi_C b^\alpha \bar{b}^\beta \hat{E}_{\alpha A}\hat{E}_{\beta B} - (\partial\phi_A \bar{b}^\alpha - \bar{\partial}\phi_A b^\alpha)\hat{E}^A_\alpha \right). \tag{6.3.73}$$

The path integral of the complex fields b^α and $\bar{b}^\alpha$ resembles a Gaussian integral which can be performed using (6.1.10). By comparing (6.3.73) with (6.1.10), one obtains

$$M_{\alpha\beta} = 2qif^{ABC}\phi_B \hat{E}_{\alpha A}\hat{E}_{\beta C}, \qquad J_\alpha = -2\hat{E}^A_\alpha \partial\phi_A, \qquad \bar{J}_\alpha = 2\hat{E}^A_\alpha \bar{\partial}\phi_A. \tag{6.3.74}$$

Consequently, it is not hard to find that the effective Lagrangian with respect to the scalar field ϕ is

$$\mathcal{L}_{\text{eff}}(\phi) = \frac{-i}{2q}\bar{J}_\alpha (\widetilde{M}^{-1})^\alpha{}_\beta J^\beta \;=\; \frac{2i}{q}\partial\phi_A \bar{\partial}\phi_B \left(\hat{E}^A_\alpha (\widetilde{M}^{-1})^\alpha{}_\beta \hat{E}^B_\beta \right) \tag{6.3.75}$$

where we used $M^\alpha{}_\beta = 2qi\widetilde{M}^\alpha{}_\beta$.

A general expression for an inverse matrix $\widetilde{M}^\alpha{}_\beta$ is

$$(\widetilde{M}^{-1})^\alpha{}_\beta = \frac{\text{adj}(\widetilde{M})^\alpha{}_\beta}{\det(\widetilde{M})} \tag{6.3.76}$$

where

$$\begin{aligned} \text{adj}(\widetilde{M})^\alpha{}_\beta &= \delta^{\alpha j_2 \ldots j_n}_{\beta i_2 \ldots i_n}\, \widetilde{M}^{i_2}{}_{j_2} \widetilde{M}^{i_3}{}_{j_3} \ldots \widetilde{M}^{i_n}{}_{j_n}, \\ \det(\widetilde{M}) &= \delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}\, \widetilde{M}^{i_1}{}_{j_1} \widetilde{M}^{i_2}{}_{j_2} \ldots \widetilde{M}^{i_n}{}_{j_n}. \end{aligned} \tag{6.3.77}$$

$\delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}$ is a generalised Kronecker delta which is related to an anti-symmetrization of ordinary Kronecker deltas as

$$\delta^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} = n!\delta^{j_1}_{[i_1}\delta^{j_2}_{i_2} \ldots \delta^{j_n}_{i_n]}. \tag{6.3.78}$$

The integer n is the number of Weyl generators. In the case of SU(N), n is equal to $N^2 - N$.

To obtain the adjugate matrix and the matrix determinant expressed in (6.3.77), the matrices $\widetilde{M}^i{}_j$ are contracted with each other depending on the permutations implicit by (6.3.78). For the adjugate matrix $\mathrm{adj}(\widetilde{M})^\alpha{}_\beta$, the contractions lead to two types of terms. First, the matrices $\widetilde{M}^i{}_j$ are contracted in such a way that they form a new matrix with indices α and β. This contraction generates a chain of matrix multiplications, for instance, $\widetilde{M}^\alpha{}_{j_2}\widetilde{M}^{j_2}{}_{j_3}\widetilde{M}^{j_4}{}_{j_4}\widetilde{M}^{j_4}{}_\beta$. In this example, the matrices $\widetilde{M}^{i_2}{}_{j_2}\widetilde{M}^{i_3}{}_{j_3}\widetilde{M}^{i_4}{}_{j_4}\widetilde{M}^{i_5}{}_{j_5}$ are contracted with $\delta^\alpha_{i_2}\delta^{j_2}_{i_3}\delta^{j_3}_{i_4}\delta^{j_4}_{i_5}\delta^{j_5}_\beta$. Second, the contraction forms a trace of matrix products, i.e. $\mathrm{tr}(\widetilde{M}\cdot\widetilde{M}\ldots\widetilde{M})$. For example, when the same matrices $\widetilde{M}^{i_2}{}_{j_2}\widetilde{M}^{i_3}{}_{j_3}\widetilde{M}^{i_4}{}_{j_4}\widetilde{M}^{i_5}{}_{j_5}$ are contracted with $\delta^{j_2}_{i_3}\delta^{j_3}_{i_4}\delta^{j_4}_{i_5}\delta^{j_5}_{i_2}$. However, only the latter case contributes to the matrix determinant $\det(\widetilde{M})$.

In addition, the trace term vanishes when the number of matrices $\widetilde{M}$ inside is odd. This can be seen explicitly by considering

$$\begin{aligned}\mathrm{tr}(\widetilde{M}\cdot\widetilde{M}\ldots\widetilde{M}) =&\widetilde{M}^\alpha{}_{i_1}\widetilde{M}^{i_1}{}_{i_2}\ldots\widetilde{M}^{i_{k-2}}{}_{i_{k-1}}\widetilde{M}^{i_{k-1}}{}_\alpha\\ =&f^{A_1B_1C_1}\phi_{B_1}\eta_{C_1A_2}f^{A_2B_2C_2}\phi_{B_2}\eta_{C_2A_3}\cdot\ldots\cdot f^{A_kB_kC_k}\phi_{B_k}\eta_{C_kA_1}.\end{aligned} \tag{6.3.79}$$

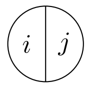
We used the completeness relation (6.3.63) to obtain the last line. When we swap the first and the third indices of each structure constant f^{ABC}, it gives an extra (-1) to the last line so the whole expression vanishes.

The calculation of the inverse matrix (6.3.76) involves a lot of contractions corresponding to chains of matrix multiplications. To facilitate the calculation, it is sensible to develop a set of diagrams to represent them. These diagrams are presented in the next section.

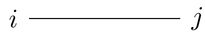
6.4 Diagrammatic Representation of the Inverse Matrix $\widetilde{M}$

According to the previous section, the inverse of the matrix $\widetilde{M}$ is an essential ingredient of the $SU(N)$ effective Lagrangian (6.3.75). To compute this object, the relation (6.3.76) is used. However, this is complicated by the large number of terms.

For this reason, we would like to develop a set of diagrams to capture the con-

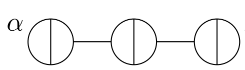


$$\equiv \widetilde{M}^i{}_j \qquad = \hat{E}^i_A (f^{ABC}\phi_B)\hat{E}_{jC}$$

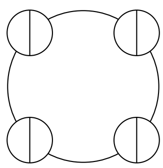


$$\equiv \delta^i_j.$$

Figure 6.1: Diagrammatic representation for matrix element $\widetilde{\mathcal{M}}$ and Kronecker delta



$$= \widetilde{M}^\alpha{}_i \widetilde{M}^i{}_j \widetilde{M}^j{}_\beta \qquad = \hat{E}^\alpha_A (f^{ABC}\phi_B \eta_{CD} f^{DEF}\phi_E \eta_{FG} f^{GHI}\phi_H)\hat{E}_{\beta I}$$



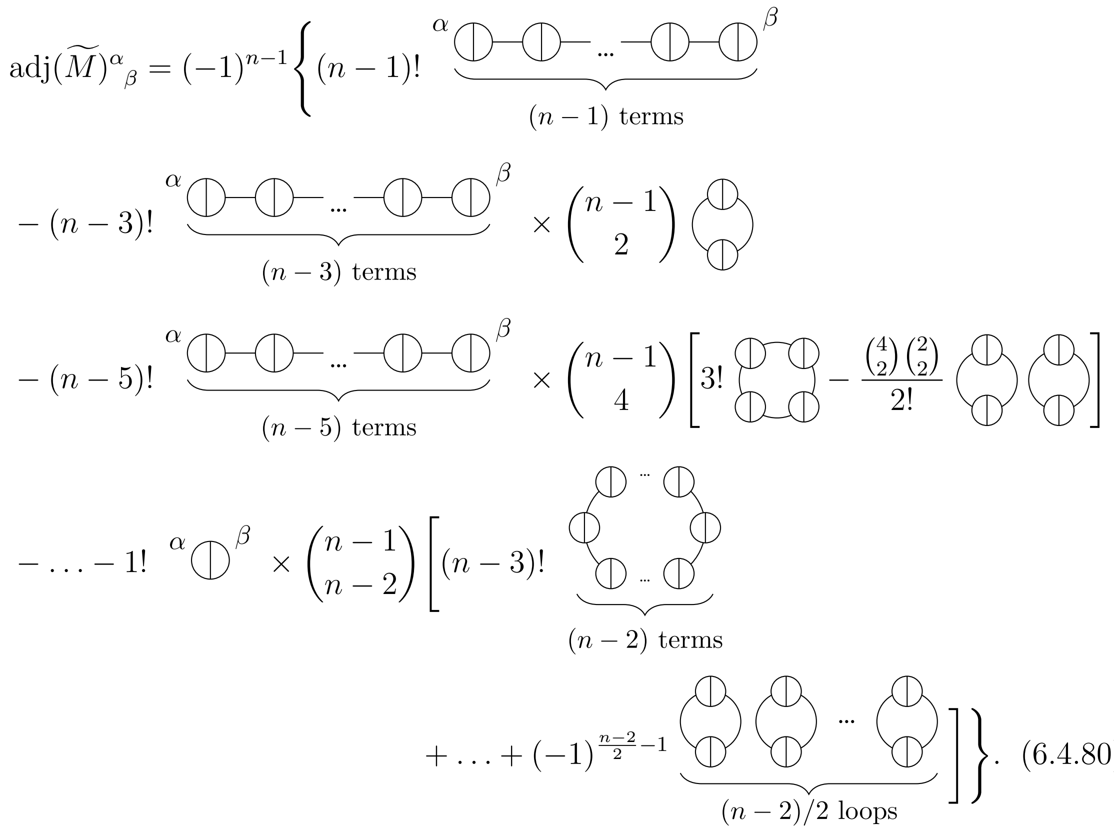
$$\begin{aligned} = \widetilde{M}^i{}_j \widetilde{M}^j{}_k \widetilde{M}^k{}_l \widetilde{M}^l{}_i = {} & f^{ABC}\phi_B \eta_{CD} f^{DEF}\phi_E \eta_{FG} \\ & \times f^{GHI}\phi_H \eta_{IJ} f^{JKL}\phi_K \eta_{LA} \end{aligned}$$

Figure 6.2: Examples for a strand and loop diagram representing certain matrix multiplications

tractions between matrix elements $\widetilde{M}^\alpha{}_\beta$ and Kronecker deltas δ^i_j. We represent these two objects as the vertices and lines shown in figure 6.1.

