Resurgence in Supersymmetric Localisable Quantum Field Theories

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A Thesis presented for the degree of
Doctor of Philosophy

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July 2020

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Abstract: In this thesis we consider the application of resurgence and Picard-Lefschetz theory to supersymmetric localisable quantum field theories in 2, 3 and 4 dimensions. We consider two problems. First, in the theories we study, observables can be calculated exactly using localization methods, and written in the form of a transseries. However in each non-perturbative sector, the associated perturbation series is not asymptotic, seemingly rendering the application of resurgence theory impossible. This problem is solved by deploying a Cheshire Cat analysis; we slightly deform the theory rendering the series asymptotic, perform a resurgence analysis in the deformed theory, and analytically continue the deformation back to 0, returning the non-perturbative data in the undeformed theories. This is achieved in $\mathcal{N} = (2,2)$ theories in 2 dimensions, and $\mathcal{N} = 2$ theories in 3 dimensions. Comments are made about how we might generalize this to 4 dimensional theories. The second problem is the disappearance of the resurgence triangle structure in $\mathcal{N} = 2$ theories on a 3-sphere. This structure is recovered by means of introducing a complex squashing parameter, uncovering a hidden topological angle present in the theory. Finally, in the two above mentioned theories and $\mathcal{N} = 2$ theories in 4 dimensions, a method is given for how to combine a resurgence analysis with additional non-perturbative structures present in these theories to compute non-perturbative contributions with different topological charge from the perturbative data.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Chapter 3 and parts of chapter 2 are based on [1], which has been published in SciPost Phys. 4, 012 (2018).

Chapter 4, chapter 5 and parts of chapter 2 are based on [2], which has been published in the Journal of High Energy Physics, article number 85 (2019).

Acknowledgements

I would like to thank my wife, Sunao, and my family for all their love and support over the course of the last four years. They have encouraged me through the rough patches, and always loved and supported me. I am also extremely grateful for the Christian fellowship and support my family has received from the congregation and ministers of Christchurch Durham.

I am greatly indebted to my supervisor Daniele Dorigoni for his supervision, guidance, and collaboration on the work contained in this thesis. It is thanks to him that I have been introduced to the fascinating topic of resurgence, and been able to complete this work.

Over the course of my PhD I have greatly benefited from conversations with Mathew Bullimore, Stefano Cremonesi, Gerald Dunne, Masazumi Honda, Tatsuhiro Misumi, Vasilis Niarchos, Matthew Renwick and Mithat Ünsal, for which I am very grateful. For the work on 2D $\mathcal{N} = (2, 2)$ theories, I would also like to thank Sungjay Lee for initial work on earlier stages of this project.

Finally I would like to thank my heavenly Father, our creator and sustainer. To him belongs all the glory.

Great are the works of the LORD,
studied by all who delight in them.

— from *Psalm 111:2*

Soli Deo Gloria

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Chapter 1

Introduction

1.1 Towards understanding non-perturbative quantum field theory

Quantum Field Theory (QFT) is the culmination of the reductionist attempt to describe all phenomena found in nature using only a few, simple mathematical laws. It has seen huge success in experiments at CERN describing fundamental particles, but also finds uses as a mathematical language across the spectrum of physics, from quantum gravity, through to condensed matter. On top of this, in recent years new results in QFT have directly lead to new results in pure mathematics, and QFT is even being used in economics and finance modeling. What started life as a theory for trying to make sense of the dynamics of extremely small objects traveling extremely quickly has become a rich mathematical framework finding applications right across the mathematical sciences.

Despite its importance, QFT still lacks a general and mathematically rigorous formulation. Coupled to this, many important aspects of the framework are poorly understood, in particular, non-perturbative aspects of QFT. Such non-perturbative phenomena are found all over nature. The most famous example is the mass gap of Yang-Mills[3]. Another example is the topic of topological insulators, where due to non-perturbative phenomena, the interior of the material is insulating, whilst the surface contains conducting states[4].

QFTs are generally analysed using perturbation theory in a coupling "constant" g. The reason for the quotation marks is that in real world theories, g is rarely constant, but runs to different values depending on the energy in the system. It has been known since Achilles raced his tortoise friend that a series like $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ eventually converges to 2. However, for many Qts present in nature, the perturbation series in g encountered when computing observables are asymptotic, and so have zero radius of convergence in g. Thus perturbation theory can only give us, at best, approximations for results at small g, and cannot tell us anything about strongly coupled theories where non-perturbative effects dominate.

Developing a full, non-perturbative, well-defined formulation of QFT, is an important outstanding problem in both physics and mathematics. In mathematics the problem is important for putting QFT on a rigorous footing. For physics it is important for explaining many observed phenomena, in particular non-perturbative phenomena that take place at strong coupling. It is with the aim of bettering our understanding of non-perturbative QFT that the research contained within this thesis has been carried out.

In recent years it has been realised that the diverging asymptotic perturbation series encountered when analysing a quantum system is in fact not a bug of the analysis, but a feature. By utilising Ecalle resurgence theory, one can actually determine non-perturbative contributions to a theory from the way the series diverges. This has caused much excitement, as it means a daunting non-perturbative analysis of a particular system can be reduced to a much more simple perturbative analysis, followed by an application of resurgence theory.

Whilst promising, there are however many technical obstacles to overcome before the dream of understanding non-perturbative QFT might be realised using resurgence theory. To name a few such difficulties, it is still poorly understood how one might apply resurgence in a regime where the coupling is running. For many important QFTs like Yang-Mills theory, it is typically extremely hard to calculate many orders of perturbation theory, meaning access to the asymptotic expansion of observables is very limited. A further problem, which is the subject of this thesis, is how to apply resurgence to theories which have non-perturbative effects, but no asymptotic perturbation expansion.

Over the last 30 years huge progress has been made in understanding non-perturbative contributions to supersymmetric (susy) QFTs. In increasingly many cases various observables in particular theories can be calculated exactly, including all the non-perturbative contributions. This has provided a new laboratory to test and develop the application of resurgence methods to QFT.

The last problem mentioned above; the lack of asymptotic series required for a resurgence analysis to be performed, is observed in many of these susy QFTs. In particular theories, observables have been computed exactly, including all their non-perturbative contributions, but the perturbation series encountered when analysing the observable, is not asymptotic. The question then is, do these theories *not* have a resurgent structure, or is it present but somehow hidden? In this thesis we tackle this question, and find the answer to be the latter option. These theories exhibit what is known as Cheshire Cat resurgence, a phenomena we will explore more in the remaining chapters of this thesis.

A further topic studied here relates to the topology of non-perturbative contributions. Such contributions typically come with a topological charge. Resurgence will usually only allow us to calculate non-perturbative data with equal topological charge to perturbation series in consideration. Thus we see the emergence of a structure, called the resurgence triangle, where elements of certain sets of non-perturbative data are related to other elements of the same set, but not to elements of other sets. Two related problems emerge. In certain theories, like the 3-dimensional theories we will study here, this resurgence triangle structure disappears. What has happened to it? We will see that despite the lack of topological theta angle in these theories, there is in fact a hidden topological angle which we will uncover by deforming our theory. The second question is can some additional non-perturbative relations be found, and used in addition to resurgence, to calculate non-perturbative data with different topological charge. We will answer this in the affirmative, and provide examples.

1.2 Outline

In this thesis we will be considering the application of resurgence methods, and its topological partner Picard-Lefschetz theory, in the analysis of supersymmetric localisable quantum field theories.

In chapter 2 we will begin by giving a review of the relevant literature regarding resurgence and Picard-Lefschetz methods, the supersymmetric theories we will be studying, and prior works applying resurgence methods to supersymmetric quantum field theories. We will then turn to review resurgence and Picard-Lefschetz theory themselves, and discuss some of the non-perturbative structures that emerge, in particular the transseries and the resurgence triangle. Will will conclude this chapter by giving a 0-dimensional example, where we will apply both resurgence and Picard-Lefschetz theory, and see our first example of Cheshire Cat resurgence.

In chapter 3 we will then turn our attention to analysing 2-d $\mathcal{N} = (2, 2)$ theory. This will be our first example of Cheshire Cat resurgence in a QFT, and we will be able to use the Cheshire Cat method to do a full resurgence analysis.

In chapter 4 we will analyse 3-d $\mathcal{N} = 2$ theories, using resurgence and Picard-Lefschetz theory. We will again see a manifestation of Cheshire Cat resurgence, but a new phenomena will be observed; we will need to deform the theory further to recover the resurgence triangle, and see the appearance of a hidden topological angle. At the end of this chapter we will also comment on how we might generalise the Cheshire Cat method to be used with 4-dimensional susy localisable theories.

In chapter 5 we will explore additional structures across the resurgence triangle. We will observe how in these theories in 2, 3, and 4 dimensions, there are non-perturbative relations that, when applied in addition to resurgence theory, allow us to calculate all the data in the resurgence triangle, rather than just in a single column.

In chapter 6 we will then draw some conclusions, and comment on future directions. At the end of this thesis there are appendices detailing the properties of some functions and regularization techniques we make use of throughout the thesis.

Chapter 2

Background

In this chapter we will first introduce some known results from the literature. This will lead us to understand the problems that this thesis aims to tackle. We will then cover some background concepts in resurgence and Picard-Lefschetz theory that will be needed in subsequent chapters. This will conclude with a simple example of an integral that can be solved by applying a resurgence or Picard-Lefschetz analysis, to illustrate the ideas.

2.1 Literature Review

Given a generic quantum field theory, where there is nothing miraculous happening (e.g. integrability or susy localization), one is typically forced to calculate observables in the theory using perturbation theory. The result will be something like

$$\mathcal{O}(g) = c_0 + c_1 g + c_2 g^2 + c_3 g^3 + ... \tag{2.1.1}$$

Dating back to 1952, an old argument by Dyson [5] suggests that, despite its acclaimed and experimentally confirmed success, the perturbative expansion of QED must have vanishing radius of convergence. The reason for this lack of convergence lies in the asymptotic nature of perturbation theory usually attributed to the rapid factorial growth of Feynman diagrams [6, 7]. Generically, in the absence of magic cancellations, we expect that all the diagrams of a given order will contribute somewhat equally, so that when we sum over all of them we will obtain just from combinatorics a factorial growth for the perturbative coefficients. We must also have that the contribution from the individual integrals does not counter the factorial growth. For example if the number Feynman graphs goes like $n!$, but the contribution from each graph goes like $\frac{g^n}{n!}$, there will be no asymptoticity. Again, back in 1952, Hurst showed that this was not the case in scalar theories like ϕ^3 and ϕ^4 theory, and there is indeed asymptoticity.

The asymptotic behaviour of the perturbative series is something very general, deeply rooted within the singular nature of perturbation theory, and an extremely recurrent feature (rather than a bug) present not only in quantum field theory but also in quantum mechanics [8, 9, 10] and in string theory [11, 12].

Ecalle resurgence theory [13] is the perfect mathematical framework to address the problem of resummation of asymptotic series. The resummation method is Borel resummation, and if we only focus on perturbation theory we do not quite get a unique physical answer but rather a family of different analytic continuations; there is ambiguity in the resummed answer.

The reason behind this is that perturbation theory is not the end of the story; these ambiguities in resummation generate new non-analytic, i.e. non-perturbative, contributions. Resurgence theory tells us how the global properties of the full solution are intimately linked to these ambiguities [13, 14, 15, 16, 17]. Our series expansion has to be replaced by a *transseries* expansion in which we add on top of the formal power series in the coupling constant these new exponentially suppressed, non-perturbative terms accompanied with their own formal power series. In other words the exact computation of an observable should not be in the form of (2.1.1) but instead in the form:

$$\mathcal{O}(g) = \sum_i e^{-S_i/g} \sum_{n=0}^{\infty} c_n^{(i)} g^n \quad . \tag{2.1.2}$$

These exponentially suppressed terms cannot be captured by perturbation theory, since $e^{-S/g}$ will vanish to all orders in perturbation theory. They should remind us of contributions we expect we need to include coming from finite action solutions of the equations of motion, such as instantons. Though a proof that this is always the case does not exist, it is noteworthy that in all the cases analysed in the literature to date it is possible to find a weak coupling limit of the theory where these contributions do indeed correspond to non-perturbative objects present in the microscopic theory.

In general transseries can contain not only power series multiplied by exponentials of power series, but also iterated exponentials, logarithms and iterated logarithms. They were developed independently in various parts of mathematics to solve a variety of different problems, and have their own rich body of mathematical literature. We won't delve much into this at all, and will be concerned with transseries of the form (2.1.2). For a nice introduction, see [18], and references therein.

In summary, resurgence theory tells us in practice how to decode from the perturbative data the non-perturbative pieces necessary to construct a unique resummed physical observable. It is possible to disentangle from the perturbative coefficients the fluctuations around different non-perturbative saddle points. A point that is sometimes overlooked is the vice-versa is also true; we can use the series around a particular non-perturbative contribution to construct the perturbative data, as well as all the other non-perturbative data. The zero radius of convergence of our original perturbation series has indeed turned out to be a feature rather than a bug in our analysis.

One important caveat must be made here. Non-perturbative contributions to a typical QFT usually come with some topology, and thus a topological charge. As has been noted in [19, 20], we can only use resurgence to calculate non-perturbative data within a topological sector (contributions with equal topological charge). Thus the resurgence structure arranges itself into what is called the resurgence triangle (see figure 2.1). We can use a resurgence analysis to calculate all the contributions within a single column of the triangle from just one element of that column[1].

For example, a resurgence analysis of the perturbative data of Yang-Mills, $\Phi_0^{(0)}$, could produce information about the instanton-anti-instanton contribution, $\Phi_0^{(1)}$, but not the

[1]In theories with a topological angle we often work with the complexified coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. Resurgence utilises the analytic continuation of g, and the imaginary part of g should not be confused with θ. It is the explicit dependence on θ that splits the triangle up into its constituent columns. We will comment further on this in section 2.2.1.

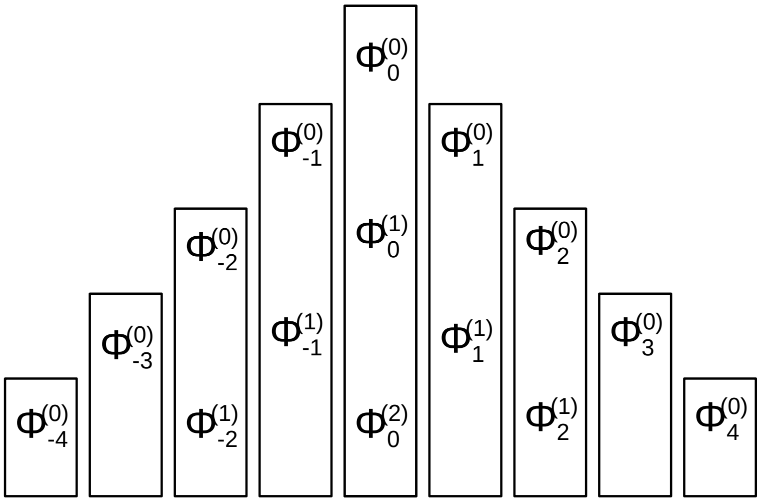


Figure 2.1: The Resurgence triangle. The k^{th} non-perturbative contribution part of the N^{th} topological sector is denoted schematically by $\Phi^{(k)}_N$. Resurgence theory allows us to reconstruct from any $\Phi^{(k)}_N$ all the other contributions in the *same* column, i.e. $\Phi^{(k')}_N$.

instanton contribution, $\Phi^{(0)}_1$.

When considering path-integrals, complementary to resurgence analysis [21] is the Picard-Lefschetz theory or complexified Morse homological decomposition in steepest descent contours [22], see also [23, 24, 25]. Rather than an analysis of a perturbation series, Picard-Lefschetz theory utilises an analysis of the path-integral, which can be thought of as a generalization of the saddle point decomposition method for ordinary integrals to path-integrals. We start off with an integral that will be something like

$$\mathcal{O}(g) \;=\; \int_\Gamma [D\phi] e^{-S[\phi]/g} \;. \tag{2.1.3}$$

Here we have denoted the fields by ϕ, the effective action by $S[\phi]$, and the path-integral "contour" of integration by Γ.

The key idea is that one has to deform the path-integral contour of integration into a suitable complexification of field space. In this complexified field space one then has to find all the saddles; these are the finite action solutions to the equations of motion.

Associated to each complex saddle point there is a privileged, steepest descent contour of integration, usually called a Lefschetz thimble or J cycle, and at a generic value of the coupling constant one can rewrite the original contour as a linear combination of these thimbles with integer coefficients[2], i.e. intersection numbers. Thus we find a result in a form very reminiscent of a transseries:

$$\mathcal{O}(g) = \sum_i n_i e^{-S_i/g} \int_{\mathrm{J}_i} [D\phi] e^{-S[\phi]/g} \quad . \tag{2.1.4}$$

Here i runs over the saddles, n_i are the intersection numbers, J_i are the J cycles associated to each saddle, and a change of variables has been made to pull out an explicit factor of $e^{-S_i/g}$ associated to each saddle.

The link between resurgence and Picard-Lefschetz decompositions comes from Stokes phenomena. For special arguments of the complexified coupling constant, i.e. Stokes directions, we have that a thimble can connect different saddles. This is usually forbidden given the fact that the imaginary part of the action is constant along a thimble, i.e. they are stationary phase contours. Across a Stokes direction some of the thimbles will undergo non-trivial monodromies and the aforementioned intersection numbers will jump. Simultaneously resurgence analysis tells us that the resummation of the asymptotic series around the saddles involved will also jump; the ambiguities in the Borel resummations mentioned before. These two discontinuities, of intersection numbers and resummations, are tightly related.

We again see the appearance of the resurgence triangle from Picard-Lefschetz theory. Whenever the theory in question contains a topological θ angle this will contribute to the classical action of the various saddle points by an imaginary part weighted by θ times the topological number. Thus even before complexifying the coupling constant (note that one should not confuse the imaginary part of the complexified coupling constant with the theta angle) we have that steepest descent paths can only connect saddles coming from the same topological sector. Thus generally whenever a theta angle is present the path integral will

[2]Note that although when dealing with finite dimensional integrals the intersection numbers will always be integers, in infinite dimensions this is not necessarily guaranteed, see for example [26].

first split into a sum over topological sectors; then upon complexification of the coupling constant we will see Stokes phenomena between saddles in the same topological sector. Hence we again see the resurgence triangle structure as in Figure 2.1.

Thus we see resurgence and Picard-Lefschetz theory are two sides of the same coin. Resurgence gives us a framework to work in, where we can analyse perturbative series to calculate non-perturbative data. Picard-Lefschetz decompositions of path-integrals are harder to work with, as the integrals are infinite dimensional, but give us simultaneously a topological explanation for what is going on in resurgence theory, and a semi-classical explanation for the exponentially suppressed contributions to the transseries.

A particularly fortuitous class of examples where we can try to apply resurgence theory are supersymmetrically localisable field theories. Starting with Pestun's seminal work [27] for $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories on S^4, many quantities like partition functions and Wilson loops have been computed exactly using supersymmetric localisation; see [28] for a pedagogical introduction and a more complete set of references.

The central observation of localisation is that, given a supercharge $\mathcal{Q}$ that squares to a bosonic symmetry of the theory, the path integral only gets contributions from classical configurations that are fixed points of $\mathcal{Q}$, and small fluctuations around these configurations. Heuristically, denoting the fields of a theory by ϕ and the action by $S[\phi]$, we see that adding a $\mathcal{Q}$ exact term, $\mathcal{Q}(V[\phi])$, to the action, where $V[\phi]$ is chosen to be invariant under the bosonic symmetry, doesn't change the result of the path integral:

$$
\begin{aligned}
\frac{d}{dt}\int [D\phi] e^{-S[\phi]+tQ(V)} &= \int [D\phi] Q(V) e^{-S[\phi]+tQ(V)} \\
&= \int [D\phi] Q\left(V e^{-S[\phi]+tQ(V)}\right) = 0 \ .
\end{aligned}
\tag{2.1.5}
$$

The second equality here holds because $\mathcal{Q}(S[\phi]) = 0$, as does $\mathcal{Q}^2(V[\phi])$. The final equality holds as it is just a total derivative. We can thus take the $t \to \infty$ limit without changing the result of the path integral. In this limit, assuming we pick a sensible negative definite $\mathcal{Q}(V[\phi])$, only configurations in the neighbourhood of solutions of $\mathcal{Q}(V[\phi]) = 0$ will contribute. Calling these solutions ϕ_{cr}, we can expand the fields as $\phi = \phi_{\text{cr}} + \frac{1}{\sqrt{t}}\delta\phi$. Then in the $t \to \infty$ limit only the quadratic terms will survive, leaving us with a Gaussian integral to perform over $\delta\phi$. The final answer is in the form of a sum over $\phi_{\rm cr}$, each term containing a ratio of bosonic and fermionic one-loop determinants, weighted by $e^{-S[\phi_{\rm cr}]}$, with an integral over any bosonic zero modes of each $\phi_{\rm cr}$. Thus the complicated path integral is simplified greatly to a sum of finite dimensional integrals.

This method is very general and can be applied to theories living on different manifolds, in various numbers of dimensions, and with various amounts of supercharges. For example one can consider $\mathcal{N}=2$ theories on a squashed S^4 [29], or in three dimensional $\mathcal{N}=2$ on a round [30] or squashed sphere [31], or similarly going to two dimensional $\mathcal{N}=(2,2)$ theories on a sphere [32, 33] or an ellipsoid [34].

Importantly in all these cases the exact localised partition functions and other observables can be written as a perturbative part plus non-perturbative sectors, i.e. as a transseries [18]. Hence a very natural question is whether or not one can apply resurgent methods to these quantities and reconstruct the complete answers from the purely perturbative data. This question was analysed in [35, 36, 37, 38, 39] and it was realised that in supersymmetric field theories the resurgence story is not as straightforward.

This constructive resurgence program is a very powerful method allowing us to reconstruct non-perturbative physics from perturbative data but ultimately it relies on the asymptotic nature of the perturbative coefficients. However there exist interesting theories for which magic cancellations between diagrams take place, effectively making perturbation theory a convergent expansion, or even better cases for which there are only a finite number of non-vanishing perturbative coefficients. For this class of “special” theories it seems impossible that we can extract non-perturbative information from perturbation theory via a straightforward use of the resurgence program as we do not have an asymptotic series to begin with.

One might think that, due to cancellations between bosons and fermions, supersymmetric theories would be the perfect candidates for this “good” but “bad” scenarios. However just requiring the theory to be supersymmetric is not a guarantee of a convergent perturbative expansion for every physical observable. In [35, 36] the authors considered different supersymmetric theories in 3 and 4 dimensions (see also [37, 38]) and analysed in great details the weak coupling expansion of particular observables obtained from supersymmetric localization. Despite supersymmetry the authors showed that the perturbative expansions of the considered observables in 4-d $\mathcal{N}=2$ super Yang-Mills (SYM) were asymptotic but Borel summable, a consequence of the absence of neutral bions configurations as argued in [40, 41]. However using resurgent calculus the authors of [36] were able to extract important non-perturbative information from the perturbative data, although a semi-classical interpretation in terms of microscopic physics for some of these non-perturbative effects is still missing, while for the 3-d $\mathcal{N}=2$ case discussed in [38] the semi-classical origin of these non-perturbative contributions was recently understood [42] in terms of complexified supersymmetric solutions.

If we consider $\mathcal{N}=4\,SU(N)$ SYM in the planar limit the situation changes slightly as we can compute exact quantities using integrability and, thanks to the large number of supersymmetries, the weak coupling expansions of various physical quantities, for example the cusp anomalous dimension [43] and the dressing phase [44], have indeed finite radius of convergence. However not everything is lost from the resurgence point of view since it now happens that the strong coupling expansions of these two observables give rise to asymptotic series; see [43] and [45] respectively. The full resurgence machinery can be then applied to the strong coupling side of planar $\mathcal{N}=4$ SYM to obtain the complete transseries for the cusp anomaly [46, 47] and the dressing phase [48] leading to important implications for weak/strong coupling interpolation with the stringy $\mathrm{AdS}_5\times\mathrm{S}^5$ side, although the semi-classical origin of the non-perturbative effects predicted in [48] is still somewhat mysterious.

The strong coupling side of planar $\mathcal{N}=4\,SU(N)$ SYM can also be studied within the context of the AdS/CFT correspondence. In particular it was realised in [49] that the hydrodynamic gradient series for the strongly coupled $\mathcal{N}=4$ super Yang-Mills plasma is only an asymptotic expansion leading to the works [50, 51, 52] dealing with resurgence and resummation issues in the fluid context of $\mathrm{AdS}_5/\mathrm{CFT}_4$.

There are however cases for which we only have access to a convergent weak coupling expansion but we do nonetheless expect non-perturbative physics to be present and for which we do not know an easy way to tackle the strong coupling side with the hope to be able to apply resurgence there. Perhaps the most emblematic example of this sort can be found in supersymmetric quantum mechanics [53] where we can construct simple models for which the ground state energy is zero to all orders in perturbation theory but we do expect non-perturbative physics to play a role. It would seem that in these cases perturbative and non-perturbative data cannot possibly have anything in common with one another, contrary to what is usually advertised in the resurgence program. The authors of [54, 55] started precisely from this puzzle and considered two very simple supersymmetric quantum mechanical models: the double Sine-Gordon (DSG) and the tilted double well (TDW). The DSG ground state energy has a trivial perturbative expansion and a normalizable ground state, i.e. susy is preserved and $E_0 = 0$ exactly, however the system has real instantons that somehow do not give rise to the expected exponentially suppressed contributions. For the TDW the ground state energy in perturbation theory still vanishes but the model does not have a supersymmetric ground state and its energy should be lifted non-perturbatively although the model does not possess real non-perturbative saddles.

The solutions to both puzzles come from a particular realisation of resurgence theory that the authors of [55] named Cheshire Cat resurgence because very much like the magical Wonderland creature, the lingering grin of resurgence can be still seen from perturbation theory even when its entire body has completely disappeared. The solution is as elegant as simple; we just need to break slightly supersymmetry by declaring that the fermion number in each superselection sector is not an integer anymore but a complex parameter ζ. Once the fermion number is an arbitrary parameter the perturbative expansion of the ground state energy in both systems becomes immediately asymptotic; the body of the cat has appeared once more.

In the deformed DSG case we can apply standard resurgent calculus and obtain a complete transseries expression for the ground state energy that contains not just the perturbative series but also contributions from real as well as complex saddle points [56, 57]. As we send the deformation parameter $\zeta \to 0$ the perturbative series truncates, the contribution coming from the real and complex bions cancel one another because of a hidden topological angle [58], and the ground state energy is exactly zero thanks to the topological quantum interference[3] between different saddle contributions [60]. The role of complex saddles is crucial for this cancellation and their contribution can be really obtained from a semi-classical calculation [61, 62]. However these results have not been yet compared against the predictions coming from resurgent analysis of [63], in which the exact same deformed DSG model is obtained from dimensionally reducing the two dimensional $SU(2)$ η-deformed principal chiral model and for which the complex bions can be promoted to soliton solutions in the complexified QFT.

A similar story holds for the deformed TDW case: when the deformation parameter ζ is non zero the perturbative expansion is asymptotic and we can use resurgent analysis to construct from the perturbative data the contribution of the complex bions to the ground state energy. As we send $\zeta \to 0$ the perturbative expansion reduces to zero, while the complex bions remain, as there are no real bions to cancel them, producing non-perturbative contributions to the ground state energy, i.e. supersymmetry is indeed broken. Even if the perturbative expansion truncates both in DSG and TDW we can still use Cheshire Cat resurgence to extract non-perturbative physics from perturbative data.

The question is can we generalise this Cheshire Cat resurgence method to be used in susy QFTs where the perturbative series also seems to be oblivious to the non-perturbative data? We study this in relation to 2-d $\mathcal{N} = (2,2)$ theories on a sphere [32, 33], 3-d $\mathcal{N} = 2$ theories on a squashed sphere [31], and make comments on how it might be generalised to 4-d $\mathcal{N} = 2$ theories on a sphere [27].

There are two further questions we would like to tackle in this thesis. As discussed already, in [19, 20] the authors noted the existence of the resurgence triangle. In our analysis of 3-d $\mathcal{N} = 2$ theories we will see that this structure seems to have disappeared. We will ask how this structure might be recovered, and develop a deformation for doing this in chapter 4.

[3]In [59] the authors presented a realisation of the same effect in a very nice and simple example involving Bessel functions. We thank Gerald Dunne for discussions on this point.

Finally, in [64, 65, 66], the authors managed to develop a method for calculating data in all columns of the resurgence triangle from data in a single column, called the Dunne-Ünsal relation. In chapter 5 we will use properties of these susy theories to develop similar relations for these theories.

2.2 A review of resurgence and Picard-Lefschetz decomposition methods

We now review the Resurgence and Picard-Lefschetz decomposition methods. In this section we will give an overview of the procedures, and in the next section we will give a zero-dimensional example that illustrates these methods, and also the Cheshire Cat principle. For other excellent expositions of these methods (save the Cheshire Cat method) the reader may like to consult [67, 68, 21, 69]. See also [23, 24] for more on Picard-Lefschetz theory.

2.2.1 Resurgence

Let us consider a familiar scenario in physics. We are trying to calculate some observable that depends on some coupling constant, $\mathcal{O}(g)$, and we have managed to calculate a series expansion[4] for it;

$$\mathcal{O}(g) = \sum_{n=0}^{\infty} c_n g^n \quad . \tag{2.2.1}$$

Now let us assume that c_n goes as $\Gamma(n+\Delta)$, i.e. the series has zero radius of convergence. In order to get a finite answer we need to use a resummation procedure, and the one we turn to is Borel resummation[5].

[4]In this review we will do everything in the weak coupling regime, i.e. small g. In situations where we have a strong coupling expansion in inverse g, a simple generalization of this is easily found by sending $g \to 1/g$ in all the equations in this section.

[5]Of course there are many different resummation procedures we could use. The reason for using Borel resummantion is that it turns out, as we will see, to give us a way of computing non-perturbative data from perturbative data. This is not the case with other resummation methods.

First we define the *directional Laplace transform*

$$\mathcal{S}_\theta[\Phi](g) = \int_0^{e^{i\theta}\infty} dx\, e^{-x/g}\Phi(x) \quad . \tag{2.2.2}$$

It is important to note that

$$g^{-\Delta}\mathcal{S}_0[x^{n-1+\Delta}](g) = \Gamma(n+\Delta)g^n \quad . \tag{2.2.3}$$

Using this, we now see the following

$$\begin{aligned} \mathcal{O}(g) &= \sum_{n=0}^{\infty} c_n g^n \\ &= g^{-\Delta}\sum_{n=0}^{\infty} c_n \frac{\mathcal{S}_0[x^{n-1+\Delta}](g)}{\Gamma(n+\Delta)} \\ \text{“}=\text{”} &\ g^{-\Delta}\mathcal{S}_0\left[\sum_{n=0}^{\infty} c_n \frac{x^{n-1+\Delta}}{\Gamma(n+\Delta)}\right](g) \\ &= g^{-\Delta}\mathcal{S}_0\left[\Phi^{(0)}\right](g) \quad . \end{aligned} \tag{2.2.4}$$

There are two things to note here. First, in the last line, we have defined the *Borel resummed* observable to be

$$\Phi^{(0)}(x) = \sum_{n=0}^{\infty} c_n \frac{x^{n-1+\Delta}}{\Gamma(n+\Delta)} \quad . \tag{2.2.5}$$

In $\Phi^{(0)}(x)$ we have divided out by the factorial divergence, rendering it to have finite radius of convergence. In principle, we can now perform the sum returning a function, and then do the inverse Laplace transform of this function to get back to our observable.

The second thing to note is that the step taken across the “ = ” is obviously not allowed mathematically. We cannot swap an integral and a sum when the series is asymptotic. For the purposes of this review, we will not take steps towards making a more rigorous argument (for such steps the reader may like to consult [13]). For now we define

$$\zeta_0(g) = g^{-\Delta}\mathcal{S}_0\left[\Phi^{(0)}\right](g) \quad , \tag{2.2.6}$$

which is considered to be the perturbative answer after Borel resummation.

We now have a finite answer for $\mathcal{O}(g)$. We typically find, however, that $\zeta_0(g)$ is not unique, due to the presence of poles and branch cuts on the real x axis of $\Phi^{(0)}$. Thus the contour for the Laplace transform is forced into the complex x plane, and the way this is done introduces some ambiguity. From now on we refer to the complex x plane as the Borel plane. Put another way, $\mathcal{S}_0\left[\Phi^{(0)}\right](g)$ is in fact not well defined, but we need to consider it as a limit as $\theta \to 0$, and we will get different answers as we do this from above or below 0.[6] We expect our theory to be real for real g, and smooth as we cross $\mathrm{Im}(g) = 0$ for the real part of g positive. Thus we need some way of removing these ambiguities.

We can already see that the locations of the branch points/poles in the Borel plane are tightly related to the coefficients of the asymptotic perturbation expansion. For example, consider the perturbation series

$$\sum_{n=0}^{\infty}(-1)^n\Gamma(n+1)g^{n+1} \quad . \tag{2.2.7}$$

This resums to

$$\Phi^{(0)}(x) = \frac{1}{1+x} \quad . \tag{2.2.8}$$

Meanwhile the perturbation series

$$\sum_{n=0}^{\infty}\Gamma(n+1)g^{n+1} \quad , \tag{2.2.9}$$

resums to

$$\Phi^{(0)}(x) = \frac{1}{1-x} \quad . \tag{2.2.10}$$

This illustrates a general feature of asymptotic perturbative series, that series with coefficients of alternating sign will resum to a functions with poles or branch points on the negative real axis. Meanwhile series with coefficients all positive will resum to functions with poles or branch cuts on the positive real axis. When there are no poles or singularities on the positive real axis of the Borel plane, the series is called *Borel summable.*

[6]Because of the presence of presence of poles and branch cuts on the real axis, we are forced to complexify x. This in turn forces us to consider other non-analytic parts of the Borel plane. The upshot is that all non-analyticities in the Borel plane will correspond to non-perturbative contributions to the theory. Thus we are forced to consider complex contributions with complex action.

These ambiguities mean that the directional Laplace transformation $\mathcal{S}_\theta$ of $\Phi^{(0)}(x)$ will jump as we vary θ. These jumps are encoded by the *Stokes automorphism* $\mathfrak{S}^\theta$, which is defined by the equation

$$\begin{aligned} \mathcal{S}_{\theta^+} &= \mathcal{S}_{\theta^-} \circ \mathfrak{S}_\theta \\ &= \mathcal{S}_{\theta^-} \circ (\mathrm{Id} + \mathrm{Disc}_\theta) \ . \end{aligned} \tag{2.2.11}$$

Here Disc_θ is the discontinuity of the Laplace transformation in the θ direction, and θ^+ and θ^- denote taking the limit as the direction goes to θ from above and below respectively.

Let us consider an example where we have one branch cut, with Branch point on the real positive axis in the Borel plane, at $x = S_1$. We now have the non-trivial Stokes automorphism given by

$$\begin{aligned} \mathcal{S}_{\theta^+}\left[\Phi^{(0)}\right](g) &= \mathcal{S}_{\theta^-}\left[\Phi^{(0)}\right](g) + \left(\int_{S_1}^{\infty+i\epsilon} - \int_{S_1}^{\infty-i\epsilon}\right) dx\, e^{-x/g}\Phi^{(0)}(x) \\ &= \mathcal{S}_{\theta^-}\left[\Phi^{(0)}\right](g) + \int_{S_1}^{\infty} dx\, e^{-x/g}\mathrm{Disc}_0(\Phi^{(0)})(x) \\ &= \mathcal{S}_{\theta^-}\left[\Phi^{(0)}\right](g) + e^{-S_1/g}\int_0^{\infty} dx\, e^{-x/g}\mathrm{Disc}_0(\Phi^{(0)})(x+S_1) \ . \end{aligned} \tag{2.2.12}$$

Here we have defined $\mathrm{Disc}_0(\Phi^{(0)})(x) = \Phi^{(0)}(x+i\epsilon) - \Phi^{(0)}(x-i\epsilon)$, defined initially only along the cut, and in the final line we have simply made a change of variables. Let us take the function $\mathrm{Disc}_0(\Phi^{(0)})(x)$ and use it as a seed defining a function on the whole complex x plane, and let us call the function $\Phi^{(1)}(x - S_1)$. Equation (2.2.12) then becomes

$$\mathcal{S}_{\theta^+}\left[\Phi^{(0)}\right](g) = \mathcal{S}_{\theta^-}\left[\Phi^{(0)}\right](g) + e^{-S_1/g}\int_0^{\infty} dx\, e^{-x/g}\Phi^{(1)}(x) \ . \tag{2.2.13}$$

Here we hit the heart of resurgence. $\Phi^{(0)}(x)$ is not an analytic function in the Borel plane, and so $\zeta_0(g)$ is not a uniquely defined function of g. But consider the following *transseries*:

$$\Phi^{(0)}(x) + \sigma_1 e^{-S_1/g}\Phi^{(1)}(x) \ . \tag{2.2.14}$$

This does not have discontinuities in θ upon directional Laplace transformation, so long as σ_1 jumps appropriately. σ_1 is called a *Stokes constant*, and it is dependant on θ. For general θ they are constant, but along *Stokes directions* where $\Phi^{(0)}(x)$ has a discontinuity, σ_1 jumps. For example, in the above example, as θ passes from just below 0, 0^-, to just above 0, 0^+, $\sigma_1 \to \sigma_1 + 1$.

Now in the theories we will be discussing in this thesis there will be multiple sources of non-analyticity in the Borel plane for $\Phi^{(0)}(x)$. Thus we will end up with more complicated transseries with infinite numbers of exponentially suppressed contributions, all being multiplied by their own Stokes constants. This will just be a simple generalisation of the above procedure. Each branch cut will be used as the seed for another function of x, and will encode the contribution coming from some non-perturbative background with classical action S_i, and will contribute to the transseries as $\sigma_i e^{-S_i/g}\Phi^{(i)}(x)$.

Thus, by demanding that there are no discontinuities in the complete resummed transseries, we can determine the full transseries with all the non-perturbative data from just the perturbative part of the transseries. This is the punchline of resurgence; we have used the asymptoticity of (2.2.1) to determine all the non-perturbative contributions.

The final thing to review here is how the perturbative series for the perturbative sector relates to the perturbative series for the non-perturbative sectors. These relations are encoded by what are know as the large-order relations. In order to to see this we now use a Cauchy-like argument to relate the branch cuts in the Borel plane encoding the non-perturbative effects to the perturbative expansion near the origin. Recall Cauchy's integral equation

$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma dz \frac{f(z)}{z - z_0} \quad . \tag{2.2.15}$$

Here Γ is a closed anti-clockwise curve encircling the point z_0, and $f(z)$ is an analytic function. For simplicity let us again assume that our function $\Phi^{(0)}(x)$ only has one branch cut on the positive real axis. Again, the following procedure can be generalised to other cases in obvious ways.

