Distortion coefficients and exponential map in sub-Riemannian geometry

BORZA, SAMUEL

Distortion coefficients and exponential map in sub-Riemannian geometry

Topics in metric geometry and Carnot–Carathéodory theories

Samuël Borza

A Thesis presented for the degree of
Doctor of Philosophy

Pure Mathematics
Department of Mathematical Sciences
Durham University
United Kingdom

October 2021

Distortion coefficients and exponential map in sub-Riemannian geometry

Topics in metric geometry and Carnot–Carathéodory theories

Samuël Borza

Submitted for the degree of Doctor of Philosophy

October 2021

Abstract: The aim of this thesis is to explore the fields of sub-Riemannian and metric geometry.

We compute the distortion coefficients of the α-Grushin plane. These distortion coefficients are expressed in terms of generalised trigonometric functions. Estimates for the distortion coefficients are then obtained and a conjecture of a synthetic curvature bound for the α-Grushin plane is proposed.

We then prove a version of Warner's properties for the sub-Riemannian exponential map. The regularity property is established by considering sub-Riemannian Jacobi fields while the continuity property follows from studying the Maslov index of Jacobi curves. We show how this implies that the exponential map of the Heisenberg group is not injective in any neighbourhood of a conjugate vector.

In the appendix, we prove that the curvature-dimension for negative effective dimension fails to hold in any strict and complete sub-Riemannian manifold.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Acknowledgements

First of all, I would like to thank my supervisor, Prof Wilhelm Klingenberg, for his guidance, support and encouragements over the past four years. Working with him has been a great pleasure, and I have been continually inspired by his passion and approach to mathematics. I also thank my second supervisors Prof David Bourne and Prof Djoko Wirosoetisno.

I am immensely grateful to the UK Engineering and Physical Sciences Research Council (EPSRC) for financially supporting my PhD degree, within the grant EP/N509462/1 (1888382) for Durham University. They have put their trust in me and my project, and in doing so have given me the chance to fulfil my long-held ambition of completing my doctorate.

I am thankful to Durham University as a whole, and to the Department of Mathematical Sciences in particular, for giving me the opportunity to study and work for this degree. I thank Prof Norbert Peyerimhoff, Prof Fernando Galaz-Garcia, Prof Megan Griffin-Pickering, Prof Liviana Palmisano, Prof Anna Felikson, Prof Pavel Tumarkin, and many others: throughout the years and over the course of many seminars and shared meals, they have become more than colleagues to me.

I want to express my deepest gratitude to Prof Karl-Theodor Sturm for hosting me at the Hausdorff Center for Mathematics in Bonn during my PhD studies. I also thank him for inviting me to the HCM Summer School in Piz Buin, and for the interesting question which he raised during one of the sessions and which I have addressed in the appendix of this work. The entire group at the institute welcomed me warmly and I thank them for their friendship and the time spent together. It was the place where many of the ideas written in this thesis matured.

I am particularly grateful to Prof Alpár Mészáros, Prof Nicolas Juillet, Prof Luca Rizzi and Prof Christian Ketterer for the stimulating mathematical discussions, for their support and for their friendship. I also benefited greatly from instructive discussions with Prof Shingo Takeuchi on the topic of generalised trigonometry. I am thankful to Prof Li Yutian for introducing me to the study of the α-Grushin plane and for his hospitality at the Chinese University of Hong Kong in Shenzhen in the summer of 2019, and to Prof Andrea Mondino who gave me a grounding in the geometry of metric measure spaces with his work on the Lévy–Gromov inequality a few months before the start my PhD degree.

I am very thankful to the Department of Mathematics at the University of Mons where I gained so many formative experiences as a mathematician. I would not have come this far without Prof Christan Michaux, Prof Maja Volkov, Prof Thomas Brihaye, Prof Catherine Finet, Prof Christophe Troestler, and so many others. I owe so much to them.

My daily life at Durham university was brightened by Daniel Ballesteros, David Cushing, Stephen Harrap, Oliver Singh, Phil Supanat, John Blackman, Matthew Northey, to name just a few of my dear friends. I am grateful for my church family at King's Church Durham and at the *Église Évangélique de Quaregnon*. My spiritual life has especially been blessed by Carolyn and Derek Rochester, Rhoda Clarke, Chris Morgan, Tom Judge, Mark Bonnington, Rich and Isobel Harvey, Laura and James Brenchley, Candy Yan and her family, as well as my beloved church small group. Above all, my closest friends Henry Silva, Charlie Evans, George Platt, Emeka Abakasanga, Yao Li, Cynthia Colin, Gatwa Tshinsele-Van Bellingen, Naomi Hacks, Perside and Achim Niyonizera have brought me so much joy and comfort. I have also spent cherished time with all my housemates over the years in Durham.

Lastly, I want to express my love and appreciation to my family. My sisters Déborah and Jennyfer have been through so much with me and I have them in my heart in everything I do. My mother and father have given everything they had in their hands and their heart to bring me to where I am today. I will be forever thankful to my *Tantine*, my *Mémé*, my *Pépé*, my *Mémé* Téa, my *Tante* Nelly and *Tonton* Tony.

Thank you to my wife and love of my life Weiling for being by my side through every circumstance. I will not forget her sister Yinling, as well as my new family, who now hold a special place in my life too.

I am dedicating this work to my *Mamie* and my *zia* Terenzia. The LORD has recently decided to take them home. They were gifted with immeasurable, selfless and sacrificial love for everyone, and to me in particular.

> *You are my hiding place; you will protect me from trouble and surround me with songs of deliverance.*
>
> — The Book of Psalms

What is it that a mathematician wants as an artist? I believe that he wishes merely to understand and to create. He wishes to understand, simply, if possible – but in any case to understand; and to create, beautifully, if possible – but in any case to create.

— Marston Morse

Dedicated to

Francesca Biondolillo

and

Suore Terenzia

Contents

List of Figures

Chapter 1

Introduction

In recent decades, considerable progress has been made in the theory of metric measure spaces. This thesis fits into this context: the aim is to dive deep into the topic of sub-Riemannian geometry, while being motivated by questions arising from metric geometry. This chapter introduces the main themes of the thesis, as well as presenting our main results.

The geometry of metric measure spaces is outlined in Chapter 2. One of the main reasons for the successful development of the theory was the introduction by Sturm, Lott and Villani of a synthetic theory of curvature bounds. After describing length spaces and their properties, the so-called *curvature-dimension condition* is defined. It is based on optimal transport, and generalises to length spaces a notion of lower bound on the Ricci curvature tensor encountered in Riemannian geometry. We also review the *metric contraction property* and the *Brunn–Minkowski inequality*: they are weaker versions of the curvature-dimension condition, and very important in the synthetic study of sub-Riemannian curvature. The distortion coefficients are also introduced. It is explained that they contain information about the curvature of the space.

In Chapter 3, sub-Riemannian geometry will be introduced via the *Hamiltonian viewpoint*. Sub-Riemannian manifolds are manifolds equipped with a singular metric, i.e. an inner product only defined on a subspace of the tangent space at each point. Optimal control theory is the modern way of formulating sub-Riemannian geometry. With the help of Pontryagin's maximum principle, some length minimisers, the ones that are called *normal*, are described as the solution of Hamilton's differential equation. These are essential in this thesis as the sub-Riemannian exponential map is the projection from the cotangent bundle to the manifold of the Hamiltonian flow that generates normal geodesics. After showing how the optimal transportation problem is solved in sub-Riemannian geometry, the chapter will end with an analysis of the sub-Riemannian distortion coefficients.

The study of the distortion coefficients of the α-Grushin plane will be the focus of Chapter 4. The α-Grushin plane, also denoted by $\mathbb{G}_\alpha$, is a generalisation of the classical Grushin plane, corresponding to $\alpha = 1$. Its geometry corresponds to the sub-Riemannian structure of $\mathbb{R}^2$ equipped with the global vector fields $X = \partial_x$ and $Y_\alpha = |x|^\alpha \partial_y$.

The study of the normal geodesics of $\mathbb{G}_\alpha$ is carried out by solving Hamilton's equation. In this case, special functions defined with the help of the inverse of the incomplete beta function appears naturally.

Theorem A (Geometry of the α-Grushin plane)**.**
Let $\gamma : I \to \mathbb{G}_\alpha$ be a horizontal path with initial value $\gamma(0) = (x_0, y_0)$, and $\lambda(t) = u(t)\mathrm{d}x|_{\gamma(t)} + v(t)\mathrm{d}y|_{\gamma(t)}$ be the cotangent lift with initial covector $(u(0), v(0)) = (u_0, v_0)$.

In the case where $v_0 \neq 0$ and $(x_0, u_0) \neq 0$, the curve γ is a geodesic if and only if

$$\begin{cases} x(t) &= A \sin_\alpha(\omega t + \phi) \\ y(t) &= y_0 + v_0 \dfrac{A^{2\alpha}}{(\alpha+1)\omega^2} \Big[\omega^2 t + \omega \cos_\alpha(\phi) \sin_\alpha(\phi) \\ & \qquad\qquad -\omega \cos_\alpha(\omega t + \phi) \sin_\alpha(\omega t + \phi)\Big] \\ u(t) &= A\omega \cos_\alpha(\omega t + \phi) \\ v(t) &= v_0 \end{cases}$$

for uniquely determined parameters $A, \omega \in \mathbb{R} \setminus \{0\}$ *and* $\phi \in [0, 2\pi_\alpha)$ *satisfying*

$$A\omega > 0, \quad A^2\omega^2 = u_0^2 + v_0^2 x_0^{2\alpha}, \quad \omega^2 = v_0^2 A^{2(\alpha-1)},$$

$$x_0 = A \sin_\alpha(\phi) \text{ and } u_0 = A\omega \cos_\alpha(\phi).$$

If $v_0 = 0$ *or* $(x_0, u_0) = 0$*, the geodesic is* $(x(t), y(t)) = (u_0 t + x_0, y_0)$ *with its lift being constant:* $(u(t), v(t)) = (u_0, v_0)$*.*

We also determine the cotangent injectivity domain of the sub-Riemannian exponential map, and the corresponding cut locus.

Theorem B (Cut locus of the α-Grushin plane)**.**
Let $\alpha \geqslant 1$ *and* $\gamma(t) = (x(t), y(t))$ *be a geodesic of* $\mathbb{G}_\alpha$ *with initial value* $\gamma(0) = (x_0, y_0)$ *and initial covector* $u_0 \mathrm{d}x|_{(x_0,y_0)} + v_0 \mathrm{d}y|_{(x_0,y_0)}$*.*

If $v_0 = 0$*, there are no singularities along* γ*,*

$$t_{\mathrm{cut}}[\gamma] = +\infty \text{ and } \mathrm{Cut}(x_0, y_0) = \varnothing.$$

If $v_0 \neq 0$*, then the cut time is*

$$t_{\mathrm{cut}}[\gamma] = \frac{\pi_\alpha}{|\omega|},$$

while the cut locus is

$$\mathrm{Cut}(x_0, y_0) = \left\{ (-x_0, y) \in \mathbb{G}_\alpha \mid |y - y_0| \geqslant |x_0|^{\alpha+1} \frac{\pi_\alpha}{(\alpha+1)} \right\}.$$

A method to disprove Brunn–Minkowski inequalities is then used to show that the α-Grushin plane fails to satisfy the curvature-dimension condition.

Theorem C (Failure of the $\mathrm{CD}(K, N)$ condition for the α-Grushin plane)**.**
The α*-Grushin plane* $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$ *does not satisfy the* $\mathrm{CD}(K, N)$ *condition, for any* $K \in \mathbb{R}$ *and* $N \geqslant 1$*.*

Since the CD condition is not suited for $\mathbb{G}_\alpha$, the analysis of its curvature must be made from another angle. As explained earlier, we choose to analyse its distortion coefficients. Using a characterisation specific to sub-Riemannian geometry, we are able to compute them explicitly, in terms of the $(2, 2\alpha)$-trigonometric functions. The use of the sub-Riemannian exponential map is crucial here.

Theorem D (Distortion coefficients of the α-Grushin plane).

Let q_0 and q be two points of $\mathbb{G}_\alpha$ such that $q \notin \mathrm{Cut}(q_0)$. For all $t \in [0,1]$, we have

$$\beta_t(q_0, q) = \frac{\mathrm{J}(t, x_0, u_0, v_0)}{\mathrm{J}(1, x_0, u_0, v_0)},$$

with

$$\mathrm{J}(t, x_0, u_0, v_0) := t\left[u_0 x(t) - (u_0 t + x_0) u(t)\right], \tag{1.0.1}$$

and where $\gamma(t) := (x(t), y(t)) : [0,1] \to \mathbb{G}_\alpha$ denotes the unique constant speed minimising geodesic joining $q_0 = (x_0, y_0)$ to q and $u(t)\mathrm{d}x|_{\gamma(t)} + v(t)\mathrm{d}y|_{\gamma(t)} \in \mathrm{T}^*_{\gamma(t)}(\mathbb{G}_\alpha)$ is the corresponding cotangent lift with initial covector $u_0\mathrm{d}x|_{q_0} + v_0\mathrm{d}y|_{q_0}$.

We then provide estimates that are relevant to the measure contraction property.

Theorem E (Relevant curvature-dimension estimates).

Let $q_0 := (x_0, y_0) \in \mathbb{G}_\alpha$ with $x_0 \neq 0$ and $q \in \mathbb{G}$ lying on the same horizontal line or with $x_0 = 0$ and $q \notin \mathrm{Cut}(q_0)$. We have that

$$\beta_t(q_0, q) \geqslant t^N \text{ for all } t \in [0,1]$$

if and only if

$$N \geqslant N_\alpha := 2\left[\frac{(\alpha+1)m_\alpha + 1}{m_\alpha + 1}\right],$$

where $m_\alpha \in [-3, -2]$ the unique non zero solution of

$$(m+1)^{2\alpha}(m+1) - ((2\alpha+1)m + 1) = 0.$$

We therefore conjecture that the α-Grushin plane satisfies the measure contraction property condition $\mathrm{MCP}(K, N)$ if and only if $K \leqslant 0$ and $N \geqslant N_\alpha$. If this is confirmed, this could be the first example of a metric measure space satisfying a curvature-dimension condition with an non integer optimal effective dimension.

This work on the α-Grushin plane is the object of a preprint [Bor20] and has been accepted for publication to the Journal of Geometric Analysis.

We continue with a less specific and more abstract topic. Warner's regularity properties are a set of sufficient conditions in Riemannian geometry for a map such as the exponential map to fail to be injective in any neighbourhood of a singularity. This result was originally shown by Morse and Littauer and is also important to metric geometry, as it enables us to characterise conjugate points along a geodesic in a synthetic way, i.e. without reference to the smooth structure of the space.

Therefore, in Chapter 5, it is shown that the sub-Riemannian exponential map satisfies a cotangent version of Warner's regularity conditions, of which there are three. The first follows the constant speed property that normal extremals verify. The second is proved by studying sub-Riemannian Jacobi fields along a normal geodesic. Finally, the third is an adaptation of Morse's index theory, making use of the Maslov index of Jacobi curves.

Theorem F (Regularity of the sub-Riemannian exponential map)**.**
Let M be a sub-Riemannian manifold and $p \in M$. Then, the corresponding exponential map $\exp_p$ with domain $\mathscr{A}_p \subseteq \mathrm{T}^*_p(M)$ satisfies the following properties.

- ***(R1)*** *The map $\exp_p$ is $\mathcal{C}^\infty$ on $\mathscr{A}_p$ and, for all $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ and all $t \in I_{p,\lambda_0}$, we have $\mathrm{d}_{t\lambda_0}\exp_p(\dot{\mathrm{r}}_{p,\lambda_0}(t)) \neq 0_{\exp_p(t\lambda_0)}$.*
- ***(R2)*** *For every $\lambda_0 \in \mathscr{A}_p \setminus \{0\}$ and every symplectic moving frame along the cotangent lift $\lambda(t)$ of the normal geodesic $\gamma(t) := \exp_p(t\lambda_0)$, the map*

$$\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p) \to \mathrm{T}_{\exp_p(\lambda_0)}(M) / \, \mathrm{d}_{\lambda_0}\exp_p(\mathrm{T}_{\lambda_0}(\mathrm{T}^*_p(M))),$$

  sending A to $\nabla J_A(1) + \mathrm{d}_{\lambda_0}\exp_p(\mathrm{T}_v(\mathrm{T}^*_p(M)))$, is a linear isomorphism.
- ***(R3)*** *When M is an ideal sub-Riemannian manifold, there exists a convex neighbourhood $\mathcal{V}$ of every $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ such that for every ray $\mathrm{r}_{p,\overline{\lambda}_0}$ which intersects $\mathcal{V}$, the number of singularities of $\exp_p$ (counted with multiplicities) on $\mathrm{Im}(\mathrm{r}_{p,\overline{\lambda}_0}) \cap \mathcal{V}$ is constant and equals the order of λ_0 as a singularity of $\exp_p$, i.e. $\dim(\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p))$.*

In sub-Riemannian geometry, it would seem that the characterisation of conjugate vector established by Morse and Littauer does not follow easily from the cotangent version of the three regularity conditions. However, we are able to write a proof of this characterisation in the case of the three-dimensional Heisenberg group.

Theorem G (Non local injectivity of the Heisenberg exponential map)**.**
The Heisenberg sub-Riemannian exponential map $\exp_p : \mathscr{A}_p \to \mathbb{H}$ *is not injective on any neighbourhood of a conjugate vector* $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$*.*

This study on the sub-Riemannian exponential map is also the subject of preprint [BK21], and it has been submitted for publication in a peer-reviewed journal.

Finally, in the appendix, we turn again to the question of the validity of the curvature-dimension condition in sub-Riemannian geometry. It has been proven that this synthetic curvature bound never holds for parameters of positive effective dimension. We show that a slight adaptation of the proof yields that the curvature-dimension with negative effective dimension does not hold as well for strict and complete sub-Riemannian manifolds.

Theorem H (Failure of the curvatuve-dimension condition with negative effective dimension in sub-Riemannian geometry)**.**
Let M *be a complete sub-Riemannian manifold such that, for any* $p \in M$*, we have that* $\operatorname{rank}(\mathcal{D}_p) < \dim T_p(M)$*. If* μ *is any smooth measure on* M*, then the metric measure space* $(M, \mathrm{d}_{CC}, \mu)$ *does not satisfy* $\mathrm{CD}(K, N)$*, for any* $K \in \mathbb{R}$ *and any* $N < 0$*.*

Chapter 2

Metric geometry and optimal transport

This chapter aims to describe recent developments of notions of synthetic curvature that have been introduced over the past few decades, using ideas from optimal transport theory.

Optimal transport was first set out in [Mon81] by the French mathematician Gaspard Monge, the father of descriptive geometry, in 1781. The author was interested in a practical engineering problem: how do you move a pile of earth from a given area, called the Déblai, to another given area of equal measure, called the Remblai, in the most effective manner, i.e. with least effort or carriage? This problem, coined the Monge problem, was revisited by the mathematician, economist and Nobel prize winner Leonid Kantorovich in his work [Kan42] in 1942. He reformulated and brilliantly expanded Monge's ideas in a more modern fashion, contributing to a renewed interest in the Monge transport problem in the second half of the 20th century.

A dramatic turn for the theory came at the start of the new century, when optimal transport problems were linked to curvature in differential geometry. Lott, Sturm, and Villani pioneered in [LV09], [Stu06a], and [Stu06b] a notion of Ricci curvature bounds, via optimal transport, equivalent to the one used in

Riemannian geometry. This characterisation of Ricci curvature bounded from below and effective dimension bounded from above is *synthetic* and therefore can be taken as a definition of curvature-dimension bounds for general metric measure spaces.

We will particularly focus on the metric geometry of sub-Riemannian manifolds in this thesis. They are close enough to the Riemannian world to make an extensive use of some differential tools but at the same time singular enough to exhibit key features in metric geometry. The general theory of optimal transport is developed in [Vil09], [Vil03] and [San15]. We refer to [BBI01] for an introduction to metric geometry.

2.1 Monge–Kantorovich problem

The modern formulation of optimal transport theory can be made in the context of Polish probability spaces. We say that X is a Polish space if it is a separable completely metrisable topological space, X being equipped with its Borel σ-algebra and a Borel measure $\mathfrak{m}$.

If X and Y are two Polish spaces, a cost function is a map $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$. In the original memoir [Mon81], we read that Monge was originally interested in $X = Y = \mathbb{R}^n$ for $n = 2$ or 3, and $c(x, y) = |x - y|$. Consider two probability measures $\mu \in \mathrm{P}(X)$ and $\nu \in \mathrm{P}(X)$, modelising the *Déblai* and *Remblai*, Monge's problem consists of finding a measurable map $T : X \to Y$ realising the minimum in

$$\min \left\{ \mathrm{M}(T) := \int_X c(x, T(x)) \mathrm{d}x \mid T : X \to Y \text{ measurable}, T_\sharp \mu = \nu \right\}. \quad (2.1.1)$$

Such a map, when it exists, is called an *optimal transport map*.

At first, proving the existence of a minimiser in Equation (2.1.1) would seem very difficult. Indeed, suppose that $X = Y = \mathbb{R}^n$ and μ, ν are induced by densities

$f, g : \mathrm{R}^n \to \mathbb{R}$ respectively. Assume also that f, g, T have enough regularity and that T is injective. Then, the condition $T_\sharp\mu = \nu$ is equivalent to

$$g(T(x))\det(\mathrm{D}_x T) = f(x),$$

a non-linear equation which is not dissimilar to the Monge–Ampère equation in the field of partial differential equations.

In fact, Monge did not address the question of the existence of such optimal transport maps. Rather, he brought to light geometric properties of the solution to Equation (2.1.1). It is Kantorovich who answered key questions of the existence and characterisation of minimisers in his seminal paper [Kan42] from 1942. In order to do that, he generalised the Monge problem to a form more natural and easier to deal with: given $\mu \in \mathrm{P}(X)$ and $\nu \in \mathrm{P}(Y)$, find a probability measure $\pi \in \mathrm{P}(X \times Y)$, called *optimal transport plan*, realising the minimum in

$$\min\left\{\mathrm{K}(\pi) := \int_{X\times Y} c(x,y)\mathrm{d}\pi(x,y) \mid \pi \in \Pi(\mu,\nu)\right\}, \tag{2.1.2}$$

where

$$\Pi(\mu,\nu) := \left\{\pi \in \mathrm{P}(X \times Y) \mid (P_1)_\sharp\pi = \mu \text{ and } (P_2)_\sharp\pi = \mu\right\}$$

is called the set of transport plans. Here, P_i denotes the projection onto the ith coordinate. The minimising problem (2.1.2) is called the Monge–Kantorovich problem, or simply the Kantorovich problem. Indeed, the Monge problem is included in Equation (2.1.2): if $T : X \to Y$ is an optimal transport map, then $(\mathrm{id}_X, T)_\sharp\mu$ is an optimal transport plan. However, an optimal transport plan is not always induced from an optimal transport map. In the next section, we will see examples of optimal transport plans induced by optimal transport maps.

The following existence theorem can then be established, following Kantorovich's work in [Kan42].

Theorem 2.1.1. *Let X and Y be Polish spaces and $c : X \times Y \to [0, +\infty]$ be a lower semi-continuous cost function. For every $\mu \in \mathrm{P}(X)$ and $\nu \in \mathrm{P}(Y)$, the Monge–Kantorovich* *problem* (2.1.2) *has a solution.*

The Monge–Kantorovich problem admits a dual formulation, formalised by the next theorem. Denote by $C_b(X)$ the space of bounded continuous functions from X to $\mathbb{R}$. Given $\mu \in P(X)$ and $\nu \in P(Y)$, the dual problem consists of finding $\psi \in C_b(X)$ and $\varphi \in C_b(Y)$, such that $\varphi(y) - \psi(x) \leqslant c(x, y)$ for every $(x, y) \in X \times Y$, realising the supremum in

$$\sup \left\{ D(\psi, \varphi) := \int_Y \varphi(y) \mathrm{d}\nu(y) - \int_X \psi(x) \mathrm{d}\mu(x) \mid (\psi, \varphi) \in C_b(\mu, \nu) \right\}. \quad (2.1.3)$$

where $C_b(\mu, \nu) := \{(\psi, \varphi) \in C_b(X) \times C_b(Y) \mid \forall (x, y) \in X \times Y,\ \varphi(y) - \psi(x) \leqslant c(x, y)\}$.

Theorem 2.1.2. *Let X, Y be Polish spaces and $c : X \times Y \to [0, +\infty]$ be a lower semi-continuous cost function. Then, for every $\mu \in P(X)$ and $\nu \in P(Y)$, we have the duality formula*

$$\min_{\pi \in \Pi(\mu,\nu)} K(\pi) = \sup_{(\psi,\varphi) \in C_b(\mu,\nu)} D(\psi, \varphi).$$

Note that in the previous theorem, the existence of a solution to the dual problem is not guaranteed. This is because the set $C_b(\mu, \nu)$ is not compact. However, the optimal plan $\pi \in \Pi(\mu, \nu)$ can be characterised with the help of c-cyclical monotonicity and c-convexity.

Definition 2.1.3. A set $\Gamma \subseteq X \times Y$ is said to be c-cyclically monotone if for every $N \in \mathbb{N}$ and any $(x_1, y_1), \dots, (x_N, y_N) \in \Gamma$, we have

$$\sum_{k=1}^{N} c(x_i, y_i) \leqslant \sum_{k=1}^{N} c(x_i, y_{i+1})$$

with the convention $y_{N+1} := y_1$. A transport plan $\pi \in \Pi(\mu, \nu)$ is c-cyclically monotone if it is concentrated on a c-cyclically monotone set.

A function $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be c-convex if it is not identically $+\infty$, and if there exists $\overline{\psi} : Y \to \mathbb{R} \cup \{\pm\infty\}$ satisfying

$$\psi(x) = \sup_{y \in Y} \left[\overline{\psi}(y) - c(x, y) \right], \text{ for all } x \in X.$$

If this is the case, the c-transform of ψ is given by

$$\psi^c(y) := \inf_{x \in X} [\psi(x) + c(x,y)], \text{ for all } y \in Y.$$

Finally, the c-subdifferential of ψ is the c-cyclically monotone set defined by

$$\partial_c \psi := \{(x,y) \in X \times Y \mid \psi^c(y) - \psi(x) = c(x,y)\}.$$

By considering the cost function $c(x,y) = -x \cdot y$ in $\mathbb{R}^n$, we can observe that the concepts of c-convexity/transform are generalisations of the convexity and the Legendre transform. The notion of c-concavity can be defined in a similar way.

Theorem 2.1.4. *Let X, Y be Polish spaces and $c : X \times Y \to [0, +\infty)$ be a lower semi-continuous cost function. Let $\mu \in \mathrm{P}(X)$ and $\nu \in \mathrm{P}(Y)$ and assume that the optimal cost $\min_{\pi \in \Pi(\mu,\nu)} K(\pi)$ is finite. Then, there exists a c-cyclically monotone set $\Gamma \subseteq X \times Y$ such that the following assertions are equivalent, for any $\pi \in \Pi(\mu, \nu)$:*

(i) *π is an optimal transport plan;*

(ii) *π is c-cyclically monotone;*

(iii) *There is a c-convex function $\psi : X \to \mathbb{R} \cup \{+\infty\}$ such that*

$$\psi^c(y) - \psi(x) = c(x,y), \textit{ for } \pi\textit{–almost every } (x,y) \in X \times Y;$$

(iv) *For some functions $\psi : X \to \mathbb{R} \cup \{+\infty\}$ and $\varphi : Y \to \mathbb{R} \cup \{-\infty\}$, we have*

$$\varphi(y) - \psi(x) \leqslant c(x,y), \textit{ for } \pi\textit{–almost every } (x,y) \in X \times Y;$$

(v) *π is concentrated on Γ.*

We finally address the existence of a solution to the dual problem (2.1.3).

Theorem 2.1.5. *Let X, Y be Polish spaces and $c : X \times Y \to [0, +\infty)$ be a lower semi-continuous cost function. Let $\mu \in \mathrm{P}(X)$ and $\nu \in \mathrm{P}(Y)$ and assume that the optimal cost $\min_{\pi \in \Pi(\mu,\nu)} K(\pi)$ is finite. Suppose that for π–almost every $(x,y) \in X \times Y$, we have*

$$c(x,y) \leqslant C_X(x) + C_Y(y), \textit{ for some } C_X \in \mathrm{L}^1(\mu) \textit{ and } C_Y \in \mathrm{L}^1(\nu).$$

Then the Monge–Kantorovich problem (2.1.2) *and its dual formulation* (2.1.3) *have solutions and*

$$\min_{\pi\in\Pi(\mu,\nu)} \mathrm{K}(\pi) = \max_{(\psi,\varphi)\in \mathrm{C}_b(\mu,\nu)} \mathrm{D}(\psi,\varphi).$$

Furthermore, there is a closed c-cyclically monotone set $\Gamma \subseteq X \times Y$ *such that*

(i) $\pi \in \Pi(\mu,\nu)$ *is an optimal transport plan if and only if* $\pi(\Gamma) = 1$*;*

(ii) *a c-convex function* $\psi : \mathrm{X} \to \mathbb{R} \cup \{+\infty\}$ *is a solution to the dual problem if and only if* $\Gamma \subseteq \partial_c \psi$*.*

More details and the proofs of the results stated in this section can be found in [Vil09, Chapter 5.] and [San15, Chapter 1.]. In these sources and the references therein, more effort is put into finding the minimal hypothesis for the well-posedness of the Monge–Kantorovich problem, its dual problem, as well as a presentation of further results, examples and applications.

2.2 Length and geodesic spaces

In this section, we give an overview of metric geometry which is the most general framework to deal with notions of synthetic curvature. We start with the definition of length structure.

Let (X, τ_X) be a topological space (with more than one point). A class of paths is a set $\mathcal{A}$ of continuous paths γ from intervals of $\mathbb{R}$ into X, i.e.

$$\mathcal{A} \subseteq \{\gamma : I \to X \mid I \text{ is an interval of } \mathbb{R}, \text{ and } \gamma \text{ is continuous}\}.$$

A class of reparametrisations is a set R of maps φ from intervals to intervals of $\mathbb{R}$ containing all the linear maps. In this work, we also assume that

$$R \subseteq \{\varphi : J \to I \mid I, J \text{ intervals}, \varphi \text{ is monotonic and surjective}\}.$$

We will often write $(\gamma, I) \in \mathcal{A}$ to say that $\gamma : I \to X$ is a path in $\mathcal{A}$ and $(\varphi, J, I) \in R$ for a reparametrisation in R.

Definition 2.2.1. We say that $\mathcal{A}$ is a class of admissible paths closed under a set R of reparametrisations if

(i) $\mathcal{A}$ is closed under restriction: $(\gamma|_J, J) \in \mathcal{A}$ whenever $(\gamma, I) \in \mathcal{A}$ and J is a subinterval of I;

(ii) $\mathcal{A}$ is closed under concatenation: $(\gamma, I) \in \mathcal{A}$ if (γ_1, I_1) and $(\gamma_2, I_2) \in \mathcal{A}$ are such that I is the disjoint union of I_1 and I_2, $\gamma|_{I_1} = \gamma_1$ and $\gamma|_{I_2} = \gamma_2$;

(iii) $\mathcal{A}$ is closed under reparametrisations of R: if $(\gamma, I) \in \mathcal{A}$ and $(\varphi, J, I) \in R$, then $(\gamma \circ \varphi, J) \in \mathcal{A}$;

(iv) For every $x \in X$, there exists $(\gamma, I) \in \mathcal{A}$ with $x \in \gamma(I)$.

Such a class of admissible paths and set of reparametrisations are the building blocks for the definition of a length structure.

Definition 2.2.2. Let $\mathcal{A}$ be a class of admissible paths closed under a set R of reparametrisations. We say that

$$\mathrm{L} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$$

is a length map, and that $(R, \mathcal{A}, \mathrm{L})$ is a length structure on the space X, if

(i) L is additive: if (γ_1, I_1) and $(\gamma_2, I_2) \in \mathcal{A}$ are such that I is the disjoint union of I_1 and I_2, $\gamma|_{I_1} = \gamma_1$ and $\gamma|_{I_2} = \gamma_2$, then $\mathrm{L}(\gamma) = \mathrm{L}(\gamma_1) + \mathrm{L}(\gamma_2)$;

(ii) L depends continuously on pieces of paths: if $(\gamma, I) \in \mathcal{A}$ and $\mathrm{L}(\gamma) < +\infty$, then the map $\mathrm{L}(\gamma, \cdot) : I \to \mathbb{R} : t \mapsto \mathrm{L}(\gamma, t) := \mathrm{L}(\gamma|_{I \cap (-\infty, t]})$ is continuous;

(iii) L is invariant under the reparametrisations of R: $\mathrm{L}(\gamma \circ \varphi) = \mathrm{L}(\gamma)$ whenever $(\gamma, I) \in \mathcal{A}$ and $(\varphi, J, I) \in R$;

(iv) L agrees with the topology of X: for all $x \in X$ and all open neighbourhood $\mathcal{U}$ of x, we have

$$\inf \{\mathrm{L}(\gamma) \mid (\gamma, [a, b]) \in \mathcal{A}, \gamma(a) = x \text{ and } \gamma(b) \in X \setminus \mathcal{U}\} > 0;$$

(v) X is connected by rectifiable paths: for every $x, y \in X$ there exists $(\gamma, [a, b]) \in \mathcal{A}$ such that $\gamma(a) = x, \gamma(b) = y$ and $\mathrm{L}(\gamma) < +\infty$.

This very general setting covers most geometries that we encounter in the differential world: (reversible) Finsler geometry, Riemannian geometry and of particular importance for this thesis, sub-Riemannian geometry. A length map induces a metric structure on X.

Theorem 2.2.3. *Let $(R, \mathcal{A}, \mathrm{L})$ be a length structure on X. Define the induced distance map*

$$\mathrm{d}_{\mathrm{L}}(x,y) := \inf\left\{\mathrm{L}(\gamma) \mid (\gamma,[a,b]) \in \mathcal{A}, \gamma(a) = x, \gamma(b) = y\right\} \textit{ for } x,y \in X.$$

Then, the structure $(X, \mathrm{d}_{\mathrm{L}})$ is a metric space.

Because of Definition 2.2.2 (iv), the topology induced by d_{L} can only be finer than the original topology of X. If we do not take the assumption that X is connected by rectifiable paths (Definition 2.2.2 (v)), then $(X, \mathrm{d}_{\mathrm{L}})$ will be an extended metric space. Also, a rectifiable path $(\gamma, I) \in \mathcal{A}$ is always continuous with respect to d_{L}.

A distance function on a set X might originate from a length structure. This motivates the next definition.

Definition 2.2.4. We say that a metric space (X, d) is a length space if $\mathrm{d} = \mathrm{d}_{\mathrm{L}}$ where $(R, \mathcal{A}, \mathrm{L})$ is a length structure on X.

Conversely, we can construct a length structure out of a distance function.

Definition 2.2.5. Let (X, d) be a metric space and $\gamma : [a,b] \to X$ a continuous path. The d-induced length of γ is

$$\mathrm{L}_{\mathrm{d}}(\gamma) := \sup\left\{\sum_{k=0}^{n-1} \mathrm{d}(\gamma(t_i), \gamma(t_{i+1})) \mid n \in \mathbb{N} \text{ and } a = t_0 < t_1 < \cdots < t_n = b\right\}.$$

If I is an interval of $\mathbb{R}$ and $\gamma : I \to X$ a continuous path, then

$$\mathrm{L}_{\mathrm{d}}(\gamma) := \sup\left\{\mathrm{L}_{\mathrm{d}}(\gamma|_J) \mid J \text{ is a closed and bounded subinterval of } I\right\}.$$

If $\mathcal{A}$ is a subset of the space of d-continuous maps $\mathcal{C}_{\mathrm{d}}(X)$ and if it is a class of admissible paths closed under a set of reparametrisations R, then $(R, \mathcal{A}, \mathrm{L}_{\mathrm{d}_{\mathrm{L}}})$ is a length structure for (X, d). The length function L_{d} is lower semi-continuous.

