Beyond Mathieu Moonshine: a look at large $N = 4$ Algebras.

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Abstract: The conformal field theory approach to calculate the elliptic genus of $K3$ surfaces has revealed the Mathieu moonshine phenomenon, which highlights relations between the 'small' $\mathcal{N} = 4$ superconformal algebra at central charge $c = 6$, the sporadic group Mathieu 24 and mock modular forms. Here we take a look at a family of 'large' $\mathcal{N} = 4$ superconformal algebras, labelled $\mathcal{A}_\gamma, \gamma \in [\frac{1}{2}, \infty[$ (from which one can recover the small $\mathcal{N} = 4$ algebras in some limit), in the hope that a moonshine-like phenomenon might be observed. We consider realizations of $\mathcal{A}_\gamma$ and its closely related family of non-linear algebras $\widetilde{\mathcal{A}}_\gamma$ on $SU(3) = WS(3) \times SU(2) \times U(1)$, where $WS(3)$ is a 4-dimensional Wolf space, i.e. a quaternionic symmetric space. The underlying physical models are supersymmetric Wess-Zumino-Novikov-Witten models describing superstring propagation on the $SU(3)$ group manifold, for which explicit partition functions can be constructed. In order to exhibit the $\widetilde{\mathcal{A}}_\gamma$ (and $\mathcal{A}_\gamma$) symmetries of these models at the level of partition functions, we construct character sum rules which encode how products of affine $\widehat{su(3)}$ characters with a character for four 'Wolf space' fermions decompose as sums of $\widetilde{\mathcal{A}}_\gamma$ characters. We find close analytic forms for the corresponding branching functions in a theory with $\widetilde{\mathcal{A}}_\gamma$ symmetry where the levels of the two affine $\widehat{su(2)}$ subalgebras of $\widetilde{\mathcal{A}}_\gamma$ are $\tilde{k}^+ = 2$ and $\tilde{k}^- = 1$, and we discover that they form a vector-valued mock modular form of weight $1/2$. To arrive at this result, we used the transformation laws of the $\widetilde{\mathcal{A}}_\gamma$ characters under the modular group $SL(2, \mathbb{Z})$, which we derive in the twisted Ramond sector.

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The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

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This thesis is dedicated to

My Parents: YongQiu Tang and
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and

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Chapter 1

Introduction

The driving force in theoretical particle physics is the construction of a consistent quantized theory of the four fundamental forces of Nature. Experimentally, the Standard Model of particle physics is the most successful theory which describes the combination of special relativity and quantum mechanics. It gives a clue that the Universe is described by an effective theory of quantum fields at low energy level. However, the combination of general relativity and quantum mechanics is still mysterious. String theory is the most likely candidate for the quantization theory of gravity.

One remarkable concept in particle physics is supersymmetry, which pairs bosons and fermions. It is an extension of Poincaré symmetry and can also be viewed as an extension of spacetime describing new quantum mechanical degrees of freedom. It could explain why the masses of the fundamental particles of the Standard Model are so small compared to the Planck Mass, provide unification of the electroweak and strong forces and offer clues on the nature of dark matter. Moreover, supersymmetry arises naturally in string theory at the Planck scale. However, because supersymmetry requires each known elementary particle to have a superpartner of the same mass and opposite statistics, and since not a single superpartner has been found experimentally to date, one must conclude that if supersymmetry exists in Nature, it must be broken. But even if supersymmetry is broken at some lower scale than the Planck scale, an explanation of the small masses of elementary particles typically requires at least one of the superpartners to have a mass comparable to that of the heaviest Standard Model particles. Such a particle has so far failed to be detected in experiments at the Large Hadron Collider. Nevertheless, supersymmetry remains a powerful tool that has led to invaluable insights in other areas of physics and mathematics. This thesis is very much inspired by the more mathematical consequences of supersymmetry.

Indices, in the context of operator algebras and functional analysis, provide very helpful information in theories governed by elliptic differential operators (EDO). The index of an EDO is the difference between the dimension of its kernel and the dimension of its cokernel, which is an invariant quantity. As such, any deformation of an EDO by a compact operator may change the dimension of the kernel and the dimension of the cokernel, but the difference is unchanged. The most famous example of EDO in physics is the Dirac operator, which encodes the dynamics of a particle with spin, or the supercharge Q of a system with supersymmetric quantum mechanics, and its index is in fact the partition function of such a system. This index was further studied by Friedan and Windey [FW84], who related it to the chiral anomaly for fermion in background gauge and gravitational fields. Slightly prior to that, Witten had been studying constraints on supersymmetry breaking in supersymmetric theories defined in a spatial box [Wit82] and introduced a topological invariant to count the difference between the number of bosonic states and the number of fermionic states of zero energy in the Hilbert space $\mathcal{H}$ of such theories, modelled by $\mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right), \beta \to 0, F = 2\pi i J_z$ with $J_z \in \frac{1}{2}\mathbb{Z}$ being the generator of infinitesimal rotation and β being the inverse of temperature. He pointed out that, since the bosonic zero modes satisfy $Q|\mathrm{bos}\rangle = 0$ and the fermionic zero modes satisfy $Q^\dagger|\mathrm{ferm}\rangle = 0$, one actually deals with the 'analytic' index of the supercharge $\mathfrak{I}(Q) := \ker(Q) - \mathrm{coker}(Q) = \mathrm{Tr}_{\mathcal{H}}\left((-1)^F e^{-\beta H}\right), \beta \to 0$ when studying supersymmetry breaking. There is a priori another notion of index associated with an EDO , called the 'topological' index, but in fact, the Atiyah-Singer index theorem [AS68] states that the analytic index and the topological index of an EDO are equal. Using the path integral method on supersymmetric quantum mechanics, Alvarez-Gaume presented a derivation of the Atiyah-Singer index theorem [Alv83].

Index theorems in quantum field theories relate the spectrum of massless particles and the topology of the spacetime in which they evolve. Generalisations of such index theorems have been derived in string theory, where the Dirac operator is replaced by the Ramond operator, which acts on the configuration space of a closed string, known as the infinite-dimensional loop space $\mathcal{L}(M)$ of the target space M [AKMW87a] [AKMW87b]. These generalised index theorems relate the topology of $\mathcal{L}(M)$ to the string spectrum and the elliptic genus of the quantum field theory on the string world sheet. In particular, Witten [Wit87] showed that the large volume limit of the partition function of type IIB superstrings reproduces the topological elliptic genus introduced by the algebraic geometer Ochanine [Och87]. In this thesis, we call this object the conformal field-theoretic elliptic genus to emphasize that it is calculated using conformal field theory techniques, as in Witten's approach.

Local conformal symmetry appears naturally in the study of Polyakov's action for a bosonic string, and in Euclidean spacetime, one can use the sophisticated technology of two-dimensional conformal field theory (CFT) to analyse the string dynamics, which is governed by a non-linear σ model, i.e. a quantum CFT on the string world sheet. In string theory, including fermions needs supersymmetry, and the Witten index arises naturally in the supersymmetric generalization of conformal field theory (SCFT). It is well-known that a bosonic string lives in a 26-dimensional spacetime while a superstring emerges in a 10-dimensional spacetime, but the real world is 4-dimensional; hence the wish to compactify the 10-dimensional spacetime to a 4-dimensional manifold that would describe the physics of Nature. One popular compactification scheme uses Calabi-Yau manifolds of complex dimension 3, for which one has $\mathcal{N} = 2$ supersymmetry on the worldsheet and $\mathcal{N} = 1$ supersymmetry in $4d$ spacetime. The worldsheet supersymmmetry is encoded in a $2d$ superconformal algebra (SCA) with $\mathcal{N} = 2$ supercharges and central charge $c = 9$, and the conformal field-theoretic elliptic genus is expressible in terms of the characters of specific representations of the SCA. Other compactification schemes exist, and the one of interest when discussing Mathieu Moonshine is the compactification of the type IIB superstring on a K3 surface, which is a Calabi-Yau manifold of complex dimension 2. The corresponding $2d$, chiral SCA is the small $\mathcal{N} = 4$ algebra with four supercharges and central charge $c = 6$. In this case too, the elliptic genus can be derived using characters [ET88b] [ET88a] of this small $\mathcal{N} = 4$ SCA [EOTY89]. When one approaches the calculation of the elliptic genus from the string theory perspective, one arrives naturally at a formula where the elliptic genus is decomposed in small $\mathcal{N} = 4$ characters, which highlights a phenomenon that escapes detection when one takes the expression of the elliptic genus of K3 found in the mathematics literature, and which involves Jacobi theta functions. This phenomenon was first observed by Eguchi, Ooguri and Tachikawa in 2010 [EOT11] and was since coined 'Mathieu Moonshine'. It suggests an action of the sporadic group Mathieu 24 (M_{24}) within type IIB superstring theory, but to date this action has not been properly understood.

We are not making progress in that direction here, but instead investigate another type of SCFT, with an associated doubly-extended SCA called 'large' $\mathcal{N} = 4$ SCA (also known as $\mathcal{A}_\gamma$ algebra). The words 'doubly-extended' and 'large' refer to the fact that this SCA has an affine $\widehat{su(2)}_{k^+} \times \widehat{su(2)}_{k^-}$ subalgebra (as well as a $\widehat{u(1)}$ subalgebra) in contrast to the 'small' $\mathcal{N} = 4$ SCA, which only has one $\widehat{su(2)_k}$ subalgebra. It was discovered by Sevrin et al [SSTV88a; SSTV88b; STV88] and emerged from a systematic study of the restrictions imposed by extended supersymmetry on a Wess-Zumino-Novikov-Witten model (WZNW). Realizations of the large $\mathcal{N} = 4$ SCA exist, in particular, when the target space of the WZNW model is a manifold without curvature but with completely antisymmetric torsion, i.e. an absolutely parallelizable manifold [SSTV88a]. Of all such manifolds, the 8-dimensional group manifold of $SU(3)$ is, after the almost trivial $SU(2) \times U(1)$ manifold, the simplest manifold which admits an (almost) quaternionic complex structure which is necessary for $\mathcal{N} = 4$ supersymmetry. Moreover, as shown by Sevrin et al, any other absolutely parallelizable manifold leads to the existence of more than one energy-momentum tensor in the theory, and we will not study these here. From the physics point of view, we therefore concentrate on superstrings propagating on an $SU(3)$ group manifold, which could correspond to a compactification of type IIB superstrings leading to a $(1+1)$-dimenssional spacetime, but even that interpretation is not straightforward, as the central charge of the associated large $\mathcal{N}=4$ SCA in this case is $c = 12k^+/(k^++2), k^+ \in \mathbb{Z}_{\geq 2}$, which cannot be interpreted in the usual manner, for any value of k^+, i.e. as the contribution of d bosons and d fermions, namely $c := d + d/2$, when $d = 8$. We are not claiming this is a realistic string model, but instead that it is worth exploring the more mathematical aspects of the model, especially from an algebraic and number theoretic point of view.

We will in fact mainly concentrate on a SCA which is closely related to $\mathcal{A}_\gamma$, i.e. the non-linear SCA $\widetilde{\mathcal{A}}_\gamma$ which is obtained from $\mathcal{A}_\gamma$ after factorization of a free boson and four free fermions. $\widetilde{\mathcal{A}}_\gamma$ possesses an $\widehat{su(2)}_{\tilde{k}^+} \times \widehat{su(2)}_{\tilde{k}^-}$ affine subalgebra with levels $\tilde{k}^{\pm} = k^{\pm} - 1$ and central charge $\tilde{c} = c - 3$. The subtraction of 3 is due to the decoupling of the boson ($c = 1$) and the four fermions ($c = 4 \times \frac{1}{2}$). We will exploit the fact that $\widetilde{\mathcal{A}}_\gamma$ at $\tilde{k}^- = 1$ admits realizations on manifolds corresponding to group cosets $SU(3)/U(1)$ for any positive integer value of $\tilde{k}^+$ [GPTV89] and look in detail at the model with $\tilde{k}^+ = 2$ and $\tilde{k}^- = 1$. Character sum rules associated with these realizations were presented in [PT93], but failed to provide a complete analytic description of some of their constituents, which are the branching functions $\widehat{F}_i(\tau), i \in \mathcal{I}$ (for some model-dependent discrete set $\mathcal{I}$) occurring in the decomposition of the products of $\widehat{su(3)}_{\tilde{k}^+}$ characters with the character of a four-free-fermion system into products of $\widetilde{\mathcal{A}}_\gamma$ characters and a rational torus character emerging from the coset realization. We obtain analytic formulas for these branching functions in the model $\tilde{k}^+ = 2, \tilde{k}^- = 1$ and find that they form a vector-valued mock modular form (different for different realizations), a fact not anticipated in [PT93]. In order to pin down the information that was missing in that paper, we rely crucially on the transformation of $\widetilde{\mathcal{A}}_\gamma$ characters under the modular group, which we have derived using Appell functions and their modular transformations. The latter were worked out in [STT05] and also in [Zwe08]. The calculation was particularly challenging technically and we have provided full details for reference. As anticipated, and in analogy with the small $\mathcal{A}_\gamma$ case, the characters corresponding to short representations (massless) transform under S as a finite sum of massless characters and an infinite sum of characters for long representations (massive), while the massive characters transform into massive characters in a straightforward manner under $SL(2,\mathbb{Z})$.

The conformal field-theoretic elliptic genus for theories with small $\mathcal{N}=4$ symmetry was calculated using, in particular, a realization of the SCA in terms of Gepner models. In the calculation of the elliptic genus, the model-dependent branching functions of (sums of) tensor products of minimal $\mathcal{N}=2$ characters into small $\mathcal{N}=4$ massive characters would combine linearly in a model-dependent way but would, whatever the model, always produce the same mock modular form $h^{(2)}(\tau)$ encoding the information on M_{24}. This is a consequence of the topological invariance of the elliptic genus of $K3$. One might expect a similar phenomenon to occur in realizations of either the $\mathcal{A}_\gamma$ or $\widetilde{\mathcal{A}}_\gamma$ SCA, since the small $\mathcal{N}=4$ can be reached from $\mathcal{A}_\gamma$ in the limit where one of the two $\widehat{su(2)}$ levels tends to infinity [SSTV88a; SSTV88b]. However the conformal field-theoretic elliptic genus for $\mathcal{A}_\gamma$ theories vanishes due to the structure of the characters, and prompted Gukov et al [GMMS04] to introduce a new type of index, which they calculate for the symmetric product $\mathrm{Sym}^n(\mathcal{S})$ where $\mathcal{S}$ is a realization of $\mathcal{A}_\gamma$ in terms of a free boson and four Majorana fermions. This symmetric product has $\mathcal{A}_\gamma$ symmetry at $k^+=k^-=n$. Their motivation was the search for a holographic dual to $AdS_3\times S^3\times S^3\times S^1$. Our work on the other hand provides a sound preparation for the calculation of this new index for the WZNW coset realizations of $\mathcal{A}_\gamma$ and for exploring patterns that go beyond Mathieu Moonshine.

The structure of the thesis is as follows. Each chapter, apart from the Introduction and the Conclusion, is accompanied by a number of appendices which gather notations, some definitions and more technical information supporting the main text. In chapter 2, we review the Witten index of a $(0+1)$-dimensional quantum spinning particle theory, and present a physical understanding of the Atiyah-Singer index theorem by the path integral approach. We then generalise the Witten index for a $(1+1)$-dimensional quantum field theory, inspired by the work presented in [AKMW87a] and show the link between the Witten index and the elliptic genus.

We briefly introduce the small $\mathcal{N}=4$ superconformal algebra and its unitary representations in chapter 3. Using the characters of small $\mathcal{N}=4$ SCAs, we recall the elliptic genus for theories with small $\mathcal{N}=4$ symmetry and explain what the Mathieu moonshine is, after a brief description of Monstrous moonshine.

In chapter 4, we review the unitary representations of $\mathcal{A}_\gamma$ and $\widetilde{\mathcal{A}}_\gamma$ SCAs and give the corresponding massive and massless characters. We then rewrite the $\widetilde{\mathcal{A}}_\gamma$ massless characters in the twisted Ramond sector in terms of the higher level Appell function and write the massive characters as products of $\widehat{su(2)}$ characters. It is not difficult to derive the modular S-transformation of the massive characters in the twisted Ramond sector, but for the massless case, we first rewrite the massless characters in a particular way so that we can use some properties of higher level Appell functions, then proceed with the calculation of the S-transformation of the massless characters for coprime levels $\tilde{k}^+, \tilde{k}^-$. We end the chapter with an explanation of why the elliptic genus for $\mathcal{A}_\gamma$ theories is zero. A new topological invariant due to Gukov et. al [GMMS04] is introduced alongside a straightforward counterpart, which exploits an isomorphism of the algebra when $\bar{T}_0^{-3} \to -\bar{T}_0^{-3}$. We briefly comment on which states contribute to these two new indices.

We review the $\widetilde{\mathcal{A}}_\gamma$ character sum rules for $\tilde{k}^+ = 2, \tilde{k}^- = 1$ in chapter 5 and derive the S-transformation of the 6 branching functions $\widehat{F}_i(\tau)$ appearing in the sum rules for this model, alongside analytic expressions for all of them.

Finally, in chapter 6, we conclude this thesis and present ideas for future research.

The material presented in Chapter 4, Section 4.4, together with appendices 4.B.4, 4.B.2 and 4.D leading to the central result (4.4.47) is original, although we have benefitted, at an early stage of this project, from a private communication from Dr Y. Sugawara, who had derived a formula for the S-transformation of the massless $\widetilde{\mathcal{A}}_\gamma$ characters in the Neveu-Schwarz sector using a different method. We agree partially with his unpublished result after spectral flow (i.e. we agree with the contribution from massless characters after S-transformation, but the structure of the massive character contribution to the S transformation differs. We performed several consistency checks on our formula (4.4.47), in particular in the process of calculating the S-transformation of the character sum rules in Chapter 5. We are therefore confident in our results. The new perspective on the character sum rules, which we acquired thanks to the precise knowledge of the S-transformation of $\widetilde{\mathcal{A}}_\gamma$ characters, in particular the confirmation that the branching functions $\widehat{F}_i(\tau)$ form a vector-valued mock modular form and that they may be expressed in terms of functions known in number theory is also original. A paper in collaboration with Sam Fearn and Anne Taormina reporting on these discoveries is being prepared.

Chapter 2

From Index Theorem to Elliptic Genus

Atiyah and Singer have provided us with a profound result on elliptic partial differential operators on a compact manifold M, i.e. on those operators $\mathcal{D}$ such that $\dim \ker \mathcal{D}$ (the dimension of the space of solutions to $\mathcal{D}f = 0$) and $\dim \operatorname{coker} \mathcal{D}$ (the dimension of the space of constraints g in $\mathcal{D}f = g$, or equivalently the dimension of the adjoint operator $\mathcal{D}^\dagger$) are finite. Although $\dim \ker \mathcal{D}$ and $\dim \operatorname{coker} \mathcal{D}$ are usually difficult to derive without detailed knowledge of the operator $\mathcal{D}$, Atiyah and Singer showed that their difference, called the *index* of $\mathcal{D}$, may be computed using tools from topology, namely cohomology classes on the background manifold M. Interestingly, they also showed that it is sufficient to analyse the index problem for the class of first order elliptic operators - the so-called *Dirac operators* - which are of great importance in physics. From the physicists' point of view, the Atiyah-Singer index theorem [AS68] governs the spectrum of massless particles. Indeed, the index of the Dirac operator [Wit82] can be derived from a modified partition function of the spinning particle [FW84][Alv83], and one can probe the topology of the configuration space M, i.e. the spacetime in which such spinning particles evolve, with the help of the index theorem. A natural generalisation of such an index theorem has been derived in string theory, where the Dirac operator is replaced by the Ramond operator which acts on the configuration space of a closed string, known as the infinite-dimensional loop space $\mathcal{L}(M)$ of M. This generalised index theorem relates the topology of $\mathcal{L}(M)$ to the string spectrum. In this chapter, we will recall how the Atiyah-Singer index theorem can be recovered in the context of field theory by using path integral techniques in the case of a suitably modified spinning particle. We then review how to generalise the calculation to a closed string, which gives a version of the index theorem for string theory and links it to the elliptic genus introduced by Witten [Wit87].

2.1 Index Theorem in Field Theory

2.1.1 Witten Index

It is remarkable that the Atiyah-Singer index theorem can be derived in the framework of supersymmetric quantum mechanics, as we sketch below. An $\mathcal{N}$-supersymmetric quantum mechanical system is a (0+1)-dimensional supersymmetric quantum field theory with $\mathcal{N}$ supersymmetry (SUSY) generators. The corresponding algebra can be written as,

$$\begin{aligned} \{Q_i, Q_j^\dagger\} &= 2\delta_{ij}H, \\ \{Q_i, Q_j\} &= \{Q_i^\dagger, Q_j^\dagger\} = 0, \\ \{Q_i, (-1)^F\} &= 0, \quad i,j = 1, ..., \mathcal{N}, \end{aligned} \tag{2.1.1}$$

where Q_i are the SUSY generators (or SUSY charges) and $Q_i^\dagger$ their adjoint, H is the Hamiltonian of the system and $(-1)^F$ is the fermion number operator (also known as the fermion parity operator) defined as anticommuting with all the elementary fermion fields.

We follow [AKMW87a; AKMW87b] and consider an $\mathcal{N} = 1$ theory, introducing a Hermitian operator $S = \frac{1}{\sqrt{2}}(Q + Q^\dagger)$, and by the SUSY algebra we have $S^2 = H$. Let $|E\rangle$ be an arbitrary eigenstate of the Hamiltonian H with eigenvalue E. If $|E\rangle \neq 0$, we have another eigenstate $S|E\rangle$ with the same energy as $|E\rangle$ due to the supersymmetry (e.g. if $|E\rangle$ is a bosonic state, then $S|E\rangle$ is the corresponding fermionic state, and vice versa.), i.e. in non-zero energy states, we have boson-fermion pairs in the spectrum of the Hilbert space $\mathcal{H}$ of the SUSY system and states can only leave or reach ground energy by pairs. Witten introduced a quantity, which became known as the Witten index, to count the difference between the number of bosonic states ($n_B^{E=0}$) and the number of fermionic states ($n_F^{E=0}$)

in the Hilbert space of a supersymmetric system as [Wit82]

$$WI(Q) \equiv \mathrm{Tr}\left((-1)^F \exp(-\beta H) \right) = n_B^{E=0} - n_F^{E=0}. \tag{2.1.2}$$

Due to the supersymmetry, the Witten index depends only on the ground states. The above definition implies the Witten index is invariant under continuous deformations of the Hamiltonian and hence is a topological index for the full quantum theory, which is useful for calculating the index later on. Note that non-zero Witten index means the supersymmetry cannot be broken.

In this case, the Hilbert space $\mathcal{H}$ of a SUSY quantum mechanics may split into two subspaces: $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$ such that $(-1)^F$ has $+1(-1)$ eigenvalue on $\mathcal{H}_B(\mathcal{H}_F)$ and $Q : \mathcal{H}_B \to \mathcal{H}_F$, i.e. $(-1)^F|\mathrm{boson}\rangle = |\mathrm{boson}\rangle$ and $(-1)^F|\mathrm{fermion}\rangle = -|\mathrm{fermion}\rangle$, where $|\mathrm{boson}\rangle \in \mathcal{H}_B$ and $|\mathrm{fermion}\rangle \in \mathcal{H}_F$. For the ground states, we have $Q|\mathrm{boson}\rangle = 0$ and $Q^\dagger|\mathrm{fermion}\rangle = 0$, and hence they are the supersymmetric states. Therefore we could consider the Witten index as a topological invariant (i.e. an integer)

$$WI(Q) = \mathrm{Tr}\left((-1)^F \exp(-\beta H)\right) = \dim\ker\, Q - \dim\ker\, Q^\dagger, \tag{2.1.3}$$

where the $\ker Q = \{\psi; Q\psi = 0\}$ with ψ are states in the Hilbert space $\mathcal{H}$.

From a statistical mechanics perspective, $\mathrm{Tr}\left((-1)^F \exp(-\beta H) \right)$ can be considered as a partition function for an ensemble at finite temperateure β^{-1} with the density matrix $\rho = (-1)^F \exp(-\beta H)$. Hence in the path integral language it could be evaluated as [Alv83]

$$Tr\left((-1)^F \exp(-\beta H) \right) = \int_{PBC} d\phi(t) d\psi(t) \exp(-S_E(\phi, \psi)), \tag{2.1.4}$$

where $\phi(t), \psi(t)$ are bosonic and fermionic fields respectively, the acronym PBC means periodic boundary conditions with period β i.e. $\phi(0) = \phi(\beta)$ and $\psi(0) = \psi(\beta)$, and S_E is the action in the Euclidean space. In this perspective, we can perform an expansion of the action for small β ($\beta \to 0$), i.e. in the high temperature limit, to get an explicit topological index in a physical system.

2.1.2 The Index of the Dirac Operator

In this subsection, we will identify the analytic index of the Dirac operator with the Witten index for a supersymmetric theory.

Let M be a $2n$-dimensional compact oriented Riemannian manifold without boundary, and let $S_\pm(M)$ be chiral spin bundles[1] over M with positive and negative chirality respectively. Let $\Gamma(S_\pm(M))$ be the smooth sections of the chiral spin bundles $S_\pm(M)$. Then we define $\not{D}$ as the covariant Dirac operator on $S_\pm(M)$, namely

$$\not{D} := \begin{pmatrix} 0 & \not{D}_- \\ \not{D}_+ & 0 \end{pmatrix}, \qquad \not{D}_- = -\not{D}_+^\dagger, \tag{2.1.5}$$

where $\not{D}_+ : \Gamma(S_+(M)) \to \Gamma(S_-(M))$ and $\not{D}_- : \Gamma(S_-(M)) \to \Gamma(S_+(M))$.

The analytic index is defined to be

$$I(\not{D}) = \dim\ker\not{D}_+ - \dim\ker\not{D}_+^\dagger, \tag{2.1.6}$$

i.e. the number of positive chirality spinor fields annihilated by $\not{D}$ minus the number of negative chirality spinor fields annihilated by $\not{D}$.

Let $\Delta_+ \equiv \not{D}_+^\dagger\not{D}_+$ be the Laplacian. If $\not{D}_+\psi = 0$ where ψ is a spinor field, then we have $\Delta_+\psi = \not{D}_+^\dagger\not{D}_+\psi = 0$, and

$$\Delta_+\psi = 0 \Rightarrow \langle\psi, \Delta_+\psi\rangle = \langle\psi, \not{D}_+^\dagger\not{D}_+\psi\rangle = \langle\not{D}_+\psi, \not{D}_+\psi\rangle = 0 \Rightarrow \not{D}_+\psi = 0, \tag{2.1.7}$$

hence we have $\ker\not{D}_+ = \ker\Delta_+$, and similarly $\ker\not{D}_- = \ker\Delta_-$.

Let λ be the nonzero eigenvalue of the Laplacian, i.e. $\Delta_+\psi = \lambda\psi$. We then have $\Delta_-\not{D}_+\psi = \lambda\not{D}_+\psi$, as

$$\Delta_-\not{D}_+\psi = \not{D}_-^\dagger\not{D}_-\not{D}_+\psi = \not{D}_+\not{D}_+^\dagger\not{D}_+\psi = \not{D}_+\Delta_+\psi = \not{D}_+\lambda\psi \tag{2.1.8}$$

[1] A spin bundle needs the base manifold M to have a spin structure, i.e. the transition function $\tilde{\Lambda}_{ij} \in \mathrm{Spin}(m)$, where m is the dimension of M and Spin(m) is the double cover of SO(m), satisfying $\tilde{\Lambda}_{ij}\tilde{\Lambda}_{jk}\tilde{\Lambda}_{ki} = 1$ and $\tilde{\Lambda}_{ii} = 1$ with $\phi(\tilde{\Lambda}_{ij}) = \Lambda_{ij}$ where ϕ is the double covering Spin(m) $\to$ SO(m). One may think of the transition function, for instance, as $e_i(x) = \Lambda_{ij}(x)e_j(x)$, where $e_i(x)$ is a tetrad, and this relation is a local Lorentz transformation.

and

$$\langle \psi, \Delta_+ \psi \rangle = \lambda \langle \psi, \psi \rangle = \langle \not{D}_+ \psi, \not{D}_+ \psi \rangle \neq 0. \Rightarrow \not{D}_+ \psi \neq 0. \tag{2.1.9}$$

Therefore we proved that the nonzero eigenvalues of the two Laplacians are exactly the same, which means the index is invariant under a small deformation of $\not{D}_+$ since $\ker \Delta_+ (\ker \not{D}_+)$ and $\ker \Delta_- (\ker \not{D}_-)$ are paired. The Hamiltonian of the spinor system could be defined as

$$H = \not{D}^\dagger \not{D} = \begin{pmatrix} 0 & \not{D}_+^\dagger \\ \not{D}_-^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & \not{D}_- \\ \not{D}_+ & 0 \end{pmatrix} = \begin{pmatrix} \Delta_+ & 0 \\ 0 & \Delta_- \end{pmatrix}. \tag{2.1.10}$$

Hence the index of the Dirac operator (also known as the heat kernel formula for the index) can be written as

$$I(\not{D}) = \mathrm{Tr}\,\left(\gamma_5\, e^{-\beta H}\right) = \mathrm{Tr}\left(e^{-\beta \not{D}_+^\dagger \not{D}_+}\right) - \mathrm{Tr}\left(e^{-\beta \not{D}_-^\dagger \not{D}_-}\right), \tag{2.1.11}$$

$$= \dim\ker \Delta_+ - \dim\ker \Delta_-,$$

$$= \dim\ker \not{D}_+ - \dim\ker \not{D}_-, \quad \forall \beta > 0, \tag{2.1.12}$$

where γ_5 is $\mathrm{diag}\,(-1, 1)$, which satisfies $\{\not{D}, \gamma_5\} = 0$. Note that the index is independent of β, since the nonzero eigenvalues of $\ker(\not{D}_\pm)$ are paired.

Finally the identification between supersymmetry generator and the Dirac operator is given by

$$Q = \not{D}, \qquad (-1)^F = \gamma_5, \qquad \text{and} \qquad H = Q^\dagger Q = \not{D}^\dagger \not{D}. \tag{2.1.13}$$

2.1.3 Non-linear Sigma Model

In this subsection we will introduce a realization of the SUSY algebra (2.1.1), namely a $(0+1)$-dimensional supersymmetric **nonlinear σ-model** (a generalisation of SUSY quantum mechanics) whose lagrangian is given by [FW84] [Alv83]

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}(\phi) \dot{\phi}^\mu \dot{\phi}^\nu + \frac{i}{2} g_{\mu\nu}(\phi) \psi^\mu (\dot{\psi}^\nu + \dot{\phi}^\rho \Gamma^\nu_{\rho\sigma} \psi^\sigma), \quad \mu, \nu, \rho, \sigma = 1, \cdots, d, \tag{2.1.14}$$

where $\dot{\phi}^\mu$=$\frac{d}{dt}\phi^\mu(t)$ with $\phi^\mu(t)$ a scalar, $\psi^\mu(t)$ is a two component real spinor, $g_{\mu\nu}(\phi)$ is the metric on the d-dimensional spacetime manifold M, and $\Gamma^\nu_{\rho\sigma}$ is the Christoffel symbol. The basic (anti-)commutation relations are $[\phi_\mu, p_\nu] = ig_{\mu\nu}$ and $\{\psi_\mu, \psi_\nu\} = g_{\mu\nu}$. Noether theorem implies that in this case the supersymmetry charge is $Q = ip_\mu\psi^\mu = \frac{1}{\sqrt{2}}\gamma^\mu\partial_\mu$, as we define $\psi^\mu = \frac{1}{\sqrt{2}}\gamma^\mu$ with $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$.

When introducing the Cartan Formalism, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}(\phi)\dot{\phi}^\mu\dot{\phi}^\nu + \frac{i}{2}\eta_{ab}\psi^a(\dot{\psi}^b + \dot{\phi}^\mu\omega^a_{\mu a}\psi^b), \tag{2.1.15}$$

where

$$\omega^a_{\mu b} = e^a{}_\rho\partial_\mu e_b{}^\rho + e^a{}_\rho\Gamma^\rho_{\mu\sigma}e_b{}^\sigma = -e_b{}^\rho\partial_\mu e^a{}_\rho + e^a{}_\rho\Gamma^\rho_{\mu\sigma}e_b{}^\sigma, \tag{2.1.16}$$

is the spin connection, and the Vierbein $e^a{}_\mu$ is defined by $e^a{}_\mu e^b{}_\nu\eta_{ab} = g_{\mu\nu}$ with $e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu$ so that the spinor in curved spacetime is $\psi^a = e^a{}_\mu\psi^\mu$. In this case the supersymmetry charge becomes

$$\sqrt{2}Q = \not{D} \equiv \gamma^\mu(\partial_\mu + \omega_\mu) \equiv \gamma^a e_a{}^\mu(\partial_\mu + \frac{1}{2}\omega_{\mu ab}\sigma^{ab}), \sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b], \tag{2.1.17}$$

where we have used that $\{\psi^a, \psi^b\} = \eta^{ab}$ (after canonically quantising the theory) and $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ which implies $\gamma^a = \sqrt{2}\psi^a$, and $p_\mu = -i\partial_\mu$. The operator $\not{D}$ is called the massless **Dirac operator** of the spacetime manifold M.

We next couple the non-linear σ-model (2.1.15) to the external gauge field $A_\mu(\phi) \equiv A^\alpha_\mu T_\alpha$, $\alpha = 1, ..., dim(\mathfrak{g})$, where T_α are the generators of the Lie algebra $\mathfrak{g}$ of the non-abelian gauge symmetry. The Dirac operator then becomes

$$\not{D} = \gamma^\mu(\partial_\mu + \omega_\mu + A_\mu). \tag{2.1.18}$$

To retain supersymmetry in the Lagrangian, one has to introduce a pair of fermionic creation and annihilation operators κ_A and $\bar{\kappa}_B$ to couple with gauge fields, where $A, B = 1, ..., N$ with N the dimension of the representation of the Lie algebra $\mathfrak{g}$, satisfying

$$\{\kappa_A, \kappa_B\} = \{\bar{\kappa}_A, \bar{\kappa}_B\} = 0, \quad \{\bar{\kappa}_A, \kappa^B\} = \delta^B_A. \tag{2.1.19}$$

The Dirac operator may be recast as

$$\not{D} = \gamma^\mu(\partial_\mu + \omega_\mu + \bar{\kappa}A_\mu\kappa), \tag{2.1.20}$$

where the indices of $\kappa(\bar{\kappa})$ are omitted. The final Lagrangian, including background gauge and gravitational fields, is of the form

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu}(\phi)\dot{\phi}^\mu\dot{\phi}^\nu + \frac{i}{2}\eta_{ab}\psi^a(\dot{\psi}^b + \dot{\phi}^\mu\omega^a_{\mu a}\psi^b) + i\bar{\kappa}_A(\dot{\kappa}^B - iA^\alpha_\mu T^{AB}_\alpha\dot{\phi}^\mu\kappa_B) \\ - \frac{i}{2}\bar{\kappa}_A F^\alpha_{ab}T^{AB}_\alpha\psi^a\psi^b\kappa_B, \tag{2.1.21}$$

where $F^\alpha_{ab}T_\alpha \equiv F^\alpha_{\mu\nu}T_\alpha e_a{}^\mu e_b{}^\nu$ is the gauge strength of gauge field A_μ, which plays the same role as the spacetime curvature $R_{\mu\nu cd}$ of ϕ_μ. Notice that this theory has a mismatch between the number of bosons and fermions caused by the introduction of the fermions $\bar{\kappa}_A$ and κ_A; however, as mentioned before, the Dirac operator anticommutes with γ_5, and we then can write the index of the Dirac operator as

$$\mathrm{Tr}\left(\gamma_5\, e^{-\beta\not{D}^2}\right) = n^{E=0}(\gamma_5 = +1) - n^{E=0}(\gamma_5 = -1), \tag{2.1.22}$$

where $n^{E=0}(\gamma_5 = \pm 1)$ refers to the number of the zero eigenvalues of the Hamiltonian $H = \not{D}^\dagger\not{D}$ with $\gamma_5 = \pm 1$. Therefore when the index $(2.1.22)$ is computed, we have the equivalent Witten index $(2.1.4)$ of the supersymmetric system.

We now turn to calculate the functional integral of Lagrangian $(2.1.21)$ for the partition function $(2.1.22)$ in the limit $\beta \to 0$ as the index is a topological invariant which does not depend on β. In this limit, the functional integral is dominated by time-independent constant configurations, which indicates that the higher order interaction terms will drop out from the action, i.e.

$$\phi^\mu(t) = \phi^\mu_0 + \xi^\mu(t), \qquad \psi^a(t) = \psi^a_0 + \zeta^a(t), \qquad c_A = \bar{c}_A = 0, \tag{2.1.23}$$

where $\xi^\mu(t)(\zeta^a(t))$ are the fluctuations of $\phi(\psi)$ around $\phi_0(\psi_0)$, which are not constant configurations, and we also expand around $A_\mu(\phi_0)$ for last two terms of $(2.1.21)$. The functional integral nicely splits into constant and non-constant configurations. Then in the saddle point approximation (see Appendix 1) by using Riemann normal coordinate expansion around ϕ_0 (which means local coordinates are chosen such that the form of a generic metric is as close as possible to the flat metric), we obtain the Atiyah-Singer index of the Dirac operator $(2.1.20)$ of the $2n$-dimensional spacetime manifold M,

$$I(\not{D}) = (\frac{i}{2\pi})^n \int_M (\mathrm{Tr}\, e^{-\frac{i}{2}\psi_0^a\psi_0^b F_{ab}}) \prod_{i=1}^n \frac{ix_i/2}{\sinh(ix_i/2)}, \tag{2.1.24}$$

where x_i are the eigenvalues of the skew matrix $R_{ab} = \frac{1}{2}R_{abcd}\psi_0^c\psi_0^d$.
More geometrically, we define the curvature and the gauge strength as the two-forms

$$R_{ab} = \frac{1}{2}R_{abcd}dx^c \wedge dx^d, \tag{2.1.25}$$
$$F = \frac{1}{2}F_{ab}dx^a \wedge dx^b. \tag{2.1.26}$$

Here the states with one fermion $\psi^i|\Omega\rangle$ or their constant configurations $\psi_0^i|\Omega\rangle$ correspond to one-forms on the spacetime manifold M due to the quantisation algebra $\{\psi_i, \psi_j\} = g_{ij}(\phi)$. Therefore, the Hilbert space of the supersymmetric quantum theory can be represented by the exterior algebra $\Lambda^*(M)$ on M. The index of the Dirac operator can be recast as

$$I(\not{D}) = (\frac{i}{2\pi})^n \int_M ch(F)\hat{A}(M) \tag{2.1.27}$$

in terms of

$$ch(F) = \mathrm{Tr}\, e^{F/2\pi}, \quad \text{and} \quad \hat{A}(M) = \prod_{i=1}^n \frac{x_i/2}{\sinh(x_i/2)}, \tag{2.1.28}$$

where $\mathrm{ch}(F)$ is called the Chern character of the principal bundle in terms of the gauge field A_μ and $\hat{A}(M)$ the A-roof genus of the manifold M [EGH80].

2.2 Index Theorem in String Theory

2.2.1 Witten index in Superstring Theory

We saw that the Atiyah-Singer index theorem may be recast by computing the partition function of a supersymmetric field theory (with the supercharge playing the role of the Dirac operator) using path integral techniques. We also saw that index theorems probe topological properties of the configuration space M of a specified field theory, so one may expect to find similar properties of the configuration space of a closed string theory, i.e. the loop space $\mathcal{L}(M)$ of M, by studying index theorems in string theory [AKMW87a] [AKMW87b] [Wit88]. It was therefore natural for these authors to consider the generalisation to string theory of the partition function used in the case of point particles, and to calculate it with path integral techniques, while identifying which operator would play the role that the Dirac operator plays in field theory. It turns out that this operator is the supersymmetry generator G_0, also known as the **Ramond operator**. It is the zero mode of $T_F(z)$, the fermionic part of the energy-momentum superfield $T(z,\theta) = T_F(z) + \theta T_B(z)$ in the superconformal field theory associated with the closed superstring, while L_0 is the zero mode of its bosonic counterpart. Its index $WI(G_0)$ is given by the partition function of a closed superstring sweeping a two-dimensional torus embedded in a $2n$-dimensional spacetime manifold M (in the point particle case, the particle moves around a closed loop on the spacetime manifold M). One has,

$$WI(G_0) = \mathrm{Tr}\left((-1)^F \exp(-2\pi\tau_2 H + 2\pi i \tau_1 P) \right), \tag{2.2.1}$$

where the complex variable $\tau = \tau_1 + i\tau_2$, $\tau_2 > 0$ parametrises the $2d$-torus, which is the world sheet swept by the closed string, (we ignore that the string can slide on itself during its time evolution), P is the momentum operator generating a rotation between the initial and final states of the closed string, signalling the existence of an S^1 action on $\mathcal{L}(M)$. The momentum P and Hamiltonian H commute, i.e $[H, P] = 0$, and therefore, the states of the system can be labelled by both the eigenvalues of H and P.

In the radial quantisation, the Hamiltonian is given by

$$H = L_0 + \bar{L}_0 - \frac{1}{24}(c + \bar{c}) \tag{2.2.2}$$

and the momentum is of the form

$$P = L_0 - \bar{L}_0 - \frac{1}{24}(c - \bar{c}), \tag{2.2.3}$$

where L_0 ($\bar{L}_0$) are the left (right)-moving Virasoro generators and c (left-moving) and $\bar{c}$ (right-moving) are the central charges of the corresponding superconformal field theory. The Ramond operator commutes with P and H and anticommutes with the fermion number operator, i.e. $\{(-1)^F, G_0\} = 0$. The underlying superconformal algebra reveals that

$$G_0^2 = L_0 - c/24, \tag{2.2.4}$$

which implies that the states $|\psi_0\rangle$ invariant under supersymmetry satisfy $G_0|\psi_0\rangle = 0$ and $L_0|\psi_0\rangle = \frac{c}{24}|\psi_0\rangle$ (these are the supersymmetric ground states), and that the Ramond operator may be used to pair a bosonic and a fermionic state if the eigenvalue of G_0^2 is strictly greater than zero. Indeed starting with a state $|\psi_h\rangle$ such that

$$L_0|\psi_h\rangle = h|\psi_h\rangle \quad \text{and} \quad (-1)^F\,|\psi_h\rangle = |\psi_h\rangle,\ h \neq 0, \tag{2.2.5}$$

the state $|\widetilde{\psi}_h\rangle := G_0|\psi_h\rangle$ satisfies

$$L_0|\widetilde{\psi}_h\rangle = h|\widetilde{\psi}_h\rangle \quad \text{and} \quad (-1)^F\,|\widetilde{\psi}_h\rangle = -|\widetilde{\psi}_h\rangle, \tag{2.2.6}$$

so that, for all $h \neq 0$, the pair $(|\psi_h\rangle, |\widetilde{\psi}_h\rangle)$ consists of a bosonic and a fermionic state of same conformal dimension h. This has an important consequence on the index of G_0: the presence of the operator $(-1)^F$ in the trace means that all these pairs effectively disappear from the counting and only supersymmetric states of type $|\psi_0\rangle$ may contribute to the index. It is therefore possible to rewrite the index (2.2.1) in a more precise way, first with the help of (2.2.2), (2.2.3) and (2.2.4), namely

$$WI(G_0) = \mathrm{Tr}\left((-1)^F\, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \bar{c}/24}\right) = \mathrm{Tr}\left((-1)^F q^{G_0^2}\, \bar{q}^{\bar{L}_0 - \bar{c}/24}\right), \tag{2.2.7}$$

where $q = \exp(2\pi i\tau)$ and $\bar{q} = \exp(-2\pi i\bar{\tau}), \tau \in \mathfrak{H}$, where $\mathfrak{H}$ is the complex upper half-plane. Since the only states contributing are zero-eigenvalue states of the operator $L_0 - \frac{c}{24}$, the index $WI(G_0)$, viewed as a trace restricted to these contributing states, takes the form

$$\begin{aligned} WI(G_0) &= \mathrm{Tr}\left((-1)^F\, \bar{q}^{-P}\right) = \mathrm{Tr}\left((-1)^F\, e^{2\pi i(\tau_1 - i\tau_2)P}\right) \\ &= \mathrm{Tr}\left((-1)^F\, e^{2\pi i\tau_1 P - 2\pi\tau_2 H}\right) \end{aligned} \tag{2.2.8}$$

since

$$P = (L_0 - \frac{c}{24}) - (\bar{L}_0 - \frac{\bar{c}}{24}) = -(\bar{L}_0 - \frac{\bar{c}}{24}) \quad \text{and} \quad H = -P \tag{2.2.9}$$

with the trace restricted to the supersymmetric states $|\psi_0\rangle$. Finally, one may write

$$WI(G_0) = \bar{q}^{\bar{h}-\frac{\bar{c}}{24}} \sum_{n=0}^{\infty} I_n \, \bar{q}^n, \tag{2.2.10}$$

where the eigenvalues of P are of type $-(\bar{h} - \frac{\bar{c}}{24} + n), n \in \mathbb{N}$ and I_n for each value of n is an integer interpreted as an index on the subspace of states with momentum eigenvalue $-(\bar{h} - \frac{\bar{c}}{24} + n)$. Hence $WI(G_0)$ is an example of a *character-valued* index.

2.2.2 The Index of the Ramond Operator

In this subsection we will realize the Ramond operator G_0 in terms of a (1+1)-dimensional supersymmetric non-linear σ-model [AKMW87b], which generalizes (2.1.14), namely

$$\mathcal{L} = g_{\mu\nu}(\phi)\bar{\partial}\phi^\mu\partial\phi^\nu - g_{\mu\nu}(\phi)(\bar{\partial}\psi^\mu + \bar{\partial}\phi^\alpha\Gamma^\mu_{\alpha\beta}\psi^\beta)\psi^\nu, \qquad \alpha,\beta,\mu,\nu = 1,\ldots d, \tag{2.2.11}$$

where $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$, and $\phi^\mu(z,\bar{z})$ are $2d$-scalars with their superpartners $\psi^\mu(z)$ which are $2d$-, right-moving Majorana-Weyl spinors. Then the supersymmetric charge is defined as

$$Q \equiv G_0 = \int dz g_{\mu\nu}\psi^\mu\partial\phi^\nu. \tag{2.2.12}$$

As for the Dirac operator, the index of the Ramond operator $I(G_0)$ can be calculated in a similar approach, i.e. using path integral techniques on the partition function of the theory with Lagrangian (2.2.11), but some modular functions appear. One expects the index to only carry information about the loop space $\mathcal{L}(M)$, i.e. it should only depend on τ_1 and not τ_2 (since P is the 'loop' generator) but (2.2.8), which is the character-valued index, clearly depends on τ_2. As pointed out in [AKMW87a], the character-valued index is the boundary value of the index, which is an analytic function of $\bar{q}$. Therefore one calculates the path integral in the small τ_2 limit and then analytically continues the result to the associated analytic function to obtain the index. In order to do so, only the quadratic approximation of

(2.2.11) is needed. We therefore consider the Riemann normal coordinate expansion about the configurations $\phi^\mu = \phi_0^\mu + \xi^\mu$ and $\psi^\mu = \psi_0^\mu + \zeta^\mu$, where ϕ_0^μ and ψ_0^μ are the classical constant solutions while ξ^μ and ζ^μ are the fluctuations about these classical solutions. We have

$$g_{\mu\nu}(x_0) = \delta_{\mu\nu}, \quad \partial_\lambda g_{\mu\nu}(x_0) = 0, \quad \partial_\rho \partial_\lambda g_{\mu\nu}(x_0) = -\frac{1}{3}(R_{\mu\rho\nu\lambda} + R_{\mu\lambda\nu\rho}). \tag{2.2.13}$$

Then the quadratic approximation to (2.2.11) is

$$\mathcal{L} = \bar{\partial}\xi^\mu \partial \xi_\mu + R_{\mu\nu}\bar{\partial}\xi^\mu \xi^\nu + \bar{\partial}\zeta^\mu \zeta_\mu, \tag{2.2.14}$$

where $R_{\mu\nu} = \frac{1}{2} R_{\alpha\beta\mu\nu}\psi^\alpha \psi^\beta$. The index of the Ramond operator could be reduced to that of the Dirac operator

$$I(G_0) = \frac{1}{(2\pi)^{d/2}} \int_M d^d\phi \, d^d\psi \frac{[\mathrm{Det}'(\bar{\partial})]^{1/2}}{[\mathrm{Det}'(-\bar{\partial}\partial + \mathcal{R}\bar{\partial})]^{1/2}}, \tag{2.2.15}$$

where Det' means that the zero modes are not included in the calculation of the determinant, and $\mathcal{R}$ is the matrix with entries $R_{\mu\nu}$. In calculations, the torus considered has periods $\omega_1 = 1$ and $\omega_2 = \tau$ (recall that the modulus of the torus is defined as $\tau := \frac{\omega_2}{\omega_1}$, $\tau \in \mathfrak{H}$) and we consider the period lattice $\Gamma := \{w = m + n\tau | m, n \in \mathbb{Z}\}$ (corresponding to periodicity under $z \to z + 1$ and $z \to z + \tau$) alongside its complex conjugate version $\bar{\Gamma} := \{\bar{w} = m + n\bar{\tau} | m, n \in \mathbb{Z}\}$. In order to calculate $I(G_0)$ it is helpful to rewrite the ratio of determinants in (2.2.15) as

$$\frac{[\mathrm{Det}'(\bar{\partial})]^{1/2}}{[\mathrm{Det}'(\bar{\partial}\partial + \mathcal{R}\bar{\partial})]^{1/2}} = \frac{[\mathrm{Det}'(-\bar{\partial}\partial)]^{1/2}}{[\mathrm{Det}'(\bar{\partial}\partial + \mathcal{R}\bar{\partial})]^{1/2}} \frac{[\mathrm{Det}'(\bar{\partial})]^{1/2}}{[\mathrm{Det}'(-\bar{\partial}\partial)]^{1/2}} = \mathrm{Det}'(-\partial + \mathcal{R})^{-1/2}. \tag{2.2.16}$$

We first evaluate the ratio $\dfrac{[\mathrm{Det}'(\bar{\partial})]^{1/2}}{[\mathrm{Det}'(-\bar{\partial}\partial)]^{1/2}}$. The regularised determinant of the Laplacian $-\bar{\partial}\partial$ for the flat metric on the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is

$$\mathrm{Det}'(-\bar{\partial}\partial) = 4\tau_2^2 \, |\eta(\tau)|^4. \tag{2.2.17}$$

A detailed analytic derivation of this result may be found in [RS73] and Siegel's Lecture Notes on Advanced Analytic Number Theory [Sie65], where zeta function regularisation is used. We have presented the zeta function regularisation of the determinant of the Laplacian in one dimension in Appendix 2.A to highlight the technique in a simpler setting. As explained there, one starts by defining a zeta function for the differential operator of interest. Here, $\hat{O} := -\bar{\partial}\partial$, the Laplacian operator with eigenfunctions given by

$$e^{\frac{\pi}{\tau_2}(-\bar{w}z+w\bar{z})}, \quad \omega \in \Gamma,\ \bar{\omega} \in \bar{\Gamma} \tag{2.2.18}$$

and eigenvalues $(\frac{\pi}{\tau_2}\bar{w})(\frac{\pi}{\tau_2}w) = (\frac{\pi}{\tau_2})^2 \, |m+n\tau|^2 =: \lambda_{m,n}$ so that ($s \in \mathbb{C}$),

$$\begin{aligned}
\zeta_{-\bar{\partial}\partial}(s) &:= \sum_{(m,n)\neq(0,0)} \lambda_{m,n}^{-s} = \sum_{(m,n)\neq(0,0)} (\frac{\pi}{\tau_2})^{-2s} \, |m+n\tau|^{-2s} \\
&= 2(\frac{\pi}{\tau_2})^{-2s} \sum_{m=1}^{\infty} m^{-2s} + 2(\frac{\pi}{\tau_2})^{-2s} \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} |m+n\tau|^{-2s} \\
&= 2(\frac{\pi}{\tau_2})^{-2s}\, \zeta(2s) + 2(\frac{\pi}{\tau_2})^{-2s} \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} |m+n\tau|^{-2s},
\end{aligned} \tag{2.2.19}$$

where $\zeta(s)$ is the Riemann zeta function. It is well-established that $\zeta(s)$ can be continued analytically into the half- plane $\mathrm{Re}(s) > 0$ and that the continuation is regular on that half-plane, except for a simple pole at s = 1 with residue 1 (for a proof, see [Sie65] for example). So $\zeta(2s)$ has an analytic continuation on the half- plane $\mathrm{Re}(s) > 0$, which is regular except at $s = \frac{1}{2}$, where it has a simple pole. For the regularisation of determinants, one needs an analytic continuation of $\zeta_{-\bar{\partial}\partial}(s)$. The sought analytic continuation is [Sie65; RS73]

$$\zeta_{-\bar{\partial}\partial}(s) \;=\; -1 - 2s\ln|\prod_{m=1}^{\infty}(1-q^m)|^2 - 2s\ln\tau_2 - 2s\ln 2 + \frac{\pi}{3}\tau_2\, s, \tag{2.2.20}$$

with $\zeta_{-\bar{\partial}\partial}(0) = -1$ and

$$\zeta'_{-\bar{\partial}\partial}(0) = -2\ln|\prod_{m=1}^{\infty}(1-q^m)|^2 - 2\ln\tau_2 - 2\ln 2 + \frac{\pi}{3}\tau_2. \tag{2.2.21}$$

Since $e^{-\frac{\pi}{3}\tau_2} = e^{-\frac{\pi}{6i}(\tau-\bar{\tau})} = q^{\frac{1}{12}}\bar{q}^{\frac{1}{12}}$, and $\eta(\tau) = q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m)$, one gets

$$\mathrm{Det}'(-\bar{\partial}\partial) = e^{-\zeta'_{-\bar{\partial}\partial}(0)} = 4\tau_2^2\,|\eta(\tau)|^4 \tag{2.2.22}$$

as announced. This is consistent with the familiar partition function of a free boson on a torus ($c = \bar{c} = 1$), given by

$$\mathrm{Tr} q^{L_0 - 1/24} \bar{q}^{\bar{L}_0 - 1/24} = \eta(\tau)^{-1} \eta(\bar{\tau})^{-1} = 2\mathrm{Im}\tau \, \mathrm{Det}'(-\bar{\partial}\partial)^{-1/2}. \tag{2.2.23}$$

The regularised determinant for the operator $\bar{\partial}$ may be obtained by similar methods. The eigenvalues are $\frac{\pi}{\tau_2} w,\ w \in \Gamma$ and, after the appropriate definition of a zeta function $\zeta_{\bar{\partial}}(s)$ for $-\pi < \arg w < \pi$, one gets (see for instance [QHS93]),

$$\widetilde{\mathrm{Det}'}(\bar{\partial}) = -2i\tau_2 \, q^{-1/12} \, \eta(\tau)^2 = -iq^{-1/12} \mathrm{Det}'(\bar{\partial}) \tag{2.2.24}$$

This result is closely related to the calculation of the partition function of a free periodic (R, R) fermion on a torus. If one integrates the ($c = \frac{1}{2}$) holomorphic field only, ignoring the zero mode, one gets

$$\mathrm{Tr}' \, (-1)^F q^{L_0 - 1/48} = q^{-1/48} q^{1/16} \prod_{n=1}^{\infty} (1 - q^n) = \eta(\tau) \propto (2\tau_2)^{-1/2} \mathrm{Det}'(\bar{\partial})^{1/2}. \tag{2.2.25}$$

In d dimensions the result is [AKMW87b]

$$\frac{[\mathrm{Det}'(\bar{\partial})]^{1/2}}{[\mathrm{Det}'(-\bar{\partial}\partial)]^{1/2}} = [\frac{1}{(2\tau_2)^{1/2} \eta(\bar{\tau})}]^d. \tag{2.2.26}$$

We now briefly comment on the first ratio of regularised determinants in (2.2.16). Let us denote by $\pm 2\pi\lambda_i, i = 1, \ldots \frac{d}{2}$ the eigenvalues of the skew matrix $\mathcal{R}$, i.e. we set $\mathcal{R}_{2i-1,2i} = -\mathcal{R}_{2i,2i-1} = 2\pi\lambda_i$. Then,

$$\frac{[\mathrm{Det}'(-\bar{\partial}\partial)]^{1/2}}{[\mathrm{Det}'(-\bar{\partial}\partial + \mathcal{R}\bar{\partial})]^{1/2}} = \prod_{i=1}^{d/2} [\prod_{(m,n) \neq (0,0)} (1 - \frac{2i\tau_2 \lambda_i}{m + n\bar{\tau}})]^{-1} \tag{2.2.27}$$

$$= (\prod_{i=1}^{d/2} \frac{2i\tau_2 \lambda_i}{\sigma(2i\tau_2 \lambda_i)}) \exp[\frac{1}{2} G_2(\bar{w}) \sum_i \lambda_i^2], \tag{2.2.28}$$

where we use the Weierstrass sigma function associated to a $2d$-lattice $\bar{\Lambda} \in \mathbb{C}$, namely

$$\sigma(z) := z \prod_{\bar{w} \in \bar{\Lambda} \setminus (0,0)} (1 - \frac{z}{\bar{w}}) \, \exp(\frac{z}{\bar{w}} + \frac{1}{2} \frac{z^2}{\bar{w}^2}), \tag{2.2.29}$$

together with the weight-2 Eisenstein series

$$G_2(\bar{w}) := \sum_{\bar{w}\in\bar{\Gamma}\backslash\{(0.0)\}} \frac{1}{\bar{w}^2} = \sum_m \left(\sum_n \frac{1}{(m+n\bar{\tau})^2}\right)\Bigg|_{(m,n)\neq(0,0)}. \tag{2.2.30}$$

Note that the above expression is not absolutely convergent since the $G_2(\bar{w})$ is not absolutely convergent. A nice way to avoid this is to let the coefficient of $G_2(\bar{w})$ vanish, i.e. $\sum \lambda_i^2 = p_1(M) = TrR^2 = 0$, which means the first Pontryagin class of the manifold vanishes.