Let us define the function

$$\hat{\mathcal{O}}(g) = \mathcal{S}_{-\arg(g)}\left[\Phi^{(0)}\right](g) \quad . \tag{2.2.16}$$

Now this function has a branch cut along the ray $\arg(g) = 0$. Now taking Cauchy's integral equation, and widening the path so that it loops around at infinity, then from infinity to the origin just under the cut and then back out to infinity just over the cut (see Figure 1 of [9] for an illustration of such a contour), we find

$$\hat{\mathcal{O}}(g) = \frac{1}{2\pi i}\int_0^\infty dw \frac{\mathrm{Disc}_0(\hat{\mathcal{O}})(w)}{w-g} \quad . \tag{2.2.17}$$

Here $\mathrm{Disc}_0(\hat{\mathcal{O}})(w)$ is defined as $\hat{\mathcal{O}}(w+i\epsilon) - \hat{\mathcal{O}}(w-i\epsilon)$, again only along the positive real axis, but then analytically continued to be a function for the entire w plane. We now expand

$$\begin{aligned} \hat{\mathcal{O}}(g) &= \sum_{n=0}^{\infty} \hat{\mathcal{O}}_n g^n \quad , \\ \frac{1}{w-g} &= \sum_{n=0}^{\infty} g^n w^{-n-1} \quad . \end{aligned} \tag{2.2.18}$$

Plugging these into (2.2.17) and equating powers of g we thus find

$$\hat{\mathcal{O}}_n \sim \frac{1}{2\pi i}\int_0^\infty dw\; w^{-n-1}\mathrm{Disc}_0(\hat{\mathcal{O}})(w) \quad . \tag{2.2.19}$$

Note this is an asymptotic expression, due to us swapping the sum and the integral on the RHS when we substituted (2.2.19) into (2.2.18). The final steps are to swap $\mathrm{Disc}_0(\hat{\mathcal{O}})(w)$ for $e^{-S_1/w}\mathcal{S}_0\left[\Phi^{(1)}\right]$, where the g in the definition of $\mathcal{S}_\theta$ is swapped for w, and set $\arg(g) = 0$. Now for $\arg(g) = 0$ we have $\hat{\mathcal{O}}_n = c_n$. Moreover $\mathcal{S}_0\left[\Phi^{(1)}\right]$ can be expanded out to give a perturbation series around the non-perturbative background;

$$\mathcal{S}_0\left[\Phi^{(1)}\right](w) = \sum_{n=0}^{\infty} c_n^{(1)} w^n \quad . \tag{2.2.20}$$

We thus conclude that

$$\begin{aligned} c_n &\sim \sum_{m=0}^{\infty} \frac{1}{2\pi i}\int_0^\infty dw\; e^{-S_1/w} c_m^{(1)} w^{m-n-1} \\ &\sim \sum_{m=0}^{\infty} \frac{1}{2\pi i} c_m^{(1)} \Gamma(n-m) S_1^{m-n} \quad . \end{aligned} \tag{2.2.21}$$

We thus see that for large n, c_n is dominated by $c_0^{(1)}$, and so on[7]. Thus we can use the

[7]The reader may be worried that $\Gamma(n-m)$ is actually infinite for large enough m. The reason for this

large order coefficients of the perturbative expansion to read off the low order coefficients of the non-perturbative data.

The final thing to comment on in this section is the emergence of the resurgence triangle. Our resurgence calculations are in the form of an analysis in the perturbation series in g. Let us consider a theory with topological θ angle. Now we have an additional parameter in our theory, and non-perturbative contributions will typically come with a topological charge. Of course, an analysis of a perturbation series in g can't tell us any information about θ dependency, and thus about contributions with different topological charge. We thus see that resurgence will only tell us about non-perturbative contributions from non-perturbative effects with equal topological charge, recovering the structure of Figure 2.1. From a single element of a particular column we can use a resurgence analysis to calculate all the other data in that column. However, we cannot use resurgence to tell us about the data in any of the other columns.

2.2.2 The Picard-Lefschetz Method

Let us now suppose that we had our observable to calculate in the form of a path integral given by

$$\mathcal{O}(g) = \int [\mathcal{D}\phi] e^{-S[\phi]/g} \quad , \tag{2.2.22}$$

where ϕ is short-hand for the fields in the theory, and $S[\phi]$ is the effective action for the theory. We now describe the Picard-Lefschetz decomposition method for solving this path integral. This method can be thought of as a generalization of the method of steepest decent decomposition for finite dimensional integrals.

The first step is to holomorphize the fields,

$$\phi \to \hat{\phi} = \phi_R + i\phi_I \quad , \quad S[\phi] \to S[\hat{\phi}] = S_R + iS_I \quad . \tag{2.2.23}$$

Here S_R is a real *Morse function*; that is a real valued function whose critical points are

divergence is that the expression is asymptotic, and no longer valid for such large m.

non-degenerate[8]. S_I is the imaginary part of the holomorphic action. In doing this we have doubled the dimension of the field step, but we will be integrating over half dimensional cycles withing this holomorphic field space.

The next step is to find all the finite action critical points of the holomorphic action; that is the finite action solutions of

$$\frac{\delta S[\hat{\phi}]}{\delta \hat{\phi}} = 0 \ . \tag{2.2.24}$$

Put another way, these are the solutions of the Euler-Lagrange equations of motion whose action is finite. We label the i^{th} critical point by $\hat{\phi}_i^{cr}$.

The third step is to find the steepest decent and ascent cycles cycles from these critical points. These cycles are half dimensional cycles which are solutions to the complex *Morse flow equations*, given by

$$\frac{d\hat{\phi}}{d\tau} = \pm \overline{\frac{\delta S[\hat{\phi}]}{\delta \hat{\phi}}} \ , \quad \lim_{\tau \to -\infty} \hat{\phi}(\tau) = \hat{\phi}_i^{cr} \ . \tag{2.2.25}$$

Here τ is a "time" parameter parametrising the flow. The solution with the $+$ sign attached to the i^{th} critical point is called the unstable manifold, or the downward manifold, or the steepest decent manifold, or as we shall call it here, the J_i cycle. The solution with the $-$ sign is called the stable/upward/steepest ascent manifold, or as we shall call it here, the K_i cycle.

An important thing to notice about these solutions is that the imaginary part of the action, S_I, is constant along the J and K cycles. Let us see this for the J cycles:

$$\begin{aligned}
\frac{dS_I}{d\tau} &= \frac{1}{2}\left(\frac{dS_I[\hat{\phi}]}{d\tau} - \frac{d\bar{S}_I[\hat{\phi}]}{d\tau}\right) \\
&= \frac{1}{2}\left(\frac{d\hat{\phi}}{d\tau}\frac{\delta S_I[\hat{\phi}]}{\delta\hat{\phi}} - \overline{\frac{d\hat{\phi}}{d\tau}}\,\overline{\frac{\delta S_I[\hat{\phi}]}{\delta\hat{\phi}}}\right) \\
&= \frac{1}{2}\left(\frac{d\hat{\phi}}{d\tau}\overline{\frac{d\hat{\phi}}{d\tau}} - \overline{\frac{d\hat{\phi}}{d\tau}}\frac{d\hat{\phi}}{d\tau}\right)
\end{aligned}$$

[8]The use of Morse functions in physics has a rich history. Initiated by Witten in [70], where a proof of the Morse inequalities was given using supersymmetric quantum mechanics, they have found many applications in supersymmetric QFT, leading to deep connections between QFT and pure mathematics.

$$= \quad 0 \quad . \tag{2.2.26}$$

The calculation for the K cycles is very similar. This fact will be important later when we come to discuss the emergence of the resurgence triangle.

A further important point about these solutions is the value of the real part of the action is monotonically increasing as we move along the J cycles associated to a particular critical point. This is a simple variation of the above calculation:

$$\begin{aligned}
\frac{dS_R}{d\tau} &= \frac{1}{2}\left(\frac{dS_I[\hat{\phi}]}{d\tau} + \frac{d\bar{S}_I[\hat{\phi}]}{d\tau}\right) \\
&= \frac{1}{2}\left(\frac{d\hat{\phi}}{d\tau}\frac{\delta S_I[\hat{\phi}]}{\delta\hat{\phi}} + \overline{\frac{d\hat{\phi}}{d\tau}\frac{\delta S_I[\hat{\phi}]}{\delta\hat{\phi}}}\right) \\
&= \frac{1}{2}\left(\frac{d\hat{\phi}}{d\tau}\frac{d\overline{\hat{\phi}}}{d\tau} + \frac{d\overline{\hat{\phi}}}{d\tau}\frac{d\hat{\phi}}{d\tau}\right) \\
&= \left|\frac{d\hat{\phi}}{d\tau}\right|^2 \quad .
\end{aligned} \tag{2.2.27}$$

This will be important in just a moment when we consider the convergence of the integrals we decompose (2.2.22) into.

The final step is to decompose the path integral into a sum over integrals over the J cycles. We call the original contour Γ. For this to be a *good* contour, it must start and end in, or on the boundary of, *good regions*. These are regions where the integrand of the path integral goes to 0 at infinity. In *bad regions* the integrand diverges at infinity. Because of (2.2.27), the J cycles will always end in good regions (assuming we are not on a Stokes wall; more on this in a moment). If Γ is good and we are not on a Stokes wall, we decompose Γ as

$$\Gamma = \sum_i n_i J_i \quad . \tag{2.2.28}$$

Here J_i are the various J cycles, which are attached to the i^{th} critical points. n_i are known as the intersection numbers. They are defined as

$$n_i = (K_i, \Gamma) \quad . \tag{2.2.29}$$

This is the number of times (counted with sign) the K cycle from the same critical point associated with J_i intersects the original contour. We now write the partition function as

$$\begin{aligned} \mathcal{O}(g) &= \int_\Gamma [\mathcal{D}\phi] e^{-S[\phi]/g} \\ &= \sum_i n_i \int_{J_i} [\mathcal{D}\phi] e^{-S[\phi]/g} \quad . \end{aligned} \tag{2.2.30}$$

Importantly we have already seen that the real part of the action is monotonically increasing as we move along the J cycles. We see therefore that $e^{-S[\phi]/g}$ will become exponentially suppressed as we move along the integration cycles and the integrals in (2.2.30) will converge. This is the Picard-Lefschetz decomposition. We have taken our path integral, and written it as a sum of integrals that we know absolutely converge.

How does this relate to the resurgence framework we have looked at in the previous section? The answer stems from the fact that there is an ambiguity in how the Picard-Lefschetz decomposition can be performed. Whilst the real part of the action is monotonically increasing as we move along the J cycles, it is not necessarily true that it will go all the way to infinity. In particular the J cycle may end on another saddle point, rather than going out all the way to infinity. In this case our decomposition breaks down; not all the J cycles end in the good regions, and so Γ can't be written as a sum of J cycles.

When does this ambiguity occur? Recall that the imaginary part of the action is constant as we move along the J cycles. Thus, if the action at two distinct saddles has the same imaginary part, it is possible (in principle, but not guaranteed) for a J cycle to connect them. Conversely, if the imaginary parts of two saddles are not equal then it is not possible for them to be connected by a J cycle. Now if two saddles are connected by a J cycle, typically a small adjustment of the argument of g, thinking of g as complex now, will change this. Thus, as we vary the argument of g, the J cycle will pass along one side of the saddle, then at a special value for the argument it will connect the saddles, and the beyond that it will pass along the other side of the saddle.

Thus, as we vary the argument of g, we pass through special values of the argument where the J cycles from particular saddles will jump across other saddles, and the intersection numbers will jump. This is another example of *Stokes phenomena* which we saw in the discussion surrounding (2.2.14). The jump in the intersection numbers can be described by a monodromy matrix. Suppose that we have a saddle labeled i, and its contribution to the path integral coming with the intersection number n_i, and a further saddle labeled j with associated intersection number n_j. Let us also suppose that at $\arg(g) = \theta_0$ there is a J cycle associated to the i saddle that connects these two saddles. Then as θ passes from just less that θ_0, θ_{0^-}, to just above θ_0, θ_{0^+}, we have

$$\begin{pmatrix} n_i \\ n_j \end{pmatrix} \rightarrow \begin{pmatrix} n_i \\ n_j + n_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n_i \\ n_j \end{pmatrix} \quad . \tag{2.2.31}$$

The matrix here is called the *monodromy matrix*. It describes how the intersection numbers jump as we pass through critical arguments of g.

It is these Stokes phenomena that allow us to make the connection with resurgence theory. In the previous section we saw that at particular directions of directional Laplace transformation, we hit branch cuts or poles, leading to a jump in the Stokes constants. Here we see similar jumps in the intersection numbers as we vary the argument of g. For finite dimensional integrals, the J cycle integrals of Picard-Lefschetz theory, and the directional Laplace transformation of a resurgence analysis, are indeed equivalent[13, 71, 72]. It remains an open problem to see if this carries over to the infinite dimensional case, where various aspects of both resurgence and Picard-Lefschetz theory are still poorly understood (e.g. the fact that couplings run, or that there can be infinite numbers of intersections between cycles).

Some important things emerge from this decomposition of the path integral. Firstly we see the appearance of the transseries. Each integral in (2.2.30) can be calculated perturbatively for small g, and the answer we will get is of the form

$$\mathcal{O}(g) = \sum_i n_i e^{-S(\phi_i^{cr})/g} \sum_{n=0}^{\infty} c_n^{(i)} g^n \quad . \tag{2.2.32}$$

This is of course just the usual form of a transseries. However the Picard-Lefschetz method of deriving this transseries has given us an interpretation of the Stokes coefficients. Here they are the intersection numbers of the K_i cycle associated to a critical point, and the original cycle.

The structure we want to discuss once again, that emerges from the Picard-Lefschetz decomposition of the path-integral, is the resurgence triangle. Let us consider the application of Picard-Lefschetz theory to a theory with a topological θ angle. Saddle points with different topology will have different topological charge, and thus the action of the such saddles will have a different imaginary part. Thus the J cycles will never link them, and so we never observe Stokes phenomena between saddles with different topological charge. Thus, we can split the saddles into different topological sectors, and again observe the structure pictured in Figure 2.1. Within these columns we expect to see Stokes phenomena, but not between saddles in different columns.

2.3 A 0-Dimensional Toy Example

In this section we will now put these tools together that we have explained in the previous section, and consider a simple example directly related to (2.2.1). In the next two subsections we look at two different ways of solving the 1-dimensional integral

$$Z(\xi) = \int_{-\infty}^{\infty} dx\, e^{-i\xi x + (N+\Delta)\log\Gamma(-ix)} \quad . \tag{2.3.1}$$

Throughout N will be a positive integer, and Δ some real number greater than $-N$. Though Z depends on N and Δ we will suppress this and simply write it as $Z(\xi)$ so as not to clutter equations. The coupling has changed from g to $\frac{1}{\xi}$ to match notation later on in this thesis, where the coupling will the the Fayet-Iliopoulos constant. We will be interested in the large ξ expansion of this integral, and its resummation. (2.3.1) can be considered to be a very simple toy model of the 2-d and 3-d models that we will be considering later, that captures many of the interesting points of those models.

Before we start our analysis there are three important things to note about this integral. The first is that $Z(\xi)$ is real. This is easy to see by taking the complex conjugate and making the substitution $x \to -x$.

Secondly we note that this integrand has branch cuts (or poles if Δ is an integer). We take the contour of integration to go above the branch point at the origin, and arrange the branch cuts so they go vertically down the imaginary x axis. The exponent has infinite negative real part for $x = re^{i\theta}$, $-\pi < \theta < 0$, $r \to \infty$, which means contributions from the integrand are infinitely suppressed around infinity in the lower half plane. Combining this with Cauchy's integral theorem means that we can rewrite the contour of integration as coming from $-\epsilon - i\infty$ up and around the origin, and back to $+\epsilon - i\infty$, where ϵ is some infinitesimal positive real number.

Thirdly note that when Δ is an integer, as mentioned above, the branch cuts become poles and the integral can easily be calculated from the residues of the poles. As an example, consider the case when $N = 1$ and $\Delta = 0$. In this case a simple residue calculation renders

$$Z(\xi)|_{N=1,\Delta=0} = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-n\xi} \quad . \tag{2.3.2}$$

Note that this is a sum over exponentially suppressed terms, a transseries, but that there is no asymptotic series associated to each term. $N = 1$, $\Delta = 0$ is a Cheshire Cat point. We will comment more on this in the next subsection.

2.3.1 Resurgence and Cheshire Cat points

In this subsection we will look at this zero dimensional example in full detail, in order to get a grip on the resurgence method, and to see a simple example of Cheshire Cat resurgence. Let us take (2.3.1) and try and solving it using resurgence.

We first need to get it in the form of a perturbation series. Closing the contour as described above and substituting $x \to -ix$ one finds

$$Z(\xi) = -i \left(\int_0^{\infty+i\epsilon} - \int_0^{\infty-i\epsilon} \right) dx \; e^{-\xi x + (N+\Delta)\log\Gamma(-x)} \quad . \tag{2.3.3}$$

We now use

$$\log\Gamma(-x \pm i\epsilon) = \log\Gamma(1 - x \pm i\epsilon) - \log(x) \mp i\pi \quad , \tag{2.3.4}$$

to shift the $\log\Gamma(-x)$ function to $\log\Gamma(1-x)$. This shift gives us a factor of $e^{\pm(N+\Delta)\pi i}$ for the integral above or below the cut. We then Taylor expand $e^{(N+\Delta)\log\Gamma(1-x)}$ as

$$e^{(N+\Delta)\log\Gamma(1-x)} = \sum_{n=0}^{\infty} a_n x^n \quad . \tag{2.3.5}$$

Here a_n are the coefficients of the Taylor expansion. The first few of these coefficients are given by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= (N+\Delta)\gamma \\ a_2 &= \frac{1}{2}\left(\frac{1}{6}\pi^2(N+\Delta) + \gamma^2(N+\Delta)^2\right) \quad . \end{aligned} \tag{2.3.6}$$

Here γ is the Euler-Mascheroni constant. Swapping the sum and the integral, we are thus left with

$$Z(\xi)“=”\sum_{n=0}^{\infty} -i\left(e^{(N+\Delta)\pi i}\int_0^{\infty+i\epsilon} - \; e^{-(N+\Delta)\pi i}\int_0^{\infty-i\epsilon}\right) dx\; e^{-\xi x} a_n x^{n-N-\Delta} \quad . \tag{2.3.7}$$

We now take the $\epsilon \to 0$ limit and perform the integrals [9]. We thus find the perturbative expansion for (2.3.1) is given by

$$Z(\xi)“=”\zeta_0(\xi) = 2\sin(\pi(N+\Delta))\xi^{N+\Delta-1}\sum_{n=0}^{\infty} a_n \xi^{-n}\Gamma(n+1-N-\Delta) \quad . \tag{2.3.8}$$

The coefficients in the expansion in (2.3.8) are clearly increasing factorially, and the series has zero radius of convergence, with a particular exception. If Δ is an integer $\sin(\pi(N+\Delta)) = 0$. However we also see that $\Gamma(n+1-N-\Delta)$ is infinite for the first $N+\Delta$ terms. When this limit is taken we are left with a truncating series containing just $N+\Delta$ terms. For example, in the $N=1$ and $\Delta \to 0$ case we are just left with one term, the first term of (2.3.2).

We here see our theory has some special points, when Δ is an integer, where the perturbation series is rendered non-asymptotic. These points are known as *Cheshire Cat points*. If

[9] In expanding the exponential and swapping the sum and the integral we have lost the branch cut structure of the integrands. This step in necessary in getting a perturbative expansion. Resurgence takes a perturbative expansion and reverses this procedure to uncover the branch cut structure of the Borel plane.

we started out wanting to do a resurgent analysis at one of these points we would not be able to proceed, as there is no asymptotic series to analyse. However a tiny deformation away from this point, setting Δ to be non-integer, renders the series asymptotic, and the full resurgence framework can be utilized. In the QFTs we will be studying in this thesis, the theories sit at precisely these kinds of Cheshire Cat points. In order to uncover the resurgence structure of the theory, we will need to deform the theory. As we will see shortly, once such a deformation is done, and a resurgence analysis performed, the results we find will be dependent on the deformation. We will then be able to return the deformation to 0, and keep all the non-perturbative data we have calculated.

Let us now leave Δ non-integer, and use resurgence to calculate the full non-perturbative transseries. The first step we take is to Borel resum this series. Dividing each term by the factorial contribution, and using (2.3.8), we find

$$\zeta_0(\xi) = 2\sin(\pi(N+\Delta)) \int_0^\infty dx \; e^{-\xi x + (N+\Delta)\log\Gamma(1-x)} x^{-(N+\Delta)} \quad . \tag{2.3.9}$$

We have $\zeta_0(\xi)$ as the Laplace transformation of a function that has branch cuts starting from the positive integers on the real axis and going out to infinity.

The next step is to use the branch cut structure of the Borel plane to to calculate the non-perturbative data for this integral. In order to do this we need to calculate the discontinuity from taking the Laplace transform contour to go either side of each of these cuts. The formula for the discontinuity across the cuts for the $\log\Gamma$ function is

$$\log\Gamma(-x+i\epsilon) - \log\Gamma(-x-i\epsilon) = -2\pi i(\lfloor x \rfloor + 1) \quad . \tag{2.3.10}$$

Here $\lfloor x \rfloor$ is the floor of x. The discontinuity we are after is given by

$$\mathrm{Disc}_0(\zeta_0)(\xi) = 2\sin(\pi(N+\Delta)) \left(\int_1^{\infty+i\epsilon} - \int_1^{\infty-i\epsilon} \right) dx \; e^{-\xi x + (N+\Delta)\log\Gamma(1-x)} x^{-(N+\Delta)} \quad . \tag{2.3.11}$$

Note that the discontinuity is completely imaginary; it is easy to see that complex conjugation simply gives a factor of -1. In order to get all the non-perturbative data separated into contributions coming from each background, we need to separate this discontinuity in to the discontinuity coming from each of the cuts. To do this we rewrite the integrals as

$$\int_1^\infty dx \quad = \quad \sum_{n=1}^\infty \int_n^{n+1} dx \quad . \tag{2.3.12}$$

Doing this, and making use of (2.3.10), we find

$$\begin{aligned}\mathrm{Disc}_0(\zeta_0)(\xi) = 2\sin(\pi(N+\Delta))\sum_{n=1}^\infty \int_n^{n+1} dx \qquad & e^{-\xi x+(N+\Delta)\log\Gamma(1-x-i\epsilon)} \\ & x^{-(N+\Delta)}\left(1-e^{-2\pi i n(N+\Delta)}\right) \ . \end{aligned} \tag{2.3.13}$$

We now need to get these integrals into the form of Laplace transforms. In order to do this we note that

$$\int_n^{n+1} dx = \int_n^\infty dx - \int_{n+1}^\infty dx \quad . \tag{2.3.14}$$

We use this to rewrite our integrals, and then we will need to make changes to variables to set the lower limit on all of the integrals to be 0, for which we will need to make use of (2.3.4).

Putting this all together, we find

$$\begin{aligned}\mathrm{Disc}_0(\zeta_0)(\xi) = 2\sin(\pi(N+\Delta))\sum_{n=1}^\infty e^{-n\xi}\int_0^\infty dx \qquad & e^{-\xi x+(N+\Delta)\log\Gamma(1-x-i\epsilon)} \\ & (x(x+1)...(x+n))^{-(N+\Delta)} \\ & \left(e^{-\pi i(n-2)(N+\Delta)}-e^{-\pi i n(N+\Delta)}\right) \ . \end{aligned} \tag{2.3.15}$$

We are now in a position to construct the transseries. Let's start by looking at the $n=1$ part. The contribution to the discontinuity from the $n=1$ part is given by

$$\begin{aligned} & 2\sin(\pi(N+\Delta))e^{-\xi}\int_0^\infty dx e^{-\xi x+(N+\Delta)\log\Gamma(1-x-i\epsilon)} \\ & \qquad (x(x+1))^{-(N+\Delta)}\left(e^{\pi i(N+\Delta)}-e^{-\pi i(N+\Delta)}\right) \\ = \ & 4i\sin^2(\pi(N+\Delta))e^{-\xi}\int_0^\infty dx e^{-\xi x+(N+\Delta)\log\Gamma(1-x-i\epsilon)}(x(x+1))^{-(N+\Delta)} \ . \end{aligned} \tag{2.3.16}$$

Now recall that the integral we started with is real, so we want our resummation to be real. This contribution to the discontinuity from the $n = 1$ sector is completely imaginary (up to further exponentially suppressed terms which we'll deal with in a moment), and we need it to be canceled by the first non-perturbative contribution to the transseries, which fixes the imaginary part of the transseries parameter; the imaginary part of the transseries parameter times the first non-perturbative contribution must be half the discontinuity from the perturbative sector, with sign picked appropriate to the direction of Laplace transform, such that the resummation is real. We thus have that our transseries to this order is given by

$$\begin{aligned} Z(\xi) &= 2\sin(\pi(N+\Delta)) \int_0^{\infty \pm i\epsilon} dx\; e^{-\xi x + (N+\Delta)\log\Gamma(1-x)} x^{-(N+\Delta)} \\ &\quad + (\sigma_1^R \mp i\sin(\pi(N+\Delta)))2\sin(\pi(N+\Delta))e^{-\xi} \int_0^{\infty \pm i\epsilon} dx\; e^{-\xi x + (N+\Delta)\log\Gamma(1-x)} \\ &\qquad (x(x+1))^{-(N+\Delta)} \\ &\quad + O(e^{-2\xi}) \quad . \end{aligned} \tag{2.3.17}$$

Here σ_1^R is some undetermined real constant. Resurgence will not typically fix the real part of the Stokes parameters, as we we are just requiring that the imaginary discontinuity be canceled. However in this case we are not working with an asymptotic series given out of the blue but one that is the solution to an integral, namely (2.3.1). In cases such as these, where the series is the solution to some known integral or differential equation say, we often can fix the real part of the Stokes parameter. In this case we have

$$\sigma_1^R = \cos(\pi(N+\Delta)) \quad . \tag{2.3.18}$$

This follows from splitting the contributions to (2.3.1) into contributions from each branch cut, when the contour is closed as described below (2.3.1).

To find the $n = 2$ contribution to the transseries, we need to not only consider the discontinuity from $\zeta_0(\xi)$, but we need to consider the discontinuity from the $n = 1$, $\zeta_1(\xi)$, contribution as well. Specifically, in (2.3.17), there is a discontinuity in both integrals, which the $n = 2$ non-perturbative contribution must cancel. Note that the jump in the $n = 2$ sector will depend on the value of σ_1^R, and so we find σ_2 depends on σ_1. We see that the Stokes parameters are highly intertwined.[10].

The calculation to find the $n = 2, 3, ...$ contributions is very similar to the above calculation. Once we have found the real parts of the Stokes parameters by comparing with (2.3.1), we find.

$$Z(\xi) = \sum_{n=0}^{\infty} e^{\mp n\pi i(N+\Delta)} 2\sin(\pi(N+\Delta)) \int_0^{\infty\pm i\epsilon} dx \quad e^{-\xi x+(N+\Delta)\log\Gamma(1-x)} \\ (x(x+1)...(x+n))^{-(N+\Delta)} \ . \tag{2.3.19}$$

The result of our efforts is that, using resurgence, we have managed to take the asymptotic series (2.3.8) and calculate the full non-perturbative transseries (2.3.19). The final thing to note is that, having done this, we can set Δ back to an integer, and keep all the non-perturbative data we have calculated.[11] This is *Cheshire Cat* resurgence. For special values of Δ it appeared that the perturbative series knew nothing about the non-perturbative contributions. However, upon even a slight deviation from these points, a full resurgence analysis can be performed, and the data calculated kept in the limit where we go back to one of the Cheshire Cat points.

2.3.2 Picard-Lefschetz Decomposition of the Path Integral

In this section we now want to shift our attention to the Picard-Lefschetz decomposition of the path integral, and in the same line of thought as the previous subsection we will consider the zero dimensional example (2.3.1). Now this has effective action

$$S[x] = i\xi x - (N+\Delta)\log\Gamma(-ix) \ . \tag{2.3.20}$$

This effective action has singularities when $x = 0, -i, -2i, ...$, and branch cuts starting from these points that we take to flow out to negative imaginary infinity.

[10] In this case in fact it turns out once we find the real parts of the Stokes parameters by comparing with (2.3.1) we only have a one-parameter family of solutions; $\sigma_i = (\sigma_1)^i$. Such transseries are referred to as a one-parameter transseries. For more on these see [73]

[11] Using (2.3.5) and expanding the integrals in (2.3.19) we will again get an asymptotic series in each non-perturbative sector. In the limit Δ goes to an integer we again find $\sin(\pi(N+\Delta))$ goes to 0, and an infinite part from the $\Gamma(n+1-N-\Delta)$ coefficients, rendering a truncating series in each sector with $N+\Delta$ terms.

The first step is to holomorphize the fields. We thus send $x \to z = x + iy$. The "equations of motion" are then given by

$$\frac{\partial S[z]}{\partial z} = 0 = i\xi + i(N+\Delta)\psi^{(0)}(-iz) \quad . \tag{2.3.21}$$

Likewise the equations for the J and K cycles are given by

$$\begin{aligned} \frac{dz(\tau)}{d\tau} &= \pm\left(-i\xi - i(N+\Delta)\psi^{(0)}(i\bar{z})\right) \quad , \\ \lim_{\tau\to-\infty} z(\tau) &= z_{\text{crit}} \quad . \end{aligned} \tag{2.3.22}$$

These equations are quite difficult to solve analytically. However numerically it is very easy. We can find the locations of the saddles perturbatively as well. For example, there is a saddle just above $z = 0$, and it is quickly found that

$$z_{cr} = \frac{i(N+\Delta)}{\xi} + \mathcal{O}\left(\frac{1}{\xi^2}\right) \quad . \tag{2.3.23}$$

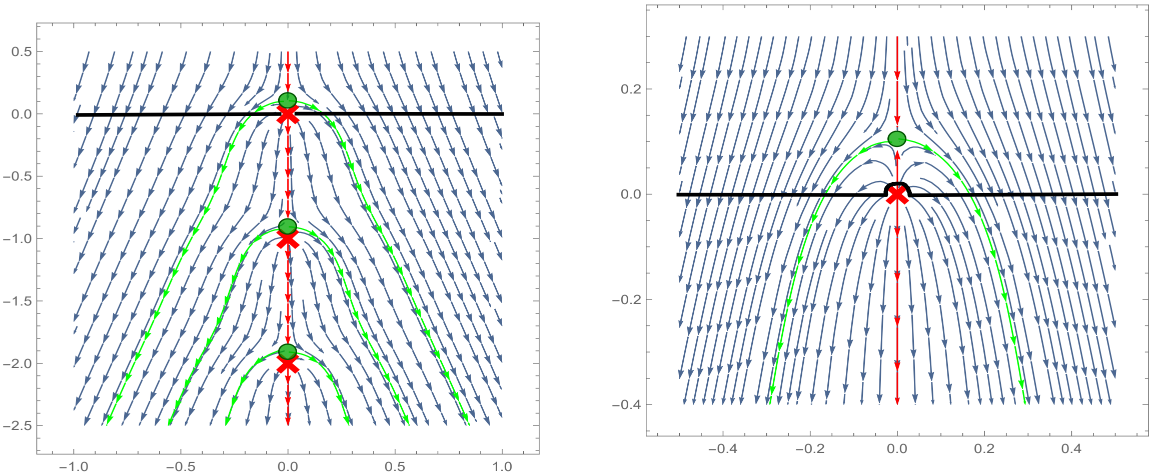
Let us look at some plots of the solutions.

We will consider, for simplicity, the case with $N = 1$ and $\Delta = 0$. In figure 2.2 we see the plots of the flows for $\xi = 10$. There are poles at $z = 0, -i, -2i, -3i, ...$, and saddle points just above each of these. The J cycles associated to each saddle flow to the sides and out to negative imaginary infinity. The K cycles flow vertically in the z plane from the saddles till they hit one of the poles.

Now the original contour of integration was the real axis, passing just above the pole at the origin. The only K cycle that intersects this is the K cycle from the perturbative saddle, that is the saddle just above the pole at the origin. This K cycle connects the saddle to the pole at the origin. Thus here we see that the integral can be written as

$$Z(\xi) = \int_{\mathrm{J}_0} dx\; e^{-i\xi x + (N+\Delta)\log\Gamma(-ix)} \quad . \tag{2.3.24}$$

Here J_0 is the J cycle associated to the perturbative saddle. We will not actually perform this integral as it won't prove particularly instructive. We see here (2.3.1) is written in the form of a transseries, but here the only non-zero Stokes parameter is n_0, the parameter associated to the perturbative saddle.



(a) Flows from the first three saddles.

(b) Closer view of the flows from the perturbative saddle.

Figure 2.2: Morse flow for (2.3.1) with $N = 1$, $\Delta = 0$, and $\xi = 10$. The black line shows the original integral contour; that is the real axis, passing infinitesimally above the pole at the origin. The green circles are saddles and the red crosses are poles. From each saddle the J cycles (in green) go off to the sides and eventually off to ∞. The K cycles (in red) flow up and down from the saddles to the nearest pole. Only the K cycle from the perturbative saddle hits the original integral contour.

Let us now consider what happens when we vary the argument of ξ. This is shown in figure 2.3.

We see that at approximately $\theta \approx 0.32$ Stokes phenomenon occurs, and the intersection numbers all jump. n_0, the intersection number for the perturbative saddle remains the same. The intersection numbers for all the other saddles jump:

$$n_i|_{\theta<0.32,\xi=10} = 0 \to n_i|_{\theta>0.32,\xi=10} = 1 \quad . \tag{2.3.25}$$

Viewing this in terms of the monodromies we discussed earlier is that $n_1 \to n_1 + n_0$, and then simultaneously $n_2 \to n_2 + n_1$, and so on. Thus for $\theta > 0.32$ we wee that (2.3.1) may be written as

$$Z(\xi) = \sum_{i=0}^{\infty} \int_{\mathrm{J}_i} dx \; e^{-i\xi x + (N+\Delta)\log\Gamma(-ix)} \quad . \tag{2.3.26}$$

Here again J_i is the J cycle attached to the i^{th} non-perturbative saddle. We see that the integral is in the form of a transseries, now with non-zero Stokes parameters for all of the non-perturbative contributions.

2.3.3 Drawing the strings together

In this section we have seen an example of how (2.3.1) can be solved with a resurgence analysis and a Picard-Lefschetz analysis. Here we want to summarise what has happened, and see more precisely how these two analyses are related.

We saw that using a resurgence analysis of the perturbative series we were able to determine the non-perturbative contributions, and sum them up in a transseries. We were also able to determine the jumps of the Stokes parameters completely. We weren't able to fix the real parts of the Stokes parameters with resurgence alone. However in this example we saw that the parameters highly intertwined, and in this case we were left with a one-parameter transseries. Starting with a perturbative series produced from nowhere, this would be the end of the road, but as we had our transseries as the solution to an integral, we were able to fix the real parts of the transseries parameters as well.

Now as we varied the argument of ξ we observed Stokes phenomenon. In this case the Stokes phenomenon occurred as $\arg(\xi)$ passed through 0, and the transseries parameters jumped between $e^{\pm n\pi i(N+\Delta)}$ as we crossed $\arg(\xi) = 0$. How do we relate this jump to the jump in intersection numbers we observed in Picard-Lefschetz analysis? We will answer this in just a moment.

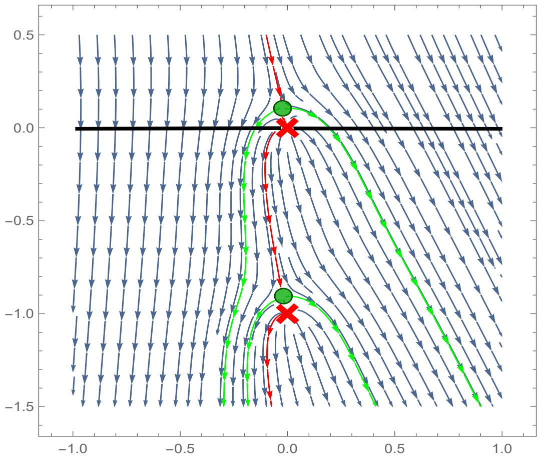
In our Picard-Lefschetz analysis we were able to take the original integral and decompose it into integrals over J cycles starting from saddles, and sum them together in a transseries. We also observed Stokes phenomena as we varied the argument of ξ. As we passed through critical values of the argument the intersection numbers jumped, and we had to include a different combination of the integrals in the transseries.

Now to see how these two analyses are related we need to think about the $\xi \to \infty$ limit of the Picard-Lefschetz decomposition. In this case the saddles move infinitesimally close to the singularities they are associated to. The J cycles associated to each saddle pass around the associated pole, infinitely close on either side to the associated pole, and flow straight to $-ie^{-\arg(\xi)}\infty$.

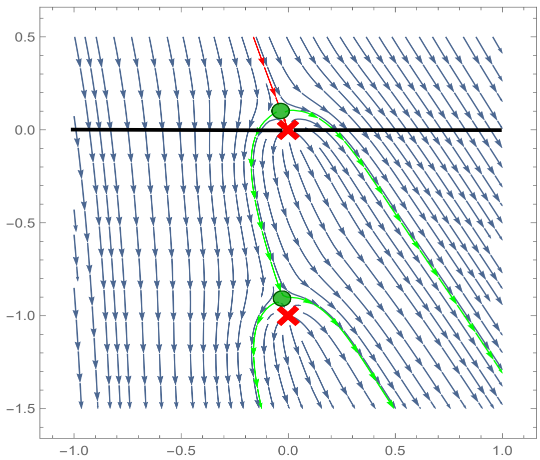
Now here there is again Stokes phenomenon as we pass through $\arg(\xi) = 0$, as we saw in the resurgence analysis. For $\arg(\xi) = 0^+$ J_0 passes just to the left of all the other saddles, J_1 just to the left of the saddles below it, and so on. The K^i cycles pass vertically upwards, just to the right of all the saddles above them, and all pass through the real axis. Thus we sum over all the cycles, as in (2.3.26). For $\arg(\xi) = 0^-$ everything is the same, but swapping left and right in the discussion. It may look like no Stokes phenomenon has occurred, but this is not so.

Because of the branch cut structure of the effective action, we should not think about there being a single saddle associated to each pole, but in fact lots, all living on different Riemann sheets. As $\arg(\xi)$ passes through 0, the intersection number of a saddles we were including (except the perturbative saddle) go from 1 to 0, but the intersection number for the saddles associated to the same poles on the neighbouring sheet go from 0 to 1. From combining (2.3.10) and (2.3.20) we see the action, evaluated at the same saddle on neighbouring Riemann sheets, differs by $2\pi i n(N + \Delta)$.

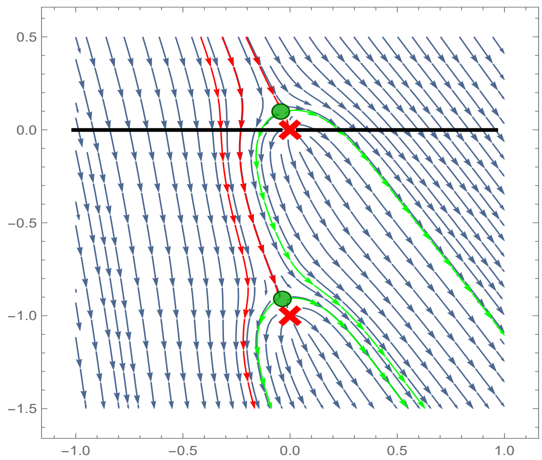
Thus we see that the jumping Stokes parameters in our resurgence analysis are explained in our Picard-Lefschetz analysis by jumping intersections numbers of saddles, taking into account which Riemann sheet the saddle is on. Note that to relate the two analyses we needed to consider the $\xi \to \infty$ limit of our Picard-Lefschetz decomposition, the same limit that we had to consider when calculating the perturbative series which we used to perform the resurgence analysis.



(a) Flows from the first two saddles for $\xi = 10e^{0.2i}$.



(b) Flows from the first two saddles for $\xi = 10e^{0.32i}$.



(c) Flows from the first two saddles for $\xi = 10e^{0.4i}$.

Figure 2.3: Morse flow for (2.3.1) with $N = 1$, $\Delta = 0$, and $\xi = 10e^{i\theta}$, for various values of θ. The black line shows the original integral contour; that is the real axis, passing infinitesimally above the pole at the origin. The green circles are saddles and the red crosses are poles. The J cycles are shown in green and the K cycles in red. In figure 2.3a the only K cycle that passes through the original contour is the cycle connecting the perturbative saddle to the pole at the origin. As θ is increased, in figure 2.3b a Stokes crossing occurs. The J cycle from the perturbative saddle coincides with the K cycle from the first non-perturbative saddle, and likewise with subsequent saddles. Here the contour decomposition breaks down. As θ is increased further we move to the situation in figure 2.3c, where now the K cycles from all the saddles flow down to the nearest pole, and up through the real axis to positive imaginary infinity. (Only the K cycles from the first 3 saddles are included.)

Chapter 3

Cheshire Cat Resurgence in 2 Dimensions

This chapter is based on the work [1], done in collaboration with Daniele Dorigoni.