Proposition 2.2.6. *Let (X, d) be a metric space and $(R, \mathcal{A}, \mathrm{L}_{\mathrm{d_L}})$ a length structure on X induced from* d. *If $\gamma_n, \gamma : [a,b] \to X$ are paths in $\mathcal{A}$ such that $\gamma_n \to \gamma$ when $n \to +\infty$ with respect to the uniform convergence, then*

$$\mathrm{L_d}(\gamma) \leqslant \liminf_{n \to +\infty} \mathrm{L_d}(\gamma_n).$$

A length structure $(R, \mathcal{A}, \mathrm{L})$ on a topological space X induces a metric structure on $(X, \mathrm{d_L})$. The previous discussion shows that this metric structure in turn induces a length structure $(R, \mathcal{A} \cap \mathcal{C}_{\mathrm{d_L}}(X), \mathrm{L}_{\mathrm{d_L}})$ on $(X, \mathrm{d_L})$. In fact, we must have $\mathrm{d}_{\mathrm{L}_{\mathrm{d_L}}}(x,y) = \mathrm{d_L}(x,y)$ for every $x, y \in X$ and $\mathrm{L}(\gamma) = \mathrm{L}_{\mathrm{d_L}}(\gamma)$ for every $\gamma \in \mathcal{A} \cap \mathcal{C}_{\mathrm{d_L}}$. So, if (X, d) is a length space induced by a length structure $(R, \mathcal{A}, \mathrm{L})$, then $\mathrm{d}(x,y) = \mathrm{d}_{\mathrm{L_d}}(x,y)$ for every $x, y \in X$ and $\mathrm{L}(\gamma) = \mathrm{L_d}(\gamma)$ for every $\gamma \in \mathcal{A}$. This observation allows the following characterisation of length spaces. A metric space (X, d) is a length space induced by a length structure $(R, \mathcal{A}, \mathrm{L})$ if and only if for every $x, y \in X$ and all $\epsilon > 0$, there exists a path $(\gamma, [a,b]) \in \mathcal{A}$ such that $\gamma(a) = x$, $\gamma(b) = y$ and $\mathrm{L_d}(\gamma) < \mathrm{d}(x,y) + \epsilon$.

Paths can potentially represent the same curve under different parametrisations. A path $\gamma : I \to X$ is parametrised by the constant speed $v > 0$ if $\mathrm{L_d}(\gamma|_{[t,t']}) = v|t - t'|$ for every $t, t' \in I$. In differential geometry of curves and surfaces, it is well known that a smooth path can always be reparametrised by any constant speed $v > 0$. The next proposition generalises this result to metric geometry.

Theorem 2.2.7. *Let (X, d) be a metric space, $\gamma : I \to X$ a continuous path such that $\mathrm{L_d}(\gamma) < +\infty$, and $v > 0$. The map*

$$\psi_v : I \to \mathbb{R} : t \mapsto \frac{\mathrm{L_d}(\gamma, t)}{v}$$

is a non-decreasing continuous function. Furthermore, the curve

$$\overline{\gamma}_v : \psi_v(I) \to X : \tau \mapsto \overline{\gamma}_v(\tau) := \gamma(t),$$

where t is chosen arbitrarily in $\psi_v^{-1}(\tau)$, is a well-defined Lipschitz continuous curve, parametrised on the interval $\psi_v(I)$ by the constant speed v, and $\gamma = \overline{\gamma}_v \circ \psi_v$.

We finally turn our attention to geodesics.

Definition 2.2.8. In a length space (X, d) induced by a length structure $(R, \mathcal{A}, \mathrm{L})$, a minimising geodesic is a path $(\gamma, [a, b]) \in \mathcal{A}$ such that $\mathrm{L}(\gamma) = \mathrm{d}(\gamma(a), \gamma(b))$. A path $(\gamma, I) \in \mathcal{A}$ is a locally minimising geodesic, often simply called a geodesic, if for every $t \in I$, there exists an $\delta > 0$ such that $[t-\delta, t+\delta] \subseteq I$ and $(\gamma|_{[t-\delta,t+\delta]}, [t-\delta, t+\delta])$ is a minimising geodesic.

We now introduce a particular class of length spaces.

Definition 2.2.9. A length space (X, d) induced by a length structure $(R, \mathcal{A}, \mathrm{L})$ is called a *geodesic space* if every two points in X can be joined by at least one minimising geodesic parametrised by constant speed.

We end this section by mentioning that the classical Hopf–Rinow theorem admits a generalisation in this setting: a locally compact length space is complete if and only if it is proper, i.e. every closed metric ball in X is compact (see [BBI01, Theorem 2.5.28]). For a longer exposition on the subject of metric geometry, we refer the reader to the textbooks [BBI01] and [Pap14].

2.3 Distortion coefficients in metric geometry

We introduce the general definition of *(volume) distortion coefficients*.

Definition 2.3.1. Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space and $x, y \in X$. The distortion coefficient from x to y at time $t \in [0, 1]$ is

$$\beta_t(x, y) = \limsup_{r \to 0^+} \frac{\mathfrak{m}(\mathrm{Z}_t(x, B_r(y)))}{\mathfrak{m}(B_r(y))} \tag{2.3.1}$$

where $\mathrm{Z}_t(x, B_r(y))$ stands for the set of t-intermediate points from x to the ball centred in y of radius r;

$$\mathrm{Z}_t(A, B) := \{\gamma(t) | \gamma \in \mathrm{Geo}(X), \gamma(0) \in A \text{ and } \gamma(1) \in B\}$$

whenever A and B are $\mathfrak{m}$-measurable subsets of X.

Note that $Z_t(x, B_r(y))$ might not be measurable. In that case, the measure $\mathfrak{m}$ in the numerator of (2.3.1) is understood as the outer measure of $\mathfrak{m}$.

There is an intuitive physical interpretation of the distortion coefficients (quoted from [Vil09, Chapter 14.]):

> $[\beta_t(x,y)]$ compares the volume occupied by the light rays emanating from the light source $[x]$, when they arrive close to $\gamma(t)$, to the volume that they would occupy in a flat space.

In particular, we can thus heuristically expect that the distortion coefficients are related to the curvature of the space. The distortion coefficients of the Riemannian space forms of constant curvature $K \in \mathbb{R}$ and dimension $N > 1$ are given by

$$\beta_t^{(K,N)}(x,y) = \begin{cases} t\Big(\dfrac{\sin(t\alpha)}{\sin(\alpha)}\Big)^{N-1} & \text{if } K > 0; \\ t^N & \text{if } K = 0; \\ t\Big(\dfrac{\sinh(t\alpha)}{\sinh(\alpha)}\Big)^{N-1} & \text{if } K < 0, \end{cases} \tag{2.3.2}$$

where $\alpha := \sqrt{\dfrac{|K|}{N-1}}\mathrm{d}(x,y)$. Furthermore, if (M,g) is a Riemannian manifold equipped with the Riemannian volume $\mathrm{d}vol_g$, and if $\mathrm{Ric} \geqslant K$ and $\dim(M) \leqslant N$, then we have $\beta_t(x,y) \geqslant \beta_t^{(K,N)}(x,y)$ for all $x,y \in M$ and $t \in [0,1]$. The comparison of the curvature properties of a space to the model spaces of constant curvature K and dimension N is at the basis of the making of synthetic curvature-dimension conditions.

2.4 Synthetic curvature-dimension conditions

The theory of synthetic curvature was developed by Lott, Sturm, and Villani (see [LV09], [Stu06a], and [Stu06b]). Here we summarise some of the points from their works.

Let (X, d) be a geodesic space, induced by a length strucutre $(R, \mathcal{A}, \mathrm{L})$, and we equip it with a Radon measure $\mathfrak{m}$. We say that $(X, \mathrm{d}, \mathfrak{m})$ is a *metric measure space* (or *measured geodesic space*, to be more precise). We write $\mathrm{Geo}(X)$ for the set of all minimising geodesics of X parametrised by constant speed on $[0, 1]$.

Two minimising geodesics (parametrised by constant speed) $\gamma_1, \gamma_2 : [0, 1] \to X$ are said to be *branching* if there is $t \in [0, 1]$ such that $\gamma_1(s) = \gamma_2(s)$ for all $s \in [0, t)$ while $\gamma_1 \neq \gamma_2$. In Riemannian geometry, geodesics never branch as they are the solution to a second-order differential equation, but generally it can be more complicated in other geometric structures (see [MR20] for example). A set of geodesics $G \subseteq \mathrm{Geo}(X)$ is said to be non-branching if there are no branching geodesics in G. The geodesic space (X, d) is itself called non-branching if $\mathrm{Geo}(X)$ is non-branching.

We denote by $\mathrm{P}(X)$ the set of Borel probability measures and for $p \in [1, +\infty)$, the subset $\mathrm{P}_p(X)$ of those with finite pth-moment. For all $t \in [0, 1]$, the evaluation map is defined as

$$\mathrm{e}_t : \mathrm{Geo}(X) \to X : \gamma \mapsto \gamma(t).$$

A dynamical transference plan Π is a Borel probability measure on $\mathrm{Geo}(X)$ while a displacement interpolation associated to Π is a path $(\mu_t)_{t \in [0,1]} \subseteq \mathrm{P}^2(X)$ such that $\mu_t = (\mathrm{e}_t)_\# \Pi$ for all $t \in [0, 1]$.

We equip $\mathrm{P}_p(X)$ with the L_p-Wasserstein distance $\mathcal{W}_p$: for any $\mu, \nu \in \mathrm{P}^2(X)$,

$$\mathcal{W}_p(\mu_0, \mu_1) := \left[\inf_{\pi \in \Pi(\mu, \nu)} \int_X \mathrm{d}(x, y)^p \mathrm{d}\pi(x, y) \right]^{1/p}$$

with $\Pi(\mu, \nu) := \{\pi \in \mathrm{P}(X^2) \mid (P_1)_\# \pi = \mu \text{ and } (P_2)_\# \pi = \nu\}$. In other words, we define a distance between two probabilities as the value realising the infimum in the Monge–Kantorovich problem (2.1.2) with the cost function $c(x, y) = \mathrm{d}(x, y)^p$. It can be shown that $(\mathrm{P}_p(X), \mathcal{W}_p)$ is a geodesic space, from the assumption that (X, d) is a geodesic space (see [Vil09, Chapter 7.]).

For $\mu_0, \mu_1 \in \mathrm{P}_p(X)$, the set $\mathrm{OptGeo}_p(\mu_0, \mu_1)$ is the space of all measures $\nu \in$

$\mathrm{P}(\mathrm{Geo}(X))$ such that $(e_0, e_1)_\# \nu$ realises the minimum for the Wasserstein distance $\mathcal{W}_p$. A measure $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$ is called a p-dynamical optimal plan.

The metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is said to be p-essentially non-branching for all $\mu_0, \mu_1 \in \mathrm{P}_p(X)$, any p-dynamical optimal plan $\nu \in \mathrm{OptGeo}_p(\mu_0, \mu_1)$ is concentrated on a Borel non-branching set $G \subseteq \mathrm{Geo}(X)$.

We now introduce the (K, N)-distortion coefficients. For $K \in \mathbb{R}, N \in [1, +\infty], \theta \in (0, +\infty)$ and $t \in [0, 1]$, we set

$$\tau_{K,N}^{(t)}(\theta) = t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{1-1/N},$$

with

$$\sigma_{K,N}^{(t)}(\theta) = \begin{cases} +\infty & K\theta^2 \geqslant N\pi^2 \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0 < K\theta^2 < N\pi^2 \\ t & \text{if } K\theta^2 = 0 \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 \leqslant 0 \end{cases}.$$

Comparing with Equation (2.3.2), we see that the definition of the coefficients $\tau_{K,N}$ corresponds to $(\beta^{(K,N)})^{1/N}$. We are ready to introduce a first notion of synthetic curvature: the curvature-dimension condition.

Definition 2.4.1. Let $K \in \mathbb{R}$ and $N \in [1, +\infty)$. We say that a geodesic metric measure space $(X, \mathrm{d}, \mathfrak{m})$ has curvature bounded from below by K and dimension bounded from above by N, or satisfies the (K, N)-curvature-dimension condition – $\mathrm{CD}(K, N)$ for short – if, for any $\mu_0, \mu_1 \in \mathrm{P}_2(X)$ absolutely continuous with respect to $\mathfrak{m}$ and with bounded support, there exists $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ and $\pi \in \mathrm{P}(X^2)$ a $\mathcal{W}_2$-optimal plan such that $\mu_t := (\mathrm{e}_t)_\# \nu \ll \mathfrak{m}$ and for any $N' \geqslant N$,

$$\mathcal{E}_{N'}(\mu_t) \geqslant \int_{X^2} \tau_{K,N}^{(1-t)}(\mathrm{d}(x,y)) \rho_0^{-1/N'} + \tau_{K,N}^{(t)}(\mathrm{d}(x,y)) \rho_1^{-1/N'} \pi(\mathrm{d}x\mathrm{d}y), \qquad (2.4.1)$$

where $\mathcal{E}_N$ stands for the Rényi functional

$$\mathcal{E}_N : \mathcal{P}(X) \to [0, +\infty] : \rho\mathfrak{m} + \mu_s \mapsto \int_X \rho^{1-1/N} \mathfrak{m}(\mathrm{d}x).$$

The CD condition is built on the Wasserstein space of order 2. This can be seen as being rooted in the fact that the theory of Ricci curvature in Riemannian geometry is a quadratic differential theory. Nevertheless, it has recently been proved in [Akd+20] that under the essentially non-branching condition, an arbitrary p in the definition will give rise to the same curvature-dimension condition.

The $\mathrm{CD}(K,N)$ condition has been very successful in generalising classical theorems of differential geometry to the non-smooth setting such as the Bonnet–Myers theorem, Sobolev/log-Sobolev inequalities, Poincaré inequalities, Talagrand inequalities, Lévy–Gromov isoperimetric inequality, to name just a few.

We mention in more detail one implication of the $\mathrm{CD}(K,N)$ condition: the *Brunn–Minkowski inequality*.

Proposition 2.4.2. *Let $(X,\mathrm{d},\mathfrak{m})$ be a geodesic metric measure space satisfying the $\mathrm{CD}(K,N)$ condition for some $K\in\mathbb{R}$ and $N\in[1,+\infty]$. Then, the following Brunn–Minkowski inequality, denoted by $\mathrm{BM}(K,N)$, holds in $(X,\mathrm{d},\mathfrak{m})$. For every Borel subsets A_0 and A_1 of X and any $t\in[0,1]$, we have*

$$\mathfrak{m}(Z_t(A_0,A_1))^{1/N} \geqslant \tau_{K,N}^{(1-t)}(\Theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\Theta)\mathfrak{m}(A_1)^{1/N},$$

if $N<+\infty$, and if $N=+\infty$, we have

$$\begin{aligned}\log\left[\frac{1}{\mathfrak{m}(Z_t(A_0,A_1))}\right] \leqslant (1-t)\log\left[\frac{1}{\mathfrak{m}(A_0)}\right] &+ t\log\left[\frac{1}{\mathfrak{m}(A_1)}\right]\\ &- \frac{Kt(1-t)}{2}\sup_{(x,y)\in A_0\times A_1}\mathrm{d}(x,y)^2,\end{aligned}$$

where Θ is the minimal or maximal geodesic length from A_0 to A_1:

$$\Theta := \begin{cases} \inf_{(x,y)\in A_0\times A_1}\mathrm{d}(x,y) & \text{if } K\geqslant 0;\\ \sup_{(x,y)\in A_0\times A_1}\mathrm{d}(x,y) & \text{if } K<0.\end{cases}$$

Alongside this notion of synthetic curvature, a weaker condition was developed independently by Sturm and Ohta: the measure contraction property – $\mathrm{MCP}(K,N)$ for short –, see [Stu06b] and [Oht07].

Definition 2.4.3. Let $K \in \mathbb{R}$ and $N \in [1, +\infty)$. A geodesic metric measure space $(X, \mathrm{d}, \mathfrak{m})$ satisfies MCP(K, N) if, for every $x \in X$ and measurable set $A \subseteq X$ with $\mathfrak{m}(A) \in (0, +\infty)$, there exists $\nu \in \mathrm{OptGeo}\,(\mu_A, \delta_x)$ such that for all $t \in [0, 1]$

$$\mu_A \geqslant (\mathrm{e}_t)_{\#} \left(\tau_{K,N}^{(1-t)}(\mathrm{d}(\gamma(0), \gamma(1)))\nu(\mathrm{d}\gamma) \right)$$

where $\mu_A := \frac{1}{\mathfrak{m}(A)}\mathfrak{m} \in \mathcal{P}(X)$ is the normalisation of μ to A.

The MCP(K, N), although weaker, does imply interesting geometric inequalities: for example, the Bonnet–Myers theorem and the Bishop–Gromov volume comparison theorem.

Proposition 2.4.4 (Bishop–Gromov volume comparison theorem)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a geodesic metric measure space satisfying the* CD(K, N) *condition for some $K \in \mathbb{R}$ and $N \in [2, +\infty)$. Then, for all $x \in X$, the map*

$$[0, +\infty) \to R : r \mapsto \frac{\mathfrak{m}(B(x, r))}{V_{K,N}(r)}$$

is non-increasing. Here, $B(x, r)$ denotes the open ball centred at x of radius $r > 0$ while $V^{K,N}(r)$ is the volume of a ball of radius r in the (K, N)-model space.

Both the CD and MCP conditions generalise the notion of Ricci curvature bounded from below by $K \in \mathbb{R}$ and dimension bounded from above by $N \geqslant 1$ from Riemannian geometry.

Theorem 2.4.5. *If (M, g) is a Riemannian manifold and ψ a positive $\mathcal{C}^2$ function on M, the metric measure space $(M, \mathrm{d}_g, \psi \cdot \mathrm{vol}_g)$ satisfies the* CD(K, N) *condition if and only if* $\mathrm{Ric}_{g,\psi,N} \geqslant Kg$ *where*

$$\mathrm{Ric}_{g,\psi,N} := \mathrm{Ric}_g - (N - n)\frac{\nabla_g^2 \psi^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}}.$$

Note that in the case where $N = n$, it only makes sense to consider constant functions ψ in the definition of the generalised Ricci tensor. The proof of the equivalence with the CD condition can be found in [Stu06b] and [LV09]. Furthermore, it is also proved in [Oht07, Corollary 3.3.] that, in the Riemannian setting, the MCP(K, N) condition is equivalent to CD(K, N) if N is greater than the topological dimension of (M, g).

For general metric measure spaces, the two notions of synthetic curvature are not equivalent. For *non-branching* spaces, the CD condition does imply the MCP condition however (see [Oht07] and [CS12]). As we will see later (see Section 4.4 and Appendix A), this already appears in sub-Riemannian geometry.

Chapter 3

Sub-Riemannian geometry

As we have seen in the previous chapter, the rich geometry of metric measure spaces was developed to study the structure of singular spaces such as limits of Riemannian manifolds, Alexandrov spaces or Finsler manifolds. In this thesis, we will focus on sub-Riemannian manifolds. Roughly speaking, they are built by constraining the directions in which a path is allowed to travel.

The theory of sub-Riemannian geometry has received a lot of attention in recent years. In particular, it has successfully provided smooth examples and counter-examples which are helpful for solving some open problems in metric geometry. For example, it was proven by Juillet in [Jui21] that no (strict and complete) sub-Riemannian structure satisfies the $\mathrm{CD}(K, N)$ condition (see Definition 2.4.1).

The geodesics associated with sub-Riemannian structures are also interesting. The study of length minimisers reveals the existence of geodesics which can be normal, i.e. solving a differential equation, or abnormal. Although abnormal geodesics are at the heart of hard open problems in the theory, this thesis will focus on normal extremals.

In this chapter therefore, we introduce the basics of sub-Riemannian geometry. We rely on [ABB20] and [Agr08] for the general theory, and we introduce sub-Riemannian optimal transportation problems and distortion coefficients in the spirit of [FR10] and [BR19].

3.1 Carnot–Carathéodory structure

A manifold is a set equipped with an equivalence class of atlases such that its manifold topology is Hausdorff and second-countable. In this work, manifolds are considered without a boundary. All the objects are assumed to be of class $\mathcal{C}^r$ for $r \in [\![2, \infty]\!] \cup \{\omega\}$. We begin with the definition of a *sub-Riemannian manifold*, also called a *Carnot–Carathéodory space*.

Definition 3.1.1. A triple $(E, \langle\cdot,\cdot\rangle_E, f_E)$ induces a sub-Riemannian structure on a manifold M if

(i) E is a vector bundle over M,

(ii) $\langle\cdot,\cdot\rangle_E$ is a metric on E,

(iii) $f_E : E \to \mathrm{T}(M)$ is a morphism of vector bundles.

We say that $(M, E, \langle\cdot,\cdot\rangle_E, f_E)$, or simply M when the context is clear, is a sub-Riemannian manifold.

The family of *horizontal vector fields* is defined as

$$\mathcal{D} := \{f_E \circ u \mid u \text{ is a section of } E\}.$$

They correspond to vector fields that are compatible with the sub-Riemannian structure of M. The *horizontal distribution* at a point $p \in M$ is $\mathcal{D}_p := \{v(p) \mid v \in \mathcal{D}\}$. The *rank* of the sub-Riemannian structure at $p \in M$ is $\mathrm{rank}(p) := \dim(\mathcal{D}_p)$. Observe that in our definition, a sub-Riemannian manifold can be rank-varying.

When considering paths in M, we are mainly interested in those that have their tangent vector field in the horizontal distribution.

Definition 3.1.2. We say that curve $\gamma : [0, T] \to M$ is *horizontal* or *admissible* if γ is Lipschitz in charts and if there exists a *control* $u : [0, T] \to E$ such that $u \in L^2([0, T], E)$ with $\dot{\gamma}(t) = f_E(u(t))$ for almost every $t \in [0, T]$.

The *sub-Riemannian length* and the *sub-Riemannian energy* of γ are defined by

$$\mathrm{L}(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_{\mathcal{D}_{\gamma(t)}} \, \mathrm{d}t, \;\; \mathrm{J}(\gamma) = \frac{1}{2}\int_0^T \|\dot{\gamma}(t)\|^2_{\mathcal{D}_{\gamma(t)}} \, \mathrm{d}t \tag{3.1.1}$$

where $\|v\|_{\mathcal{D}_p} := \min\left\{\sqrt{\langle u,u\rangle_{E_p}} \mid u \in E_p \text{ and } f_E(u) = (p,v)\right\}$ for $v \in \mathcal{D}_p$ and $p \in M$.

Remark 3.1.3. The norm $\|\cdot\|_{\mathcal{D}_p}$ is well-defined, induced by an inner product $\langle\cdot,\cdot\rangle_{\mathcal{D}_p}$ and the map $t \mapsto \|\dot{\gamma}(t)\|_{\mathcal{D}_{\gamma(t)}}$ is measurable.

Given a horizontal curve $\gamma : [0,T] \to M$, we define at every differentiability point of γ the *minimal control* $\overline{u}$ associated with γ

$$\overline{u}(t) := \arg\min\left\{\sqrt{\langle u,u\rangle_{E_p}} \mid u \in E_p \text{ and } f_E(u) = \dot{\gamma}(t)\right\}.$$

It can be established that the minimal control $\overline{u}$ corresponding to a horizontal curve is always measurable and essentially bounded. Therefore, Equation 3.1.1 can be written as

$$\mathrm{L}(\gamma) = \int_0^T \|\overline{u}(t)\|_E \, \mathrm{d}t, \;\; \mathrm{J}(\gamma) = \frac{1}{2}\int_0^T \|\overline{u}(t)\|^2_E \, \mathrm{d}t.$$

The right-hand side of the previous equality can also serve as a definition for length and energy functionals defined on the space of controls $\mathrm{L}^2([0,T],E)$ which we will still denote by $\mathrm{L},\mathrm{J} : \mathrm{L}^2([0,T],E) \to \mathbb{R}$.

We can define a distance on a sub-Riemannian manifold in the same way as in Riemannian geometry.

Definition 3.1.4. The distance between two points x and y of a sub-Riemannian manifold M, also called the *Carnot–Carathéodory distance*, is defined by

$$\mathrm{d}_{\mathrm{CC}}(x,y) := \inf\{\mathrm{L}(\gamma) \mid \gamma : [0,T] \to M \text{ is horizontal, } \gamma(0) = x \text{ and } \gamma(T) = y\}.$$

When the context is clear enough, we will simply write d for the sub-Riemannian distance. However, this function will not induce a structure of metric space on (M,d) unless we can join every two points in M with a horizontal path.

For this reason, it is conventional to assume that the sub-Riemannian structure satisfies the *Hörmander condition*. This states that $\mathrm{Lie}_p(\mathcal{D}) = \mathrm{T}_p(M)$ for all $p \in M$ where $\mathrm{Lie}_p(\mathcal{D})$ denotes the smallest vector subspace of $\Gamma(M)$ containing $\mathcal{D}$ and satisfying

$$[X, Y] \in \mathrm{Lie}_p(\mathcal{D}) \text{ whenever } X \in \mathcal{D} \text{ and } Y \in \mathrm{Lie}_p(\mathcal{D}).$$

We also say that $\mathcal{D}$ is *bracket-generating* in this case. This definition is motivated by the following well-known result.

Theorem 3.1.5 (Chow–Rashevskii theorem)**.** *Let M be a sub-Riemannian manifold such that its distribution $\mathcal{D}$ is $\mathcal{C}^\infty$ and satisfies the Hörmander condition. Then, (M, d) is a metric space and the manifold and metric topology of M coincide.*

Unless explicitly stated (e.g. when studying the α-Grushin plane in Chapter 4), a sub-Riemannian manifold will be assumed to be bracket-generating. Horizontal curves can then be characterised with the Carnot–Carathéodory distance.

Proposition 3.1.6. *A curve $\gamma : [0, T] \to M$ is horizontal if and only if it is Lipschitz with respect to* d_{CC}.

The *horizontal distribution* of a sub-Riemannian manifold M is defined by $\mathrm{H}(M) := \bigsqcup_{p \in M} \mathcal{D}_p$. If the sub-Riemannian manifold has constant rank, then $\mathrm{H}(M)$ is a subbundle of $\mathrm{T}(M)$.

3.2 Free sub-Riemannian manifolds and first examples

We now introduce the notion of isometry in sub-Riemannian geometry.

Definition 3.2.1. Let $(M_1, E_1, \langle \cdot, \cdot \rangle_{E_1}, f_{E_1})$ and $(M_2, E_2, \langle \cdot, \cdot \rangle_{E_2}, f_{E_2})$ be two sub-Riemannian manifolds. We say that they are isometric if there exists a diffeomorphism $F : M_1 \to M_2$ and an isomorphism $\overline{F} : E_1 \to E_2$ such that $f_{E_2} \circ \overline{F} = f_{E_1} \circ \mathrm{d}F$.

When equipping a manifold with different distributions, we could also end up with equivalent sub-Riemannian structures.

Definition 3.2.2. Let $(E_1, \langle \cdot, \cdot \rangle_{E_1}, f_{E_1})$ and $(E_2, \langle \cdot, \cdot \rangle_{E_2}, f_{E_2})$ be two sub-Riemannian structures on a same manifold M. They are said to be equivalent if:

(i) There exists a Euclidean vector bundle $(E, \langle \cdot, \cdot \rangle_E)$ and surjective vector bundle morphisms $p_1 : E \to E_1$ and $p_2 : E \to E_2$ such that $f_{E_1} \circ p_1 = f_{E_2} \circ p_2$.

(ii) For all $u_1 \in E_1$ and all $u_2 \in E_2$, we have

$$|u_1|_{E_1} = \min \{|u|_E \mid p_1(u) = u_1\} \text{ and } |u_1|_{E_2} = \min \{|u|_E \mid p_2(u) = u_2\}.$$

Given m global vector fields $X_1, \ldots, X_m : M \to \mathrm{T}(M)$ on a manifold M, we can build on M a sub-Riemannian structure in the following way: Set $E = M \times \mathbb{R}^m$ the trivial bundle of rank m, $f_E : E \to \mathrm{T}(M) : (p, (u_1, \ldots, u_m)) \mapsto \sum_{k=1}^m u_k X_k(p)$ and finally consider the Euclidean metric on E. In this way, we induce an inner product on $\mathcal{D}_p = \mathrm{span}\{X_1(p), ..., X_m(p)\}$ by the polarisation formula applied to the norm

$$\|u\|^2_{\mathcal{D}_p} := \min \left\{ \sum_{k=1}^m u_i^2 \;\middle|\; \sum_{k=1}^m u_i X_k(p) = u \right\}. \tag{3.2.1}$$

The family $(X_1, \ldots, X_m)$ is said to be a *generating family* of the sub-Riemannian manifold. A *free* sub-Riemannian structure is one that is induced from a generating family. Every sub-Riemannian structure is *equivalent* to a free one (see [ABB20, Section 3.1.4]). From now on, we will therefore assume, without loss of generality, that every sub-Riemannian manifold is free.

Example 3.2.3. The Heisenberg group $\mathbb{H}$ is the sub-Riemannian structure induced on $\mathbb{R}^3$ by the global vector fields

$$X_1 = \partial_x - \frac{y}{2}\partial_\tau \text{ and } X_2 = \partial_y + \frac{x}{2}\partial_\tau$$

where (x, y, τ) denotes the usual global chart of $\mathbb{R}^3$. This structure will be studied in Section 5.5.

The Grushin plane $\mathbb{G}$ is a sub-Riemannian manifold on $\mathbb{R}^2$ generated by

$$X = \partial_x \text{ and } Y = x\partial_y.$$

Chapter 4 is devoted to a generalisation of the Grushin plane.

3.3 End-point map and length minimisers

Consider a sub-Riemannian manifold M for which the family $(X_1, \dots, X_m)$ is generating. From an optimal control point of view, a curve $\gamma : [0, T] \to M$ with initial value $\gamma(0) = p \in M$ is horizontal if there exists $u \in \mathrm{L}^2([0, T], \mathbb{R}^m)$, called a *control*, such that $\dot{\gamma}(t) = \sum_{k=0}^m u_k(t) X_k(\gamma(t))$.

In fact, from Carathéodory's theorem for ordinary differential equations, we know that there exists a unique maximal Lipschitz solution to the Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = \sum_{k=0}^m u_k(t) X_k(\gamma(t)) \\ \gamma(t_0) = p \end{cases} \tag{3.3.1}$$

for every $u \in \mathrm{L}^2([0, T], \mathbb{R}^m)$, $p \in M$ and $t_0 \in [0, T]$. We denote such a solution by $\gamma_{t_0,p,u}$ and we have a well-defined family of diffeomorphisms $P_{t_0,t,u}(q) := \gamma_{t_0,q,u}(t)$ that satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} P_{t_0,t,u}(q) = \sum_{k=0}^m u_k(t) X_k(P_{t_0,t,u}(q)) \\ P_{t,t,u} = \mathrm{Id} \\ P_{t_2,t_3,u} \circ P_{t_1,t_2,u} = P_{t_1,t_3,u} \\ (P_{t_1,t_2,u})^{-1} = P_{t_2,t_1,u} \end{cases}$$

whenever these objects are well-defined.

We can now introduce the end-point map.

Definition 3.3.1. Let $p \in M$ and $T > 0$. The *end-point map at time* $T > 0$ of the system (3.3.1) is the smooth map

$$\mathrm{E}_{p,T} : \mathcal{U} \to M : u \mapsto \gamma_{0,p,u}(T)$$

where $\mathcal{U} \subseteq \mathrm{L}^2([0,T], \mathbb{R}^m)$ is the open subset of controls such that $\gamma_{0,p,u}$, the solution to the Cauchy problem (3.3.1), is defined on the whole interval $[0,T]$.

The end-point map is smooth and its Fréchet differential can be computed (see [ABB20, Section 8.1]) as follows: for all $v \in \mathrm{L}^2([0,T], \mathbb{R}^m)$,

$$D_u \mathrm{E}_{p,T}(v) = \int_0^T \sum_{k=1}^m v_k(t) (P_{t,T,u})_* X_k(\gamma_{0,p,u}(t))\, \mathrm{d}t.$$

The concept of length minimisers is central to metric geometry. In sub-Riemannian geometry, it is natural to consider minimisers of the length functional that induces the Carnot–Carathéodory distance.

Definition 3.3.2. A horizontal path $\gamma : [0,T] \to M$ is a length minimiser if

$$\mathrm{L}(\gamma) = \mathrm{d}_{\mathrm{CC}}(\gamma(0), \gamma(T)).$$

Length minimisers are thus horizontal curves whose length minimises the distance between its endpoints. A length minimiser between two points might or might not exist in general. Furthermore, even if one does exist, it may or may not be unique. The existence of a length minimiser can however be guaranteed locally, i.e. in a small neighbourhood around an arbitrary point.

Proposition 3.3.3. *For every $p \in M$, there exists $\epsilon > 0$ such that for every $q \in B(p, \epsilon)$, there exists a horizontal path $\gamma : [0,T] \to M$ such that $\gamma(0) = p$, $\gamma(T) = q$ and γ is a length minimiser.*

In view of the Cauchy problem (3.3.1), it is clear that finding a length minimiser for L among the horizontal curves with fixed end-points $\gamma(0) = p$ and $\gamma(T) = q$ is equivalent to finding a minimal control for L for which the associated path joins p and q. Furthermore, we have the following classical correspondence (from the Cauchy–Schwarz inequality): a horizontal curve $\gamma : [0, T] \to M$ joining p to q is a minimiser of E if and only if it is a minimiser of L and is parametrised by constant speed. A necessary condition for horizontal paths to be length minimisers is given in the next result.

Theorem 3.3.4 (Pontryagin extremals)**.** *If $\gamma : [0, T] \to M$ is a horizontal curve that is length minimising and parametrised by constant speed and $\overline{u} : [0, T] \to \mathbb{R}^m$ is the minimal control corresponding to γ, i.e. $|\overline{u}|$ is constant almost everywhere, and*

$$\dot{\gamma}(t) = \sum_{k=1}^{m} \overline{u}_k(t) X_k(\gamma(t)), \quad \mathrm{L}(\gamma) = \int_0^T |\overline{u}(s)| \mathrm{d}s = \mathrm{d}_{\mathrm{CC}}(\gamma(0), \gamma(T)).$$

Then, there exists a covector $\lambda_0 \in \mathrm{T}^*_{\gamma(0)}(M)$ such that the curve

$$\lambda : [0, T] \to \mathrm{T}^*(M) : t \mapsto (P^{-1}_{0,t,\overline{u}})^*[\lambda_0]$$

satisfies one and only one of the following

(P-N) $\overline{u}_k(t) = \langle \lambda(t), X_k(\gamma(t)) \rangle$ *for all* $k \in [\![1, m]\!]$ *and every* $t \in [0, T]$*;*

(P-A) $\lambda_0 \neq 0$ *and* $\langle \lambda(t), X_k(\gamma(t)) \rangle = 0$ *for all* $k \in [\![1, m]\!]$ *and every* $t \in [0, T]$.

A curve $\lambda : [0, T] \to \mathrm{T}^*(M)$ satisfying (P-N) or (P-A) is sometimes called a Pontryagin extremal corresponding to the length minimiser γ. If λ satisfies (P-N) (resp. (P-A)), we say that λ is *normal* (resp. *abnormal*) and γ is then normal (resp. abnormal). There could be different Pontryagin extremals associated with the same length minimiser. In other words, although a Pontryagin extremal is either normal or abnormal, a length minimiser can be both normal and abnormal at the same time.

In terms of the end-point map, the problem of finding the minimisers joining two fixed points $p, q \in M$ is also equivalent to solving the constrained variational problem

$$\min \left\{ \mathrm{J}(u) \;\middle|\; u \in \mathrm{E}^{-1}_{p,T}(q) \right\}. \tag{3.3.2}$$

The Lagrange multipliers rule provides an alternative necessary condition to be satisfied by a control u which is a constrained critical point for (3.3.2).