Finally, we use the expansion of $\sigma(z)/z$ in terms of Eisenstein series

$$G_{2k}(\tau) = \sum_{\{m,n\}\neq\{0,0\}} (m+n\tau)^{-2k}, \tag{2.2.31}$$

namely

$$\frac{\sigma(z)}{z} = \exp[-\sum_{k=2}^{\infty} z^{2k} G_k(\bar{w}) \tag{2.2.32}$$

in (2.2.27) and, after inserting (2.2.26) and (2.2.27) in (2.2.15), we obtain the final result [AKMW87b]

$$\begin{aligned} I(G_0) = q^{h-c/24}\sum_{k\in\mathbb{N}} I_k q^k &= \int_M d^d x \prod_{i=1}^{d/2} \frac{ix_i/2\pi}{\sigma(ix_i/2\pi,\omega)} \frac{1}{\eta(q)^d} \\ &= \int_M d^d x \prod_{i=1}^{d/2} \exp(\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau) x_i^{2k}) \eta(q)^{-d}. \end{aligned} \tag{2.2.33}$$

2.3 Elliptic Genus

Witten predicted that the role of the supercharge of the supersymmetric nonlinear σ-model in elliptic cohomology might be similar to the role of the Dirac operator in K-theory. Consider the previous case of the index of the Ramond operator (2.2.12). We may define the **elliptic genus** $EG(Q)$ of the supercharge of the 2d supersymmetric nonlinear σ-model with supersymetric right-moving Ramond sector as

$$EG(Q)(q) := I(Q)\,\eta(q)^d. \tag{2.3.1}$$

$EG(Q)(q)$ is a modular form of $SL(2,\mathbb{Z})/\mathbb{Z}_2$ of weight $d/2$ which may be written as

$$EG(Q)(q) = \prod_i^{d/2} \exp(\sum_{k=2}^{\infty} \frac{2}{(2k)!} G_{2k}(\tau)\lambda_i^{2k}), \tag{2.3.2}$$

where λ_i are the eigenvalues of the curvature two-form $R_{\mu\nu}$, and G_{2k} is the Eisenstein series of weight $2k$.

We may generalise the theory to the case of a closed type-II superstring with Ramond and Neveu-Schwarz sectors for right and left moving modes respectively. The elliptic genus is given by

$$I(Q) = (\frac{\eta(-q^{-1/2})}{\eta(q)\eta(-q)})^d EG(Q)(q) = Trq^{\bar{L}_0}\,\bar{q}^{L_0}(-1)^{F_R}, \tag{2.3.3}$$

where $EG(Q)(q)$ is a modular form of weight $d/2$ for the congruence subgroup $\Gamma_0(2)$ of $SL(2,\mathbb{Z})$, and $(-1)^{F_R}$ is the fermion number operator for the right-moving Ramond sector.

Hence, one might conclude that alternative supersymmetric nonlinear σ-models will have alternative way to define elliptic genera.

2.A Partition Function and Path Integral

2.A.1 Partition Function of Bosonic Harmonic Oscillators

In this appendix, the determinant of the Laplacian operator is calculated in the case of a one-dimensional harmonic oscillator. The aim is to highlight the technique of zeta-function regularisation of determinants, which is used in Subsection 2.2.2 in the case of the Laplacian for the flat metric on the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The Lagrangian of a one-dimensional simple harmonic oscillator with mass $m=1$ is given by

$$\mathcal{L}_{HO} = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \tag{2.A.1}$$

with energy eigenvalue $E_n = (n+\frac{1}{2})\omega$. The partition function is easy to compute as

$$Z_{HO} = \mathrm{Tr} e^{-\beta H} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\omega} = \frac{1}{2\sinh(\frac{1}{2}\beta\omega)}. \tag{2.A.2}$$

We now evaluate the partition function by using path integral methods, and the partition function may be written as

$$\begin{aligned} Z_{HO} = \mathrm{Tr} e^{-\beta H} &= \int dy \langle x|e^{-\beta H}|x\rangle = \int_{x(0)=y(\beta))} [\mathcal{D}y] \exp[-\int_0^\beta dt \frac{1}{2}(\dot{x}^2+\omega^2x^2)] \\ &= \int_{x(0)=y(\beta))} [\mathcal{D}y] \exp[-\int_0^\beta dt \frac{1}{2} x(-\frac{d^2}{dt^2}+\omega^2)x] \\ &= [\mathrm{Det}_{PBC}(-\frac{d^2}{dt^2}+\omega^2)]^{-1/2}, \end{aligned} \tag{2.A.3}$$

where 'PBC' stands for 'periodic boundary condition'. We used the Gaussian integral for the last step above, i.e.

$$\int [\mathcal{D}x] \exp(-\frac{1}{2}\sum A_{ij}x_i x_j) = (\det A)^{-1/2} \tag{2.A.4}$$

where A is a real symmetric positive definite matrix. To evaluate the determinant with PBC, we first write the Fourier expansion of $x(t)$ in the form

$$x(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} x_n e^{(\frac{2\pi i n}{\beta})t}, \tag{2.A.5}$$

then we have

$$\begin{aligned}
\mathrm{Det}_{PBC}(-\frac{d^2}{dt^2}+\omega^2) &= \prod_n((\frac{2\pi n}{\beta})^2+\omega^2) = \omega^2(\prod_{n=1}^{\infty}[(\frac{2\pi n}{\beta})^2+\omega^2])^2 \\
&= [\prod_{n=1}^{\infty}(\frac{2\pi n}{\beta})^2]^2(\omega\prod_{p=1}^{\infty}(1+(\frac{\beta\omega}{2\pi p})^2)^2] \\
&= \mathrm{Det}'_{PBC}(-\frac{d^2}{dt^2})\times(\omega\prod_{p=1}^{\infty}(1+(\frac{\beta\omega}{2\pi p})^2))^2],
\end{aligned} \tag{2.A.6}$$

where Det' means the determinant with the 'zero modes' excluded. To calculate $\mathrm{Det}'_{PBC}(-\frac{d^2}{dt^2})$ we will use the zeta function regularisation.

Let $\hat{O}$ be a positive definite operator with eigenvalues $\{\lambda_n\}$, and formally we have

$$\log\det{}' \hat{O} = \mathrm{Tr}'\log\hat{O} = \sum{}' \log\lambda_n, \tag{2.A.7}$$

where the prime denotes the omission of the 'zero mode'. Now we define a zeta function of the operator $\hat{O}$ as

$$\zeta_{\hat{O}}(s) := \sum{}' \frac{1}{\lambda_n^s}. \tag{2.A.8}$$

The right-hand side of above zeta function converges if $\mathrm{Re}(s)$ is sufficienty large and $\zeta(s)$ is analytic in this region. Note also that the zeta function can be analytically continued to the whole s-plane except at some (finite number) positive points. Hence we have

$$\frac{d\zeta_{\hat{O}}(s)}{ds}|_{s=0} = -\sum{}' \log\lambda_n, \tag{2.A.9}$$

and then

$$\det{}' \hat{O} = \exp[-\frac{d\zeta_{\hat{O}}(s)}{ds}|_{s=0}]. \tag{2.A.10}$$

Now we take $\hat{O}$ to be $\hat{O} = -d^2/dt^2$, hence

$$\zeta_{-d^2/dt^2}(s) = \sum_{n\neq 0}(\frac{2\pi n}{\beta})^{-2s} = 2(\frac{\beta}{2\pi})^{2s}\zeta(2s), \tag{2.A.11}$$

where $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function with $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. Then we have

$$\frac{d\zeta_{-d^2/dt^2}(s)}{ds}|_{s=0} = 2\times 2\times\zeta(0)\times\log\frac{\beta}{2\pi} + 2\times 2\times\zeta'(0) = -2\log\beta, \tag{2.A.12}$$

hence the final result is

$$\mathrm{Det}_{PBC}(-\frac{d^2}{dt^2}+\omega^2) = [(\beta\omega)\prod_{p=1}^{\infty}(1+(\frac{\beta\omega}{2\pi p})^2)]^2 = (2\sinh(\frac{1}{2}\beta\omega))^2, \qquad (2.A.13)$$

and

$$\mathrm{Tr} e^{-\beta H} = \frac{1}{2\sinh(\frac{1}{2}\beta\omega)}, \qquad (2.A.14)$$

which agrees with $(2.A.2)$. Note we used the formula

$$\frac{\sinh(x)}{x} = \prod_{n=1}[1+(\frac{x}{n\pi})^2]. \qquad (2.A.15)$$

2.A.2 Partition Function of Fermionic Harmonic Oscillators

Recall that the Hamiltonian of a Fermionic Harmonic Oscillator is given by

$$H = \frac{1}{2}(b^\dagger b - bb^\dagger)\omega, \qquad (2.A.16)$$

where the operators $b, b^\dagger$ satisfy the anticommutation relations

$$\{b, b^\dagger\} = bb^\dagger + b^\dagger b = 1, \quad \{b, b\} = \{b^\dagger, b^\dagger\} = 0. \qquad (2.A.17)$$

Then we could write the Hamiltonian as

$$H = \frac{1}{2}(b^\dagger b - (1 - b^\dagger b))\omega := (N - \frac{1}{2})\omega, \qquad (2.A.18)$$

where the number operator is defined as $N = b^\dagger b$ with eigenvalue 0 or 1 since $N^2 = b^\dagger b b^\dagger b = b^\dagger b = N$. Therefore we could derive the partition function for a Fermionic Harmonic Oscillator in the form

$$\mathrm{Tr} e^{-\beta H} = \sum_{n=0}^{1}\langle n|e^{-\beta H}|n\rangle = e^{-\beta\omega/2} + e^{\beta\omega/2} = 2\cosh(\beta\omega/2). \qquad (2.A.19)$$

Now we evaluate the partition function from fermionic path integral formula, i.e.

$$\mathrm{Tr} e^{-\beta H} = \int d\theta^* d\theta \langle -\theta|e^{-\beta H}|\theta\rangle e^{-\theta^*\theta}, \qquad \theta, \theta^* \text{Grassman variables} \qquad (2.A.20)$$

where the completeness relation $\int d\theta^* d\theta|\theta><\theta|e^{-\theta^*\theta} = 1$ is useful. Note that we impose anti-periodic condition $\theta(\beta) = -\theta(0)$ in the trace. Inserting the completeness relation into the partition function and with the help of the expression

$$e^{-\beta H} = \lim_{N\to\infty}(1-\beta H/N)^N, \tag{2.A.21}$$

we have

$$\begin{aligned}
\mathrm{Tr}e^{-\beta H} &= \lim_{N\to\infty}\int d\theta^* d\theta e^{-\theta^*\theta}\prod_{k=1}^{N-1} d\theta_k^* d\theta_k e^{-\sum_{n=1}^{N-1}\theta_n^*\theta_n} \\
&\times \langle -\theta|1-\beta H/N|\theta_{N-1}\rangle\langle\theta_{N-1}|...|\theta_1\rangle\langle\theta_1|1-\beta H/N|\theta\rangle \\
&= \lim_{N\to\infty}\int\prod_{k=1}^{N} d\theta_k^* d\theta_k e^{-\sum_{n=1}^{N}\theta_n^*\theta_n} \\
&\times \langle\theta_N|1-\beta H/N|\theta_{N-1}\rangle\langle\theta_{N-1}|...|\theta_1\rangle\langle\theta_1|1-\beta H/N|-\theta_N\rangle.
\end{aligned} \tag{2.A.22}$$

Let us focus on one matrix element, i.e.

$$\begin{aligned}
\langle\theta_k|(1-\beta H/N)|\theta_{k-1}\rangle &= \langle\theta_k|\theta_{k-1}\rangle[1-\frac{\beta}{N}\frac{\langle\theta_k|H|\theta_{k-1}\rangle}{\langle\theta_k|\theta_{k-1}\rangle}] \\
&\simeq \langle\theta_k|\theta_{k-1}\rangle\exp[-\frac{\beta}{N}\frac{\langle\theta_k|H|\theta_{k-1}\rangle}{\langle\theta_k|\theta_{k-1}\rangle}] \\
&= \exp[\theta_k^*\theta_{k-1}]\exp[-\frac{\beta}{N}\omega(\theta_k^*\theta_{k-1}-1/2)] \\
&= \exp[\frac{\beta\omega}{2N}]\exp[(1-\frac{\beta}{N}\omega)\theta_k^*\theta_{k-1}],
\end{aligned} \tag{2.A.23}$$

where we used $\langle\theta_k|\theta_{k-1}\rangle = 1+\theta_k^*\theta_{k-1} = \exp[\theta_k^*\theta_{k-1}]$. Then the partition function could be expressed as

$$\begin{aligned}
\mathrm{Tr}e^{-\beta H} &= e^{\beta\omega/2}\lim_{N\to\infty}\prod_{k=1}^{N}\int d\theta_k^* d\theta_k e^{-\sum_{n=1}^{N}\theta_n^*\theta_n} e^{\sum_{n=1}^{N}(1-\frac{\beta}{N}\omega)\theta_n^*\theta_{n-1}} \\
&= e^{\beta\omega/2}\lim_{N\to\infty}\prod_{k=1}^{N}\int d\theta_k^* d\theta_k e^{-[\sum_{n=1}^{N}\theta_n^*(\theta_n-\theta_{n-1})+(\frac{\beta}{N}\omega)\theta_n^*\theta_{n-1}]} \\
&= e^{\beta\omega/2}\int\mathcal{D}\theta^*\mathcal{D}\theta\exp[-\int_0^\beta dt\theta^*((1-\frac{\beta\omega}{N})\frac{d}{dt}+\omega)\theta] \\
&= e^{\beta\omega/2}Det_{APBC}((1-\frac{\beta\omega}{N})\frac{d}{dt}+\omega),
\end{aligned} \tag{2.A.24}$$

where 'APBC' denotes anti-periodic boundary condition $\theta(\beta) = -\theta(0)$. The Fourier expansion of $\theta(t)$ is $\theta(t) = \sum_{n=-\infty}^{\infty}\theta_n e^{\frac{\pi i(2n+1)t}{\beta}}$, then

$$\mathrm{Tr}e^{-\beta H} = e^{\beta\omega/2}\lim_{N\to\infty}\prod_{k=-N/4}^{N/4}[i(1-\frac{\beta}{N}\omega)\frac{\pi(2k-1)}{\beta}+\omega]$$

$$= e^{\beta\omega/2} e^{-\beta\omega/2} \prod_{k=1}^{\infty} [(\frac{2\pi(k-1/2)}{\beta})^2 + \omega^2]$$
$$= \prod_{k=1}^{\infty} [\frac{\pi(2k-1)}{\beta}]^2 \prod_{n=1}^{\infty} [1 + (\frac{\beta\omega}{\pi(2n-1)})^2]. \quad (2.A.25)$$

Note that the first product is divergent and need regularisation. Let us denote

$$\log P = \sum_{k=1}^{\infty} 2\log[\frac{2\pi(k-1/2)}{\beta}], \quad (2.A.26)$$

and the corresponding zeta function is of the form

$$\hat{\zeta}(s) = \sum_{k=1}^{\infty} [\frac{2\pi(k-1/2)}{\beta}]^{-s} = (\frac{\beta}{2\pi})^s \zeta(s, 1/2), \quad (2.A.27)$$

with $P = e^{-2\hat{\zeta}'(0)}$, where the generalised ζ-function is defined as

$$\zeta(s,a) = \sum_{k=0} \frac{1}{(k+a)^s}, \quad 0 < a < 1, \quad (2.A.28)$$

with $\zeta(0, 1/2) = 0$ and $\zeta'(0, 1/2) = -\frac{1}{2}\log 2$ as the derivative of $\hat{\zeta}(s)$ at $s = 0$ is

$$\hat{\zeta}'(0) = \log(\frac{\beta}{2\pi})\zeta(0/1/2) + \zeta'(0, 1/2) = -\frac{1}{2}\log 2. \quad (2.A.29)$$

Therefore we have P=2 and the final version of the partition function is of the form

$$\mathrm{Tr}e^{-\beta H} = 2\prod_{n=1}^{\infty} [1 + (\frac{\beta\omega}{\pi(2n-1)})^2] = 2\cosh(\frac{\beta\omega}{2}), \quad (2.A.30)$$

where we used $\cosh(x/2) = \prod_{n=1}^{\infty} [1 + \frac{x^2}{\pi^2(2n-1)^2}]$.

For later use, we next introduce the'twisted' trace

$$\mathrm{Tr}(-1)^F e^{-\beta H} = e^{-\beta(-\frac{1}{2}\omega)} + (-1)e^{-\beta(\frac{1}{2})\omega} = 2\sinh\frac{\beta\omega}{2}, \quad (2.A.31)$$

where $(-1)^F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and H is the Hamiltonian of a fermion. And from above calculation we may express the twisted trace as path integral of a fermion with periodic boundary condition.

2.A.3 The Supersymmetry Path Integral

To put the bosonic and fermionic path integrals on an equal footing, we may impose periodic boundary conditions on the fermionic part of the partition function, i.e.

$$
\begin{aligned}
\mathrm{Tr}(-1)^F e^{-\beta H} &= \sum_{n=0}^{1} \langle \theta | (-1)^F e^{-\beta H} | \theta \rangle \\
&= \int d\theta^* d\theta \langle -\theta | (-1)^F e^{-\beta H} | \theta \rangle e^{-\theta^* \theta} \\
&= \int d\theta^* d\theta \langle \theta | e^{-\beta H} | \theta \rangle e^{-\theta^* \theta},
\end{aligned} \tag{2.A.32}
$$

where $(-1)^F|\theta\rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \theta \\ 1 \end{pmatrix} = \begin{pmatrix} -\theta \\ 1 \end{pmatrix} = |-\theta\rangle$ and $\langle\theta|(-1)^F = \langle\theta|$. Therefore the supersymmetry path integral in the Euclidean time is of the form

$$
\mathrm{Tr}(-1)^F \exp(-\beta H) = \int_{PBC} d\phi(t) d\psi(t) \exp(-S_E(\phi, \psi)), \tag{2.A.33}
$$

where $\phi(t), \psi(t)$ are bosonic and fermionic field respectively, the PBC is the periodic boundary conditions with period β for short which means $\phi(0) = \phi(\beta)$ and $\psi(0) = \psi(\beta)$, and S_E is the action in the Euclidean space.

2.B Saddle point approximation to calculate indices

We will give detailed calculation of the index of the Dirac operator $(2.1.20)$ by using the saddle point approximation in the Riemann normal coordinate expansion about the constant configuration ϕ_0 which is of the form

$$
g_{\mu\nu}(\phi_0) = \eta_{\mu\nu}, \quad \partial_\lambda g_{\mu\nu}(\phi_0) = 0, \quad \eta_{ab}\omega^a_{\mu a}(\phi_0) = \frac{1}{2} R_{ab\mu\nu}\xi^\nu. \tag{2.B.1}
$$

Then the second-order term in the expansion of the Lagrangian $(2.1.21)$ will be given as

$$
\mathcal{L}_2 = \frac{1}{2}\eta_{\mu\nu}\dot{\xi}^\mu\dot{\xi}^\nu + \frac{i}{2}R_{\mu\nu}\dot{\xi}^\mu\xi^\nu + \frac{i}{2}\eta_{ab}\dot{\zeta}^a\zeta^b + i\dot{\kappa}_A\bar{\kappa}_A - \frac{i}{2}\psi_0^a\psi_0^b F^\alpha_{ab}(\phi_0)\bar{\kappa}_A T^{AB}_\alpha \kappa_B. \tag{2.B.2}
$$

where $R_{\mu\nu} \equiv \frac{1}{2}\psi_0^a\psi_0^b R_{ab\mu\nu}$. Let us first consider the first three terms, and we will evaluate the index of $\not{D}$, i.e.

$$I(\not{D}) = \int_M \mathcal{D}\xi\mathcal{D}\zeta\; exp(-\int_0^\beta dt\mathcal{L}_2), \tag{2.B.3}$$

where we have used that the measure of the path integral $\mathcal{D}\xi\mathcal{D}\zeta = \mathcal{D}\phi\mathcal{D}\psi$ is invariant under the translation of ϕ and ψ. The Fourier expansions of ξ^μ and ζ^μ (for simplicity, we assume $\beta = 1$) are

$$\xi^\mu = \sum_{n=-\infty}^{\infty} \xi_n^\mu e^{-2\pi int}, \tag{2.B.4}$$

$$\zeta^\mu = \sum_{n=-\infty}^{\infty} \zeta_n^\mu e^{-2\pi int}. \tag{2.B.5}$$

Then we could rewrite the $I(\not{D})$ by using Gaussian integrals of Grassmann even and odd numbers as

$$I(\not{D}) = \mathcal{N}\int_M \prod_{\mu=1}^{2n} \frac{1}{\sqrt{2\pi}} d\xi_0^\mu d\zeta_0^\mu \frac{[\mathrm{Det}'_{PBC}(\delta_{\mu\nu}\frac{d}{dt})]^{1/2}}{[\mathrm{Det}'_{PBC}(-\delta_{\mu\nu}\frac{d^2}{dt^2} + R_{\mu\nu}(\phi_0)\frac{d}{dt})]^{1/2}} \tag{2.B.6}$$

$$= \mathcal{N}\int_M \prod_{\mu=1}^{2n} \frac{1}{\sqrt{2\pi}} d\xi_0^\mu d\zeta_0^\mu [\mathrm{Det}'_{PBC}(-\delta_{\mu\nu}\frac{d}{dt} + R_{\mu\nu}(\phi_0))]^{-1/2}, \tag{2.B.7}$$

where $\mathcal{N}$ is the the normalisation factor which will be i^n[7], $1/\sqrt{2\pi}$ is from the Feynman measure for the constant modes, and Det' means we omit the zero modes in the determinant. As $R_{\mu\nu} = -R_{\nu\mu}$ is antisymmetric, we may write block diagonalized $R_{\mu\nu}$ in $2n$-dim manifold M as

$$R_{\mu\nu} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_n \\ & & & -x_n & 0 \end{pmatrix}, \tag{2.B.8}$$

and we focus on a 2×2 block,

$$\mathrm{Det}' \begin{pmatrix} -\frac{d}{dt} & x_1 \\ -x_1 & -\frac{d}{dt} \end{pmatrix} = \mathrm{Det}'(-\frac{d^2}{dt^2} + x_1^2) = \prod_{n\neq 0}(x_1^2 + (2\pi n)^2) \tag{2.B.9}$$

$$= [\prod_{n\geq 1}(2\pi n)^2 \prod_{n\geq 1}[1 + (\frac{x_1}{2\pi n})^2]]^2 \tag{2.B.10}$$

$$= \mathrm{Det}'(-\frac{d^2}{dt^2})(\frac{\sinh\ x_1/2}{x_1/2})^2 \tag{2.B.11}$$

$$= (\frac{\sinh\ x_1/2}{x_1/2})^2, \tag{2.B.12}$$

where we used $\mathrm{Det}'(-\frac{d^2}{dt^2}) = 1$ by Riemann zeta regularisation. Therefore the index of the Dirac operator could be write as

$$I(\not{D}) = (\frac{i}{2\pi})^n \int_M \prod_{i=1}^{n} \frac{x_i/2}{\sinh\ x_i/2}. \tag{2.B.13}$$

Then we consider the last two terms of the Lagrangian $(2.B.2)$. As we did for the first three terms by expanding around $A_\mu(\phi_0)$, we will compute the fermion determinant

$$\mathrm{Det}'(\frac{d}{dt} - \frac{1}{2}F_{ab}\psi_0^a\psi_0^b). \tag{2.B.14}$$

By using the formula

$$\mathrm{Tr}\ e^{-\bar{\kappa}\omega\kappa} = det(1+e^{\omega}), \tag{2.B.15}$$

where ω is an antisymmetric matrix, we have the final expression of the index

$$I(\not{D}) = (\frac{i}{2\pi})^n \int_M ch(F)\hat{A}(M), \tag{2.B.16}$$

where $\mathrm{ch}(F) = \mathrm{Tr}\ e^{F/2\pi}$ with $F = \frac{1}{2}F_{ab}\psi_0^a\psi_0^b$ is called the **Chern character** of the principal bundle defined by the gauge field, and $\hat{A}(M) = \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)}$ is known as the **$\hat{A}$-roof genus** [EGH80]. This (analytical) index of $\not{D}$ is the celebrated **Atiyah-Singer formula** for index of the Dirac operator in even dimensional oriented compact manifold.

Chapter 3

Superconformal Algebras and Elliptic Genera

In this chapter, we first review the Virasoro algebra and its representations, and then discuss a natural extension of the Virasoro algebra, i.e. the superconformal algebras of 2-dimensional superconformal field theories. In 1971, P. Ramond introduced fermions in the Dual Resonance Model [Ram71], which could be considered as the first example of the super-extension of the Virasoro algebra. After that, Neveu and Schwarz considered this model further by introducing fermionic operators [NS71a][NS71b]. Ten years later, L. Alvarez-Gaume and DZ. Freedman [AF81] studied the superconformal σ-model, and physicists started learning superconformal field theory as the world-sheet theory of string theory with two different superconformal algebras introduced by Ramond, and Neveu and Schwarz, respectively.

We will briefly discuss the $\mathcal{N} = 1$ superconformal algebras when introducing a supercharge operator, and then if there exists a current operator (R-symmetry) in a superconformal field theory, we will have the extended $\mathcal{N} = 2$ superconformal algebras. From $\mathcal{N} = 2$ to small $\mathcal{N} = 4$ we will use Odake's [Oda89] approach, i.e. introducing a spectral flow generator $S(z)$. We then study the representation theory of the small $\mathcal{N} = 4$ superalgebra and give the characters of unitary, highest weight state representations of this algebra. Finally we will introduce the conformal field-theoretic elliptic genus of a small $\mathcal{N} = 4$ theory whose calculation requires knowledge of the characters, and recall the Mathieu Moonshine phenomenon which was observed by Eguchi, Ooguri and Tachikawa in 2011 [EOT11].

3.1 Virasoro Algebra and Representations

In this section, we will review the Viraroso algebra and its representations. The general form of the operator product expansion for the energy-momentum tensor is given by

$$T(z)T(w) = \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} + ... \tag{3.1.1}$$

where c denotes the central charge of a conformal field theory. We perform a Laurent expansion (aka mode expansion) around the origin $z = 0$, namely

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z). \tag{3.1.2}$$

Then we could calculate the commutator of two generators L_m and L_n with $m, n \in \mathbb{Z}$ by the help of the OPE (3.1.1), i.e.

$$\begin{aligned}
[L_m, L_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z)T(w)] \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \oint_w \frac{dz}{2\pi i} z^{m+1} T(z)T(w) \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \oint_w \frac{dz}{2\pi i} z^{m+1} (\frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}) \\
&= \oint \frac{dw}{2\pi i} w^{n+1} \oint_w ((m+1)m(m-1)w^{m-2}\frac{c}{2\cdot 3!} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w)) \\
&= \oint \frac{dw}{2\pi i} (\frac{c}{12}(m^3-m)w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + w^{m+n+2}\partial_w T(w)) \\
&= \frac{c}{12}(m^3-m)\delta_{m+n,0} + 2(m+1)L_{m+n} + 0 - (m+n+2)L_{m+n} \\
&= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0},
\end{aligned} \tag{3.1.3}$$

where $\oint$ and $\oint_w$ means the contour integral around origin and point w respectively, and we denote $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$ and $L_m = \oint \frac{dw}{2\pi i} w^{m+1} T(w)$. We call the above algebra the Virasoro Algebra. One could also obtain the Virasoro algebra by the central extension (the term containing the central charge) of the Witt algebra. Note that there may exist different CFTs with the same central charge, which will be important in the future discussion.

In a closed string theory, two decoupled Virasoro algebras with central charge $(c, \bar{c})$ play an important role and are given below

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
&[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}(m^3 - m)\delta_{m+n,0}, \\
&[L_m, \bar{L}_n] = 0.
\end{aligned} \tag{3.1.4}
$$

In order to build a quantum field theory with conformal symmetry, we need a Hilbert space $\mathcal{H}$, a vacuum vector $|0\rangle$ and a bunch of observables. Consider a primary field $\Phi(z, \bar{z})$ with conformal weights $(h, \bar{h})$. Then it satisfies the following OPEs with the energy-momentum tensors $T(w)$ and $\bar{T}(\bar{w})$,

$$
\begin{aligned}
T(w)\Phi(z,\bar{z}) &= \frac{h}{(w-z)^2}\Phi(z,\bar{z}) + \frac{1}{w-z}\partial_z\Phi(z,\bar{z}) + \ldots. \\
\bar{T}(\bar{w})\Phi(z,\bar{z}) &= \frac{\bar{h}}{(\bar{w}-\bar{z})^2}\Phi(z,\bar{z}) + \frac{1}{\bar{w}-\bar{z}}\partial_{\bar{z}}\Phi(z,\bar{z}) + \ldots.
\end{aligned} \tag{3.1.5}
$$

We can perform a Laurent expansion around the origin of the complex plane $\mathcal{C}^2$ as

$$
\Phi(z,\bar{z}) = \sum_{n,m\in\mathbb{Z}} z^{-n-h}\bar{z}^{-m-\bar{h}}\Phi_{n,m}. \tag{3.1.6}
$$

Then we define an asymptotic in-state $|\Phi\rangle$ as

$$
|\Phi\rangle := \lim_{z,\bar{z}\to 0}\Phi(z,\bar{z})|0\rangle = \Phi_{-h,-\bar{h}}|0\rangle, \tag{3.1.7}
$$

as $z, \bar{z} \to 0$ at the origin of the complex plane implies the infinite time (past infinite) $\sigma_0 = -\infty$ on the cylinder. The above relation is often called state-operator correspondence that maps a field $\Phi(z, \bar{z})$ to a state $|\Phi\rangle$. Note that the above map is bijective which means every state corresponds uniquely to a single local operator while for different fields one can find the same asymptotic in-state. Note also that

$$
\Phi_{n,m}|0\rangle = 0 \quad \text{for} \quad \mathrm{n} > -\mathrm{h},\ \mathrm{m} > -\bar{\mathrm{h}}. \tag{3.1.8}
$$

The Hermitian conjugation of a primary field Φ is defined by

$$\Phi^\dagger(z,\bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \Phi(\frac{1}{\bar{z}}, \frac{1}{z}), \tag{3.1.9}$$

with its Laurent expansion

$$\Phi^\dagger(z,\bar{z}) = \sum_{n,m\in\mathbb{Z}} \bar{z}^{+n-h} z^{+m-\bar{h}} \Phi_{n,m}, \tag{3.1.10}$$

hence we have $\Phi^\dagger_{n.m} = \Phi_{-n,-m}$. Then the Hermitian conjugate of the asymptotic in-states, namely the asymptotic out-states is given by

$$\langle\Phi| = \lim_{z,\bar{z}\to 0} \langle 0|\Phi^\dagger(z,\bar{z}) = \lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}} \langle 0|\Phi(w,\bar{w}) = \langle 0|\Phi_{h,\bar{h}}. \tag{3.1.11}$$

Note that

$$\langle 0|\Phi_{n,m} = 0, \quad for \quad n < h,\ m < \bar{h}. \tag{3.1.12}$$

Asymptotic states are created by primary fields acting on the vacuum. Indeed, using (3.1.2), (3.1.5) and residue analysis, one shows that, for $n \geq -1$,

$$[L_n, \Phi(z,\bar{z})] = \frac{1}{2\pi i}\oint_z dw\, w^{n+1} T(w)\Phi(z,\bar{z}) = h(n+1)\, z^n\, \Phi(z,\bar{z}) + z^{n+1}\,\partial\Phi(z,\bar{z}), \tag{3.1.13}$$

with analogous commutation relations when $\bar{T}(\bar{w})$ is used. If one considers the asymptotic in-state state (3.1.7), and introduce the notation $|h,\bar{h}\rangle \coloneqq \Phi(0,0)|0\rangle$, the commutation relations above reveal that $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian since

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \quad \text{and analogously } \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle. \tag{3.1.14}$$

Furthermore, for $n > 0$, the same commutation relations yield

$$L_n|h,\bar{h}\rangle = 0 \quad \text{and analogously } \bar{L}_n|h,\bar{h}\rangle = 0. \tag{3.1.15}$$

So for instance, the conformal weight h is the energy of the state $|h,\bar{h}\rangle$ with respect to the energy-momentum tensor $T(w)$. Although the full conformal field theory contains holomorphic and antiholomorphic degrees of freedom, they very often decouple from each other in physics systems and one can for instance 'ignore' the antiholomorphic degrees of freedom when performing complicated calculations since it is very easy to restore the dependence on these degrees of freedom at the end. In this spirit, we now consider the following commutation relations for the modes $\Phi_n = \oint \frac{dw}{2\pi i} w^{n+h-1}\Phi(w)$ of a chiral primary field $\Phi(z)$ of conformal weight h, namely,

$$\begin{aligned}
[L_m, \Phi_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+h-1} [T(z)\Phi(w)] \\
&= \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_w \frac{dz}{2\pi i} z^{m+1} T(z)\Phi(w) \\
&= \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_w \frac{dz}{2\pi i} z^{m+1} \left(\frac{h}{(z-w)^2}\Phi(w) + \frac{1}{z-w}\partial_w \Phi(w)\right) \\
&= \oint \frac{dw}{2\pi i} (h(m+1) w^{m+n+h-1}\Phi(w) + w^{m+n+h}\partial_w \Phi(w)) \\
&= h(m+1)\Phi_{m+n} - (m+n+h)\Phi_{m+n} \\
&= ((h-1)m - n)\Phi_{m+n}.
\end{aligned} \tag{3.1.16}$$

Let $|h\rangle$ be an eigenstate of L_0 with eigenvalue h, i.e. $L_0|h\rangle = h|h\rangle$ which is also annihilated by $L_n, \text{for all}\, n > 0$. Then (3.1.16) shows that $\Phi_{-n}, n > 0$ acts as a raising operator on $|h\rangle$. Indeed,

$$L_0 \Phi_{-n}|h\rangle = [L_0, \Phi_{-n}]|h\rangle + \Phi_{-n} L_0 |h\rangle = (n+h)\Phi_{-n}|h\rangle. \tag{3.1.17}$$

Furthermore for a state $|\Psi\rangle$ of conformal weight H,

$$L_0 L_{\pm n}|\Psi\rangle = [L_0, L_{\pm n}]|\Psi\rangle + L_{\pm n} L_0|\Psi\rangle = (\mp n + H) L_{\pm n}|\Psi\rangle, \tag{3.1.18}$$

which means the generators L_n and L_{-n} are lowering and raising operators respectively (as they decrease (resp. increase) the energy H of the state $|\Psi\rangle$). We can construct a set of states by acting with products of the raising operators $L_{-n}, n \in \mathbb{Z}_{>0}$ on the state $|h\rangle$, which is commonly called the ‘highest weight state’ of the Virasoro representation. The terminology of ‘highest weight state’ for $|h\rangle$ is misleading here, given that the module is built on $|h\rangle$ by the action of operators $L_{-n}, n > 0$ which *increase* the energy from h to $n + h$. However we follow the common practice here and will *not* call $|h\rangle$ a *lowest* weight state. We have

$$L_{-1}|h\rangle, \quad L_{-2}|h\rangle, \quad L_{-1}L_{-1}|h\rangle, \quad L_{-3}|h\rangle, \ldots \tag{3.1.19}$$

The hermitian conjugate of L_n is given by

$$\langle h|L_{-n} = 0, \forall n > 0 \tag{3.1.20}$$

where $L_n^\dagger = L_{-n}$

We now exploit the so-called state-field correspondence further. Note that the asymptotic in-state of the energy- momentum $T(z)$ is given by

$$L_{-2}|0\rangle, \quad T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n. \tag{3.1.21}$$

and hence the asymptotic in-state of the first derivative of $T(z)$ is of the form

$$L_{-3}|0\rangle, \quad \partial T(z) = \sum_{n\in\mathbb{Z}} (-n-2) z^{-n-3} L_n. \tag{3.1.22}$$

Also the asymptotic in-state $L_1|0\rangle$ corresponds to the field $\partial\Phi(z)$ while $(L_1)^2|0\rangle$ corresponds to the field $\partial^2\Phi(z)$. Table 3.1 summarizes what we have just described. Although

State	Field	Level
$\|h\rangle$	$\Phi(z)$	0
$L_{-1}\|h\rangle$	$\partial\Phi(z)$	1
$L_{-1}L_{-1}\|h\rangle$	$\partial^2\Phi(z)$	2
$L_{-2}\|h\rangle$	$T(z)\Phi(z)$	2
$L_{-1}L_{-1}L_{-1}\|h\rangle$	$\partial^3\Phi(z)$	3
$L_{-2}L_{-1}\|h\rangle$	$T(z)\partial\Phi(z)$	3
$L_{-3}\|h\rangle$	$\partial(T(z)\Phi(z))$	3
...	...	...

Table 3.1: *State-field correspondence up to level 3*

above, $\Phi(z)$ was considered a primary field, $\partial\Phi(z)$ and other fields in Table 3.1 are not primary: they are called descendants of the primary field $\Phi(z)$ and the corresponding states are called the descendant states. In general (if there are no null states) the descendant states are defined as

$$L_{-n_1}L_{-n_2}...L_{-n_p}|h\rangle, \quad 0 \le n_1 \le n_2 \le ... \le n_k, \quad n_i \in \mathbb{Z}_{>0} \tag{3.1.23}$$

and the level of a descendant state is defined as $N = n_1 + n_2 + ... + n_k$. We call the set of descendant states along with the primary state $|h\rangle$ of conformal weight h the Verma module $\mathcal{V}_{h,c}$ where c is the central charge of the conformal system. The primary state $|h\rangle$ is then called the highest-weight state of the module.

Given a Verma module $\mathcal{V}_{h,c}$ we can construct the character

$$\mathrm{ch}_{\mathcal{V}_{h,c}}(\tau) := \mathrm{Tr}_{\mathcal{V}_{h,c}} q^{L_0 - c/24} = q^{h-c/24} \sum_{n\in\mathbb{Z}_{\geq 0}} a(n) q^n = \sum_{n\in\mathbb{Z}_{\geq 0}} \dim(\mathcal{V}_{h,c}) q^{n+h-c/24}, \tag{3.1.24}$$

where $q = e^{2\pi i\tau}, \tau \in \mathcal{H}^+$, $\mathcal{H}^+$ is known as the upper half plane and $a(n)$ is the number of states at this level. The second equality implies that the character is the generating function for the degeneracy of states at each energy level, which means a character encodes the information on the states in a conformal system in dimension two. Moreover, denote by $P(n)$ the number of partitions of n, and the generating function for $P(n)$ reads

$$\sum_{n\in\mathbb{Z}_{\geq 0}} P(n) q^n = \prod_{N=1}^{\infty} \frac{1}{1-q^N} =: q^{1/24} \frac{1}{\eta(\tau)} \tag{3.1.25}$$

where $\eta(\tau)$ is known as the Dedekind eta function and $a(n) = P(n)$. Then we can rewrite the character of a Verma module as

$$\mathrm{ch}_{\mathcal{V}_{h,c}}(\tau) = q^{h-c/24} \eta^{-1}(\tau). \tag{3.1.26}$$

We now study the null states (aka zero-norm states) in a Verma module. The inner product of two generic states in a Verma module is given by

$$\langle h| L_{m_q} ... L_{m_1} L_{-n_1} ... L_{-n_p} |h\rangle, \tag{3.1.27}$$

where $p, q \in \mathbb{Z}_{\geq 0}$. At level-1, with the help of the Virasoro algebra one gets

$$\| L_{-1}|h\rangle \|^2 = \langle h| L_1 L_{-1} |h\rangle = 2\langle h| L_0 |h\rangle = 2h \langle h|h\rangle, \tag{3.1.28}$$

which is non-negative when $h \geq 0$. It then follows for the case $n > 0$ that

$$\| L_{-n}|h\rangle \|^2 = \langle h| L_n L_{-n} |h\rangle = \langle h| [L_n, L_{-n}] |h\rangle = n(2h + \frac{c}{12}(n^2-1)\langle h|h\rangle, \tag{3.1.29}$$

which is non-negative when $h, c \geq 0$. Next we discuss the null states at level two. For $n = 1$, either $h > 0$ or $h = 0$. In the latter case, $\|L_{-1}|h\rangle\| = 0$ i.e. $|h\rangle = |0\rangle$. Let us act with L_1 on a general state at level two, given by,

$$L_{-2}|h\rangle + aL_{-1}L_{-1}|h\rangle, \quad a \in \mathbb{R}. \tag{3.1.30}$$

Then

$$[L_1, L_{-2}]|h\rangle + a[L_1, L_{-1}L_{-1}]|h\rangle = (3 + 2a(2h+1))L_{-1}|h\rangle = 0 \tag{3.1.31}$$

when $a = \frac{-3}{2(2h+1)}$. Then we act on the state (3.1.30) with L_2, i.e.

$$[L_2, L_{-2}]|h\rangle + a[L_2, L_{-1}L_{-1}]|h\rangle = (4h + \frac{c}{2} + 6ah)|h\rangle = 0 \tag{3.1.32}$$

when $c = \frac{2h(5-8h)}{2h+1}$. Hence for a conformal field theory with central charge $c = \frac{2h(5-8h)}{2h+1}$, the null state at level-2 is given by

$$(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2)|h\rangle. \tag{3.1.33}$$

The detailed knowledge of all null states in a Verma module $\mathcal{V}_{c,h}$ is crucial in the construction of characters of irreducible, unitary representations of the Virasoro algebra.

3.2 $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Superconformal Algebras

In this section, we will introduce the two dimensional superconformal algebras, which play important roles in the discussion of the properties of two dimensional superconformal field theories. Suppose we have a two dimensional conformal field theory with Hilbert space $\mathcal{H}$ (we assume the spectrum to be discrete). The left-moving and right-moving Virasoro algebras with central charge $c, \bar{c}$ are defined in (3.1.4). From now on, we only consider the holomorphic (left-moving) Virasoro.

The super Virasoro algebra [FQS84][FQS+85] [GKO86] (aka $\mathcal{N} = 1$ superconformal algebra) is a supersymmetric extension of the Virasoro algebra, described by a superfield

$$T_F(z, \vartheta) := G(z) + \vartheta T(z) \tag{3.2.1}$$

where ϑ is a Grassman variable, $T(z)$ is the generator of conformal symmetry and the supersymmetry generator $G(z)$ has Laurent expansion

$$G(z)=\sum_{r\in\mathbb{Z}+\epsilon} G_r z^{-r-3/2}, \tag{3.2.2}$$

with $\epsilon \in \{0,\frac{1}{2}\}$ ($\epsilon=0$ corresponds to the Ramond sector [Ram71] and $\epsilon=\frac{1}{2}$ corresponds to the Neveu-Schwarz sector (see [NS71a] and [NS71b]) of the theory. The algebra of modes is of the form

$$[L_m, G_r]=(\frac{m}{2}-r)G_{m+r}, \quad m\in\mathbb{Z} \tag{3.2.3}$$

$$\{G_r, G_s\}=2L_{r+s}+\frac{c}{3}(r^2-\frac{1}{4})\delta_{r+s,0}. \tag{3.2.4}$$

We now consider a conformal field theory with $\mathcal{N}=2$ supersymmetry [Ade+76]. The $\mathcal{N}=2$ indicates we include 2 fermionic (super-)currents, denoted $G^{\pm}(z)$ in the algebra of the theory, and we also introduce a new symmetry, known as R-symmetry, which rotates the fermionic supercurrents onto each other. We denote the generator of the $U(1)$ R-symmetry in $\mathcal{N}=2$ as $J(z)$. As the Virasoro generators transform a primary field ϕ with conformal dimension d_ϕ as (3.1.16),

$$[L_m,\phi_n]=[(d_\phi-1)m-n]\phi_{m+n}, \quad m\in\mathbb{Z} \text{ and } n\in\mathbb{Z} \text{ or } \mathbb{Z}+\frac{1}{2}, \tag{3.2.5}$$

the commutation relations

$$[L_m,J_n]=-nJ_{m+n}, \qquad [L_m,G_r^{\pm}]=(\frac{n}{2}-r)G^{\pm}_{n+r}, \tag{3.2.6}$$

correctly encode the conformal weights of the Virasoro primaries $J(z)$ and $G^{\pm}(z)$, i.e. $d_{G^{\pm}}=3/2$, $d_J=1$. Here again, the modes of the supercharges $G^{\pm}$ are integer (Ramond sector) or half-integer (Neveu-Schwarz sector). The $U(1)$ R-symmetry generators satisfy

$$[J_m,J_n]=\frac{c}{3}m\delta_{m+n,0}, \tag{3.2.7}$$

and

$$[J_m,G_r^{\pm}]=\pm G^{\pm}_{m+r}, \tag{3.2.8}$$

which indicates the supercharge $G^{\pm}$ have charges ± 1under this affine $\widehat{u(1)}$ algebra. Finally, the supersymmetry algebra is given by

$$\{G^{\pm}_r, G^{\pm}_s\} = 0, \tag{3.2.9}$$

$$\{G^{\pm}_r, G^{\mp}_s\} = 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}. \tag{3.2.10}$$

The $\mathcal{N}=4$ superconformal algebra can be generalised from the $\mathcal{N}=2$ case by introducing a spectral flow generator $S(z)$ [Oda89]. The OPEs for the $\mathcal{N}=2$ superconformal algebra are given by

$$\begin{aligned}
J(z)J(w) &= \frac{c}{3(z-w)^2} + \dots, \\
J(z)G^{+}(w) &= \frac{1}{z-w}G^{+}(w) + \dots, \qquad J(z)G^{-}(w) = \frac{-1}{z-w}G^{-}(w) + \dots, \\
G^{+}(z)G^{-}(w) &= \frac{2c}{3(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{\partial J(w) + 2T(w)}{z-w} + \dots, \\
G^{+}(z)G^{+}(w) &= G^{-}(z)G^{-}(w) = 0 + \dots.
\end{aligned} \tag{3.2.11}$$

where $z, w \in \mathbb{C}$, $T(z)$ is the energy-momentum tensor as before, $J(z)$ is the $U(1)$ current and $G^{\pm}(z)$ are the supercurrents. As usual, the $\dots$ stand for regular terms in the OPE when $z \to w$. The spectral flow generator $S(z)$, which is a primary field with conformal dimension $d_S = s/2$ has the following OPE with the $U(1)$ current,

$$J(z)S(w) = \frac{s}{z-w}S(w) + \dots \tag{3.2.12}$$

We then define the OPEs of the spectral flow generator with supercurrents as

$$G^{+}(z)S(w) =) + \dots, \qquad G^{-}(z)S(w) = \frac{R(w)}{z-w} + \dots \tag{3.2.13}$$

where the new generator $R(w)$ is a primary field with conformal dimension $d_R = \frac{s+1}{2}$ and

$$\begin{aligned}
J(z)R(w) &= \frac{s-1}{z-w}R(w) + \dots, \\
G^{+}(z)R(w) &= \frac{sS(w)}{(z-w)^2} + \frac{\partial S(w)}{z-w} + \dots, \qquad G^{+}(z)\bar{R}(w) = 0 + \dots.
\end{aligned} \tag{3.2.14}$$

We also normalise $S(z)\bar{S}(w) = \frac{2}{(z-w)^s} + \dots$.