Based on this diagrammatic representation, matrix multiplication is represented by vertices connecting by a line. Note that no more than two lines are allowed to be connected to each vertex. This fact implies that a diagram involved in the calculation is either a strand or a loop which corresponds to a chain of matrix multiplications and its trace respectively. Just for clarification, we show some examples for a loop diagram and a strand diagram as well as their corresponding matrix representations in figure 6.2.

According to (6.3.77), the adjugate matrix, $\mathrm{adj}(\widetilde{M})^\alpha{}_\beta$, can be expressed diagrammatically as a summation of all possible products between a strand diagram and loop diagrams. The diagram includes $n-1$ vertices in total where $n = N^2 - N$ for $SU(N)$ (n is always even for $N \geq 2$). In order to obtain all possible combinations of a strand and loops without overcounting, we can start by listing all possible strand diagrams which simply are the strand with different numbers of vertices ranging from 1 to $n-1$. Then, for each strand, loop diagrams can be created using the remaining vertices. Therefore, we can expand the adjugate matrix as



(6.4.80)

There is no contribution from loops with odd vertices as they are zero as discussed previously. The minus sign factor comes from an antisymmetric permutation of the generalised Kronecker delta. Each time the diagram collapses to form smaller loops, an extra (-1) appears which corresponds to an odd permutation of the lower indices of the Kronecker delta in (6.3.78). The numbers in front of the diagrams count the multiplicities.

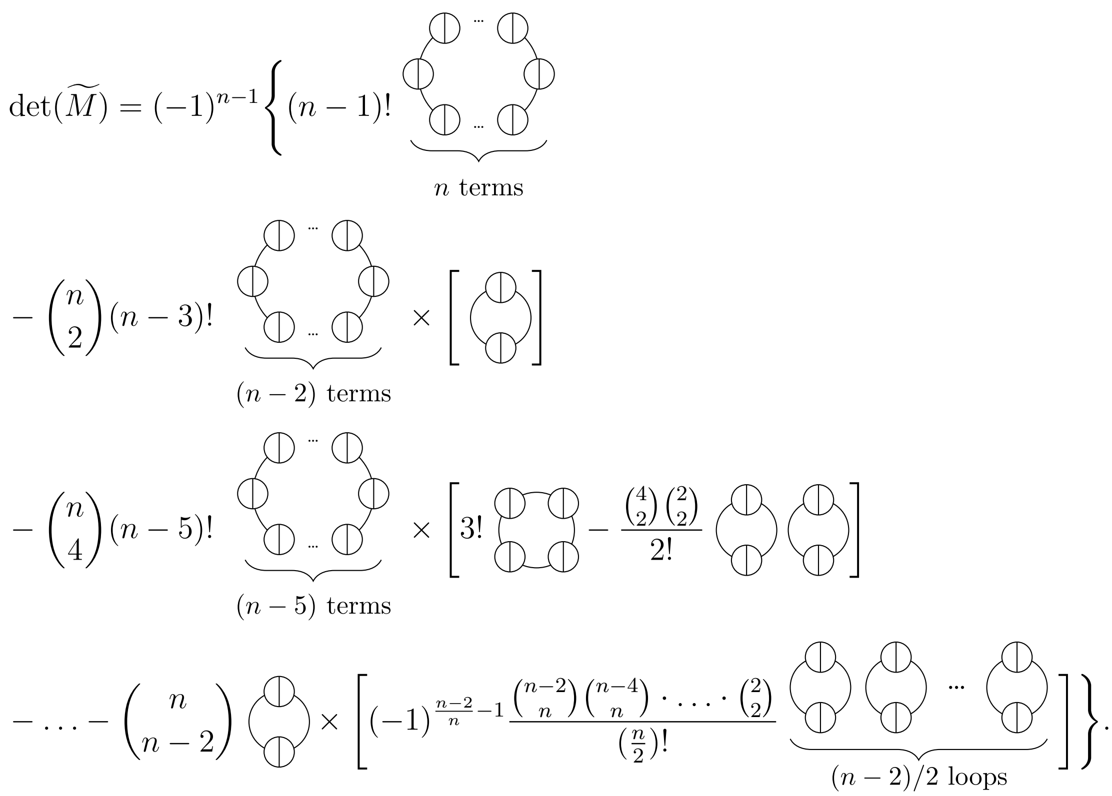
One can also see that the indices α and β from the $\mathrm{adj}(\widetilde{M})^{\alpha}{}_{\beta}$ are embedded at the ends of the strands corresponding to the basis $\hat{E}^{A\alpha}$. Consequently, we can always factor out these bases to write the adjugate matrix as

$$\mathrm{adj}(\widetilde{M})^{\alpha}{}_{\beta} = \hat{E}^{\alpha}_{A}\Theta^{AB}\hat{E}_{\beta B} \tag{6.4.81}$$

or equivalently $\Theta^{AB} = \hat{E}^{A}_{\alpha}(\mathrm{adj}(\widetilde{M})^{\alpha}{}_{\beta})\hat{E}^{\beta B}$. Due to the above relation, it is not hard to see that the effective Lagrangian takes the form

$$\mathcal{L}_{\text{eff}}(\phi) = \frac{2i}{q}\frac{1}{\det(\widetilde{M})}\partial\phi_A\Theta^{AB}\bar{\partial}\phi_B. \tag{6.4.82}$$

Unlike the adjugate matrix, only loop diagrams contribute to the matrix determinant $\det(\widetilde{M})$. There are n vertices involve in the expression of the matrix determinant. $\mathrm{Det}(\widetilde{M})$ is expressed as the sum over all product of loops. To obtain these, we can start with the biggest loop of n vertices and then cut it down to form smaller loops. The expression for $\det(\widetilde{M})$ is shown in the equation (6.4.83). To avoid overcounting, all diagrams in the squared brackets contain the same number for fewer vertices than the loop in front of the bracket. Therefore, the general expression for the determinant is

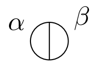


(6.4.83)

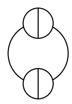
6.5 Explicit Expressions for Effective $SU(2)$ and $SU(3)$ Lagrangians

In this section, we show the explicit calculation to obtain the effective Lagrangians for 2D topological field theory for $SU(2)$ and $SU(3)$ using the expression (6.4.82) together with the diagrammatic representation for adjugate matrix and matrix determinant expressed in (6.4.80) and (6.4.83) respectively.

For $SU(2)$, the adjugate matrix is

$\mathrm{adj}(\widetilde{M})^{\alpha}{}_{\beta} = (-1)$  $= -\hat{E}^{\alpha}_{A}\epsilon^{ACB}\phi_C\hat{E}_{\beta B}$ (6.5.84)

where $f^{ABC} = \epsilon^{ABC}$ for SU(2). Therefore, $\Theta^{AB} = -\epsilon^{ACB}\phi_C$. The matrix determinant is

$\det(\widetilde{M}) = (-1)$  $= -\epsilon^{ABC}\phi_B\eta_{CD}\epsilon^{DEF}\phi_E\eta_{AF}$

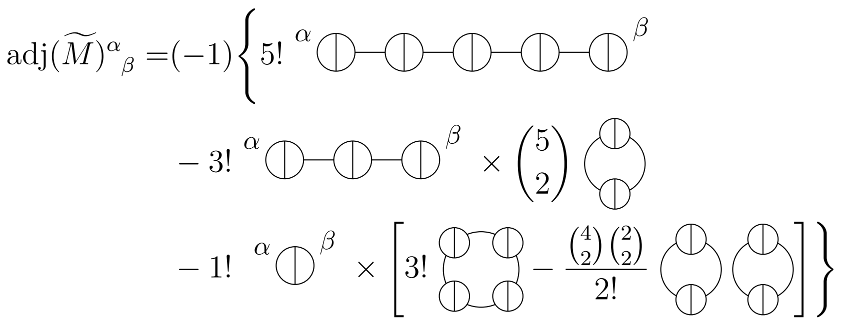
$$= 2\eta^{BD}\phi_B\phi_D = 2|\phi|^2. \tag{6.5.85}$$

Thus, when substituting the above relations into (6.4.82), we obtain

$$\mathcal{L}_{\text{eff}}(\phi) = \frac{-i}{q|\phi|^2}\partial\phi_A\epsilon^{ABC}\phi_B\bar{\partial}\phi_C \tag{6.5.86}$$

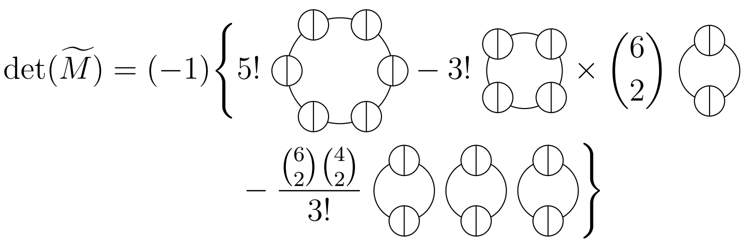
which is identical to what we found earlier in the equation (6.1.15).