3.1 Introduction

In this chapter we apply the Cheshire Cat method, following the idea developed in [55], to a two-dimensional quantum field theory. We consider the $\mathbb{CP}^{N-1}$ model, whose resurgent properties have been studied in [19, 20] (see also the recent [74] for connections with 4-d physics), written as a two-dimensional gauged linear sigma model (GLSM) with $\mathcal{N} = (2,2)$ supersymmetry. The S^2 partition function of this model can be computed exactly via localization [32, 33] and its weak coupling expansion can be decomposed as an infinite sum over topological sectors. Each topological sector corresponds to a column in the resurgence triangle [19] and can be written as a perturbative piece plus an infinite tower of non-perturbative terms corresponding to instanton-anti-instanton events, each one of them multiplied by its own perturbative series of fluctuations around it. Due to the supersymmetric nature of the observable under investigation every one of these perturbative series truncates after a finite number of terms, so it would seem that the resurgence program does not allow us to reconstruct the whole column in the resurgence triangle, i.e. the non-perturbative instanton-anti-instanton corrections, from the perturbative expansion in a given topological sector. Following the works of [54, 55] we deform the localized theory by introducing an unbalance, $\Delta = N_f - N_b$, between the number of fermions and bosons present in the theory and immediately we see the full transseries, the body of the Cheshire Cat resurgence, popping out. Whenever Δ is an integer all the perturbative expansions truncate, suggesting that our deformation corresponds to the insertion of some supersymmetric operator; this is very similar to the case ζ integer and quasi-exact solvability considered in [54, 55]. However as soon as Δ is kept generic the perturbative expansion becomes asymptotic and we can fully reconstruct the non-perturbative physics out of it. Only at the very end we remove the deformation by considering $\Delta \to 0$ and reconstruct the full supersymmetric result from the perturbative data providing a nice example of Cheshire Cat resurgence in a supersymmetric quantum field theory.

We further show that a similar structure can be obtained from a more supersymmetric deformation of the model that amounts to an analytic continuation in the number of chiral multiplets[1] from $N \in \mathbb{N}$ (the same N of $\mathbb{CP}^{N-1}$) to a real (or complex) number r. Unlike what happens when we introduce Δ, formally in the presence of this deformation the observable under consideration remains supersymmetric. However as soon as r is kept generic, i.e. non-integer, the perturbative expansion becomes asymptotic. We can apply resurgent analysis to reconstruct non-perturbative information from the perturbative data eventually sending $r \to N \in \mathbb{N}$ to recover the original supersymmetric results.

This chapter is organised as follows. We first introduce in Section 3.2 a few generalities about the supersymmetric formulation of the $\mathbb{CP}^{N-1}$ as a gauged linear sigma model (GLSM) and subsequently use localization to compute the partition function of the model when put on S^2. As a check in Section 3.3 we show that the partition function does indeed reproduce the correct twisted chiral ring structure and comment on its connection with the topological-anti-topological partition function. Due to the supersymmetric nature of the observable under consideration we find a perturbative expansion which is far from

[1] We thank Stefano Cremonesi for the origin of this idea.

asymptotic: the perturbative coefficients actually truncate after finitely many orders. For this reason, in Section 3.4 we deform the theory by introducing an unbalance between the number of bosons and fermions present in the model thus effectively breaking supersymmetry. The deformation considered has a dramatic effect: the perturbative coefficients are not finite in number anymore and perturbation theory becomes an asymptotic expansion. In Section 3.5 we apply the full machinery of resurgent analysis to the deformed model, where we also show how the intricate set of resurgent relations between the perturbative and non-perturbative sectors survives as we send the deformation parameter to zero. In Section 3.6 we show that a similar structure can be obtained also when considering a more supersymmetric (at least formally) deformation studying the $\mathbb{CP}^{r-1}$ model defined via analytic continuation from $N \to r \in \mathbb{R}$. We finally draw some conclusions from this chapter in Section 3.7.

3.2 Supersymmetric $\mathbb{CP}^{N-1}$ as a GLSM

It is useful to briefly review the gauged linear sigma model formulation of the $\mathbb{CP}^{N-1}$ theory with $\mathcal{N} = (2,2)$ supersymmetry; we refer to [75] for all the details.

2-dimensional $\mathcal{N} = (2,2)$ theories have 4 real supercharges, $\mathcal{Q}_\pm$ and $\bar{\mathcal{Q}}_\pm$. Fields are called *chiral* if they commute with $\bar{\mathcal{Q}}_\pm$, and *anti-chiral* if they commute with $\mathcal{Q}_\pm$. We also have *twisted chiral* fields which commute with $\bar{\mathcal{Q}}_+$ and $\mathcal{Q}_-$.

In 2-d the $U(1)$ gauge multiplet is given by a twisted chiral superfield Σ containing a complex scalar $\sigma(x)$, as its lowest component, and a $U(1)$ gauge potential $A_\mu(x)$, plus of course fermions. The theory also contains N chiral superfields Φ_i, $i = 1, ..., N$, each charged $+1$ under the gauge group whose lowest components are the complex scalars $\phi_i(x)$. The parameters of the theory are the gauge coupling e, which has the dimension of a mass, a dimensionless Fayet-Iliopoulos (FI) term ξ, and a vacuum angle θ, that can be combined in the complex coupling $\tau = i\xi + \frac{\theta}{2\pi}$.

The action for these theories is given by

$$S = \int d^2x \ \left(\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm mat} + \mathcal{L}_{\rm FI} + \mathcal{L}_{\theta}\right) \quad . \tag{3.2.1}$$

In superfield notation (see for example [76] if this is not familiar; here we follow the conventions of [75]), with vector superfield V, chiral superfields Φ_i, and the field strength twisted chiral superfield Σ, these Lagrangians are given by

$$\begin{aligned} \mathcal{L}_{\rm YM} &= \frac{-1}{4e^2}\int d^4\theta \ \bar{\Sigma}\Sigma \ \ , \\ \mathcal{L}_{\rm mat} &= \sum_i \int d^4\theta \ \bar{\Phi}_i e^{2V}\Phi_i \ \ , \\ \mathcal{L}_{\rm FI} + \mathcal{L}_{\theta} &= \frac{i\tau}{2\sqrt{2}}\int d\theta^+ d\bar{\theta}^- \ \Sigma \ + \ \text{h.c.} \ \ . \end{aligned} \tag{3.2.2}$$

Here h.c. is short for hermitian conjugate. One can add other terms to the Lagrangian, in particular a more general twisted superpotential, but we will not be concerned with these theories in this thesis. It is also possible to add additional *twisted mass* terms by coupling the theory to a background vector multiplet, and giving a VEV to the background field's scalar components. However we also won't be concerned with these in this thesis.

The theory contains vortices, which are solutions to the vortex or antivortex equations:

$$D + \frac{\sigma}{r} = -i(\phi\phi^\dagger - \chi) = \pm iF_{12} \ , \quad D_{\mp} = 0 \ \ . \tag{3.2.3}$$

Here D is the auxiliary field in the vector multiplet, χ is some free parameter, and upper/lower of the $\pm$ and $\mp$ are for vortices/antivortices respectively. These are saddle points of the classical action.

The D-term conditions for having a supersymmetric vacuum fix $\sigma(x) = 0$ and force the complex scalars $\phi_i(x)$ to satisfy

$$\sum_{i=1}^N |\phi_i(x)|^2 = \xi \, . \tag{3.2.4}$$

At energies much smaller than the gauge coupling e the gauge potential is essentially frozen and becomes non-dynamical. We must then identify field configurations

$$\phi_i(x) \sim e^{i\alpha}\phi_i(x) \, , \qquad\qquad \forall \, i = 1, ..., N. \tag{3.2.5}$$

The two conditions (3.2.4-3.2.5) are precisely the conditions specifying the sigma model with target space $\mathbb{CP}^{N-1}$, so in the infrared the $\mathcal{N} = (2,2)$ gauged linear sigma model becomes the $\mathbb{CP}^{N-1}$ with coupling constant

$$g^2_{\mathbb{CP}^{N-1}} = \frac{1}{\xi}\,. \tag{3.2.6}$$

At this scale the vortices transmute into the usual instanton solutions of the $\mathbb{CP}^{N-1}$ model[75]. In all that follows we will express everything in terms of the FI term ξ, so that the weak coupling expansion of the $\mathbb{CP}^{N-1}$ model will correspond to the regime $\xi \gg 1$, while the strong coupling expansion will be $\xi \sim 0$[2].

3.2.1 Supersymmetric partition function on $\mathbf{S}^2$

In [32, 33] the authors studied the Euclidean path integral of two-dimensional $\mathcal{N} = (2,2)$ theories with vector and chiral multiplets, placed on a round sphere S^2. In the S^2 theory the authors constructed a supercharge $\mathcal{Q}$ whose square is a bosonic symmetry and used localization techniques to show that the path integral only receives contribution from classical configurations that are fixed points of $\mathcal{Q}$ and from small quadratic fluctuations around them, i.e. *one-loop determinants*. This set of fixed points is generically discrete or with finite dimension so the path integral reduces dramatically to a sum over topological sectors of ordinary integrals over the Cartan subalgebra of the gauge group dressed by one-loop determinants[3]. We refer to the original works [32, 33] for all the details in the computations and to [78] for a recent review on supersymmetric localization in two dimensions.

[2]From our analysis we will be also able to consider $\xi \to -\infty$, however this regime does not directly relate to the geometric $\mathbb{CP}^{N-1}$ phase. The reason is that as soon as $\xi < 0$ the D-term equation (3.2.4) cannot be solved anymore and one needs to use the mirror Landau-Ginzburg theory, see [75, 77]

[3]The localisation calculation can in fact be performed in multiple ways, each using a different $\mathcal{Q}$ exact deformation. For *Higgs branch* localisation the scalar components of the chiral multiplets become massive, and the path integral localises to a sum over vortex and antivortex contributions. For *Coulomb branch* localisation the path integral localises onto configurations with constant quantized magnetic flux F_{12} which must be summed over, and with constant σ which needs to be integrated over. We will be concerned with the expression for the partition function arising from Coulomb branch localisation. By performing the σ integral we can recover an expression for the partition function as a sum over contributions from different non-perturbative backgrounds, which we will see shortly.

We can specialise the work of these authors to the case of a $U(1)$ gauge theory with a FI parameter ξ[4], a θ-term, and with $N_f = N$ chiral multiplet with charge $+1$ and no multiplets with charge -1. In absence of twisted masses the localized partition function can be then written as:

$$Z_{\mathbb{CP}^{N-1}} = \sum_{B\in\mathbb{Z}} e^{-i\theta B} \int_{-\infty}^{+\infty} \frac{d\sigma}{2\pi} e^{-4\pi i\xi\,\sigma} \left(\frac{\Gamma(-i\sigma - B/2)}{\Gamma(1+i\sigma - B/2)} \right)^N , \tag{3.2.7}$$

where the full path integral is reduced to a sum over topological sectors with quantized magnetic flux B times an ordinary integral over the lowest component of the twisted chiral field Σ constrained to take the constant value $\sigma(x) = \sigma$ over which we integrate. The first term in the integrand corresponds to the classical action evaluated on shell while the second term in parenthesis comes precisely from the one-loop determinants. Note that when the gamma functions are moved into the exponent, this partition function takes a very similar form to that of the 0-d toy model we considered in section 2.3.

The gamma function in the numerator has poles at locations $\sigma = \sigma_k = -i(k - B/2)$ with $k \in \mathbb{N}$. However for $B \geq 0$ and $k < B$ the integrand is regular because the pole from the numerator is cancelled by the pole from the gamma function in the denominator. For this reason the poles of the integrand are at locations $\sigma = \sigma_k = -i(k + |B|/2)$, and the zeroes are at $\sigma = \sigma_n^{(0)} = +i(n + 1 + |B|/2)$, in particular for $B = 0$ the integrand has a pole at $\sigma = 0$ that has to be included (see [32, 33]) and the integration contour is understood as circling around the pole at the origin in the upper-half complex σ plane, see Figure 3.1.

To evaluate the integral we notice that we can close the contour of integration in the lower-half complex σ plane since for $|\sigma| \to \infty$ with $\operatorname{Im}\sigma < 0$ the one-loop determinant provides a converging factor, so that we can then rewrite the integral as a sum of residues at the N^{th} order pole locations $\sigma = \sigma_k = -i(k + |B|/2)$. The residue of an N^{th} order pole can be computed as

$$2\pi i \operatorname{Res}_{z=z_0} f(z) = \frac{2\pi i}{\Gamma(N)} \frac{d^{N-1}}{d\alpha^{N-1}} \left[f(z_0 + \alpha)\alpha^N \right]_{\alpha=0} \tag{3.2.8}$$

[4]Note that the FI parameter runs, and needs to be regularized. There is a natural scale here given by $1/r$ where r is the radius of the sphere. The partition function is then found as a function of ξ which is the value of the FI parameter at the scale $1/r$ (see [32, 33] for details).

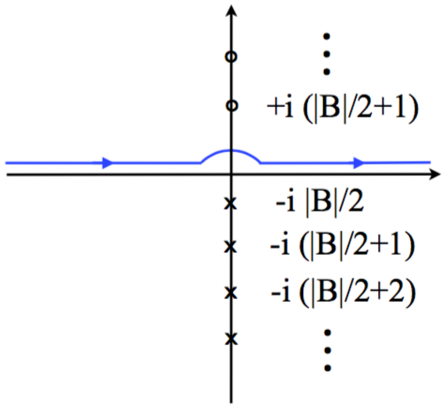


Figure 3.1: Location of the poles (negative imaginary axis) and of the zeroes (positive imaginary axis) of the one-loop determinant. The contour of integration is deformed in the upper-half plane to avoid the pole at $\sigma = 0$ present in the $B = 0$ case.

and replacing $\alpha \to i\alpha$ we can write the partition function as

$$Z_{\mathbb{CP}^{N-1}} = \frac{1}{(N-1)!} \sum_{B\in\mathbb{Z}} \sum_{k=max(0,B)}^{\infty} \frac{d^{N-1}}{d\alpha^{N-1}} \left[\alpha^N \frac{\Gamma(-k+\alpha)^N}{\Gamma(1+k-B-\alpha)^N} e^{-4\pi\xi(k-B/2-\alpha)} e^{-i\theta B} \right]_{\alpha=0} . \tag{3.2.9}$$

We can rewrite the above formula introducing the parameter $t = e^{2\pi i\tau} = e^{-2\pi\xi + i\theta}$ written in terms of the complex coupling $\tau = i\xi + \frac{\theta}{2\pi}$ and after simple manipulations with sums indices we get:

$$Z_{\mathbb{CP}^{N-1}} = \frac{1}{(N-1)!} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{d^{N-1}}{d\alpha^{N-1}} \left[\left(\frac{(-1)^n \pi\alpha / \sin(\pi\alpha)}{\Gamma(1+n-\alpha)\Gamma(1+m-\alpha)} \right)^N t^{n-\alpha}\, \bar{t}^{m-\alpha} \right]_{\alpha=0} , \tag{3.2.10}$$

where we also made use of the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. We can introduce the regularised generalised hypergeometric function

$$ {}_1\tilde{F}_N(a; b_1, b_2, \dots, b_N|z) = \frac{1}{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_N)} \sum_{n=0}^{\infty} \frac{(a)_n}{(b_1)_n\,(b_2)_n \cdots (b_N)_n} \frac{z^n}{n!} \tag{3.2.11}$$

where $(a)_n$ denotes the Pochhammer symbol, and our partition function can be written in the compact form:

$$Z_{\mathbb{CP}^{N-1}} = \frac{1}{(N-1)!} \frac{d^{N-1}}{d\alpha^{N-1}} \left[\left(\frac{\pi\alpha}{\sin(\pi\alpha)} \right)^N (t\bar{t})^{-\alpha}\ {}_1\tilde{F}_N(1; 1-\alpha, \dots, 1-\alpha|(-1)^N t) \right.$$
$$\left. {}_1\tilde{F}_N(1; 1-\alpha, \dots, 1-\alpha|\,\bar{t}) \right]_{\alpha=0} . \tag{3.2.12}$$

It is useful to rewrite the above equation as a sum over instanton sectors each one weighted by the instanton counting parameter $\exp(-2\pi\xi|B|+i\theta B)$ where $B\in\mathbb{Z}$ denotes the instanton number, or equivalently the magnetic flux as above. To this end we can go back to equation (3.2.9) and by isolating the instanton counting parameter we obtain

$$Z_{\mathbb{CP}^{N-1}}=\sum_{B\in\mathbb{Z}}e^{-2\pi\xi|B|+i\theta B}\,\zeta_B(N,\xi)\,, \tag{3.2.13}$$

where the Fourier mode $\zeta_B(N,\xi)$ takes the form

$$\zeta_B(N,\xi)=\frac{(-1)^{NB\,\theta(B)}}{(N-1)!}\sum_{k=0}^{\infty}e^{-4\pi\xi\,k}\frac{d^{N-1}}{d\alpha^{N-1}}\left[\left(\frac{(-1)^k\,\pi\alpha/\sin(\pi\alpha)}{\Gamma(1+k-\alpha)\Gamma(1+k+|B|-\alpha)}\right)^N e^{4\pi\xi\alpha}\right]_{\alpha=0}\,, \tag{3.2.14}$$

with $\theta(B)$ the Heaviside function.

The equations (3.2.13)-(3.2.14) are very suggestive: the supersymmetric localized partition function for the $\mathbb{CP}^{N-1}$ model on S^2 takes the form of an infinite series over instantons sectors, each one of them denoted by an integer $B\in\mathbb{Z}$ and weighted by the instanton counting parameter $\exp(-2\pi\xi|B|+i\theta B)$. Each B-instanton sector produces a contribution $\zeta_B(N,\xi)$, function only of the Fayet-Iliopoulos term ξ, i.e. the coupling constant, and not of the θ angle. Every Fourier mode $\zeta_B(N,\xi)$ gives rise to a purely perturbative piece, i.e. the $k=0$ term in (3.2.14), plus an infinite sum over exponentially suppressed terms of the form $e^{-4\pi\xi\,k}$ with $k\in\mathbb{N}^\star$, corresponding to instantons-anti-instantons events, each one of them multiplied by a perturbative expansion. Fixing the instanton number B corresponds to fixing the column in the resurgence triangle diagram of [19] so that $\zeta_B(N,\xi)$ can be interpreted as the transseries containing the perturbative part plus all the instantons-anti-instantons corrections, together with their own perturbative series, on top of a B-instanton event.

At this stage we would like to apply resurgent analysis within each instanton sector, i.e. studying separately each transseries $\zeta_B(N,\xi)$ (3.2.14) seen as some suitably defined analytic function in some wedge of the complex ξ-plane. However it is simple to see that the coefficient of each $e^{-4\pi\xi\,k}$ term in the infinite sum (3.2.14) is actually a polynomial of degree $N-1$ in ξ meaning that both the perturbative expansion around a B-instanton event and the perturbative expansions around k instantons-anti-instantons on top of the B-instanton event are all entire functions of ξ. For a generic observable in a generic field theory we would expect all of these perturbative expansions to be asymptotic series rather than finite degree polynomials. The reason for this truncation is clearly the supersymmetric nature of the quantity under consideration. Being an observable protected by supersymmetry we expect only the first few orders in perturbation theory not to vanish. The same goes for the perturbative expansion on top of non-trivial but still supersymmetric saddles. We are then left with the question: in these lucky situation where the perturbative expansion truncates after a finite number of terms can resurgent analysis tell us anything at all about the non-perturbative completion of the physical observable? At first sight this would seem unlikely, how can an entire function tell you anything about non-perturbative terms? However we will shortly see that Cheshire Cat resurgence is at play here: when we focus on this supersymmetric quantity the cat seems to have disappeared but its footprints can still be seen!

3.2.2 The $\mathbb{CP}^1$ case

Instead of working with general N in this Section we specialise equation (3.2.12) to the case $N = 2$ so that we can give shorter and less cluttered equations, the discussions however can be repeated for the general case. To compute the partition function on S^2 for the $\mathbb{CP}^1$ model we simply take (3.2.12) and substitute $N = 2$:

$$Z_{\mathbb{CP}^1} = \frac{d}{d\alpha}\left[\left(\frac{\pi\alpha}{\sin(\pi\alpha)}\right)^2 (t\bar{t})^{-\alpha}\ {}_1\tilde{F}_2(1;1-\alpha,1-\alpha|t)\ {}_1\tilde{F}_2(1;1-\alpha,1-\alpha|\bar{t})\right]_{\alpha=0}. \tag{3.2.15}$$

When the derivative with respect to α does not act on the hypergeometric function we obtain terms of the form ${}_1\tilde{F}_2(1;1,1|t) = I_0(2\sqrt{t})$, where I_0 denotes the modified Bessel function of 0-th order, whilst when the derivative acts on the hypergeometric parameters, we obtain terms of the form

$$\frac{d}{db_1}\ {}_1F_2(a;b_1,b_2|z) = \psi(b_1)\,{}_1F_2(a;b_1,b_2|z) - \sum_{k=0}^{\infty}\frac{(a)_k\,\psi(k+b_1)}{(b_1)_k(b_2)_k}\frac{z^k}{k!}\,, \tag{3.2.16}$$

where $\psi(x)$ denotes the digamma function and $-\psi(1) = \gamma$ gives the Euler-Mascheroni constant. So we obtain

$$Z_{\mathbb{CP}^1} = -\log(t\bar{t})_0F_1(1|t)_0F_1(1|\bar{t}) - 2\,_0F_1(1|\bar{t})\frac{d}{db}\,_0\tilde{F}_1(b|t)|_{b=1} - 2\,_0F_1(1|t)\frac{d}{db}\,_0\tilde{F}_1(b|\bar{t})|_{b=1}\,,$$

and changing variable $b = 1 + a$ we can write

$$_0\tilde{F}_1(1 + a|z) = \frac{1}{(\sqrt{z})^a} I_a(2\sqrt{z})\,,$$

that together with the relation

$$\frac{d}{da} I_a(z)|_{a=0} = -K_0(z)$$

brings us to the final form

$$Z_{\mathbb{CP}^1} = 2\left(I_0(2\sqrt{t})K_0(2\sqrt{\bar{t}}) + K_0(2\sqrt{t})I_0(2\sqrt{\bar{t}})\right)\,. \tag{3.2.17}$$

It is also useful to specialise the Fourier mode decomposition (3.2.13) to the $\mathbb{CP}^1$ case

$$Z_{\mathbb{CP}^1} = \sum_{B\in\mathbb{Z}} e^{-2\pi\xi|B|+i\theta B}\zeta_B(2,\xi)\,, \tag{3.2.18}$$

with

$$\begin{aligned}\zeta_B(2,\xi) &= \sum_{k=0}^{\infty} e^{-4\pi\xi\, k}(4\pi\xi)^2\left[\frac{1}{[k!\,(k+|B|)!]^2}(4\pi\xi)^{-1} + \frac{2\psi(k+1)+2\psi(k+|B|+1)}{[k!\,(k+|B|)!]^2}(4\pi\xi)^{-2}\right]\\ &= \frac{4\pi\xi - 2\gamma + 2\psi(|B|+1)}{|B|!^2} + \frac{4\pi\xi + 2(1-\gamma)) + 2\psi(|B|+1)}{[(|B|+1)!]^2} \times e^{-4\pi\xi} + \mathrm{O}(e^{-8\pi\xi})\,.\end{aligned} \tag{3.2.19}$$

Since in what follows we will mostly consider the $B = 0$ sector we can specialise the above equation even further obtaining the very simple expression

$$\zeta_0(2,\xi) = \sum_{k=0}^{\infty} e^{-4\pi\xi\, k}(4\pi\xi)^2\left[\frac{1}{(k!)^4}(4\pi\xi)^{-1} + \frac{4H_k - 4\gamma}{(k!)^4}(4\pi\xi)^{-2}\right]\,, \tag{3.2.20}$$

where $H_k = \psi(1 + k) + \gamma$ denotes the k^{th} harmonic number. Equation (3.2.20) can be written in terms of Meijer G function and it is neither asymptotic in the weak coupling regime $\xi \to \infty$ nor in the strong coupling one $\xi \to 0$.

As mentioned above each topological sector can be written as a purely perturbative expansion, given by a very simple degree 1 polynomial in ξ, plus an infinite tower of instanton-anti-instanton events, weighted by $e^{-4\pi\xi}$, each one of them accompanied by a simple perturbative expansion given by a different degree 1 polynomial in ξ. Due to the supersymmetric nature of the observable under consideration perturbation theory is not asymptotic at all, it actually truncates after finitely many terms so that there is no need to apply Borel resummation and the perturbative expansion appears to be completely oblivious of the non-perturbative sectors. We will see later on that this is precisely an example of Cheshire Cat resurgence at play in quantum field theory.

3.3 Chiral ring structure

Having obtained the partition function for $\mathbb{CP}^1$ (3.2.17) and more generically for $\mathbb{CP}^{N-1}$ (3.2.12) we can compute the chiral ring for these models, see [77][5]. For $\mathbb{CP}^{N-1}$ the ring is generated by one element Σ that at the classical level satisfies $\Sigma^N = 0$, but receives instantons corrections and it gets modified to

$$\Sigma^N = \Lambda^N_{\mathbb{CP}^{N-1}}\,, \tag{3.3.1}$$

where $\Lambda_{\mathbb{CP}^{N-1}} = \mu e^{-2\pi\xi/N+i\theta/N} = \mu\, t^{1/N}$. The top component of Σ is related to the action itself via:

$$S = \log t \int d^2x\, d^2\theta\, \Sigma + h.c. = 2\pi i\, \tau \int d^2x\, d^2\theta\, \Sigma + h.c. \tag{3.3.2}$$

so we can generate the full chiral ring by considering[6]

$$\langle \Sigma^n \bar{\Sigma}^m \rangle = \frac{1}{Z_{\mathbb{CP}^{N-1}}} (t\partial_t)^n (\bar{t}\partial_{\bar{t}})^m Z_{\mathbb{CP}^{N-1}} = \frac{1}{(2\pi i)^n(-2\pi i)^m} \partial^n_\tau \partial^m_{\bar{\tau}} \log Z_{\mathbb{CP}^{N-1}}\,. \tag{3.3.3}$$

[5]The chiral ring is defined as the $\bar{\mathcal{Q}}$ cohomology; that is the equivalence class of operators $\mathcal{O}$ given by

$$\frac{\{\mathcal{O} : \bar{\mathcal{Q}}_{\dot{\alpha}}\mathcal{O} = 0\}}{\{\mathcal{O} = \bar{\mathcal{Q}}_{\dot{\alpha}}\mathcal{O}'\}}\,,$$

where $\mathcal{O}'$ is some other operator.

[6]By slight abuse of notation from this point onward we denote insertions of the top component of Σ with Σ itself.

3.3.1 Chiral ring for $\mathbb{CP}^1$

Let us start with $N = 2$ and use (3.2.17) to compute $\langle\Sigma\rangle$ obtaining:

$$\langle\Sigma\rangle = \frac{t}{Z_{\mathbb{CP}^1}}\,\partial_t Z_{\mathbb{CP}^1} = 2\sqrt{t}\left(I_1(2\sqrt{t})K_0(2\sqrt{\bar{t}}) - K_1(2\sqrt{t})I_0(2\sqrt{\bar{t}})\right)/Z_{\mathbb{CP}^1}\,. \tag{3.3.4}$$

We can easily compute $\langle\Sigma^2\rangle = 1/Z_{\mathbb{CP}^1}\, t\,\partial_t(t\,\partial_t Z_{\mathbb{CP}^1})$ and making use of the relations for the modified Bessel:

$$\begin{aligned} I'_\nu(z) &= I_{\nu-1}(z) - \frac{\nu}{z}I_\nu(z)\,, \\ K'_\nu(z) &= -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z)\,, \end{aligned} \tag{3.3.5}$$

we obtain

$$\langle\Sigma^2\rangle = t = \Lambda^2_{\mathbb{CP}^1}\,, \tag{3.3.6}$$

as expected from the chiral ring structure (3.3.1). The S^2 localized partition function and its derivatives with respect to the the (anti-)holomorphic coupling give rise to a representation of the chiral ring in terms of modified Bessel functions.

It was shown in [34] that in the superconformal case, where the sigma model target space is a Calabi-Yau manifold rather than $\mathbb{CP}^{N-1}$, the supersymmetric localized partition function on the round two-sphere matched precisely the exact Kähler potential on the quantum Kähler moduli space of the Calabi-Yau emerging in the infrared. This means that in the superconformal case the localized partition function coincides with the seemingly unrelated topological-anti-topological construction of Cecotti and Vafa [79].

The model we are considering is however an asymptotically free theory rather than a superconformal one and it is not clear how to relate the two-sphere localized calculations to the $\mathbb{CP}^1$ topological-anti-topological results obtained in [77]. To this end, we first complete the chiral ring (3.3.6) (similarly for $\bar{\Sigma}$) and study correlators with multiple Σ, $\bar{\Sigma}$ insertions obtaining the functions:

$$\langle\Sigma\rangle = \frac{t}{Z_{\mathbb{CP}^1}}\,\partial_t Z_{\mathbb{CP}^1} = 2\sqrt{t}\left(I_1(2\sqrt{t})K_0(2\sqrt{\bar{t}}) - K_1(2\sqrt{t})I_0(2\sqrt{\bar{t}})\right)/Z_{\mathbb{CP}^1}\,, \tag{3.3.7}$$

$$\langle\bar{\Sigma}\rangle = \frac{\bar{t}}{Z_{\mathbb{CP}^1}}\,\partial_{\bar{t}} Z_{\mathbb{CP}^1} = 2\sqrt{\bar{t}}\left(K_0(2\sqrt{t})I_1(2\sqrt{\bar{t}}) - I_0(2\sqrt{t})K_1(2\sqrt{\bar{t}})\right)/Z_{\mathbb{CP}^1}\,, \tag{3.3.8}$$

$$\langle\bar{\Sigma}\Sigma\rangle = \frac{t\bar{t}}{Z_{\mathbb{CP}^1}}\,\partial_t\partial_{\bar{t}}Z_{\mathbb{CP}^1} = -2\sqrt{t\,\bar{t}}\left(I_1(2\sqrt{t})K_1(2\sqrt{\bar{t}}) + K_1(2\sqrt{t})I_1(2\sqrt{\bar{t}})\right)/Z_{\mathbb{CP}^1}\,. \quad (3.3.9)$$

With these functions we can construct the hermitian metric

$$g = \begin{pmatrix} \langle 1\rangle & \langle\bar{\Sigma}\rangle \\ \langle\Sigma\rangle & \langle\bar{\Sigma}\Sigma\rangle \end{pmatrix}, \quad (3.3.10)$$

and note that it is manifestly not diagonal unlike the metric considered in [77]. The reason is that on S^2, compared to $\mathbb{R}^2$, we have operators mixing with lower dimensional ones[7], see [80], in particular Σ and $\bar{\Sigma}$ mix with the identity. The determinant of the matrix g removes this mixing and produces the only relevant correlator for $\mathbb{CP}^1$ given by the connected correlator $\langle\Sigma\bar{\Sigma}\rangle_C = \langle\Sigma\bar{\Sigma}\rangle - \langle\Sigma\rangle\langle\bar{\Sigma}\rangle$. This determinant can be easily computed using the relation

$$I_{\nu+1}(z)K_\nu(z) + I_\nu(z)K_{\nu+1}(z) = \frac{1}{z} \quad (3.3.11)$$

arriving at $\det g = -1/Z^2_{\mathbb{CP}^1}$, so the only function we need to consider for the $t\bar{t}$-equations of [77] is precisely $Z_{\mathbb{CP}^1}$.

It is now a matter of calculation to show that our result does not quite solve the topological-anti-topological equation of [77] but rather satisfies a simple modification of it

$$t\bar{t}\,\partial_t\partial_{\bar{t}}\log Z_{\mathbb{CP}^1} = 0\times t\bar{t}\,Z^2_{\mathbb{CP}^1} - \frac{1}{Z^2_{\mathbb{CP}^1}}\,, \quad (3.3.12)$$

or using the same notation as [77] we can define $q_0 = -q_1 = \log Z_{\mathbb{CP}^1} + \frac{1}{4}\log|t|^2$ and using the variable $z = 2\sqrt{t}$ (our t corresponds to their β) we can rewrite (3.3.12) as

$$\partial_z\partial_{\bar{z}}q_0 = 0\times e^{2q_0} - e^{-2q_0}\,. \quad (3.3.13)$$

Had the coefficient of the first term on the right-hand side in (3.3.12-3.3.13) been 1 instead of 0 we would have found precisely the Toda equation of [77], however we do not know why we obtain this modification. It is possible that because of UV divergences one has to regulate insertions of the composite operator $\Sigma\bar{\Sigma}$ to correctly reproduce the topological-anti-topological results from supersymmetric localization, or it could also happen that the

[7] We thank Vasilis Niarchos for discussions on this point.

localization calculation in the non-superconformal case is computing something genuinely different from [77]. These are very interesting questions deserving more studies however they fall outside of (and will not affect) the main message of this thesis.

3.3.2 Chiral ring for $\mathbb{CP}^{N-1}$

Starting from equation (3.2.12) we want to show that the chiral ring structure $\langle\Sigma^N\rangle = \Lambda^N_{\mathbb{CP}^{N-1}}$ can be obtained from the supersymmetric localized partition function. To this end we need to show that the equation

$$\frac{1}{Z_{\mathbb{CP}^{N-1}}}(t\,\partial_t)^N Z_{\mathbb{CP}^{N-1}} = \langle\Sigma^N\rangle = t \tag{3.3.14}$$

holds. Instead of working with (3.2.12) we can use the power series expansion (3.2.10) and when we act with the operator $(t\,\partial_t)^N$ on $Z_{\mathbb{CP}^{N-1}}$ we can commute the derivatives with the series and the only term we have to consider is $t^{n-\alpha}$ for which we obtain the simple action

$$(t\,\partial_t)^N \frac{t^{n-\alpha}}{\Gamma(1+n-\alpha)^N} = \frac{(n-\alpha)^N}{\Gamma(1+n-\alpha)^N} t^{n-\alpha} = \frac{t^{n-\alpha}}{\Gamma(n-\alpha)^N}\,. \tag{3.3.15}$$

We can thus shift $n \to n+1$ and obtain

$$\begin{aligned}(t\,\partial_t)^N Z_{\mathbb{CP}^{N-1}} =& t Z_{\mathbb{CP}^{N-1}} + \\ &+ \frac{1}{(N-1)!}\sum_{m=0}^{\infty}\frac{d^{N-1}}{d\alpha^{N-1}}\left[\left(\frac{\pi\alpha/\sin(\pi\alpha)}{\Gamma(-\alpha)\Gamma(1+m-\alpha)}\right)^N t^{-\alpha}\,\bar{t}^{m-\alpha}\right]_{\alpha=0}\,,\end{aligned} \tag{3.3.16}$$

where the second term comes from the $n=0$ contribution in (3.2.12) after we use (3.3.15). This second term vanishes because the $1/\Gamma(-\alpha)^N$ term has an N^{th} order zero when $\alpha \to 0$ and at most $N-1$ derivatives with respect to α can act upon it. Hence we obtain the expected chiral ring structure

$$\langle\Sigma^N\rangle = \frac{1}{Z_{\mathbb{CP}^{N-1}}}(t\,\partial_t)^N Z_{\mathbb{CP}^{N-1}} = t = \Lambda^N_{\mathbb{CP}^{N-1}}\,. \tag{3.3.17}$$

It would be interesting to construct general correlation functions of the form $\langle\Sigma^n\bar{\Sigma}^m\rangle$ to see if we can find a solution to some modification of the affine Toda equations presented in [77], similar to what we obtained for $\mathbb{CP}^1$ in equation (3.3.13), however this is beyond the purpose of the present thesis.

The reader should now be convinced that the S^2 partition function does indeed capture various physical properties of the supersymmetric $\mathbb{CP}^{N-1}$ model. However due to supersymmetry, the weak coupling expansion does not give rise to any asymptotic series but it does nonetheless contain infinitely many non-perturbative corrections, seemingly defying the resurgence program whose task is to reconstruct non-perturbative information out of perturbative data. In the next Section we will see how to get around these superficial negative results by breaking supersymmetry in a controlled way.

3.4 Away from the supersymmetric point

Since each instanton sector in (3.2.14) gives rise, due to supersymmetry, to a convergent rather than an asymptotic expansion it would appear that resurgent analysis cannot be applied in the model at hand. However motivated by the works [54, 55] we decided to modify slightly the localized path integral by unbalancing the number of bosons and fermions in the one-loop determinants so that supersymmetry is broken but in a very tamed manner.

To obtain via supersymmetric localization the partition function presented in (3.2.7), after having found the localized critical points one has to compute the one-loop determinant for the quadratic fluctuations around these BPS configurations. For the $\mathbb{CP}^{N-1}$ model the one-loop determinant for the vector multiplet is just 1 while it becomes non-trivial for the chiral multiplet. For a single chiral multiplet the matter one-loop determinant is given by

$$Z_{matter} = \frac{\det\mathcal{O}_\psi}{\det\mathcal{O}_\phi}\,, \tag{3.4.1}$$

where ϕ and ψ denote respectively the complex scalar and the Dirac fermion in the multiplet and the one-loop determinants are given by (see [32, 33])

$$\det\mathcal{O}_\phi = \prod_{j=\frac{|B|}{2}}^{\infty} (j-i\sigma)^{2j+1}(j+1+i\sigma)^{2j+1}\,, \tag{3.4.2}$$

$$\det \mathcal{O}_\psi = (-1)^{B\theta(B)} \prod_{k=\frac{|B|}{2}}^{\infty} (k - i\sigma)^{2k} (k + 1 + i\sigma)^{2k+2} . \tag{3.4.3}$$

As discussed in Section 3.2, the GLSM realisation of the $\mathbb{CP}^{N-1}$ model contains N chiral multiplets so that the matter one-loop determinant contribution to the partition function amounts to Z^N_{matter}. However at this point, in a similar way to the works [54, 55], we want to introduce a small unbalance between the bosonic and fermionic contributions to the matter one-loop determinant by declaring that after having localized on the susy critical points we have $N_f = N$ fermions but only $N_b = N - \Delta$ bosons so that

$$\tilde{Z}_{matter}(\sigma) = \frac{(\det \mathcal{O}_\psi)^{N_f}}{(\det \mathcal{O}_\phi)^{N_b}} = Z^N_{matter} \, (\det \mathcal{O}_\phi)^{\Delta} \ , \tag{3.4.4}$$

and when $\Delta = N_f - N_b$ vanishes we go back to the undeformed, supersymmetric case.

By using (3.4.2-3.4.3) we can rewrite $\tilde{Z}_{matter}$ as

$$\tilde{Z}_{matter}(\sigma) = (-1)^{N\,B\theta(B)} \prod_{j=0}^{\infty} \left(\frac{j+b}{j+a} \right)^{N-\Delta(2i\sigma+1)} \cdot (j+a)^{2\Delta(j+a)} \cdot (j+b)^{2\Delta(j+b)} , \tag{3.4.5}$$

where we defined $a = |B|/2 - i\sigma$ and $b = 1 + i\sigma + |B|/2$.

We can use zeta-function regularisation to define these infinite products, see details in Appendix A.1, and using equations (A.1.7)-(A.1.12) we obtain a regularised version of the modified matter one-loop determinant

$$\begin{aligned} \tilde{Z}_{matter}(\sigma) = & \left[(-1)^{B\theta(B)} \frac{\Gamma(-i\sigma + |B|/2)}{\Gamma(1 + i\sigma + |B|/2)} \right]^N e^{-2\Delta\left(2\zeta'(-1) + \zeta'(0)(|B|+1) + |B|^2/4 + i\sigma - \sigma^2\right)} \\ & \times \exp\left[\Delta(2i\sigma + 1) \left(\log\Gamma(1 + i\sigma + |B|/2) - \log\Gamma(-i\sigma + |B|/2) \ \right) \right] \\ & \times \exp\left[-2\Delta \left(\psi^{(-2)}(1 + i\sigma + |B|/2) + \psi^{(-2)}(-i\sigma + |B|/2 \right) \right] . \end{aligned} \tag{3.4.6}$$

One can recognise that the above one-loop determinants are very similar to the effective actions for bosonic and fermionic fields on the hyperbolic manifold H^2 used as building blocks to study the strong-coupling expansions for the Wilson loop minimal surfaces in $\mathrm{AdS}_5 \times \mathrm{S}^5$ (see e.g. [81]). The very same effective actions have been studied in [82] using generalised dyadic identities for the polygamma function to obtain inverse factorial series expansion. It would be interesting to understand how to apply the results of [82] to the

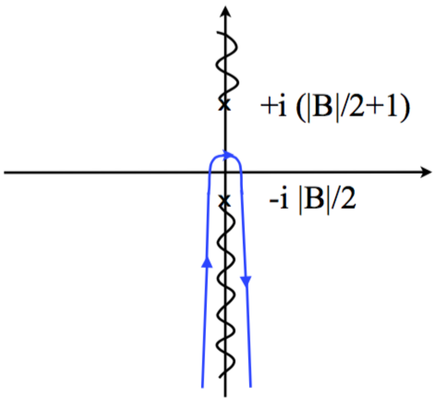


Figure 3.2: The contour of integration $\mathcal{C}$ comes from $-i\infty - \epsilon$, circles around the branch cut and then goes back to $-i\infty + \epsilon$.

current problem.