Proposition 3.3.5 (Lagrange multipliers). *Let $u \in \mathcal{U}$ be an optimal control for the variation problem* (3.3.2). *Then at least one of the following statements holds true:*

(L-N) there exists $\lambda(T) \in \mathrm{T}_q^*(M)$ such that $\lambda(T) \circ D_u \mathrm{E}_{p,T} = \mathrm{d}_u \mathrm{J}$;

(L-A) there exists $\lambda(T) \in \mathrm{T}_q^*(M) \setminus \{0\}$ such that $\lambda(T) \circ D_u \mathrm{E}_{p,T} = 0$.

An optimal control is called *normal* (resp. *abnormal*) when it satisfies the condition (L-N) (resp. (L-A)). This terminology is consistent with the vocabulary of Pontryagin extremals: a control u and a Pontryagin extremal λ are normal (resp. abnormal), in the sense of Proposition 3.3.5 and Theorem 3.3.4 respectively, if and only if $q = \mathrm{E}_{p,T}(u)$ and (L-N) (resp. (L-A)) is satisfied with $\lambda(t) = (P^{-1}_{t,T,\overline{u}})^*[\lambda_0]$.

A normal trajectory $\gamma : [0, T] \to M$ is called *strictly normal* if it is not abnormal. If, in addition, the restriction $\gamma|_{[0,s]}$ is strictly normal for every $s > 0$, we say that γ is *strongly normal*. It can be seen that γ is strongly normal if and only if the normal geodesic γ does not contain any abnormal segment.

3.4 Characterisation of sub-Riemannian geodesics

Now that we can turn a sub-Riemannian manifold into a metric space, we move on to studying the geodesics associated with its distance function. These are horizontal curves, parametrised by constant speed, that are locally minimising the sub-Riemannian length functional. Because of the lack of a torsion-free metric connection, we can not have a geodesic equation through a covariant derivative. Rather, sub-Riemannian geodesics are characterised via Hamilton's equations.

We recall that the Hamiltonian vector field of a map $a \in C^\infty(\mathrm{T}^*(M))$ is the unique vector field $\overrightarrow{a} \in \Gamma^\infty(\mathrm{T}^*(M))$ that satisfies

$$\sigma(\cdot, \overrightarrow{a}(\lambda)) = \mathrm{d}_\lambda a, \qquad \forall \lambda \in \mathrm{T}^*(M),$$

where σ denotes the canonical symplectic form on the cotangent bundle $\mathrm{T}^*(M)$.

The smooth *control-dependent Hamiltonian* of a sub-Riemannian structure is the map $h : \mathbb{R}^m \times \mathrm{T}^*(M) \to \mathbb{R}$ defined as

$$h_u(\lambda) = \sum_{k=1}^{m} u_k \langle \lambda, X_k(\pi(\lambda)) \rangle - \frac{1}{2} \sum_{k=1}^{m} u_k^2.$$

It is easy to see that, by strict convexity, there exists a unique maximum $\overline{u}(\lambda)$ of $u \mapsto h_u(\lambda)$ for every $\lambda \in \mathrm{T}^*(M)$. Therefore, a *maximised Hamiltonian*, or simply *Hamiltonian*, is well-defined:

$$H : \mathrm{T}^*(M) \to \mathbb{R} : \lambda \mapsto H(\lambda) := \max_{u \in \mathbb{R}^m} h_u(\lambda).$$

Furthermore, the Hamiltonian H can be written in terms of the generating family of the sub-Riemannian structure $(X_1, \ldots, X_m)$, as follows

$$H(\lambda) = \frac{1}{2} \sum_{k=1}^{m} \langle \lambda, X_k(\pi(\lambda)) \rangle^2, \qquad \forall \lambda \in T^*(M).$$

For $k = 1, \ldots, m$ and $(p, \lambda_0) \in \mathrm{T}^*(M)$, we also write $h_k(p, \lambda_0) := \langle \lambda_0, X_k(p) \rangle$.

The Lagrange multiplier rule can be further developed to characterise normal extremals as curves that satisfy Hamilton's differential equation. Alternatively, the following result can also be seen as an application of Pontryagin's maximum principle to the sub-Riemannian length minimisation problem.

Theorem 3.4.1. *Let $\gamma : [0, T] \to M$ be a horizontal curve which is a length minimiser, parametrised by constant speed and let $\overline{u} : [0, T] \to \mathbb{R}^m$ be the corresponding minimal control. Then, there exists a Lipschitz curve $(\gamma(t), \lambda(t)) \in \mathrm{T}^*(M)$ such that*

$$\dot{\lambda}(t) = \sum_{k=1}^{m} \overline{u}_k(t) \overrightarrow{h_k}(\gamma(t), \lambda(t)) \textit{ for almost every } t \in [0, T],$$

such that one of the following is satisfied

(N) $h_k(\gamma(t), \lambda(t)) = \overline{u}_k(t)$;

(A) $h_k(\gamma(t), \lambda(t)) = 0$.

Moreover, in the case (A), we have $\lambda(t) \neq 0$ for every $t \in [0, T]$

The last theorem can be rewritten in the following more concise form, which does not refer to the minimal control $\overline{u}$ of γ.

Theorem 3.4.2 (Pontryagin's maximum principle). *Let $\gamma : [0, T] \to M$ be a horizontal curve which is a length minimiser and parametrised by constant speed. Then, there exists a Lipschitz curve $\lambda(t) \in \mathrm{T}^*_{\gamma(t)}(M)$ such that one and only one of the following is satisfied:*

(N) $\dot{\lambda} = \overrightarrow{H}(\lambda)$;

(A) $\sigma_{\lambda(t)}(\dot{\lambda}(t), \cap_{k=1}^{n} \ker(\mathrm{d}_{\lambda(t)} h_k)) = 0$ *for all* $t \in [0, T]$.

Moreover, in the case (A), we have $\lambda(t) \neq 0$ for every $t \in [0, T]$.

If λ satisfies (N) (resp. (A)), we will also say that λ is a normal extremal (resp. abnormal extremal). Again, this is consistent with Proposition 3.3.5 and Theorem 3.4.1. The projection of a normal extremal onto M is locally minimising, that is to say, it is a normal geodesic parametrised by constant-speed. However, the projection of an abnormal extremal onto M may not be locally minimising.

The study of abnormal geodesics is an area of intensive research. There are some cases where a sub-Riemannian structure does not have any non-trivial abnormal geodesic (the trivial geodesic is always abnormal as soon as $\mathrm{rank}(p) < \dim(M)$). In this case, a sub-Riemannian manifold is said to be ideal.

If γ is a normal geodesic associated with a normal extremal λ, then (N) is nothing but Hamilton's equation. In the natural coordinates of the cotangent bundle, (N) becomes

$$\begin{cases} \partial x_i &= \dfrac{\partial H}{\partial p_i} \\ \partial p_i &= -\dfrac{\partial H}{\partial x_i}. \end{cases} \tag{3.4.1}$$

The theory of ordinary differential equations provides the existence of a maximal solution to (3.4.2) for any given initial condition $\lambda(0) = (p, \lambda_0) \in \mathrm{T}^*(M)$. The flow of Hamilton's equation is denoted by $\mathrm{e}^{t\overrightarrow{H}}$.

3.5 Sub-Riemannian exponential map

We finally turn our attention to a central object of this thesis: the sub-Riemannian *exponential map*.

Definition 3.5.1. The sub-Riemannian *exponential map* at $p \in M$ is the map

$$\exp_p : \mathscr{A}_p \to M : \lambda \mapsto \pi(\mathrm{e}^{\vec{H}}(\lambda))$$

where $\mathscr{A}_p \subseteq \mathrm{T}_p^*(M)$ is the open set of covectors such that the corresponding solution of (3.4.2) is defined on the whole interval $[0,1]$.

The sub-Riemannian exponential map $\exp_p$ is smooth. If $\lambda : [0,T] \to \mathrm{T}^*(M)$ is the normal extremal that satisfies the initial condition $\lambda(0) = (p, \lambda_0) \in \mathrm{T}^*(M)$, then the corresponding normal extremal path $\gamma(t) = \pi(\lambda(t))$ satisfies $\gamma(t) = \exp_p(t\lambda_0)$ for all $t \in [0,T]$. If M is complete for the Carathéodory distance, then $\mathscr{A}_p = \mathrm{T}_p^*(M)$, and if in addition there are no strictly abnormal length minimisers, the exponential map $\exp_p$ is surjective. Contrary to the Riemannian case, the sub-Riemannian exponential map is not necessarily a diffeomorphism of a small ball in $\mathrm{T}_p^*(M)$ onto a small geodesic ball in M. In fact, $\mathrm{Im}(\mathrm{d}_0\exp_p) = \mathcal{D}_p$ and $\exp_p$ is a local diffeomorphism at 0 if and only if $\mathcal{D}_p = \mathrm{T}_p(M)$.

A relatively different version of Gauss' lemma is also available in sub-Riemannian geometry.

Proposition 3.5.2 ([ABB20, Proposition 8.42])**.** *Let $p \in M$, $\lambda_0 \in \mathrm{T}_p^*(M)$ and $(q, \lambda_1) := \mathrm{e}^{\vec{H}}(p, \lambda_0)$. Then, for every $w_0 \in \mathrm{T}_p^*(M)$, we have*

$$\langle \lambda_1, \mathrm{d}_{\lambda_0}\exp_p(w_0)\rangle = 2H(\lambda_0, w_0)$$

where we have identified $\mathrm{T}_p^*(M)$ with $\mathrm{T}_{\lambda_0}(\mathrm{T}_p^*(M))$ when necessary.

Remark 3.5.3. With a slight abuse of notation, we have written H to denote the symmetric bilinear form related to the Hamiltonian:

$$H : \mathrm{T}^*(M) \times \mathrm{T}^*(M) : (\lambda, \nu) \mapsto \frac{1}{2}\sum_{k=1}^{m} \langle \lambda, X_k(\pi(\lambda))\rangle\langle \nu, X_k(\pi(\nu))\rangle.$$

A sub-Riemannian manifold is said to be *geodesically complete* if its exponential map $\exp_p$ is defined on all of $\mathrm{T}_p^*(M)$, for every $p \in M$.

Theorem 3.5.4 (Sub-Riemannian Hopf–Rinow theorem). *For a sub-Riemannian manifold M, the following are equivalent:*

(i) $(M, \mathrm{d}_{\mathrm{CC}})$ *is a complete metric space;*

(ii) *For all* $r > 0$*, the closed ball* $B[p, r]$ *is compact for every* $p \in M$*;*

(iii) *There exists* $r > 0$ *such that the closed ball* $B[p, r]$ *is compact for every* $p \in M$*.*

Moreover, if M is ideal, i.e. it does not contain any non-trivial abnormal minimisers, then any of the previous conditions are equivalent to

(iv) *M is geodesically complete;*

(v) *The Hamiltonian vector field* $\overrightarrow{H}$ *of M is complete on* $\mathrm{T}^*(M)$*.*

The first three equivalences are not specific to sub-Riemannian geometry but are true for length spaces in general, as we have seen in Section 2.2. It comes down to the fact that $(M, \mathrm{d}_{\mathrm{CC}})$ is indeed a locally compact metric space (under the bracket generating assumption). Furthermore, if $p \in M$, then for some $r > 0$, every $q \in B(p, r)$ can be joined to p by a minimising curve.

3.6 Sub-Riemannian optimal transport

We now address the optimal transport problem for ideal sub-Riemannian manifolds. We present a generalisation of Brenier and McCann's theorems proved by Figalli and Rifford [FR10] (see also [Rif14]).

Equipping a sub-Riemannian manifold M with a Radon measure $\mathfrak{m}$ turns the structure $(M, \mathrm{d}, \mathfrak{m})$ into a metric measure space. Usually, the measure $\mathfrak{m}$ is a smooth measure, that is to say $\mathfrak{m}$ is induced from a positive density.

The transport cost of a measurable map $T : M \to M$, relative to a measurable cost function $c : M \times M \to \mathbb{R}$, is given by

$$\mathrm{M}(T) := \int_M c(x, T(x)) \mathrm{d}\mathfrak{m}(x).$$

Given two probability measures μ_0 and μ_1 on M, the Monge transportation problem consists of finding a map that minimises the transport cost:

$$\mathrm{M}(\mu_0, \mu_1) := \min_{T \in \tau(\mu_0, \mu_1)} \int_M c(x, T(x)) \mathrm{d}\mathfrak{m}(x) \tag{3.6.1}$$

where $\tau(\mu_0, \mu_1) := \{T : M \to M \mid T_\sharp \mu_0 = \mu_1\}$. Such a map is called an *optimal transport map*.

The study of the Monge problem in sub-Riemannian geometry was first examined by Ambrosio and Rigot for the Heisenberg group in [AR04] (see also [DR11] for an approach involving the MCP condition). Most recently, Figalli and Rifford [FR10] provided a complete solution to the Monge problem related to a squared-distance cost in a general ideal sub-Riemannian manifold. The original statement of their result only applies to constant-rank sub-Riemannian manifold. However, it can easily be adapted to more general distribution, as noted in [BR19] for example.

Theorem 3.6.1. *Let M be an ideal and complete sub-Riemannian manifold, $\mu_0 \in \mathcal{P}_c^{ac}$ and $\mu_1 \in \mathcal{P}_c$. Then, there exists a unique optimal transport map $T : M \to M$, solution to the minimisation problem* (3.6.1)*, with respect to the cost function*

$$c(x, y) := \frac{1}{2} \mathrm{d}_{CC}^2(x, y).$$

There exists a function $\psi : M \to \mathbb{R}$, a closed set $\mathcal{S}^\psi$ and an open set $\mathcal{M}^\psi = M \setminus \mathcal{S}^\psi$, respectively called the static and the moving set, such that

1. *ψ is locally semi-convex in a neighbourhood of $\mathcal{M}^\psi \cap \operatorname{supp}(\mu_0)$;*
2. *For μ_0-almost every $x \in \mathcal{S}^\psi$, we have $T(x) = x$;*
3. *For μ_0-almost every $x \in \mathcal{M}^\psi$, the value of $T(x) = y$ is characterised by the condition*

$$\psi(x) + c(x, y) \leqslant \psi(z) + c(z, y), \text{ for all } z \in M.$$

Furthermore, for μ_0-almost every $x \in M$, there exists a unique minimising geodesic between x and $T(x)$ and its expression is given by

$$[0,1] \to M : t \mapsto T_t(x) := \begin{cases} \exp_x(t\mathrm{d}_x\psi) & \text{if } x \in \mathcal{M}^\psi \cap \operatorname{supp}(\mu_0) \\ x & \text{if } x \in \mathcal{S}^\psi \cap \operatorname{supp}(\mu_0) \end{cases}.$$

Finally, there exists a unique Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ joining μ_0 to μ_1, given by $\mu_t := (T_t)_\sharp \mu_0$.

We also note that the regularity of the transport map is studied in detail in [FR10] (and [BR19]), as well as a way to relax the compactness assumption on μ_0 and μ_1.

3.7 Distortion coefficients in sub-Riemannian geometry

The $\mathrm{CD}(K,N)$ condition is never satisfied for ideal sub-Riemannian manifolds M such that $\operatorname{rank}(p) < \dim(M)$ at every point $p \in M$. Indeed, Juillet showed in [Jui21] that they do not satisfy the Brunn–Minkowski inequality of Proposition 2.4.2. However, by studying the distortion coefficients in sub-Riemannian geometry, it is possible to prove a modified version of the Brunn–Minkowki inequality.

In order to state the result, we need to review the two notions of cut locus in sub-Riemannian geometry (see [FR10] and [BR19]).

Firstly, the cut time of a maximal geodesic $\gamma : [0,T] \to M$ is defined as

$$t_{\mathrm{cut}}[\gamma] := \sup \left\{ t > 0 \mid \gamma|_{[0,t]} \text{ is a minimising geodesic} \right\},$$

and the optimal cut locus at a point $p \in M$ is

$$\mathrm{Cut}(p) := \{\gamma(t_{\mathrm{cut}}[\gamma]) \mid \gamma \text{ is a maximal geodesic starting at } p\}.$$

In Section 4.3, we will use a method, called the *extended Hadamard technique*, to compute the optimal cut locus of a specific sub-Riemannian structure: the α-Grushin plane.

Secondly, if p, q are points of M such that there exists a unique minimising normal geodesic, which is not abnormal, from p to q, and q is not conjugate to p along that curve, we say that (p, q) is a smooth pair of points. The cut locus of $p \in M$ is given by $\underline{\mathrm{Cut}}(p) := M \setminus \{q \in M \mid (p, q) \text{ is a smooth pair of points}\}$. The global cut locus is then defined as $\underline{\mathrm{Cut}}(M) := \{(p, q) \in M \times M \mid q \in \underline{\mathrm{Cut}}(p)\}$.

If M is an ideal sub-Riemannian manifold, then $\mathrm{Cut}(p) = \underline{\mathrm{Cut}}(p) \setminus \{p\}$. However, if abnormal minimisers are present, the relation between the two definitions of cut locus is more complicated and some related questions are still open (see [BR19, Section 4.2.1.]).

Theorem 3.7.1 ([BR19, Theorem 7.]). *If M is an n-dimensional ideal sub-Riemannian manifold equipped with a smooth measure $\mathfrak{m}$, then for every Borel set A_0 and A_1 of M and every $t \in [0, 1]$, we have*

$$\mathfrak{m}(Z_t(A_0, A_1)) \geqslant \beta_{1-t}(A_1, A_0)^{1/n}\mathfrak{m}(A_0)^{1/n} + \beta_t(A_0, A_1)^{1/n}\mathfrak{m}(A_1)^{1/n},$$

where $\beta_t(A_0, A_1) := \inf\{\beta_t(x, y) \mid (x, y) \in (M \times M) \setminus \underline{\mathrm{Cut}}(M)\}$.

Although the classical $\mathrm{CD}(K, N)$ condition is never satisfied in sub-Riemannian geometry, it is known that sub-Riemannian manifolds do often satisfy a MCP condition, such as the Heisenberg groups (see [Jui09]), generalised H-type groups, Sasakian manifolds (see [BR19, Section 7.]), etc. To relate the MCP condition to a lower bound on the distortion coefficients of an ideal sub-Riemannian manifold, we will keep the following result in mind.

Theorem 3.7.2 ([BR19, Theorem 9.]). *Let M be an ideal sub-Riemannian manifold equipped with a smooth measure μ. When $N \geqslant 1$, the following conditions are equivalent:*

(i) $\beta_t(q_0, q) \geqslant t^N$ *for all* $q_0, q \notin \mathrm{Cut}(M)$ *and* $t \in [0, 1]$;

(ii) *The measure contraction property* $\mathrm{MCP}(0, N)$ *is satisfied, i.e. for all non-empty Borel sets* $B \subseteq M$ *and* $q \in M$ *we have* $\mu(\mathrm{Z}_t(q, B)) \geqslant t^N \mu(B)$.

This observation is a motivation for the study of the α-Grushin plane that will be carried out in the next chapter.

Chapter 4

Distortion coefficients of the α-Grushin plane

Grushin structures first appeared in the work of Grushin on hypoelliptic operators in the seventies, for example, see [Gru70]. The α-Grushin plane, denoted by $\mathbb{G}_\alpha$ in this thesis, consists of equipping the two-dimensional Euclidean space with the sub-Riemannian structure generated by the global vector fields $X = \partial_x$ and $Y_\alpha = |x|^\alpha \partial_y$.

These structures form a class of rank-varying sub-Riemannian manifolds. In this work, we will focus on the case $\alpha \geqslant 1$. The α-Grushin plane has Hausdorff dimension $\alpha + 1$ and is not bracket-generating unless α is an integer. Furthermore, the α-Grushin planes constitute a natural generalisation of the traditional Grushin plane, corresponding to the case $\alpha = 1$. Along with the Heisenberg groups $\mathbb{H}_n$, they are considered as fundamental examples of sub-Riemannian geometry, exhibiting key characteristics of the theory.

Since the work first set out by Juillet in [Jui09] and extended by the same author in [Jui21], it is known that, unlike Riemannian manifolds, no sub-Riemannian manifold satisfies the curvature-dimension conditions introduced by Sturm, Lott and Villani. It has been shown by Barilari and Rizzi in [BR19] that they can however support interpolation inequalities and even a Brunn–Minkowski inequality.

For the Heisenberg group, this was in fact first proved by Balogh, Kristály and Sipos in [BKS18]. Distortion coefficients, which capture some curvature information, play a key role in these results. The present chapter studies the distortion coefficients for the α-Grushin plane.

To achieve this goal, it will be important to study the geodesics of $\mathbb{G}_\alpha$ in depth. Because of the lack of a natural connection in sub-Riemannian geometry, geodesics are obtained with Pontryagin's maximum principle. This is the Hamiltonian point of view: a normal minimising path between two points can be lifted to one on the cotangent bundle that satisfies Hamilton's equation.

The geodesics of the α-Grushin plane were first studied by Li and Chang in [CL12]. They are expressed with a generalisation of trigonometric functions, defined as inverses of some special functions.

Section 4.1 and Section 4.2 are devoted to these topics while in Section 4.3, we use an extended Hadamard technique to find the cut loci of $\mathbb{G}_\alpha$. The notation $\mathrm{Cut}(q_0)$ stands for the set of cut loci of q_0, i.e. the set of points in $\mathbb{G}_\alpha$ where the geodesics starting at q_0 stop being minimising.

Theorem 4.0.1 (Distortion coefficients of the α-Grushin plane)**.** *Let q_0 and q be two points of $\mathbb{G}_\alpha$ such that $q \notin \mathrm{Cut}(q_0)$. For all $t \in [0,1]$, we have*

$$\beta_t(q_0, q) = \frac{\mathrm{J}(t, x_0, u_0, v_0)}{\mathrm{J}(1, x_0, u_0, v_0)},$$

with

$$\mathrm{J}(t, x_0, u_0, v_0) := t\left[u_0 x(t) - (u_0 t + x_0) u(t)\right], \tag{4.0.1}$$

and where $\gamma(t) := (x(t), y(t)) : [0,1] \to \mathbb{G}_\alpha$ denotes the unique constant speed minimising geodesic joining $q_0 = (x_0, y_0)$ to q and $u(t)\mathrm{d}x|_{\gamma(t)} + v(t)\mathrm{d}y|_{\gamma(t)} \in \mathrm{T}^*_{\gamma(t)}(\mathbb{G}_\alpha)$ is the corresponding cotangent lift with initial covector $u_0\mathrm{d}x|_{q_0} + v_0\mathrm{d}y|_{q_0}$.

Because of the analyticity of the geodesic flow, the case $v_0 = 0$ can be seen as taking the limit of $\beta_t(q_0, q)$ as v_0 tends to 0. Geometrically, this means that the points q_0 and q are joined by a straight horizontal line.

Proposition 4.0.2. *Let q_0 and q be two points of $\mathbb{G}_\alpha$ such that $q \notin \mathrm{Cut}(q_0)$. When v_0 tends to* 0*, we have*

$$\beta_t(q_0, q) = t\frac{(u_0 t + x_0)^{2\alpha}(u_0 t + x_0) - x_0^{2\alpha} x_0}{(u_0 + x_0)^{2\alpha}(u_0 + x_0) - x_0^{2\alpha} x_0},$$

for all $t \in [0, 1]$.

Although the CD condition is not suited to this type of spaces (see Section 4.4), the weaker measure contraction property introduced independently by Ohta and Sturm in [Oht07] and [Stu06b] seems more adapted to sub-Riemannian geometry. Indeed, there are numerous examples of sub-Riemannian manifolds that do satisfy a MCP condition, including the Heisenberg group $\mathbb{H}_n$ (see [Jui09]) and the Grushin plane $\mathbb{G}_1$ (see [BR19]).

We therefore investigate the measure contraction property for the α-Grushin plane and we obtain a relevant estimate on the distortion coefficients for singular points, that is to say, those on the y-axis, and for those lying on the same horizontal line. We therefore propose the following conjecture.

Conjecture 4.0.3 (Curvature-dimension of the α-Grushin plane)**.** *For $\alpha \geqslant 1$, the α-Grushin plane satisfies the measure contraction property $\mathrm{MCP}(K, N)$ if and only if $K \leqslant 0$ and*

$$N \geqslant 2\left[\frac{(\alpha+1)m_\alpha + 1}{m_\alpha + 1}\right]$$

with $m_\alpha \in [-3, -2]$ the unique non-zero solution of

$$(m+1)^{2\alpha}(m+1) - ((2\alpha+1)m + 1) = 0.$$

As we have seen in Theorem 3.7.2, the $\mathrm{MCP}(0, N)$ condition is equivalent to a lower bound for the distortion coefficients of the form $\beta_t(q_0, q) \geqslant t^N$.

We will provide evidence in favour of this conjecture in Section 4.6. It will be proven that the lower bound holds for singular points. Furthermore, it seems to be sharp for the points lying on the same horizontal line.

4.1 Generalised trigonometric functions

In this section, we give an account of (p,q)-trigonometry. The generalised sine and cosine functions will be essential in the study of the geometry of the α-Grushin plane, as shown by Li in [CL12]. Generalised trigonometry has a long history. The theory as presented here was pioneered by Edmunds in [EGL12]. For recent developments, we point out the work of Takeuchi [Tak17] and the references therein, as well as [Lok20] for a related approach via convex geometry.

Consider

$$F_{p,q} : [0,1] \to \mathbb{R} : x \mapsto \int_0^x \frac{1}{\sqrt[p]{1-t^q}} \mathrm{d}t.$$

The map $F_{p,q}$ being strictly increasing, we may define its inverse

$$\sin_{p,q} : \left[0, \frac{\pi_{p,q}}{2}\right] \to \mathbb{R} : x \mapsto F_{p,q}^{-1}(x),$$

where the (p,q)-pi constant is defined as

$$\pi_{p,q} := 2\int_0^1 \frac{1}{\sqrt[p]{1-t^q}} \mathrm{d}t = \mathrm{B}\left(\frac{1}{p}, 1-\frac{1}{q}\right).$$

Here the function $\mathrm{B}(\cdot,\cdot)$ stands for the complete beta function.

We will extend the (p,q)-sine function to the whole real line. We first note that $\sin_{p,q}(0) = 0$ and $\sin_{p,q}(\pi_{p,q}/2) = 1$. For $x \in [\pi_{p,q}/2, \pi_{p,q}]$, we set $\sin_{p,q}(x) := \sin_{p,q}(\pi_{p,q} - x)$. The (p,q)-sine is then extended to $[-\pi_{p,q}, \pi_{p,q}]$ by requiring that it is odd and finally to $\mathbb{R}$ by $2\pi_{p,q}$-periodicity. We then define the (p,q)-cosine by setting $\cos_{p,q} := (\sin_{p,q})'$. These two functions are of class $\mathcal{C}^1$. In fact, they are also of class $\mathcal{C}^\infty$ except at the points $x = k\pi_{p,q}$ for $k \in \mathbb{Z}$.

We have the following identities:

$$\begin{cases} |\sin_{p,q}|^q + |\cos_{p,q}|^p = 1, \\ (\sin_{p,q})'' = (\cos_{p,q})' = \dfrac{-q}{p} |\cos_{p,q}|^{2-p} |\sin_{p,q}|^{q-2} \sin_{p,q}. \end{cases} \tag{4.1.1}$$

Therefore, the (p,q)-sine function can be alternatively defined as the solution to the following ordinary differential equation

$$-\left(|f'|^{p-2}f'\right)' = \frac{(p-1)q}{p}|f|^{q-2}f, \quad f(0)=0, \quad f'(0)=1. \tag{4.1.2}$$

As for the usual sine and cosine functions, we have $\sin_{p,q}(x+\pi_{p,q}) = -\sin_{p,q}(x)$ and $\cos_{p,q}(x+\pi_{p,q}) = -\cos_{p,q}(x)$. However, unlike the case of classical trigonometric functions, general addition formulas are not known for $\sin_{p,q}(x+y)$ and $\cos_{p,q}(x+y)$ (except for very specific values of p and q). This problem ultimately comes down to finding a function $F_{p,q}$ that solves the integral equation

$$\int_0^{F_{p,q}(x,y)} \frac{1}{\sqrt[p]{1-t^q}}\mathrm{d}t = \int_0^x \frac{1}{\sqrt[p]{1-t^q}}\mathrm{d}t + \int_0^y \frac{1}{\sqrt[p]{1-t^q}}\mathrm{d}t.$$

We would then have $\sin_{p,q}(x+y) = F_{p,q}(\sin_{p,q}(x), \sin_{p,q}(y))$. This is a very difficult problem, even for integer values of p and q. For $(p,q) = (2,2)$, the classical addition formula for the sine functions emerges. When $(p,q) = (2,4)$, the corresponding addition formula is the one used for the lemniscate function that Euler investigated in [Eul61]: let $\mathrm{sl}(x) := \sin_{2,4}(x)$ (resp. $\mathrm{sl}'(x) := \cos_{2,4}(x)$) stand for the sinlem function (resp. the sinlem′ function), then we have

$$\mathrm{sl}(x+y) = \frac{\mathrm{sl}(x)\mathrm{sl}'(y) + \mathrm{sl}(y)\mathrm{sl}'(x)}{1+\mathrm{sl}^2(x)\mathrm{sl}^2(y)},$$

with an analogous formula for $\mathrm{sl}'(x+y)$. Note that Euler's coslem function is defined as $\mathrm{cl}(\mathrm{x}) := \mathrm{sl}(x+\pi_{(2,4)}/2)$, which is different from our $(2,4)$-cosine function.

We mention a useful expansion of the (p,q)-sine and cosine functions.

Theorem 4.1.1. *Let $p,q \in (1,+\infty)$. For every $x \in \left(-\frac{\pi_{p,q}}{2}, \frac{\pi_{p,q}}{2}\right)$, we have*

$$\sin_{p,q}(x) = \sum_{k=0}^{+\infty} a_k |x|^{kq} x, \text{ and } \cos_{p,q}(x) = \sum_{k=0}^{+\infty} a_k (kq+1)|x|^{kq}, \tag{4.1.3}$$

where the first values of a_k are given by

$$a_0 = 1, \; a_1 = -\frac{1}{p(q+1)}, \text{and } a_2 = \frac{1-p+3q-pq}{2p^2(q+1)(2q+1)}.$$

The proof of this theorem essentially follows from [PU03, Theorem 3.2.] (see also [MT21, Lemma 3.1.]). It seems to us that the question of convergence for $x = \pm\frac{\pi_{p,q}}{2}$ still remains open.

4.2 Geodesics of the α-Grushin plane

For $\alpha \in [1, +\infty)$, the α-Grushin plane $\mathbb{G}_\alpha$ is defined as the sub-Riemannian structure on $\mathbb{R}^2$ generated by the global vector fields $X = \partial_x$ and $Y_\alpha = |x|^\alpha \partial_y$, as explained in Section 3.2. This generating family of vector fields are $\mathcal{C}^{\lfloor \alpha \rfloor}$ if α is not an integer and $\mathcal{C}^\infty$ otherwise. We always write $(\cdot)^{2\alpha}$ in place of $((\cdot)^2)^\alpha$.

The horizontal space at $p \in \mathbb{G}_\alpha$ is $\mathcal{D}_p(\mathbb{G}_\alpha) = \operatorname{span}\{X(p), Y_\alpha(p)\}$ and the horizontal distribution is the disjoint union of these $\mathrm{H}(\mathbb{G}_\alpha) = \sqcup_{p \in \mathbb{G}_\alpha} \mathcal{D}_p(\mathbb{G}_\alpha)$. The rank of $\mathcal{D} = \operatorname{span}\{X, Y_\alpha\}$ is not constant: it is a singular distribution if $x = 0$ and Riemannian otherwise. We then consider the scalar metric $\langle \cdot, \cdot \rangle_{\mathcal{D}_p}$ on $\mathcal{D}_p$ as described in (3.2.1). If for example $uX(x,y) + vY_\alpha(x,y) \in \mathcal{D}_{(x,y)}$ and $x \neq 0$, then

$$\langle u, v \rangle_{\mathcal{D}_{(x,y)}} = u^2 + \frac{1}{x^{2\alpha}} v^2.$$

This turns the α-Grushin plane $\mathbb{G}_\alpha$ into a sub-Riemannian manifold. It is easy to see that it does not satisfy the Hörmander condition unless $\alpha \in \mathbb{N} \setminus \{0\}$.

Let I be a non-empty interval of $\mathbb{R}$. As we have seen in the previous chapter, a path $\gamma : I \to \mathbb{G}_\alpha$ is said to be horizontal if, for almost every $t \in I$, the equality $\dot{\gamma}(t) = u(t)X(\gamma(t)) + v(t)Y_\alpha(\gamma(t))$ holds for some L^2-maps $u, v : I \to \mathbb{R}$. In particular, this implies that $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for almost every $t \in I$. We can compute the length of a horizontal curve with the formula $\mathrm{L}_\alpha(\gamma) = \int_I \|\dot{\gamma}(t)\|_{\mathcal{D}_{\gamma(t)}} \mathrm{d}t$. We denote the Carnot–Carathéodory distance associated with L_α by d_α. Equipping the α-Grushin plane with the Lebesgue measure $\mathcal{L}^2$, we obtain a metric measure space $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$.

The theory of sub-Riemannian geometry informs us that the geodesics of the space are found by solving Hamilton's equations. Here, the Hamiltonian is

$$H : \mathrm{T}^*(\mathbb{G}_\alpha) \to \mathbb{R} : (x, y, u\mathrm{d}x|_{(x,y)} + v\mathrm{d}y|_{(x,y)}) \mapsto \frac{1}{2}(u^2 + v^2 x^{2\alpha}).$$

A simple calculation shows that there are no non-trivial abnormal geodesics in the α-Grushin plane.

Lemma 4.2.1. *There are no non-trivial abnormal extremals in $\mathbb{G}_\alpha$.*

Proof. The condition of abnormality in Theorem 3.4.2 states that if $\gamma : [0, T] \to M$ is an abnormal extremal, its cotangent lift $\lambda(t) \in \mathrm{T}^*(\mathbb{G}_\alpha)$ should satisfy $\lambda_0 \neq 0$ and

$$\langle \lambda(t), X(\gamma(t)) \rangle = \langle \lambda(t), Y_\alpha(\gamma(t)) \rangle = 0,$$

for every $t \in [0, T]$. This means that $u(t) = 0$ and $v(t)|x(t)|^\alpha = 0$. Since $\lambda(t) \neq 0$ for all $t \in [0, T]$, we must have $x(t) = 0$. But unless we also have $y(t) = 0$ for every $t \in [0, T]$, such a curve cannot be horizontal. □

Consequently, the sub-Riemannian manifold $\mathbb{G}_\alpha$ is ideal. In this context, Hamilton's equations (3.4.1) becomes

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v x^{2\alpha} \\ \dot{u} = -\alpha v^2 x^{2(\alpha-1)} x \\ \dot{v} = 0 \end{cases}. \tag{4.2.1}$$

We observe that $\ddot{x} = -\alpha v^2 x^{2(\alpha-1)}$. When $v_0 = 1$, this is just the equation (4.1.2) for $(p, q) = (2, 2\alpha)$. The $(2, 2\alpha)$-trigonometric functions will therefore be essential and in what follows, we will denote $\sin_\alpha$ instead of $\sin_{2,2\alpha}$ (and respectively $\cos_\alpha$, π_α) for simplicity.

Theorem 4.2.2. *Let $\gamma : I \to \mathbb{G}_\alpha$ be a horizontal path with initial value $\gamma(0) = (x_0, y_0)$ and $\lambda(t) = u(t)\mathrm{d}x|_{\gamma(t)} + v(t)\mathrm{d}y|_{\gamma(t)}$ be the cotangent lift with initial covector $(u(0), v(0)) = (u_0, v_0)$.*

In the case where $v_0 \neq 0$ and $(x_0, u_0) \neq 0$, the curve γ is a geodesic if and only if

$$\begin{cases} x(t) &= A\sin_\alpha(\omega t+\phi) \\ y(t) &= y_0 + v_0\dfrac{A^{2\alpha}}{(\alpha+1)\omega^2}\Big[\omega^2 t + \omega\cos_\alpha(\phi)\sin_\alpha(\phi) \\ & \qquad\qquad -\omega\cos_\alpha(\omega t+\phi)\sin_\alpha(\omega t+\phi)\Big] \\ u(t) &= A\omega\cos_\alpha(\omega t+\phi) \\ v(t) &= v_0 \end{cases} \tag{4.2.2}$$

for uniquely determined parameters $A, \omega \in \mathbb{R}\setminus\{0\}$ and $\phi \in [0, 2\pi_\alpha)$ satisfying

$$\begin{gathered} A\omega > 0, \quad A^2\omega^2 = u_0^2 + v_0^2 x_0^{2\alpha}, \quad \omega^2 = v_0^2 A^{2(\alpha-1)}, \\ x_0 = A\sin_\alpha(\phi) \text{ and } u_0 = A\omega\cos_\alpha(\phi). \end{gathered} \tag{4.2.3}$$

If $v_0 = 0$ or $(x_0, u_0) = 0$, the geodesic is $(x(t), y(t)) = (u_0 t + x_0, y_0)$ with its lift being constant: $(u(t), v(t)) = (u_0, v_0)$.