The small $\mathcal{N}=4$ superconformal algebra with central charge $c = 6k$ is the $s = 2$ case with

$$J^+(z)=\sqrt{\frac{k}{2}}S(z),\quad J^-(z)=\sqrt{\frac{k}{2}}\bar{S}(z),\quad J^3(z)=\frac{1}{2}J(z),$$
$$G^1(z)=-\sqrt{2k}\bar{R}(z),\quad \bar{G}^1(z)=-\sqrt{2k}R(z),\quad G^2(z)=G^+(z),\quad \bar{G}^2(z)=G^-(z). \tag{3.2.15}$$

Hence we receive the (anti-) commutation relations of the small $\mathcal{N}=4$ superconformal algebra, namely

$$\begin{aligned}
[L_m,L_n]&=(m-n)L_{m+n}+\frac{k}{2}(m^3-m)\delta_{m+n,0},\\
\{G^a_r,G^b_s\}&=\{\bar{G}^a_r,\bar{G}^b_s\}=0,\quad a,b=1,2,\\
\{G^a_r,\bar{G}^b_s\}&=2\delta^{ab}L_{r+s}-2(r-s)\sigma^i_{ab}J^i_{r+s}+\frac{k}{2}(4r^2-1)\delta_{r+s,0}\delta^{ab},\\
[J^i_m,J^j_n]&=i\epsilon^{ijk}J^k_{m+n}+\frac{k}{2}m\delta_{m+n,0}\delta^{ij},\\
[J^i_m,G^a_r]&=-\frac{1}{2}\sigma^i_{ab}G^b_{m+r},\quad [J^i_m,\bar{G}^a_r]=-\frac{1}{2}\sigma^{i*}_{ab}\bar{G}^b_{m+r},\\
[L_m,G^a_r]&=(\frac{m}{2}-r)G^a_{m+r},\quad [L_m,\bar{G}^a_r]=(\frac{m}{2}-r)\bar{G}^a_{m+r}, &(3.2.16)\\
[L_m,J^i_n]&=-nJ^i_{m+n}, &(3.2.17)
\end{aligned}$$

where $k\in\mathbb{N}$ is the level of the affine $\widehat{su(2)}$ algebra. In particular, $k=1$ refers to the $K3$ SCFT [EOTY89].

3.3 Representation Theory of the Small $\mathcal{N}=4$ Algebra

In this section, we briefly discuss the representation theory of the $2d$ small $\mathcal{N}=4$ algebra which encodes the information of states of a $2d$ superconformal field theory with small $\mathcal{N}=4$ supersymmetry.

The small $\mathcal{N}=4$ superconformal algebra (SCA) with central charge $c=6k$ contains a level-k affine Lie algebra $\widehat{su(2)}_k$ as the enhanced algebra from $\widehat{u(1)}$ affine symmetry in $\mathcal{N}=2$ SCA [Tao87] [ET88a]. The highest weight states are labeled by the conformal dimension h and isospin ℓ (i.e. the eigenvalue of the Cartan generator J^3_0 of $\widehat{su(2)}_k$). The unitarity constraint on the representations of the small $\mathcal{N}=4$ SCA provides a bound on the conformal dimension of both NS and R sector (due to the supersymmetry), i.e.

$$R: \qquad h \geq \frac{k}{4}, \tag{3.3.1}$$

$$NS: \qquad h \geq \ell. \tag{3.3.2}$$

There are two different types of representations in each sector, namely

• massive(or non-BPS, or long) representations:

$$R: \qquad h > \frac{k}{4}, \quad \ell = 1/2, 1, \ldots, k/2, \tag{3.3.3}$$

$$NS: \qquad h > \ell, \quad \ell = 0, 1/2, \ldots, (k-1)/2. \tag{3.3.4}$$

• massless (or BPS, or short) representations:

$$R: \qquad h = \frac{k}{4}, \quad \ell = 1/2, 1, \ldots, k/2, \tag{3.3.5}$$

$$NS: \qquad h = \ell, \quad \ell = 0, 1/2, \ldots, k/2. \tag{3.3.6}$$

Characters of the small $\mathcal{N} = 4$ SCA are defined as the trace over the representationspace (Hilbert space $\mathcal{H}$), formally

$$Ch_{k,h,\ell}(\tau; z) \equiv Tr_{\mathcal{H}}\, q^{L_0 - c/24} y^{J_0^3}, \quad y := e^{2\pi i z},\ z \in \mathbb{C}. \tag{3.3.7}$$

The explicit formulae in the Ramond sector are given by

• massive character (for non-BPS, or long representations) [Tao87] [ET88a]:

$$Ch^R_{k,h,\ell}(\tau; z) = q^{h - \frac{\ell^2}{k+1} - \frac{k}{4}} \frac{\vartheta_2(\tau; z)^2}{\eta(\tau)^3} \chi_{k-1, \ell-1/2}(\tau; z). \tag{3.3.8}$$

• massless character (for BPS, or short representations) [Tao87] [ET88a]:

$$Ch^R_{k,k/4,l}(\tau; z) = -\frac{1}{\vartheta_1(\tau, 2z)} \frac{\vartheta_2(\tau; z)^2}{\eta(\tau)^3} \sum_{m \in \mathbb{Z}} \left(\frac{y^{2(k+1)m+2l} q^{(k+1)m^2+2lm}}{(1 + y^{-1}q^{-m})^2} - \frac{y^{-2(k+1)m-2l} q^{(k+1)m^2+2lm}}{(1 + yq^{-m})^2} \right) \tag{3.3.9}$$

Note that the denominator in the massless character originates from the BPS condition that the supercharge annihilates the BPS states, and if we ignore the denominator, the massless character becomes the massive character at the limit $h = k/4$. Further, one may observe a recursion relation by simple calculation [Tao87] [ET88a], namely

$$Ch^R_{k,k/4,\ell}(\tau;z)+2Ch^R_{k,k/4,\ell-1/2}(\tau;z)+Ch^R_{k,k/4,\ell-1}(\tau;z)=q^{-\frac{\ell^2}{k+1}}\frac{\vartheta_2(\tau;z)^2}{\eta(\tau)^3}\chi_{k-1,\ell-1/2}(\tau;z), \tag{3.3.10}$$

which shows how the non-BPS characters (at threshold, i.e. taken at the unitarity bound) decompose into a sum of BPS character. There exists an isomorphism of the small $\mathcal{N}=4$ SCA, called spectral flow, whereby the primed generators below satisfy the commutation relations of small $\mathcal{N}=4$ if the unprimed generators do, and this is true for any real value of the parameter θ. In the following, $n\in\mathbb{Z}$,

$$\begin{aligned}
L'_n &= L_n+2\theta J^3_n+\theta^2\frac{c}{6}\delta_{n,0},\\
J'^3_n &= J^3_n+\theta\frac{c}{6}\delta_{n,0}, \qquad J'^{\pm}_n=T^{\pm}_{n\pm 2\theta},\\
G'^{\pm}_r &= G^{\pm}_{n\mp\theta}, \qquad \bar{G}'^{\pm}_r=\bar{G}^{\pm}_{n\pm\theta},
\end{aligned} \tag{3.3.11}$$

where $\theta\in[0,1)$. One flows between the R sector and the NS sector by letting $\theta\to\theta+1/2$ and taking $\theta=0$, while letting $\theta\to\theta+1$ (and also taking $\theta=0$), one flows from the R sector to the R sector. The spectral flow is implemented in the characters by shifting $z\to z+\theta\tau$ or $z\to z+\theta$. If $\theta=\frac{1}{2}$, one gets,

$$Ch^{NS}_{k,h+\ell/2+k/4,k/2-\ell}(\tau;z) = q^{k/4}y^k Ch^R_{k,h,\ell}(\tau;z+\tau/2), \tag{3.3.12}$$

$$Ch^{\widetilde{R}}_{h,k,\ell}(\tau;z) = Ch^R_{k,h,\ell}(\tau;z+1/2), \tag{3.3.13}$$

$$Ch^{\tilde{NS}}_{h,k,\ell}(\tau;z) = Ch^{NS}_{k,h,\ell}(\tau;z+1/2). \tag{3.3.14}$$

Note that the massless character in $\widetilde{R}$ sector $Ch^{\widetilde{R}}_0(\tau;0)$ is the Witten Index.

Consider the case of $k=1$, i.e. the small $mathcalN=4$ SCA with central charge $c=6k=6$, whose corresponding non-linear sigma model has $K3$ as its target space. The massless character in the R sector becomes

$$Ch^R_{k=1,h=1/4,\ell=0}(\tau;z)=\frac{1}{\vartheta_1(\tau,2z)}\frac{\vartheta_2(\tau;z)^2}{\eta(\tau)^3}\sum_{m\in\mathbb{Z}}q^{2m^2}y^{4m}\frac{1-yq^m}{1+yq^m}. \tag{3.3.15}$$

By spectral flow (3.3.13) (under the shift $z \to z + 1/2$, $y \to -y$, $\vartheta_2(\tau; z) \to \vartheta_1(\tau; z)$), the massless character in the $\widetilde{R}$ sector could be written as

$$Ch^{\widetilde{R}}_{k=1,h=1/4,\ell=0}(\tau; z) = \frac{1}{\vartheta_1(\tau; 2z)} \frac{\vartheta_1(\tau; z)^2}{\eta(\tau)^3} \sum_{m\in\mathbb{Z}} q^{2m^2} y^{4m} \frac{1 + yq^m}{1 - yq^m}, \tag{3.3.16}$$

and the relation between non-BPS and BPS characters reduces to

$$Ch^{\widetilde{R}}_{k=1,h=1/4,\ell=1/2}(\tau; z) + 2Ch^{\widetilde{R}}_{k=1,h=1/4,l=0}(\tau; z) = -q^{-1/8} \frac{\vartheta_1(\tau; z)^2}{\eta(\tau)^3}. \tag{3.3.17}$$

Define the level-l Appell function as [STT05]

$$\mathcal{K}_l(\tau; z) = \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n^2 l}{2}} y^{nl}}{1 - yq^n}, \tag{3.3.18}$$

then

$$Ch^{\widetilde{R}}_{k=1,h=1/4,\ell=0}(\tau; z) = \frac{1}{\vartheta_1(\tau; 2z)} \frac{\vartheta_1(\tau; z)^2}{\eta(\tau)^3} (\mathcal{K}_4(\tau; z) - \mathcal{K}_4(\tau; -z)). \tag{3.3.19}$$

The general case is

$$Ch^{\widetilde{R}}_{k,h=1/4,\ell=0}(\tau; z) = \frac{1}{\vartheta_1(\tau; 2z)} \frac{\vartheta_1(\tau; z)^2}{\eta(\tau)^3} (\mathcal{K}_{2(k+1)}(\tau; z) - \mathcal{K}_{2(k+1)}(\tau; -z)). \tag{3.3.20}$$

Another expression of the R sector massless character could be written as [**eguchi1988 unitary**; ET88a]

$$Ch^{R}_{k=1,h=1/4,\ell=0}(\tau; z) = \frac{y^{1/2}\vartheta_2(\tau; z)}{\eta(\tau)^3} \sum_{m\in\mathbb{Z}} \frac{q^{\frac{1}{2}m(m+1)} y^m}{1 + yq^m}, \tag{3.3.21}$$

and in the $\widetilde{R}$ sector,

$$\begin{aligned} Ch^{\widetilde{R}}_{k=1,h=1/4,\ell=0}(\tau; z) &= \frac{y^{1/2}\vartheta_1(\tau; z)}{\eta(\tau)^3} \sum_{m\in\mathbb{Z}} \frac{(-1)^m q^{\frac{1}{2}m(m+1)} y^m}{1 - yq^m}, \\ &= -\frac{\vartheta_1(\tau; z)^2}{\eta(\tau)^3} \mu(\tau; z) \end{aligned} \tag{3.3.22}$$

where $\mu(\tau; z)$ is known as the **Appell-Lerch sum** [EOTY89]

$$\mu(\tau; z) = \frac{-iy^{1/2}}{\vartheta_1(\tau; z)} \sum_{m\in\mathbb{Z}} \frac{(-1)^m q^{\frac{1}{2}m(m+1)} y^m}{1 - yq^m}. \tag{3.3.23}$$

One can use complex analysis to prove the equivalence of the two different expressions forthe massless character by comparing poles and corresponding residues.

3.4 Elliptic Genera and Mathieu Moonshine

We briefly discussed a topological invariant named conformal field-theoretic elliptic genus [Wit87] for a $2d$ non-linear σ-model in Section 2.3 and in this section we will continue to study the properties of the elliptic genus for a $2d$ non-linear σ model with supersymmetry $\mathcal{N}=(2,2)$ or $\mathcal{N}=(4,4)$.

We first review the conformal field theory on the torus and its partition function. The general procedure for constructing a conformal field theory on the torus is to map the local information of the CFT operators constructed on the plane to a cylinder with the boundaries identified. More precisely, consider a pair (α_1,α_2) of complex numbers which are free over $\mathbb{R}$, i.e. such that, for $r_1,r_2\in\mathbb{R}$, $r_1\alpha_1+r_2\alpha_2=0\Rightarrow r_1=r_2=0$. One may define a two-dimensional lattice (which is a discrete abelian group in $\mathbb{C}$) using (α_1,α_2) in the following way,

$$\Lambda\coloneqq\{m\alpha_1+n\alpha_2|m,n\in\mathbb{Z}\}. \tag{3.4.1}$$

Then the quotient space $\mathbb{T}\coloneqq\mathbb{C}/\Lambda$ is a 2-torus with periods α_1 and α_2. An element of $\mathbb{T}$ is an equivalence class of complex numbers for the equivalence relation

$$z\equiv z'\Leftrightarrow\exists m,n\in\mathbb{Z}|z'=z+m\alpha_1+n\alpha_2. \tag{3.4.2}$$

The shape of the torus is measured by the modular parameter τ defined by

$$\tau=\frac{\alpha_2}{\alpha_1}\coloneqq\tau_1+i\tau_2,\quad \tau_1,\tau_2\in\mathbb{R}. \tag{3.4.3}$$

Note that a lattice could be generated by different choices of the pair $(\alpha_!,\alpha_2)$, hence under a deformation of τ, namely

$$\tau\mapsto\frac{a\tau+b}{c\tau+d},\quad ad-bc=1,\quad a,b,c,d\in\mathbb{Z}, \tag{3.4.4}$$

the shape of torus is unchanged. We call such transformation the modular transformation. Choosing $\{a=1,b=1,c=0,d=1\}$, we have the T-transformation $\tau\mapsto\tau+1$, while the S-transformation $\tau\mapsto-\frac{1}{\tau}$ corresponds to the choice $\{a=0,b=-1,c=1,d=0\}$. The transformations T and S generate the modular group and satisfy $S^2=1,(ST)^3=1$.

Before we exploit the partition function of a conformal field theory on the torus, we first discuss the relation between the energy-momentum tensor on the plane and on the cylinder. Under the conformal transformation $z \to f(z)$ the energy-momentum tensor transforms as

$$T(z) \mapsto (\partial f)^2 T(f(z)) + \frac{c}{12} S(f,z), \tag{3.4.5}$$

where $S(f,z) = \frac{\partial f \partial^3 f - \frac{3}{2}(\partial f)^2}{(\partial f)^2}$ with $\partial = \partial_z$ is known as Schwarzian derivative. Setting $f(z) = e^z$, we have

$$T_{\text{cyl}}(z) = z^2 T(z) - \frac{c}{24}, \tag{3.4.6}$$

with mode expansion given by

$$T_{\text{cyl}}(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n} - \frac{c}{24} = \sum_{n\in\mathbb{Z}} (L_n - \frac{c}{24}\delta_{n,0}) e^{-nz}. \tag{3.4.7}$$

In particular, the zero mode is then $(L_{\text{cyl}})_0 = L_0 - c/24$. As we mentioned before the (real) Hamiltonian H of a conformal filed theory is the generator for time translation while the (real) momentum operator P is the generator for space translation. As $L_0 + \bar{L}_0$ and $L_0 - \bar{L}_0$ generate dilations and rotations on the plane respectively, when considering a conformal field theory on a torus, the Hamiltonian H_{cyl} and momentum operator P_{cyl} are given by

$$H \equiv H_{cyl} = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}, \tag{3.4.8}$$

$$P \equiv P_{cyl} = L_0 - \bar{L}_0 - \frac{c-\bar{c}}{24}. \tag{3.4.9}$$

Let us now study the partition function of a conformal field theory defined as

$$Z(\tau_1, \tau_2) = \text{Tr}_{\mathcal{H}}(e^{-2\pi\tau_2 H} e^{2\pi i \tau_1 P}) \tag{3.4.10}$$

where $\mathcal{H}$ is the Hilbert space of the theory. When using (3.4.8) and (3.4.9), we obtain the partition function for a conformal field theory on a torus with parameter $\tau = \tau_1 + i\tau_2$

$$Z(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}}(q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \bar{c}/24}) \tag{3.4.11}$$

where $q = e^{2\pi i \tau}$ as usual. Note that τ_1 and τ_2 correspond to space and time on the torus respectively.

One can also explain the partition function from the characters of primary fields and descendants, namely

$$Z(\tau,\bar{\tau}) = \sum_{i,j} F_{ij}\mathrm{ch}_{i\mathcal{V}_{h,c}}(\tau)\bar{\mathrm{ch}}_{j\mathcal{V}_{h,c}}(\bar{\tau}), \tag{3.4.12}$$

where $F_{ij} \in \mathbb{Z}_{\geq 0}$ are multiplicities.

In a superconformal field theory with $\mathcal{N} = (2,2)$ symmetry, the full partition function is defined as

$$Z(\tau,\bar{\tau},z,\bar{z}) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0-c/24}y^{J_0}\,\bar{q}^{\bar{L}_0-\bar{c}/24}\bar{y}^{\bar{J}_0}) \tag{3.4.13}$$

where $y = e^{2\pi i z}, \bar{y} = e^{-2\pi i \bar{z}}, z, \bar{z} \in \mathbb{C}$ and $J_0(\bar{J}_0)$ denotes the zero modes of the $U(1)$ current with integral charges in the left-(right-)moving sector. In (3.4.13), the Hilbert space of states $\mathcal{H}$ encodes information from the R and NS sectors as well as their twisted versions.

We first introduce the conformal field-theoretic elliptic genus for an $\mathcal{N} = (2,2)$ superconformal field theory,

$$\mathcal{EG}_M(\tau;z) := Tr_{\mathcal{H}^R\otimes\mathcal{H}^R}\left((-1)^F y^{J_0} q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right), \tag{3.4.14}$$

where $(-1)^F = e^{\pi i(J_0-\bar{J}_0)}$ with $F = F_L + F_R$, M is the target space of the non-linear sigma model, i.e. SCFT, and $\mathcal{H}^R$ denotes the Hilbert space of the Ramond sector. Note that the elliptic genus is independent of $\bar{q}$ due to the $(-1)^{F_R}$ insertion. We notice that the relation between $Z_{\mathcal{H}^{\widetilde{R}}}$, the partition function restricted to the twisted Ramond ($\widetilde{R}$) sector and the elliptic genus for a $\mathcal{N} = (2,2)$ theory is

$$\mathcal{EG}_M(\tau;z) = Z_{\mathcal{H}^{\widetilde{R}}}(\tau,\bar{\tau},z,\bar{z}=0). \tag{3.4.15}$$

Similarly, for a theory with $\mathcal{N} = (4,4)$ supersymmetry, the conformal field-theoretic elliptic genus is defined by taking the sum over all states in the left-moving sector of the theory while the right-moving part is fixed at the Ramond ground states

$$\mathcal{EG}_M(\tau;z) := Tr_{\mathcal{H}^R\otimes\mathcal{H}^R}(-1)^F y^{2J_0^3} q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}, \tag{3.4.16}$$

where and $J_0^3(\bar{J}_0^3)$ denotes the zero mode of the Cartan generators of the affine $\widehat{su(2)}$ subalgebra of the $\mathcal{N}=4$ algebra in the left-(right-)moving sector, $(-1)^F = e^{\pi i(2J_0^3 - 2\bar{J}_0^3)}$ with $F = F_L + F_R$.

Let us now study the modular and elliptic properties of the elliptic genus for an $\mathcal{N}=(2,2)$ or $\mathcal{N}=(4,4)$ theory on a Calabi-Yau manifold [KYY94] (In the following we will only discuss the elliptic genus for a Calabi-Yau manifold.). The first property of the elliptic genus is easy to read from the definition of an elliptic genus, namely

$$\mathcal{EG}_M(\tau;-z) = \mathcal{EG}_M(\tau;z). \tag{3.4.17}$$

which means the spectrum of the Ramond sector is symmetric under charge conjugation. The modular (τ) and elliptic (z) transformations of the elliptic genus are of the form [KYY94]

$$\begin{aligned}
\mathcal{EG}_M(\tau+1;z) &= \mathcal{EG}_M(\tau;z), \quad \mathcal{EG}_M(-\frac{1}{\tau};\frac{z}{\tau}) = e^{2\pi i \frac{c}{6}\frac{z^2}{\tau}}\mathcal{EG}_M(\tau;z), \\
\mathcal{EG}_M(\tau;z+1) &= (-1)^{\frac{c}{3}}\mathcal{EG}_M(\tau;z), \; \mathcal{EG}_M(\tau;z+\tau) = (-1)^{\frac{c}{3}} q^{-\frac{c}{6}} e^{-2\pi i \frac{c}{3} z}\mathcal{EG}_M(\tau;z).
\end{aligned} \tag{3.4.18}$$

Recall that a weight k ($k \in \mathbb{Z}$), index m ($m \in \mathbb{Z}_{\geq 0}$) **weak Jacobi Form** [EZ85] on $SL(2,\mathbb{Z})$ is a holomorphic function $\varphi : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ satisfying the two transformation properties

$$\begin{aligned}
\varphi(\frac{a\tau+b}{c\tau+d};\frac{z}{c\tau+d}) &= (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}}\varphi(\tau;z), \; a,b,c,d \in \mathbb{Z}, \; ad-bc=1, \\
\varphi(\tau;z+\lambda\tau+\mu) &= e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(\tau;z), \lambda,\mu \in \mathbb{Z}
\end{aligned} \tag{3.4.19}$$

as well as an extra condition given below. By taking special values of (a,b,c,d) and (λ,μ), one finds

$$\begin{aligned}
\varphi(-\frac{1}{\tau};\frac{z}{\tau}) &= \tau^k e^{2\pi i m \frac{z^2}{\tau}}\varphi(\tau;z); \quad (0,-1,1,0) \\
\varphi(\tau;z+\tau) &= q^{-m}e^{-4\pi i m z}\varphi(\tau;z); \quad (1,0).
\end{aligned} \tag{3.4.20}$$

and

$$\varphi(\tau+1;z)=\varphi(\tau;z+1)=\varphi(\tau;z); \quad (1,1,0,1)\ (0,1). \tag{3.4.21}$$

Note that the invariance (3.4.21) implies a Fourier expansion

$$\varphi(\tau;z)=\sum_{n,l\in\mathbb{Z}} c(n,l)q^n y^l. \tag{3.4.22}$$

A weak Jacobi form is a holomorphic function $\varphi(\tau,z)$ satisfying the modularity and ellipticity properties (3.4.19) and whose Fourier coefficients $c(n,l)=(-1)^k c(n,-l)$ vanish when $n<0$.

One useful weak Jacobi form of weight 0 and index 1 is

$$\varphi_{0,1}(\tau,z):=4\sum_{j=2}^{4}\frac{\vartheta_j(\tau,z)^2}{\vartheta_j(\tau,0)^2}. \tag{3.4.23}$$

The elliptic genus is a weight 0, index $c/6$ (when $c/6\in\mathbb{Z}$) weak Jacobi form[1]. Note that from the geometric viewpoint, the conformal field-theoretic elliptic genus we have discussed is the elliptic genus of a complex manifold M with complex dimension $c/3$, which could be considered as the target space of a non-linear sigma model, and is therefore a topological invariant. One may read topological information by setting different special values of the elliptic genus, namely [EOTY89],

$$\begin{aligned}\mathcal{EG}_M(\tau;z=0) &= \chi_M,\\ \mathcal{EG}_M(\tau;z=\frac{1}{2}) &= \sigma_M+O(q),\\ q^{c/12}\mathcal{EG}_M(\tau;z=\frac{1+\tau}{2}) &= \hat{A}_M+O(q),\end{aligned} \tag{3.4.24}$$

where $\chi_M,\sigma_M,\hat{A}_M$ are the Euler characteristic, Hirzebruch signature and A-roof genus, respectively.

3.4.1 Mathieu Moonshine

In Mathematics, the term ‘moonshine’ is used to describe an unexpected relation between two areas of mathematics arising from speculation. The word was coined by the group

[1] Some special cases appeared in [KYY94]

theorist John Conway in the 1980's when it was observed by John McKay that $J(\tau)$, the modular function for the group $SL(2,\mathbb{Z})$, admits a q-expansion

$$J(\tau) = q^{-1} + \sum_{n\in\mathbb{Z}_{>0}} j_n q^n = q^{-1} + 196884q + 21493760q^2 + 8642909970q^3 + \ldots, \quad a_n \in \mathbb{Z}_{\geq 0} \tag{3.4.25}$$

where $196884 = 1 + 196883$, with 1 and 196883 being the dimensions of the first two smallest irreducible representations of the Monster group $\mathbb{M}$, which is the largest sporadic group. After checking several coefficients $j_n, n > 1$, and showing that they could be decomposed in dimensions of irreducible representations of $\mathbb{M}$ as well, Conway and Norton conjectured that there should exist an infinite-dimensional graded module V for the Monster group, $V = \oplus_{r=0}^{\infty} V_r$ with graded dimension

$$\dim_V(\tau) = \sum_{r=0}^{\infty} q^r \dim(V_r) = qJ(\tau). \tag{3.4.26}$$

This conjecture is known as the 'monstrous moonshine conjecture' and it was proved by Borcherds in 1995. For details, see [Gan06] for example. Since modular functions arise as torus partition functions in $2d$ CFT, it is natural to ask if $J(\tau)$ could be the partition function of a chiral CFT with $\mathbb{M}$ symmetry, i.e. whether

$$J(\tau) = \mathrm{Tr}_{\mathcal{H}} \left(q^{L_0 - c/24} \right) \tag{3.4.27}$$

for some Hilbert space $\mathcal{H}$ and some central charge c, with $\mathcal{H} = \oplus_{r=0}^{\infty}\mathcal{H}_r$ and $\dim \mathcal{H}_r = j_r$. In [FLM89], Frenkel, Lepowsky and Meurman constructed a $c = 24$ bosonic CFT based on one of the twenty-four 24-dimensional Euclidean even, self-dual lattices called the Leech lattice Γ_{Leech}, which provides the densest sphere packing in 24 dimensions. This construction is almost what is needed for a realization of the Monstrous Moonshine module. Indeed, the corresponding partition function is

$$Z_{\Gamma_{\text{Leech}}}(\tau) = \frac{\Theta_{\text{Leech}}(\tau)}{\eta^{24}(\tau)} = J(\tau) + 24, \tag{3.4.28}$$

with $\Theta_{\text{Leech}}(\tau)$ providing the sum over momenta and windings on the Leech lattice, and $1/\eta^{24}(\tau)$ providing the partition function for 24 bosonic oscillators. In superconformal field theory, there exists a different type of moonshine, observed by Eguchi, Ooguri and

Tachikawa in 2010 [EOT11] in the context of a $2d$ sigma model with target space the hyperkähler manifold $K3$, and hence with $\mathcal{N} = (4,4)$ symmetry [AF81].

The conformal field-theoretic elliptic genus of such a theory was calculated in [EOTY89] using realizations of the sigma model as Gepner models and orbifolds, and was expressed in terms of the $\mathcal{N} = 4$ characters reviewed in Section 3.3, namely

$$\mathcal{EG}_{K3}(\tau;z) = 20\mathrm{Ch}^{\widetilde{R}}_{h=1/4,\ell=0}(\tau,z) - 2\mathrm{Ch}^{\widetilde{R}}_{h=1/4,\ell=1/2}(\tau,z) + \sum_{n\geq 1} a_n \mathrm{Ch}^{\widetilde{R}}_{h=n+1/4,\ell=1/2}(\tau,z), \tag{3.4.29}$$

where the coefficents a_n are positive integers that were first enumerated explicitly up to $n =$ in [Oog89]. They are given in Table 3.2.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
90	462	1540	4554	11592	27830	61686	131100

Table 3.2: *First few coefficients of massive $\mathcal{N} = 4$ characters in the decomposition of $\mathcal{EG}_{K3}$.*

The observation made in [EOT11] is that these coefficients are, up to an overall factor of 2, either the dimensions of irreducible representations of the sporadic group Mathieu 24 (M_{24}), or sums of dimensions of irreducible representations of that same group. For instance, $a_1 = 45 + 45$, $a_2 = 231 + 231$ while $a_6 = 2 \times (3520 + 10395)$. This unexpected relation between the conformal field-theoretic elliptic genus $\mathcal{EG}_{K3}$ and M_{24} is called 'Mathieu Moonshine'. Despite a lot of efforts since 2010, the deep reason why the Mathieu 24 symmetry appears to play a role in $2d$ non-linear sigma models with target space $K3$ remains a mystery to this date.

Finally, one can show that the conformal field-theoretical elliptic genus (3.4.29) is equal to the 'geometric' elliptic genus of $K3$, given by

$$\mathcal{EG}^{\mathrm{geom}}_{K3}(\tau;z) = 2\varphi_{0,1}(\tau,z). \tag{3.4.30}$$

One checks that with the definition (3.4.23), the Euler characteristic of $K3$, which is given by $\mathcal{EG}_{K3}(\tau;0) = 2\varphi_{0,1}(\tau,0)$ is indeed 24. Note that the other Calabi-Yau 2-fold is $T4$, and the elliptic genus for $T4$ is 0.

3.A Theta Functions

We give the definitions of the different types of theta functions that arise in this work, and summarize the properties that are useful to us.

3.A.1 Jacobi Theta functions

The Jacobi Theta Functions are defined as

$$
\begin{aligned}
\vartheta_1(\tau;\zeta) &:= i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = 2\sin(\pi\zeta) q^{\frac{1}{8}} \prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m), \\
&= i q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1})(1-y^{-1}q^m), \\
\vartheta_2(\tau;\zeta) &:= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}} = 2\cos(\pi\zeta) q^{\frac{1}{8}} \prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m) \\
&= q^{\frac{1}{8}} y^{\frac{1}{2}} \prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^{m-1}), \\
\vartheta_3(\tau;\zeta) &:= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} y^n = \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-\frac{1}{2}})(1+y^{-1}q^{m-\frac{1}{2}}) = \vartheta(\tau,\zeta), \\
\vartheta_4(\tau;\zeta) &:= \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}n^2} y^n = \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-\frac{1}{2}})(1-y^{-1}q^{m-\frac{1}{2}}),
\end{aligned}
\tag{3.A.1}
$$

where $y = e^{2\pi i\zeta} =: \mathbf{e}(\zeta)$, $\zeta \in \mathbb{C}$.

3.A.2 $\widehat{su(2)}_k$ theta functions

1. **Definition**: the $\widehat{su(2)}_k$ generalised theta functions are defined as,

$$\theta_{m,k}(\tau,\zeta) := \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2} z^{kn}, \tag{3.A.2}$$

2. **Transformation properties under the modular group** $SL(2,\mathbb{Z})$:

$$\theta_{m,k}(-1/\tau,\zeta/\tau) = (-i\tau)^{1/2}\frac{1}{\sqrt{2k}}\,\mathbf{e}(\frac{k\zeta^2}{4\tau}) \sum_{m'\in\mathbb{Z}_{2k}} e^{-\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta) \tag{3.A.3a}$$

$$\theta_{m,k}(\tau+1,\zeta) = \mathbf{e}(\frac{\mu^2}{4k})\theta_{m,k}(\tau,\zeta) \tag{3.A.3b}$$

where $\mathbf{e}(x) := e^{2\pi i x}$.

Remark: We note that (3.A.3a), which may be derived with the help of the standard technique of Poisson resummation, is in agreement with the result obtained in a much more general context by Kac, and thoroughly studied in [FMS12]. In particular, we recover the prefactor $\mathbf{e}(\frac{k\zeta^2}{4\tau})$ starting with their definition (14.161) of generalized theta function and specializing to the $su(2)$ case. The authors write the generalized level k theta functions associated with the affine $\widehat{su(2)}_k$ weight $\hat{\lambda}$ as

$$\Theta_{\hat{\lambda}}(\hat{\xi}) = \mathbf{e}(-kt) \sum_{\alpha^\vee \in Q^\vee} \mathbf{e}(-\frac{1}{2}(2k(\alpha^\vee, \xi) + 2(\lambda, \xi) - \tau k\, |\alpha^\vee + \frac{\lambda}{k}|^2)), \quad (3.A.4)$$

where $\hat{\xi} = -2\pi i(\xi, \tau, t)$.

In the case of $su(2)$, one takes $\alpha^\vee = n\alpha_1$ with $\alpha_1^2 = 2$ (α_1 is the highest root for $su(2)$) and $\xi = z\omega_1$ with $\omega_1 = \frac{1}{2}\alpha_1$ (ω_1 is the $su(2)$ fundamental weight, so that $(\alpha_1, \omega_1) = 1$)[2]. Therefore, one arrives at the expression (14.176) in di Francesco et al:

$$\Theta^{(k)}_{\lambda_1}(\xi, \tau, t) = \mathbf{e}(-kt) \sum_{n \in \mathbb{Z} + \frac{\lambda_1}{2k}} \mathbf{e}(k(n^2\tau - nz)) \quad (3.A.5)$$

The S-transformation on (ξ, τ, t) is $(\frac{\xi}{\tau}, -\frac{1}{\tau}, t + \frac{|\xi|^2}{2\tau})$ (see (14.214)) in [FMS12], so we see that $\frac{|\xi|^2}{2\tau} = \frac{z^2}{4\tau}$ (since $\omega_1^2 = \frac{1}{4}\alpha_1^2 = \frac{1}{2}$) and the factor we wish to check is

$$\mathbf{e}(\frac{|\xi|^2}{2\tau}) = \mathbf{e}(\frac{z^2}{4\tau}) = e^{\pi i z^2/2\tau}. \quad (3.A.6)$$

3. **Product formulae**: the generalised theta functions satisfy the product formula

$$\theta_{m,k}(\tau,\zeta)\,\theta_{m',k'}(\tau,\zeta') = \sum_{j\in\mathbb{Z}_{k+k'}} \theta_{mk'-m'k+2jkk',kk'(k+k')}(\tau, \frac{\zeta-\zeta'}{k+k'})\,\theta_{m+m'+2jk,k+k'}(\tau, \frac{k\zeta+k'\zeta'}{k+k'}), \quad (3.A.7)$$

or again, since $\theta_{m,k}(\tau,\zeta) = \theta_{-m,k}(\tau,-\zeta)$,

$$\theta_{m,k}(\tau,\zeta)\,\theta_{m',k'}(\tau,\zeta') = \sum_{j\in\mathbb{Z}_{k+k'}} \theta_{mk'+m'k+2jkk',kk'(k+k')}(\tau, \frac{\zeta+\zeta'}{k+k'})\,\theta_{m-m'+2jk,k+k'}(\tau, \frac{k\zeta-k'\zeta'}{k+k'}). \quad (3.A.8)$$

[2] See [FMS12] text under (14.87) and (14.175).

4. **Relation to the Jacobi ϑ function**: the following relation between the Jacobi theta function ϑ and the generalised theta functions is useful,

$$\theta_{b,\ell}(\tau,\zeta) = z^{b/2}\, q^{b^2/4\ell}\, \vartheta(2\ell\tau, \ell\zeta + b\tau). \tag{3.A.9}$$

5. **Odd generalised theta functions**: we introduce odd generalised theta functions,

$$\theta^-_{m,k}(\tau,\zeta) \coloneqq \theta_{m,k}(\tau,\zeta) - \theta_{m,k}(\tau,-\zeta). \tag{3.A.10}$$

Upon using the periodic conditions $\theta_{m,k}(\tau,-\zeta) = \theta_{-m,k}(\tau,\zeta) = \theta_{2k-m,k}(\tau,\zeta)$, the S-transformation of the odd generalised theta functions reads

$$\theta^-_{m,k}(-1/\tau,\zeta/\tau) = -i(-i\tau)^{1/2}\,\mathbf{e}(\frac{k\zeta^2}{4\tau})\sum_{m'=1}^{k-1}\sqrt{\frac{2}{k}}\sin(\pi\frac{mm'}{k})\theta^-_{m',k}(\tau,\zeta). \tag{3.A.11}$$

Proof: We write

$$\begin{aligned}
&\theta^-_{m,k}(-1/\tau,\zeta/\tau) = \theta_{m,k}(-1/\tau,\zeta/\tau) - \theta_{-m,k}(-1/\tau,\zeta/\tau)\\
&= (-i\tau)^{1/2}\frac{1}{\sqrt{2k}}\,\mathbf{e}(\frac{k\zeta^2}{4\tau})(\sum_{m'=0}^{2k-1} e^{-\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta) - \sum_{m'=0}^{2k-1} e^{\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta))\\
&= (-i\tau)^{1/2}\frac{1}{\sqrt{2k}}\,\mathbf{e}(\frac{k\zeta^2}{4\tau})(\sum_{m'=1}^{k-1} e^{-\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta) - \sum_{m'=1}^{k-1} e^{\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta)\\
&\qquad\qquad + \sum_{m'=k+1}^{2k-1} e^{-\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta) - \sum_{m'=k+1}^{2k-1} e^{\pi i\frac{mm'}{k}}\theta_{m',k}(\tau,\zeta))\\
&= (-i\tau)^{1/2}\frac{1}{\sqrt{2k}}\,\mathbf{e}(\frac{k\zeta^2}{4\tau})\sum_{m'=1}^{k-1}(e^{-\pi i\frac{mm'}{k}} - e^{\pi i\frac{mm'}{k}})\,\theta_{m',k}(\tau,\zeta)\\
&\qquad\qquad + \sum_{m''=2k-m'=1}^{k-1}(e^{-\pi i\frac{m(2k-m'')}{k}} - e^{\pi i\frac{m(2k-m'')}{k}})\,\theta_{2k-m'',k}(\tau,\zeta))\\
&= -i(-i\tau)^{1/2}\,\mathbf{e}(\frac{k\zeta^2}{4\tau})\sum_{m'=1}^{k-1}\sqrt{\frac{2}{k}}\sin(\pi\frac{mm'}{k})(\theta_{m',k}(\tau,\zeta) - \theta_{m',k}(\tau,-\zeta))\\
&= -i(-i\tau)^{1/2}\,\mathbf{e}(\frac{k\zeta^2}{4\tau})\sum_{m'=1}^{k-1}\sqrt{\frac{2}{k}}\sin(\pi\frac{mm'}{k})\,\theta^-_{m',k}(\tau,\zeta)
\end{aligned} \tag{3.A.12}$$

3.B $\widehat{su(2)}_k$ characters

The $\widehat{su(2)}_k$ characters of irreducible representations of spin ℓ $(0 \le 2\ell \le k)$ are[3],

$$\chi^k_\ell(\tau,\zeta) = \frac{1}{i\vartheta_1(\tau,2\zeta)}\,\theta^-_{2\ell+1,k+2}(\tau,2\zeta). \tag{3.B.1}$$

The following rewriting of the $\widehat{su(2)}_k$ characters is also useful,

$$\chi^k_\ell(\tau,\zeta) = \frac{q^{(\ell+1/2)^2/(k+2)-1/8}}{(y-y^{-1})\prod_{m=1}^\infty(1-q^m)(1-yq^m)(1-y^{-1}q^m)} \times \sum_{n\in\mathbb{Z}} q^{(k+2)n^2+(2\ell+1)n}\{y^{2(k+2)n+2\ell+1} - y^{-2(k+2)n-2\ell-1}\}, \tag{3.B.2}$$

Using (3.A.11), their modular S-transformation is easily found to be

$$\begin{aligned}\chi^k_l(-\frac{1}{\tau},\frac{\zeta}{\tau}) &= \mathbf{e}(\frac{k\zeta^2}{\tau})\sqrt{\frac{2}{k+2}}\sum_{2\ell'=0}^{k}\sin\left(\frac{(2\ell+1)(2\ell'+1)\pi}{k+2}\right)\chi^k_{\ell'}(\tau,\zeta)\\ &:= \mathbf{e}(\frac{k\zeta^2}{\tau})\sum_{2\ell'=0}^{k} S^{(k)}_{\ell,\ell'}\,\chi^k_{\ell'}(\tau,\zeta)\end{aligned} \tag{3.B.3}$$

The following formulae are useful and immediate, for all $n\in\mathbb{Z}$ and for $p\in\{1,\dots,k+2\}$,

$$\chi^k_{\frac{1}{2}(2n(k+2)+p-1)}(\tau,\zeta) = \chi^k_{\frac{1}{2}(p-1)}(\tau,\zeta) = -\chi^k_{\frac{1}{2}(2n(k+2)-p-1)}(\tau,\zeta). \tag{3.B.4}$$

In particular,

$$\chi^k_{\frac{1}{2}(k+1)}(\tau,\zeta) = 0 \qquad \text{and} \quad \chi^k_{\frac{1}{2}(k+p+1)}(\tau,\zeta) = -\chi^k_{\frac{1}{2}(k-p+1)}(\tau,\zeta). \tag{3.B.5}$$

[3]The angle variable ζ is associated with the zero-mode insertion $e^{2\pi i\cdot 2\zeta J^3_0}$ with the current normalisation $J^3(z)J^3(0)\sim\frac{k/2}{z^2}$

Chapter 4

$\mathcal{A}_\gamma$ Algebras, Characters and Refined Index

In this chapter, we discuss a one-parameter family of conventional $d = 2$ superconformal algebras with $\mathcal{N} = 4$ supersymmetry [STV88]. In the literature these algebras are also known as maximal or large[1] $\mathcal{N} = 4$ superconformal algebras or again $\mathcal{A}_\gamma$ algebras, where the parameter γ, alongside the central charge, encodes information about the levels of the two affine $\widehat{su(2)}$ subalgebras embedded in $\mathcal{A}_\gamma$.

The large $\mathcal{N} = 4$ symmetry might provide a link between the instanton moduli space and perturbations of the superconformal field theory discussed in [CHS91]. In that paper, the authors are concerned with the construction of heterotic string solitons and focus on 5-brane solutions, in particular some with extended world sheet supersymmetry. These solutions present a throat geometry and are exact solutions of the string theory. The non-linear sigma model associated to these is the supersymmetric Wess-Zumino-Witten model with $SU(2) \times U(1)$ target group manifold (with the radius of the $U(1)$ tending to infinity). It was first established by Sevrin et.al [STV88] that this model provides a realization of the large $\mathcal{N} = 4$ algebra. The first affine $\widehat{su(2)}$ (at level 1) is associated with the usual (small) $\mathcal{N} = 4$ superconformal algebra while the second (at level n, say, related to the area

[1]The term 'large' is used in contrast to the 'small' $\mathcal{N} = 4$ superconformal algebras studied in [ET88b; Tao87]

of the cross section of the throat) is associated with the affine $\widehat{su(2)}$ algebra of the WZW model and hence with the geometry of the throat. However, standard analyses of sigma models enjoying extended supersymmetry require the canonically defined supercharges and energy-momentum tensor to satisfy the small $\mathcal{N} = 4$ algebra [NW89], and this is not happening in $\mathcal{A}_\gamma$. However, Callan et al. show that one may improve the energy-momentum tensor by adding the derivative of a $U(1)$ current $J^0(z)$ (and the supercharges by adding a suitable derivative in the free fermion fields) so that the new generators close on the small $\mathcal{N} = 4$, with central charge $c = 6$ (as opposed to the central charge $c = 6(n + 1)(n + 2)$ of $\widetilde{\mathcal{A}}_\gamma$). So there is a suitable subalgebra of the large $\mathcal{N} = 4$ algebra which is equivalent to the small $\mathcal{N} = 4$ algebra and this is a good evidence for the WZW-Feigin-Fuks conformal field theory of the throat whose central charge is $c = 6$.

More recently, Gaberdiel and Gopakumar showed that the large $\mathcal{N} = 4$ cosets are dual to the supersymmetric higher spin theory on AdS_3 [GG14]. In this paper, the authors focus on the symmetric product of large $\mathcal{N} = 4$ superconformal field theories whose tensionless limit is plausible to be the small $\mathcal{N} = 4$ which is better studied [Dij99]. The symmetric product of large $\mathcal{N} = 4$ superconformal field theories was studied in 2004 [GMMS05].

In this chapter, we first introduce the $\mathcal{A}_\gamma$ algebras and the non linear (W-type) $\widetilde{\mathcal{A}}_\gamma$ algebras obtained when factoring out four dimension-$\frac{1}{2}$ operators and one bosonic $U(1)$ generator [GS88] from $\mathcal{A}_\gamma$ algebras. Then we discuss the unitary representation theory of $\mathcal{A}_\gamma$ algebras [GPTV89] and the corresponding superconformal massive and massless characters [PT90a; PT90b]. After introducing the characters, one quickly establishes that a conformal field-theoretic elliptic genus constructed from a partition function with $\mathcal{A}_\gamma$ symmetry, in analogy with the construction of the elliptic genus in the context of strings propagating on $K3$ [EOTY89], is trivial. From a mathematical perspective, the root of the problem is that the massless (and not just the massive) $\mathcal{A}_\gamma$ characters have trivial Witten index. However, Gukov et.al. [GMMS04] introduced a refined index for large $\mathcal{N} = 4$ theories. We review the construction of this refined index and comment on its potential use in the context of coset constructions of $\widetilde{\mathcal{A}}_\gamma$.

4.1 Large $\mathcal{N} = 4$ Superconformal Algebras

The $\mathcal{A}_\gamma$ algebra contains an energy-momentum tensor $L(z)$ with conformal dimension 2, four dimension-$\frac{3}{2}$ supercharges $G_a(z)$ where $a = \{\pm, \pm K\}$ and four corresponding dimension-$\frac{1}{2}$ operators $Q_a(z)$, six dimension-1 commuting affine $\widehat{SU(2)_{k^\pm}}$ generators $T^{\pm,i}(z)$, $i \in \{+,-,3\}$ at level k^+ and k^- with $k^\pm \in \mathbb{Z}_+$, one dimension-1 affine $\widehat{u(1)}$ generator $U(z)$. The central charge c of the Virasoro algebra is given by

$$c = \frac{6k^+k^-}{k^+ + k^-} = 6k^+\gamma = 6k\gamma(1-\gamma), \tag{4.1.1}$$

with

$$k := k^+ + k^- = \frac{c}{6\gamma(1-\gamma)}, \tag{4.1.2}$$

where the parameter $\gamma := \frac{k^-}{k}$ labels the $\mathcal{A}_\gamma$ theory. The commutation relations of $\mathcal{A}_\gamma$ are given in Appendix 4.A.1 for reference. We also reproduce in Appendix 4.A.2 two of its $\mathcal{N} = 2$ subalgebras and comment on the spectral flow induced by the affine $\hat{u}(1)$ subalgebras of these $\mathcal{N} = 2$ superconformal algebras.

Given a superconformal theory with $\mathcal{A}_\gamma$ symmetry, one can gauge a $U(1)$ subalgebra, i.e. a $\widehat{u(1)}$ boson and four fermions, to produce a new theory with a non-linear $\widetilde{\mathcal{A}}_\gamma$ symmetry [GS88]. The $\widetilde{\mathcal{A}}_\gamma$ contains an energy-momentum tensor $\tilde{L}(z)$ with conformal dimension 2, four dimension-$\frac{3}{2}$ supercharges $\tilde{G}_a(z)$ where $a = \{\pm, \pm K\}$, six dimension-$\frac{1}{2}$ affine $\widehat{SU(2)_{\tilde{k}^\pm}}$ generators $\tilde{A}^{\pm,i}(z)$ at level $\tilde{k}^+ = k^+ - 1$ and $\tilde{k}^- = k^- - 1$ where $i \in \{+,-,3\}$ and $\tilde{k}^\pm \in \mathbb{Z}_+$, and the central charge is given by

$$\tilde{c} = c - 3, \tag{4.1.3}$$

where $3 = 1 + 4 \times \frac{1}{2}$ is the central charge contribution from one boson and four fermions. The relations between operators in $\mathcal{A}_\gamma$ and $\widetilde{\mathcal{A}}_\gamma$ are below

$$\begin{aligned}
\tilde{L} &= L + \frac{1}{k}(UU + \partial Q^a Q_a) \\
\tilde{Q}_a &= Q_a + \frac{2}{k}UQ_a - \frac{2}{3^2}\epsilon_{abcd}Q^bQ^cQ^d + \frac{4}{k}Q^b(\alpha^{+i}_{ba}\tilde{A}^+_i - \alpha^{-i}_{ba}\tilde{A}^-_i) \\
\tilde{A}^{\pm i} &= A^{\pm i} - \frac{1}{k}\alpha^{\pm i}_{ab}Q^aQ^b, \quad \tilde{Q} = Q, \quad \tilde{U} = U.
\end{aligned} \tag{4.1.4}$$

The representations of the $\mathcal{A}_\gamma$ and $\widetilde{\mathcal{A}}_\gamma$ algebras are related. From (4.1.4), it is clear that once a representation of $\mathcal{A}_\gamma$ is known, it provides a representation of $\widetilde{\mathcal{A}}_\gamma$ [STV88], but given a representation of $\widetilde{\mathcal{A}}_\gamma$, one can add four free fermions and a boson and obtain a representation of $\mathcal{A}_\gamma$ through the inverse relations to (4.1.4) [GS88].

4.2 The Unitary Representations of $\mathcal{A}_\gamma$ and $\widetilde{\mathcal{A}}_\gamma$

We will discuss the unitary representations of $\mathcal{A}_\gamma$ and $\widetilde{\mathcal{A}}_\gamma$ algebras in both Neveu-Schwarz and Ramond sectors. From phycisists' viewpoint, one may be interested in the unitary highest weight representations (uhwr) with integer levels $k^\pm$ whose spectrum is bounded from below. As there is no unique highest weight state in the Ramond sector of $\mathcal{A}_\gamma$ algebras from which to build the representation [GPTV89], we will first consider the uhwr in the Neveu-Schwarz sector and use the spectral flow to exploit the uhwr of the $\mathcal{A}_\gamma$ algebra in the Ramond sector [PT90a].