For $SU(3)$, the diagrammatic expressions for the adjugate matrix and matrix determinant are



(6.5.87)

and



(6.5.88)

According to the above expressions, one can write the effective Lagrangian in the form (6.4.82) with

$$\begin{aligned}\Theta^{AB} =& -5!(\mathcal{F}^{A}{}_{C}\mathcal{F}^{C}{}_{D}\mathcal{F}^{D}{}_{E}\mathcal{F}^{E}{}_{F}\mathcal{F}^{FB}) + 3!\cdot 10\cdot(\mathcal{F}^{A}{}_{C}\mathcal{F}^{C}{}_{D}\mathcal{F}^{DB})(\mathcal{F}^{E}{}_{F}\mathcal{F}^{F}{}_{E}) \\ &+ \mathcal{F}^{AB}\Big[3!(\mathcal{F}^{C}{}_{D}\mathcal{F}^{D}{}_{E}\mathcal{F}^{E}{}_{F}\mathcal{F}^{E}{}_{C}) - 3(\mathcal{F}^{C}{}_{D}\mathcal{F}^{D}{}_{C})(\mathcal{F}^{E}{}_{F}\mathcal{F}^{F}{}_{E})\Big]\end{aligned} \tag{6.5.89}$$

and

$$\det(\widetilde{M}) = -\,5!(\mathcal{F}^{A}{}_{B}\mathcal{F}^{B}{}_{C}\mathcal{F}^{C}{}_{D}\mathcal{F}^{E}{}_{F}\mathcal{F}^{F}{}_{G}\mathcal{F}^{G}{}_{A}) + 3!\cdot 15\cdot(\mathcal{F}^{A}{}_{B}\mathcal{F}^{B}{}_{C}\mathcal{F}^{C}{}_{D}\mathcal{F}^{D}{}_{A})(\mathcal{F}^{E}{}_{F}\mathcal{F}^{F}{}_{E}) + 15(\mathcal{F}^{A}{}_{B}\mathcal{F}^{B}{}_{A})(\mathcal{F}^{C}{}_{D}\mathcal{F}^{D}{}_{C})(\mathcal{F}^{E}{}_{F}\mathcal{F}^{F}{}_{E}) \tag{6.5.90}$$

where we used the notation $\mathcal{F}^{AB} = f^{ACB}\phi_C$.

We can further simplify the above terms by expanding them explicitly in the Cartan-Weyl basis for $SU(3)$. The generators are

$$I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$U^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad U^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$V^+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{6.5.91}$$

We can determine the structure constants by considering all matrix commutators between the elements. The generators I^3 and Y are the Cartan subalgebra elements satisfying

$$[I^3, Y] = 0. \tag{6.5.92}$$

The plus and minus superscripts of the generators denote the raising and lowering operators within the three su(2) subalgebras given by

$$[I^+, I^-] = 2I^3, \qquad [U^+, U^-] = \frac{3}{2}Y - I^3, \qquad [V^+, V^-] = \frac{3}{2}Y + I^3. \tag{6.5.93}$$

Note that the Hermitian conjugation of generators switches the plus and minus superscripts of the generators within each $SU(2)$ subgroup, i.e. $(I^{\pm})^\dagger = I^{\mp}$, $(U^{\pm})^\dagger = U^{\mp}$, $(V^{\pm})^\dagger = V^{\mp}$.

Apart from (6.5.93), the remaining non-zero commutators are

$$
\begin{aligned}
[I^3, I^\pm] &= \pm I^\pm, & [I^3, U^\pm] &= \mp\frac{1}{2}U^\pm, \\
[I^3, V^\pm] &= \pm\frac{1}{2}U^\pm, & [Y, U^\pm] &= \pm U^\pm, \\
[Y, V^\pm] &= \pm V^\pm, & [I^\pm, U^\pm] &= \pm V^\pm, \\
[I^\pm, V^\mp] &= \mp U^\mp, & [U^\pm, V^\mp] &= \pm I^\mp.
\end{aligned}
\tag{6.5.94}
$$

For convenience, we denote $\{I^3, Y, I^+, I^-, U^+, U^-, V^+, V^-\}$ by $\{T^1, T^2, \ldots, T^8\}$ respectively. In this notation, the metric tensor η^{AB} can be written as

$$
\eta^{AB} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}
\tag{6.5.95}
$$

which is directly from $\eta^{AB} = 2\ \mathrm{tr}(T^A T^B)$.

The field ϕ is an element of the Cartan subalgebra, i.e. $\phi = \phi_1 T^1 + \phi_2 T^2$. Consequently, one can find the adjoint representation of the field ϕ as

$$
\mathrm{ad}(\phi) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2\phi_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\phi_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \phi_1 - 2\phi_2 & 0 & 0 \\
0 & 0 & 0 & 0 & -\phi_1 + 2\phi_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\phi_1 - 2\phi_2 \\
0 & 0 & 0 & 0 & 0 & 0 & \phi_1 + 2\phi_2 & 0
\end{pmatrix}
\tag{6.5.96}
$$

where $\mathrm{ad}(\phi) = if^{ABC}\phi_B = i\mathcal{F}^{AC}$. With this matrix (6.5.96), one can compute all loop and strand diagrams appearing in the equations (6.5.89) and (6.5.90). Chains of matrix multiplications of the matrix $\mathcal{F}$ are shown in the Appendix A.3.

From the calculation, we can further simplify the loop terms. The loop diagram with two vertices $\mathcal{F}^A{}_B\mathcal{F}^B{}_C$ can be replaced by the absolute square of the field ϕ as

$$\begin{aligned}\mathcal{F}^A{}_B\mathcal{F}^B{}_A &= -2(\phi_1)^2 - \frac{1}{2}(\phi_1 - 2\phi_2)^2 - \frac{1}{2}(\phi_1 + 2\phi_2)^2 \\ &= -3((\phi_1)^2 + \frac{4}{3}(\phi_2)^2) \ = \ -3|\phi|^2. \end{aligned} \tag{6.5.97}$$

A similar pattern appears in the four-vertex loop as it is proportional to $|\phi|^4$:

$$\begin{aligned}\mathcal{F}^A{}_B\mathcal{F}^B{}_C\mathcal{F}^C{}_D\mathcal{F}^D{}_A &= 2(\phi_1)^4 + \frac{1}{8}(\phi_1 - 2\phi_2)^4 + \frac{1}{8}(\phi_1 + 2\phi_2)^4 \\ &= \frac{9}{4}((\phi_1)^2 + \frac{4}{3}(\phi_2)^2)^2 \ = \ \frac{9}{4}|\phi|^4. \end{aligned} \tag{6.5.98}$$

The six-vertex loop can be expressed in terms of two invariant objects, $|\phi|^6$ and $d^{ABC}\phi_A\phi_B\phi_C$ as

$$\begin{aligned}\mathcal{F}^A{}_B\mathcal{F}^B{}_C\mathcal{F}^C{}_D\mathcal{F}^D{}_E\mathcal{F}^E{}_F\mathcal{F}^F{}_A &= -2(\phi_1)^6 - \frac{1}{32}(\phi_1 - 2\phi_2)^6 - \frac{1}{32}(\phi_1 + 2\phi_2)^6 \\ &= -\frac{33}{16}|\phi|^6 + \frac{9}{8}\left(2(\phi_1)^2(\phi_3) - \frac{8}{9}(\phi_2)^3\right)^2 \end{aligned} \tag{6.5.99}$$

where the quantity inside the parenthesis is $d^{ABC}\phi_A\phi_B\phi_C$ where d^{ABC} is the totally symmetric third rank tensor defined in (6.3.68).

In consequence, we can rewrite the terms (6.5.89) and (6.5.90) as

$$\Theta^{AB} = -120(\mathcal{F}^A{}_C\mathcal{F}^C{}_D\mathcal{F}^D{}_E\mathcal{F}^E{}_F\mathcal{F}^{FB}) - 180(\mathcal{F}^A{}_C\mathcal{F}^C{}_D\mathcal{F}^{DB})|\phi|^2 - \frac{27}{2}\mathcal{F}^{AB}|\phi|^4 \tag{6.5.100}$$

and

$$\det(\widetilde{M}) = -765|\phi|^6 - 135(d^{ABC}\phi_A\phi_B\phi_C). \tag{6.5.101}$$

6.6 The Topological Field Action with a Source Term and the Expectation Value of the Wilson Loop

In this section we would like to generalise the BF action (1.1.44) further by adding a source term for the gauge field $\mathcal{A}$. Consider the action given by

$$S[\mathcal{J}] = 2\int_{\mathcal{M}} d^2\xi\ \epsilon^{ij}\mathrm{tr}(\phi\mathcal{F}_{ij} + \mathcal{J}_i\mathcal{A}_j). \tag{6.6.102}$$

To obtain the effective Lagrangian for the field ϕ, we will integrate out the gauge field $\mathcal{A}$ as before. By doing so, we expand the field $\mathcal{A}$ in terms of the unit basis defined by (6.3.64). This gives the partition function as

$$Z[\mathcal{J}] = \frac{1}{\mathrm{Vol}}\int D\phi_A Da_i^\alpha D\chi_{ja}\ \exp(-\tilde{S}[\mathcal{J}]). \tag{6.6.103}$$

with

$$\begin{aligned}\tilde{S}[\mathcal{J}] = \int_{\mathcal{M}} d^2\xi\Big(& iqf^{ABC}\phi_C a_i^\alpha a_j^\beta \hat{E}_{\alpha A}\hat{E}_{\beta B} - (2\partial_i\phi_A - \mathcal{J}_{iA})a_j^\alpha \hat{E}_\alpha^A \\ & - (2\partial_i\phi_A - \mathcal{J}_{iA})\chi_{ja}\hat{H}^{Aa}\Big)\epsilon^{ij}.\end{aligned} \tag{6.6.104}$$

It is not hard to see that the path integration of the last line leads to a constraint on the theory. This appears in the form of a Dirac delta function

$$\prod_{a=1}^{N-1}\prod_{i=1}^{2}\delta(\mathrm{tr}(2\partial_i\phi - \mathcal{J}_i)H^a). \tag{6.6.105}$$

The constraint implies that the difference between $2\partial\phi$ and $\mathcal{J}$ does not lie in the Cartan subalgebra.