Remark 4.2.3. Since the right-hand side of the equation is continuous with respect to the initial condition v_0, the normal extremals corresponding to $v_0 = 0$ can be obtained by letting v_0 tend to 0 in (4.2.2).

Proof. The case when $v_0 = 0$ or $(x_0, u_0) = 0$ is straightforward. We assume that $v_0 \neq 0$ and $(x_0, u_0) \neq 0$.

For $A, \omega \in \mathbb{R}\setminus\{0\}$ such that $Aw > 0$ and $\phi \in [0, 2\pi_\alpha)$, we have

$$\begin{aligned} (A\sin_\alpha(\omega t+\phi))'' &= (A\omega\cos_\alpha(\omega t+\phi))' \\ &= -\alpha A\omega^2 \sin_\alpha(\omega t+\phi)^{2(\alpha-1)}\sin_\alpha(\omega t+\phi) \\ &= -\alpha\frac{\omega^2}{A^{2(\alpha-1)}}(A\sin_\alpha(\omega t+\phi))^{2(\alpha-1)}(A\sin_\alpha(\omega t+\phi)). \end{aligned}$$

By the uniqueness of solutions to the differential equation (4.2.1), we get

$$\begin{cases} x(t) &= A\sin_\alpha(\omega t+\phi), \\ u(t) &= A\omega\cos_\alpha(\omega t+\phi), \end{cases} \tag{4.2.4}$$

where we set $\omega^2 = v_0^2 A^{2(\alpha-1)}$, $x_0 = A\sin_\alpha(\phi)$ and $u_0 = A\omega\cos_\alpha(\phi)$.

Considering the constant of motion $u^2 + v^2 x^{2\alpha}$ at $t = 0$ yields

$$u_0^2 + v_0^2 x_0^{2\alpha} = (A\omega \cos_\alpha(\phi))^2 + \frac{\omega^2}{A^{2(\alpha-1)}}(A \sin_\alpha(\phi))^{2\alpha} = A^2\omega^2.$$

Since $\ddot{x} = -\alpha v_0^2 x^{2(\alpha-1)} x$, we deduce that $x^{2\alpha} = -x\ddot{x}/\alpha v_0^2$ and thus, integrating by part, we have

$$\int_0^t x^{2\alpha} = \int_0^t \frac{-x\ddot{x}}{\alpha v_0^2} = \frac{-1}{\alpha v_0^2}\left([x\dot{x}]_0^t - \int_0^t (\dot{x})^2\right) = \frac{-1}{\alpha v_0^2}\left([xu]_0^t - \int_0^t u^2\right).$$

We use the identity $u^2 = A^2\omega^2 - v^2 x^{2\alpha}$ to find

$$\begin{aligned}\int_0^t x^{2\alpha} &= \frac{-1}{\alpha v_0^2}\left(x(t)u(t) - x(0)u(0) - \int_0^t A^2\omega^2 + \int_0^t v^2 x^{2\alpha}\right) \\ &= \frac{A^2}{\alpha v_0^2}\Big(\omega^2 t + \omega \cos_\alpha(\phi)\sin_\alpha(\phi) \\ &\qquad -\omega\cos_\alpha(\omega t + \phi)\sin_\alpha(\omega t + \phi) - \frac{v_0^2}{A^2}\int_0^t x^{2\alpha}\Big).\end{aligned}$$

Finally, we isolate $\int_0^t x^{2\alpha}$ and integrate $\dot{y} = v_0 x^{2\alpha}$ to get

$$\begin{aligned}y(t) = y_0 + v_0\frac{A^{2\alpha}}{(\alpha+1)\omega^2}\Big(&\omega^2 t + \omega\cos_\alpha(\phi)\sin_\alpha(\phi) \\ &- \omega\cos_\alpha(\omega t + \phi)\sin_\alpha(\omega t + \phi)\Big).\end{aligned}$$

It remains to prove that there is a one-to-one and continuous correspondence between the variables (A, ω, ϕ) and (x_0, u_0, v_0) via (4.2.3). Going from (A, ω, ϕ) to (x_0, u_0, v_0) is clear. The other direction is given by

$$\begin{cases} A = \operatorname{sgn}(v_0)\left(\dfrac{u_0^2 + v_0^2 x_0^{2\alpha}}{v_0^2}\right)^{1/2\alpha}, \\ \omega = v_0\left(\dfrac{u_0^2 + v_0^2 x_0^{2\alpha}}{v_0^2}\right)^{(\alpha-1)/2\alpha}, \\ \sin_\alpha(\phi) = \operatorname{sgn}(v_0)x_0\left(\dfrac{v_0^2}{u_0^2 + v_0^2 x_0^{2\alpha}}\right)^{1/2\alpha}, \\ \cos_\alpha(\phi) = \dfrac{u_0}{(u_0^2 + v_0^2 x_0^{2\alpha})^{1/2}}. \end{cases} \tag{4.2.5}$$

□

By differentiating the relations (4.2.3) with respect to x_0, u_0 and v_0, we find the following useful identities:

$$\begin{aligned} A_{x_0} &= \frac{1-\cos_\alpha^2(\phi)}{\sin_\alpha(\phi)}, \quad A_{u_0} = \frac{\cos_\alpha(\phi)}{\alpha\omega}, \quad A_{v_0} = \frac{-\cos_\alpha^2(\phi)A}{\alpha v_0}; \\ \phi_{x_0} &= \frac{\cos_\alpha(\phi)}{A}, \quad \phi_{u_0} = \frac{-\sin_\alpha(\phi)}{\alpha\omega A}, \quad \phi_{v_0} = \frac{\sin_\alpha(\phi)\cos_\alpha(\phi)}{\alpha v_0}; \\ \omega_{x_0} &= (\alpha-1)\left(\frac{\omega}{A}\right)\left(\frac{1-\cos_\alpha^2(\phi)}{\sin_\alpha(\phi)}\right), \quad \omega_{u_0} = \left(\frac{\alpha-1}{\alpha}\right)\frac{\cos_\alpha(\phi)}{A}, \\ \omega_{v_0} &= \frac{\omega}{v_0}\left(1-\left(\frac{\alpha-1}{\alpha}\right)\cos_\alpha^2(\phi)\right). \end{aligned} \tag{4.2.6}$$

We mention here the work of Li and Chang (see [CL12]). They obtained the expressions of the geodesics joining every two points in the α-Grushin plane by solving the boundary value problem corresponding to the differential equation in Theorem 4.2.2. We note that their results are stated for $\alpha \in \mathbb{N} \setminus \{0\}$. However, if we carefully define sub-Riemannian manifolds of class $\mathcal{C}^k$, we can see that their conclusions remain valid in the case $\alpha \geqslant 1$. In particular, their detailed study of the geodesics was used to derive an expression for the Carnot–Carathéodory distance of $\mathbb{G}_\alpha$ between every two points.

4.3 Cut locus of the α-Grushin plane

When we look at the the geodesics of $\mathbb{G}_\alpha$, we observe three types of behaviours: the straight horizontal lines corresponding to an initial covector with $v_0 = 0$; the geodesics for which $x_0 = 0$ (called *singular* or *Grushin points*); and those for which $x_0 \neq 0$ (called *Riemannian points*). In this section, we investigate the *sub-Riemannian cut loci* and *times* of the α-Grushin plane. The techniques used here were developed in [ABS08, Section 3.2], [Riz18, Appendix A] and [ABB20, Section 13.5].

The case when $v_0 = 0$ is trivial: the corresponding geodesic is a straight horizontal line and is length-minimising for all times. Its cut locus is empty and its cut time is infinite.

We now look at a geodesic γ starting from a singular point $x_0 = 0$.

Since $A^2\omega^2 = u_0^2 + v_0^2 x_0^{2\alpha} = u_0^2 = \kappa^2$, where the positive parameter $\kappa > 0$ is the constant speed of the geodesic γ, we can parametrise u_0, v_0 and the corresponding parameters A and ω with respect to $t \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$:

$$u_0 = \pm\kappa, \;\; v_0 = \beta, \;\; \phi = 0 \text{ or } \pi_\alpha,$$
$$A = \operatorname{sgn}(\beta) \left(\frac{\kappa}{|\beta|}\right)^{1/\alpha} \text{ and } \omega = \beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}}$$

The geodesic starting at $(0, y_0)$ can then be written as follows:

$$\begin{cases} x^{\pm}(t,\beta) &= \pm\operatorname{sgn}(\beta) \left(\frac{\kappa}{|\beta|}\right)^{1/\alpha} \sin_\alpha \left(\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t\right) \\ y(t,\beta) &= y_0 + \frac{1}{(\alpha+1)} \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha+1}{\alpha}} \left[\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t \right. \\ & \left. \quad - \cos_\alpha \left(\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t\right) \sin_\alpha \left(\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t\right)\right] \end{cases} \tag{4.3.1}$$

and in the case $\beta = 0$, the system can be interpreted as $x^{\pm}(t,\beta) = \pm\kappa t$ and $y(t,\beta) = y_0$.

After some computations, we find that the corresponding Jacobian determinant $D(t,\beta)$ of the exponential map $E_{(0,y_0)}(t,\beta) := (x^{\pm}(t,\beta), y(t,\beta))$ is given by

$$\begin{aligned} &\pm\frac{\kappa^3}{\alpha\beta^3} \left[1 - \cos_\alpha^2 \left(\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t\right)\right] \\ &\times \left[\left(\frac{|\beta|}{\kappa}\right)^{\frac{\alpha-1}{\alpha}} \sin_\alpha \left(\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t\right) - \beta t \cos_\alpha \left(\beta \left(\frac{\kappa}{|\beta|}\right)^{\frac{\alpha-1}{\alpha}} t\right)\right]. \end{aligned} \tag{4.3.2}$$

From (4.3.1), we see that the geodesic $(x^+(\cdot,\beta), y(\cdot,\beta))$ is a reflection of $(x^-(t,\beta), y(t,\beta))$ with respect to the y-axis. Furthermore, these two intersect the y-axis for the first time when $t = \pi_\alpha/|\omega|$.

Therefore, a geodesic γ starting at a singular point $(0, y_0)$ must lose its optimality after $t = \pi_\alpha/|\omega|$.

The following lemma guarantees the optimality of γ when $t \leqslant \pi_\alpha/|\omega|$. It is analogous to the case $\alpha = 1$ (see [ABB20, Section 13.5.2]).

Lemma 4.3.1. *A geodesic γ starting at a singular point $(0, y_0) \in \mathbb{G}_\alpha$ is minimising when $t \leqslant \pi_\alpha/|\omega|$.*

Proof. Let $(x_1, y_1) := \gamma(t^*)$ for a fixed $t^* \in [0, \pi_\alpha/|\omega|)$. From [CL12, Theorem 12], we know that there is a finite number of geodesics joining the singular point $(0, y_0)$ to a point (x_1, y_1), only one among them being minimising. We claim that there is a unique $\beta \in \mathbb{R}$ and unique $t \in [0, \pi_\alpha/|\omega|)$ such that $(x^\pm(t, \beta), y(t, \beta)) = (x_1, y_1)$. By the symmetries of the α-Grushin and since $x_1 = 0$ corresponds to γ being a horizontal line, we can assume that $x_1 > 0$ and $y_1 \geqslant y_0$ without loss of generality. In particular, this implies that $\beta > 0$ and the geodesic to consider is $(x^+(\cdot, \beta), y(\cdot, \beta))$. The first equation in (4.3.1) implies that for a solution to exist, we must have $\beta \leqslant \kappa/x_1^\alpha$. When that is the case, there are two solutions:

$$\begin{cases} t_1(\beta) &= \dfrac{1}{\beta}\left(\dfrac{\beta}{\kappa}\right)^{\frac{\alpha-1}{\alpha}} \arcsin_\alpha\left(\dfrac{x_1\beta^{1/\alpha}}{\kappa^{1/\alpha}}\right) \\ t_2(\beta) &= \dfrac{1}{\beta}\left(\dfrac{\beta}{\kappa}\right)^{\frac{\alpha-1}{\alpha}} \left[\pi_\alpha - \arcsin_\alpha\left(\dfrac{x_1\beta^{1/\alpha}}{\kappa^{1/\alpha}}\right)\right]. \end{cases} \tag{4.3.3}$$

The function $t_1(\beta)$ is increasing from x_1/κ as β goes to 0, to $\pi_\alpha/|\omega|$ when $\beta = \kappa/x_1^\alpha$. The function $t_2(\beta)$ is decreasing from $+\infty$ when β tends to 0, to $\pi_\alpha/|\omega|$ when $\beta = \kappa/x_1^\alpha$. We substitute these two into the second equation in (4.3.1) and use the identity $\cos_\alpha^2(x) = 1 - \sin_\alpha^{2\alpha}(x)$. The assumption $y_1 \geqslant y_0$ enables us to choose the positive sign when taking the square root:

$$\begin{cases} y_1(\beta) &= y_0 + \dfrac{(\kappa/\beta)^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)}\left[\arcsin_\alpha\left(\dfrac{x_1\beta^{1/\alpha}}{\kappa^{1/\alpha}}\right) - \sqrt{1 - \dfrac{x_1^{2\alpha}\beta^2}{\kappa^2}}\,\dfrac{x_1\beta^{1/\alpha}}{\kappa^{1/\alpha}}\right] \\ y_2(\beta) &= y_0 + \dfrac{(\kappa/\beta)^{\frac{\alpha+1}{\alpha}}}{(\alpha+1)}\left[\pi_\alpha - \arcsin_\alpha\left(\dfrac{x_1\beta^{1/\alpha}}{\kappa^{1/\alpha}}\right) + \sqrt{1 - \dfrac{x_1^{2\alpha}\beta^2}{\kappa^2}}\,\dfrac{x_1\beta^{1/\alpha}}{\kappa^{1/\alpha}}\right]. \end{cases} \tag{4.3.4}$$

The function $y_1(\beta)$ is increasing (resp. $y_2(\beta)$ is decreasing) and behaves in the following way. When β tends to 0, y_1 goes to y_0 (resp. y_2 goes to $+\infty$) and when $\beta = \kappa/x_1^\alpha$, the function y_1 (resp. y_2) takes the value $y_0 + x_1^{\alpha+1}\pi_\alpha/[2(\alpha+1)]$. Therefore, given $x_1 > 0$ and $y_1 \geqslant y_0$, if $y_1 \leqslant y_0 + x_1^{\alpha+1}\pi_\alpha/[2(\alpha+1)]$ (resp.

$y_1 \geqslant y_0 + x_1^{\alpha+1}\pi_\alpha/[2(\alpha+1)]$), we use (4.3.4) to deduce the existence of a unique $\beta > 0$ such that $y_1(\beta) = y_1$ (resp. $y_2(\beta) = y_1$) and (4.3.3) provides the unique $t = t_1(\beta) \in [0, \pi_\alpha/|\omega|)$ (resp. $t = t_2(\beta)$) such that $(x^+(t,\beta), y(t,\beta)) = (x_1, y_1)$. The geodesic γ is consequently minimising before $t = \pi_\alpha/|\omega|$. □

It remains to study the case of a geodesic γ starting at a Riemannian point (x_0, y_0), i.e. with $x_0 \neq 0$. We will use an extended Hadamard technique, as described in [ABB20, Section 13.4]:

Theorem 4.3.2 (Extended Hadamard technique)**.** *Let M be an ideal sub-Riemannian manifold and $q_0 \in M$ be a Riemannian point (resp. a singular point). Let $\mathrm{Cut}^*(q_0) \subseteq M$ be the conjectured cut locus and $t^*_{q_0}[\lambda_0] \in [0, +\infty]$ be the conjectured cut time at q_0 for an initial covector $\lambda_0 \in \mathrm{T}^*_{q_0}(M) \cap H^{-1}(1/2)$.*

Set N as the set of covectors in $\mathrm{T}^*_{q_0}(M)$ for which the corresponding geodesics are conjectured to be optimal up to time 1.

In other words,

$$N := \{t\theta \mid \lambda_0 \in \mathrm{T}^*_{q_0}(M) \cap H^{-1}(1/2) \text{ and } t \in [0, t^*_{q_0}[\lambda_0]) \ (\text{resp. } t \in (0, t^*_{q_0}[\lambda_0]))\}.$$

Assume that the set N is shown to satisfy the following conditions:

(i) $\exp_{q_0}(N) = M \setminus \mathrm{Cut}^*(q_0)$*;*

(ii) *The restriction of the sub-Riemannian exponential $\exp_{q_0}|_N$ is a proper map, invertible at every point of N;*

(iii) *The set $\exp_{q_0}(N)$ is simply-connected (resp. $\exp_{q_0}|_N$ is a diffeomorphism).*

Then, $\exp_{q_0}|_N$ is a diffeomorphism and the conjectured cut locus and cut times are the right ones: $\mathrm{Cut}(q_0) = \mathrm{Cut}^*(q_0)$ and $t_{q_0} = t^*_{q_0}$.

Remark 4.3.3. The restriction of $\mathrm{T}^*_{q_0}(M)$ to $H^{-1}(1/2)$ results from considering geodesics parametrised by arclength.

We firstly observe that

$$\gamma\left(\frac{\pi_\alpha}{|\omega|}\middle| A, \omega, \phi\right) = \gamma\left(\frac{\pi_\alpha}{|\omega|}\middle| A, \omega, \pi_\alpha - \phi\right).$$

This means that the points

$$\left(-x_0, y_0 + \mathrm{sgn}(\omega)\left(\frac{x_0}{\sin_\alpha(\phi)}\right)^{\alpha+1}\frac{\pi_\alpha}{(\alpha+1)}\right)$$

are joined from (x_0, y_0) by two distinct geodesics unless $\phi = \pi_\alpha/2$ or $3\pi_\alpha/2$ in which case there is only one.

This leads us to conjecture that the cut time should be $t^*_{\mathrm{cut}}(u_0, v_0) = \pi_\alpha/|\omega|$ and that the cut locus should be

$$\mathrm{Cut}^*(x_0, y_0) = \left\{(-x_0, y) \in \mathbb{G}_\alpha \mid |y - y_0| \geqslant |x_0|^{\alpha+1}\frac{\pi_\alpha}{(\alpha+1)}\right\}.$$

Here, the set of covectors in $\mathrm{T}^*_{q_0}(\mathbb{G}_\alpha)$ for which the corresponding geodesics are conjectured to be optimal up to time 1 is

$$\begin{aligned} N &:= \left\{t\lambda_0 \mid \lambda_0 \in \mathrm{T}^*_{(x_0,y_0)}(\mathbb{G}_\alpha) \cap H^{-1}(1/2),\ t \in [0, t^*_{\mathrm{cut}}[\lambda_0])\right\} \\ &= \left\{u_0 \mathrm{d}x|_{(x_0,y_0)} + v_0 \mathrm{d}y|_{(x_0,y_0)} \in \mathrm{T}^*_{(x_0,y_0)}(\mathbb{G}_\alpha) \mid |\omega| < \pi_\alpha\right\}, \end{aligned} \tag{4.3.5}$$

and thus $\exp_{(x_0,y_0)}(N) = \{(x, y) \in \mathbb{G}_\alpha \mid (x, y) \notin \mathrm{Cut}^*(q_0)\}$.

Let us show that the equality in Equation (4.3.5) indeed holds. When considering a covector λ_0 (resp. $\overline{\lambda}_0$), we write A and ω (resp. $\overline{A}$ and $\overline{\omega}$) for the corresponding coordinates given by (4.2.5). If v_0 tends to 0, then ω tends to 0 and $t^*_{\mathrm{cut}}[\lambda_0] = +\infty$ which implies that covectors with $v_0 = 0$ belong to both sets in (4.3.5). We can now assume that $v_0 \neq 0$ (resp. $\overline{v}_0 \neq 0$). If $\overline{\lambda}_0 = t\lambda_0$ is a vector in N for some $t \in [0, t^*_{\mathrm{cut}}[\lambda_0])$, then, with the help of (4.2.5), we find that $|\overline{\omega}| = t|\omega|$ and therefore $|\overline{\omega}| < \pi_\alpha$. On the other hand, if $\overline{\lambda_0}$ is a covector such that $|\overline{\omega}| < \pi_\alpha$, we can express it as $\overline{\lambda}_0 = t\lambda_0$ with $t := \overline{A}\overline{\omega} > 0$ and $\lambda_0 := \overline{\lambda}_0/t$. Using (4.2.5) again, we deduce that $A\omega = 1$ and thus $\lambda_0 \in H^{-1}(1/2)$. Furthermore, the coefficient t satisfies

$$0 \leqslant t = |\overline{A}||\overline{\omega}| = |A||\overline{\omega}| = \frac{|\overline{\omega}|}{|\omega|} < \frac{\pi_\alpha}{|\omega|},$$

since $|\overline{\omega}| < \pi_\alpha$ by hypothesis.

Remark 4.3.4. The set (4.3.5) corresponds to what is called the *(cotangent) injectiv-*

ity domain. If $x_0 = 0$, the cotangent injectivity domain will be as in (4.3.5) but with $H^{-1}(0)$ being removed, since this time $t \in (0, t^*_{q_0}[\lambda_0])$ by Theorem 4.3.2. When $\alpha = 1$, the condition defining N reduces to $|v_0| \leqslant \pi$. Geometrically, this is a horizontal strip in the cotangent space. The shape of the cotangent injectivity domain for $\alpha > 1$ is different than when $\alpha = 1$: see Figure 4.1.

We know that $A^2\omega^2 = u_0^2 + v_0^2 x_0^{2\alpha} = \kappa^2 = 2H(u_0, v_0)$, where the positive parameter $\kappa > 0$ is the constant speed of the geodesic γ. We can then parametrise u_0, v_0 and the corresponding parameters A and ω with respect to $t \in [0, t^*_{\text{cut}}(u_0, v_0)]$ and $\phi \in (0, 2\pi_\alpha) \setminus \{\pi_\alpha\}$:

$$u_0 = \kappa \cos_\alpha(\phi), \quad v_0 = \kappa \frac{\sin_\alpha(\phi)}{x_0} \left| \frac{\sin_\alpha(\phi)}{x_0} \right|^{\alpha-1}, A = \frac{x_0}{\sin_\alpha(\phi)} \text{ and } \omega = \kappa \frac{\sin_\alpha(\phi)}{x_0}.$$

The expression of the geodesics from Theorem 4.2.2 can thus be written as

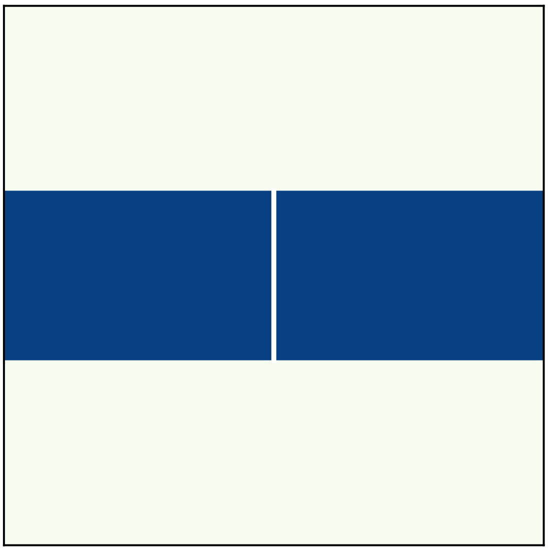
$$\begin{cases} x(t,\phi) &= \dfrac{x_0}{\sin_\alpha(\phi)} \sin_\alpha\left(\kappa \dfrac{\sin_\alpha(\phi)}{x_0} t + \phi\right) \\ y(t,\phi) &= y_0 + \dfrac{1}{(\alpha+1)} \left| \dfrac{x_0}{\sin_\alpha(\phi)} \right|^{\alpha+1} \left[\kappa \dfrac{\sin_\alpha(\phi)}{x_0} t + \cos_\alpha(\phi)\sin_\alpha(\phi) \right. \\ & \qquad \left. - \cos_\alpha\left(\kappa \dfrac{\sin_\alpha(\phi)}{x_0} t + \phi\right) \sin_\alpha\left(\kappa \dfrac{\sin_\alpha(\phi)}{x_0} t + \phi\right)\right]. \end{cases} \tag{4.3.6}$$

In fact, $\phi = 0$ or π_α correspond to the geodesic starting at (x_0, y_0) with initial covector $(\kappa, 0)$ and $(-\kappa, 0)$ respectively. In that case, the geodesics are parametrised by

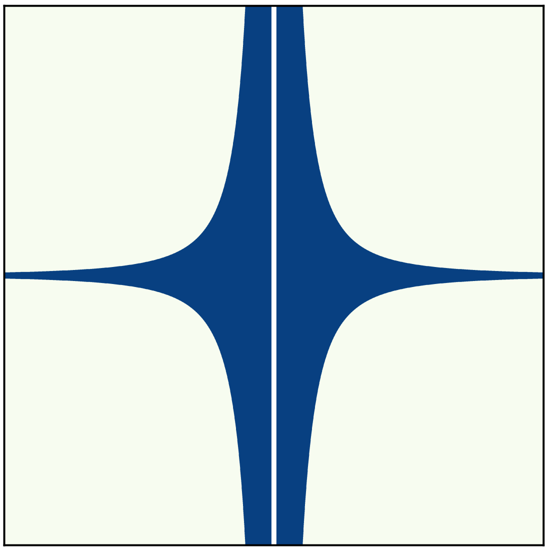
$$\begin{cases} x(t,0) &= \kappa t + x_0 \\ x(t,\pi_\alpha) &= -\kappa t + x_0 \\ y(t,0) &= y_0 \\ y(t,\pi_\alpha) &= y_0 \end{cases}. \tag{4.3.7}$$

Given a constant speed $\kappa > 0$ and an initial point $p := (x_0, y_0)$ with $x_0 \neq 0$, we can compute the determinant of the differential of the corresponding *exponential map* $E_{(x_0,y_0)} : (t,\phi) \mapsto (x(t,\phi), y(t,\phi))$:

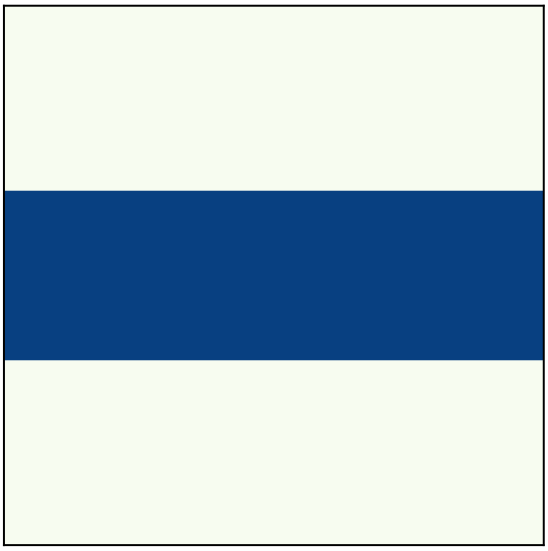
$$\begin{aligned} D(t,\phi) &= \frac{\kappa}{x_0 \sin_\alpha(\phi)} \left| \frac{x_0}{\sin_\alpha(\phi)} \right|^{\alpha+1} \left[x_0 \sin_\alpha\left(\kappa \frac{\sin_\alpha(\phi)}{x_0} t + \phi\right) \cos_\alpha(\phi) \right. \\ & \qquad \left. - \sin_\alpha(\phi)(x_0 + \kappa t \cos_\alpha(\phi)) \cos_\alpha\left(\kappa \frac{\sin_\alpha(\phi)}{x_0} t + \phi\right)\right]. \end{aligned} \tag{4.3.8}$$



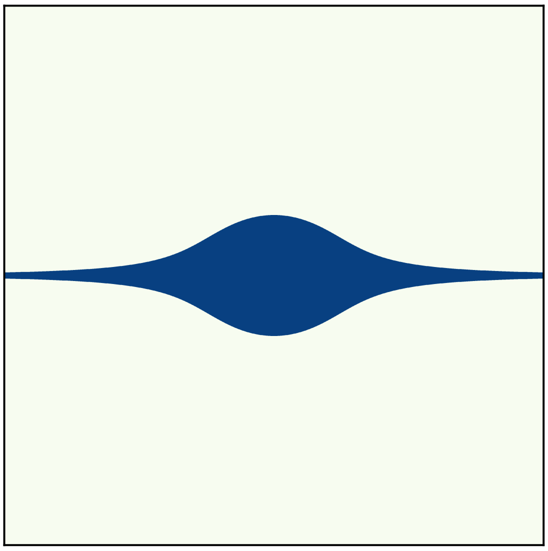
(a) $x_0 = 0, \alpha = 1$



(b) $x_0 = 0, \alpha > 1$



(c) $x_0 \neq 0, \alpha = 1$



(d) $x_0 \neq 0, \alpha > 1$

Figure 4.1: Cotangent injectivity domain for different values of x_0 and α

The cotangent injectivity domain is an open star-shaped region if $x_0 \neq 0$. If $x_0 = 0$, it looks like a star-shaped region but with the starting point and the annihilator of the distribution removed.

One can check that $\lim_{\phi\to 0} D(t,\phi) = \lim_{\phi\to\pi_\alpha} D(t,\phi) = 0$ unless $\alpha = 1$, in which case we have

$$\lim_{\phi\to 0} D(t,\phi) = \frac{\kappa^4 t}{3x_0^5}\left(\kappa^2 t^2 + 3\kappa t x_0 + 3x_0^2\right)$$

and

$$\lim_{\phi\to 0} D(t,\phi) = \frac{\kappa^4 t}{3x_0^5}\left(\kappa^2 t^2 - 3\kappa t x_0 + 3x_0^2\right).$$

We now claim that the exponential map has no singularities before $t = \pi_\alpha/|\omega|$. Indeed, we observe firstly that $D(0,\phi)$ vanishes for every ϕ. Secondly, with the help of the derivative of D with respect to t;

$$\begin{aligned}\partial_t D(t,\phi) &= \alpha\frac{\kappa^2}{x_0^2}(x_0 + \kappa t\cos_\alpha(\phi))\left|\frac{x_0}{\sin_\alpha(\phi)}\right|^{\alpha+1}\sin_\alpha(\phi)\\ &\quad\times \sin_\alpha^{2(\alpha-1)}\left(\kappa\frac{\sin_\alpha(\phi)}{x_0}t+\phi\right)\sin_\alpha\left(\kappa\frac{\sin_\alpha(\phi)}{x_0}t+\phi\right),\end{aligned}$$

we see that $\partial_t D(t,\phi) = 0$ if and only if

$$t = -\frac{x_0}{\kappa\cos_\alpha(\phi)} \text{ or } t = \frac{x_0}{\kappa\sin_\alpha(\phi)}(l\pi_\alpha - \phi),\ l\in\mathbb{Z}.$$

The former is a local minimum that is positive while the later is a local maximum that is also positive. Thirdly, we observe that

$$D(t^*_{\text{cut}},\phi) = \kappa\frac{\pi_\alpha}{\sin_\alpha(\phi)}\left|\frac{x_0}{\sin_\alpha(\phi)}\right|^{\alpha+1}\cos_\alpha^2(\phi),$$

which is zero if and only if $\phi = \pi_\alpha/2$ or $3\pi_\alpha/2$. So, the function D is never zero on $(0, t^*_{\text{cut}})$ and the exponential map is invertible at every point of N.

Finally, we need to make some topological considerations in order to conclude. Consider the set N for which the corresponding geodesics are conjectured to be optimal up to time 1 and its image M under the sub-Riemannian exponential map at (x_0, y_0). The map $\exp_{(x_0,y_0)} : N \to \exp(N)$ is proper: if a sequence of points $(u_i, v_i) \in N$ escapes to infinity, we must have $u_i \to \pm\infty$ and therefore $\exp_{(x_0,y_0)}(u_i, v_i)$ will also escape to infinity. Therefore, $\exp|_N$ is indeed proper, its differential is not singular at any point and furthermore $\exp(N)$ is simply connected. We can conclude that exp is a diffeomorphism and the extended

Hadamard technique (Theorem 4.3.2) implies that the conjectured cut loci and time are thus the true ones.

To summarise the findings of this section, we have proved the following result:

Theorem 4.3.5. *Let $\alpha \geqslant 1$ and $\gamma(t) = (x(t), y(t))$ be a geodesic of $\mathbb{G}_\alpha$ with initial value $\gamma(0) = (x_0, y_0)$ and initial covector $u_0 \mathrm{d}x|_{(x_0,y_0)} + v_0 \mathrm{d}y|_{(x_0,y_0)}$, as described in Theorem 4.2.2.*

If $v_0 = 0$, there are no singularities along γ,

$$t_{\text{cut}}[\gamma] = +\infty \text{ and } \mathrm{Cut}(x_0, y_0) = \emptyset.$$

If $v_0 \neq 0$, then the cut time is

$$t_{\text{cut}}[\gamma] = \frac{\pi_\alpha}{|\omega|},$$

while the cut locus is

$$\mathrm{Cut}(x_0, y_0) = \left\{ (-x_0, y) \in \mathbb{G}_\alpha \mid |y - y_0| \geqslant |x_0|^{\alpha+1} \frac{\pi_\alpha}{(\alpha+1)} \right\}.$$

The cut loci and geodesics of $\mathbb{G}_\alpha$ are illustrated in Figure 4.2. With this in mind, we now turn to the analysis of the distortion coefficients of the α-Grushin plane.

4.4 CD condition for the α-Grushin plane

In this section, we prove that the α-Grushin plane does not satisfy the $\mathrm{CD}(K, N)$ condition, for any $K \in \mathbb{R}$ and $N \geqslant 1$. We will use the techniques developed by Juillet in [Jui08], [Jui09], [Jui10] and [Jui21].

Firstly, we note that $\mathrm{CD}(K, N)$ spaces with $K > 0$ are bounded and the α-Grushin plane is not. We will therefore investigate $\mathrm{CD}(0, N)$ and this will be enough.