A unitary highest weight state in the Neveu-Schwarz sector of the $\mathcal{A}_\gamma$ algebra is a state $|\Omega\rangle$ satisfying the following conditions:

$$\begin{aligned}L_n|\Omega\rangle = T_n^{\pm i}|\Omega\rangle = U_n|\Omega\rangle = 0, \quad \forall n \in \mathbb{Z}_+, \quad T_0^{\pm+}|\Omega\rangle = 0,\\ Q_{a,r}|\Omega\rangle = G_{a,r}|\Omega\rangle = 0, \forall r \in \mathbb{Z}_{\geq 0} + \frac{1}{2}, a \in \{\pm, \pm K\}.\end{aligned} \tag{4.2.1}$$

Each highest weight representation (hwr) is characterised by the eigenvalues of $|\Omega\rangle$ under the zero modes of the $\mathcal{A}_\gamma$ algebra,

$$L_0|\Omega\rangle = h_{NS}|\Omega\rangle, \quad U_0|\Omega\rangle = -iu|\Omega\rangle, \quad T_0^{\pm 3}|\Omega\rangle = \ell^\pm_{\mathrm{NS}}|\Omega\rangle, \tag{4.2.2}$$

with $u \in \mathbb{R}$. Unitarity puts restrictions on the conformal dimension h and the isospins $\ell^\pm_{NS}$ [GPTV89] and these depend on whether one considers massless (also called short) or massive (also called long) representations. In the Neveu-Schwarz sector, the masslessness conditions are

$$(G_+)_{-\frac{1}{2}}|\Omega\rangle = 0 \quad \text{and} \quad kh_{NS} = u^2 + (\ell^+_{NS} - \ell^-_{NS})^2 + k^+\ell^-_{NS} + k^-\ell^+_{NS} =: kh^0_{NS} \tag{4.2.3}$$

while the conformal dimension of a uhws for a massive representation satisfies $kh_{NS} > kh^0_{NS}$. Moreover the isospin quantum numbers belong to $\frac{1}{2}\mathbb{Z}$ and are bounded,

$$\begin{aligned} 0 \leq \ell^{\pm}_{NS} \leq \frac{1}{2}(k^{\pm} - 1) \qquad & \text{for massless representations,} \\ 0 \leq \ell'^{\pm}_{NS} \pm \leq \frac{1}{2}(k^{\pm} - 2) \qquad & \text{for massive representations,} \end{aligned} \tag{4.2.4}$$

where we have introduced the $\ell'^{\pm}_{NS}$ notation for the isospin quantum numbers of the massive representations as a reminder that their range is different from the range for massless representations.

We next analyse the quantum numbers of unitary representations in the Ramond sector by using the spectral flow property (see Appendix 4.A) to flow from the Neveu-Schwarz to the Ramond sector. Consider first the hws $|\Omega\rangle$ of a unitary massive representation of $\mathcal{A}_\gamma$ in the NS sector with isospin quantum numbers $(\ell^{+\prime}_{NS}, \ell^{-\prime}_{NS})$, where $\ell'^{\pm}_{NS} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and

$$0 \leq \ell^{\pm\prime}_{NS} \leq \frac{1}{2}(k^{\pm} - 2) \tag{4.2.5}$$

as already stated in (4.2.4). Then, following [PT90a],

1. spectral flow in $SU(2)^+$:
   $|\Omega\rangle$ flows to $|\Omega_+\rangle$ of quantum numbers $(\ell^{+\prime}, \ell^{-\prime}) := (\frac{k^+}{2} - \ell^{+\prime}_{NS}, \ell^{-\prime}_{NS})$.

2. spectral flow in $SU(2)^-$:
   $|\Omega\rangle$ flows to $|\Omega_-\rangle$ of quantum numbers $(\ell^{+\prime}, \ell^{-\prime}) := (\ell^{+\prime}_{NS}, \frac{k^-}{2} - \ell^{-\prime}_{NS})$.

• The ranges of NS values (4.2.5) allowed by unitarity translate into the following ranges for the quantum numbers of $|\Omega_+\rangle$: $1 \leq \ell^{+\prime} \leq \frac{k^+}{2}$ and $0 \leq \ell^{-\prime} \leq \frac{1}{2}(k^- - 2)$. Note however that the massive representation built on $|\Omega_+\rangle$ is not labelled by the quantum numbers of $|\Omega_+\rangle$, but instead by the highest value of $SU(2)^+$ and $SU(2)^-$ isospins present in the representation, which are $T_0^{+3}|\Omega_+\rangle := \ell^{+\prime}_R|\Omega_+\rangle$ and $T_0^{-3}(G_{-K})_0(Q_{-K})_0|\Omega_+\rangle := \ell^{-\prime}_R(G_{-K})_0(Q_{-K})_0|\Omega_+\rangle$ with $\ell^{+\prime}_R = \frac{k^+}{2} - \ell^{+\prime}$ and $\ell^{-\prime}_R = \ell^{-\prime} + 1$. Hence $1 \leq \ell^{\pm\prime}_R \leq \frac{k^{\pm}}{2}$.

• The ranges of NS values (4.2.5) allowed by unitarity translate into the following ranges for the quantum numbers of $|\Omega_-\rangle$: $0 \leq \ell^{+\prime} \leq \frac{1}{2}(k^+ - 2)$ and $1 \leq \frac{k^-}{2} - \ell^{-\prime} \leq \frac{k^-}{2}$. The massive representation built on $|\Omega_-\rangle$ is not labelled by the quantum numbers of $|\Omega_-\rangle$, but instead by the highest value of $SU(2)^+$ and $SU(2)^-$ isospins present in the representation, which are $T_0^{+3}(G_{+K})_0(Q_{+K})_0|\Omega_-\rangle := \ell_R^{+\prime}(G_{+K})_0(Q_{+K})_0|\Omega_-\rangle$ and $T_0^{-3}|\Omega_-\rangle := \ell_R^{-\prime}|\Omega_-\rangle$ with $\ell_R^{+\prime} = \ell^{+\prime} + 1$ and $\ell_R^{-\prime} = \frac{k^-}{2} - \ell^{-\prime}$. Hence $1 \leq \ell_R^{\pm\prime} \leq \frac{k^\pm}{2}$.

In summary, the massive characters in the NS sector will be labelled by the quantum numbers of the hws $|\Omega\rangle$, namely $\ell_{NS}^{\pm\prime}$, while the massive characters in the R sector will be labelled by either the eigenvalue of $|\Omega_+\rangle$ under T_0^{+3} ($\ell_R^{+\prime}$) and the eigenvalue of $(G_{-K})_0(Q_{-K})_0|\Omega_+\rangle$ under T_0^{-3} ($\ell_R^{-\prime}$), or the eigenvalue of $(G_{+K})_0(Q_{+K})_0|\Omega_-\rangle$ under T_0^{+3} ($\ell_R^{+\prime}$) and the eigenvalue of $|\Omega_-\rangle$ under T_0^{-3} ($\ell_R^{-\prime}$). The two corresponding representations are isomorphic.

Now consider the hws $|\Omega\rangle$ of a unitary massless representation of $\mathcal{A}_\gamma$ in the NS sector with isospin quantum numbers $(\ell_{NS}^+, \ell_{NS}^-)$, where $\ell_{NS}^\pm \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and

$$0 \leq \ell_{NS}^\pm \leq \frac{1}{2}(k^\pm - 1), \tag{4.2.6}$$

as already stated in (4.2.4). In this case, the NS hws $|\Omega\rangle$ flows to $|\Omega_+\rangle$ under the $SU(2)^+$ spectral flow, where $|\Omega_+\rangle$ (with isospin quantum numbers (ℓ^+, ℓ^-)) satisfies the following masslessness conditions,

$$(G_+)_0|\Omega_+\rangle = (G_{+K})_0\Omega_+\rangle = 0. \tag{4.2.7}$$

Analogously, $|\Omega\rangle$ flows to $|\Omega_-\rangle$ under the $SU(2)^-$ spectral flow, where $|\Omega_-\rangle$ (with isospin quantum numbers (ℓ^+, ℓ^-)) satisfies the following masslessness conditions,

$$(G_-)_0|\Omega_-\rangle = (G_{-K})_0|\Omega_-\rangle = 0. \tag{4.2.8}$$

• The ranges of NS values (4.2.6) translate into the following ranges for the quantum numbers of $|\Omega_+\rangle$: $\frac{1}{2} \leq \frac{k^+}{2} - \ell^+ \leq \frac{k^+}{2}$ and $0 \leq \ell^- \leq \frac{1}{2}(k^- - 1)$. The massless representation built on $|\Omega_+\rangle$ is not labelled by the quantum numbers of $|\Omega_+\rangle$, but instead by the highest value of $SU(2)^+$ and $SU(2)^-$ isospins present in the representation, which are $T_0^{+3}|\Omega_+\rangle := \ell_R^+|\Omega_+\rangle$ and $T_0^{-3}(Q_{-K})_0|\Omega_+\rangle := \ell_R^-(Q_{-K})_0|\Omega_+\rangle$ with $\ell_R^+ = \frac{k^+}{2} - \ell^+$ and $\ell_R^- = \ell^- + \frac{1}{2}$. Hence $\frac{1}{2} \leq \ell_R^\pm \leq \frac{k^\pm}{2}$.

type of rep	sector	$SU(2)^{\pm}$ isospin	conformal weight
$\mathcal{A}_\gamma$ massive	NS	$0 \leq \ell^{\pm\prime}_{NS} \leq \frac{1}{2}(k^{\pm}-2)$	$kh_{NS} > u^2 + (\ell^{+\prime}_{NS} - \ell^{-\prime}_{NS})^2 + k^-\ell^{+\prime} + k^+\ell^{-\prime}_{NS}$
	R	$1 \leq \ell^{\pm}_{R}{}' \leq \frac{1}{2}k^{\pm}$	$kh_R > u^2 + (\ell^{+}_{R}{}' + + \ell^{-}_{R}{}' - 1)^2 + \frac{1}{4}k^+k^-$
$\mathcal{A}_\gamma$ massless	NS	$0 \leq \ell^{\pm}_{NS} \leq \frac{1}{2}(k^{\pm}-1)$	$kh_{NS} = u^2 + (\ell^{+}_{NS} - \ell^{-}_{NS})^2 + k^-\ell^{+}_{NS} + k^+\ell^{-}_{NS}$
	R	$\frac{1}{2} \leq \ell^{\pm}_{R} \leq \frac{1}{2}k^{\pm}$	$kh_R = u^2 + (\ell^{+}_{R} + \ell^{-}_{R} - \frac{1}{2})^2 + \frac{1}{4}k^+k^-$

Table 4.1: $\mathcal{A}_\gamma$ *quantum numbers - unitary bounds.*

• The ranges of NS values (4.2.6) translate into the followingranges for the quantum numbers of $|\Omega_-\rangle$: $0 \leq \ell^+ \leq \frac{1}{2}(k^+ - 1)$ and $\frac{1}{2} \leq \frac{k^-}{2} - \ell^- \leq \frac{k^-}{2}$. The massless representation built on $|\Omega_-\rangle$ is not labelled by the quantum numbers of $|\Omega_-\rangle$, but instead by the highest value of $SU(2)^+$ and $SU(2)^-$ isospins present in the representation, which are $T_0^{+3}(Q_{+K})_0|\Omega_-\rangle := \ell^+_R(Q_{+K})_0|\Omega_-\rangle$ and $T_0^{-3}|\Omega_-\rangle := \ell^-_R|\Omega_-\rangle$ with $\ell^+_R = \ell^+ + \frac{1}{2}$ and $\ell^-_R = \frac{k^-}{2} - \ell^-$. Hence $\frac{1}{2} \leq \ell^{\pm}_R \leq \frac{1}{2}k^{\pm}$.

In summary, the massless characters in the NS sector will be labelled by the quantum numbers of the hws $|\Omega\rangle$, namely $\ell^{\pm}_{NS}$, while the massless characters in the R sector will be labelled by either the eigenvalue of $|\Omega_+\rangle$ under T_0^{+3} (ℓ^+_R) and the eigenvalue of $(Q_{-K})_0|\Omega_+\rangle$ under T_0^{-3} (ℓ^-_R), or the eigenvalue of $(Q_{+K})_0|\Omega_-\rangle$ under T_0^{+3} (ℓ^+_R) and the eigenvalue of $|\Omega_-\rangle$ under T_0^{-3} (ℓ^-_R). The two corresponding representations are isomorphic.

We collect the data on quantum numbers labelling the representations in both NS and R sector for massive and massless representations in Table 4.1.

Let us move on to the $\widetilde{\mathcal{A}}_\gamma$ quantum numbers. First we discuss the massive representations. In the NS sector, the hws $|\Omega\rangle$ is unique with $\widetilde{T_0}^{\pm 3}|\Omega\rangle := \widetilde{\ell}^{\pm\prime}_{NS}|\Omega\rangle$ and $0 \leq \widetilde{\ell}^{\pm\prime}_{NS} \leq \frac{1}{2}(\widetilde{k}^{\pm} - 1)$.

1. spectral flow in $SU(2)^+$:

   $|\Omega\rangle$ flows to $|\Omega_+\rangle$ of quantum numbers $(\widetilde{\ell}^{+\prime}, \widetilde{\ell}^{-\prime}) := (\frac{\widetilde{k}^+}{2} - \widetilde{\ell}^{+\prime}_{NS}, \widetilde{\ell}^{-\prime}_{NS})$.

2. spectral flow in $SU(2)^-$:

   $|\Omega\rangle$ flows to $|\Omega_-\rangle$ of quantum numbers $(\widetilde{\ell}^{+\prime}, \widetilde{\ell}^{-\prime}) := (\widetilde{\ell}^{+\prime}_{NS}, \frac{\widetilde{k}^-}{2} - \widetilde{\ell}^{-\prime}_{NS})$.

• A massive representation built on $|\Omega_+\rangle$ is not labelled by the quantum numbers of $|\Omega_+\rangle$, but instead by $\widetilde{\ell}^{+}_{R}{}' := \frac{\widetilde{k}^+}{2} - \widetilde{\ell}^{+\prime}_{NS}$ and $\widetilde{\ell}^{-}_{R}{}' := \widetilde{\ell}^{-\prime}_{NS} + \frac{1}{2}$ with $\frac{1}{2} \leq \widetilde{\ell}^{\pm}_{R}{}' \leq \frac{\widetilde{k}^{\pm}}{2}$, where

$\widetilde{T_0}^{+3}|\Omega_+\rangle := \widetilde{\ell}_R^+{}'|\Omega_+\rangle$ and $\widetilde{T_0}^{-3}(\widetilde{G}_{-K})_0|\Omega_+\rangle := \widetilde{\ell}_R^-{}'(\widetilde{G}_{-K})_0|\Omega_+\rangle$.

• A massive representation built on $|\Omega_-\rangle$ is not labelled by the quantum numbers of $|\Omega_-\rangle$, but instead by $\widetilde{\ell}_R^+{}' := \widetilde{\ell}_{NS}^+{}' + \frac{1}{2}$ and $\widetilde{\ell}_R^-{}' := \frac{\widetilde{k}^-}{2} - \widetilde{\ell}_{NS}^-{}'$ with $\frac{1}{2} \leq \widetilde{\ell}_R^\pm{}' \leq \frac{\widetilde{k}^\pm}{2}$, where $\widetilde{T_0}^{+3}(\widetilde{G}_{+K})_0|\Omega_-\rangle := \widetilde{\ell}_R^+{}'(\widetilde{G}_{+K})_0|\Omega_-\rangle$ and $\widetilde{T_0}^{-3}|\Omega_-\rangle := \widetilde{\ell}_R^-{}'|\Omega_-\rangle$. The two representations are isomorphic.

So for massive Ramond $\widetilde{\mathcal{A}}_\gamma$ unitary representations of labels $\widetilde{\ell}_R^\pm{}'$, the bounds are

$$\frac{1}{2} \leq \widetilde{\ell}_R^\pm{}' \leq \frac{\widetilde{k}^\pm}{2}.$$

Now consider the hws $|\Omega\rangle$ of a unitary massless representation of $\widetilde{\mathcal{A}}_\gamma$ in the NS sector with isospin quantum numbers $(\widetilde{\ell}_{NS}^+, \widetilde{\ell}_{NS}^-)$, where $\widetilde{\ell}_{NS}^\pm \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and

$$0 \leq \widetilde{\ell}_{NS}^\pm \leq \frac{1}{2}\widetilde{k}^\pm. \tag{4.2.9}$$

In this case, the NS hws $|\Omega\rangle$ flows to $|\Omega_+\rangle$ under the $SU(2)^+$ spectral flow, where $|\Omega_+\rangle$ (with isospin quantum numbers $(\widetilde{\ell}^+, \widetilde{\ell}^-)$) satisfies the following masslessness condition,

$$(\widetilde{G}_{-K})_0|\Omega_+\rangle = 0. \tag{4.2.10}$$

Analogously, $|\Omega\rangle$ flows to $|\Omega_-\rangle$ under the $SU(2)^-$ spectral flow, where $|\Omega_-\rangle$ (with isospin quantum numbers $(\widetilde{\ell}^+, \widetilde{\ell}^-)$) satisfies the following masslessness condition,

$$(\widetilde{G}_{+K})_0|\Omega_-\rangle = 0. \tag{4.2.11}$$

• A massless representation built on $|\Omega_+\rangle$ is labelled by the quantum numbers of $|\Omega_+\rangle$. The labels are thus $\widetilde{\ell}_R^+ := \frac{\widetilde{k}^+}{2} - \widetilde{\ell}_{NS}^+$ and $\widetilde{\ell}_R^- := \widetilde{\ell}_{NS}^-$ with $0 \leq \widetilde{\ell}_R^\pm \leq \frac{\widetilde{k}^\pm}{2}$.

• A massless representation built on $|\Omega_-\rangle$ is labelled by the quantum numbers of $|\Omega_-\rangle$. The labels are thus $\widetilde{\ell}_R^+ := \widetilde{\ell}_{NS}^+$ and $\widetilde{\ell}_R^-{}' := \frac{\widetilde{k}^-}{2} - \widetilde{\ell}_{NS}^-$ and with $0 \leq \widetilde{\ell}_R^\pm \leq \frac{\widetilde{k}^\pm}{2}$. The two representations are isomorphic.

So for massless Ramond $\widetilde{\mathcal{A}}_\gamma$ representations of labels $\widetilde{\ell}_R^\pm$, the bounds are

$$0 \leq \widetilde{\ell}_R^\pm \leq \frac{\widetilde{k}^\pm}{2}.$$

We summarize the information on quantum numbers of $\widetilde{\mathcal{A}}_\gamma$ unitary representations in

type of rep	sector	$SU(2)^\pm$ isospin	conformal weight
$\widetilde{\mathcal{A}}_\gamma$ massive	NS	$0 \leq \widetilde{\ell}^{\pm\prime}_{NS} \leq \frac{1}{2}(\widetilde{k}^\pm - 1)$	$k\widetilde{h}_{NS} > (\widetilde{\ell}^{+\prime}_{NS} - \widetilde{\ell}^{-\prime}_{NS})^2 + k^-\widetilde{\ell}^+_{NS} + k^+\widetilde{\ell}^-_{NS}$
		with $\widetilde{\ell}^{\pm\prime}_{NS} = \ell^{\pm\prime}_{NS}$	with $k\widetilde{h}_{NS} = kh_{NS} - u^2$
	R	$\frac{1}{2} \leq \widetilde{\ell}^{\pm\prime}_R \leq \frac{1}{2}\widetilde{k}^\pm$	$k\widetilde{h}_R > (\widetilde{\ell}^{+\prime}_R + \widetilde{\ell}^{-\prime}_R + \frac{1}{2})(\widetilde{\ell}^{+\prime}_R + \widetilde{\ell}^{-\prime}_R - \frac{1}{2}) + \frac{1}{4}\tilde{k}^+\tilde{k}^-$
		with $\widetilde{\ell}^{\pm\prime}_R = \ell^{\pm\prime}_R - \frac{1}{2}$	with $k\widetilde{h}_R = kh_R - u^2 - \frac{1}{4}k$
$\widetilde{\mathcal{A}}_\gamma$ massless	NS	$0 \leq \widetilde{\ell}^\pm_{NS} \leq \frac{1}{2}\widetilde{k}^\pm$	$k\widetilde{h}^0_{NS} = (\widetilde{\ell}^+_{NS} - \widetilde{\ell}^-_{NS})^2 + k^-\widetilde{\ell}^+_{NS} + k^+\widetilde{\ell}^-_{NS}$
	R	$0 \leq \widetilde{\ell}^\pm_R \leq \frac{1}{2}\widetilde{k}^\pm$	$k\widetilde{h}^0_R = (\widetilde{\ell}^+_R + \widetilde{\ell}^-_R)(\widetilde{\ell}^+_R + \widetilde{\ell}^-_R + 1) + \frac{1}{4}\tilde{k}^+\tilde{k}^-$

Table 4.2: $\widetilde{\mathcal{A}}_\gamma$ *quantum numbers - unitary bounds.*

Table 4.2.

4.3 Characters for $\mathcal{A}_\gamma$ and $\widetilde{\mathcal{A}}_\gamma$

The character of a module $V(c, h, \ell^+, \ell^-)$ of the $\mathcal{A}_\gamma$ algebra is defined formally by the trace over V [PT90a], i.e.

$$Ch^{\mathcal{A}_\gamma,\mathrm{I}}(k^+, k^-, h, \ell^+, \ell^-; \tau, \omega_+, \omega_-) := \mathrm{Tr}_V(q^{L_0 - c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}) \tag{4.3.1}$$

where I denotes NS or R, and $q = e^{2\pi i\tau}$ with $\tau \in \mathfrak{H}$ and $z_\pm = e^{2\pi i w_\pm}$ for $w_\pm \in \mathbb{C}$ throughout correspond to the variables of the affine $\widehat{SU(2)}_{k^\pm}$ charges. Note that in principle, one could include a third angular variable $y := e^{2\pi i\xi}, \xi \in \mathbb{C}$ to keep track of the $U(1)$ quantum number of states in the module V in the characters, but we set $\xi = 0$ in this work, and most of the time choose the eigenvalue u of the hws under the $U(1)$ generator U_0 to be zero. The spectral flow property reviewed in Appendix 4.A.2 provides an isomorphism between characters in the NS and the R sectors and is implemented at the level of characters through specific shifts in the angular variables $z_\pm$, namely

$$\omega^\pm \to \omega^\pm \pm \frac{\tau}{2} \ \text{ or } \ z_\pm \to q^{\pm\frac{1}{2}} z_\pm. \tag{4.3.2}$$

For a spectral flow in the $SU(2)^+$ direction one has, for instance,

$$Ch^{\mathcal{A}_\gamma,\mathrm{NS}}(k^+, k^-, h, \ell^+, \ell^-; \tau, \omega_+, \omega_-) = \mathrm{Tr}(q^{L_0^{\mathrm{NS}} - c/24} z_+^{2T_0^{+3,\mathrm{NS}}} z_-^{2T_0^{-3,\mathrm{NS}}})$$

$$
\begin{aligned}
&= \mathrm{Tr}(q^{L_0^{\mathrm{R}}+T_0^{+3,\mathrm{R}}+k^+/4-c/24} z_+^{2T_0^{+3,\mathrm{R}}+k^+/2} z_-^{2T_0^{-3,\mathrm{R}}}) \\
&= q^{k^+/4} z_+^{k^+/2} \mathrm{Tr}(q^{L_0^{\mathrm{R}}-c/24} (q^{1/2} z_+)^{2T_0^{+3,\mathrm{R}}} z_-^{2T_0^{-3,\mathrm{R}}}) \\
&= q^{k^+/4} z_+^{k^+/2} Ch^{\mathcal{A}_\gamma,\mathrm{R}}(k^+,k^-,h-\ell^+ + k^+/4, k^+/2-\ell^+, \ell^- +1; \tau, \omega_+ + \frac{\tau}{2}, \omega_-),
\end{aligned}
\tag{4.3.3}
$$

while in the $SU(2)^-$ direction we have the similar result,

$$
\begin{aligned}
&Ch^{\mathcal{A}_\gamma,\mathrm{NS}}(k^+,k^-,h,\ell^+,\ell^-;q,z^+,z^-) \\
&= q^{k^-/4} z_+^{k^-/2} Ch^{\mathcal{A}_\gamma,\mathrm{R}}(k^+,k^-,h-\ell^- + k^-/4, \ell^+ + 1, k^-/2-\ell^-; \tau, \omega_+, \omega_- + \frac{\tau}{2}).
\end{aligned}
\tag{4.3.4}
$$

We will mainly consider characters in the Ramond sector and using the spectral flow isomorphism, one can easily get the characters in the NS sector. We very briefly recall how to construct such characters and refer to [PT90a; PT90b] for details. The $\mathcal{A}_\gamma$ character for the full *reducible* module built on a massive highest weight state in the Ramond sector can be written as

$$
\begin{aligned}
&Ch^{\mathcal{A}_\gamma,\mathrm{R}}_{\mathrm{full}}(k^+,k^-,h_R,\ell_R^+,\ell_R^-;\tau,\omega_+,\omega_-) = q^{h_R-\frac{c}{24}} z_+^{2\ell_R^+} z_-^{2\ell_R^- -2} \times \\
&B(q,z_+,z_-) \left(F^{\mathrm{R}}(q,z_+,z_-)\right)^2 \times (1+z_+^{-1}z_-)^2(1+z_+^{-1}z_-^{-1})^2 \prod_{n=1}^{\infty}(1-q^n)^{-2},
\end{aligned}
\tag{4.3.5}
$$

where

$$
B(q,z_+,z_-) := \prod_{n=1}^{\infty}(1-z_+^2q^n)^{-1}(1-z_+^{-2}q^{n-1})^{-1}(1-z_-^2q^n)^{-1}(1-z_-^{-2}q^{n-1})^{-1}(1-q^n)^{-2}
\tag{4.3.6}
$$

and

$$
F^{\mathrm{R}}(q,z_+,z_-) := \prod_{n=1}^{\infty}(1+z_+z_-q^n)(1+z_+z_-^{-1}q^n)(1+z_+^{-1}z_-q^n)(1+z_+^{-1}z_-^{-1}q^n). \tag{4.3.7}
$$

Explanation of the reducible character:

1. The states built from the hws $|\Omega_+\rangle$ by negative modes of the Virasoro generator $L_{-n}, n \in \mathbb{Z}_{>0}$ are counted by $\prod_{n=1}^{\infty}(1-q^n)^{-1}$, as are the states built with negative modes of the operators U and $T^{\pm 3}$.

2. The factors $\prod_{n=1}^{\infty}(1-z_+^2q^n)^{-1}(1-z_+^{-2}q^{n-1})^{-1}(1-z_-^2q^n)^{-1}(1-z_-^{-2}q^{n-1})^{-1}$ are

from the raising and lowering negative modes of operators $T_n^{\pm,+}$ and $T_n^{\pm,-}$ and the two zero modes $T_0^{\pm,-}$ of the two affine $\widehat{SU(2)}_{k^\pm}$ subalgebras on the highest weight state. Hence the above $B^+(q,z_+,z_-)$ refers to the two affine $\widehat{SU(2)}_{k^\pm}$ subalgebras contribution.

3. The negative modes of the four fermionic operator $Q_\pm$ and $Q_{\pm K}$ give the contribution $F^{\mathrm{R}}(q,z_+,z_-)$; while the other factor $F^{\mathrm{R}}(q,z_+,z_-)$ corresponds to the negative modes of the four fermionic operator $G_\pm$ and $G_{\pm K}$, respectively.

4. The zero modes of the four fermionic operators Q_-, Q_{-K}, G_-, G_{-K} give the contribution $(1+z_+^{-1}z_-)^2(1+z_+^{-1}z_-^{-1})^2$.

Note also that in the NS sector, we have

$$F^{\mathrm{NS}}(q,z_+,z_-) := \prod_{n\in\mathbb{Z}_{\geq 0}+1/2} (1+z_+z_-q^n)(1+z_+z_-^{-1}q^n(1+z_+^{-1}z_-q^n))(1+z_+^{-1}z_-^{-1}q^n). \tag{4.3.8}$$

The construction of irreducible characters requires the knowledge of all singular vectors appearing in the reducible module V. A singular vector is a vector satisfying the same conditions as the state hws, but unlike the hws a singular vector has zero norm and its scalar product with all other states in the representation vanishes. One then subtracts modules built on these singular vectors and adds back the states that have been subtracted twice according to a technique advocated by Kac and which is quite involved in the case of $\mathcal{A}_\gamma$. We will not say more here, referring to [PT90a; PT90b] for details.

One may consider the singular vectors of $\widetilde{\mathcal{A}}_\gamma$ instead of the full $\mathcal{A}_\gamma$. Note that

$$Ch^{\mathcal{A}_\gamma,I} = Ch^{A_{QU},I} \times Ch^{\widetilde{\mathcal{A}}_\gamma,I}, \tag{4.3.9}$$

where $Ch^{A_{QU},I}$ is the algebra for four fermionic operators $Q_\pm$ and $Q_{\pm K}$ and one $U(1)$ bosonic generator, and in the twisted Ramond sector, we have

$$\begin{aligned} Ch^{A_{QU},\widetilde{\mathrm{R}}}(u;\tau,\omega_+,\omega_-) &= q^{u^2/k+1/8} F^{\widetilde{\mathrm{R}}}(q,z_+,z_-) \times \prod_{n=1}^{\infty}(1-q^n)^{-1}(1-z_+^{-1}z_-^{-1})(z_+-z_-) \\ &= -q^{u^2/k}\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta(\tau)^3}, \end{aligned} \tag{4.3.10}$$

where $F^{\widetilde{\mathrm{R}}}(q,z_+,z_-) = F^R(q,z_+,-z_-)$, the central charge $c=3$ is from four fermion contribution $4c_f = 4\times\frac{1}{2} = 2$ and one boson $c_b = 1$ and u^2/k is from bosonic $\widehat{u(1)}$ contribution.

The characters for massive irreducible representations of $\widetilde{\mathcal{A}}_\gamma$ in the Ramond sector are given by [PT90b], formula (5.4),

$$\begin{aligned} &Ch^{\widetilde{\mathcal{A}}_\gamma,\mathrm{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\widetilde{\ell}_R^+{}',\widetilde{\ell}_R^-{}',x),\widetilde{\ell}_R^+{}',\widetilde{\ell}_R^-{}';\tau,\omega_+,\omega_-) = \\ &\qquad q^{\widetilde{h}_R-\tilde{c}/24}F^{\mathrm{R}}(q,z_+,z_-)B(q,z_+,z_-)\prod_{n=1}^{\infty}(1-q^n)^{-1}(z_+^{-1}+z_-^{-1})(1+z_+^{-1}z_-^{-1}) \\ &\qquad\times\sum_{m,n=-\infty}^{\infty} q^{n^2k^++2n\widetilde{\ell}_R^+{}'+m^2k^-+2m\widetilde{\ell}_R^-{}'}\sum_{\epsilon_\pm=\pm1}\epsilon_+\epsilon_- z_+^{2\epsilon_+(\widetilde{\ell}_R^+{}'+nk^+)}z_-^{2\epsilon_-(\widetilde{\ell}_R^-{}'+mk^-)}, \end{aligned} \tag{4.3.11}$$

where $x\in\mathbb{R}\setminus\{0\}$ and $\widetilde{h}_R(\widetilde{\ell}_R^+{}',\widetilde{\ell}_R^-{}',x)$ is a continuous real parameter satisfying

$$\widetilde{h}_R(\widetilde{\ell}_R^+{}',\widetilde{\ell}_R^-{}',x) > \frac{1}{k}(\widetilde{\ell}_R^+{}'+\widetilde{\ell}_R^-{}'+\frac{1}{2})(\widetilde{\ell}_R^+{}'+\widetilde{\ell}_R^-{}'-\frac{1}{2})+\frac{1}{4k}\widetilde{k^+}\widetilde{k^-} =: \widetilde{h_R^0}(\widetilde{\ell}_R^+{}',\widetilde{\ell}_R^-{}'-\frac{1}{2}), \tag{4.3.12}$$

and we recall that $\tilde{k}^\pm = k^\pm - 1$. The characters for massless irreducible representations of $\widetilde{\mathcal{A}}_\gamma$ in the Ramond sector are (see [PT90b], formula (5.6))

$$\begin{aligned} &Ch_0^{\tilde{A}_\gamma,\mathrm{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\widetilde{\ell}_R^+,\widetilde{\ell}_R^-);\tau,\omega_+,\omega_-) = \\ &\qquad q^{\widetilde{h_R^0}-\tilde{c}/24}F^{\mathrm{R}}(q,z_+,z_-)B(q,z_+,z_-)\prod_{n=1}^{\infty}(1-q^n)^{-1}(z_+^{-1}+z_-^{-1})(1+z_+^{-1}z_-^{-1}) \\ &\qquad\times\sum_{m,n=-\infty}^{\infty} q^{n^2k^++2n\widetilde{\ell}_R^++m^2k^-+2m\widetilde{\ell}_R^-} \\ &\qquad\times\sum_{\epsilon_\pm=\pm1}\epsilon_+\epsilon_-(-1)^{2\widetilde{\ell}_R^-}z_+^{2\epsilon_+(\widetilde{\ell}_R^++nk^+)}z_-^{2\epsilon_-(\widetilde{\ell}_R^-+mk^-)}(z_+^{-\epsilon_+}q^{-n}+z_-^{-\epsilon_-}q^{-m})^{-1}, \end{aligned} \tag{4.3.13}$$

with

$$\widetilde{h_R^0}(\widetilde{\ell}_R^+,\widetilde{\ell}_R^-) = (\widetilde{\ell}_R^++\widetilde{\ell}_R^-)(\widetilde{\ell}_R^++\widetilde{\ell}_R^-+1)+\frac{\tilde{k}^+\tilde{k}^-}{4}, \tag{4.3.14}$$

as in Table 4.2. In the next section, we present a detailed calculation of how the $\widetilde{\mathcal{A}}_\gamma$ characters transform under the modular group $SL(2,\mathbb{Z})$. We work in the twisted Ramond sector, which may be obtained from the expressions in the Ramond sector ((4.3.11) and

(4.3.13)) by letting $z_- \to -z_-$ (or equivalently, $\omega_- \to \omega_- \pm \frac{1}{2}$). Note that using spectral flow, we may obtain the massive and massless characters for $\widetilde{\mathcal{A}}_\gamma$ in the Neveu-Schwarz sector.

When the angular variables $z_\pm$ are related by $z_- = z_+$ or $z_- = z_+^{-1}$, the massless characters in the twisted Ramond sector reduce to $\widehat{su(2)}_{k-2}$ characters, namely

$$Ch_0^{\tilde{A}_\gamma,\tilde{R}}(\tilde{k}^\pm, \widetilde{h_0^R}, \ell_R^+, \ell_R^-; q, z_+, z_+) = (-1)^{2\ell_R^- + 1}\chi^{k-2}_{2\ell_R^+ + 2\ell_R^-}(q, z_+), \tag{4.3.15}$$

$$Ch_0^{\tilde{A}_\gamma,\tilde{R}}(\tilde{k}^\pm, \widetilde{h_0^R}, \ell_R^+, \ell_R^-; q, z_+, z_+^{-1}) = (-1)^{2\ell_R^-}\chi^{k-2}_{2\ell_R^+ + 2\ell_R^-}(q, z_+), \tag{4.3.16}$$

where

$$\chi^k_\ell(\tau, \zeta) = \frac{i}{\vartheta_1(\tau, 2\zeta)}\{\theta_{2\ell+1,k+2}(\tau, 2\zeta) - \theta_{-(2\ell+1),k+2}(\tau, 2\zeta)\}, \tag{4.3.17}$$

is the $\widehat{SU(2)}_k$ character of irreducible representations of isospin ℓ $(0 \leq \ell \leq k/2)$, see Appendix 3.B. One may derive the identity $(4.3.15)$ by setting $n = m$ and $\epsilon_+ = \epsilon_-$ in $(4.3.13)$ and the identity $(4.3.16)$ by setting $n = m$ and $\epsilon_+ = -\epsilon_-$ in the same character formula $(4.3.13)$.

4.4 S-transformation of $\widetilde{\mathcal{A}}_\gamma$ Characters

In this section, we first present expressions for the $\widetilde{\mathcal{A}}_\gamma$ characters in the twisted Ramond ($\tilde{R}$) sector by replacing ω_- by $\omega_- + \frac{1}{2}$ in the $\widetilde{\mathcal{A}}_\gamma$ Ramond characters (4.3.11) and (4.3.13). Our motivation for studying the $\widetilde{\mathcal{A}}_\gamma$ $\tilde{R}$ characters is driven by the wish to explore the possibility of a Moonshine phenomenon in theories enjoying $\widetilde{\mathcal{A}}_\gamma$ symmetry. By analogy with the Mathieu Moonshine observation, a potential Moonshine phenomenon in the present context is expected to be easily recognisable in the $\tilde{R}\tilde{R}$ sector of some relevant partition function with $\widetilde{\mathcal{A}}_\gamma$ symmetry. In a subsequent chapter we present some preliminary results of this exploration, which rely crucially on the transformations of the $\widetilde{\mathcal{A}}_\gamma$ characters under the modular group $SL(2, \mathbb{Z})$ generated by the two transformations $T : \tau \to \tau + 1$ and $S : \tau \to -\frac{1}{\tau}$. Although the T-transformation is usually straightforward to derive, the S-transformation is much more involved, and our first task is to rewrite the massless

$\widetilde{\mathcal{A}}_\gamma$ characters calculated in [PT90b] in terms of Appell functions, whose modular transformations at higher integer level than one were first derived in the mathematical physics community by Semikhatov, Taormina and Tipunin [STT05].[2]

4.4.1 $\widetilde{\mathcal{A}}_\gamma$ characters in the twisted Ramond sector

Recall that $\tilde{k}^+ = k^+ - 1$ and $\tilde{k}^- = k^- - 1$ are the levels of the two $\widehat{su(2)}$ subalgebras of $\widetilde{\mathcal{A}}_\gamma$ and the central charge be given by,

$$\tilde{c} = c - 3, \qquad c = \frac{6k^+k^-}{k}, \qquad k := k^+ + k^-. \tag{4.4.1}$$

Since we are only working in the $\widetilde{R}$ sector of the $\widetilde{\mathcal{A}}_\gamma$ algebra from now on, we shall suppress the subscript R and the tilde symbol over the isospin quantum numbers in an attempt to make the formulas easier to read. So we set $\ell^{\pm\prime} := \widetilde{\ell}^{\pm\prime}_R$ in the case of massive characters, and we set $\ell^\pm := \widetilde{\ell}^\pm_R$ in the case of massless characters from now on.

Massive characters

Starting with (4.3.11) and letting $\omega_- \to \omega_- + \frac{1}{2}$, one may obtain the $\tilde{A}_\gamma$ massive characters in the $\tilde{R}$ sector,

$$\begin{aligned}
&Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-) = (-1)^{2\ell^{-\prime}} q^{\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)-\tilde{c}/24} \\
&\times \prod_{n=1}^{\infty}(1-z_+z_-q^n)(1-z_+^{-1}z_-^{-1}q^n)(1-z_+^{-1}z_-q^n)(1-z_+z_-^{-1}q^n)(z_+^{-1}-z_-^{-1})(1-z_+^{-1}z_-^{-1}) \\
&\times \prod_{n=1}^{\infty}(1-q^n)^{-3}(1-z_+^2q^n)^{-1}(1-z_-^2q^n)^{-1}(1-z_+^{-2}q^{n-1})^{-1}(1-z_-^{-2}q^{n-1})^{-1} \\
&\times \sum_{m,n=-\infty}^{\infty} q^{n^2k^++2n\ell^{+\prime}+m^2k^-+2m\ell^{-\prime}} \sum_{\epsilon_+,\epsilon_-=\pm1} \epsilon_+\epsilon_- z_+^{2\epsilon_+(\ell^{+\prime}+nk^+)} z_-^{2\epsilon_-(\ell^{-\prime}+mk^-)} \\
&= (-1)^{2\ell^-+1}\, q^{\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)-\tilde{c}/24+\frac{1}{8}} \,\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)} \\
&\times \sum_{m,n=-\infty}^{\infty} q^{n^2k^++2n\ell^{+\prime}+m^2k^-+2m\ell^{-\prime}} \sum_{\epsilon_+,\epsilon_-=\pm1} \epsilon_+\epsilon_- z_+^{2\epsilon_+(\ell^{+\prime}+nk^+)} z_-^{2\epsilon_-(\ell^{-\prime}+mk^-)} \\
&= (-1)^{2\ell^{-\prime}+1} q^{\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)-\tilde{c}/24+1/8-\frac{(\ell^{+\prime})^2}{k^+}-\frac{(\ell^{-\prime})^2}{k^-}} \frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)}
\end{aligned}$$

[2] See also Zwegers' PhD thesis [Zwe08].

$$\times \sum_{\epsilon_+,\epsilon_-=\pm 1} \epsilon_+\epsilon_-\theta_{2\ell^{+\prime},k^+}(\tau, 2\epsilon_+ w_+)\theta_{2\ell^{-\prime},k^-}(\tau, 2\epsilon_- w_-)$$

$$\stackrel{(3.A.8)}{=} (-1)^{2\ell^{-\prime}} q^{\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)-\tilde{c}/24+1/8-\frac{(\ell^{+\prime})^2}{k^+}-\frac{(\ell^{-\prime})^2}{k^-}} \frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)}$$
$$\times \sum_{\epsilon_+,\epsilon_-=\pm 1}\sum_{r\in\mathbb{Z}_k} \epsilon_+\epsilon_-\theta_{2\ell^{+\prime}k^-+2\ell^{-\prime}k^++2k^+k^-r,k^+k^-k}(\tau, \frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\,\theta_{2\ell^{+\prime}-2\ell^{-\prime}-2k^-r,k}(\tau,\zeta_{\epsilon_+,\epsilon_-}) \tag{4.4.2}$$

$$=(-1)^{2\ell^{-\prime}} q^{\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)-\tilde{c}/24+1/8-\frac{(\ell^{+\prime})^2}{k^+}-\frac{(\ell^{-\prime})^2}{k^-}} \frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)}$$
$$\times \chi^{k^+-2}_{\ell^{+\prime}-1/2}(\tau,w_+)\chi^{k^--2}_{\ell^{-\prime}-1/2}(\tau,w_-), \tag{4.4.3}$$

where $\chi^{k^+-2}_{\ell^{+\prime}-1/2}(\tau,w_+)$ and $\chi^{k^--2}_{\ell^{-\prime}-1/2}(\tau,w_-)$ are affine $\widehat{su(2)}$ characters (see Appendix 3.B), $1 \le 2\ell^{\pm\prime} \le \tilde{k^{\pm}}$, $x \in \mathbb{R}_+ \setminus \{0\}$ and $\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)$ is the continuous real parameter introduced in (4.3.12).

The lower bound, which is never attained for massive characters, corresponds to the conformal dimension of massless Ramond characters, which is strictly positive. We write

$$\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x) := \widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}) + \frac{x^2}{2}, \qquad x\in\mathbb{R}\setminus\{0\} \tag{4.4.4}$$

with h^0_R defined in (4.3.14) and define

$$\begin{aligned}\Delta(\ell^{+\prime},\ell^{-\prime}) &:= \widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x)-\tilde{c}/24+1/8-\frac{(\ell^{+\prime})^2}{k^+}-\frac{(\ell^{-\prime})^2}{k^-}-\frac{x^2}{2}\\ &= -\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}\ell^{+\prime}-\sqrt{\frac{k^+}{k^-}}\ell^{-\prime}\right)^2.\end{aligned} \tag{4.4.5}$$

Under the modular transformation $T\colon \tau\to\tau+1$, the $\widetilde{\mathcal{A}}_\gamma$ massive characters transform as,

$$Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x),\ell^{+\prime},\ell^{-\prime};\tau+1,\omega_+,\omega_-) =$$
$$e^{\pi i\{x^2+\frac{1}{2k}(2\ell^{+\prime}+2\ell^{-\prime})^2-\frac{1}{4}\}}\,Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-). \tag{4.4.6}$$

In the next section, we study the transformation of the massive characters under $S\colon\ \tau\to-\frac{1}{\tau}$, $\omega_\pm\to\frac{\omega_\pm}{\tau}$, with the characters expressed as

$$Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},x),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-)$$

$$= (-1)^{2\ell^{-\prime}} \frac{\vartheta_1(\tau,\omega_+ + \omega_-)\vartheta_1(\tau,\omega_+ - \omega_-)}{\eta^3(\tau)}\, q^{\Delta(\ell^{+\prime},\ell^{-\prime})+\frac{x^2}{2}}\, \chi^{k^+-2}_{\ell^{+\prime}-1/2}(\tau;w_+)\chi^{k^--2}_{\ell^{-\prime}-1/2}(\tau;w_-), \tag{4.4.7}$$

where $x \in \mathbb{R}\setminus\{0\}$.

In the context of some applications of the S-transformation, to be discussed in the next chapter, it will also be helpful to introduce the definition of 'massive character at threshold',

$$\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-) := \\ Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R(\ell^{+\prime},\ell^{-\prime},x),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-)\Big|_{x=0}. \tag{4.4.8}$$

Note that the massive characters at threshold defined above are *not* characters of representations of the $\widetilde{\mathcal{A}}_\gamma$ superconformal algebra. One has

$$Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R(\ell^{+\prime},\ell^{-\prime},x),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-) = \\ q^{\frac{x^2}{2}}\,\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-), \quad x\in\mathbb{R}\setminus\{0\}. \tag{4.4.9}$$

Massless characters

The massless $\widetilde{\mathcal{A}}_\gamma$ characters for unitary and irreducible representations in the $\tilde{R}$ sector may be obtained by replacing ω_- by $\omega_- + \frac{1}{2}$ in formula (5.6) of [PT90b] (reproduced in (4.3.13)) and by imposing the chirality condition on $|\Omega_+\rangle$. This yields

$$Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^+,\ell^-),\ell^+,\ell^-;\tau,\omega_+,\omega_-) = (-1)^{2\ell^-+1} \\ \times q^{\widetilde{h^0_R}(\ell^+,\ell^-)-\tilde{c}/24+1/8}\,\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)} \sum_{\epsilon_+,\epsilon_-=\pm1}\epsilon_+\epsilon_- S^{\tilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-). \tag{4.4.10}$$

Defining $\widetilde{T_0}^{\pm 3}|\Omega_+\rangle := \ell^\pm|\Omega_+\rangle$ with $0 \le 2\ell^\pm \le \tilde{k}^\pm$, the conformal weight $\widetilde{h^0_R}$ of the massless state $|\Omega_+\rangle$ on which we choose to build the representation is given by

$$\widetilde{h^0_R}(\ell^+,\ell^-) = \frac{1}{k}(\ell^++\ell^-)(\ell^++\ell^-+1)+\frac{\tilde{k}^+\tilde{k}^-}{4k}, \tag{4.4.11}$$

in accordance with Table 4.2, and therefore,

$$\widetilde{h^0_R}(\ell^+,\ell^-)-\tilde{c}/24+1/8 = \frac{1}{k}(\ell^++\ell^-+1)(\ell^++\ell^-)+\frac{1}{4k}. \tag{4.4.12}$$

The infinite sum $S^{\tilde{R}}_{\epsilon_+,\epsilon_-}$ is given by

$$\begin{aligned}
& S^{\tilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-) \\
= \quad & \sum_{m,n=-\infty}^{\infty} q^{m^2k^+ + n^2k^- + 2\ell^+ m + 2\ell^- n}\, \frac{z_+^{2\epsilon_+(\ell^+ + mk^+)}\; z_-^{2\epsilon_-(\ell^- + nk^-)}}{z_+^{-\epsilon_+} q^{-m} - z_-^{-\epsilon_-} q^{-n}} \\
\overset{n\to -n}{=} \quad & \sum_{m,n=-\infty}^{\infty} q^{m^2k^+ + n^2k^- + (2\ell^+ +1)m - 2\ell^- n}\, \frac{z_+^{2\epsilon_+(\ell^+ + mk^+ + \frac{1}{2})}\; z_-^{2\epsilon_-(\ell^- - nk^-)}}{1 - z_+^{\epsilon_+} z_-^{-\epsilon_-} q^{m+n}} \\
\overset{p=m+n}{=} \quad & \sum_{m,p=-\infty}^{\infty} q^{m^2k^+ + (p-m)^2k^- + (2\ell^+ +1)m - 2\ell^-(p-m)}\, \frac{z_+^{2\epsilon_+(\ell^+ + mk^+ + \frac{1}{2})}\; z_-^{2\epsilon_-(\ell^- - (p-m)k^-)}}{1 - z_+^{\epsilon_+} z_-^{-\epsilon_-} q^{p}} \\
= \quad & \sum_{m,p=-\infty}^{\infty} q^{k(m+\frac{1}{k}(\ell^+ + \ell^- + \frac{1}{2} - pk^-))^2 - \frac{1}{k}(\ell^+ + \ell^- + \frac{1}{2} - pk^-)^2 + p^2k^- - 2\ell^- p} \\
& \qquad \times \frac{z_+^{2\epsilon_+(\ell^+ + mk^+ + \frac{1}{2})}\; z_-^{2\epsilon_-(\ell^- - (p-m)k^-)}}{1 - z_+^{\epsilon_+} z_-^{-\epsilon_-} q^{p}}.
\end{aligned} \tag{4.4.13}$$

We are interested in the behaviour of the massless $\widetilde{\mathcal{A}}_\gamma$ characters under the modular group $SL(2,\mathbb{Z})$. Under the T-transformation, the characters transform as

$$\begin{aligned}
& Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;\tau+1,\omega_+,\omega_-) = \\
& \quad e^{\pi i\{\frac{1}{2k}[(2\ell^+ + 2\ell^- + 2)(2\ell^+ + 2\ell^-)+1] - \frac{1}{4}\}}\, Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;\tau,\omega_+,\omega_-).
\end{aligned} \tag{4.4.14}$$

In order to derive how they behave under the S-transformation, we now partition the summation index p as $p = kr + s, r \in \mathbb{Z}$ and $s \in \mathbb{Z}_k$ and write

$$\begin{aligned}
S^{\tilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-) = & \sum_{s=0}^{k-1} \sum_{m,r=-\infty}^{\infty} q^{k(m-k^-r+\frac{1}{k}(\ell^+ + \ell^- + \frac{1}{2} - sk^-))^2 - \frac{1}{k}(\ell^+ + \ell^- + \frac{1}{2})^2 + k^-k^+k(r+\frac{s}{k})^2} \\
& \times\; q^{-(r+\frac{s}{k})(2k^+\ell^- - 2k^-\ell^+ - k^-)}\, \frac{z_+^{2\epsilon_+(\ell^+ + mk^+ + \frac{1}{2})}\; z_-^{2\epsilon_-(\ell^- + mk^- - k^-k(r+\frac{s}{k}))}}{1 - z_+^{\epsilon_+} z_-^{-\epsilon_-} q^{k(r+\frac{s}{k})}},
\end{aligned} \tag{4.4.15}$$

and note that

$$\begin{aligned}
& z_+^{2\epsilon_+(\ell^+ + mk^+ + \frac{1}{2})}\; z_-^{2\epsilon_-(\ell^- + mk^- - k^-k(r+\frac{s}{k}))} = \\
& \left(z_+^{2\epsilon_+k^+/k} z_-^{2\epsilon_-k^-/k}\right)^{k(m-k^-r+\frac{\ell^+ + \ell^- + \frac{1}{2} - sk^-}{k})} \left(z_+^{2\epsilon_+} z_-^{-2\epsilon_-}\right)^{k^+k^-r + \frac{k^-}{k}\ell^+ - \frac{k^+}{k}(\ell^- + \frac{1}{2} - sk^-) + \frac{1}{2}}.
\end{aligned} \tag{4.4.16}$$

We define

$$\nu_{\epsilon_+,\epsilon_-} := \epsilon_+\omega_+ - \epsilon_-\omega_- \quad \text{and} \quad \zeta_{\epsilon_+,\epsilon_-} := \frac{2\epsilon_+k^+\omega_+ + 2\epsilon_-k^-\omega_-}{k} \tag{4.4.17}$$

and express $S^{\tilde{R}}_{\epsilon_+,\epsilon_-}$ in terms of

$$y_{\epsilon_+,\epsilon_-} := e^{2\pi i\nu_{\epsilon_+,\epsilon_-}} = z_+^{\epsilon_+} z_-^{-\epsilon_-}, \qquad z_{\epsilon_+,\epsilon_-} := e^{2\pi i\zeta_{\epsilon_+,\epsilon_-}} = z_+^{2\epsilon_+k^+/k} z_-^{2\epsilon_-k^-/k}. \tag{4.4.18}$$

In (4.4.15), the infinite sum over m is a theta function (see Appendix 3.A.2) namely

$$\sum_{m=-\infty}^{\infty} q^{k(m-k^-r+\frac{1}{k}(\ell^++\ell^-+\frac{1}{2}-sk^-))^2}\, z_{\epsilon_+,\epsilon_-}^{k(m-k^-r+\frac{1}{k}(\ell^++\ell^-+\frac{1}{2}-sk^-))} = \theta_{2\ell^++2\ell^-+1-2sk^-,k}(\tau,\zeta_{\epsilon_+,\epsilon_-}). \tag{4.4.19}$$

We therefore have

$$\begin{aligned} &S^{\tilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-) \\ &\quad = q^{-\frac{1}{k}(\ell^++\ell^-+\frac{1}{2})^2}\, y_{\epsilon_+,\epsilon_-}^{\frac{1}{k}(2k^-\ell^+-2k^+\ell^-+k^-)} \sum_{s=0}^{k-1}\sum_{r=-\infty}^{\infty} \theta_{2\ell^++2\ell^-+1-2sk^-,k}(\tau,\zeta_{\epsilon_+,\epsilon_-}) \\ &\qquad\qquad \times q^{k^-k^+k(r+\frac{s}{k})^2-(r+\frac{s}{k})(2k^+\ell^--2k^-\ell^+-k^-)}\, \frac{y_{\epsilon_+,\epsilon_-}^{2k^+k^-(r+\frac{s}{k})}}{1-y_{\epsilon_+,\epsilon_-}q^{k(r+\frac{s}{k})}}. \end{aligned} \tag{4.4.20}$$

The rest of this chapter is original work. The next step is to rewrite the infinite series in terms of a higher level Appell function (see [STT05] and Appendix 4.B)

$$\mathcal{K}_\ell(\tau,\nu,\mu) := \sum_{m=-\infty}^{\infty} \frac{q^{\frac{\ell}{2}m^2}y^{\ell m}}{1-tyq^m} \tag{4.4.21}$$

with $y := e^{2\pi i\nu}$ and $t := e^{2\pi i\mu}$, $\mu,\nu\in\mathbb{C}$. We introduce the notations

$$\begin{aligned} \kappa &:= 2kk^+k^- \\ b(\ell^+,\ell^-) &:= 2\ell^-k^+ - 2\ell^+k^- - k^- \end{aligned} \tag{4.4.22}$$

and define the function

$$\begin{aligned} &X(b(\ell^+,\ell^-),s;\tau,\nu) \\ &\quad := q^{\frac{s}{k}(\frac{\kappa}{2k}s-b(\ell^+,\ell^-))}\, y^{\frac{1}{k}(\frac{\kappa}{k}s-b(\ell^+,\ell^-))}\, \mathcal{K}_{\frac{\kappa}{k}}(k\tau,\, \nu+(s-\frac{k}{\kappa}b(\ell^+,\ell^-))\tau,\, \frac{k}{\kappa}b(\ell^+,\ell^-)\tau) \end{aligned} \tag{4.4.23}$$

so that $S^{\tilde{R}}_{\epsilon_+,\epsilon_-}$ is rewritten as

$$S^{\tilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-) = q^{-\frac{1}{k}(\ell^+ + \ell^- + \frac{1}{2})^2} \sum_{s=0}^{k-1} \theta_{2\ell^+ + 2\ell^- + 1 - 2sk^-,k}(\tau,\zeta_{\epsilon_+,\epsilon_-}) X(b(\ell^+,\ell^-), s; \tau, \nu_{\epsilon_+,\epsilon_-}). \tag{4.4.24}$$

The $\widetilde{\mathcal{A}}_\gamma$ massless characters in the $\tilde{R}$ sector are thus,

$$\begin{aligned} & Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^+,\ell^-),\ell^+,\ell^-;\tau,\omega_+,\omega_-) \\ &= (-1)^{2\ell^- + 1} \frac{\vartheta_1(\tau,\omega_+ + \omega_-)\vartheta_1(\tau,\omega_+ - \omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)} \\ &\times \sum_{\epsilon_+,\epsilon_- = \pm 1} \epsilon_+\epsilon_- \sum_{s=0}^{k-1} \theta_{2\ell^+ + 2\ell^- + 1 - 2sk^-,k}(\tau,\zeta_{\epsilon_+,\epsilon_-})\, X(b(\ell^+,\ell^-), s; \tau, \nu_{\epsilon_+,\epsilon_-}). \end{aligned} \tag{4.4.25}$$

Three properties of the function $X(b(\ell^+,\ell^-), s; \tau, \nu)$, which will be useful when deriving the modular transformations of the massless characters can be easily checked from the explicit form or properties of the higher level Appell functions. The first one is a 'reflection' property in the variable ν,

$$X(b(\ell^+,\ell^-), s; \tau, -\nu) = -X(-(k + b(\ell^+,\ell^-)), -s; \tau, \nu). \tag{4.4.26}$$

The second is the periodicity in the variable s, namely

$$X(b(\ell^+,\ell^-), s + k; \tau, \nu) = X(b(\ell^+,\ell^-), s; \tau, \nu). \tag{4.4.27}$$

The third is the quasi-periodicity in the variable $b(\ell^+,\ell^-)$. We consider here the case where k^+ and k^- are coprime[3] and we assume, without loss of generality, that $k^+ > k^-$. With this in mind, we introduce, for each pair (ℓ^+,ℓ^-) in the ranges $0 \leq 2\ell^\pm \leq \tilde{k}^\pm$, an integer $N(\ell^+,\ell^-)$

$$N(\ell^+,\ell^-) := \left\lceil \frac{b(\ell^+,\ell^-)}{k} \right\rceil, \qquad N(\ell^+,\ell^-) \in \mathbb{Z} \tag{4.4.28}$$

so that

$$b(\ell^+,\ell^-) = N(\ell^+,\ell^-)k - b'(\ell^+,\ell^-), \qquad 0 \leq b'(\ell^+,\ell^-) \leq k - 1, \tag{4.4.29}$$

[3] This property allows us to express characters in terms of Appell functions having their third argument equal to zero (4.4.32).

noting that, given the definition (4.4.22), $b'(\ell^+, \ell^-)$ cannot be zero (see Lemma in Appendix 4.C).