We can proceed with the calculation as in previous sections by changing from spacetime coordinates (ξ^1, ξ^2) to the complex coordinates $(z, \bar{z})$. The partition function now resembles a Gaussian path integral with respect to the complex fields b and $\bar{b}$ expressed in (6.1.7) which is

$$Z[\mathcal{J}] = \frac{\mathcal{N}}{\mathrm{Vol}}\int D\phi_A Db^\alpha D\bar{b}^\alpha\ \prod_{i=1}^{2}\delta^{(N-1)}(\mathrm{tr}((2\partial_i\phi - \mathcal{J}_i)\hat{\phi}))\exp(-S[\phi, b, \bar{b}, \mathcal{J}]). \tag{6.6.106}$$

where

$$S[\phi,b,\bar{b},\mathcal{J}]=\int_{\mathcal{M}}d^2z\Big(2iqf^{ABC}\phi_C b^\alpha\bar{b}^\beta\hat{E}_{\alpha A}\hat{E}_{\beta B}-((2\partial\phi_A-\mathcal{J}_A)\bar{b}^\alpha-(2\bar{\partial}\phi_A-\bar{\mathcal{J}}_A)b^\alpha)\hat{E}^A_\alpha\Big). \tag{6.6.107}$$

Integrating out the b and $\bar{b}$ using (6.1.10) yields

$$Z[\mathcal{J}]=\frac{\mathcal{N}}{\mathrm{Vol}}\int D\phi_A\prod_{i=1}^{2}\delta^{(N-1)}(\mathrm{tr}((2\partial_i\phi-\mathcal{J}_i)\hat{\phi}))\mathrm{exp}\Big(-S_{\mathrm{eff}}(\phi,\mathcal{J})\Big) \tag{6.6.108}$$

with

$$S_{\mathrm{eff}}(\phi,\mathcal{J})=\int d^2z\,\frac{i}{2q}\frac{1}{\det(\widetilde{M})}(2\partial\phi_A-\mathcal{J}_A)\Theta^{AB}(2\bar{\partial}\phi_B-\bar{\mathcal{J}}_B). \tag{6.6.109}$$

Turning back to the (ξ^1,ξ^2) coordinates, the effective action takes the form

$$S_{\mathrm{eff}}(\phi,\mathcal{J})=\int d^2\xi\,\frac{i}{4q}\frac{1}{\det(\widetilde{M})}(2\partial_i\phi_A-\mathcal{J}_{iA})\Theta^{AB}(2\partial_j\phi_B-\mathcal{J}_{jB})\epsilon^{ij}. \tag{6.6.110}$$

It is known that one can relate a BF theory to $2D$ Yang-Mills theory by introducing the quadratic term for the field ϕ which is

$$S_{qd}=e^2\int d^2\xi\sqrt{g}|\phi|^2. \tag{6.6.111}$$

As a result, the partition function for the 2D gauge theory with the gauge field source $\mathcal{J}$ can be expressed as

$$Z[\mathcal{J}]=\frac{\mathcal{N}}{\mathrm{Vol}}\int D\phi_A\prod_{i=1}^{2}\delta^{(N-1)}(\mathrm{tr}((2\partial_i\phi-\mathcal{J}_i)\hat{\phi}))\mathrm{exp}\Big(-(S_{\mathrm{eff}}+S_{qd})\Big). \tag{6.6.112}$$

With a suitable choice of the source term $\mathcal{J}$, in principle we are able to compute the expectation value of a Wilson loop in 2D Yang-Mills theory based on our effective BF theory. However, we have to deal with the issue of path-ordering.

The non-Abelian Wilson loop can be expressed as the trace of the path-ordered exponential of a line integral of the gauge field $\mathcal{A}$ along a closed loop C,

$$W[C]=\mathrm{tr}\Big(\mathcal{P}\Big(e^{-q\oint_C\mathcal{A}\cdot d\xi}\Big)\Big). \tag{6.6.113}$$

The trace together with the path-ordering operator can be replaced by a functional integral over a complex anti-commuting field ψ [87, 88] as

$$W[C]=\int D\psi^\dagger D\psi\,\mathrm{exp}\Big(\int d\tau\psi^\dagger\dot{\psi}-q\mathcal{A}_{iR}\dot{\xi}^i\psi^\dagger T^R\psi\Big) \tag{6.6.114}$$

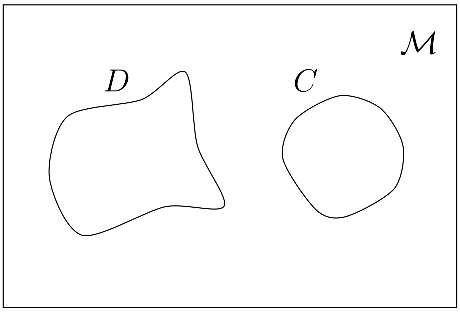


Figure 6.3: Two-dimensional manifold $\mathcal{M}$ with a region D and a closed loop C

where the loop C is now parameterized by τ. Therefore, the expectation value of the Wilson loop takes the form

$$\langle W[C]\rangle = \frac{1}{Z'}\int D\phi D\psi^\dagger D\psi \prod_{i=1}^{2}\delta^{(N-1)}(\mathrm{tr}((2\partial_i\phi - \mathcal{J}_i)\hat{\phi}))\exp\left(-(S_{\text{eff}} + S_{qd}) + \int d\tau\psi^\dagger\dot{\psi}\right) \tag{6.6.115}$$

with

$$\mathcal{J}^A_{\ i}(\xi) = -q\oint_C \psi^\dagger(\tilde{\xi})T^A\psi(\tilde{\xi})\delta^{(2)}(\xi - \tilde{\xi})\epsilon_{ij}d\tilde{\xi}^j \tag{6.6.116}$$

and the action S_{eff} and S_{qd} are expressed in (6.6.110) and (6.6.111) respectively. The term Z' in the denominator is a normalization factor such that $\langle 1\rangle = 1$.

However, it turns out that the solution for the equation $2\partial_i\phi - \mathcal{J}_i = 0$ with the source term expressed above is not consistent as the line integral of $\mathcal{J}_i$ is not path independent which contradicts to the equation itself. To deal with this, we will exploit gauge symmetry.

For simplicity, we will proceed with the calculation in the context of $SU(2)$ theory. In this setting, we choose the gauge fixing such that the unit vector $\hat{\phi}$ is constant everywhere outside a region D. Therefore, the manifold $\mathcal{M}$ now consists of the region D where the value of $\hat{\phi}$ varies and the rest of the manifold where the $\hat{\phi}$ is constant. We can further choose that the region D does not intercept the loop C as depicted in the figure 6.3.

According to this gauge choice, the effective action term (6.6.110) becomes

$$\begin{aligned} S_{\text{eff}}(\phi,\mathcal{J}) = &\int_D d^2\xi\ \frac{i|\phi|}{2q}\partial_i\hat{\phi}_A\partial_j\hat{\phi}_B\hat{\phi}_C\epsilon^{ABC}\epsilon^{ij} \\ &+ \int_{\mathcal{M}/D} d^2\xi\ \frac{i}{8q|\phi|^2}(2\partial_i\phi_A - \mathcal{J}_{iA})(2\partial_j\phi_B - \mathcal{J}_{jB})\phi_C\epsilon^{ABC}\epsilon^{ij}. \end{aligned} \tag{6.6.117}$$

If we consider the case when the manifold $\mathcal{M}$ has a topology of unit sphere S^2, the first term can be related to a winding number as discussed in the earlier section. Note that $|\phi|$ is constant due to the absence of the source in D. Moreover, since the $\hat{\phi}$ is constant in $\mathcal{M}/D$, the non-vanishing contribution to the second line is

$$\int_{\mathcal{M}/D} d^2\xi \; \frac{i}{8q|\phi|^2}\mathcal{J}_{iA}\mathcal{J}_{jB}\phi_C\epsilon^{ABC}\epsilon^{ij}$$
$$= \frac{iq}{8}\oint_C\oint_C d\tilde{\xi}^i d\xi'^j\left(\psi^\dagger T^A\psi\Big|_{\tilde{\xi}}\right)\left(\psi^\dagger T^B\psi\Big|_{\xi'}\right)\delta^{(2)}(\tilde{\xi}-\xi')\epsilon_{ij}\frac{\phi^C}{|\phi|^2}\epsilon_{ABC}. \quad (6.6.118)$$

The term $\oint_C\oint_C d\tilde{\xi}^i d\xi'^j\delta^{(2)}(\tilde{\xi}-\xi')\epsilon_{ij}$ counts the number of times the loop C intersects itself. Therefore, the above term can be set to zero provided that the loop C does not have a self-intersection. Subsequently, the effective action (6.6.117) turns into

$$S_{\text{eff}}(\phi) = \frac{i}{q}|\phi|(4\pi n) \quad (6.6.119)$$

where n is the winding number of the map $\hat{\phi}$.