Indeed, note that the α-Grushin plane can be endowed with the following family of dilations:

$$\delta_\lambda^\alpha : \mathbb{G}_\alpha \to \mathbb{G}_\alpha : (x, y) \mapsto (\lambda x, \lambda^{\alpha+1} y), \text{ for all } \lambda > 0.$$

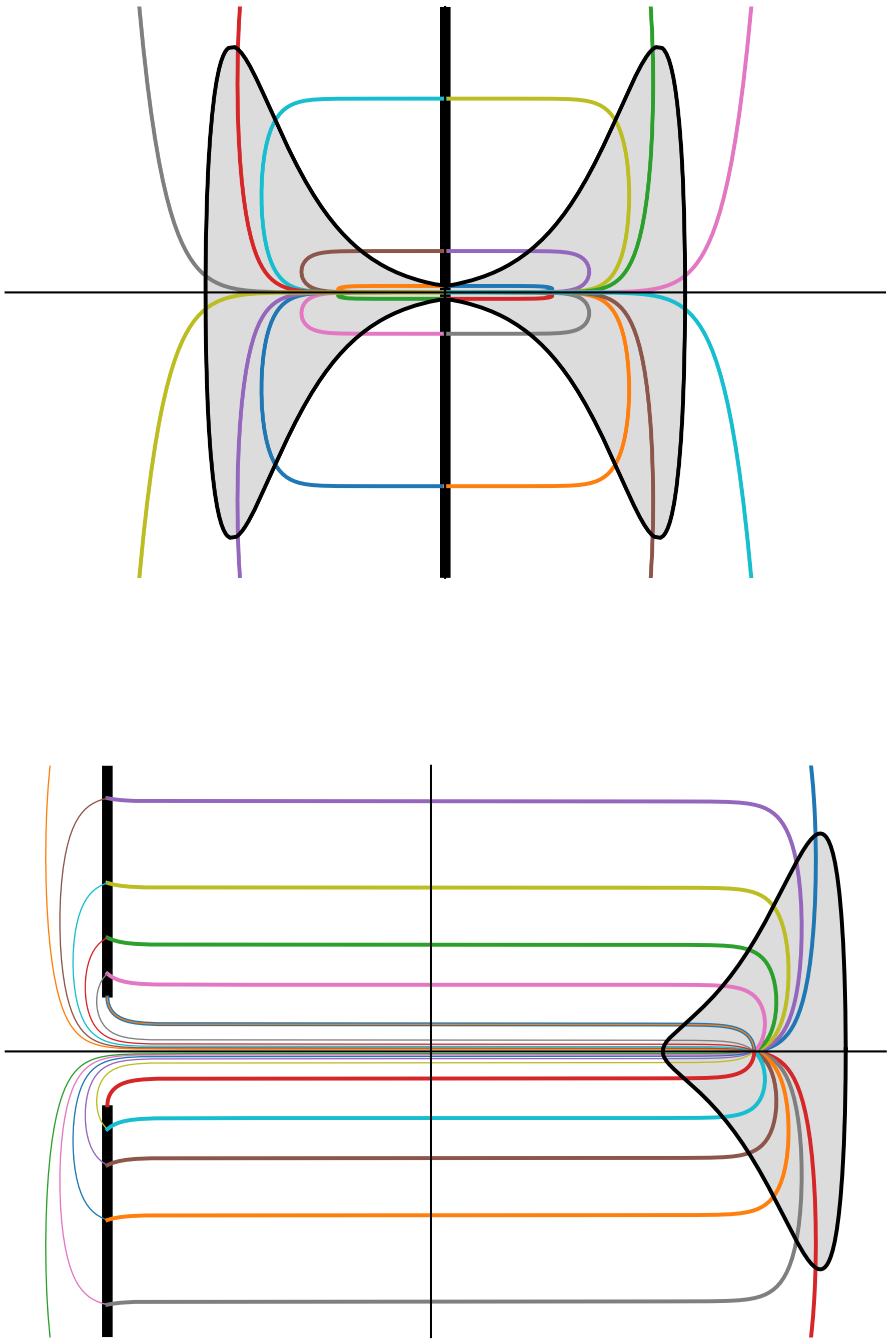


Figure 4.2: Geometry of $\mathbb{G}_\alpha$

Illustration of the geodesics of the α-Grushin plane from a singular point (on the left) and from a Riemannian point (on the right). The shaded area represents a ball around the starting point and the thick line is the cut locus.

The dilations satisfy $\mathrm{d}_\alpha(\delta^\alpha_\lambda[p], \delta^\alpha_\lambda[q]) = \lambda \mathrm{d}_\alpha(p,q)$ for every $p, q \in \mathbb{G}_\alpha$ and $\mathcal{L}^2(\delta^\alpha_\lambda[E]) = \lambda^{\alpha+1}\mathcal{L}^2(E)$ for all measurable sets $E \subseteq \mathbb{G}_\alpha$. If $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$ satisfies $\mathrm{CD}(K,N)$, then the space $(\mathbb{G}_\alpha, \lambda^{-1}\mathrm{d}_\alpha, \lambda^{-(\alpha+1)}\mathcal{L}^2)$ would satisfy $\mathrm{CD}(\lambda^2 K, N)$ (see [Stu06b, Proposition 1.4.]). Moreover $(\mathbb{G}_\alpha, \lambda^{-1}\mathrm{d}_\alpha, \lambda^{-(\alpha+1)}\mathcal{L}^2)$ is isomorphic to $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$ via the dilation δ^α_λ. Therefore, the α-Grushin plane would satisfy $\mathrm{CD}(K,N)$ for every $K < 0$, which implies the $\mathrm{CD}(0,N)$ condition by taking the limit of (2.4.1) as K goes to 0. So, if we prove that $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$ is not $\mathrm{CD}(0,N)$, we would have shown that it can not be $\mathrm{CD}(K,N)$ for any $K \in \mathbb{R}$ and $N \geqslant 1$.

Theorem 4.4.1. *The α-Grushin plane $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$ does not satisfy $\mathrm{CD}(K,N)$ for any $K \in \mathbb{R}$ and $N \geqslant 1$.*

Proof. As we have argued above, it is enough to show that $\mathrm{CD}(0,N)$ does not hold in $(\mathbb{G}_\alpha, \mathrm{d}_\alpha, \mathcal{L}^2)$. Assuming that it does, then so does $\mathrm{BM}(0,N)$, according to Proposition 2.4.2. In particular, the weaker condition $\mathrm{BM}(0,+\infty)$ must be satisfied too. For every $t \in [0,1]$ and all Borel sets $A_0, A_1 \subseteq \mathbb{G}_\alpha$, it should hold

$$\log\left[\frac{1}{\mathfrak{m}(Z_t(A_0,A_1))}\right] \leqslant (1-t)\log\left[\frac{1}{\mathfrak{m}(A_0)}\right] + t\log\left[\frac{1}{\mathfrak{m}(A_1)}\right] - \frac{Kt(1-t)}{2}\sup_{(x,y)\in A_0\times A_1} \mathrm{d}(x,y)^2.$$

For $t = 1/2$, we would have

$$\mathfrak{m}(Z_{1/2}(A_0,A_1)) \geqslant \sqrt{\mathfrak{m}(A_0)\mathfrak{m}(A_1)}. \tag{4.4.1}$$

Let $q_0 = (0,0)$, $q_{1/2} = (1,0)$ and $q_1 = (2,0)$ and consider the map

$$\begin{aligned} \mathrm{I}_{q_{1/2}}\colon \mathbb{G}_\alpha \setminus \mathrm{Cut}(q_{1/2}) &\longrightarrow \mathbb{G}_\alpha \\ (x(t,\phi), y(t,\phi)) &\longmapsto (x(-t,\phi), y(-t,\phi)) = (x(t,\phi+\pi_\alpha), y(t,\phi+\pi_\alpha)), \end{aligned}$$

where $(x(t,\phi), y(t,\phi))$ are given in (4.3.6) and (4.3.7) with $(x_0,y_0) = q_{1/2}$ and $\kappa = 1$. In other words, the map $\mathrm{I}_{q_{1/2}}$ takes a point joined by a unique geodesic through $q_{1/2}$ and sends it to a point the other side of the geodesic so that $q_{1/2}$ is their midpoint.

For a small enough $\epsilon > 0$, consider a small compact ball $A_0^\epsilon := B_\epsilon[q_1]$ with centre q_1 and $A_1^\epsilon := \mathrm{I}_{q_{1/2}}(A_0^\epsilon)$. The Brunn–Minkowski inequality (4.4.1) implies that

$$\frac{\mathfrak{m}(Z_{1/2}(B_\epsilon[q_1], \mathrm{I}_{q_{1/2}}(B_\epsilon[q_1])))}{\mathfrak{m}(B_\epsilon[q_1])} \geqslant \sqrt{\frac{\mathfrak{m}(\mathrm{I}_{q_{1/2}}(B_\epsilon[q_1]))}{\mathfrak{m}(B_\epsilon[q_1])}}. \tag{4.4.2}$$

We are going to perform the limit of (4.4.2) as ϵ tends to 0. The right-hand side of (4.4.2) is evaluated with the Jacobian determinant of $\mathrm{I}_{q_{1/2}}$:

$$\left|\det(D_{q_1}\mathrm{I}_{q_{1/2}})\right| = \lim_{\epsilon\downarrow 0} \frac{\mathfrak{m}(\mathrm{I}_{q_{1/2}}(B_\epsilon[q_1]))}{\mathfrak{m}(B_\epsilon[q_1])} = \frac{1}{2}\lim_{\phi\to 0}\left|\frac{D(-1,\phi)}{D(1,\phi)}\right|,$$

where $D(t,\phi)$ is given by (4.3.8) with $(x_0, y_0) = q_{1/2}$ and $\kappa = 1$.

Thanks to the expansion (4.1.3) that we can use for small enough $\phi > 0$, we find that as ϕ tends to 0, we have

$$D(t,\phi) - \Big[\big((2\alpha+1)(t+1) - k(t,\phi)\big) - 2\alpha\Big] k(t,\phi)^{2\alpha} \frac{\phi^{2\alpha+1}}{2(2\alpha+1)} \in o(\phi^{4\alpha+1}),$$

where we have set $k(t,\phi) := 1 + t\sin_\alpha(\phi)/\phi$. Since $\sin_\alpha(\phi)/\phi$ tends to 1 as ϕ tends to 0, we obtain

$$\lim_{\phi\to 0}\left|\frac{D(-1,\phi)}{D(1,\phi)}\right| = \frac{1}{2^{2\alpha+1}-1}.$$

To estimate the limit of the left hand-side of (4.4.2) when ϵ goes to 0, we use the following result of Juillet, e.g. [Jui10, Theorem 1.]:

$$\limsup_{\epsilon\downarrow 0} \frac{\mathfrak{m}(Z_{1/2}(B_\epsilon[q_1], \mathrm{I}_{q_{1/2}}(B_\epsilon[q_1])))}{\mathfrak{m}(B_\epsilon[q_1])} \leqslant 2^2 \times |\det(D_{q_1}\mathrm{M}_{q_0})|$$

where M_{q_0} is the midpoint map

$$\mathrm{M}_{q_0} : B_\epsilon[q_1] \to \mathbb{G}_\alpha : p \mapsto E_p\left(\frac{1}{2}t_p[q_0], \phi_p[q_0]\right) \text{ with } (t_p[q_0], \phi_p[q_0]) = E_p^{-1}(q_0).$$

Using again the expansion and the Jacobian determinant (4.3.2), we obtain

$$|\det(D_{q_1}\mathrm{M}_{q_0})| = \frac{1}{2} \times 4\lim_{\beta\to 0}\left|\frac{D_{(1/2,\beta)}E_{q_0}}{D_{(1,\beta)}E_{q_0}}\right| = \frac{1}{2^{4\alpha}}.$$

Equation (4.4.2) then implies that $\frac{1}{2^{4\alpha-2}} \geqslant \sqrt{\frac{1}{2^{2\alpha+1}-1}}$, which never happens when $\alpha \geqslant 1$. Therefore, the assumption that the α-Grushin plane satisfies the

$\mathrm{BM}(0,+\infty)$ condition leads to a contradiction. Consequently, the $\mathrm{CD}(K,N)$ condition does not hold for the α-Grushin plane, for any $N \in [1,+\infty]$ and any $K \in \mathbb{R}$. □

With this in mind, we can now turn to the analysis of the distortion coefficients of the α-Grushin plane.

4.5 Distortion coefficients of the α-Grushin plane

We present here our main result: an explicit computation of the distortion coefficient of $\mathbb{G}_\alpha$. To this aim, we use the techniques established by Balogh, Kristály and Sipos in [BKS18] and generalised by Barilari and Rizzi in [BR19]. In the latter, the authors prove interpolation inequalities of optimal transport for ideal sub-Riemannian manifolds. They are expressed in terms of the distortion coefficients for which the expression is obtained through a fine analysis of sub-Riemannian Jacobi fields.

Theorem 4.5.1. *Let $q, q_0 \in \mathbb{G}_\alpha$ such that $q \notin \mathrm{Cut}(q_0)$. Assume that q and q_0 do not lie on the same horizontal line. Under the correspondence of Theorem 4.2.2 and the relations* (4.2.5)*, we have*

$$\beta_t(q,q_0) = \frac{\mathrm{J}(t,A,\omega,\phi)}{\mathrm{J}(1,A,\omega,\phi)} \text{ for all } t \in [0,1],$$

where

$$\mathrm{J}(t,A,\omega,\phi) = t\Big[\sin_\alpha(\omega t+\phi)\cos_\alpha(\phi) - \cos_\alpha(\omega t+\phi)\big(\sin_\alpha(\phi)+\omega t\cos_\alpha(\phi)\big)\Big]. \tag{4.5.1}$$

Remark 4.5.2. We consider geodesics parametrised by constant speed on $[0,1]$. Consequently, since Theorem 4.3.5 states that $t_{\mathrm{cut}} = \pi_\alpha/|\omega|$, we always have $|\omega| \leqslant \pi_\alpha$ when $q \notin \mathrm{Cut}(q_0)$.

Proof. We let $\lambda_0 = u_0 \mathrm{d}x|_{q_0} + v_0 \mathrm{d}y|_{q_0} \in \mathrm{T}^*_{q_0}(\mathbb{G}_\alpha)$ be the covector corresponding to the unique minimising geodesic joining $q_0 = (x_0,y_0)$ to $q = (x,y)$ in $\mathbb{G}_\alpha$. The assumption that q and q_0 do not lie on the same horizontal line means that $v_0 \neq 0$.

By choosing the global Darboux frame induced by the sections of $\mathrm{T}(\mathrm{T}^*(\mathbb{G}_\alpha))$; $E_1 = \partial_u, E_2 = \partial_v, F_1 = \partial_x, F_2 = \partial_y$, Lemma 44 in [BR19] yields that $\beta_t(q, q_0) = J(t)/J(1)$ where the function J is the determinant of the exponential map $(u, v) \to \exp_{(x_0,y_0)}(u, v)$ in these coordinates, computed at (u_0, v_0).

Taking the derivatives of (4.2.2), we find

$$\begin{aligned} x_{u_0}(t) &= A_{u_0} \sin_\alpha(\omega t + \phi) + A\,(\omega_{u_0} t + \phi_{u_0}) \cos_\alpha(\omega t + \phi); \\ x_{v_0}(t) &= A_{v_0} \sin_\alpha(\omega t + \phi) + A\,(\omega_{v_0} t + \phi_{v_0}) \cos_\alpha(\omega t + \phi), \end{aligned}$$

and

$$\begin{aligned} y_{u_0}(t) &= \frac{v_0 A^{2(\alpha-1)} A}{(\alpha+1)\omega^2} \Big[\alpha\omega t\big(2A_{u_0}\omega + A\omega_{u_0}\big) \\ &\qquad + A\omega(\alpha+1)(\phi_{u_0}\cos_\alpha^2(\phi) - (\omega_{u_0} t + \phi_{u_0})\cos_\alpha^2(\omega t + \phi)) \\ &\qquad + \big(2\alpha A_{u_0}\omega - A\omega_{u_0}\big) \\ &\qquad\qquad \times \big(\sin_\alpha(\phi)\cos_\alpha(\phi) - \sin_\alpha(\omega t + \phi)\cos_\alpha(\omega t + \phi)\big)\Big]; \\ y_{v_0}(t) &= \frac{v_0 A^{2(\alpha-1)} A}{(\alpha+1)\omega^2} \Big[\omega t\big(2\alpha v_0 \omega A_{v_0} + A(\omega + \alpha v_0 \omega_{v_0})\big) \\ &\qquad + A\omega v_0(\alpha+1)(\phi_{u_0}\cos_\alpha^2(\phi) - \big(\omega_{v_0} t + \phi_{v_0}\big)\cos_\alpha^2(\omega t + \phi)) \\ &\qquad + \big(2\alpha\omega v_0 A_{v_0} + A(\omega - v_0\omega_{v_0})\big) \\ &\qquad\qquad \times \big(\sin_\alpha(\phi)\cos_\alpha(\phi) - \sin_\alpha(\omega t + \phi)\cos_\alpha(\omega t + \phi)\big)\Big]. \end{aligned}$$

To make things clearer, set $[f, g] := f_{u_0} g_{v_0} - f_{v_0} g_{u_0}$ and we obtain

$$\begin{aligned} [x, y](t) = &\frac{A^{2\alpha}}{(\alpha+1)\omega^2} \Big[\sin_\alpha^2(\omega t + \phi)\cos_\alpha(\phi)\big(v[A, \omega] - \omega A_{u_0}\big) \\ &+ \alpha v_0 \omega \sin_\alpha^{2\alpha}(\omega t + \phi)\sin_\alpha(\omega t + \phi)\big([A, \omega]t + [A, \phi]\big) \\ &+ \sin_\alpha(\omega t + \phi)\Big(\sin_\alpha(\phi)\cos_\alpha(\phi)(\omega A_{u_0} - v_0[A, \omega]) \\ &\quad - \alpha v_0 \omega \sin_\alpha^{2\alpha}(\phi)[A, \phi] + \omega(\omega A_{u_0} t + v_0 \cos_\alpha^2(\phi)[A, \phi])\Big) \\ &+ \sin_\alpha(\omega t + \phi)\cos_\alpha^2(\omega t + \phi)\Big(t\big((2\alpha - 1)v_0\omega[A, \omega] - A\omega\omega_{u_0}\big) \\ &\quad + (2\alpha - 1)v_0\omega[A, \phi] + Av_0[\phi, \omega] - A\omega\phi_{u_0}\Big) \end{aligned}$$

$$
\begin{aligned}
&+ \cos_\alpha(\omega t+\phi)\Big(\omega^2 t^2 (2\alpha v_0[\omega, A] + A\omega_{u_0})\\
&\quad + \sin_\alpha(\phi)\cos_\alpha(\phi)(A\omega\phi_{u_0} + 2\alpha v_0 \omega[\phi, A] + A v_0[\omega,\phi])\\
&\quad - \omega t(\sin_\alpha(\phi)\cos_\alpha(\phi)(2\alpha v_0[A,\omega]) - A\omega_{u_0})\\
&\quad + \omega(2\alpha v_0[A,\phi]) - A\phi_{u_0} + \alpha v_0 A \sin_\alpha^{2\alpha}(\phi)[\omega,\phi]\\
&\quad + \alpha v_0 \cos_\alpha^2(\phi)[\phi,\omega]\Big)\Big].
\end{aligned}
$$

Using the identities $\sin_\alpha^{2\alpha}(x) + \cos_\alpha^2(x) = 1$ and (4.2.6), we find that $[\omega,\phi] = \frac{\sin_\alpha(\phi)}{\alpha v_0 A}$, $[A,\omega] = \frac{\cos_\alpha(\phi)}{\alpha v_0}$, and $[A,\phi] = 0$. Consequently, we have

$$
[x,y](t) = t\frac{A^{2\alpha}}{\alpha\omega}\Big[\sin_\alpha(\omega t+\phi)\cos_\alpha(\phi) - \cos_\alpha(\omega t+\phi)\big(\sin_\alpha(\phi) + \omega t\cos_\alpha(\phi)\big)\Big]. \tag{4.5.2}
$$

By performing $\beta_t(q,q_0) = [x,y](t)/[x,y](1)$, we finally obtain the desired expression. □

We can transform (4.5.1) from the set of coordinates A, ω and ϕ to x_0, u_0 and v_0 via the identities (4.2.5). This leads to (4.0.1) and concludes the proof of Theorem 4.0.1.

It is interesting to look at the limit of (4.0.1) when α tends to 1. In this case, the α-Grushin plane is the traditional Grushin plane, while $\sin_\alpha$ and $\cos_\alpha$ are the usual sine and cosine functions. The formula (4.5.2) simplifies to

$$
\begin{aligned}
[x,y](t) &= t\frac{A^2}{\omega}\Big[\sin(\omega t+\phi)\cos(\phi) - \cos(\omega t+\phi)\big(\sin(\phi)+\omega t\cos(\phi)\big)\Big]\\
&= t\frac{(u_0^2 + t u_0 v_0^2 x_0 + v_0^2 x_0^2)\sin(t v_0) - t u_0^2 v_0 \cos(t v_0)}{v_0^3},
\end{aligned}
$$

and thus, we find what was already established in [BR19, Proposition 61]: the distortion coefficients of the usual Grushin plane are

$$
\beta_t(q,q_0) = t\frac{(u_0^2 + t u_0 v_0^2 x_0 + v_0^2 x_0^2)\sin(t v_0) - t u_0^2 v_0\cos(t v_0)}{(u_0^2 + u_0 v_0^2 x_0 + v_0^2 x_0^2)\sin(v_0) - u_0^2 v_0 \cos(v_0)}, \text{ for all } t \in [0,1].
$$

We now want to investigate the behaviour of $\beta_t(q_0, q)$ when q_0 and q do lie on the same horizontal line, that is to say, when v_0 tends to 0.

Proposition 4.5.3. *In the same setting of Theorem 4.5.1, when $q, q_0 \in \mathbb{G}_\alpha$, with $q \notin \mathrm{Cut}(q_0)$, and that are on the same horizontal line, we have*

$$\beta_t(q_0, q) = t\frac{(u_0 t + x_0)^{2\alpha}(u_0 t + x_0) - x_0^{2\alpha} x_0}{(u_0 + x_0)^{2\alpha}(u_0 + x_0) - x_0^{2\alpha} x_0}, \text{ for all } t \in [0, 1].$$

Remark 4.5.4. Considering q_0 and q on the same horizontal line corresponds to starting from q_0 with an initial covector $u_0 \mathrm{d}x|_{q_0} + v_0 \mathrm{d}y|_{q_0}$ such that $v_0 = 0$. By continuity with respect to initial conditions, the distortion coefficient in Proposition 4.5.3 is the limit of (4.0.1) when v_0 tends to 0. In particular, the parameter u_0 cannot vanish. Indeed, we would otherwise have a trivial (constant) geodesic since $v_0 = 0$. Furthermore, this implies that the denominator in the expression above is also never vanishing.

Proof. We aim to perform $\lim_{v_0 \to 0} \mathrm{J}(t)/\mathrm{J}(1)$, where J is defined by (4.0.1). We already know from Theorem 4.2.2 that $\lim_{v_0 \to 0} u(t) = u_0$ and $\lim_{v_0 \to 0} x(t) = u_0 t + x_0$. Let us make the following preliminary calculations:

$$\begin{aligned}
u_{v_0}(t) &= -\alpha A\omega(\omega_{v_0} t + \phi_{v_0}) \sin_\alpha^{2(\alpha-1)}(\omega t + \phi) \sin_\alpha(\omega t + \phi) \\
&\qquad + \cos_\alpha(\omega t + \phi)(\omega A_{v_0} + A\omega_{v_0}) \\
&= \frac{A\omega}{\alpha}\Big[\sin_\alpha^{2(\alpha-1)}(\omega t + \phi) \sin_\alpha(\omega t + \phi) \\
&\qquad \times \Big(\omega t((\alpha-1)\cos_\alpha^2(\phi) - \alpha) - \sin_\alpha(\phi)\cos_\alpha(\phi)\Big) \cos_\alpha(\omega t + \phi) \\
&\qquad\quad + (1 - \cos_\alpha^2(\phi))\Big] \\
&= v_0 \frac{x_0^{2\alpha} \cdot u(t) - \left[t(u_0^2 + \alpha v_0^2 x_0^{2\alpha}) + u_0 x_0\right] \cdot x(t)^{2(\alpha-1)} x(t)}{(u_0^2 + v_0^2 x_0^{2\alpha})},
\end{aligned}$$

and also

$$\begin{aligned}
x_{v_0}(t) &= A_{v_0} \sin_\alpha(\omega t + \phi) + A\,(\omega_{v_0} t + \phi_{v_0}) \cos_\alpha(\omega t + \phi) \\
&= \frac{A}{\alpha v_0}\Big[\cos_\alpha(\omega t + \phi)\Big(\sin_\alpha(\phi)\cos_\alpha(\phi) + \omega t\big(\alpha - (\alpha-1)\cos_\alpha^2(\phi)\big)\Big) \\
&\qquad - \sin_\alpha(\omega t + \phi)\cos_\alpha^2(\phi)\Big] \\
&= \frac{\left[t(u_0^2 + \alpha v_0^2 x_0^{2\alpha}) + u_0 x_0\right] \cdot u(t) + u_0^2 \cdot x(t)}{\alpha v_0 (u_0^2 + v_0^2 x_0^{2\alpha})}.
\end{aligned}$$

Since simply replacing v_0 with 0 in $\beta_t(q_0, q)$ will lead to $0/0$, we use L'Hôpital's rule as many times as needed, and we find:

$$\begin{aligned}\beta_t(q_0,q) &= \lim_{v_0\to 0}\frac{\mathrm{J}(t,x_0,u_0,v_0)}{\mathrm{J}(1,x_0,u_0,v_0)} = \lim_{v_0\to 0}\frac{\partial_{v_0}\mathrm{J}(t,x_0,u_0,v_0)}{\partial_{v_0}\mathrm{J}(1,x_0,u_0,v_0)}\\ &= \lim_{v_0\to 0}\frac{\partial^2_{v_0}\mathrm{J}(t,x_0,u_0,v_0)}{\partial^2_{v_0}\mathrm{J}(1,x_0,u_0,v_0)} = t\frac{(u_0t+x_0)^{2\alpha}(u_0t+x_0)-x_0^{2\alpha}x_0}{(u_0+x_0)^{2\alpha}(u_0+x_0)-x_0^{2\alpha}x_0}.\end{aligned}$$

□

4.6 Relevant curvature-dimension estimates

Now that we have the expressions for the distortion coefficients, we would like to find appropriate bounds for them. In [Jui21], Juillet proved that a sub-Riemannian manifold never satisfies the $\mathrm{CD}(K,N)$ condition when $\operatorname{rank}(\mathcal{D}_p) < \dim(M)$ for all $p \in M$. This result does not apply directly to the α-Grushin plane as its distribution has full rank away from the singular set. However, a variant of the technique [Jui10] presented in [Jui08], as shown in Section 4.4, is valid here and we can still conclude that $\mathbb{G}_\alpha$ does not satisfy the CD condition. However, there is a possibility that the weaker synthetic curvature condition $\mathrm{MCP}(K,N)$ can hold for the α-Grushin plane.

In particular, the traditional Grushin plane, equivalent to $\mathbb{G}_\alpha$ when $\alpha = 1$, is $\mathrm{MCP}(K,N)$ if and only if $N \geqslant 5$ and $K \leqslant 0$. We expect the α-Grushin plane to satisfy the MCP property for a minimal value of N that would depend on α. According to Theorem 3.7.2, the related bound on the distortion coefficients should be of the form $\beta_t(q_0,q) \geqslant t^N$.

In this section, we provide a bound in the case where q_0 and q lie on the same horizontal line and when q_0 is a Grushin point. In what follows, we will still parametrise the geodesics of the α-Grushin plane by constant speed and on the interval $[0,1]$.

For $\alpha \geqslant 1$, let $m_\alpha \in [-3, -2]$ be the unique non-zero solution of

$$(m+1)^{2\alpha}(m+1) - ((2\alpha+1)m+1) = 0.$$

If $\alpha = 1$, the value of the root is $m = -3$.

Proposition 4.6.1. *Let $q_0 := (x_0, y_0) \in \mathbb{G}_\alpha$ with $x_0 \neq 0$ and $q \in \mathbb{G}_\alpha$ lying on the same horizontal line. We have that*

$$\beta_t(q_0, q) \geqslant t^N \textit{ for all } t \in [0, 1]$$

if and only if

$$N \geqslant 2\left[\frac{(\alpha+1)m_\alpha + 1}{m_\alpha + 1}\right].$$

Proof. We are looking for the optimal $N \in [1, +\infty]$ such that

$$\frac{(u_0 t + x_0)^{2\alpha}(u_0 t + x_0) - x_0^{2\alpha} x_0}{(u_0 + x_0)^{2\alpha}(u_0 + x_0) - x_0^{2\alpha} x_0} \geqslant t^{N-1} \tag{4.6.1}$$

for all $t \in [0, 1]$ and $x_0, u_0 \in \mathbb{R}$. The function $f_{x_0}(z) := (z + x_0)^{2\alpha}(z + x_0) - x_0^{2\alpha} x_0$ is positive (resp. negative) when $z > 0$ (resp. $z < 0$) and $f_{x_0}(0) = 0$. Therefore, the left-hand side of (4.6.1) is always non-negative. If we take the logarithm of the above, we find that the inequality is equivalent to

$$\int_{u_0 t}^{u_0} \frac{\mathrm{d}}{\mathrm{d}z} \log\left|f_{x_0}(z)\right| \mathrm{d}z \leqslant (N-1) \int_{u_0 t}^{u_0} \frac{\mathrm{d}}{\mathrm{d}z} \log|z| \mathrm{d}z.$$

Since this must hold for every $t \in [0, 1]$, it is equivalent to the same inequality for the integrands:

$$\pm \frac{(2\alpha+1)(z+x_0)^{2\alpha}}{(z+x_0)^{2\alpha}(z+x_0) - x_0^{2\alpha} x_0} \leqslant \pm(N-1)\frac{1}{z}, \text{ when } \pm z > 0.$$

Consequently, Equation (4.6.1) is equivalent to

$$N \geqslant z \frac{(2\alpha+1)(z+x_0)^{2\alpha}}{(z+x_0)^{2\alpha}(z+x_0) - x_0^{2\alpha} x_0} + 1,$$

for all $z \in \mathbb{R}$ and all $x_0 \in \mathbb{R} \setminus \{0\}$. When $z \to 0$, we find that since $x_0 \neq 0$,

$$\frac{(2\alpha+1)(z+x_0)^{2\alpha}}{(z+x_0)^{2\alpha}(z+x_0) - x_0^{2\alpha} x_0} \to 1.$$

We are therefore looking for the global maximum of the map

$$\begin{aligned} f\colon \big(\mathbb{R}\setminus\{0\}\big)\times\mathbb{R} &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto x\frac{(2\alpha+1)(x+y)^{2\alpha}}{(x+y)^{2\alpha}(x+y)-y^{2\alpha}y}+1. \end{aligned}$$

We use polar coordinates: for $r>0$ and $\theta\in[0,2\pi)\setminus\{\pi/2,-3\pi/2\}$, we have

$$f(r\cos(\theta),r\sin(\theta)) = \frac{(2\alpha+1)\cos(\theta)\big[\cos(\theta)+\sin(\theta)\big]^{2\alpha}}{\big(\cos(\theta)+\sin(\theta)\big)\big[\cos(\theta)+\sin(\theta)\big]^{2\alpha}-\sin(\theta)^{2\alpha}\sin(\theta)}+1,$$

which does not depend on r. In particular, the limit of f when $(x,y)\to(0,0)$ does not exist.

Firstly, let us compute the critical points of $\theta\mapsto f(r\cos(\theta),r\sin(\theta))$. We find that $\frac{\partial}{\partial\theta}f(r\cos(\theta),r\sin(\theta))$ is given by

$$\begin{aligned} &\frac{(2\alpha+1)\big(\cos(\theta)+\sin(\theta)\big)\big[\cos(\theta)+\sin(\theta)\big]^{2(\alpha-1)}}{\Big[\big(\cos(\theta)+\sin(\theta)\big)\big[\cos(\theta)+\sin(\theta)\big]^{2\alpha}-\sin(\theta)^{2\alpha}\sin(\theta)\Big]^2} \\ &\quad\times\Big[\sin^{2\alpha}(\theta)\big((2\alpha+1)\cos(\theta)+\sin(\theta)\big) \\ &\qquad -\big(\cos(\theta)+\sin(\theta)\big)\big[\cos(\theta)+\sin(\theta)\big]^{2\alpha}\Big], \end{aligned}$$

which vanishes when $\cos(\theta)+\sin(\theta)=0$, i.e. $\theta=3\pi/4, 7\pi/4$, or when

$$\sin^{2\alpha}(\theta)\left((2\alpha+1)\cos(\theta)+\sin(\theta)\right)=\left(\cos(\theta)+\sin(\theta)\right)\left[\cos(\theta)+\sin(\theta)\right]^{2\alpha}.$$

In the first case, we simply get $f(r\cos(\theta),r\sin(\theta))=1$.

The second case implies that $\sin(\theta)\neq 0$, and thus, setting $m=\cot(\theta)$, we obtain

$$(m+1)^{2\alpha}(m+1)-((2\alpha+1)m+1)=0. \tag{4.6.2}$$

Equation (4.6.2) has two roots: $m=0$, which we reject since it corresponds to $\theta=\pi/2,3\pi/2$ and another root in the interval $[-3,-2]$, denoted by m_α.

With a second derivative test, it is easy to see that the $\theta\in[0,2\pi)\setminus\{\pi/2,-3\pi/2\}$ such that $\cot^{-1}(m_\alpha)=\theta$ gives the local maximum values of $\theta\mapsto f(r\cos(\theta),r\sin(\theta))$:

at these points, we have

$$\begin{aligned} f(r\cos(\theta), r\sin(\theta)) &= \frac{(2\alpha+1)\cos(\theta)\,((2\alpha+1)\cos(\theta)+\sin(\theta))}{(\cos(\theta)+\sin(\theta))\,[((2\alpha+1)\cos(\theta)+\sin(\theta))-\sin(\theta)]}+1 \\ &= \frac{(m_\alpha+1)^{2\alpha}(2(\alpha+1)m_\alpha+1)-1}{(m_\alpha+1)^{2\alpha}(m_\alpha+1)-1} \\ &= 2\left[\frac{(\alpha+1)m_\alpha+1}{m_\alpha+1}\right]. \end{aligned}$$

They are in fact global maximums because $f(r\cos(\theta), r\sin(\theta)) \to 1$ when $\theta \to \pi/2$ or $3\pi/2$. Since $f(r\cos(\theta), r\sin(\theta))$ does not depend on r, this upper bound will not be exceeded either when r escapes to $+\infty$ or when r approaches 0.

We have therefore established that

$$\max_{(x,y)\in\left(\mathbb{R}\setminus\{0\}\right)\times\mathbb{R}} \frac{(2(\alpha+1)x+y)(x+y)^{2\alpha}-yy^{2\alpha}}{(x+y)(x+y)^{2\alpha}-yy^{2\alpha}} = 2\left[\frac{(\alpha+1)m_\alpha+1}{m_\alpha+1}\right].$$

This maximum provides the desired optimal N in the inequality (4.6.1). □

It seems that the Grushin structures behave in such a way that points q_0 and q lying on the same horizontal line (with $x_0 \neq 0$) provide the sharpest N where $\beta_t(q_0, q) \geqslant t^N$ holds for all $t \in [0,1]$. This is also what happens when $\alpha = 1$ (see [BR19, Proposition 62.] and [Riz18, Theorem 8.] for Grushin half-planes).

We thus expect that the optimal N obtained in Proposition 4.6.1 is sharp. We are able to verify this intuition for singular points, i.e. when $q_0 = (0, y_0)$.

Proposition 4.6.2. *Let* $q_0 = (x_0, y_0) \in \mathbb{G}_\alpha$ *with* $x_0 = 0$ *and* $q \notin \mathrm{Cut}(q_0)$. *The inequality*

$$\beta_t(q, q_0) \geqslant t^N$$

holds for all $t \in [0,1]$ *and every* $N \geqslant 2\left[\dfrac{(\alpha+1)m_\alpha+1}{m_\alpha+1}\right]$.

Proof. We firstly observe that

$$N_\alpha := 2\left[\frac{(\alpha+1)m_\alpha+1}{m_\alpha+1}\right] \geqslant 2(\alpha+1), \tag{4.6.3}$$

since $a \geqslant 1 > 0$.

If $x_0 = 0$ and $v_0 = 0$, the formula of the distortion coefficients in Proposition 4.5.3 and Equation (4.6.3) yield $\beta_t(q_0, q) = t^{2(\alpha+1)} \geqslant t^{N_\alpha}$.

Assume now that $x_0 = 0$ and $v_0 \neq 0$, i.e. $\phi = 0$ or $\phi = \pi_\alpha$, the Jacobian determinant (4.5.2) is given by

$$[x, y](t) = t\frac{A^{2\alpha}}{\alpha\omega}\Big[\sin_\alpha(\omega t) - \omega t \cos_\alpha(\omega t)\Big]. \tag{4.6.4}$$

It follows from (4.6.4) that

$$\beta_t(q, q_0) = t\frac{g(\omega t)}{g(\omega)}. \tag{4.6.5}$$

where we have set $g(z) := \sin_\alpha(z) - z\cos_\alpha(z)$. We first note that $g(0) = 0$. Then, we compute

$$g'(z) = \alpha z \sin_\alpha^{2(\alpha-1)}(z) \sin_\alpha(z)$$

to find that $g'(z) > 0$ for every $z \in (0, \pi_\alpha)$ and $\alpha \geqslant 1$. Therefore, the functions g is strictly increasing and positive.