Since a given pair (ℓ^+, ℓ^-) labels a unique $\widetilde{\mathcal{A}}_\gamma$ massless character, we often use b, b' and N instead of $b(\ell^+, \ell^-)$, $b'(\ell^+, \ell^-)$ and $N(\ell^+, \ell^-)$ in formulas.

We use the quasi-periodicity of Appell functions [STT05] reproduced in (4.B.8), as well as the relation between the Jacobi ϑ-function and the generalised theta-functions (see (3.A.9)) to obtain

$$X(b, s; \tau, \nu) = X(Nk - b', s; \tau, \nu) = X(-b', s; \tau, \nu) + T(N, b', s; \tau, \nu) \tag{4.4.30}$$

where

$$T(N, b', s; \tau, \nu) :=$$
$$\begin{cases} T_>(N, b', s; \tau, \nu) = \sum_{r=1}^{N} q^{-\frac{1}{2\kappa}(kr-b')^2}\, \theta_{\frac{\kappa}{k}s-kr+b', \frac{\kappa}{2}}(q, y^{2/k}) & \text{for } N > 0 \\ T_<(N, b', s; \tau, \nu) = -\sum_{r=N+1}^{0} q^{-\frac{1}{2\kappa}(kr-b')^2}\, \theta_{\frac{\kappa}{2}s-kr+b', \frac{\kappa}{2}}(q, y^{2/k}) & \text{for } N < 0 \\ 0 & \text{for } N = 0. \end{cases} \tag{4.4.31}$$

Using the average (4.B.2) with $\ell \to \frac{\kappa}{k}, \ell' \to k, a \to s$ and $b' \to b'$, we further write

$$X(-b', s; \tau, \nu) = \frac{1}{k}\sum_{s'=0}^{k-1} e^{-2\pi i b's'/k} q^{\frac{\kappa}{2k^2}s^2} y^{\frac{\kappa}{k^2}s}\, \mathcal{K}_\kappa(\tau, \frac{\nu + s\tau + s'}{k}, 0). \tag{4.4.32}$$

4.4.2 S transformation of $\widetilde{\mathcal{A}}_\gamma$ characters in $\widetilde{R}$ sector

Consider the massive character (4.4.7). First note that the S-transformation of $q^{\frac{p^2}{2}}$ is

$$e^{-\frac{\pi i p^2}{\tau}} = (-i\tau)^{\frac{1}{2}} \int_{-\infty}^{\infty} dr\, e^{-2\pi i p r} e^{\pi i \tau r^2}, \tag{4.4.33}$$

and

$$\vartheta_1(-\frac{1}{\tau}, \frac{\zeta}{\tau}) = (-i)(-i\tau)^{\frac{1}{2}}\, \mathbf{e}(\frac{\zeta^2}{2\tau})\, \vartheta_1(\tau, \zeta), \tag{4.4.34}$$

where we introduce the notation $\mathbf{e}(x) := e^{2\pi i x}$. Using (3.B.3), the S-transformed massive characters read

$$
\begin{aligned}
Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\ell^{+\prime},\ell^{-\prime},p),\,\ell^{+\prime},\,\ell^{-\prime};-\frac{1}{\tau},\frac{z_+}{\tau},\frac{z_-}{\tau}) &= (-1)^{2\ell^{-\prime}+1}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau}) \\
\times\,\widetilde{q}^{\Delta(\ell^{+\prime},\ell^{-\prime})}\sum_{2\lambda^{+\prime}=1}^{\tilde{k}^+}\sum_{2\lambda^{-\prime}=1}^{\tilde{k}^-}(-1)^{2\lambda^{-\prime}}\,q^{-\Delta(\lambda^{+\prime},\lambda^{-\prime})}\,S^{(k^+-2)}_{\ell^{+\prime}-\frac{1}{2},\lambda^{+\prime}-\frac{1}{2}}\,S^{(k^--2)}_{\ell^{-\prime}-\frac{1}{2},\lambda^{-\prime}-\frac{1}{2}} \\
\times\int_{-\infty}^{\infty}dr\cos(2pr\pi)\,Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h}_R(\lambda^{+\prime},\,\lambda^{-\prime},\,r),\lambda^{+\prime},\,\lambda^{-\prime};\tau,\omega_+,\omega_-),
\end{aligned}
\tag{4.4.35}
$$

where $S^{(k)}_{\ell,\ell'}$ is defined in (3.B.3) and $\widetilde{q}^{\Delta}:=e^{-\frac{2\pi i}{\tau}\Delta}$.

Remark: The S-transformation of massive characters at threshold reads

$$
\begin{aligned}
\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\,\ell^{+\prime},\,\ell^{-\prime};-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau}) &= \frac{(-1)^{2\ell^{-\prime}+1}}{(-i\tau)^{\frac{1}{2}}}\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau}) \\
\times\,\widetilde{q}^{\Delta(\ell^{+\prime},\ell^{-\prime})}\sum_{2\lambda^{+\prime}=1}^{\tilde{k}^+}\sum_{2\lambda^{-\prime}=1}^{\tilde{k}^-}(-1)^{2\lambda^{-\prime}}q^{-\Delta(\lambda^{+\prime},\lambda^{-\prime})}\,S^{(k^+-2)}_{\ell^{+\prime}-\frac{1}{2},\lambda^{+\prime}-\frac{1}{2}}\,S^{(k^--2)}_{\ell^{-\prime}-\frac{1}{2},\lambda^{-\prime}-\frac{1}{2}} \\
\times\,\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\lambda^{+\prime},\lambda^{-\prime}-\frac{1}{2}),\,\lambda^{+\prime},\lambda^{-\prime};\tau,\omega_+,\omega_-).
\end{aligned}
\tag{4.4.36}
$$

Massless characters

1. **Infinite product**

   The S-transformation of the infinite product in (4.4.25) is standard, and reads

$$
\begin{aligned}
&\frac{\vartheta_1(-\frac{1}{\tau},\frac{\omega_++\omega_-}{\tau})\vartheta_1(-\frac{1}{\tau},\frac{\omega_+-\omega_-}{\tau})}{\eta^3(-\frac{1}{\tau})\vartheta_1(-\frac{1}{\tau},\frac{2\omega_+}{\tau})\vartheta_1(-\frac{1}{\tau},\frac{2\omega_+}{\tau})} \\
&\qquad=(-i\tau)^{-3/2}e^{-\frac{2\pi i}{\tau}(\omega_+^2+\omega_-^2)}\,\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)}.
\end{aligned}
\tag{4.4.37}
$$

2. **Generalized theta function**

   Following (3.A.2), the S-transformation of the generalised theta function $\theta_{2\ell^++2\ell^-+1-2sk^-,k}(q,z)$ appearing in (4.4.25) with $z=e^{2\pi i\zeta}$, ζ as in (4.4.17), is given by

$$
\theta_{2\ell^++2\ell^-+1-2sk^-,k}(-\frac{1}{\tau},\frac{\zeta}{\tau})=\frac{(-i\tau)^{1/2}}{\sqrt{2k}}\mathbf{e}(\frac{k\zeta^2}{4\tau})\sum_{n'=0}^{2k-1}e^{-\pi i\frac{n'}{k}(2\ell^++2\ell^-+1-2sk^-)}\theta_{n',k}(\tau,\zeta).
\tag{4.4.38}
$$

3. S**-transformation of** $X(b,s;\tau,\nu)$ Recall that b in (4.4.22) is parametrized as $b = Nk - b'$ with $1 \leq b' \leq k-1$ and $N \in \mathbb{Z}$. The S-transformation of (4.4.30) involves that of $X(-b',s;\tau,\nu)$ and of $T(N,b',s;\tau,\nu)$. We first consider the S-transformation of the function $X(-b',s;\tau,\nu)$, and then give the S-transformation of the generalized theta function term in (4.4.31).

Consider the S-transformation of the function (4.4.32),

$$X(-b',s;-\frac{1}{\tau},\frac{\nu}{\tau}) = \frac{1}{k}\sum_{s'=0}^{k-1} e^{-2\pi i b's'/k}\, e^{-\pi i\frac{\kappa}{k^2\tau}s^2} y^{\frac{\kappa}{k^2\tau}s}\,\mathcal{K}_\kappa(-\frac{1}{\tau},\frac{1}{k\tau}(\nu - s + s'\tau),0). \tag{4.4.39}$$

We use the S-transformation of Appell functions at even positive integer level as given in Appendix 4.B.4, formula (4.B.12). Using (4.B.12) with $2m \to \kappa, \nu \to \frac{1}{k}(\nu - s + s'\tau)$ and $\mu \to 0$, we write

$$X(-b',s;-\frac{1}{\tau},\frac{\nu}{\tau}) = X_1(-b',s;\tau,\nu) + X_2(-b',s;\tau,\nu) \tag{4.4.40}$$

where

$$X_1(-b',s;\tau,\nu) = \\ \frac{\tau}{k}\sum_{s'=0}^{k-1} e^{-2\pi i b's'/k}\, e^{-\pi i\frac{\kappa}{k^2\tau}s^2} y^{\frac{\kappa}{k^2\tau}s}\, e^{\pi i\frac{\kappa}{k^2\tau}(\nu-s+s'\tau)^2}\,\mathcal{K}_\kappa(\tau,\frac{1}{k}(\nu - s + s'\tau),0), \tag{4.4.41}$$

which, using (4.B.7) with $\ell \to k, \ell' \to \frac{\kappa}{k}, a \to s', \nu \to \nu$ and $c \to -s$, becomes

$$X_1(-b',s;\tau,\nu) = \frac{\tau}{k}\mathbf{e}(\frac{\kappa\nu^2}{2k^2\tau})\sum_{a',\,s'=0}^{k-1} e^{-2\pi i\frac{1}{k}(sa'+b's'+2k^+k^-ss')}\, X(-a',s';\tau,\nu). \tag{4.4.42}$$

We also have

$$X_2(-b',s;\tau,\nu) \\ = -\frac{\tau}{k}\mathbf{e}(\frac{\kappa\nu^2}{2k^2\tau})\sum_{s'=0}^{k-1} e^{-2\pi i b's'/k}\sum_{\ell=0}^{\kappa-1} e^{-2\pi i\frac{s}{k}(\ell+\frac{\kappa}{k}s')}\, h_\ell^{(\frac{\kappa}{2})}(0,\tau)\theta_{\ell+2k^+k^-s',\frac{\kappa}{2}}(\tau,\frac{2\nu}{k}), \tag{4.4.43}$$

where $h_\ell^{(\frac{\kappa}{2})}(0,\tau)$ is given by (4.B.22) for $2m = \kappa$, while the S-transformation of

$T(N, b', s; \tau, \nu)$ is given by

$$T_>(N, b', s; -\frac{1}{\tau}, \frac{\nu}{\tau}) = \\ (-i\tau)^{\frac{1}{2}} \frac{1}{\sqrt{\kappa}} \mathbf{e}(\frac{\kappa\nu^2}{2k^2\tau}) \sum_{r=1}^{N} \widetilde{q}^{-\frac{1}{2\kappa}(kr-b')^2} \sum_{m=0}^{\kappa-1} e^{-\frac{2\pi i}{\kappa}(\frac{\kappa}{k}s-kr+b')m} \theta_{m,\frac{\kappa}{2}}(\tau, \frac{2\nu}{k}) \quad (4.4.44)$$

for $N > 0$ and

$$T_<(N, b', s; -\frac{1}{\tau}, \frac{\nu}{\tau}) = \\ -(-i\tau)^{\frac{1}{2}} \frac{1}{\sqrt{\kappa}} \mathbf{e}(\frac{\kappa\nu^2}{2k^2\tau}) \sum_{r=N+1}^{0} \widetilde{q}^{-\frac{1}{2\kappa}(kr-b')^2} \sum_{m=0}^{\kappa-1} e^{-\frac{2\pi i}{\kappa}(\frac{\kappa}{k}s-kr+b')m} \theta_{m,\frac{\kappa}{2}}(\tau, \frac{2\nu}{k}) \quad (4.4.45)$$

for $N < 0$.

Let us define

$$Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+, \tilde{k}^-, \widetilde{h_R^0}(\ell^+, \ell^-), \ell^+, \ell^-; -\frac{1}{\tau}, \frac{\omega_+}{\tau}, \frac{\omega_-}{\tau})\Big|_{\text{massless}} \\ := (-1)^{2\ell^-+1} \frac{\vartheta_1(-\frac{1}{\tau}, \frac{\omega_++\omega_-}{\tau})\vartheta_1(-\frac{1}{\tau}, \frac{\omega_+-\omega_-}{\tau})}{\eta^3(-\frac{1}{\tau})\vartheta_1(-\frac{1}{\tau}, \frac{2\omega_+}{\tau})\vartheta_1(-\frac{1}{\tau}, \frac{2\omega_+}{\tau})} \\ \times \sum_{\epsilon_+,\epsilon_-=\pm 1} \epsilon_+\epsilon_- \sum_{s=0}^{k-1} \theta_{2\ell^++2\ell^-+1-2sk^-,k}(-\frac{1}{\tau}, \frac{\zeta_{\epsilon_+,\epsilon_-}}{\tau})\, X_1(-b', s; \tau, \nu_{\epsilon_+,\epsilon_-}). \quad (4.4.46)$$

Then, from (4.4.37), (4.4.38) and (4.4.42), one obtains

$$\mathbf{e}(-\frac{\tilde{k}^+\omega_+^2 + \tilde{k}^-\omega_-^2}{\tau})\, Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+, \tilde{k}^-, \widetilde{h_R^0}(\ell^+, \ell^-), \ell^+, \ell^-; -\frac{1}{\tau}, \frac{\omega_+}{\tau}, \frac{\omega_-}{\tau})\Big|_{\text{massless}} \\ = \sqrt{\frac{2}{k}}(-1)^{2\ell^-} \sum_{\substack{0\le 2\lambda^\pm \le \tilde{k}^\pm \\ 1\le 2\lambda^+k^- - 2\lambda^-k^+ + k^- \le k-1}} (-1)^{2\lambda^-} \sin\left(\frac{\pi}{k}(2\ell^+ + 2\ell^- + 1)(2\lambda^+ + 2\lambda^- + 1)\right) \\ \times Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+, \tilde{k}^-, \widetilde{h_R^0}(\lambda^+, \lambda^-), \lambda^+, \lambda^-; \tau, \omega^+, \omega^-) \quad (4.4.47)$$

after a lengthy calculation reproduced in Appendix 4.D.1.

Now define

$$Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+, \tilde{k}^-, \widetilde{h_R^0}(\ell^+, \ell^-), \ell^+, \ell^-; -\frac{1}{\tau}, \frac{\omega_+}{\tau}, \frac{\omega_-}{\tau})\Big|_{\text{massive}} :=$$

$$
\begin{aligned}
&Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau}) \\
&\qquad - Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massless}} = \\
&\qquad (-1)^{2\ell^-+1}\frac{\vartheta_1(-\frac{1}{\tau},\frac{\omega_++\omega_-}{\tau})\vartheta_1(-\frac{1}{\tau},\frac{\omega_+-\omega_-}{\tau})}{\eta^3(-\frac{1}{\tau})\vartheta_1(-\frac{1}{\tau},\frac{2\omega_+}{\tau})\vartheta_1(-\frac{1}{\tau},\frac{2\omega_+}{\tau})} \\
&\qquad \times \sum_{\epsilon_+,\epsilon_-=\pm1}\epsilon_+\epsilon_-\sum_{s=0}^{k-1}\theta_{2\ell^++2\ell^-+1-2sk^-,k}(-\frac{1}{\tau},\frac{\zeta_{\epsilon_+,\epsilon_-}}{\tau}) \\
&\qquad \times \left\{X_2(-b',s;\tau,\nu)+T(N,b',s;-\frac{1}{\tau},\frac{\nu_{\epsilon_+,\epsilon_-}}{\tau})\right\}, \qquad (4.4.48)
\end{aligned}
$$

which can be rewritten in terms of massive $\widetilde{\mathcal{A}}_\gamma$ characters, as shown in Appendix 4.D.2. We arrive at

$$
\begin{aligned}
&Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R,\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massive}} \\
&\qquad = Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R,\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massive}, N=0} \\
&\qquad\qquad + \begin{cases} \mathcal{T}_{N>0}(\tilde{k}^+,\tilde{k}^-,b';\tau,\omega_+,\omega_-) & \text{if } N>0, \\ \mathcal{T}_{N<0}(\tilde{k}^+,\tilde{k}^-,b';\tau,\omega_+,\omega_-) & \text{if } N<0, \\ 0 & \text{if } N=0, \end{cases} \qquad (4.4.49)
\end{aligned}
$$

where $Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R,\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massive}, N=0}$ is given in Appendix 4.D.2, (4.D.28), while $\mathcal{T}_{N>0}(\tilde{k}^+,\tilde{k}^-,b';\tau,\omega_+,\omega_-)$ (resp. $\mathcal{T}_{N<0}(\tilde{k}^+,\tilde{k}^-,b';\tau,\omega_+,\omega_-)$) is given in Appendix 4.D.2, (4.D.38) (resp. (4.D.39)).

4. **Summary**

The main result of this chapter is the derivation of the S-transformation of massive and massless $\widetilde{\mathcal{A}}_\gamma$ characters in the twisted Ramond sector. The final formula for the S-transformation of massive characters is straightforward and given in (4.4.35). The final formula for the S-transformation of massless characters given in terms of massless and massive $\widetilde{\mathcal{A}}_\gamma$ characters is more involved and given below for future reference. We first recall the relevant definitions and notations.

With

$$\alpha k^- + \beta k^+ = 1,\ \alpha, \beta \in \mathbb{Z}, |\alpha| < k^+,\ |\beta| < k^-, \tag{4.4.50}$$

$$2\ell^{+\prime}_{n,p} := n+\alpha p,\ 2\ell^{-\prime}_{n,p} := n+\beta p,\ 2\ell^{+\prime\prime}_{n,p} := n+\alpha p+\beta k^+ \text{ and } 2\ell^{-\prime\prime}_{n,p} := n+\beta p-\beta k^-, \tag{4.4.51}$$

and

$$2\ell^{\pm\prime}_{n,p} = \nu^{\pm\prime}_{n,p} k^\pm + \rho^\pm_{n,p}, \quad 2\ell^{\pm\prime\prime}_{n,p} = \nu^{\pm\prime\prime}_{n,p} k^\pm + \rho^\pm_{n,p}, \qquad \rho^\pm_{n,p} \in \{0, \dots, \tilde{k}^\pm\} \tag{4.4.52}$$

where

$$\nu^{\pm\prime}_{n,p} := \left\lfloor \frac{2\ell^{\pm\prime}_{n,p}}{k^\pm} \right\rfloor,\ \nu^{\pm\prime\prime}_{n,p} := \left\lfloor \frac{2\ell^{\pm\prime\prime}_{n,p}}{k^\pm} \right\rfloor = \nu^{\pm\prime}_{n,p} \pm \beta, \tag{4.4.53}$$

the quantum numbers $L^{\pm\prime}_{n,p}$ and $L^{\pm\prime\prime}_{n,p}$ appearing in the S-transformation of massless characters are (see (4.D.26))

$$2L^{\pm\prime}_{n,p} := \frac{1-(-1)^{\nu^{\pm\prime}_{n,p}}}{2}k^\pm+(-1)^{\nu^{\pm\prime}_{n,p}}\rho^\pm_{n,p}-1,\ 2L^{\pm\prime\prime}_{n,p} := \frac{1-(-1)^{\nu^{\pm\prime\prime}_{n,p}}}{2}k^\pm+(-1)^{\nu^{\pm\prime\prime}_{n,p}}\rho^\pm_{n,p}-1. \tag{4.4.54}$$

Also, with

$$n = \mu^{\pm\prime}_n k^\pm + \rho^{\pm\prime}_n, \quad \mu^{\pm\prime}_n := \left\lfloor \frac{n}{k^\pm} \right\rfloor, \quad \rho^{\pm\prime}_n \in \{0, \dots, \tilde{k}^\pm\}, \tag{4.4.55}$$

and

$$\begin{aligned} n+1 &= \mu^{+\prime\prime}_n k^+ + \rho^{+\prime\prime}_n, \quad \mu^{+\prime\prime}_n := \left\lfloor \frac{n+1}{k^+} \right\rfloor, \\ n &= \mu^{-\prime\prime}_n k^- + \rho^{-\prime\prime}_n, \quad \mu^{-\prime\prime}_n := \left\lfloor \frac{n}{k^-} \right\rfloor, \quad \rho^{\pm\prime\prime}_n \in \{0, \dots, \tilde{k}^\pm\}, \end{aligned} \tag{4.4.56}$$

the quantum numbers $\lambda^{\pm\prime}_n$ and $\lambda^{\pm\prime\prime}_n$ also appearing in the S-transformation of massless characters are,

$$\begin{aligned} 2\lambda^{\pm\prime}_n &:= \frac{1}{2}(1-(-1)^{\mu^{\pm\prime}_n})k^\pm + (-1)^{\mu^{\pm\prime}_n}\rho^{\pm\prime}_n, \\ 2\lambda^{\pm\prime\prime}_n &:= \frac{1}{2}(1-(-1)^{\mu^{\pm\prime\prime}_n})k^\pm + (-1)^{\mu^{\pm\prime\prime}_n}\rho^{\pm\prime\prime}_n. \end{aligned} \tag{4.4.57}$$

Let us consider the large $N = 4$ $\widetilde{\mathcal{A}}_\gamma$ superconformal algebra at central charge $\tilde{c} = \frac{6k^+k^-}{k} - 3$. The modular S-transformation of characters corresponding to unitary massless representations in the twisted Ramond sector with isospins (ℓ^+, ℓ^-),

$\ell^\pm \in \{0, \ldots, \tilde{k}^\pm\}$, is given by[4]

$$
\begin{aligned}
&(-1)^{2\ell^-}\, \mathbf{e}(-\frac{\tilde{k}^+\omega_+^2 + \tilde{k}^-\omega_-^2}{\tau})\, Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau}) \\
&= \sum_{\substack{0\le 2\lambda^\pm \le \tilde{k}^\pm \\ 1\le 2\lambda^+k^- - 2\lambda^-k^+ + k^- \le k-1}} (-1)^{2\lambda^-}\sqrt{\frac{2}{k}}\,\sin\left(\frac{\pi}{k}(2\ell^+ + 2\ell^- + 1)(2\lambda^+ + 2\lambda^- + 1)\right) \\
&\qquad\qquad \times Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\lambda^+,\lambda^-),\lambda^+,\lambda^-;\tau,\omega^+,\omega^-) \\
&- \frac{i}{\sqrt{2k}} \sum_{\substack{n=0 \\ n+\alpha p\notin \mathbb{Z}k^+,\, n+\beta p\notin\mathbb{Z}k^-}}^{2k^+k^- -1}\ \sum_{p=0}^{k-1} e^{-\frac{\pi i}{k}(\alpha-\beta)p(2\ell^+ + 2\ell^- + 1)}\, h^{(\frac{\kappa}{2})}_{nk+p}(0,\tau) \\
&\times\Bigg\{ (-1)^{2L^{-\prime}_{n,p} + \nu^{+\prime}_{n,p} + \nu^{-\prime}_{n,p}} q^{\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}(L^{+\prime}_{n,p}+\frac12) - \sqrt{\frac{k^+}{k^-}}(L^{-\prime}_{n,p}+\frac12)\right)^2} \\
&\qquad \times \widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(L^{+\prime}_{n,p}+\frac12, L^{-\prime}_{n,p}), L^{+\prime}_{n,p}+\frac12, L^{-\prime}_{n,p}+\frac12; q, z_+, z_-) \\
&\quad + (-1)^{2L^{-\prime\prime}_{n,p} + \nu^{+\prime\prime}_{n,p} + \nu^{-\prime\prime}_{n,p} + \beta(2\ell^+ + 2\ell^- + 1)}\, q^{\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}(L^{+\prime\prime}_{n,p}+\frac12) - \sqrt{\frac{k^+}{k^-}}(L^{-\prime\prime}_{n,p}+\frac12)\right)^2} \\
&\qquad \times \widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(L^{+\prime\prime}_{n,p}+\frac12, L^{-\prime\prime}_{n,p}), L^{+\prime\prime}_{n,p}+\frac12, L^{-\prime\prime}_{n,p}+\frac12; \tau, \omega_+, \omega_-) \Bigg\}
\end{aligned}
$$

$$
+\begin{cases}
\begin{aligned}
&(-i\tau)^{-\frac12}\sqrt{\tfrac{k}{2\kappa}}\ \textstyle\sum_{r=1}^{N}\sum_{n=0}^{2k^+k^- -1}\ \widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\, e^{\frac{2\pi i}{\kappa}kn(kr-b')} \\
&\times\Big\{(-1)^{2\lambda_n^{-\prime}+\mu_n^{+\prime}+\mu_n^{-\prime}+1}\, q^{-\Delta(\lambda_n^{+\prime},\lambda_n^{-\prime})}\, \widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\lambda_n^{+\prime},\lambda_n^{-\prime}-\tfrac12),\lambda_n^{+\prime},\lambda_n^{-\prime};\tau,\omega_+,\omega_-) \\
&+(-1)^{2\lambda_n^{-\prime\prime}+\mu_n^{+\prime\prime}+\mu_n^{-\prime\prime}+1}\, e^{-\frac{\pi i}{k}(2\ell^+ + 1 + 2\ell^-)}\, e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\, q^{-\Delta(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime})} \\
&\qquad\times\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime}-\tfrac12),\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime};\tau,\omega_+,\omega_-)\Big\}
\end{aligned} & \text{if } N>0 \\[2ex]
\begin{aligned}
&(-i\tau)^{-\frac12}\sqrt{\tfrac{k}{2\kappa}}\ \textstyle\sum_{r=N+1}^{0}\sum_{n=0}^{2k^+k^- -1}\ \widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\, e^{\frac{2\pi i}{\kappa}kn(kr-b')} \\
&\times\Big\{(-1)^{2\lambda_n^{-\prime}+\mu_n^{+\prime}+\mu_n^{-\prime}}\, q^{-\Delta(\lambda_n^{+\prime},\lambda_n^{-\prime})}\, \widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\lambda_n^{+\prime},\lambda_n^{-\prime}-\tfrac12),\lambda_n^{+\prime},\lambda_n^{-\prime};\tau,\omega_+,\omega_-) \\
&+(-1)^{2\lambda_n^{-\prime\prime}+\mu_n^{+\prime\prime}+\mu_n^{-\prime\prime}}\, e^{-\frac{\pi i}{k}(2\ell^+ + 1 + 2\ell^-)}\, e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\, q^{-\Delta(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime})} \\
&\qquad\times\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime}-\tfrac12),\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime};\tau,\omega_+,\omega_-)\Big\}
\end{aligned} & \text{if } N<0 \\[2ex]
0 & \text{if } N=0.
\end{cases}
\tag{4.4.58}
$$

[4] We have suppressed the dependence on $\tilde{k}^\pm$ of the characters to avoid over-complication of formulas.

4.5 Indices for $\mathcal{A}_\gamma$

4.5.1 Conformal field-theoretic elliptic genus for $\mathcal{A}_\gamma$

We reviewed the conformal field-theoretic elliptic genus for a $\mathcal{N} = (4, 4)$ superconformal field theory on $T4$ and $K3$ in the previous chapter, and we expected there would be a well-defined analogous elliptic genus for a theory enjoying $\mathcal{A}_\gamma$ symmetry and that this might lead to a new moonshine phenomenon. First recall the formal definition of a character of $\mathcal{A}_\gamma$,

$$Ch^{\mathcal{A}_\gamma,\mathrm{I}}(k^+, k^-, h, \ell^+, \ell^-; q, z_+, z_-) := \mathrm{Tr}_{\mathcal{H}_I}(q^{L_0 - c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}) \tag{4.5.1}$$

where I denotes NS or R, and $q = e^{2\pi i\tau}$ and $z_\pm = e^{2\pi i w_\pm}$ for $\tau \in \mathfrak{H}$,and $w_\pm \in \mathbb{C}$. The variables $z_\pm$ keep track of the isospin quantum numbers of the states stemming from the affine $\widehat{SU(2)}_{k^\pm}$ symmetries. For the $\mathcal{N} = 2$ and small $\mathcal{N} = 4$ cases, the conformal field-theoretic elliptic genus is given by the partition function in the $\widetilde{R}$ sector, where the antiholomorphic (right-moving) $\widetilde{R}$ characters are specialized to $\bar{\omega} = 0$ (recall (3.4.16)). Since the partition function is a sesquilinear expression in the characters, calculating the elliptic genus amounts to evaluate the Witten index of these characters, which is zero for massive representations and non-zero for massless ones. Hence the elliptic genus counts short right-moving representations only, so only counts supersymmetric (or BPS) states. Viewed as the index of the supercharge, it is constant through smooth deformations of the moduli of the theory. If $Z_{\mathcal{H}_{\widetilde{R}}}$ denotes the partition function in the twisted Ramond sector of a theory with $\mathcal{A}_\gamma$ symmetry, a possible generalisation of the elliptic genus to $\mathcal{A}_\gamma$ is

$$\begin{aligned} \mathcal{EG}_{\mathcal{A}_\gamma}(q, \bar{q}; z_+, z_-) &:= \mathrm{Tr}_{\mathcal{H}_R}(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}, & (4.5.2)\\ &= Z_{\mathcal{H}_{\widetilde{R}}}(q, z_+, z_-, u = 0, \bar{q}, \bar{z}_+ = 1, \bar{z}_- = 1, \bar{u} = 0), & (4.5.3) \end{aligned}$$

where $F = 2T_0^{-3} + 2\bar{T}_0^{-3}$. It counts states annihilated by $(\bar{Q}_a)_0(\bar{G}_a)_0$ for $a \in \{\pm, \pm K\}$. One could be a little bit more general and define indices

$$\mathcal{EG}'_{\mathcal{A}_\gamma}(q, \bar{q}; z_+, z_-) := \mathrm{Tr}_{\mathcal{H}_R}(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \bar{z}^{2(\pm\bar{T}_0^{+3} + \bar{T}_0^{-3})}, \tag{4.5.4}$$

$$= Z_{\mathcal{H}_{\widetilde{R}}}(q, z_+, z_-, u=0, \bar{q}, \bar{z}_+ = \bar{z}^{\pm 1}, \bar{z}_- = \bar{z}, \bar{u} = 0), \tag{4.5.5}$$

which count only states annihilated by $(\bar{Q}_a)_0(\bar{G}_a)_0$ for either $a \in \{\pm K\}$ (when $\bar{z}_+ = \bar{z}$) or $a \in \{\pm\}$ (when $\bar{z}_+ = \bar{z}^{-1}$).

However, all these specializations of $\bar{z}_\pm$ lead to vanishing elliptic genera, as a consequence of the fact that massless $\widetilde{R}$ characters have a simple zero at $z_+ = z_-$ and $z_+ = z_-^{-1}$, while massive representations have a double zero at these values of z_+. Indeed, we have noticed previously (4.3.9) that the $\mathcal{A}_\gamma$ characters (massive and massless) factorize in the twisted Ramond sector according to

$$Ch^{\mathcal{A}_\gamma, \widetilde{R}} = Ch^{A_{QU}, \widetilde{R}} \times Ch^{\widetilde{\mathcal{A}}_\gamma, \widetilde{R}}, \tag{4.5.6}$$

where we recall (4.3.10), namely

$$\begin{aligned} &Ch^{A_{QU}\,\widetilde{\mathrm{R}}}(u; \tau, \omega_+, \omega_-) = \\ &\qquad q^{u^2/k+1/8} F^{\widetilde{\mathrm{R}}}(q, z_+, z_-) \times \prod_{n=1}^{\infty} (1-q^n)^{-1}(1 - z_+^{-1} z_-^{-1})(1 - z_+^{-1} z_-) z_+, \end{aligned} \tag{4.5.7}$$

which vanishes when we set $z_+ = z_-^{\pm 1}$. In fact, the zero-mode contribution from $(1 - z_+^{-1} z_-^{-1})(z_+ - z_-)$ is responsible for the vanishing, which means both massless and massive characters in the twisted Ramond sector are zero when we set $z_+ = z_-^{\pm 1}$. Therefore we have the following important statement: The conformal field-theoretic elliptic genus of a theory with $\mathcal{A}_\gamma$ symmetry vanishes.

4.5.2 New Indices for $\mathcal{A}_\gamma$

Motivated by the search of a holographic dual to string theory on an $AdS_3 \times S^3 \times S^3 \times S^1$ background, which enjoys $\mathcal{A}_\gamma$ symmetry, Gukov et. al. [GMMS04] introduced a new index for conformal field theories where both left and right movers enjoy $\mathcal{A}_\gamma$ symmetry. Let $\mathcal{C}$ denote such a conformal field theory whose partition function restricted to the twisted Ramond sector is $Z^{\mathcal{C}}_{\mathcal{H}_{\widetilde{R}}}$. Then the new index of $\mathcal{C}$ is defined by

$$\begin{aligned}\mathfrak{I}(\mathcal{C})(\tau,\bar\tau,\omega_+,\omega_-,\bar\omega) &:= \left\{-\bar z_+\partial_{\bar z_-}\, Z^{\mathcal{C}}_{\mathcal{H}_{\widetilde{R}}}(q,\bar q,z_+,z_-,\overline{z_+},\overline{z_-})\right\}\Bigg|_{\overline{\omega_+}=\overline{\omega_-}=\bar\omega} \\ &= \mathrm{Tr}_{\mathcal{H}_{\widetilde{R}}}\left(-F_R(-1)^F\, q^{L_0-c/24}\bar q^{\bar L_0-\bar c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}\, \bar z^{2(\bar T_0^{+3}+\bar T_0^{-3})}\right),\end{aligned} \tag{4.5.8}$$

where $(-1)^{F_R} = e^{2\pi i \bar T_0^{-3}}$, $(-1)^F = e^{2\pi i(\bar T_0^{+3}+\bar T_0^{-3})}$ and $\bar z = e^{2\pi i\bar\omega}$. This is a generalisation of the index introduced in [CFIV92]. Given the structure of $Z^{\mathcal{C}}_{\mathcal{H}_{\widetilde{R}}}$ as a sesquilinear combination of $\mathcal{A}_\gamma$ characters in the twisted Ramond sector, to calculate the index (4.5.8), one needs to act with the derivation operator $-\bar z_+\partial_{\bar z_-}$ on the right-moving characters and then set $\overline{\omega_+} = \overline{\omega_-} = \bar\omega$. We therefore review the contribution to the index $\mathfrak{I}$ from the $\mathcal{A}_\gamma$ characters. From the structure of (4.3.9) and (4.3.10), and from the expressions for the massive and massless characters of $\widetilde{\mathcal{A}}_\gamma$ ((4.4.2) and (4.4.10)), one can see that the $\mathcal{A}_\gamma$ massive characters have a double zero at $\omega_+ = \pm\omega_-$ while the massless characters have a simple zero at these values of ω_+. Therefore the action of $-z_+\partial_{z_-}$ on a massive character will yield a zero answer once one sets $\omega_+ = \omega_- = \omega$, while the action of this differential operator on massless characters is more interesting. We define

$$\mathfrak{D}^{\mathcal{A}_\gamma,\tilde R}_{+-}(k^\pm,\ell^\pm_R,u)(\tau,\omega) := \left\{-z_+\partial_{z_-} Ch_0^{\mathcal{A}_\gamma,\tilde R}(k^+,k^-,h_0^R,\ell_R^+,\ell_R^-,u;\tau,\omega_+,\omega_-)\right\}\Bigg|_{\omega_+=\omega_-=\omega}. \tag{4.5.9}$$

It turns out that the only non-zero contribution to the index from a massless representation of $\mathcal{A}_\gamma$ comes from the derivative of the zero mode factor in A_{QU}. Indeed, the key observation is that the A^{QU} character $(4.3.10)$ has a simple zero at $\omega_+ = \omega_-$, and thanks to this, the action (4.5.9) can be simply calculated as follows,

$$\begin{aligned}&\mathfrak{D}^{\mathcal{A}_\gamma,\tilde R}_{+-}(k^\pm,\ell^\pm_R,u)(\tau,\omega) = \\ &\qquad \mathfrak{D}^{A^{QU},\tilde R}_{+-}(k^\pm,u)(\tau,\omega)\, Ch_0^{\tilde A_\gamma,\tilde R}(k^+,k^-,\widetilde{h_0^R},\ell_R^+,\ell_R^-;\tau,\omega_+,\omega_-)\Bigg|_{\omega_+=\omega_-=\omega}\end{aligned} \tag{4.5.10}$$

where

$$\begin{aligned}
\mathfrak{D}^{A^{QU},\tilde{R}}_{+-}(k^\pm,u)(\tau,\omega) &:= \left\{-z_+\partial_{z_-}Ch^{QU,\widetilde{R}}(u;\tau,\omega_+,\omega_-)\right\}\Bigg|_{\omega_+=\omega_-=\omega} \\
&= q^{u^2/k+1/8}\prod_{n=1}^{\infty}(1-q^n)(z-z^{-1})(1-z^2q^n)(1-z^{-2}q^n) \\
&= q^{u^2/k+1/8}\sum_{m=0}^{\infty}(-1)^m q^{\frac{1}{2}m(m+1)}(z^{2m+1}-z^{-2m-1}) \qquad (4.5.11a) \\
&= q^{u^2/k}\theta^-_{1,2}(\tau,2\omega) = -iq^{u^2/k}\vartheta_1(\tau,2\omega), \qquad (4.5.11b)
\end{aligned}$$

where $z = e^{2\pi i\omega}, \omega \in \mathbb{C}$, the second quality is derived from the Jacobi triple product identity, namely

$$\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}y)(1+q^{n-1/2}y^{-1}) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2}y^n, \qquad (4.5.12)$$

in which we set $y = -q^{1/2}z^2$ to reach $(4.5.11a)$. In the third equality we used the fact that $\theta^-_{1,2}(\tau,2\omega) := \theta_{1,2}(\tau,2\omega) - \theta_{-1,2}(\tau,2\omega) = -i\vartheta_1(\tau,2\omega)$, where

$$\theta_{\mu,k}(\tau;2\omega) := \sum_{\ell\in\mathbb{Z},\ell\equiv\mu \mod 2k} q^{\frac{\ell^2}{4k}}z^\ell = \sum_{n\in\mathbb{Z}} q^{k(n+\frac{\mu}{2k})^2}z^{\mu+2kn} = q^{\frac{\mu^2}{4k}}z^\mu\sum_{n\in\mathbb{Z}} q^{kn^2+n\mu}z^{2kn}, \qquad (4.5.13)$$

see Appendix 3.A.2. From the point of view of (4.5.11b), the QU theory is a special $\mathcal{A}_\gamma$ theory with $k^\pm = 1$ and $\ell^\pm_R = 1/2$.

Now using (4.3.15) for $Ch^{\tilde{A}_\gamma,\tilde{R}_0}(k^+,k^-,\widetilde{h^R_0},\ell^+_R,\ell^-_R;\tau,\omega_+,\omega_-)|_{\omega_+=\omega_-=\omega}$ in (4.5.10) together with the definition of affine $\widehat{su(2)}$ characters (4.3.17), one obtains

$$\mathfrak{D}^{\mathcal{A}_\gamma,\tilde{R}}_{+-}(k^\pm,\ell^\pm_R,u)(\tau,\omega) = (-1)^{2\ell^-_R+1}q^{u^2/k}\theta^-_{2\ell^+_R+2\ell^-_R-1,k}(\tau,2\omega) \qquad (4.5.14)$$

as the contribution to the index $\mathfrak{I}^{\mathcal{C}}_{+-}$ from a massless $\mathcal{A}_\gamma$ character with labels $(h_R,\ell^+_R,\ell^-_R,u)$. Note that $\mu := 2\ell^+_R + 2\ell^-_R - 1$ is the Witten index for a representation of $\tilde{A}_\gamma$ with labels (ℓ^+_R,ℓ^-_R) as the number of bosonic ground states is $(2(\ell^+_R-1/2)+1)(2(\ell^-_R-1/2)+1)$ and the number of fermionic ground states is $(2(\ell^+_R-1)+1)(2(\ell^-_R-1)+1)$ for that same representation .

One can also check consistency of the action of the differential operator $-z_+\partial_{z_-}$ by acting on the linear relation between massless characters and massive character at threshold of

$\mathcal{A}_\gamma$ in the twisted Ramond sector [PT90b],

$$Ch_0^{\mathcal{A}_\gamma,\tilde{R}}(\ell_R^+,\ell_R^-)+Ch_0^{\mathcal{A}_\gamma,\tilde{R}}(\ell_R^++1/2,\ell_R^--1/2)=\widehat{Ch}^{\mathcal{A}_\gamma,\tilde{R}}(h_R,\ell_R^+,\ell_R^-+1/2). \quad (4.5.15)$$

Indeed, and it is easy to find [GMMS04]

$$\begin{aligned}&\mathfrak{D}_{+-}^{\mathcal{A}_\gamma,\tilde{R}}(k^\pm,\ell_R^\pm,u)(\tau,\omega)+\mathfrak{D}_{+-}^{\mathcal{A}_\gamma,\tilde{R}}(k^\pm,\ell_R^\pm\pm\frac{1}{2},u)(\tau,\omega)=\\&q^{u^2/k}\left((-1)^{2\ell_R^-+1}\theta^-_{2\ell_R^++2\ell_R^--1,k}(\tau,2\omega)+(-1)^{2\ell_R^-}\theta^-_{2\ell_R^++2\ell_R^--1,k}(\tau,2\omega)\right)=0. \quad (4.5.16)\end{aligned}$$

Hence the new index (4.5.8) only encodes information from right-moving massless representations of $\mathcal{A}_\gamma$ in the twisted Ramond sector. We note that, unlike what happens for the small $N=4$ case, the contribution to the new index (4.5.8) from a massless representation of $\mathcal{A}_\gamma$ is a $\bar{q}$-series as opposed to an integer. In fact, from the (antiholomorphic) contribution analogous to (4.5.14), we immediately read the quantum numbers of the right-moving states counted by the index $\mathfrak{I}_{+-}^{\mathcal{C}}$. Indeed, since

$$\theta^-_{\bar{\mu},k}(\bar{\tau},2\bar{\omega})=\sum_{n\in\mathbb{Z}}\left\{\bar{q}^{\,k(n+\frac{\bar{\mu}}{2k})^2}\,\bar{z}^{2k(n+\frac{\bar{\mu}}{2k})}-\bar{q}^{\,k(n-\frac{\bar{\mu}}{2k})^2}\,\bar{z}^{2k(n-\frac{\bar{\mu}}{2k})}\right\}, \quad (4.5.17)$$

where $\bar{\mu}:=2(\bar{\ell}_R^++\bar{\ell}_R^-)-1$, we see that the states contributing have charge $\pm\bar{\mu} \pmod{2k}$ under the operator $2(\bar{T}_0^{+3}+\bar{T}_0^{-3})$ and eigenvalue $\frac{\bar{u}^2}{k}+kn^2\pm n\bar{\mu}+\frac{\bar{\mu}^2}{4k}$ under $\bar{L}_0-\bar{c}/24$. In other words, the states that contribute are eigenstates of $\bar{L}_0-\bar{c}/24=\frac{\bar{U}_0^2}{k}+\frac{1}{k}(\bar{T}_0^{+3}+\bar{T}_0^{-3})^2$. This is exactly the condition for masslessness, which means that all right-moving states contributing to $\mathfrak{I}_{+-}^{\mathcal{C}}$ are annihilated by $(\bar{Q}_{-K})_0(\bar{G}_{-K})_0$.

In particular, it was found in [Sau05] that the right-moving states contributing to $\mathfrak{I}_{+-}^{\mathcal{C}}$ belong to orbits of special RR ground states under a spectral flow with $(\rho,\sigma)=(2n,2n), n\in\mathbb{Z}$ (see (4.A.7)). Indeed, under this flow one has,

$$\begin{aligned}(\bar{T}_0^{+3}+\bar{T}_0^{-3})^{2n,2n}&=(\bar{T}_0^{+3}+\bar{T}_0^{-3})-kn.\\ \bar{L}_0^{2n,2n}-\bar{c}/24&=\bar{L}_0-\bar{c}/24-2n(\bar{T}_0^{+3}+\bar{T}_0^{-3})+kn^2,\\&=\frac{\bar{U}_0^2}{k}+\frac{1}{k}(\bar{T}_0^{+3}+\bar{T}_0^{-3})^2-2n(\bar{T}_0^{+3}+\bar{T}_0^{-3})+kn^2,\\&=\frac{\bar{U}_0^2}{k}+\frac{1}{k}\left(\bar{T}_0^{+3;2n,2n}+\bar{T}_0^{-3;2n,2n}\right)^2, \quad (4.5.18)\end{aligned}$$

so the masslessness condition is preserved through this 'symmeric' spectral flow.

