Regular representations of GL _n(O) and the inertial Langlands correspondence

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Matematykiem jest kto umie znajdować analogie
między twierdzeniami; lepszym, kto widzi
analogie dowodów; jeszcze wyższym, kto dostrzega
analogie teoryj; a można sobie wyobrazić i
takiego, co między analogiami widzi analogie.

Stefan Banach

Part II of this thesis is a joint work with Mikołaj Frączyk.

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Abstract

This thesis is divided into two parts. The first one comes from the representation theory of reductive $\mathfrak{p}$-adic groups. The main motivation behind this part of the thesis is to find new explicit information and invariants of the types in general linear groups. Let F be a non-Archimedean local field and let $\mathcal{O}_F$ be its ring of integers. We give an explicit description of cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$, with p prime, in terms of orbits. We determine which of them are regular representations and we provide an example which shows that an orbit of a representation does not always determine whether it is a cuspidal type or not. At the same time we prove that a cuspidal type for a representation π of $GL_p(F)$ is regular if and only if the normalised level of π is equal to m or $m - \frac{1}{p}$ for $m \in \mathbb{Z}$.

The second part of the thesis comes from the theory of integer-valued polynomials and simultaneous $\mathfrak{p}$-orderings. This is a joint work with Mikołaj Frączyk. The notion of simultaneous $\mathfrak{p}$-ordering was introduced by Bhargava in his early work on integer-valued polynomials. Let k be a number field and let $\mathcal{O}_k$ be its ring of integers. Roughly speaking a simultaneous $\mathfrak{p}$-ordering is a sequence of elements from $\mathcal{O}_k$ which is equidistributed modulo every power of every prime ideal in $\mathcal{O}_k$ as well as possible. Bhargava asked which subsets of Dedekind domains admit simultaneous $\mathfrak{p}$-ordering. Together with Mikołaj Frączyk we proved that the only number field k with $\mathcal{O}_k$ admitting a simultaneous $\mathfrak{p}$-ordering is $\mathbb{Q}$.

Key words

Representation theory of $\mathfrak{p}$-adic reductive group, cuspidal types, number theory, integer-valued polynomials, simultaneous $\mathfrak{p}$-orderings

Résumé

Cette thèse contient deux parties. La première porte sur la théorie des représentations des groupes $\mathfrak{p}$-adiques. Le but est de trouver de nouvelles informations et de nouveaux invariants des types cuspidaux de groupes linéaires généraux. Soit F un corps local non archimédien et soit $\mathcal{O}_F$ son anneau des entiers. Nous décrivons les types cuspidaux sur $\mathrm{GL}_p(\mathcal{O}_F)$ (où p est un nombre premier) en termes d'orbites. Nous déterminons quels types cuspidaux sont réguliers et donnons un exemple qui montre que l'orbite de la représentation ne suffit pas à déterminer si la représentation est un type cuspidal ou non. Nous montrons qu'un type cuspidal pour une représentation π de $\mathrm{GL}_p(F)$ est régulier si et seulement si le niveau normalisé de π est égal à m ou $m - \frac{1}{p}$ pour un certain $m \in \mathbb{Z}$.

La deuxième partie porte sur les polynômes à valeurs entières, les $\mathfrak{p}$-rangements simultanés (au sens de Bhargava) et l'équidistribution dans les corps des nombres. C'est un projet joint avec Mikołaj Frączyk. La notion de $\mathfrak{p}$-rangement provient des travaux de Bhargava sur les polynômes à valeurs entières. Soit k un corps de nombres et soit $\mathcal{O}_k$ son anneau des entiers. Une suite d'élements de $\mathcal{O}_k$ est un $\mathfrak{p}$-rangement simultané si elle est équidistribuée modulo tous les idéaux premieres de $\mathcal{O}_k$ du mieux possible. Nous prouvons que le seul corps de nombres k tel que $\mathcal{O}_k$ admette des $\mathfrak{p}$-rangements simultanés est $\mathbb{Q}$.

Mots-clés

Théorie des représentations des groupes $\mathfrak{p}$-adiques, types cuspidaux, théorie des nombres, polynômes à valeurs entières, $\mathfrak{p}$-rangements

Contents

Introduction

In this thesis we distinguish two main projects. The first one comes from the representation theory of reductive $\mathfrak{p}$-adic groups. The main motivation behind this project is to find new invariants and information about the types in general linear groups. Let F be a non-Archimedean local field and let $\mathcal{O}_F$ be its ring of integers. Let π be an irreducible cuspidal representation of $\mathrm{GL}_n(F)$. By $\mathfrak{I}(\pi)$ we will denote the inertial support of π (see (2.1.1)). A **cuspidal type (on $K = \mathrm{GL}_n(\mathcal{O}_F)$)** for $\mathfrak{I}(\pi)$ is an irreducible representation λ of $\mathrm{GL}_n(\mathcal{O}_F)$ which satisfies the following condition: an irreducible representation π_1 of $\mathrm{GL}_n(F)$ contains λ if and only if the inertial support of π_1 coincides with that of π. The existence of cuspidal types on $\mathrm{GL}_n(\mathcal{O}_F)$ easily follows from Bushnell and Kutzko's work [9]. This is explained by Paskunas in [30]. Paskunas also showed the unicity of cuspidal types on $\mathrm{GL}_n(\mathcal{O}_F)$. More precisely he proved that for any π irreducible cuspidal representation of $\mathrm{GL}_n(F)$ there exists λ an irreducible representation of K depending only on $\mathfrak{I}(\pi)$ which is a cuspidal type on K. Moreover λ is unique up to isomorphism. The regular representations of $\mathrm{GL}_n(\mathcal{O}_F)$ were introduced by Shintani ([31]). Those are in certain sense the best behaved representations of $\mathrm{GL}_n(\mathcal{O}_F)$. In **Chapter 2** we determine which cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$ (where p is a prime number) are regular.

The second part of the thesis comes from the theory of integer-valued polynomials, simultaneous $\mathfrak{p}$-orderings and equidistribution in number fields. This is a joint work with Mikołaj Frączyk. It comes from our preprint [18]. Let k be a number field with ring of integers $\mathcal{O}_k$. In **Chapter 3** we study how well finite subsets of $\mathcal{O}_k$ can be equidistributed modulo powers of all primes ideals in $\mathcal{O}_k$. We deduce that the only number field k whose ring of integers $\mathcal{O}_k$ has a simultaneous $\mathfrak{p}$-ordering is $\mathbb{Q}$.

In the following subsections we give an overview of the contents of Chapters 1, 2 and 3. This overview will be short since both Chapter 2 and Chapter 3 has its own introduction with the motivation, the structure of paper and the notation.

Chapter 1 - Representation theory of $\mathfrak{p}$-adic groups and the theory of types

In this chapter we recall basic notions from the representation theory which we use in Chapter 2. We recall the notions of a smooth representation, induction and compact induction. Later we focus on irreducible smooth representations of $\mathrm{GL}_n(F)$. We also mention results of Paskunas on types.

Chapter 2 - Cuspidal types

Let $\mathfrak{p}_F$ be the maximal ideal in $\mathcal{O}_F$. Any irreducible smooth representation ρ of $\mathrm{GL}_n(\mathcal{O}_F)$ factors through a finite group $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^r)$ where r is a natural number bigger than or equal to 1. The minimal natural number r with this property is called the conductor of the representation ρ. Let ρ be an irreducible smooth representation of $\mathrm{GL}_n(\mathcal{O}_F)$ with conductor $r > 1$. Sometimes it will be convenient to see ρ as a representation of $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^r)$. In this case we will denote it by $\bar{\rho}$. Let $l = \lfloor \frac{r+1}{2} \rfloor$ and let K^l be the kernel of the projection from $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^r)$ onto $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^l)$. Note that K^l/K^r is an abelian group. Fix an additive character $\psi : F \to \mathbb{C}^\times$. Denote by $\mathrm{M}_n(\mathcal{O}_F)$ the set of all $n \times n$ - matrices with entries in $\mathcal{O}_F$. By Clifford's theorem

$$\bar{\rho} \mid_{K^l} = m \bigoplus_{\bar{\alpha} \sim \bar{\alpha}_1} \bar{\psi}_{\bar{\alpha}}, \tag{0.0.1}$$

where $\bar{\alpha}, \bar{\alpha}_1 \in \mathrm{M}_n(\mathcal{O}_F/\mathfrak{p}_F^{r-l})$, the equivalence classes of $\sim$ are $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}^{r-l})$-conjugacy classes, $m \in \mathbb{N}$ and the characters $\bar{\psi}_{\bar{\alpha}} : K^l \to \mathbb{C}^\times$ are defined as follows: $\bar{\psi}_{\bar{\alpha}}(1+x) = \psi(\mathrm{tr}(\widehat{\alpha}\widehat{x}))$ for some lifts of $x, \bar{\alpha}$ to elements in $\mathrm{M}_n(\mathcal{O}_F)$. The definition of $\bar{\psi}_{\bar{\alpha}}$ does not depend on the choice of lifts. If a matrix $\alpha \in \mathrm{M}_n(\mathcal{O}_F)$ is such that its image in $\mathrm{M}_n(\mathcal{O}_F/\mathfrak{p}_F^{r-l})$ appears in the decomposition (2.1.2) we say that α **is in the orbit of** ρ. We say that a representation is regular if its orbit contains a matrix whose image in $\mathrm{M}_n(\mathcal{O}_F/\mathfrak{p}_F)$ has abelian centralizer. Krakovski, Onn and Singla [25] constructed all such representations under the condition that the characteristic of the residue field of F is different than 2. Stasinski and Stevens in [34] described all regular representations of $\mathrm{GL}_n(\mathcal{O}_F)$ in terms of orbits. In [33] Stasinski asked which cuspidal types are regular. In Chapter 2 we will obtain the following description of orbits of cuspidal types on $\mathrm{GL}_2(\mathcal{O}_F)$.

Theorem 0.0.1. *A cuspidal type on $\mathrm{GL}_2(\mathcal{O}_F)$ is exactly a one-dimensional twist of one of the following:*

1. *a representation inflated from some irreducible cuspidal representation of $\mathrm{GL}_2(k)$;*
2. *a representation whose orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$;*
3. *a representation whose orbit is equivalent to an orbit containing a matrix β whose characteristic polynomial is Eisenstein and which satisfies one of the following conditions:*

   (a) *it has conductor at least 4;*

(b) it has conductor $r = 2$ or 3 and is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_K(\bar{\psi}_{\bar{\beta}})}^{K}\theta$ *where* $\theta \mid_{U_{\mathfrak{M}}^{\lfloor\frac{r+1}{2}\rfloor}} = m\psi_\beta$ *for certain* $m \in \mathbb{Z}$ *and* θ *does not contain the trivial character of* $\begin{pmatrix} 1 & \mathfrak{p}_F^{r-2} \\ 0 & 1 \end{pmatrix}$.

Let p be a prime number. We also obtained a description of orbits of cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$ with conductor bigger than 3. Let $\mathfrak{I}$ be the chain order consisting of matrices that are upper triangular modulo $\mathfrak{p}_F$, let $U_{\mathfrak{I}}$ be the group of invertible elements of $\mathfrak{I}$ and let $\mathfrak{P}_{\mathfrak{I}}$ be the Jacobson radical in $\mathfrak{I}$. Let $\Pi_{\mathfrak{I}}$ be a generator of $\mathfrak{P}_{\mathfrak{I}}$. Denote by k_F the residue field of F. We prove the following result:

Theorem 0.0.2. *If λ is a cuspidal type on $K = \mathrm{GL}_p(\mathcal{O}_F)$, then it is a one-dimensional twist of one of the following:*

1. *a representation inflated from an irreducible cuspidal representation of $\mathrm{GL}_p(k_F)$;*
2. *a representation whose orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$;*
3. *a representation whose orbit contains a matrix of the form $\Pi_{\mathfrak{I}}^j B$ where $0 < j < p$ and $B \in U_{\mathfrak{I}}$.*

Moreover if a representation is a one-dimensional twist of a representation of the form (3) *and has conductor bigger than* 3*, or is of the form* (1) *or* (2)*, then it is a cuspidal type.*

In particular this implies that a cuspidal type on $\mathrm{GL}_p(\mathcal{O}_F)$ of conductor $r \geqslant 4$ is regular if and only if its orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$ or a matrix whose characteristic polynomial is Eisenstein. In particular for $p > 2$ even for big conductors there are cuspidal types which are not regular.

Chapter 3 - Optimal rate of equidistribution in number fields

Let k be a number field. Denote by $\mathcal{O}_k$ its ring of integers and let S be a subset of $\mathcal{O}_k$. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_k$. We say that $(a_i)_{i\in\mathbb{N}}$ is a **$\mathfrak{p}$-ordering** in S if for every $n \in \mathbb{N}$

$$v_S(\mathfrak{p}, n) := v_{\mathfrak{p}}\left(\prod_{i=0}^{n-1}(a_i - a_n)\right) = \min_{s\in S} v_{\mathfrak{p}}\left(\prod_{i=0}^{n-1}(a_i - s)\right),$$

where $v_{\mathfrak{p}}$ is $\mathfrak{p}$-adic additive valuation in $\mathcal{O}_k$. The value $v_S(\mathfrak{p}, n)$ does not depend on the choice of $\mathfrak{p}$-ordering. Bhargava defined the generalized factorial as the ideal $n!_S = \prod_{\mathfrak{p}} \mathfrak{p}^{v_S(\mathfrak{p},n)}$ where $\mathfrak{p}$ runs over all prime ideals in $\mathcal{O}_k$. A sequence is called a simultaneous $\mathfrak{p}$-ordering if it is a $\mathfrak{p}$-ordering for all prime ideals $\mathfrak{p}$ in $\mathcal{O}_k$. Simultaneous $\mathfrak{p}$-ordering are also called Newton sequences [13]. Bhargava in [5] asked which subsets of Dedekind domains contain a simultaneous $\mathfrak{p}$-ordering. Wood in [36] has proved that there are no simultaneous $\mathfrak{p}$-orderings in $\mathcal{O}_k$ for k imaginary quadratic number field. This was generalized by Adam and Cahen in [1] to all quadratic number fields $\mathbb{Q}(\sqrt{d})$ besides $d = 2, 3, 5$ and $d \equiv 1 \bmod 8$.

In a joint work with Mikołaj Frączyk we have proved that this true for any number field [18]. The precise statement we prove is stronger. To state it properly we need to define n-optimal sets. A finite set $S \subseteq \mathcal{O}_k$ is **almost uniformly distributed modulo** $\mathfrak{p}$ if for every $a, b \in \mathcal{O}_k$ we have

$$| \{s \in S \mid \ s - a \in \mathfrak{p}\} | - | \{s \in S \mid \ s - b \in \mathfrak{p}\} | \in \{-1, 0, 1\}.$$

We say that a finite subset $S \subseteq \mathcal{O}_k$ with $n+1$ elements is n**-optimal** if it is almost uniformly equidistributed modulo every power of $\mathfrak{p}$ for every prime ideal $\mathfrak{p}$. If $(a_i)_{i \in \mathbb{N}}$ is a simultaneous $\mathfrak{p}$-ordering, then $\{a_i \mid \ 0 \leqslant i \leqslant n\}$ forms an n-optimal set. In particular the non-existence of n-optimal sets for n big enough implies non-existence of simultaneous $\mathfrak{p}$-orderings. The idea for study n-optimal sets comes from the theory of integer valued polynomials. We say that a polynomial $f \in k[x]$ is **integer valued** if $f(\mathcal{O}_k) \subseteq \mathcal{O}_k$ (see [12]). The n-optimal sets are in some sense the smallest testing sets for finding such polynomials. Let $n \in \mathbb{N}$. An equivalent definition of n-optimal sets is the following: a set S is n-optimal if and only if for every polynomial $f \in k[x]$ of degree at most n the following condition is satisfied: $f(S) \subseteq \mathcal{O}_k$ implies that f is integer valued. In [11] together with Mikołaj Frączyk and Jakub Byszewski we have proved that for every k imaginary quadratic number field there exists $N \in \mathbb{Z}$ such that for $n > N$ there are no n-optimal sets in the ring of integers of k. In Chapter 3 we have proved that for every number field k different that $\mathbb{Q}$ there exsists $N \in \mathbb{Z}$ such that for $n > N$ there are no n-optimal sets in $\mathcal{O}_k$. Let us explain very briefly what is the idea of the proof. We assume the contrary, i.e. that there exists $\mathcal{S}_{n_i}$ a sequence of n_i-optimal sets with n_i tending to infinity. Let $V = k \otimes_{\mathbb{Q}} \mathbb{R}$. First we show that for every n there exists a cylinder in V of the volume $O(n)$ which contains $\mathcal{S}_n$. Something similar was proved in the case of imaginary quadratic number field in [11] but the general case was much more complicated because the norm in this case is not always convex. In the proof of that we used some number theoretical input, for example Ikehara's Tauberian theorem and Baker–Wüstholz's theorem. Then we deduce that there exists a compact set Ω and sequences $(s_n)_{n \in \mathbb{N}}$, $(t_n)_{n \in \mathbb{N}}$ in V with some restriction on the norm of s_n such that sets $s_n^{-1}(\mathcal{S}_n - t_n)$ are contained in Ω. Define $\mu_n = \frac{1}{n} \sum_{x \in \mathcal{S}_n} \delta_{s_n^{-1}(x - t_n)}$. Since Ω is compact we consider weak -* limits of μ_n. We call them limit measures. They provide information about geometry of large n-optimal sets. We study properties of n-optimal sets to show that limit measures cannot exist.