Using the properties of the gamma function, see [32], one can rewrite the first parenthesis in the above expression to put it back into the form $\Gamma(-i\sigma - B/2)/\Gamma(1 + i\sigma - B/2)$ which appears in the undeformed one-loop determinant as in (3.2.7). All the remaining terms have the form $e^{\Delta(\dots)}$ clearly tending to 1 as $\Delta \to 0$.

It is crucial for what follows to analyse the analytic properties of $\tilde{Z}_{matter}$ as a function of σ. In the undeformed case, see Figure 3.1 and the discussion below equation (3.2.7), we had poles for $\sigma = -i(k + |B|/2)$ and zeroes for $\sigma = +i(1 + n + |B|/2)$ with $k, n \in \mathbb{N}$. However due to the presence of $\log\Gamma$ and $\psi^{(-2)}$ instead of poles and zeroes we have two branch cuts running along the positive and negative imaginary axis. The functions $\log\Gamma(z)$ and $\psi^{(-2)}(z)$ are analytic throughout the complex z-plane, except for a single branch cut discontinuity along the negative real axis[8]. The discontinuities of $\log\Gamma$ and $\psi^{(-2)}$ can be easily computed

$$\log\Gamma(-x + i\epsilon) - \log\Gamma(-x - i\epsilon) = -2\pi i\,(\lfloor x \rfloor + 1)\,, \tag{3.4.7}$$

$$\psi^{(-2)}(-x + i\epsilon) - \psi^{(-2)}(-x - i\epsilon) = \pi i(\lfloor x \rfloor + 1)(2x - \lfloor x \rfloor)\,, \tag{3.4.8}$$

for $x \geq 0$, where $\lfloor x \rfloor$ denotes the floor of x.

[8]In here we use a specific determination of $\log\Gamma$, what Mathematica calls `LogGamma[z]`. The function $\log(\Gamma(z))$ has a more complex branch cut structure.

We are now in position to study our deformed localized path integral taking the form

$$Z(N,\Delta)=\sum_{B\in\mathbb{Z}}e^{-i\theta B}\int_{\mathcal{C}}\frac{d\sigma}{2\pi}e^{-4\pi i\xi\,\sigma}\tilde{Z}_{matter}(\sigma)\,,\tag{3.4.9}$$

where $\mathcal{C}$ is a suitably defined contour in the complex σ plane. In the supersymmetric case $\mathcal{C}$ is given by the real line that we subsequently close in the lower-half complex plane collecting the residues from all the poles of the integrand. If we repeat for the case at hand and push the contour of integration in the lower-half complex plane, due to the presence of the branch cut, we end up with a contour $\mathcal{C}$ along the negative imaginary axis, coming from $\sigma=-i\infty-\epsilon$, circling around the origin of the branch cut at $\sigma=-i|B|/2$ and the continuing back to infinity in the direction $\sigma=-i\infty+\epsilon$ as depicted in Figure 3.2, we will comment later on the analytic properties of these branch cuts.

Note that for any $\Delta\neq 0$ this is just a formal definition[9] since the integrand in (3.4.9) behaves as $\tilde{Z}_{matter}(\sigma)\sim\exp[-2\Delta\cos(2\theta)\,R^2\log R]$ when $|\sigma|=R\to\infty$, with $\theta=\arg\sigma$ and closing the contour in the lower-half plane will produce a different analytic continuation. However if we insist on taking the contour $\mathcal{C}$ to be the one presented in Figure 3.2 and consider the integral (3.4.9) only as a formal object, we will see that as we send $\Delta\to 0$ everything will be well-defined[10].

Once the contour is fixed we can make the change of variable $\sigma=-iy$ and rewrite (3.4.9) as

$$Z(N,\Delta)=\sum_{B\in\mathbb{Z}}e^{-i\theta B}\int_0^\infty\frac{dy}{2\pi i}e^{-4\pi\xi\,y}\left(\tilde{Z}_{matter}(-iy+\epsilon)-\tilde{Z}_{matter}(-iy-\epsilon)\right)\,.\tag{3.4.10}$$

The integral in the above expression is nothing but the Laplace transform of the discontinuity of $\tilde{Z}_{matter}$ along the negative imaginary axis. This discontinuity starts at $y=|B|/2$, so after shifting $y=x+|B|/2$ we obtain a Fourier mode expansion of the same form as

[9]If one does not want to work with formal objects we can add a quartic twisted superpotential allowing us to close the contour on the imaginary axis. Now everything becomes well defined and convergent so we can check numerically that all the formal equations derived using resurgent analysis are indeed correct, and only at the very end we send this auxiliary quartic coupling to zero.

[10]This situation is similar to the case [83] of bad $\mathcal{N}=4$ theories in 3-d where it can be shown that the localized matrix integral over the "original" contour of integration diverges but can be regularised by modifying the contour in the complex (fields) space. It is only with this deformed contour of integration that one obtains a well defined integral that can be understood in terms of infrared physics [84]. We thank Stefano Cremonesi for discussions on this point.

the original one (3.2.13)

$$Z(N,\Delta) = \sum_{B\in\mathbb{Z}} e^{-2\pi\xi|B|-i\theta B}\,\tilde{\zeta}_B(N,\xi,\Delta)\,, \tag{3.4.11}$$

and in each topological sector we can make use of the discontinuities equations (3.4.7-3.4.8) to obtain

$$\begin{aligned}\tilde{\zeta}_B(N,\xi,\Delta) &= \sum_{k=0}^{\infty}\int_k^{k+1}\frac{dx}{2\pi i}e^{-4\pi\xi\, x}\tilde{Z}_{matter}(-ix-i|B|/2-\epsilon)\left[e^{-2\pi i\Delta(k+1)(k+|B|+1)}-1\right]\\ &= \sum_{k=0}^{\infty}\int_k^{k+1}\frac{dx}{2\pi i}e^{-4\pi\xi\, x}\tilde{Z}_{matter}(-ix-i|B|/2+\epsilon)\left[1-e^{+2\pi i\Delta(k+1)(k+|B|+1)}\right]\,.\end{aligned} \tag{3.4.12}$$

We can rewrite each integral as $\int_k^{k+1}=\int_k^{\infty}-\int_{k+1}^{\infty}$ and then shift integration variables so that every integral becomes between $[0,\infty)$, arriving at

$$\begin{aligned}\tilde{\zeta}_B(N,\xi,\Delta) = \sum_{k=0}^{\infty} e^{-4\pi\xi\, k}\int_0^{\infty}\frac{dx}{2\pi i}e^{-4\pi\xi\, x}&\left[(-1)^{B\theta(B)}\frac{\Gamma(-x-k)}{\Gamma(1+x+k+|B|)}\right]^N f(x,\Delta)\\ \times \exp&\left[-\Delta(2x+2k+|B|+1)\log\Gamma(-x-k+i\epsilon)-2\Delta\psi^{(-2)}(-x-k+i\epsilon)\right]\\ \times \exp&\left[\Delta(2x+2k+|B|+1)\log\Gamma(x+k+|B|+1)-2\Delta\psi^{(-2)}(x+k+|B|+1)\right]\\ \times &\left[e^{-2\pi i\Delta(k+1)(k+|B|+1)}-e^{-2\pi i\Delta k(k+|B|)}\right]\,,\end{aligned} \tag{3.4.13}$$

where $f(x,\Delta)$ is an entire function of x that goes to 1 as $\Delta\to 0$ given by $f(x,\Delta) = \exp[-2\Delta(x^2+x+c)]$ with c an x independent constant. Note that a similar equation can be straightforwardly derived for $\epsilon\to-\epsilon$.

This equation will be the starting point of our resurgent analysis of the deformed theory: the B instanton sector contribution $\tilde{\zeta}_B(N,\xi,\Delta)$ has been written as the sum over instanton-anti-instanton events, weighted by $e^{-4\pi\xi\, k}$, each one of them multiplied by the Laplace transform of a function with branch cuts in the directions $\arg x = 0$, coming from the first exponential in the integrand, and $\arg x = \pi$, coming from the second exponential in the integrand, these being the only two *Stokes* directions. As we will shortly see, a weak-coupling expansion of the Laplace integral in (3.4.13) will give rise to asymptotic series in ξ^{-1} with Δ dependent coefficients. Furthermore since $f(x,\Delta)$ is an entire function of x it will not change the asymptotic nature of the perturbative expansion, so for this function we can safely set $\Delta = 0$ and replace $f(x,\Delta) \to f(x,0) = 1$ without modifying the resurgence structure[11].

We can rewrite (3.4.13)

$$\tilde{\zeta}_B(N,\xi,\Delta) = (-1)^{NB\,\theta(B)} \sum_{k=0}^{\infty} e^{-4\pi\xi\,k}\, e^{\pm i\pi k(k+|B|)\Delta}\, \mathcal{S}_{\pm}\left[\Phi_B^{(k)}\right](\xi,\Delta) \tag{3.4.14}$$

where $\mathcal{S}_{\pm}$ denote the lateral Laplace transforms

$$\mathcal{S}_{\pm}\left[\Phi_B^{(k)}\right](\xi,\Delta) = \int_0^{\infty \pm i\epsilon} dx\, e^{-4\pi\xi\,x}\, x^{-N+\Delta(2k+1+|B|)}\, \Phi_B^{(k)}(x,\Delta)\,, \tag{3.4.15}$$

obtained as the limiting case approaching a Stokes line of the directional Borel resummation

$$\mathcal{S}_{\theta}\left[\Phi_B^{(k)}\right](\xi,\Delta) = \int_0^{\infty\, e^{i\theta}} dx\, e^{-4\pi\xi\,x}\, x^{-N+\Delta(2k+1+|B|)}\, \Phi_B^{(k)}(x,\Delta)\,. \tag{3.4.16}$$

The Borel transform $\Phi_B^{(k)}(x,\Delta)$ appearing in the above equation can be rewritten from (3.4.13) as

$$\begin{aligned}
\Phi_B^{(k)}(x,\Delta) = &-\frac{\sin[\pi\Delta(2k+|B|+1)]}{\pi}\left[\frac{(-1)^{k+1}\,\pi x/\sin(\pi x)}{\Gamma(1+x+k)\Gamma(1+x+k+|B|)}\right]^N \\
&\times \exp\Big[-\Delta(2x+2k+|B|+1)(\log\Gamma(1-x) - \log((x+1)_k) - 2\Delta\,(\,\psi^{(-2)}(1-x)+ \\
&-\psi^{(-2)}(k+1) - (k+1)(x+k) + k\log k + \sum_{j=1}^{k}\left[(x+j)\log(x+j) + (k-j)\log(k-j)\right]\,)\Big] \\
&\times \exp\Big[\Delta(2x+2k+|B|+1)\log\Gamma(x+k+|B|+1) - 2\Delta\psi^{(-2)}(x+k+|B|+1)\Big]\,,
\end{aligned} \tag{3.4.17}$$

after repeated use of the formulas

$$\log\Gamma(-x \pm i\epsilon) = \log\Gamma(1-x\pm i\epsilon) - \log x \mp i\pi\,, \tag{3.4.18}$$

$$\psi^{(-2)}(-x\pm i\epsilon) = \psi^{(-2)}(1-x\pm i\epsilon) - \psi^{(-2)}(1) + x[\log\,x - 1] \pm i\pi x\,, \tag{3.4.19}$$

valid for $x \geq 0$. For example the purely perturbative contribution $k = 0$ in the trivial

[11] If we expand for z large the Laplace transform of the product of two functions $\int_0^\infty e^{-x\,z} f(x)g(x) = \sum_{n=0}^\infty n! c_n z^{-n-1}$ we have that the coefficients c_n are given by the convolution sum $c_n = \sum_{k=0}^n a_{n-k}b_k$ where $f(x) = \sum_{n=0}^\infty a_n x^n$ and $g(x) = \sum_{n=0}^\infty b_n x^n$. This convolution amounts to a change in the definition of coupling constant, i.e. $z^{-1} \to w^{-1} = F(z^{-1})$, and this change of variable is analytic in the neighbourhood of $z = \infty$ when the function $f(x)$ is entire, so that the resurgence properties remain the same. See [67] for more details on this. The most intuitive way of seeing that $f(x,\Delta)$ does not effect the resurgence properties is to note that it does not effect the location of the branch cuts in the Borel plane.

topological sector $B = 0$ can be obtain from the directional Laplace transform of

$$\Phi_0^{(0)}(x,\Delta) = -\frac{(-1)^N \sin(\pi\Delta)}{\pi} \left[\frac{\pi x/\sin(\pi x)}{\Gamma(1+x)^2}\right]^N \exp\left[2\Delta(x+\psi^{(-2)}(1))\right] \times \qquad (3.4.20)$$
$$\exp\left[\Delta(2x+1)\left(\log\Gamma(1+x) - \log\Gamma(1-x)\right) - 2\Delta\left(\psi^{(-2)}(1+x) + \psi^{(-2)}(1-x)\right)\right] .$$

It is now clear from (3.4.17) or (3.4.20) that we cannot naively take the limit $\Delta \to 0$ since the overall factor $\sin[\pi\Delta(2k+|B|+1)]$ coming from the discontinuity (3.4.12) vanishes. However, as we will shortly see, precisely in this limit this factor multiplies an asymptotic series in inverse powers of ξ with singular coefficients.

From equation (3.4.16) we can understand the branch structure introduced by our deformed one-loop determinant (3.4.6) that was schematically depicted in Figure 3.2. In the directional Borel resummation (3.4.16) we have split the branches into two separate structures. First we notice the $x^{\Delta(2k+1+|B|)}$ term that for generic Δ introduces a cut starting from the origin $x = 0$. This non-analytic term will serve as a regulator and it will be crucial to properly recover the supersymmetric result from the deformed theory. Secondly the modified Borel transforms, i.e the functions $\Phi_B^{(k)}(x,\Delta)$, have branch cuts starting at $x = +1$ running to $x \to +\infty$ and at $x = -1 - |B|$ running to $x \to -\infty$, so that their only singular directions, i.e. their *Stokes lines*, are $\arg x = 0$ and $\arg x = \pi$. We will shortly see the consequences of these facts.

3.5 Cheshire Cat Resurgence

The first thing we can check from our expansion (3.4.14) is that we reproduce the undeformed case (3.2.14) or (3.2.19) in the case of $\mathbb{CP}^1$, i.e. $N = 2$. The key point is that our Borel transform (3.4.17) for $x \sim 0$ behaves as

$$\Phi_B^{(k)}(x,\Delta) \sim -\frac{\sin[(2k+1+|B|)\pi\Delta]}{\pi}\left(\sum_{n=0}^{\infty} c_{B,n}^{(k)}(\Delta)x^n\right) , \qquad (3.5.1)$$

where the coefficients $c^{(k)}_{B,n}(\Delta)$ can be easily obtained from (3.4.17). For $\Delta = 0$ these coefficients are simply the Taylor coefficients of the function $\left[\frac{(-1)^{k+1}\,\pi x/\sin(\pi x)}{\Gamma(1+x+k)\Gamma(1+x+k+|B|)}\right]^N$:

$$
\begin{aligned}
c^{(k)}_{B,0}(0) &= \left(\frac{(-1)^{k+1}}{k!(k+|B|)!}\right)^N , \\
c^{(k)}_{B,1}(0) &= -N[\psi(k+1)+\psi(k+|B|+1)]\, c^{(k)}_{B,0}(0) \,.
\end{aligned}
\tag{3.5.2}
$$

So if we consider a weak coupling expansion, i.e. $\xi \to \infty$, of the lateral Borel resummation (3.4.15) we obtain the power series

$$
\mathcal{S}_\pm\left[\Phi^{(k)}_B\right](\xi,\Delta) \sim -(4\pi\xi)^{N-\tilde{\Delta}} \sum_{n=0}^{\infty} c^{(k)}_{B,n}(\Delta) \frac{\Gamma(n+1+\tilde{\Delta}-N)\,\sin(\pi\tilde{\Delta})}{\pi} (4\pi\xi)^{-n-1} \,,
\tag{3.5.3}
$$

where $\tilde{\Delta} = (2k+|B|+1)\Delta$. If we plug this expansion in (3.4.14) we obtain the transseries representation

$$
\tilde{\zeta}_B(N,\xi,\Delta) = (-1)^{NB\,\theta(B)} \sum_{k=0}^{\infty} e^{-4\pi\xi\, k}\, e^{\pm i\pi k(k+|B|)\Delta} (4\pi\xi)^{N-\tilde{\Delta}} \left(\sum_{n=0}^{\infty} \frac{C^{(k)}_{B,n}(\Delta)}{(4\pi\xi)^{n+1}}\right) ,
\tag{3.5.4}
$$

where the perturbative coefficients $C^{(k)}_{B,n}(\Delta)$ in the k instanton-anti-instanton background on top of the B-instanton topological sector are given by

$$
C^{(k)}_{B,n}(\Delta) = -c^{(k)}_{B,n}(\Delta) \frac{\Gamma(n+1+\tilde{\Delta}-N)\,\sin(\pi\tilde{\Delta})}{\pi} \,.
\tag{3.5.5}
$$

These coefficients, as well as the $c^{(k)}_{B,n}(\Delta)$, are all real numbers whenever $\Delta \in \mathbb{R}$. The reason for this lies in how we rewrote the integrand (3.4.17) of the directional Laplace transform (3.4.15). In the transseries (3.4.14) we have factorised out the complex phase from the integrand, so that the function (3.4.17) appearing in the lateral Laplace transform (3.4.15) is manifestly real for $x \in [0,1)$ and $\xi, \Delta \in \mathbb{R}$. However there is still a branch cut starting at $x = 1$ and that is why in (3.4.14) we need to take lateral Borel resummations where the factor $e^{\pm i\pi k(k+|B|)\Delta}$ coming from the discontinuity is just the transseries parameter[12].

Once we have the expression (3.5.5) we note that for generic Δ the factor $\Gamma(n+1+\tilde{\Delta}-N)$

[12] The sign $\pm$ of the phase is correlated with the direction of the lateral Laplace resummation as in (3.4.14). In here we use the same symbol to denote the formal transseries and its appropriate directional Borel-Ecalle resummation.

gives a factorial growth of the perturbative coefficients thus making the above expression (3.5.4) a purely formal object, i.e. a transseries representation. However as we send $\Delta \to 0$ we see that the $\sin(\pi\tilde{\Delta}) \to 0$ but $\Gamma(n+1+\tilde{\Delta}-N)$ develops a pole for every $n = 0, ..., N-1$, thus effectively truncating the expansion (3.5.3) to a degree $N-1$ polynomial in ξ as already seen previously in the undeformed equation (3.2.14). For example if we take the $\Delta \to 0$ limit for the $\mathbb{CP}^1$ case $\sin(\pi\tilde{\Delta})\,\Gamma(n+1+\tilde{\Delta}-2)$ gives a finite non-zero contribution for $n = 0$ and $n = 1$ while vanishing for $n \geq 2$ so the transseries expansion (3.5.4) effectively reduces to

$$\begin{aligned}\tilde{\zeta}_B(2,\xi,0) &= \sum_{k=0}^{\infty} e^{-4\pi\xi\, k}(4\pi\xi)^2 \left(c^{(k)}_{B,0}(0)\,(4\pi\xi)^{-1} - c^{(k)}_{B,1}(0)\,(4\pi\xi)^{-2}\right)\\ &= \sum_{k=0}^{\infty} e^{-4\pi\xi\, k}\,\frac{4\pi\xi + 2\psi(k+1) + 2\psi(k+|B|+1)}{[k!\,(k+|B|)!]^2} = \zeta_B(2,\xi)\ ,\end{aligned} \tag{3.5.6}$$

where we used the explicit form for the coefficients (3.5.2) to obtain precisely the same expression (3.2.19). One can easily check for different values of N that the limit of the transseries (3.5.4) when $\Delta \to 0$ reproduces the same topological sector contribution $\zeta_B(N,\xi)$ written in equation (3.2.14) obtained from localization.

If we start from the very beginning with $\Delta = 0$, as we did in the supersymmetric case (3.2.7) leading to (3.2.14), we do not generate a transseries and there is no direct way to exploit resurgent analysis to extract non-perturbative information out of the purely perturbative, asymptotic power series. As a matter of fact there is not even an asymptotic power series to begin with since perturbation theory truncates after a finite number of loops due to the supersymmetric nature of the physical quantity under consideration. However, as soon as we break slightly supersymmetry by introducing this Δ-deformation we immediately generate an infinite perturbative expansion, and in fact the full transseries, out of thin hair. As the Cheshire Cat says [85]:

"You may have noticed that I'm not all there myself." .

Once we realise that for generic Δ we do indeed have a transseries we know from resurgent analysis that obviously the splitting of perturbative and non-perturbative part in (3.5.4) give rise to ambiguities as the directional Borel integral (3.4.16) is ill-defined for $\theta = 0$

since $\arg x = 0$ (and $\arg x = \pi$) is a Stokes direction for $\Phi_0^{(0)}(x)$. The branch cut begins at $x = 1$ and depending on how we dodge it, either from above or from below, we will generate non-perturbative "ambiguities" that are exactly compensated for by the non-perturbative terms in (3.5.4). We will promptly show that the resummation of the full transseries (3.5.4) gives rise to an unambiguous result.

3.5.1 Cancellation of ambiguities

As just mentioned if instead of working with the full transseries (3.4.14)-(3.5.4) we were only to focus on the purely perturbative piece, i.e. the $k = 0$ term, in a given topological sector B, according to which resummation we decided to pick $\mathcal{S}_+ \left[\Phi_B^{(0)}\right](\xi, \Delta)$ or $\mathcal{S}_- \left[\Phi_B^{(0)}\right](\xi, \Delta)$ we would find two different analytic continuations of the formal asymptotic expansion (3.5.3). Furthermore, even if the formal power series (3.5.3) is manifestly real for ξ and Δ real, neither of the analytic continuation $\mathcal{S}_\pm \left[\Phi_B^{(0)}\right](\xi, \Delta)$ is, the difference between the two is purely imaginary and usually called an "ambiguity" in the resummation procedure. The presence of these "ambiguities" is due to the fact that we decided to split the full transseries (3.4.14) into perturbative and non-perturbative part. We can now show that if we consider the Borel-Ecalle resummation of the complete transseries (3.4.14)-(3.5.4), the ambiguities $(\mathcal{S}_+ - \mathcal{S}_-)\left[\Phi_B^{(k)}\right](\xi, \Delta)$ in each non-perturbative sector together with the jump in the transseries parameter $e^{\pm i\pi k(k+|B|)}$ precisely conspire to cancel out and give an unambiguous and real answer when ξ and Δ are real.

To this end let us start with the ambiguity in the resummation of the purely perturbative piece in the trivial topological sector $B = 0$:

$$
\begin{aligned}
(\mathcal{S}_+ - \mathcal{S}_-)\left[\Phi_0^{(0)}\right] &= \int_0^\infty dx\, e^{-w\,x} x^{-N+\Delta} \left(\Phi_0^{(0)}(x + i\epsilon, \Delta) - \Phi_0^{(0)}(x - i\epsilon, \Delta)\right) \\
&= \int_0^\infty dx\, e^{-w\,x} x^{-N+\Delta} \Phi_0^{(0)}(x + i\epsilon, \Delta)\left(1 - e^{2\pi i \Delta \lfloor x \rfloor (\lfloor x \rfloor + 2)}\right) \\
&= \int_1^2 dx\, e^{-w\,x} x^{-N+\Delta} \Phi_0^{(0)}(x + i\epsilon, \Delta)\left(1 - e^{6\pi i \Delta}\right) + \\
&\quad + \int_2^3 dx\, e^{-w\,x} x^{-N+\Delta} \Phi_0^{(0)}(x + i\epsilon, \Delta)\left(1 - e^{16\pi i \Delta}\right) + \mathrm{O}(e^{-3w}),
\end{aligned}
$$

where we used the discontinuity equations (3.4.7-3.4.8) and defined $w = 4\pi\xi$. In each of the above integrals we can shift the integration variables to make manifest the exponentially suppressed factor, furthermore we also extend the integration all the way to infinity making sure that we are consistent with the order of the instanton counting parameter e^{-w} at which we are working with. Proceeding as just outlined and using the connection formulas (3.4.18-3.4.19) we can rewrite the above equation as

$$\begin{aligned}(\mathcal{S}_+ - \mathcal{S}_-)\left[\Phi_0^{(0)}\right] &= -2i\sin(\pi\Delta)\, e^{-w}\int_0^\infty dx\, e^{-w\,x} x^{-N+3\Delta}\Phi_0^{(1)}(x+i\epsilon,\Delta) + \\ &\quad - 2i\sin(\pi\Delta)e^{3i\pi\Delta}\, e^{-2w}\int_0^\infty dx\, e^{-w\,x}x^{-N+5\Delta}\Phi_0^{(2)}(x+i\epsilon,\Delta) + \mathrm{O}(e^{-3w}) \\ &= -2i\sin(\pi\Delta)e^{-w}\mathcal{S}_+\left[\Phi_0^{(1)}\right] - 2i\sin(\pi\Delta)e^{3i\pi\Delta}\, e^{-2w}\mathcal{S}_+\left[\Phi_0^{(2)}\right] + \mathrm{O}(e^{-3w})\,.\end{aligned} \tag{3.5.7}$$

We were able to relate the difference between the two lateral resummations of the perturbative series to the resummation of the non-perturbative sectors, this relation is usually called *Stokes automorphism* (for more details see [15, 16]). Note that the "ambiguity" in the resummation of the perturbative series is purely non-perturbative, i.e. it starts with e^{-w} plus higher instantons sectors. This ambiguity does not look manifestly imaginary, however this is only due to the fact that the right-hand side is written in terms of the lateral resummation $\mathcal{S}_+\left[\Phi_0^{(k)}\right]$ of higher instanton sectors which is not a real quantity due to the branch cut running on the real axis for each $\Phi_0^{(k)}(x)$. We will obtain a manifest purely imaginary expression later on.

In a similar manner we can study what happens to the first non-perturbative sector, $k=1$, in the transseries (3.4.14) and repeating a similar calculation we find:

$$\begin{aligned}&e^{-w}\left(e^{i\pi\Delta}\mathcal{S}_+ - e^{-i\pi\Delta}\mathcal{S}_-\right)\left[\Phi_0^{(1)}\right] = \\ &= +2i\sin(\pi\Delta)e^{-w}\mathcal{S}_+\left[\Phi_0^{(1)}\right] - 2i\sin(3\pi\Delta)e^{-i\pi\Delta}e^{-2w}\mathcal{S}_+\left[\Phi_0^{(2)}\right] + \mathrm{O}(e^{-3w})\,.\end{aligned} \tag{3.5.8}$$

Note that, unlike (3.5.7), the difference in lateral resummation of the $k=1$ sector contains a term (the first one in the above expression) exactly of the same non-perturbative order e^{-w}. The reason for this is that we are not quite computing the ambiguity $(\mathcal{S}_+ - \mathcal{S}_-)\left[\Phi_0^{(1)}\right]$ but rather the joint combination of the jump in resummation together with the jump in the transseries parameter $e^{\pm i\pi\Delta}$.

Finally, to order $\mathrm{O}(e^{-3w})$ in the instanton counting parameter, we need to compute the "ambiguity" in the $k=2$ non-perturbative sector of the transseries (3.4.14) given by

$$e^{-2w}\left(e^{4i\pi\Delta}\mathcal{S}_+ - e^{-4i\pi\Delta}\mathcal{S}_-\right)\left[\Phi_0^{(2)}\right] = +2i\sin(4\pi\Delta)e^{-2w}\mathcal{S}_+\left[\Phi_0^{(2)}\right] + \mathrm{O}(e^{-3w})\,. \qquad (3.5.9)$$

Putting together the three pieces (3.5.7),(3.5.8), and (3.5.9) we obtain what expected

$$\begin{aligned}(\mathcal{S}_+ - \mathcal{S}_-)\left[\Phi_0^{(0)}\right] + e^{-w}\left(e^{i\pi\Delta}\mathcal{S}_+ - e^{-i\pi\Delta}\mathcal{S}_-\right)\left[\Phi_0^{(1)}\right] + e^{-2w}\left(e^{4i\pi\Delta}\mathcal{S}_+ - e^{-4i\pi\Delta}\mathcal{S}_-\right)\left[\Phi_0^{(2)}\right]\\ = \mathrm{O}(e^{-3w})\,,\end{aligned}$$

namely the difference in lateral resummation together with the correct jump in the transseries parameter combine and cancel out, giving a unique and unambiguous Borel-Ecalle resummation of the transseries (3.4.14).

From equations (3.5.7),(3.5.8), and (3.5.9) we can also rewrite the transseries (3.4.14) in a form which is manifestly real for real ξ and Δ and absolutely unambiguous

$$\begin{aligned}\tilde{\zeta}_0(N,\xi,\Delta) =& \mathcal{S}_0\left[\mathrm{Re}\left(\Phi_0^{(0)}\right)\right](\xi,\Delta) + \cos(\pi\Delta)e^{-4\pi\xi}\,\mathcal{S}_0\left[\mathrm{Re}\left(\Phi_0^{(1)}\right)\right](\xi,\Delta) +\\ &+\cos(\pi\Delta)\cos(3\pi\Delta)e^{-8\pi\xi}\,\mathcal{S}_0\left[\mathrm{Re}\left(\Phi_0^{(2)}\right)\right](\xi,\Delta) + \mathrm{O}\left(e^{-12\pi\xi}\right)\,. \qquad (3.5.10)\end{aligned}$$

Note that we do not need to take any lateral resummation now as the real part of the Borel transform $\mathrm{Re}\left(\Phi_0^{(0)}\right)$ does not have a branch cut along the positive real axis allowing us to safely perform the directional Borel transform $\mathcal{S}_0$ (3.4.16) without any ambiguity. We can repeat this analysis for generic topological sector B obtaining a manifestly real and unambiguous resummation for the transseries (3.4.14)

$$\begin{aligned}\tilde{\zeta}_B(N,\xi,\Delta) =& \mathcal{S}_0\left[\mathrm{Re}\left(\Phi_B^{(0)}\right)\right](\xi,\Delta) + \cos[(|B|+1)\pi\Delta]e^{-4\pi\xi}\,\mathcal{S}_0\left[\mathrm{Re}\left(\Phi_B^{(1)}\right)\right](\xi,\Delta) +\\ &+\cos[(|B|+1)\pi\Delta]\cos[(|B|+3)\pi\Delta]e^{-8\pi\xi}\,\mathcal{S}_0\left[\mathrm{Re}\left(\Phi_B^{(2)}\right)\right](\xi,\Delta) + \mathrm{O}\left(e^{-12\pi\xi}\right)\,. \qquad (3.5.11)\end{aligned}$$

3.5.2 Large orders relations

Now that we understand how the ambiguities in the resummation procedure cancel out when we consider the full transseries, we can try and use the purely perturbative data to retrieve some non-perturbative information, following the procedure laid out in section 2.2.1. To proceed we consider Δ generic and use the transseries expansion (3.5.4) to extract the purely perturbative sector, now asymptotic, and only at the very end we will send $\Delta \to 0$ to learn something about the supersymmetric case. For simplicity let us focus on the perturbative part, $k = 0$, of the $B = 0$ topological sector in (3.5.4):

$$\tilde{\zeta}_{\text{pert}}(N, \xi, \Delta) = \int_0^\infty dx\, e^{-4\pi\xi\, x}\, x^{-N+\Delta}\, \Phi_0^{(0)}(x, \Delta) \sim (4\pi\xi)^{N-\Delta} \sum_{n=0}^{\infty} \frac{C_{0,n}^{(0)}(\Delta)}{(4\pi\xi)^{n+1}}\,, \tag{3.5.12}$$

where the Borel transform obtained in (3.4.20) is

$$\Phi_0^{(0)}(x, \Delta) = -\frac{(-1)^N \sin(\pi\Delta)}{\pi} \left[\frac{\pi x/\sin(\pi x)}{\Gamma(1+x)^2}\right]^N \exp\left[2\Delta(x + \psi^{(-2)}(1))\right] \times$$
$$\exp\left[\Delta(2x+1)\left(\log\Gamma(1+x) - \log\Gamma(1-x)\right) - 2\Delta\left(\psi^{(-2)}(1+x) + \psi^{(-2)}(1-x)\right)\right]\,,$$

and the perturbative coefficients (3.5.5)

$$C_{0,n}^{(0)}(\Delta) = -c_{0,n}^{(0)}(\Delta)\frac{\Gamma(n+1+\Delta-N)\sin(\pi\Delta)}{\pi} \tag{3.5.13}$$

can be obtained from (3.5.1) and grow factorially with n for Δ generic.

Let us consider the particular determination of the resummation of the purely perturbative series (3.5.12), that we denote with the same symbol, where we anti-correlate $\arg\xi = \theta$ with the argument of the integration variable x as:

$$(4\pi\xi)^{-N+\Delta}\tilde{\zeta}_{\text{pert}}(N, \xi, \Delta) = \int_0^{\infty e^{-i\theta}} dx\, e^{-4\pi\xi\, x}\, (4\pi\xi\, x)^{-N+\Delta}\, \Phi_0^{(0)}(x, \Delta)\,. \tag{3.5.14}$$

Note that this is not the correct physical quantity but rather it is the best we could do if we only had access to perturbation theory.

From the explicit expression (3.4.20) for $\Phi_0^{(0)}(x, \Delta)$ we know that the integrand of the above equation has two branch cuts along the Stokes directions $\arg x = 0$ starting at $x = +1$, and $\arg x = \pi$ starting at $x = -1$, thus forcing the determination for $\tilde{\zeta}_{\text{pert}}(N, \xi, \Delta)$ to have branch cuts along $\arg\xi = 0$ and $\arg\xi = \pi$. Using a standard Cauchy-like contour argument (see [9, 86], and Section 2.2.1) we can relate the perturbative coefficients $C_{0,n}^{(0)}(\Delta)$ in the expansion (3.5.12), or more generically the one appearing in (3.5.4), to the discontinuities in the $\theta = 0$ and $\theta = \pi$ direction of the determination (3.5.14):

$$C^{(0)}_{0,n}(\Delta) \sim -\frac{1}{2\pi i}\int_0^{\infty} dw\,\mathrm{Disc}_0(w)\,w^n - \frac{1}{2\pi i}\int_0^{\infty e^{i\pi}} dw\,\mathrm{Disc}_\pi(w) w^n\,. \tag{3.5.15}$$

The discontinuities across the cuts of (3.5.14) can be easily computed using the discontinuity equations (3.4.7-3.4.8) for the $\log\Gamma$ and $\psi^{(-2)}$; in particular $\mathrm{Disc}_0(w)$ vanishes for $0 < w < 1$ while $\mathrm{Disc}_\pi(w)$ vanishes for $-1 < w < 0$. For example if we focus on

$$\mathrm{Disc}_0(w) = \int_0^{\infty} dx\, e^{-w\,x}(w\,x)^{-N+\Delta}\,\left(\Phi_0^{(0)}(x - i\epsilon,\Delta) - \Phi_0^{(0)}(x + i\epsilon,\Delta)\right)\,, \tag{3.5.16}$$

we can use multiple times (3.4.7-3.4.8) proceeding as we did in Section 3.5.1, and rewrite this expression as

$$\begin{aligned}\mathrm{Disc}_0(w) =& 2i\sin(\pi\Delta)\,e^{-w}w^{-2\Delta}\int_0^{\infty} dx\,e^{-w\,x}(w\,x)^{-N+3\Delta}\,\mathrm{Re}\left(\Phi_0^{(1)}(x,\Delta)\right) \\ &+ 2i\sin(\pi\Delta)\cos(3\pi\Delta)\,e^{-2w}w^{-4\Delta}\int_0^{\infty} dx\,e^{-w\,x}(w\,x)^{-N+5\Delta}\,\mathrm{Re}\left(\Phi_0^{(2)}(x,\Delta)\right) \\ &+ \mathrm{O}\left(e^{-3w}\right)\,.\end{aligned} \tag{3.5.17}$$

One can also derive an expression for $\mathrm{Disc}_\pi(w)$ and subsequently use equation (3.5.15) to obtain the asymptotic expansion valid at large $n \gg 1$ of the perturbative coefficients

$$\begin{aligned}C^{(0)}_{0,n}(\Delta) \sim& -\frac{1}{2\pi}\,2\sin(\pi\Delta)\,\frac{\Gamma(n-2\Delta)}{(+1)^{n-2\Delta}}\left(C^{(1)}_{0,0}(\Delta) + \frac{C^{(1)}_{0,1}(\Delta)}{n-2\Delta-1} + \mathrm{O}(n^{-2})\right) + \\ &-\frac{1}{2\pi}\,2\sin(\pi\Delta)\,\frac{\Gamma(n-4)}{(-1)^n}\left(C^{(-1)}_{0,0}(\Delta) + \mathrm{O}(n^{-1})\right) + \\ &-\frac{1}{2\pi}\,2\sin(\pi\Delta)\cos(3\pi\Delta)\,\frac{\Gamma(n-4\Delta)}{2^{n-4\Delta}}\left(C^{(2)}_{0,0}(\Delta) + \frac{2\,C^{(2)}_{0,1}(\Delta)}{n-4\Delta-1} + \mathrm{O}(n^{-2})\right) + \dots\,.\end{aligned} \tag{3.5.18}$$

From the large order perturbative coefficients coefficient $C^{(0)}_{0,n}$ we can disentangle the $C^{(k)}_{0,n}$ which are precisely the perturbative coefficients at order n in the k^{th} non-perturbative sector given in equation (3.5.5) and appearing in the transseries expansion (3.5.4). From perturbative data we can reconstruct non-perturbative physics. The second term in the asymptotic expansion would correspond to the $k = -1$ instanton-anti-instanton sector, i.e. something weighted by $e^{+4\pi\,\xi}$, and the first perturbative coefficient in this sector is given by

$$C_{0,0}^{(-1)}(\Delta) = \frac{\sin(\pi\Delta)}{\pi}\,(2\pi)^{-\Delta}\,\Gamma(3+\Delta)\,. \tag{3.5.19}$$

However we do not particularly care about these sectors as we specialised our transseries (3.5.4) to the wedge of the complex ξ plane $\mathrm{Re}\,\xi > 0$ and terms of the form $e^{+4\pi\,\xi}$ are unphysical here. The large order perturbative coefficients do nonetheless know about these terms because if we were to analytically continue to the wedge $\mathrm{Re}\,\xi < 0$ terms of the form $e^{+4\pi\,\xi}$ would become exponentially suppressed and the most general transseries would contain both terms of the form $e^{\pm 4\pi\,k\,\xi}$. In particular, to be consistent, we should have written in (3.5.18) a term going as $\Gamma(n-\alpha)/(-2)^n$ however, as we will shortly see, in the supersymmetric limit $\Delta \to 0$ the $k<0$ sectors will disappear completely as expected from the discussion in Section 3.2, while the footprints of the non-perturbative $k \geq 1$ sectors will still be present. The dots at the end of equation (3.5.18) represent all higher instanton sectors going as $\Gamma(n-\alpha_k)/(\pm k)^n$ for some constant α_k, possibly Δ dependent.