At this point, the appearance of the fermionic field ψ in the effective action S_{eff} has been removed due to the gauge choice. Therefore, according to (6.6.115), the only term that is subject to the path-ordering operation is the source term $\mathcal{J}$ in the constraint. This allows us to rewrite (6.6.115) as

$$\langle W[C]\rangle = \frac{1}{Z}\int D\phi D\chi \;\mathrm{tr}\Bigg[\mathcal{P}\Bigg(\exp\Bigg(\frac{q}{2}\oint_C\hat{\phi}\chi_i d\xi^i\Bigg)\Bigg)\Bigg]$$
$$\times\exp\Bigg(-(S_{\text{eff}}+S_{qd})+\int d^2\xi(\partial_i\phi_A)\hat{\phi}^A\chi_j\epsilon^{ij}\Bigg) \quad (6.6.120)$$

where the Dirac delta function is replaced by the functional integral over the field χ. Since the field $\hat{\phi}$ is constant and commutes with itself throughout the loop, the path-ordering operator $\mathcal{P}$ can be dropped. Denoting the eigenvalue of $\hat{\phi}$ by λ, the trace of the exponential in the first line takes the form

$$\sum_\lambda \exp\Bigg(\frac{q\lambda}{2}\int d^2\xi\oint_C\delta^{(2)}(\xi-\tilde{\xi})\chi_i(\xi)d\tilde{\xi}^i\Bigg). \quad (6.6.121)$$

We then proceed with the calculation by integrating out the field χ. This generates a constraint via a Dirac delta function as

$$\langle W[C]\rangle = \frac{1}{Z}\int D\phi\;\sum_\lambda\prod_{i=1}^{2}\delta\Bigg(\partial_i|\phi|+\frac{q\lambda}{2}\oint_C\delta^{(2)}(\xi-\tilde{\xi})\epsilon_{ij}d\tilde{\xi}^j\Bigg)e^{-(S_{\text{eff}}+S_{qd})}. \quad (6.6.122)$$

It is not hard to see that the solution for the constraint,

$$\partial_i|\phi| + \frac{q\lambda}{2}\oint_C \delta^{(2)}(\xi - \tilde{\xi})\epsilon_{ij}d\tilde{\xi}^j = 0. \tag{6.6.123}$$

takes the form

$$\varphi_\lambda - \varphi_0 = -\frac{q\lambda}{2}\Big(\int_O^\xi \oint_C \delta^{(2)}(\xi' - \tilde{\xi})\epsilon_{ij}d\tilde{\xi}^i d\xi'^j\Big). \tag{6.6.124}$$

where φ_λ and φ_0 are the scalar fields at arbitrary point ξ and a reference point O respectively. The object in the parenthesis counts the number of oriented intersections between two curves [85]. The solution above is independent of path, hence, it depends only on the reference point O. If we set the point O to be outside the loop C,

$$\varphi_\lambda - \varphi_0 = \begin{cases} -\frac{q\lambda}{2}, & \text{if } \xi \text{ is inside the loop } C \\ 0, & \text{otherwise.} \end{cases} \tag{6.6.125}$$

This allow us to compute the expectation value of the Wilson loop in 2D Yang-Mills theory (6.6.122) as

$$\langle W[C]\rangle = \frac{1}{Z}\sum_\lambda \int_0^\infty d\varphi_0 \sum_{n=-\infty}^{\infty} \exp\left[\frac{-i}{q}(4\pi n)\varphi_0 - e^2\int_{\mathcal{M}} d^2\xi\sqrt{g}\varphi_\lambda^2\right] \tag{6.6.126}$$

The infinite m limit of the Dirichlet kernel, $D_m(x)$, represents the Dirac delta function as

$$\lim_{m\to\infty} D_m(x) = \lim_{m\to\infty}\sum_{k=-m}^{m} e^{imx} = 2\pi\delta(x) \tag{6.6.127}$$

where $x \in [0, 2\pi]$. Therefore, (6.6.126) becomes

$$\langle W[C]\rangle = \frac{1}{Z}\sum_\lambda \int_0^\infty d\varphi_0\ \frac{q}{2}\delta(\varphi_0 \bmod \frac{q}{2})\exp\Big[-e^2\Big(\int_\Gamma d^2\xi\sqrt{g}\varphi_\lambda^2 + \int_{\mathcal{M}/\Gamma} d^2\xi\sqrt{g}\varphi_\lambda^2\Big)\Big] \tag{6.6.128}$$

In the above expression, we separate the region $\mathcal{M}$ into Γ and $\mathcal{M}/\Gamma$ where Γ is all the region inside the loop C with the boundary $\partial\Gamma = C$. Denoting the surface area of the region Γ and $\mathcal{M}/\Gamma$ by A_1 and A_2 subsequently together with (6.6.125), the relation (6.6.128) takes the form

$$\langle W[C]\rangle = \frac{q}{2Z}\sum_\lambda\sum_{N=0}^{\infty}\exp\Big[-\Big(\frac{eq}{2}\Big)^2\Big(A_1(N-\lambda)^2 + A_2N^2\Big)\Big]. \tag{6.6.129}$$

In the case of $SU(2)$, if we consider the eigenvalues of $\hat{\phi}$ in the fundamental representation, $\lambda = \pm 1/2$. This turns the expression (6.6.129) into

$$\begin{aligned}\langle W[C]\rangle &= \frac{q}{2Z}\left(\sum_{N=-\infty}^{\infty} \exp\left[-e_{\mathrm{YM}}^2\left(A_1(N+1/2)^2 + A_2 N^2\right)\right] + \exp\left[-\frac{e_{\mathrm{YM}}^2}{4}A_1\right]\right) \\ &= \frac{q}{2Z}\left(\vartheta\left(\frac{ie^2A_1}{2\pi};\frac{ie^2A}{\pi}\right)+1\right)\exp\left[-\frac{e_{\mathrm{YM}}^2}{4}A_1\right] \end{aligned} \tag{6.6.130}$$

where we re-define the Yang-Mills coupling constant e_{YM} as $\frac{eq}{2}$ and $\vartheta(z;\tau)$ is the Jacobi's third theta function defined as

$$\vartheta(z;\tau) = \sum_{N=-\infty}^{\infty} \exp(2\pi i N z + \pi i N^2 \tau). \tag{6.6.131}$$

In the case that $\mathcal{M}$ is an infinitely large sphere, i.e. $A_2 \to \infty$, the vacuum expectation value of the Wilson loop (6.6.129) turns into

$$\langle W[C]\rangle = \frac{q}{Z}\exp\left[-\frac{e_{\mathrm{YM}}^2}{4}A_1\right] \tag{6.6.132}$$

as the theta function becomes unity at this limit.

The result (6.6.132) shows that the expectation value of the Wilson loop for 2D Yang-Mills theory obtained by the effective topological BF theory satisfies the area law. This agrees with known results [109–111] as far as the exponent is concerned, which is the dominant piece. To compute the prefactor would require the computing the determinants arising from the Guassian integrals generalising the argument given above for the $SU(2)$ partition function.

Chapter 7

Concluding Remarks

In this thesis we have examined connections between Yang-Mills and string theories in the two different limits which are the limits of infinite and zero string tensions. It was long known that, in the first limit, string theory reproduces scattering amplitudes of the Yang-Mills theories. To be more precise, it leads to α' corrections to the Yang-Mills Lagrangians. The calculation for obtaining the effective field theory upto the order of α'^2 based on Tseytlin's work [40] was reviewed in chapter 2. This method is known as the S-matrix approach in which the coefficients of the effective field theory are determined by comparing with scatting amplitudes in string theory.

In chapter 3, geometrical diagrams based on linear monodromy relations between open string amplitudes, namely Plahte diagrams, were explored. Colour-ordered open string amplitudes and kinematic variables are represented in these diagrams as polygonal sides and external angles respectively. We have generalised the diagrams to complex momenta when the amplitudes have a meromorphic continuation. For complex momenta, the diagrams are deformed such that external angles are shifted by the difference between internal phases of adjacent amplitudes and the sides themselves are re-scaled based on the imaginary components of kinematic variables.

The Plahte diagrams for five-particle scattering are depicted as quadrilaterals. By combining different quadrilaterals together, we were able to express the KLT relations relating closed and open string amplitudes for five-point scattering as geometrical expressions. Furthermore, we used the diagrams to re-derive the fact that all five-point amplitudes can be expressed in terms of two selected amplitudes [37,58].

Mixed open and closed string amplitudes were also investigated using the geometrical expression of the KLT relations. It was found that the five-point closed tachyon string amplitudes with any two momenta set equal can be expressed as a quadratic in the disk amplitudes describing three open string and one closed string. This result holds for all excited states.

We described a connection between Plahte diagrams and BCFW on-shell recursion relations. We noticed that a triangle obtained from a diagonal line of the diagrams for five-gluon scattering coincides with the BCJ relations derived from the BCFW recursion relations of the five-gluon scattering amplitudes.

The formulation of Abelian Yang-Mills theory as a tensionless string with contact interactions was discussed in chapter 4. In this correspondence, the expectation value of the Wilson loop in Yang-Mills theory is equal to the worldsheet average of the exponential of the contact interaction (4.3.56) with the help of worldsheet supersymmetry [8, 9].

In chapter 5, we discussed possible string models intending to describe non-Abelian Yang-Mills theories in the tensionless limit based on [85] and [90]. The first model suggests insertions of Lie algebra valued field, $\phi^A = \psi^\dagger T^A \psi$, into the string worldsheet whose boundary propagator is described by [87]. The intersection number of curves (5.1.11) was used to generalise the dynamics of ϕ^A in the worldsheet interior. Similar to the previous model, the Lie algebra valued fields are also introduced into the worldsheet but this time the field dynamics are described by the topological BF action (1.1.41). However, both models lack the correct structure to reproduce three-point self-interaction terms in the Yang-Mills theory. At the end of the chapter, we provided some suggestions towards a further modification of the contact interaction term which may include the self-interaction contributions.

In chapter 6, an effective theory for $2D$ non-Abelian topological BF theory is investigated. The calculation was implemented by expanding the fields in the Cartan-Weyl basis. By performing a Gaussian functional integration, we obtained the effective theory with the Lagrangian (6.4.82) together with the constraint addressed in (6.3.66). The constraint implies that the magnitude of a scalar field, $|\phi|$, as well as the quantity $d^{ABC}\phi_A\phi_B\phi_C$ are constant throughout the space.