Part I

Cuspidal types

Chapter 1

Representation theory of $\mathfrak{p}$-adic groups and theory of types.

1.1 Representation theory of $\mathfrak{p}$-adic groups

In this chapter we give an introduction to representation theory of $\mathfrak{p}$-adic groups and we also provide an introduction to the theory of types.

1.1.1 Basics

Let G be a locally profinite group and let (π, V) be a representation of G over $\mathbb{C}$. Let F be a local non-Archimedean field.

Definition 1.1.1. *We say that π is **smooth** if*

$$V = \bigcup_{H} V^H$$

where the union runs over open compact subgroups H of G and

$$V^H := \{v \in V : \quad \pi(h)v = v \quad \text{for all} \quad h \in H\}.$$

One of the ways to construct a representation is by induction. Let H be a closed subgroup of G. Let (σ, W) be a smooth representation of H.

Definition 1.1.2. *Define X to be set of functions $f : G \to W$ such that*

1. *$f(hg) = \sigma(h)f(g)$, for all $h \in H$, $g \in G$;*

2. *there exists a compact open subgroup K_f of G depending on f such that $f(gk) = f(g)$ for $k \in K_f$.*

We define the action of G on X by translation:

$$\Sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X)$$
$$\Sigma(g)f : g_1 \mapsto f(g_1 g) \qquad g, g_1 \in G.$$

The representation (Σ, X) is called ***induction*** *and we denote it by $\mathrm{Ind}_H^G \sigma$.*

Induction defined as above is a smooth representation.

Definition 1.1.3. *Take X and Σ as before. Define*

$$X_c := \{f \in X : \quad \mathrm{supp} f \subseteq HC \quad \textit{for a compact subset}\, C \textit{ of}\, G\}.$$

The representation $(\Sigma\,|_{X_c}, X_c)$ is called ***compact induction*** *and is denoted by* c-$\mathrm{Ind}_H^G\,\sigma$.

Let F be a local non-Archimedean field. Write $\mathcal{O}_F$ for its ring of integers and $\mathfrak{p}_F$ for the maximal ideal in $\mathcal{O}_F$.

1.1.2 Characters of $F^\times$

This section is a recap of [10, 1.6,1.8].

A continuous homomorphism $G \to \mathbb{C}^\times$ is called a **character** of G.

Lemma 1.1.4. [10, 1.6 Proposition] *Let $\chi : G \to \mathbb{C}^\times$ be a group homomorphism. Then the following conditions are equivalent*

1. *the kernel of χ is open;*
2. *χ is continuous.*

By $F^\times$ we denote the group of invertible elements in F. Write $U_F^m := 1 + \mathfrak{p}_F^m$ for $m \geqslant 1$ and $U_F^0 := \mathcal{O}_F^\times$. The group $F^\times$ is locally profinite. Let χ be a character of $F^\times$. By the above lemma χ is trivial on U_F^m for some m. Fix an additive character ψ of F such that $\mathfrak{p}_F$ is the biggest fractional ideal in F which is contained in the kernel of ψ. For $a \in F$ we define a function $\psi_a : F^\times \to \mathbb{C}^\times$ as follows

$$\psi_a(x) = \psi(a(x-1)).$$

Lemma 1.1.5. [10, 1.8 Proposition] *Let $m, n \in \mathbb{Z}$ be such that $0 \leqslant m < n \leqslant 2m+1$. Denote by $\widehat{U_F^{m+1}/U_F^{n+1}}$ the group of characters of U_F^{m+1}/U_F^{n+1}. The map*

$$\mathfrak{p}_F^{-n}/\mathfrak{p}_F^{-m} \to \widehat{U_F^{m+1}/U_F^{n+1}}$$
$$a \mapsto \psi_a\,|_{U_F^{m+1}}$$

is an isomorphism.

1.1.3 Smooth representations of $\mathrm{GL}_n(F)$

Irreducible smooth representations of $\mathrm{GL}_n(F)$ are well studied. We can divide them into two classes. Representations from one of the classes are called cuspidal.

Let (π, V) be an irreducible smooth representation of G. Let P a parabolic subgroup in G. Denote by L a Levi subgroup in P and by N the unipotent radical in P. Define

$$V_N := V/\langle \pi(n)v - v : \quad n \in N,\ v \in V\rangle$$

We call V_N a Jacquet module.

Definition 1.1.6. *Let (π, V) be an irreducible smooth representation of G. We say that π is* ***cuspidal*** *if $V_N = 0$ for every proper parabolic subgroup P in G.*

A smooth irreducible representation of G is not cuspidal then it is a **parabolic induction** $\mathrm{Ind}_P^G \sigma$ for certain parabolic subgroup P in G and a cuspidal representation of L. Therefore cuspidal representations are building blocks of irreducible smooth representations of G.

1.2 Types on $\mathrm{GL}_n(\mathcal{O}_F)$.

The following section is based on [30]. The existence of cuspidal types on K relatively easily follows from the work of Bushnell and Kutzko [9]. We recall the explanation of that fact given by Paskunas [30]. Paskunas also showed the unicity of cuspidal types on K but the proof of that is much more involved.

Let $N \geqslant 1$ be any natural number. For this section we fix an irreducible cuspidal representation π of $\mathrm{GL}_N(F)$. Write $G = \mathrm{GL}_N(F)$ and $K := \mathrm{GL}_N(\mathcal{O}_F)$. Let (J, λ) be a simple type [9, 5.5.10] occuring in π and coming from a simple stratum $[\mathfrak{A}, n, 0, \beta]$. Define

$$\rho \cong \mathrm{Ind}_J^K \lambda.$$

Proposition 1.2.1. [30, Proposition 3.1] *The representation ρ defined as above is a type on K for $\mathfrak{I}(\pi)$.*

proof First we show that ρ is irreducible. Denote $E = F[\beta]$. By [9, 5.5.11] $g \in G$ intertwines λ if and only if $g \in E^\times J$ (for a definition of intertwining see Section 2.2). Since J is the unique maximal compact open subgroup of $E^\times J$ we have $E^\times J \cap K = J$ and ρ is irreducible.

By Mackey's formula π contains ρ. If $\pi_1 \in \mathfrak{I}(\pi)$, then $\pi_1 \mid_K \cong \pi \mid_K$. Therefore if $\pi_1 \in \mathfrak{I}(\pi)$ then π_1 contains ρ.

For the reverse implication assume that an irreducible smooth representation π_1 of G contains ρ. Then $\pi_1 \mid_J$ contains λ. By [9, 6.2.3], $\pi_1 \in \mathfrak{I}(\pi)$.

□

Chapter 2

Cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$

2.1 Introduction

2.1.1 Cuspidal types

The main motivation behind this chapter is to find new explicit information and invariants of types in general linear groups over a local non-Archimedean field. Let us recall the definition of a cuspidal type. Let F be a non-Archimedean local field and let $\mathcal{O}_F$ be its ring of integers. Denote by k_F the residue field of F. All representations we consider are smooth and over $\mathbb{C}$. Let $n \in \mathbb{N}$, $n \geqslant 1$ and let π be an irreducible cuspidal representation of $\mathrm{GL}_n(F)$. Let

$$\mathfrak{I}(\pi) = \{\pi_2 \mid \pi_2 \cong \pi \otimes \chi \circ \det \quad \text{for some unramified character } \chi \text{ of } F^\times\} \tag{2.1.1}$$

be the inertial support of π.

Definition 2.1.1. *Let H be a compact open subgroup of $\mathrm{GL}_n(F)$, π an irreducible cuspidal representation of $\mathrm{GL}_n(F)$. We say that an irreducible smooth representation λ of H is a* ***cuspidal type*** *on H for $\mathfrak{I}(\pi)$ if the following condition is satisfied: for any irreducible smooth representation π_1 of $\mathrm{GL}_n(F)$*

$$\pi_1 \mid_H \quad \textit{contains} \quad \lambda \quad \textit{if and only if} \quad \mathfrak{I}(\pi_1) = \mathfrak{I}(\pi).$$

In this chapter we mostly consider types on $\mathrm{GL}_n(\mathcal{O}_F)$ so we will supress K from the notation. We say that a representation is a cuspidal type when it is a cuspidal type on $\mathrm{GL}_n(\mathcal{O}_F)$ for $\mathfrak{I}(\pi)$ for some irreducible cuspidal representation π of $\mathrm{GL}_n(F)$. Henniart gave an explicit description of cuspidal types on $\mathrm{GL}_2(\mathcal{O}_F)$ in [7]. Bushnell–Kutzko's construction of irreducible cuspidal representations of $\mathrm{GL}_n(F)$ easily implies existence of cuspidal types on $\mathrm{GL}_n(\mathcal{O}_F)$. It is explained by Paskunas in [30]. Moreover Paskunas [30] proved that for any irreducible cuspidal representation π of $\mathrm{GL}_n(F)$ there exists λ a unique up to isomorphism irreducible smooth representation of $\mathrm{GL}_n(\mathcal{O}_F)$ depending only on $\mathfrak{I}(\pi)$ which is a cuspidal type on $\mathrm{GL}_n(\mathcal{O}_F)$ for $\mathfrak{I}(\pi)$. Using that and the local Langlands correspondence he deduced an inertial Langlands correspondence. In rough terms the inertial Langlands correspondence is a correspondence between cuspidal types on $\mathrm{GL}_n(\mathcal{O}_F)$ and certain irreducible representations of the inertia group of F. For a precise statement see 2.1.3.

The regular representations of $\mathrm{GL}_n(\mathcal{O}_F)$ were introduced by Shintani [31]. They were rediscovered by Hill [20]. Those are in certain sense the best behaved representations of $\mathrm{GL}_n(\mathcal{O}_F)$. In this chapter we determine which cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$ (where p is a prime number) are regular. Moreover we provide a precise description of all orbits which can give cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$ with conductor at least 4. We precisely determine orbits of cuspidal types in small conductor case for $p = 2$. We use tools from Clifford theory, the classification of cuspidal representations of $\mathrm{GL}_n(F)$ due to Bushnell and Kutzko specialized to $n = p$ and the properties of the actions of subgroups of $\mathrm{GL}_2(F)$ on their Bruhat–Tits buildings.

2.1.2 Cuspidal types in terms of orbits

Any irreducible smooth representation ρ of $\mathrm{GL}_n(\mathcal{O}_F)$ factors through a finite group $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^r)$ where r is a natural number bigger than or equal to 1 and $\mathfrak{p}_F$ is the maximal ideal in $\mathcal{O}_F$. The minimal natural number r with this property is called the conductor of the representation ρ. Let ρ be an irreducible smooth representation of $\mathrm{GL}_n(\mathcal{O}_F)$ with conductor $r > 1$. Sometimes it will be convenient to view ρ as a representation of $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^r)$. In this case we will denote it by $\bar{\rho}$. Let $l = \lfloor \frac{r+1}{2} \rfloor$ and let K^l be the kernel of the projection from $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^r)$ onto $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^l)$. Note that K^l is an abelian group. We fix once and for all an additive character $\psi : F \to \mathbb{C}^\times$ with conductor $\mathfrak{p}_F$ i.e., $\mathfrak{p}_F$ is the biggest fractional ideal of F on which ψ is trivial. Denote by $\mathrm{M}_n(\mathcal{O}_F)$ the set of all $n \times n$ - matrices with entries in $\mathcal{O}_F$. By Clifford's theorem (see [23, 6.2])

$$\bar{\rho}\mid_{K^l} = m \bigoplus_{\bar{\alpha} \sim \bar{\alpha_1}} \bar{\psi}_{\bar{\alpha}}, \tag{2.1.2}$$

where $\bar{\alpha}_1 \in \mathrm{M}_n(\mathcal{O}_F/\mathfrak{p}_F^{r-l})$, $\bar{\alpha}$ runs over the conjugacy class of $\bar{\alpha_1}$ under $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}^{r-l})$, $m \in \mathbb{N}$ and the characters $\bar{\psi}_{\bar{\alpha}} : K^l \to \mathbb{C}^\times$ are defined as follows: $\bar{\psi}_{\bar{\alpha}}(1+x) = \psi(\mathrm{tr}(\widehat{\alpha}\widehat{x}))$ for some lifts $\widehat{x}, \widehat{\alpha}$ of $x, \bar{\alpha}$ to elements in $\mathrm{M}_n(\mathcal{O}_F)$. The definition of $\bar{\psi}_{\bar{\alpha}}$ does not depend on the choice of lifts. If a matrix $\alpha \in \mathrm{M}_n(\mathcal{O}_F)$ is such that its image in $\mathrm{M}_n(\mathcal{O}_F/\mathfrak{p}_F^{r-l})$ appears in the decomposition (2.1.2) we say that α **is in the orbit of** ρ. The recalled description is a recap of a part of [33]. We say that a representation is **regular** if its orbit contains a matrix whose image in $\mathrm{M}_n(\mathcal{O}_F/\mathfrak{p}_F)$ has abelian centralizer in $\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F)$. Krakovski, Onn and Singla [25] constructed all such representations under the condition that the characteristic of the residue field of F is odd. Stasinski and Stevens in [34] constructed all regular representations of $\mathrm{GL}_n(\mathcal{O}_F)$. In [33] Stasinski asked which cuspidal types are regular.

We give a full description of cuspidal types on $\mathrm{GL}_2(\mathcal{O}_F)$ in terms of orbits. For a character $\bar{\psi}_{\bar{\alpha}}$ on K^l let $\mathrm{Stab}_{\mathrm{GL}_2(\mathcal{O}_F)}\bar{\psi}_{\bar{\alpha}}$ be the preimage of $\mathrm{Stab}_{\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^r)}\bar{\psi}_{\bar{\alpha}}$ through the canonical projection $\mathrm{GL}_2(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}^r)$. Recall that a polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is called **Eisenstein** if $a_1, \ldots, a_{n-1} \in \mathfrak{p}_F$ and $a_0 \in \mathfrak{p}_F \setminus \mathfrak{p}_F^2$. The following theorem gives a full description of cuspidal types on $\mathrm{GL}_2(\mathcal{O}_F)$ in terms of orbits.

Theorem 2.1.2. *A cuspidal type on $K_2 := \mathrm{GL}_2(\mathcal{O}_F)$ is precisely a one-dimensional twist of one of the following:*

1. *a representation inflated from some irreducible cuspidal representation of $\mathrm{GL}_2(k_F)$;*
2. *a representation whose orbit contains a matrix whose characteristic polynomial is irreducible mod $\mathfrak{p}_F$;*
3. *a representation whose orbit contains a matrix β whose characteristic polynomial is Eisenstein and which satisfies one of the following conditions:*
   - *(a) it has conductor at least 4;*
   - *(b) it has conductor $r = 2$ or 3 and is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})}^{K_2}\theta$ where $\theta\mid_{U_{\mathfrak{M}}^{\lfloor\frac{r+1}{2}\rfloor}} = m\psi_\beta$ for certain $m \in \mathbb{Z}$ and θ does not contain the trivial character of $\begin{pmatrix} 1 & \mathfrak{p}_F^{r-2} \\ 0 & 1 \end{pmatrix}$.*

We also give a description of cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$ with p prime. Let $\mathfrak{I}$ be the $\mathcal{O}_F$-order consisting of matrices that are upper triangular modulo $\mathfrak{p}_F$. Let $U_{\mathfrak{I}}$ be the group of invertible elements of $\mathfrak{I}$ and let $\mathfrak{P}_{\mathfrak{I}}$ be the Jacobson radical in $\mathfrak{I}$. We choose $\Pi_{\mathfrak{I}}$ such that $\Pi_{\mathfrak{I}}\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}$. We prove the following result:

Theorem 2.1.3. *If λ is a cuspidal type on $K := \mathrm{GL}_p(\mathcal{O}_F)$, then it is a one-dimensional twist of one of the following:*

1. *a representation which is inflated from an irreducible cuspidal representation of $\mathrm{GL}_p(k_F)$;*
2. *a representation whose orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$;*
3. *a representation whose orbit contains a matrix of the form $\Pi_{\mathfrak{I}}^j B$ where $0 < j < p$ and $B \in U_{\mathfrak{I}}$.*

Moreover if a representation is a one-dimensional twist of a representation of the form (3) and has conductor at least 4, or is of the form (1) or (2), then it is a cuspidal type.