We wrote equation (3.5.18) in a way that makes the Stokes constants for each non-perturbative sector manifest. For example for the $k=1$ sector the Stokes constant is given by $A_1^{(0)} = 2\sin(\pi\Delta) = 2\,\mathrm{Im}\,e^{i\pi\Delta}$, i.e. the Stokes constant is exactly equal to the jump of the transseries parameter in the $k=1$ instanton sector in equation (3.5.4) as expected since the Borel-Ecalle resummation of the transseries (3.4.14) should give the same result if we resum for $\arg\xi = +\epsilon$ or $\arg\xi = -\epsilon$. The Stokes constant for the $k=2$ sector is however $A_2^{(0)} = 2\sin(\pi\Delta)\cos(3\pi\Delta)$ and does not equal the jump of $2\,\mathrm{Im}\,e^{4i\pi\,\Delta}$ in the transseries parameter for the $k=2$ sector in (3.5.4). The reason is that the jump in the two instanton sector is compensated partly from the term e^{-2w} in the discontinuity in the $k=0$ sector in (3.5.17) but also from a term e^{-w} in the discontinuity for the $k=1$ sector, see (3.5.8). It is only the sum of these two pieces that reproduces the jump of $2\,\mathrm{Im}\,e^{4i\pi\,\Delta}$ in the transseries parameter for the $k=2$ sector. To show that this is indeed the case we can first easily repeat the large order analysis for the perturbative coefficients $C_{0,n}^{(1)}$ in the $k=1$ non-perturbative sector obtaining

$$C_{0,n}^{(1)}(\Delta) \sim -\frac{1}{2\pi}2\sin(3\pi\Delta)\frac{\Gamma(n+2\Delta)}{(-1)^n}\left(C_{0,0}^{(0)}(\Delta) + \frac{(-1)\,C_{0,1}^{(0)}(\Delta)}{n+2\Delta-1} + \mathrm{O}(n^{-2})\right) +$$

$$- \frac{1}{2\pi} 2\sin(3\pi\Delta) \frac{\Gamma(n)}{(+1)^n} \left(C^{(2)}_{0,0}(\Delta) + \frac{C^{(2)}_{0,1}(\Delta)}{n-1} + \mathrm{O}(n^{-2}) \right) + \dots \,, \qquad (3.5.20)$$

where the dots represent higher non-perturbative contributions as above. The $k=1$ sector "sees" the perturbative sector with a relative action of -1 hence the alternating factor $(-1)^n$ in the first term multiplying exactly the purely perturbative coefficients $C^{(0)}_{0,n}$ with Stokes constant $A^{(1)}_{-1} = 2\sin(3\pi\Delta)$. The relative action between the $k=1$ sector and the $k=2$ sector is instead equal to $+1$ hence the second term in the above equation does not have an alternating factor and multiplies the perturbative coefficients $C^{(2)}_{0,n}$ of the $k=2$ sector with Stokes constant $A^{(1)}_1 = 2\sin(3\pi\Delta)$. We can now see that the jump $2\,\mathrm{Im}\, e^{4i\pi\,\Delta}$ of the $k=2$ transseries parameter in (3.5.4) is exactly controlled by the Stokes constant $A^{(0)}_2 = 2\sin(\pi\Delta)\cos(3\pi\Delta)$ of the perturbative sector plus the Stokes constant $A^{(1)}_1 = 2\sin(3\pi\Delta)$ of the $k=1$ sector multiplied by the real part $\mathrm{Re}\, e^{i\pi\Delta}$ of the transseries parameter[13] for the $k=1$ sector

$$2\,\mathrm{Im}\, e^{4i\pi\,\Delta} = A^{(0)}_2 + A^{(1)}_1 \mathrm{Re}\, e^{i\pi\Delta} = 2\sin(\pi\Delta)\cos(3\pi\Delta) + 2\sin(3\pi\Delta)\cos(\pi\Delta) = 2\sin(4\pi\Delta)\,.$$

The large order coefficients (3.5.18-3.5.20) are a genuine factorial asymptotic expansion for generic Δ. As a numerical check we can fix Δ to some value and read from the large order perturbative coefficients (3.5.18) the low order non-perturbative sector coefficients. A curious incident happens whenever we pick a rational $\Delta = p/q$ for some coprime integers $p, q \in \mathbb{Z}$. From equation (3.5.1) we see that in all the instanton sectors where $(2k+|B|+1) = 0 \,(\mathrm{mod}\, q)$ due to the $\sin((2k+|B|+1)\pi\Delta)$ factor we have a truncation and the perturbative coefficients $C^{(k)}_{B,n}$ in those non-perturbative sectors are not asymptotic but rather finite in number. In all the sectors for which $(2k+|B|+1) \neq 0 \,(\mathrm{mod}\, q)$, in particular the purely perturbative one, the coefficients remain asymptotic and this truncation seems of accidental nature. However as we will comment later on in Section 3.5.3 whenever $\Delta \in \mathbb{Z}$ we have that this truncation happens in *all* sectors giving rise to some "exact"

[13] In (3.4.14) one considers the jump of the $k=1$ sector $e^{i\pi\Delta}\,\mathcal{S}_+[\Phi^{(1)}_0] - e^{-i\pi\Delta}\,\mathcal{S}_-\left[\Phi^{(1)}_0\right]$ and the only term in this expression contributing to the $k=2$ sector is given by $\mathrm{Re}\left(e^{i\pi\Delta}\right) \times (\mathcal{S}_+ - \mathcal{S}_-)[\Phi^{(1)}_0] \sim 2i\,\mathrm{Re}\left(e^{i\pi\Delta}\right) A^{(1)}_1 e^{-8\pi\,\xi}$. See the thorough discussion in Section 3.5.1 and in particular equation (3.5.8).

observable, as in the case $\Delta = 0$ discussed in detail above. As a nice example of this accidental truncation we can pick the large order expansion (3.5.18) and specialise it to the case $\Delta = 1/3$. Fixing for concreteness $N = 2$, i.e. $\mathbb{CP}^1$, and for the particular value $\Delta = 1/3$, we have that the $k = 1$ sector truncates dramatically

$$\begin{aligned} C^{(1)}_{0,0}(1/3) &= \sqrt[3]{\frac{e^4}{2\pi}}\,, \\ C^{(1)}_{0,n}(1/3) &= 0\,, \qquad\qquad n \geq 1\,, \end{aligned}$$

note that for larger N these coefficients would truncate after $N-1$ orders. Using (3.5.18) we obtain the asymptotic form of the perturbative coefficients

$$C^{(0),\mathrm{as}}_{0,n}(1/3) = -\frac{\sin(\pi/3)}{\pi}\,\Gamma(n-2/3)\,C^{(1)}_{0,0}(1/3) \tag{3.5.21}$$

using (3.5.19) for $\Delta = 1/3$, so according to (3.5.18) for $n \gg 1$ the difference between the perturbative coefficients and (3.5.21) will tell us about the first sub-leading correction:

$$\begin{aligned} \left[\frac{-\pi\,\left(C^{(0)}_{0,n}(1/3) - C^{(0),\mathrm{as}}_{0,n}(1/3)\right)}{\sin(\pi/3)\Gamma(n-2/3)}\right] &\sim \frac{(-1)^n}{n^{10/3}} C^{(-1)}_{0,0}(1/3) + \mathrm{O}(n^{-13/3}) \\ &\sim \frac{(-1)^n}{n^{10/3}}\frac{\sin(\pi/3)}{\pi}(2\pi)^{-1/3}\,\Gamma(10/3)\,. \end{aligned}$$

In Figure 3.3 we plot the difference between the perturbative coefficients $C^{(0)}_{0,n}(1/3)$ computed via (3.5.5) and their asymptotic form $C^{(0),\mathrm{as}}_{0,n}(1/3)$ just presented in (3.5.21):

$$\begin{aligned} d_n &= \left[\frac{-\pi\,\left(C^{(0)}_{0,n}(1/3) - C^{(0),\mathrm{as}}_{0,n}(1/3)\right)}{\sin(\pi/3)\Gamma(n-2/3)}\right](-1)^n n^{10/3} \qquad (3.5.22) \\ &\sim \frac{\sin(\pi/3)}{\pi}(2\pi)^{-1/3}\,\Gamma(10/3) + \mathrm{O}(n^{-1})\,. \qquad (3.5.23) \end{aligned}$$

For a generic value of Δ we can read the non-perturbative coefficients from the large order perturbative ones.

We want to understand now what happens to the asymptotic forms (3.5.18-3.5.20) when $\Delta \to 0$, i.e. when we reach the supersymmetric point. As we already saw below equation (3.5.5), when we send $\Delta \to 0$ in every non-perturbative sector only the first two perturbative coefficients $C^{(k)}_{0,0}(0)$ and $C^{(k)}_{0,1}(0)$ survive, while all the others vanish. It would seem that

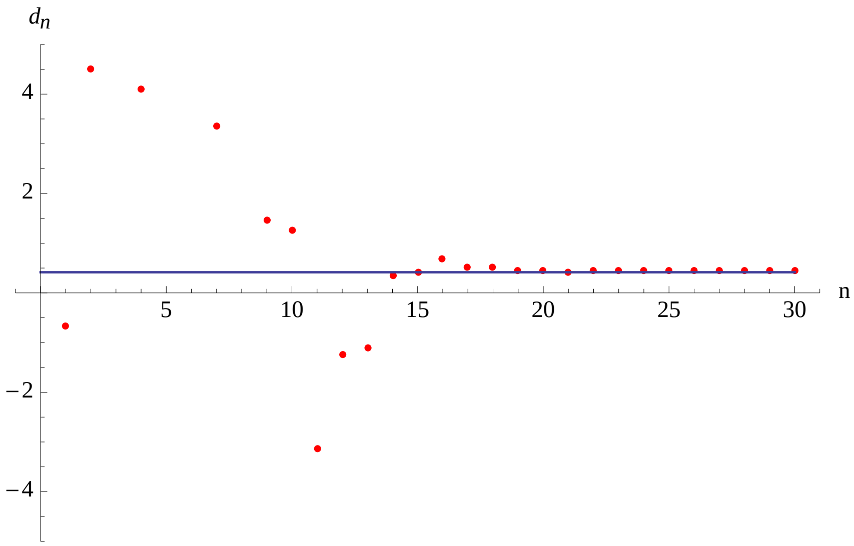


Figure 3.3: Difference d_n between the perturbative coefficients $C^{(0)}_{0,n}(1/3)$ and their asymptotic form $C^{(0),\mathrm{as}}_{0,n}(1/3)$. The blue line is given by the equation $y = C^{(-1)}_{0,0}(1/3) = \frac{\sin(\pi/3)}{\pi}(2\pi)^{-1/3}\,\Gamma(10/3) \simeq 0.415$.

there is no way to reconstruct from the perturbative coefficients some non-perturbative physics and vice versa because the asymptotic forms (3.5.18-3.5.20) do not hold; the series are not asymptotic but they drastically truncate. However the footprints of the Cheshire Cat resurgence are still there! If we consider the asymptotic form (3.5.18) but rather study the coefficients $-c^{(0)}_{0,n}(\Delta)$ using (3.5.13) we have

$$\begin{aligned} c^{(0)}_{0,n}(\Delta) &= \frac{-\pi\, C^{(0)}_{0,n}(\Delta)}{\sin(\pi\Delta)\,\Gamma(n+1+\Delta-N)} \\ &\sim \frac{\Gamma(n-2\Delta)}{\Gamma(n+1+\Delta-N)\,(+1)^{n-2\Delta}}\left(C^{(1)}_{0,0}(\Delta)+\frac{C^{(1)}_{0,1}(\Delta)}{n-2\Delta-1}+\mathrm{O}(n^{-2})\right)+ \\ &\quad+\frac{\Gamma(n-4)}{\Gamma(n+1+\Delta-N)\,(-1)^n}\left(C^{(-1)}_{0,0}(\Delta)+\mathrm{O}(n^{-1})\right)+ \\ &\quad+\cos(3\pi\Delta)\,\frac{\Gamma(n-4\Delta)}{\Gamma(n+1+\Delta-N)\,2^{n-4\Delta}}\left(C^{(2)}_{0,0}(\Delta)+\frac{2\,C^{(2)}_{0,1}(\Delta)}{n-4\Delta-1}+\mathrm{O}(n^{-2})\right). \end{aligned} \tag{3.5.24}$$

We can now safely send $\Delta \to 0$ and the coefficients $c^{(0)}_{0,n}(0)$ will not truncate. As we set $\Delta = 0$ in the right-hand side the first thing to notice is that the contributions of the form $\Gamma(n-\alpha_k)/(-k)^n$, corresponding to the presence of exponentially enhanced terms $e^{+4\pi k\,\xi}$ in the transseries, all disappear since all the coefficients $C^{(-k)}_{0,n}(0) = 0$ when $k > 0$, see equation (3.5.19). This is expected since in the supersymmetric case $\Delta = 0$ these terms were not present in (3.2.19). Furthermore on physical grounds we do not expect terms exponentially enhanced to appear in the expansion of any physical quantity. On the other hand for $k \in \mathbb{N}$ we know that the perturbative coefficients $C^{(k)}_{0,n}(0)$ in the $\mathbb{CP}^{N-1}$ model are non-vanishing only for $n \leq N-1$. For concreteness in the $\mathbb{CP}^1$ case in each non-perturbative sector only the first two perturbative terms are non-vanishing as we already saw in equation (3.2.19), and the asymptotic expansion (3.5.24) reduces to

$$c^{(0)}_{0,n}(0) \sim \frac{(n-1)}{1^n}\left(C^{(1)}_{0,0}(0) + \frac{C^{(1)}_{0,1}(0)}{n-1}\right) + \frac{(n-1)}{2^n}\left(C^{(2)}_{0,0}(0) + \frac{2\,C^{(2)}_{0,1}(0)}{n-1}\right) + \mathrm{O}(n\,3^{-n})\,, \tag{3.5.25}$$

where the non-perturbative coefficients $C^{(k)}_{0,0}(0)$, and $C^{(k)}_{0,1}(0)$ can be obtained from (3.5.5) and (3.5.2) and reproduce precisely the coefficients in the supersymmetric expansion (3.2.20)

$$C^{(k)}_{0,0}(0) = \frac{1}{(k!)^4}\,, \qquad\qquad C^{(k)}_{0,1}(0) = \frac{4H_k - 4\gamma}{(k!)^4}\,. \tag{3.5.26}$$

Note that equation (3.5.25) is actually not an asymptotic expansion and could have been derived in the supersymmetric case by considering the undeformed integrand of (3.2.7), writing the coefficient $c^{(0)}_{0,n}(0)$ as a Cauchy integral around the origin and then closing the contour so to get the contribution from all the other poles, i.e. the non-perturbative sectors. This is of course possible because the partition function (3.2.7) does contain all the information, perturbative and non-perturbative. However had we been given *only* the perturbative coefficients in the supersymmetric case it would have been impossible to reconstruct the non-perturbative data without the aid of Cheshire Cat resurgence.

As a numerical check we can define the asymptotic approximation

$$c^{(0),\,\mathrm{as}}_{0,n}(0) = \frac{(n-1)}{(+1)^n}\left(1 + \frac{4-4\gamma}{n-1}\right) + \frac{(n-1)}{(+2)^n}\frac{1}{16}\,, \tag{3.5.27}$$

where we made explicit use of (3.5.26). From the difference between the perturbative coefficients $c^{(0)}_{0,n}$, that we can easily generate from (3.4.20) and (3.5.1), and the asymptotic

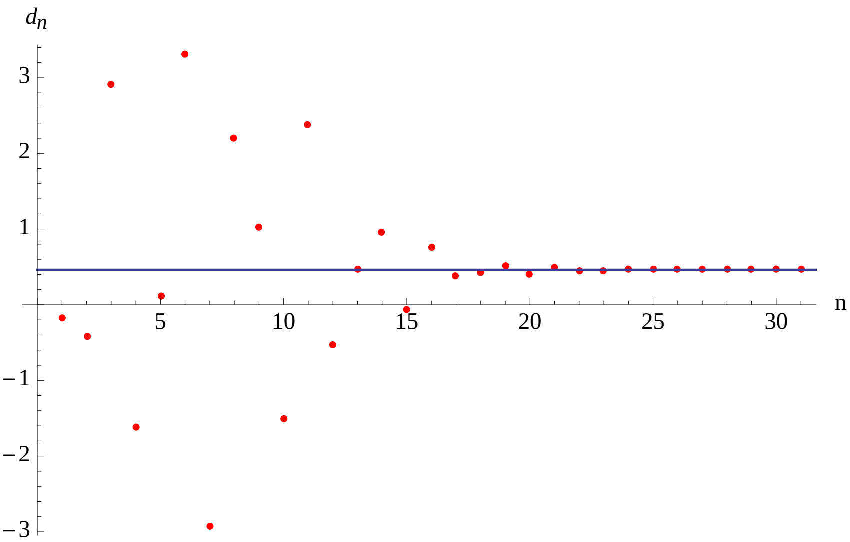


Figure 3.4: Difference d_n between the perturbative coefficients $c^{(0)}_{0,n}(0)$ and their asymptotic form $c^{(0),\mathrm{as}}_{0,n}(0)$. The blue line is given by the equation $y = 2\,C^{(2)}_{0,1}(0) = \frac{1}{4}(3 - 4\gamma) \simeq 0.461$.

form (3.5.27) we can extract non-perturbative information out of perturbative data

$$d_n = \left(c^{(0)}_{0,n}(0) - c^{(0),\,\mathrm{as}}_{0,n}(0)\right) 2^n \sim 2C^{(2)}_{0,1}(0) + \mathrm{O}\left((2/3)^n\right) \sim \frac{1}{4}(3 - 4\gamma)\,. \tag{3.5.28}$$

In Figure 3.4 we show how this difference d_n tends to $2C^{(2)}_{0,1}(0)$ allowing us to reconstruct the perturbative coefficients of the non-perturbative sectors. Surprisingly enough it is still possible to extract non-perturbative data from perturbation theory even when the perturbative expansion truncates: the Cheshire Cat's grin still lingers on even when his body has completely disappeared.

We can repeat this story also for the large order form of the non-perturbative sectors coefficients. We can consider the $k = 1$ sector and rewrite equation (3.5.20) using (3.5.5)

$$\begin{aligned} c^{(1)}_{0,n}(0) &= \frac{-\pi\, C^{(1)}_{0,n}(\Delta)}{\sin(3\pi\Delta)\,\Gamma(n+1+3\Delta-N)} \\ &\sim \frac{\Gamma(n+2\Delta)}{\Gamma(n+1+3\Delta-N)\,(-1)^n}\left(C^{(0)}_{0,0}(\Delta) + \frac{(-1)\,C^{(0)}_{0,1}(\Delta)}{n+2\Delta-1} + \mathrm{O}(n^{-2})\right) + \\ &\quad + \frac{\Gamma(n)}{\Gamma(n+1+3\Delta-N)\,(+1)^n}\left(C^{(2)}_{0,0}(\Delta) + \frac{C^{(2)}_{0,1}(\Delta)}{n-1} + \mathrm{O}(n^{-2})\right) + \dots\,. \end{aligned} \tag{3.5.29}$$

For concreteness we fix once more $N = 2$, i.e. $\mathbb{CP}^1$, so that when we take the limit $\Delta \to 0$ we have only two non-vanishing perturbative coefficients in each sector and in this limit

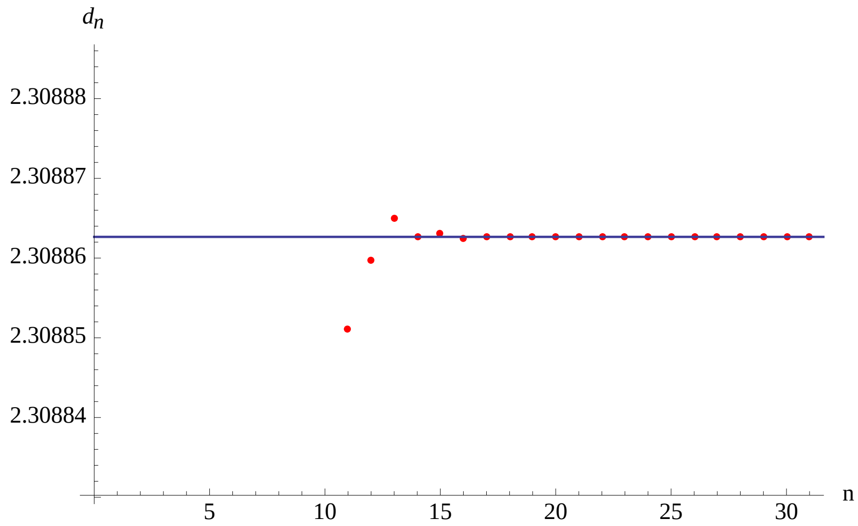


Figure 3.5: Difference d_n between the perturbative coefficients $c^{(1)}_{0,n}(0)$ and their asymptotic form $c^{(1),\mathrm{as}}_{0,n}(0)$. The blue line is given by the equation $y = -C^{(0)}_{0,1}(0) = 4\gamma \simeq 2.308$.

the above equation becomes

$$c^{(1)}_{0,n}(0) \sim \frac{(n-1)}{(-1)^n}\left(C^{(0)}_{0,0}(0) - \frac{C^{(0)}_{0,1}(0)}{n-1}\right) + \frac{(n-1)}{1^n}\left(C^{(2)}_{0,0}(0) + \frac{C^{(2)}_{0,1}(0)}{n-1}\right) + \mathrm{O}(n\,2^{-n})\,.$$

Since the $k=1$ sector "sees" the perturbative sector with a relative action of -1, while the $k=2$ sector with a relative action of $+1$, we have two competing saddles here and find an oscillating behaviour. We can define the asymptotic approximation

$$c^{(1),\,\mathrm{as}}_{0,n}(0) = \frac{(n-1)}{(-1)^n} + \frac{(n-1)}{(+1)^n}\left(\frac{1}{16} - \frac{3-4\gamma}{8(n-1)}\right)\,, \tag{3.5.30}$$

where we made explicit use of (3.5.26). If we consider the difference between the perturbative coefficients in the $k=1$ non-perturbative sector, easily obtained from (3.4.17-3.5.1), and the asymptotic approximation just defined we have

$$d_n = \left(c^{(1)}_{0,n}(0) - c^{(1),\,\mathrm{as}}_{0,n}(0)\right)(-1)^n \sim -C^{(0)}_{0,1}(0) + \mathrm{O}(2^{-n}) \sim 4\gamma\,, \tag{3.5.31}$$

and in Figure 3.5 we see how we can reconstruct the purely perturbative coefficients out of the perturbative data in a given non-perturbative sector even when all the perturbative expansions truncate to a finite number of terms.

3.5.3 Other solvable observables

So far we have considered in detail only the limit $\Delta \to 0$ for which the body of the Cheshire Cat disappears and we find once more the convergent supersymmetric result. However from equation (3.5.5) we can see that more generically we just need Δ to approach an integer and *all* the topological sectors perturbative expansions will truncate after a finite number of terms. For example in the perturbative sector $k = 0, B = 0$, we read from (3.5.5) that whenever $\Delta \to n \in \mathbb{N}$ the perturbative coefficients truncate after $N-n-1$ orders, so fewer orders than the $\Delta \to 0$ case. From (3.5.5) we see that in higher topological number sectors we obtain even fewer perturbative coefficients. It is suggestive to go back to our modified one-loop determinant (3.4.4) and reinterpret this truncation when $\Delta = N_f - N_b \to n \in \mathbb{N}$ as perhaps the insertion of some supersymmetric fermionic operator.

Similarly when Δ approaches a negative integer, $-\Delta = m \in \mathbb{N}$, the perturbative coefficients in the $k = 0, B = 0$, truncate after $N+m-1$ orders hence we find more coefficients than the $\Delta = 0$ case. Contrary to before we can see from (3.5.5) that in higher and higher topological number sectors we obtain more and more perturbative coefficients. Again this increase in perturbative coefficients can be seen from the modified one-loop determinant (3.4.4) interpreting the limit $-\Delta = N_b - N_f \to m \in \mathbb{N}$ as the insertion of some supersymmetric bosonic operator.

It would be tempting to interpret these results as the genuine modification of the original path integral with an unequal (but integer) number of bosons and fermions. However we should stress once more that our modification to the one-loop determinant (3.4.4) effectively takes place only after having heavily exploited the supersymmetry of the model to localize the path integral. It is nonetheless striking to notice the similarity of our truncation of the perturbative coefficients when $\Delta \to n \in \mathbb{Z}$ with the quasi-solvability discussed in [55]. As mentioned in the Introduction the authors of [55] consider an analytic continuation in the number of fermions ζ and they found that in the double Sine-Gordon quantum mechanics the lowest ζ states are algebraically solvable when $\zeta \in \mathbb{N}$ and the exact energies of these levels can be exactly computed and are algebraic functions of the coupling constant.

3.6 Resurgence from analytic continuation in N

An alternative way to obtain Cheshire Cat resurgence for the $\mathbb{CP}^{N-1}$ model is to turn off the supersymmetry breaking deformation Δ and instead consider an analytic continuation in the number of chiral multiplets from $N \in \mathbb{N}$ to $r \in \mathbb{R}$ (or $\mathbb{C}$) thus studying the undeformed partition function (3.2.7) but for a $\mathbb{CP}^{r-1}$ model (one can also consider both deformations at once). Unlike the previously discussed case $\Delta \neq 0$, this deformation is of a more supersymmetric nature and the supersymmetry algebra is still formally unchanged and satisfied. Nonetheless for generic $r \in \mathbb{R}$ we will show that perturbation theory is asymptotic and truncates precisely when $r \to N \in \mathbb{N}$.

When N is replaced by $r \in \mathbb{R}$ the poles and zeroes of the original partition function become branch cuts for the undeformed one-loop determinant and for $r > 0$ (or $\operatorname{Re} r > 0$) we can write the partition function as we did in (3.4.9)

$$Z(r) = \sum_{B\in\mathbb{Z}} e^{-i\theta B} \int_{\mathcal{C}} \frac{d\sigma}{2\pi} e^{-4\pi i\xi\,\sigma}\, \tilde{Z}_{matter}(\sigma)\,, \tag{3.6.1}$$

where the deformed one-loop determinant can be obtained from (3.4.6) after having set $\Delta = 0$

$$\tilde{Z}_{matter}(\sigma) = e^{i\pi B\theta(B)\,r} \exp\left[r\left(\log\Gamma(-i\sigma + |B|/2) - \log\Gamma(1 + i\sigma + |B|/2)\right)\right]\,, \tag{3.6.2}$$

which reproduces the original supersymmetric result whenever $r = N \in \mathbb{N}$. As previously discussed the contour of integration $\mathcal{C}$ comes from $\sigma \to -i\infty - \epsilon$, circles around the origin and then goes back to $\sigma \to -i\infty + \epsilon$, for $r < 0$ (or $\operatorname{Re} r < 0$) we simply close the contour around the positive imaginary axis.

At this point we can repeat the same procedure we followed in Section 3.4, realising that the discontinuity in (3.6.2) now comes only from the $\log\Gamma$ function and, after using the discontinuity property (3.4.7), we obtain

$$Z(r) = \sum_{B\in\mathbb{Z}} e^{-2\pi\xi|B|-i\theta B}\,\tilde{\zeta}_B(r,\xi)\,, \tag{3.6.3}$$

where each Fourier mode can be written as

$$\begin{aligned}\tilde{\zeta}_B(r,\xi) &= \sum_{k=0}^{\infty}\int_k^{k+1}\frac{dx}{2\pi i}e^{-4\pi\xi\,x}\tilde{Z}_{matter}(-ix-i|B|/2-\epsilon)\left[e^{2\pi ikr}-1\right]\\ &= \sum_{k=0}^{\infty}\int_k^{k+1}\frac{dx}{2\pi i}e^{-4\pi\xi\,x}\tilde{Z}_{matter}(-ix-i|B|/2+\epsilon)\left[1-e^{-2\pi ikr}\right]\,.\end{aligned} \tag{3.6.4}$$

Similarly to what we did before we rewrite $\int_k^{k+1}=\int_k^{\infty}-\int_{k+1}^{\infty}$ and shift variables so that every integral becomes between $[0,\infty)$ arriving at the transseries expansion

$$\tilde{\zeta}_B(r,\xi) = e^{i\pi B\theta(B)\,r}\sum_{k=0}^{\infty}e^{-4\pi\xi\,k}\,e^{\mp i\pi kr}\,\tilde{\mathcal{S}}_{\pm}\left[\tilde{\Phi}_B^{(k)}\right](\xi,r)\,, \tag{3.6.5}$$

where $\tilde{\mathcal{S}}_{\pm}$ denote the modified lateral Laplace transforms

$$\tilde{\mathcal{S}}_{\pm}\left[\tilde{\Phi}_B^{(k)}\right](\xi,r) = \int_0^{\infty\pm i\epsilon}dx\,e^{-4\pi\xi\,x}\,x^{-r}\,\tilde{\Phi}_B^{(k)}(x,r)\,, \tag{3.6.6}$$

and, after repeated use of the connection formula (3.4.18), the Borel transform $\tilde{\Phi}_B^{(k)}(x,r)$ is given by

$$\tilde{\Phi}_B^{(k)}(x,r) = \frac{\sin(\pi r)}{\pi}\,\exp\left[r\left(\log\Gamma(1-x)-\log\Gamma(x+k+|B|+1)-\sum_{j=1}^{k}\log(x+j)\right)\right]\,. \tag{3.6.7}$$

Comparing these equations to (3.4.14)-(3.4.15)-(3.4.17) obtained in Section 3.4, we see that the role played by the deformation parameter Δ is now taken by r. If we expand the Borel transform for $x\sim 0$ we get

$$\tilde{\Phi}_B^{(k)}(x,r) \sim \frac{\sin(\pi r)}{\pi}\left(\sum_{n=0}^{\infty}\tilde{c}_{B,n}^{(k)}(r)x^n\right)\,. \tag{3.6.8}$$

We see that in the limit $r\to N\in\mathbb{N}$ we obtain precisely the same coefficients (3.5.2) previously found in the limit $\Delta\to 0$[14].

[14]In the $r\to 0$ limit, the localised partition function becomes a delta function of ξ. However we cannot see this in (3.6.8). The reason is that the way we have closed the contour (see discussion around (3.2.8)) to find (3.6.6) is no longer valid in the $\xi\to 0$ limit, so we need to revert to the original contour of integration to see how the delta function is generated.

So if we consider a weak coupling expansion, i.e. $\xi \to \infty$, of the lateral Borel resummation (3.6.6) we obtain the power series

$$\tilde{\mathcal{S}}_{\pm}\left[\tilde{\Phi}_B^{(k)}\right](\xi, r) \sim (4\pi\xi)^r \sum_{n=0}^{\infty} \tilde{c}_{B,n}^{(k)}(r) \frac{\Gamma(n+1-r)\,\sin(\pi r)}{\pi} (4\pi\xi)^{-n-1}\,. \tag{3.6.9}$$

If we plug this expansion in (3.6.5) we obtain the transseries representation

$$\tilde{\zeta}_B(r,\xi) = e^{i\pi B\theta(B)\,r} \sum_{k=0}^{\infty} e^{-4\pi\xi\,k}\, e^{\mp i\pi k r} (4\pi\xi)^r \left(\sum_{n=0}^{\infty} \frac{\tilde{C}_{B,n}^{(k)}(r)}{(4\pi\xi)^{n+1}} \right), \tag{3.6.10}$$

where the perturbative coefficients $\tilde{C}_{B,n}^{(k)}(r)$ in the k instanton-anti-instanton background on top of the B-instanton topological sector are given by

$$\tilde{C}_{B,n}^{(k)}(r) = \tilde{c}_{B,n}^{(k)}(r) \frac{\Gamma(n+1-r)\,\sin(\pi r)}{\pi}\,, \tag{3.6.11}$$

and the sign of the transseries parameter $e^{\mp i\pi k r}$ is correlated with the direction of the Lateral resummation as in (3.6.5).

These coefficients (3.6.11) are, for generic $r \in \mathbb{R}$, factorially diverging and the above expression (3.6.10) is a purely formal object, i.e. a transseries representation. However as we send $r \to N \in \mathbb{N}$ we see that the $\sin(\pi r) \to 0$ but $\Gamma(n+1-r)$ develops a pole for every $n = 0, ..., N-1$, thus effectively truncating the expansion (3.6.9) to a degree $N-1$ polynomial in ξ reproducing the undeformed equation (3.2.14) for $\mathbb{CP}^{N-1}$, in an identical fashion to the limit $\Delta \to 0$ for deformed case (3.5.3). Although formally still supersymmetric, the $\mathbb{CP}^{r-1}$ model with $r \in \mathbb{R}$ produces asymptotic perturbative expansions, truncating only in the limit $r \to N \in \mathbb{N}$.

3.6.1 Cancellation of ambiguities

Using the formulas just derived we can repeat also in the present $\mathbb{CP}^{r-1}$, $r \in \mathbb{R}$, case the same analysis carried out in Section 3.5 for the Δ deformed model. In particular we can show that the ambiguities in resummation cancel out in (3.6.5) and that the discontinuity for the resummation of the purely perturbative sector contains all the non-perturbative data. To this end we can analyse the difference in lateral resummations, i.e. the Stokes automorphism,

$$\begin{aligned}(\tilde{\mathcal{S}}_+ - \tilde{\mathcal{S}}_-)\left[\tilde{\Phi}_B^{(k)}\right] &= \int_0^\infty dx\, e^{-w\,x} x^{-r}\left(\tilde{\Phi}_B^{(k)}(x+i\epsilon, r) - \tilde{\Phi}_B^{(k)}(x-i\epsilon, r)\right)\\ &= 2i\sin(\pi r)\sum_{n=1}^{\infty} e^{-nw} e^{\mp i\pi(n-1)r}\tilde{\mathcal{S}}_{\pm}\left[\tilde{\Phi}_B^{(k+n)}\right]\end{aligned} \tag{3.6.12}$$

where we made intensive use of the discontinuity property (3.4.7) and connection formula (3.4.18) for the $\log\Gamma$ function and denoted $4\pi\xi = w$.

We can now prove that all the ambiguities cancel out in (3.6.5) by considering the difference between the two lateral resummations together with the jump in the transseries parameter:

$$\begin{aligned}&\sum_{k=0}^{\infty} e^{-kw}\left(e^{-i\pi kr}\tilde{\mathcal{S}}_+ - e^{i\pi kr}\tilde{\mathcal{S}}_-\right)\left[\tilde{\Phi}_B^{(k)}\right]\\ &= \sum_{k=0}^{\infty} -2i\sin(\pi kr) e^{-kw}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(k)}\right] + \sum_{k=0}^{\infty} e^{i\pi kr} e^{-kw}(\tilde{\mathcal{S}}_+ - \tilde{\mathcal{S}}_-)\left[\tilde{\Phi}_B^{(k)}\right]\\ &= \sum_{k=0}^{\infty} -2i\sin(\pi kr) e^{-kw}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(k)}\right] + \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} 2i\sin(\pi r) e^{-(k+n)w} e^{i\pi(k-n+1)r}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(k+n)}\right]\\ &= \sum_{k=0}^{\infty} -2i\sin(\pi kr) e^{-kw}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(k)}\right] + \sum_{m=1}^{\infty} 2i\sin(\pi r) e^{-mw}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(m)}\right]\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\delta_{k+n,m} e^{i\pi(k-n+1)r}\\ &= \sum_{k=0}^{\infty} -2i\sin(\pi kr) e^{-kw}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(k)}\right] + \sum_{m=1}^{\infty} 2i\sin(\pi r)\frac{\sin(\pi mr)}{\sin(\pi r)} e^{-mw}\tilde{\mathcal{S}}_+\left[\tilde{\Phi}_B^{(m)}\right] = 0\,,\end{aligned}$$

where we made use of the Stokes automorphism (3.6.12).

Similarly to Section 3.5, see equation (3.5.14), we can also define the analytic continuation obtained from the purely perturbative coefficients

$$(4\pi\xi)^{-r}\tilde{\zeta}_{\text{pert}}(r,\xi) = \int_0^{\infty\, e^{-i\theta}} dx\, e^{-4\pi\xi\, x}\,(4\pi\xi\, x)^{-r}\,\tilde{\Phi}_0^{(0)}(x,r)\,, \tag{3.6.13}$$

with $\theta = \arg\xi$. From equation (3.6.7) we deduce that this function has two branch cuts along the complex directions $\arg\xi = 0$ and π, and it is a matter of simple calculations to show that its discontinuity across the real positive axis is given by

$$\begin{aligned}\text{Disc}_0(w) &= \int_0^\infty dx\, e^{-w\,x}(w\,x)^{-r}\left(\tilde{\Phi}_0^{(0)}(x-i\epsilon, r) - \tilde{\Phi}_0^{(0)}(x+i\epsilon, r)\right)\\ &= -2i\sin(\pi r)\, e^{-w}\int_0^\infty dx\, e^{-w\,x}(w\,x)^{-r}\,\text{Re}\left(\tilde{\Phi}_0^{(1)}(x,r)\right)\\ &\quad - i\sin(2\pi r)\, e^{-2w}\int_0^\infty dx\, e^{-w\,x}(w\,x)^{-r}\,\text{Re}\left(\tilde{\Phi}_0^{(2)}(x,r)\right) + \text{O}\left(e^{-3w}\right)\,,\end{aligned} \tag{3.6.14}$$

similar to what we obtained in the Δ deformed case (3.5.17). An analog equation can be obtained for the discontinuity across the negative real axis.

From the discontinuity we can read the Stokes constants and as expected the Stokes constant $\tilde{A}_1^{(0)} = -2\sin(\pi r)$ is exactly equal to the jump $2\,\mathrm{Im}\, e^{-i\pi r}$ of the transseries parameter in the $k = 1$ instanton sector in equation (3.6.5). Furthermore, similarly to the Δ deformed case, the Stokes constant $\tilde{A}_2^{(0)} = -\sin(2\pi r)$ for the $k = 2$ sector does not equal the jump $2\,\mathrm{Im}\, e^{-2i\pi r}$ in the transseries parameter for the $k = 2$ sector in (3.6.10). The reason is of course that the jump in the two instanton sector is compensated partly from the term e^{-2w} in the discontinuity for the $k = 0$ sector in (3.6.14) but also from a term e^{-w} in the discontinuity for the $k = 1$ sector that can be similarly computed and produces a Stokes constant $\tilde{A}_1^{(1)} = -2\sin(\pi r)$. The jump $2\,\mathrm{Im}\, e^{-2i\pi r}$ of the $k = 2$ transseries parameter in (3.6.10) is exactly controlled by the Stokes constant $\tilde{A}_2^{(0)}$ of the perturbative sector plus the Stokes constant $\tilde{A}_1^{(1)}$ of the $k = 1$ sector multiplied by the real part $\mathrm{Re}\, e^{-i\pi r}$ of the transseries parameter for the $k = 1$ sector

$$2\,\mathrm{Im}\, e^{-2i\pi r} = \tilde{A}_2^{(0)} + \tilde{A}_1^{(1)}\,\mathrm{Re}\, e^{-i\pi r} = -\sin(2\pi r) - 2\sin(\pi r)\cos(\pi r) = -2\sin(2\pi r)\,.$$

From the above discussion it is a simple exercise to obtain the large order behaviour of the perturbative coefficients, as we did in Section 3.5.2, allowing us to reconstruct non-perturbative physics out of perturbative data. However since these relations are very similar to the ones obtained in Section 3.5.2 we will not present them here.

The key message is that as soon as the number of chiral multiplets $r \in \mathbb{R}$ is kept generic, although the supersymmetry algebra is still formally respected, we have that all the perturbative series appearing in (3.6.10) are just asymptotic expansions. At this point we can make use of resurgent analysis to extract from the purely perturbative data non-perturbative information and only at the very end send the parameter $r \to N \in \mathbb{N}$ obtaining precisely the $\mathbb{CP}^{N-1}$ model result.

3.7 Summary of chapter 3 and open problems

In this chapter we have considered the S^2 partition function of the supersymmetric $\mathbb{CP}^{N-1}$ computed using localization and checked that we can reconstruct the expected chiral ring structure. The weak coupling expansion of this observable can be decomposed according to the resurgence triangle [19] and in each topological sector we find a perturbative series that truncates after finitely many orders making it seemingly impossible to exploit the resurgence machinery to reconstruct non-perturbative physics out of perturbative data. To this end we introduce, after having localized the path integral, a non-supersymmetric deformation that amounts to an unequal number of bosons and fermions. With this deformation in place we can reconstruct the full transseries representation of the deformed partition function and check that perturbation theory does indeed become asymptotic. This is an example of Cheshire Cat resurgence. We can use resurgent analysis to reconstruct from perturbative data the entire non-perturbative sectors previously completely hidden. Once we remove the deformation parameter we go back to the original undeformed case but we can still see the presence of resurgence at work.

Similarly we also consider a supersymmetry preserving deformation where we modify the number of chiral fields from $N \to r \in \mathbb{R}$ and study the $\mathbb{CP}^{r-1}$ model via analytic continuation. Although formally we still retain supersymmetry we immediately generate asymptotic transseries whenever r is kept generic. We show that also in this case from the perturbative asymptotic series we can reconstruct the full transseries via resurgent analysis and only at the very end we send $r \to N \in \mathbb{N}$ to recover the $\mathbb{CP}^{N-1}$ result for which in each topological sector all the perturbative series truncate after finitely many orders.