We want to prove that (4.6.5) is greater than t^{N_α}. Similarly as we did in the proof of Proposition 4.6.1, we know that the desired inequality holds if and only if we have

$$G(z) := (N_\alpha - 1)g(z) - zg'(z) \geqslant 0 \text{ for all } z \in [0, \pi_\alpha].$$

We can see that $G(0) = 0$, and

$$G'(z) = \alpha z \sin_\alpha^{2(\alpha-1)}(z)\left[(N-3)\sin_\alpha(z) - (2\alpha-1)z\cos_\alpha(z)\right].$$

From Equation (4.6.3), we deduce that

$$G'(z) \geqslant \alpha(2\alpha-1)z\sin_\alpha^{2(\alpha-1)}(z)\left[\sin_\alpha(z) - z\cos_\alpha(z)\right].$$

Therefore, $G'(z)$ is non-negative and so is $G(z)$, for all $\in [0, \pi_\alpha]$. □

By analysing in more detail and looking at the graph of the distortion coefficients (4.5.1), it would indeed seem to us that the relevant condition is also satisfied when $x_0 \neq 0$. We therefore propose Conjecture 4.0.3. A proof of this could require further work, potentially involving a more comprehensive study of the $(2, 2\alpha)$-trigonometric functions.

Chapter 5

Regularity properties of the sub-Riemannian exponential map

5.1 Warner's regularity conditions

In the work [War65] on the conjugate locus in Riemannian geometry, Warner introduced the notion of regular exponential map, a map $F : \mathrm{T}_p(M) \to M$, where M is a finite dimensional smooth manifold, that satisfies a non-vanishing speed condition along rays, a regularity condition and a continuity condition. Furthermore, Warner showed that such a map is non-injective in any neighbourhood of any singularities of F. This is done through studying the normal forms of F around singularities, namely the points where the Jacobian determinant of F vanishes. Warner then proves that the exponential map of a Finsler manifold is regular in this sense, giving an alternative proof of a result due to Morse and Littauer [ML32]. Warner's theorem was adapted to Lorentzian structures in [Ros83] and then to semi-Riemannian manifolds in [Sze08].

In the present work, we adapt Warner's conditions for the exponential map in sub-Riemannian geometry. Because of the lack of a Levi-Civita connection, the study of geodesics is carried out from the Hamiltonian viewpoint (see Chapter 3 for a summary of the theory). Length minimisers are found to be normal and/or abnormal. Normal geodesics are solutions of a Hamiltonian system of differential equations with initial conditions taking values in the cotangent space. The sub-Riemannian exponential map is the projection of the corresponding Hamiltonian flow. Abnormal geodesics can also appear: they are length minimisers that satisfy a condition not characterised by a differential equation. When a sub-Riemannian manifold has no non-trivial abnormal geodesics, we say that it is ideal.

Let us introduce some notation to state our main results. The normal geodesic starting at $p \in M$ with initial covector $\lambda_0 \in \mathrm{T}^*_p(M)$ is denoted by γ_{p,λ_0}, and I_{p,λ_0} is its maximal domain. We denote by H_p the restriction of the Hamiltonian to a fiber $\mathrm{T}^*_p(M)$ (see again Chapter 3). If $A \in \mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p)$, the sub-Riemannian Jacobi field J_A along γ_{p,λ_0} is the one with initial values $(0, A)$. Choosing a symplectic moving frame along γ_{p,λ_0} allows us to introduce ∇J_A, a (non-canonical) derivative of J_A, along the curve.

Theorem 5.1.1 (Regularity of the sub-Riemannian exponential map)**.** *Let M be a sub-Riemannian manifold and $p \in M$. Then, the corresponding exponential map* $\exp_p$ *with domain $\mathscr{A}_p \subseteq \mathrm{T}^*_p(M)$ satisfies the following properties.*

(R1) *The map* $\exp_p$ *is C^∞ on $\mathscr{A}_p$ and, for all $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ and all $t \in I_{p,\lambda_0}$, we have* $\mathrm{d}_{t\lambda_0}\exp_p(\dot{\mathrm{r}}_{p,\lambda_0}(t)) \neq 0_{\exp_p(t\lambda_0)}$.

(R2) *For every $\lambda_0 \in \mathscr{A}_p \setminus \{0\}$ and every symplectic moving frame along the cotangent lift $\lambda(t)$ of the normal geodesic $\gamma(t) := \exp_p(t\lambda_0)$, the map*

$$\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p) \to \mathrm{T}_{\exp_p(\lambda_0)}(M)/\, \mathrm{d}_{\lambda_0}\exp_p(\mathrm{T}_{\lambda_0}(\mathrm{T}^*_p(M))),$$

sending A to $\nabla J_A(1) + \mathrm{d}_{\lambda_0}\exp_p(\mathrm{T}_v(\mathrm{T}^*_p(M)))$, is a linear isomorphism.

(R3) *When M is an ideal sub-Riemannian manifold, there exists a convex neighbourhood $\mathcal{V}$ of every $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ such that for every ray $\mathrm{r}_{p,\bar{\lambda}_0}$ which intersects $\mathcal{V}$, the number of singularities of* $\exp_p$ *(counted with multiplicities) on* $\mathrm{Im}(\mathrm{r}_{p,\bar{\lambda}_0}) \cap \mathcal{V}$ *is constant and equals the order of λ_0 as a singularity of* $\exp_p$*, i.e.* $\dim(\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p))$.

Condition (R1) follows from the constant speed property of normal geodesics (see Section 5.2). The theory of sub-Riemannian Jacobi fields, developed for example in [BR17], will help us to prove condition (R2) in Section 5.3. In Riemannian geometry, condition (R3) is a consequence of Morse's index theory. His ideas are adapted to this context with the Maslov index and the condition (R3) will be obtained in Section 5.4.

Warner uses these three conditions in [War65] to conclude that the Riemannian exponential map is not locally injective around a singularity. This result, originally shown by Morse and Littauer [ML32], is extended to the three-dimensional Heisenberg group $\mathbb{H} := \mathbb{H}_1$ in Section 5.5.

Theorem 5.1.2. *For $p \in \mathbb{H}$, the sub-Riemannian exponential map $\exp_p : \mathscr{A}_p \to \mathbb{H}$ is not injective on any neighbourhood of a conjugate vector $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$.*

The proof of this theorem uses direct computations and an analysis of the exponential map near its singularities.

5.2 Normal extremals

As pointed out in the previous section, the normal geodesic $\gamma(t)$ of the sub-Riemannian manifold M, with initial point $\gamma(0) = p$ and initial covector $\lambda_0 \in \mathscr{A}_p$ is the projection of the normal extremal $\lambda : I_{p,\lambda_0} \to \mathrm{T}(M)$, the solution to Hamilton's equation with initial value $\lambda(0) = (p, \lambda_0)$.

The ray in $\mathscr{A}_p$ through λ_0 is the map

$$\begin{aligned} \mathrm{r}_{p,\lambda_0} \colon I_{p,\lambda_0} &\longrightarrow \mathrm{T}_p^*(M) \\ t &\longmapsto t\lambda_0, \end{aligned}$$

where $I_{p,\lambda_0} \subseteq \mathbb{R}^+$ is the maximal interval containing 0 such that $t\lambda_0 \in \mathscr{A}_p$ for every $t \in I_{t,\lambda_0}$. In this way, $\dot{\mathrm{r}}_{p,\lambda_0}(t) \in \mathrm{T}_{t\lambda_0}(\mathrm{T}_p^*(M))$, and identifying $\mathrm{T}_{t\lambda_0}(\mathrm{T}_p^*(M))$ with $\mathrm{T}_p^*(M)$ in the usual way, we have $\dot{\mathrm{r}}_{p,\lambda_0}(t) = \lambda_0$ for every $t \in I_{p,\lambda_0}$.

Proposition 5.2.1 (see [ABB20], Theorem 4.25). *Let $\lambda : [0,T] \to \mathrm{T}^*(M)$ be a normal extremal, i.e. a solution to Hamilton's equation*

$$\dot{\lambda} = \overrightarrow{H}(\lambda).$$

The corresponding normal geodesic $\gamma(t) = \pi(\lambda(t))$ has constant speed and

$$\frac{1}{2}\|\dot{\gamma}(t)\|^2 = H(\lambda(t))$$

for every $t \in [0,T]$.

Proof. The Hamiltonian is constant along a normal trajectory:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\lambda(t)) = \mathrm{d}_\lambda H \circ \dot{\lambda}(t) = \sigma(\dot{\lambda}(t), \overrightarrow{H}(\lambda(t))) = 0.$$

The minimal control for the curve $\gamma = \pi \circ \lambda$ is given by $u_i(t) = \langle \lambda(t), X_k(\pi(\lambda)) \rangle$ and therefore

$$\frac{1}{2}\|\dot{\gamma}(t)\|^2 = \frac{1}{2}\sum_{i=1}^{m} u_i(t)^2 = \frac{1}{2}\sum_{i=1}^{m}\langle \lambda(t), X_k(\pi(\lambda))\rangle^2 = H(\lambda(t)).$$

□

In view of this result, we observe that, contrary to the Riemannian case, there might exist initial covectors $\lambda_0 \in \mathrm{T}_p^*(M)$ such that the corresponding normal geodesic is trivial, i.e. constant. This can happen if $\lambda_0 \in H_p^{-1}(0)$.

Since the normal geodesic γ has constant speed, we have

$$\dot{\gamma}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\exp_p(t\lambda_0) = \mathrm{d}_{t\lambda_0}\exp_p(\lambda_0) = \mathrm{d}_{t\lambda_0}\exp_p(\dot{\mathrm{r}}_{p,\lambda_0}(t))$$

which is non-zero as long as $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$. This proves a cotangent version of the first condition of Warner.

Theorem 5.2.2 (Constant speed property). *The map $\exp_p$ is $\mathcal{C}^\infty$ on $\mathscr{A}_p$, and for all $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ and all $t \in I_{p,\lambda_0}$, we have $\mathrm{d}_{t\lambda_0}\exp_p(\dot{\mathrm{r}}_{p,\lambda_0}(t)) \neq 0_{\exp_p(t\lambda_0)}$.*

The set $\mathscr{A}_p \setminus H_p^{-1}(0)$ is open and radially convex in the following sense: if $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$, then $\left\{t \in \mathbb{R} \mid t\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)\right\}$ is an open interval.

5.3 Jacobi fields and the regularity property

As will be shown, the so-called regularity property is a feature of Jacobi fields, the theory of which we outline here in the sub-Riemannian context.

Let $\gamma : [0, T] \to M$ be a normal geodesic and $\lambda(t)$ be its cotangent lift. We can write $\gamma(t) = \exp_p(t\lambda_0)$ for some initial covector $\lambda_0 \in \mathrm{T}_p^*(M)$. Consider a variation of $\gamma(t)$ through normal geodesics

$$\Gamma(t,s) = \exp_{\sigma(s)}(tV(s)),$$

where $\Lambda(s) = (\sigma(s), V(s))$ is a curve in $\mathrm{T}^*(M)$ with $\Lambda(0) = (p, \lambda_0)$. The curve Λ is well-defined on a small interval $(-\epsilon, \epsilon)$. A sub-Riemannian Jacobi field J along the normal geodesic γ can be seen as the variation field of a variation γ through normal geodesics:

$$J(t) = \frac{\partial}{\partial s} \exp_{\sigma(s)}(tV(s))\Big|_{s=0}.$$

Remembering that $\exp_p(tv) = \pi \circ \mathrm{e}^{t\overrightarrow{H}}(p, v)$, we have the equalities

$$\begin{aligned} J(t) &= \frac{\partial}{\partial s} \pi \left(\mathrm{e}^{t\overrightarrow{H}}(\sigma(s), V(s)) \right) \Big|_{s=0} = \frac{\partial}{\partial s} \pi \left(\mathrm{e}^{t\overrightarrow{H}}(\Lambda(s)) \right) \Big|_{s=0} \\ &= \mathrm{d}_{\lambda(t)} \pi \left(\mathrm{d}_{\lambda_0} \mathrm{e}^{t\overrightarrow{H}} \dot{\Lambda}(0) \right). \end{aligned}$$

The Jacobi field J along γ is therefore uniquely determined by its initial value $\dot{\Lambda}(0) \in \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M)) \cong \mathrm{T}_p(M) \oplus \mathrm{T}^*_{\lambda(0)}(M)$. This implies that the space of Jacobi fields along the geodesic γ, which we denote by $\mathscr{J}(\gamma)$, is a vector space of dimension $2n$.

On the other hand, the space of Jacobi fields along the extremal λ, denoted this time by $\mathscr{J}(\lambda)$, is the collection of vector fields along λ of the form

$$\mathcal{J}(t) := \mathrm{d}_{\lambda_0} \mathrm{e}^{t\overrightarrow{H}} \dot{\Lambda}(0),$$

also uniquely determined by $\dot{\Lambda}(0) \in \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$. The space $\mathscr{J}(\gamma)$ is linearly isomorphic to $\mathscr{J}(\lambda)$ through the pushforward of the bundle projection

$\pi : \mathrm{T}^*(M) \to M$. Equivalently, a vector field $\mathcal{J}$ is a Jacobi field along the extremal λ if it satisfies

$$\dot{\mathcal{J}} := \mathcal{L}_{\vec{H}} \mathcal{J} = 0, \tag{5.3.1}$$

where $\mathcal{L}_{\vec{H}} \mathcal{J}$ is the Lie derivative of a vector field along λ in the direction of $\vec{H}$:

$$\begin{aligned} \mathcal{L}_{\vec{H}} \mathcal{J}(t) &= \lim_{\epsilon \to 0} \frac{(\mathrm{d}_{\lambda(t+\epsilon)} \mathrm{e}^{-\epsilon \vec{H}})[\mathcal{J}(t+\epsilon)] - \mathcal{J}(t)}{\epsilon} \\ &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} (\mathrm{d}_{\lambda(t+\epsilon)} \mathrm{e}^{-\epsilon \vec{H}})[\mathcal{J}(t+\epsilon)]. \end{aligned}$$

The equation (5.3.1) can be rewritten using the symplectic structure of $\mathrm{T}^*(M)$ and a symplectic moving frame generalising, in a a non-canonical way, the Riemannian parallel transport (see also [BR17] for more details).

Theorem 5.3.1. *Let $\gamma : [0, T] \to M$ be a normal geodesic and $\lambda(t)$ its cotangent lift. There exists a frame $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t)$ along $\lambda(t)$ such that*

(i) $\mathrm{Ver}_{\lambda(t)} = \mathrm{span}\,\{E_1(t), \dots, E_n(t)\}$, *where*

$$\mathrm{Ver}_{\lambda(t)} := \mathrm{Ker}(\mathrm{d}_{\lambda(t)}\pi) \subseteq \mathrm{T}_{\lambda(t)}(\mathrm{T}^*(M))$$

is the vertical subspace along λ;

(ii) *it is a symplectic moving frame:*

$$\omega(E_i, E_j) = 0, \qquad \omega(F_i, F_j) = 0, \qquad \omega(E_i, F_j) = \delta_{i,j}.$$

Furthermore, given such a moving frame, a vector field $\mathcal{J}(t) = \sum_{i=1}^n p_i(t)E_i(t) + x_i(t)F_i(t)$ is a Jacobi field along λ if and only the following Hamilton's equation for Jacobi fields is satisfied:

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -A(t)^T & R(t) \\ B(t) & A(t) \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix} \tag{5.3.2}$$

for some matrix $A(t)$ with $\mathrm{rank}(A(t)) = \dim(\mathcal{D}_{\gamma(t)})$ and some symmetric matrices $B(t), R(t)$.

Let $\gamma : [0, T] \to M$ be a normal geodesic, its cotangent lift $\lambda(t)$, and fix a symplectic moving frame $E_1(t), \dots, E_n(t), F_1(t), \dots, F_n(t)$ along λ such as given by Theorem 5.3.1. The function $t \mapsto (p(t), x(t))$ from (5.3.2) will be called the coordinates of the Jacobi field $\mathcal{J}(t)$ along λ (resp. $J(t)$ along γ). The family $\mathrm{d}_{\lambda(t)}\pi\left(F_1(t)\right), \dots, \mathrm{d}_{\lambda(t)}\pi\left(F_n(t)\right)$ forms a basis for $\mathrm{T}_{\gamma(t)}(M)$. The scalar product $\langle \cdot, \cdot \rangle_{\gamma(t)} : \mathrm{T}_{\gamma(t)}(M) \times \mathrm{T}_{\gamma(t)}(M) \to \mathbb{R}$ designates the positive quadratic form that turns this family into an orthonormal basis along γ. Furthermore, it coincides with $\langle \cdot, \cdot \rangle_{\mathcal{D}_{\gamma(t)}}$ on $\mathcal{D}_{\gamma(t)}$. All these definitions, including the next one, are dependant on the choice of symplectic moving frame along λ.

Definition 5.3.2. The derivative of a Jacobi field J along γ with coordinates (p, x) is defined as

$$\nabla J(t) = \sum_{k=1}^{n} p_k(t)\, \mathrm{d}_{\lambda(t)}\pi\left(F_k(\lambda(t))\right).$$

The next two results are at the heart of the second condition of Warner.

Lemma 5.3.3. *Let $\gamma : [0, T] \to M$ be a normal geodesic. Suppose that J and $\overline{J}$ are two Jacobi fields along γ. Once we fix a symplectic moving frame $E_1(t), \dots, E_n(t)$, $F_1(t), \dots, F_n(t)$ along the cotangent lift $\lambda(t)$ as given by Theorem 5.3.1, we have that*

$$\langle \nabla J(t), \overline{J}(t) \rangle_{\gamma(t)} - \langle J(t), \nabla \overline{J}(t) \rangle_{\gamma(t)}$$

is constant in $t \in [0, T]$.

Remark 5.3.4. This result is a generalisation of a well-known fact in Riemannian geometry: If J_1 and J_2 are two Riemannian Jacobi fields along a geodesic γ, then $\langle D_t J_1, J_2 \rangle - \langle J_1, D_t J_2 \rangle$ is constant along γ, where D_t stands for the covariant derivative.

Proof. Let (p, x) (resp. $(\overline{p}, \overline{x})$) be the coordinates of the Jacobi field J (resp. $\overline{J}$) with respect to the moving frame. Hamilton's equation for Jacobi fields (5.3.2) states that $\dot{p}(t) = -A(t)^T p(t) + R(t)x(t)$ and $\dot{x}(t) = B(t)p(t) + A(t)x(t)$ and we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\langle p(t), \overline{x}(t) \rangle_{\mathbb{R}^n} - \langle \overline{p}(t), x(t) \rangle_{\mathbb{R}^n}\right] = \langle x(t), Q(t)\overline{x}(t) \rangle_{\mathbb{R}^n} - \langle Q(t)x(t), \overline{x}(t) \rangle_{\mathbb{R}^n}$$

$$
\begin{aligned}
&+ \langle p(t), B\overline{p}(t)\rangle_{\mathbb{R}^n} - \langle Bp(t), \overline{p}(t)\rangle_{\mathbb{R}^n} \\
&+ \langle A^T\overline{p}(t), x(t)\rangle_{\mathbb{R}^n} - \langle \overline{p}(t), Ax(t)\rangle_{\mathbb{R}^n} \\
&+ \langle p(t), A\overline{x}(t)\rangle_{\mathbb{R}^n} - \langle A^T p(t), \overline{x}(t)\rangle_{\mathbb{R}^n} = 0.
\end{aligned}
$$

This holds for every $t \in [0, T]$, which concludes the proof. □

Let $\gamma : I \to M$ be a normal geodesic and $a, b \in I$. We use the notation $\mathscr{J}_a(\gamma)$ for the vector space of Jacobi fields along γ vanishing at time $t = a$ and $\mathscr{J}_{a,b}(\gamma)$ for the subspace of $\mathscr{J}_a(\gamma)$ of Jacobi fields along $\gamma : I \to M$ vanishing at both $t = a$ and $t = b$.

Proposition 5.3.5. *Let $\gamma : [0, T] \to M$ be a normal geodesic with initial covector $\lambda_0 \in \mathrm{T}^*_p(M)$ and such that $\gamma(0) = p \in M$, and fix a symplectic moving frame along γ as provided by Theorem 5.3.1. Then, the sets $A_{\gamma(t)} := \{J(t) \mid J \in \mathscr{J}_0(\gamma)\}$ and $B_{\gamma(t)} := \{\nabla J(t) \mid J \in \mathscr{J}_{0,t}(\gamma)\}$ are orthogonal complements (with respect to $\langle \cdot, \cdot \rangle_{\gamma(t)}$) in $\mathrm{T}_{\gamma(t)}(M)$, i.e. $\mathrm{T}_{\gamma(t)}(M) = A_{\gamma(t)} \oplus B_{\gamma(t)}$ and $A_{\gamma(t)} = B^{\perp}_{\gamma(t)}$.*

Proof. Choose a basis $(J_1, \dots, J_k)$ of $\mathscr{J}_{0,t}(\gamma)$ and complete it into a basis for $\mathscr{J}_0(\gamma)$ with $(\overline{J}_1, \dots \overline{J}_{n-k})$. The family $(\nabla J_1(t), \dots, \nabla J_k(t))$ is linearly independent. Indeed, assuming it is not the case, then there exists $a_1, \dots, a_k \in R$ such that $\sum_{i=1}^k a_i \nabla J_i(t) = 0$. The Jacobi field $J := \sum_{i=1}^k a_i J_i$ satisfies $J(0) = 0$, $J(t) = 0$ and $\nabla J(t) = 0$. From Hamilton's equation for Jacobi fields, we can conclude that J is identically zero and therefore $a_1 = \dots = a_k = 0$.

Similarly, the family $(\overline{J}_1, \dots \overline{J}_{n-k})$ is linearly independent. If it is not the case, then there exists $b_1, \dots, b_{n-k}$ such that $\sum_{i=1}^k b_i \overline{J}_i(t) = 0$. Let $\overline{J} = \sum_{i=1}^k b_i \overline{J}_i$. We have $\overline{J}(0) = 0$ and $\overline{J}(t) = 0$. So, the Jacobi field $\overline{J} \in \mathscr{J}_{0,t}(\gamma)$ and hence $b_1 = \dots = b_{n-1} = 0$.

Finally, Lemma 5.3.3 implies that

$$
\begin{aligned}
\langle \nabla J_i(t), \overline{J}_j(t)\rangle_{\gamma(t)} - \langle J_i(t), \nabla \overline{J}_j(t)\rangle_{\gamma(t)} &= \langle \nabla J_i(0), \overline{J}_j(0)\rangle_{\gamma(t)} - \langle J_i(0), \nabla \overline{J}_j(0)\rangle_{\gamma(t)} \\
&= 0.
\end{aligned}
$$

and hence $\langle J_i(t), \nabla \bar{J}_j(t)\rangle_{\gamma(t)} = \langle \nabla J_i(t), \bar{J}_j(t)\rangle_{\gamma(t)} = 0$. Therefore, we have $A_{\gamma(t)} = B^{\perp}_{\gamma(t)}$, $A_{\gamma(t)} \cap B_{\gamma(t)} = \{0\}$ and $\mathrm{T}_{\gamma(t)}(M) = A_{\gamma(t)} \oplus B_{\gamma(t)}$. □

It remains to link Jacobi fields and the exponential map.

The following two results are analogous to the Riemannian context. Firstly we examine Jacobi fields along γ vanishing at its initial time.

Proposition 5.3.6. *Let $\gamma : [0, T] \to M$ be a normal geodesic with initial covector $\lambda_0 \in \mathrm{T}^*_p(M)$ and such that $\gamma(0) = p \in M$. For every $w \in \mathrm{T}^*_p(M)$, the unique Jacobi field along γ with initial value $(0, w) \in \mathrm{T}_p(M) \oplus \mathrm{T}^*_p(M) \cong \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$ is given by*

$$J(t) = \mathrm{d}_{t\lambda_0}\exp_p(tw),$$

where we view tw as an element of $\mathrm{T}_{tv}(\mathrm{T}^*_p(M)) \cong \mathrm{T}^*_p(M)$.

Proof. Consider a variation of a normal geodesic $\gamma(t) = \exp_p(t\lambda_0)$ of the form

$$\Gamma(t, s) = \exp_{\sigma(s)}(tV(s)),$$

where $V(s) \in \mathrm{T}^*_{\gamma(s)}(M)$ is a covector field along γ satisfying $V(0) = \lambda_0$ and $\dot{V}(0) = w$ and $\sigma(s) = p \in M$ is constant. The variation field of Γ is a Jacobi field along γ

$$\frac{\partial}{\partial s}\exp_p(tV(s))\Big|_{s=0} = \mathrm{d}_{t\lambda_0}\exp_p(tw)$$

uniquely determined by its initial value $(\dot{\sigma}(s), \dot{V}(s)) = (0, w)$. □

Secondly, we consider the singularities of the exponential map, and we study their relationship with sub-Riemannian Jacobi fields.

Definition 5.3.7. Given $p \in M$, a (co)vector $\lambda_0 \in \mathscr{A}_p$ is said to be conjugate to p if it is a critical value of $\exp_p$, i.e. if $\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p)$ is not trivial. The point $\exp_p(\lambda_0)$ is then also said to be conjugate to p.

Proposition 5.3.8. *Let $\gamma : [0, 1] \to M$ be a normal geodesic with initial covector $\lambda_0 \in \mathscr{A}_p$ and such that $\gamma(0) = p \in M$. The covector λ_0 is conjugate to p if and only if there exists a non-trivial Jacobi field J along γ such that $J(0) = 0$ and $J(1) = 0$.*

Proof. If λ_0 is a singularity of $\exp_p$, then there exists a vector $\overline{\lambda}_0$ such that $\mathrm{d}_{\lambda_0}\exp_p(\overline{\lambda}_0) = 0$. In this case, from the previous proposition, the vector field

$$J(t) = \mathrm{d}_{t\lambda_0}\exp_p(t\overline{\lambda}_0)$$

is a non-trivial Jacobi field such that $J(0) = 0$ and $J(1) = 0$. The converse implication can be similarly proved. □

Remark 5.3.9. In light of the two previous results, we can see that

$$A_{\gamma(1)} = \mathrm{d}_v\exp_p(\mathrm{T}_v(\mathrm{T}_p^*(M)))$$

and

$$\begin{aligned} B_{\gamma(1)} &\cong \left\{ \sum_{k=1}^{n} p_k(1)E_k(1) \mid (p(t), x(t)) \text{ satisfies (5.3.2) with } x(0) = x(1) = 0 \right\} \\ &= \mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p). \end{aligned}$$

In particular, the subspace $B_{\gamma(1)}$ does not depend on the choice of moving frame along $\lambda(t)$.

For $A \in \mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p)$, we denote by $J_A(t) \in \mathscr{J}_{0,1}(\gamma)$ the Jacobi field along γ with initial value $(0, A)$. We finish this section by proving the following cotangent version of Warner's second condition.

Proposition 5.3.10 (Regularity property)**.** *For every $\lambda_0 \in \mathscr{A}_p \setminus \{0\}$ and every symplectic moving frame along the cotangent lift $\lambda(t)$ of the normal geodesic $\gamma(t) := \exp_p(t\lambda_0)$, as given by Theorem 5.3.1, the map*

$$\mathrm{Ker}(\mathrm{d}_v\exp_p) \to \mathrm{T}_{\exp_p(v)}(M) / \, \mathrm{d}_v\exp_p(\mathrm{T}_v(\mathrm{T}_p^*(M)))$$

sending A to $\nabla J_A(1) + \mathrm{d}_v\exp_p(\mathrm{T}_v(\mathrm{T}_p^*(M)))$ is a linear isomorphism.

Proof. Let $\lambda_0 \in \mathrm{T}_p^*(M)$ and $(A_1, \ldots, A_k)$ be a basis for $\mathrm{Ker}(\mathrm{d}_v\exp_p)$. We can view them as elements of $\mathrm{T}_p^*(M)$ via the identification $\mathrm{T}_v(\mathrm{T}_p^*(M)) \cong \mathrm{T}_p^*(M)$ when necessary.

For $i = 1, \dots, k$, the Jacobi fields

$$J_i(t) := \mathrm{d}_{t\lambda_0} \exp_p(tA_i)$$

vanish at their initial time and have the the initial value $(0, A_i) \in \mathrm{T}_p(M) \oplus \mathrm{T}_p^*(M) \cong \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$ (see Proposition 5.3.6). They also vanish at the final time $t = 1$ since $A_1, \dots, A_k \in \mathrm{Ker}(\mathrm{d}_v \exp_p)$. Therefore, $J_i \in \mathscr{J}_{0,1}(\gamma)$ for every $i = 1, \dots, k$. Using a moving frame along λ given by Theorem 5.3.1, we define a linear map

$$\theta : \mathrm{Ker}(\mathrm{d}_v \exp_p) \to \mathrm{T}_{\exp_p(v)}(M) / \, \mathrm{d}_v \exp_p(\mathrm{T}_v(\mathrm{T}_p^*(M)))$$

via $\theta(A_i) := \nabla J_i(1) + \mathrm{d}_v \exp_p(\mathrm{T}_v(\mathrm{T}_p^*(M)))$. Proposition 5.3.5 implies that the family $(\nabla J_1(1), \dots, \nabla J_k(1))$ is linearly independent and that $\mathrm{Ker}(\mathrm{d}_v \exp_p) \cong B_{\gamma(1)}$ by the orthogonal decomposition. Therefore, the map θ is a linear isomorphism. □

5.4 Maslov index and the continuity property

We now approach Jacobi fields and conjugate points via Lagrangian Grassmannian geometry. For the definitions and properties related to Jacobi curves, we refer the reader to [ABB20, Chapter 15], while the Maslow index, in its full generality, is developed for example in [PT08, Chapter 5].

We start with Lagrangian Grassmannian geometry.

Let (Σ, ω) be a symplectic vector space of dimension $2n$. The Lagrangian Grassmannian of Σ, denoted by $L(\Sigma)$, is the compact manifold of dimension $n(n+1)/2$ consisting of all Lagrangian subspaces of Σ. Furthermore, if $\Lambda \in L(\Sigma)$, there is a linear isomorphism between the tangent space $\mathrm{T}_\Lambda(L(\Sigma))$ and the space of bilinear forms $Q(\Lambda)$ on Λ. The linear isomorphism is given by

$$\mathrm{T}_\Lambda(L(\Sigma)) \to Q(\Lambda) : \dot{\Lambda}(0) \mapsto \underline{\dot{\Lambda}},$$

where $\underline{\dot{\Lambda}}(z) := \sigma(z(0), \dot{z}(0))$, and where we consider a smooth curve $\Lambda(t)$ in $L(\Sigma)$ such that $\Lambda(0) = \Lambda$ and a smooth extension $z(t) \in \Lambda(t)$ such that $z(0) = 0$. It can be shown that $\underline{\dot{\Lambda}}$ is a well-defined quadratic map, independent of the extension considered.

For $k = 0, \ldots, n$, we define the following subsets of $L(\Sigma)$

$$\Lambda^k(L_0) := \{L \in L(\Sigma) \mid \dim(L \cap L_0) = k\} \text{ and } \Lambda^{\geqslant k}(L_0) := \bigcup_{i=k}^{n} \Lambda^i(L_0).$$

For each $k = 0, \cdots, n$, the spaces $\Lambda^k(L_0)$ are embedded manifolds of $L(\Sigma)$ with codimension $\frac{1}{2}k(k+1)$. Their tangent space is given by

$$T_L(\Lambda^k(L_0)) = \{B \in B_{\text{sym}}(L) \mid B_{(L_0 \cap L) \times (L_0 \cap L)} = 0\},$$

for all $L \in \Lambda^k(L_0)$.

Let $\Lambda(\cdot) : [a, b] \to L(\Sigma)$ be a curve of class $\mathcal{C}^1$. We say that $\Lambda(\cdot)$ intercepts $\Lambda^{\geqslant 1}(L_0)$ transversally at the instant $t = t_0$ if $\Lambda(t_0) \in \Lambda^1(L_0)$ and if the symmetric bilinear form $\underline{\dot{\Lambda}}(t_0)$ is non-zero in the space $\Lambda(t_0) \cap L_0$. This intersection is positive (resp. negative, non-degenerate) if $\underline{\dot{\Lambda}}(t_0)$ is positive definite (resp. negative definite, non-degenerate).

We recall that the first group of relative homology of the pair $(L(\Sigma), \Lambda^0(L_0))$ is an infinite cyclic group ([PT08, Corrolary 1.5.3.]) and is denoted by $H_1(L(\Sigma), \Gamma^0(L_0))$.

Definition 5.4.1. Let $L_0 \in L(\Sigma)$. We say that a curve $\Lambda(\cdot) : [a, b] \to L(\Sigma)$ of class $\mathcal{C}^1$ with endpoints in $\Lambda^0(L_0)$ is a positive generator of $H_1(L(\Sigma), \Lambda^0(L_0))$ if $\Lambda(\cdot)$ transversally and positively intercepts $\Lambda^{\geqslant 1}(L_0)$ only once.

Positive generators are homologous in $H_1(L(\Sigma), \Lambda^0(L_0))$ and thus any of these curves defines a generator of $H_1(L(\Sigma), \Lambda^0(L_0)) \cong \mathbb{Z}$ ([PT08, Lemma 5.1.11]). This fact assures us that the next object is well-defined.

Definition 5.4.2. An isomorphism

$$\mu_{L_0} : H_1(L(\Sigma), \Lambda^0(L_0)) \to \mathbb{Z} \tag{5.4.1}$$

is defined by requiring that any positive generator of $H_1(L(\Sigma), \Lambda^0(L_0))$ is sent to $1 \in \mathbb{Z}$. If $\Lambda(\cdot) : [a,b] \to L(\Sigma)$ is a continuous curve with endpoints in $\Lambda^0(L_0)$, we denote by $\mu_{L_0}(\Lambda(\cdot)) \in \mathbb{Z}$ the integer number that corresponds to the homology class of $\Lambda(\cdot)$ by the isomorphism (5.4.1). The number $\mu_{L_0}(\Lambda(\cdot))$ is called the Maslov index of the curve $\Lambda(\cdot)$ relative to the Lagrangian L_0.

The key properties of the Maslov index, including the fact that it is homotopy invariant, are summarised in the next theorem.

