On Compact Hyperbolic Coxeter Polytopes with Few Facets

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by

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A thesis presented for the degree of
Master of Science by Research in Mathematical Sciences

Department of Mathematical Sciences
Durham University
September 2021

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Abstract. This thesis is concerned with classifying and bounding the dimension of compact hyperbolic Coxeter polytopes with few facets. We derive a new method for generating the combinatorial type of these polytopes via the classification of point set order types. We use this to complete the classification of d-polytopes with $d+4$ facets for $d=4$ and 5. In dimensions 4 and 5, there are 341 and 50 polytopes, respectively, yielding many new examples for further study. By previous work of Felikson and Tumarkin, the only remaining dimension where new polytopes may arise is $d=6$. We furthermore show that any polytope of dimension 6 must have a missing face of size 3 or 4.

The second portion of this thesis provides new upper bounds on the dimension of compact hyperbolic Coxeter d-polytopes with $d+k$ facets for $k \leq 10$. It was shown by Vinberg in 1985 that for any k, we have $d \leq 29$, and no better bounds have previously been published for $k \geq 5$. In the process of proving the present bounds, we additionally show that there are no compact hyperbolic Coxeter 3-free polytopes of dimension higher than 13. As a consequence of our bounds, we prove that a compact hyperbolic Coxeter 29-polytope has at least 40 facets.

Contents

1. Acknowledgments

The author would like to sincerely thank her advisor, Pavel Tumarkin, for his expert support throughout the course of this research and for introducing her to this fascinating topic. His thorough reading and helpful suggestions have been crucial to the development of this work. She is also grateful to Anna Felikson for interesting discussions, for helping to share information about this research, and for maintaining the dynamic webpage on hyperbolic Coxeter polytopes.

The author thanks Wenyu Jin, Max Kontorovich, and Eric Winsor for their friendship, in addition to their help with implementing the coding portion of this project. She also thanks her family for their constant encouragement and love.

Additional thanks are due to the Marshall Commission for providing financial support and advice, making the author's study in the UK possible.

2. Introduction

Let $\mathbb{H}^d$ be the d-dimensional real hyperbolic space. A hyperbolic Coxeter polytope is a domain in $\mathbb{H}^d$ bounded by a collection of geodesic hyperplanes, such that each intersecting pair of hyperplanes meets at dihedral angle $\frac{\pi}{m}$ for some integer $m \geq 2$. Hyperbolic Coxeter polytopes are precisely the fundamental domains of discrete hyperbolic reflection groups. These polytopes also have relevance to the construction of orbifolds and manifolds, in particular some of minimal volume [18].

While Euclidean and spherical Coxeter polytopes were classified by Coxeter in 1934 [7], no complete classification is known in the hyperbolic case. Henceforth, all polytopes are assumed to be hyperbolic unless otherwise specified.

The partial classification of compact Coxeter polytopes has been obtained by restricting either the dimension, combinatorial type, or number of facets. A dynamic summary of this progress is maintained by Anna Felikson on her webpage `www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html`.

2.1. Classification of compact Coxeter d-polytopes with $d+4$ facets for $d \neq 6$

We first focus on restricting the number of facets with respect to the dimension. Compact Coxeter simplices, i.e., d-polytopes with $d+1$ facets, were classified by Lannér in 1950 [19]. These arise only in dimensions 2, 3, and 4. The compact Coxeter d-polytopes with $d+2$ facets are classified in [17] and [9]; these arise in dimensions 3 through 5 and must be a simplicial prism except in dimension 4. Esselmann [8] showed in 1994 that a compact Coxeter d-polytope with $d+3$ facets must satisfy $d \leq 8$, and that there is a unique polytope of dimension 8 (first constructed by Bugaenko [6]). In 2007, Tumarkin [23] completed the classification of compact Coxeter d-polytopes with $d+3$ facets, which arise in dimensions 2 through 6 and 8.

The first portion of this thesis is dedicated to furthering the classification of compact Coxeter d-polytopes with $d+4$ facets. In 2008, Felikson and Tumarkin [10] showed that such polytopes arise only in dimension at most 7, and furthermore that there is a unique compact Coxeter 7-polytope with 11 facets (originally constructed by Bugaenko [6]). We complete this classification in dimensions 4 and 5, with 341 polytopes in dimension 4 and 50 of dimension 5. This includes the first known Coxeter polytope in dimension > 3 with an angle of less than $\frac{\pi}{10}$ and the first known Coxeter polytope in dimension > 3 with an angle of $\frac{\pi}{7}$, along with many new essential polytopes. A polytope is *essential* if it is minimal with respect to the operations of taking the fundamental domain of a finite index reflection subgroup of the corresponding reflection group, or gluing two Coxeter polytopes along congruent facets (see [13] for further details). The present work combined with that of Felikson and Tumarkin yields a classification in all dimensions except 6, where the only known polytope was constructed by Bugaenko [6]. We show that a compact Coxeter 6-polytope with 10 facets must contain a missing face of size 3 or 4.

In order to obtain this classification, we develop a new method for restricting the possible combinatorial types of these polytopes. The Gale diagram of a d-dimensional polytope with n facets is an $n - d - 1$ arrangement of points in Euclidean space that encodes the combinatorial type of the polytope. In studying d-polytopes with $d + 4$ facets, this means their combinatorial types can be studied in terms of a 3-dimensional point arrangement. Moreover, each Gale diagram can be transformed into an affine Gale diagram, an arrangement of positive and negative points, to further reduce the dimension by 1 [24]. We show that the (affine) Gale diagrams of compact d-dimensional hyperbolic polytopes with $d+4$ facets can be generated by bipartitioning the points in all point set order types with $d + 4$ points (see Theorem 4.5 for further details). The point set order types of sizes up to 10 have been enumerated and made available through the Point Set Order Type Database [2]. Using this database, we produce a reasonably short list of possible combinatorial types for the polytopes of interest in dimensions 4 and 5. In dimension 6, the same methods can be applied, but the number of point set order types makes the process rather computationally demanding.

Having greatly restricted the possible combinatorial types in dimensions 4 and 5, we then determine whether each combinatorial type can be realised as one or more polytopes. This involves enumerating weighted graphs with restrictions on certain subgraphs and the spectral properties of their adjacency matrices. Though the search space is infinite, combinatorial and linear algebraic techniques (see, e.g., [24]) have previously been successful in reducing this to a computational problem. In particular, Tumarkin handled the analogous task for polytopes with $d+3$ facets by inspecting local determinants, face structures, and gluings of Lannér diagrams [23]. These methods are only partially effective for the polytopes with $d+4$ facets, due to the greater complexity of the polytopes and less restrictive missing face structures. We then utilise the computer algebra system `Mathematica` to check a finite number of cases in order to list all polytopes of a given combinatorial type, a technique which was recently used in classifying the compact Coxeter cubes [16].

2.2. **Bounding the dimension of polytopes with few facets**

In Sections 10 and 11, we shift our focus to bounding the dimension of certain compact Coxeter polytopes. It was shown in 1984 by Vinberg [25] that compact Coxeter polytopes do not arise in dimensions higher than 29. Vinberg proceeded by constructing certain weightings on the edges of the polytopes, and utilised a result of Nikulin [21] on the average number of vertices along a 2-dimensional face. In Section 10, we show that a slight modification of Vinberg's argument yields a stronger bound for 3-free polytopes, that is, polytopes having missing faces only of order 2. In particular, we show that compact Coxeter 3-free polytopes do not arise in dimension higher than 13.

In Section 11, we improve the bounds on the dimension of compact Coxeter d-polytopes with $d+k$ facets for $5 \leq k \leq 10$. In order to obtain an initial bound, we examine certain faces which must themselves be compact Coxeter polytopes, similar to the methods used by Felikson and Tumarkin [10] to bound the dimension when $k=4$. These ideas combined with the results in Section 10 yield the bounds in Theorem 11.7. The rest of the section is devoted to improving these bounds in certain cases, frequently referring to the classification of polytopes with fewer facets. One corollary of our improved bounds is that any compact Coxeter polytopes of dimension 29, i.e., the threshold of Vinberg's bound, must have at least $29+11=40$ facets.

3. Convex Polytopes and Their Combinatorial Types

3.1. **Convex polytope preliminaries**

A convex polytope in $\mathbb{H}^d$ can be defined as $\bigcap_{i \in I} H_i^-$, where I is an index set and H_i^- is a half-space containing P. Throughout this paper we will consider only polytopes of finite volume, or equivalently, those which can be obtained as the convex hull of a finite point set. We furthermore assume that the set of bounding hyperplanes is chosen minimally to define P. The intersection of a convex polytope with a bounding hyperplane is called a *facet* of the polytope. We identify the index set I with the set $\{0, 1, \ldots, d+k-1\}$, where $d+k$ is the number of facets of P. If the convex polytope P has a vertex at infinity, then P is said to be *non-compact*, otherwise we say P is *compact*. For the purposes of this section, we do not assume P is compact, though we only consider compact polytopes in the remainder of this paper. If each vertex is formed by the intersection of precisely d half spaces, then P is said to be *simple*. An equivalent condition is that every $(d-j)$-face is contained in precisely j facets.

Fix an enumeration of the facets of P as f_0, f_1, $\ldots$, f_{d+k-1}. Then each face $P \cap f_{i_0} \cap \cdots \cap f_{i_s}$ for $0 \leq i_0 < \cdots < i_s < d+k$ is denoted by the string $i_0 \ldots i_s$. A *missing face* of P is a list of facets whose intersection is empty, but such that the intersection of every proper subset of these facets is non-empty. That is, a missing face is an intersection of facets $i_0 \ldots i_s$ that is empty, but such that $i_0 \ldots i_{m-1} i_{m+1} \ldots i_s$ is non-empty for each $0 \leq m \leq s$. We refer to the set of missing faces of P as the *missing face list*. Note that every missing face list is an antichain by inclusion, i.e., no missing face is a subset of another. Two polytopes are said to be *isomorphic*, or of the same *combinatorial type*, provided there exists a bijective correspondence between their faces, such that two faces of the first polytope meet if and only if the corresponding faces of the second meet. In particular, two polytopes have the same combinatorial type if and only if they have the same set of missing faces, up to relabelling of the facets.

3.2. **Gale diagrams and affine Gale diagrams**

An important technique in classifying convex polytopes is representing a polytope by a "diagram" from which one can read off the face structure. Throughout this paper, we frequently reference *Gale diagrams*, introduced in a 1956 paper by David

Gale [14]. Note that while Gale diagrams are often defined using the vertices of a polytope, we look at the dual construction defined on the facets. We will consider only simple polytopes in this section, which include all compact Coxeter polytopes (see Remark 5.5).

Given a simple d-polytope P with $d+k$ facets, a Gale diagram of P consists of a set of $d+k$ points on $S^{k-2} \subset \mathbb{R}^{k-1}$ corresponding to the facets of P. These points can be obtained by applying a *Gale transform* to the affine dependences of the normal vectors to the facets (see [15] for details of the construction). These points encode the face structure of P in the following way: a set of facets $\{f_i : i \in I\}$ of P has a non-trivial intersection if and only if the points in the Gale diagram corresponding to $\{f_j : j \in [d+k] \setminus I\}$ have a convex hull containing the origin. Two Gale diagrams are *isomorphic* if their corresponding polytopes are combinatorially equivalent.

A set of $d+k$ points on S^{k-2} corresponds to a Gale diagram of a simple convex d-polytope with $d+k$ facets if and only if every half space bounded by a hyperplane through the origin contains at least two of the points. One can see that the latter condition is necessary by taking the convex hull of the points corresponding to $\{f_j : j \neq i\}$ for any $i \in [d+k]$; this convex hull should contain the origin, as any facet is itself a non-empty face.

In the case $k = 4$, Gale diagrams consist of point configurations on $S^2 \subseteq \mathbb{R}^3$. These seem rather difficult to classify, and thus a crucial step in our analysis is passing to the affine Gale diagrams. These affine variants encode the same information as Gale diagrams, but using partitioned sets of points in $\mathbb{R}^{k-2}$. Thus, the configurations are reduced to points on the Euclidean plane when $k = 4$, which can be classified using point set order types (further details in Section 4).

An *affine Gale diagram* of a d-polytope with $d+k$ facets consists of two (not necessarily disjoint) point sets, called "positive" and "negative", in $\mathbb{R}^{k-2}$ containing $d+k$ points in total (counted with multiplicity). An affine Gale diagram is obtained from a Gale diagram G by taking a hyperplane H through the origin not containing any points of G, and projecting the points orthogonally onto H, with the projections of points from the open half space H^+ being labelled "positive" and those from H^- labelled "negative". The face structure of P is determined in the following way: a set of facets $\{f_i : i \in I\}$ of P has a non-trivial intersection if and only if the convex hull of the positive points in $\{f_j : j \notin I\}$ non-trivially intersects the convex hull of the negative points in $\{f_j : j \notin I\}$.

A configuration of positive and negative points then corresponds to an affine Gale diagram of a polytope if and only if, for every hyperplane through the origin H in $\mathbb{R}^{k-2}$, the total number of positive points contained in the open half space H^+ and negative points contained in H^- is at least 2. This classification follows directly from the analogous half space condition for Gale diagrams.

4. Combinatorial Types of Simple d-Polytopes with $d+4$ Facets

4.1. Affine Gale diagrams of simple d-polytopes with $d+4$ facets

In order to determine the compact hyperbolic d-polytopes with $d+4$ facets in dimension 4 and 5, our methods involve first limiting their possible combinatorial types. This was accomplished for compact hyperbolic d-polytopes with $d+3$ facets by Tumarkin [23], using *standard Gale diagrams* described by Esselmann [8]. However, Tumarkin's methods do not seem immediately generalisable to Gale diagrams of higher dimension. We develop a new method for generating affine Gale diagrams using a classification of point set order types by Aichholzer, Aurenhammer, and Krasser [1]. These are reduced to a representative list of Gale diagrams, from which one can easily determine the set of missing faces. Later sections are devoted to determining which of these combinatorial types are realisable.

We now describe how to obtain the potential combinatorial types of compact hyperbolic d-polytopes with $d+4$ facets, listed in Appendix A. First, we show that every combinatorial type of such a polytope has a corresponding affine Gale diagram satisfying certain properties (see Subsection 3.2 for details on affine Gale diagrams). In fact, the conditions we consider hold for the more general case of simple hyperbolic d-polytopes with $d+4$ facets.

Remark 4.1. Let P be a simple d-polytope with $d+k$ facets for $k \geq 2$. Observe that the positive and negative points of an affine Gale diagram for P are necessarily disjoint. If not, let $v \in \mathbb{R}^2$ be a point which is both positive and negative. The facets not corresponding to this point then intersect non-trivially in a face of codimension $d+k-2 \geq d$, which is not possible.

A set of points in $\mathbb{R}^d$ is said to be in *general position* if no m points lie in a subspace of dimension $m-2$ for $m = 2, \dots, d+1$. In particular, when $d = 2$ this is equivalent to requiring that no three points are collinear.

Proposition 4.2. *For $d \geq 4$, every simple d-polytope with $d+4$ facets admits an affine Gale diagram where all points are in general position.*

Proof. Let $A = A_+ \cup A_-$ be an affine Gale diagram for P, where $A_+, A_- \subseteq \mathbb{R}^2$ denote the set of positive points and the set of negative points, respectively. Our aim will be to slightly perturb the points of A to ensure they are in general position while preserving the associated combinatorial type. By Remark 4.1, A_+ and A_- are necessarily disjoint.

Since every vertex of P is obtained as the intersection of exactly d facets, then no set of fewer than 4 positive and negative vertices can have intersecting convex hulls. In particular, no positive vertex lies on the line segment between two negative vertices, and no negative vertex lies on the line segment between two positive vertices. Thus, the convex hulls of a set of positive and negative vertices intersect non-trivially if and only if their interiors intersect.

Thus, each point of A can be moved within a small neighbourhood of its original position without changing whether a given set of positive and negative points have intersecting convex hulls, i.e., without changing the combinatorial type associated to A. We can thus slightly perturb the points within this small neighbourhood to obtain a set of points in general position. This process will yield an affine diagram $A' = A'_+ \cup A'_-$ of the same combinatorial type as A but with all points in general position, along with the additional property that $|A'_+| = |A_+|$. □

It has been shown by Tumarkin and Felikson [11, 12] that for $k \geq 4$, every compact d-polytope with $d + k$ facets has at least two pairs of non-intersecting facets (see Theorem 6.1). Given a polytope P with $d+4$ facets, let f and f' be a non-intersecting pair of facets. In any Gale diagram associated to P, the points corresponding to f and f' can be separated from the remaining points by a hyperplane through the origin. Take the affine Gale diagram obtained by orthogonal projection onto this hyperplane and choosing the half-space containing f and f' to be positive. Applying the results of Proposition 4.2, in particular the last line of the proof, we obtain an affine Gale diagram associated to P where all points are in general position and with exactly two positive points.

Proposition 4.3. *Let $A = A_+ \cup A_-$ be an affine Gale diagram for a simple d-polytope with $d + 4$ facets, where A is in general position in $\mathbb{R}^2$ and $|A_+| = 2$. Then A_+ is contained in the interior of the convex hull of A_-.*

Proof. Suppose that $u \in A_+$ is not in the interior of the convex hull of A_-. Then there is some $v \in A_-$ such that no two points of A_- are separated by the line through

u and v. In other words, all points of A_- lie in a closed half space $H \cup H^+$ bounded by the line H through u and v. Observe then that the number of positive points in H^+ is at most one, as $|A_+| = 2$ and $u \in H$. Moreover, there are no negative points in H^-. As discussed in Subsection 3.2, $|A_+ \cap H^+| + |A_- \cap H^-|$ must be at least 2 for A to be an affine Gale diagram, so we reach a contradiction. □

In summary, every simple d-polytope with $d+4$ facets admits an affine Gale diagram $A = A_+ \cup A_-$ where

(i) all points of A are in general position,
(ii) $|A_+| = 2$, and
(iii) A_+ is contained in the interior of the convex hull of A_-.

In the sequel, we use these restrictions to show that all affine Gale diagrams of compact d-polytopes with $d + 4$ facets can be obtained by bipartitioning the $d + k$ points of a point set order type.

4.2. **Point set order types**

In order to enumerate the possible Gale diagrams of compact d-polytopes with $d + 4$ facets, we utilise a classification of the point set order types of size at most 10 [1]. This classification was obtained with the aid of extensive computer search in 2002 and is restricted to point sets in general position. The order type of a point set records the relative orientations of triples of points, from which one can determine many other combinatorial properties, such as whether a set of points is in convex position or whether line segments between points intersect.

The *orientation* of an ordered triple $((x_1, x_2), (y_1, y_2), (z_1, z_2)) \in (\mathbb{R}^2)^3$ of points in general position is determined by the sign of the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}.$$

The points are said to be oriented *counter-clockwise* if the determinant is positive and *clockwise* if the determinant is negative. The *order type* of a set of points $\{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ in general position is a mapping that assigns to each ordered triple (i, j, k) of distinct elements in $\{1, \ldots, n\}$ the orientation of the point triple (p_i, p_j, p_k).

One can also define the order type of points not in general position, but we are not concerned with those in the present setting. Two finite subsets $P, Q \subset \mathbb{R}^2$ are said to be *combinatorially equivalent* if they have the same order type, i.e., if there exists a bijection $f : P \to Q$ that preserves orientations.

The following lemma shows that the order type of a point set records the same intersection properties of convex hulls that the affine Gale diagrams use. This is a crucial step in seeing that all affine Gale diagrams of compact d-polytopes with $d+4$ facets can be obtained by partitioning the points of each order type.

Lemma 4.4. *Fix two combinatorially equivalent point sets $P, Q \subseteq \mathbb{R}^2$, each in general position, with an orientation-preserving bijection $\sigma : P \to Q$ between them. Then for any $P_1, P_2 \subseteq P$ and $Q_1 = \sigma(P_1), Q_2 = \sigma(P_2) \subseteq Q$, we have*

$$\operatorname{conv}(P_1) \cap \operatorname{conv}(P_2) = \emptyset \iff \operatorname{conv}(Q_1) \cap \operatorname{conv}(Q_2) = \emptyset\,.$$

Proof. Note that we can assume P_1 and P_2, hence Q_1 and Q_2, are disjoint, since otherwise both intersections are clearly non-empty. Since the sets are in general position, $\operatorname{conv}(P_1)$ and $\operatorname{conv}(P_2)$ intersect non-trivially if and only if their interiors intersect. If their interiors intersect, there are two possibilities, up to swapping P_1 and P_2:

(i) there exist points $u, v \in P_1$ and $x, y \in P_2$ such that the line segment $\overline{uv}$ transversally intersects the line segment $\overline{xy}$, i.e, the segments intersect at an interior point, or

(ii) $\operatorname{conv}(P_1)$ is contained in the interior of $\operatorname{conv}(P_2)$.

We can detect the first condition by looking at the orientations of certain triples. The line segments $\overline{uv}$ and $\overline{xy}$ intersect transversally if and only if (u, v, x) and (u, v, y) have opposite orientations, and additionally (x, y, u) and (x, y, v) have opposite orientations.

The second condition can also be detected by the order type. Recall that a set of points in $\mathbb{R}^2$ is in convex position, i.e., form the vertices of a convex polygon, if and only if every set of four points is in convex position (this follows directly from Caratheodory's theorem; see, e.g., [20]). The latter condition is equivalent to saying that every four points define precisely one pair of intersecting line segments, which can be detected by the order type as described in the previous paragraph. Moreover, given a set of points $(p_1, \ldots, p_n)$ forming cyclically adjacent vertices of a convex polygon, hence in convex position, a point p is contained in their convex hull if and only if the orientations of (p_i, p_{i+1}, p) are all the same, where the subscripts are taken modulo n. Now, if $\text{conv}(P_1)$ is contained in $\text{conv}(P_2)$, then there is a subset $P_2' \subset P_2$ in convex position such that $\text{conv}(P_1) \subseteq \text{conv}(P_2')$. Using the order type, we can then detect if each vertex of P_1 is contained in the convex hull of P_2', hence the convex hull of P_2.

Therefore, whether $\text{conv}(P_1) \cap \text{conv}(P_2)$ is empty is determined by the orientations of triples in $P_1 \cup P_2$. That is, whether each such intersection is non-empty is determined by the order type of P. □

We now obtain our method for generating the desired affine Gale diagrams using the point set order types.

Theorem 4.5. *Every compact Coxeter d-polytope P with $d+4$ facets admits an affine Gale diagram obtained by taking an arrangement of $d+4$ points $A \subseteq \mathbb{R}^2$ in general position and choosing two points from the interior of* $\text{conv}(A)$ *to be positive. Moreover, the combinatorial type of P is completely determined by the order type of X.*

Proof. This follows directly from Proposition 4.2, Proposition 4.3, and Lemma 4.4. □

An example of such an affine Gale diagram and a polytope realising its combinatorial type are depicted in Figure 1.

Using the order type database for up to 10 points [1], one can classify the combinatorial types of all polytopes with $d+4$ facets in dimension at most 6. There are 3,315 order types on 8 points, which yield 34 possible combinatorial types of 4-polytopes with 8 facets, listed in Table A. There are 158,817 order types on 9 points, which yield 186 possible combinatorial types of 5-polytopes with 9 facets; the 111 combinatorial types with at least two pairs of disjoint facets are listed in Appendix A. There are 14,309,547 order types on 10 facets; the significant jump in the number of order types, recalling that we must also iterate over all choices of two positive vertices for each order type, makes determining the possible combinatorial types for this class computationally infeasible for the current author. However, we have been able to glean some partial information about 6-polytopes with 10 facets; see Section 9 for more details.

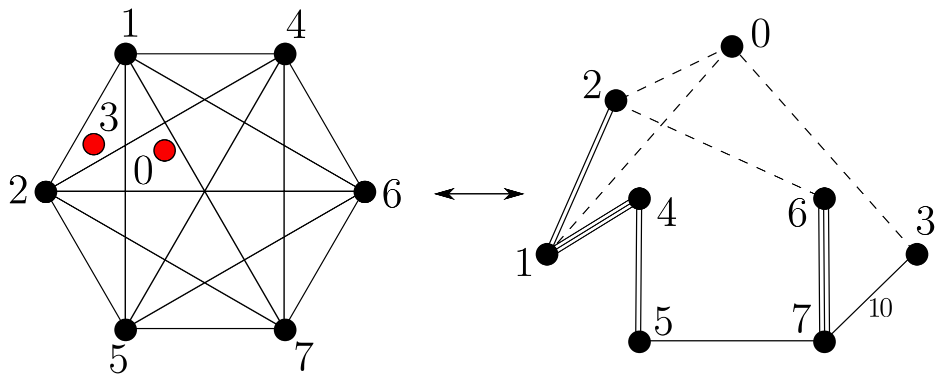


FIGURE 1. On the left is an affine Gale diagram of combinatorial type G_6, with the positive points depicted in red and the negative points in black. The line segments between negative points have been included to aid in identifying the convex hull of each subset. On the right, we have a Coxeter diagram of a compact 4-polytope with combinatorial type G_6 (see Appendix A for a description of this type); the labelling of the vertices corresponding to the 8 facets is preserved (see Section 5 for details of how to interpret Coxeter diagrams).

5. Coxeter Diagrams and Gram Matrices

In this section, we define (abstract) Coxeter diagrams and Gram matrices. We also illustrate several properties of these structures when associated to a compact hyperbolic Coxeter d-polytope with $d+4$ facets, which are used in our classification of these polytopes in the subsequent sections.