This 2-dimensional example has shed some light on the role that resurgence plays in quantum field theories with convergent perturbative expansions. As in quantum mechanical examples [54, 55] we can immediately see that a full transseries is hiding behind the “deceptive” convergent supersymmetric result as soon as an appropriate deformation is implemented. This Δ deformation we introduce is not fully satisfactory as it is not a genuine path integral deformation but rather corresponds to a mismatch between the number of bosons and fermions only *after* having localized the path integral. It would be interesting to see if a similar result can be obtained from a bona fide deformation of the original path integral and perhaps understand how it relates to the thimble decomposition discussed in [87]. Whilst not fully satisfactory, we will now move on and try and implement the same type of Δ deformation in higher dimensional theories.

Chapter 4

Cheshire Cat Resurgence in 3 Dimensions

This chapter is based on the work [2], done in collaboration with Daniele Dorigoni.

4.1 Introduction

In this chapter we turn our interest to 3-d $\mathcal{N} = 2$ gauge theories. As mentioned in Section 2.1 this class of theories is amenable to localisation on S^3 [30] and, when a Chern-Simons term is present, their partition functions can be directly written [38] in the form of a resurgent transseries in terms of a small coupling $g = 1/k \ll 1$ given by the inverse of the Chern-Simons level k.

In [39][1] the authors performed the complete resurgent analysis and thimble decomposition for 3-d $\mathcal{N} = 2$ Chern-Simons matter theories showing that, as one varies the argument of the coupling $g = 1/k$, the thimble decomposition of the path-integral exhibits Stokes phenomenon. As expected the ambiguities in resummation of the Borel transform are directly related to the jump in thimbles attached to non-perturbative saddles. Furthermore the analysis of these authors provided a nice interpretation of these non-perturbative effects

[1] We thank Tatsuhiro Misumi for making us aware of this interesting and relevant work.

as contributions coming from new supersymmetric solutions [42] living in a complexification of field space but not on the original path-integral contour.

In this chapter we continue the studies of [39] by considering abelian gauge theories without a Chern-Simons term for which both asymptoticity and topological angle will turn out to be absent. At first it looks like the resurgence structure found in [38, 39] disappears completely, and although we have these complex non-perturbative saddles [42], their classical actions will nonetheless be real, hence the Picard-Lefschetz decomposition of [39] somehow becomes degenerate. The same holds even when considering a squashed sphere [31] with squashing parameter $b > 0$. However, by complexifying the squashing parameter $b = e^{i\theta}$ we will be able to identify "would-be" different topological sectors, i.e. we can distinguish from the thimble point of view the topologically trivial sector from a vortex and an anti-vortex.

The complexification of the squashing parameter can be seen as the introduction of a chemical potential for the $U(1)$ rotation of the S^2 where the vortices are living on when we write the S^3 as a Hopf fibration. It is also interesting to notice that since the building blocks to compute the 3-d $\mathcal{N} = 2$ partition functions are directly related [88, 31] to the structure constants in 2-d Liouville with central charge $c = 1+6(b+b^{-1})^2$ we have that the complexification $b = e^{i\theta}$ interpolates precisely between space-like and time-like Liouville.

However, regardless of the interpretation, the important point is that complexifying the squashing parameter generates a topological angle which was hidden before, and this is very reminiscent of the hidden topological angle studied in [58, 89, 56, 57, 54]. We can thus introduce a Cheshire Cat deformation very similar to the previous chapter and restore the asymptotic nature of the perturbative series around each saddle. This allows us to use the full resurgence machinery to reconstruct from just one element in a column of the resurgence triangle all other elements in the same column, i.e. in the same topological sector. In the next chapter we will comment on how this topological decomposition, combined with the known [90] vortex/anti-vortex factorisation of the partition function which introduces an extra structure on top of resurgence, can be used calculate data from one column of the resurgence triangle and relate it to different columns.

The chapter is organised as follows. Firstly in Section 4.2 we will briefly give an overview of $\mathcal{N} = 2$ supersymmetric field theories on a squashed 3-sphere and present the localised partition functions for different matter contents.

In Section 4.3 we will perform a Picard-Lefschetz decomposition of the localised path-integral and show how a complexification of the squashing parameter b will give rise to a hidden topological angle, allowing us to decompose the theory in a resurgence triangle structure. We will also discuss the physical interpretation of this complexification and the arising of Stokes phenomenon in the Picard-Lefschetz decomposition.

We continue in Section 4.4 with the analysis of the localised path integral using Cheshire Cat resurgence methods. Similarly to the 2-d case we will see that for the original theory there is no asymptotic, factorially growing perturbative series. However after introducing a suitable deformation the asymptotic nature of perturbation theory is reinstated and the resurgence framework can finally be applied. We also discuss how to connect the non-perturbative data to the perturbative data, and at the very end smoothly continue all of our results back to the undeformed original theory while retaining the non-perturbative information acquired from perturbation theory.

Finally we draw some conclusions from this chapter in Section 4.6. Some useful identities for the double sine function, which are used extensively in this chapter, are presented in Appendix A.2.

4.2 $\mathcal{N} = 2$ theories on squashed S^3

It will be useful to recall some facts about $\mathcal{N} = 2$ gauge theories on S^3 and how to calculate their partition functions from localisation methods. For a nice review of 3D $\mathcal{N} = 2$ theories see [91], and for all the details of the localisation calculation see [31] and [92].

Three dimensional field theories with $\mathcal{N} = 2$ supersymmetry have 4 real supercharges, $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot\alpha}$ where α and $\dot\alpha$ run from 1 to 2. They can be seen as a reduction of 4-d

$\mathcal{N}=1$ theories down to three dimensions. Fields are called *chiral* if they commute with $\bar{\mathcal{Q}}_{\dot{\alpha}}$ and anti-chiral if they commute with $\mathcal{Q}_\alpha$. Vector multiplets contain a vector field, two Weyl-fermions and a scalar, while chiral or anti-chiral matter multiplets are made of two complex scalars, and two Weyl-fermions.

We will be interested in theories with an abelian vector multiplet and various chiral and/or anti-chiral matter multiplets charged under the gauge symmetry; hence supersymmetric Lagrangians will then contain a Yang-Mills part, a matter part, a Fayet-Iliopoulos term (*FI* in what follows) with parameter ξ, and finally a Chern-Simons term (which we will not be concerned with[2]):

$$S=\int d^3x\ \left(\mathcal{L}_{\rm YM}+\mathcal{L}_{\rm mat}+\mathcal{L}_{\rm FI}\right)\quad . \tag{4.2.1}$$

In superfield notation (see for example [76] if this is not familiar, here we follow the conventions of [91]), with vector superfield V, matter superfield Q_f in representations R_f of the gauge group, and chiral field strength superfield W_α, these Lagrangians are given by

$$\begin{aligned}
\mathcal{L}_{\rm YM} &= \frac{1}{e^2}\int d^2\theta\ \bar{W}_\alpha W_\alpha\quad ,\\
\mathcal{L}_{\rm mat} &= \sum_f\int d^4\theta\ \bar{Q}_f e^V Q_f\quad ,\\
\mathcal{L}_{\rm FI} &= \xi\int d^4\theta\ V\quad .
\end{aligned}\tag{4.2.2}$$

In the rest of this thesis we will express everything in terms of the FI parameter, and thus the weak coupling expansion will correspond to large $|\xi|\gg 1$, and the strong coupling expansion will have $\xi\sim 0$[3].

The vortex solutions (3.2.3), which are finite actions solutions to the equations of motion in the 2-dimensional case, get promoted to finite energy solutions of the equations of motion in the 3-dimensional case. However for the theory on a squashed 3-sphere, our

[2]When the difference between the number of chiral and anti-chiral multiplets is odd one is forced to include a bare Chern-Simons term to cancel the parity anomaly[93, 94]. Throughout this chapter we will choose matter content such that we do not need to consider the Chern-Simons term.

[3]As in the 2-dimensional case the FI parameter runs and needs to be regularised. There is a natural scale when the theory is put on a squashed 3-sphere, namely $1/r$, and our partition functions are written as a function of the FI parameter at this scale.

manifold can be seen as an S^1 fibration over S^2. Hence the 3-d vortex action can be understood as its energy timesed by the length of the S^1 fiber.

We will be concerned with theories defined on a squashed 3-sphere which can be embedded in 4-d via

$$\frac{b^2}{r^2}\left(x_0^2 + x_1^2\right) + \frac{1}{b^2 r^2}\left(x_2^2 + x_3^2\right) = 1 \ , \tag{4.2.3}$$

and will be denoted as S_b^3 where b is our squashing parameter, and r is the radius of the sphere that we will set to 1 in appropriate units. Note that in here b is thought of as a positive real number and $b = 1$ corresponds to the round sphere case.

The partition function can be computed following the procedure laid out in [27], and performed in [31] and [92][4]. In the case where we do not turn on any mass parameters, the squashed S^3 partition function can be written as

$$Z_{S_b^3} = \int d\hat{x}\, e^{2\pi i \xi\, \mathrm{Tr}(\hat{x})} Z_{vec}(\hat{x}) Z_{matter}(\hat{x}) \ , \tag{4.2.4}$$

where the integral is over the Cartan subalgebra of the gauge group. The parameter ξ is the usual FI term, and the one-loop determinants are for the vector multiplet

$$Z_{vec}(\hat{x}) = \prod_{\alpha \in \Delta_+} \sinh(\pi b \alpha(\hat{x})) \sinh(\pi b^{-1} \alpha(\hat{x})) \quad , \tag{4.2.5}$$

while for the chiral/anti-chiral multiplets

$$Z_{matter}(\hat{x}) = \prod_{w \in R} s_b\left(\frac{iQ}{2}(1 - \Delta) - w(\hat{x})\right) \quad . \tag{4.2.6}$$

Note here we have used Δ_+ to denote the positive roots, w to denote the weights in representation R of the matter multiplet, b is once again the squashing parameter and $Q = b + 1/b$, and Δ is the R-charge of the scalar in the chiral multiplet. For abelian gauge theories the vector multiplet one-loop determinant will simply be one.

[4] As in 2-dimensions there are multiple ways of performing the localisation calculation, each using a different $\mathcal{Q}$ exact deformation. Once again we have Higgs branch localisation, where the scalars in the matter multiplets become massive, and the field strength is 0 everywhere except for at the poles where there are point like vortices. We also have Coulomb branch localisation, where the scalar in the vector multiplet is localised to a constant which needs to be integrated over. We will be concerned with expressions for the partition function that arise from Coulomb branch localisation.

The matter one-loop determinants can be all written in terms of the double sine function $s_b(x)$ presented here in terms of an infinite product

$$s_b\,(x) \;\; = \;\; \prod_{m,n\geq 0} \frac{(mb + n/b + Q/2 - ix)}{(mb + n/b + Q/2 + ix)} \quad . \tag{4.2.7}$$

This function is closely related to the hyperbolic gamma function [95, 96] and the multiple sine function [97], and we present some of its properties in Appendix A.2. The only property of $s_b(x)$ we want to stress here is that for generic b this function has simple zeroes on the lattice $\Lambda_+ = -iQ/2 - ib\,\mathbb{Z}_{\geq 0} - i/b\,\mathbb{Z}_{\geq 0}$ and simple poles on the lattice $\Lambda_- = +iQ/2 + ib\,\mathbb{Z}_{\geq 0} + i/b\,\mathbb{Z}_{\geq 0}$. In the physical squashing limit $b \in \mathbb{R}^+$ the poles and zeroes of the one-loop determinant fall on the imaginary axis and correspond to the appearance of bosonic and fermionic, respectively, massless states.

For the sake of simplicity in the present work we will only be concerned with $U(1)$ gauge theories, though all our conclusions should carry over to the non-abelian case quite simply. For us therefore the vector multiplet one-loop determinant Z_{vec} will always be equal to 1, and the partition function will depend on the FI parameter ξ, the squashing b, and the number of chiral multiplets of charge $+1$, N_c, and the number of chiral multiplets of charge -1 (i.e. anti-chiral multiplets), N_a[5]. For this class of theories we have

$$Z^{(N_c,N_a)}_{S^b_3}(\xi) = \int_\Gamma dx\, e^{2\pi i \xi x}\, \frac{\prod_{i=1}^{N_c} s_b(x + iQ/2)}{\prod_{i=1}^{N_a} s_b(x - iQ/2)} \quad , \tag{4.2.8}$$

where the contour Γ runs along the real x axis and circles around the origin passing in the lower complex x half-plane. Note that we chose the R-charge of the scalars to be $\Delta = 0$.

These integrals can be calculated by closing the contour in the upper half plane (assuming ξ is real and positive) and picking up contributions from all the poles thus leaving a sum over the residues of these poles. General results with non-zero vector and axial masses can be found in [90].

We will shortly analyse the Picard-Lefschetz decomposition and the Cheshire Cat deform-

[5]In the present thesis we will also work with $N_c - N_a$ an even number as to avoid having to introduce a bare Chern-Simons term to cancel the parity anomaly [93, 94].

ation of these types of theories, and to this end it will be useful to better understand the analytic properties of the integrands and their dependence on the matter content and squashing parameter. For this reason we will now discuss some particular examples in more detail.

4.2.1 Round sphere

We start off by considering the theory on the round S^3, i.e. $b=1$, with any matter content. Using equations (4.2.7) and (4.2.8) we have that the partition function on the sphere is given by

$$Z_{S^3}^{(N_c,N_a)}(\xi)=\int_\Gamma dx\, e^{2\pi i\xi x}\prod_{m,n\geq 0}^{\infty}\left(\frac{m+n+2-ix}{m+n+ix}\right)^{N_c}\left(\frac{m+n+2+ix}{m+n-ix}\right)^{N_a}. \quad (4.2.9)$$

We can now rearrange the product by defining $L=m+n$, and realising that for fixed $L\in\mathbb{N}$ we have $L+1$ distinct pairs (m,n) such that $L=m+n$, so we can write

$$Z_{S^3}^{(N_c,N_a)}(\xi) = \int_\Gamma dx\, e^{2\pi i\xi x}\prod_{L=0}^{\infty}\left(\frac{L+2-ix}{L+ix}\right)^{N_c(L+1)}\left(\frac{L+2+ix}{L-ix}\right)^{N_a(L+1)}. \quad (4.2.10)$$

One can evaluate this infinite products using zeta-regularisation (see Appendix A.1) or alternatively for the non-chiral theory $N_c=N_a=N$ one can use equation (A.2.6) to obtain

$$\begin{aligned} Z_{S^3}^{(N,N)}(\xi) &= \int_\Gamma dx\, e^{2\pi i\xi x}\left(\frac{1}{2\sinh(\pi x)}\right)^{2N} = \frac{(-1)^N}{\Gamma(2N)}\sum_{n=0}^{\infty}e^{-2\pi n\xi}\,\xi\prod_{k=1}^{N-1}(\xi^2+k^2) \\ &= \frac{(-1)^N}{\Gamma(2N)}\frac{\xi}{1-e^{-2\pi\xi}}\prod_{k=1}^{N-1}(\xi^2+k^2)\,, \end{aligned} \quad (4.2.11)$$

which can be obtained as the limit $b\to 1$ and vanishing vector and axial masses of the general expression obtained in [90].

Note that the partition function has simple poles at $\xi=ik$ for $k\in\mathbb{Z}$; however for $k\in\{\pm1,...,\pm(N-1)\}$ these are cancelled by the simple zeroes coming from the product. This can be understood from the mirror theory [91, 98] as due to the presence of a single bosonic zero mode for the monopole operators for $\xi=ik$ with $k\in\mathbb{Z}$. However when

$k \in \{\pm 1, ..., \pm(N-1)\}$ the monopole operators acquire also a fermionic zero mode thus giving a finite, non-zero, contribution[6].

A particular case that will shortly be useful is when $N_c = N_a = 1$ and the above expression simplifies to

$$Z^{(1,1)}_{S^3}(\xi) \;=\; \int_\Gamma dx\, e^{2\pi i \xi x} \frac{1}{4\sinh(\pi x)^2} = -\frac{\xi}{1-e^{-2\pi\xi}} \,. \tag{4.2.12}$$

It is manifest both in the above equation as well as in the general case (4.2.11) that the S^3 partition function takes the form of a transseries for which the perturbative expansion in each non-perturbative sector truncates because of supersymmetry after $2N-1$ orders, where again $N = N_c = N_a$.

4.2.2 Non-Chiral theory on squashed S^3

In the case of the non-chiral theory, i.e. when $N_c = N_a$, on the squashed 3-sphere we have the identity given in equation (A.2.6), which enables us to write

$$\frac{s_b\,(x+iQ/2)}{s_b\,(x-iQ/2)} = \frac{1}{4\sinh(\pi x b)\sinh(\pi x/b)} \quad . \tag{4.2.13}$$

Hence we can write the partition function for the non-chiral theory as

$$Z^{(N,N)}_{S^3_b}(\xi) = \int_\Gamma dx\; e^{2\pi i x\xi} \left(\frac{1}{4\sinh(\pi x b)\sinh(\pi x/b)}\right)^N \quad . \tag{4.2.14}$$

For $N=1$ it is fairly simple to compute the residues and obtain

$$Z^{(1,1)}_{S^b_3}(\xi) = -\xi + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \left[e^{-2\pi n\xi b}\csc(n\pi b^2)b + e^{-2\pi n\xi/b}\frac{\csc(n\pi/b^2)}{b}\right] , \tag{4.2.15}$$

which reproduces (4.2.12) when we take the $b \to 1$ limit. As is well known the reason for these two different types of exponentially suppressed corrections comes from the fact that vortices are finite action solutions in 2-d and finite energy solutions in 3-d. However since our 3-d manifold can be seen as an S^1 fibration over S^2 we can understand the 3-d vortex action as its energy timed by the length of the S^1 fibre, hence precisely either $2\pi\xi n \times b$

[6]We thank Stefano Cremonesi for clarifications on this point.

or $2\pi\xi n \times b^{-1}$ depending on which S^1 we are fibering. We note also that due to the supersymmetric nature of the observable under consideration the perturbative expansion in $\xi \gg 1$ around the vacuum, as well as all the non-perturbative sectors, does truncate after finitely many orders.

4.2.3 Chiral theory on squashed S^3

For $N_c \neq N_a$ on the squashed 3-sphere things are a bit harder and one has to introduce the q-Pochhammer symbol, denoted by $(a;q)_\infty$, to obtain a regularised formula for $s_b(x)$ given in equation (A.2.9). From this we can write the partition function as

$$Z_{S_b^3}^{(N_c,N_a)}(\xi) = \int_\Gamma dx\, e^{2\pi i x\xi} \left(e^{-i\pi\frac{(x+iQ/2)^2}{2}} \frac{\left(e^{2\pi bx+2\pi i b^2}; e^{2\pi i b^2}\right)_\infty}{\left(e^{2\pi x/b}; e^{-2\pi i/b^2}\right)_\infty} \right)^{N_c} \left(e^{-i\pi\frac{(x-iQ/2)^2}{2}} \frac{\left(e^{2\pi bx}; e^{2\pi i b^2}\right)_\infty}{\left(e^{2\pi x/b-2\pi i/b^2}; e^{-2\pi i/b^2}\right)_\infty} \right)^{-N_a} \alpha^{N_c-N_a}\,, \tag{4.2.16}$$

where again $Q = b + 1/b$ and we introduce the constant $\alpha = \exp(-i\pi\frac{Q^2-2}{24})$. Note that the q-Pochhammer $(a;q)_\infty$ has a natural boundary of analyticity at $|q| = 1$. We will shortly see that our complexification of the squashing parameter $b \to e^{i\theta}$ will bring us to work within the unit disk for q-Pochhammers. This expression will be useful when analysing the Picard-Lefschetz decompositions.

4.3 Picard-Lefschetz decomposition and hidden topological angle

We start our analysis of the Picard-Lefschetz decomposition of the localised path-integral by considering first theories with a real squashing parameter $b > 0$, and subsequently complexifying it. As it will become clear later on the combination $\Theta = -i(b - 1/b)$ will play the role of hidden topological angle, and hence, by abuse of notation, in this Section we will say that a give saddle belong to the N^{th} topological sector if the imaginary part of its action goes like $N\Theta$; for example the perturbative saddle and the vortex-anti-vortex saddle both have $N = 0$ while the vortex and the anti-vortex have $N = 1$ and $N = -1$ respectively.

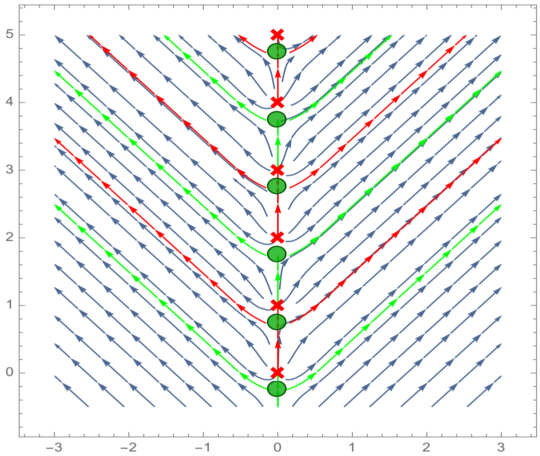
For concreteness let us consider a theory with 1 chiral and 1 anti-chiral multiplet on a round S^3, i.e. $b = 1$. The partition function is given by equation (4.2.12) and it is simple to note that the integrand has double order poles at $x = in$ for $n \in \mathbb{Z}$. Let us now look at this path integral from a Picard-Lefschetz point of view. To this end we exponentiate the one-loop determinant and write the integrand in terms of an effective action

$$\begin{aligned} Z^{(1,1)}_{S^3}(\xi) &= \int_\Gamma dx\, e^{2\pi i x\xi} \frac{1}{4\sinh(\pi x)^2} = \int_\Gamma dx\, e^{-S^{(1,1)}_{eff}(x)} \quad , \\ S^{(1,1)}_{eff}(x) &= -2\pi i \xi x + 2\log(2\sinh(\pi x)) \quad . \end{aligned} \tag{4.3.1}$$

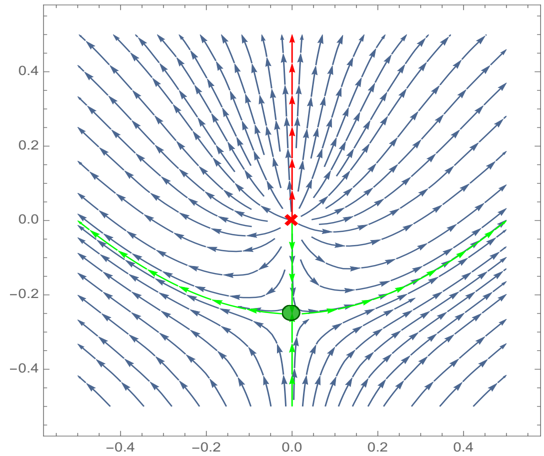
Following the steps outlined in section 2.2.2 we now try and perform the Picard-Lefschetz decomposition of the integration contour Γ using the Morse flow induced by $S^{(1,1)}_{eff}(x)$ for the example above (4.3.1). Since the effective action is basically the logarithm of the one-loop determinant we have that both zeroes and poles of the one-loop determinant will produce singularities of the effective action. Since we are interested in the $\xi \gg 1$ expansion of the path integral we have that each one of the saddle points will live close to each one of the singularities of the effective action (i.e. zeroes and poles of the one-loop determinant) and steepest descent and ascent cycles can now terminate at singular points of the effective action. This is shown in Figure 4.1. Note the similarities between this and the 0-d toy model considered in section 2.3.2.

A very similar analysis was already carried out for 3-d $\mathcal{N} = 2$ Chern-Simons matter theories in [39], see in particular equation (II.21) and our (4.3.1). Notice however some key differences with our results. In particular that when a Chern-Simons term is present the J thimble attached to the perturbative vacuum, noted with $\mathcal{J}_{\text{pt}}$ in [39], passes through the lattice of saddles and poles, see for example their Figure 3, while in our Figure 4.1-(a) the perturbative thimble envelopes all the singularities and saddles.

In the extremely thorough analysis of [39] the authors noted that as the real mass parameter is increased, or equivalently the argument of the coupling $g = 1/k$ is varied, more and



(a) The flow in the upper half plane.



(b) Perturbative saddle and its nearest pole.

Figure 4.1: Morse flow for the theory with one chiral multiplet of charge +1 and one chiral multiplet of charge −1, $b = 1$, $\xi = 1$. The green circles are saddle points while the red crosses are poles of the effective action. From each saddle the downward flows (the J cycles) go off to ∞ while the upward flows (K cycles) flow vertically until they hit a pole.

more non-perturbative thimbles cross the perturbative one and Stokes phenomenon take place, presented in their Figure 7 and 9. These jumps are directly correlated with the jumps in the resummation of the asymptotic expansion for small $g \ll 1$.

At a first glance in our case none of these phenomena happen, the key difference being the absence of a Chern-Simons term. As we have already shown in Section 4.2 the original contour of integration, which can be straightforwardly deformed to the perturbative thimble of Figure 4.1-(a), simply reduces the integral to a sum over residues hence not giving rise to any asymptotic perturbative expansion.

The presence of a Chern-Simons term changes completely the asymptotic form of the effective action for large Coulomb branch parameter $|x| \gg 1$ from the case at hand where $S_{eff}(x) \sim -2\pi i\xi_{\text{eff}}x$ to $S_{eff}(x) \sim -ix^2/(4\pi g)$ with $g = 1/k$ being the inverse Chern-Simons level. As the level goes to zero, i.e. $g \to \infty$, we have a discontinuous jump in the asymptotic regions $\operatorname{Re} S_{eff}(x) > 0$, usually referred to as good regions [23]. As explained in details in [23] the J thimbles are non-compact and their tails must lie in the good regions. It is then the asymptotic behaviour of $\operatorname{Re} S_{eff}(x)$ for $|x| \to \infty$ that dictates the topology of the thimbles. When a Chern-Simons level is present the good regions asymptote two quadrants $\operatorname{Re}[-ix^2/(4\pi g)] > 0$ in the complex x-plane while in our case they asymptote a half plane $\text{Re}\,[2\pi i\xi_{\text{eff}}x] > 0$. This is the reason why the perturbative thimble found in [39] passes through the singular points and the perturbative expansion in small $g = 1/k \ll 1$ becomes asymptotic, as already found in [38], while for us the perturbative thimble circles around the singularities and the perturbative expansion in $\xi \gg 1$, being just a residue calculation, is truncating after finitely many orders[7].

We will shortly see that Stokes phenomenon and an asymptotic perturbative expansion are present also in our case although both very different in nature from the analysis of [39]. We first focus on the thimble decomposition.

As just discussed, in the present case the following puzzle emerges. It is clear from Figure 4.1-(b) that the only non-zero intersection number between the original contour of integration Γ (which was running along the real line and circling around the origin in the lower complex x half-plane) and the K thimbles is when we consider the upward manifold associated to the perturbative saddle, i.e. $x_{cr} = -i/(\pi\xi) + O(\xi^{-3})$. So in order to compute the path integral from the Picard-Lefschetz decomposition we only need to include the integral over the J cycle that is attached to the perturbative saddle and this contour picks up contributions from all the poles in the upper half plane. Contrary to what usually happens in 2-d and 4-d, this includes not only contributions from non-perturbative parts in the same topological sector (vortex-anti-vortex, 2-vortex-2-anti-vortex etc.), but also the contributions from all the other non-perturbative sectors (vortex, anti-vortex, 2-vortex, etc.). For example the second order pole at $x = i$ contains the contributions from the vortex and anti-vortex parts; likewise the pole at $x = 2i$ contains the contributions from the 2-vortex, and the 2-antivortex parts, together with the vortex-anti-vortex, and so on.

We would like to have a decomposition that allows us to discern one topological sector from another. One might try to move away from the round sphere case, i.e. $b \neq 1$, and indeed if we consider the squashed sphere case we do see the poles splitting. This is easiest seen by looking at the definition of $s_b(x)$ in equation (4.2.7). The poles are at $x = imb + in/b$ for $n, m \in \mathbb{Z}$ so for $b \neq 1$ we find first order poles in general, each encoding the contribution coming from a single non-perturbative background. However the Picard-

[7]We thank Masazumi Honda and Tatsuhiro Misumi for useful discussions on these points.

Lefschetz decomposition still has the same problem: we only need to keep the one thimble attached to the saddle corresponding to the perturbative background. Integrating over this J cycle we will pick up all the poles for all the different non-perturbative backgrounds, in every topological sector.

How do we get a decomposition of the localised path integral in terms of different thimbles, each one associated to a would-be different topological sector hence giving us a manifest resurgence triangle structure? The solution to this puzzle comes from considering a complexified squashing parameter $b \in \mathbb{C}$ and $|b| = 1$, i.e. $b = e^{i\theta}$. We will provide a physical interpretation for this complexification in Section 4.3.2 but for the moment let us see what happens to the Picard-Lefschetz decomposition when we consider $b = e^{i\theta}$.

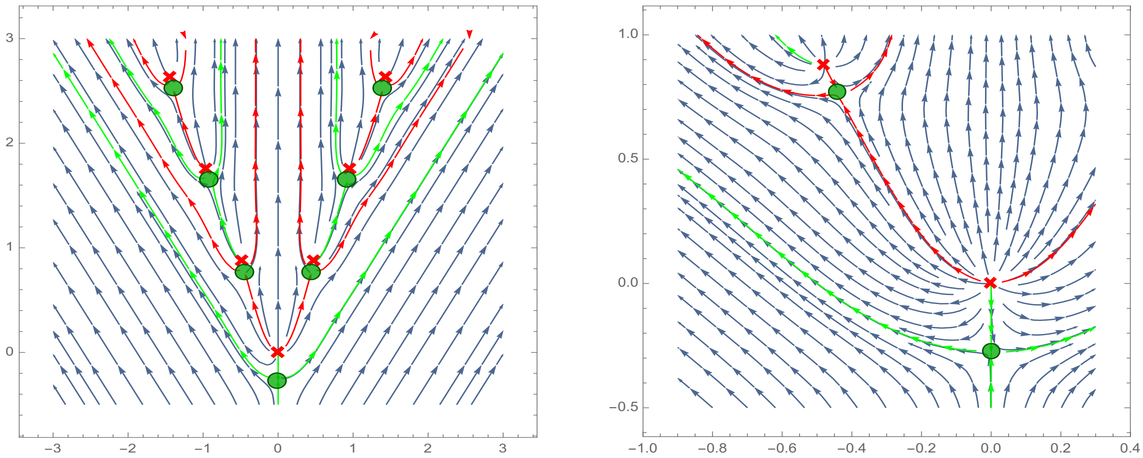
The first effect is that although the poles are still located at $x = imb + in/b$ they no longer are confined to the positive imaginary axis but form a lattice and the only poles found on the positive imaginary axis are those coming from what will form the trivial topological sector.

To be concrete let us re-examine the case with one chiral and one anti-chiral multiplet. The partition function (4.2.14) and effective action, now with general b, are given by

$$\begin{aligned} Z^{(1,1)}_{S^3_b}(\xi) &= \int_\Gamma dx e^{2\pi i x\xi} \frac{1}{4\sinh(\pi x b)\sinh(\pi x/b)} , \\ S^{(1,1)}_{eff}(x) &= -2\pi i x\xi + \log\left(2\sinh(\pi x b)\right) + \log\left(2\sinh(\pi x/b)\right) . \end{aligned} \tag{4.3.2}$$

The singularities are obviously at imb and in/b for $m, n \in \mathbb{Z}$, and while in (4.3.1) these were second order poles for the partition function we see that now the poles split up and separately carry information about the vortices and the anti-vortices. We notice that in this example there are no contributions from poles with both vortices and anti-vortices, e.g. for example a pole at $b + 1/b$. This is very likely because fermion zero modes for these saddles conspire to cancel all their contributions from the path integral. It would be interesting to understand this from the mirror theory.

If we perform a Picard-Lefschetz decomposition as before we obtain Figure 4.2, where we have chosen $b = e^{i/2}$ and $\xi = 1$. For this choice of parameters we can easily see the splitting



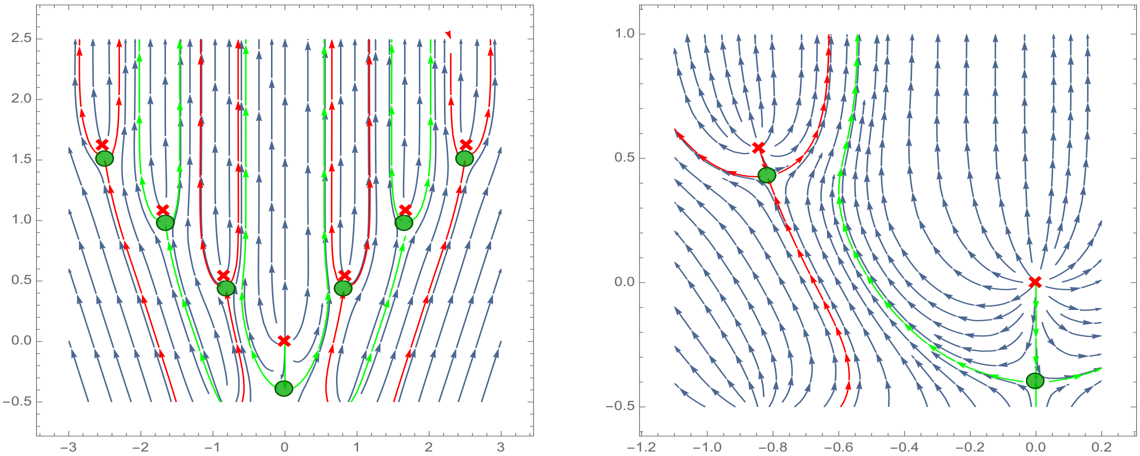
(a) The flow in the upper half plane. (b) Perturbative and 1-vortex saddles.

Figure 4.2: Morse flow for the theory with one chiral multiplet of charge +1 and one chiral multiplet of charge −1, $b = e^{i/2}$, $\xi = 1$. The green circles are saddles and the red crosses are poles. From each saddle the downward flows (the J cycles) go off to the sides and eventually off to ∞. The upward flows (K cycles) flow up or down to the nearest pole. Only the K cycle from the perturbative saddle hits the real axis.

of the poles into contributions from different topological sectors, and as we will shortly discuss in Section 4.3.1, complexifying b will effectively introduce a hidden topological angle so we can distinguish between all the sectors with different topological number; for example the vortex sector from the anti-vortex sector. However it is clear from Figure 4.2 that with this choice of parameters we still only need to integrate over the J cycle from the perturbative saddle as its K cycle is the only one having non-zero intersection number with the original integration contour Γ, i.e. we still have not achieved a complete splitting of the path-integral in thimbles for each topological sectors.

The reason for this lies in our choice of parameters b and ξ. Let us repeat the Picard-Lefschetz decomposition but this time with $b = e^{i}$, without changing $\xi = 1$, shown in Figure 4.3. All the K cycles intersect the original contour of integration Γ, hence following our discussion at the beginning of this Section we must include the contributions from the J cycles coming from all the saddles. We moved from the decomposition in Figure 4.2 to the one in Figure 4.3 by making the argument of b larger. However we could have obtained the same result by cranking up the FI parameter ξ.

As we increase the FI parameter, or alternatively the argument of b, more and more K



(a) The flow in the upper half plane. (b) Perturbative saddle and 1-vortex saddles.

Figure 4.3: Morse flow for the theory with one chiral multiplet of charge +1 and one chiral multiplet of charge −1, $b = e^i$, $\xi = 1$. The green circles are saddles and the red crosses are poles. From each saddle the downward flows (the J cycles) go off to the sides and eventually off to ∞. The upward flows (K cycles) flow up to the nearest pole and they all intersect the real axis.

cycles will eventually intersect the original contour Γ, and hence we have to include in the path-integral more and more J cycles coming from new saddles. This discontinuous transition is called Stokes phenomenon and its presence is tightly connected with the physical interpretation of the complexification of the squashing parameter. We will expand on this in Section 4.3.2. Note however that since we are interested in a weak coupling, semi-classical expansion for the path-integral we are actually interested in the limit $\xi \to \infty$. For this reason, in this limit it is sufficient to include any non-zero complexification of $b = e^{i\theta}$ in order to split the path-integral into the sum of integrals over all of the J cycles in each topological sector as in Figure 4.3.

Let us look at yet another more interesting example given by the theory with two chiral multiplets. The partition function is now

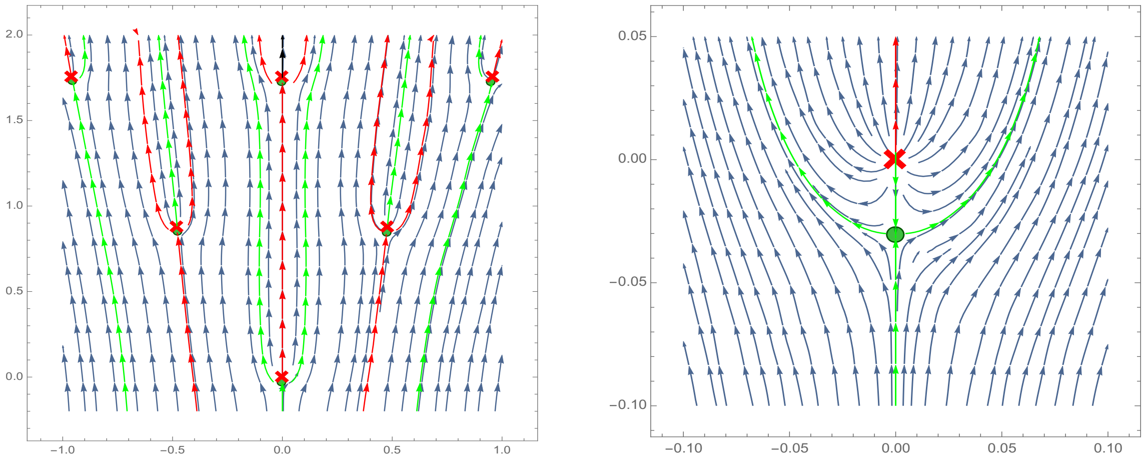
$$\begin{aligned} Z^{(2,0)}_{S^3_b}(\xi) &= \int_\Gamma dx\, e^{2\pi i \xi x} \left(s_b\left(x + iQ/2\right)\right)^2 \\ &= \int_\Gamma dx\, e^{2\pi i\xi x - i\pi(x+iQ/2)^2 - i\pi\frac{Q^2-2}{12}} \left(\frac{\left(e^{2\pi(bx+ib^2)}, e^{2\pi i b^2}\right)_\infty}{\left(e^{2\pi x/b}, e^{-2\pi i/b^2}\right)_\infty}\right)^2 , \end{aligned} \tag{4.3.3}$$

giving us the effective action

$$S^{(2,0)}_{eff}(x) = -2\pi i x\xi + i\pi\left(x + \frac{iQ}{2}\right)^2 + i\pi\frac{Q^2-2}{12} - 2\log\left[\left(e^{2\pi(bx+ib^2)}, e^{2\pi i b^2}\right)_\infty\right]$$

$$+2\log\left[\left(e^{2\pi x/b}, e^{-2\pi i/b^2}\right)_\infty\right] \quad . \tag{4.3.4}$$

Note that the q-Pochhammer $(a;q)_\infty$ is only defined when the modulus of the second argument is less than one. Thus for the above expression for S_{eff} to make sense we must have $e^{2\pi i b^2}$ and $e^{-2\pi i/b^2}$ both with modulus less than one. Thus we will only consider the case where $b = e^{i\theta}$ for $0 < \theta < \pi/2$ (or alternatively $-\pi < \theta < -\pi/2$). We will be easily able to relate this to the case $-\pi/2 < \theta < 0$ (respectively $\pi/2 < \theta < \pi$) by the vortex $\leftrightarrow$ anti-vortex symmetry, i.e. $b \to b^{-1}$.



(a) The flow in the upper half plane. (b) Perturbative saddle and its nearest pole.

Figure 4.4: Morse flow for the theory with two chirals of charge +1, $b = e^{i/2}$, $\xi = 12$. The green circles are saddles and the red crosses are poles. From each saddle the downward flows (the J cycles) go off to the sides and eventually off to ∞. The upward flows (K cycles) flow up to the nearest pole, and down to either the nearest pole, or past the real axis if they are the lowest saddle in their given topological sector.