Theorem 5.4.3 ([PT08, Lemma 5.1.13 and Corollary 5.1.18])**.** *Let $\Lambda : [a,b] \to L(\Sigma)$ be a curve with endpoints in $\Lambda^0(L_0)$. The Maslov index satisfies the following properties.*

(i) *If $\sigma : [c,d] \to [a,b]$ is a continuous map with $\sigma(c) = a$ and $\sigma(d) = b$, then $\mu_{L_0}(\Lambda(\sigma(\cdot))) = \mu_{L_0}(\Lambda(\cdot))$.*

(ii) *If $\Lambda'(\cdot) : [c,d] \to L(\Sigma)$ is a curve such that $\Lambda(a) \cap L_0 = \Lambda(b) \cap L_0 = \emptyset$ and $\Lambda(b) = \Lambda'(c)$, and if $\star$ denotes the concatenation of curves, then we have*

$$\mu_{L_0}((\Lambda \star \Lambda')(\cdot)) = \mu_{L_0}(\Lambda(\cdot)) + \mu_{L_0}(\Lambda'(\cdot)).$$

(iii) *$\mu_{L_0}(\Lambda(\cdot)^{-1}) = -\mu_{L_0}(\Lambda(\cdot))$ where $\cdot^{-1}$ denotes the reversed curve.*

(iv) *If $\mathrm{Im}(\Lambda(\cdot)) \subset \Lambda^0(L_0)$, then $\mu_{L_0}(\Lambda(\cdot)) = 0$.*

(v) *If $\Lambda'(\cdot) : [a,b] \to L(\Sigma)$ is a curve homotopic to $\Lambda(\cdot)$ with free endpoints in $\Lambda^0(L_0)$, i.e. there is exists a continuous function $H : [0,1] \times [a,b] \to L(\Sigma)$ such that $H(0,t) = \Lambda'(t)$, $H(1,t) = \Lambda'(t)$ for every $t \in [a,b]$ and $H(s,a), H(s,b) \in \Lambda^0(L_0)$ for every $s \in [0,1]$, then $\mu_{L_0}(\Lambda'(\cdot)) = \mu_{L_0}(\Lambda(\cdot))$;*

(vi) *We have $\mu_{L_0}(\Lambda(\cdot)) = \mu_{S(L_0)}(S \circ \Lambda(\cdot))$ if $S : (\Sigma, \omega) \to (\Sigma', \omega')$ is a symplectomorphism.*

(vii) *If $\Lambda(\cdot) : [a,b] \to L(\Sigma)$ is of class C^1 with endpoints in $\Lambda^0(L_0)$ that has only non-degenerate intersection with $\Lambda^{\geqslant 1}(L_0)$ and if $\Lambda(t) \in \Lambda^{\geqslant 1}(L_0)$ only at a finite number of $t \in [a,b]$, then*

$$\mu_{L_0}(\Lambda(\cdot)) = \sum_{t \in [a,b]} \mathrm{sgn}(\underline{\dot{\Lambda}}(t)|_{(L_0 \cap \Lambda(t)) \times (L_0 \cap \Lambda(t))}),$$

where $\operatorname{sgn}(B)$ *is the signature of B, that is to say,* $\operatorname{sgn}(B) = \eta_+(B) - \eta_-(B)$ *with*

$$\eta_{+/-}(B) := \sup\{\dim(W) | B_{W\times W} \text{ is negative/positive definite}\}.$$

We now turn our attention to the concept of Jacobi curves and to the continuity property. As seen in the previous section, a vector is conjugate when there is a non-trivial Jacobi field along the geodesic that vanishes both at its initial and endpoint. It seems therefore natural to study the evolution of the space of all Jacobi fields at a time t that vanish at its initial time:

$$\mathrm{L}_{(p,\lambda_0)}(t) := \{\mathcal{J}(t) \mid \mathcal{J} \text{ is a Jacobi field along } \lambda(t) := \mathrm{e}^{t\overrightarrow{H}}(p,\lambda_0) \text{ and } J(0) = 0\}, \tag{5.4.2}$$

for $p \in M$ and $\lambda_0 \in \mathscr{A}_p$. For every $t \in [0,1]$, the set $\mathrm{L}_{(p,\lambda_0)}(t)$ is a Lagrangian subspace of $\mathrm{T}_{\lambda(t)}(\mathrm{T}^*(M))$. In order to be able to use the geometry and theory of Lagrangian Grassmannian, we will work with an alternative curve that lives in the fixed Grassmannian $\mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$.

Definition 5.4.4. Let $\gamma : [0,T] \to M$ be a normal geodesic with initial covector $\lambda_0 \in \mathrm{T}^*_p(M)$. The Jacobi curve along the cotangent lift $\lambda : [0,T] \to \mathrm{T}^*(M)$ is defined by

$$\mathrm{J}_{(p,\lambda_0)}(t) := \mathrm{d}_{\lambda(t)}\mathrm{e}^{-t\overrightarrow{H}}[\mathrm{Ver}_{\lambda(t)}]$$

where $\mathrm{Ver}_\lambda := \mathrm{T}_\lambda(\mathrm{T}^*_{\pi(\lambda)}(M))$ is the vertical subspace of $\mathrm{T}_\lambda(\mathrm{T}^*(M))$.

Remark 5.4.5. In particular, we see that the Jacobi curve $\mathrm{J}_{(p,\lambda_0)}(t)$ is a subspace of $\mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$ for every $t \in [0,T]$ and $\mathrm{J}_{(p,\lambda_0)}(0) = \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$. The subscript (p,λ_0) on $\mathrm{J}_{(p,\lambda_0)}(t)$ is a reminder of the fact that in order to define the Jacobi curve, one only needs to specify a point $p \in M$ and a covector $\lambda_0 \in \mathrm{T}^*_p(M)$, the cotangent lift being $\lambda(t) = \mathrm{e}^{-t\overrightarrow{H}}(p,\lambda_0)$ and the normal geodesic $\gamma(t) = \pi(\lambda(t))$. The Jacobi curve can be seen as the evolution of the space of all Jacobi fields at time 0 that vanish at time t:

$$\mathrm{J}_{(p,\lambda_0)}(t) = \{\mathcal{J}(0) \mid \mathcal{J} \text{ is a Jacobi field along } \lambda(s) := \mathrm{e}^{s\overrightarrow{H}}(p,\lambda_0) \text{ and } J(t) = 0\}.$$

We state basic properties of Jacobi curves.

Proposition 5.4.6 ([ABB20, Proposition 15.2.]). *The Jacobi curve $J_{(p,\lambda_0)}(t)$ satisfies the following properties:*

(i) $J_{(p,\lambda_0)}(t+s) = d_{\lambda(t+s)} e^{-t\vec{H}}[J_{(p,\lambda_0)}(s)]$;

(ii) $\dot{\underline{J}}_{(p,\lambda_0)}(0) = -2H_p$ *as quadratic forms on* $\mathrm{Ver}_\lambda \cong T_p^*(M)$;

(iii) $\operatorname{rank} J_{(p,\lambda_0)}(t) = \operatorname{rank} H_p$,

for every $t, s \in \mathbb{R}$ such that both sides of the statements are well-defined.

The previous result tells us that the Jacobi curve $J_{(p,\lambda_0)}(t)$ is a monotone non-increasing curve, i.e. $\dot{\underline{J}}_{(p,\lambda_0)}(t)$ non-positive quadratic form for every $t \in [0, T]$. The relationship between the Jacobi curve and the conjugates vectors is given in the next proposition.

Proposition 5.4.7 ([ABB20, Proposition 15.6.]). *For every $p \in M$, a cotangent vector $s\lambda_0 \in \mathscr{A}_p$ is a conjugate vector if and only if*

$$J_{(p,\lambda_0)}(s) \cap J_{(p,\lambda_0)}(0) \neq \emptyset.$$

Furthermore, the order of $s\lambda_0$ as a singularity of $\exp_p$ is equal to $\dim(J_{(p,\lambda_0)}(s) \cap J_{(p,\lambda_0)}(0))$.

In other words, $s\lambda_0$ is conjugate if and only if the Jacobi curve $J_{(p,\lambda_0)}(t)$ is in $\Lambda^{\geqslant 1}(L_0)$ for $t = s$. Proposition 5.4.7 together with the condition of abnormality for geodesics show that if we have a segment of points that are conjugate to the initial one, then the segment is also abnormal.

Corollary 5.4.8 ([ABB20, Proposition 15.7.]). *Let $J_{(p,\lambda_0)}(t)$ be a Jacobi curve associated with $(p, \lambda_0) \in T_p^*(M)$ and let $\gamma(t)$ be the corresponding normal geodesic. Then, $\gamma|_{[0,s]}$ is abnormal if and only if $J_{(p,\lambda_0)}(t) \cap J_{(p,\lambda_0)}(0) \neq \emptyset$ for every $t \in [0, s]$.*

In particular, a geodesic that does contain an abnormal segment has an infinite number of conjugate points while a strongly normal geodesic has a discrete number of conjugate times. In order to use the theory of Maslov indices, we will therefore make the assumption, from this point, that the sub-Riemannian manifold M is ideal, i.e. there is no non-trivial abnormal geodesic in M.

The manifold $\Sigma := \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$ has dimension $2n$ and the cotangent symplectic form on $\mathrm{T}^*(M)$ induces a symplectic bilinear form on Σ. The Jacobi curve $\mathrm{J}_{(p,\lambda_0)}(t)$ defines a one-parameter family of n-dimensional subspaces of Σ. This is because the Hamiltonian flow $\mathrm{e}^{-t\overrightarrow{H}}$ is a symplectic transformation and the vertical space Ver_λ is a Lagrangian subspace of Σ. As a consequence, the Jacobi curve $\mathrm{J}_{(p,\lambda_0)}(t)$ is a smooth curve in the Lagrangian Grassmannian $L(\Sigma)$.

The Maslov index can therefore be computed for Jacobi curves by applying Theorem 5.4.3, proving the following proposition.

Proposition 5.4.9. *For $\Sigma := \mathrm{T}_{(p,\lambda_0)}(\mathrm{T}^*(M))$ with $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$, $L_0 := \mathrm{J}_{(p,\lambda_0)}(r)$ and $\Lambda_{r,s}(t) := \mathrm{J}_{(p,\lambda_0)}|_{[r,s]}(t)$ where $r < s$ are chosen such that $r\lambda_0$ and $s\lambda_0$ are not conjugate vectors of $\exp_p$, we have*

$$\mu_{L_0}(\Lambda_{r,s}(\cdot)) = -\sum_{\substack{t\in[r,s]\\ t\lambda_0\in\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p)}} \dim(\Lambda_{r,s}(t)\cap\Lambda_{r,s}(r)).$$

We are now ready to prove Warner's third condition of regularity.

Proposition 5.4.10 (Continuity property). *Let M be an ideal sub-Riemannian manifold. For every $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$, there exists a convex neighbourhood $\mathcal{V}$ of λ_0 such that for every ray $\mathrm{r}_{p,\overline{\lambda}_0}$ which intersects $\mathcal{V}$, the number of singularities of $\exp_p$ (counted with multiplicities) on $\mathrm{Im}(\mathrm{r}_{p,\overline{\lambda}_0}) \cap \mathcal{V}$ is constant and equals the order of λ_0 as a singularity of $\exp_p$, i.e. $\dim(\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p))$.*

We will break down the proof of this proposition in a series of lemmas.

Lemma 5.4.11. *For $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$, there exists $\delta > 0$ and $\epsilon \in (0,1)$ such that for every $\overline{\lambda_0} \in B(\lambda_0, \delta)$, there is no conjugate vector in $\mathrm{Im}(\mathrm{r}_{p,\overline{\lambda_0}}|_{(0,\epsilon]})$.*

Proof. For $\overline{\lambda}_0$ close enough to λ_0, the ray $\mathrm{r}_{p,\overline{\lambda}_0}$ does not intersect the subspace $H_p^{-1}(0)$. Because we have assumed that there are no non-trivial abnormal geodesics in M, the geodesic $\gamma_{p,\overline{\lambda}_0}$ corresponding to the initial covector $\overline{\lambda}_0$ is strongly normal and therefore there exists $\epsilon(\overline{\lambda}_0)$ such that $\gamma_{p,\overline{\lambda}_0}(t)$ is not conjugate to p for every $t \in (0, \epsilon(\overline{\lambda}_0)]$. According to Corollary 5.4.8, the number of conjugate times on $\gamma_{p,\overline{\lambda}_0}(t)$ is discrete, and by consequence, by taking $\delta > 0$ small enough we can find $\epsilon \in (0,1)$ uniformly of $\overline{\lambda}_0 \in B(\lambda_0, \delta)$, that is to say $\gamma_{p,\overline{\lambda}_0}(t)$ is not conjugate to p for every $t \in (0, \epsilon]$. □

Lemma 5.4.12. *Let $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ be a covector that is not a conjugate vector of $\exp_p$. There exists $\delta > 0$ and $\epsilon \in (0,1)$ such that every $\overline{\lambda_0} \in B(\lambda_0, \delta)$ is not a conjugate vector of $\exp_p$ and $\mathrm{Im}(\mathrm{r}_{p,\overline{\lambda_0}}|_{(0,\epsilon]})$ does not contain any conjugate vector of $\exp_p$.*

Proof. From Proposition 5.4.7, the hypothesis that λ_0 is not a conjugate vector is equivalent to

$$\mathrm{J}_{(p,\lambda_0)}(1) \cap \mathrm{J}_{(p,\lambda_0)}(0) = \emptyset. \tag{5.4.3}$$

Since solutions of differential equations are regular with respect to initial conditions, the Jacobi curve $\mathrm{J}_{(p,\lambda_0)}(\cdot)$ is smooth with respect to λ_0. Thus the transversality condition (5.4.3) must remain valid for small variations of λ_0, which concludes the proof. □

Lemma 5.4.13. *Let $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ be a covector that is not a conjugate vector of $\exp_p$. There exists $\delta > 0$ and $\epsilon \in (0,1)$ such that*

$$\mu_{L_0}(\mathrm{J}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}) = \mu_{\overline{L}_0}(\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,1]})$$

for every $\overline{\lambda}_0 \in B(\lambda_0, \delta)$, where we have denoted $L_0 := \mathrm{J}_{(p,\lambda_0)}(0)$ and $\overline{L}_0 := \mathrm{J}_{(p,\overline{\lambda}_0)}(0)$.

Proof. Consider the geodesic $\gamma(t)$ starting at p with initial covector λ_0 and its lift $\lambda(t) \in \mathrm{T}^*(M)$. According to Theorem 5.3.1, there exists a symplectic moving frame along λ such that a vector field

$$\mathcal{J}(t) = \sum_{i=1}^{n} p_i(t) E_i(t) + x_i(t) F_i(t)$$

is a Jacobi field if

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -A_{\lambda_0}(t)^T & R_{\lambda_0}(t) \\ B_{\lambda_0}(t) & A_{\lambda_0}(t) \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}, \tag{5.4.4}$$

for some matrix $A_{\lambda_0}(t)$ and some symmetric matrices $B_{\lambda_0}(t)$ and $R_{\lambda_0}(t)$. Such a choice of moving frame can enable us to identify the Jacobi curve $\mathrm{J}_{(p,\lambda_0)}(t)$ with

$$\widetilde{\mathrm{J}}_{(p,\lambda_0)}(t) := \{(p(0), x(0)) \mid (p, x) \text{ is a solution of (5.4.4) such that } x(t) = 0\}.$$

With the same moving frame we can identify $\mathrm{L}_{(p,\lambda_0)}(t)$ with

$$\widetilde{\mathrm{L}}_{(p,\lambda_0)}(t) := \{(p(t), x(t)) \mid (p, x) \text{ is a solution of (5.4.4) such that } x(0) = 0\}.$$

Both $\widetilde{\mathrm{J}}_{(p,\lambda_0)}(\cdot)$ and $\widetilde{\mathrm{L}}_{(p,\lambda_0)}(\cdot)$ are smooth curves in $L(\mathbb{R}^{2n})$ and

$$\mu_{L_0}(\mathrm{J}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}) = \mu_{\widetilde{L}_0}(\widetilde{\mathrm{J}}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}) = \mu_{\widetilde{L}_0}(\widetilde{\mathrm{L}}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}) \tag{5.4.5}$$

with $\widetilde{L}_0 := \mathbb{R}^n \times \{0\}$. Indeed, the moving frame $(E_1, \dots, E_n, F_1, \dots, F_n)$ along the lift $\lambda(t)$ induces a symplectomorphism between $\mathrm{T}_{\lambda(t)}(\mathrm{T}^*(M))$ and $\mathbb{R}^{2n}$. Theorem 5.4.3 then justifies the first equality in (5.4.5). The second equality follows when observing that $\widetilde{\mathrm{J}}_{(p,\lambda_0)}(t) \cap \widetilde{L}_0 \neq \emptyset$ if and only if $\widetilde{\mathrm{L}}_{(p,\lambda_0)}(t) \cap \widetilde{L}_0 \neq \emptyset$ and for these conjugate times

$$\dim(\widetilde{\mathrm{J}}_{(p,\lambda_0)}(t) \cap \widetilde{L}_0) = \dim(\widetilde{\mathrm{L}}_{(p,\lambda_0)}(t) \cap \widetilde{L}_0).$$

We then extend the moving frame along the extremal $\lambda(t)$ to a symplectic frame on an open set $\mathcal{U} \subseteq \mathrm{T}^*(M)$ containing $\mathrm{Im}(\lambda)$. This is possible for two reasons: firstly $\mathrm{T}^*(M)$ is orientable, and secondly, an intersection appears only if $\lambda(t)$ is a closed curve. We can now choose a $\delta > 0$ small enough and $\epsilon \in (0,1)$ such that Lemma 5.4.11 and Lemma 5.4.12 are satisfied and $\mathrm{Im}(\overline{\lambda}) \subseteq \mathcal{U}$ where $\overline{\lambda}(t) = \mathrm{e}^{t\overrightarrow{H}}(\overline{\lambda}_0)$, for every $\overline{\lambda}_0 \in B(\delta, \lambda_0)$.

For $\overline{\lambda}_0 \in B(\delta, \lambda_0)$, let σ be straight line joining λ_0 to $\overline{\lambda}_0$, i.e.

$$\sigma\colon [0,1] \longrightarrow \mathrm{T}^*_p(M)\colon s \longmapsto t\lambda_0.$$

The smooth symplectic frame $(E_1(q),\dots,E_n(q),F_1(q),\dots,F_n(q))$ for $q \in \mathcal{U}$ is a moving frame along $\lambda(t)$ when $q=\lambda(t)$ and along $\overline{\lambda}(t)$ when $q=\overline{\lambda}(t)$, in the sense of Theorem 5.3.1. With respect to them, we construct the curves $\widetilde{\mathrm{L}}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}$ and $\widetilde{\mathrm{L}}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,1]}$ and we claim that they are homotopic with free endpoints in $\Lambda^0(\widetilde{L}_0)$. Set

$$H:[0,1]\times[\epsilon,1]:(s,t)\mapsto \widetilde{\mathrm{L}}_{(p,\sigma(s))}(t)|_{[\epsilon,1]}.$$

The curves $\widetilde{\mathrm{L}}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}$ and $\widetilde{\mathrm{L}}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,1]}$ have endpoint in $\Lambda^0(\widetilde{L}_0)$ since $\gamma(\epsilon)$, $\overline{\gamma}(\epsilon)$, $\gamma(1)$, $\overline{\gamma}(1)$ are not conjugate to p by the two previous lemmas. Since $\sigma(s)\in B(\delta,\lambda_0)$ for all $s\in[0,1]$, the same applies to the geodesic starting at p with initial covector $\sigma(s)$. Therefore, the curve $\widetilde{\mathrm{L}}_{(p,\sigma(s))}(\cdot)|_{[\epsilon,1]}$ has its endpoints in $\Lambda^0(\widetilde{L}_0)$ too. It remains to prove the continuity of H. Because the solutions of the geodesic equation depend continuously on the initial covector λ_0, the matrices in (5.4.4) and therefore $\widetilde{\mathrm{L}}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}$ also depend continuously on λ_0. Consequently, H is a homotopy and we can use Theorem 5.4.3 and (5.4.5) to conclude that

$$\begin{aligned}\mu_{L_0}(\mathrm{J}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]}) &= \mu_{\widetilde{L}_0}(\widetilde{\mathrm{L}}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1]})\\ &= \mu_{\widetilde{L}_0}(\widetilde{\mathrm{L}}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,1]}) = \mu_{\overline{L}_0}(\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,1]}).\end{aligned}$$

□

We can now prove the continuity property. We illustrate the proof in Figure 5.1.

Proof of Proposition 5.4.10. Let $\lambda_0\in\mathscr{A}_p\setminus H_p^{-1}(0)$ be a conjugate covector for $\exp_p$. Because of Corollary 5.4.8, we know that λ_0 is an isolated singularity on the ray r_{p,λ_0}. In particular, there exists a $\delta\in(0,1)$ such that λ_0 is the only conjugate vector on the ray r_{p,λ_0} on the interval $[(1-\delta)\lambda_0,(1+\delta)\lambda_0]$.

From Lemma 5.4.12, there exists $\delta_1,\delta_2>0$ and $\epsilon\in(0,1-\delta)$ for which $B_1:=B((1-\delta)\lambda_0,\delta_1)$ and $B_2:=B((1+\delta)\lambda_0,\delta_2)$ do not contain a singularity of $\exp_p$ and $\mathrm{Im}(\mathrm{r}_{p,\overline{\lambda_0}}|_{(0,\epsilon]})$ does not contain any conjugate vector for every $\overline{\lambda_0}\in B_1\cup B_2$. Without loss of generality, we choose δ_1 and δ_2 such that the previous few lemmas apply and in such a way that the rays from the origin going through B_1 also pass through B_2 and vice versa.

Let C be the union of B_1, B_2 and all the rays between B_1 to B_2. The set C is clearly an open convex subset of $\mathrm{T}_p^*(M)$ and we claim that it has the desired property. Let $\mathrm{r}_{p,\overline{\lambda}_0}$ be a ray in $\mathrm{T}_p^*(M)$ intersecting C. The Jacobi curve corresponding to the initial values (p, λ_0) is the concatenation of the following smooths curves

$$\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot) = \mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[0,\epsilon]} \star \mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,t_1]} \star \mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[t_1,t_2]}$$

where $t_1, t_2 \in I_{p,\overline{\lambda}_0}$ are such that $\mathrm{r}_{p,\overline{\lambda}_0}(t_1) \in B_1$ and $\mathrm{r}_{p,\overline{\lambda}_0}(t_2) \in B_2$. From Lemma 5.4.13, we deduce that $\mu_{L_0}(\mathrm{J}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1+\delta]}) = \mu_{\overline{L}_0}(\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,t_2]})$ as well as $\mu_{L_0}(\mathrm{J}_{(p,\lambda_0)}(\cdot)|_{[\epsilon,1-\delta]}) = \mu_{\overline{L}_0}(\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[\epsilon,t_1]})$.

The Maslov index of a concatenation being the sum of the Maslov indices of the pieces, we obtain that

$$\mu_{\overline{L}_0}(\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[t_1,t_2]}) = \mu_{L_0}(\mathrm{J}_{(p,\lambda_0)}(\cdot)|_{[1-\delta,1+\delta]}) = \dim \mathrm{Ker}(\mathrm{d}_{\lambda_0}\mathrm{exp}_p).$$

The last equality comes from Proposition 5.4.9 and the proof is complete, as the same proposition states that $\mu_{\overline{L}_0}(\mathrm{J}_{(p,\overline{\lambda}_0)}(\cdot)|_{[t_1,t_2]})$ must equal the sum of singularities of exp_p, counted with multiplicities, on $\mathrm{Im}(\mathrm{r}_{p,\overline{\lambda}_0}) \cap C$. □

This also completes the proof of Theorem 5.1.1.

It is reasonable to ask whether the cotangent version of Warner's regularity conditions implies a sub-Riemannian analogue of Morse–Littauer's theorem, i.e. the non-injectivity of the sub-Riemannian exponential map on any neighbourhood of a conjugate covector. Warner's approach involves giving the normal forms of the exponential map on neighbourhood of (regular) conjugate vectors. It is not obvious to us that Theorem 5.1.1 would easily provide such a local description of the sub-Riemannian exponential map about its singularities. However, we are able to pursue Warner's program for a specific example: the three dimensional Heisenberg group.

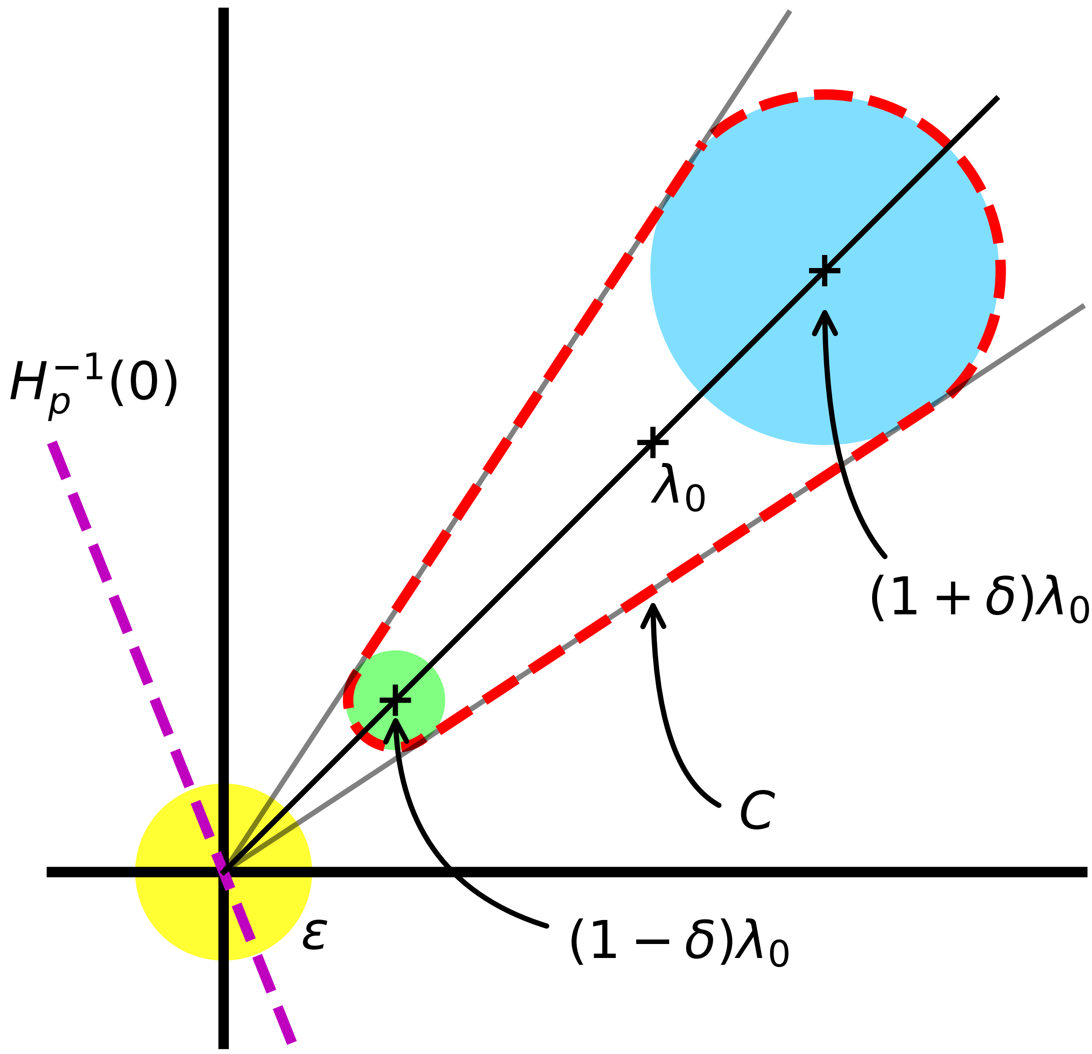


Figure 5.1: Illustration of the proof of Proposition 5.4.10

The singularity λ_0 is the only one on the segment joining $(1-\delta)\lambda_0$ to $(1+\delta)\lambda_0$, where $\delta > 0$ is a small parameter. The green and blue ball represent B_1 and B_2, small balls around $(1-\delta)\lambda_0$ and $(1+\delta)\lambda_0$ respectively. The yellow ball illustrates the $\epsilon > 0$ provided by Lemma 5.4.12. There are no singularities in these three balls. The convex set C of the continuity property, the region delimited by the red dotted lines, is constructed by considering the union of B_1 with B_2 and all the rays between them.

5.5 Local non-injectivity of the Heisenberg exponential map

In this section, we study the three dimensional Heisenberg group and prove Theorem 5.1.2.

The Heisenberg group $\mathbb{H}$ is the sub-Riemannian structure, defined on $\mathbb{R}^3$, that is generated by the global vector fields

$$X_1 = \partial_x - \frac{y}{2}\partial_\tau,\ X_2 = \partial_y + \frac{x}{2}\partial_\tau.$$

The Heisenberg group $\mathbb{H}$ has a structure of Lie group when equipped with the law

$$(x,y,\tau)\cdot(x',y',\tau') = (x+x', y+y', \tau+\tau' - \frac{1}{2}I[z\overline{z'}]),$$

where $z := (x,y)$ and $z' := (x',y')$ are elements of $\mathbb{R}^2$ that we will identify with $\mathbb{C}$ for convenience ($\bar{\cdot}$ denotes the complex conjugation, $R[\cdot]$ and $I[\cdot]$ the real and complex part respectively). The neutral element of this operation is $(0,0,0)$ and the inverse of (x,y,τ) is $(-x,-y,-\tau)$. The Hamiltonian $H : \mathrm{T}^*(\mathbb{H}) \to \mathbb{R}$ is thus given by

$$H(\lambda) = \frac{1}{2}\left(\left(v + \frac{\alpha x}{2}\right)^2 + \left(u - \frac{\alpha y}{2}\right)^2\right),$$

for $\lambda = \left(p, u\mathrm{d}x|_p + v\mathrm{d}y|_p + \alpha\mathrm{d}\tau|_p\right)$ and $p = (x,y,\tau)$.

We can solve Hamilton's equations explicitly to find the expression of a normal geodesic starting from an arbitrary point $p = (x_0, y_0, z_0)$ of $\mathbb{H}$ with an initial covector $\lambda_0 = u_0\,\mathrm{d}x|_p + v_0\,\mathrm{d}y|_p + z_0\,\mathrm{d}z|_p \in \mathrm{T}^*_p(\mathbb{H})$:

$$\lambda(t) = \begin{cases} z(t) &= \dfrac{1}{i\alpha_0}w_0(\mathrm{e}^{i\alpha_0 t} - 1) + \dfrac{1}{2}z_0(\mathrm{e}^{i\alpha_0 t} + 1) \\ \tau(t) &= \tau_0 + \dfrac{I[\overline{z_0}w_0]}{2}t + \dfrac{1}{2\alpha_0}R[\overline{z_0}w_0](1 - \cos(\alpha_0 t)) \\ & \qquad + \dfrac{1}{8}|z_0|^2(\alpha_0 t + \sin(\alpha_0 t)) + \dfrac{1}{2\alpha_0^2}|w_0|^2(\alpha_0 t - \sin(\alpha_0 t)) \\ w(t) &= \dfrac{1}{2}w_0(\mathrm{e}^{i\alpha_0 t} + 1) + \dfrac{\alpha_0}{4}iz_0(\mathrm{e}^{i\alpha_0 t} - 1) \\ \alpha(t) &= \alpha_0 \end{cases}$$

To simplify the notation, we also set

$$\xi_0 := u_0 - \frac{\alpha_0 y_0}{2},\ \widetilde{\xi_0} := u_0 + \frac{\alpha_0 y_0}{2},\ \eta_0 := v_0 + \frac{\alpha_0 x_0}{2} \text{ and } \widetilde{\eta_0} := v_0 - \frac{\alpha_0 x_0}{2}.$$

The symplectic moving frame $(E_1, E_2, E_3, F_1, F_2, F_3) = (\partial_u, \partial_v, \partial_\alpha, \partial_x, \partial_y, \partial_\tau)$ along $\lambda(t)$ induced by the global coordinates of $\mathbb{H}^n$ satisfies Theorem 5.3.1. The Jacobi fields along γ are thus determined a the differential equation of the form

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} -A(t)^T & R(t) \\ B(t) & A(t) \end{pmatrix} \begin{pmatrix} p \\ x \end{pmatrix}.$$

After some computation, we find that in the Heisenberg group, and for the chosen symplectic frame, the block matrices $R(t)$, $A(t)$ and $B(t)$ in the above differential equation are respectively given by

$$\begin{pmatrix} -\frac{\alpha_0^2}{4} & 0 & 0 \\ 0 & -\frac{\alpha_0^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -\frac{\alpha_0}{2} & 0 \\ \frac{\alpha_0}{2} & 0 & 0 \\ -\frac{\widetilde{\eta_0} - 3(\eta_0 \cos(\alpha_0 t) + \xi_0 \sin(\alpha_0 t))}{4} & \frac{\widetilde{\xi_0} - 3(\xi_0 \cos(\alpha_0 t)) - \eta_0 \sin(\alpha_0 t)}{4} & 0 \end{pmatrix},$$

and the symmetric matrix

$$\begin{pmatrix} 1 & 0 & -\frac{\widetilde{\xi_0} - \xi_0 \cos(\alpha_0 t) + \eta_0 \sin(\alpha_0 t)}{2\alpha_0}) \\ 0 & 1 & -\frac{\widetilde{\eta_0} - \eta_0 \cos(\alpha_0 t) - \xi_0 \sin(\alpha_0 t)}{2\alpha_0} \\ \star & \star & \frac{4|w_0|^2(1-\cos(\alpha_0 t)) + \alpha_0^2 |z_0|^2 (1+\cos(\alpha_0 t)) + 4\alpha_0 R[\overline{z_0} w_0] \sin(\alpha_0 t)}{8\alpha_0^2} \end{pmatrix}.$$

Solving this ordinary differential equation yields the general form of a Jacobi field along $\lambda(t)$

$$\mathcal{J}(t) = \sum_{i=1}^{n} p_i(t) E_i(t) + x_i(t) F_i(t),$$

with initial condition $(p(0), x(0))$.

This differential equation can be solved explicitly in the specific context of the Heisenberg group, and we find

$$\begin{pmatrix} p(t) \\ x(t) \end{pmatrix} = M(t) \begin{pmatrix} p(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} M_1(t) & M_2(t) \\ M_3(t) & M_4(t) \end{pmatrix} \begin{pmatrix} p(0) \\ x(0) \end{pmatrix}, \tag{5.5.1}$$

with the expression of $M(t)$ given by

$$\begin{pmatrix}
\frac{1+\cos(\alpha_0 t)}{2} & -\frac{\sin(\alpha_0 t)}{2} & f_1(t) & -\frac{\alpha_0 \sin(\alpha_0 t)}{4} & \frac{\alpha_0(1-\cos(\alpha_0 t))}{4} & 0 \\
\frac{\sin(\alpha_0 t)}{2} & \frac{1+\cos(\alpha_0 t)}{2} & f_2(t) & \frac{\alpha_0(1-\cos(\alpha_0 t))}{4} & -\frac{\alpha_0 \sin(\alpha_0 t)}{4} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\frac{\sin(\alpha_0 t)}{\alpha_0} & -\frac{1-\cos(\alpha_0 t)}{\alpha_0} & f_3(t) & \frac{1+\cos(\alpha_0 t)}{2} & -\frac{\sin(\alpha_0 t)}{2} & 0 \\
\frac{1-\cos(\alpha_0 t)}{\alpha_0} & \frac{\sin(\alpha_0 t)}{\alpha_0} & f_4(t) & \frac{\sin(\alpha_0 t)}{2} & \frac{1+\cos(\alpha_0 t)}{2} & 0 \\
f_5(t) & f_6(t) & f_7(t) & f_8(t) & f_9(t) & 1
\end{pmatrix},$$

where

$$\begin{cases}
f_1(t) = \dfrac{y_0 - (2t\eta_0 + y_0)\cos(\alpha_0 t) - (2t\xi_0 + x_0)\sin(\alpha_0 t)}{4}; \\
f_2(t) = \dfrac{-x_0 + (2t\xi_0 + x_0)\cos(\alpha_0 t) - (2t\eta_0 + y_0)\sin(\alpha_0 t)}{4}; \\
f_3(t) = \dfrac{-v_0 + (v_0 - t\alpha_0\widetilde{\xi_0})\cos(t\alpha_0) + (u_0 + t\alpha_0\eta_0)\sin(t\alpha_0)}{\alpha_0^2}; \\
f_4(t) = \dfrac{-u_0 + (u_0 + t\alpha_0\eta_0)\cos(\alpha_0 t) + (u_0 - t\alpha_0\widetilde{\xi_0})\sin(\alpha_0 t)}{\alpha_0^2}; \\
f_5(t) = \dfrac{2t\xi_0 + \alpha_0 x_0(1-\cos(\alpha_0 t)) - 2u_0\sin(t\alpha_0)}{2\alpha_0^2}; \\
f_6(t) = \dfrac{2t\eta_0 + \alpha_0 y_0(1-\cos(\alpha_0 t)) - 2v_0\sin(t\alpha_0)}{2\alpha_0^2}; \\
f_7(t) = \dfrac{1}{8\alpha_0^3}\Big[\alpha_0\big[t(\alpha_0^2|z_0|^2 - 4|w_0|^2)(1+\cos(\alpha_0 t)) - 4\, z_0 \cdot w_0(1-\cos(\alpha_0 t))\big] \\
\qquad\qquad + 4(2|w_0|^2 + t\alpha_0^2\, z_0 \cdot w_0)\sin(\alpha_0 t)\Big]; \\
f_8(t) = \dfrac{2\alpha_0 t\eta_0 + 2u_0(1-\cos(\alpha_0 t)) + \alpha_0 x_0 \sin(\alpha_0 t)}{4\alpha_0}; \\
f_9(t) = \dfrac{-2\alpha_0 t\xi_0 + 2v_0(1-\cos(\alpha_0 t)) + \alpha_0 y_0 \sin(\alpha_0 t)}{4\alpha_0}.
\end{cases}$$

With this explicit description of sub-Riemannian Jacobi fields along normal geodesics of $\mathbb{H}$, we are now able to study conjugate vectors. Alternatively, this could also be done by computing the determinant of the sub-Riemannian exponential map. We use the Jacobi fields characterisation of the kernel of $\exp_p$ to illustrate our work.