We begin by introducing the notion of an *abstract Coxeter diagram*, a weighted graph which can, under certain conditions, describe the dihedral angles of a polytope.

Definition 5.1. An *abstract Coxeter diagram* is a finite simple graph (i.e., one-dimensional simplicial complex) with weighted edges. The weights w_{ij} must be positive and satisfy the following condition: if $w_{ij} < 1$, then $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some integer $m_{ij} \geq 3$.

We often denote abstract Coxeter diagrams by the letter Σ. The order $|\Sigma|$ of the diagram Σ is the number of vertices of Σ. A *subdiagram* Σ' of Σ is a vertex-induced subgraph of Σ, where the edge weights are preserved; this relation is written as $\Sigma' \subset \Sigma$. For any two subdiagrams Σ_1, Σ_2 of Σ, we let $\langle \Sigma_1, \Sigma_2 \rangle$ denote the subdiagram induced by the vertices contained in either Σ_1 or Σ_2. When drawing abstract Coxeter diagrams, we adhere to the following conventions:

(i) If $w_{ij} < 1$, hence $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some integer $m_{i,j} \geq 2$, then the corresponding edge is drawn as a straight line labelled by m_{ij}. In the special cases where m_{ij} is equal to 3, 4, or 5, we draw the corresponding edge as an unlabelled single, double, or triple line, respectively. If $m_{ij} = 2$, we leave the edge empty.

(ii) If $w_{ij} > 1$, the corresponding edge is drawn as a dashed line with label w_{ij}.

(iii) If $w_{ij} = 1$, the corresponding edge is drawn as an unlabelled bold line. Since we are considering only compact polytopes, this case does not arise in our setting.

An ordinary edge of weight $\cos\left(\frac{\pi}{m}\right)$ for some $m \geq 2$ is said to have *multiplicity* $m-2$. We refer to an ordinary edge of weight $\cos\left(\frac{\pi}{m_{ij}}\right)$ for $2 \leq m_{ij} \leq 5$ as having *low weight*, and an ordinary edge of weight $\cos\left(\frac{\pi}{m_{ij}}\right)$ for $m_{ij} \geq 6$, i.e., having multiplicity at least 4, as being *multi-multiple*. While abstract Coxeter diagrams encode the information of the dihedral angles in a graph theoretic context, it is also often useful to treat these linear-algebraically. We do so by considering a certain weighted adjacency matrix, known as a Gram matrix.

Definition 5.2. The *Gram matrix* $M(\Sigma)$ of an abstract Coxeter diagram Σ on n vertices is an $n \times n$ matrix with entries m_{ij} as prescribed:

$$m_{ij} = \begin{cases} 1 & \text{if } i = j\,, \\ -w_{ij} & \text{if vertices } i, j \in \Sigma \text{ are connected by an edge of weight } w_{ij}\,, \\ 0 & \text{if vertices } i, j \in \Sigma \text{ are not adjacent}\,. \end{cases}$$

Now that we have introduced the notions of Coxeter diagrams and Gram matrices abstractly, we describe their relation to Coxeter polytopes. The *Coxeter diagram* $\Sigma(P)$ of P is a weighted graph with vertices corresponding to facets $\{f_i\}_{i\in I}$, where two vertices corresponding to f_i and f_j are connected by

- an ordinary edge of weight $\cos(\frac{\pi}{k})$ if f_i and f_j meet at angle $\frac{\pi}{k}$;
- a dashed edge of weight $\cosh(\rho)$ if f_i and f_j diverge and are at distance ρ apart;
- a bold edge of weight 1 if f_i and f_j are parallel.

The *Gram matrix* $M(P)$ of a Coxeter polytope P is the Gram matrix of the corresponding Coxeter diagram, i.e., $M(\Sigma(P))$. Note that $M(P)$ coincides with the Gram matrix of unit normal vectors to the facets of P.

Recall that the *signature* of a real symmetric matrix is a triple (n_+, n_-, n_0) of non-negative integers where n_+ is the number of positive eigenvalues (the *positive inertia index*), n_- is the number of negative eigenvalues (the *negative inertia index*), and n_0 is the multiplicity of 0 as an eigenvalue, i.e., the nullity. Thus, a real symmetric matrix is *positive definite* if it has signature $(n, 0, 0)$. A matrix is called *indecomposable* if it cannot be transformed into a block-diagonal matrix via simultaneous permutations of columns and rows.

An abstract Coxeter diagram Σ is called

- *elliptic* if $M(\Sigma)$ is positive definite (the connected elliptic diagrams are listed in Figure 5);
- *parabolic* if any indecomposable component of $M(\Sigma)$ is singular, and every proper subdiagram of an indecomposable component is elliptic;
- *Lannér* if Σ is connected and is neither elliptic nor parabolic, and any proper subdiagram of Σ is elliptic (all Lannér diagrams are listed in Figure 3);
- *hyperbolic* if $M(\Sigma)$ has negative inertia index $n_- = 1$;
- *superhyperbolic* if $M(\Sigma)$ has negative inertia index $n_- > 1$;
- *admissible*, following the terminology from [23], if $M(\Sigma)$ is not superhyperbolic and contains no parabolic subdiagrams.

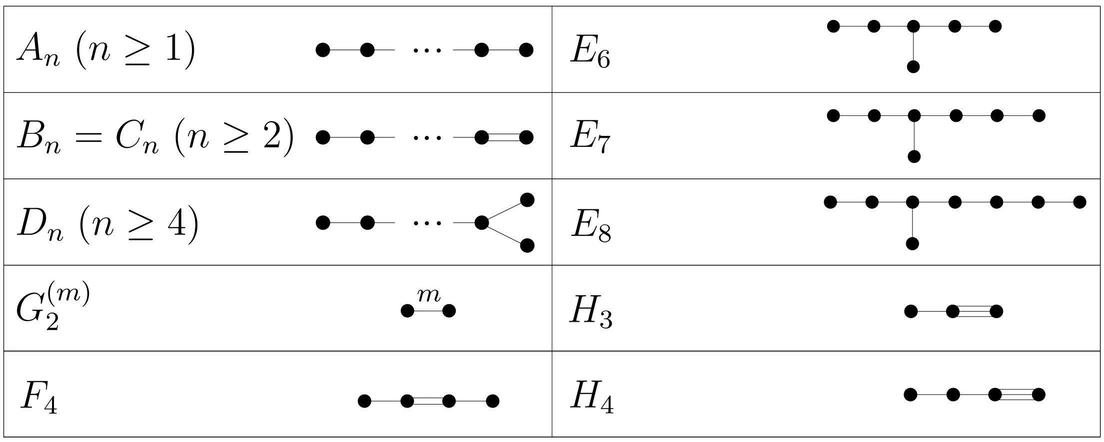


Figure 2. The list of all connected elliptic diagrams along with their type, where the subscript corresponds to the number of vertices.

Remark 5.3. In [24], Vinberg shows that if $\Sigma = \Sigma(P)$ is the Coxeter diagram of a compact hyperbolic d-polytope P, then Σ is an admissible connected hyperbolic

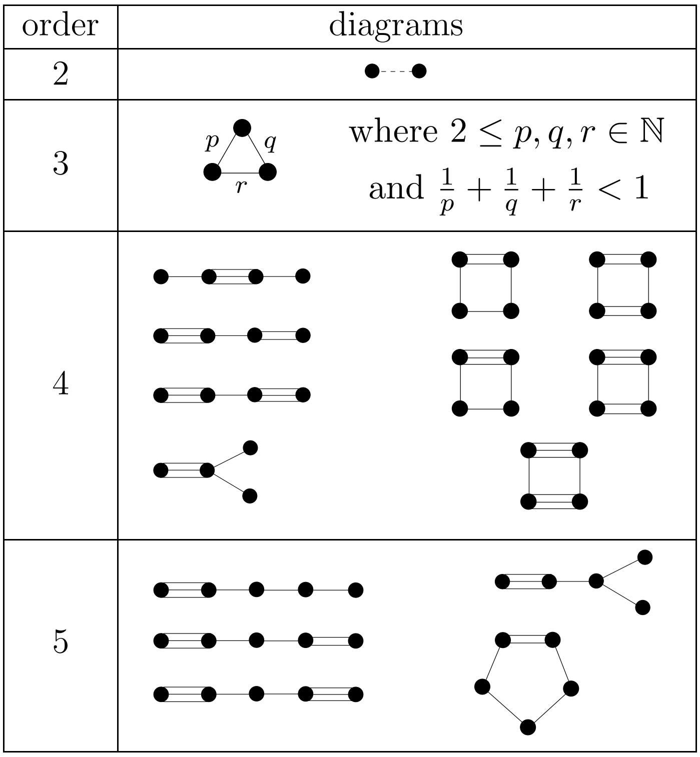


FIGURE 3. The list of all Lannér diagrams, classified by Lannér in [19, Table 3].

diagram with positive inertia index d. Note that this implies that Σ does not contain any parabolic subdiagrams nor bold edges.

We can see from the above remark that $M(\Sigma)$ has rank $d+1$, so any principal minor of order $d+2$ or larger is singular. Though we often omit the weights of dashed edges when drawing Coxeter diagrams, these can be determined from the remaining weights by applying this rank condition. The face structure of $\Sigma(P)$ can be determined by the following proposition.

Proposition 5.4 ([24, Proposition 3.1]). *Let $P = \bigcap_{i \in I} f_i^- \subset \mathbb{H}^d$ be an acute-angled polytope and M its Gram matrix. Let $J \subset I$ be a subset such that the induced submatrix M_J is positive semidefinite. Then $P^J = P \cap \left(\bigcap_{i \in J} f_i\right)$ is a face of P if and only if M_J is positive definite. If so, P^J is a face of codimension $|J|$.*

Thus, the faces of codimension m in P correspond to the elliptic subdiagrams of order m in $\Sigma(P)$, and hence the missing faces of P correspond to the Lannér subdiagrams of $\Sigma(P)$. In particular, a compact hyperbolic polytope has no missing faces of order greater than 5. Thus, using the data of the positive semidefinite submatrices of a Gram matrix, one can determine the combinatorial type of the corresponding polytope.

Remark 5.5. Note that Proposition 5.4 implies that every compact hyperbolic Coxeter d-polytope is simple, i.e., every $(d-j)$-face is contained in precisely j facets.

5.1. **Polytope faces and Coxeter subdiagrams**

Given a face of a compact Coxeter polytope, we discuss restrictions on the combinatorial structure of the face and, if the face is a Coxeter polytope, the structure of its Coxeter diagram. Let P be a compact Coxeter d-polytope with Coxeter diagram $\Sigma = \Sigma(P)$, and S_0 an elliptic subdiagram of order m. By Proposition 5.4, S_0 corresponds to a face of codimension m in P; denote this face by $P(S_0)$. While $P(S_0)$ is necessarily an acute-angled polytope, it may not be Coxeter in certain cases. Borcherds provided the following sufficient condition for $P(S_0)$ to be Coxeter.

Proposition 5.6 ([5, Example 5.6])**.** *Suppose P is a Coxeter polytope with diagram Σ, and $S_0 \subset \Sigma$ is an elliptic subdiagram containing no connected component of type A_n or D_5. Then $P(S_0)$ is a Coxeter polytope.*

We now examine what can be said about the Coxeter diagram of $P(S_0)$ for S_0 satisfying the conditions of Proposition 5.6. Supposing that $P(S_0)$ is indeed Coxeter, denote its Coxeter diagram by Σ_{S_0}. The following terminology was introduced in [10, Section 1], which contains further details of the computation of the dihedral angles in $P(S_0)$. A vertex of Σ *attaches* to S_0 if it is joined with any vertex of S_0 by an edge of any type, in which case w is called a *neighbour* of S_0. Then w is called a *good neighbour* if $\langle S_0, w\rangle$ is elliptic, and *bad* otherwise. Note that in the latter case $\langle S_0, w\rangle$ must contain a Lannér diagram.

Let $\overline{S_0}$ denote the subdiagram of Σ induced by the vertices corresponding to facets of $P(S_0)$, i.e., the good neighbours of S_0 along with all vertices not attached to S_0. We now use the following results of Felikson and Tumarkin, based on the analysis of Allcock [3, Theorem 2.2], to describe the possible differences between the diagrams Σ_{S_0} and $\overline{S_0}$. A *simple edge* refers to an ordinary edge of weight $\cos\left(\frac{\pi}{3}\right)$, denoted by an ordinary unlabelled edge of multiplicity 1 in a Coxeter diagram.

Proposition 5.7 ([10, Corollary 1.1])**.** *Under the hypotheses of Proposition 5.6,* *(a) If S_0 is of the type H_4, F_4, or $G_2^{(m)}$ for $m \geq 6$, or any other diagram having no good neighbours, then $\overline{S_0} = \Sigma_{S_0}$.*

(b) If S_0 is of the type H_3, then $\overline{S_0}$ may be obtained by replacing some dashed edges by ordinary edges.

(c) If S_0 is of the type $G_2^{(5)}$, then $\overline{S_0}$ may be obtained from Σ_{S_0} by replacing some edges labelled by 10 by simple edges, and some dashed edges by ordinary edges.

(d) If S_0 is of the type B_n for $n \geq 3$, then $\overline{S_0}$ may be obtained from Σ_{S_0} by replacing some double edges by simple edges, and some dashed edges by ordinary edges.

(e) If S_0 is of the type $B_2 = G_2^{(4)}$, then $\overline{S_0}$ may be obtained from Σ_{S_0} by replacing some double edges by simple edges, and some dashed edges by ordinary or empty edges.

We make frequent use of these restrictions in the second portion of this paper, when we bound the dimension of polytopes with few facets.

5.2. **Local determinants**

A technical tool that can help to identify superhyperbolic subdiagrams is the local determinant, for which the theory was developed in [25]. Let Σ be an abstract Coxeter diagram, and let T be a subdiagram of Σ such that $\det(\Sigma \setminus T) \neq 0$. The *local determinant* of Σ on the subdiagram T is given by

$$\det(\Sigma, T) = \frac{\det(\Sigma)}{\det(\Sigma \setminus T)}\,.$$

When Σ is composed of two subdiagrams joined in a simple way, we have the following two results to simplify the calculation of the local determinant.

Proposition 5.8 ([25, Proposition 12]). *If a Coxeter diagram Σ consists of two subdiagrams Σ_1 and Σ_2 having a unique vertex v in common, and if no vertex of Σ_1 attaches to Σ_2, then*

$$\det(\Sigma, v) = \det(\Sigma_1, v) + \det(\Sigma_2, v) - 1\,.$$

Proposition 5.9 ([25, Proposition 13]). *If a Coxeter diagram Σ is spanned by two disjoint subdiagrams Σ_1 and Σ_2 joined by a unique edge v_1v_2 of weight a, then*

$$\det(\Sigma, \langle v_1, v_2\rangle) = \det(\Sigma_1, v_1)\det(\Sigma_2, v_2) - a^2\,.$$

In particular, under the hypotheses of Proposition 5.9, if $M(\Sigma)$ is singular and $\Sigma \setminus \{v_1, v_2\}$ is elliptic, then we must have $\det(\Sigma_1, v_1)\det(\Sigma_2, v_2) = a^2$. We are particularly interested in the case when Σ_1 and Σ_2 are Lannér triangles. It is straightforward to check that the local determinant of a Lannér triangle (which is necessarily negative) increases in magnitude as a function of its edge weights (see, e.g., [25]).

6. Properties of Compact Hyperbolic d-Polytopes with $d+4$ Facets

Before delving into the characterisation of compact hyperbolic Coxeter d-polytopes with $d+4$ facets, we first mention several restrictions on these polytopes that we refer to throughout the classification.

The first restrictions we consider are related to the number of pairs of disjoint facets. The compact Coxeter polytopes with no pairs of disjoint facets are precisely the d-simplices, which have $d+1$ facets and were classified by Lannér [19], along with the seven Esselmann polytopes, which are 4-polytopes with 6 facets that were constructed by Esselmann in [9]; this list was shown to be complete by Felikson and Tumarkin [11, Theorem A]. In 2009, Felikson and Tumarkin studied compact Coxeter polytopes with precisely one pair of disjoint facets.

Theorem 6.1 ([12, Main Theorem])**.** *A compact hyperbolic Coxeter d-polytope with exactly one pair of non-intersecting facets has at most $d+3$ facets.*

Thus, in the setting of d-polytopes with $d+4$ facets, we can restrict to looking at polytopes with at least two pairs of disjoint facets. This can be strengthened in dimension 4. Felikson and Tumarkin [13] later studied d polytopes with n facets having at most $n-d-2$ pairs of disjoint facets. They gave a finite algorithm for listing these polytopes, and carried out this algorithm in dimension 4. The algorithm produced no previously-unknown polytopes of dimension 4, yielding the following theorem.

Theorem 6.2 ([13, Theorem 7.1])**.** *Any compact hyperbolic Coxeter 4-polytope with n facets having at most $n-6$ pairs of disjoint facets satisfies $n \leq 7$.*

For the special case $d=4$ and $n=8$, we immediately obtain the following result.

Corollary 6.3. *Any compact hyperbolic Coxeter 4-polytope with 8 facets must have at least three pairs of disjoint facets.*

As we shall see in Section 7, there do exist such polytopes with exactly three pairs of disjoint facets, in particular, those of combinatorial type G_{10}, G_{12}, or G_{14} (see Appendix A for their description).

Following the methods of Tumarkin in [23], we can use a geometric property of polytopes to reduce our search. If a polytope has a facet f that meets precisely d other facets, we call this facet a *prism facet*, and the corresponding vertex of the Coxeter diagram a *prism vertex*. Note that this condition is equivalent to f being a simplex, since as a $(d-1)$-polytope it would have precisely d facets. We additionally refer to all edges incident to a prism vertex as *prism edges*. At each prism facet f, we can truncate the polytope by a hyperplane H to obtain a new polytope P' with the following properties:

- The new facet $f' = P \cap H$ of P' does not intersect f,
- f' is either disjoint from or orthogonal to each facet of P', and
- P' is combinatorially equivalent to P.

In other words, we can obtain P from a combinatorially equivalent polytope P', where all facets of P' corresponding to prism facets of P meet any incident facets at dihedral angle $\frac{\pi}{2}$, by gluing Coxeter prisms onto P'. The hyperplane H can be obtained by choosing successive facets of f and transforming the bounding hyperplane along each chosen facet of f so that they meet at dihedral angle $\frac{\pi}{2}$. This process terminates with P' being combinatorially equivalent to P since each transformation preserves the face structure, as P is acute-angled. In our classification, we first assume prism facets already satisfy the dihedral angle condition, and then later obtain any combinatorially equivalent polytopes by gluing compact Coxeter prisms. The compact Coxeter prisms have been classified by Kaplinskaja [17].

Lastly, we mention a rank condition on the Gram matrix. As discussed in Section 5, the Gram matrix of a compact Coxeter d-polytope has rank $d+1$. This is because the normal vectors to the facets are in $\mathbb{H}^d$, which can be embedded in $\mathbb{R}^{d+1}$. Thus, we have the following condition on the principal minors.

Proposition 6.4. *If P is a compact Coxeter d-polytope with $d+4$ facets, then every subdiagram Σ_0 obtained by deleting two vertices of $\Sigma(P)$ has a singular Gram matrix.*

6.1. **The set of multi-multiple edges**

In this section, we mention several restrictions on which ordinary edges can be multi-multiple, i.e., have weight $\cos\left(\frac{\pi}{m}\right)$ for $m \geq 6$, in a compact Coxeter diagram.

Restricting this set is critical to making the classification in Sections 7 and 8 computationally feasible.

Lemma 6.5. *Let v_1v_2 be an ordinary edge in an admissible abstract Coxeter diagram Σ. Suppose that*

(a) Σ has a Lannér diagram of order greater than 3 containing v_1 and v_2; or

(b) Σ has no Lannér diagram containing v_1 and v_2, Σ has a Lannér diagram L of order greater than 2 containing v_2, and there is no dashed edge from v_1 to any vertex of L .

Then v_1v_2 has low weight.

Proof. The first part is an immediate consequence of the characterisation of Lannér diagrams given in Figure 3.

For the second part, note that v_2 must be incident to an ordinary edge induced by L; call this edge v_2v_3. If v_1v_2 was multi-multiple, then $\{v_1, v_2, v_3\}$ would induce a parabolic diagram or Lannér triangle, since v_1v_3 is not dashed by assumption. The former cannot happen because Σ is admissible, and the latter cannot happen because we assume Σ has no Lannér diagram containing both v_1 and v_2. □

Lemma 6.6. *Suppose that* 01234 *is a subdiagram of an admissible abstract Coxeter diagram. If the induced set of missing faces is* $\{012, 234\}$*, then* 02*,* 12*,* 23*, and* 24 *are not multi-multiple.*

Proof. Note that one of 23 or 24 must be non-empty in order for 234 to be a missing face of size 3; without loss of generality suppose 23 is non-empty. Thus, if either 02 or 12 is multi-multiple, it would induce a missing face 023 or 123, which is forbidden. Hence neither 02 nor 12 is multi-multiple; the analogous result for 23 and 24 can be obtained by swapping the roles of 0,1 and 3,4, respectively. □

Lemma 6.7 ([23, Lemma 4.14])**.** *There is no compact Coxeter* 4*-polytope containing a subdiagram with induced missing face list isomorphic to* $\{0123, 014, 235\}$*.*

6.2. **Admissible partial weightings**

Now that we have discussed some general properties of compact Coxeter d-polytopes with $d+4$ facets and the multi-multiple edges of their Coxeter diagrams, we focus on the possible structure of the ordinary low-weight edges of their Coxeter diagrams.

Definition 6.8. We define the *partial low-weighting* of an abstract Coxeter diagram Σ as the image $\Sigma^{\leq 6}$ of a forgetful map φ to a new weighted diagram on the same vertex set where all edge weights of the form $\cos\left(\frac{\pi}{m}\right)$ for $m \geq 6$ or $\cosh(\rho)$ for any $\rho \in \mathbb{R}$ are forgotten, though the information of whether these edges are ordinary or dashed remains. For a multi-multiple edge whose weight was forgotten, we denote its new weight by $*$.

Figure 4 depicts the partial low-weighting of a Coxeter diagram, the same one shown in Figure 1.

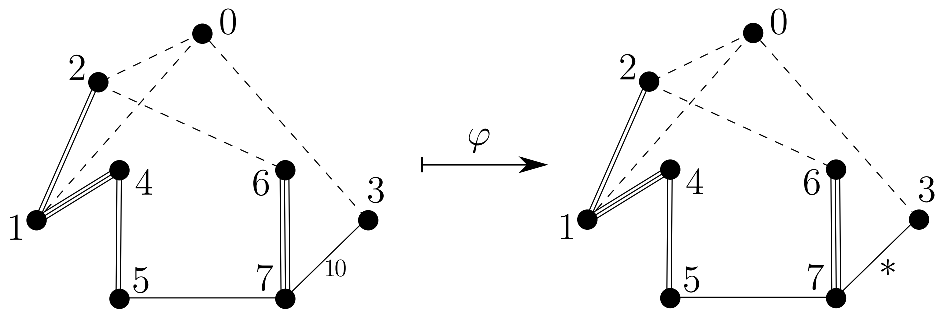


FIGURE 4. This illustrates the action of the forgetful map φ on an abstract Coxeter diagram Σ (on the left) to obtain its partial low-weighting (on the right). Note that the unique multi-multiple edge changes weight from 10 to $*$. Moreover, the dashed edges in the left diagram have weight induced by Proposition 6.4, while the dashed edges on the right have no associated weight.

Note that there are finitely many partial low-weightings of a given order, as each edge is either simple, double, triple, dashed, or multi-multiple with weight $*$. The forgetful map φ also retains some information about the elliptic subdiagrams: we say that a subdiagram S_0 of a partial low-weighting is *elliptic* if it remains elliptic when replacing the edges weighted by $*$ with any weights of the form $\cos\left(\frac{\pi}{m_{ij}}\right)$ for $m_{ij} \geq 6$. Since the only connected elliptic diagrams containing a multi-multiple edge are $G_2^{(m)}$ for $m \geq 6$, this is well-defined. In particular, a subdiagram S_0 of a partial low-weighting is elliptic if and only if any connected component of S_0 is either comprised of ordinary edges and is elliptic, or consists of a single edge with weight $*$. A *missing face* of a partial low-weighting is a set of vertices S_0 (corresponding to a set of facets) such that the subdiagram S_0 is not elliptic, but every proper subdiagram is elliptic.

We say that a partial low-weighting $\Sigma_1^{\leq 6}$ *has the same combinatorial type* as (the partial low-weighting of) a Coxeter diagram Σ_2 if their sets of missing faces are the same, up to a relabelling of the vertices. We call a partial low-weighting on the vertex set $\{0, 1, \ldots, d+k-1\}$ *admissible* if it can be obtained as the image of an admissible abstract Coxeter diagram. In particular, note that an *admissible* partial low-weighting must not contain any parabolic subdiagrams.

Though we initially define partial low-weightings as the images of Coxeter diagrams under the map φ, we can conversely consider constructing an abstract Coxeter diagram from a low-partial weighting. We call an abstract Coxeter diagram Σ_1 an *admissible extension* of a partial low-weighting $\Sigma_2^{\leq 6}$ if Σ_1 is admissible and $\Sigma_2^{\leq 6} = \Sigma_1^{\leq 6}$.