We now perform the Picard-Lefschetz decomposition as above, which is shown in Figure 4.4. As it is manifest from Figure 4.4 when we decompose the path-integral we need to include all of the J cycles coming from the lowest saddle point in each topological sector. The main novelty in this example is that now we do have contributions coming for all the non-perturbative solutions, i.e. we get contributions from m-vortex-n-anti-vortex saddles for $m, n \in \mathbb{N}$. We do not need to include all of their J cycles, we just need the J thimble coming from the lowest (real part of the) action solution in each topological sector. For example integrating over the J cycle from the perturbative saddle will pick up the contributions from all the saddles in the trivial topological sector, i.e. all the k-vortex-k-anti-vortex saddles. We have just recovered the full resurgence triangle structure.

4.3.1 Recovering the resurgence triangle

In this Section we kept on referring to the critical points of the effective action as different topological sectors despite our 3-dimensional theory not having a topological theta angle characterising the usual 4-d decomposition of the path-integral into different instantonic sectors. The reason for our "abuse" of terminology lies in the complexification of the squashing parameter and the subsequent appearance of what seems to be very similar to a topological angle.

Let us go back to the general partition function (4.2.8) and look more closely at the poles of the one-loop determinant. Here the poles lie at $x = imb + in/b$ for $m, n \in \mathbb{N}$ and the classical action term in the integrand, $e^{2\pi i\xi x} = e^{-S_c}$, evaluated at these locations is $e^{2\pi i\xi(imb+in/b)}$. When $b = e^{i\theta}$ we can define

$$Q = b + \frac{1}{b} = 2\cos\theta\,, \qquad \Theta = -i\left(b - \frac{1}{b}\right) = 2\sin\theta\,. \tag{4.3.5}$$

Now we see that the classical action evaluated at each of the poles can suggestively be rewritten as

$$\begin{aligned} S_c(m,n) &= -2\pi i\xi(imb + in/b) = \pi\xi[(m+n)Q + i(m-n)\Theta] \\ &= \pi\xi\left(|N|Q + i\Theta N\right) + 2\pi\xi Qk\,, \end{aligned} \tag{4.3.6}$$

where $N = m - n$ and $k = \min(m, n)$. In terms of these new variables (N, k) it is now clear that $S_c(m, n)$ corresponds to the k vortex-anti-vortex solution on top of the N-vortex topological sector (anti-vortex sector if $N < 0$). The case $N = 0$, i.e. $m = n$, is then related to the topologically trivial sector, directly connected to the usual perturbative vacuum.

Importantly we notice that the classical actions of these solutions are now complex: the imaginary part of the action is related to a hidden topological angle (HTA) Θ. When b

is real Θ vanishes and we cannot decompose the path-integral into different topological sectors but as soon as we complexify b, Θ becomes non-zero and the HTA allows us to identify a column of non-perturbative contributions topological sector by topological sector. This is reminiscent of the theories studied in [58, 89, 56, 57, 54].

Note however a key difference: for theories with a genuine topological angle the action of non-perturbative objects, being that for example instantons in 4-d or vortices in 2-d, takes the schematic form $S = |N|/g + i\Theta N$ for some coupling constant g and topological number N. In particular the real and imaginary part of the on-shell actions are not correlated, i.e. the θ angle has nothing to do with the coupling constant. In the present case however we have that both the real and imaginary part of the saddles action (4.3.6) depend from the coupling ξ, this will have important repercussions on the resurgent structure of the theory.

Forgetting this issue for the moment we can thus split the partition function into a sum over topological sectors in a transseries:

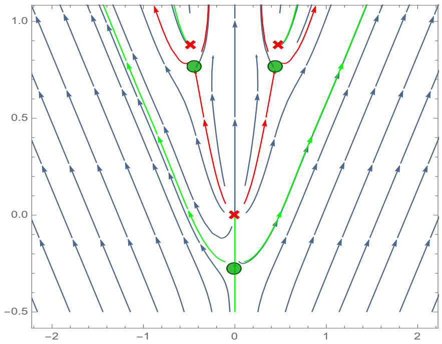
$$Z_{S_b^3}^{(N_c,N_a)}(\xi) = \sum_{N=-\infty}^{\infty} e^{-\pi\xi Q|N| - i\pi\xi\Theta N} \zeta_N(\xi) \, , \tag{4.3.7}$$

where $\zeta_N(\xi)$ contains the contributions from all the k vortex-anti-vortex saddles in the N^{th} topological sector.

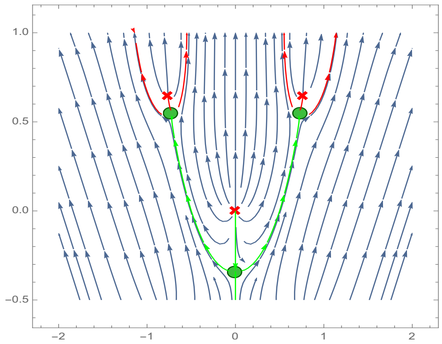
The function $\zeta_N(\xi)$ precisely corresponds to the N^{th} column of the resurgence triangle presented in Figure 2.1

$$\zeta_N(\xi) = \sum_{k=0}^{\infty} e^{-2\pi\xi Q k} \Phi_N^{(k)}(\xi) \, , \tag{4.3.8}$$

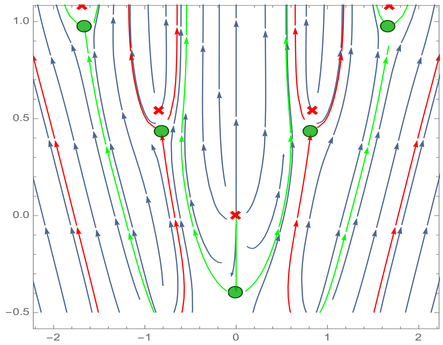
a sum of perturbative expansions, $\Phi_N^{(k)}(\xi)$, on top of a k vortex-anti-vortex background in the N^{th} topological sector. In Section 4.4 we will show how one can use resurgent theory to extract from just one of the $\Phi_N^{(k)}(\xi)$ all the other $\Phi_N^{(k')}(\xi)$ belonging to the same topological sector.



(a) $b = e^{i/2}$



(b) $b = e^{0.87i}$, at Stokes line.



(c) $b = e^{i}$

Figure 4.5: Morse flow for the theory with one chiral multiplet of charge $+1$ and one chiral multiplet of charge -1, for $b = e^{i/2}$, $b = e^{0.87i}$ and $b = e^{i}$ all with $\xi = 1$. The green circles are saddles and the red crosses are poles. As θ increases the position of the saddles and the J and K cycles change. Figure (b) shows when the Stokes crossing happens. At this point the J cycle from the perturbative saddle connects with the K cycle from the vortex and anti-vortex. As θ increases beyond this value, Figure (c), the K cycles from vortex/ anti-vortex no longer flow to the perturbative pole, but cross through the real axis.

4.3.2 Complexified squashing and Stokes phenomenon

We would like to understand now the physical interpretation of this complexified squashing parameter. In [99] (see also [100]) the authors studied the rigid limit of 3-d new minimal supergravity to find all possible backgrounds (metric and auxiliary fields) admitting rigid supercharges. In particular for theories with four supercharges and for which the three dimensional manifold is an S^1 fibration over S^2 the metric takes the form

$$ds^2 = h^2(d\psi + 2\sin^2\frac{\theta}{2}d\phi)^2 + (d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{4.3.9}$$

where (θ, ϕ) are the usual coordinates on S^2, ψ is the angular coordinate parametrising the S^1 Hopf fibre over S^2 and $h \in \mathbb{R}\setminus\{0\}$ that we can parametrise as $h = (b + 1/b)/2$.

However to have rigid supersymmetry we must also turn on some background fields, in particular a vector field V^μ must be present and it takes the form

$$V^\mu \partial_\mu = \frac{b - b^{-1}}{b + b^{-1}} \partial_\psi \,, \tag{4.3.10}$$

hence a never vanishing Killing vector associated to the $U(1)$ isometry of the S^1 fibre. Note that this supersymmetric background has actually two branches. When $b > 0$ both h and V_μ are real and this corresponds precisely to the squashed S^3_b case discussed so far. However we can also pick $b = e^{i\theta}$ for some $\theta \in [0, 2\pi]$ (not to be confused with one of the coordinates of the S^2). The metric is still completely real however the Killing vector has now become purely imaginary.

We can understand this complex squashing as turning on a chemical potential for the $U(1)$ rotation or equivalently, thanks to the non-trivial fibering (4.3.9), for the J_z rotation of S^2. The branch $b > 0$ corresponding to real squashing is continuously connected to the branch $b = e^{i\theta}$ corresponding to the introduction of an omega-deformation, effectively rotating the S^2 along its axis. When this chemical potential is turned on we have that vortices will become weighted by $(b - 1/b)/(b + 1/b) = i\Theta/Q$ while anti-vortices will be weighted by $-(b - 1/b)/(b + 1/b) = -i\Theta/Q$ exactly as shown in equation (4.3.6). For real b we cannot distinguish between different topological sectors via Picard-Lefschetz decomposition, but the moment we include a phase in b the topological sectors split and we can distinguish between them in our Picard-Lefschetz decomposition.

Furthermore we can also understand the reason for the appearance of Stokes phenomenon as we vary the argument of b at fixed FI ξ, or similarly modifying the value of the FI parameter for fixed, non-zero argument of b. The reason is that the FI parameter is regulating the size of the vortices; for infinite FI parameter the vortices are point-like objects perfectly localised at the north and south poles of the S^2. On the other hand for finite FI parameter vortices have a size and they are not perfectly localised at the poles, and have some overlap at the equator.

There is now some play off between the FI parameter and the phase of b. If the value of ξ is not large enough we cannot immediately distinguish between topological sectors via

Picard-Lefschetz decomposition the moment we switch on a phase for b. For a given phase of b, the FI parameter needs to be sufficiently large, i.e. the vortices need to be sufficiently localised at the north and south poles, before we can distinguish between sectors. For small argument of b the imaginary part of the action at the perturbative saddle of the effective action (the saddle just below the pole at the origin) is small but non-zero, and it will generically be different from the imaginary part of the action at the non-perturbative saddles of the effective action. As the argument of b (or the FI) increases, so does the imaginary part of the action of the saddles in the non-perturbative sectors. At some point the imaginary part of the classical action of the perturbative saddle will equal the vortex and the anti-vortex one and it will be possible to construct a thimble joining these different saddles, i.e. we will be at a Stokes line, see Figure 4.5 - (b). At this point the J cycle from the perturbative saddle hits the vortex and anti-vortex saddles. Increasing b even more and we will cross this Stokes line, the J cycle jumps over the saddle from the vortex saddle, and in our decomposition we now have to include the J cycles from the vortex and the anti-vortex as well, 4.5 - (c).

It should be in principle possible to derive our analysis as the limit of vanishing Chern-Simons level, i.e. strong coupling $g = 1/k \to \infty$, and vanishing real masses of the thimble decomposition carried out in [39]. However it is very likely that this is a singular limit since the tails of the thimbles, i.e. the relative homology of good regions ($\mathrm{Re}\, S_{eff}(x) > 0$) in the complex x-plane, change discontinuously for $g > 0$ and $g = 0$. It would also be interesting to analyse more in details the monodromy structure of these thimbles for intermediate values of ξ and understand the connection between these Stokes jumps and the analysis carried out in [101].

Furthermore it was observed in [88] that the building blocks (4.2.7) to compute the 3-d $\mathcal{N} = 2$ partition functions on a round sphere, i.e. $b = 1$, are directly related to the structure constants in 2-d Liouville with central charge $c = 25$. Roughly speaking our 3-d theory is realised on the domain wall of two S-dual $\mathcal{N} = 4$, 4-d gauge theories which are in turn related to 2-d Liouville via AGT correspondence. Subsequently in [31] this correspondence was generalised to the 3-d squashed sphere case, i.e. $b > 1$, and the structure constants of Liouville with central charge $c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2$.

Our complexification of the squashing parameter would now allow us to interpolate continuously between "standard" space-like Liouville, for which $b \in \mathbb{R}$ and $c = 1+6(b+b^{-1})^2 \geq 25$, and time-like Liouville, for which $b = i\hat{b}$ with $\hat{b} \in \mathbb{R}$ $c = 1-6(\hat{b}-\hat{b}^{-1})^2 \leq 1$. We just need to use $b = e^{i\theta}$ with $\theta \in [0, \pi/2]$ to connect $b = 1$ to $\hat{b} = 1$. It would be extremely interesting to follow the analytic continuation of the integration contours of the path-integral for Liouville along this path in the complex b plane following the works [25, 102].

4.4 Resurgence analysis

Now that we have understood how to decompose the localised path integral as a sum over thimbles, each one of them associated to a different topological sector, we want to analyse whether or not, in each topological sector, one can retrieve higher non-perturbative corrections from the purely perturbative data by means of resurgent analysis. We will shortly see that it will be necessary to introduce a Cheshire Cat deformation to make this resurgent structure manifest. However we will first start our discussion with the undeformed theory to clarify the necessity of this deformation.

4.4.1 Undeformed theory

Let us analyse more in detail, and thimble by thimble, the analytic structure of the localised path-integral and for concreteness we will focus to the case of two chirals although it is easy to repeat the analysis in theories with any other matter content. As argued in the previous Section we can decompose the path integral into contours as shown in Figure 4.6.

Thus we can write the path integral as

$$\begin{aligned} Z^{(2,0)}_{S^3_b}(\xi) &= \int_\Gamma dx\, e^{2\pi i \xi x} \left(s_b(x + iQ/2)\right)^2 \\ &= \sum_{n\in\mathbb{Z}} \int_{\Gamma_n} dx\, e^{2\pi i \xi x} \left(s_b(x + iQ/2)\right)^2 \ , \end{aligned} \tag{4.4.1}$$

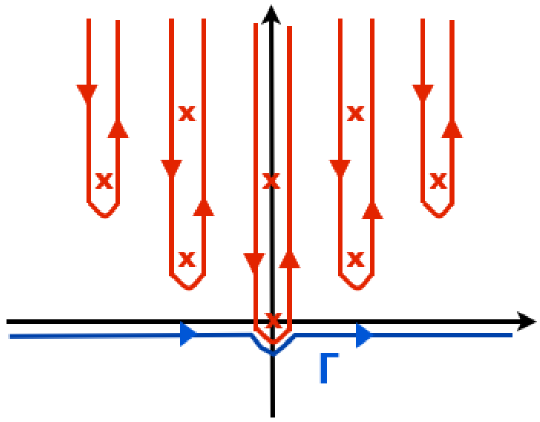


Figure 4.6: Picard-Lefschetz decomposition of the original contour of integration Γ. Each thimble identifies a different topological sector.

where Γ_N is the contour associated to the N^{th} topological sector, $N \leq 0$ being the $-N$ vortex sector, while $N > 0$ being the N anti-vortex sector, as schematically depicted in Figure 4.6.

The contour Γ_{-N}, for $N \geq 0$, runs vertically starting from $+i\infty + \text{Re}(iNb) - \epsilon$ circles around the point iNb and goes back to $+i\infty + \text{Re}(iNb) + \epsilon$. Similarly the contour Γ_N, with $N > 0$, runs vertically starting from $+i\infty + \text{Re}(iN/b) - \epsilon$ circles around the point iN/b and goes back to $+i\infty + \text{Re}(iN/b) + \epsilon$. The first pole in each topological sector is to be found at iNb or iN/b for the contour Γ_{-N} or Γ_N respectively. For each one of these integrals we can shift the integration variable to move the first pole in its topological sector to the origin, namely we rewrite the contours as $\Gamma_{-N} = iNb + \Gamma_0$ or $\Gamma_N = iN/b + \Gamma_0$, for $N \geq 0$. This shift in integration variable will bring out an explicit exponential of the classical action factor as the weight of each topological sector. The partition function is then

$$\begin{aligned}
Z^{(2,0)}_{S^3_b}(\xi) &= \int_{\Gamma_0} dx\, e^{2\pi i \xi x} \left(s_b(x + iQ/2)\right)^2 \\
&\quad + \sum_{N>0} e^{-2\pi\xi N b} \int_{\Gamma_0} dx\, e^{2\pi i \xi x} \left(s_b(x + iNb + iQ/2)\right)^2 \\
&\quad + \sum_{N>0} e^{-2\pi\xi N/b} \int_{\Gamma_0} dx\, e^{2\pi i \xi x} \left(s_b(x + iN/b + iQ/2)\right)^2 \\
&= \zeta_0(\xi, b) + \sum_{N>0} e^{-2\pi\xi N b} \zeta_N(\xi, b) + \sum_{N<0} e^{2\pi\xi N/b} \zeta_N(\xi, b)
\end{aligned}$$

$$= \sum_{N=-\infty}^{\infty} e^{-\pi\xi Q|N|-i\pi\xi\Theta N}\zeta_N(\xi,b)\,, \tag{4.4.2}$$

where again $Q = b + 1/b$ while $\Theta = -i(b - 1/b)$. Upon complexification of the squashing parameter we have that the Picard-Lefschetz decomposition of the path-integral can be directly seen as a manifestation of the resurgence triangle precisely as presented in equation (4.3.7) and the related discussion.

Now let us zoom in on the topologically trivial sector, i.e. $N = 0$, and analyse its corresponding contour integral. We have

$$\begin{aligned} \zeta_0(\xi,b) &= \int_{\Gamma_0} dx\; e^{2\pi i x\xi} s_b\left(x + iQ/2\right)^2 \\ &= \int_{\Gamma_0} dx\; e^{2\pi i x\xi} \prod_{n,m\geq 0}^{\infty} \frac{((m+1)b + (n+1)/b - ix)^2}{(mb + n/b + ix)^2} \\ &= \int_{\Gamma_0} dx\; e^{2\pi i x\xi} \prod_{m=0}^{\infty} \frac{((m+1)(b+1/b) - ix)^2}{(m(b+1/b) + ix)^2} H_0(x)\,, \end{aligned} \tag{4.4.3}$$

where we define $H_0(x)$ by

$$H_0(x) = \prod_{m\neq n} \frac{((m+1)b + (n+1)/b - ix)^2}{(mb + n/b + ix)^2} \quad . \tag{4.4.4}$$

$H_0(x)$ can be regularised using q-Pochhammers, but the important property is that it is entire along the contour Γ_0 as well as in the region define by its interior. On the other hand, the remaining infinite product in the integral can be regularised (see Appendix A.1) to give

$$\zeta_0(\xi,b) \;\; = \;\; \int_{\Gamma_0} dx\; e^{2\pi i x\xi} \frac{\Gamma\left(\frac{ix}{Q}\right)^2}{\Gamma\left(1-\frac{ix}{Q}\right)^2} H_0(x) \quad . \tag{4.4.5}$$

Note that if we were to replace $H_0(x) \to 1$ we would obtain precisely the contribution from the topologically trivial sector to the partition function of the $\mathcal{N} = (2,2)$ $\mathbb{CP}^1$ model on S^2 discussed in the previous chapter where the chiral fields have effective charge $q = 1/Q$. Perhaps not surprisingly this function $H_0(x)$ is storing all the information regarding the additional S^1 and all the different topological sectors. Note also the similarities between (4.4.5) and the 0-d toy model we considered in section 2.3, when the gamma functions are moved into the exponent.

This integral can be performed by summing over the residues of the poles on the positive imaginary axis and the answer we get is of the form

$$\zeta_0(\xi,b) = \sum_{k=0}^{\infty} e^{-2\pi\xi k Q}\zeta_{0,k}(\xi,b) \quad . \tag{4.4.6}$$

We have denoted by $\zeta_{N,k}(\xi,b)$ the contribution from the k vortex-anti-vortex saddle on top of the N^{th} topological sector. For example we have

$$\begin{aligned}
\zeta_{0,0}(\xi,b) &= (2\pi Q)^2\xi H_0(0)\left(1+\frac{iH_0'(0)}{H_0(0)}(2\pi\xi)^{-1}-\frac{4\gamma}{Q}(2\pi\xi)^{-1}\right)\,, \\
\zeta_{0,1}(\xi,b) &= (2\pi Q)^2\xi H_0(iQ)\left(1-\frac{iH_0'(iQ)}{H_0(iQ)}(2\pi\xi)^{-1}+\frac{4(1-\gamma)}{Q}(2\pi\xi)^{-1}\right)\,.
\end{aligned} \tag{4.4.7}$$

The values of the function H_0 and its derivatives at these special points can be computed making use of the functional relations (A.2.6) and the known residues for s_b, for example from (A.2.3) one can easily see that $H_0(0) = 1/(2\pi Q)^2$. The actual values will not play any role in what follows so we will keep them in this implicit form.

Note importantly that these are the perturbative expansions around each of the classical non-perturbative backgrounds, and they are not asymptotic series in ξ, but in fact they truncate after finitely many orders. Thus at first sight it looks like we cannot apply resurgence analysis to this theory.

We can of course repeat this analysis in all of the topological sectors. For the N^{th} topological sector we find

$$\zeta_N(\xi,b) = \int_{\Gamma_0} dx\, e^{2\pi i x\xi}\frac{\Gamma\left(\frac{ix}{Q}\right)^2}{\Gamma\left(1-\frac{ix}{Q}+|N|+iN\frac{\Theta}{Q}\right)^2}H_N(x+i\,|N|\,\frac{Q}{2}-N\frac{\Theta}{2}) \quad . \tag{4.4.8}$$

Here we have defined $H_N(x)$ by

$$H_N(x) = \prod_{m\neq n+N}\frac{((m+1)b+(n+1)/b-ix)^2}{(mb+n/b+ix)^2}\,. \tag{4.4.9}$$

Note that the reason for this splitting into ratio of gamma functions and $H_N(x)$ arises quite naturally by using equations (4.3.5)-(4.3.6) to rewrite the one-loop determinant

(4.2.7)

$$s_b(x + iQ/2) = \prod_{B\in\mathbb{Z}} \prod_{k=0}^{\infty} \frac{|B|Q + iB\,\Theta + 2(k+1)Q - 2ix}{|B|Q + iB\,\Theta + 2kQ + 2ix} \tag{4.4.10}$$

$$= \prod_{B\in\mathbb{Z}} \frac{\Gamma\left(\frac{ix}{Q} + \frac{|B|}{2} + i\frac{B\,\Theta}{2Q}\right)}{\Gamma\left(1 - \frac{ix}{Q} + \frac{|B|}{2} + i\frac{B\,\Theta}{2Q}\right)}\,, \tag{4.4.11}$$

where we defined $B = m - n$ and $k = \min(m, n)$. In the given N^{th} topological sector we factorise out the ratio of gamma functions coming from the $B = N$ term which will be the only singular factor along the corresponding contour of integration; everything else is collected in these auxiliary functions $H_N(x)$. Once more if we were to set $H_N(x) \to 1$ we would obtain precisely the contribution coming from the topological sector with magnetic flux $B = N$ and θ angle directly related to our Θ for the two-dimensional supersymmetric $\mathbb{CP}^1$ model discussed in the previous chapter.

Here as well we can regulate the function $H_N(x)$ using q-Pochhammers, but as it is entire along the contour and in its interior we will not need its precise form. When we can evaluate these integrals we get an expansion of the form

$$\zeta_N(\xi, b) = \sum_{k=0}^{\infty} e^{-2\pi\xi kQ} \zeta_{N,k}(\xi, b) \quad , \tag{4.4.12}$$

precisely as expected from our resurgence triangle discussion for equation (4.3.8). As seen for the topologically trivial sector, when we write $\zeta_{N,k}(\xi, b)$ as a perturbative series in ξ and we find that it truncates after finitely many orders. In the present case of two chirals the truncation happens precisely after two orders, so we will need to deform the theory before we are able to apply the resurgence framework to reconstruct non-perturbative information from perturbative data.

4.4.2 Cheshire Cat deformation

To re-introduce the (general) asymptotic nature of every perturbative expansion we now want to add a Cheshire cat deformation to the theory. Following what we did in the previous chapter, we have two options to consider. One possibility is to analytically deform the integrand of the localised partition function to mimic a non-supersymmetric unbalance between the number of bosonic and fermionic degrees of freedom. To do this we note that the matter one-loop determinant for the chiral theory $(N_c, 0)$ can easily be written as

$$Z_{matter} = \left(\frac{\det O_\psi}{\det O_\phi}\right)^{N_c} \quad . \tag{4.4.13}$$

Thus this supersymmetry breaking deformation would look something like

$$\tilde{Z}_{matter} = \frac{(\det \mathcal{O}_\psi)^{N_f}}{(\det \mathcal{O}_\phi)^{N_b}} = Z_{matter}\ (\det \mathcal{O}_\phi)^{\Delta}\ , \tag{4.4.14}$$

where we have set $N_b = N_c - \Delta$ and $N_f = N_c$. To proceed we would need to keep and regulate the full one-loop determinant written as an infinite product over eigenvalues with degeneracies, which can be found in [31], [92], without all the cancellations between pairing of bosonic and fermionic modes that take place when $\Delta = 0$ producing the simpler expression (4.2.7).

The second option, which turns out to be nicer, is to deform the number of chiral multiplets to be non-integer, $N_c \to N_c + \Delta$. Everything we will discus in this thesis works perfectly fine in both cases, but the expressions are much shorter for this latter deformation, and just as illuminating. In this case we have

$$\tilde{Z}_{matter} = \left(\frac{\det \mathcal{O}_\psi}{\det \mathcal{O}_\phi}\right)^{N_c+\Delta} \quad . \tag{4.4.15}$$

Because both the fermionic and bosonic determinants are raised to the same power we still have the same cancellations between the determinants, and so we can stick with the one-loop determinant expressions we already have.

In this Section we will focus only on this second type of Cheshire Cat deformation where we analytically continue in the number of chiral fields to non-integer values. In the previous chapter we have already shown in a 2-d context how the introduction of an unbalance between bosons and fermions, effectively breaking supersymmetry, produces very similar results.

However a striking point we want to stress is how almost any conceivable deformation of the theory will immediately make the perturbative expansions asymptotic, allowing us to use the full machinery of resurgent analysis. In many cases, supersymmetrically localised theories are effectively sitting at very special points in the space of "physical functions" where miraculous cancellations hide the resurgence structure. Whenever a Cheshire Cat deformation is re-instated we can instantly see reappearing the complete resurgent body, and when taking the vanishing limit of this deformation only its grin will remain.

For simplicity we will now work in the topologically trivial sector, but everything follows through in the other sectors in exactly the same manner, one has just to replace $H_0(x)$ by $H_N(x+iNb)$ or $H_N(x-iN/b)$ and the ratio of gamma functions according to (4.4.8). Now applying the deformation instead of equation (4.4.3) we get

$$\begin{aligned}\tilde{\zeta}_0(\xi,b,\Delta) &= \int_{\Gamma_0} dx\, e^{2\pi i x\xi}\frac{\Gamma\left(\frac{ix}{Q}\right)^{\Delta+2}}{\Gamma\left(1-\frac{ix}{Q}\right)^{\Delta+2}}H_0(x) \\ &= \int_{\Gamma_0} dx\, e^{2\pi i x\xi}H_0(x)e^{(\Delta+2)\left[\log\Gamma\left(\frac{ix}{Q}\right)-\log\Gamma\left(1-\frac{ix}{Q}\right)\right]}\,.\end{aligned} \tag{4.4.16}$$

Note that in principle the deformation would also alter the function $H_0(x)\to H_0(x)^{1+\Delta/2}$. However this turns out to be superfluous since the deformation of $H_0(x)$ will not add anything new and to recover the resurgence structure it will be sufficient to just deform the ratio of gamma functions. The only change we want to point out is that both the poles and the zeroes of $H_0(x)$ will become branching points for $H_0(x)^{1+\Delta/2}$.

Now the contour Γ_0 comes down from $+i\infty-\epsilon$, circles the origin and goes back up to $+i\infty+\epsilon$. We make the change of variables $x\to ix$ so the integral is now along the positive real axis and we note that the function $\log\Gamma(-\frac{x}{Q})$ has a branch cut precisely on the contour of integration so we obtain the integral along the real axis of its discontinuity

$$\tilde{\zeta}_0(\xi,b,\Delta) = i\int_0^\infty dx\, e^{-2\pi\xi x}H_0(ix)e^{-(\Delta+2)\log\Gamma\left(1+\frac{x}{Q}\right)} \tag{4.4.17}$$

$$\left(e^{(\Delta+2)\log\Gamma\left(-\frac{x}{Q}+i\epsilon\right)}-e^{(\Delta+2)\log\Gamma\left(-\frac{x}{Q}-i\epsilon\right)}\right)\,. \tag{4.4.18}$$

We can now use the discontinuity formula,

$$\log\Gamma(-x+i\epsilon)-\log\Gamma(-x-i\epsilon) = -2\pi i\left(\lfloor x\rfloor+1\right)\,, \tag{4.4.19}$$

where $\lfloor x \rfloor$ denotes the floor of x, to write $\tilde{\zeta}_0(\xi, b, \Delta)$ in the form

$$\tilde{\zeta}_0(\xi, b, \Delta) = i \int_0^\infty dx\ e^{-2\pi\xi x} H_0(ix) e^{-(\Delta+2)\log\Gamma\left(1+\frac{x}{Q}\right)+(\Delta+2)\log\Gamma\left(-\frac{x}{Q}\pm i\epsilon\right)}$$
$$e^{\pm\pi i(\Delta+2)\left(\lfloor\frac{x}{Q}\rfloor+1\right)}\left(e^{-\pi i(\Delta+2)\left(\lfloor\frac{x}{Q}\rfloor+1\right)} - e^{+\pi i(\Delta+2)\left(\lfloor\frac{x}{Q}\rfloor+1\right)}\right) \quad . \tag{4.4.20}$$

Next to make manifest the transseries nature of this integral we rewrite the domain of integration as

$$\int_0^\infty dx\ f(x) = \sum_{k=0}^\infty \int_{kQ}^{(k+1)Q} dx\ f(x)\,,$$

evaluate the floor function on each interval and then use the identity

$$\sum_{k=0}^\infty \int_{kQ}^{(k+1)Q} dx\ f(x) = \sum_{k=0}^\infty \left(\int_{kQ}^\infty dx\ f(x) - \int_{(k+1)Q}^\infty dx\ f(x)\right) \quad . \tag{4.4.21}$$

Finally we change variables to make all the integrals start from the origin. In this way we can write

$$\tilde{\zeta}_0(\xi, b, \Delta) = \sum_{k=0}^\infty e^{-2\pi\xi kQ}\tilde{\zeta}_{0,k}(\xi, b, \Delta) \quad . \tag{4.4.22}$$

For the moment we specialise to $\tilde{\zeta}_{0,0}(\xi, b, \Delta)$ which takes the form

$$\tilde{\zeta}_{0,0}(\xi, b, \Delta) = 2\sin(\pi\Delta) e^{\pm i\pi\Delta} \int_0^\infty dx\, e^{-2\pi\xi x} H_0(ix) e^{-(\Delta+2)\left[\log\Gamma\left(1+\frac{x}{Q}\right)-\log\Gamma\left(-\frac{x}{Q}\pm i\epsilon\right)\right]}\,,$$

and by making use of the shift formula

$$\log\Gamma(-x \pm i\epsilon) = \log\Gamma(1 - x \pm i\epsilon) - \log(x) \mp i\pi \quad , \tag{4.4.23}$$

we obtain

$$\begin{aligned}
\tilde{\zeta}_{0,0}(\xi, b, \Delta) &= 2\sin(\pi\Delta) \int_0^\infty dx\, e^{-2\pi\xi x} H_0(ix) \left(\frac{x}{Q}\right)^{-(\Delta+2)} e^{-(\Delta+2)\left[\log\Gamma\left(1+\frac{x}{Q}\right)-\log\Gamma\left(1-\frac{x}{Q}\pm i\epsilon\right)\right]} \\
&= \int_0^\infty dx\, e^{-2\pi\xi x} x^{-(\Delta+2)} \Phi_0^{(0)}(x \mp i\epsilon) = \tilde{\mathcal{S}}_\mp \left[\Phi_0^{(0)}\right](\xi, b, \Delta)\,.
\end{aligned} \tag{4.4.24}$$

In the last line we introduced the modified lateral Laplace transform whose explicit definition is given by

$$\begin{aligned}
\tilde{\mathcal{S}}_{\mp}\left[\Phi\right](\xi) &= \lim_{\epsilon\to 0^+}\int_0^{\infty\mp i\epsilon} dx\, e^{-2\pi\xi x}x^{-(\Delta+2)}\Phi(x) \\
&= \lim_{\epsilon\to 0^+}\int_0^{\infty} dx\, e^{-2\pi\xi x}x^{-(\Delta+2)}\Phi(x\mp i\epsilon)\,. \qquad (4.4.25)
\end{aligned}$$

The Borel transform of the purely perturbative part in the topologically trivial sector $\Phi_0^{(0)}(x)$ can be read off from (4.4.24)

$$\Phi_0^{(0)}(x) = 2\sin(\pi\Delta)\, H_0(ix)\, Q^{\Delta+2}\, e^{-(\Delta+2)\left[\log\Gamma\left(1+\frac{x}{Q}\right)-\log\Gamma\left(1-\frac{x}{Q}\right)\right]}\,. \qquad (4.4.26)$$

Importantly this has finite radius of convergence around the origin and can be expanded as a power series in x

$$\Phi_0^{(0)}(x) = \sin(\pi\Delta)\sum_{m=0}^{\infty} c_{0,0,m}(b,\Delta)x^m\,. \qquad (4.4.27)$$

After commuting this series with the Laplace integral we finally obtain

$$\tilde{\zeta}_{0,0}(\xi,b,\Delta) = \sin(\pi\Delta)(2\pi\xi)^{\Delta+2}\sum_{m=0}^{\infty}\frac{c_{0,0,m}(b,\Delta)}{(2\pi\xi)^{m+1}}\Gamma(m-1-\Delta)\quad . \qquad (4.4.28)$$

It is simple to note that for generic, i.e. non-integer Δ this series is asymptotic. Precisely as anticipated after having performed this Cheshire Cat deformation the perturbative expansion is not truncating any longer and we are left with a factorially growing asymptotic series. Furthermore when we take the limit $\Delta\to 0$ we have $\sin(\pi\Delta)\to 0$ but simultaneously the $\Gamma(m-1-\Delta)$ develops poles for $m=0,1$, hence in this limit we reproduce exactly the undeformed result $\tilde{\zeta}_{0,0}(\xi,b,\Delta)\overset{\Delta\to 0}{\longrightarrow}\zeta_{0,0}(\xi,b)$ of equation (4.4.7).

We can also find the general expression for all $\tilde{\zeta}_{0,k}(\xi,b,\Delta)$. Starting from equation (4.4.20), rewriting the integral as we did before, and using the shift formula (4.4.23) we find

$$\begin{aligned}
\tilde{\zeta}_{0,k}(\xi,b,\Delta) \;&=\; 2\sin(\pi\Delta)e^{\pm i\pi k\Delta}\int_0^{\infty} dx\, e^{-2\pi\xi x}H_0(ix+ikQ)\prod_{n=0}^{k}\left(\frac{x}{Q}+n\right)^{-(\Delta+2)} \\
&\qquad e^{-(\Delta+2)\left[\log\Gamma\left(1+k+\frac{x}{Q}\right)-\log\Gamma\left(1-\frac{x}{Q}\pm i\epsilon\right)\right]} \\
&=\; e^{\pm i\pi k\Delta}\int_0^{\infty} dx\; e^{-2\pi\xi x}x^{-(2+\Delta)}\Phi_0^{(k)}(x\mp i\epsilon)
\end{aligned}$$

$$= \quad e^{\pm i\pi k\Delta}\tilde{\mathcal{S}}_{\mp}\left[\Phi_0^{(k)}\right](\xi, b, \Delta) \quad , \tag{4.4.29}$$

where once more we used the modified lateral Laplace transform (4.4.25) to integrate the Borel transform $\Phi_0^{(k)}(x)$ of the k^{th} vortex-anti-vortex non-perturbative sector that reads

$$\begin{aligned}\Phi_0^{(k)}(x) =& 2\sin(\pi\Delta)\, H_0(ix+ikQ)\, Q^{\Delta+2}\prod_{n=1}^{k}\left(\frac{x}{Q}+n\right)^{-(\Delta+2)} \\ & e^{-(\Delta+2)\left[\log\Gamma\left(1+k+\frac{x}{Q}\right)-\log\Gamma\left(1-\frac{x}{Q}\right)\right]}\,,\end{aligned} \tag{4.4.30}$$

reducing to (4.4.26) for $k=0$.

Similarly to the purely perturbative series also in the non-perturbative sectors one can expand the Borel transform as a convergent power series at the origin $x=0$ and commute the sum with the integral to obtain an asymptotic, factorially growing power series for generic Δ. Taking the limit $\Delta\to 0$ reproduces precisely the truncating perturbative series (4.4.7) of the undeformed case.

Putting everything together we arrive at the complete transseries expression for (4.4.22)

$$\tilde{\zeta}_0(\xi, b, \Delta) = \sum_{k=0}^{\infty} e^{-2\pi\xi kQ} e^{\pm i\pi k\Delta}\tilde{\mathcal{S}}_{\mp}\left[\Phi_0^{(k)}\right](\xi, b, \Delta)\,, \tag{4.4.31}$$

where the factor $e^{\pm i\pi k\Delta}$ is called the transseries parameter. Note that a similar analysis can be carried out in each topological sector.

The expression (4.4.31) for the full, perturbative and non-perturbative, set of contributions to the topologically trivial sector tells us that we are working with what is called a one parameter transseries. One might think that according to our choice of sign $e^{i\pi k\Delta}\tilde{\mathcal{S}}_-$ or $e^{-i\pi k\Delta}\tilde{\mathcal{S}}_+$ we would find two different results for real and positive ξ; however as was shown in full details in [73] for the most general one parameter the jump in this transseries parameter is precisely needed to cancel the ambiguity in the resummation $(\tilde{\mathcal{S}}_+-\tilde{\mathcal{S}}_-)[\Phi_0^{(k)}]$, also called Stokes automorphism.

Our transseries (4.4.31) is completely real and unambiguous for real and positive ξ: one can use the analysis[8] of Section 3.6 or the more general expressions in [73] to show that the

[8]Note that in the present case the function H_0 does not really play any role and it is just carried along the way. The one-parameter nature of the transseries under consideration comes from the particular

would-be ambiguity cancels order by order in the vortex-anti-vortex counting parameter $e^{-2\pi\xi k Q}$.

The imaginary part of the transseries parameter $\mathrm{Im} e^{\pm i\pi k\Delta} = \pm\sin(\pi k\Delta)$ is exactly (anti-)correlated with the discontinuity $(\tilde{\mathcal{S}}_+ - \tilde{\mathcal{S}}_-)[\Phi_0^{(k)}]$. Hence (4.4.31) is the real solution corresponding to what is called median resummation (see [103] and the general discussion in [73]). Using the results of Section 6 of [1] we can also rewrite (4.4.31) in the manifestly real and unambiguous form

$$\begin{aligned}\tilde{\zeta}_0(\xi,b,\Delta) &= \sum_{k=0}^{\infty} e^{-2\pi\xi k Q}\cos^k(\pi\Delta)\,\tilde{\mathcal{S}}_0\left[\mathrm{Re}\left(\Phi_0^{(k)}\right)\right](\xi,b,\Delta)\\ &= \tilde{\mathcal{S}}_0\left[\mathrm{Re}\left(\Phi_0^{(0)}\right)\right](\xi,b,\Delta) + e^{-2\pi\xi Q}\cos(\pi\Delta)\,\tilde{\mathcal{S}}_0\left[\mathrm{Re}\left(\Phi_0^{(1)}\right)\right](\xi,b,\Delta)\\ &\quad + e^{-4\pi\xi Q}\cos^2(\pi\Delta)\,\tilde{\mathcal{S}}_0\left[\mathrm{Re}\left(\Phi_0^{(2)}\right)\right](\xi,b,\Delta) + O\left(e^{-6\pi\xi Q}\right)\,, \qquad (4.4.32)\end{aligned}$$

where $\tilde{\mathcal{S}}_0$ denotes the modified Laplace transform (4.4.25) where the integration contour is the positive real axis which we can do now given the fact that $\mathrm{Re}\left(\Phi_0^{(k)}\right)(x)$ is completely regular for $x > 0$.