The adjugate and the determinant of the matrix $\widetilde{\mathcal{M}}$ play an important part in (6.4.82) where $\widetilde{\mathcal{M}}$ is defined as (6.3.74). We developed a diagrammatic approach to represent these objects. The diagrams are constructed from vertices connected to each other by lines. No more than two line are allowed to connect with one vertex. There are two type of diagrams, i.e. a strand and a loop. The adjugate matrix and the matrix determinant were expressed as summations over products of these diagrams as in (6.4.80) and (6.4.83) respectively.

For the case of $SU(2)$ and the manifold having the topology of a unit sphere, the effective action (6.1.16) contains the winding number of the field $\hat{\phi}$ which maps a point on the manifold into a point on S^2. By using the $SU(2)$ effective action and summing over this winding number, we re-formulated the partition function on a sphere of $SU(2)$ Yang-Mills theory.

At last, we investigated the BF theory coupled to a source term for the gauge field. We exploited the gauge symmetry to deal with the path-ordering of the Wilson loop. The result showed that the vacuum expectation value of the Wilson loop exhibits the area law agreeing with the well-known results [109–111].

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Appendix A

A.1 KLT relations

In this section, we review a derivation of the KLT relations from string theory following [53, 112] by factorizing a closed string into a sum of products between two open strings. Consider an expression for n-point tree-level closed string amplitude

$$\mathcal{A}_n^{\rm cl} = i g_s^n C_{S^2} \int \prod_{i=2}^{n-2} d^2 z_i |z_i|^{4\alpha' k_1 \cdot k_i} |z_i - 1|^{4\alpha' k_{n-1}\cdot k_i} \prod_{i<j\leq n-2} |z_j - z_i|^{4\alpha' k_i \cdot k_j} f(z_i) g(\bar{z}_i) \tag{A.1.1}$$

where we fix the points $z_1 = 0, z_{n-1} = 1$ and $z_n = \infty$. The functions $f(z_i)$ and $g(\bar{z}_i)$ contains no branch cuts. They come from the operator product expansion of vertex operators. The explicit forms vary depending on external states of strings. Note that the amplitude (A.1.1) describes the scattering of n closed strings with momenta $2k_1, 2k_2, \ldots, 2k_n$.

If we write $z_i = x_i + iy_i$, then the complex variables y_i are integrated along the real axis from $-\infty$ to ∞. There exists branch points at $y_i = \pm i x_i, \pm i(1-x_i), \ldots$. All the branch points are located along the imaginary axis. Therefore, we can deform a contour integral of y_i from the original contour C_1 along the real axis to the contour C_2 along (almost) the pure imaginary axis as

$$y_i \to i e^{-2i\epsilon} y_i \simeq i(1 - 2i\epsilon) y_i. \tag{A.1.2}$$

We insert the exponential term to ensure that the new contour line avoids all the branch points located along the imaginary axis. This changes the integrand of

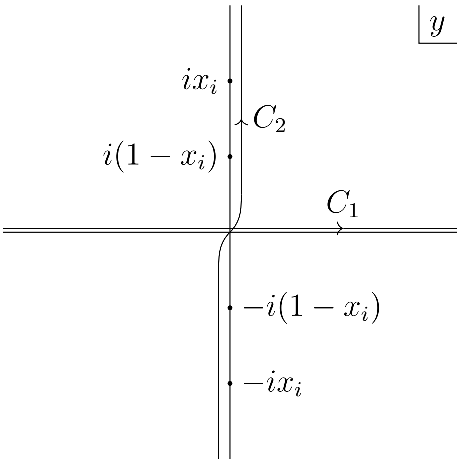

Figure A.1: The complex y plane for $0 < x < 1$ showing the original contour C_1 and the deformed one C_2.

(A.1.1) to be

$$\begin{aligned}
|z_i|^{4\alpha' k_1\cdot k_i} \to& \Big[(x_i)^2 - (y_i)^2 + 4i\epsilon(y_i)^2\Big]^{2\alpha' k_1\cdot k_i}, \\
|z_i - 1|^{4\alpha' k_{n-1}\cdot k_i} \to& \Big[(x_i)^2 - (y_i)^2 - 2x_i + 1 + 4i\epsilon(y_i)^2\Big]^{2\alpha' k_{n-1}\cdot k_i}, \\
|z_j - z_i|^{4\alpha' k_i\cdot k_j} \to& \Big[(x_j - x_i)^2 - (y_j - y_i)^2(1 - 4i\epsilon)\Big]^{2\alpha' k_i\cdot k_j}.
\end{aligned} \tag{A.1.3}$$

If we now introduce new variables

$$\xi = x_i + y_i, \qquad \eta = x_i - y_i. \tag{A.1.4}$$

and define $\delta_i = \xi_i - \eta_i$, we can re-express the right-hand side of (A.1.3) as

$$\begin{aligned}
&\big[(\xi_i - i\epsilon\delta_i)(\eta_i + i\epsilon\delta_i)\big]^{2\alpha' k_1\cdot k_i}, \big[(\xi_i - 1 - i\epsilon\delta_i)(\eta_i - 1 + i\epsilon\delta_i)\big]^{2\alpha' k_{n-1}\cdot k_i}, \\
&\qquad\qquad \big[(\xi_i - \xi_j - i\epsilon(\delta_i - \delta_j))(\eta_i - \eta_j + i\epsilon(\delta_i - \delta_j))\big]^{2\alpha' k_i\cdot k_j}
\end{aligned} \tag{A.1.5}$$

respectively. Putting everything together, the closed string amplitude (A.1.1) takes the form

$$\begin{aligned}
\mathcal{A}_n^{\text{cl}} =& \Big(\frac{i}{2}\Big)^{n-3} i(g_s^{(l)})^n C_{S^2} \int_{-\infty}^{\infty} \prod_{i=2}^{n-2} d\xi_i d\eta_i \; f(\eta_i)\; g(\xi_i) \\
&\times (\xi_i - i\epsilon\delta_i)^{2\alpha' k_1\cdot k_i}(\eta_i + i\epsilon\delta_i)^{2\alpha' k_1\cdot k_i}(\xi_i - 1 - i\epsilon\delta_i)^{2\alpha' k_{n-1}\cdot k_i}(\eta_i - 1 + i\epsilon\delta_i)^{2\alpha' k_{n-1}\cdot k_i} \\
&\times \prod_{i<j\le n-2} (\xi_i - \xi_j - i\epsilon(\delta_i - \delta_j))^{2\alpha' k_i\cdot k_j}(\eta_i - \eta_j + i\epsilon(\delta_i - \delta_j))^{2\alpha' k_i\cdot k_j}
\end{aligned} \tag{A.1.6}$$

where the factor $(i/2)^{n-3}$ is the Jacobian appearing from changing the integration variables.

To evaluate the integral, let first assume that at least one $\xi_i \in (-\infty, 0)$ and consider the integration with respect to η_i

$$\int_{-\infty}^{\infty} d\eta_i\; f(\eta_i)(\eta_i+i\epsilon\delta_i)^{2\alpha' k_1\cdot k_i}(\eta_i-1+i\epsilon\delta_i)^{2\alpha' k_{n-1}\cdot k_i}\prod_{i<j\leq n-2}(\eta_i-\eta_j+i\epsilon(\delta_i-\delta_j))^{2\alpha' k_i\cdot k_j}. \tag{A.1.7}$$

As $\xi_i < 0$, one can determine the values of imaginary ϵ terms near the branch points as

$$\eta_i \sim 0 \longrightarrow \delta_i \sim \xi_i < 0 \tag{A.1.8}$$

$$\eta_i \sim 1 \longrightarrow \delta_i \sim \xi_i - 1 < 0 \tag{A.1.9}$$

$$\eta_i \sim \eta_j \longrightarrow \delta_i - \delta_j \sim \xi_i - \xi_j < 0 \;\text{ when } \xi_i < \xi_j. \tag{A.1.10}$$

Consequently, to avoid all the branch points, the contour line of η_i is deformed below the real axis at the points $\eta_i = 0$ and $\eta_i = 1$. If we further assume that ξ_i is the smallest value among ξ_j, at the branch points $\eta_i = \eta_j$, the contour also deforms below the real axis. This means we can close the contour at infinity in the lower half plane and because there exists no possible poles inside the contour, the integral vanishes. Moreover, if one apply the similar argument to the case that at least one $\xi_i > 1$, one encounters the same result. This means that to obtain non-zero amplitudes, values of all ξ_i must lie between 0 and 1.

As a result, we can write (A.1.6) as

$$\mathcal{A}_n^{\text{cl}} = \sum_P \mathcal{A}_n^{\text{op}}(P)\mathcal{M}_n(P) \tag{A.1.11}$$

where $\mathcal{A}_n^{\text{op}}(P)$ is the n-point color-ordered open string amplitude of the ordering P resulted from integrating ξ_i. $\mathcal{M}_n(P)$ refers to the remaining integral with respect to η_i subjecting to the ordering P. The sum over P denotes sum over all permutations of ξ_i in the region $(0, 1)$. For example, we can write a certain ordering P as $\{0, (\xi_2, \xi_3, \ldots, \xi_{n-2}), 1\}$ where the order of all variables ξ_i in the bracket is to be permuted.