A critical step in our method is ensuring that we have a system of polynomial restrictions on our abstract Coxeter diagrams such that any valid partial low-weighting is the image of at most one admissible Coxeter diagram satisfying the polynomial restrictions. For each combinatorial type G_i, we denote the corresponding system of equations by $V(G_i)$. In our case, each system is determined by checking that certain principal submatrices of $M(\Sigma)$ of size at least $d+2$ are singular, which is satisfied for any abstract Coxeter diagram corresponding to a d-polytope by Proposition 6.4. For an $n \times n$ matrix M and $A, B \subseteq \{0, 1, \ldots, n-1\}$, let $M|_{A,B}$ denote the submatrix formed by restricting to the rows with indices in A and the columns with indices in B (where rows and columns are zero-indexed). Thus, $M|_{A,B}$ is a $|A| \times |B|$ submatrix. The elements of $V(G_i)$ are represented by strings $i_1 \cdots i_m$, representing the equation

$$\det(M|_{S,S}) = 0 \text{ where } S = \{0, \ldots, n-1\} \setminus \{i_1, i_2, \ldots, i_m\}.$$

Note that while this conflicts with our notation for subdiagrams/facets, these strings are only ever written as members of a set $V(G)$ for some combinatorial type G, so the usage is made clear by the context.

Example 6.9. Fix a 4×4 matrix $M = (a_{i,j})_{0 \le i,j \le 3}$. Then using our notation, $V(G) = \{01, 13\}$ is the system of equations

$$\begin{cases} \det\left(\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}\right) = 0\,, \\ \\ \det\left(\begin{bmatrix} a_{0,0} & a_{0,2} \\ a_{2,0} & a_{2,2} \end{bmatrix}\right) = 0\,, \end{cases}$$

or, equivalently,

$$\begin{cases} a_{2,2}a_{3,3} - a_{2,3}a_{3,2} = 0\,, \\ a_{0,0}a_{2,2} - a_{0,2}a_{2,0} = 0\,. \end{cases}$$

We now discuss our main method for classifying compact Coxeter polytopes, which involves finding a sufficient set of restrictions such that all possible compact Coxeter polytopes of a given combinatorial type can be listed by a finite algorithm.

Proposition 6.10. *Fix a combinatorial type G, a set of restrictions $R(G)$ on the dihedral angles, and a system of equations on the edge weights $V(G)$. Suppose $V(G)$ is chosen such that for every admissible partial low-weighting of type G satisfying R, there are finitely many solutions to $V(G)$. Then all Coxeter diagrams corresponding to compact polytopes of type G satisfying $V(G)$ and $R(G)$ can be listed by iterating over all admissible partial low-weightings of G and testing whether the solutions to $V(G)$ yield a diagram satisfying the properties of Remark 5.3.*

In our classification, we first fix a combinatorial type G and then deduce certain properties of the dihedral angles that must hold for any compact Coxeter polytope of type G. This set of restrictions will comprise $R(G)$. Moreover, our set of equations $V(G)$ will be a subset of the restrictions guaranteeing that the Gram matrix of a d-polytope has rank $d+1$. Hence, $V(G)$ is satisfied by the Gram matrix of any compact Coxeter d-polytope. Imposing the conditions of $R(G)$ and $V(G)$ does not eliminate any compact Coxeter polytopes of combinatorial type G. Thus, we obtain the following corollary, stating that Proposition 6.10 yields all compact Coxeter polytopes of type G when $R(G)$ and $V(G)$ are as described.

Corollary 6.11. *Suppose G, $R(G)$, and $V(G)$ are chosen satisfying the hypotheses of Proposition 6.10, and additionally that all equations in $V(G)$ come from setting principal minors of size $\geq d+2$ to zero, and that the properties in $R(G)$ are satisfied for all compact Coxeter polytopes of type G. Then the process in Proposition 6.10 yields all Coxeter diagrams corresponding to polytopes of type G.*

Proof. Since the rank of the Gram matrix $M(P)$ for any compact Coxeter d-polytope P is $d+1$, then every principal minor of size $d+2$ or larger must vanish. Thus, all Coxeter diagrams corresponding to compact polytopes of type G satisfy $V(G)$. By assumption, all compact Coxeter polytopes of type G also satisfy $R(G)$. The process in Proposition 6.10 yields the set of all compact Coxeter polytopes of type G satisfying $V(G)$ and $R(G)$, which is merely equivalent to the set of all compact Coxeter polytopes of type G. □

We emphasize that in Proposition 6.10, our conditions on $V(G)$ require iterating over all admissible partial low-weightings of type G satisfying $R(G)$. In practice, this involves checking that the system $V(G)$ has finitely many solutions for each such low-weighting, which can be done with a computer algebra system such as `Mathematica`. We often choose $V(G)$ to be minimal or nearly minimal to satisfy the hypotheses of Proposition 6.10, since in practice this speeds up the necessary computations. However, we are not aware of a less computationally challenging method for checking the sufficiency of the choice of $V(G)$, as the system may (and often does) have infinitely many solutions when considering non-admissible partial low-weightings, and enumerating the set of admissible partial low-weightings is itself computationally complex.

6.3. **Computational methods**

In this section, we describe the computational steps used to classify compact Coxeter polytopes of a given combinatorial type G, following the methods described in Proposition 6.10 and Corollary 6.11. We begin with a set of $d+k$ vertices with unknown edge weights, and describe the process by which we assign edge weights such that every admissible partial low-weighting of type G is constructed.

1. Select a set of multi-multiple edges. For each combinatorial type, we begin by restricting the set of edges which can be multi-multiple according to the results in

Subsection 6.1. In practice, each combinatorial type considered in this paper can be limited to having at most five multi-multiple edges. We then iterate over all subsets of the possibly multi-multiple edges, fix such a subset H, and examine the Coxeter diagrams for which the set of multi-multiple edges is precisely H.

2. Assign edge weights within Lannér diagrams of size 4 *and* 5*.* Recall that there are finitely many Lannér diagrams of size 4 or 5 (see Figure 3). In particular, if we restrict an admissible partial low-weighting to only those vertices contained in Lannér diagrams of size 4 or 5, the resulting subdiagram is obtained by gluing Lannér diagrams from this list. We can thus iterate over the possible Lannér diagrams and over permutations of the vertices within each Lannér diagram to assign weightings to edges within such a subdiagram.

3. Assign weights to the remaining ordinary low-weight edges. There are finitely many assignments of the remaining low-weight edges, i.e., those not contained within a Lannér diagram of size 4 or 5, since each must have weight 0, 1, 2, or 3. We iterate over all these possibilities, with the additional restriction that all subdiagrams not containing a missing face must be elliptic (see Figure 5). We furthermore require that subdiagram induced by any two Lannér diagrams is connected, lest this subdiagram be superhyperbolic.

4. Solve for the remaining edge weights. At this stage, the only edge weights which have not been assigned are the weights of the multi-multiple edges and dashed edges. The weights of the multi-multiple edges must be real numbers in the range $\left[\cos\left(\frac{\pi}{6}\right), 1\right) = \left[\frac{\sqrt{3}}{2}, 1\right)$, and the weights of the dashed edges must be real numbers in the range $(1, \infty)$. Restricting to these ranges, we find all solutions to the system of equations $V(G)$, where the unknowns are the weights of the multi-multiple and dashed edges. We do so using the computer algebra system `Mathematica`. By assumption, $V(G)$ is chosen so that this solution set is finite.

5. Check the signatures of the resulting matrices. For each of the solutions to the system of equations, we obtain a potential Gram matrix of a compact Coxeter polytope. By Remark 5.3, it is sufficient to check whether the signature of the resulting matrix is $(d, 1, k-1)$. If so, this is the Gram matrix of a compact Coxeter d-polytope with $d+k$ facets of combinatorial type G.

6. Glue prisms onto any prism facets. Recall that we initially assumed the weights of all prism edges were 0. If any polytopes are constructed by the process above, we can obtain the complete list of compact Coxeter polytopes of combinatorial type G by gluing compact Coxeter prisms to this prism facet. There are finitely many compact Coxeter prisms of dimension 4 or 5, see [17] for a complete list.

The code implementing the process described above is publicly available at `https://github.com/agburcroff/Cox_d-Polytopes_with_dplus4_Facets.git`.

7. Classification of Compact Coxeter 4-Polytopes with 8 Facets

In this section, we explain how to determine all compact Coxeter 4-polytopes with 8 facets. We handle each of the 34 possible combinatorial types determined in Section 4 and listed in Appendix A individually.

For each such combinatorial type, we prove a sufficient number of restrictions on the Coxeter diagram of any polytope realising this type so that the possible Coxeter diagrams can be listed by a finite algorithm. To be more specific, we claim that given any possible assignment of weights on the ordinary edges of Σ, there are a finite number of solutions for the weights of the remaining edges given the system of equations for $\mathrm{rank}(M(\Sigma))$ i.e, that every principal minor of order 6 in $M(\Sigma)$ has determinant 0.

Given a combinatorial type, the task of enumerating all compact polytopes with this type is a rather involved task. Due to the presence of multi-multiple edges, initially there is an infinite number of potential Gram matrices that must be searched. In this section, we detail how to use information about the elliptic and Lannér subdiagrams to reduce the number of multi-multiple edges, then, once we have reduced to a system of algebraic equations with finitely many solutions, computationally determine those solutions which yield compact polytopes.

We first eliminate several combinatorial types by applying the previous work of Felikson and Tumarkin [11–13, 23].

Corollary 7.1. *There are no compact hyperbolic Coxeter* 4*-polytopes of combinatorial types* G_{31}, G_{32}, G_{33}, *or* G_{34}.

Proof. These combinatorial types all have exactly one pair of disjoint facets, and thus cannot be realised by Theorem 6.1. □

Using Corollary 6.3 (following from results in [13]) we can immediately exclude six more combinatorial types, each having precisely two pairs of disjoint facets.

Corollary 7.2. *There are no compact hyperbolic Coxeter* 4*-polytopes of combinatorial types* G_{25}, G_{26}, G_{27}, G_{28}, G_{29}, *or* G_{30}.

We can furthermore eliminate three combinatorial types by considering a certain forbidden subdiagram.

Corollary 7.3. *There are no compact hyperbolic Coxeter* 4*-polytopes of combinatorial types* G_{22}*,* G_{23}*, or* G_{24}*.*

Proof. Each of these has a subdiagram with induced missing face list isomorphic to $\{0123, 014, 235\}$, hence cannot be realised as a polytope by Lemma 6.7. □

It remains to determine whether there are polytopes of combinatorial type G_i for $i = 1, 2, \dots, 21$, and if so, to classify these polytopes. We handle each of these combinatorial types individually in the following subsections. In each subsection, we prove a sufficient set of restrictions to ensure that any polytopes realising a given combinatorial type can be listed by a finite algorithm. One such initial restriction is that we can set the weight of all prism edges to 0 (see Section 6 for further explanation), and then glue prisms to any polytopes obtained. In each case, the algorithm has been successfully implemented via the program described in Subsection 6.3, though this program can in some cases take several days to check one combinatorial type on a standard machine.

Combinatorial type G_1

Observe that this is the combinatorial type of a simplex which has been truncated at three distinct vertices. In particular, vertices v_0, v_1, and v_2, which correspond to the facets obtained by the truncation, are all prism vertices. There are no possible multi-multiple edges by Lemma 6.5, since all non-prism edges are contained in a Lannér diagram of order 4. This combinatorial type satisfies the conditions of Corollary 6.11 and can be handled via computer computation without further restriction using the following system of equations:

$$V(G_1) = \{01, 02, 04, 12, 13, 23\}\,.$$

This yields 130 compact Coxeter polytopes of type G_1.

Combinatorial type G_2

This is the combinatorial type of a simplex that has been truncated at two vertices and one edge, disjoint from the two vertices. The vertices v_0 and v_1 are both prism vertices. There are no possible multi-multiple edges, as all non-prism edges satisfy the hypotheses of Lemma 6.5. Similarly to the previous case, this combinatorial type satisfies the conditions of Corollary 6.11 with:

$$V(G_2) = \{01, 03, 04, 12, 13, 23\}\,.$$

This yields 105 compact Coxeter polytopes of type G_2.

Combinatorial type G_3

This combinatorial type, like the previous two, is also relatively straightforward to classify. Note that the prism vertices are v_0 and v_3. There are no possible multi-multiple edges by Lemma 6.5. The system of equations

$$V(G_3) = \{02, 03, 07, 12, 13, 23\}$$

provides sufficient restrictions to satisfy Corollary 6.11. This yields 52 compact Coxeter polytopes of type G_3.

Combinatorial type G_4

Since this combinatorial type consists of four pairs of disjoint facets, it is the combinatorial type of a 4-cube. The compact 4-cubes were recently classified by Jacquemet and Tschantz [16], which, similarly to the present methods, utilised significant computer searches. There are 12 such polytopes.

Combinatorial type G_5

This is the first example of a combinatorial type for which we need to apply extra restrictions before passing to computer computation.

Note that there is at most one multi-multiple edge within the subdiagram 567, and one multi-multiple edge within the subdiagram 01234. There are no further possible multi-multiple edges. By symmetry, we can assume the multi-multiple edges are limited to 57 and 23.

Suppose that both arise, say 57 and 23 are multi-multiple. Note then that 46, 06, and 16 are non-empty to ensure no two Lannér subdiagrams are disjoint, and in fact they must be single edges. Moreover, 56 and 57 are either both have multiplicity 1, or 0 and 1, respectively. The only other additional non-empty edges are 12 and 03, each with multiplicity at most 3. Up to symmetry, this leaves $2(1+3+9) = 24$ diagrams to check. We can do so using Corollary 6.11 and the system of equations

$$V(G_5) = \{03, 04, 12, 13, 14, 24, 34\}.$$

Now suppose just 57 is multi-multiple. The previous check actually encompasses the case where precisely three vertices of 01234 are adjacent to v_6. Suppose there are four such vertices, say 0123 are. Then all edges between the subdiagram 0123 and v_6 are simple edges, and 34 and 45 either both have multiplicity 1, or 0 and 1, respectively. Then the only additional edges are 02 and 27, each with multiplicity at most 3. As in the previous case, there are 24 partial low-weightings to test, we can complete this using Corollary 6.11 and the system of equations $V(G_5)$ defined above. Lastly, suppose there are five such vertices, then we can proceed similarly with just 2 diagrams to check.

The remaining cases, namely when only 23 is multi-multiple or when there are no multi-multiple edges, can be handled with the same choice of $V(G_5)$ as above. This yields 3 compact Coxeter polytopes of type G_5.

Combinatorial type G_6

Note that 15 and 35 are not multi-multiple by applying Lemma 6.6 to the subdiagram 13457.

If 36 is multi-multiple, then so is 37. Suppose it is, and 37 is not. Then 46, 56, and 35 are empty to prevent inducing a forbidden missing face of size 3. Then 57 has multiplicity at least 2 - looking at the Lannér subdiagrams, there are only two possibilities for the rank-4 Lannér diagram (where in fact 57 has multiplicity 3). It can be checked that the left triangle 145 cannot be admissibly completed in either case.

Suppose 36 and 37 are multi-multiple. Then 4567 is a Lannér path. We can then use local determinants on 134567, along with Proposition 5.9, to show that there are no possibilities.

Otherwise, just 37, 12, 24, and 14 can be multi-multiple. Under this assumption, G_6 satisfies the hypotheses of Corollary 6.11 with

$$V(G_6) = \{01, 02, 06, 12, 14, 16, 23, 26, 27\}.$$

This yields 2 compact Coxeter polytopes of type G_6.

Combinatorial type G_7

Note that vertices v_0 and v_1 are both prism vertices. The only possible multi-multiple edges are 25 and 34 by Lemma 6.5. Under these restrictions, we can apply Corollary 6.11 with

$$V(G_7) = \{02, 03, 07, 12, 13, 23\}.$$

This yields 2 compact Coxeter polytopes of type G_7.

Combinatorial type G_8

This has prism vertices v_0 and v_3, with the only possible multi-multiple edges being 16 and 27 by Lemma 6.5. The system of equations

$$V(G_8) = \{03, 05, 06, 07, 13, 23, 67\}$$

provides sufficient restrictions to satisfy Corollary 6.11. This yields 1 compact Coxeter polytope of type G_8.

Combinatorial type G_9

Note that the vertex v_0 is the only prism vertex. By Lemma 6.5, the only possible multi-multiple edges are 36, 37, and 15. Note that 36 and 37 cannot simultaneously be multi-multiple, otherwise any assignment of weights to the rank-4 Lannér diagram 4567 induces a forbidden Lannér triangle with 3. By symmetry, we can assume 37 is not multi-multiple.

Suppose 36 and 15 are multi-multiple. Note that 46, 56, and 57 must then be empty, hence the rank-4 Lannér diagram 4567 is a path. For any assignment of weights within this Lannér diagram, there is no additional non-empty edge connecting Lannér diagram 25 to the Lannér diagram 367. Thus, this cannot occur, as we cannot have two disjoint Lannér diagrams.

Suppose only edge 15 is multi-multiple. The hypotheses of Corollary 6.11 are satisfied with

$$V(G_9) = \{01, 04, 05, 12, 13, 15, 45\}.$$

Suppose only edge 36 is multi-multiple. We can again look at the structure of the rank-4 Lannér diagram (noting 46 and 56 are empty) - if it is a path, then we can repeat the same analysis as two paragraphs higher with connecting 25 to 367. The only remaining option is that the rank-4 diagram consists of the star with one edge of multiplicity 3 and two edges of multiplicity 1, with two possible positions (up to symmetry). Note that the only additional edges are 15 and 24. We complete the analysis of this case with $V(G_9)$ as above.

Similarly, if all ordinary edges have low weight, then the same system of equations $V(G_9)$ suffices. This yields 15 compact Coxeter polytopes of type G_9.

Combinatorial type G_{10}

By Lemma 6.6, the only possible multi-multiple edges are in the subdiagrams 256 and 034. Note that not both 25 and 26 can be multi-multiple lest 156 cannot be admissibly completed, so assume 26 has low weight. A similar analysis can be used to show that at least one of 03 or 04 has low weight, hence we can assume 04 has low weight.

Suppose both 34 and 56 are multi-multiple. Note that we cannot have all edges 13, 14, 15, 16 non-empty, less we form a 4-cycle in the subdiagram 134567. This subdiagram cannot contain a 4-cycle since it contains no Lannér diagrams of size at least 4, and all elliptic diagrams are acyclic. Thus, we can assume by symmetry that the edge 13 is empty. Applying Proposition 5.9 to the subdiagram 013456, we obtain that at least one of 56 or 34 has multiplicity at most 13. Under this assumption, we can complete the analysis via Corollary 6.11 with

$$V(G_{10}) = \{02, 03, 05, 06, 12, 13, 23\}.$$

Otherwise, assuming the multi-multiple edges are limited to 03, 25, and at most one of 34 or 56, Corollary 6.11 similarly applies with $V(G_{10})$ defined as above. This yields 1 compact Coxeter polytope of type G_{10}.

Combinatorial type G_{11}

Observe that the only possible multi-multiple edges are those within the subdiagrams 0167 or 2345.

Suppose that 45 is multi-multiple. Observe that no other edges of 2345 are multi-multiple, lest 67 cannot connect to one of the rank-3 Lannér diagrams. Moreover, by symmetry cases we can assume the multi-multiple edges among 0267 are limited to 02 and 67 or 02 and 27. Suppose first that these two edges are 02 and 67. We can apply Corollary 6.11 with

$$V(G_{11}) = \{01, 02, 03, 04, 05, 06, 16, 17, 27, 67\}.$$

The same analysis can be applied supposing that these two edges are 02 and 27. Thus, we can now assume 45 is not multi-multiple.

Suppose 06 is multi-multiple. In order to ensure that 67 is connected to 245 and 345, we have three cases:

(1) Suppose 26 and 37 are non-empty. The only other possible multi-multiple edge is 17.
(2) Suppose 27 and 37 are non-empty. The only other possible multi-multiple edge is 27.
(3) Suppose 57 is non-empty. The other possible multi-multiple edges are 24, 34, and 16. We cannot have 24 and 34 simultaneously multi-multiple, lest we create a forbidden Lannér triangle, so assume by symmetry that 24 is not multi-multiple.

In all three cases, we can apply Corollary 6.11 with $V(G_{11})$ as above. We can now assume the multi-multiple edges are among 1345 excluding 34, i.e. either 13, 45 or 35, 45 or just one of these, and again the same analysis applies. This yields 8 compact Coxeter polytopes of type G_{11}.

Combinatorial type G_{12}

Observe that 57, 37, 47, 67, and 27 are not multi-multiple by Lemma 6.6.

Suppose 03 is multi-multiple, and 34 is not. Then 47 is empty, and 34 and 37 have multiplicities $2, 3$ or $3, 3$, respectively. Thus, 67 has multiplicity 1. Considering local determinants on 013457, the only two possibilities are that 03 has multiplicity 4, 34 has multiplicity 2 or 3, 37 has multiplicity 3, and 56 has multiplicity 5. Note that now 03 and 13 must be empty. If edge 27 is empty, we can consider local determinants on 234567 using Proposition 5.9 to see that we must have multiplicity 2 on 34, and repeating local determinant analysis on 123467 we find that the multiplicity of 26 is 5, 6, or 7. Then considering local determinants on the subdiagram 234567 yieldsthat 17 is empty, and no other can arise. This leaves us with three diagrams to check - none of these yield polytopes. Now we can assume that 27 is non-empty - in fact, it must have multiplicity 1. Considering local determinants on the subdiagram 123467 using Proposition 5.9 yields that the respective weights of 45, 34, 67 are $2, 3, 2$; $2, 3, 3$; $3, 3, 2$; or $3, 2, 2$. Moreover, the only other possible non-empty edges are 25 or 12, each with multiplicity 1. This leaves us with 16 diagrams to check - none of these yield polytopes.

Suppose 03 and 34 are multi-multiple. Then 07 is empty. So we can consider local determinants on 034567, noting that if 37 has weight larger than 1 then 67 has multiplicity 1. From this and the previous paragraph, we can determine that there are no admissible diagrams with 03 multi-multiple.

We are now left with considering multi-multiple edges 34, 56, 26, 25. We can then apply Corollary 6.11 with

$$V(G_{12}) = \{05, 12, 15, 16, 25, 26, 35, 56\}\,.$$

This yields 4 compact Coxeter polytopes of type G_{12}.

Combinatorial type G_{13}

Observe by Lemmas 6.5 and 6.6 that the only possible multi-multiple edges are 12, 56, 16, 25, 34, 03, and 04. We cannot simultaneously have 03 and 04 multi-multiple lest 37 and 47 are both empty, so assume 04 is not. We cannot simultaneously have 16 and 25 multi-multiple lest 57 and 47 are both empty, so assume 25 is not. Thus, we are limited to 12, 56, 16, 34, and 03 being multi-multiple.

Suppose 56 and 03 are multi-multiple. Then 37 and 03 are empty, so we can consider local determinants on 034567 using Proposition 5.9. Assume 57 is non-empty. Note that either 67 is non-empty, or 2 is attached to 347 to ensure 26 is attached to 347. If

67 is non-empty, the local determinant on 034567 with respect to 07 cannot be zero, which is a contradiction. If 2 is attached to 7 or 4, then 47 cannot have multiplicity 3 so is 2. Considering the same local determinant shows that this is not possible. So 23 is non-empty, and by local determinants we must have that 03 has multiplicity 4 or 5, 34 has multiplicity 2 or 3, 47 has multiplicity 3, and both 57 and 23 have multiplicity 1, with no other edges being nonempty. Now take the determinant of the subdiagram 034567; this is non-zero, a contradiction.

Suppose 34 and 03 are multi-multiple (we can assume 56 is not from previous paragraph). Then 37 is empty, so 47 is not. We can now use local determinants on 034567. From this, we determine that 567 must have one empty edge. Thus, one of 67, 57 has multiplicity at least 2, so 47 has multiplicity 1. Repeating a similar local determinant argument, we find that there are no admissible diagrams of this type.

The remaining cases can be handled by Corollary 6.11 with

$$V(G_{13}) = \{01, 02, 05, 06, 12, 13, 16, 25, 26\}.$$

This yields 2 compact Coxeter polytopes of type G_{13}.

Combinatorial type G_{14}

By similar considerations to type G_{12}, if any of 27, 37, 26, 36, 02, or 03 is multi-multiple then so is 23.

Suppose 27 and 23 are multi-multiple. Note that the other possible edges (up to symmetry) can be limited to within 145, 56, 04, 03, and 36.

- Suppose 56 and 04 are non-empty. Then the only other possible multi-multiple edge is 14.
- Suppose 56 is non-empty, 04 is empty. Then the only other possible multi-multiple edges are 45 and 14.
- Suppose 04 is non-empty, 56 is empty. Then the only other possible multi-multiple edges are 15 and 14.
- Suppose 56 and 04 are empty. Then the other possible multi-multiple edges are 45, 14, and 15. The only additional edges are 03 and 36, one of which has multiplicity 1 or 2 and the other having multiplicity 1.

In all these cases, we can apply Corollary 6.11 with

$$V(G_{14}) = \{01, 14, 15, 17, 23, 24, 47, 57\}\,.$$

We can now restrict the multi-multiple edges to 23, 45, 14, and 15, and the same analysis applies along with the same choice of $V(G_{14})$. This yields 2 compact Coxeter polytopes of type G_{14}.