As already stressed if we were to expand each Laplace integral as a power series we would obtain a factorially divergent perturbative expansion in $1/\xi$ in each non-perturbative sector, however when we take the limit $\Delta \to 0$ all of these will truncate to finitely many perturbative coefficients thus reproducing (4.4.7). We will now show that having made the body of the Cheshire Cat visible by considering generic Δ will allow us to reconstruct the non-perturbative sectors from the asymptotic perturbative one and vice versa.

4.4.3 Non-perturbative data from perturbation theory

What we would like to do now is using the resurgence machinery to reconstruct the deformed non-perturbative sectors (4.4.29) and eventually the undeformed contributions (4.4.7) from the deformed resummed perturbative data (4.4.24) or equivalently from the deformed asymptotic perturbative series (4.4.28).

combination of $\log\Gamma$ functions.

A standard method is to start from the perturbative asymptotic power series (4.4.28) and resum it by performing a directional Laplace integral of its Borel transform (4.4.26)

$$\begin{aligned}(2\pi\xi)^{-(\Delta+2)}\tilde{\zeta}_{0,0}(\xi,b,\Delta) &= \int_0^\infty \frac{dy}{2\pi\xi}e^{-y}y^{-(\Delta+2)}\Phi_0^{(0)}\left(\frac{y}{2\pi\xi}\right) \\ &= \int_0^{\infty e^{-i\theta}} dx\, e^{-2\pi\xi x}(2\pi\xi x)^{-(\Delta+2)}\Phi_0^{(0)}(x) = \mathcal{S}_\theta[\Phi_0^{(0)}](\xi,b,\Delta)\,,\end{aligned} \tag{4.4.33}$$

where $\theta = \arg\xi$ and $\mathcal{S}_\theta$ denotes the modified directional Laplace transform, similar to equation (4.4.25) (in here we added an additional factor $(2\pi x)^{-(\Delta+2)}$ for convenience).

The above equation does define a function on the complex variable ξ by anti-correlating its argument with the direction of the Laplace transform. This function is defined everywhere on the complex ξ plane save some cuts where there is a discontinuity in the directed Laplace transform because of singularities of the integrand, i.e. the Stokes directions of the Borel transform.

The well known dispersion like arguments [104, 86] discussed in Section 2.2.1 applied to the function just constructed from the purely perturbative data, i.e. $\tilde{\zeta}_{0,0}(\xi,b,\Delta)$, would generically allow us to relate the asymptotic form of the perturbative coefficient (4.4.28) to the discontinuities of this function, which in turn directly relates to all the non-perturbative contributions (4.4.29) (and their associated perturbative expansions) coming from the tower of k vortex-anti-vortex configurations in the same topological sector.

In the present case however we cannot straightforwardly use this standard method because of presence of the function $H_0(ix)$ within the Borel transform (4.4.26). This function has poles, or alternatively its Cheshire Cat deformation, $H_0(ix)^{1+\Delta/2}$, has branch cuts going out horizontally to infinity in the positive real direction starting at $x = mb$ and $x = m/b$ for $m \in \mathbb{N}^*$ as one can read from the denominator of (4.4.4). For this reason in equation (4.4.33) there are no straight rays emanating from the origin $x = 0$ in a direction θ with $-\arg b \leq \theta \leq \arg b$ without intersecting any of the singular directions.

This suggests that we just need to find a different way from (4.4.33) to define a function of the complex ξ variable with countably many branch cuts. One such way is as follows. We define this function by gluing analytic functions defined in different wedges of the complex

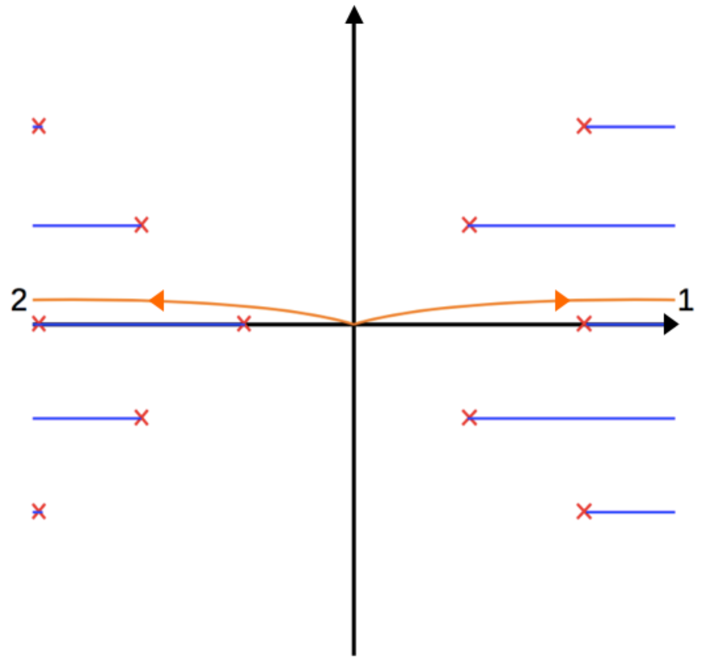


Figure 4.7: Contours of integration for the directed Laplace transformations: The red crosses show the locations of the branch points corresponding to the different non-perturbative contributions, and the blue lines the branch cuts associated to them. The yellow contour 1 gives a contour of integration for the directed Laplace transformations used to define an analytic function of ξ for $-\pi/2 < \arg\xi < \pi/2$. Contour 2 gives a contour of integration for the directed Laplace transformations used to define an analytic function of ξ for the wedge $\pi/2 < \arg\xi < \pi$ union with $-\pi < \arg\xi < -\pi/2$.

ξ plane. First we consider the directional Laplace contour along the first integration contour shown in Figure 4.7. This defines a function of ξ analytic for $-\pi/2 < \arg\xi < \pi/2$. Likewise we use the second path shown in Figure 4.7 to define a function of ξ in the wedge $\pi/2 < \arg\xi < \pi$ union with $-\pi < \arg\xi < -\pi/2$. The function thus obtained will have branch cuts along the directions $\arg\xi = \pm\pi/2$ and its discontinuities will be related to the discontinuity of the Borel transform along the directions $\arg x = 0$ and $\arg x = \pi$ which in turn are related to all the k vortex-anti-vortex non-perturbative sectors, but also infinitely many other discontinuities associated with $H_0(ix)^{1+\Delta/2}$ with starting points either the poles or the zeroes of (4.4.4).

This is somewhat unexpected from the resurgence point of view since these additional branch cuts are associated with *different* topological sectors from the one we were focusing on! In resurgence theory when we work in a given topological sector, say for example the trivial one, we complexify the coupling constant to understand the analytic properties of the resummed perturbative series and from this reconstruct the non-perturbative contributions in the *same* topological sector. Said in other words the imaginary part of the complexified coupling constant has nothing to do with the topological angle. Hence on resurgent ground we generically expect the Borel transform of the purely perturbative data to know "everything" about non-perturbative saddles in the same topological sector and "nothing" about different topological sectors. This is of course if no other structure is present as we will discuss in the next Session.

The present case is entirely different and the reason behind it lies in the unusual appearance of the hidden topological angle and the path-integral decomposition in topological sectors (4.3.7). The imaginary part of the action in the N^{th} topological sector is given by $\mathrm{Im}S \propto \xi\Theta N \propto (b-1/b)\xi$. Now it is clear that what we just said is not true anymore; if we keep fixed $\Theta = -i(b-1/b)$, complexify the coupling constant ξ and vary its imaginary part we will inevitably vary the theta angle, i.e. the imaginary part of the action of each topological sector. Hence in the case at hand we have some additional structure (see more in the next Section), making so that the Borel transform of the purely perturbative data knows also of different topological sectors.

We found however two different methods that can be applied to these standard dispersion arguments to disentangle from the Borel transform the branches coming from the same topological sector and the ones coming from other sectors. As a proof of principle we will now present both but will not dwell too much on the consequences.

A first possibility is to impose that, as a genuine theta angle would do, indeed $\mathrm{Im}S = \pi\xi\Theta N \sim (b-1/b)\xi$ is independent from the complexification of ξ. If we assume the double scaling limit $\xi \to \infty$ and simultaneously $b = e^{i\vartheta/\xi}$ we have that $\mathrm{Im}S \propto \vartheta$ is independent from ξ. In this regime when we complexify ξ we have that b is not of unit modulus anymore; however the background geometry discussed in Section 4.3.2 still makes sense[9]. The price to pay is that now the weak coupling expansion $\xi \to \infty$ of (4.4.24) will not be as

[9]Recall for the background geometry to make sense we need $h = (b+1/b)/2$ to be real. In this double scaling limit we have $h = 1 + O(1/\xi^2)$, which is real in the $\xi \to \infty$ limit.

straightforward as when we computed the factorially growing perturbative series (4.4.28) since in this double scaling limit b is no longer an independent parameter and the Borel transform does depend from the coupling through b.

An alternative method is to define something similar to (4.4.33) but not holomorphic:

$$\begin{aligned}(2\pi\xi)^{-(\Delta+2)}\tilde{\zeta}_{0,0}(\xi,b,\Delta) &= 2\sin(\pi\Delta)\int_0^\infty \frac{dy}{2\pi\xi}\,e^{-y}\left(\frac{y}{Q}\right)^{-(\Delta+2)} H_0\left(\frac{iy}{|\xi|}\right) \\ &\qquad e^{-(\Delta+2)\left(\log\Gamma\left(1+\frac{y}{2\pi\xi Q}\right)-\log\Gamma\left(1-\frac{y}{2\pi\xi Q}\right)\right)} \\ &= 2\sin(\pi\Delta)\int_0^{\infty e^{-i\theta}} dx\,e^{-2\pi\xi x}\left(\frac{2\pi\xi x}{Q}\right)^{-(\Delta+2)} H_0(ixe^{+i\theta}) \\ &\qquad e^{-(\Delta+2)\left(\log\Gamma\left(1+\frac{x}{Q}\right)-\log\Gamma\left(1-\frac{x}{Q}\right)\right)},\end{aligned} \tag{4.4.34}$$

where again $\theta = \arg\xi$ and we anti-correlate the direction of the Laplace transform with the argument of the complexified coupling constant. The difference is that as we rotate the argument of ξ we simultaneously rotate the branches of the function $H_0(ix)$ so that they never cross our contour of integration, or equivalently in the y variable as we rotate the argument of ξ the only branches crossing the contour of integration are the ones coming from the $\log\Gamma$ functions and not from $H_0(iy)$. Hence as a function of ξ we only have two discontinuities now, one across the $\arg\xi = 0$ direction which will persist the $\Delta \to 0$ limit and one across the $\arg\xi = \pi$ direction which will disappear in the $\Delta \to 0$ limit.

With this definition we still have exactly the same perturbative asymptotic series (4.4.28) since for $\xi > 0$ we trivially have that $\xi = |\xi|$. However when performing the Borel transform we treat differently terms belonging to the same topological sector from terms belonging to others in what effectively seems like an extremely ad-hoc prescription.

As mentioned before these discontinuities will be related to non-perturbative contributions and with this, once again very a posteriori, prescription we can isolate only the non-perturbative saddles in the same topological sector. It would be nice to provide some numerical examples of large order relations similar to [104, 86], but unfortunately this turns out to be quite non-trivial. The main issue we have with running some numerics lies in evaluating the function $H_0(ix)$, at $x = 0, Q, 2Q, ...$ and so on. Using the results outlined in Appendix A.2 this should be a doable task but we have decided to be content with the analytic results derived and defer the numerics to future works.

4.5 Comments on Cheshire Cat resurgence in 4-d

So far in this thesis we have considered using Cheshire Cat deformations to perform a resurgence analysis of theories in 2-d and 3-d. The obvious next question is can we do the same in 4-d. This is the subject of current work which has not been completed. Here though we will make some observations and comments on how this might be realised in 4-d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories. Although not fully satisfactory, lacking a path integral explanation, the same type of deformations we have considered here can surely be implemented in basically all the supersymmetric localized theories?

If we compute the S^4 partition function of $\mathcal{N} = 4$ $SU(N)$ SYM via localization [27, 105], since both the one-loop determinant and the instanton factor are trivial [105], the partition function is simply given by a Gaussian matrix model so it would seem that resurgence does not play any role. It would be very interesting to see if the deformation introduced in the present thesis can be used to "deconstruct" this "1" in $\mathcal{N} = 4$ similarly to what the authors of [54] did to deconstruct the "0" of a vanishing ground state energy to uncover a Cheshire cat resurgence structure.

For the case of $\mathcal{N} = 2$ theories on S^4 the situation is different. The resurgence analysis for this class of theories with different matter content has been discussed in details in [36] (see also the earlier [35]) and the authors showed that it is not however possible to reconstruct in this way the instanton-anti-instanton sectors from perturbation theory. The singularities of the Borel transform for the purely perturbative sector are not directly related to instanton-anti-instanton configurations. It was subsequently realised [42] (at least for the three dimensional case) that these singularities come from new finite action complexified supersymmetric solutions.

The reason for this is subtle: although the Borel transform of the perturbative series has poles, these are coming from the one-loop determinant of matter multiplets, i.e.

hypermultiplets of $\mathcal{N} = 2$, and the fields involved in these complexified supersymmetric solutions come precisely from the matter sector. However we know that instantons are present even in absence of hypermultiples. This suggests that the instanton-anti-instanton poles are hidden by a Cheshire Cat structure inside the one-loop determinant of the $\mathcal{N} = 2$ vector multiplet.

We have analysed the localised one-loop determinant for the vector and hyper multiplets, but failed so far to find a suitable Cheshire Cat deformation that would allow us to carry on the program outlined above. It does indeed seem that, unlike vortices in 2 and 3 dimension, the instantons are only sensitive to vector multiplet. The $N \to r$ deformation that we have used in the 3-d and 2-d cases is not a strong enough deformation in the 4-d case. The next obvious steps to try deforming $N_f - N_b = \Delta$, as we did in the 2-d case. However for this calculation we will need access to the complete 1-loop determinants, before cancellations have taken place between the bosonic and fermionic parts. This will thus require a technical susy localisation analysis of the theory.

4.6 Summary of chapter 4 and open problems

In this chapter we have considered the partition function for abelian $\mathcal{N} = 2$ supersymmetric theories with different matter content living on a squashed S^3. This problem was first analysed in [39] for $\mathcal{N} = 2$ Chern-Simons matter theories where the authors showed that the presence of Stokes phenomenon in the thimble decomposition was directly related to the ambiguities in resummation of the asymptotic perturbative expansion in the small coupling $g = 1/k$, with k the Chern-Simons level.

In our work we have set to zero the Chern-Simons level and considered the perturbative expansion in large FI parameter. Firstly we have analysed the Picard-Lefschetz decomposition of the localised path-integral into steepest descent contours and we have shown that if a suitable complexification of the squashing parameter b is introduced, a hidden topological angle seems to appear and a steepest descent contour can be associated to each topological sector.

Physically this complexified squashing parameter can be seen as adding a chemical potential for rotation of the S^2 so that vortices and anti-vortices rotate oppositely. The FI term on the other hand regulates the size of the vortices localised at north and south poles, thus we have a play off between these two parameters.

As we vary the complexified squashing b and the Fayet-Iliopoulos we observe the splitting of saddle points into different topological sectors as well as Stokes phenomenon whenever a saddle crosses the steepest descent cycle coming from another saddle. For large enough FI parameter these saddles can be associated to point-like vortex (or anti-vortex) solutions and the path-integral can be decomposed into a sum of contour integrals, one for each topological sector.

Having split the path integral into a sum over topological sector we first perform a semi-classical expansion $\xi \gg 1$ showing that, due to the supersymmetric nature of the observable under consideration, every perturbative series truncates after finitely many orders. These $\mathcal{N} = 2$ theories provide another interesting example of a field theory that lies at a very special point in theory space where lots of miraculous cancellations hide the resurgence structure rendering the perturbative expansions in each of the non-perturbative sectors as truncating series.

To use the resurgence machinery we then introduce a Cheshire Cat deformation by analytically continuing the number of chiral fields. As soon as the deformation parameter is generic we immediately re-introduce the asymptotic nature of perturbation theory. Thus we work at a generic point and using resurgent analysis reconstruct the vortex-anti-vortex contributions from the deformed, factorially growing and purely perturbative data. Once the deformation parameter is set back to its physical value we have that the asymptotic tail of the perturbative series vanishes but the non-perturbative contributions still stand.

Finally we posed the question of how to extend this Cheshire Cat deformation combined with Nekrasov partition function to the case of theories on S^4, say the pure $SU(2)\,\mathcal{N} = 2$ supersymmetric theory, where on the one hand we do expect infinitely many instanton-anti-instanton contributions but on the other hand these are somehow completely hidden from perturbation theory.

Chapter 5

Moving Sideways in the Resurgence Triangle

5.1 Introduction

In chapters 3 and 4 we have been analysing supersymmetric theories in 2-d and 3-d using a Cheshire Cat resurgence analysis. As explained in chapter 2, a resurgence analysis can typically only tell us about non-perturbative contributions within a single column of the resurgence triangle. As stressed there, the theta angle can be seen as introducing a grading in the partition function, a sort of Fourier modes decomposition. Working within a topological sector we can complexify the coupling constant and use resurgence theory to understand its analytic properties[1]. To be able to move between different topological sectors we need some additional structure that somehow links the theta angle to the complexified coupling constant.

In this chapter we will remark on additional structures going beyond resurgence theory. These additional structures are very common in supersymmetrically localised partition functions in various dimensions, and they are very reminiscent of the quantum mechanical Dunne-Ünsal relations [64, 65, 66], which allow us to move "horizontally" in the resurgence triangle, thus deriving data in different topological sectors just by analysing the

[1]Recall the theta angle and the imaginary part of the coupling must not be confused with one another.

perturbative one. In [64, 65] the authors used global boundary conditions to determine such relations for various potentials, including the double-well potential,the Sine-Gordon potential, the Fokker-Planck Potential and the symmetric AHO Potential. In [66] this was generalized to cubic and quartic anharmonic oscillators. Here we look at similar structures that we find in 2-d, 3-d and 4-d supersymmetric theories.

5.2 Relation between different topological sectors

So far we have understood that the localised partition function can be written as a transseries over different topological sectors (for which the imaginary part of the squashing parameter plays the role of a hidden topological angle in the 3-d case). Each topological sector can be furthermore written as a transseries capturing the perturbative series in the given topological sector, plus the infinitely many non-perturbative contributions coming from vortex-anti-vortex configurations on top of it. Upon Cheshire Cat deformation from a given perturbative series we can reconstruct every element in the same topological sector, i.e. from one element of the resurgent triangle of Figure 5.1 we can reconstruct all the other elements in the same column. In this Section we wish to discuss the relation between the topological sectors and additional structures allowing us to move "horizontally" in the resurgence triangle.

In many supersymmetric QFTs we indeed have this type of additional structure which allows us to use the data contained in the transseries in the trivial topological sector, e.g. for the 3-d case (4.4.6), to calculate the data in different topological sectors.

In 2-dimensional $\mathcal{N} = (2, 2)$ supersymmetric field theories we have seen in Section 3.3.1 how the $tt^\star$ structure of Cecotti and Vafa [79] is modified but still imposes that the partition function must satisfy a differential equation in the holomorphic coupling $\tau \sim \xi + i\theta$. This is precisely the extra structure needed. With resurgence theory we complexify ξ to reconstruct for example the topologically trivial sector from perturbation theory and then use the $tt^\star$ differential equation to obtain the data for sectors with non-trivial θ dependence; i.e. the data for the instanton and anti-instanton sectors are intimately tied up with the

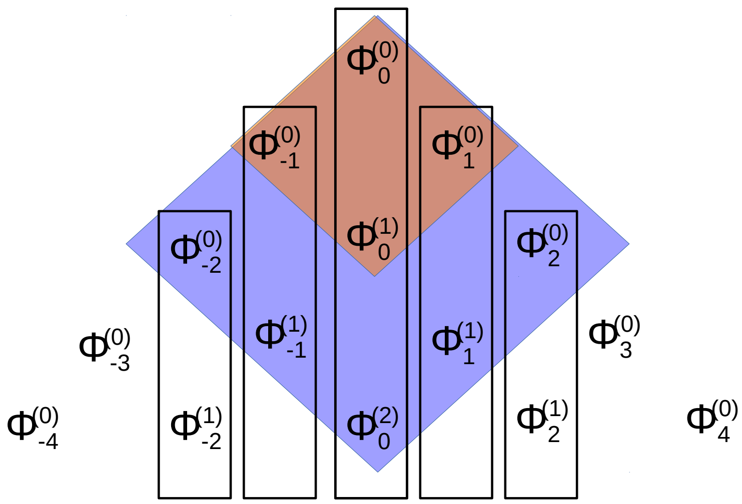


Figure 5.1: Resurgence theory allows us to move vertically along each rectangle. From just one of the contributions in a topological sector, i.e. $\Phi_B^{(k)}$ we can get all the $\Phi_B^{(k')}$ with $k' \neq k$.

instanton-anti-instanton contributions, and so on. From the n-instanton-n-anti-instanton contribution we can use this additional structure to calculate the contributions from different sectors in the resurgence triangle moving "horizontally" across the resurgence triangle as shown in Figure 5.1.

Schematically, if we were to reconstruct from the perturbative data $\Phi_0^{(0)}$ say the first instanton-anti-instanton contribution $\Phi_0^{(1)}$ we would then be able to retrieve all the data in the red square of Figure 5.1. Similarly once we reconstruct the 2-instanton-2-anti-instanton sector $\Phi_0^{(2)}$ out of perturbation theory we would have access to the entire blue square of Figure 5.1.

This is very reminiscent of the Dunne-Ünsal relation in quantum mechanics [64, 65, 66]. There the same could be achieved; the data in the instanton-anti-instanton, or even just in the purely perturbative series, can be related to the data in the instanton sector. In that case the relationship was derived using boundary conditions on the non-perturbative effects.

In the 3-d $\mathcal{N} = 2$ case we have a very similar story. Here we have what is usually called vortex-anti-vortex factorisation discussed in [90]. Reminiscent of the 2-d case [32, 33], these three dimensional theories, say for example with $2N$ chirals, have a partition function that factorises schematically as

$$Z^{(2N,0)}_{S^3_b}(\xi) = \sum_{i=1}^{N} Z^{(i)}_{cl} \times \left(Z^{(i)}_{1-loop} Z^{(i)}_{V} \right) \times \left(\bar{Z}^{(i)}_{1-loop} \bar{Z}^{(i)}_{V} \right) \quad . \tag{5.2.1}$$

Here $Z^{(i)}_{cl} = \exp(-i\pi\xi_{\text{eff}}\,\mu_i)$ is the classical part of the action, with μ_i axial mass for the i^{th} chiral and ξ_{eff} the effective FI parameter, while $Z^{(i)}_{V}$ and its complex conjugate are the abelian vortex and anti-vortex partition functions with $2N$ chirals, dressed by $Z^{(i)}_{1-loop}$ and its conjugate.

As discussed in [90] the vortex partition function Z_V can be better understood in the degenerate $b \to 0$ limit where the background geometry becomes $\mathbb{R}^2 \times S^1$ and the partition function counts finite-energy configurations on $\mathbb{R}^2$, i.e. vortices. Similarly in the $1/b \to 0$ limit the squashed sphere degenerates to a different $\mathbb{R}^2 \times S^1$ and the partition function $\bar{Z}_V$ counts anti-vortices.

From this factorised form it is now not surprising that the transseries in different topological sectors are related to one another. Hence we have the following method to obtain all the non-perturbative data in all the topological sectors from the perturbative data alone. First we deform the theory with some Cheshire Cat deformation as to re-introduce all the asymptotic tails in the various perturbative expansions. Next we use usual resurgence methods on the deformed theory to calculate all the non-perturbative vortex-anti-vortex contributions in the trivial topological sector. Then we send the deformation back to zero, retrieving all the non-perturbative data in this sector for the undeformed theory. Finally we use the factorisation formula (5.2.1) to compute the data for all the other topological sectors from the non-perturbative data in the trivial topological sector.

It is interesting to push this idea to higher dimensions. In fact it was already noted in [90] that this factorised form for the partition function (5.2.1) is very reminiscent of the Nekrasov structure in 4-d. If we focus for example to 4-d $\mathcal{N} = 2$ theories on S^4 we have

Pestun's celebrated partition function [27]

$$Z_{S^4}(g,\theta) = \int d\mu_\alpha \, e^{-\frac{S_{cl}(\alpha)}{g^2}} \left|Z_{1-loop}(\alpha)\right|^2 \left|Z_{inst}(\alpha,\tau)\right|^2 , \tag{5.2.2}$$

where g is the gauge coupling and θ the topological angle, while $d\mu_\alpha$ is the measure over the Cartan subalgebra of the gauge group and $Z_{inst}(\alpha,\tau)$ denotes Nekrasov [106, 107] instanton partition function with $\tau = \frac{i}{g^2} + \frac{\theta}{2\pi}$.

The vortex partition function is now replaced by Nekrasov partition function, the 3-d FI parameter translates into the 4-d coupling $1/g^2$, and the discrete sum of (5.2.1) becomes an integral over the Cartan subalgebra of the gauge group.

As a concrete example let us consider pure $\mathcal{N}=2$ with gauge group $SU(2)$ so that the integral over the Cartan subalgebra reduces to an integral over $\alpha \in \mathbb{R}$. In this case following [36] we can rewrite the path-integral in the topological sector form

$$Z^{SU(2)}_{S^4}(g,\theta) = \sum_{B\in\mathbb{Z}} e^{-\frac{2\pi}{g^2}|B| + \frac{i\theta}{2\pi}B} \zeta_B(g) , \tag{5.2.3}$$

where for $B \geq 0$ we have

$$\zeta_B(g) = \sum_{N\geq 0} e^{-\frac{4\pi}{g^2}N} \int_{-\infty}^{\infty} d\alpha \, e^{-\frac{S_{cl}(\alpha)}{g^2}} \left|Z_{1-loop}(\alpha)\right|^2 Z^{(B+N)}_{inst}(i\alpha) \, Z^{(N)}_{inst}(-i\alpha) , \tag{5.2.4}$$

while for $B<0$ we just need to take the complex conjugate of this. These two equations should be compared with their 3-d counterparts (4.4.2) and (4.4.12). Note the function $Z^{(k)}_{inst}(i\alpha)$ corresponds to the k-instanton Nekrasov partition function, e.g. $Z^{(0)}_{inst}(i\alpha) = 1$, and can be explicitly found in [36] for $k \leq 8$.

As for the three dimensional case, in this S^4 example we have some extra structure. It is clear from the argument outlined above that if we were able to compute with resurgence methods from the purely perturbative expansion, i.e. $B=0, N=0$ above, all the contributions from the instanton-anti-instanton sectors, i.e. $B=0, N>0$, we would then be able to calculate all the perturbative and non-perturbative data in all the other topological sectors.

5.3 Summary of chapter 5 and open problems

In this chapter we have commented on the strange nature of the other topological sectors which should be in principle completely disconnected from perturbation theory and the trivial topological sector but in practice they are not. This suggests the existence of additional structures, beyond the standard resurgence framework. We have seen such structures are present in 2-d, 3-d and 4-d supersymmetric theories and allow us to use non-perturbative data in the topologically trivial sector to obtain non-perturbative data in other topological sectors. These structures are the tt^* equations in 2-d, the vortex/anti-vortex factorization in 3-d, and Nekrasov partition function in 4-d.

It would be extremely interesting to see if these kind of holomorphic/anti-holomorphic structures, intertwining the complexified coupling constant with the theta angle, can be extended to less supersymmetric theories as for example just pure Yang-Mills, thus allowing us to extend resurgence to the whole triangle of Figure 5.1.

Chapter 6

Conclusions

The purpose of this thesis has been to explore the application of resurgence and Picard-Lefschetz method to supersymmetric localisable quantum field theories. In particular we have been interested in theories where the usual resurgence structures appear to be absent, and to see if these structures are in fact present, and if so develop tools for uncovering them.

The first question we have addressed is the lack of asymptotic series in localisable theories. We have seen for $\mathcal{N} = (2,2)$ theories on a 2-sphere, and for $\mathcal{N} = 2$ theories on a 3-sphere, that an asymptotic series is not present due to cancellations between the fermionic and bosonic 1-loop determinants. Using a Cheshire Cat deformation, either changing the number of fermions to be different from the number of bosons, or by setting the number of particles to be non-integer, we can recover an asymptotic perturbative series. In these deformed theories we can do a full resurgence analysis, including finding all the non-perturbative data from the perturbative data. We can also return the deformation smoothly back to zero, and keep all the data we have calculated. We have thus seen that a full resurgence structure is present, and a resurgent analysis can be performed using the Cheshire Cat method. These theories sit very special points where this is hidden, and a Cheshire Cat deformation is required to uncover it.

Further to this, in 3 dimensional $\mathcal{N} = 2$ theories, we have seen that the resurgence triangle structure also appears at first site to have disappeared. Here we have seen that we can deform the squashing parameter for the 3-sphere, and we see the appearance of a hidden topological angle. With the hidden topological angle present, the full resurgence triangle reappears. This has been analysed using Picard-Lefschetz theory. Having recovered the resurgence triangle, we can again use the Cheshire Cat method to uncover a full resurgence structure, and perform a complete resurgence analysis.

In the introduction to this thesis we reviewed the reasons for the existence of the resurgence triangle, and saw that we could only use resurgence to calculate non-perturbative data within a single column of the triangle from an analysis of a particular entry in that column. In the process of analysing theories in 2, 3 and 4 dimensions however, we have seen the existence of non-perturbative relations which can be combined, in addition to a resurgence analysis, to calculate data in different columns. Thus we have demonstrated how, in these theories, one can calculate all the non-perturbative data in the theory from the perturbative data alone.

Although not fully satisfactory, the same type of Δ deformation can surely be implemented in basically all the supersymmetric localized theories. The most obvious future direction for building on the results of this thesis is to generalise the Cheshire Cat method to theories in 4 dimensions. This will involve a careful analysis of the 1-loop determinants found following a localisation calculation. A suitable deformation will then need to be found to uncover the resurgence structure. For example if we compute the S^4 partition function of $\mathcal{N} = 4\, SU(N)$ SYM via localization [27, 105], since both the one-loop determinant and the instanton factor are trivial [105], the partition function is simply given by a Gaussian matrix model so it would seem that resurgence does not play any role. It would be very interesting to see if the deformation introduced in the present paper can be used to "deconstruct" this "1" in $\mathcal{N} = 4$ similarly to what the authors of [54] did to deconstruct the "0" of a vanishing ground state energy to uncover a Cheshire Cat resurgence structure. Similarly, we would like to uncover a Cheshire Cat resurgence structure in $\mathcal{N} = 2$ theories on S^4.

An interesting question would be to study the large-N expansion of the $\mathbb{CP}^{N-1}$ partition function discussed in chapter 3. It is not clear how the resurgence properties discussed in

[20, 19] would arise from localization in the large-N limit and what role the deformation has to play. Furthermore once the large-N limit is computed we would like to understand, perhaps using similar methods to the one introduced in [108], how to interpolate this result with the finite N case discussed in the present paper. It would also be interesting to understand how this large-N limit is attained whether from taking N over the natural numbers or over the reals since for finite N the resurgence properties change dramatically as shown in this paper.

A further direction along the line of investigating Cheshire Cat resurgence is to analyse other theories that exhibit the phenomena. Whilst a number of other supersymmetric theories have this property, one would also want to consider topological QFTs such as 2D Yang-Mills theory. Some recent studies [109, 110] have also revealed connections between the Cheshire cat resurgence phenomena and number theory and modular forms. In particular the authors observed the phenomena in modular graph functions which have applications in higher derivative corrections to type IIB string theory, and Lambert series and iterated Eisenstein integrals which have applications in string scattering amplitudes.

One final direction for future research leading from this thesis is the development of non-perturbative structures that can be combined with resurgence to calculate non-perturbative data with different topological charge. Here we have explored such structures in a number of supersymmetric quantum field theories. It would be interesting to find ways of generalizing this to non-supersymmetric theories. Such development would be very exciting, as it would allow us to calculate all the non-perturbative data in a theory from the perturbative data alone.

Appendix A

Appendices

A.1 ζ-function regularisation

Infinite products of the form

$$\prod_{k=0}^{\infty}(k+a)^{f(k)} \tag{A.1.1}$$

arise naturally when computing one-loop determinants, with $f(k)$ representing the degeneracy of the k^{th} eigenvalue $(k+a)$. A standard way to regularise these type of products is to rewrite them in terms of the logarithm of the above expression using

$$\prod_{k=0}^{\infty}(k+a)^{f(k)} = \exp\left(\sum_{k=0}^{\infty} f(k)\log(k+a)\right) . \tag{A.1.2}$$

Let us specialise now to the case

$$\prod_{k=0}^{\infty}(k+a) = \exp\left(\sum_{k=0}^{\infty}\log(k+a)\right) , \tag{A.1.3}$$

which can be formally written as

$$\prod_{k=0}^{\infty}(k+a) = \exp\left(-\partial_s\zeta(s,a)|_{s=0}\right) , \tag{A.1.4}$$

where $\zeta(s,a)$ denotes the Hurwitz-zeta function which is defined for complex arguments s with $\mathrm{Re}(s) > 1$ and a with $\mathrm{Re}(a) > 0$ via the series

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} , \tag{A.1.5}$$

and can be then extended to a meromorphic function defined for all $s \neq 1$. In particular one can show (for a short proof see [111])

$$\zeta'(0,a) - \zeta'(0) = \log \Gamma(a)\,, \tag{A.1.6}$$

where $\zeta'(0) = d\zeta(s)/ds|_{s=0} = -\log\sqrt{2\pi}$ is the derivative of the Riemann-zeta at the origin. We can then rewrite a regularised version of the infinite product

$$\prod_{k=0}^{\infty}(k+a)\text{``} = \text{''}\frac{\sqrt{2\pi}}{\Gamma(a)}\,. \tag{A.1.7}$$

We need another regularised infinite product where the degeneracy $f(k)$ grows linearly with k, i.e. $f(k) = k + a$. We consider

$$\prod_{k=0}^{\infty}(k+a)^{k+a} = \exp\left(\sum_{k=0}^{\infty}(k+a)\log(k+a)\right) = \exp\left(-\partial_s\zeta(s,a)|_{s=-1}\right)\,; \tag{A.1.8}$$

we need then a formula for $\partial_s\zeta(s,a)|_{s=-1}$, see [112, 113].

We can proceed by first writing the asymptotic form (see http://dlmf.nist.gov/25.11.44 or [114])

$$\zeta'(-1,a) = \frac{1}{12} - \frac{a^2}{4} + \log a\left(\frac{1}{12} - \frac{a}{2} + \frac{a^2}{2}\right) - \sum_{k=1}^{\infty}\frac{B_{2k+2}}{(2k+2)(2k+1)2k}a^{-2k}\,, \tag{A.1.9}$$

with B_n the Bernoulli numbers. By taking the derivative with respect to a we obtain

$$\frac{\partial}{\partial a}\zeta'(-1,a) = a - \frac{1}{2} + \log\Gamma(a) + \zeta'(0)\,, \tag{A.1.10}$$

which upon integration gives us the desired formula

$$\zeta'(-1,a) - \zeta'(-1) = \frac{1}{2}a(a-1) + a\zeta'(0) + \psi^{(-2)}(a)\,, \tag{A.1.11}$$

where $\zeta'(-1) = d\zeta(s)/ds|_{s=-1} = 1/12 - \log G$ and G denotes Glaisher constant $G = 1.282...$, while $\psi^{(-2)}(a) = \int da \log\Gamma(a)$. We can then rewrite a regularised version of the infinite product

$$\prod_{k=0}^{\infty}(k+a)^{k+a}\text{``} = \text{''}\exp\left(-\zeta'(-1) - \frac{1}{2}a(a-1) - a\zeta'(0) - \psi^{(-2)}(a)\right)\,. \tag{A.1.12}$$

A.2 Double Sine Function Identities

We here state a number of useful formulae for the double sine function. The Appendices of [90, 115] contain comprehensive lists of properties for this function; otherwise we refer to [95, 97, 96].

We define the double sine function as

$$s_b\,(x) \quad = \quad \prod_{m,n\geq 0} \frac{(mb+n/b+Q/2-ix)}{(mb+n/b+Q/2+ix)} \quad . \tag{A.2.1}$$

Following [96] we can introduce the double gamma function Γ_2 defined by the analytic continuation

$$\Gamma_2(z;\omega_1,\omega_2) = \exp\left[\partial_s \left(\sum_{m,n\geq 0} (m\omega_1 + n\omega_2 + z)^{-s}\right)\Bigg|_{s=0}\right] , \tag{A.2.2}$$

so that the formal infinite product (A.2.1) can be rewritten as

$$s_b(x+iQ/2) = \frac{\Gamma_2(ix;b,b^{-1})}{\Gamma_2(Q-ix;b,b^{-1})} = \Gamma_h(ix;b,b^{-1}) = S_2(ix|b,b^{-1})^{-1} = G(-ib,-ib^{-1};ix-Q/2) \, ,$$

where Γ_h is van de Bult [96] hyperbolic gamma function, S_2 is the double Sine function of Kurokawa and Koyama [97] and G is Ruijsenaars hyperbolic gamma [95].

Obviously $s_b(0) = 1$ and furthermore $s_b(x)$ has zeroes on the lattice $\Lambda_+ = -iQ/2 - ib\,\mathbb{Z}_{\geq 0} - i/b\,\mathbb{Z}_{\geq 0}$ and poles on the lattice $\Lambda_- = +iQ/2 + ib\,\mathbb{Z}_{\geq 0} + i/b\,\mathbb{Z}_{\geq 0}$. Both the zeroes and poles are simple provided that b^2 is not rational. In particular the pole at $x = iQ/2$ is always simple and we have

$$s_b(x) = \frac{i}{2\pi(x-iQ/2)} + O(1)\, , \qquad\qquad x \to iQ/2\, . \tag{A.2.3}$$

From the known functional equations

$$s_b(x+ib/2)s_b(-x+ib/2) = \frac{1}{2\cosh(\pi bx)}\, , \tag{A.2.4}$$

$$s_b(x+iQ/2) = \frac{s_b(x+iQ/2-ib)}{2i\sinh(\pi bx)}\, , \tag{A.2.5}$$

one can derive the general expressions

$$\begin{aligned} \frac{s_b(x+iQ/2+imb+in/b)}{s_b(x+iQ/2)} &= \frac{(-1)^{mn}}{\prod_{k=1}^{n} 2i\sinh[\pi b(x+ikb)] \prod_{l=1}^{m} 2i\sinh[\pi/b(x+il/b)]} , \quad \text{(A.2.6)} \\ \frac{s_b(x-iQ/2+imb+in/b)}{s_b(x-iQ/2)} &= \frac{(-1)^{mn}}{\prod_{k=1}^{n} 2i\sinh[\pi b(x-iQ+ikb)] \prod_{l=1}^{m} 2i\sinh[\pi/b(x-iQ+il/b)]} , \end{aligned}$$

allowing us to obtain the residue at different poles from the residue at zero (A.2.3).

A useful infinite product identity for $s_b(x)$ is given by

$$s_b(x) \quad = \quad e^{-i\pi\frac{x^2}{2}-i\pi\frac{b^2+b^{-2}}{24}} \frac{\prod_{k=0}^{\infty}\left(1+e^{2\pi bx}e^{2\pi ib^2(k+1/2)}\right)}{\prod_{k=0}^{\infty}\left(1+e^{2\pi x/b}e^{-2\pi i(k+1/2)/b^2}\right)} \quad , \tag{A.2.7}$$

which can be regularised using q-Pochhammers symbols. Recall that the q-Pochhammer $(a;q)_\infty$ is defined as

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right) \quad . \tag{A.2.8}$$

Using this we can thus write

$$s_b(x) = e^{-i\pi\frac{x^2}{2}-i\pi\frac{b^2+b^{-2}}{24}} \frac{\left(-e^{2\pi bx+\pi ib^2};e^{2\pi ib^2}\right)_\infty}{\left(-e^{2\pi x/b-\pi i/b^2};e^{-2\pi i/b^2}\right)_\infty} \quad , \tag{A.2.9}$$

valid for $\mathrm{Im}(b^2)>0$ so that $|e^{2\pi ib^2}|<1$ as well as $|e^{-2\pi i/b^2}|<1$.

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