Theorem 2.1.3 for $p = 2$ coincides with Theorem 2.1.2 as long as a representation is of the conductor $r \geqslant 4$. Theorem 2.1.2 for representations of conductor $r = 2$ or 3 gives a more precise description of cuspidal types on $\mathrm{GL}_2(\mathcal{O}_F)$.

Representations whose orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$ are regular. In Subsection 2.3.3 we prove that a matrix of the form $\Pi_{\mathfrak{J}}^j B$ with $0 < j < p$ and $B \in U_{\mathfrak{J}}$ is regular if and only if $j = 1$. The characteristic polynomial of a matrix of the form $\Pi_{\mathfrak{J}} B$ is Eisenstein. However the characteristic polynomial of a matrix of the form $\Pi_{\mathfrak{J}}^j B$ with $1 < j < p$ is not Eisenstein. Therefore a cuspidal type on $\mathrm{GL}_p(\mathcal{O}_F)$ of conductor $r \geq 4$ is regular if and only if its orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$ or a matrix whose characteristic polynomial is Eisenstein. In particular, for $p > 2$ even for big conductors there are cuspidal types which are not regular. Indeed, if a representation has conductor at least 4 and is of the form (3) from the above theorem with $j > 1$ then it is a cuspidal type but it is not regular.

2.1.3 Perspectives

To the best of our knowledge the regular representations of $\mathrm{GL}_p(\mathcal{O}_F)$ form the biggest family of irreducible smooth representations of $\mathrm{GL}_p(\mathcal{O}_F)$ which has been described in terms of orbits so far. Our description of cuspidal types in terms of orbits suggests that even though the cuspidal types are not always regular they can be described in terms of orbits.

It could be also interesting to study representations which correspond to the regular cuspidal types under the inertial Langlands correspondence. We recall the precise statement of the inertial Langlands correspondence. Denote by W_F the Weil group of F and by I_F the inertia subgroup. For an infinite-dimensional irreducible smooth representation π of $\mathrm{GL}_n(F)$ we denote by $WD(\pi)$ the Weil–Deligne representation of W_F which corresponds to π through the local Langlands correspondence. Paskunas in [30] proved the following result (**the inertial Langlands correspondence**): for a smooth n-dimensional representation τ of I_F which extends to a smooth irreducible Frobenius semisimple representation of W_F there exists a unique up to isomorphism smooth irreducible representation ρ of $\mathrm{GL}_n(\mathcal{O}_F)$ which satisfies the following condition: for every irreducible smooth infinite-dimensional representation π of $\mathrm{GL}_n(F)$ we have π contains ρ if and only if $WD(\pi)\mid_{I_F}$ is isomorphic to τ. Moreover ρ has multiplicity at most one in π. Having a description of cuspidal types in terms of orbits, it would also be interesting to look how the properties of orbits of cuspidal types translate to properties of the corresponding representations of I_F.

Finally, the problem of describing cuspidal types can be also studied in other cases of maximal compact subgroups of other reductive $\mathfrak{p}$-adic groups.

2.1.4 Outline of the Chapter

In **Section 2.2** we recall the properties of hereditary orders and simple strata which in our case are specific because we consider $\mathrm{GL}_p(F)$ with p prime. In 2.2.3 we give an explicit description of simple strata which is one of crucial ingredients in the proof of Theorem 2.1.3. In 2.2.4 we recall the classification of irreducible cuspidal representations of $\mathrm{GL}_p(F)$ and we study twists of cuspidal representations with minimal level. In 2.2.5 we recall some basic notions from Clifford theory. In these terms we describe cuspidal types.

In **Section 2.3** we prove the two main results of this chapter: Theorem 2.1.2 and Theorem 2.1.3. Then we determine which of cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$ are regular.

In **Section 2.4** we give an example of two representations of $\mathrm{GL}_2(\mathcal{O}_F)$ with the same orbit but one representation is a cuspidal type and the other is not.

2.1.5 Notation

We will write $\lfloor a \rfloor$ for the biggest integer less than or equal to a and $\mathrm{tr}A$ for the trace of a matrix A. For any local non-Archimedean field E we will denote by $\mathcal{O}_E$ its ring of integers, by $\mathfrak{p}_E$ the maximal ideal in $\mathcal{O}_E$, by ϖ_E a prime element in E, by $\mathcal{O}_E^\times$ invertible elements of $\mathcal{O}_E$ and by k_E the residue field of E. We fix a non-Archimedean local field F and a prime number p. Let E/F be a finite field extension. Write $e(E/F)$ for the ramification index and $f(E/F)$ for the residue class degree. Let $G := \mathrm{GL}_p(F)$ and $K := \mathrm{GL}_p(\mathcal{O}_F)$. We write Z for the center of G. We denote by V a vector space over F of dimension p and $A := \mathrm{End}_F(V)$. For a local field E we denote by ν_E the additive valuation which takes 1 on a uniformizer. Write π for a representation of $\mathrm{GL}_n(F)$ and let χ be a character of $F^\times$. Set $\chi\pi := (\chi \circ \det) \otimes \pi$. For B a subgroup of G we denote by $N_G(B)$ the normalizer of B in G.

2.2 Simple strata and cuspidal representations

2.2.1 Cuspidal types on K

Paskunas in [30] has proven the unicity of (cuspidal) types:

Theorem 2.2.1 (cf [30], Theorem 1.3)**.** *Let π be an irreducible cuspidal representation of G. Then there exists a smooth irreducible representation ρ of K depending on $\mathfrak{I}(\pi)$, such that ρ is a cuspidal type on K for $\mathfrak{I}(\pi)$. Moreover, ρ is unique (up to isomorphism) and it occurs in $\pi \mid_K$ with multiplicity 1.*

Denote by $X_F(G)$ the group of F-rational characters of G. Denote by $\|\cdot\|_F$ normalized absolute value on F. Define

$$^\circ G = \textstyle\bigcap_{\phi \in X_F(G)} \mathrm{Ker}(\| \phi \|_F).$$

The following proposition will be a useful tool while describing cuspidal types in terms of orbits.

Proposition 2.2.2 (cf [9], 5.4 Proposition). *Let π be an irreducible cuspidal representation of G of the form $\pi \cong \text{c-Ind}_J^G \tau_1$ for τ_1 a representation of some compact mod Z open subgroup J. Let $J^\circ = J \cap {}^\circ G$ and let τ be an irreducible component of $\tau_1 |_{J^\circ}$. Then J° is the unique maximal compact subgroup of J and τ is a cuspidal type on J° for $\mathfrak{I}(\pi)$.*

Remark 2.2.3 ([9]). *Every irreducible cuspidal representation of $\mathrm{GL}_p(F)$ is of the form as in Proposition 2.2.2.*

Remark 2.2.4. *Theorem 2.2.1, Proposition 2.2.2 and Remark 2.2.3 do not use the assumption that p is prime.*

2.2.2 Hereditary orders

Let $A = \mathrm{End}_F(V)$ where V is of prime dimension p. In this section we recall basic notions associated to hereditary orders in A. The given description of principal orders relies on the fact that V is of a prime dimension. In the general case things are more complicated. For more detailed discussion on hereditary orders we refer to ([9], 1.1). Lemmas 2.2.5 and 2.2.8 play an important role for us and they are not true for non-principal hereditary orders. We call a finitely generated $\mathcal{O}_F$-submodule of V containing an F-basis of V an *$\mathcal{O}_F$-lattice* in A. An *$\mathcal{O}_F$-order* in A is an $\mathcal{O}_F$-lattice in A which is also a subring of A (with the same identity element). A sequence $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ of $\mathcal{O}_F$-lattices satisfying the following conditions:

1. $L_{i+1} \subsetneq L_i$, $i \in \mathbb{Z}$
2. there exists $e \in \mathbb{Z}$ such that $\mathfrak{p}_F L_i = L_{i+e}$ for every $i \in \mathbb{Z}$

is called an *$\mathcal{O}_F$-lattice chain* in V. We call $e = e(\mathcal{L}) = e(\mathfrak{A}(\mathcal{L}))$ the *$\mathcal{O}_F$-period* of $\mathcal{L}$. For $n \in \mathbb{Z}$ and an $\mathcal{O}_F$-lattice chain $\mathcal{L}$ define

$$\mathrm{End}^n_{\mathcal{O}_F}(\mathcal{L}) = \{g \in A : gL_i \subseteq L_{i+n}, i \in \mathbb{Z}\}.$$

Taking $n = 0$ we get $\mathrm{End}^0_{\mathcal{O}_F}(\mathcal{L}) =: \mathfrak{A}(\mathcal{L}) = \mathfrak{A}$ an $\mathcal{O}_F$-order in A. We call such $\mathcal{O}_F$-order a hereditary order. A hereditary order $\mathfrak{A}(\mathcal{L})$ is called principal if $\dim_{k_F}(L_i/L_{i+1}) = \dim_{k_F}(L_j/L_{j+1})$ for every $i, j \in \mathbb{Z}$. Let $\mathcal{L}$ be an $\mathcal{O}_F$ lattice chain. Then $\mathrm{End}^1_{\mathcal{O}_F}$ is the Jacobson radical of $\mathfrak{A}(\mathcal{L})$. We denote it by $\mathfrak{P}_{\mathfrak{A}}$ or by $\mathfrak{P}$ if the order is clear form the context. It is an invertible fractional ideal and we have

$$\mathfrak{P}^n_{\mathfrak{A}} = \mathfrak{P}^n = \mathrm{End}^n_{\mathcal{O}_F}(\mathcal{L}) \quad \text{for any } n \in \mathbb{Z}.$$

We also have $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}_{\mathfrak{A}}^{e(\mathfrak{A})}$. We denote by $U(\mathfrak{A}) = U_{\mathfrak{A}}^0$ the group of invertible elements in $\mathfrak{A}$ and we define the subgroups

$$U_{\mathfrak{A}}^n = 1 + \mathfrak{P}_{\mathfrak{A}}^n \quad \text{for any } n \in \mathbb{N}, n \geqslant 1.$$

We define the *normalizer* of $\mathfrak{A}$ as

$$\mathfrak{K}(\mathfrak{A}) = \{g \in G : g\mathfrak{A}g^{-1} = \mathfrak{A}\}$$

or equivalently if $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ for some lattice chain $\mathcal{L}$ as

$$\mathfrak{K}(\mathfrak{A}) = \{g \in G : gL \in \mathcal{L} \quad \text{for any} \quad L \in \mathcal{L}\}.$$

We now restrict our attention to principal orders.

Lemma 2.2.5. *Any principal order is* $\mathrm{GL}_p(F)$*-conjugate to* $\mathfrak{M} = \mathrm{M}_p(\mathcal{O}_F)$ *or to order* $\mathfrak{I}$ *which consists of matrices with coefficients in* $\mathcal{O}_F$ *and uppertriangular modulo* $\mathfrak{p}_F$*:*

$$\mathfrak{M} = \begin{pmatrix} \mathcal{O}_F & \cdots & \cdots & \mathcal{O}_F \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \mathcal{O}_F & \cdots & \cdots & \mathcal{O}_F \end{pmatrix} \quad \textit{and} \quad \mathfrak{I} = \begin{pmatrix} \mathcal{O}_F & \cdots & \cdots & \mathcal{O}_F \\ \mathfrak{p}_F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathfrak{p}_F & \cdots & \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}.$$

Proof. The proof is based on the notion of an $\mathcal{O}_F$-basis of an $\mathcal{O}_F$-lattice chain. For the reference see ([9], 1.1). Let $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ be an $\mathcal{O}_F$-lattice chain in V. An $\mathcal{O}_F$*-basis* of $\mathcal{L}$ is an F-basis $\{v_1, \ldots, v_p\}$ of V such that it is an $\mathcal{O}_F$-basis of some $L_j \in \mathcal{L}$ and $L_i = \amalg_{l=1}^{p} \mathfrak{p}_F^{f(i,l)} v_i$, $i \in \mathbb{Z}$, for some integers $f(i,1) \leqslant f(i,2) \leqslant \ldots \leqslant f(i,p)$. Any $\mathcal{O}_F$-lattice chain has an $\mathcal{O}_F$-basis.

Take $\mathfrak{A}(\mathcal{L})$ to be a principal order with $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ an $\mathcal{O}_F$-lattice chain. We want to show that $\mathfrak{A}$ is $\mathrm{GL}_p(F)$-conjugate to $\mathfrak{M}$ or $\mathfrak{I}$. Let $\{v_1, \ldots, v_p\}$ be an $\mathcal{O}_F$-basis of $\mathcal{L}$. We use this basis to identify A with $\mathrm{M}_p(F)$. Let $\mathcal{L}_{\max}$ be the $\mathcal{O}_F$-lattice chain formed by $\mathcal{O}_F$-lattices of the form

$$\mathfrak{p}_F^j(\mathcal{O}_F v_1 + \ldots + \mathcal{O}_F v_l + \mathfrak{p}_F v_{l+1} + \ldots + \mathfrak{p}_F v_p)$$

where $1 \leqslant l \leqslant p$ and $j \in \mathbb{Z}$. The $\mathcal{O}_F$-lattice chain $\mathcal{L}$ is contained in $\mathcal{L}_{\max}$ (see [9, 1.1]). Since $\mathcal{L}$ is principal $\dim_{k_F}(L_i/L_{i+1}) = \dim_{k_F}(L_l/L_{l+1})$ for any $i, l \in \mathbb{Z}$. It is easy to see that $\sum_{i=1}^{e(\mathcal{L})} \dim_{k_F}(L_i/L_{i+1}) = p$. Therefore $e(\mathcal{L}) = 1$ or p. With this identification we are going to deduce that $\mathfrak{A}(\mathcal{L})$ is either $\mathfrak{M}$ or $\mathfrak{I}$. If $e(\mathcal{L}) = 1$ then $\mathcal{L}$ consists of $\mathcal{O}_F$-lattices of the form $\mathfrak{p}_F^j(\mathcal{O}_F v_1 + \ldots + \mathcal{O}_F v_p)$ for $j \in \mathbb{Z}$ and $\mathfrak{A} = \mathfrak{M}$. If $e(\mathcal{L}) = p$ then $\mathcal{L} = \mathcal{L}_{\max}$ and $\mathfrak{A}(\mathcal{L}) = \mathfrak{I}$. $\square$

Denote

$$\Pi_{\mathfrak{M}} = \varpi_F \mathrm{Id}_{p\times p} \quad \text{and} \quad \Pi_{\mathfrak{J}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 \\ \varpi_F & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

where $\mathrm{Id}_{p\times p}$ denotes the identity matrix of size $p \times p$.

Corollary 2.2.6. *For a principal order $\mathfrak{A}$ there exists an element a such that $\mathfrak{P}_{\mathfrak{A}} = a\mathfrak{A} = \mathfrak{A}a$.*

Proof. By Lemma 2.2.5 it is enough to check the statement for $\mathfrak{M}$ and $\mathfrak{J}$. By simple computation we see that taking $a = \Pi_{\mathfrak{M}}$ for $\mathfrak{M}$ and $a = \Pi_{\mathfrak{J}}$ for $\mathfrak{J}$ we obtain the desired equalities. □

We call an element a from Corollary 2.2.6 a prime element in $\mathfrak{A}$.