Remark: We could introduce a new index inspired by the index (4.5.9). Another observation on the A^{QU} character is

$$z_-^{-1}\frac{d}{dz_+}Ch^{QU,\tilde{R}}(u;q,z_+,z_-)|_{z_+=z_-^{-1}=z} = q^{u^2/k}\theta^-_{1,2}(\tau,2\omega); \tag{4.5.19}$$

hence by combining with the identity (4.3.16), we define a new index I' as

$$\begin{aligned}
I'(\mathcal{A}_\gamma;k^+,k^-,\ell_R^+,\ell_R^-,u)(\tau,\omega) &= z_-^{-1}\frac{d}{dz_+}Ch_0^{\mathcal{A}_\gamma,\tilde{R}}(k^+,k^-,h_0^R,\ell_R^+,\ell_R^-,u;q,z_+,z_-)|_{z_+=z_-^{-1}=z}\\
&= \left(z_-^{-1}\frac{d}{dz_+}Ch^{QU,\widetilde{R}}(u;q,z_+,z_-)|_{z_+=z_-^{-1}=z}\right)\times Ch_0^{\tilde{A}_\gamma,\tilde{R}}(k^+,k^-,\widetilde{h_0^R},\ell_R^+,\ell_R^-;q,z_+,z_-)|_{z_+=z_-^{-1}=z}\\
&= -iq^{u^2/k}\vartheta_1(\tau,2\omega)\times Ch_0^{\tilde{A}_\gamma,\tilde{R}}(k^+,k^-,\widetilde{h_0^R},\ell_R^+,\ell_R^-;q,z_+,z_-)|_{z_+=z_-^{-1}=z}\\
&= (-1)^{2\ell_R^-}q^{u^2/k}\theta^-_{2\ell_R^++2\ell_R^--1,k}(\tau,2\omega).
\end{aligned} \tag{4.5.20}$$

4.A Commutation Relations and Spectral Flow

4.A.1 The $\mathcal{A}_\gamma$ Algebras

The Virasoro subalgebra of $\mathcal{A}_\gamma$ is as usual,

$$[L_m,L_n] = (m-n)L_{m+n} + \frac{c}{12}\delta_{m+n,0}m(m^2-1),\ \ m,n\in\mathbb{Z} \tag{4.A.1}$$

and the Virasoro generator $L(z)$ transforms a primary field ϕ with conformal dimension d_ϕ according to

$$[L_m,\phi_n] = [(d_\phi-1)m-n]\phi_{m+n}. \tag{4.A.2}$$

In the $\mathcal{A}_\gamma$ case, one has $d_G = 3/2$, $d_{T^{\pm,i}} = d_U = 1$ and $d_Q = 1/2$. The $\mathcal{A}_\gamma$ supercharges anti-commutation relations are of the form

$$\begin{aligned}
\{G_{+,m},G_{-,n}\} &= L_{m+n} + \frac{c}{6}(m^2-\frac{1}{4})\delta_{m+n,0} + (m-n)[\gamma^+T^{+3}_{m+n} + \gamma^-T^{-3}_{m+n}],\\
\{G_{+K,m},G_{-K,n}\} &= L_{m+n} + \frac{c}{6}(m^2-\frac{1}{4})\delta_{m+n,0} + (m-n)[\gamma^+T^{+3}_{m+n} - \gamma^-T^{-3}_{m+n}],\\
\{G_{\pm,m},G_{+K,n}\} &= \pm\gamma^\pm(n-m)T^{\pm\pm}_{m+n},
\end{aligned}$$

$$\{G_{\pm,m}, G_{-K,n}\} = \pm\gamma^{\mp}(n-m)T^{\mp\pm}_{m+n}, \tag{4.A.3}$$

where $\gamma^{\pm} \coloneqq k^{\mp}/k$ and $\{T^{\pm i}_m, i = \pm, 3\}$ are generators of the affine Kac-Moody algebra $\widehat{SU(2)}_{k^{\pm}}$ at level $k^{\pm}$, whose commutation rules are given by

$$\begin{aligned}
&[T^{\pm+}_m, T^{\pm-}_n] = 2T^{\pm3}_{m+n} + mk^{\pm}\delta_{m+n,0}, \quad [T^{\pm3}_m, T^{\pm3}_n] = \frac{1}{2}mk^{\pm}\delta_{m+n,0},\\
&[T^{\pm3}_m, T^{\pm+}_n] = T^{\pm+}_{m+n}, \quad [T^{\pm3}_m, T^{\pm-}_n] = -T^{\pm-}_{m+n}.
\end{aligned} \tag{4.A.4}$$

One can explore the commutation relations between $\widehat{SU(2)}_{k^{\pm}}$ generators and supercharges and fermion operators, namely

$$\begin{aligned}
&[T^{+\pm}_m, Q_{\pm,n}] = 0, \quad [T^{+\pm}_m, Q_{\mp,n}] = \pm Q_{\pm K,m+n}, \quad [T^{+3}_m, Q_{\pm,n}] = \pm\frac{1}{2}Q_{\pm,m+n},\\
&[T^{+\pm}_m, Q_{\pm K,n}] = 0, \quad [T^{+\pm}_m, Q_{\mp K,n}] = \mp Q_{\pm,m+n}, \quad [T^{+3}_m, Q_{\pm K,n}] = \pm\frac{1}{2}Q_{\pm K,m+n},\\
&[T^{-\pm}_m, Q_{\pm,n}] = 0, \quad [T^{-\pm}_m, Q_{\mp,n}] = \pm Q_{\mp K,m+n},\\
&[T^{-\pm}_m, Q_{\mp K,n}] = 0, \quad [T^{-\pm}_m, Q_{\pm K,n}] = \mp Q_{\pm,m+n},\\
&[T^{-3}_m, Q_{\pm,n}] = \pm\frac{1}{2}Q_{\pm,m+n}, \quad [T^{-3}_m, Q_{\pm K,n}] = \mp\frac{1}{2}Q_{\pm K,m+n},\\
&[T^{+\pm}_m, G_{\pm,n}] = 0, \quad [T^{+\pm}_m, G_{\mp,n}] = \pm G_{\pm K,m+n} \mp 2\gamma^{-}mQ_{\pm K,m+n},\\
&[T^{+\pm}_m, G_{\pm K,n}] = 0, \quad [T^{+\pm}_m, G_{\mp K,n}] = \mp G_{\pm,m+n} \pm 2\gamma^{-}mQ_{\pm,m+n},\\
&[T^{+3}_m, G_{\pm,n}] = \pm\frac{1}{2}G_{\pm,m+n} \mp \gamma^{-}mQ_{\pm,m+n},\\
&[T^{+3}_m, G_{\pm K,n}] = \pm\frac{1}{2}G_{\pm K,m+n} \mp \gamma^{-}mQ_{\pm K,m+n},\\
&[T^{-\pm}_m, G_{\pm,n}] = 0, \quad [T^{-\pm}_m, G_{\mp,n}] = \pm G_{\mp K,m+n} \pm 2\gamma^{+}mQ_{\mp K,m+n},\\
&[T^{-\pm}_m, G_{\mp K,n}] = 0, \quad [T^{-\pm}_m, G_{\pm K,n}] = \mp G_{\pm,m+n} \mp 2\gamma^{+}mQ_{\mp,m+n},\\
&[T^{-3}_m, G_{\pm,n}] = \pm\frac{1}{2}G_{\pm,m+n} \pm \gamma^{+}mQ_{\pm,m+n},\\
&[T^{-3}_m, G_{\pm K,n}] = \mp\frac{1}{2}G_{\pm K,m+n} \mp \gamma^{+}mQ_{\pm K,m+n}.
\end{aligned} \tag{4.A.5}$$

The remaining (anti-)commutation rules are given by

$$\begin{aligned}
&\{Q_{\pm,m}, G_{\mp,n}\} = \mp\frac{1}{2}(T^{+3}_{m+n} - T^{-3}_{m+n}) + \frac{1}{2}U_{m+n},\\
&\{Q_{\pm K,m}, G_{\mp K,n}\} = \mp\frac{1}{2}(T^{+3}_{m+n} + T^{-3}_{m+n}) + \frac{1}{2}U_{m+n},
\end{aligned}$$

$$
\begin{aligned}
&\{Q_{\pm,m}, G_{+K,n}\} = \frac{1}{2}T^{\pm\pm}_{m+n}, \quad \{Q_{\pm,m}, G_{-K,n}\} = -\frac{1}{2}T^{\mp\pm}_{m+n},\\
&\{Q_{+K,m}, G_{\pm,n}\} = -\frac{1}{2}T^{\pm\pm}_{m+n}, \quad \{Q_{-K,m}, G_{\pm,n}\} = \frac{1}{2}T^{\mp\pm}_{m+n},\\
&[U_m, G_{a,n}] = mQ_{a,m+n}, \quad [U_m, U_n] = -m\frac{k}{2}\delta_{m+n,0},\\
&\{Q_{+,m}, Q_{-,n}\} = \{Q_{+K,m}, Q_{-K,n}\} = -\frac{k}{4}\delta_{m+n,0}.
\end{aligned}
\tag{4.A.6}
$$

where U_m is the generator of $\widehat{u(1)}$.

There exists an isomorphism of $\mathcal{A}_\gamma$ known as spectral flow [DST88], which relates different modings of the Laurent mode algebra, namely the generators with superscripts $\rho, \sigma \in \mathbb{R}$ below satisfy the same $\mathcal{A}_\gamma$ commutation relations as the ones given above,

$$
\begin{aligned}
L^{\rho,\sigma}_m &= L_m - (\rho T^{+3}_m + \sigma T^{-3}_m) + \left(\frac{k^+}{4k}\rho^2 + \frac{k^-}{4k}\sigma^2\right)\delta_{m,0}, & U^{\rho,\sigma}_m &= U_m,\\
G^{\rho,\sigma}_{\pm,m\pm(\rho+\sigma)/2} &= G_{\pm,m} \pm \left(\rho\frac{k^+}{k} - \sigma\frac{k^-}{k}\right)Q_{\pm,m}, & Q^{\rho,\sigma;\pm}_{\pm,m\pm(\rho+\sigma)/2} &= Q_{\pm,m},\\
G^{\rho,\sigma;\pm}_{\pm K,m\pm(\rho-\sigma)/2} &= G_{\pm K,m} \pm \left(\rho\frac{k^+}{k} + \sigma\frac{k^-}{k}\right)Q_{\pm K,m}, & Q^{\rho,\sigma;\pm}_{\pm K,m\pm(\rho-\sigma)/2} &= Q_{\pm K,m},\\
T^{\rho,\sigma;+3}_m &= T^{+3}_m - \left(\rho\frac{k^+}{2}\right)\delta_{m,0}, & T^{\rho,\sigma;+\pm}_{m\pm\rho} &= T^{+\pm}_m,\\
T^{\rho,\sigma;-3}_m &= T^{-3}_m - \left(\sigma\frac{k^-}{2}\right)\delta_{m,0}, & T^{\rho,\sigma;-\pm}_{m\pm\sigma} &= T^{-\pm}_m.
\end{aligned}
\tag{4.A.7}
$$

In particular, the algebras characterised by $(\rho, \sigma) = (0, 0)$ and $(\rho, \sigma) = (-1, 0)$ are isomorphic and are known as the $\mathcal{A}_\gamma$ algebras in the Ramond (R) and Neveu-Schwarz (NS) sector respectively. Analogously, $(\rho, \sigma) = (0, 0)$ and $(\rho, \sigma) = (0, -1)$ are isomorphic $\mathcal{A}_\gamma$ R and NS algebras. Actually, as long as $\gamma \neq \frac{1}{2}$, all $\mathcal{A}_\gamma$ algebras are isomorphic.

4.A.2 $\mathcal{N} = 2$ Subalgebras of $\mathcal{A}_\gamma$

One of the $\mathcal{N} = 2$ subalgebras of $\mathcal{A}_\gamma$ has generators $\{L'_m, J'_m, G'_{\pm,m}\}$ defined as

$$
L'_m := L_m, \quad J'_m := 2\gamma^+ T^{+3}_m + 2\gamma^- T^{-3}_m, \quad G'_{\pm,m} := G_{\pm,m},
\tag{4.A.8}
$$

and the generators defined for all $\theta \in \mathbb{R}$ as

$$
L'^{\theta}_m := L'_m + 2\theta(\gamma^+ T^{+3}_m + \gamma^- T^{-3}_m) + \frac{1}{6}\theta^2 c'\delta_{m,0} = L'_m + \theta J'_m + \frac{1}{6}\theta^2 c'\delta_{m,0},
$$

$$J'^{\theta}_m := J'_m + \frac{1}{3}\theta c'\delta_{m,0} = 2\gamma^+ T^{+3}_m + 2\gamma^- T^{-3}_m + \frac{1}{3}\theta c'\delta_{m,0}, \tag{4.A.9}$$

$$G'^{\theta}_{\pm,m} = G_{\pm,m\pm\theta} \tag{4.A.10}$$

satisfy the same commutation relations as the generators (4.A.8). This isomorphism extends to $\mathcal{A}_\gamma$ with

$$G^{\theta}_{\pm K,m} = G_{\pm K,m\pm\theta(\gamma^+-\gamma^-)} \mp 4\theta\gamma^+\gamma^- Q_{\pm K,m+\theta(\gamma^+-\gamma^-)},$$
$$Q^{\theta}_{\pm,m} = Q_{\pm,m\pm\theta}, \quad Q^{\theta}_{\pm K,m} = Q_{\pm K,m\pm\theta(\gamma^+-\gamma^-)}, \quad U^{\theta}_m = U_m,$$
$$T^{\pm\pm,\theta}_m = T^{\pm\pm}_{m+\theta(\gamma^+-\gamma^-\pm 1)}, \quad T^{\pm\mp,\theta}_m = T^{\pm\mp}_{m-\theta(\gamma^+-\gamma^-\mp 1)}.$$

Another $\mathcal{N}=2$ subalgebra has generators

$$L'_m = L_m, \quad J'_m = 2\gamma^+ T^{+3}_m - 2\gamma^- T^{-3}_m, \quad G'_{\pm K,m} = G_{\pm K,m}.$$

The spectral flow

$$L'^{\theta}_m = L'_m + 2\theta(\gamma^+ T^{+3}_m - \gamma^- T^{-3}_m) + \frac{1}{6}\theta^2 c'\delta_{m,0} = L'_m + \theta J'_m + \frac{1}{6}\theta^2 c'\delta_{m,0},$$
$$J'^{\theta}_m = J'_m + \frac{1}{3}\theta c'\delta_{m,0} = 2\gamma^+ T^{+3}_m - 2\gamma^- T^{-3}_m + \frac{1}{3}\theta c'\delta_{m,0}, \tag{4.A.11}$$
$$G'^{\theta}_{\pm K,m} = G'_{\pm K,m\pm\theta},$$

and extends to $\mathcal{A}_\gamma$ as

$$G^{\theta}_{\pm,m} = G_{\pm,m\pm\theta(\gamma^+-\gamma^-)} \mp 4\theta\gamma^+\gamma^- Q_{\pm,m+\theta(\gamma^+-\gamma^-)},$$
$$Q^{\theta}_{\pm,m} = Q_{\pm,m\pm\theta(\gamma^+-\gamma^-)}, \quad Q^{\theta}_{\pm K,m} = Q_{\pm K,m\pm\theta}, \quad U^{\theta}_m = U_m,$$
$$T^{\pm\pm,\theta}_m = T^{\pm\pm}_{m+\theta(\gamma^+-\gamma^-\pm 1)}, \quad T^{\pm\mp,\theta}_m = T^{\pm\mp}_{m-\theta(\gamma^+-\gamma^-\mp 1)},$$

Note that one may check the consistency of spectral flow of the $\mathcal{N}=2$ SCA by considering the invariance of operator $\frac{2c}{3}L_0 - J_0^2$.

4.B Higher level Appell functions

4.B.1 Definition

Level $\ell \in \mathbb{N}$ Appell functions are defined as [STT05]

$$\mathcal{K}_\ell(\tau,\nu,\mu) \coloneqq \sum_{m=-\infty}^{\infty} \frac{q^{\frac{\ell}{2}m^2} y^{\ell m}}{1-tyq^m} \tag{4.B.1}$$

with $q = e^{2\pi i\tau}, y = e^{2\pi i\nu}$ and $t = e^{2\pi i\mu}$, where $\tau \in \mathfrak{H}, \mu,\nu \in \mathbb{C}$.

4.B.2 An average formula

We show that, for $0 \le b' < \ell'$, one can rewrite the Appell function at level ℓ as an 'average' of Appell functions at higher level $\ell\ell'$:

$$\mathcal{K}_\ell(\ell'\tau, \nu + a\tau + \frac{b'}{\ell}\tau, -\frac{b'}{\ell}\tau) = \frac{1}{\ell'}\sum_{c=0}^{\ell'-1} e^{-2\pi i\frac{b'c}{\ell'}} y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau+c}{\ell'}, 0). \tag{4.B.2}$$

Proof. Let us start from the right hand side

$$\frac{1}{\ell'}\sum_{c=0}^{\ell'-1} e^{-2\pi i\frac{b'c}{\ell'}} y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau+c}{\ell'}, 0) =$$
$$\frac{1}{\ell'}\sum_{m\in\mathbb{Z}}\sum_{c=0}^{\ell'-1} e^{-2\pi i b'\frac{c}{\ell'}} y^{-\frac{b'}{\ell'}+\ell m} q^{-a\frac{b'}{\ell'}+\ell a m} q^{\frac{\ell\ell'}{2}m^2} \frac{1}{1-y^{\frac{1}{\ell'}}q^{\frac{a}{\ell'}}e^{\frac{2\pi i c}{\ell'}}q^m} \tag{4.B.3}$$

and rewrite

$$\frac{1}{1-y^{\frac{1}{\ell'}}q^{\frac{a}{\ell'}}e^{\frac{2\pi i c}{\ell'}}q^m} = \frac{1}{1-yq^{a+\ell' m}}\sum_{d=0}^{\ell'-1} y^{\frac{d}{\ell'}} q^{\frac{ad}{\ell'}+dm} e^{\frac{2\pi i cd}{\ell'}} \tag{4.B.4}$$

so that

$$\frac{1}{\ell'}\sum_{c=0}^{\ell'-1} e^{-2\pi i\frac{b'c}{\ell'}} y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau+c}{\ell'}, 0) =$$
$$\sum_{m\in\mathbb{Z}}\sum_{d\in\mathbb{Z}_{\ell'}} \left\{\sum_{c=0}^{\ell'-1}\frac{1}{\ell'}\, e^{-2\pi i(b'-d)\frac{c}{\ell'}}\right\} y^{-\frac{b'-d}{\ell'}+\ell m} q^{-a\frac{b'-d}{\ell'}+\ell a m+dm} q^{\frac{\ell\ell'}{2}m^2}\frac{1}{1-yq^{a+\ell' m}}. \tag{4.B.5}$$

Since the right hand side is zero unless $d = b'$, one has

$$\frac{1}{\ell'}\sum_{c=0}^{\ell'-1} e^{-2\pi i\frac{b'c}{\ell'}} y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau+c}{\ell'}, 0) = \mathcal{K}_\ell(\ell'\tau, \nu + a\tau + \frac{b'}{\ell}\tau, -\frac{b'}{\ell}\tau). \tag{4.B.6}$$

■

We also note, for further reference, that

$$\mathcal{K}_{\ell\ell'}(\tau, \frac{\nu + a\tau + c}{\ell'}, 0) = \sum_{b'=0}^{\ell'-1} e^{2\pi i \frac{b'c}{\ell'}} y^{\frac{b'}{\ell'}} q^{\frac{ab'}{\ell'}} \mathcal{K}_{\ell}(\ell'\tau, \nu + a\tau + \frac{b'}{\ell}\tau, -\frac{b'}{\ell}\tau). \qquad (4.B.7)$$

Remark: the summand in (4.B.2) is periodic in c of period ℓ'. Indeed one has

- For $c = 0$, the summand is $y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau}{\ell'}, 0)$
- For $c = \ell'$, the summand is $y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau}{\ell'}+1, 0) = y^{-\frac{b'}{\ell'}} q^{-\frac{ab'}{\ell'}} \mathcal{K}_{\ell\ell'}(\tau, \frac{\nu+a\tau}{\ell'}, 0)$ by (2.7) in [STT05]

However, the summand in (4.B.7) is not periodic in b' of period ℓ', as is manifest in [STT05] eq(2.4).

4.B.3 Some quasi-periodicity properties

One has (see [STT05])

$$\mathcal{K}_{\ell}(q, xq^{-\frac{n}{\ell}}, yq^{\frac{n}{\ell}}) = (xy)^n \, \mathcal{K}_{\ell}(q, x, y) + \begin{cases} \sum_{r=1}^{n} (xy)^{n-r} \, \vartheta(q^{\ell}, x^{\ell} q^{-r}) & \text{for } n > 0 \\ -\sum_{r=n+1}^{0} (xy)^{n-r} \, \vartheta(q^{\ell}, x^{\ell} q^{-r}) & \text{for } n < 0. \end{cases} \qquad (4.B.8)$$

4.B.4 S-transformation

Formula (1.4) in [STT05] yields the following S-transformation for the Appell function at non-zero, even integer level $2m, m \in \mathbb{N} \setminus \{0\}$; with $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$ and $u \in \mathbb{C}$,

$$\begin{aligned} &\mathcal{K}_{2m}(\tau, z, -u) - \frac{1}{\tau} e^{2\pi i m \frac{u^2 - z^2}{\tau}} \mathcal{K}_{2m}(-\frac{1}{\tau}, \frac{z}{\tau}, -\frac{u}{\tau}) = \\ &\quad - e^{2\pi i m \frac{u^2-z^2}{\tau}} \sum_{\ell=0}^{2m-1} e^{\pi i \frac{2m}{\tau}(z + \frac{\ell}{2m}\tau)^2} \Phi(2m\tau, -2mu - \ell\tau) \vartheta(2m\tau, 2mz + \ell\tau), \end{aligned} \qquad (4.B.9)$$

where

$$\Phi(\tau,u) := -\int_{\substack{\mathbb{R}-it\\ 0<t<1}} dx\, e^{-\pi x^2}\,\frac{e^{-2\pi i x u/\sqrt{-i\tau}}}{1-e^{-2\pi x\sqrt{-i\tau}}} \tag{4.B.10}$$

and where, using (3.A.9),

$$\vartheta(2m\tau, 2mz+\ell\tau) = e^{-2\pi i\ell z}e^{-\pi i\tau\ell^2/2m}\theta_{\ell,m}(\tau,2z). \tag{4.B.11}$$

<u>Claim</u>: the S-transformation of the Appell function $\mathcal{K}_{2m}(\tau,z,-u)$ can be rewritten as

$$\mathcal{K}_{2m}(\tau,z,-u) - \frac{1}{\tau}e^{2\pi i m\frac{u^2-z^2}{\tau}}\mathcal{K}_{2m}(-\frac{1}{\tau},\frac{z}{\tau},-\frac{u}{\tau}) = \sum_{\ell=0}^{2m-1} h_\ell(u,\tau)\,\vartheta^{ZW}_{m,\ell}(z,\tau), \tag{4.B.12}$$

where

$$h_\ell(u,\tau) := ie^{-\pi i\ell^2\tau/2m-2\pi i\ell u}\int_{\substack{\mathbb{R}-it\\ 0<t<1}} dx\, e^{2\pi i m\tau x^2}\,\frac{e^{-2\pi x(2mu+\ell\tau)}}{1-e^{2\pi x}} \tag{4.B.13}$$

and

$$\begin{aligned}\vartheta^{ZW}_{m,\ell}(z,\tau) &:= \sum_{\lambda\in\mathbb{Z},\,\lambda\equiv\ell\,(2m)} e^{\pi i\lambda^2\tau/2m+2\pi i\lambda z},\\ &= \sum_{n\in\mathbb{Z}} e^{\pi i(2mn+\ell)^2\tau/2m+2\pi i(2mn+\ell)z} = \sum_{n\in\mathbb{Z}} q^{m(n+\frac{\ell}{2m})^2}y^{2m(n+\frac{\ell}{2m})} = \theta_{\ell,m}(\tau,2z),\end{aligned} \tag{4.B.14}$$

with $y=e^{2\pi i z}$.

<u>Justification of claim</u>: the claim is nothing else than a consistency check between the work of [STT05] and the work of Zwegers in his thesis [Zwe08], Chapter 3, Proposition 3.3 (6). There, Zwegers uses the following definition of the Appell function at level $2m$,

$$f_u(z,\tau) = f_u^{(m)}(z,\tau) := \sum_{\lambda\in\mathbb{Z}}\frac{e^{2\pi i m\lambda^2\tau+4\pi i m\lambda z}}{1-e^{2\pi i\lambda\tau+2\pi i(z-u)}}. \tag{4.B.15}$$

We note that

$$f_u^{(m)}(z,\tau) = \mathcal{K}_{2m}(\tau,z,-u). \tag{4.B.16}$$

Now use formula (2.38) of [STT05] with $\tau\to 2m\tau$, $\mu=-2mu+(2m-\ell)\tau$, which gives

$$\Phi(2m\tau,-2mu-\ell\tau) = \Phi(2m\tau,-2mu+(2m-\ell)\tau-2m\tau)$$

$$= \frac{i}{\sqrt{-2im\tau}}\, e^{-\frac{\pi i}{2m\tau}(-2mu+(2m-\ell)\tau)^2}\, \Phi(-\frac{1}{2m\tau}, -\frac{u}{\tau}+\frac{2m-\ell}{2m}), \quad (4.B.17)$$

where

$$\Phi(-\frac{1}{2m\tau}, -\frac{u}{\tau}+\frac{2m-\ell}{2m}) = -\int_{\substack{\mathbb{R}-it\\0<t<1}} dx\, e^{-\pi x^2}\, \frac{e^{2\pi x(2mu-(2m-\ell)\tau)/\sqrt{-2im\tau}}}{1-e^{-2\pi x/\sqrt{-2im\tau}}}. \quad (4.B.18)$$

Now set $x' = -\frac{x}{\sqrt{-2im\tau}}$ in (4.B.18) so that

$$\Phi(-\frac{1}{2m\tau}, -\frac{u}{\tau}+\frac{2m-\ell}{2m}) = -\sqrt{2im\tau}\int_{\substack{\mathbb{R}+it\\0<t<1}} dx\, e^{2\pi im\tau x^2}\, \frac{e^{-2\pi x(2mu+(\ell-2m)\tau)}}{1-e^{2\pi x}} \quad (4.B.19)$$

$$\begin{aligned} &= -\sqrt{2im\tau}\int_{\substack{\mathbb{R}+it\\0<t<1}} dx\, e^{2\pi im\tau(x-i)^2+2\pi im\tau}\, \frac{e^{-2\pi x(2mu+\ell\tau)}}{1-e^{2\pi x}} \\ &= -\sqrt{2im\tau}\, e^{2\pi i(m-\ell)\tau}\, e^{-4\pi imu}\int_{\substack{\mathbb{R}-it\\0<t<1}} dx\, e^{2\pi im\tau x^2}\, \frac{e^{-2\pi x(2mu+\ell\tau)}}{1-e^{2\pi x}}. \end{aligned} \quad (4.B.20)$$

We infer from the above that the right hand side of (4.B.9) reads

$$-e^{2\pi im\frac{u^2}{\tau}}\sum_{\ell=0}^{2m-1}\Phi(2m\tau, -2mu-\ell\tau)\vartheta^{ZW}_{m,\ell}(z,\tau) = \sum_{\ell=0}^{2m-1} h^{(m)}_\ell(u,\tau)\,\vartheta^{\mathrm{ZW}}_{m,\ell}(z,\tau), \quad (4.B.21)$$

which provides the sought link between the results in [STT05] and [Zwe08].

We remark that with the change of variable $p = \sqrt{2m}x + i\frac{\ell}{\sqrt{2m}}$, the specialised function $h^{(m)}_\ell(0,\tau)$ is given by,

$$h^{(m)}_\ell(0,\tau) = -\frac{i}{\sqrt{2m}}\int_{\substack{\mathbb{R}+i\frac{\ell}{\sqrt{2m}}-it\\0<t<1}} dp\, q^{\frac{p^2}{2}}\, \frac{e^{-2\pi(\frac{p}{\sqrt{2m}}-i\frac{\ell}{2m})}}{1-e^{-2\pi(\frac{p}{\sqrt{2m}}-i\frac{\ell}{2m})}}. \quad (4.B.22)$$

Furthermore, using [STT05], formulae (2.28) and (2.30),

$$h^{(m)}_\ell(0,\tau) + h^{(m)}_{2m-\ell}(0,\tau) = -\frac{i}{\sqrt{2m}}\,(-i\tau)^{-\frac{1}{2}}. \quad (4.B.23)$$

4.C Lemmas

Lemma 1: *Let k^+ and k^- be coprime and assume, without loss of generality, that $k^+ > k^-$.*

Then, with $0 \leq 2\ell^\pm \leq k^\pm - 1$*, the quantity* $b(\ell^+, \ell^-) := 2\ell^- k^+ - 2\ell^+ k^- - k^-$ *is never a multiple of* $k := k^+ + k^-$.

Proof: It suffices to rewrite

$$b(\ell^+, \ell^-) = 2\ell^-(k - k^-) - 2\ell^+ k^- - k^- = 2\ell^- k - k^-(2\ell^- + 2\ell^+ + 1)$$

and to remark that k^- does not divide k, and that $2\ell^- + 2\ell^+ + 1$ is strictly positive and strictly smaller than kk^-.

Lemma 2: *Let* k^+ *and* k^- *be coprime and assume, without loss of generality, that* $k^+ > k^-$. *With* $0 \leq 2\ell^\pm \leq k^\pm - 1$*, let* $b(\ell^+, \ell^-) := 2\ell^-(k-k^-) - 2\ell^+ k^- - k^- = Nk - b'$, for $1 \leq b' \leq k - 1$ and $N \in \mathbb{Z}$. *Then there exists a unique pair* (ℓ_L^+, ℓ_L^-) *in the range* $0 \leq 2\ell_L^\pm \leq k^\pm - 1$ *that satifies*

$$\begin{aligned} 2\ell_L^+ + 2\ell_L^- &= L \\ b(\ell_L^+, \ell_L^-) &= -b'_L, \end{aligned} \tag{4.C.1}$$

where $1 \leq b'_L \leq k - 1$.

4.D S-transformation of $\widetilde{\mathcal{A}}_\gamma$ characters - Intermediate steps

4.D.1 Derivation of formula (4.4.47)

From (4.4.37), (4.4.38) and (4.4.46), one obtains

$$Ch_0^{\widetilde{\mathcal{A}}_\gamma, \tilde{R}}(\tilde{k}^+, \tilde{k}^-, \tilde{h}_R, \ell^+, \ell^-; -\frac{1}{\tau}, \frac{\omega_+}{\tau}, \frac{\omega_-}{\tau})\Big|_{\text{`massless'}} =$$
$$\frac{i}{k\sqrt{2k}}(-1)^{2\ell^-+1} e^{\frac{2\pi i}{\tau}(\tilde{k}^+\omega_+^2 + \tilde{k}^-\omega_-^2)} \frac{\vartheta_1(\tau, \omega_+ + \omega_-)\vartheta_1(\tau, \omega_+ - \omega_-)}{\eta^3(\tau)\vartheta_1(\tau, 2\omega_+)\vartheta_1(\tau, 2\omega_-)}$$

$$\times \sum_{\epsilon_+,\epsilon_-=\pm 1} \epsilon_+\epsilon_- \sum_{s,s',a'=0}^{k-1} \sum_{n'=0}^{2k-1} e^{-\frac{\pi i}{k}[n'(2\ell^+ + 2\ell^- + 1 - 2sk^-) + 2(sa' + b's' + 2k^+k^- ss')]}$$

$$\times\, \theta_{n',k}(\tau, \zeta_{\epsilon_+,\epsilon_-}) X(-a', s'; \tau, \nu_{\epsilon_+,\epsilon_-}). \quad (4.D.1)$$

Note that $\nu_{1,1} = -\nu_{-1,-1}$ and similarly $\zeta_{1,1} = -\zeta_{-1,-1}$, $\nu_{1,-1} = -\nu_{-1,1}$ and $\zeta_{1,-1} = -\zeta_{-1,1}$. Therefore, the explicit summation over $\epsilon_\pm$ above yields

$$S := \sum_{s,s',a'=0}^{k-1} \sum_{n'=0}^{2k-1} e^{\frac{2\pi i}{k}(k^- n' - 2k^+k^- s' - a')s} e^{-\frac{\pi i}{k} n'(2\ell^+ + 2\ell^- + 1)} e^{-\frac{2\pi i}{k} b's'}$$

$$\times \{\, \theta_{n',k}(\tau, \zeta_{1,1}) X(-a', s'; \tau, \nu_{1,1}) + \theta_{n',k}(\tau, -\zeta_{1,1}) X(-a', s'; \tau, -\nu_{1,1})$$

$$-\theta_{n',k}(\tau, \zeta_{1,-1}) X(-a', s'; \tau, \nu_{1,-1}) - \theta_{n',k}(\tau, -\zeta_{1,-1}) X(-a', s'; \tau, -\nu_{1,-1})\}\,. \quad (4.D.2)$$

Let us first concentrate in (4.D.2) on the sub-contribution S_1 defined by

$$S_1 := \sum_{s,s',a'=0}^{k-1} \sum_{n'=0}^{2k-1} e^{\frac{2\pi i}{k}(k^- n' - 2k^+k^- s' - a')s} e^{-\frac{\pi i}{k} n'(2\ell^+ + 2\ell^- + 1)} e^{-\frac{2\pi i}{k} b's'}$$

$$\times \{\, \theta_{n',k}(\tau, \zeta_{1,1}) X(-a', s'; \tau, \nu_{1,1}) + \theta_{n',k}(\tau, -\zeta_{1,1}) X(-a', s'; \tau, -\nu_{1,1})\,. \quad (4.D.3)$$

Using the property (4.4.26) on $X(-a', s'; \tau, -\nu_{1,1})$ as well as (4.4.27), one gets

$$S_1 = \sum_{s,s'=0}^{k-1} \sum_{a'=1}^{k-1} \sum_{n'=0}^{2k-1} e^{\frac{2\pi i}{k}(k^- n' - 2k^+k^- s' - a')s} e^{-\frac{\pi i}{k} n'(2\ell^+ + 2\ell^- + 1)} e^{-\frac{2\pi i}{k} b's'}$$

$$\times \{\, \theta_{n',k}(\tau, \zeta_{1,1}) X(-a', s'; \tau, \nu_{1,1}) - \theta_{n',k}(\tau, -\zeta_{1,1}) X(-(k - a'), -s'; \tau, \nu_{1,1})\}$$

$$+\, 2k \sum_{s'=0}^{k-1} e^{-\frac{\pi i}{k} 2k^+k^- s'(2\ell^+ + 2\ell^- + 1)} e^{-\frac{2\pi i}{k} b's'}$$

$$\times \{\theta_{2k^+s',k}(\tau, \zeta_{1,1}) X(0, s'; \tau, \nu_{1,1}) - \theta_{-2k^+s',k}(\tau, \zeta_{1,1}) X(-k, -s'; \tau, \nu_{1,1})\}\,, \quad (4.D.4)$$

where the last sum corresponds to the terms $a' = 0$ in S_1 and where we have summed over the variable s. Now, using (4.4.22) with $b = Nk - b'$ and (4.4.30), this 'boundary' contribution becomes

$$2k \sum_{s'=0}^{k-1} \theta_{2k^+k^- s', \frac{\kappa}{2}}(\tau, \frac{2\nu_{1,1}}{k})\, \theta_{2k^+s',k}(\tau, \zeta_{1,1}). \quad (4.D.5)$$

We have isolated this boundary contribution since, according to Lemma 1 (Appendix 4.C), for $a' = 0$ or a non zero multiple of k, the function $X(-a', s'; \tau, \nu)$ does not lead to an $\widetilde{\mathcal{A}}_\gamma$

massless character.

Upon the changes of variables $a'' = k - a'$, $n'' = 2k - n'$, $s'' = k - s'$ in the sum involving $\theta_{n',k}(\tau, -\zeta_{1,1})X(-(k-a'), -s'; \tau, \nu_{1,1})$, $a' \neq 0$ in S_1, one obtains for this sum (after using (4.4.27))

$$-\sum_{s,s'=0}^{k-1}\sum_{a'=1}^{k-1}\sum_{n'=1}^{2k} e^{\frac{2\pi i}{k}(-k^-n'+2k^+k^-s'+a')s} e^{\frac{\pi i}{k}n'(2\ell^++2\ell^-+1)} e^{\frac{2\pi i}{k}b's'}\theta_{n',k}(\tau,\zeta_{1,1})X(-a',s';\tau,\nu_{1,1}) \tag{4.D.6}$$

Now recall that, according to Lemma 2 (Appendix 4.C), to each value a' in the range $1 \leq a' \leq k-1$ corresponds a unique pair (λ^+, λ^-) in the range $0 \leq 2\lambda^\pm \leq \tilde{k}^\pm$ such that $a' = 2\lambda^+k^- - 2\lambda^-k^+ + k^-$.

With this in mind, and after summing over the variable s, which only yields a non zero value when

$$-n' + 2\lambda^+ + 2\lambda^- + 1 - 2k^-s' = \mu\, k, \qquad \mu \in \mathbb{Z}, \tag{4.D.7}$$

(4.D.4) becomes

$$\begin{aligned} S_1 = -4ik\sum_{s'=0}^{k-1}\sideset{}{'}\sum_{0\leq 2\lambda^\pm\leq\tilde{k}^\pm} &\sin\left(\frac{\pi}{k}(2\ell^++2\ell^-+1)(2\lambda^++2\lambda^-+1)\right)\\ &\times\theta_{2\lambda^++2\lambda^-+1-2k^-s',k}(\tau,\zeta_{1,1})X(2\lambda^-k^+-2\lambda^+k^--k^-,s';\tau,\nu_{1,1})\\ &\qquad + 2k\sum_{s'=0}^{k-1}\theta_{2k^+k^-s',\frac{\kappa}{2}}(\tau,\frac{2\nu_{1,1}}{k})\theta_{2k^+s',k}(\tau,\zeta_{1,1}), \end{aligned} \tag{4.D.8}$$

where $\sideset{}{'}\sum$ indicates that the variables summed over obey the constraint

$$1 \leq 2\lambda^+k^- - 2\lambda^-k^+ + k^- \leq k-1.$$

An identical treatment of the sub-contribution S_2 in (4.D.2), defined by

$$\begin{aligned} S_2 := -\sum_{s,s',a'=0}^{k-1}\sum_{n'=0}^{2k-1} &e^{\frac{2\pi i}{k}(k^-n'-2k^+k^-s'-a')s} e^{-\frac{\pi i}{k}n'(2\ell^++2\ell^-+1)} e^{-\frac{2\pi i}{k}b's'}\\ \times\{\,&\theta_{n',k}(\tau,\zeta_{1,-1})X(-a',s';\tau,\nu_{1,-1}) + \theta_{n',k}(\tau,-\zeta_{-1,1})X(-a',s';\tau,-\nu_{-1,1})\} \end{aligned} \tag{4.D.9}$$

leads to an expression for S_2, which, up to an overall sign and the angular dependence being in $\nu_{1,-1}, \nu_{-1,1}, \zeta_{1,-1}$ and $\zeta_{-1,1}$, is of exactly the same form as (4.D.8). This yields

$$\begin{aligned} S &= S_1 + S_2 = \\ -2ik & \sum_{\epsilon^\pm=\pm1} \epsilon_+\epsilon_- \sum_{s'=0}^{k-1} \sum_{0\le 2\lambda^\pm \le \tilde{k}^\pm}' \sin\Big(\frac{\pi}{k}(2\ell^+ + 2\ell^- + 1)(2\lambda^+ + 2\lambda^- + 1)\Big) \\ & \qquad \times \theta_{2\lambda^+ + 2\lambda^- + 1 - 2k^- s', k}(\tau, \zeta_{\epsilon_+,\epsilon_-}) X(2\lambda^- k^+ - 2\lambda^+ k^- - k^-, s'; \tau, \nu_{\epsilon_+,\epsilon_-}) \end{aligned} \tag{4.D.10}$$

$$+ \quad k \sum_{\epsilon^\pm=\pm1} \epsilon_+\epsilon_- \sum_{s'=0}^{k-1} \theta_{\frac{\kappa}{k}s', \frac{\kappa}{2}}(\tau, \frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\, \theta_{2k^+ s',k}(\tau, \zeta_{\epsilon_+,\epsilon_-}), \tag{4.D.11}$$

where the sum (4.D.11) vanishes. Indeed, using the change of variable $s'' = k - s'$ and the periodicity of theta functions in the term corresponding to $(\epsilon_+, \epsilon_-) = (-1,-1)$, one finds a contribution identical to the contribution from $(\epsilon_+, \epsilon_-) = (1,1)$. So together they yield

$$2k \sum_{s'=0}^{k-1} \theta_{2k^+ s',k}(\tau, \zeta_{1,1}) \theta_{\frac{\kappa}{k}s', \frac{\kappa}{2}}(\tau, \frac{2\nu_{1,1}}{k}) = 2k\theta_{0,\,k^+}(\tau, \omega_+)\theta_{0,\,k^-}(\tau, \omega_-), \tag{4.D.12}$$

where one uses the theta product formula (3.A.7). A similar analysis for the contributions $(\epsilon_+, \epsilon_-) = (1,-1)$ and $(\epsilon_+, \epsilon_-) = (-1,1)$ yields a contribution of

$$-\, 2k\theta_{0,\,k^+}(\tau, \omega_+)\theta_{0,\,k^-}(\tau, \omega_-), \tag{4.D.13}$$

so that the sum (4.D.11) over all values of (ϵ_+, ϵ_-) vanishes as announced.

The massless contribution to the S-transformation of massless characters is thus indeed (4.4.47).

4.D.2 Derivation of formula (4.4.49)

Our starting point is the definition (4.4.48)

$$\begin{aligned} & Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+, \tilde{k}^-, \tilde{h}_R, \ell^+, \ell^-; -\frac{1}{\tau}, \frac{\omega_+}{\tau}, \frac{\omega_-}{\tau})\Big|_{\text{massive}} = \\ & \qquad (-1)^{2\ell^- + 1} \frac{\vartheta_1(-\frac{1}{\tau}, \frac{\omega_+ + \omega_-}{\tau})\vartheta_1(-\frac{1}{\tau}, \frac{\omega_+ - \omega_-}{\tau})}{\eta^3(-\frac{1}{\tau})\vartheta_1(-\frac{1}{\tau}, \frac{2\omega_+}{\tau})\vartheta_1(-\frac{1}{\tau}, \frac{2\omega_+}{\tau})} \sum_{\epsilon_+,\epsilon_-=\pm1} \epsilon_+\epsilon_- \\ & \times \sum_{s=0}^{k-1} \theta_{2\ell^+ + 2\ell^- + 1 - 2sk^-, k}(-\frac{1}{\tau}, \frac{\zeta_{\epsilon_+,\epsilon_-}}{\tau}) \left\{ X_2(-b', s; \tau, \nu) + T(N, b', s; -\frac{1}{\tau}, \frac{\nu_{\epsilon_+,\epsilon_-}}{\tau}) \right\}, \end{aligned} \tag{4.D.14}$$

where

$$0 \leq 2\ell^{\pm} \leq \tilde{k}^{\pm},$$
$$b := 2\ell^- k^+ - 2\ell^+ k^- - k^- = Nk - b', \; N \in \mathbb{Z}, \; 0 \leq b' \leq k-1 \quad (4.D.15)$$

and $T(N, b', s; \tau, \nu)$ is defined in (4.4.31).

First case: $N = 0$: ℓ^+ and ℓ^- are chosen so that $b = -b'$.

From (4.4.37), (4.4.38) and (4.4.43), we obtain

$$\begin{aligned}\mathcal{M} := \sqrt{2k}\,(-1)^{2\ell^-+1}\frac{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)}{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}\,\mathbf{e}(-\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})\\ \times Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R,\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massive},N=0} =\\ -\frac{i}{k}\sum_{\epsilon_+,\epsilon_-=\pm1}\epsilon_+\epsilon_-\sum_{\ell=0}^{\kappa-1}\sum_{n'=0}^{2k-1}\sum_{s'=0}^{k-1}e^{-\frac{2\pi i}{k}b's'}e^{-\frac{\pi i}{k}n'(2\ell^++2\ell^-+1)}\left\{\sum_{s=0}^{k-1}e^{\frac{2\pi i}{k}s(n'k^--\ell-2k^+k^-s')}\right\}\\ \times h_\ell^{(\frac{\kappa}{2})}(\tau,0)\,\theta_{\ell+2k^+k^-s',\frac{\kappa}{2}}(\tau,\frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\,\theta_{n',k}(\tau,\zeta_{\epsilon_+,\epsilon_-}),\end{aligned} \quad (4.D.16)$$

with $h_\ell^{(\frac{\kappa}{2})}(\tau,0)$ given by (4.B.22) for $2m = \kappa$.

The sum over s in (4.D.16) is non-zero (and with value k) for $(n'k^- - \ell - 2k^+k^-s')$ a multiple of k. Within the ranges of ℓ, s' and n', there are exactly $2k\kappa$ triplets (ℓ, s', n') yielding a non-zero sum. In order to reconstruct massive character contributions from (4.D.16), the following reparametrization is useful,

$$\begin{aligned} n' &= 2\ell^{+\prime} - 2\ell^{-\prime} - 2k^-s' \\ \ell &= 2\ell^{+\prime}k^- + 2\ell^{-\prime}\,k^+, \end{aligned} \quad (4.D.17)$$

with $0 \leq s' \leq k-1$ and

either

$$2\ell^{+\prime} = n + \alpha p \quad \text{and} \quad 2\ell^{-\prime} = n + \beta p \quad (4.D.18)$$

for $0 \leq n \leq 2k^+k^- - 1$ and $0 \leq p \leq k-1$,

or

$$2\ell^{+\prime} = n + \alpha p + 1 \quad \text{and} \quad 2\ell^{-\prime} = n + \beta p \quad (4.D.19)$$

for $0 \leq n \leq 2k^+k^- - 1$ and $-k^- \leq p \leq k^+ - 1$.

In the above, and for fixed (k^-, k^+), the values of α and β are fixed by Bézout's lemma and the extended Euclidean algorithm[5]. Indeed, exploiting the fact that k^+ and k^- are coprime, we write $\alpha k^- + \beta k^+ = 1$ for the Bézout coefficients (α, β) obtained by applying the extended Euclidean algorithm. In particular, $|\alpha| < k^+$ and $|\beta| < k^-$.

This parametrisation produces $2k\kappa$ triplets (ℓ, s', n') yielding a non-zero value for the s-sum in (4.D.16). Note that some triplets have their n' component out of the range $0 \leq n' \leq 2k - 1$ but (4.D.16) is periodic in n' of period $2k$ so it is always possible to replace an out-of-range n' value by one in-range at no cost.

We thus have

$$\begin{aligned}\mathcal{M} = &-i \sum_{\epsilon_+,\epsilon_-=\pm 1} \epsilon_+\epsilon_- \sum_{n=0}^{2k^+k^--1} \sum_{p=0}^{k-1} e^{-\frac{\pi i}{k}(\alpha-\beta)p(2\ell^+ + 2\ell^- + 1)}\, h^{(\frac{\kappa}{2})}_{(n+\alpha p)k^- + (n+\beta p)k^+}(\tau, 0) \\ &\times \sum_{s'=0}^{k-1} \theta_{(n+\alpha p)k^- + (n+\beta p)k^+ + 2k^+k^- s', \frac{\kappa}{2}}(\tau, \frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\, \theta_{(\alpha-\beta)p - 2k^- s', k}(\tau, \zeta_{\epsilon_+,\epsilon_-}) \\ &- i \sum_{\epsilon_+,\epsilon_-=\pm 1} \epsilon_+\epsilon_- \sum_{n=0}^{2k^+k^--1} \sum_{p=-k^-}^{k^+-1} e^{-\frac{\pi i}{k}((\alpha-\beta)p+1)(2\ell^+ + 2\ell^- + 1)}\, h^{(\frac{\kappa}{2})}_{(n+\alpha p+1)k^- + (n+\beta p)k^+}(\tau, 0) \\ &\times \sum_{s'=0}^{k-1} \theta_{(n+\alpha p+1)k^- + (n+\beta p)k^+ + 2k^+k^- s', \frac{\kappa}{2}}(\tau, \frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\, \theta_{(\alpha-\beta)p + 1 - 2k^- s', k}(\tau, \zeta_{\epsilon_+,\epsilon_-}). \end{aligned} \tag{4.D.20}$$

Taking advantage of (4.4.2) and (4.4.3), one may rewrite (4.D.20) in terms of affine $su(2)$ characters as

$$\begin{aligned}\mathcal{M} &= -i\vartheta_1(\tau, 2\omega_+)\vartheta_1(\tau, 2\omega_-)\times \\ &\sum_{n=0}^{2k^+k^--1} \Bigg\{ \sum_{p=0}^{k-1} e^{-\frac{\pi i}{k}(\alpha-\beta)p(2\ell^+ + 2\ell^- + 1)}\, h^{(\frac{\kappa}{2})}_{(n+\alpha p)k^- + (n+\beta p)k^+}(\tau, 0)\chi^{k^+-2}_{\frac{1}{2}(n+\alpha p-1)}(\omega_+)\, \chi^{k^--2}_{\frac{1}{2}(n+\beta p-1)}(\omega_-) \\ &+ \sum_{p=-k^-}^{k^+-1} e^{-\frac{\pi i}{k}((\alpha-\beta)p+1)(2\ell^+ + 1 + 2\ell^-)}\, h^{(\frac{\kappa}{2})}_{(n+\alpha p+1)k^- + (n+\beta p)k^+}(\tau, 0)\chi^{k^+-2}_{\frac{1}{2}(n+\alpha p)}(\omega_+)\, \chi^{k^--2}_{\frac{1}{2}(n+\beta p-1)}(\omega_-) \Bigg\} \\ &= -i\vartheta_1(\tau, 2\omega_+)\vartheta_1(\tau, 2\omega_-) \sum_{n=0}^{2k^+k^--1} \sum_{p=0}^{k-1} e^{-\frac{\pi i}{k}(\alpha-\beta)p(2\ell^+ + 2\ell^- + 1)}\, h^{(\frac{\kappa}{2})}_{nk+p}(\tau, 0)\end{aligned}$$

[5] This lemma states that if a and b are two integers with greatest common divisor d, then there exist integers α and β such that $\alpha a + \beta b = d$. These pairs (α, β) are not unique, but the extended Euclidean algorithm produces a pair which obeys the bounds $|\alpha| \leq |\frac{b}{d}|, |\beta| \leq |\frac{a}{d}|$.

$$\left\{\chi^{k^+-2}_{\frac{1}{2}(n+\alpha p-1)}(\omega_+)\chi^{k^--2}_{\frac{1}{2}(n+\beta p-1)}(\omega_-)+e^{-\pi i\beta(2\ell^++2\ell^-+1)}\chi^{k^+-2}_{\frac{1}{2}(n+\alpha p-\alpha k^-)}(\omega_+)\chi^{k^--2}_{\frac{1}{2}(n+\beta p-\beta k^--1)}(\omega_-)\right\}. \tag{4.D.21}$$

Setting

$$2\ell^{+\prime}_{n,p}:=n+\alpha p,\quad 2\ell^{-\prime}_{n,p}:=n+\beta p,\quad 2\ell^{+\prime\prime}_{n,p}:=n+\alpha p+\beta k^+ \text{ and } 2\ell^{-\prime\prime}_{n,p}:=n+\beta p-\beta k^-, \tag{4.D.22}$$

we rewrite the above as

$$\begin{aligned}\mathcal{M}&=-i\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)\times\\ &\sum_{n=0}^{2k^+k^--1}\sum_{p=0}^{k-1}\left\{e^{-\frac{\pi i}{k}(2\ell^{+\prime}_{n,p}-2\ell^{-\prime}_{n,p})(2\ell^++2\ell^-+1)}\,h^{(\frac{\kappa}{2})}_{2\ell^{+\prime}_{n,p}k^-+2\ell^{-\prime}_{n,p}k^+}(\tau,0)\chi^{k^+-2}_{\ell^{+\prime}_{n,p}-\frac{1}{2}}(\omega_+)\chi^{k^--2}_{\ell^{-\prime}_{n,p}-\frac{1}{2}}(\omega_-)\right.\\ &\left.+e^{-\frac{\pi i}{k}(2\ell^{+\prime\prime}_{n,p}-2\ell^{-\prime\prime}_{n,p})(2\ell^++2\ell^-+1)}\,h^{(\frac{\kappa}{2})}_{2\ell^{+\prime\prime}_{n,p}k^-+2\ell^{-\prime\prime}_{n,p}k^+}(\tau,0)\chi^{k^+-2}_{\ell^{+\prime\prime}_{n,p}-\frac{1}{2}}(\omega_+)\chi^{k^--2}_{\ell^{-\prime\prime}_{n,p}-\frac{1}{2}}(\omega_-)\right\}.\end{aligned} \tag{4.D.23}$$

Although elegant, the expression (4.D.23) for $\mathcal{M}$ does not provide all the $\widehat{su(2)}$ characters with (twice) their isospin quantum numbers in the unitary ranges $\{0,\ \ldots,\ k^+-2\}$ and $\{0,\ldots,k^--2\}$, which is necessary to formulate $\mathcal{M}$ in terms of massive $\widetilde{\mathcal{A}}_\gamma$ characters. In order to achieve this, we exploit the properties of affine $su(2)$ characters (3.B.4). For each pair (n,p) where $n\in\{0,\ldots,2k^+k^--1\}$ and $p\in\{0,\ldots,k-1\}$, we introduce the notations

$$\nu^{\pm\prime}_{n,p}:=\left\lfloor\frac{2\ell^{\pm\prime}_{n,p}}{k^\pm}\right\rfloor,\ \nu^{\pm\prime\prime}_{n,p}:=\left\lfloor\frac{2\ell^{\pm\prime\prime}_{n,p}}{k^\pm}\right\rfloor=\nu^{\pm\prime}_{n,p}\pm\beta, \tag{4.D.24}$$

and write

$$2\ell^{\pm\prime}_{n,p}=\nu^{\pm\prime}_{n,p}k^\pm+\rho^\pm_{n,p},\quad 2\ell^{\pm\prime\prime}_{n,p}=\nu^{\pm\prime\prime}_{n,p}k^\pm+\rho^\pm_{n,p}, \tag{4.D.25}$$

where $\rho^+_{n,p}$ is an integer in the range $\{0,\ldots,k^+-1\}$ and $\rho^-_{n,p}$ an integer in the range $\{0,\ldots,k^--1\}$. We note that when any of $\rho^\pm_{n,p}$ vanishes, the corresponding term in (4.D.23) vanishes as a consequence of the properties of affine $su(2)$ characters. We use repeatedly the relation $\alpha k^-+\beta k^+=1$ and the properties of $su(2)$ affine characters, and introduce the isospin quantum numbers $L^{\pm\prime}_{n,p}$ and $L^{\pm\prime\prime}_{n,p}$ for $\widehat{su(2)}_{k^\pm-2}$ as follows,

$$2L^{\pm\prime}_{n,p}:=\frac{1-(-1)^{\nu^{\pm\prime}_{n,p}}}{2}k^\pm+(-1)^{\nu^{\pm\prime}_{n,p}}\rho^\pm_{n,p}-1,\quad 2L^{\pm\prime\prime}_{n,p}:=\frac{1-(-1)^{\nu^{\pm\prime\prime}_{n,p}}}{2}k^\pm+(-1)^{\nu^{\pm\prime\prime}_{n,p}}\rho^\pm_{n,p}-1. \tag{4.D.26}$$

This yields

$$\mathcal{M} = -i\vartheta_1(\tau, 2\omega_+)\vartheta_1(\tau, 2\omega_-) \sum_{n=0}^{2k^+k^--1} \sum_{p=0}^{k-1}$$
$$\Big\{ e^{-\frac{\pi i}{k}(2\ell^{+\prime}_{n,p}-2\ell^{-\prime}_{n,p})(2\ell^++2\ell^-+1)} h^{(\frac{\kappa}{2})}_{2\ell^{+\prime}_{n,p}k^-+2\ell^{-\prime}_{n,p}k^+}(\tau,0)\,(-1)^{\nu^{+\prime}_{n,p}+\nu^{-\prime}_{n,p}} \chi^{k^+-2}_{L^{+\prime}_{n,p}}(\omega_+)\chi^{k^--2}_{L^{-\prime}_{n,p}}(\omega_-)$$
$$+e^{-\frac{\pi i}{k}(2\ell^{+\prime\prime}_{n,p}-2\ell^{-\prime\prime}_{n,p})(2\ell^++2\ell^-+1)} h^{(\frac{\kappa}{2})}_{2\ell^{+\prime\prime}_{n,p}k^-+2\ell^{-\prime\prime}_{n,p}k^+}(\tau,0)\,(-1)^{\nu^{+\prime\prime}_{n,p}+\nu^{-\prime\prime}_{n,p}} \chi^{k^+-2}_{L^{+\prime\prime}_{n,p}}(\omega_+)\chi^{k^--2}_{L^{-\prime\prime}_{n,p}}(\omega_-)\Big\}\,. \tag{4.D.27}$$

One notes that, if $\rho^+_{n,p}$ (resp. $\rho^-_{n,p}$) vanishes, the corresponding characters $\chi^{k^+-2}_{L^{+\prime}_{n,p}}(\omega_+)$, $\chi^{k^+-2}_{L^{+\prime\prime}_{n,p}}(\omega_+)$ (resp. $\chi^{k^--2}_{L^{-\prime}_{n,p}}(\omega_-)$, $\chi^{k^--2}_{L^{-\prime\prime}_{n,p}}(\omega_-)$) vanish, so that effectively one restricts to $\rho^+_{n,p} \in \{1,\ldots,k^+-1\}$ and $\rho^-_{n,p} \in \{1,\ldots,k^--1\}$ and $0 \leq 2L^{+\prime}_{n,p}, 2L^{+\prime\prime}_{n,p} \leq k^+-2$, while $0 \leq 2L^{-\prime}_{n,p}, 2L^{-\prime\prime}_{n,p} \leq k^- - 2$.