From now, let first content ourselves to consider the specific ordering $P' =$

$\{0, \xi_2, \xi_3, \ldots, \xi_{n-2}, 1\}$ giving the color-ordered open string amplitude in the form

$$\mathcal{A}_n^{\mathrm{op}}(P') = \int_{0<\xi_2<\ldots<\xi_{n-2}<1} \prod_{i=2}^{n-2} d\xi_i g(\xi_i)(\xi_i)^{2\alpha' k_1\cdot k_i} \\ \times (1-\xi_i)^{2\alpha' k_{n-1}\cdot k_i} \prod_{i<j\leq n-2} (\xi_j - \xi_i)^{2\alpha' k_i\cdot k_j} \tag{A.1.12}$$

where the infinitesimal ϵ terms are omitted. Comparing to (A.1.6), we wrote $(1-\xi_i)^{2\alpha' k_{n-1}\cdot k_i}$ instead of $(\xi_i-1)^{2\alpha' k_{n-1}\cdot k_i}$ and $(\xi_j-\xi_i)^{2\alpha' k_i\cdot k_j}$ instead of $(\xi_i-\xi_j)^{2\alpha' k_i\cdot k_j}$ just to satisfy the definition of color-ordered open string amplitude. However, to compensate our action, we have to make a similar change in η_i-integration. Therefore, the η_i-integration is written by

$$\int_{-\infty}^{\infty} \prod_{i=2}^{n-2} d\eta f(\eta_i)(\eta_i + i\epsilon\delta_i)^{2\alpha' k_1\cdot k_i}(1-\eta_i - i\epsilon\delta_i)^{2\alpha' k_{n-1}\cdot k_i} \prod_{i<j\leq n-2} (\eta_j - \eta_i + i\epsilon(\delta_j - \delta_i))^{2\alpha' k_i\cdot k_j}. \tag{A.1.13}$$

We can determine the behavior of ϵ terms near branch points. As $\eta_i \sim 0$, $i\epsilon\delta_i \sim i\epsilon\xi_i$ giving a positive imaginary number, the contour goes around this point above the real axis to avoid the branch point. For $\eta_i \sim 1$, $i\epsilon\delta_i \sim i\epsilon(\xi_i - 1)$, it yields negative imaginary number, so the contour goes below the real axis. Finally, as $\eta_i \sim \eta_j$, $i\epsilon(\delta_j - \delta_i) \sim i\epsilon(\xi_j - \xi_i)$. In case $i < j$, the contour lies below the real axis. Conversely, if $i > j$, the contour goes above the real axis instead.

The next step is to decide how to close the η_i-contours. There are two possible ways to do. First, If we choose to close the contour in the lower half plane, the contour is then deformed to close the point $\eta_i = 0$ to the left. Second, the contour is close the point $\eta_i = 1$ to right. We have freedom to choose ways to deform the contours (left or right).

To obtain the expression of the colour-ordered amplitude, some terms of the integrand (A.1.13) need to be corrected with phase factors using the relation

$$z^c = \begin{cases} e^{i\pi c}(-z)^c, & \mathrm{Im}(z) \geq 0 \\ e^{-i\pi c}(-z)^c, & \mathrm{Im}(z) < 0 \end{cases} \tag{A.1.14}$$

where Re$(z)< 0$.

To compute the η_i-integration. There are totally $n-3$ variables to be integrated. As stated earlier, we have two ways to close the contours leading to the deformations

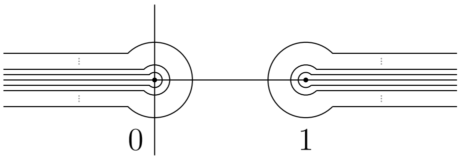

Figure A.2: The contours of integration for the η_i variables. The contours enclose the point 0 to the left for $2 \le i \le m-1$ and enclose the point 1 to the right for $m \le i \le n-1$.

of contours either to the left or right around the point 0 and 1 respectively. With this fact, we pick η_i where $i = 2, 3, \ldots, m-1$ whose contour lines are closed to the left while the remaining η_i-contours are closed to the right as illustrated in figure A.2. The number m is arbitrary. If we set $m = 2$ or $m = n-1$, it means all are deformed to the right or to the left.

Determine the contour integral (A.1.13) which involves only variables η_2. We find that

$$\begin{aligned}
&\int_{C_2} d\eta_2 f(\eta_2)(\eta_2)^{2\alpha' k_1\cdot k_2}(1-\eta_2)^{2\alpha' k_{n-1}\cdot k_2}\prod_{j=3}^{n-2}(\eta_j-\eta_2)^{2\alpha' k_2\cdot k_j}\\
&= 2i\sin(2\pi\alpha' k_1\cdot k_2)\int_{-\infty}^{0} d\eta_2 f(\eta_2)(-\eta_2)^{2\alpha' k_1\cdot k_2}(1-\eta_2)^{2\alpha' k_{n-1}\cdot k_2}\prod_{j=3}^{n-2}(\eta_j-\eta_2)^{2\alpha' k_2\cdot k_j}
\end{aligned} \tag{A.1.15}$$

where (A.1.14) was used to obtain the second line.

The contour integral of variables η_3 is

$$\begin{aligned}
&\int_{C_3} d\eta_3 f(\eta_3)(\eta_3)^{2\alpha' k_1\cdot k_3}(1-\eta_3)^{2\alpha' k_{n-1}\cdot k_3}(\eta_3-\eta_2)^{2\alpha' k_2\cdot k_3}\prod_{j=4}^{n-2}(\eta_j-\eta_3)^{2\alpha' k_3\cdot k_j}\\
&= 2i\sin(2\pi\alpha' k_1\cdot k_3)\int_{\eta_2}^{0} d\eta_3 f(\eta_3)(-\eta_3)^{2\alpha' k_1\cdot k_3}(1-\eta_3)^{2\alpha' k_{n-1}\cdot k_3}(\eta_3-\eta_2)^{2\alpha' k_2\cdot k_3}\\
&\qquad\qquad\times\prod_{j=4}^{n-2}(\eta_j-\eta_3)^{2\alpha' k_2\cdot k_j}\\
&+ 2i\sin(2\pi\alpha'(k_1+k_2)\cdot k_3)\int_{-\infty}^{\eta_2} d\eta_3 f(\eta_3)(-\eta_3)^{2\alpha' k_1\cdot k_3}(1-\eta_3)^{2\alpha' k_{n-1}\cdot k_3}(\eta_2-\eta_3)^{2\alpha' k_2\cdot k_3}\\
&\qquad\qquad\times\prod_{j=4}^{n-2}(\eta_j-\eta_3)^{2\alpha' k_2\cdot k_j}.
\end{aligned} \tag{A.1.16}$$

The integral splits into two terms representing two possible cases which are the η_3 is grater and lower than η_2. We then do the calculation iteratively until the η_{m-1}-integration is done.

The calculation of η_i-integral where $i = m, m+1, \ldots, n-2$ whose contours are pulled to the right can be proceed in the similar manner. For example, integration of η_{n-2} takes the form

$$\int_{C_{n-2}} d\eta_{n-2} f(\eta_{n-2})(\eta_{n-2})^{2\alpha' k_1\cdot k_{n-2}}(1-\eta_{n-2})^{2\alpha' k_{n-1}\cdot k_{n-2}} \prod_{j=2}^{n-3}(\eta_{n-2}-\eta_j)^{2\alpha' k_{n-2}\cdot k_j}$$
$$= 2i\sin(2\pi\alpha' k_{n-1}\cdot k_{n-2}) \int_1^\infty d\eta_{n-2} f(\eta_{n-2})(\eta_{n-2})^{2\alpha' k_1\cdot k_{n-2}}(\eta_{n-2}-1)^{2\alpha' k_{n-1}\cdot k_{n-2}}$$
$$\prod_{j=2}^{n-3}(\eta_{n-2}-\eta_j)^{2\alpha' k_{n-2}\cdot k_j}. \quad \text{(A.1.17)}$$

As before, we perform the calculation until the η_m contour is executed. By combining all η_i-integral terms together, the η-integral (A.1.13) is written as summation of color-ordered open string amplitudes, $\tilde{\mathcal{A}}_n^{\text{op}}(\gamma(2,3,\ldots,m-1),1,$
$n-1,\beta(m,m+1,\ldots,n-2),n)$ weighted by products of sine functions. The set of numbers in the parenthesis denotes the ordering of η_i where γ and β represent permutation of variables inside.

Generally, the expression for the total η_i integral part is

$$\sim \sum_{\gamma,\beta} \mathcal{S}_{\alpha'}[\gamma(2,...,m-1)|2,...,m-1]_{k_1} \mathcal{S}_{\alpha'}[\beta(m,...,n-2)|m,...,n-2]_{k_{n-1}}$$
$$\times \tilde{\mathcal{A}}_n^{\text{op}}(\gamma(2,3,\ldots,m-1),1,n-1,\beta(m,m+1,\ldots,n-2),n). \quad \text{(A.1.18)}$$

$\mathcal{S}_{\alpha'}[\gamma|\sigma]_p$ is called momentum kernel which contains sine terms from the η_i integration. It is defined as

$$\mathcal{S}_{\alpha'}[i_1,\ldots,i_k|j_1,\ldots,j_k]_p \equiv \left(\frac{1}{\pi\alpha'}\right)^k \prod_{t=1}^{k} \sin\left(2\pi\alpha'\left(p\cdot k_{i_t} + \sum_{q>t}^{k}\Theta(i_t,i_q)k_{i_t}\cdot k_{i_q}\right)\right) \quad \text{(A.1.19)}$$

where $\Theta(i_t,i_q)$ equal to 1 if the ordering of i_t and i_q in $\{i_1,\ldots,i_k\}$ is opposite to $\{j_1,\ldots,j_k\}$, and 0 if the ordering of both is the same. Besides, $\mathcal{S}_{\alpha'}[\emptyset|\emptyset]_p = 1$ for the empty set.