Remark 2.2.7. *In particular, $\mathfrak{P}_{\mathfrak{M}} = \Pi_{\mathfrak{M}}\mathfrak{M} = \mathfrak{M}\Pi_{\mathfrak{M}}$ and $\mathfrak{P}_{\mathfrak{J}} = \Pi_{\mathfrak{J}}\mathfrak{J} = \mathfrak{J}\Pi_{\mathfrak{J}}$.*

For a principal order we can deduce a more specific form of the normalizer:

Lemma 2.2.8. *Let $\mathfrak{A}$ be a principal order. Then $\mathfrak{K}(\mathfrak{A}) = U_{\mathfrak{A}} \rtimes \langle \Pi_{\mathfrak{A}} \rangle$.*

Proof. By definition $U_{\mathfrak{A}}$ is contained in $\mathfrak{K}(\mathfrak{A})$. Since $\Pi_{\mathfrak{A}}\mathfrak{A} = \mathfrak{P}_{\mathfrak{A}} = \mathfrak{A}\Pi_{\mathfrak{A}}$ also the subgroup generated by $\Pi_{\mathfrak{A}}$ is contained in $\mathfrak{K}(\mathfrak{A})$. Therefore the group generated by $U_{\mathfrak{A}}$ and $\Pi_{\mathfrak{A}}$ is contained in $\mathfrak{K}(\mathfrak{A})$. On the other hand the group $U_{\mathfrak{A}}\langle \Pi_{\mathfrak{A}} \rangle$ contains the center. It is compact modulo center and it is a maximal subgroup of G with this property. Therefore $\mathfrak{K}(\mathfrak{A})$ is generated by $U_{\mathfrak{A}}$ and $\Pi_{\mathfrak{A}}$. By ([9], section 1.1) the subgroup $U_{\mathfrak{A}}$ is normal in $\mathfrak{K}(\mathfrak{A})$. The intersection $U_{\mathfrak{A}} \cap \langle \Pi_{\mathfrak{A}} \rangle$ is trivial. □

A normalizer $\mathfrak{K}(\mathfrak{A})$ is an open compact modulo center subgroup of G (see [9], section 1.1).

Define

$$\nu_{\mathfrak{A}}(a) = \max\{n \in \mathbb{Z} : a \in \mathfrak{P}_{\mathfrak{A}}^n\}.$$

2.2.3 Simple strata

A simple stratum is a notion used in the classification of irreducible cuspidal representations of $\mathrm{GL}_p(F)$. We recall the definition and then we prove its properties which are crucial in the description of cuspidal types. We focus on simple strata which come from principal orders as these ones are used in the classification of irreducible cuspidal representations of $\mathrm{GL}_p(F)$. Again the given properties rely on the fact that the dimension of V is prime. We use a definition (Definition 2.2.12) of a simple stratum which is not a standard one (comes from [8]) but we prove that in the cases interesting for us it is equivalent with the one used in [9]. The goal of this subsection is to prove Proposition 2.2.16.

Definition 2.2.9. *A 4-tuple $[\mathfrak{A}, n, r, \beta]$ is called a stratum in A if $\mathfrak{A}$ is a hereditary $\mathcal{O}_F$-order in A, n, r are integers such that $n > r$ and $\beta \in A$ is such that $\nu_{\mathfrak{A}}(\beta) \geqslant -n$.*

We say that two strata $[\mathfrak{A}_1, n_1, r_1, \beta_1]$ and $[\mathfrak{A}_2, n_2, r_2, \beta_2]$ are equivalent if

$$\beta_1 + \mathfrak{P}_1^{r_1} = \beta_2 + \mathfrak{P}_2^{r_1}$$

where $\mathfrak{P}_1$ (resp. $\mathfrak{P}_2$) is the Jacobson radical of $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$). We will keep this notation for the rest of the chapter. If $n > r \geqslant \lfloor \frac{n}{2} \rfloor \geqslant 0$, then we can associate with a stratum $[\mathfrak{A}, n, r, \beta]$ a character $\psi_\beta : U_{\mathfrak{A}}^{r+1} \to \mathbb{C}^\times$ which is trivial on $U_{\mathfrak{A}}^{n+1}$ and defined as follows $\psi_\beta(x) = \psi(\mathrm{tr}(\beta(1-x)))$. We say that a representation π of $\mathrm{GL}_p(F)$ contains a stratum $[\mathfrak{A}, n, r, \alpha]$ if π contains the character ψ_α of $U_{\mathfrak{A}}^{r+1}$. We define the normalized level of a representation π as

$$l(\pi) = \min\{\frac{n}{e(\mathfrak{A})} : \quad (\mathfrak{A}, n) \text{ such that } \mathfrak{A} \text{ is a hereditary order, } n \in \mathbb{N}, n \geqslant 0$$
$$\text{and } \pi \text{ contains a trivial character of } U_{\mathfrak{A}}^{n+1}\}.$$

We say that a stratum $[\mathfrak{A}, n, n-1, \beta]$ is fundamental if $\beta + \mathfrak{P}_{\mathfrak{A}}^{1-n}$ does not contain nilpotents from A. We say that two strata $[\mathfrak{A}_1, n_1, r_1, \beta_1]$ and $[\mathfrak{A}_2, n_2, r_2, \beta_2]$ **intertwine in** G if there exists $x \in G$ such that $x(\beta_2 + \mathfrak{P}_{\mathfrak{A}_2}^{-r_2})x^{-1} \cap (\beta_1 + \mathfrak{P}_{\mathfrak{A}_1}^{-r_1}) \neq \emptyset$.

Let H_1, H_2 be two compact open subgroups of G and let π_1 (resp. π_2) be an irreducible smooth representation of H_1 (rep. H_2). Take $g \in G$. Write $H_1^g := g^{-1}H_1 g$. Define π_1^g to be a representation of H_1^g such that $\pi_1^g(h) = \pi_1(ghg^{-1})$ for any $h \in H_1$. We say that $g \in G$ intertwines π_1 with π_2 if $\mathrm{Hom}_{H_1^g \cap H_2}(\pi_1^g, \pi_2) \neq 0$.

Lemma 2.2.10. *(see* [10, 11.1 Proposition 1] *and* [2, Lemma 1.13.5]*) Let π be an irreducible cuspidal representation of G. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be principal orders and let $[\mathfrak{A}_1, n_1, n_1 - 1, \beta_1]$ and $[\mathfrak{A}_2, n_2, n_2 - 1, \beta_2]$ be two strata contained in π. Then they intertwine.*

In order to introduce the simple stratum we first define a notion of a minimal element over F.

Definition 2.2.11. *Let E/F be a finite field extension with $E = F[\beta]$. We say that β is minimal over F if the following is satisfied:*

- *$gcd(\nu_E(\beta), e(E/F)) = 1$ and*
- *$\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)} + \mathfrak{p}_E$ generates the extension of the residue fields k_E/k_F.*

Definition 2.2.12. *A stratum $[\mathfrak{A}, n, n-1, \beta]$ is called simple if*

1. $E = F[\beta]$ *is a field*

2. $\beta\mathfrak{A} = \mathfrak{P}_{\mathfrak{A}}^{-n}$

3. β *is minimal over* F

Let π be an irreducible cuspidal representation of G which contains a simple stratum $[\mathfrak{A}, n, n-1, \beta]$. Since we consider $\mathrm{GL}_p(F)$ with p prime there are only two possibilities for the degree $[F[\beta] : F]$. Namely it is 1 or p.

Lemma 2.2.13. *(see* [9, 1.5.6 Exercise]*) Let* $[\mathfrak{A}, n, n-1, \beta]$ *be a simple stratum with* $\mathfrak{A} = \mathfrak{M}$ *or* $\mathfrak{I}$*. Denote* $E = F[\beta]$*. Then* $E^\times \subseteq \mathfrak{K}(\mathfrak{A})$*.*

Proof. First we prove that $\beta \in \mathfrak{K}(\mathfrak{A})$. Take an $\mathcal{O}_F$-lattice chain $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ such that $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. Take arbitrary $L_i \in \mathcal{L}$. We want to have $\beta L_i \in \mathcal{L}$. The fractional ideal $\mathfrak{P}_{\mathfrak{A}}^n$ is invertible and $\mathfrak{P}_{\mathfrak{A}}^n\mathfrak{P}_{\mathfrak{A}}^{-n} = \mathfrak{A}$. We have

$$L_{i-n} = \mathfrak{A}L_{i-n} = \mathfrak{P}_{\mathfrak{A}}^{-n}\mathfrak{P}_{\mathfrak{A}}^{n}L_{i-n} \subseteq \mathfrak{P}_{\mathfrak{A}}^{-n}L_i = \beta\mathfrak{A}L_i = \beta L_i \subseteq L_{i-n}.$$

Therefore $\beta L_i = L_{i-n} \in \mathcal{L}$ for any $i \in \mathbb{Z}$ and $\beta \in \mathfrak{K}(\mathfrak{A})$.

Since β is minimal over F the value $\nu_E(\beta)$ is coprime with $e(E/F)$. Therefore there exist $n_1, n_2 \in \mathbb{Z}$ such that $1 = n_1\nu_E(\beta) + n_2 e(E/F) = \nu_E(\beta^{n_1}\varpi_F^{n_2})$. We can write any element from $E^\times$ as $u(\beta^{n_1}\varpi_F^{n_2})^m$ for some $u \in \mathcal{O}_E^\times$, $m \in \mathbb{Z}$ and $\beta^{n_1}\varpi_F^{n_2} \in \mathfrak{K}(\mathfrak{A})$. To finish the proof it is enough to show that $\mathcal{O}_E^\times \subseteq \mathfrak{K}(\mathfrak{A})$. First we want to show that $\mathcal{O}_E \subseteq \mathfrak{A}$.

By the definition of a minimal element $\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)} + \mathfrak{p}_E$ generates k_E/k_F so $\mathcal{O}_E = \mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \varpi_E\mathcal{O}_E$. Iterating

$$\begin{aligned}\mathcal{O}_E =&\mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \varpi_E\mathcal{O}_E = \mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \\ &\varpi_E\mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \ldots + \varpi_E^{p-1}\mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \mathfrak{p}_F\mathcal{O}_E\end{aligned}$$

By Nakayama's Lemma,

$$\begin{aligned}\mathcal{O}_E =&\mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \varpi_E\mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}] + \ldots \\ &+\varpi_E^{p-1}\mathcal{O}_F[\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}].\end{aligned}$$

We can take $\varpi_E = \beta^{n_1}\varpi_F^{n_2} \in \mathfrak{K}(\mathfrak{A})$. Since $1 = \nu_E(\varpi_E) = \frac{e(E/F)}{[E:F]}\nu_F(\det(\beta^{n_1}\varpi_F^{n_2}))$, $\nu_F(\det(\beta^{n_1}\varpi_F^{n_2})) > 0$ and $\varpi_E \in \mathfrak{A}$. Similarly $\nu_E(\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)}) = 0$ so $\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)} \in \mathfrak{A}$ and $\mathcal{O}_E \subseteq \mathfrak{A}$.

To sum up we proved $\mathcal{O}_E \subseteq \mathfrak{A}$. Therefore $\mathcal{O}_E^\times \subseteq U_{\mathfrak{A}} \subseteq \mathfrak{K}(\mathfrak{A})$. □

Remark 2.2.14. *Assume* $[\mathfrak{A}, n, n-1, \beta]$ *is not equivalent to a stratum* $[\mathfrak{A}, n, n-1, \beta']$ *with* β' *a scalar matrix. By Lemma 2.2.13 and* [9, 1.4.15] *our definition of a simple stratum* *coincides with the standard definition of a simple stratum in which the hereditary order is* $\mathfrak{M}$ *or* $\mathfrak{I}$ *and* $r = n-1$ *(see* [9, 1.5.5]*).*

Lemma 2.2.15. *Let* $\mathfrak{A} = \mathfrak{M}$ *or* $\mathfrak{I}$*. Let* $\beta \in \mathfrak{A}$ *be such that* $\beta \in \mathfrak{K}(\mathfrak{A})$ *and* $E = F[\beta]$ *is a field. Assume* $[\mathfrak{A}, n, n-1, \beta]$ *is not equivalent to* $[\mathfrak{A}, n, n-1, \beta']$ *with* β' *a scalar matrix and assume* $E^\times \subseteq \mathfrak{K}(\mathfrak{A})$*. Then*

- $e(E/F) = e(\mathfrak{A})$
- $\nu_E(\beta) = \nu_{\mathfrak{A}}(\beta)$.

Proof. For the first equality observe that by [9, 1.2.4 Proposition] $e(E/F)$ divides $e(\mathfrak{A})$. Pick an $\mathcal{O}_F$-lattice chain $\mathcal{L} = \{L_i : i \in \mathbb{Z}\}$ such that $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. Fix i a natural number. L_i/L_{i+1} is a vector space over k_F. Define $f(\mathfrak{A})$ to be the dimension of L_i/L_{i+1} over k_F. Since $\mathfrak{A}$ is a principal order the number $f(\mathfrak{A})$ does not depend on the choice of i and $e(\mathfrak{A})f(\mathfrak{A}) = p$. We also have $e(E/F)f(E/F) = p$. Therefore to finish the proof it is enough to prove that $f(E/F)$ divides $f(\mathfrak{A})$. By [9, 1.2.1 Proposition], L_i/L_{i+1} is a vector space over k_E and $f(E/F)$ divides $f(\mathfrak{A})$.

For the second equality write $\nu_{\mathfrak{A}}(\beta) = n$. Then by the definition $\beta \in \mathfrak{P}_{\mathfrak{A}}^n \setminus \mathfrak{P}_{\mathfrak{A}}^{n+1}$. Since β is an element of the normalizer $\mathfrak{K}(\mathfrak{A}) = \langle \Pi_{\mathfrak{A}} \rangle \ltimes U_{\mathfrak{A}}$ the matrix β is of the form $\beta = \Pi_{\mathfrak{A}}^n C$ where C is an element of $U_{\mathfrak{A}}$. Therefore $\nu_F(\det(\beta)) = \frac{np}{e(\mathfrak{A})}$ and $\nu_E(\beta) = \frac{e(E/F)}{[E:F]}\nu_F(\det(\beta)) = n = \nu_{\mathfrak{A}}(\beta)$. □

The following description will be useful in the proofs of the main theorems.

Proposition 2.2.16. *Let* $[\mathfrak{A}, n, n-1, \beta]$ *be a stratum with* $\mathfrak{A} = \mathfrak{M}$ *or* $\mathfrak{I}$ *which is not equivalent to a stratum* $[\mathfrak{A}, n, n-1, \beta']$ *with* $[F[\beta'] : F] = 1$*. The stratum* $[\mathfrak{A}, n, n-1, \beta]$ *is simple if and only if* $n = -\nu_{\mathfrak{A}}(\beta)$ *and*

1. $\mathfrak{A} = \mathfrak{M}$ *and the characteristic polynomial of* $\varpi_F^n\beta$ *is irreducible modulo* $\mathfrak{p}_F$ *or*
2. $\mathfrak{A} = \mathfrak{I}$ *and* $\varpi_F^{\lfloor \frac{n}{p} \rfloor + 1}\beta$ *is of the form* $\Pi_{\mathfrak{I}}^j B$ *where* $1 \leqslant j \leqslant p-1$, $B \in U_{\mathfrak{I}}$.

Proof. Assume that $[\mathfrak{M}, n, n-1, \beta]$ is a simple stratum. We want to prove that the characteristic polynomial of $\varpi_F^n\beta$ is irreducible modulo $\mathfrak{p}_F$ and $\nu_{\mathfrak{M}}(\beta) = -n$. The second follows from the definition of a simple stratum. By the definition β is minimal over F and in particular $\varpi_F^{-\nu_E(\beta)}\beta^{e(E/F)} + \mathfrak{p}_E$ generates the extension k_E/k_F. By Lemma 2.2.13 and Lemma 2.2.15, $-\nu_E(\beta) = -\nu_{\mathfrak{M}}(\beta) = n$ and $e(E/F) = e(\mathfrak{M}) = 1$. Therefore $\varpi_F^n\beta + \mathfrak{p}_E$ generates k_E/k_F. This means that the minimal polynomial of $\varpi_F^n\beta$ is irreducible modulo $\mathfrak{p}_F$ and is of degree p. This implies that the minimal polynomial modulo $\mathfrak{p}_F$ is equal to the characteristic polynomial modulo $\mathfrak{p}_F$. Therefore the characteristic polynomial of $\varpi_F^n\beta$ is irreducible modulo $\mathfrak{p}_F$.