Combinatorial type G_{15}

Look at the subdiagram generated by 234567. It can quickly be checked that the only possible multi-multiple edge is one of 26 or 27, suppose by symmetry it is 26. Then 36 must be empty, so 37 and 67 must have multiplicities 2 and 3 or both 3. From this information, we can then complete the remainder of the rank-4 Lannér diagram, keeping in mind that 46 must have multiplicity 0 as well. After doing so, it can be easily checked that the triangle 345 cannot be completed to an admissible diagram. Hence there are no compact Coxeter polytopes of this type.

Combinatorial type G_{16}

Considering overlapping triangles, the only possible multi-multiple edges are 15, 12, 03, 04, and 34. We cannot have 03 and 04 simultaneously multi-multiple, so assume 03 has low weight. By similar considerations to G_{12}, if 04 is multi-multiple then so is 34.

We can apply Corollary 6.11 with

$$V(G_{16}) = \{01, 02, 07, 12, 17, 24, 25\}$$

to show that there are no compact Coxeter polytopes of this type.

Combinatorial type G_{17}

By Lemma 6.6, the only possible multi-multiple edges are 36, 37, 14, and 15. Note that 36 and 37 cannot be simultaneously multi-multiple; assume by symmetry that 36 has low weight. Moreover, 14 and 15 cannot simultaneously be multi-multiple.

With these assumptions, we can apply Corollary 6.11 with

$$V(G_{17}) = \{03, 05, 07, 13, 34, 35, 57\}$$

to show that there are no compact Coxeter polytopes of this type.

Combinatorial type G_{18}

Considering overlapping triangles, the only possible multi-multiple edges are 12, 37, 36, 25, and 14.

Suppose all five of these edges are multi-multiple. Then the rank-4 Lannér diagram must be a path, and there are no additional non-empty edges. We can check that there are no solutions to the determinants of the subdiagrams 123456 and 012345 being zero, assuming that the undetermined edges have weight in the range $\left[-\cos\left(\frac{\pi}{6}\right), 0\right)$. Thus, we can assume at most four of these edges are multi-multiple. We can now complete the analysis with Corollary 6.11, setting

$$V(G_{18}) = \{01, 02, 03, 04, 05, 12, 23, 13\},$$

to show that there are no compact Coxeter polytopes of this type.

Combinatorial type G_{19}

By Lemma 6.6, the only possible multi-multiple edges are 15, 14, 35, 37, 24, and 26. Note 35 and 15 cannot simultaneously be multi-multiple, and similarly for 24 and 14. We can now complete the analysis with Corollary 6.11, setting

$$V(G_{19}) = \{02, 03, 05, 06, 07, 12, 13, 23\},$$

to show that there are no compact Coxeter polytopes of this type.

Combinatorial type G_{20}

By Lemma 6.6, the only possible multi-multiple edges are contained in the subdiagram 167. We can then apply Corollary 6.11 with

$$V(G_{20}) = \{05, 06, 12, 15, 16, 56\}.$$

This shows that there are no compact Coxeter polytopes of this type.

Combinatorial type G_{21}

By Corollary 6.11, the only possible multi-multiple edges are 05, 07, 15, 13, 34, 46, 26, and 27. Note that 05 and 07 cannot simultaneously be multi-multiple, lest 57 cannot connect to 346. By symmetry, assume 07 is not multi-multiple. We can then apply Corollary 6.11 with

$$V(G_{21}) = \{01, 02, 03, 04, 05, 06, 07, 12, 15, 25\}\,,$$

to show that there are no compact Coxeter polytopes of this type.

8. Classification of Compact Coxeter 5-Polytopes with 9 Facets

We now proceed to the classification of compact Coxeter 5-polytopes with 9 facets. Though the process detailed in Section 4 yields more possible combinatorial types in dimension 5 than in dimension 4, only five of these (H_i for $1 \leq i \leq 5$) are realised by compact Coxeter polytopes. Moreover, these combinatorial types in general have a more restrictive face structure than in the lower-dimensional cases, thus in most cases find a set of equations for which the hypotheses of Corollary 6.11 are satisfied without first proving additional restrictions. The main challenges are limiting the set of multi-multiple edges and determining the set of equations $V(H)$ for each combinatorial type H. We examine this process in detail for a few combinatorial types, but we often omit the argument for routine cases. In these cases, Lemmas 6.5 and 6.6 are sufficient to limit the multi-multiple edges, and of course the claimed properties of $V(H)$ are checked computationally.

Combinatorial type H_1. Note that v_0 and v_1 are both prism vertices. By [8, Lemma 5.3], it is easily checked that there are four possible subdiagrams with the proper missing face structure on 2345678. We can thus obtain all diagrams by taking these four subdiagrams and gluing prisms onto facets 0 and 1. This process yields 22 compact polytopes of type H_1.

Combinatorial type H_2. Note that v_0 and v_3 are both prism vertices. It is straightforward to check that there are only four possible subdiagrams for 1245678 by gluing together subdiagrams from [8, Lemma 5.3]. We thus obtain all 18 compact polytopes of type H_2 by gluing prisms as appropriate to these subdiagrams.

Combinatorial type H_3. At most one of the edges 25 or 26 can be multi-multiple. By symmetry, we can assume that only 25 can be multi-multiple. We can then apply Corollary 6.11 with $V(H_3) = \{23, 15, 12, 04, 02, 01, 13, 15\}$. This yields 6 compact Coxeter polytopes of type H_3.

Combinatorial type H_4. Only possible multi-multiple edges are 37 or 38. These can not be multi-multiple simultaneously. We can then take $V(H_4) = \{02, 03, 07, 08, 23, 27, 38\}$. This yields 3 compact Coxeter polytopes of type H_4.

Combinatorial type H_5. The possible multi-multiple edges are restricted to 25, 04, 03, 14, and 13, some of which cannot be multi-multiple simultaneously. By symmetry between 3 and 4, we can restrict to 25, 04, and 14 possibly being multi-multiple. We then take $V(H_5) = \{01, 03, 05, 12, 14, 45\}$. This yields 1 compact Coxeter polytope of type H_5.

Combinatorial type H_6. There are no possible multi-multiple edges. We can take $V(H_6) = \{01, 02, 04, 12, 13, 23\}$.

Combinatorial type H_7. At most one of the edges 27 or 28 can be multi-multiple. By symmetry, we can assume that only 28 can be multi-multiple. We can take $V(H_7) = \{02, 03, 12, 13, 23\}$.

Combinatorial type H_8. The only possible multi-multiple edge is 28. We can take $V(H_8) = \{02, 03, 08, 13, 23, 27, 28, 38\}$.

Combinatorial type H_9. The only possible multi-multiple edge is 34. We can take $V(H_9) = \{01, 02, 03, 04, 12, 13, 14, 23\}$.

Combinatorial type H_{10}. The only possible multi-multiple edge is 12. We can take $V(H_{10}) = \{01, 02, 03, 12, 18, 28\}$.

Combinatorial type H_{11}. The only possible multi-multiple edges are 36 or 37, but not both. As these cannot be multi-multiple simultaneously, we can assume by symmetry that 37 has low weight. We can take $V(H_{11}) = \{02, 03, 07, 08, 12, 23, 28, 38\}$.

Combinatorial type H_{12}. The only possible multi-multiple edges either 04 and 14, or 03 and 13. By symmetry, we can assume the possible multi-multiple edges are only 03 and 13. We can take $V(H_{12}) = \{01, 02, 03, 05, 12, 13, 35\}$.

Combinatorial type H_{13}. The only possible multi-multiple edges are one of 26 or 27, and one of 14 or 15. By symmetry, we can assume the multi-multiple edges are limited to 14 and 26. We can take $V(H_{13}) = \{01, 02, 04, 06, 12, 13, 23\}$.

Combinatorial type H_{14}. The only possible multi-multiple edges are one of 14 or 15. By symmetry, we can assume 15 has low weight. We can take $V(H_{14}) = \{02, 05, 12, 13, 23, 24\}$.

Combinatorial type H_{15}. The only possible multi-multiple edges are one of 14 or 15. We can take $V(H_{15}) = \{01, 03, 04, 05, 12, 13, 15, 34, 35\}$.

Combinatorial type H_{16}*.* The multi-multiple edges limited to one of 03 or 04, and one of 13 or 15. We can take $V(H_{16}) = \{01, 03, 05, 06, 12, 13, 16, 36\}$.

Combinatorial type H_{17}*.* The multi-multiple edges are limited to one of 03 or 04, and one of 27 or 28. By symmetry can limit ourselves to considering only 03 and 27. We can take $V(H_{17}) = \{02, 06, 07, 12, 26, 67\}$.

Combinatorial type H_{18}*.* The multi-multiple edges are limited to 12 and 14. We can take $V(H_{18}) = \{02, 04, 12, 13, 23\}$.

Combinatorial type H_{19}*.* The multi-multiple edges are limited to 27, 28, 37, and 38. We can take $V(H_{19}) = \{02, 03, 07, 08, 12, 13, 23\}$.

Combinatorial type H_{20}*.* The multi-multiple edges are limited to 14, 17, 34, and 37. We can take $V(H_{20}) = \{01, 03, 04, 07, 12, 13, 23\}$.

Combinatorial type H_{21}*.* The only possible multi-multiple edge are 37 or 38, and by symmetry we can assume 38 has low weight. We can then take $V(H_{21}) = \{06, 07, 13, 23, 67\}$.

Combinatorial type H_{22}*.* The only possible multi-multiple edges are 37 and 38. We can take $V(H_{22}) = \{07, 08, 12, 13, 23\}$.

Combinatorial type H_{23}*.* The only possible multi-multiple edges are 16 and one of 37 or 38. By symmetry, we can assume 38 has low weight. We can take $V(H_{23}) = \{01, 03, 06, 12, 13, 23\}$.

Combinatorial type H_{24}*.* The only possible multi-multiple edges are 24 and 27. We can take $V(H_{24}) = \{04, 07, 12, 13, 23\}$.

Combinatorial type H_{25}*.* The only possible multi-multiple edges are 02, 05, 17. We can take $V(H_{25}) = \{01, 02, 12, 14, 17, 23, 47\}$.

Combinatorial type H_{26}*.* The only possible multi-multiple edges are 12, 14, 15, and 24. We can take $V(H_{26}) = \{01, 02, 04, 12, 14, 16, 23, 25\}$.

Combinatorial type H_{27}*.* The only possible multi-multiple edges are 12, 24, and 25. We can take $V(H_{27}) = \{04, 05, 12, 14, 23, 45\}$.

Combinatorial type H_{28}*.* The only possible multi-multiple edges are 14, 15, and 24. We can take $V(H_{28}) = \{01, 02, 04, 12, 13, 23\}$.

Combinatorial type H_{29}*.* The only possible multi-multiple edges are 14, 24, and 27. We can take $V(H_{29}) = \{01, 02, 04, 12, 13, 23\}$.

Combinatorial type H_{30}*.* The only possible multi-multiple edge is 14. We can take $V(H_{30}) = \{05, 12, 13, 23\}$.

Combinatorial type H_{31}*.* The only possible multi-multiple edge is 38. We can take $V(H_{31}) = \{07, 12, 13, 23\}$.

Combinatorial type H_{32}*.* There are no possible multi-multiple edges. We can take $V(H_{32}) = \{12, 13, 23\}$.

Combinatorial type H_{33}*.* The only possible multi-multiple edges are 15 or 24; by symmetry we can assume 24 has low weight. We can take $V(H_{33}) = \{01, 02, 05, 12, 13, 23\}$.

Combinatorial type H_{34}*.* The only possible multi-multiple edges are one of 25 or 26, and one of 37 or 38. By symmetry, we can assume 26 and 38 have low weight. We can take $V(H_{34}) = \{02, 05, 07, 12, 13, 23\}$.

Combinatorial type H_{35}*.* The only possible multi-multiple edges are 25, 26, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{35}) = \{02, 05, 06, 12, 25, 56\}$.

Combinatorial type H_{36}*.* The only possible multi-multiple edges are 34 and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{36}) = \{07, 12, 13, 23\}$.

Combinatorial type H_{37}*.* The only possible multi-multiple edges are 27 and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{37}) = \{08, 12, 13, 23\}$.

Combinatorial type H_{38}*.* The only possible multi-multiple edges are 07, 08, 17, 18. We can take $V(H_{38}) = \{01, 03, 07, 08, 12, 17, 18, 27\}$.

Combinatorial type H_{39}*.* The only possible multi-multiple edges are 34 and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{39}) = \{08, 12, 13, 23\}$.

Combinatorial type H_{40}*.* The only possible multi-multiple edges are 15 and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{40}) = \{06, 12, 13, 23\}$.

Combinatorial type H_{41}. The only possible multi-multiple edges are 03, 04, and 15. We can take $V(H_{41}) = \{06, 12, 13, 14, 23\}$.

Combinatorial type H_{42}. The only possible multi-multiple edges are 02, 03, and 23. We can take $V(H_{42}) = \{03, 07, 12, 13, 23\}$.

Combinatorial type H_{43}. The only possible multi-multiple edges are 03 and 34. We can take $V(H_{43}) = \{01, 04, 13, 14, 23\}$.

Combinatorial type H_{44}. The only possible multi-multiple edges are 12, 15, 25, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{44}) = \{02, 05, 07, 12, 17, 23, 57\}$.

Combinatorial type H_{45}. The only possible multi-multiple edges are 03, 04, 13, 15, and 34. We can take $V(H_{45}) = \{01, 03, 05, 12, 14, 23, 24\}$.

Combinatorial type H_{46}. The only possible multi-multiple edges are 04, 15, and one of 03 or 13. By symmetry, we can assume 13 has low weight. We can take $V(H_{46}) = \{05, 06, 12, 13, 14, 24\}$.

Combinatorial type H_{47}. The only possible multi-multiple edges are 03, 04, 13, 15, and 34. We can take $V(H_{47}) = \{01, 03, 04, 12, 14, 23, 24\}$.

Combinatorial type H_{48}. The only possible multi-multiple edges are 03, 04, and 34. We can take $V(H_{48}) = \{05, 12, 13, 14, 23\}$.

Combinatorial type H_{49}. The only possible multi-multiple edges are 27, 28, and 78. We can take $V(H_{49}) = \{07, 08, 12, 27, 28, 78\}$.

Combinatorial type H_{50}. The only possible multi-multiple edges are 34 and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{50}) = \{02, 04, 12, 13, 23\}$.

Combinatorial type H_{51}. The only possible multi-multiple edges are 26, 27, and 67. We can take $V(H_{51}) = \{06, 07, 12, 26, 27, 67\}$.

Combinatorial type H_{52}. The only possible multi-multiple edges are 27, 28, 67. We can take $V(H_{52}) = \{02, 07, 08, 27, 78\}$.

Combinatorial type H_{53}. The only possible multi-multiple edges are 17, 18, 28, and 78. We can take $V(H_{53}) = \{07, 08, 12, 17, 18, 78\}$.

Combinatorial type H_{54}. The only possible multi-multiple edges are 16, 27, and 67. We can take $V(H_{54}) = \{02, 06, 07, 17, 26\}$.

Combinatorial type H_{55}. The only possible multi-multiple edges are 34 and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{55}) = \{08, 12, 13, 23\}$.

Combinatorial type H_{56}. The only possible multi-multiple edges are 13, 16, and 67. We can take $V(H_{56}) = \{01, 03, 06, 16, 36\}$.

Combinatorial type H_{57}. The only possible multi-multiple edges are 07, 26, 27, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{57}) = \{02, 06, 07, 17, 23, 27, 37\}$.

Combinatorial type H_{58}. The only possible multi-multiple edges are 25, 28, and 68. We can take $V(H_{58}) = \{05, 08, 12, 28, 58\}$.

Combinatorial type H_{59}. The only possible multi-multiple edges are 03, 04, 15, 16. We can take $V(H_{59}) = \{01, 05, 06, 13, 14, 24, 34\}$.

Combinatorial type H_{60}. The only possible multi-multiple edges are 02, 03, 13, 15. We can take $V(H_{60}) = \{01, 03, 08, 13, 18, 23\}$.

Combinatorial type H_{61}. The only possible multi-multiple edges are 12, 15, 26, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{61}) = \{01, 02, 06, 12, 13, 23\}$.

Combinatorial type H_{62}. The only possible multi-multiple edges are 02, 04, and 24. We can take $V(H_{62}) = \{02, 07, 14, 24, 27\}$.

Combinatorial type H_{63}. The only possible multi-multiple edges are 03 and 04. We can take $V(H_{63}) = \{12, 13, 14, 24\}$.

Combinatorial type H_{64}. The only possible multi-multiple edges are 67 and one of 17 or 26. By symmetry, we can assume 26 has low weight. We can take $V(H_{64}) = \{06, 07, 12, 16, 17, 67\}$.

Combinatorial type H_{65}. The only possible multi-multiple edge is 67. We can take $V(H_{65}) = \{01, 17, 26\}$.

Combinatorial type H_{66}. The only possible multi-multiple edges are 02, 04, 14, 23, and one of 15 or 16. By symmetry, we can assume 16 has low weight. We can take $V(H_{66}) = \{01, 04, 08, 12, 14, 15, 24\}$.

Combinatorial type H_{67}. The only possible multi-multiple edges are 13 and 35. We can take $V(H_{67}) = \{02, 12, 13, 23\}$.

Combinatorial type H_{68}. The only possible multi-multiple edges are 34, 03, 13, 04, and 14. Note that if 03 is multi-multiple, then 04 and 14 have low weight, lest a forbidden Lannér diagram be induced among 01348. Similar arguments hold when swapping the roles of vertices $3, 4$ or $0, 1$, so by symmetry we can assume 04 and 14 have low weight. Then we can take $V(H_{68}) = \{01, 03, 05, 12, 13, 23\}$.

Combinatorial type H_{69}. The only possible multi-multiple edges are 03, 04, 13, and 15. We can take $V(H_{69}) = \{03, 05, 12, 13, 14, 23\}$.

Combinatorial type H_{70}. The only possible multi-multiple edges are 03, 04, 15, and 28. We can take $V(H_{70}) = \{05, 08, 12, 13, 23, 24\}$.

Combinatorial type H_{70}. The only possible multi-multiple edges are 03, 04, 15, and 28. We can take $V(H_{70}) = \{05, 08, 12, 13, 23, 24\}$.

Combinatorial type H_{71}. The only possible multi-multiple edges are 03, 04, and one of 12 or 25. By symmetry, we can assume 25 has low weight. We can take $V(H_{71}) = \{01, 08, 12, 18, 23, 58\}$.

Combinatorial type H_{72}. The only possible multi-multiple edges are 03, 13, and 15. We can take $V(H_{72}) = \{03, 05, 12, 13, 23\}$.

Combinatorial type H_{73}. The possible multi-multiple edges are 03, 04, 13, 15, 67. We can take $V(H_{73}) = \{01, 03, 06, 12, 13, 14, 35, 36\}$.

Combinatorial type H_{74}. The only possible multi-multiple edges are 16, 27, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{74}) = \{06, 07, 12, 13, 23\}$.

Combinatorial type H_{75}. The only possible multi-multiple edges are 03, 04, 16, and 27. We can take $V(H_{75}) = \{06, 07, 12, 13, 14, 23\}$.

Combinatorial type H_{76}. The only possible multi-multiple edges are 03, 04, 27, and 28. We can take $V(H_{76}) = \{07, 08, 12, 13, 14, 23\}$.

Combinatorial type H_{77}. The only possible multi-multiple edges are one of 03 or 04, and one of 26 or 27. By symmetry, we can assume 04 and 26 have low weight. We can take $V(H_{77}) = \{06, 08, 12, 13, 23\}$.

Combinatorial type H_{78}. The only possible multi-multiple edges are 26, 27, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{78}) = \{07, 08, 12, 13, 23\}$.

Combinatorial type H_{79}. The only possible multi-multiple edges are 16, one of 03 or 04, and one of 27 or 28. By symmetry, we can assume 04 and 28 have low weight. We can take $V(H_{79}) = \{02, 06, 07, 12, 13, 23\}$.

Combinatorial type H_{80}. The only possible multi-multiple edges are 02, 03, and 24. We can take $V(H_{80}) = \{04, 08, 12, 13, 23\}$.

Combinatorial type H_{81}. The only possible multi-multiple edges are 16, 26, and 28. We can take $V(H_{81}) = \{02, 06, 08, 26, 68\}$.

Combinatorial type H_{82}. The only possible multi-multiple edges are 02, 04, 15, and 17. We can take $V(H_{82}) = \{01, 05, 07, 12, 14, 24\}$.

Combinatorial type H_{83}. The only possible multi-multiple edges are 03, 04, 15, and 16. We can take $V(H_{83}) = \{05, 06, 12, 12, 13, 24\}$.

Combinatorial type H_{84}. The only possible multi-multiple edges are 02, 04, 15, and 16. We can take $V(H_{84}) = \{01, 04, 06, 12, 14, 24\}$.

Combinatorial type H_{85}. The only possible multi-multiple edges are 02, 03, and 13. We can take $V(H_{85}) = \{01, 08, 12, 13, 23\}$.

Combinatorial type H_{86}. The only possible multi-multiple edges are 03, 04, and 13. We can take $V(H_{86}) = \{08, 12, 13, 14, 23\}$.

Combinatorial type H_{87}. The only possible multi-multiple edges are 04, 17, and one of 02 or 24. By symmetry, we can assume 24 has low weight. We can take $V(H_{87}) = \{01, 04, 12, 14, 24\}$.

Combinatorial type H_{88}. The only possible multi-multiple edges are 03, 04, 28, and 34. We can take $V(H_{88}) = \{04, 08, 12, 14, 23, 24\}$.

Combinatorial type H_{89}. The only possible multi-multiple edges are 03, 04, 27, and 28. We can take $V(H_{89}) = \{07, 08, 12, 13, 23, 24\}$.

Combinatorial type H_{90}. The only possible multi-multiple edges are 25, 27, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{90}) = \{05, 07, 12, 13, 23\}$.

Combinatorial type H_{91}. The only possible multi-multiple edges are 02, 04, 16, and 24. We can take $V(H_{91}) = \{02, 04, 06, 14, 24, 26\}$.

Combinatorial type H_{92}. The only possible multi-multiple edges are 03, 04, 28, and 34. We can take $V(H_{92}) = \{03, 08, 12, 14, 23, 24\}$.

Combinatorial type H_{93}. The only possible multi-multiple edges are 03, 04, 15, 16, and 25. We can take $V(H_{93}) = \{01, 05, 06, 12, 13, 14, 23\}$.

Combinatorial type H_{94}. The only possible multi-multiple edges are 05, 13, and 15. We can take $V(H_{94}) = \{01, 05, 08, 15, 18, 25\}$.

Combinatorial type H_{95}. The only possible multi-multiple edges are 04, 08, 14, and 28. We can take $V(H_{95}) = \{01, 04, 08, 14, 24, 28, 48\}$.

Combinatorial type H_{96}. The only possible multi-multiple edges are 02, 04, 23, and 24. We can take $V(H_{96}) = \{02, 04, 08, 12, 14, 24\}$.

Combinatorial type H_{97}. The only possible multi-multiple edges are 03, 04, 13, 24, and one of 16, 17, 26, or 27. By symmetry between vertices 6 and 7, we can assume 17, 26, and 27 have low weight. We can take $V(H_{97}) = \{01, 02, 06, 12, 14, 23, 24\}$.

Combinatorial type H_{98}. The only possible multi-multiple edge is one of 03 or 04; by symmetry, we can assume 04 has low weight. We can take $V(H_{98}) = \{12, 13, 23\}$.

Combinatorial type H_{99}. The only possible multi-multiple edges are 12, 16, 25, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{99}) = \{01, 02, 05, 06, 12, 13, 16, 25\}$.

Combinatorial type H_{100}. The only possible multi-multiple edges are 03, 04, 13, and 34. We can take $V(H_{100}) = \{01, 04, 12, 13, 14, 24\}$.

Combinatorial type H_{101}. The only possible multi-multiple edges are one of 03 or 04, one of 15 or 16, and one of 27 or 28. By symmetry, we can assume 04, 16, and 28 have low weight. We can take $V(H_{101}) = \{01, 02, 05, 12, 13, 23\}$.

Combinatorial type H_{101}. The only possible multi-multiple edges are one of 03 or 04, one of 15 or 16, and one of 27 or 28. By symmetry, we can assume 04, 16, and 28 have low weight. We can take $V(H_{101}) = \{01, 02, 05, 12, 13, 23\}$.

Combinatorial type H_{102}. Note that there cannot be multi-multiple edges incident to any of 5, 6, 7, or 8 since these are all contained in a Lannér diagram of size 4 but no Lánner diagram of size 3. Moreover, no two vertices joined by a dashed edge can be incident to a multi-multiple edge, lest the Lannér diagram of size 2 containing these vertices cannot be connected to the Lannér diagram 5678. Thus there can be at most one multi-multiple edge in any given weighting, and by symmetry we can assume 03 is the only multi-multiple edge. We can then take $V(H_{102}) = \{01, 03, 04, 12, 13, 23\}$.

Combinatorial type H_{103}. The only possible multi-multiple edges are 14 and one of 02, 03, 26, or 36. By symmetry between vertices 2 and 3, we can assume 03, 26, and 36 have low weight. We can take $V(H_{103}) = \{01, 06, 14, 24\}$.