We now use (4.4.7) to rewrite the products of $su(2)$ affine characters in (4.D.27) in terms of massive $\widetilde{\mathcal{A}}_\gamma$ characters, and given the definition of $\mathcal{M}$ in (4.D.16) and of $h^{(m)}_\ell(0,\tau)$ in (4.B.22), one obtains

$$Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R,\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massive},N=0} = \frac{(-1)^{2\ell^-+1}}{\sqrt{2k\kappa}}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})$$
$$\times \sum_{n=0}^{2k^+k^--1}\sum_{p=0}^{k-1} e^{-\frac{\pi i}{k}(\alpha-\beta)p(2\ell^++2\ell^-+1)} \int_{\mathbb{R}+\frac{i}{\sqrt{\kappa}}(nk+p)-i0} dx\, \frac{e^{-\frac{2\pi}{\sqrt{\kappa}}(x-\frac{i}{\sqrt{\kappa}}(nk+p))}}{1-e^{-\frac{2\pi}{\sqrt{\kappa}}(x-\frac{i}{\sqrt{\kappa}}(nk+p))}}$$
$$\times\Bigg\{(-1)^{2L^{-\prime}_{n,p}+\nu^{+\prime}_{n,p}+\nu^{-\prime}_{n,p}} q^{\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}(L^{+\prime}_{n,p}+\frac{1}{2})-\sqrt{\frac{k^+}{k^-}}(L^{-\prime}_{n,p}+\frac{1}{2})\right)^2}$$
$$\times Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R(L^{+\prime}_{n,p}+\frac{1}{2},L^{-\prime}_{n,p}+\frac{1}{2},x),L^{+\prime}_{n,p}+\frac{1}{2},L^{-\prime}_{n,p}+\frac{1}{2};\tau,\omega_+,\omega_-)$$
$$+(-1)^{2L^{-\prime\prime}_{n,p}+\nu^{+\prime\prime}_{n,p}+\nu^{-\prime\prime}_{n,p}+\beta(2\ell^++2\ell^-+1)} q^{\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}(L^{+\prime\prime}_{n,p}+\frac{1}{2})-\sqrt{\frac{k^+}{k^-}}(L^{-\prime\prime}_{n,p}+\frac{1}{2})\right)^2}$$
$$\times Ch^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R(L^{+\prime\prime}_{n,p}+\frac{1}{2},L^{-\prime\prime}_{n,p}+\frac{1}{2},x),L^{+\prime\prime}_{n,p}+\frac{1}{2},L^{-\prime\prime}_{n,p}+\frac{1}{2};\tau,\omega_+,\omega_-)\Bigg\}\,, \tag{4.D.28}$$

or again, using (4.4.9) and the definition (4.B.22) with $\kappa = 2m$,

$$Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\tilde{h}_R,\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau})\Big|_{\text{massive},N=0} =$$

$$\frac{i}{\sqrt{2k}}(-1)^{2\ell^-+1}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})\times\sum_{n=0}^{2k^+k^--1}\sum_{p=0}^{k-1}e^{-\frac{\pi i}{k}(\alpha-\beta)p(2\ell^++2\ell^-+1)}\,h_{nk+p}(0,\tau)$$
$$\times\Bigg\{(-1)^{2L^{-\prime}_{n,p}+\nu^{+\prime}_{n,p}+\nu^{-\prime}_{n,p}}q^{\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}(L^{+\prime}_{n,p}+\frac{1}{2})-\sqrt{\frac{k^+}{k^-}}(L^{-\prime}_{n,p}+\frac{1}{2})\right)^2}$$
$$\times\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(L^{+\prime}_{n,p}+\frac{1}{2},L^{-\prime}_{n,p}),L^{+\prime}_{n,p}+\frac{1}{2},L^{-\prime}_{n,p}+\frac{1}{2};\tau,\omega_+,\omega_-)$$
$$+(-1)^{2L^{-\prime\prime}_{n,p}+\nu^{+\prime\prime}_{n,p}+\nu^{-\prime\prime}_{n,p}+\beta(2\ell^++2\ell^-+1)}\,q^{\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}(L^{+\prime\prime}_{n,p}+\frac{1}{2})-\sqrt{\frac{k^+}{k^-}}(L^{-\prime\prime}_{n,p}+\frac{1}{2})\right)^2}$$
$$\times\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(L^{+\prime\prime}_{n,p}+\frac{1}{2},L^{-\prime\prime}_{n,p}),L^{+\prime\prime}_{n,p}+\frac{1}{2},L^{-\prime\prime}_{n,p}+\frac{1}{2};\tau,\omega_+,\omega_-)\Bigg\}.\tag{4.D.29}$$

<u>Second case: $N>0$</u>

If $N>0$, (4.D.14) also receives a contribution from (4.4.44), which reads,

$$\begin{aligned}&\mathcal{T}_{N>0}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-;\tau,\omega^+,\omega^-):=\\&\quad(-i\tau)^{-\frac{1}{2}}\,\frac{(-1)^{2\ell^-+1}}{\sqrt{2k\kappa}}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})\,\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega^+)\vartheta_1(\tau,2\omega^-)}\\&\qquad\times\sum_{\epsilon_+,\epsilon_-=\pm1}\epsilon_+\epsilon_-\sum_{n'=0}^{2k-1}\sum_{m=0}^{\kappa-1}\left\{\sum_{s=0}^{k-1}e^{\frac{2\pi is}{k}(k^-n'-m)}\right\}e^{-\frac{\pi in'}{k}(2\ell^++2\ell^-+1)}\\&\qquad\quad\times\sum_{r=1}^{N}\widetilde{q}^{-\frac{1}{2\kappa}(kr-b')^2}e^{\frac{2\pi i}{\kappa}(kr-b')m}\,\theta_{m,\frac{\kappa}{2}}(\tau,\frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\,\theta_{n',k}(\tau,\zeta_{\epsilon_+,\epsilon_-}).\end{aligned}\tag{4.D.30}$$

The sum over s is non-zero (and with value k) for $(k^-n'-m)$ a multiple of k. Within the ranges of n' and m, there are exactly 2κ pairs (n',m) yielding a non-zero sum. In order to reconstruct massive character contributions from (4.D.30), the following reparametrization is useful,

$$\begin{aligned}n'&=2\ell^{+\prime}-2\ell^{-\prime}-2k^-r',\\m&=2\ell^{+\prime}k^-+2\ell^{-\prime}k^++2k^+k^-r',\qquad 0\le r'\le k-1,\end{aligned}\tag{4.D.31}$$

with

$$2\ell^{+\prime}=2\ell^{-\prime}=n,\qquad\text{and}\qquad 2\ell^{+\prime}=2\ell^{-\prime}+1=n+1,\tag{4.D.32}$$

for $0\le n\le 2k^+k^--1$. This produces 2κ pairs (n',m) yielding a non-zero value for the s-sum $\sum_{s=0}^{k-1}e^{\frac{2\pi i}{k}s(n'k^--m)}$, but some of these pairs are out of the ranges $0\le n'\le 2k-1$

and $0 \leq m \leq \kappa - 1$ dictated by (4.D.30). However, (4.D.30) is periodic in n' of period $2k$ and m is periodic of period κ, so that one can replace out-of-range n' and m values by some in-range at no cost. This leads to

$$\begin{aligned}
&\mathcal{T}_{N>0}(\tilde{k}^+, \tilde{k}^-, \ell^+, \ell^-; \tau, \omega^+, \omega^-) = \\
&(-i\tau)^{-\frac{1}{2}} (-1)^{2\ell^-+1} \sqrt{\frac{k}{2\kappa}}\, \mathbf{e}(\frac{\tilde{k}^+\omega_+^2 + \tilde{k}^-\omega_-^2}{\tau}) \frac{\vartheta_1(\tau, \omega_+ + \omega_-)\vartheta_1(\tau, \omega_+ - \omega_-)}{\eta^3(\tau)} \sum_{r=1}^{N} \sum_{n=0}^{2k^+k^--1} \\
&\times \widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\, e^{\frac{2\pi i}{\kappa}kn(kr-b')} \sum_{\epsilon_+,\epsilon_-=\pm 1} \epsilon_+\epsilon_- \sum_{r'=0}^{k-1} \Big\{ \theta_{nk+2k^+k^-r',\frac{\kappa}{2}}(\frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\, \theta_{-2k^-r',k}(\zeta_{\epsilon_+,\epsilon_-}) \\
&\qquad + e^{-\frac{\pi i}{k}(2\ell^++2\ell^-+1)}\, e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\, \theta_{nk+k^-+2k^+k^-r',\frac{\kappa}{2}}(\tau, \frac{2\nu_{\epsilon_+,\epsilon_-}}{k})\, \theta_{1-2k^-r',k}(\tau, \zeta_{\epsilon_+,\epsilon_-}) \Big\},
\end{aligned} \tag{4.D.33}$$

or again, using (4.4.3),

$$\begin{aligned}
&\mathcal{T}_{N>0}(\tilde{k}^+, \tilde{k}^-, \ell^+, \ell^-; \tau, \omega^+, \omega^-) = \\
&\quad (-i\tau)^{-\frac{1}{2}} (-1)^{2\ell^-+1} \sqrt{\frac{k}{2\kappa}}\, \mathbf{e}(\frac{\tilde{k}^+\omega_+^2 + \tilde{k}^-\omega_-^2}{\tau}) \frac{\vartheta_1(\tau, \omega_+ + \omega_-)\vartheta_1(\tau, \omega_+ - \omega_-)}{\eta^3(\tau)} \\
&\quad \times \sum_{r=1}^{N} \sum_{n=0}^{2k^+k^--1} \widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\, e^{\frac{2\pi i}{\kappa}kn(kr-b')} \Big\{ \chi^{k^+-2}_{\frac{1}{2}(n-1)}(\omega_+)\chi^{k^--2}_{\frac{1}{2}(n-1)}(\omega_-) \\
&\qquad + e^{-\frac{\pi i}{k}(2\ell^++2\ell^-+1)}\, e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\, \chi^{k^+-2}_{\frac{n}{2}}(\omega_+)\chi^{k^--2}_{\frac{1}{2}(n-1)}(\omega_-) \Big\}.
\end{aligned} \tag{4.D.34}$$

In order to take full advantage of the properties of $su(2)$ affine characters, we rewrite

$$n = \mu_n^{\pm\prime} k^\pm + \rho_n^{\pm\prime}, \quad \mu_n^{\pm\prime} := \left\lfloor \frac{n}{k^\pm} \right\rfloor, \quad 0 \leq \rho_n^{\pm\prime} \leq k^\pm - 1 \tag{4.D.35}$$

in $\chi^{k^+-2}_{\frac{n-1}{2}}(\omega_+)$ and $\chi^{k^--2}_{\frac{n-1}{2}}(\omega_-)$ respectively and

$$\begin{aligned}
n+1 &= \mu_n^{+\prime\prime} k^+ + \rho_n^{+\prime\prime}, \quad \mu_n^{+\prime\prime} := \left\lfloor \frac{n+1}{k^+} \right\rfloor, \quad 0 \leq \rho_n^{+\prime\prime} \leq k^+ - 1, \\
n &= \mu_n^{-\prime\prime} k^- + \rho_n^{-\prime\prime}, \quad \mu_n^{-\prime\prime} := \left\lfloor \frac{n}{k^-} \right\rfloor, \quad 0 \leq \rho_n^{-\prime\prime} \leq k^- - 1,
\end{aligned} \tag{4.D.36}$$

in $\chi^{k^+-2}_{\frac{n}{2}}(\omega_+)$ and $\chi^{k^--2}_{\frac{n-1}{2}}(\omega_-)$[6] We note that if any of $\rho_n^{\pm\prime}, \rho_n^{\pm\prime\prime}$ vanishes, the corresponding term in (4.D.34) vanishes as a consequence of the properties of affine $su(2)$ characters.

[6]Note that $\mu_n^{-\prime\prime} = \mu_n^{-\prime}$ and that $\rho_n^{-\prime\prime} = \rho_n^{-\prime}$.

We also introduce the notations

$$
\begin{aligned}
2\lambda_n^{\pm\prime} &:= \frac{1}{2}(1-(-1)^{\mu_n^{\pm\prime}})k^{\pm} + (-1)^{\mu_n^{\pm\prime}}\rho_n^{\pm\prime}, \\
2\lambda_n^{\pm\prime\prime} &:= \frac{1}{2}(1-(-1)^{\mu_n^{\pm\prime\prime}})k^{\pm} + (-1)^{\mu_n^{\pm\prime\prime}}\rho_n^{\pm\prime\prime}.
\end{aligned} \tag{4.D.37}
$$

The contribution (4.D.34) thus becomes

$$
\begin{aligned}
&\mathcal{T}_{N>0}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-;\tau,\omega^+,\omega^-) = \\
&\quad (-i\tau)^{-\frac{1}{2}}(-1)^{2\ell^-}\sqrt{\frac{k}{2\kappa}}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})\,\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)} \\
&\quad \times\sum_{r=1}^{N}\sum_{n=0}^{2k^+k^--1}\widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\,e^{\frac{2\pi i}{\kappa}kn(kr-b')}\,\Big\{(-1)^{\mu_n^{+\prime}+\mu_n^{-\prime}+1}\,\chi^{k^+-2}_{\lambda_n^{+\prime}-\frac{1}{2}}(\omega_+)\chi^{k^--2}_{\lambda_n^{-\prime}-\frac{1}{2}}(\omega_-) \\
&\qquad +(-1)^{\mu_n^{+\prime\prime}+\mu_n^{-\prime\prime}+1}e^{-\frac{\pi i}{k}(2\ell^++1+2\ell^-)}\,e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\,\chi^{k^+-2}_{\lambda_n^{+\prime\prime}-\frac{1}{2}}(\omega_+)\chi^{k^--2}_{\lambda_n^{-\prime\prime}-\frac{1}{2}}(\omega_-)\Big\} \\
&\quad = (-i\tau)^{-\frac{1}{2}}(-1)^{2\ell^-}\sqrt{\frac{k}{2\kappa}}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})\sum_{r=1}^{N}\sum_{n=0}^{2k^+k^--1}\widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\,e^{\frac{2\pi i}{\kappa}kn(kr-b')} \\
&\times\Big\{(-1)^{2\lambda_n^{-\prime}+\mu_n^{+\prime}+\mu_n^{-\prime}+1}\,q^{-\Delta(\lambda_n^{+\prime},\lambda_n^{-\prime})}\,\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\lambda_n^{+\prime},\lambda_n^{-\prime}-\frac{1}{2}),\lambda_n^{+\prime},\lambda_n^{-\prime};\tau,\omega_+,\omega_-) \\
&\qquad +(-1)^{2\lambda_n^{-\prime\prime}+\mu_n^{+\prime\prime}+\mu_n^{-\prime\prime}+1}\,e^{-\frac{\pi i}{k}(2\ell^++1+2\ell^-)}\,e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\,q^{-\Delta(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime})} \\
&\qquad\qquad \times\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime}-\frac{1}{2}),\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime};\tau,\omega_+,\omega_-)\Big\}.
\end{aligned} \tag{4.D.38}
$$

<u>Third case: $N<0$</u>

If $N<0$, the contribution to (4.D.14) from (4.4.45) is analogous to (4.D.38) with

$$
\begin{aligned}
&\mathcal{T}_{N<0}(\tilde{k}^+,\tilde{k}^-,\ell^+,\ell^-;\tau,\omega^+,\omega^-) = \\
&\quad = (-i\tau)^{-\frac{1}{2}}(-1)^{2\ell^-}\sqrt{\frac{k}{2\kappa}}\,\mathbf{e}(\frac{\tilde{k}^+\omega_+^2+\tilde{k}^-\omega_-^2}{\tau})\sum_{r=N+1}^{0}\sum_{n=0}^{2k^+k^--1}\widetilde{q}^{\,-\frac{1}{2\kappa}(kr-b')^2}\,e^{\frac{2\pi i}{\kappa}kn(kr-b')} \\
&\times\Big\{(-1)^{2\lambda_n^{-\prime}+\mu_n^{+\prime}+\mu_n^{-\prime}}\,q^{-\Delta(\lambda_n^{+\prime},\lambda_n^{-\prime})}\,\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\lambda_n^{+\prime},\lambda_n^{-\prime}-\frac{1}{2}),\lambda_n^{+\prime},\lambda_n^{-\prime};\tau,\omega_+,\omega_-) \\
&\qquad +(-1)^{2\lambda_n^{-\prime\prime}+\mu_n^{+\prime\prime}+\mu_n^{-\prime\prime}}\,e^{-\frac{\pi i}{k}(2\ell^++1+2\ell^-)}\,e^{\frac{2\pi i}{\kappa}k^-(kr-b')}\,q^{-\Delta(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime})} \\
&\qquad\qquad \times\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime}-\frac{1}{2}),\lambda_n^{+\prime\prime},\lambda_n^{-\prime\prime};\tau,\omega_+,\omega_-)\Big\}.
\end{aligned} \tag{4.D.39}
$$

4.E Expansion of $\mathcal{A}_\gamma$ massless characters

The characters for unitary representations in the twisted Ramond sector of $\mathcal{A}_\gamma$ are labelled by ℓ_R^+, ℓ_R^- and h_R with

$$\frac{1}{2} \leq \ell_R^\pm \leq \frac{1}{2}k^\pm \quad \text{and} \quad kh_R = u^2 + (\ell_R^+ + \ell_R^- - \frac{1}{2})^2 + \frac{1}{4}k^+k^-. \tag{4.E.1}$$

We present here for reference the first few terms of the q-expansion of these characters, where the coefficients of the q-powers are expressed in terms of $SU(2)$ characters. For an irreducible representation of isospin ℓ, the $SU(2)$ character is given by

$$\chi_\ell(z) = \frac{z^{2\ell-1} - z^{-2\ell-1}}{z - z^{-1}} = z^{-2\ell} + z^{-2\ell+2} + \cdots + z^{+2\ell}. \tag{4.E.2}$$

Note that it is understood that $\chi_{-1/2} = 0$. Gukov et.al. [GMMS04] provided the leading (q^0) term. With $k^- \geq k^+ \geq 2$, and introducing the shortcut notation $\chi_\ell^\pm := \chi_\ell(z_\pm)$, one has

$$\begin{aligned}
&Ch_0^{A_\gamma,\tilde{R}}(k^+, k^-, h_R, \ell_R^+, \ell_R^-, u = 0; \tau, \omega_+, \omega_-) = (-1)^{2\ell_R^- - 1} q^{h - \tilde{c}/24} (\chi_{1/2}^+ - \chi_{1/2}^-) \\
&\quad \times \Big\{ (\chi^+_{\ell_R^+ - 1/2}\chi^-_{\ell_R^- - 1/2} - \chi^+_{\ell_R^+ - 1}\chi^-_{\ell_R^- - 1}) q^0 \\
&\qquad + (\chi_{1/2}^+ - \chi_{1/2}^-)^3 (\chi^+_{\ell_R^+ - 1/2}\chi^-_{\ell_R^- - 1/2} - \chi^+_{\ell_R^+ - 1}\chi^-_{\ell_R^- - 1}) q \\
&\qquad + (\chi_{1/2}^+ - \chi_{1/2}^-)^3 (\chi^+_{\ell_R^+ - 1/2}\chi^-_{\ell_R^- - 1/2} - \chi^+_{\ell_R^+ - 1}\chi^-_{\ell_R^- - 1}) [(\chi_1^+ + \chi_1^-) + 3] q^2 \\
&\qquad + (\chi_{1/2}^+ - \chi_{1/2}^-)^2 (\chi^+_{\ell_R^+ - 1/2}\chi^-_{\ell_R^- - 1/2} - \chi^+_{\ell_R^+ - 1}\chi^-_{\ell_R^- - 1}) \\
&\qquad\quad \times [(\chi_2^+ + \chi_2^-) + 3(\chi_1^+ + \chi_1^-) + \chi_1^+\chi_1^- - 4\chi_{1/2}^+\chi_{1/2}^- + 7] q^3 \\
&\qquad\quad + (\chi_{1/2}^+ - \chi_{1/2}^-)^2 (\chi^+_{\ell_R^+ - 1/2}\chi^-_{\ell_R^- - 1/2} - \chi^+_{\ell_R^+ - 1}\chi^-_{\ell_R^- - 1}) \\
&\qquad\quad \times [(\chi_3^+ + \chi_3^-) + 3(\chi_2^+ + \chi_2^-) + 12(\chi_1^+ + \chi_1^-) + 4\chi_1^+\chi_1^- + \chi_2^+\chi_1^- + \chi_1^+\chi_2^- \\
&\qquad\qquad - 16\chi_{1/2}^+\chi_{1/2}^- - 4\chi_{3/2}^+\chi_{1/2}^- - 4\chi_{1/2}^+\chi_{3/2}^- + 19] q^4 \\
&\qquad + (\chi_{1/2}^+ - \chi_{1/2}^-)^2 (\chi^+_{\ell_R^+ - 1/2}\chi^-_{\ell_R^- - 1/2} - \chi^+_{\ell_R^+ - 1}\chi^-_{\ell_R^- - 1}) \\
&\qquad\quad \times [(\chi_4^+ + \chi_4^-) + 3(\chi_3^+ + \chi_3^-) + 12(\chi_2^+ + \chi_2^-) + 39(\chi_1^+ + \chi_1^-) \\
&\qquad\qquad + 22\chi_1^+\chi_1^- + 4\chi_2^+\chi_1^- + 4\chi_1^+\chi_2^- + \chi_3^+\chi_1^- + \chi_1^+\chi_3^- + \chi_2^+\chi_2^- \\
&\qquad\qquad - 60\chi_{1/2}^+\chi_{1/2}^- - 20\chi_{3/2}^+\chi_{1/2}^- - 20\chi_{1/2}^+\chi_{3/2}^- - 4\chi_{5/2}^+\chi_{1/2}^-
\end{aligned} \tag{4.E.3}$$

$$
\begin{aligned}
&\quad - 4\chi^+_{1/2}\chi^-_{5/2} - 4\chi^+_{3/2}\chi^-_{3/2} + 47]q^5 \\
&+ (\chi^+_{1/2} - \chi^-_{1/2})^2(\chi^+_{\ell^+_R-1/2}\chi^-_{\ell^-_R-1/2} - \chi^+_{\ell^+_R-1}\chi^-_{\ell^-_R-1}) \\
&\quad \times [(\chi^+_5 + \chi^-_5) + 3(\chi^+_4 + \chi^-_4) + 12(\chi^+_3 + \chi^-_3) + 44(\chi^+_2 + \chi^-_2) \\
&\quad + 123(\chi^+_1 + \chi^-_1) + 93\chi^+_1\chi^-_1 + 23\chi^+_2\chi^-_1 + 23\chi^+_1\chi^-_2 + 4\chi^+_3\chi^-_1 + 4\chi^+_1\chi^-_3 \\
&\quad + \chi^+_4\chi^-_1 + \chi^+_1\chi^-_4 + \chi^+_4\chi^-_1 + \chi^+_1\chi^-_4 + \chi^+_2\chi^-_3 + \chi^+_3\chi^-_2 + 4\chi^+_2\chi^-_2 \\
&\quad - 188\chi^+_{1/2}\chi^-_{1/2} - 80\chi^+_{3/2}\chi^-_{1/2} - 80\chi^+_{1/2}\chi^-_{3/2} - 20\chi^+_{5/2}\chi^-_{1/2} - 20\chi^+_{1/2}\chi^-_{5/2} \\
&\quad - 4\chi^+_{7/2}\chi^-_{1/2} - 4\chi^+_{1/2}\chi^-_{7/2} - 24\chi^+_{3/2}\chi^-_{3/2} - 4\chi^+_{3/2}\chi^-_{5/2} - 4\chi^+_{5/2}\chi^-_{3/2} + 127]q^6 \\
&+(\chi^+_{1/2} - \chi^-_{1/2})^2(\chi^+_{\ell^+_R-1/2}\chi^-_{\ell^-_R-1/2} - \chi^+_{\ell^+_R-1}\chi^-_{\ell^-_R-1}) \\
&\quad \times [(\chi^+_6 + \chi^-_6) + 3(\chi^+_5 + \chi^-_5) + 12(\chi^+_4 + \chi^-_4) + 44(\chi^+_3 + \chi^-_3) \\
&\quad + 144(\chi^+_2 + \chi^-_2) + 362(\chi^+_1 + \chi^-_1) + 336\chi^+_1\chi^-_1 + 103\chi^+_2\chi^-_1 + 103\chi^+_1\chi^-_2 \\
&\quad + 23\chi^+_3\chi^-_1 + 23\chi^+_1\chi^-_3 + \chi^+_3\chi^-_3 + 24\chi^+_2\chi^-_2 + 4\chi^+_3\chi^-_2 + 4\chi^+_2\chi^-_3 + 4\chi^+_4\chi^-_1 \\
&\quad + 4\chi^+_1\chi^-_4 + \chi^+_4\chi^-_2 + \chi^+_2\chi^-_4 + \chi^+_5\chi^-_1 + \chi^+_5\chi^-_1 - 564\chi^+_{1/2}\chi^-_{1/2} \\
&\quad - 284\chi^+_{3/2}\chi^-_{1/2} - 284\chi^+_{1/2}\chi^-_{3/2} - 84\chi^+_{5/2}\chi^-_{1/2} - 84\chi^+_{1/2}\chi^-_{5/2} - 20\chi^+_{7/2}\chi^-_{1/2} \\
&\quad - 20\chi^+_{1/2}\chi^-_{7/2} - 4\chi^+_{9/2}\chi^-_{1/2} - 4\chi^+_{1/2}\chi^-_{9/2} - 108\chi^+_{3/2}\chi^-_{3/2} - 24\chi^+_{3/2}\chi^-_{5/2} \\
&\quad - 24\chi^+_{5/2}\chi^-_{3/2} - 4\chi^+_{7/2}\chi^-_{3/2} - 4\chi^+_{3/2}\chi^-_{7/2} - 4\chi^+_{5/2}\chi^-_{5/2} + 323]q^7\Big\} + o(q^8).
\end{aligned}
\tag{4.E.4}
$$

Note that the above q-coefficients have constraints $k^- \geq 2\ell^-_R + k^+ - 2\ell^+_R$.

Chapter 5

Sum rules for $\widetilde{\mathcal{A}}_\gamma$ characters

In this chapter, we focus on a realization of $\widetilde{\mathcal{A}}_\gamma$ at levels $(\tilde{k}^+, \tilde{k}^-) = (2,1)$ on a manifold based on the group coset $SU(3)/U(1)$. As the simplest example of realizations at levels $(\tilde{k}^+, \tilde{k}^-) = (n,1), n \in \mathbb{Z}_{>1}, \gcd(\tilde{k}^+ + 1, \tilde{k}^- + 1) = 1$, it provides a toy model for superstrings propagating on group cosets $SU(\tilde{k}^- + 2)/SU(\tilde{k}^-)$ where $\tilde{k}^+$ enters as the level of the affine $\widehat{su}(\tilde{k}^- + 2)$ algebra emerging from the associated supersymmetric Wess-Zumino-Novikov-Witten model. The motivation is to investigate in the simplest possible but not trivial setting whether a moonshine-like phenomenon is present in such theories. Although we cannot conclude at this stage whether or not such a phenomenon occurs, this chapter should be read as a preparation for a future in-depth investigation of this question. Our guiding principle is that in this set-up, one can write explicit modular invariant partition functions based on products of rational $\widehat{su}(\tilde{k}^- + 2)_{\tilde{k}^+}$ characters and characters of free fermions, which can be branched into products of $\widetilde{\mathcal{A}}_\gamma$ characters and $\widehat{u(1)}$ torus characters, hence the name 'sum rules for $\widetilde{\mathcal{A}}_\gamma$ characters'. One may write general formulas for such sum rules, and this was done for $\tilde{k}^- = 1$ and arbitrary $\tilde{k}^+$ in [PT93]. However, analytic expressions for the branching functions, which we generically label $\widehat{F}_i(\tau), i \in \mathcal{I}$ with $\mathcal{I}$ a discrete set of labels, remained elusive then, not the least because it was not fully appreciated in 1993 that these branching functions may be viewed, for each theory considered, as the components of a theory-dependent vector-valued mock modular form. For more information on mock modular forms, see [DMZ14; Zwe08]. A better grasp on these branching functions is welcome, as in analogy with the Mathieu Moonshine observation, one might expect that, once one calculates the new index $\mathfrak{I}(\mathcal{C})$ (4.5.8) for $\mathcal{C}$ being the superconformal field theories we are interested in, the branching functions organise themselves in such a way that interesting patterns emerge. Of course, when considering Gepner models at central charge $c = 6$ in the Mathieu Moonshine case, the relevant index is the conformal field-theoretic elliptic genus, whose simplicity relies on the Witten indices of the small $\mathcal{N} = 4$ characters being integers. In that case, the dimensions of representations of M_{24} (3.2) appearing in a specific decomposition of the elliptic genus into characters of the small $\mathcal{N} = 4$ algebra are the coefficients of a mock modular form (called $h^{(2)}(\tau)$ in [DMZ14], formula (7.16)) obtained as a linear combination of the components $\widehat{F}_i(\tau)$ of the vector-valued mock modular form. In the context of theories enjoying $\mathcal{A}_\gamma$ symmetry, the situation is more challenging, first of all because the conformal field-theoretic elliptic genus vanishes, and second of all because the only other type of indices that could be exploited are those constructed in [GMMS04]. As pointed out in (4.5.11b), one of the crucial ingredient in the evaluation of the index $\mathfrak{I}(\mathcal{C})$ is the generalized Witten index for $\mathcal{A}_\gamma$ characters, which, for massless representations, is essentially a Jacobi theta function, so definitely not a pure number. Identifying patterns will therefore be more involved but nevertheless interesting.

5.1 Coset realizations of $\mathcal{A}_\gamma$

In [SSTV88a; SSTV88b], the authors constructed the supersymmetric extension of the Wess-Zumino-Novikov-Witten action as

$$\begin{aligned} S[\phi,\psi] = -\frac{1}{2\pi}\int \mathrm{d}^2x\ \Big(& g_{\mu\nu}(\phi)\partial_a\phi^\mu\partial^a\phi^\nu + \lambda_{\mu\nu}(\phi)\epsilon^{ab}\partial_a\phi^\mu\partial_b\phi^\nu \\ &+ g_{\mu\nu}\bar{\psi}^\mu \not{D}\psi^\nu - \frac{1}{4}\bar{\psi}^\mu_+\gamma_a\psi^\nu_+\bar{\psi}^\rho_-\gamma^a\psi^\sigma_- R_{\mu\nu\rho\sigma}(\Gamma_+)\Big),\ a,b\in\{0,1\}, \end{aligned} \tag{5.1.1}$$

where $\phi^\mu, \mu \in \{1,\dots d\}$ are the coordinates on the target manifold M, their fermionic superpartners ψ^μ are tangent vector fields on M and their covariant derivative is calculated as

$$D_a\psi^\mu = \partial_a\psi^\mu + (\Gamma_+{}^\mu{}_{\nu\rho}\psi_+^\nu + \Gamma_-{}^\mu{}_{\nu\rho}\psi_-^\nu)\partial_a\phi^\rho, \tag{5.1.2}$$

with a connection including a totally antisymmetric torsion term $\mathcal{T}$, namely

$$\Gamma_\pm{}^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \pm 2\mathcal{T}_{\sigma\nu\rho}), \tag{5.1.3}$$

where locally, $\mathcal{T}_{\sigma\nu\rho} - -\frac{3}{2}\lambda_{[\sigma\nu,\rho]}$. This action is invariant under the supersymmetry defined by

$$\begin{aligned} \delta\phi^\mu &= \bar{\epsilon}\,\mathfrak{J}_0{}^\mu{}_\nu\psi^\nu, \quad \mathfrak{J}_0{}^\mu{}_\nu := \delta^\mu_\nu, \\ \delta\psi^\mu &= \partial\!\!\!/\phi^\mu\epsilon - \psi_+^\nu(\bar{\epsilon}_+\Gamma_+{}^\mu{}_{\nu\rho}\psi_-^\rho + \bar{\epsilon}_-S_+{}^\mu{}_{\nu\rho}\psi_+^\rho) + \psi_-^\nu(\bar{\epsilon}_-\Gamma_-{}^\mu{}_{\nu\rho}\psi_+^\rho + \bar{\epsilon}_+S_-{}^\mu{}_{\nu\rho}\psi_-^\rho), \end{aligned} \tag{5.1.4}$$

where the rank 3 tensor $(S_\pm)_{\mu\nu\rho}$ must be antisymmetric and covariantly constant. Sevrin et al show that solutions exist on group manifolds, and that, in that case, S can be chosen as proportional to the torsion $\mathcal{T}$ [SSTV88a]. The action (5.1.1) thus enjoys $\mathcal{N} = 1$ superconformal symmetry. In order to have $\mathcal{N} = N$ supersymmetries, one must be able to define $N-1$ almost complex structures $\mathfrak{J}_i, i \in \{1, \dots N-1\}$, which, together with $\mathfrak{J}_0$ (see (5.1.4)), form a Clifford algebra, i.e.

$$\mathfrak{J}_i{}^\mu{}_\nu\mathfrak{J}_j{}^\nu{}_\rho + \mathfrak{J}_j{}^\mu{}_\nu\mathfrak{J}_i{}^\nu{}_\rho = -2\delta^\mu_\rho\delta_{ij} \tag{5.1.5}$$

and must satisfy the Nijenhuis conditions

$$\mathfrak{N}_{ij}{}^\mu{}_{\nu\rho} := \mathfrak{J}_{(i}{}^\sigma{}_{[\nu}\mathfrak{J}_{j)}{}^\mu{}_{\rho],\sigma} + \mathfrak{J}_{(i}{}^\mu{}_{\sigma,[\rho}\mathfrak{J}_{j)}{}^\sigma{}_{\nu]} = 0, \quad i,j \in \{0,1,\dots,N-1\}, \tag{5.1.6}$$

where (i,j) means symmetrization in the indices i and j, while $[\mu,\nu]$ means antisymmetrization in the indices μ and ν. We restrict ourselves to $N = 4$ here, although [SSTV88a] discuss cases where $N > 4$, which is possible when the target manifolds have no curvature but totally antisymmetric torsion, i.e. when these manifolds are absolutely parallelizable. According to these authors, realizations of $\mathcal{A}_\gamma$ may be constructed as follows. One starts with the Lie group $G := SU(\tilde{k}^-{+}2)$ and considers the coset $G/H := SU(\tilde{k}^-{+}2)/SU(\tilde{k}^-)$

so that

$$WS(\tilde{k}^- + 2) := \frac{SU(\tilde{k}^- + 2)}{SU(\tilde{k}^-) \times SU(2) \times U(1)} \tag{5.1.7}$$

is a Wolf space (i.e. a quaternionic symmetric space) of dimension $\dim WS(\tilde{k}^- + 2) = 4(\tilde{g} - 2)$, where $\tilde{g}$ is the dual Coxeter number of $SU(\tilde{k}^- + 2)$. For $\tilde{k}^- > 1$, one obtains a realization of $\mathcal{A}_\gamma$ for each so-called 'stage' in the decomposition of the root system Δ of the Lie algebra $su(\tilde{k}^- + 2)$ into Δ_θ (the set of roots comprising the highest root θ of $su(\tilde{k}^- + 2)$, its negative and two zero roots corresponding to elements of the Cartan subalgebra into which the generators $E_{\pm\theta}$ transform under the complex structure $\mathfrak{J}_1$, as well as all the roots non perpendicular to θ) and $\Delta_{\perp\theta}$ (all the roots perpendicular to θ). A second stage would be provided by

$$WS(\tilde{k}^-) := \frac{SU(\tilde{k}^-)}{SU(\tilde{k}^- - 2) \times SU(2) \times U(1)} \tag{5.1.8}$$

with $\Delta_{\perp\theta}$ decomposed into a set $\Delta_{\theta'}$ of roots (comprising $\pm\theta'$ where θ' is the highest root in $\Delta_{\perp\theta}$, two zero roots corresponding to Cartan generators into which $E_{\pm\theta'}$ transform under the complex structure and all the roots non perpendicular to θ'), as well as a set $\Delta_{\perp\theta'}$ of roots perpendicular to θ', and so on for subsequent stages. We are not interested here in $\tilde{k}^- > 1$, as in this case, one would obtain an energy-momentum tensor at each stage (since $\mathcal{A}_\gamma$ is realized at each stage). This motivates our analysis of superstrings propagating on $SU(3)$, as there is only one stage, and hence one realization of $\mathcal{A}_\gamma$.

In the case of $SU(3)$, the corresponding Lie algebra is $\mathfrak{su}(3)$ of rank 2. If we label the simple roots as α_1 and α_2, the highest root is $\theta = \alpha_1 + \alpha_2$ and $\Delta_{\perp\theta} = \emptyset$. Let us call $E_{\pm\alpha}, \alpha \in \{\alpha_1, \alpha_2, \theta\}$ the generators associated with the nonzero roots, and H_1, H_2 the Cartan generators. $SU(3)$ is an absolutely parallelizable manifold on which one can define an almost quaternionic structure as follows,

$$\begin{aligned}
&\mathfrak{J}_1 H_1 = H_1, \quad \mathfrak{J}_1 H_2 = -H_2, \\
&\mathfrak{J}_1 E_\alpha = i E_\alpha, \quad \mathfrak{J}_1 E_{-\alpha} = -i E_{-\alpha}, \\
&\mathfrak{J}_2 E_{\pm\alpha} = \frac{1}{4}(1 + i\mathfrak{J}_1)\,[E_{-\theta}, E_{\pm\alpha}] - \frac{1}{4}(1 - i\mathfrak{J}_1)\,[E_\theta, E_{\pm\alpha}], \\
&\mathfrak{J}_3 = \mathfrak{J}_2 \mathfrak{J}_1.
\end{aligned} \tag{5.1.9}$$

The generators $\{E_{\pm\theta}, H_1, H_2\}$ satisy an $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ algebra and the corresponding group $SU(2)\times U(1)$ is such that

$$WS(3) = \frac{SU(3)}{SU(2)\times U(1)} \tag{5.1.10}$$

is the 4-dimensional Wolf space associated with the $\mathcal{A}_\gamma$ realization. If we label the bosonic fields $\phi^A, A \in \{m, \theta, \alpha_1, \alpha_2\}$ and their complex conjugates ϕ_A, their superpartners are ψ^A and ψ_A. Among these, the four fermions $\{\psi^{\alpha_1}, \psi_{\alpha_1}, \psi^{\alpha_2}, \psi_{\alpha_2}\}$ are associated with the nonzero roots which are not perpendicular to the highest root θ and they are called 'Wolf space fermions' (see Fig. 5.1). The corresponding $\mathcal{A}_\gamma$ realization for $\tilde{k}^- = 1$ (i.e. $k^- = 2$)

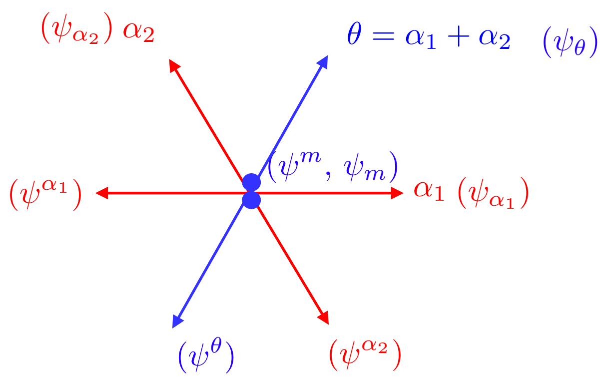

Figure 5.1: $\mathfrak{su}(3)$ *root system and Wolf space fermions*

and $\tilde{k}^+ = n$ (i.e $k^+ = n+1$) is given in terms of the eight bosonic and eight fermionic fields described above, where the integer n appears in the bosonic OPE

$$V^A(z)V_B(w) \sim -\frac{4n}{n+3}\delta^A_B + F^A_{BC}V^C(w) + F^{A\ C}_{B}V_C(w), \tag{5.1.11}$$

with F^A_{BC} are complex structure constants for $\mathfrak{su}(3)$. This realization is provided in [GPTV89], Appendix B.

In view of (4.1.4), it is straightforward to obtain a realization of $\tilde{\mathcal{A}}_\gamma$ once we have a realization of $\mathcal{A}_\gamma$. In the following section, we consider character sum rules associated with a realization of $\mathcal{A}_\gamma$ at levels $\tilde{k}^+ = 2, \tilde{k}^- = 1$.

5.2 Analytic Structure of Sum Rules for $\widetilde{\mathcal{A}}_\gamma$ Characters

In this section, we review the analytic structure of the sum rules for $\widetilde{\mathcal{A}}_\gamma$ characters in the twisted Ramond sector and study the example of $\tilde{k}^+ = 2$ and $\tilde{k}^- = 1$.

For a given value of $\tilde{k}^+$, the $\widetilde{\mathcal{A}}_\gamma$ realization is based on Hilbert spaces

$$\mathcal{H}^{WS} \otimes \mathcal{H}_\Lambda^{\widehat{SU(3)}_{\tilde{k}^+}}, \tag{5.2.1}$$

where $\mathcal{H}^{WS}$ is the Fock space for the four free fermions associated with the Wolf space $WS(3)$ of (5.1.10) and $\mathcal{H}_\Lambda^{\widehat{SU(3)}_{\tilde{k}^+}}$ is the representation space for the affine Lie algebra $\widehat{SU(3)}_{\tilde{k}^+}$ with highest weight $\Lambda = a_1\lambda_1 + a_2\lambda_2$, where $\lambda_i, i = 1, 2$ are the fundamental weights and (a_1, a_2) are the Dynkin labels, i.e. a_i are non-negative integers such that $a_1 + a_2 \leq 2$. The Hilbert spaces (5.2.1) provide more than the representations of $\widetilde{\mathcal{A}}_\gamma$; they also provide representations for the rational torus algebra $\mathcal{A}_{3k}$, which extends the $\widehat{u(1)}$ Lie subalgebra of $\mathcal{A}_\gamma$ by an operator of dimension $3k$ (see [PT93]). Here, $3k = 3(\tilde{k}^+ + 3)$.

In the twisted Ramond sector, the above information can be schematically encoded as

$$\chi^{WS,\widetilde{R}} \otimes \chi_\Lambda^{\widehat{SU(3)}_{\tilde{k}^+}} = \left\{ \oplus_i \left(\mathcal{H}^{\widetilde{\mathcal{A}}_\gamma, \widetilde{R}}_{0,\ell_i^+,\ell_i^-} \otimes \mathcal{H}^{\mathcal{A}_{3k}}_{m_i} \right) \right\} \oplus \left\{ \oplus_j \oplus_n \left(\mathcal{H}^{\widetilde{\mathcal{A}}_\gamma, \widetilde{R}}_{h_n,\ell_j^+,\ell_j^-=0} \otimes \mathcal{H}^{\mathcal{A}_{3k}}_{m_j} \right) \right\}, \tag{5.2.2}$$

where the first bracket on the right collects the contributions from massless representations of $\widetilde{\mathcal{A}}_\gamma$ while the second bracket collects the massive contributions. The branching functions we wish to pin down appear in this second bracket.

5.2.1 Building blocks

In the case $\tilde{k}^+ = 2, \tilde{k}^- = 1$ which is our focus here, the sum rules are built from

1. The $\widehat{su(3)}_2$ affine characters.

   For $\hat{g}$ an affine algebra of rank r and level ℓ with g the associated Lie algebra, let us define

$$\widehat{\chi}_\Lambda^{(\ell)}(\tau, \omega) := \mathrm{Tr}_\Lambda q^{L_0 - \frac{c}{24}} \, e^{-2\pi i \sum_{i=1}^r \omega_i h^i} = q^{-\frac{c}{24}} \, \chi_\Lambda^{(\ell)}(\tau, z) \tag{5.2.3}$$

as the character of the unitary, irreducible representation of $\hat{g}$ built on the highest weight state Λ, where $h_i, i \in \{1,\dots,r\}$ are the Cartan generators of g, $\omega := (\omega_1, \omega_2, .., \omega_r) \in \mathbb{C}^r$ and the central charge is $c = \frac{\ell \dim g}{\ell + h}$, h being the dual Coxeter number of g.

Here we are interested in $\hat{g} = \widehat{su(3)}_2$, the affine Lie algebra of rank 2 and level $\ell = 2$ based on the Lie algebra $su(3)$. The $\widehat{su(3)}_2$ affine characters are labelled by the Dynkin labels (a_1, a_2) of the 6 dominant highest weights of $su(3)$ and we write them as

$$\widehat{\chi}^{(2)}_{(a_1,a_2)}(\tau, \omega_1, \omega_2), \quad a_i \in \mathbb{N},\ a_1 + a_2 \leq 2. \tag{5.2.4}$$

It is helpful to organize these characters according to *triality*. Consider $t \coloneqq a_1 - a_2 \pmod 3$ with $t \in \{-1, 0, 1\}$, and define three doublets of $\widehat{su(3)}_2$ affine characters as

$$\vec{\chi}^{(2)}_{-1} := \begin{pmatrix} \widehat{\chi}^{(2)}_{(2,0)} \\ \\ \widehat{\chi}^{(2)}_{(0,1)} \end{pmatrix}, \quad \vec{\chi}^{(2)}_{0} := \begin{pmatrix} \widehat{\chi}^{(2)}_{(0,0)} \\ \\ \widehat{\chi}^{(2)}_{(1,1)} \end{pmatrix}, \quad \vec{\chi}^{(2)}_{1} := \begin{pmatrix} \widehat{\chi}^{(2)}_{(0,2)} \\ \\ \widehat{\chi}^{(2)}_{(1,0)} \end{pmatrix}. \tag{5.2.5}$$

Note that

$$L_0|\Lambda\rangle \coloneqq h(a_1,a_2)|\Lambda\rangle = \frac{1}{3(\ell+3)}\{a_1^2 + a_2^2 + a_1 a_2 + 3(a_1 + a_2)\}|\Lambda\rangle, \tag{5.2.6}$$

so that the factor $q^{h(a_1,a_2)-c/24}$ in the $\widehat{su(3)}_\ell$ affine characters defined in (5.2.3) for $r = 2$ has a fractional q-power clearly depending on the conformal weight of the highest weight state considered. We call these fractional powers 'q-offsets'. In our case, the q-offsets for the six irreducible highest weight $\widehat{su(3)}_2$ characters are given in Table 5.1. In the sum rules we are considering, the $\widehat{su(3)}_2$ characters are expressed in terms of two angular variables ω_+ and ω_y, which are related to ω_1 and ω_2 through

$$\omega_1 = \omega_+ + \omega_y, \qquad \omega_2 = \omega_+ - \omega_y. \tag{5.2.7}$$

(a_1,a_2)	$h(a_1,a_2)$	$-c/24$	total q-offset
$(2,0)$	$2/3$	$-2/15$	$8/15$
$(0,1)$	$4/15$	$-2/15$	$2/15$
$(0,0)$	0	$-2/15$	$-2/15$
$(1,1)$	$9/15$	$-2/15$	$7/15$
$(0,2)$	$2/3$	$-2/15$	$8/15$
$(1,0)$	$4/15$	$-2/15$	$2/15$

Table 5.1: *Fractional powers of q in level 2 $su(3)$ characters*

It is known (see [PT93]) that, with the order 3 transformation

$$\phi(a_1,a_2) = (a_2, \ell - a_1 - a_2), \tag{5.2.8}$$

the $\widehat{su(3)}_\ell$ characters enjoy the flow property

$$\widehat{\chi}^{(\ell)}_{(a_1,a_2)}(\tau,\omega_+,\omega_y+\tau) = q^{-\ell/3} z_y^{-2\ell/3}\, \widehat{\chi}^{(\ell)}_{\phi(a_1,a_2)}(\tau,\omega_+,\omega_y). \tag{5.2.9}$$

2. The Wolf-space fermions characters

$$\chi^{WS,\widetilde{R}}(\tau,\omega_-,\omega_y) = \frac{\vartheta_1(\tau,\omega_-+\omega_y)\,\vartheta_1(\tau,\omega_--\omega_y)}{\eta^2(\tau)} \tag{5.2.10}$$

with q-offset $q^{1/6}$.