Proposition 5.5.1. *The covector $\lambda_0 = u_0\,\mathrm{d}x|_p + v_0\,\mathrm{d}y|_p + \alpha_0\,\mathrm{d}\tau|_p \in \mathrm{T}^*_p(\mathbb{H})$ is a conjugate covector of $p = (x_0, y_0, \tau_0) \in \mathbb{H}$ with $H(\lambda_0) \neq 0$ if and only if $\alpha_0 \sin(\alpha_0) + 2\cos(\alpha_0) - 2 = 0$ and $\alpha_0 \neq 0$. Furthermore, the conjugate vectors are all of order one, they form a two-dimensional submanifold of $\mathrm{T}^*_p(\mathbb{H})$ and*

$$\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p) = \begin{cases} \mathrm{span}\{(-\chi_0, \xi_0, 0)\} & \text{if } \sin(\alpha_0) = 0 \\ \mathrm{span}\left\{\begin{pmatrix} -\dfrac{\eta_0(\cos(\alpha_0) - 1) - 2y_0}{4} \\ \dfrac{\xi_0(\cos(\alpha_0) - 1) - 2x_0}{4} \\ 1 \end{pmatrix}\right\} & \text{if } \sin(\alpha_0) \neq 0 \end{cases}$$

Proof. We have seen in Section 5.3 that

$$\begin{aligned} \mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p) &= \left\{ \sum_{k=1}^{3} p_k(1)E_k(1) \mid (p, x) \text{ satisfies (5.5.1) with } x(0) = x(1) = 0 \right\} \\ &\cong \left\{ \nabla J(1) \mid J \in \mathscr{J}_{0,1}(\gamma_{p,\lambda_0}) \right\}. \end{aligned}$$

We thus assume that $x(0) = 0$ in Equation (5.5.1) and we look for the image through $M_1(1)$ of elements $p(0)$ in the kernel of the 3×3 bottom-left block matrix $M_3(1)$ of $M(1)$. The covector λ_0 will be conjugate if this kernel is non-trivial. Furthermore, the image of that kernel through $M_1(1)$ will give $\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p)$.

When α_0 tends to 0, the block $M_3(1)$ is similar to

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2}(v_0 + y_0) \\ 0 & 1 & \frac{1}{2}(u_0 + x_0) \\ 0 & 0 & \frac{1}{12}|w_0|^2 \end{pmatrix}.$$

Since we assume that $H(\lambda_0) \neq 0$, the matrix above has a trivial kernel and $\alpha_0 = 0$ does not produce a conjugate vector.

We assume now that $\alpha_0 \neq 0$. If $\sin(\alpha_0) = 0$ and $\cos(\alpha_0) = 1$, then we have

$$\begin{pmatrix} 0 & 0 & \xi_0 \\ 0 & 0 & \eta_0 \\ \xi_0 & \eta_0 & \frac{-4|w_0|^2+\alpha_0^2|z_0|^2}{4\alpha_0} \end{pmatrix}.$$

Since $H(\lambda_0) \neq 0$, the numbers ξ_0 and χ_0 cannot vanish at the same time. This case thus yields a conjugate vector of dimension 1 with $\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p) = \mathrm{span}\{(-\chi_0, \xi_0, 0)\}$.

If $\sin(\alpha_0) = 0$ and $\cos(\alpha_0) = -1$, the block matrix $M_3(1)$ is similar to

$$\begin{pmatrix} 0 & 1 & \frac{\xi_0}{2} - \frac{v_0}{\alpha_0} \\ 1 & 0 & -\frac{\eta_0}{2} - \frac{u_0}{\alpha_0} \\ 0 & 0 & \frac{1}{2\alpha_0^2} H(\lambda_0) \end{pmatrix},$$

which has a trivial kernel.

If $\sin(w_0) \neq 0$, the matrix is equivalent to

$$\begin{pmatrix} 1 & 0 & \star \\ 0 & 1 & \star \\ 0 & 0 & H(\lambda_0)\left[\alpha_0 \cot\left(\frac{\alpha_0}{2}\right) - 2\right] \end{pmatrix},$$

which has a non-trivial kernel if only if $\alpha_0 \sin(\alpha_0) + 2\cos(\alpha_0) - 2 = 0$. In this case, we obtain

$$\mathrm{Ker}(\mathrm{d}_{\lambda_0}\exp_p) = \mathrm{span}\left\{ \begin{pmatrix} -\frac{\eta_0(\cos(\alpha_0)-1) - 2y_0}{4} \\ \frac{\xi_0(\cos(\alpha_0)-1) - 2x_0}{4} \\ 1 \end{pmatrix} \right\}.$$

We finally observe that the collection of conjugate vectors consists of $u_0\,\mathrm{d}x|_p + v_0\,\mathrm{d}y|_p + \alpha_0\,\mathrm{d}\tau|_p \in \mathrm{T}_p^*(\mathbb{H})$ such that $\sin(\alpha_0)\,(\alpha_0 \cot(\alpha_0/2) - 2) = 0$, $\alpha_0 \neq 0$ and $(u_0, v_0) \neq \frac{\alpha_0}{2}(y_0, -x_0)$.

They form planes in $\mathrm{T}_p^*(\mathbb{H})$, parallel to the plane ∂_u-∂_v, where the covector $\frac{\alpha_0}{2}(y_0\partial_u - x_0\partial_v) + \alpha_0\partial_\alpha$ has been removed. □

The structure of the conjugate locus being established, we can finally prove Morse–Littauer's theorem for the Heisenberg group, following Warner's approach in [War65, Theorem 3.4.].

Proof of Theorem 5.1.2. Let $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$ be a conjugate vector of $p \in \mathbb{H}$. By $\mathrm{Conj}_p(\mathbb{H})$, we denote the conjugate locus at p. By Proposition 5.5.1, we can thus choose a one-dimensional open connected submanifold C of $\mathrm{Conj}_p(\mathbb{H})$ in $\mathrm{T}_p^*(\mathbb{H})$ containing λ_0. In particular, the conjugate vectors in C all have order 1. We also write C^0 (resp. C^1) for the set of covectors $\overline{\lambda}_0 \in C$ such that

$$\dim\left[\mathrm{Ker}(\mathrm{d}_{\overline{\lambda}_0}\exp_p) \cap \mathrm{T}_{\overline{\lambda}_0}(\mathrm{Conj}_p(\mathbb{H}))\right] = 0 \text{ (resp. = 1).}$$

It can be seen clearly that the set C^1 (resp. C^0) corresponds to the case $\sin(\alpha_0) = 0$ (resp. $\sin(\alpha_0) \neq 0$). Both C^1 and C^0 are thus open sets.

Case 1 : $\lambda_0 \in C^1$**.**

The subspaces $\mathrm{Ker}(\mathrm{d}_{\overline{\lambda}_0}\exp_p)$ with $\overline{\lambda_0} \in C^1$ form a one-dimensional and involutive smooth distribution of C^1. This is because it corresponds to the distribution induced by the kernels of $\mathrm{d}_{\overline{\lambda}_0}\exp_p$. According to Frobenius' theorem, there exists a unique integral manifold passing through λ_0. This is a one-dimensional connected submanifold N of C^1 such that $\mathrm{T}_{\overline{\lambda}_0}(N) = \mathrm{Ker}(\mathrm{d}_{\overline{\lambda}_0}\exp_p)$ for all $\overline{\lambda}_0 \in N$. We then have that the restriction of $\exp_p$ to N satisfies $\mathrm{d}_{\overline{\lambda}_0}(\exp_p|_N) = 0$ for every $\overline{\lambda}_0 \in N$. Since N is connected, this implies that the sub-Riemannian exponential map maps every elements of N into a single point and hence $\exp_p$ is not injective in any neighbourhood of $\lambda_0 \in C^1$.

Case 2 : $\lambda_0 \in C^0$

Firstly, the hypothesis that $\lambda_0 \in C^0$ means that $\mathrm{Ker}(\mathrm{d}_{\overline{\lambda}_0}\exp_p) \not\subseteq \mathrm{T}_{\overline{\lambda}_0}(\mathrm{Conj}_p(\mathbb{H}))$, i.e. λ_0 is a *fold singularity* of $\exp_p$. We also have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=1}\left[\mathrm{d}_{t\lambda_0}\exp_p\right] = \frac{H(\lambda_0)}{2\alpha_0^2}\left[2 - (2+\alpha_0^2)\cos(\alpha_0)\right]. \tag{5.5.2}$$

We know that $H(\lambda_0) \neq 0$ since $\lambda_0 \in \mathscr{A}_p \setminus H_p^{-1}(0)$. Furthermore, the covector

$\lambda_0 \in C^0$ satisfies $\alpha_0 \sin(\alpha_0) + 2\cos(\alpha_0) - 2 = 0$, $\alpha_0 \neq 0$ and $\sin(\alpha_0) \neq 0$. Therefore, the expression (5.5.2) is not zero. This means that λ_0 is a *good singularity* of $\exp_p$. Using Whitney's singularity theory [Whi55] (see also Warner [War65, Theorem 3.3. c)]), we deduce that there exist coordinate systems $(\mathcal{U}, \xi)$ and $(\mathcal{V}, \eta)$ around λ_0 and $\exp_p(\lambda_0)$ respectively, such that

(i) $\eta^k \circ \exp_p = \xi^k$ for all $k = 1, 2$;

(ii) $\eta^3 \circ \exp_p = \xi^3 \cdot \xi^3$.

This normal form of the sub-Riemannian exponential map implies that $\exp_p$ can not be injective in any neighbourhood of $\lambda_0 \in C^0$. □

We have aimed to describe a cotangent version of Warner's method, via normal forms, to prove the non-injectivity of the sub-Riemannian exponential map in a neighbourhood of a singularity. The failure to be injective may in practice be characterised in a more precise way, by further analysing the Jacobian determinant and counting how many preimages the exponential map has at each critical value. We point the reader to [LR17] for the details of this analysis in the case of the Heisenberg group.

Appendix A

CD condition in sub-Riemannian geometry

This appendix answers a question from Prof Karl-Theodor Sturm.

In July 2019, while attending the summer school of the Hausdorff Center for Mathematics in Piz Buin (Austria), Sturm asked whether the curvature-dimension condition with negative effective dimension in [Oht16] could hold in sub-Riemannian geometry. He suggested that, similarly to the case of positive effective dimension, this should not be the case. I realised afterwards that this was correct and is an adaptation of Juillet's result that appeared in [Jui21].

I thus take the opportunity here to summarise Juillet's method to disprove Brunn–Minkowski inequalities and to explain how it can be used to prove that they do not hold in sub-Riemannian geometry, even for negative effective dimension.

A.1 Curvature-dimension condition for negative effective dimension

Recently, the curvature-dimension condition $\mathrm{CD}(K,N)$ was generalised by Ohta in [Oht16], to allow for negative values of the dimensional parameter N. The definition is analogous to Definition 2.4.1. We recall that $\mathcal{E}_N$ stands for the Rényi functional:

$$\mathcal{E}_N : \mathcal{P}(X) \to [0, +\infty] : \rho\mathfrak{m} + \mu_s \mapsto \int_X \rho^{1-1/N} \mathfrak{m}(\mathrm{d}x).$$

The (K, N)-distortion coefficients for $K \in \mathbb{R}$ and $N \in (-\infty, 0)$ are defined as follows: let $\theta \in (0, +\infty)$ and $t \in [0, 1]$, we have

$$\tau_{K,N}^{(t)}(\theta) = t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{1-1/N},$$

where

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} +\infty & K\theta^2 \leqslant N\pi^2 \\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } N\pi^2 < K\theta^2 < 0 \\ t & \text{if } K\theta^2 = 0 \\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2 > 0. \end{cases}$$

We can now write the definition of the curvature-dimension condition with negative effective dimension.

Definition A.1.1. Let $K \in \mathbb{R}$ and $N \in (-\infty, 0)$. A geodesic metric measure space $(X, \mathrm{d}, \mathfrak{m})$ verifies $\mathrm{CD}(K, N)$ if, for any $\mu_0, \mu_1 \in \mathrm{P}_2(X)$ absolutely continuous with respect to $\mathfrak{m}$ and with bounded support, there exist $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ and a $\mathcal{W}_2$-optimal plan $\pi \in \mathrm{P}(X^2)$ such that for $\mu_t := (\mathrm{e}_t)_\# \nu \ll \mathfrak{m}$ and for any $N' \in [N, 0)$, we have

$$\mathcal{E}_{N'}(\mu_t) \leqslant \int_{X^2} \tau_{K,N}^{(1-t)}(\mathrm{d}(x,y)) \rho_0^{-1/N'} + \tau_{K,N}^{(t)}(\mathrm{d}(x,y)) \rho_1^{-1/N'} \pi(\mathrm{d}x\mathrm{d}y),$$

for every $t \in [0, 1]$.

The n-dimensional sphere, equipped with the harmonic measure

$$\mu_x^{n,\alpha} \propto \frac{1}{|y - x|^{n+\alpha}},$$

is an example of space satisfying $\mathrm{CD}(n - 1 - (n + \alpha)/4, -\alpha)$ whenever $n \geqslant 1$

and $\alpha \geqslant -n$ (see [Mil17b]).

The $\mathrm{CD}(K,N)$ condition for $N<0$ has been studied in the Riemannian setting by Milman in [Mil17a]. More recently, questions of convergence, existence of optimal maps and the local-to-global property were investigated in [MRS21] and [MR21]. The curvature-dimension with negative effective dimension can also be used to prove a Brunn–Minkowski inequality.

Proposition A.1.2. *Let $(X,\mathrm{d},\mathfrak{m})$ be a geodesic metric measure space satisfying the $\mathrm{CD}(K,N)$ condition for some $K\in\mathbb{R}$ and $N\in(-\infty,0)$. Then, the following Brunn–Minkowski inequality, denoted by $\mathrm{BM}(K,N)$, holds in $(X,\mathrm{d},\mathfrak{m})$: for every Borel sets A_0 and A_1 of X and $t\in[0,1]$, we have*

$$\mathfrak{m}(Z_t(A_0,A_1))^{1/N} \leqslant \tau_{K,N}^{(1-t)}(\Theta)\mathfrak{m}(A_0)^{1/N} + \tau_{K,N}^{(t)}(\Theta)\mathfrak{m}(A_1)^{1/N}, \tag{A.1.1}$$

where

$$\Theta := \begin{cases} \inf_{(x,y)\in A_0\times A_1} \mathrm{d}(x,y) & \text{if } K\geqslant 0 \\ \sup_{(x,y)\in A_0\times A_1} \mathrm{d}(x,y) & \text{if } K<0 \end{cases}$$

and

$$Z_t(A_0,A_1) := \{\gamma(t) \mid \gamma\in\mathrm{Geo}(X), \gamma(0)\in A_0 \text{ and } \gamma(1)\in A_1\}$$

is the t-intermediate set from A_0 to A_1.

We also need to introduce the *geodesic dimension* of a metric measure space. It was first studied in sub-Riemannian geometry and then extended to metric measure space in [Riz16].

Definition A.1.3. The *geodesic dimension* of the metric measure space $(X,\mathrm{d},\mathfrak{m})$ at $x\in X$ is defined by

$$\mathcal{N}(p) := \sup\{N>0 \mid C_N(p)=+\infty\}, \text{ where}$$

$$C_N(p) := \sup\Big\{\limsup_{t\to 0^+} \frac{1}{t^N}\frac{\mathfrak{m}(Z_t(p,B))}{\mathfrak{m}(B)} \mid B \text{ measurable, bounded,} \\ \text{and } \mathfrak{m}(B)\in(0,+\infty)\Big\}.$$

It was proved in [ABR18, Proposition 5.49.] that for a sub-Riemannian manifold M, we also have $\mathcal{N}(p) \geqslant \dim M$ for every $p \in M$. Furthermore, we have equality if and only if M is a Riemannian manifold.

In the next section, we use Juillet's method to show that when $N < 0$, the $\mathrm{BM}(K, N)$ condition from Proposition A.1.2 is not satisfied in sub-Riemannian geometry. The argument closely follows [Jui21] (see also [Jui08], [Jui09] and [Jui10]).

A.2 Brunn–Minkowski inequality in sub-Riemannian geometry

Let M be a complete sub-Riemannian manifold. Remember from Section 3.5 that we call $q \in M$ a smooth point with respect to $p \in M$ if there exists a unique minimising geodesic joining p to q, which is not abnormal, and such that p and q are not conjugate along that curve. The set $\underline{\mathrm{Cut}}(M) \subseteq M \times M$ denotes the set of pairs of smooth points of M. We also define $\underline{\mathrm{Cut}}(p) := \{q \in M \mid (p,q) \in M\}$. It can be shown (see [ABR18, Lemma 2.2.]) that if a pair of points (p,q) is smooth, then d_{CC}^2 is smooth at (p,q). This property is further improved in [ABR18, Theorem 5.8.]: for every $p \in M$, the set $\underline{\mathrm{Cut}}(p)$ is open, and dense, and $\mathrm{d}_{CC}^2(p,\cdot)$ is smooth precisely on $\underline{\mathrm{Cut}}(p)$.

Definition A.2.1. The *midpoint map* is given by

$$\mathcal{M} : \underline{\mathrm{Cut}}(M) \times [0,1] \to M : (p,q,t) \mapsto \exp_p(t\lambda_0),$$

where λ_0 satisfies $q = \exp_p(\lambda_0)$.

In other words, the midpoint map gives the point $\gamma(t)$ where $\gamma : [0,1] \to M$ is the unique minimising geodesic joining p to q. The midpoint map is smooth on $\underline{\mathrm{Cut}}(M) \times (0,1)$ and satisfies $\mathcal{M}(p,q,t) = \mathcal{M}(q,p,1-t)$.

Definition A.2.2. The *inverse geodesic map* is defined as

$$\mathcal{I} : \underline{\mathrm{Cut}}(M) \times [0,1) \to M : (m,q,t) \mapsto \exp_p\Big(\frac{-t}{1-t}\lambda_0\Big),$$

where λ_0 satisfies $q = \exp_m(\lambda_0)$.

Here, the inverse geodesic map is thus taken such that $\mathcal{M}(\mathcal{I}(m,q,t),q,t) = m$, that is to say $\mathcal{I}(m,q,t)$ gives the t-inverse of q with respect to m.

We are now ready to explain why the $\mathrm{CD}(K,N)$ condition is not satisfied in sub-Riemannian geometry, even for $N < 0$.

Theorem A.2.3. *Let M be a complete sub-Riemannian manifold such that, for any $p \in M$, we have that* $\mathrm{rank}(p) < \dim T_p(M)$*. If μ is any smooth measure on M, then the metric measure space $(M, \mathrm{d}_{CC}, \mu)$ does not satisfy* $\mathrm{BM}(K,N)$*, for any $K \in \mathbb{R}$ and any $N < 0$.*

Proof. Consider an arbitrary point $p \in M$. By [ABR18, Theorem 5.17.], we know that there exists a strongly normal geodesic $\gamma : [0,T] \to M$ starting at p such that the following properties hold for γ.

(i) The final time $T > 0$ can be chosen small enough such that γ is contained in a small enough coordinate chart in which we can easily compare μ and the Lebesgue measure in $\mathbb{R}^n$.

(ii) The number of conjugate points along γ is discrete.

(iii) If $s \in (0,T)$, there exists $\epsilon > 0$ such that $(\gamma(s),\gamma(t)) \in \underline{\mathrm{Cut}}(M)$ for every $t \in (s-\epsilon, s+\epsilon)$.

We set $a := \gamma(s)$ and $b := \gamma(s+\epsilon)$. Since $(a,\gamma(t))$, $(\gamma(t),b)$ and (a,b) are smooth points, we can make use of the midpoint map and of the inverse geodesic map. If we let $m(r) := \mathcal{M}(a,b,r)$ for $r \in [0,1]$, we can see by using the chain rule on the identity $\mathcal{M}(\mathcal{I}(m,q,t),q,t) = m$ that

$$\lim_{r\to 0} \mathrm{Jac}_b\big[\mathcal{I}(m(r),\cdot,r)\big] = 0 \text{ and } \lim_{r\to 1} \mathrm{Jac}_b\big[\mathcal{I}(m(r),\cdot,r)\big] = +\infty.$$

Consequently, there must be $r \in (0,1)$ such that

$$\mathrm{Jac}_b\big[\mathcal{I}(m(r),\cdot,r)\big] = 1. \tag{A.2.1}$$

By symmetry, we can in fact assume that $r \leqslant 1/2$ without loss of generality.

Consider a small ball $B(b,\rho)$ around b and $\mathcal{I}(m(r),B(b,\rho),r)$, its image under the inverse geodesic map. The key to Juillet's method is the following estimate (see [Jui21, Theorem 1.1.] as well as [Jui10, Theorem 1.]), obtained via a careful Taylor expansion of the midpoint map:

$$\limsup_{\rho\to 0} \frac{\mu\Big(Z_r\big(\mathcal{I}(m(r),B(b,\rho),r),B(b,\rho)\big)\Big)}{\mu\big(B(b,\rho)\big)} = \frac{1}{2^{\mathcal{N}-n}}, \tag{A.2.2}$$

where $\mathcal{N} := \inf_{p\in M}\mathcal{N}(p)$ denotes the minimal geodesic dimension of M.

Now, let $K \in \mathbb{R}$ and $N < 0$ and assume that the sub-Riemannian manifold M satisfies $\mathrm{BM}(K,N)$. A contradiction will follow. Equation (A.1.1) implies that

$$\Big(\frac{\mathfrak{m}(Z_r(\mathcal{I}(m(r),B(b,\rho),r)))}{\mathfrak{m}(B(b,\rho))}\Big)^{1/N} \leqslant \tau_{K,N}^{(1-r)}(\Theta)\Big(\frac{\mathfrak{m}(\mathcal{I}(m(r),B(b,\rho),r))}{\mathfrak{m}(B(b,\rho))}\Big)^{1/N} + \tau_{K,N}^{(r)}(\Theta).$$

We consider the limit of the above inequality as ρ tends to 0. In particular, $\tau_{K,N}^{(r)}(\Theta)$ tends to r, and with (A.2.1) and (A.2.2), we obtain

$$\Big(\frac{1}{2^{\mathcal{N}-n}}\Big)^{1/N} \leqslant 1-r+r = 1,$$

which is impossible, since the geodesic dimension $\mathcal{N}$ is always greater than the manifold dimension n. □

Bibliography

[ABB20] Andrei Agrachev, Davide Barilari and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Vol. 181. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2020.
doi:10.1017/9781108677325.

[ABR18] A. Agrachev, D. Barilari and L. Rizzi. 'Curvature: a variational approach'. *Mem. Amer. Math. Soc.* 256.1225 (2018), pp. v+142.
doi:10.1090/memo/1225.

[ABS08] Andrei Agrachev, Ugo Boscain and Mario Sigalotti. 'A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds'. *Discrete Contin. Dyn. Syst.* 20.4 (2008), pp. 801–822.
doi:10.3934/dcds.2008.20.801.

[Agr08] A. A. Agrachev. 'Geometry of optimal control problems and Hamiltonian systems'. *Nonlinear and optimal control theory*. Vol. 1932. Lecture Notes in Math. Springer, Berlin, 2008, pp. 1–59.
doi:10.1007/978-3-540-77653-6_1.

[Akd+20] Afiny Akdemir, Fabio Cavalletti, Andrew Colinet, Robert McCann and Flavia Santarcangelo. *Independence of synthetic Curvature Dimension conditions on transport distance exponent*. 2020.
arXiv:2007.10980 [math.DG].

[APS60] W. Ambrose, R. S. Palais and I. M. Singer. 'Sprays'. *An. Acad. Brasil. Ci.* 32 (1960), pp. 163–178.

[AR04] L. Ambrosio and S. Rigot. 'Optimal mass transportation in the Heisenberg group'. *J. Funct. Anal.* 208.2 (2004), pp. 261–301. doi:10.1016/S0022-1236(03)00019-3.

[BBI01] Dmitri Burago, Yuri Burago and Sergei Ivanov. *A course in metric geometry*. Vol. 33. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xiv+415. doi:10.1090/gsm/033.

[BE12] P. J. Bushell and D. E. Edmunds. 'Remarks on generalized trigonometric functions'. *Rocky Mountain J. Math.* 42.1 (2012), pp. 25–57. doi:10.1216/RMJ-2012-42-1-25.

[BK21] Samuël Borza and Wilhelm Klingenberg. *Regularity and Continuity properties of the sub-Riemannian exponential map*. 2021. arXiv:2106.11350 [math.DG].

[BKS18] Zoltán M. Balogh, Alexandru Kristály and Kinga Sipos. 'Geometric inequalities on Heisenberg groups'. *Calc. Var. Partial Differential Equations* 57.2 (2018), Paper No. 61, 41. doi:10.1007/s00526-018-1320-3.

[BL05] Francesco Bullo and Andrew D. Lewis. *Geometric control of mechanical systems*. Vol. 49. Texts in Applied Mathematics. Modeling, analysis, and design for simple mechanical control systems. Springer-Verlag, New York, 2005, pp. xxiv+726. doi:10.1007/978-1-4899-7276-7.

[Bor20] Samuël Borza. *Distortion coefficients of the α-Grushin plane*. 2020. arXiv:2010.16350 [math.MG].

[BR17] Davide Barilari and Luca Rizzi. 'On Jacobi fields and a canonical connection in sub-Riemannian geometry'. *Arch. Math. (Brno)* 53.2 (2017), pp. 77–92.
doi:10.5817/AM2017-2-77.

[BR19] Davide Barilari and Luca Rizzi. 'Sub-Riemannian interpolation inequalities'. *Invent. Math.* 215.3 (2019), pp. 977–1038.
doi:10.1007/s00222-018-0840-y.

[CL12] Der-Chen Chang and Yutian Li. 'SubRiemannian geodesics in the Grushin plane'. *J. Geom. Anal.* 22.3 (2012), pp. 800–826.
doi:10.1007/s12220-011-9215-y.

[CS12] Fabio Cavalletti and Karl-Theodor Sturm. 'Local curvature-dimension condition implies measure-contraction property'. *J. Funct. Anal.* 262.12 (2012), pp. 5110–5127.
doi:10.1016/j.jfa.2012.02.015.

[DR11] Luigi De Pascale and Severine Rigot. 'Monge's transport problem in the Heisenberg group'. *Adv. Calc. Var.* 4.2 (2011), pp. 195–227.
doi:10.1515/ACV.2010.026.

[EGL12] David E. Edmunds, Petr Gurka and Jan Lang. 'Properties of generalized trigonometric functions'. *J. Approx. Theory* 164.1 (2012), pp. 47–56.
doi:10.1016/j.jat.2011.09.004.

[Eul61] Leonhard Euler. 'Observationes de comparatione arcuum curvarum irrectificibilium'. *Novi commentarii academiae scientiarum Petropolitanae* (1761), pp. 58–84.

[FR10] Alessio Figalli and Ludovic Rifford. 'Mass transportation on sub-Riemannian manifolds'. *Geom. Funct. Anal.* 20.1 (2010), pp. 124–159.
doi:10.1007/s00039-010-0053-z.

[Gru70] V. V. Grušin. 'A certain class of hypoelliptic operators'. *Mat. Sb. (N.S.)* 83 (125) (1970), pp. 456–473.

[Jui08] Nicolas Juillet. 'Optimal transport and geometric analysis in Heisenberg groups'. Theses. Université Joseph-Fourier - Grenoble I ; Rheinische Friedrisch-Wilhelms-Universität Bonn, Dec. 2008. https://tel.archives-ouvertes.fr/tel-00345301.

[Jui09] Nicolas Juillet. 'Geometric inequalities and generalized Ricci bounds in the Heisenberg group'. *Int. Math. Res. Not. IMRN* 2009.13 (2009), pp. 2347–2373. doi:10.1093/imrn/rnp019.

[Jui10] Nicolas Juillet. 'On a method to disprove generalized Brunn-Minkowski inequalities'. *Probabilistic approach to geometry*. Vol. 57. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2010, pp. 189–198. doi:10.2969/aspm/05710189.

[Jui21] Nicolas Juillet. 'Sub-Riemannian structures do not satisfy Riemannian Brunn-Minkowski inequalities'. *Rev. Mat. Iberoam.* 37.1 (2021), pp. 177–188. doi:10.4171/rmi/1205.

[Kan42] L. Kantorovitch. 'On the translocation of masses'. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 37 (1942), pp. 199–201.

[Le 17] Enrico Le Donne. 'Lecture notes on sub-Riemannian geometry'. *preprint* (2017). https://sites.google.com/site/enricoledonne/lecture_notes.

[LLP19] Enrico Le Donne, Danka Lučić and Enrico Pasqualetto. 'Universal infinitesimal Hilbertianity of sub-Riemannian manifolds' (Oct. 2019). arXiv:1910.05962 [math.MG].

[Lok20] L. V. Lokutsievskiy. 'Explicit Formulae for Geodesics in Left-Invariant Sub-Finsler Problems on Heisenberg Groups via Convex Trigonometry'. *Journal of Dynamical and Control Systems* (Oct. 2020). doi:10.1007/s10883-020-09516-z.

[LR17] A. Lerario and L. Rizzi. 'How many geodesics join two points on a contact sub-Riemannian manifold?' *J. Symplectic Geom.* 15.1 (2017), pp. 247–305. doi:10.4310/JSG.2017.v15.n1.a7.

[LV09] John Lott and Cédric Villani. 'Ricci curvature for metric-measure spaces via optimal transport'. *Ann. of Math. (2)* 169.3 (2009), pp. 903–991. doi:10.4007/annals.2009.169.903.

[Mil17a] Emanuel Milman. 'Beyond traditional curvature-dimension I: new model spaces for isoperimetric and concentration inequalities in negative dimension'. *Trans. Amer. Math. Soc.* 369.5 (2017), pp. 3605–3637. doi:10.1090/tran/6796.

[Mil17b] Emanuel Milman. 'Harmonic measures on the sphere via curvature-dimension'. *Ann. Fac. Sci. Toulouse Math. (6)* 26.2 (2017), pp. 437–449. doi:10.5802/afst.1540.

[ML32] Marston Morse and S. B. Littauer. 'A Characterization of Fields in the Calculus of Variations'. *Proceedings of the National Academy of Sciences* 18.12 (1932). https://www.pnas.org/content/18/12/724.full.pdf, pp. 724–730. doi:10.1073/pnas.18.12.724.

[Mon81] Gaspard Monge. 'Mémoire sur la théorie des déblais et des remblais'. *Histoire de l'Académie Royale des Sciences de Paris* (1781).

[MR20] Thomas Mietton and Luca Rizzi. 'Branching geodesics in sub-Riemannian geometry'. *Geom. Funct. Anal.* 30.4 (2020), pp. 1139–1151. doi:10.1007/s00039-020-00539-z.

[MR21] Mattia Magnabosco and Chiara Rigoni. *Optimal maps and local-to-global property in negative dimensional spaces with Ricci curvature bounded from below*. 2021. arXiv:2105.12017 [math.DG].

[MRS21] Mattia Magnabosco, Chiara Rigoni and Gerardo Sosa. *Convergence of metric measure spaces satisfying the CD condition for negative values of the dimension parameter*. 2021. arXiv:2104.03588 [math.MG].

[MT21] Houry Melkonian and Shingo Takeuchi. 'Remarkable properties of the $\mathrm{sinc}_{p,q}$ functions and related integrals'. *J. Math. Anal. Appl.* 499.1 (2021), pp. 124981, 14. doi:10.1016/j.jmaa.2021.124981.

[Oht07] Shin-Ichi Ohta. 'On the measure contraction property of metric measure spaces'. *Comment. Math. Helv.* 82.4 (2007), pp. 805–828. doi:10.4171/CMH/110.

[Oht16] Shin-ichi Ohta. '(K, N)-convexity and the curvature-dimension condition for negative N'. *J. Geom. Anal.* 26.3 (2016), pp. 2067–2096. doi:10.1007/s12220-015-9619-1.

[Pap14] Athanase Papadopoulos. *Metric spaces, convexity and non-positive curvature*. Second. Vol. 6. IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society (EMS), Zürich,

2014, pp. xii+309.
doi:10.4171/132.

[PT08] Paolo Piccione and Daniel Victor Tausk. *A student's guide to symplectic spaces, Grassmannians and Maslov index*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2008, pp. xiv+301.

[PU03] Lorna I. Paredes and Koichi Uchiyama. 'Analytic singularities of solutions to certain nonlinear ordinary differential equations associated with p-Laplacian'. *Tokyo J. Math.* 26.1 (2003), pp. 229–240.
doi:10.3836/tjm/1244208690.

[Rif14] Ludovic Rifford. *Sub-Riemannian geometry and optimal transport*. SpringerBriefs in Mathematics. Springer, Cham, 2014, pp. viii+140.
doi:10.1007/978-3-319-04804-8.

[Riz16] Luca Rizzi. 'Measure contraction properties of Carnot groups'. *Calc. Var. Partial Differential Equations* 55.3 (2016), Art. 60, 20.
doi:10.1007/s00526-016-1002-y.

[Riz18] Luca Rizzi. 'A counterexample to gluing theorems for MCP metric measure spaces'. *Bull. Lond. Math. Soc.* 50.5 (2018), pp. 781–790.
doi:10.1112/blms.12186.

[Ros83] Kjell Rosquist. 'On the structure of space-time caustics'. *Comm. Math. Phys.* 88.3 (1983), pp. 339–355.
http://projecteuclid.org/euclid.cmp/1103922381.

[San15] Filippo Santambrogio. *Optimal transport for applied mathematicians*. Vol. 87. Progress in Nonlinear Differential Equations and their Applications. Calculus of variations, PDEs, and modeling. Birkhäuser/Springer, Cham, 2015, pp. xxvii+353.
doi:10.1007/978-3-319-20828-2.

[Stu06a] Karl-Theodor Sturm. 'On the geometry of metric measure spaces. I'. *Acta Math.* 196.1 (2006), pp. 65–131. doi:10.1007/s11511-006-0002-8.

[Stu06b] Karl-Theodor Sturm. 'On the geometry of metric measure spaces. II'. *Acta Math.* 196.1 (2006), pp. 133–177. doi:10.1007/s11511-006-0003-7.

[Sze08] D. Szeghy. 'On the conjugate locus of pseudo-Riemannian manifolds'. *Indag. Math. (N.S.)* 19.3 (2008), pp. 465–480. doi:10.1016/S0019-3577(08)80013-4.

[Tak17] Shingo Takeuchi. 'Arithmetric properties of the generalized trigonometric functions (Qualitative Theory of Ordinary Differential Equations and Related Areas)'. *RIMS Kokyuroku* 2032 (2017), pp. 76–100.

[Vil03] Cédric Villani. *Topics in optimal transportation*. Vol. 58. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2003, pp. xvi+370. doi:10.1090/gsm/058.

[Vil09] Cédric Villani. *Optimal transport*. Vol. 338. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Old and new. Springer-Verlag, Berlin, 2009, pp. xxii+973. doi:10.1007/978-3-540-71050-9.

[Wal06] Michel Waldschmidt. 'Transcendence of periods: the state of the art'. *Pure Appl. Math. Q.* 2.2, Special Issue: In honor of John H. Coates. Part 2 (2006), pp. 435–463. doi:10.4310/PAMQ.2006.v2.n2.a3.

[War65] Frank W. Warner. 'The conjugate locus of a Riemannian manifold'. *Amer. J. Math.* 87 (1965), pp. 575–604.
doi:10.2307/2373064.

[Whi55] Hassler Whitney. 'On singularities of mappings of euclidean spaces. I. Mappings of the plane into the plane'. *Ann. of Math. (2)* 62 (1955), pp. 374–410.
doi:10.2307/1970070.