Combinatorial type H_{104}. The only possible multi-multiple edges are 06, 18, 23, and one of 02, 03, 28, or 38. By symmetry, we can assume 03, 28, and 38 have low weight. We can take $V(H_{104}) = \{03, 06, 08, 12, 16, 26\}$.

Combinatorial type H_{105}. The only possible multi-multiple edges are 13, 34, and 46. We can take $V(H_{105}) = \{03, 04, 06, 13, 23\}$.

Combinatorial type H_{106}. The only possible multi-multiple edges are 18, 26, 34, and one of 03 or 04. By symmetry, we can assume 04 has low weight. We can take $V(H_{106}) = \{01, 05, 08, 12, 13, 14, 23\}$.

Combinatorial type H_{107}. The only possible multi-multiple edges are 34 and 56. We can take $V(H_{107}) = \{04, 06, 13, 26\}$.

Combinatorial type H_{108}. The only possible multi-multiple edges are one of 07 or 08, one of 25 or 26, and one of 13 or 14. By symmetry, we can assume 08, 26, and 14 have low weight. We can take $V(H_{108}) = \{03, 05, 07, 12, 17, 27\}$.

Combinatorial type H_{109}. First, observe that the multi-multiple edges must be contained either in the subdiagram 678 or in the subdiagram 012345. Moreover, there can be at most one multi-multiple edge among 678, lest the Lannér diagrams of size 2 cannot be connected to the Lannér triangle. No two vertices joined by a dashed edge can both be incident to a multi-multiple edge for this same reason. Moreover, no vertex in 012345 can be adjacent to two multi-multiple edges lest these induce a Lannér triangle or violate the previous assertion. By symmetry, we can now assume the multi-multiple edges are limited to 02 and 67.

Suppose that 67 is multi-multiple. Then neither vertex of 67 is connected to any vertex of 012345 lest these induce a Lannér triangle, so for connectivity reasons 8 must be joined to at least one vertex of each of 01, 23 and 45. However, vertex 8 must also be connected to a vertex of 67. Thus, vertex 8 has degree at least 4 in some elliptic diagram, which is not possible. Therefore, we need only consider that 02 is multi-multiple. In this case, we can take $V(H_{109}) = \{02, 04, 12, 13, 14, 24, 47, 67\}$.

Combinatorial type H_{110}. The only possible multi-multiple edges are 45, one of 34 or 45, one of 24 or 25, and any edges of 678. By symmetry of 4 and 5, we can assume 34 and 24 have low weight. Since the Lannér triangle 678 must be connected to the other Lannér triangles, the edges 25 and 35 cannot simultaneously be multi-multiple. Hence, by symmetry we can assume 35 has low weight. For the same reason at most one edge of 678 can be multi-multiple. So we can conclude, again invoking symmetry of 678, that the multi-multiple edges are limited to 25, 45, and 67. We can then take $V(H_{110}) = \{01, 02, 05, 06, 12, 15, 16, 23, 24, 26, 35\}$.

Combinatorial type H_{111}. The only possible multi-multiple edges are 23, 45, one of 02 or 03, and one of 14 or 15. By symmetry, we can assume 03 and 15 have low weight. Moreover, both 02 and 14 cannot simultaneously be multi-multiple lest 78 cannot be connected to 01, so we can assume 14 is also low weight. Thus, our set of multi-multiple edges must be among 23, 45, and 03. We can then set $V(H_{111}) = \{03, 04, 05, 07, 12, 34, 37, 47\}$.

9. Remarks on Compact Coxeter 6-Polytopes with 10 Facets

Throughout this section, we consider compact Coxeter 6-polytopes with 10 facets. Using the methods outlined in Section 4, there is a finite algorithm yielding a superset of the combinatorial types of such polytopes. Though the process was too computationally intense for the classification to be completed in the current project, we were able to list the combinatorial types in the special case where the missing faces have size only 2 and 5. In this case, there are precisely two combinatorial types:

$$I_1 \text{ with missing face list } \{01, 02, 13, 24567, 34567, 89\}$$

and

$$I_2 \text{ with missing face list } \{01, 02, 13, 24, 34, 56789\}\,.$$

The methods used to analyse these two combinatorial types are similar to those used extensively in Section 8. It is straightforward to check using an appropriate choice of $V(I_1)$ and $V(I_2)$ that there are no compact Coxeter polytopes of these combinatorial types.

First, suppose there is a polytope realising I_1, and consider its Coxeter diagram. The only multi-multiple edges are 08, 09, 18, and 19. Moreover, note that vertices 8 and 9 cannot simultaneously be incident to a multi-multiple edge, lest the Lannér diagram 89 cannot be connected to one of the Lannér diagrams of size 5. Thus, by symmetry, we can assume the multi-multiple edges are limited to 08 and 18. Then setting

$$V(I_1) = \{01, 03, 08, 09, 12, 18, 19, 28, 38\}$$

is sufficient to check computationally that there are no polytopes of type I_1.

Now consider a Coxeter diagram of combinatorial type I_2. The only possible multi-multiple edges must be contained within the subdiagram 01234. Moreover, any two vertices joined by a dashed edge in this subdiagram cannot both be incident to multi-multiple edges, lest the Lannér subdiagram corresponding to these vertices cannot be connected to the Lannér diagram of size 5. Hence, there is at most one multi-multiple edge contained within 01234, and by symmetry we can assume this edge is 03. One can then show that there are no compact Coxeter polytopes of this combinatorial type by applying Corollary 6.11 with

$$V(I_2) = \{01, 02, 03, 04, 12, 13, 14, 23, 34\}\,.$$

The analysis of these combinatorial types as described above yields the following result.

Theorem 9.1. *There are no compact Coxeter* 6*-polytopes with* 10 *facets having missing faces of orders only* 2 *and* 5*.*

There is one known compact Coxeter 6-polytope with 10 facets (see Figure 5), constructed by Bugaenko [6].

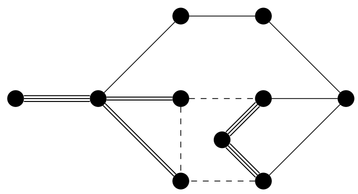


FIGURE 5. The Coxeter diagram of the compact Coxeter 6-polytope with 10 facets constructed by Bugaenko.

10. Compact 3-Free Coxeter Polytopes

In this section, we use a modification of Vinberg's proof that simple Coxeter polytopes have dimension at most 29 to find a tighter bound on the dimension of compact Coxeter 3-free polytopes, i.e., polytopes where every missing face is of size 2. In particular, we show that the dimension of compact 3-free polytopes is at most 13. While this result may be of interest in its own right, we make use of this result to bound the dimension of polytopes with few facets in Section 11. To our knowledge, the best previous bound on the dimension for this class of polytopes was Vinberg's bound $d \leq 29$ for all compact Coxeter polytopes.

Following the terminology of Esselmann [8], a polytope is called k-*free* if it has no missing face of size at least k.

Remark 10.1. Every face of a k-free polytope is itself a k-free polytope (see [8, Folgerung 1.7]). This holds because given a face f of a polytope P, any missing face of f is the intersection of a (possibly larger) missing face of P with f.

We first state a lemma from Vinberg's proof of the upper bound on the dimension of a compact polytope which relies on weightings of the planar angles. A *planar angle* of a polytope is a pair (A, F) where A is a vertex and F is a two-dimensional face containing it; (A, F) is said to be a planar angle *at* A and *on* F.

Proposition 10.2 ([24, Prop. 6.2])**.** *Let P be a d-dimensional compact Coxeter polytope and $c > 0$ a positive number. We assume that the planar angles of P can be endowed with weights in such a way that:*

(a) The sum $\sigma(A)$ of the weights of the planar angles at the vertex A does not exceed cd.

(b) The sum $\sigma(F)$ of the weights of the planar angles of any 2-dimensional face is not less than $5 - k$, where k is the number of vertices of this face.

Then $d < 8c + 6$.

We also require one of Vinberg's results on the structure of quadrilateral two-dimensional faces. We first introduce the language of star diagrams; these were referred to as "star schemes" by Vinberg. The *diagram of a planar angle* (A, F) is the Coxeter subdiagram S_A with two chosen "black" vertices corresponding to the facets containing A but not F. Note that this is an elliptic diagram of order d. The *star diagram of a face* F is the Coxeter subdiagram S_F^* whose vertices correspond to the facets having a non-trivial intersection with F. We call the vertices of S_F^* that are also vertices of S_F, i.e., that correspond to facets containing F, "white" and the remaining vertices "black". Note that when F is a two-dimensional face, the black vertices correspond to facets whose intersection with F is precisely an edge.

Proposition 10.3 ([24, Prop. 6.4])**.** *Let S_F^* be the star diagram of a quadrilateral two-dimensional face F of a polytope P. We divide the black vertices of S_F^* into pairs in such a way that the vertices corresponding to opposite sides of F correspond to a single pair. Then*

(1) the removal from S_F^* of any two black vertices in different pairs leaves the diagram of one of the planar angles of F;

(2) every hyperbolic subdiagram of S_F^* contains both the black vertices from some pair;

(3) the removal from S_F^* of the two black vertices in the same pair leaves a hyperbolic diagram.

Theorem 10.4. *There are no compact* 3*-free Coxeter hyperbolic polytopes of dimension* 14 *or higher.*

Proof. Let P be a compact 3-free Coxeter d-polytope and S its scheme. We attach weights to the planar angle P similarly to the original construction by Vinberg [24, Theorem 6.1]: the weight of a planar angle P is 1 if the black vertices in its scheme are adjacent and 0 otherwise. We now check that the hypotheses of Proposition 10.2 are satisfied with $c = 1$.

Since every elliptic Coxeter diagram is a forest (when viewed as an unweighted graph), there are at most $d - 1$ edges, i.e., pairs of vertices of distance at most $c = 1$. Thus, Condition (a) of Proposition 10.2 holds.

It remains to check Condition (b) for two-dimensional triangular and quadrilateral faces. Faces with five or more vertices vacuously satisfy this condition, as the planar angle weights are all non-negative.

First, we show that P cannot have a two-dimensional triangular face. By Remark 10.1, every face of P is 3-free as well. Note that a two-dimensional triangular face itself has three one-dimensional facets, which all together have trivial intersection but of which any two intersect non-trivially. Thus, this triangular face, viewed as a two-dimensional polytope, has a missing face of order 3. Hence, it is not 3-free, contradicting Remark 10.1.

Suppose now that F is a two-dimensional quadrilateral face of P. It follows from Proposition 10.3 that every pair of black vertices of S_F^* is contained in some hyperbolic subdiagram not containing the black vertices of the other pair. Since we have no missing faces of size other than 2, this implies that each pair of black vertices corresponds to Lannér subdiagram of size 2; call these subdiagrams L and M. Since the subdiagram induced by $L \cup M$ is connected (lest it be parabolic), there must be two black vertices from separate pairs that are adjacent. These two black vertices lie in the scheme of the corresponding angle of F at a distance of 1. Thus, we have shown $\sigma(F) \geq 1$, as desired. Therefore, by Proposition 10.2, we have $d < 8(1) + 6 = 14$. $\square$

11. Bounding the Dimension of Polytopes with Few Facets

We now utilise the classification of compact Coxeter d-polytopes with few facets to place upper bounds on the dimensions of polytopes with $d+k$ facets for $5 \leq k \leq 10$. Our methods also utilise the bound $d \leq 14$ for compact 3-free Coxeter polytopes derived in Section 10.

Definition 11.1. Let $D(k)$ denote the maximum positive integer for which a compact Coxeter $D(k)$-polytope with $D(k)+k$ facets exists.

The only previously published bound on the dimension of such polytopes is the following general result of Vinberg:

Theorem 11.2 ([25, Theorem 4])**.** *There are no compact Coxeter polytopes in dimension* 30 *or higher.*

That is, we have $D(k) \leq 29$ for all k. For $k \leq 4$, the exact value of $D(k)$ is known.

Theorem 11.3.[1] *We have*

- $D(1) = 4$*, due to Lannér* [19]*;*
- $D(2) = 5$*, due to Kaplinskaja* [17] *and Esselmann* [8]*;*
- $D(3) = 8$*, due to Esselmann* [8]*;*
- $D(4) = 7$*, due to Felikson and Tumarkin* [10]*.*

Thus, we begin our investigation with $k = 5$, and proceed until the bounds obtained by our methods are no stronger than Vinberg's Theorem 11.2. In particular, we derive the following bounds in this section:

Theorem 11.4. *We have*

- $D(5) \leq 9$*,*
- $D(6) \leq 12$*,*
- $D(7) \leq 15$*,*
- $D(8) \leq 18$*,*
- $D(9) \leq 22$*, and*

[1]Anna Felikson and Pavel Tumarkin have shown in unpublished notes that $D(5) \leq 8$ using a fairly involved argument [Tumarkin, personal communication (2021)].

- $D(10) \leq 26$.

The argument proceeds by first proving a linear bound on d having slope 4 (with respect to k), then applying the classification of polytopes with fewer facets to slightly improve these bounds in particular cases. The first bound is obtained by iterating part of the argument used by Felikson and Tumarkin to bound $D(4)$ [10].

Lemma 11.5. *If P contains a missing face of size $\ell > 2$, then*

$$D(k) \leq \max_{1 \leq i \leq k-1} D(i) + \ell - 1\,.$$

Proof. Let T be a Lannér subdiagram of order ℓ. Let S_0 be a subdiagram of the type

$$\begin{cases} G_2^{(m)} \text{ for some } m \geq 4 & \text{if } \ell = 3\,; \\ H_3 \text{ or } B_3 & \text{if } \ell = 4\,; \\ H_4 \text{ or } F_4 & \text{if } \ell = 5\,. \end{cases}$$

Since S_0 has at least one bad neighbour (the unique vertex of $T \setminus S_0$), by Proposition 5.6 we have that $P(S_0)$ is a Coxeter $(d-\ell+1)$-polytope with at most $d+k-(\ell+1) = (d-\ell+1)+k-1$ facets. Hence,

$$d - \ell + 1 \leq \max_{1 \leq i \leq k-1} D(i)\,.$$

This implies the desired inequality. □

In order to obtain a general bound from this lemma, we make use of the following result of Esselmann.

Lemma 11.6 ([8, Lemma 6.7]). *A 3-free compact Coxeter d-polytope has at least $2d$ facets, and in the case of equality, it must be a d-cube.*

Theorem 11.7. *For $k \geq 5$, we have*

$$D(k) \leq \max\left\{\max_{i<k} D(i) + 4, \min\{k-1, 13\}\right\}.$$

Proof. Fix a Coxeter d-polytope P with $d+k$ facets. If P is not 3-free, then it contains a missing face of order 3, 4, or 5. The first bound then follows from Lemma 11.5.

Now suppose P is 3-free, so by Lemma 11.6, P either has at least $2d+1$ facets or is a d-cube. Since compact Coxeter d-cubes only exist in dimension at most 5 [16], any 3-free compact Coxeter d polytope with $d+k$ facets satisfies $d \leq k-1$ for $k > 5$. For $k = 5$, we can have 3-free compact polytopes in dimension 5 or less, but this is exceeded by the first bound $\max_{i<5}\{D(i)+4\} = 12$. We furthermore have from Theorem 10.4 that such polytopes arise in dimension at most 13. Hence for $k > 5$, a 3-free compact Coxeter d-polytope with $d+k$ facets satisfies $d \leq \min\{k-1, 13\}$, from which the second bound follows. □

We now proceed to improving this bound for polytopes with few facets, which involves a more detailed argument depending on the classifications of compact Coxeter polytopes with fewer facets.

11.1. **Polytopes with $d+5$ facets**

Theorem 11.7 implies that there are no compact Coxeter d-polytopes with $d+5$ facets for $d \geq 13$. In this section, we additionally show that no such polytopes arise in dimensions 12, 11, and 10.

For the first reduction, we require some knowledge about compact Coxeter 6-polytopes with 10 facets. In fact, it has been verified that any such polytope has a missing face of size 3 or 4 using the exhaustive methods detailed in the first portion of this thesis (see Theorem 9.1).

Proposition 11.8. *There is no Coxeter d-polytope with $d+5$ facets for $d \geq 10$.*

Proof. Suppose there is a compact Coxeter polytope P of dimension $d \geq 10$ with at most $d+5$ facets; denote its Coxeter diagram by Σ.

First suppose that P has a missing face of order 3, and let S_0 be a subdiagram of type $G_2^{(m)}$ for $m \geq 4$ contained in this missing face. The face $P(S_0)$ corresponding to the subdiagram S_0 is a Coxeter polytope of dimension $d_1 \geq 8$ with at most d_1+4 facets. This must be the unique 8-polytope with 11 facets. Note that Coxeter diagram of this polytope, Σ_{S_0}, contains two subdiagrams of type H_4. These subdiagrams are preserved in the subdiagram $\overline{S_0}$, call them T_1 and T_2. Moreover, there is a vertex $v \in \Sigma$ connected to T_1 and T_2 by simple edges, as well as distinct vertices u_i connected to T_i by a simple edge for $i \in \{1,2\}$, as this is true in Σ_{S_0} and is preserved under the operations described in Proposition 5.6. In particular, T_1 has at least two bad neighbours in Σ. The face corresponding to the subdiagram T_1 in Σ is then a compact

Coxeter polytope of dimension $d_2 \geq 6$ with at most d_2+3 facets. In particular, it must have dimension 6 with 9 facets or dimension 8 with 11. There are four such polytopes, and in each such polytope every subdiagram of type H_4 has two bad neighbours joined by simple edges. Thus, T_2 must have at least 3 bad neighbours joined by simple edges in Σ, since there are at least two in each of $\overline{S_0}$ and $\overline{T_1}$, with at most one bad neighbour being common to both. Thus, the face corresponding to T_2 in Σ must be a compact Coxeter polytope of dimension $d_3 \geq 6$ with at most d_3+2 facets, but no such polytope exists by Theorem 11.3. Therefore, we can assume P has no missing faces of order 3.

Now suppose that P has a missing face of order 4. Let S_0 be a subdiagram of type B_3 or H_3 contained in this missing face. Then the face corresponding to S_0 is a compact Coxeter polytope of dimension $d_1 \geq 7$ with at most $d_1 + 4$ facets. Hence, $P(S_0)$ is either the unique compact Coxeter 7-polytope with 11 facets, or the unique compact Coxeter 8-polytope with 11 facets. If $P(S_0)$ is the 7-polytope, then Σ_{S_0} (and hence $\overline{S_0}$, by Proposition 5.6) contains a subdiagram S_1 of type H_4 with three bad neighbours, each connected by a simple edge. Thus, the face corresponding to S_1 in Σ is a compact Coxeter polytope of dimension $d_2 \geq 6$ with at most $d_2 + 2$ facets, which does not exist. Thus, $P(S_0)$ is not the unique compact Coxeter 7-polytope with 11 facets. Now supposing $P(S_0)$ is the unique compact Coxeter 8-polytope with 11 facets, the same reasoning as that used in the previous paragraph shows that Σ must contain a subdiagram of type H_4 with at least three bad neighbours. This again yields a face which is a compact Coxeter polytope of dimension $d_3 \geq 6$ with at most $d_3 + 2$ facets, which does not exist. Therefore, we can assume P has no missing faces of order 4.

Thus, we can now restrict to considering polytopes with missing faces of order only 2 and 5. By Lemma 11.6, P must contain a missing face of size 5. Let S_0 be a subdiagram of type H_4 or F_4 in this missing face. Then $P(S_0)$ is a compact Coxeter polytope of dimension $d_1 \geq 6$ with at most $d_1 + 4$ facets. Moreover, since $\Sigma_{S_0} = \overline{S_0}$ by Proposition 5.6, then $P(S_0)$ also contains missing faces of order only 2 and 5. In particular, $P(S_0)$ is either a 6-polytope with 10 facets or the unique compact Coxeter 8-polytope with 11 facets, as the unique compact Coxeter 7-polytope with 11 facets contains missing faces of order 4. If $P(S_0)$ is the unique compact Coxeter 8-polytope with 11 facets, then we reach a contradiction by the same reasoning as above.

Otherwise, $P(S_0)$ must be a compact Coxeter 6-polytope with 10 facets containing missing faces of order only 2 and 5, but no such polytopes exist by Theorem 9.1. We can now conclude that no such polytope P arises. □

Corollary 11.9. *We have $D(5) \leq 9$.*

Lemma 11.10. *If there exists a 9-polytope with 14 facets, then it contains no missing faces of size 3 nor any multi-multiple edge.*

Proof. Suppose that P is a compact 9-polytope with 14 facets, and let Σ be its Coxeter diagram. Assume P contains a missing face of size 3 or a multi-multiple edge. In the first case, let Σ_0 be a subdiagram of type $G_2^{(m)}$ for $m \geq 4$ contained in the missing face of size 3. In the latter case, let Σ_0 be a subdiagram of type $G_2^{(m)}$ for $m \geq 6$. Then the face corresponding to Σ_0 is a 7-polytope with at most 11 facets. It must be the unique 7-polytope with 11 facets constructed by Bugaenko. Moreover, by Proposition 5.6 we can obtain the diagram $\overline{\Sigma_0}$ from Σ_{S_0} by possibly replacing some double edges by simple edges, or some dashed edges by ordinary or empty edges. Thus, there is a diagram of type H_4 in $\overline{S_0}$ with at least 3 bad neighbours, call it Σ_1. Moreover, Σ_1 must have precisely three bad neighbours in Σ, or else the face corresponding to Σ_1 is a 5-polytope with at most 6 facets, which does not exist. Then the the face corresponding to Σ_1 in Σ is a 5-polytope with precisely 7 facets, furthermore containing the fixed missing face of size 3 or the multi-multiple edge. However, no such 5-polytope exists. □

Theorem 11.11. *In a 9-polytope with 14 facets, every missing face of size 5 contains a subdiagram of type H_4 or F_4 with at least two bad neighbours.*

Proof. Suppose P is such a polytope containing a missing face of size 5. Let Σ_0 be a subdiagram of type F_4 or H_4. Then the face corresponding to Σ_0 is a 5-polytope P' with at most 9 facets. Furthermore, if Σ_0 does not itself have two bad neighbours, then P' has precisely 9 facets. By Lemma 11.10, P' has no missing faces of size 3. By the classification in Section 8, all 5-polytopes with 9 facets not containing missing faces of order 3 contain a subdiagram of type H_4 with at least 2 bad neighbours. □

11.2. Polytopes with $d+6$ facets

Now that we have established $D(5) \leq 9$, we obtain from Theorem 11.7 that $D(6) \leq 13$. Using results from the previous section, we can reduce this bound by one dimension.

Lemma 11.12. *There are no* 13*-polytopes with* 19 *facets.*

Proof. Suppose such a polytope exists, call it P. By Lemma 11.5, P must have missing faces only of size 2 or 5. Moreover, by Lemma 11.6, P must contain at least one missing face of size 5. Let Σ_0 be a subdiagram of T of type H_4 or F_4. Considering the face corresponding to Σ_0, we obtain a Coxeter polytope P' of dimension 9 with at most 14 facets, and furthermore it contains only missing faces of size 2 or 5. Note that in fact P' must contain precisely 14 facets, since $D(4) < 9$. Again applying Lemma 11.6, P' must contain at least one missing face of size 5. Thus, by Theorem 11.11, P' must contain a subdiagram of type F_4 or H_4 with at least two bad neighbours. Looking at the face corresponding to this subdiagram in Σ yields a face of P that is a Coxeter 9-polytope with at most 13 facets. No such Coxeter polytopes exists, so we reach a contradiction. □

Corollary 11.13. *We have* $D(6) \leq 12$*.*

We now formulate the following lemma, where we consider faces of increasing codimension corresponding to subdiagrams of type H_4 or F_4 in polytopes with missing faces of sizes either 2 or 5.

Lemma 11.14. *Let P be a compact Coxeter d-polytope with $d+k$ facets whose missing faces have size either* 2 *or* 5*, and $k < d$. Suppose every subdiagram in P of the type H_4 or F_4 has precisely one bad neighbour (note that such a diagram must always have at least one bad neighbour). Then P has a subdiagram S_ℓ corresponding to a face $P(S_\ell)$ which is a $(d-4(\ell+1))$-polytope with $d+k-5(\ell+1)$ facets, where ℓ is the maximum non-negative integer such that*

$$d > k + 3\ell\,,$$

and such that $\Sigma_{S_\ell} = \overline{S_\ell}$*.*

Proof. Fix m such that $d \geq k + 3m$. We proceed by induction on m, showing at each step that P has a subdiagram S_m corresponding to a face $P(S_m)$ which is a $(d - 4(m + 1))$-polytope with $d + k - 5(m + 1)$ facets satisfying $\Sigma_{S_m} = \overline{S_m}$. Note that in particular, $P(S_m)$ contains missing faces of size only 2 and 5, as its Coxeter diagram is a subdiagram of the diagram for P. The base case $m = 0$ is trivial, as we can view P as a face of itself and take S_0 to be the empty diagram.