3. The rational torus characters

$$\chi^{3k}_m(\tau,\omega_y) = \frac{1}{\eta(\tau)}\,\theta_{m,3k}(\tau,\frac{2}{3}\omega_y) \tag{5.2.11}$$

with q-offset $q^{-1/24+m^2/12k}$. Note that $\chi^{3k}_{-m}(\tau,\omega_y) = \chi^{3k}_m(\tau,-\omega_y) = \chi^{3k}_{6k-m}(\tau,\omega_y)$. These characters may also be organized according to triality. Define

$$\chi^{3k,\,t}_m(\tau,\omega_y) \coloneqq \chi^{3k}_{3m+t}(\tau,\omega_y), \quad t\in\{-1,0,1\}, \tag{5.2.12}$$

and, in the case $k=5$,

$$\sigma^{15,\,t}_m(\tau,\omega_y) \coloneqq \begin{cases} \chi^{15,\,t}_{m+3t}(\tau,\omega_y) + \chi^{15,\,t}_{-m+3t}(\tau,\omega_y), & m\in\{1,2,3,4\} \\ \chi^{15,\,t}_{m+3t}(\tau,\omega_y), & m\in\{0,5\} \end{cases} \tag{5.2.13}$$

Note that

$$\sigma_m^{15,\,t}(\tau,\omega_y) = \sigma_{10-m}^{15,\,t}(\tau,\omega_y). \tag{5.2.14}$$

4. The massless $\widetilde{\mathcal{A}}_\gamma$ characters for unitary and irreducible representations in the $\widetilde{R}$ sector (4.4.10),

$$\begin{aligned} &Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;\tau,\omega_+,\omega_-) = \\ &\qquad (-1)^{2\ell^-+1} q^{\widetilde{h_R^0}(\ell^+,\ell^-)-\tilde{c}/24+1/8} \frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)\vartheta_1(\tau,2\omega_+)\vartheta_1(\tau,2\omega_-)} \\ &\qquad\qquad \sum_{\epsilon_+,\epsilon_-=\pm1} \epsilon_+\epsilon_- S^{\widetilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-) \end{aligned} \tag{5.2.15}$$

where $\tilde{k}^+ = k^+ - 1$ and $\tilde{k}^- = k^- - 1$ are the levels of the two $\widehat{su(2)}$ subalgebras of $\widetilde{\mathcal{A}}_\gamma$ and the central charge is given by

$$\tilde{c} = c - 3, \qquad c = \frac{6k^+k^-}{k}, \qquad k := k^+ + k^-. \tag{5.2.16}$$

Here $q = e^{2i\pi\tau}$, $\tau \in \mathfrak{H}$ and $z_\pm = e^{2i\pi\omega_\pm}$, $\omega_\pm \in \mathbb{C}$. The conformal weight $\widetilde{h_R^0}$ of the massless state $|\Omega_+\rangle$ on which we choose to build the representation is given by

$$\widetilde{h_R^0}(\ell^+,\ell^-) = \frac{1}{k}(\ell^+ + \ell^- + 1)(\ell^+ + \ell^-) + \frac{\tilde{k}^+\tilde{k}^-}{4k}. \tag{5.2.17}$$

The infinite sum $S^{\tilde{R}}_{\epsilon_+,\epsilon_-}$ is given by

$$S^{\widetilde{R}}_{\epsilon_+,\epsilon_-}(\tau,\omega_+,\omega_-) = \sum_{m,n=-\infty}^{\infty} q^{m^2k^+ + n^2k^- + 2\ell^+m + 2\ell^-n} \frac{z_+^{2\epsilon_+(\ell^++mk^+)}\, z_-^{2\epsilon_-(\ell^-+nk^-)}}{z_+^{-\epsilon_+}q^{-m} - z_-^{-\epsilon_-}q^{-n}}. \tag{5.2.18}$$

The q-offset is $q^{\widetilde{h_R^0}(\ell^+,\ell^-)-\tilde{c}/24}$. When $\tilde{k}^+ = 2$ and $\tilde{k}^- = 1$, there are six massless $\widetilde{\mathcal{A}}_\gamma$ characters labelled by $(\ell^+,\ell^-) = (0,0), (\frac{1}{2},0), (1,0), (0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2}), (1,\frac{1}{2})$. Their q-offsets are respectively $q^{-3/40}, q^{3/40}, q^{13/40}, q^{3/40}, q^{13/40}, q^{27/40}$.

5. The massive $\widetilde{\mathcal{A}}_\gamma$ characters **at threshold** (4.4.2),

$$\begin{aligned} &\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h_R^0}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-) = (-1)^{2\ell^{-\prime}} q^{-\frac{1}{k}\left(\sqrt{\frac{k^-}{k^+}}\ell^{+\prime} - \sqrt{\frac{k^+}{k^-}}\ell^{-\prime}\right)^2} \\ &\qquad \times \frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)} \chi^{k^+-2}_{\ell^{+\prime}-\frac{1}{2}}(\tau;w_+)\chi^{k^--2}_{\ell^{-\prime}-\frac{1}{2}}(\tau;w_-). \end{aligned} \tag{5.2.19}$$

Note that the q-prefactor in (5.2.19) is obtained as[1]

$$q^{\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2})-\tilde{c}/24+1/8-\frac{(\ell^{+\prime})^2}{k^+}-\frac{(\ell^{-\prime})^2}{k^-}} = q^{-\frac{1}{k}(\sqrt{\frac{k^-}{k^+}}\ell^{+\prime}-\sqrt{\frac{k^+}{k^-}}\ell^{-\prime})^2}. \tag{5.2.20}$$

In view of the derivation of the S-transformation of the sum rules, it is natural to define the functions

$$\begin{aligned}
&\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-) = \\
&\quad q^{\frac{1}{k}(\sqrt{\frac{k^-}{k^+}}\ell^{+\prime}-\sqrt{\frac{k^+}{k^-}}\ell^{-\prime})^2}\,\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\tilde{k}^+,\tilde{k}^-,\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\ell^{+\prime},\ell^{-\prime};\tau,\omega_+,\omega_-) \\
&\qquad = (-1)^{2\ell^{-\prime}}\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)}\chi^{k^+-2}_{\ell^{+\prime}-\frac{1}{2}}(\tau;w_+)\chi^{k^--2}_{\ell^{-\prime}-\frac{1}{2}}(\tau;w_-),
\end{aligned} \tag{5.2.21}$$

as these transform covariantly under the modular group $SL(2,\mathbb{Z})$ with weight $-1/2$.

For $\tilde{k}^+=2$ and $\tilde{k}^-=1$, there are two massive $\widetilde{\mathcal{A}}_\gamma$ characters 'at threshold' labelled by $(\ell^{+\prime},\ell^{-\prime})=(\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2})$. They share the same prefactor (5.2.20) of $q^{-1/120}$ (while their total q-offsets are $q^{3/40}$ and $q^{13/40}$ respectively). This is completely consistent with the linear relations presented in (4.5.15)[2]

$$\begin{aligned}
&\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h^0_R}(\frac{1}{2},0)=\frac{1}{4},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-) = \\
&Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\widetilde{h^0_R}(0,\frac{1}{2})=\frac{1}{4},0,\frac{1}{2};\tau,\omega_+,\omega_-)+Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\widetilde{h^0_R}(\frac{1}{2},0)=\frac{1}{4},\frac{1}{2},0;\tau,\omega_+,\omega_-) \\
&\qquad = q^{-1/120}\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)}\chi^1_0(\tau;w_+) \\
&\widehat{Ch}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h^0_R}(1,0)=\frac{1}{2},1,\frac{1}{2};\tau,\omega_+,\omega_-) = \\
&Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\widetilde{h^0_R}(\frac{1}{2},\frac{1}{2})=\frac{1}{2},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-)+Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\widetilde{h^0_R}(1,0)=\frac{1}{2},1,0;\tau,\omega_+,\omega_-) \\
&\qquad = q^{-1/120}\frac{\vartheta_1(\tau,\omega_++\omega_-)\vartheta_1(\tau,\omega_+-\omega_-)}{\eta^3(\tau)}\chi^1_{\frac{1}{2}}(\tau;w_+)
\end{aligned} \tag{5.2.22}$$

6. The branching $\widehat{F}_i(\tau), i\in\{1,\dots,6\}$ functions, each emerging from the sum rules as a power series in the variable $q=e^{2\pi i\tau}$ (typically multiplied by an overall fractional

[1] This is different from the **total** q-offset for the massive $\widetilde{\mathcal{A}}_\gamma$ characters at threshold in the twisted Ramond sector, which is $q^{\widetilde{h^0_R}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2})-\tilde{c}/24}=q^{-\frac{1}{8}+\frac{1}{k}(\ell^{+\prime}+\ell^{-\prime})^2}$.

[2] We have suppressed the labels $\tilde{k}^+=2,\tilde{k}^-=1$ in the character functions to avoid crowded formulas.

q-power). By expanding all other sum rules building blocks in powers of q, one obtains[3] information on the branching functions in the form of q-series,

$$\begin{aligned}
\widehat{F}_1(\tau) &= q^{-\frac{1}{120}}(1+q+q^2+q^4+q^5+2q^6+q^7+q^8+q^9+3q^{10}+\ldots \\
\widehat{F}_2(\tau) &= q^{\frac{47}{120}}(1+q^2+q^3+q^4+2q^6+q^7+2q^8+2q^9+2q^{10}+\ldots \\
\widehat{F}_3(\tau) &= q^{-\frac{49}{120}}(q^2+q^4+q^5+q^6+q^7+2q^8+q^9+2q^{10}+\ldots \\
\widehat{F}_4(\tau) &= q^{\frac{23}{120}}(1+q+q^2+q^3+2q^4+q^5+2q^6+2q^7+3q^8+3q^9+3q^{10}+\ldots \\
\widehat{F}_5(\tau) &= q^{-\frac{1}{120}}(q+q^2+q^3+q^4+2q^5+2q^6+2q^7+2q^8+3q^9+3q^{10}+\ldots \\
\widehat{F}_6(\tau) &= q^{-\frac{49}{120}}(q+2q^3+q^4+q^5+2q^6+2q^7+2q^8+3q^9+3q^{10}+\ldots
\end{aligned} \quad (5.2.23)$$

Our aim here is to identify analytic expressions for these functions.

5.2.2 The character sum rules

Exploiting triality ((5.2.5) and (5.2.13)), the six sum rules take the form[4]

$$\begin{aligned}
&\vec{\mathcal{S}}_t :\, \chi^{WS,\widetilde{R}}(\tau,\omega_-,\omega_y)\,\vec{\chi}_t^{(2)}(\tau,\omega_+,\omega_y) = \\
&Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\frac{1}{10},0,0;\tau,\omega_+,\omega_-)\begin{pmatrix}\sigma_1^{15,t}(\tau,\omega_y)\\ \sigma_5^{15,t}(\tau,\omega_y)\end{pmatrix} + Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\frac{1}{4},\frac{1}{2},0;\tau,\omega_+,\omega_-)\begin{pmatrix}-\sigma_0^{15,t}(\tau,\omega_y)\\ \sigma_2^{15,t}(\tau,\omega_y)\end{pmatrix} \\
&+Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\frac{1}{2},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-)\begin{pmatrix}-\sigma_5^{15,t}(\tau,\omega_y)\\ \sigma_3^{15,t}(\tau,\omega_y)\end{pmatrix} + Ch_0^{\widetilde{\mathcal{A}}_\gamma,\widetilde{R}}(\frac{17}{20},0,0;\tau,\omega_+,\omega_-)\begin{pmatrix}\sigma_4^{15,t}(\tau,\omega_y)\\ \sigma_0^{15,t}(\tau,\omega_y)\end{pmatrix} \\
&\qquad + \begin{pmatrix}\widehat{F}_1(\tau) & \widehat{F}_2(\tau) & \widehat{F}_3(\tau)\\ \widehat{F}_6(\tau) & \widehat{F}_5(\tau) & \widehat{F}_4(\tau)\end{pmatrix}\left\{\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\frac{1}{4},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-)\begin{pmatrix}\sigma_0^{15,t}(\tau,\omega_y)\\ \sigma_2^{15,t}(\tau,\omega_y)\\ \sigma_4^{15,t}(\tau,\omega_y)\end{pmatrix}\right.
\end{aligned}$$

[3]The sum rules will be written in terms of the $\widehat{\widehat{Ch}}$ functions, which are the characters at threshold ‘stripped’ of the $q^{-1/120}$ factor. This factor is now part of the definition of $\widehat{F}_i := q^{-1/120}F_i,\, i \in \{2,\ldots,6\}$ and $\widehat{F}_1 := q^{-1/120}(1+F_1)$.

[4]In the $\widetilde{\mathcal{A}}_\gamma$ character functions, the labels $\tilde{k}^+ = 2, \tilde{k}^- = 1$ have been suppressed for readibility, and the first three arguments are $(\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-)$ for massless characters, and $(\widetilde{h_R^0}(\ell^{+\prime},\ell^{-\prime}-\frac{1}{2}),\ell^{+\prime},\ell^{-\prime})$ for massive characters.

$$+\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\frac{1}{2},1,\frac{1}{2};\tau,\omega_+,\omega_-)\begin{pmatrix}\sigma_5^{15,t}(\tau,\omega_y)\\ \sigma_3^{15,t}(\tau,\omega_y)\\ \sigma_1^{15,t}(\tau,\omega_y)\end{pmatrix}\Bigg\}, \quad (5.2.24)$$

with $t \in \{-1,0,1\}$.

Under the flow

$$\omega_\pm \to \omega_\pm, \quad \omega_y \to \omega_y + \tau, \quad (5.2.25)$$

the sum rules generate two orbits

$$\vec{\mathcal{S}}_{t=-1} \longrightarrow \vec{\mathcal{S}}_{t=0} \longrightarrow \vec{\mathcal{S}}_{t=1}, \quad (5.2.26)$$

as can be checked using the analytic expressions for the $\widetilde{\mathcal{A}}_\gamma$ characters and the torus characters as well as the property (5.2.9). From (5.2.26), one sees that one orbit consists of the three sum rules associated with the sextet, singlet, and anti-sextet representations of $\widehat{su(3)}_2$, and therefore contains a representative of each triality class, while the other orbit consists of the sum rules associated with the anti-triplet, octet and triplet, which are also representatives of each triality class.

Another interesting property of the sum rules is that $\vec{\chi}_t^{(2)}(\tau,\omega_+,\omega_y)$ and $\sigma_m^{15,t}(\tau,\omega_y)$ behave analogously under the S-transformation, as we now show. We choose to perform calculations in the splitting field of the polynomial $P(x) = x^4 - 5x^2 + 5 \in \mathbb{Q}[x]$, whose positive roots are

$$a := \sqrt{\frac{5}{2} - \frac{\sqrt{5}}{2}} \qquad \text{and} \quad b := \sqrt{\frac{5}{2} + \frac{\sqrt{5}}{2}}. \quad (5.2.27)$$

Useful relations are

$$\begin{aligned}
-2\sin(\frac{2\pi}{5}) + \sin(\frac{4\pi}{5}) &= -\frac{\sqrt{5}}{2}\,a,\\
\sin(\frac{2\pi}{5}) + 2\sin(\frac{4\pi}{5}) &= \frac{\sqrt{5}}{2}\,b,\\
2\cos(\frac{2\pi}{5}) &= 2 - a^2, \quad 2\sin(\frac{2\pi}{5}) = b,\\
2\cos(\frac{4\pi}{5}) &= 2 - b^2, \quad 2\sin(\frac{4\pi}{5}) = (2-a^2)b. \quad (5.2.28)
\end{aligned}$$

The $\widehat{su(3)}_2$ characters transform under the S transformation as,

$$\vec{\chi}_t^{(2)}(-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_y}{\tau}) = \frac{1}{\sqrt{15}}\,\mathbf{e}(\frac{2\omega_+^2+\frac{2}{3}\omega_y^2}{\tau})\sum_{t'\in\{-1,0,1\}}\xi_{15}^{-10tt'}\,\widetilde{\mathbf{S}}\,\vec{\chi}_{t'}^{(2)}(\tau,\omega_+,\omega_y), \quad (5.2.29)$$

where $t\in\{-1,0,1\}$, $\xi_{15}:=e^{\frac{\pi i}{15}}$ and

$$\widetilde{\mathbf{S}} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}. \quad (5.2.30)$$

On the other hand, given that the torus characters transform under S as,

$$\chi_m^{15}(-\frac{1}{\tau},\frac{\omega_y}{\tau}) = \frac{1}{\sqrt{30}}\,\mathbf{e}(\frac{\frac{5}{3}\omega_y^2}{\tau})\sum_{m'=0}^{29}\xi_{15}^{-mm'}\,\chi_{m'}^{15}(\tau,\omega_y), \quad (5.2.31)$$

the combinations of torus characters transform under S as,

$$\sigma_m^{15,t}(-\frac{1}{\tau},\frac{\omega_y}{\tau}) = \frac{1}{\sqrt{30}}\,\mathbf{e}(\frac{\frac{5}{3}\omega_y^2}{\tau})\begin{cases} \sum_{t'\in\{-1,0,1\}}\xi_{15}^{-10tt'}\sum_{m'=0}^{5}(\xi_{15}^{9mm'}+\xi_{15}^{-9mm'})\sigma_{m'}^{t'}(\tau,\omega_y) \\ \qquad\qquad\text{for } m\in\{1,2,3\}, \\ \sum_{t'\in\{-1,0,1\}}\xi_{15}^{-10tt'}\sum_{m'=0}^{5}\begin{cases}\sigma_{m'}^{t'}(\tau,\omega_y) & \text{for } m=0, \\ (-1)^{m'}\sigma_{m'}^{t'}(\tau,\omega_y) & \text{for } m=5.\end{cases}\end{cases} \quad (5.2.32)$$

So the LHS and the RHS of the sum rules transform analogously with respect to triality (through t,t'). We therefore can choose any one of the sum rule doublet $\vec{\mathcal{S}}_t$ to extract the S-transformation of the six functions $\widehat{F}_i$, and we consider $\vec{\mathcal{S}}_0$ here, i.e. the doublet of sum rules for the singlet and octet representations of $\widehat{su(3)}_2$.

5.3 The S transformation rules for the functions $\widehat{F}_i(\tau)$

Besides the S-transformations for the $\widehat{su(3)}_2$ characters and the torus characters discussed above, we need the S-transformations of the Wolf space fermions (straightforward) and for the $\widetilde{\mathcal{A}}_\gamma$ massless and massive characters at levels $\tilde{k}^+=2,\tilde{k}^-=1$, for which we have derived formulas. Explicitly we have,

1. Wolf space fermions

$$\chi^{WS,\widetilde{R}}(-\frac{1}{\tau},\frac{\omega_-}{\tau},\frac{\omega_y}{\tau}) = -\mathbf{e}(\frac{\omega_-^2+\omega_y^2}{\tau})\,\chi^{WS,\widetilde{R}}(\tau,\omega_-,\omega_y). \quad (5.3.1)$$

2. Massive $\widetilde{\mathcal{A}}_\gamma$ characters: refer to file on S-transformations of $\widetilde{\mathcal{A}}_\gamma$ characters. In the particular case where $\tilde{k}^+ = 2$ and $\tilde{k}^- = 1$, there are two characters at threshold labelled by $(\ell^{+\prime},\ell^{-\prime}) = (\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2})$. One has

$$\begin{aligned}
&\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},\frac{1}{2},\frac{1}{2};-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau}) = -\frac{\sqrt{2}}{2}\,(-i\tau)^{-\frac{1}{2}}\,\mathbf{e}(\frac{2\omega_+^2+\omega_-^2}{\tau})\\
&\times\left\{\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-)+\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},1,\frac{1}{2};\tau,\omega_+,\omega_-)\right\},\\
&\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},1,\frac{1}{2};-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau}) = -\frac{\sqrt{2}}{2}\,(-i\tau)^{-\frac{1}{2}}\,\mathbf{e}(\frac{2\omega_+^2+\omega_-^2}{\tau})\\
&\times\left\{\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-)-\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},1,\frac{1}{2};\tau,\omega_+,\omega_-)\right\}.
\end{aligned} \quad (5.3.2)$$

3. Massless $\widetilde{\mathcal{A}}_\gamma$ characters: these characters transform into a sum of massless $\widetilde{\mathcal{A}}_\gamma$ characters and a sum of massive characters. In the particular case where $\tilde{k}^+ = 2$ and $\tilde{k}^- = 1$, define

$$\begin{aligned}
H_1(\tau) &:= h_1(0,\tau)-h_{31}(0,\tau)+h_{11}(0,\tau)-h_{41}(0,\tau),\\
H_7(\tau) &:= h_7(0,\tau)-h_{37}(0,\tau)+h_{17}(0,\tau)-h_{47}(0,\tau).
\end{aligned} \quad (5.3.3)$$

One has, for $\mathcal{I} = \{(0,0),(1,0),(1,1),(2,1)\}$ and $0 \le 2\ell^+ \le 2, 0 \le 2\ell^- \le 1$,

$$(-1)^{2\ell^-}\,\mathbf{e}(-\frac{2\omega_+^2+\omega_-^2}{\tau})\,Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\ell^+,\ell^-),\ell^+,\ell^-;-\frac{1}{\tau},\frac{\omega_+}{\tau},\frac{\omega_-}{\tau}) =$$

$$\sum_{(2\lambda^+,2\lambda^-)\in\mathcal{I}}(-1)^{2\lambda^-}\sqrt{\frac{2}{5}}\,\sin\left(\frac{\pi}{5}(2\ell^++2\ell^-+1)(2\lambda^++2\lambda^-+1)\right)$$
$$\times\,Ch_0^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h_R^0}(\lambda^+,\lambda^-),\lambda^+,\lambda^-;\tau,\omega^+,\omega^-)$$

$$+\sqrt{\frac{2}{5}}\left[\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},\frac{1}{2},\frac{1}{2};\tau,\omega_+,\omega_-)-(-1)^{2\ell^++2\ell^-}\,\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\frac{1}{4},1,\frac{1}{2};\tau,\omega_+,\omega_-)\right]$$
$$\times\left\{-\sin\left(\frac{2\pi}{5}(2\ell^++2\ell^-+1)\right)\,H_1(\tau)+\sin\left(\frac{4\pi}{5}(2\ell^++2\ell^-+1)\right)\,H_7(\tau)\right\}$$

$$+\begin{cases} (-i\tau)^{-\frac{1}{2}}\,\frac{\sqrt{2}}{2}\,\widetilde{q}^{-\frac{1}{120}}\left[\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2};\tau,\omega_+,\omega_-)+\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\tfrac{1}{4},1,\tfrac{1}{2};\tau,\omega_+,\omega_-)\right] & \text{if } (2\ell^+,2\ell^-)=(0,1), \\ (-i\tau)^{-\frac{1}{2}}\,\frac{\sqrt{2}}{2}\,\widetilde{q}^{-\frac{1}{120}}\left[\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2};\tau,\omega_+,\omega_-)-\widehat{\widehat{Ch}}^{\widetilde{\mathcal{A}}_\gamma,\tilde{R}}(\widetilde{h}_R=\tfrac{1}{4},1,\tfrac{1}{2};\tau,\omega_+,\omega_-)\right] & \text{if } (2\ell^+,2\ell^-)=(2,0), \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.4}$$

Upon S-transformation, the singlet and octet sum rules provide six relations between the functions $\widehat{F}_i(\tau), i\in\{1,2,3,4,5,6\}$, their S-transformations $\widehat{F}_i(-\frac{1}{\tau})$ and the combinations of Mordell-type integrals $H_1(\tau)$ and $H_7(\tau)$. More precisely, if we define

$$\vec{\widehat{F}} := (\widehat{F}_1,\ \widehat{F}_2,\ \widehat{F}_3,\ \widehat{F}_6,\ \widehat{F}_5,\ \widehat{F}_4)^T, \quad T \text{ transpose}, \tag{5.3.5}$$

one gets the relations

$$\mathbf{A}\,\vec{\widehat{F}}(-\frac{1}{\tau})=(-i\tau)^{\frac{1}{2}}\left\{\left\{\begin{pmatrix} a & 0\\ 0 & -a\end{pmatrix}\otimes\mathbb{1}_{3\times3}+\begin{pmatrix} 0 & b\\ b & 0\end{pmatrix}\otimes\mathbb{1}_{3\times3}\right\}\vec{\widehat{F}}(\tau)+\vec{H}(\tau)\right\}, \tag{5.3.6}$$

where

$$\mathbf{A} := \mathbb{1}_{2\times2}\otimes\begin{pmatrix} 1 & 2 & 2\\ 1 & 2-a^2 & 2-b^2\\ 1 & 2-b^2 & 2-a^2\end{pmatrix} \tag{5.3.7}$$

and

$$\vec{H} := (-aH_1+bH_7,\ bH_1,\ aH_7,\ -bH_1-aH_7,\ -aH_1,\ bH_7)^T. \tag{5.3.8}$$

It is now a matter of straightforward matrix algebra to arrive at the S-transformation of the functions $\widehat{F}_i$. We get

$$\vec{\widehat{F}}(-\frac{1}{\tau}) = \frac{1}{5}(-i\tau)^{\frac{1}{2}} \left\{ \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbf{B} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{C} \right\} \vec{\widehat{F}}(\tau) + \vec{H}'(\tau) \right\}, \quad (5.3.9)$$

with

$$\mathbf{B} := \begin{pmatrix} a & 2a & 2a \\ a & \frac{a^2}{b} & -b \\ a & -b & \frac{a^2}{b} \end{pmatrix} \quad \text{and} \quad \mathbf{C} := \begin{pmatrix} b & 2b & 2b \\ b & a & -\frac{b^2}{a} \\ b & -\frac{b^2}{a} & a \end{pmatrix} \quad (5.3.10)$$

and

$$\vec{H}' := ((2b-a)H_1 + (2a+b)H_7,\, 0,\, -ab^2H_1 + a^2bH_7,\\ -(2a+b)H_1 + (2b-a)H_7,\, -a^2bH_1 - ab^2H_7,\, 0)^T. \quad (5.3.11)$$

From the above, we immediately see that $\widehat{F}_2$ and $\widehat{F}_4$ transform covariantly under $SL(2,\mathbb{Z})$ with weight $\frac{1}{2}$ while the four other functions $\widehat{F}_i, i \in \{1,3,5,6\}$ transform 'mock-covariantly', due to the presence of Mordell-type integrals H_1, H_7 in their S-transformation law. We note in passing that their behaviour under the T transformation is immediate from their q-series expansions (5.2.23), namely,

$$\widehat{F}_1(\tau+1) = e^{-\frac{\pi i}{60}}\,\widehat{F}_1(\tau),\ \widehat{F}_2(\tau+1) = e^{\frac{47\pi i}{60}}\,\widehat{F}_2(\tau),\ \widehat{F}_3(\tau+1) = e^{-\frac{49\pi i}{60}}\,\widehat{F}_3(\tau),\\ \widehat{F}_4(\tau+1) = e^{\frac{23\pi i}{60}}\,\widehat{F}_4(\tau),\ \widehat{F}_5(\tau+1) = e^{-\frac{\pi i}{60}}\,\widehat{F}_5(\tau),\ \widehat{F}_6(\tau+1) = e^{-\frac{49\pi i}{60}}\,\widehat{F}_6(\tau). \quad (5.3.12)$$

5.4 Analytic expression for the functions $\widehat{F}_i$

In order to determine analytic expressions for the functions $\widehat{F}_i, i \in \{1,\ldots,6\}$, the first attempt has been to match the first fifteen or more coefficients in their q-expansions (see (5.2.23) without the q-offset) with coefficients of series in the On-Line Encyclopedia of Integer Sequences (OEIS). The functions $\widehat{F}_i(\tau), i \in \{2,3,4,5\}$ correspond to series listed in OEIS. They enjoy several representations, and can be expressed in terms of modular and mock modular forms of the following types:

1. The Ramanujan generalized Theta function

$$\begin{aligned} f(x,y) &:= \sum_{n\in\mathbb{Z}} x^{\frac{n}{2}(n+1)}\, y^{\frac{n}{2}(n-1)}, \quad x,y\in\mathbb{C},\ |xy|\leq 1,\ \text{with} \\ f(-x) &:= f(-x,-x^2). \end{aligned} \tag{5.4.1}$$

2. The Rogers-Ramanujan functions

$$\begin{aligned} g(\tau) &:= q^{-\frac{1}{60}}\, G(\tau), \quad G(\tau) := \sum_{n=0}^{\infty} q^{n^2} \frac{1}{\prod_{j=1}^{n}(1-q^j)} \\ h(\tau) &:= q^{\frac{11}{60}}\, H(\tau), \quad H(\tau) := \sum_{n=0}^{\infty} q^{n^2+n} \frac{1}{\prod_{j=1}^{n}(1-q^j)}, \end{aligned} \tag{5.4.2}$$

where $q = e^{2\pi i\tau}$ as usual, which transform covariantly under $SL(2,\mathbb{Z})$, with

$$g(-\frac{1}{\tau}) = \frac{1}{\sqrt{5}}\,[\,b\,g(\tau) + a\,h(\tau)\,], \qquad h(-\frac{1}{\tau}) = \frac{1}{\sqrt{5}}\,[\,a\,g(\tau) - b\,h(\tau)\,] \tag{5.4.3}$$

with a, b as in (5.2.27), and

$$g(\tau+1) = e^{-\frac{\pi i}{30}}\, g(\tau), \quad h(\tau+1) = e^{\frac{11\pi i}{30}}\, h(\tau). \tag{5.4.4}$$

3. Generalised Lambert series

$$M(r;\tau) := \frac{q^{\frac{1}{24}}}{\eta(\tau)} \sum_{n\in\mathbb{Z}} q^{\frac{3}{2}n(n+1)} \frac{(-1)^n q^{n+r}}{1-q^{n+r}}, \quad 0 < r < 1,\ r\in\mathbb{Q}, \tag{5.4.5}$$

more specifically $M(\frac{1}{5};5\tau)$ and $M(\frac{2}{5};5\tau)$, which are closely related to the functions $\Phi(q)$ and $\Psi(q)$ introduced in [Hic88], formulas (0.7) and (0.8), in order to prove the Ramanujan 'Mock Theta Conjectures', namely

$$q\,M(\frac{1}{5};5\tau) = \Phi(q), \qquad q^2\,M(\frac{2}{5};5\tau) = \Psi(q). \tag{5.4.6}$$

In fact, the generalised Lambert series are essentially Appell functions. In particular,

$$M(r;\tau) = q^{\frac{1}{24}+r}\,\frac{1}{\eta(\tau)}\,\mathcal{K}_3(\tau, \frac{1}{6}(5\tau+1), r\tau - \frac{1}{6}\,(5\tau+1)), \tag{5.4.7}$$

as per the definitions and notations in [STT05], and their behaviour under the modular group $SL(2,\mathbb{Z})$ can be read from [STT05] (formulae (1.2)-(1.4)).

We have

$$\widehat{F}_2(\tau) = q^{\frac{5}{24}}\, f(-q^5)\, h(\tau), \qquad \widehat{F}_3(\tau) = q^{-\frac{49}{120}}\, q^2\, M(\frac{2}{5}; 5\tau),$$
$$\widehat{F}_4(\tau) = q^{\frac{5}{24}}\, f(-q^5)\, g(\tau), \qquad \widehat{F}_5(\tau) = q^{-\frac{1}{120}}\, q\, M(\frac{1}{5}; 5\tau). \tag{5.4.8}$$

It turns out that the q-expansion of $\widehat{F}_1(\tau) + \widehat{F}_5(\tau)$ also matches a series in OEIS, namely

$$\widehat{F}_1(\tau) = -\widehat{F}_5(\tau) + q^{\frac{1}{40}}\, f(-q^2; -q^3)\, g(\tau)^2. \tag{5.4.9}$$

However, in order to determine $\widehat{F}_6(\tau)$, the use of one of the S-transformations in (5.3.9) is helpful. We choose to work with the S-transformation of $\widehat{F}_2(\tau)$, namely,

$$\widehat{F}_2(-\frac{1}{\tau}) =$$
$$\frac{1}{5}\,(-i\tau)^{\frac{1}{2}} \left\{ \left[a\,\widehat{F}_1(\tau) + \frac{a^2}{b}\,\widehat{F}_2(\tau) - b\,\widehat{F}_3(\tau) \right] + \left[b\,\widehat{F}_6(\tau) + a\,\widehat{F}_5(\tau) - \frac{b^2}{a}\,\widehat{F}_4(\tau) \right] \right\}, \tag{5.4.10}$$

which must be equal to the S-transformation of $\widehat{F}_2$ when calculated directly from

$$\widehat{F}_2(\tau) = q^{\frac{5}{24}}\, f(-q^5)\, h(\tau). \tag{5.4.11}$$

In order to do the direct calculation, rewrite

$$f(-q^5) := f(-q^5, -q^{10}) = \sum_{n\in\mathbb{Z}} (-1)^n\, q^{\frac{15}{2}n^2 - \frac{5}{2}n} = q^{-\frac{5}{24}}\, [\theta_{5,30}(\tau) - \theta_{25,30}(\tau)] \tag{5.4.12}$$

and use (5.4.3) together with the known S-transformations of Kac-Peterson theta functions (3.A.3a) to obtain,

$$\begin{aligned}
\widehat{F}_2(-\frac{1}{\tau}) = \frac{1}{5}\,(-i\tau)^{\frac{1}{2}} \Big\{ & [\,\theta_{1,30}(\tau) - \theta_{29,30}(\tau) + \theta_{11,30}(\tau) - \theta_{19,30}(\tau)\,] \\
& - [\,\theta_{7,30}(\tau) - \theta_{23,30}(\tau) + \theta_{13,30}(\tau) - \theta_{17,30}(\tau)\,] - [\,\theta_{5,30}(\tau) - \theta_{25,30}(\tau)] \Big\} \\
& \times \{a\, g(\tau) - b\, h(\tau)\,\} \\
= \frac{1}{5}\,(-i\tau)^{\frac{1}{2}} \Big\{ & q^{\frac{1}{120}} \left[f(-q^8, -q^7) + q\, f(-q^{13}, -q^2) \right] - q^{\frac{49}{120}} \left[f(-q^{11}, -q^4) + q^2\, f(-q^{16}, -q^{-1}) \right] \\
& - q^{\frac{5}{24}}\, f(-q^5) \Big\} \times \{a\, g(\tau) - b\, h(\tau)\,\}\,.
\end{aligned} \tag{5.4.13}$$

On the other hand, inserting (5.4.8) in the S-transformation of $\widehat{F}_2(\tau)$ as given in (5.4.10), one gets

$$\widehat{F}_2(-\frac{1}{\tau}) = \frac{1}{5}(-i\tau)^{\frac{1}{2}}\left\{ b\left[\widehat{F}_6(\tau) - q^{-\frac{49}{120}}q^2\,M(\frac{2}{5};5\tau)\right] + a\,q^{\frac{1}{40}}f(-q^2,-q^3)g(\tau)^2 - \frac{b^2}{a}\,q^{\frac{5}{24}}f(-q^5)\,g(\tau) + \frac{a^2}{b}\,q^{\frac{5}{24}}\,f(-q^5)\,h(\tau)\right\}. \quad (5.4.14)$$

Comparing the total fractional q-offsets in the various terms of (5.4.13) and (5.4.14) leads to the following relations,

$$\begin{aligned}
&q^{\frac{1}{60}}\,f(-q^2,-q^3)\,g(\tau) = f(-q^8,-q^7) + q\,f(-q^{13},-q^2),\\
&q^{\frac{1}{5}}f(-q^5)\,g(\tau) = \left[\,f(-q^8,-q^7) + q\,f(-q^{13},-q^2)\right]\,h(\tau),\\
&q^{-\frac{1}{5}}\,f(-q^5)\,h(\tau) = \left[\,f(-q^{11},-q^4) + q^2\,f(-q^{16},-q^{-1})\right]\,g(\tau),
\end{aligned} \quad (5.4.15)$$

and

$$\widehat{F}_6(\tau) = q^{-\frac{49}{120}}q^2\,M(\frac{2}{5};5\tau) + q^{\frac{49}{120}}\left[\,f(-q^{11},-q^4) + q^2\,f(-q^{16},-q^{-1})\right]\,h(\tau). \quad (5.4.16)$$

Incidentally, the first two relations allow to express the Rogers-Ramanujan functions as

$$\begin{aligned}
g(\tau) &= q^{-\frac{1}{60}}\,\frac{f(-q^8,-q^7) + q\,f(-q^{13},-q^2)}{f(-q^2,-q^3)},\\
h(\tau) &= q^{\frac{11}{60}}\,\frac{f(-q^5)}{f(-q^2,-q^3)}.
\end{aligned} \quad (5.4.17)$$

All relations obtained can be checked through q-series expansions.

In summary, the six functions $\widehat{F}_i(\tau)$ that enter the character sum rules stemming from the realization of the $\widetilde{\mathcal{A}}_\gamma$ algebra on the manifold corresponding to the group coset $SU(3)/U(1)$ with associated $\widehat{su(3)}_2$ affine algebra form a vector-valued mock modular form $\overrightarrow{\widehat{F}}(\tau)$ whose components have the following analytic expressions,

$$\begin{aligned}
\widehat{F}_1(\tau) &= -q^{-\frac{1}{120}}\,q\,M(\frac{1}{5};5\tau) + q^{\frac{1}{40}}\,f(-q^2;-q^3)\,g(\tau)^2\\
\widehat{F}_2(\tau) &= q^{\frac{5}{24}}\,f(-q^5)\,h(\tau),\\
\widehat{F}_3(\tau) &= q^{-\frac{49}{120}}\,q^2\,M(\frac{2}{5};5\tau),\\
\widehat{F}_6(\tau) &= q^{-\frac{49}{120}}q^2\,M(\frac{2}{5};5\tau) + q^{\frac{49}{120}}\left[\,f(-q^{11},-q^4) + q^2\,f(-q^{16},-q^{-1})\right]\,h(\tau),\\
\widehat{F}_5(\tau) &= q^{-\frac{1}{120}}\,q\,M(\frac{1}{5};5\tau),
\end{aligned}$$

$$\widehat{F}_4(\tau) \;=\; q^{\frac{5}{24}}\, f(-q^5)\, g(\tau), \tag{5.4.18}$$

where we note that

$$\begin{aligned}
\widehat{F}_1(\tau) &= -\widehat{F}_5(\tau) + q^{\frac{1}{40}}\, f(-q^2;-q^3)\, g(\tau)^2 \\
\widehat{F}_6(\tau) &= \widehat{F}_3(\tau) + q^{\frac{49}{120}} \Big[f(-q^{11},-q^4) + q^2\, f(-q^{16},-q^{-1}) \Big]\, h(\tau).
\end{aligned} \tag{5.4.19}$$

Under the S-transformation, this vector-valued MMF transforms according to (5.3.9), while under the T-transformation one gets

$$\overrightarrow{\widehat{F}}(\tau+1) = \mathrm{diag}(1,\, \xi_{60}^{47},\, \xi_{60}^{71},\, \xi_{60}^{71},\, 1,\, \xi_{60}^{23}\,)\, \overrightarrow{\widehat{F}}(\tau), \tag{5.4.20}$$

with $\xi_N := e^{\frac{\pi i}{N}}$.

A previous attempt at finding analytic expressions for these functions, more directly based on the structure of the sum rules, was partially successful [PT93]. There, one exploits the fact that both affine algebras $\widehat{su(3)}_{\tilde{k}^+}$ and $\widetilde{\mathcal{A}}_\gamma$ contain affine $\widehat{su(2)}$ subalgebras. More precisely, the decomposition of $\widehat{su(3)}_{\tilde{k}^+}$ characters into $\widehat{su(2)}_{\tilde{k}^+}$ characters for a regular embedding of $SU(2)$ into $SU(3)$ yields the schematic structure

$$SU(3) \cong (\,\text{parafermions}\,)\; \times U(1) \times SU(2), \tag{5.4.21}$$

or formally,

$$\widehat{\chi^{(\tilde{k}^+)}_{(a_1,a_2)}}(\tau,\omega_+,\omega_y) = \sum_{2\ell^+=0}^{\tilde{k}^+} \sum_{n=0}^{\tilde{k}^+-1} P^{(a_1,a_2)}_{2\ell^+,n}(\tau)\, \chi^{3\tilde{k}^+}_{4(a_1-a_2)+6(n+\ell^+)}(\tau,\omega_y)\, \chi^{\tilde{k}^+}_{2\ell^+}(\tau,\omega_+), \tag{5.4.22}$$

where the functions $P^{(a_1,a_2)}_{2\ell^+,n}(\tau)$ are the characters for the parafermionic theory $SU(3)/(\,SU(2)\times U(1)\,)$, see [HNY90]. On the other hand, the massive $\widetilde{\mathcal{A}}_\gamma$ characters decompose in characters for the affine $\widehat{su(2)}_{\tilde{k}^+} \times \widehat{su(2)}_{\tilde{k}^-}$ subalgebra with branching functions given by products of Virasoro characters, defined as

$$\chi^{\mathrm{Vir}(m)}_{r,s}(\tau) := \frac{1}{\eta(\tau)}\, \Big(\theta_{r(m+1)-sm,m(m+1)}(\tau) - \theta_{r(m+1)+sm,m(m+1)}(\tau) \Big), \tag{5.4.23}$$

with $m,r,s \in \mathbb{N}$, $m \geq 2$, $1 \leq r \leq m-1$, $1 \leq s \leq r$. In the context of the sum rules we are interested in, the parameter m takes the values $m = k^+$ and $m = k^-$ and an explicit formula in the Neveu-Schwarz sector can be found in [PT93], formula (2.43). This information can be read in the $\widetilde{R}$ sector using spectral flow, however the functions $\widehat{F}_i(\tau)$ we are interested in are invariant under spectral flow. The reason why only a partial identification of the set of six functions $\widehat{F}_i(\tau)$ was possible in [PT93] is that the approach relied on the knowledge of the branching functions $Y_{r,s}^{(\tilde{k}^++2)}(\tau)$ of some massless $\widetilde{\mathcal{A}}_\gamma$ characters (and difference thereof) decomposed into characters for the $\widehat{su(2)}_{\tilde{k}^+} \times \widehat{su(2)}_{\tilde{k}^-}$ subalgebra (see [PT93], formula (2.44)), and that these branching functions could not be completely determined. In fact, four of the six branching functions involved were expressed in terms of the last two, chosen to be $Y_0^{(4)}(\tau)$ and $Y_{3/5}^{(4)}(\tau)$ in the case of interest to us, namely $\tilde{k}^+ = 2, \tilde{k}^- = 1$ (see [PT93], formula (A.15)).

As a by-product of our approach, we have been able to determine $Y_0^{(4)}(\tau)$ and $Y_{3/5}^{(4)}(\tau)$, and therefore all the branching functions appearing in [PT93], formula (2.44) in the case $\tilde{k}^+ = 2, \tilde{k}^- = 1$ are now known. Interestingly, and contrary to what was suggested in [PT93], the branching functions $Y_{r,s}^{(4)}(\tau)$ do not transform covariantly under any congruent subgroup of the modular group $SL(2,\mathbb{Z})$. Indeed, first note that our functions $\widehat{F}_i(\tau)$ are related to the functions $F^{\Lambda}_{2\ell^+,n}(q)$ discussed in [PT93], formula (A.18), in the following way:

$$\begin{aligned}
\widehat{F}_1(\tau) &= F_{0,0}^{((0,0),2,0)}(q) + \frac{1}{2}q^{-1/120}, \quad \widehat{F}_2(\tau) = F_{0,1}^{((0,0),2,0)}(q), \quad \widehat{F}_3(\tau) = F_{0,2}^{((0,0),2,0)}(q) \\
\widehat{F}_6(\tau) &= F_{0,0}^{((1,1),2,0)}(q), \quad \widehat{F}_5(\tau) = F_{0,1}^{((1,1),2,0)}(q) - \frac{1}{2}q^{-1/120}, \quad \widehat{F}_4(\tau) = F_{0,0}^{((1,0),2,0)}(q).
\end{aligned} \tag{5.4.24}$$

The shifts by $\pm\frac{1}{2}$ in $\widehat{F}_1$ and $\widehat{F}_5$ are a consequence of the sum rules being written in terms of some *differences* of massless $\widetilde{\mathcal{A}}_\gamma$ characters in [PT93], while we have not used such differences in our sum rules. The freedom in the system is due to the linear relations (4.5.15) between massless and massive $\widetilde{\mathcal{A}}_\gamma$ characters 'at threshold'.

Remark: it turns out that some misprints are present in [PT93], formulae (A.18). The fourth and fifth formulae should read,

$$F_{0,0}^{((0,0),2,0)}(q) + \frac{1}{2}q^{-1/120} = \frac{1}{\chi_{\frac{1}{2}}^{\mathrm{Vir}(3)}(q)} \left\{ \left[P_{0,1}^{(0,0),2,0)}(q)\,\theta_{0,10}(q) + P_{0,0}^{(0,0),2,0)}(q)\,\theta_{10,10}(q) \right] \right.$$

$$
\begin{aligned}
&\qquad\qquad +\eta(q)\,Y^{(4)}_{3/5}(q)\,\Big\} \\
F^{((1,1),2,0)}_{0,1}(q)-\frac{1}{2}q^{-1/120} \;&=\; \frac{1}{\chi^{\mathrm{Vir}(3)}_{\frac{1}{2}}(q)}\,\Big\{\Big[\,P^{(1,1),2,0)}_{0,1}(q)\,\theta_{4,10}(q)+P^{(1,1),2,0)}_{0,0}(q)\,\theta_{6,10}(q)\,\Big] \\
&\qquad\qquad -\eta(q)\,Y^{(4)}_{3/5}(q)\,\Big\}\,. \qquad (5.4.25)
\end{aligned}
$$

Chapter 6

Conclusion

The motivation of the thesis originated in the 2010 observation of a moonshine phenomenon involving the sporadic group Mathieu 24 in the framework of type IIB superstrings compactified on the hyperkähler surface $K3$ of complex dimension 2 [EOT11]. After nine years of efforts trying to understand the significance of this Mathieu moonshine in string theory, we are still missing a deep physical explanation. At the heart of this observation is the calculation of the conformal field-theoretic elliptic genus of $K3$, using algebraic methods pioneered by Witten [Wit87], which brings to light the weakly holomorphic mock modular form $h^{(2)}$ of weight $1/2$ on $SL(2, \mathbb{Z})$ [DMZ14], whose coefficients are the dimensions of representations of the group Mathieu 24 [Gan16]. The reason this observation became quickly known as Mathieu moonshine is that a similar phenomenon had been observed by McKay in 1978 between the modular function J and the dimensions of representations of the largest sporadic group called the Monster group. Monstrous moonshine was studied extensively in mathematics and lead to the discovery of beautiful structures like Vertex Operator Algebras, and culminated in a Fields medal for Borcherds. After Mathieu moonshine was observed, other moonshine phenomena were uncovered and, together with Mathieu moonshine, are known under the collective name of umbral moonshine after the work of Cheng, Duncan and Harvey [CDH12]. In our work, we have asked the question: 'Is there a moonshine phenomenon associated with large $\mathcal{N} = 4$ superconformal algebras?'. This was a natural question to ask since one can recover the small $\mathcal{N} = 4$ from the large $\mathcal{N} = 4$ superconfomal algebra, and realizations of the latter have been constructed. In particular, coset realizations [Pro89; ST90; GPTV89] do exist.

In the Mathieu moonshine case, the calculation of the conformal field-theoretic elliptic genus relies on the knowledge of the partition function of a theory with small $\mathcal{N} = (4, 4)$ symmetry at central charge $c = \bar{c} = 6$. Since the elliptic genus is a topological invariant, one may choose any theory in the moduli space of non-linear sigma models on $K3$ where the partition function can be constructed explicitly. This is the reason why explicit calculations are done for Gepner models or $\mathbb{Z}_2$-orbifolds of toroidal CFTs. In the large $\mathcal{N} = 4$ situation, coset realizations are one class of models that can be used in an attempt to uncover a moonshine phenomenon. Our first task has been to study more closely the works of Witten [Wit82; Wit87; Wit88] and Alvarez, Killingback, Mangano and Windey [AKMW87a; AKMW87b] on indices generalized to string theory in order to appreciate better the nature of these topological invariants. This material is reviewed in Chapter 2. We then proceeded with a brief account of superconformal algebras with extended supersymmetry ($\mathcal{N} = 2$ and small $\mathcal{N} = 4$) in Chapter 3, with a description of Mathieu moonshine. This was an opportunity to introduce notations and definitions as well as provide material that one could compare and contrast with results obtained in Chapters 4 and 5. In particular, the elliptic genus that was so useful in small $\mathcal{N} = 4$ vanishes in large $\mathcal{N} = 4$, as can be checked easily once the representation theory of that algebra is provided, something we review at the beginning of Chapter 4. However, another index was introduced in 2004 [GMMS04] that might be useful in the search for a new moonshine phenomenon. This new index is defined as an index for a conformal field theory, and its calculation involves the action of a differential operator on characters of the large $\mathcal{N} = 4$ algebra. Unlike the Witten index, this action returns a theta series in the variable q rather than a pure number, signalling that the index does not only count BPS states, as was the case for the Witten index in small $\mathcal{N} = 4$. See Subsection 4.5.2 for details. In order to be able to evaluate the new index on a theory with large $\mathcal{N} = 4$ symmetry, we have concentrated on a particular example of coset realization of $\widetilde{\mathcal{A}}_\gamma$, a non-linear algebra closely related to the large $\mathcal{N} = 4$ algebra, namely the realization emerging from considering a Wess-Zumino-Novikov-Witten model on the group manifold $SU(3)$ (this fixes one of the levels, $\tilde{k}^- = 1$). This has required a detailed analysis of sum rules directly related to the coset construction, involving sums of products of characters for representations of the affine $\widehat{su(3)}$ at level $\tilde{k}^+ = 2$ with a character for four free fermions and a boson, which are decomposed in $\widetilde{\mathcal{A}}_\gamma$ characters, see eq.(5.2.24). The corresponding branching functions, which were only known so far as q-power series expansions up to q^{20} or so, are presented here in compact analytic form for the first time in Chapter 5, Subsection 5.4. Partial information was known in [PT93; Fea18], but the complete information was only obtained thanks to the knowledge of the explicit transformations of $\widetilde{\mathcal{A}}_\gamma$ characters under the modular group $SL(2, \mathbb{Z})$, which were derived in full in Chapter 4, Subsection 4.4.2. The original work in this thesis confirms that these branching functions are the components of a vector-valued mock modular form of weight $1/2$, a fact that was not anticipated in [PT93]. In the particular case where $\tilde{k}^+ = 2$ (or $k^+ = 3$) and $\tilde{k}^- = 1$ (or $k^- = 2$), some of the components are mock theta functions of order 5, while the others are not mock. Regarding the behaviour of $\widetilde{\mathcal{A}}_\gamma$ characters under the action of $SL(2, \mathbb{Z})$, we wish to stress that the massive characters are Jacobi forms, up to a power of q, while the massless characters are 'mock' in the sense that they are proportional to a sum of products of theta functions with higher level Appell functions (see eq.(4.4.25)), and that one would need to introduce a non-holomorphic correction term for these Appell functions in order for the massless characters to transform as Jacobi forms. Higher level Appell functions were first introduced in the mathematical physics literature in [STT05], and appear as well in the PhD thesis of Zwegers on mock theta functions [Zwe08]. We presented in Appendix 4.B.4 a dictionary between the two works regarding the S-transformation of these higher level Appell functions.

In this thesis, we have thus prepared the way for an in-depth analysis of new indices in theories with large $\mathcal{N} = 4$ symmetry. Early investigations of potential modular invariant partition functions reveal that a moonshine-like phenomenon will not be easy to recognize - if at all present - due to the complicated expressions obtained when calculating the new index introduced in [GMMS04] on those partition functions. However, future work might discover unsuspected patterns in such theories. One could also try to define other indices that might lead to interesting phenomena.

Independently of a search for moonshine, several immediate technical questions arise. We have derived the modular transformations of large $\mathcal{N} = 4$ characters when the levels k^+ and k^- are coprime. Although we do not expect a major difficulty if this restriction is lifted, we have decided to concentrate on models where the levels are coprime due to time constraints. A particularly intriguing situation might emerge when the levels are equal and this deserves further study. Indeed, in [OPT92] the sum rules for the case $k^+ = k^- = 2$ were analyzed in detail, and the branching functions could be identified without knowing the modular transformations of the corresponding $\widetilde{\mathcal{A}}_\gamma$ characters. Interestingly in that case, the branching functions are not mock theta functions, but Jacobi theta functions. One could investigate whether this is an accident or whether the nature of branching functions (mock or not) is related to the levels being coprime or not. A classification of vector-valued mock modular forms arising in coset realizations of $\widetilde{\mathcal{A}}_\gamma$ algebras for any values of the levels would be interesting. Finally, one could use Zwegers' prescription to complete the higher level Appell functions (see [Zwe08], Chapter 3) and write down the completed massless $\widetilde{\mathcal{A}}_\gamma$ characters that would be Jacobi forms of weight 0.

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