Therefore, the tree-level closed string amplitude (A.1.6) can be written as a product of two color-ordered open string amplitudes, $\mathcal{A}_n^{\text{op}}(P)$ and $\tilde{\mathcal{A}}_n^{\text{op}}(\tilde{P})$ which is

$$\begin{aligned}\mathcal{A}_n^{\text{cl}} &= -8i\kappa^{n-2}\sum_{\sigma}\sum_{\gamma,\beta}\mathcal{S}_{\alpha'}[\gamma(\sigma(2,...,m-1))|\sigma(2,...,m-1)]_{k_1}\\ &\times\mathcal{S}_{\alpha'}[\beta(\sigma(m,...,n-2))|\sigma(m,...,n-2)]_{k_{n-1}}\mathcal{A}_n^{\text{op}}(1,\sigma(2,\ldots,n-2),n-1,n)\\ &\times\tilde{\mathcal{A}}_n^{\text{op}}(\gamma(\sigma(2,\ldots,m-1)),1,n-1,\beta(\sigma(m,\ldots,n-2)),n).\end{aligned} \tag{A.1.20}$$

where we define the new parameter κ as $\pi g_s\alpha'^{(n-4)/(n-2)}$. This is the well-known KLT relation. The expression above comprises of $(n-3)!(m-2)!(n-m-1)!$ terms. For $m=2$ and $m=n-1$ which is the case that all contours are totally closed to the right and left correspondingly, they provide a maximum terms at $(n-3)!(n-3)!$. The choice made by the original KLT paper [53] choosing half of the contours to the left and the other half to the right, i.e. $m=\lceil n/2\rceil$ corresponds to the minimum terms possible at $(n-3)!(\lceil n/2\rceil-2)!(\lfloor n/2\rfloor-1)!$ terms.

We have obtained the KLT relations in string theory. To gain a result in field theory, the field theory limit $\alpha'\to 0$ is made. This changes all quantities to their corresponding field theory expressions, i.e. $\mathcal{A}\to A$ and $\mathcal{S}_{\alpha'}\to\mathcal{S}$. The momentum kernel in the field theory limit takes the form

$$\mathcal{S}[i_1,\ldots,i_k|j_1,\ldots,j_k]_p\equiv\prod_{t=1}^{k}\big(s_{pi_t}+\sum_{q>t}^{k}\Theta(i_t,i_q)s_{i_ti_q}\big) \tag{A.1.21}$$

where $s_{ij}\equiv(k_i+k_j)^2=2k_i\cdot k_j$. Consequently, this gives rise to the connection between gravitation amplitude as square of gluonic amplitudes in field theory

$$\begin{aligned}A_n^{\text{cl}} &= -8i\kappa^{n-2}\sum_{\sigma}\sum_{\gamma,\beta}\mathcal{S}[\gamma(\sigma(2,...,m-1))|\sigma(2,...,m-1)]_{k_1}\\ &\times\mathcal{S}[\beta(\sigma(m,...,n-2))|\sigma(m,...,n-2)]_{k_{n-1}}A_n^{\text{op}}(1,\sigma(2,\ldots,n-2),n-1,n)\\ &\times\tilde{A}_n^{op}(\gamma(\sigma(2,\ldots,m-1)),1,n-1,\beta(\sigma(m,\ldots,n-2)),n).\end{aligned} \tag{A.1.22}$$

A.2 Nambu-Goto action

In this section, we would like to show how the Nambu-Goto action (up to a divergence) arises from the first term of (4.3.57). For convenience, we call the worldsheet

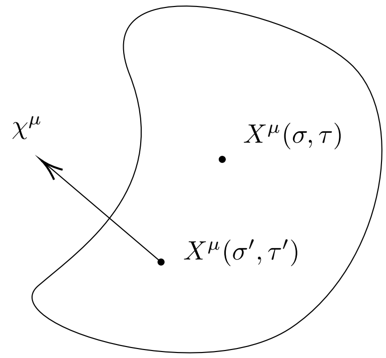

Figure A.3: Illustration of two neighbouring points on the worldsheet and a point χ^μ living outside the worldsheet

coordinates ξ^i as σ and τ to represent space-like and time-like worldsheet coordinates. Therefore, the first term of (4.3.57) can be written as

$$\frac{q^2}{2}\int d\sigma d\tau d\sigma' d\tau' \delta^4(X(\sigma,\tau)-X(\sigma',\tau')) \\ \times\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\sigma'}\frac{\partial X^\nu}{\partial\tau}\frac{\partial X_\nu}{\partial\tau'}-\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X_\mu}{\partial\tau'}\frac{\partial X^\nu}{\partial\tau}\frac{\partial X_\nu}{\partial\sigma'}\right). \tag{A.2.23}$$

Obviously, if we simply set $(\sigma,\tau)=(\sigma',\tau')$, we would obtain the divergence $\delta^4(0)$ which is hard to further analyse. Instead, we will treat the points at these two worldsheet coordinates to be two separate points yet very close to each other. Mathematically speaking, we can write

$$X^\mu(\sigma',\tau')=X^\mu(\sigma,\tau)+(\sigma'-\sigma)\frac{\partial}{\partial\sigma}X^\mu(\sigma,\tau)+(\tau'-\tau)\frac{\partial}{\partial\tau}X^\mu(\sigma,\tau). \tag{A.2.24}$$

Since the integral possesses the delta function in four dimensions, it is better to introduce two extra parameters λ_1 and λ_2 to locate a point χ^μ which lives outside the worldsheet. We define the point χ^μ as

$$\chi^\mu(\sigma',\tau',\lambda_1,\lambda_2)=X^\mu(\sigma',\tau')+\lambda_1\hat{n}_1^\mu+\lambda_2\hat{n}_2^\mu \tag{A.2.25}$$

where $\hat{n}_1$ and $\hat{n}_2$ are unit vectors. They are orthogonal to $\frac{\partial X}{\partial\sigma}$ and $\frac{\partial X}{\partial\tau}$ and each other as well as $\hat{n}_1\cdot\hat{n}_1=\hat{n}_2\cdot\hat{n}_2=-1$. Note that $\chi^\mu=X^\mu(\sigma',\tau')$ when $\lambda_1=\lambda_2=0$. The locations of $X^\mu(\sigma,\tau)$, $X^\mu(\sigma',\tau')$ and χ^μ are illustrated in the figure A.3.

We then replace the delta function in (A.2.23) by

$$\begin{aligned}\delta^4(\chi(\sigma',\tau',\lambda_1,\lambda_2)-X(\sigma,\tau)) &= \delta^4((\sigma'-\sigma)\partial_\sigma X+(\tau'-\tau)\partial_\tau X+\lambda_1\hat{n}_1+\lambda_2\hat{n}_2)\\ &= \frac{1}{\left|\det\frac{\partial\chi^\mu}{\partial\sigma^\alpha}\right|}\delta(\sigma'-\sigma)\delta(\tau'-\tau)\delta(\lambda_1)\delta(\lambda_2) \qquad \text{(A.2.26)}\end{aligned}$$

where $\sigma^\alpha=\{\sigma,\tau,\lambda_1,\lambda_2\}$. The determinant term can be obtained from the relation

$$\det\frac{\partial\chi^\mu}{\partial\sigma^\alpha}\equiv\det N=\sqrt{\frac{\det(N^\intercal\eta N)}{\det\eta}} \qquad \text{(A.2.27)}$$

where η is the Minkowski matrix. Consequently, we can find that

$$\det N=\left[\left(\frac{\partial X}{\partial\sigma}\cdot\frac{\partial X}{\partial\tau}\right)^2-\left(\frac{\partial X}{\partial\sigma}\right)^2\left(\frac{\partial X}{\partial\tau}\right)^2\right]^{1/2} \qquad \text{(A.2.28)}$$

When we substitute (A.2.26), (A.2.28) into (A.2.23) together with setting $\lambda_1=\lambda_2=0$, it gives

$$\frac{q^2}{2}\delta^2(0)\int d\sigma d\tau\sqrt{-\det N}. \qquad \text{(A.2.29)}$$

The above integral is nothing but the well-known Nambu-Goto action as claimed.

A.3 Expressions for Matrix Multiplications of the Matrix $\mathcal{F}$

The expressions for matrix multiplication of the matrix $\mathcal{F}$ are given as follows:

$$
\mathcal{F}^{A}{}_{B}\mathcal{F}^{BC} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2(\phi_1)^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -2(\phi_1)^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{2}(\phi_1 - 2\phi_2)^2 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{2}(\phi_1 - 2\phi_2)^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2}(\phi_1 + 2\phi_2)^2 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2}(\phi_1 + 2\phi_2)^2 & 0
\end{pmatrix}
\tag{A.3.30}
$$

$$\mathcal{F}^A{}_B\mathcal{F}^B{}_C\mathcal{F}^{CD} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i(\phi_1)^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2i(\phi_1)^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{4}(\phi_1 - 2\phi_2)^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{4}(\phi_1 - 2\phi_2)^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-i}{4}(\phi_1 + 2\phi_2)^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{4}(\phi_1 + 2\phi_2)^3 & 0 \end{pmatrix} \tag{A.3.31}$$

$$\mathcal{F}^A{}_B\mathcal{F}^B{}_C\mathcal{F}^C{}_D\mathcal{F}^{DE} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(\phi_1)^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2(\phi_1)^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8}(\phi_1 - 2\phi_2)^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8}(\phi_1 - 2\phi_2)^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}(\phi_1 + 2\phi_2)^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}(\phi_1 + 2\phi_2)^4 & 0 \end{pmatrix} \tag{A.3.32}$$

$$\mathcal{F}^A{}_B\mathcal{F}^B{}_C\mathcal{F}^C{}_D\mathcal{F}^D{}_E\mathcal{F}^{EF} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i(\phi_1)^5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2i(\phi_1)^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-i}{16}(\phi_1-2\phi_2)^5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{16}(\phi_1-2\phi_2)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{16}(\phi_1+2\phi_2)^5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-i}{16}(\phi_1+2\phi_2)^5 & 0 \end{pmatrix} \tag{A.3.33}$$

$$
\mathcal{F}^{A}{}_{B}\mathcal{F}^{B}{}_{C}\mathcal{F}^{C}{}_{D}\mathcal{F}^{D}{}_{E}\mathcal{F}^{E}{}_{F}\mathcal{F}^{FG} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2(\phi_1)^6 & 0 & 0 & 0 & 0 \\
0 & 0 & -2(\phi_1)^6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{-1}{32}(\phi_1 - 2\phi_2)^6 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-1}{32}(\phi_1 - 2\phi_2)^6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{32}(\phi_1 + 2\phi_2)^6 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{32}(\phi_1 + 2\phi_2)^6 & 0
\end{pmatrix}
\tag{A.3.34}
$$