Assume now that $[\mathfrak{I}, n, n-1, \beta]$ is a simple stratum. We want to show that $n = -\nu_{\mathfrak{I}}(\beta)$ and $\varpi_F^{\lfloor \frac{n}{p} \rfloor + 1}\beta$ is of the form $\Pi_{\mathfrak{I}}^j B$ where $0 < j < p$ and $B \in U_{\mathfrak{I}}$. By the definition of a simple stratum $n = -\nu_{\mathfrak{J}}(\beta)$, $\beta \in \mathfrak{K}(\mathfrak{J}) = \langle \Pi_{\mathfrak{J}} \rangle \ltimes U_{\mathfrak{J}}$ and there exists a unique $j \in \mathbb{N}$ and $B \in U_{\mathfrak{J}}$ such that $\varpi_F^{\lfloor \frac{n}{p} \rfloor + 1} \beta = \Pi_{\mathfrak{J}}^j B$. We want to show that $0 < j < p$. By the definition

$$j = \nu_{\mathfrak{J}}(\varpi_F^{\lfloor \frac{n}{p} \rfloor + 1} \beta) = -n + p(\lfloor \frac{n}{p} \rfloor + 1).$$

The element β is minimal over F and $n = -\nu_E(\beta)$ is coprime with p. Therefore $0 < j = p(\lfloor \frac{n}{p} \rfloor + 1) - n < p$.

For the opposite direction take a stratum $[\mathfrak{M}, n, n-1, \beta]$ and assume that the characteristic polynomial of $\varpi_F^n \beta$ is irreducible modulo $\mathfrak{p}_F$ and $\nu_{\mathfrak{M}}(\beta) = -n$. We want to show that the stratum $[\mathfrak{M}, n, n-1, \beta]$ is simple. E is a field because the minimal polynomial of $\varpi_F^n \beta$ is irreducible. We show that $\beta \in \mathfrak{K}(\mathfrak{M})$. By Lemma 2.2.8, $\mathfrak{K}(\mathfrak{M}) = \mathrm{GL}_p(\mathcal{O}_F) \rtimes \langle \varpi_F \mathrm{Id}_{p \times p} \rangle$. Denote the characteristic polynomial of $\varpi_F^n \beta$ by f. Since f is irreducible modulo $\mathfrak{p}_F$ the element $f(0) = \det(\varpi_F^n \beta)$ does not belong to $\mathfrak{p}_F$. By the assumption $\beta \in \mathfrak{P}^{-n}$ and $\varpi_F^n \beta \in \mathfrak{M}$. Therefore $\varpi_F^n \beta \in \mathrm{GL}_p(\mathcal{O}_F)$ and in particular $\beta \in \mathfrak{K}(\mathfrak{M})$. By the assumption, $\varpi_{\mathfrak{M}}(\beta) = -n$ and since $\beta \in \mathfrak{K}(\mathfrak{M})$ we have $\beta \mathfrak{M} = \mathfrak{P}_{\mathfrak{M}}^{-n}$. We want to show that β is minimal over F. The element $\varpi_F^n \beta + \mathfrak{p}_E$ generates the extension of the residues fields and the extension is of degree p. Therefore $f(E/F) = p$, $e(E/F) = 1$ and the first condition from the definition of a minimal element is satisfied. Compute $\nu_E(\beta) = \frac{e(E/F)}{[E:F]} \nu_F(\det(\beta)) = n$. Since the characteristic polynomial of $\varpi_F^n \beta = \varpi_F^{-\nu_E(\beta)} \beta^{e(E/F)}$ is irreducible modulo $\mathfrak{p}_F$, $\varpi_F^{-\nu_E(\beta)} \beta^{e(E/F)} + \mathfrak{p}_E$ generates the field extension k_E/k_F.

Finally consider a stratum $[\mathfrak{J}, n, n-1, \beta]$ with $n = -\nu_{\mathfrak{J}}(\beta)$ and β of the form $\varpi_F^{-\lfloor \frac{n}{p} \rfloor - 1} \Pi_{\mathfrak{J}}^j B$ where $0 < j < p$, $B \in U_{\mathfrak{J}}$. We want to prove that the stratum $[\mathfrak{J}, n, n-1, \beta]$ is simple. First we prove that $E = F[\beta]$ is a field. Denote by f the characteristic polynomial of $\Pi_{\mathfrak{J}}^j B$. If $j = 1$ then $f(x) = x^p$ modulo $\mathfrak{p}_F$ and $f(0) = \det(\Pi_{\mathfrak{J}}^j B) = u\varpi_F$ for some $u \in \mathcal{O}_F$. We deduce that $f(x)$ is Eisenstein and therefore it is irreducible. In particular, E is a field. Consider now the case when j is an arbitrary integer number $0 < j < p$. Since j is coprime with p, there exists $m_1, m_2 \in \mathbb{Z}$ such that $m_1 j + m_2 p = 1$ and $\varpi_F^{m_2} (\Pi_{\mathfrak{J}}^j B)^{m_1} = \Pi_{\mathfrak{J}} B_1$ for some $B_1 \in U_{\mathfrak{J}}$. Since $\Pi_{\mathfrak{J}} B_1$ generates a field extension of degree p this means that also $E = F[\Pi_{\mathfrak{J}}^j B]$ is a field. By definition $\beta \in \mathfrak{K}(\mathfrak{J})$, $\nu_{\mathfrak{J}}(\beta) = -n$ and $\beta \mathfrak{J} = \mathfrak{P}_{\mathfrak{J}}^n$. The characteristic polynomial $f(x)$ of $\varpi_F^{\lfloor \frac{n}{p} \rfloor + 1} \beta$ is equal to x^p modulo $\mathfrak{p}_F$. Therefore the extension k_E/k_F is trivial and to check that β is minimal it is enough to check that $\nu_E(\beta)$ is coprime with $e(E/F) = p$. By the assumption $\nu_E(\beta) = \frac{e(E/F)}{[E:F]} \nu_F(\det(\beta)) = -p(\lfloor \frac{n}{p} \rfloor + 1) + j = \nu_{\mathfrak{J}}(\beta) = -n$. If p would divide n, then $\nu_E(\beta) = -n - p + j = -n$ and $j = p$ which is impossible. Therefore β is minimal over F. $\square$

2.2.4 Cuspidal representations of G

In this subsection we recall the classification of irreducible cuspidal representations of $G = \mathrm{GL}_p(F)$. The classification originates in Carayol's work ([14]). We will follow [27] and [8]. The goal of this subsection is to recall the proof of the following theorem:

Theorem 2.2.17. *Let π be an irreducible cuspidal representation of G. Then there exists a character χ of $F^\times$ such that $\chi\pi$is of one of the following form:*

1. $\text{c-Ind}_{KZ}^G \Lambda$ *with Λ such that $\Lambda\mid_K$ is inflated from some irreducible cuspidal representation of* $\mathrm{GL}_p(k_F)$*,*
2. $l(\pi) > 0$ *and π contains a simple stratum $[\mathfrak{A}, n, n-1, \beta_1]$ with $n \geqslant 1$ and $\mathfrak{A}$ principal such that there exists a stratum $[\mathfrak{A}, n, n-1, \beta]$ equivalent to $[\mathfrak{A}, n, n-1, \beta_1]$ such that $\pi \cong \text{c-Ind}_J^G \Lambda$ where $J = F[\beta]^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ and Λ restricted to $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$ contains ψ_β*

Moreover every representation π satisfying one of the above is cuspidal.

Remark 2.2.18. *If π is an irreducible cuspidal representation with the minimal normalized level among all its one-dimensional twist $\pi \otimes \chi$ then π satisfies 1. or 2. from Theorem 2.2.17.*

Remark 2.2.19. *If an irreducible representation of G contains some stratum then it contains all strata G-conjugate to it. Therefore in Theorem 2.2.17 we can assume that $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{I}$.*

Before the proof of Theorem 2.2.17 we state some lemmas.

Lemma 2.2.20. *(see* [9, 2.4.11]*) Let $\mathfrak{A}$ be a hereditary order and let $n \in \mathbb{N}$, $n \geqslant 1$. Then any character of $U_{\mathfrak{A}}^n$ which factors through a determinant is of the form ψ_β where β is a scalar matrix.*

Lemma 2.2.21. *Let $\mathfrak{A}$ be a principal hereditary order. Let π be an irreducible cuspidal representation of* $\mathrm{GL}_p(F)$ *which contains a simple stratum $[\mathfrak{A}, n, n-1, \beta_1]$. Then the following conditions are equivalent:*

1. *there exists a stratum $[\mathfrak{A}, n, n-1, \beta]$ equivalent to $[\mathfrak{A}, n, n-1, \beta_1]$ such that $[F[\beta] : F] = 1$*
2. *there exists a character χ of $F^\times$ such that $l(\chi\pi) < l(\pi)$*

Proof of Lemma 2.2.21. First we assume that there exists a stratum $[\mathfrak{A}, n, n-1, \beta]$ equivalent to $[\mathfrak{A}, n, n-1, \beta_1]$ with $[F[\beta] : F] = 1$. We want to show that there exists a character χ of $F^\times$ such that $l(\chi\pi) < l(\pi)$. Assume $\beta =: b\mathrm{Id}_p$ is a scalar matrix. By definition $\beta\mathfrak{A} = \mathfrak{P}_{\mathfrak{A}}$ so $e(\mathfrak{A})$ divides n. Using β we define a character χ_1 of $(1 + \mathfrak{p}_F^{\frac{n}{e(\mathfrak{A})}})/(1 + \mathfrak{p}_F^{\frac{n}{e(\mathfrak{A})}+1})$:

$$\chi_1(1 + x) := \psi(bx).$$

The determinant map induce the homomorphism:

$$U^n_{\mathfrak{A}}/U^{n+1}_{\mathfrak{A}} \to (1+\mathfrak{p}_F^{\frac{n}{e(\mathfrak{A})}})/(1+\mathfrak{p}_F^{\frac{n}{e(\mathfrak{A})}-1}). \tag{2.2.1}$$

Now we will show that $\chi_1 \circ \det$ coincides with a character ψ_α of $U^n_{\mathfrak{A}}/U^{n+1}_{\mathfrak{A}}$. For this see both ψ_β and $\chi_1 \circ \det$ as characters of $U^n_{\mathfrak{A}}$. Let $x \in \mathfrak{P}^n_{\mathfrak{A}}$. By Leibniz formula $\det(1+x) = 1+\mathrm{tr}x+y$ for some $y \in \mathfrak{p}_F^{n+1}$. We have $\beta y \in \mathfrak{p}_F$ so

$$\chi_1 \circ \det(1+x) = \chi_1(1+\mathrm{tr}x+y) = \psi(a(\mathrm{tr}x+y)) = \psi(\mathrm{tr}(\beta x)) = \psi_\beta(1+x).$$

Denote by χ_2 an extension of χ_1 to $F^\times$. Define $\chi(1+x) := \chi_2(1+x)^{-1}$. Then χ is a character which satisfies the desired property.

For the converse assume that there exists a character χ of $F^\times$ such that $l(\chi\pi) < l(\pi)$. We want to prove that there exists a stratum $[\mathfrak{A}, n, n-1, \beta]$ equivalent to $[\mathfrak{A}, n, n-1, \beta_1]$ such that β is a scalar matrix. Denote $\pi_1 := \chi\pi$. Denote by χ^{-1} the character of $F^\times$ such that $\chi^{-1}(x) = \chi(x)^{-1}$ for every $x \in F^\times$.

The representation π_1 is irreducible and cuspidal. By Proposition 2.2.23, if $l(\pi_1) > 0$ then π_1 contains a simple stratum $[\mathfrak{A}_1, n_1, n_1-1, \gamma]$ with $\mathfrak{A}_1$ principal. By the assumption $l(\pi_1) < l(\chi^{-1}\pi_1)$ so $\chi^{-1} \circ \det \otimes \psi_\gamma \mid_{U^{n_1+1}_{\mathfrak{A}_1}} \neq 1$. If $l(\pi_1) = 0$ then π_1 contains the trivial character of $\mathfrak{A}_1^{n_1}$ with $\mathfrak{A}_1 = \mathfrak{M}$ and $n_1 = 0$. Therefore in both cases there exists $m \geqslant n_1+1$ such that

$$\chi^{-1} \circ \det \mid_{U^m_{\mathfrak{A}_1}} \neq 1 \quad \text{and} \quad \chi^{-1} \circ \det \mid_{U^{m+1}_{\mathfrak{A}_1}} = 1. \tag{2.2.2}$$

We can write $\chi^{-1} \circ \det \mid_{U^m_{\mathfrak{A}_1}}$ as ψ_{β_2} for some $\beta_2 \in \mathfrak{P}^{-m}_{\mathfrak{A}_1}/\mathfrak{P}^{-m+1}_{\mathfrak{A}_1}$. By Lemma 2.2.20 we can take β_2 to be a scalar matrix.

To sum up we have proven that $\pi = \chi^{-1}\pi_1$ contains a stratum $[\mathfrak{A}_1, m, m-1, b_2\mathrm{Id}_p]$. It is a fundamental stratum. By Lemma 2.2.10, the stratum $[\mathfrak{A}_1, m, m-1, b_2\mathrm{Id}_p]$ intertwines with $[\mathfrak{A}, n, n-1, \beta_1]$. By [9, 2.6.1, 2.6.4] the stratum $[\mathfrak{A}, n, n-1, \beta_1]$ is equivalent to $[\mathfrak{A}_1, m, m-1, b_2\mathrm{Id}_p]$.

□

Lemma 2.2.22 (see Corollary 7.15 and 9.3, [8]). *Let π be an irreducible smooth representation of G which contains a simple stratum $[\mathfrak{A}, n, n-1, \beta]$ with $n \geqslant 1$ and $\mathfrak{A}$ principal. Assume $[F[\beta] : F] = \dim_F(V)$. Then π is cuspidal and $\pi \cong \text{c-Ind}_J^G \Lambda_0$ where $J = F[\beta]^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$. Moreover there exists a simple stratum $[\mathfrak{A}, n, n-1, \beta']$ equivalent to $[\mathfrak{A}, n, n-1, \beta]$ and such that $\Lambda_0 \mid_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor+1}}$ contains $\psi_{\beta'}$.*

Proof. By the assumption π contains a character ψ_β of $U^n_{\mathfrak{A}}$. There exists an extension $\psi_{\beta'}$ of ψ_β which is also contained in π. We have $\beta' \equiv \beta \pmod{\mathfrak{P}^{1-n}_{\mathfrak{A}}}$. The matrix β is of the form (1) or (2) from Proposition 2.2.16 and such that $[F[\beta] : F] \neq 1$ so $F[\beta']$ is a field, β' is minimal over F and $\beta' \in \mathfrak{K}(\mathfrak{A})$. By [9, 1.5.8 Theorem] the G-intertwining of $\psi_{\beta'}$

is $J := F[\beta']^\times U_{\mathfrak{J}}^{\lfloor \frac{n+1}{2} \rfloor}$. Since J is compact modulo Z there exists an irreducible smooth representation Λ of J which is contained in π and which contains ψ_β when restricted to $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor+1}$. By [10, Theorem 11.4 and Remark 1], c-$\mathrm{Ind}_J^G \Lambda$ is irreducible and cuspidal. By Frobenius reciprocity $\pi \cong$ c-$\mathrm{Ind}_J^G \Lambda$. □

Proposition 2.2.23. [27, Theorem 3.2] *Let π be an irreducible cuspidal representation of G. Then $l(\pi) = 0$ or π contains a simple stratum $[\mathfrak{A}, n, n-1, \beta]$ with $\mathfrak{A}$ principal.*

Proof of Theorem 2.2.17. An irreducible smooth representation of G whose normalized level is 0 is cuspidal if and only if it is of the form (1) from ([9, Theorem 8.4.1]). Therefore we restrict our consideration to representations of G with normalized level strictly greater than 0.

Take an irreducible cuspidal representation π of G with $l(\pi) > 0$. We want to show that there exists a character χ of $F^\times$ such that $\chi\pi$ is of the form (1) or (2) from Theorem 2.2.17. Assume that for any character χ of $F^\times$ we have $l(\pi) \leqslant l(\chi\pi)$. By Proposition 2.2.23, π contains a simple stratum $[\mathfrak{A}, n, n-1, \beta]$ with $\mathfrak{A}$ principal. By Lemma 2.2.21 and Lemma 2.2.22, π is of the form as in (2).

Moreover if π of the form (2) then by Lemma 2.2.22 it is cuspidal and a one-dimensional twist of an irreducible cuspidal representation of G is irreducible cuspidal. □

2.2.5 Irreducible representations of $\mathrm{GL}_p(\mathcal{O}_F)$ in terms of orbits

Let ρ be an irreducible smooth representation of $K = \mathrm{GL}_p(\mathcal{O}_F)$ with conductor $r > 1$. In this subsection we adjust the notation from a description of representations of K as in subsection 2.1.2 to be more consistent with the notation from [10].