Now suppose that we have fixed a subdiagram S_{m-1} with the necessary properties. Since we have $d > k + 3m$, then we have

$$2(d - 4m) > d + k - 5m\,.$$

Thus, by Lemma 11.6, a $(d - 4m)$-polytope with $d + k - 5m$ facets cannot be 3-free. Hence the polytope $P(S_{m-1})$ must have a missing face of size 5. The corresponding Lannér diagram must contain a subdiagram T_m of type H_4 or F_4. Let $S_m = S_{m-1} \cup T_m$. By our hypotheses, T_m has precisely one bad neighbour, hence $P(S_m)$ is a compact Coxeter $(d - 4(m + 1))$-polytope with $d + k - 5(m + 1)$ facets, as desired. □

Lemma 11.15. *Suppose there exist a compact Coxeter* 12*-polytope* P *with* 18 *facets whose missing faces have size either* 2 *and* 5*. Then* P *contains a subdiagram of the type* H_4*,* F_4 *with at least* 2 *bad neighbours.*

Proof. By Lemma 11.14, P must have a subdiagram $\Sigma_{P'}$ corresponding to a face P' that is a compact Coxeter 4-polytope with 8 facets. However, it can be checked that each of these contains a subdiagram of type D_4 with 4 bad neighbours or of type B_3 with at least 3 bad neighbours, respectively. Looking at the faces corresponding to these subdiagrams in Σ would yield a compact Coxeter 8-polytope with at most 10 facets or a compact Coxeter 9-polytope with at most 12 facets, both of which do not exist. □

11.3. **Polytopes with $d + 7$ facets**

Lemma 11.16. *There is no compact Coxeter* 16*-polytope with* 23 *facets.*

Proof. By Lemma 11.14, this must have a face which is a 4-polytope with 8 facets. But by the argument in Lemma 11.15, these each contain either a subdiagram of type D_4 with 4 bad neighbours or B_3 with at least 3 bad neighbours, respectively. Looking at the face corresponding to such a subdiagram in Σ would yield a compact Coxeter 12-polytope with at most 15 facets or a compact Coxeter 13-polytope with at most 17 facets, both of which do not exist. □

Corollary 11.17. *We have $D(7) \leq 15$.*

Lemma 11.18. *Suppose there exist a compact Coxeter 15-polytope P with 22 facets whose missing faces have size either 2 and 5. Then P contains a subdiagram of the type H_4 or F_4 with at least 2 bad neighbours.*

Proof. By Lemma 11.14, P must have a subdiagram $\Sigma_{P'}$ corresponding to a face P' that is a compact Coxeter 3-polytope with 7 facets. By [8, Satz 6.9], there is precisely one possible combinatorial type. It is fairly straightforward to check that no such polytope can have only angles $\pi/2$ or $\pi/3$ (this can be accomplished with the code described in Section 6). If this contains a subdiagram of the type $G_2^{(m)}$ for $m \geq 4$, then it must have at least 3 bad neighbours (as any two vertices not joined by a dashed edge in the diagram are together adjacent to at least 3 dashed edges). So there is a face of P' which is a compact Coxeter 13-polytope with at most 17 facets, which does not exist. □

11.4. Polytopes with $d + 8$ facets

Lemma 11.19. *There is no compact Coxeter 19-polytope with 27 facets.*

Proof. By Lemma 11.5, P must have missing faces only of size 2 or 5. By Lemma 11.14, this must have a face which is a 15-polytope with 22 facets. By Lemma 11.18, this must have a subdiagram of type H_4 or F_4 with at least two bad neighbours. But then this yields a face which is a compact Coxeter 15-polytope with at most 21 facets, contradicting that $D(i) < 15$ for $i \leq 6$. □

Corollary 11.20. *We have $D(8) \leq 18$.*

While in the previous cases we were able to reduce the dimension by one via showing that there are certain subdiagrams with at least two bad neighbours, we were not able to obtain such a reduction in this case. Lemma 11.14 guarantees that such a polytope has a face which is a compact 2-polytope with 6 facets. However, by the classification in [4] there are many of these for which every subdiagram not of type A_n or D_5 has at most one bad neighbour.

11.5. Polytopes with $d+9$ or $d+10$ facets

As mentioned previously, in these case we could not reduce the dimension by one.

Corollary 11.21. *We have* $D(9) \leq 22$*.*

Proof. This follows directly from Theorem 11.7 and the results proved above that $D(k) \leq 18$ for $k < 9$. □

Corollary 11.22. *We have* $D(10) \leq 26$*.*

Proof. This follows directly from Theorem 11.7 and the results proved above that $D(k) \leq 22$ for $k < 10$. □

For $k \geq 11$, the bounds we obtain from repeated application of Theorem 11.7 are weaker than Vinberg's bound $D(k) \leq 29$ [25].

12. Further Directions

By the results presented in this thesis, the only dimension where compact Coxeter d-polytopes with $d+4$ facets have not been classified is $d = 6$. A list of potential combinatorial types can be generated in the same manner as in Section 4, but the large number of point set order types with 10 points makes this computationally challenging. With such a list in hand, the analysis of each combinatorial type seems likely to be fairly straightforward, as was the case in dimension 5. The author plans to continue this work and is hopeful that the computational difficulties can be handled.

We also provide many new examples of compact Coxeter polytopes in dimension 4 and 5. This includes the first known example in dimension higher than three with a dihedral angle of less than $\frac{\pi}{10}$, as well as the first known example in dimension higher than three with an angle of $\frac{\pi}{7}$. It may be interesting to further study the properties of many of these polytopes, especially those that are essential.

In Section 11, we provide improved upper bounds on the dimension of compact Coxeter d-polytopes with $d+k$ facets for $5 \leq k \leq 10$. Thus far, there are no compact Coxeter polytopes known in dimension larger than 8. These bounds can be viewed as further evidence that there are perhaps no higher dimensional examples, or that such polytopes may be fairly complicated. It seems quite likely that the bounds provided can be improved by a constant factor with more detailed analysis. This may also be the case with the bound $d \leq 13$ for compact Coxeter 3-free d-polytopes obtained in Section 10, since examples are known only up through dimension 5 (see, e.g., the cubes classified in [16]).

Appendix A. List of Combinatorial Types

The first table contains the missing face list of the 34 possible combinatorial types of compact hyperbolic 4-polytopes with 8 facets. The second table contains the missing face list of the 111 possible combinatorial types of compact hyperbolic 5-polytopes with 9 facets having at least two pairs of disjoint facets. Each list of missing faces is presented in its lexicographically least form, with respect to relabellings of the facets. A missing face $f = f_{i_1} \cap \cdots \cap f_{i_s}$ is denoted by the string $i_1 \ldots i_s$ where $i_1 < \cdots < i_s$.

Type	Missing Face List	Number of Polytopes
G_1	01, 02, 03, 12, 14, 25, 3467, 3567, 4567	130
G_2	01, 02, 03, 12, 14, 2567, 34, 3567, 4567	105
G_3	01, 02, 03, 1245, 1456, 27, 36, 37, 4567	52
G_4	01, 23, 45, 67	12
G_5	01, 02, 13, 24, 34, 567	3
G_6	01, 02, 03, 124, 145, 26, 357, 367, 4567	2
G_7	01, 02, 03, 12, 14, 256, 347, 3567, 4567	2
G_8	01, 02, 03, 1245, 146, 257, 36, 37, 4567	1
G_9	01, 02, 03, 14, 25, 367, 4567	15
G_{10}	01, 02, 034, 134, 156, 256, 27, 347, 567	1
G_{11}	01, 02, 13, 245, 345, 67	8
G_{12}	01, 02, 034, 134, 15, 256, 267, 347, 567	4
G_{13}	01, 02, 034, 15, 26, 347, 567	2
G_{14}	01, 023, 145, 236, 237, 46, 57	2
G_{15}	01, 02, 03, 12, 145, 267, 345, 367, 4567	0
G_{16}	01, 02, 034, 125, 156, 27, 346, 347, 567	0
G_{17}	01, 02, 03, 124, 145, 267, 35, 367, 4567	0
G_{18}	01, 02, 03, 124, 125, 146, 257, 346, 357, 367, 4567	0
G_{19}	01, 02, 03, 124, 135, 145, 246, 267, 357, 367, 4567	0
G_{20}	01, 02, 034, 135, 156, 247, 26, 347, 567	0
G_{21}	01, 02, 134, 135, 246, 267, 346, 57	0
G_{22}	01, 02, 03, 124, 1456, 27, 356, 37, 4567	0
G_{23}	01, 02, 034, 134, 15, 25, 267, 3467, 567	0
G_{24}	01, 02, 034, 135, 16, 247, 26, 3457, 567	0
G_{25}	01, 023, 024, 135, 156, 246, 247, 37, 456, 567	0
G_{26}	01, 023, 024, 156, 157, 234, 236, 356, 47, 567	0
G_{27}	01, 02, 034, 134, 135, 156, 247, 256, 267, 347, 567	0
G_{28}	01, 02, 034, 135, 136, 157, 246, 247, 257, 346, 567	0
G_{29}	01, 02, 034, 125, 136, 156, 247, 257, 346, 347, 567	0
G_{30}	01, 023, 024, 135, 156, 237, 247, 357, 46, 567	0
G_{31}	01, 023, 024, 035, 126, 157, 167, 234, 246, 345, 357, 467, 567	0
G_{32}	01, 023, 024, 035, 124, 146, 156, 237, 247, 356, 357, 467, 567	0
G_{33}	01, 023, 024, 035, 146, 157, 167, 234, 235, 246, 357, 467, 567	0
G_{34}	01, 023, 024, 035, 126, 137, 167, 245, 246, 345, 357, 467, 567	0

Type	**Missing Face List**	**Number of Polytopes**
H_1	01, 02, 03, 12, 14, 25678, 34, 35678, 45678	22
H_2	01, 02, 03, 12456, 14567, 28, 37, 38, 45678	18
H_3	01, 02, 03, 12, 14, 256, 3478, 35678, 45678	6
H_4	01, 02, 03, 1245, 1456, 27, 368, 378, 45678	3
H_5	01, 02, 034, 134, 15, 256, 2678, 3478, 5678	1
H_6	01, 02, 03, 12, 14, 25, 34678, 35678, 45678	0
H_7	01, 02, 03, 12456, 1457, 267, 268, 38, 45678	0
H_8	01, 02, 03, 12456, 1457, 268, 37, 38, 45678	0
H_9	01, 02, 03, 12, 14, 2567, 348, 35678, 45678	0
H_{10}	01, 02, 03, 124, 14567, 28, 3567, 38, 45678	0
H_{11}	01, 02, 03, 1245, 14567, 28, 367, 38, 45678	0
H_{12}	01, 02, 034, 134, 15, 25, 2678, 34678, 5678	0
H_{13}	01, 02, 03, 12, 145, 267, 3458, 3678, 45678	0
H_{14}	01, 02, 03, 12, 145, 2678, 345, 3678, 45678	0
H_{15}	01, 02, 03, 124, 145, 2678, 35, 3678, 45678	0
H_{16}	01, 02, 034, 135, 16, 2478, 26, 34578, 5678	0
H_{17}	01, 02, 034, 1345, 16, 26, 278, 34578, 5678	0
H_{18}	01, 02, 03, 124, 125, 146, 2578, 346, 3578, 3678, 45678	0
H_{19}	01, 02, 03, 1245, 1346, 1456, 257, 278, 368, 378, 45678	0
H_{20}	01, 02, 03, 124, 1256, 147, 2568, 347, 3568, 378, 45678	0
H_{21}	01, 02, 03, 1245, 1456, 278, 36, 378, 45678	0
H_{22}	01, 02, 03, 1245, 1246, 1457, 268, 357, 368, 378, 45678	0
H_{23}	01, 02, 03, 1245, 146, 2578, 36, 378, 45678	0
H_{24}	01, 02, 03, 124, 1356, 1456, 247, 278, 3568, 378, 45678	0
H_{25}	01, 02, 03, 124, 1456, 27, 3568, 378, 45678	0
H_{26}	01, 02, 03, 124, 145, 26, 3578, 3678, 45678	0
H_{27}	01, 02, 03, 124, 125, 146, 257, 3468, 3578, 3678, 45678	0
H_{28}	01, 02, 03, 124, 135, 145, 246, 2678, 3578, 3678, 45678	0
H_{29}	01, 02, 03, 124, 1356, 145, 247, 278, 3568, 3678, 45678	0
H_{30}	01, 02, 03, 124, 145, 267, 358, 3678, 45678	0
H_{31}	01, 02, 03, 124, 1456, 278, 356, 378, 45678	0
H_{32}	01, 02, 03, 124, 1456, 278, 356, 378, 45678	0
H_{33}	01, 02, 03, 14, 25, 3678, 45678	0
H_{34}	01, 02, 03, 14, 256, 378, 45678	0

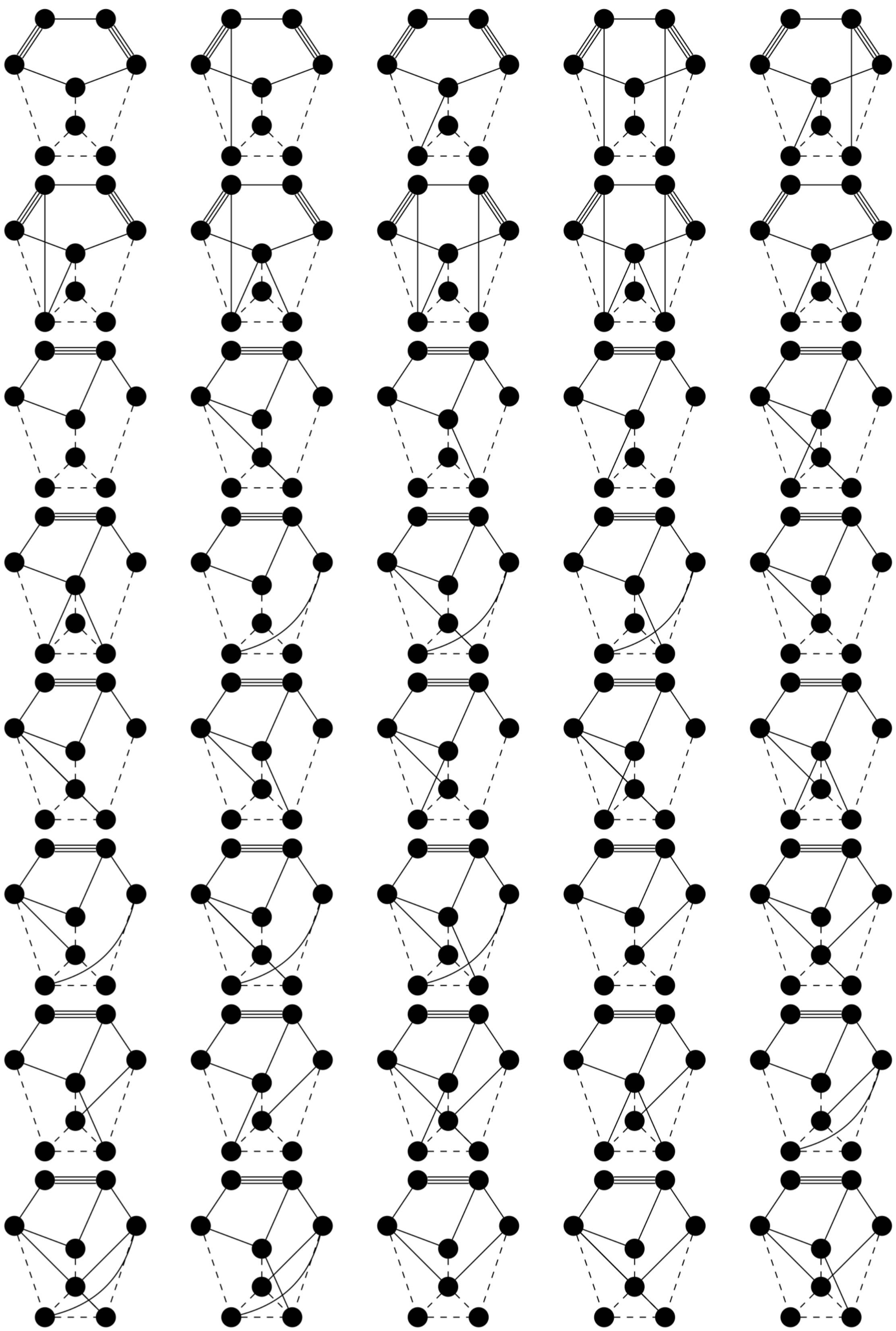
Type	**Missing Face List**	**Number of Polytopes**
H_{35}	01, 02, 034, 15, 256, 267, 3478, 5678	0
H_{36}	01, 02, 034, 1256, 157, 268, 347, 348, 5678	0
H_{37}	01, 02, 034, 1345, 156, 267, 278, 348, 5678	0
H_{38}	01, 02, 13, 2456, 3456, 78	0
H_{39}	01, 02, 034, 134, 156, 2567, 278, 348, 5678	0
H_{40}	01, 02, 034, 125, 156, 278, 346, 3478, 5678	0
H_{41}	01, 02, 034, 135, 156, 2478, 267, 348, 5678	0
H_{42}	01, 023, 1456, 27, 38, 4567, 4568	0
H_{43}	01, 02, 034, 13, 1567, 248, 2567, 348, 5678	0
H_{44}	01, 02, 034, 125, 156, 27, 3468, 3478, 5678	0
H_{45}	01, 02, 034, 135, 136, 157, 2468, 2478, 2578, 346, 5678	0
H_{46}	01, 02, 034, 135, 16, 2478, 2678, 345, 5678	0
H_{47}	01, 02, 034, 134, 135, 156, 2478, 2567, 2678, 348, 5678	0
H_{48}	01, 02, 034, 135, 1567, 248, 267, 348, 5678	0
H_{49}	01, 02, 0345, 1346, 17, 258, 278, 3456, 678	0
H_{50}	01, 02, 034, 134, 1567, 2567, 28, 348, 5678	0
H_{51}	01, 02, 0345, 1345, 16, 267, 278, 3458, 678	0
H_{52}	01, 02, 0345, 1345, 1346, 167, 258, 267, 278, 3458, 678	0
H_{53}	01, 02, 0345, 1346, 137, 178, 2456, 258, 278, 3456, 678	0
H_{54}	01, 02, 0345, 1345, 136, 167, 2458, 267, 278, 3458, 678	0
H_{55}	01, 02, 034, 1256, 1567, 28, 347, 348, 5678	0
H_{56}	01, 02, 0345, 134, 136, 167, 2458, 2578, 267, 3458, 678	0
H_{57}	01, 02, 034, 1345, 16, 267, 278, 3458, 5678	0
H_{58}	01, 02, 0345, 1346, 1347, 168, 257, 258, 268, 3457, 678	0
H_{59}	01, 02, 034, 135, 156, 2478, 26, 3478, 5678	0
H_{60}	01, 023, 024, 135, 156, 2467, 2478, 38, 4567, 5678	0
H_{61}	01, 02, 034, 125, 126, 157, 268, 3457, 3468, 3478, 5678	0
H_{62}	01, 023, 024, 1356, 1567, 238, 248, 3568, 47, 5678	0
H_{63}	01, 02, 034, 1256, 137, 1567, 248, 2568, 347, 348, 5678	0
H_{64}	01, 02, 0345, 16, 27, 3458, 678	0
H_{65}	01, 02, 0345, 134, 167, 258, 267, 3458, 678	0
H_{66}	01, 023, 024, 156, 237, 3567, 48, 5678	0
H_{67}	01, 02, 134, 135, 146, 2578, 2678, 357, 468	0
H_{68}	01, 02, 034, 134, 15, 2567, 2678, 348, 5678	0
H_{69}	01, 02, 034, 134, 135, 156, 2478, 256, 2678, 3478, 5678	0
H_{70}	01, 02, 034, 135, 1367, 158, 2467, 248, 258, 3467, 5678	0

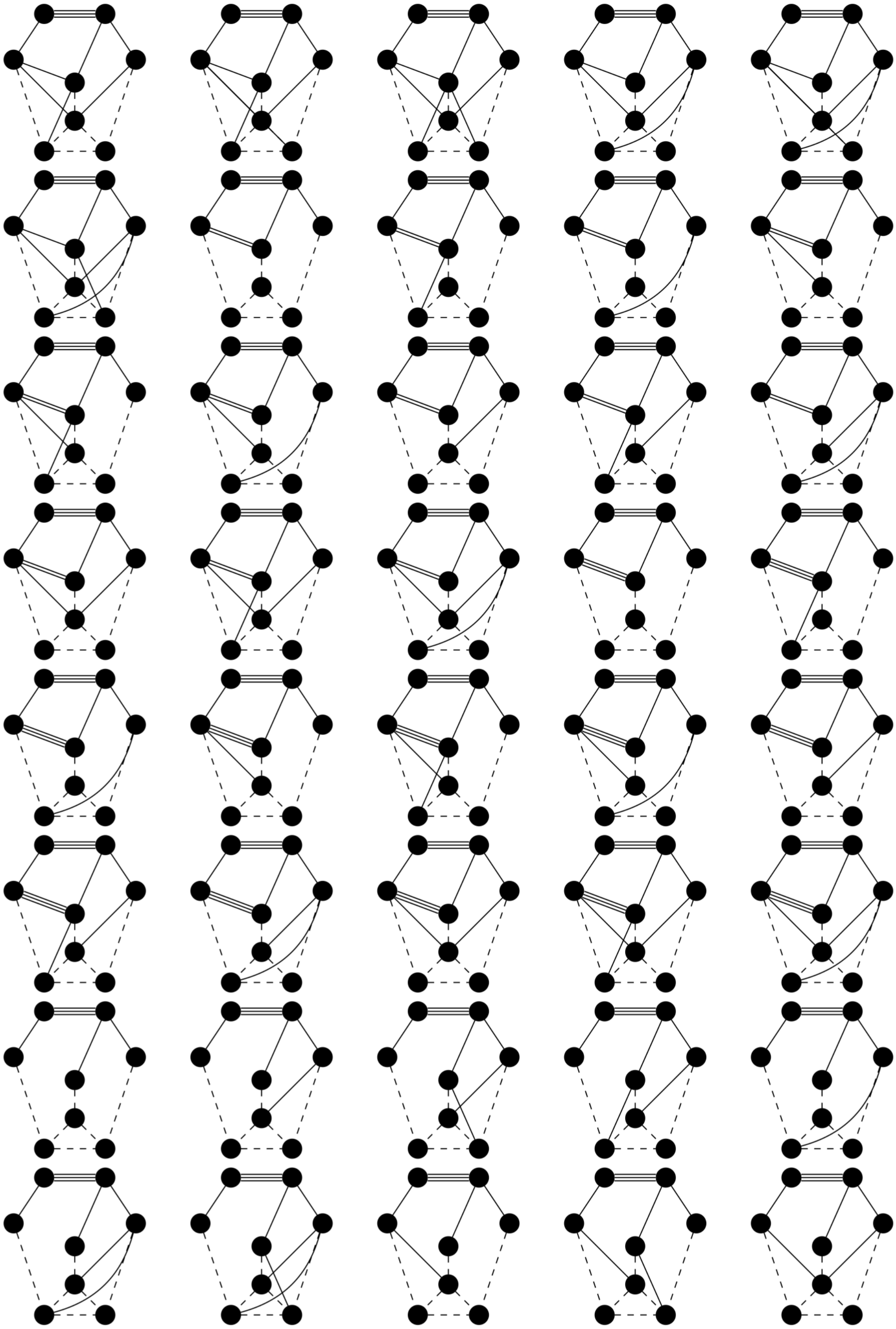
Type	**Missing Face List**	**Number of Polytopes**
H_{71}	01, 02, 034, 125, 1567, 28, 3467, 348, 5678	0
H_{72}	01, 02, 034, 135, 136, 157, 2468, 2478, 257, 3468, 5678	0
H_{73}	01, 02, 034, 135, 16, 2478, 267, 3458, 5678	0
H_{74}	01, 02, 034, 1345, 156, 167, 2348, 267, 278, 3458, 5678	0
H_{75}	01, 02, 034, 1345, 136, 167, 2458, 267, 278, 3458, 5678	0
H_{76}	01, 02, 034, 1345, 1356, 167, 248, 267, 278, 3458, 5678	0
H_{77}	01, 02, 034, 1345, 1567, 267, 28, 348, 5678	0
H_{78}	01, 02, 034, 1345, 1346, 157, 257, 268, 278, 3468, 5678	0
H_{79}	01, 02, 034, 1345, 156, 26, 278, 3478, 5678	0
H_{80}	01, 023, 024, 1356, 1567, 247, 248, 38, 4567, 5678	0
H_{81}	01, 02, 0345, 126, 1347, 167, 258, 268, 3457, 3458, 678	0
H_{82}	01, 023, 024, 156, 157, 2348, 2368, 3568, 47, 5678	0
H_{83}	01, 02, 034, 125, 136, 156, 2478, 2578, 346, 3478, 5678	0
H_{84}	01, 023, 024, 135, 156, 2378, 2478, 3578, 46, 5678	0
H_{85}	01, 023, 024, 135, 1567, 2467, 248, 38, 4567, 5678	0
H_{86}	01, 02, 034, 135, 136, 1578, 246, 2478, 2578, 346, 5678	0
H_{87}	01, 023, 024, 1356, 157, 2368, 248, 3568, 47, 5678	0
H_{88}	01, 02, 034, 1256, 1357, 1567, 248, 268, 347, 348, 5678	0
H_{89}	01, 02, 034, 1345, 1356, 1567, 248, 267, 278, 348, 5678	0
H_{90}	01, 02, 034, 125, 1346, 156, 257, 278, 3468, 3478, 5678	0
H_{91}	01, 023, 024, 156, 1578, 234, 2378, 3578, 46, 5678	0
H_{92}	01, 02, 034, 134, 1356, 1567, 248, 2567, 278, 348, 5678	0
H_{93}	01, 02, 034, 125, 136, 156, 2478, 257, 3468, 3478, 5678	0
H_{94}	01, 02, 134, 135, 2467, 2678, 3467, 58	0
H_{95}	01, 02, 134, 1356, 2567, 278, 3567, 48	0
H_{96}	01, 023, 024, 1567, 1568, 234, 237, 3567, 48, 5678	0
H_{97}	01, 02, 034, 135, 167, 248, 267, 3458, 5678	0
H_{98}	01, 02, 034, 134, 156, 256, 278, 3478, 5678	0
H_{99}	01, 02, 034, 15, 26, 3478, 5678	0
H_{100}	01, 02, 034, 134, 135, 1567, 248, 2567, 2678, 348, 5678	0
H_{101}	01, 02, 034, 12, 156, 278, 3456, 3478, 5678	0
H_{102}	01, 02, 13, 24, 34, 5678	0
H_{103}	01, 023, 145, 236, 2378, 46, 578	0
H_{104}	01, 023, 1456, 237, 238, 457, 68	0
H_{105}	01, 02, 134, 135, 246, 2678, 346, 578	0