Denote $l = [\frac{r+1}{2}]$ and $l' = r - l$. As in subsection 2.1.2 by Clifford's theorem

$$\bar{\rho} \mid_{K^l} = m \bigoplus_{\bar{\alpha_1} \sim \bar{\alpha_0}} \bar{\psi}_{\bar{\alpha_1}} \tag{2.2.3}$$

for some matrix $\bar{\alpha_0} \in \mathrm{M}_p(\mathcal{O}_F/\mathfrak{p}_F^{l'})$, $m \in \mathbb{N}$ and $\bar{\psi}_{\bar{\alpha_1}} : K^l \to \mathbb{C}^\times$ defined in the following way $\bar{\psi}_{\bar{\alpha_1}}(1+x) = \psi(\varpi_F^{-r+1}\mathrm{tr}(\alpha_1\widehat{x}))$ for some lifts $\alpha_1, \widehat{x} \in \mathrm{M}_p(\mathcal{O}_F)$. The characters $\bar{\psi}_{\bar{\alpha_1}}$ do not depend on choices of lifts. In our case it will be more convenient to look at ρ as a representation of K not $\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^r)$. By (2.2.3) we can write

$$\rho \mid_{U_{\mathfrak{M}}^l} = m \bigoplus_{\bar{\alpha_1} \sim \bar{\alpha}_0} \varphi_{\varpi_F^{-r+1}\alpha_1} \tag{2.2.4}$$

where $\varphi_{\varpi_F^{-r+1}\alpha_1} : U_{\mathfrak{M}}^l \to \mathbb{C}^\times$ and $\varphi_{\varpi_F^{-r+1}\alpha_1}(1+x) = \psi(\varpi_F^{-r+1}\mathrm{tr}\alpha_1 x)$.
Note that if $\psi_{\varpi_F^{-r+1}\alpha_1}$ defined as in subsection 2.2.2 is a character of the group $U_{\mathfrak{M}}^l$ then $\varphi_{\varpi_F^{-r+1}\alpha_1} = \psi_{\varpi_F^{-r+1}\alpha_1}$. However if $\psi_{\varpi_F^{-r+1}\alpha_1}$ is a character of another group (for example $U_{\mathfrak{J}}^m$ with any $m \in \mathbb{N}$ and $m \geqslant 1$) the last equality does not hold. We introduce this

notation to underline the importance of the group on which a given character acts.
For the sake of simplicity characters in the decomposition (2.2.4) are indexed by matrices from $\mathrm{M}_p(\mathcal{O}_F)$ instead of matrices in $\mathrm{M}_p(\mathcal{O}_F/\mathfrak{p}_F^{l'})$ as in (2.2.3) however we are still taking sums over the conjugacy class in $\mathrm{M}_p(\mathcal{O}/\mathfrak{p}_F^{l'})$.
We say that a representation ρ contains a matrix α_1 in its orbit if it admits the decomposition of the form (2.2.4). We say that two orbits $\{\alpha_i\}_{i\in I}$ and $\{\beta_i\}_{i\in J}$ are equivalent if $\{\bar{\alpha}_i\}_{i\in I} = \{\bar{\beta}_i\}_{i\in J}$ where $\bar{a}$ denotes the image of an element $a \in \mathrm{M}_p(\mathcal{O}_F)$ in $\mathrm{M}_p(\mathcal{O}_F/\mathfrak{p}_F^{l'})$. Note that the notion of equivalence depends on r. From now on we consider orbits up to equivalence.

Remark 2.2.24. *By Clifford theory, if a representation ρ admits the decomposition (2.2.3) then it is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_{\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^r)}\bar{\psi}_{\bar{\alpha}_1}}^{\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^r)}\bar{\theta}$ for some $\bar{\theta}$ irreducible representation of $\mathrm{Stab}_{\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^r)}\bar{\psi}_{\bar{\alpha}_1}$ which contains $\bar{\psi}_{\bar{\alpha}_1}$. Therefore as a representation of K, ρ is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_K\bar{\psi}_{\bar{\alpha}_1}}^{K}\theta$ where θ is an inflation of $\bar{\theta}$ to $\mathrm{Stab}_K\bar{\psi}_{\bar{\alpha}_1}$.*

2.3 Cuspidal types on K in terms of orbits

In this section we give a description of orbits of cuspidal types. We show that if a representation is a cuspidal type on $K = \mathrm{GL}_p(\mathcal{O}_F)$ then it contains an orbit of a certain form. We also determine which orbits provide cuspidal types under condition that the conductor of a cuspidal type is at least 4. This in particular allows us to determine which cuspidal types on K with conductor at least 4 are regular representations.

2.3.1 Cuspidal types on $\mathrm{GL}_p(\mathcal{O}_F)$

The goal of this subsection is to prove the following theorem.

Theorem 2.3.1. *If λ is a cuspidal type on $K = \mathrm{GL}_p(\mathcal{O}_F)$, then it is a one-dimensional twist of one of the following:*

1. *a representation which is inflated from an irreducible cuspidal representation of $\mathrm{GL}_p(k_F)$;*
2. *a representation whose orbit contains a matrix whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$;*
3. *a representation whose orbit contains a matrix of the form $\Pi_{\mathfrak{J}}^j B$ where $0 < j < p$ and $B \in U_{\mathfrak{J}}$.*

Moreover if a representation is a one-dimensional twist of a representation of the form (3) *and has conductor at least* 4*, or is of the form* (1) *or* (2)*, then it is a cuspidal type.*

Remark 2.3.2. *Let π be an irreducible cuspidal representation of G. A type for $\mathfrak{I}(\pi)$ is regular if and only if $l(\pi) = m$ or $m - \frac{1}{p}$ for certain $m \in \mathbb{Z}$.*

Before the proof of Theorem 2.3.1 we state auxiliary lemmas.

Lemma 2.3.3. *Let $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{I}$ and let J be an open compact modulo Z subgroup of $\mathfrak{K}(\mathfrak{A})$. Denote by J° the maximal compact subgroup of J. Then $J^\circ = J \cap U_{\mathfrak{A}} = J \cap K$.*

Proof. By Lemma 2.2.8, $\mathfrak{K}(\mathfrak{A}) = \langle \Pi_{\mathfrak{A}} \rangle \ltimes U_{\mathfrak{A}}$. Since any non-trivial subgroup of $\langle \Pi_{\mathfrak{A}} \rangle$ is not compact J° has to be contained in $U_{\mathfrak{A}}$ and $J^\circ \subseteq J \cap U_{\mathfrak{A}}$. The subgroup J is closed and $U_{\mathfrak{A}}$ is compact so $J \cap U_{\mathfrak{A}}$ is compact. Therefore $J^\circ = J \cap U_{\mathfrak{A}}$. Since J° is the unique maximal compact subgroup of J and $J \cap K$ is compact, $J \cap K \subseteq J^\circ$. We also have $U_{\mathfrak{A}} \subseteq K$ and $J^\circ = J \cap U_{\mathfrak{A}} \subseteq J \cap K$. □

Let H be a locally profinite group. Let π_1 and π_2 be representations of H. We write $\pi_1 \sim \pi_2$ if there exists $h \in H$ such that $\pi_1 = \pi_2^h$. The following is a variation on Clifford's theorem:

Proposition 2.3.4. *Let H be a locally profinite group, N a normal open compact subgroup of H and let (π_1, V_1) be an irreducible admissible smooth representation of H. Then*

$$\pi_1 |_N = m \bigoplus_{\rho_1 \sim \rho} \rho_1$$

for certain $m \in \mathbb{Z}$ and ρ an irreducible smooth representation of N.

Proof. Denote by $\widehat{N}$ the set of equivalence classes of irreducible smooth representations of N. Let $\rho \in \widehat{N}$. The ρ-isotypic component of V_1 is a sum of irreducible N-subspaces of V_1 of class ρ. We denote it by V_1^ρ. By [10, 2.3 Proposition]

$$V_1 = \bigoplus_{\rho \in \widehat{N}} V_1^\rho.$$

Fix some $\rho \in \widehat{N}$. Since π_1 is irreducible we have $V_1 = \sum_{g \in G} g V_1^\rho = \sum_{g \in G} g V_1^{\rho^g} = \bigoplus_{\rho_1 \sim \rho} V_1^{\rho_1}$. Since π_1 is admissible and $\ker(\rho)$ is open $V_1^\rho \subseteq V_1^{\ker(\rho)}$ is finite dimensional hence $V_1^\rho \cong m\rho$ for certain $m \in \mathbb{Z}$. Therefore $\pi_1 |_N = m \bigoplus_{\rho_1 \sim \rho} \rho_1$. □

Remark 2.3.5. *If $H \subseteq \mathrm{GL}_p(F)$ is compact modulo Z then by* [19, Theorem 2.1] *every irreducible smooth representation of H is finite dimensional and hence admissible.*

Lemma 2.3.6. *Let U be a compact open subgroup of K and let π be an irreducible cuspidal representation of G. Let ρ' be a cuspidal type on U for $\mathfrak{I}(\pi)$ and let ρ be an irreducible smooth representation of K which contains ρ'. Moreover assume that ρ is contained in $\pi |_K$. Then ρ is a cuspidal type on K for $\mathfrak{I}(\pi)$.*

Proof. Take an irreducible smooth representation π_1 of G. We want to show that $\pi_1 |_K$ contains ρ if and only if $\mathfrak{I}(\pi_1) = \mathfrak{I}(\pi)$. If $\pi_1 |_K$ contains ρ then it also contains ρ' and by the assumption $\mathfrak{I}(\pi) = \mathfrak{I}(\pi_1)$. For the reverse implication assume that $\mathfrak{I}(\pi_1) = \mathfrak{I}(\pi)$. By the definition $\pi_1 \cong \pi \otimes \chi \circ \det$ for certain unramified character χ of $F^\times$ so $\pi_1 |_K \cong \pi |_K$. Therefore π_1 contains ρ. □

Lemma 2.3.7. *Consider a stratum $[\mathfrak{A}, n, n-1, \alpha]$ with $\mathfrak{A}$ principal, $n \geqslant 1$. Let $J = F[\alpha]^\times U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ and let π be an irreducible cuspidal representation such that $\pi \cong \text{c-Ind}_J^G \Lambda$ with Λ such that $\Lambda |_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor +1}}$ contains ψ_α. Denote by J° the maximal compact subgroup of J. Then $\Lambda |_{J^\circ}$ is irreducible.*

Proof. Let $\Lambda |_{J^\circ} = \sum_{i \in I} \lambda_i$ for a certain set I and irreducible representations λ_i of J°. First we show that for any $i, l \in I$ we have $\lambda_i \cong \lambda_l$. Take $i, l \in I$. By Proposition 2.2.2 the representations λ_i and λ_l are cuspidal types on J° for $\mathfrak{I}(\pi)$. By Lemma 2.3.6 irreducible components of $\text{c-Ind}_{J^\circ}^K \lambda_i$ and $\text{c-Ind}_{J^\circ}^K \lambda_l$ are cuspidal types on K for $\mathfrak{I}(\pi)$. By Theorem 2.2.1 a cuspidal type on K for $\mathfrak{I}(\pi)$ is unique and appears in π with multiplicity one so $\text{c-Ind}_{J^\circ}^K \lambda_i$ and $\text{c-Ind}_{J^\circ}^K \lambda_l$ are irreducible and $\text{c-Ind}_{J^\circ}^K \lambda_i \cong \text{c-Ind}_{J^\circ}^K \lambda_l$. By Frobenius reciprocity and Mackey's formula this implies that there exists $k \in K$ which intertwines λ_l with λ_i, i.e.

$$\mathrm{Hom}_{J^\circ \cap (J^\circ)^k} \left(\lambda_l^k |_{J^\circ \cap (J^\circ)^k}, \lambda_i |_{J^\circ \cap (J^\circ)^k} \right) \neq 0. \tag{2.3.1}$$

On the other hand we can apply Proposition 2.3.4 and Remark 2.3.5 to the representation Λ. The group J° is the maximal compact subgroup of J so after J-conjugation it remains the maximal compact subgroup of J. Therefore J° is a normal subgroup of J. By Proposition 2.3.4 there exists $j \in J$ such that $\lambda_l \cong \lambda_i^j$. Together with (2.3.1) this implies

$$\mathrm{Hom}_{(J^\circ)^k \cap J^\circ} \left(\lambda_i^{jk} |_{(J^\circ)^k \cap J^\circ}, \lambda_i |_{(J^\circ)^k \cap J^\circ} \right) \neq 0.$$

Since J° is normal in J we have $(J^\circ)^{jk} = (J^\circ)^k$ and jk intertwines λ_i. In particular, jk intertwines Λ. An element from G intertwines Λ with itself if and only if it belongs to J so $jk \in J$ as otherwise $\text{c-Ind}_J^G \Lambda$ would not be irreducible (see [10, 11.4 Theorem and 11.4 Remark 1,2]). This means $k \in K \cap J = J^\circ$ and by (2.3.1) $\lambda_i \cong \lambda_l$.

We proved $\Lambda |_{J^\circ} = m\lambda_i$ for some $m \in \mathbb{Z}$. By Mackey formula $\pi |_K$ contains $\text{c-Ind}_{J^\circ}^K(\Lambda |_{J^\circ}) = m \,\text{c-Ind}_{J^\circ}^K \lambda_i$. The representation $\text{c-Ind}_{J^\circ}^K \lambda_i$ is a cuspidal type for $\mathfrak{I}(\pi)$ on K. By Theorem 2.2.1 $m = 1$ so $\Lambda |_{J^\circ} = \lambda_i$ is irreducible.

□

By the above lemma and by [30] we know the following.

Lemma 2.3.8. *(see* [30, Proposition 3.1]*) Let π and Λ be as in Lemma 2.3.7. Then $\text{c-Ind}_{J^\circ}^K(\Lambda |_{J^\circ})$ is a cuspidal type on K for $\mathfrak{I}(\pi)$.*

Proof of Theorem 2.3.1. First we prove that if a representation is a cuspidal type on K then it is of the form (1), (2) or (3) from Theorem 2.3.1. Let λ be a cuspidal type on K for $\mathfrak{I}(\pi)$ with some irreducible cuspidal representation π of G. Let χ be a one-dimensional character of $F^{\times}$. Since λ is a cuspidal type on K if and only if $\chi\lambda$ is a cuspidal type on K, by Theorem 2.2.17, we can assume that either $l(\pi) = 0$ or $l(\pi) > 0$ and it contains a simple stratum $[\mathfrak{A}, n, n-1, \alpha]$ with $n > 0$, $\mathfrak{A}$ principal and such that

$$\pi \cong \text{c-Ind}_J^G \Lambda, \tag{2.3.2}$$

where $J = F[\alpha]^{\times} U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ and $\Lambda \mid_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}} = m\psi_\beta$ for some $m \in \mathbb{N}$ and ψ_β some extension of ψ_α to $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$. We have $\beta \equiv \alpha \bmod \mathfrak{P}_{\mathfrak{A}}^{1-n}$ and the stratum $[\mathfrak{A}, n, n-1, \alpha]$ is equivalent to $[\mathfrak{A}, n, n-1, \beta]$. Therefore without lose of generality we can take $\beta = \alpha$ and consider ψ_α as a character of $U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}$. By Remark 2.2.19 we can assume $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{A} = \mathfrak{I}$.

The subgroup J is open, contains and is compact modulo Z. Since α is minimal, $J \subseteq \mathfrak{K}(\mathfrak{A})$.

Now we consider two cases depending on the level of π.

Case 1 Assume $l(\pi) = 0$. By Theorem 2.2.17, there exists Λ a representation of ZK which is an extension of an inflation of some irreducible cuspidal representation of $\mathrm{GL}_p(k_F)$ such that $\pi \simeq \text{c-Ind}_{ZK}^G \Lambda$. The group K is the maximal compact subgroup of G which is contained in ZK and by definition $\Lambda \mid_{J \cap K}$ is irreducible. By Proposition 2.2.2, $\Lambda \mid_K$ is a cuspidal type on K for π. By Paskunas' unicity theorem ([30], Theorem 1.3), $\lambda \cong \Lambda \mid_K$. Therefore λ is inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(k_F)$ and λ is of the form 1 from Theorem 2.3.1.

Case 2 Assume $l(\pi) > 0$. By Theorem 2.2.17, π contains a simple stratum $[\mathfrak{A}, n, n-1, \alpha]$ with $n \geqslant 1$, $\mathfrak{A}$ a principal order and such that $\pi \simeq \text{c-Ind}_J^G \Lambda$ where $J = F[\alpha]^{\times} U_{\mathfrak{A}}^{\lfloor \frac{n+1}{2} \rfloor}$ and $\Lambda \mid_{U_{\mathfrak{A}}^{\lfloor \frac{n}{2} \rfloor + 1}}$ contains ψ_α.
The proof in this case will contain two steps. In the first step we will show that $\lambda \mid_{U_{\mathfrak{A}}^{[\frac{n}{2}]+1}}$ contains ψ_α. In the second we will compute the conductor r of λ in terms of n and we will show that the orbit of λ contains the matrix $\varpi_F^{r-1}\alpha$ whose image in $\mathrm{M}_p(\mathcal{O}_F/\mathfrak{p}_F^{l'})$ satisfies the properties from the statement of Theorem 2.3.1.

Step 1: By Lemma 2.3.7, $\Lambda \mid_{J^{\circ}}$ is irreducible and by Proposition 2.2.2, $\Lambda \mid_{J^{\circ}}$ is a cuspidal type on J° for π. By Lemma 2.3.8, $\mathrm{Ind}_{J^{\circ}}^K(\Lambda \mid_{J^{\circ}})$ is a cuspidal type on K for π and by Theorem 2.2.1, $\mathrm{Ind}_{J^{\circ}}^K(\Lambda \mid_{J^{\circ}}) \cong \lambda$. By Mackey formula $\lambda \mid_{U_{\mathfrak{A}}^{[\frac{n}{2}]+1}}$ contains ψ_α.
Step 2 : We will consider two subcases depending on the choice of a hereditary order $\mathfrak{A}$.
Subcase 1 Assume $\mathfrak{A} = \mathfrak{M}$. The representation λ when restricted to $U_{\mathfrak{M}}^{n+1}$ contains $\psi_\alpha \mid_{U_{\mathfrak{A}}^{n+1}} = 1_{U_{\mathfrak{A}}^{n+1}}$. Since $U_{\mathfrak{M}}^{n+1}$ is an open normal subgroup of K and K is compact, $\lambda \mid_{U_{\mathfrak{M}}^{n+1}}$ is a direct sum of irreducible representations and each of them is conjugated to the trivial character. This means $\lambda \mid_{U_{\mathfrak{M}}^{n+1}}$ is trivial and λ factors through $\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^{n+1})$. Since $\nu_{\mathfrak{A}}(\alpha) = -n$, the character ψ_α as a character of $U_{\mathfrak{M}}^n$ is non-trivial and λ does not factor

through $\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^n)$. Therefore λ has conductor $r = n+1$.

Denote as before $l = [\frac{r+1}{2}]$ and let $\alpha_0 := \varpi_F^{r-1}\alpha$. By Step 1, $\lambda\mid_{U_{\mathfrak{M}}^l}$ contains $\psi_\alpha = \varphi_{\varpi_F^{-r+1}\alpha_0} = \varphi_{\varpi_F^{-n}\alpha_0}$. Therefore α_0 is contained in the orbit of λ. By Proposition 2.2.16, the characteristic polynomial of α_0 is irreducible mod $\mathfrak{p}_F$.

Subcase 2 Assume $\mathfrak{A} = \mathfrak{I}$. We compute the conductor of λ in terms of n. The Step 1 provides some information about restrictions of λ to subgroups $U_{\mathfrak{I}}^i$ for certain i. However to compute the conductor we need information about the restricitions to subgroups $U_{\mathfrak{M}}^j$ for certain j. To switch between these two classes of subgroups we will use the following inclusion:

$$U_{\mathfrak{I}}^{i+pm} \supseteq U_{\mathfrak{M}}^{2+m} \qquad \text{for all} \quad 1 < i \leqslant p+1 \tag{2.3.3}$$

By Step 1, $\lambda\mid_{U_{\mathfrak{I}}^{n+1}}$ contains the trivial character and by (2.3.3) $\lambda\mid_{U_{\mathfrak{M}}^{\lfloor\frac{n}{p}\rfloor+2}}$ contains the trivial character. By similar argument as in Subcase 1 this shows that λ factors through $\mathrm{GL}_p(\mathcal{O}_F/\mathfrak{p}_F^{\lfloor\frac{n}{p}\rfloor+2})$. We want to show that λ is of conductor $\lfloor\frac{n}{p}\rfloor+2$. By Step 1, $\lambda\mid_{U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}}$ contains ψ_α where $\alpha = \varpi_F^{-\lfloor\frac{n}{p}\rfloor-1}\Pi_{\mathfrak{I}}^jB$ for $0 < j := p(\lfloor\frac{n}{p}\rfloor+1)-n < p$ and $B \in U_{\mathfrak{I}}$. In particular $\lambda\mid_{U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}\cap U_{\mathfrak{M}}^{\lfloor\frac{n}{p}\rfloor+1}}$ contains $\psi_\alpha\mid_{U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}\cap U_{\mathfrak{M}}^{\lfloor\frac{n}{p}\rfloor+1}}$. Assume $\psi_\alpha\mid_{U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}\cap U_{\mathfrak{M}}^{\lfloor\frac{n}{p}\rfloor+1}}$ is trivial. Then $\alpha \in \mathfrak{P}_{\mathfrak{I}}^{-\lfloor\frac{n}{2}\rfloor} + \mathfrak{P}_{\mathfrak{M}}^{-\lfloor\frac{n}{p}\rfloor}$ and

$$B = \varpi_F^{\lfloor\frac{n}{p}\rfloor+1}\Pi_{\mathfrak{I}}^{-j}\alpha \in \Pi_{\mathfrak{I}}^{-j}\mathfrak{P}_{\mathfrak{M}} + \mathfrak{P}_{\mathfrak{I}}^{-\lfloor\frac{n}{2}\rfloor+p\lfloor\frac{n}{p}\rfloor+p-j} = \Pi_{\mathfrak{I}}^{p-j}\mathfrak{M} + \mathfrak{P}_{\mathfrak{I}}^{n-\lfloor\frac{n}{2}\rfloor} \subseteq \Pi_{\mathfrak{I}}\left(\mathfrak{M}+\mathfrak{I}\right).$$

Since $B \in U_{\mathfrak{I}}$ this gives a contradiction. Therefore $\psi_\alpha\mid_{U_{\mathfrak{M}}^{\lfloor\frac{n}{p}\rfloor+1}\cap U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}}$ is non-trivial. In particular $\lambda\mid_{U_{\mathfrak{M}}^{\lfloor\frac{n}{p}\rfloor+1}}$ is non-trivial and λ is of conductor $\lfloor\frac{n}{p}\rfloor+2$. Denote $r = \lfloor\frac{n}{p}\rfloor+2$ and as before $l = \lfloor\frac{r+1}{2}\rfloor$. Summing up we know that λ contains $\psi_\alpha = \psi_{\varpi_F^{-r+1}\Pi_{\mathfrak{I}}^jB}$ the character of $U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}$ but we would like to know that it contains ψ_β the character of $U_{\mathfrak{M}}^{\lfloor\frac{\lfloor\frac{n}{p}\rfloor+3}{2}\rfloor}$ where $\Pi_{\mathfrak{I}}^j\varpi_F^{\lfloor\frac{n}{p}\rfloor+1}\beta \in U_{\mathfrak{I}}$.

The sketch of the end of the proof of this case is as follows. Depending on n and p sometimes $U_{\mathfrak{M}}^l \subseteq U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}$ and then $\varpi_F^{r-1}\alpha$ is contained in the orbit of λ and we are done. Unfortunately this not always the case. If this inclusion does not hold then we can assume that λ contains in its orbit β such that $\psi_\beta\mid_{U_{\mathfrak{M}}^l\cap U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}} = \psi_\alpha\mid_{U_{\mathfrak{M}}^l\cap U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}}$ and from that we shall deduce that β is of the desired form.

First assume $n \equiv 1 \bmod p$. In this case we show $U_{\mathfrak{M}}^l \subseteq U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}$. If $n=1$ then $U_{\mathfrak{M}}^l = U_{\mathfrak{M}}^1 \subseteq U_{\mathfrak{I}}^1 = U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1}$. Therefore by (2.3.3) for any $n \in \mathbb{N}$, $n \geqslant 1$ we have

$$U_{\mathfrak{I}}^{\lfloor\frac{n}{2}\rfloor+1} \supseteq U_{\mathfrak{M}}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{p}\rfloor+2}. \tag{2.3.4}$$

We want to show that the index $\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{p}\rfloor+2$ is equal to l. Let $a \in \mathbb{N}$. If $n = 2ap+1$ then $\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{p}\rfloor+2 = a+1 = \lfloor\frac{\lfloor\frac{n}{p}\rfloor+3}{2}\rfloor$. On the other hand if $n = (2a+1)p+1$ then

$\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{p}\rfloor+2=\lfloor\frac{ap+\lfloor\frac{p+1}{2}\rfloor-1}{p}\rfloor+2=a+2=\lfloor\frac{\lfloor\frac{n}{p}\rfloor+3}{2}\rfloor=l$. Therefore by (2.3.4) we have $U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}\supseteq U_{\mathfrak{M}}^{l}$. Since we are considering n coprime with p in particular this ends the proof in the case $p=2$.

Assume now $p\neq 2$ and $n=b+2ap$ for $a,b\in\mathbb{N}$ such that $p+2\leqslant b\leqslant 2p-1$. In this case we also want to show $U_{\mathfrak{M}}^{l}\subseteq U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}$. Again $U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}\supseteq U_{\mathfrak{M}}^{\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{p}\rfloor+2}$ and $\lfloor\frac{\lfloor\frac{n}{2}\rfloor-1}{p}\rfloor+2=\lfloor\frac{ap+\lfloor\frac{b}{2}\rfloor-1}{p}\rfloor+2=a+2=\lfloor\frac{\lfloor\frac{n}{p}\rfloor+3}{2}\rfloor=l$.

It remains to consider the case $p\neq 2$ and $n=b+2pa$ for $a,b\in\mathbb{N}$ and $2\leqslant b\leqslant p-1$. Unfortunately in this case $U_{\mathfrak{M}}^{l}$ is not always contained in $U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}$. However we can study the behaviour of λ on $U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}$ and then deduce the desired result. Take β from the orbit of λ. We want to show that we can pick β such that $\Pi_{\mathfrak{J}}^{n}\beta\in U_{\mathfrak{J}}$. Since $\lambda\mid_{U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}}$ contains ψ_α we know that $\lambda\mid_{U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}}$ contains $\psi_\alpha\mid_{U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}}$. On the other hand by the definition of the orbit of λ

$$\lambda\mid_{U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}}=m_\lambda\bigoplus_{\bar\beta'\sim\bar\beta}\psi_{\beta'}\mid_{U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}} \tag{2.3.5}$$

for some $m_\lambda\in\mathbb{N}$. Therefore we can assume

$$\psi_\beta\mid_{U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}}=\psi_\alpha\mid_{U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}}. \tag{2.3.6}$$

We compute now the intersection $U_{\mathfrak{M}}^{l}\cap U_{\mathfrak{J}}^{\lfloor\frac{n}{2}\rfloor+1}$. Define C to be the matrix with 1's on the antidiagonal and $0's$ elsewhere:

$$C=\begin{pmatrix}0&\cdots&0&1\\0&\cdots&1&0\\\vdots&&&\vdots\\1&0&\cdots&0\end{pmatrix}$$

For $h\in\mathbb{N}$, $1\leqslant h\leqslant 2p-1$ define $B_h:=CB'_h$ where $B'_h=(b_{s,t})_{1\leqslant s,t\leqslant p}$ is such that $b_{s,t}=\mathfrak{p}_F^2$ if $s+t\leqslant h+1$ and $\mathfrak{p}_F$ otherwise. In other words for $h<2p-1$

$$B_h=\begin{pmatrix}\mathfrak{p}_F&\cdots&\cdots&\cdots&\cdots&\mathfrak{p}_F\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\\mathfrak{p}_F&\ddots&\ddots&\ddots&\ddots&\vdots\\\mathfrak{p}_F^2&\ddots&\ddots&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\\mathfrak{p}_F^2&\cdots&\mathfrak{p}_F^2&\mathfrak{p}_F&\cdots&\mathfrak{p}_F\end{pmatrix}\text{ or }B_h=\begin{pmatrix}\mathfrak{p}_F^2&\cdots&\mathfrak{p}_F^2&\mathfrak{p}_F&\cdots&\mathfrak{p}_F\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\\vdots&\ddots&\ddots&\ddots&\ddots&\mathfrak{p}_F\\\vdots&\ddots&\ddots&\ddots&\ddots&\mathfrak{p}_F^2\\\vdots&\ddots&\ddots&\ddots&\ddots&\vdots\\\mathfrak{p}_F^2&\cdots&\cdots&\cdots&\cdots&\mathfrak{p}_F^2\end{pmatrix}$$

where first h diagonals (counting from the bottom left corner) have entries $\mathfrak{p}_F^2$ and the rest have entries $\mathfrak{p}_F$.

Compute

$$\begin{aligned} U^l_{\mathfrak{M}} \cap U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathfrak{J}} =& U^{\lfloor \frac{\lfloor \frac{n}{p} \rfloor+3}{2} \rfloor}_{\mathfrak{M}} \cap U^{\lfloor \frac{n}{2} \rfloor+1}_{\mathfrak{J}} = U^{a+1}_{\mathfrak{M}} \cap U^{pa+\lfloor \frac{b}{2} \rfloor+1}_{\mathfrak{J}} \\ =& 1+\varpi_F^a(\mathfrak{P}_{\mathfrak{M}} \cap \mathfrak{P}^{\lfloor \frac{b}{2} \rfloor+1}_{\mathfrak{J}}) = 1+\varpi_F^a B_{\lfloor \frac{b}{2} \rfloor}. \end{aligned}$$

Combining this with the equality (2.3.6) we get $\mathrm{tr}((\beta-\alpha)\varpi_F^a B_{\lfloor \frac{b}{2} \rfloor}) \subseteq \mathfrak{p}_F$.Therefore $\beta \in \alpha + {\varpi_F^{-}}^a \varpi_F^{-2} B_{2p-1-\lfloor \frac{b}{2} \rfloor}$. Then

$$\Pi^n_{\mathfrak{J}}\beta \in \Pi^n_{\mathfrak{J}}\alpha + \Pi^n_{\mathfrak{J}}\varpi_F^{-a-2} B_{2p-1-\lfloor \frac{b}{2} \rfloor} \subseteq U_{\mathfrak{J}} + \Pi^b_{\mathfrak{J}}\varpi_F^{2a-a-2} B_{2p-1-\lfloor \frac{b}{2} \rfloor} \tag{2.3.7}$$

$$\subseteq \begin{cases} U_{\mathfrak{J}} + \varpi_F^{-2}\Pi^b_{\mathfrak{J}} B_{2p-1-\lfloor \frac{b}{2} \rfloor} & \text{if} \quad a=0 \\ U_{\mathfrak{J}} + \mathfrak{P}_{\mathfrak{J}} \subseteq U_{\mathfrak{J}} & \text{otherwise.} \end{cases} \tag{2.3.8}$$

Therefore if $a \geqslant 1$ then

$$\Pi^n_{\mathfrak{J}}\beta \in U_{\mathfrak{J}}.$$

Assume now that $a=0$. Then $n=b$, $r=\lfloor \frac{b}{p} \rfloor + 2 = 2$ and $l=1$. The character ψ_β is a character of $U^1_{\mathfrak{M}}$ which is trivial on $U^2_{\mathfrak{M}}$. We show that there exists $\beta_1 \in \mathfrak{P}^{-n}_{\mathfrak{J}}$ such that $\beta-\beta_1 \in \mathrm{M}_p(\mathcal{O}_F)$ and $\Pi^n_{\mathfrak{J}}\beta_1 \in U_{\mathfrak{J}}$. If $\beta-\beta_1 \in \mathrm{M}_p(\mathcal{O}_F)$ then $(\beta-\beta_1)\mathfrak{P}^1_{\mathfrak{M}} \subseteq \mathfrak{P}^1_{\mathfrak{M}}$, $\psi(\mathrm{tr}((\beta-\beta_1)\mathfrak{P}^1_{\mathfrak{M}}))=0$ and $\psi_\beta=\psi_{\beta_1}$ as characters of $U^1_{\mathfrak{M}}$. Therefore to finish the proof it is enough to prove the existence of β_1. In other words we want to show $\beta \in {\Pi^{-}_{\mathfrak{J}}}^n U_{\mathfrak{J}} + \mathrm{M}_p(\mathcal{O}_F)$. By inclusions (2.3.7) and (2.3.8) and since $n=b$ we have

$$\beta \in {\Pi^{-}_{\mathfrak{J}}}^n U_{\mathfrak{J}} + \varpi_F^{-2}\Pi^{b-n}_{\mathfrak{J}} B_{2p-1-\lfloor \frac{b}{2} \rfloor} = {\Pi^{-}_{\mathfrak{J}}}^b U_{\mathfrak{J}} + \varpi_F^{-2} B_{2p-1-\lfloor \frac{b}{2} \rfloor}.$$

Combining that with $\varpi_F^{-2} B_{2p-\lfloor \frac{b}{2} \rfloor -1} = \Pi_{\mathfrak{J}}^{-\lfloor \frac{b}{2} \rfloor}\mathfrak{J} + \mathrm{M}_p(\mathcal{O}_F)$ we get $\beta \in {\Pi^{-}_{\mathfrak{J}}}^b U_{\mathfrak{J}} + \mathrm{M}_p(\mathcal{O}_F)$. This proves that if a representation is a cuspidal type then it is of the form (1) or (2) or (3) from the theorem.