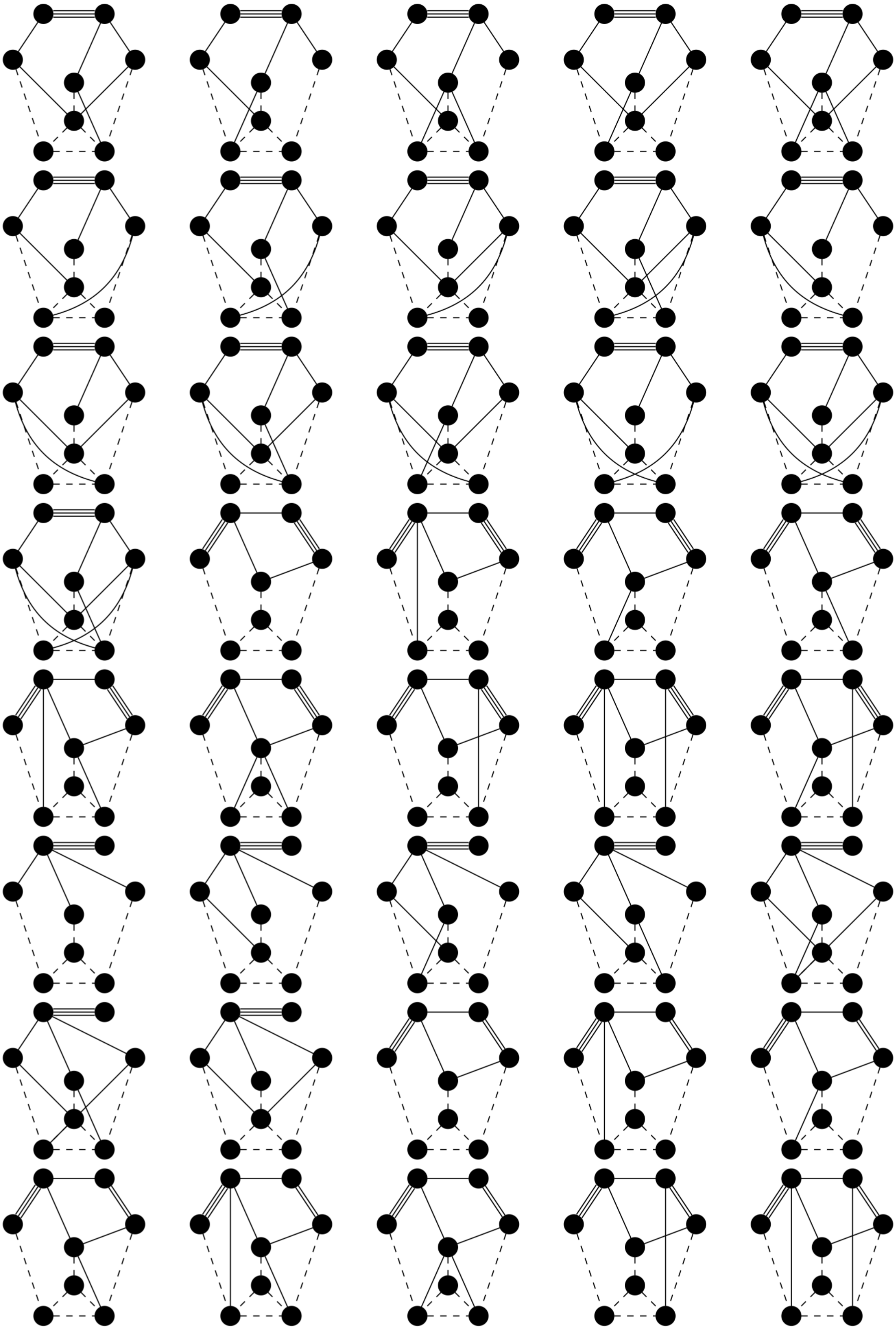
Type	**Missing Face List**	**Number of Polytopes**
H_{106}	01, 02, 034, 15, 267, 348, 5678	0
H_{107}	01, 02, 134, 156, 278, 347, 568	0
H_{108}	01, 02, 134, 256, 3456, 78	0
H_{109}	01, 23, 45, 678	0
H_{110}	01, 02, 13, 245, 345, 678	0
H_{111}	01, 023, 145, 236, 456, 78	0

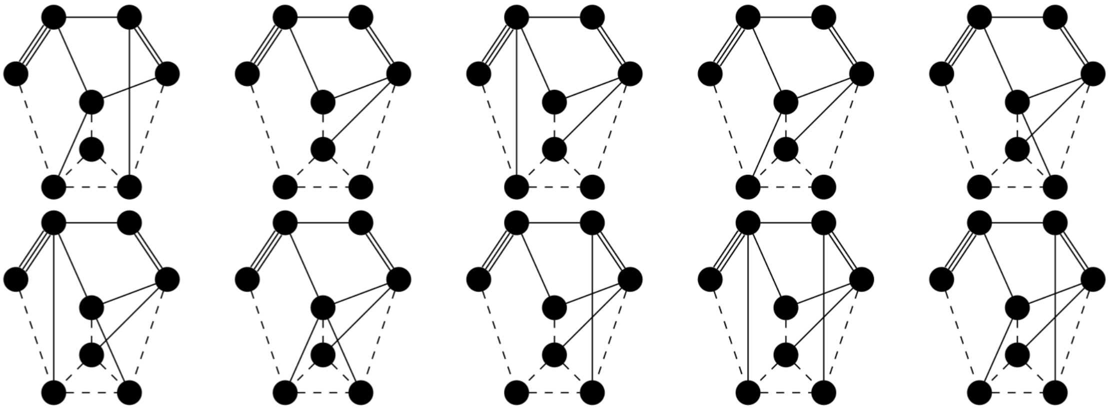
Appendix B. List of Coxeter 4-Polytopes with 8 Facets

130 Polytopes of Type G_1

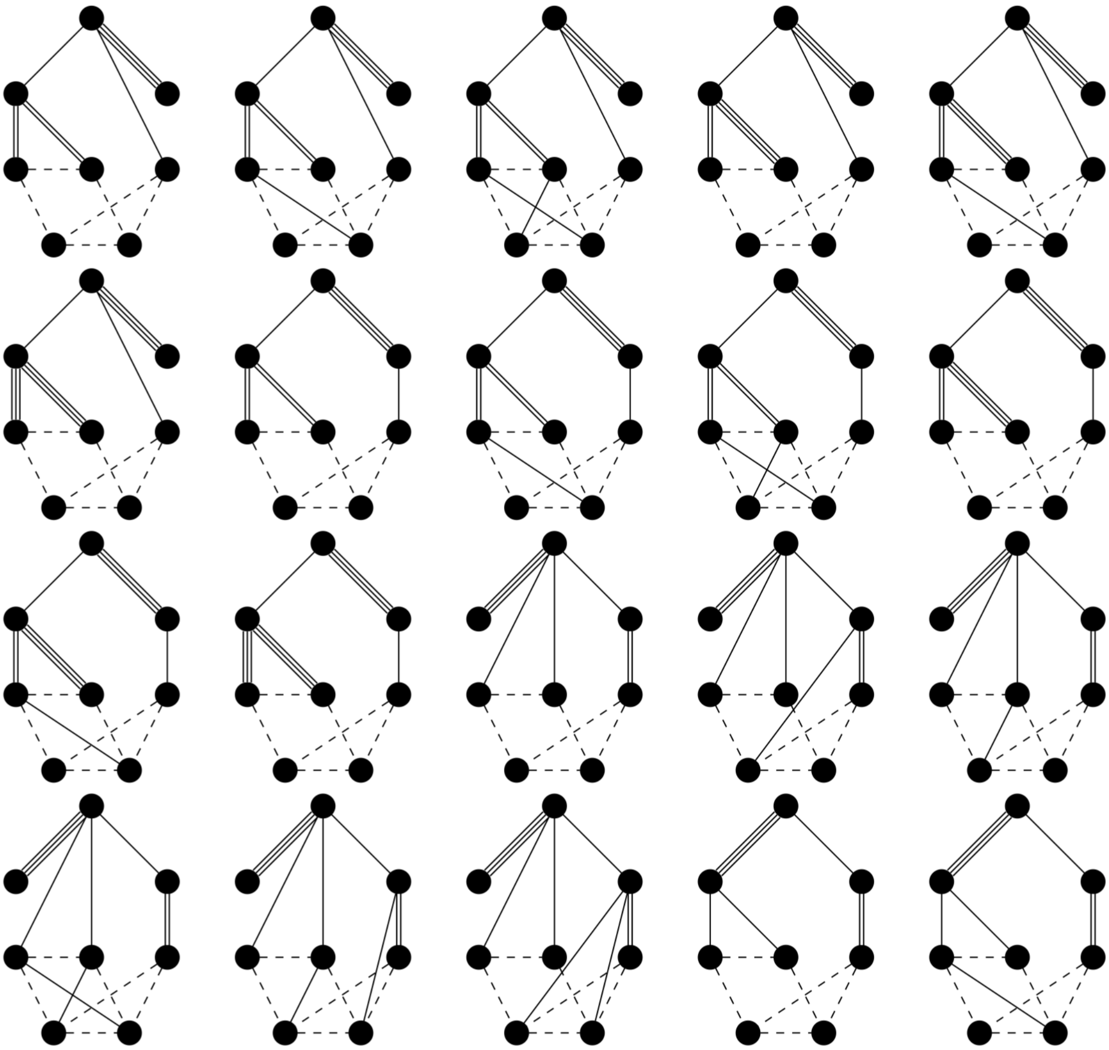


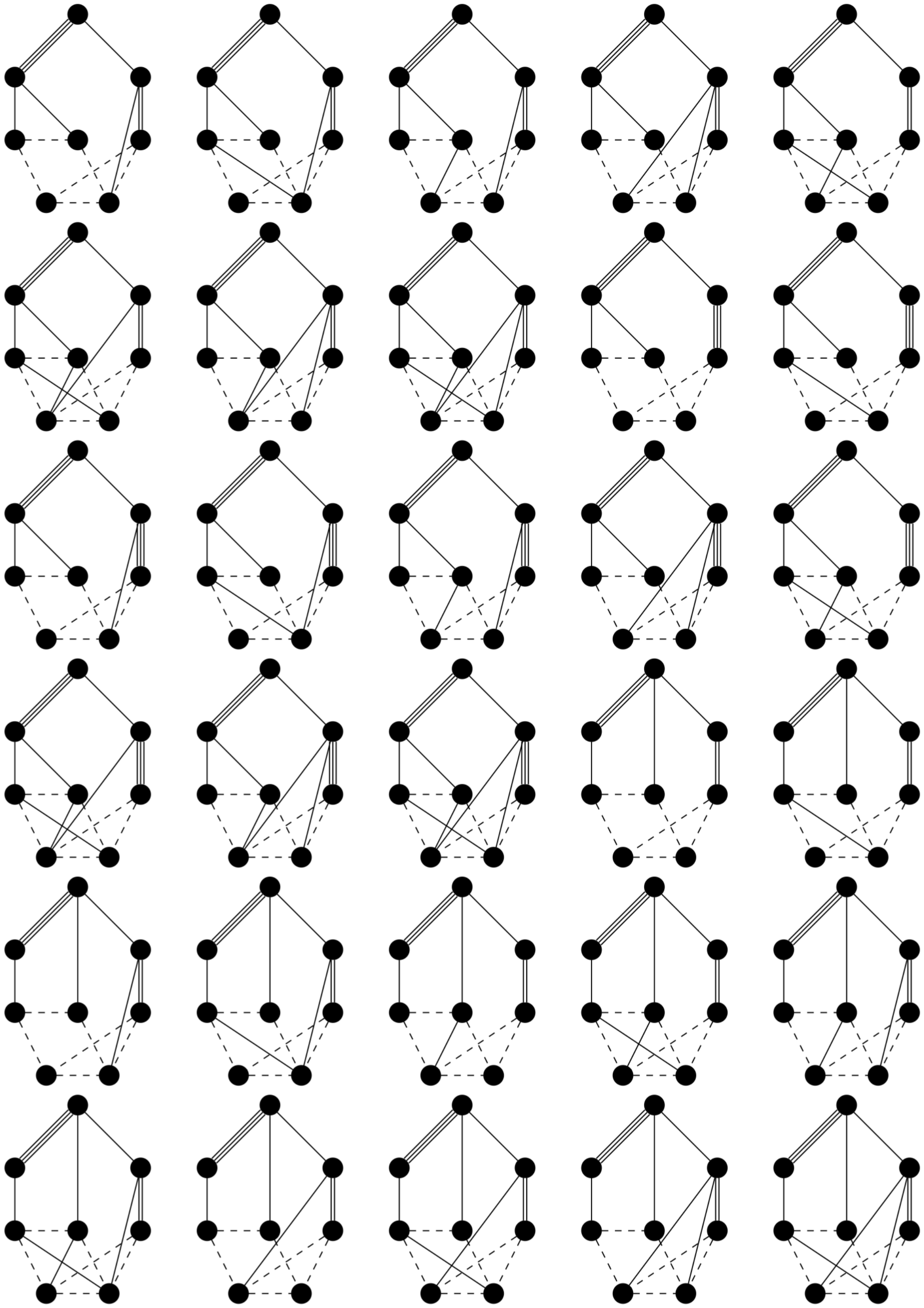


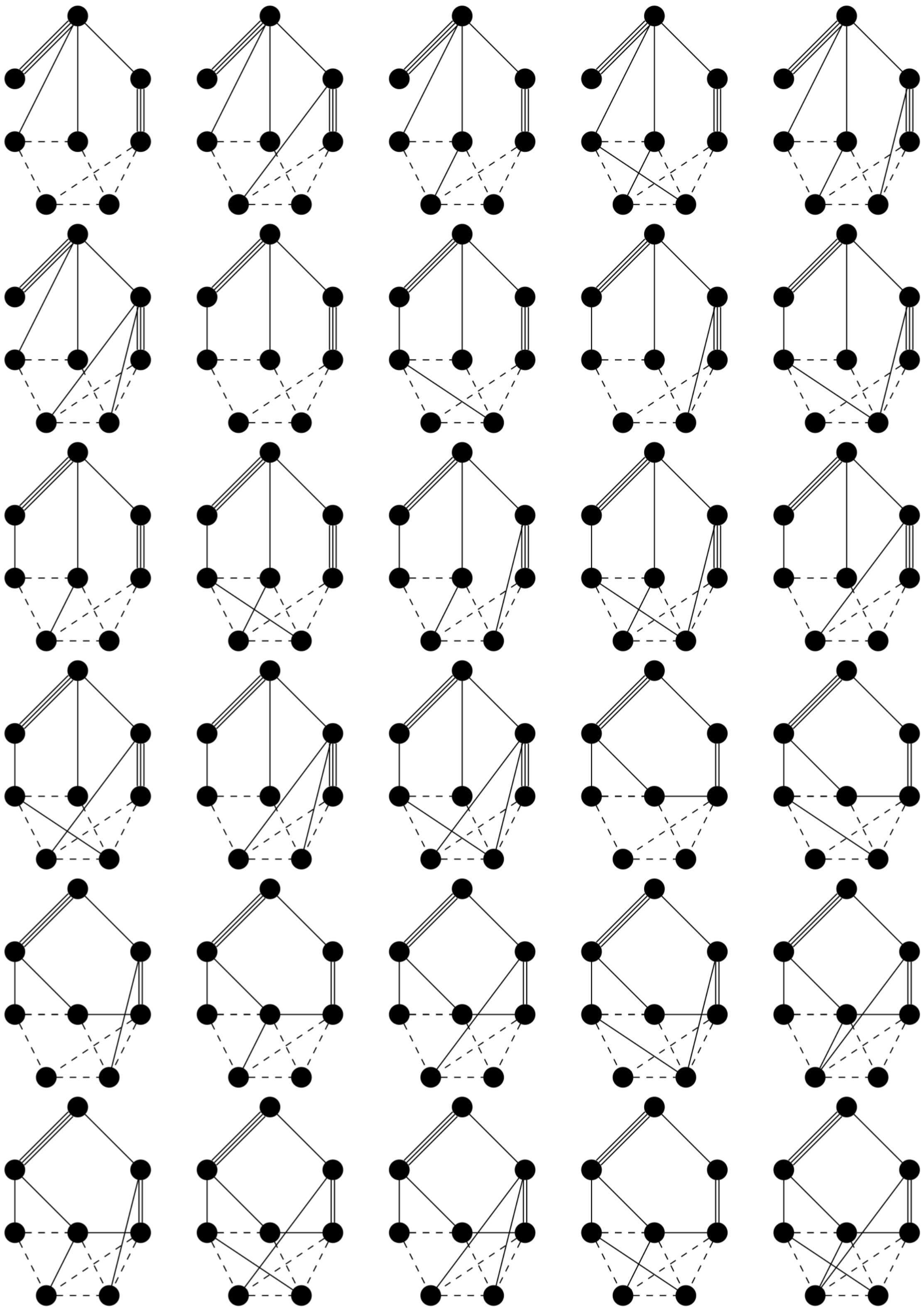


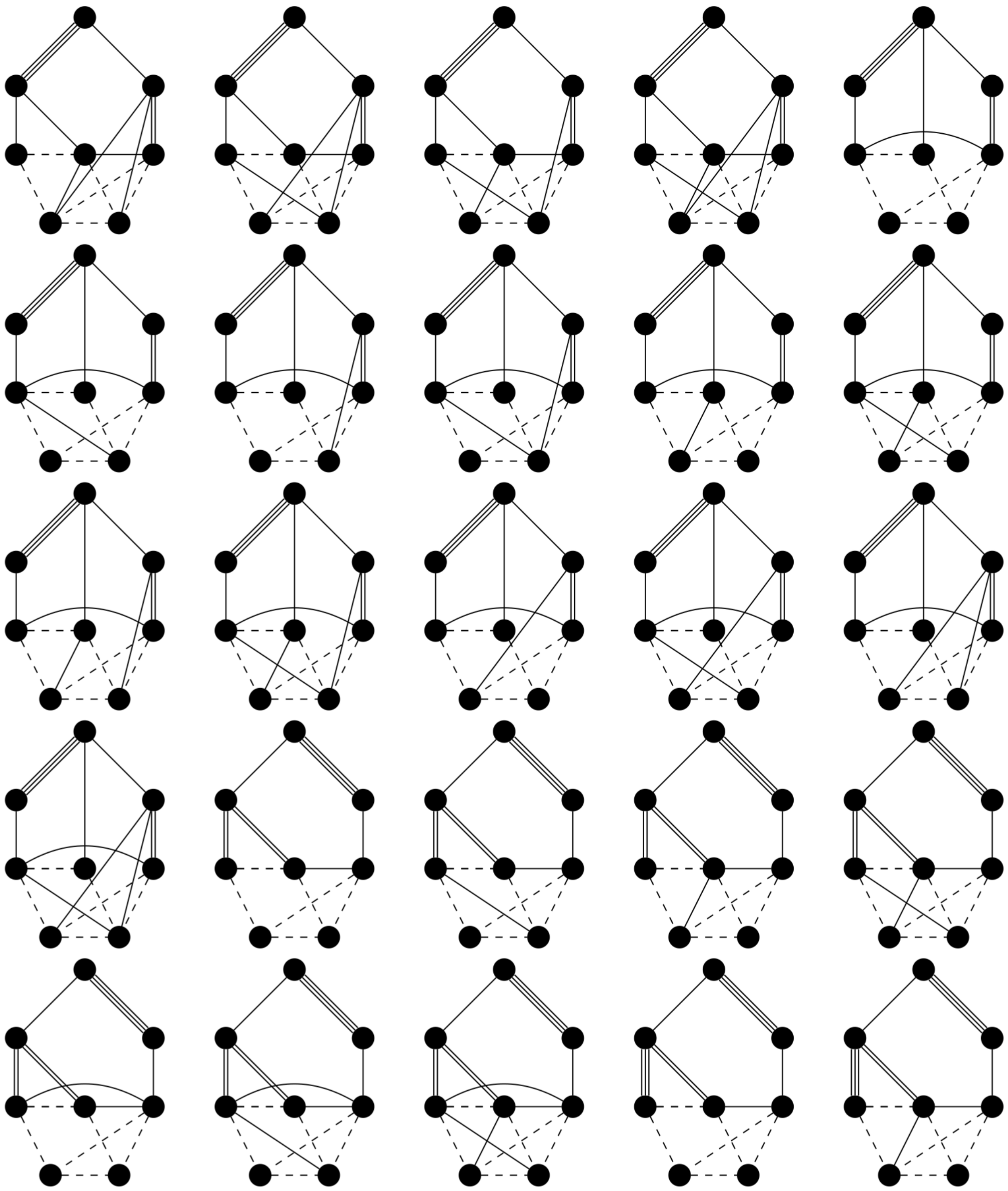


105 Polytopes of Type G_2

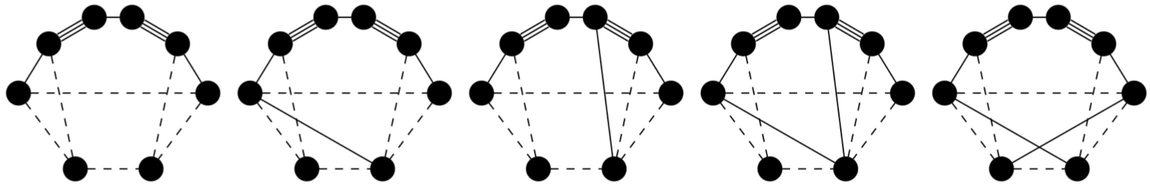


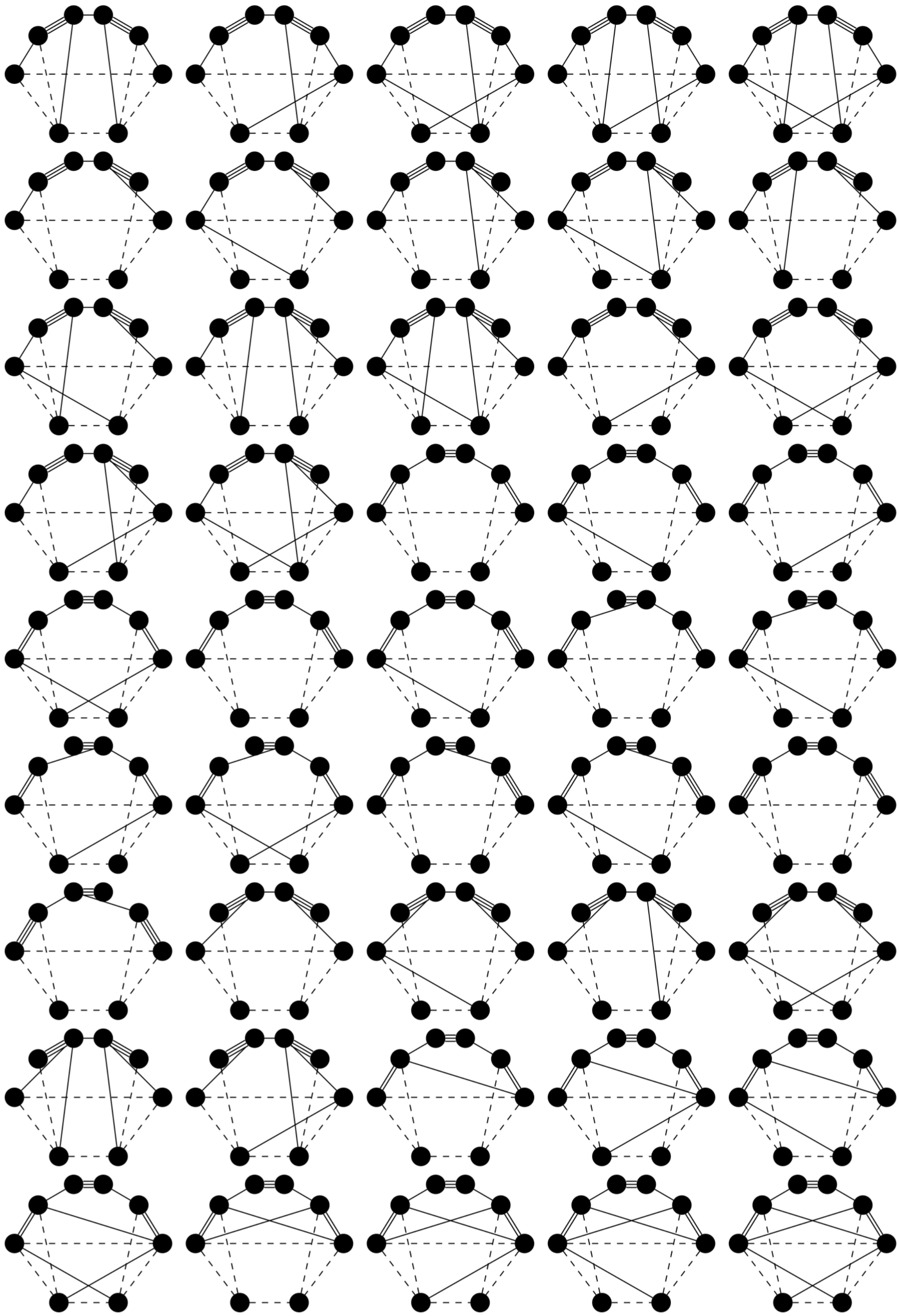




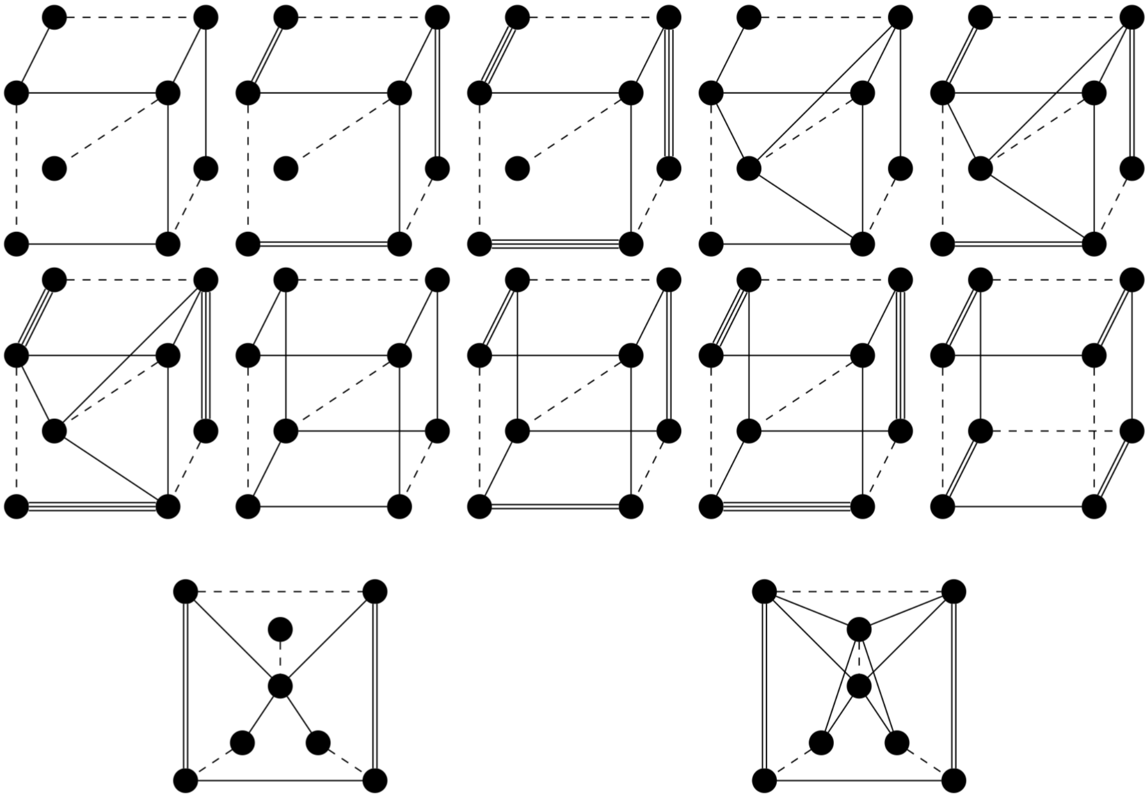


52 Polytopes of Type G_3

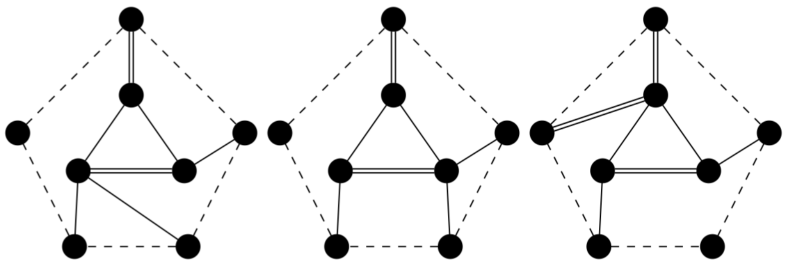




12 Polytopes of Type G_4 (due to Jacquemet and Tschantz)



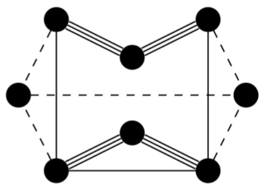
3 Polytopes of Type G_5



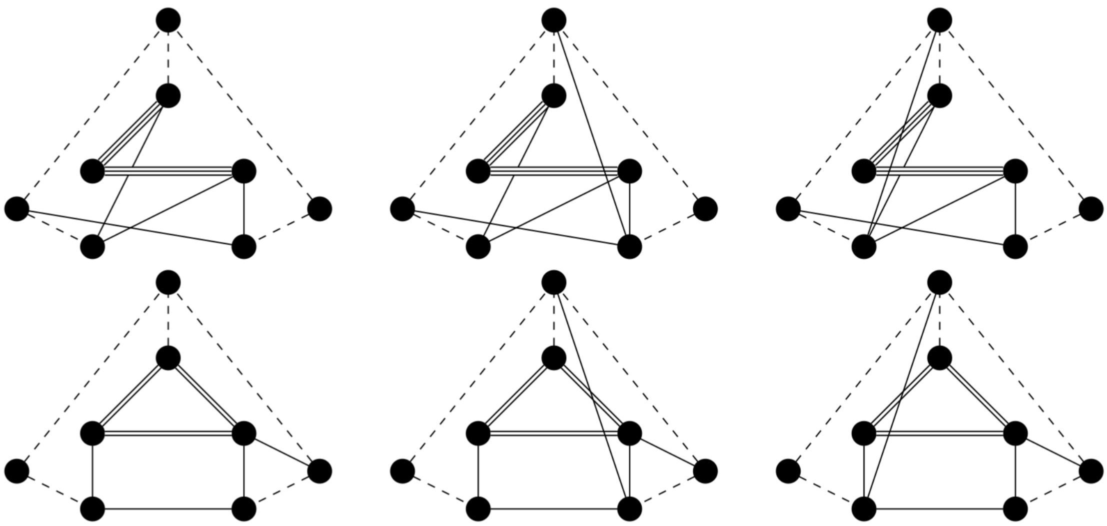
2 Polytopes of Type G_6

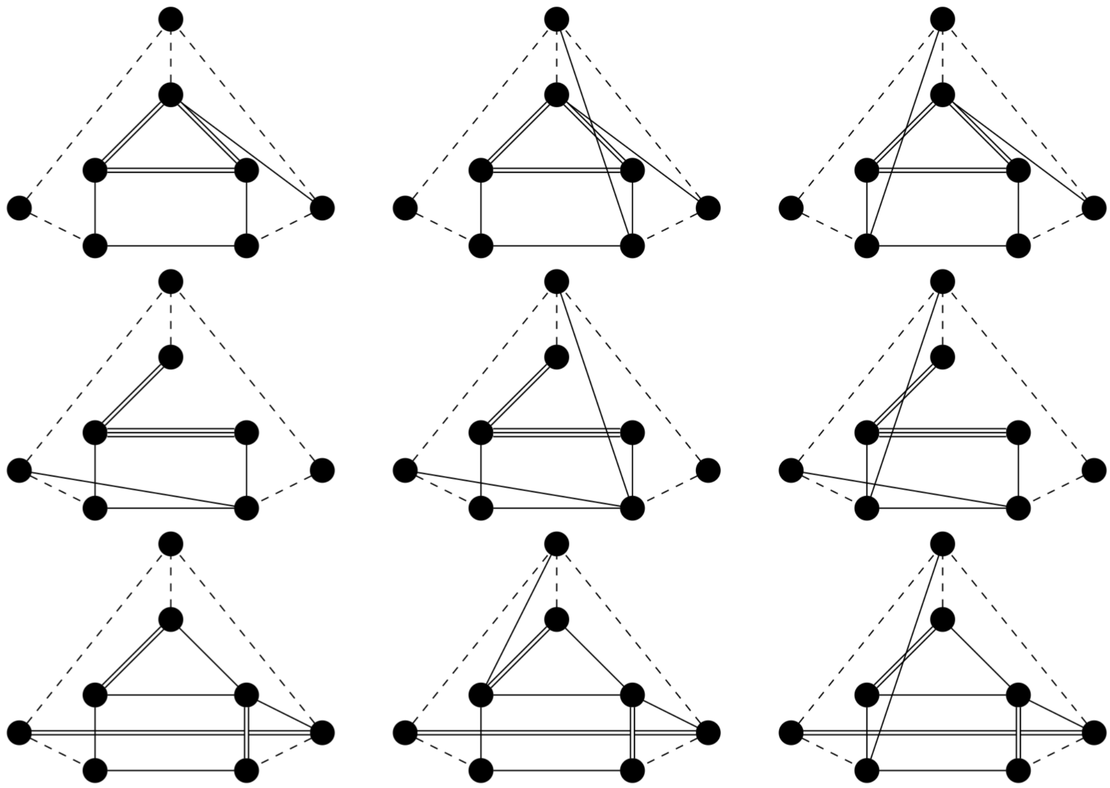
2 Polytopes of Type G_7

1 Polytope of Type G_8

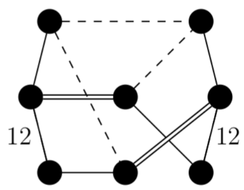


15 Polytopes of Type G_9

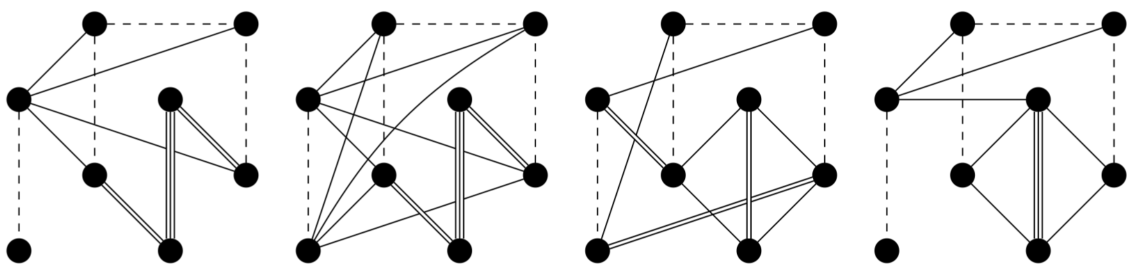


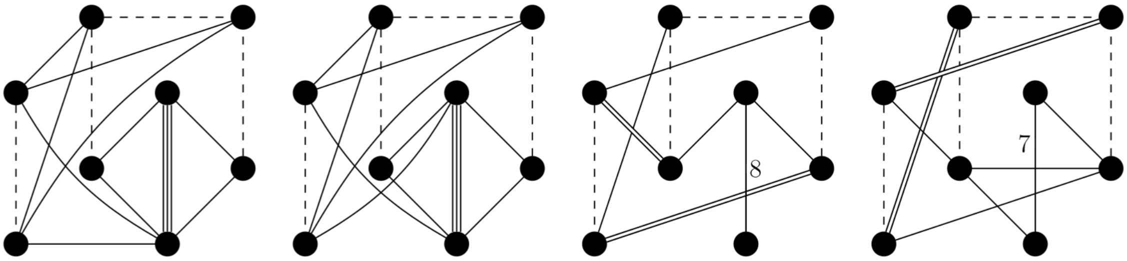


1 Polytope of Type G_{10}

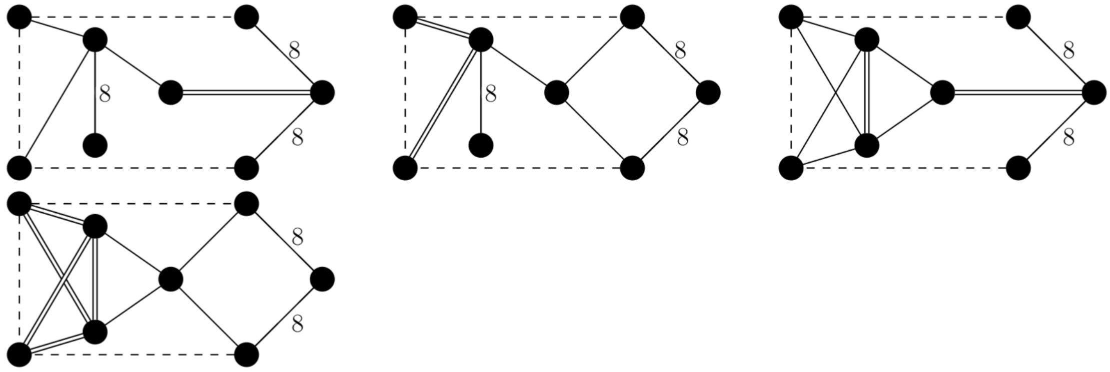


8 Polytopes of Type G_{11}

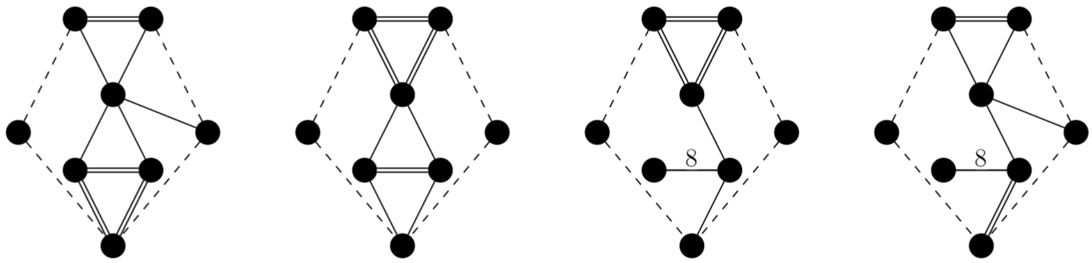




4 Polytopes of Type G_{12}



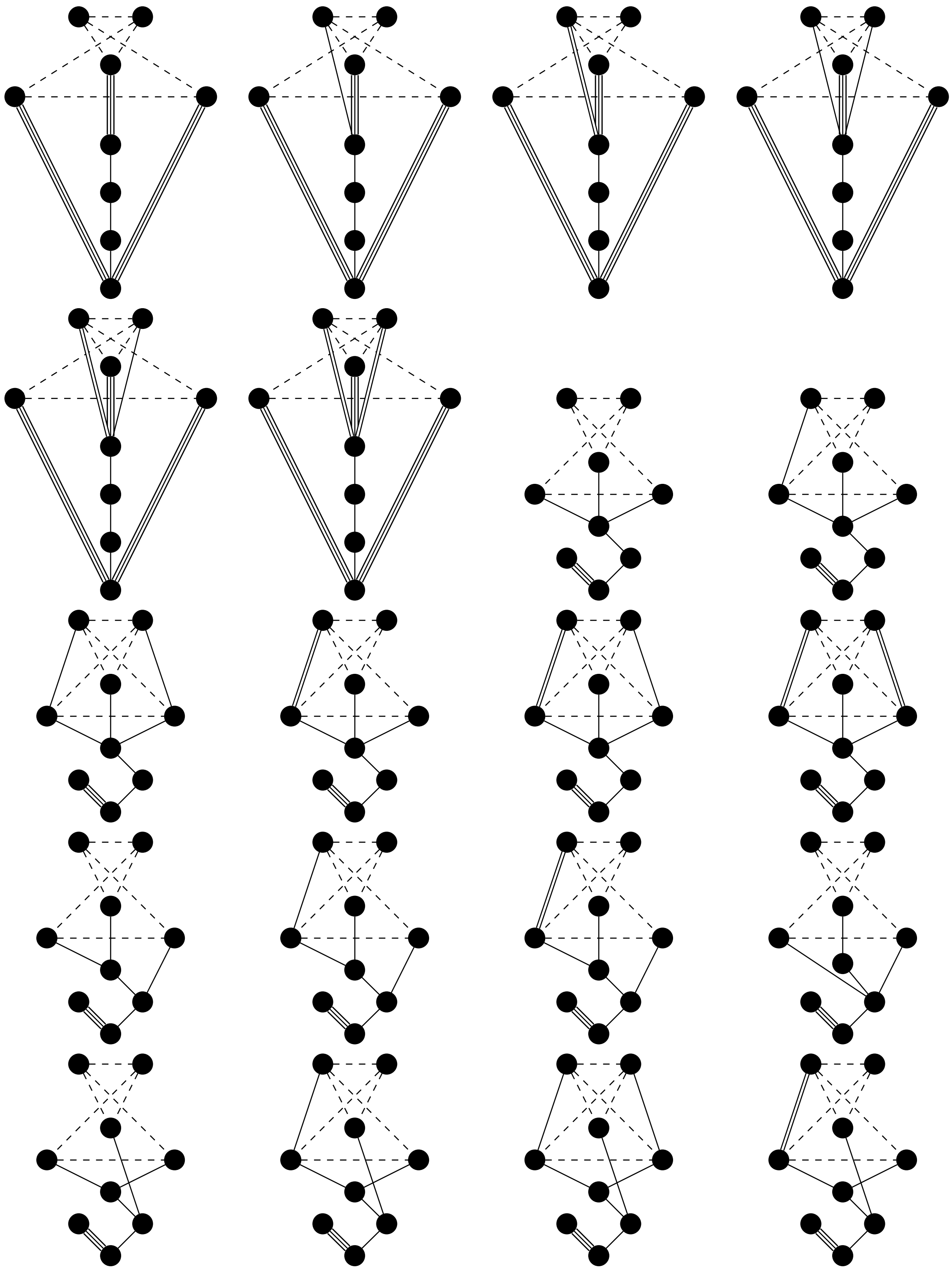
2 Polytopes of Type G_{13}

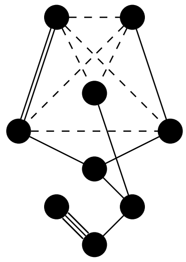


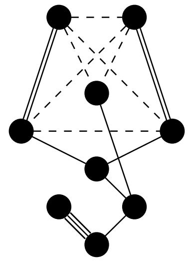
2 Polytopes of Type G_{14}

Appendix C. List of Coxeter 5-Polytopes with 9 Facets

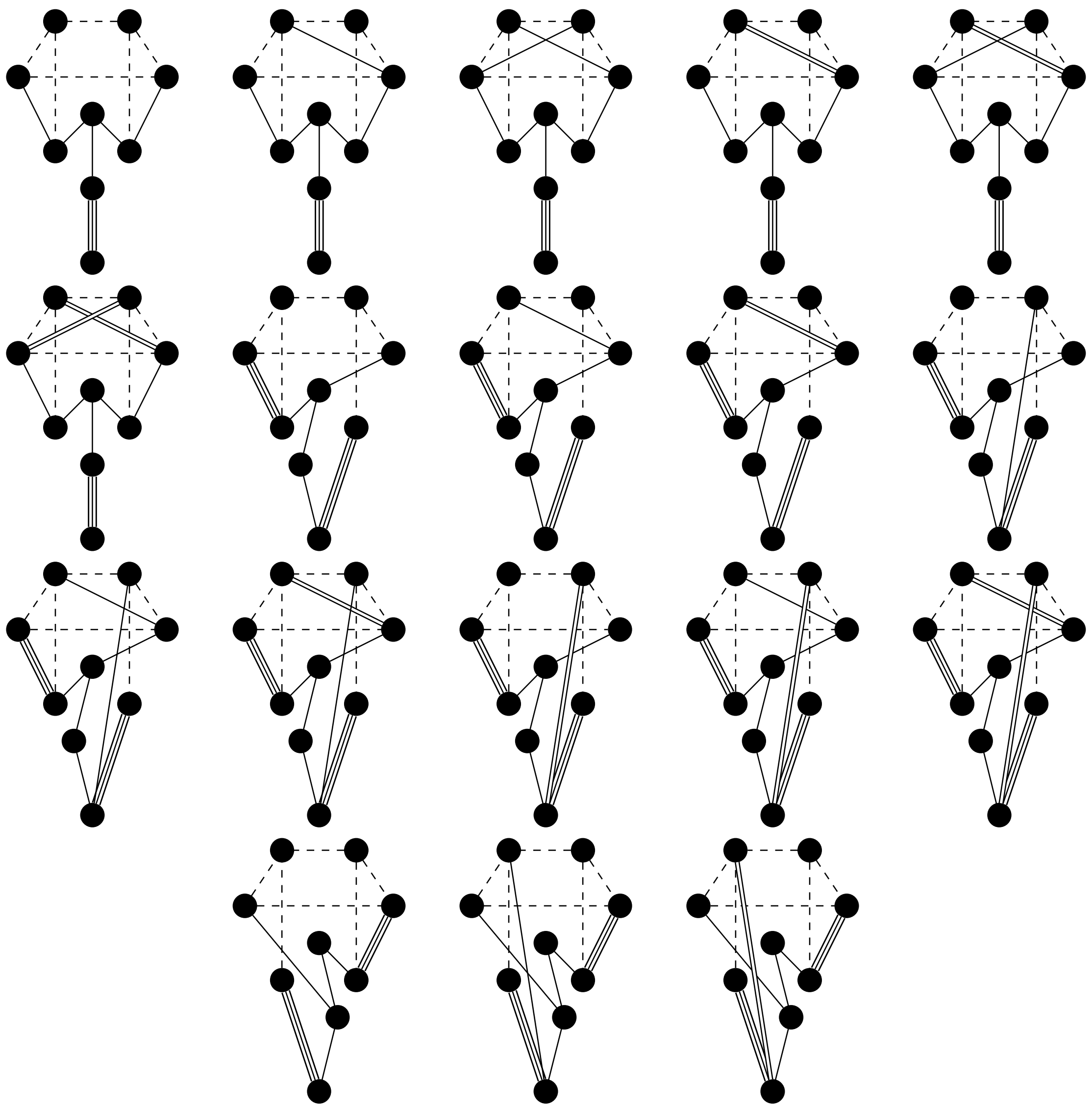
22 Polytopes of Type H_1



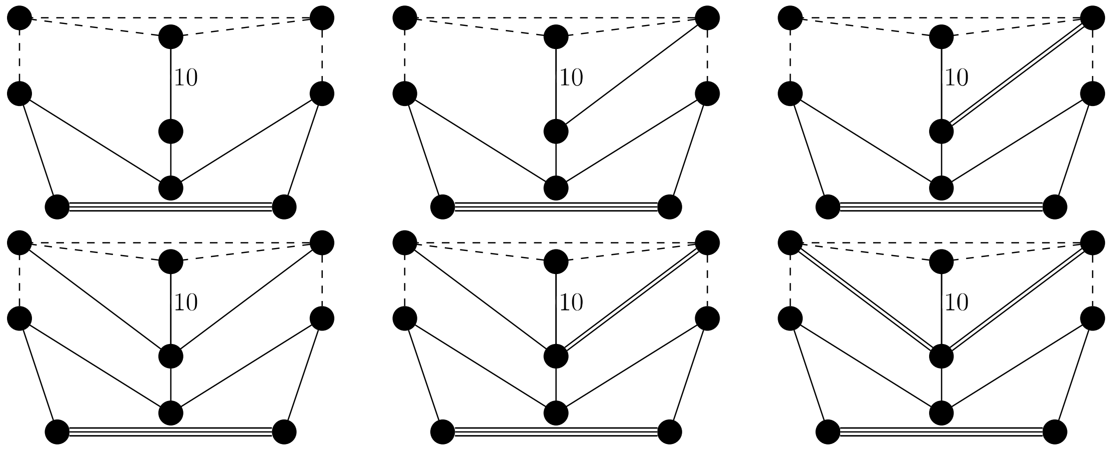




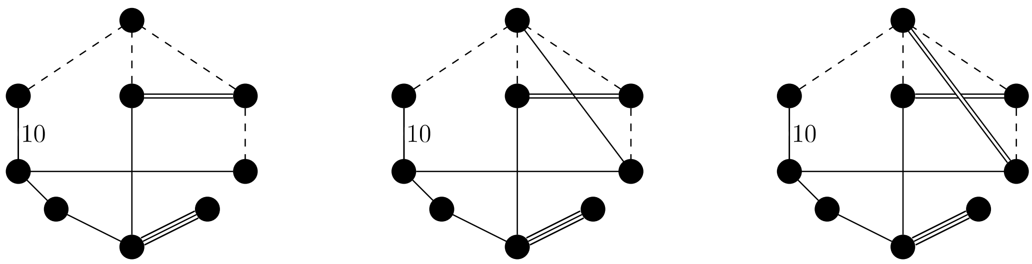
18 Polytopes of Type H_2



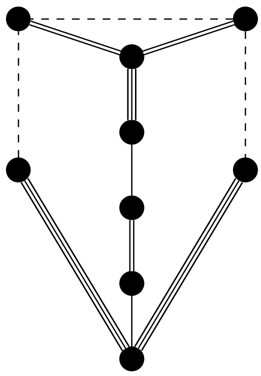
6 Polytopes of Type H_3



3 Polytopes of Type H_4



1 Polytope of Type H_5



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