Now we prove that a one-dimensional twist of a representation which is of the form 3 and has conductor at least 4 or of the form 1 or 2 is a cuspidal type. Since a one-dimensional twist of a cuspidal type is a cuspidal type we can consider given representations up to one-dimensional twists.

Case 1 Let ρ be an irreducible smooth representation of K which is inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(k_F)$. We can extend ρ to an irreducible representation of KZ. Denote this extension by Λ. By Theorem 2.2.17, $\pi = \text{c-Ind}^G_{KZ}\Lambda$ is an irreducible cuspidal representation of G. By Proposition 2.2.2, $\Lambda \mid_K$ is a cuspidal type on K for π which means that ρ is a cuspidal type on $K$for π.

Case 2 Assume ρ is an irreducible smooth representation of K either whose orbit contains a matrix with characteristic polynomial irreducible mod $\mathfrak{p}_F$ or with conductor at least 4 and an orbit containing a matrix of the form $\Pi_{\mathfrak{J}}^{j}B$ for $j \in \mathbb{N}$, $0 < j < p$ and $B \in U_{\mathfrak{J}}$. Denote the matrix with the given property by β_0 and by r the conductor of ρ. Let π be an irreducible smooth representation of G which contains ρ. First we will prove that π contains a simple stratum. We will divide the proof into two subcases depending on the property of β_0.

Subcase 1 Assume β_0 is a matrix with the characteristic polynomial irreducible mod $\mathfrak{p}_F$. Define $n = r - 1$ and $\beta = \varpi_F^{-n}\beta_0$. Since π contains the character ψ_β of $U_{\mathfrak{M}}^{\lfloor \frac{r+1}{2} \rfloor} \supseteq U_{\mathfrak{M}}^{n}$ the representation π contains the stratum $[\mathfrak{M}, n, n-1, \beta]$. By Proposition 2.2.16, the stratum is simple.

Subcase 2 Assume now that ρ has conductor $r > 3$ and has an orbit containing a matrix β_0 of the form $\Pi_{\mathfrak{J}}^{j}B$ for some $j \in \mathbb{N}$, $0 < j < p$ and $B \in U_{\mathfrak{J}}$. Denote $\beta = \varpi_F^{-r+1}\beta_0$. We have $\nu_{\mathfrak{J}}(\beta) = -p(r-1) + j$. Put $n := p(r-1) - j$. The stratum $[\mathfrak{J}, n, n-1, \beta]$ is simple. We have $U_{\mathfrak{J}}^{n} \subseteq 1 + \varpi_F^{r-2}\mathfrak{P}_{\mathfrak{J}}^{p-j} \subseteq U_{\mathfrak{M}}^{r-2}$. For $r > 3$ the last one is contained in $U_{\mathfrak{M}}^{\lfloor \frac{r+1}{2} \rfloor} = U_{\mathfrak{M}}^{l}$. Since π contains ψ_β which is a character of $U_{\mathfrak{M}}^{l} \supseteq U_{\mathfrak{J}}^{n}$, π contains a simple stratum $[\mathfrak{J}, n, n-1, \beta]$.

Therefore, we showed that in both subcases π contains some simple stratum, say $[\mathfrak{A}, n, n-1, \beta]$. Since our considerations are up to one dimensional twist we can assume that $l(\pi) \leqslant l(\chi\pi)$ for any character χ of $F^\times$. By Lemma 2.2.22 π is cuspidal.

Subcase 2a Assume now that π contains a simple stratum $[\mathfrak{M}, n, n-1, \beta]$.

Take π_1 to be an irreducible smooth representation of G such that $l(\pi_1) \leqslant l(\chi\pi_1)$ for any character χ of $F^\times$, π_1 contains ρ and such that $\pi_1 \cong \text{c-Ind}_J^G \Lambda$ with Λ such that $\Lambda \mid_{U_{\mathfrak{M}}^{\lfloor \frac{n}{2} \rfloor + 1}}$ contains ψ_β as a character of $U_{\mathfrak{M}}^{\lfloor \frac{n}{2} \rfloor + 1}$. By Lemma 2.3.8, to finish the proof it is is enough to show $\rho \cong \text{Ind}_{J \cap K}^{K} (\Lambda \mid_{J \cap K})$. Since ρ is contained in π_1, we have

$$\text{Hom}_K(\rho, (\text{c-Ind}_J^G \Lambda) \mid_K) \neq 0.$$

By Mackey formula and Frobenius reciprocity there exists $g \in J \setminus G / K$ such that

$$\text{Hom}_K(\rho, \text{c-Ind}_{J^g \cap K}^{K}(\Lambda^g \mid_{J^g \cap K})) \cong \text{Hom}_{J^g \cap K}(\rho \mid_{J^g \cap K}, \Lambda^g \mid_{J^g \cap K}) \neq 0 \tag{2.3.9}$$

where Λ^g denotes the representation $\Lambda^g(x) = \Lambda(gxg^{-1})$ for any $x \in J^g \cap K$. In particular, $\text{Hom}_{U_{\mathfrak{M}}^{\lfloor \frac{n}{2} \rfloor + 1} \cap (U_{\mathfrak{M}}^{\lfloor \frac{n}{2} \rfloor + 1})^g}(\rho, \Lambda^g) \neq 0$. Denote the subgroup $U_{\mathfrak{M}}^{\lfloor \frac{n}{2} \rfloor + 1}$ by H. By Proposition 2.3.4 and Remark 2.3.5, $\rho \mid_H$ is a multiple of a direct sum of one-dimensional representations and each of them is conjugate to ψ_β by an element of K. Therefore there exist $g_1 \in K$ such that

$$\text{Hom}_{H \cap H^g}(\psi_\beta^{g_1} \mid_{H \cap H^g}, \psi_\beta^{g} \mid_{H \cap H^g}) \neq 0.$$

Since $g_1 \in N_G(H) = U_{\mathfrak{M}} \rtimes \langle \Pi_{\mathfrak{M}} \rangle$ the previous is equivalent to

$$\text{Hom}_{H^{g_1} \cap (H^{g_1})^{g_1^{-1}g}}(\psi_{\beta^{g_1}} \mid_{H^{g_1} \cap (H^{g_1})^{g_1^{-1}g}}, (\psi_{\beta^{g_1}})^{g_1^{-1}g} \mid_{H^{g_1} \cap (H^{g_1})^{g_1^{-1}g}}) \neq 0$$

where $\beta^{g_1} = g_1^{-1}\beta g_1$. Therefore $g_1^{-1}g$ intertwines stratum $[\mathfrak{M}, n, n-1, \beta^{g_1}]$. Since $[\mathfrak{M}, n, n-1, \beta]$ is simple and $g_1 \in K$ the stratum $[\mathfrak{M}, n, n-1, \beta^{g_1}]$ is also simple. By [9, 1.5.8], $g_1^{-1}g \in g_1^{-1}Jg_1$ and therefore $g \in JK$. By (2.3.9), ρ is isomorphic to $\text{c-Ind}_{J\cap K}^{K}(\Lambda \mid_{J\cap K})$ and ρ is a cuspidal type on K for $\mathfrak{I}(\pi_1)$.

Subcase 2b Assume now that π contains a stratum $[\mathfrak{J}, n, n-1, \beta]$. The restriction $\rho \mid_{U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}}$ contains a character ψ_α such that

$$\psi_\beta \mid_{U_{\mathfrak{M}}^{l} \cap U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}} = \psi_\alpha \mid_{U_{\mathfrak{M}}^{l} \cap U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}} . \tag{2.3.10}$$

In a similar way as from (2.3.6) we deduce from (2.3.10) that α is of the form $\varpi_F^{-r+1}\Pi_{\mathfrak{J}}^{j}B'$ with some $B' \in U_{\mathfrak{J}}$. Take π_1 to be an irreducible smooth representation of G such that $l(\pi_1) \leqslant l(\chi\pi_1)$ for any character χ of $F^\times$ and which contains ρ and such that $\pi_1 \cong \text{c-Ind}_J^G \Lambda$ with Λ such that $\Lambda \mid_{U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}}$ contains ψ_α as a character of $U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}$. Let ρ_1 be an irreducible component of $\rho \mid_{U_{\mathfrak{J}}}$ such that $\rho_1 \mid_{U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}}$ contains ψ_α. Then

$$\mathrm{Hom}_{U_{\mathfrak{J}}}\left(\rho_1, \left(\text{c-Ind}_J^G \Lambda\right) \mid_{U_{\mathfrak{J}}}\right) \neq 0.$$

By Mackey formula and Frobenius reciprocity there exists $g \in J \setminus G / U_{\mathfrak{J}}$ such that

$$\mathrm{Hom}_{U_{\mathfrak{J}}}\left(\rho_1, \text{c-Ind}_{J^g\cap U_{\mathfrak{J}}}^{U_{\mathfrak{J}}}\left(\Lambda^g \mid_{J^g\cap U_{\mathfrak{J}}}\right)\right) \cong \mathrm{Hom}_{\ J^g\cap U_{\mathfrak{J}}}\left(\rho_1 \mid_{J^g\cap U_{\mathfrak{J}}}, \Lambda^g \mid_{J^g\cap U_{\mathfrak{J}}}\right) \neq 0. \tag{2.3.11}$$

In particular

$$\mathrm{Hom}_{\ U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1} \cap \left(U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}\right)^g}\left(\rho, \Lambda^g\right) \neq 0.$$

Denote by H_1 the subgroup $U_{\mathfrak{J}}^{\lfloor \frac{n}{2} \rfloor+1}$. There exists $g_1 \in U_{\mathfrak{J}}$ such that

$$\mathrm{Hom}_{H_1\cap H_1^g}\left(\psi_\alpha^{g_1} \mid_{H_1\cap H_1^g}, \psi_\alpha^{g} \mid_{H_1\cap H_1^g}\right) \neq 0.$$

Since $g_1 \in N_G(H_1) = U_{\mathfrak{J}} \rtimes \langle \Pi_{\mathfrak{J}} \rangle$ the previous one is equivalent to

$$\mathrm{Hom}_{H_1^{g_1}\cap (H^{g_1})^{g_1^{-1}g}}\left(\psi_{\alpha^{g_1}} \mid_{H_1^{g_1}\cap (H_1^{g_1})^{g_1^{-1}g}}, (\psi_{\alpha^{g_1}})^{g_1^{-1}g} \mid_{H_1^{g_1}\cap (H_1^{g_1})^{g_1^{-1}g}}\right) \neq 0.$$

Therefore $g_1^{-1}g$ intertwines stratum $[\mathfrak{J}, n, n-1, \alpha^{g_1}]$. Since $[\mathfrak{J}, n, n-1, \alpha]$ is simple and $g_1 \in U_{\mathfrak{J}}$ the stratum $[\mathfrak{J}, n, n-1, \alpha^{g_1}]$ is also simple. Therefore $g \in JU_{\mathfrak{J}}$. By (2.3.11), ρ_1 is isomorphic to $\text{c-Ind}_{J\cap U_{\mathfrak{J}}}^{U_{\mathfrak{J}}}\left(\Lambda \mid_{J\cap U_{\mathfrak{J}}}\right)$. Therefore $\rho \cong \text{c-Ind}_{J\cap U_{\mathfrak{J}}}^{K}\left(\Lambda \mid_{J\cap U_{\mathfrak{J}}}\right)$ and ρ is a cuspidal type on K for $\mathfrak{I}(\pi_1)$.

$\square$

2.3.2 Cuspidal types on $\mathrm{GL}_2(\mathcal{O}_F)$

The goal of this subsection is to prove the following theorem:

Theorem 2.3.9. *A cuspidal type on $K_2 = \mathrm{GL}_2(\mathcal{O}_F)$ is precisely a one-dimensional twist of one of the following:*

1. *a representation inflated from some irreducible cuspidal representation of $\mathrm{GL}_2(k_F)$;*
2. *a representation whose orbit contains a matrix which characteristic polynomial is irreducible mod $\mathfrak{p}_F$;*
3. *a representation whose orbit contains a matrix β whose characteristic polynomial is Eisenstein and which satisfies one of the following:*
   (a) *it has conductor at least 4;*
   (b) *it has conductor $r = 2$ or 3 and is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})}^{K_2} \theta$ where $\theta \mid_{U_{\mathfrak{M}}^{\lfloor \frac{r+1}{2} \rfloor}} = m\psi_\beta$ for certain $m \in \mathbb{Z}$ and θ does not contain the trivial character of $\begin{pmatrix} 1 & \mathfrak{p}_F^{r-2} \\ 0 & 1 \end{pmatrix}$.*

Remark 2.3.10. *Any matrix of the form $\Pi_{\mathfrak{J}} B$ for $B \in U_{\mathfrak{J}}$ has a characteristic polynomial which is Eisenstein. Moreover any matrix whose characteristic polynomial is Eisenstein is $\mathrm{GL}_p(\mathcal{O}_F)$-conjugate to one of the form $\Pi_{\mathfrak{J}} B$, $B \in U_{\mathfrak{J}}$.*

Proof. By Theorem 2.3.1, to prove the theorem it is enough to prove that a representation of K whose conductor is 2 or 3 and whose orbit is equivalent to an orbit which contains a matrix of the form $\Pi_{\mathfrak{J}} B$ for $B \in U_{\mathfrak{J}}$ is a cuspidal type if and only if 3*b* or 3*c* from Theorem 2.3.9 is satisfied.

Assume that ρ is a representation whose orbit is equivalent to an orbit containing a matrix of the form $\Pi_{\mathfrak{J}} B$ for $B \in U_{\mathfrak{J}}$. Denote by r the conductor of ρ. Assume that $r = 2$ or 3. The proof contains two steps.

Step 1 In this step we show the following statement:

A representation ρ is a cuspidal type if and only if there exists π an irreducible smooth representation of G which contains ρ and whose normalized level is $l(\pi) > r - 2$.

First we prove that if ρ with conductor 2 or 3 and an orbit containing a matrix of the form $\Pi_{\mathfrak{J}} B$ for $B \in U_{\mathfrak{J}}$ is a cuspidal type then there exists an irreducible smooth representation π of G which contains ρ and whose normalized level is strictly greater than $r - 2$. For contrary, assume that ρ is a cuspidal type and every π irreducible smooth representation of G which contains ρ has $l(\pi) \leqslant r-2$. Denote by π_1 an irreducible cuspidal representation of G for which ρ is a type. In particular, π_1 contains ρ. This means that $l(\pi_1) \leqslant r - 2$.

If $r = 2$, then $l(\pi_1) = 0$ and by ([10], 14.5 Exhaustion theorem), $\pi_1 \cong \text{c-Ind}_{K_2 Z}^{G} \Lambda$ for some Λ such that $\Lambda\mid_{K_2}$ is inflated from an irreducible cuspidal representation of $\mathrm{GL}_2(k_F)$. By Proposition 2.2.2, $\Lambda\mid_{K_2}$ is a type for π_1. By Theorem 2.2.1, $\Lambda\mid_{K_2} \cong \rho$. Since the conductor of ρ is 2 this is impossible.

Assume now $r = 3$. Then $l(\pi_1) \leqslant 1$. By analogous argument as before the normalized level of π_1 cannot be zero. Assume now that $l(\pi_1) = \frac{1}{2}$. By [10, 12.9 Theorem], the representation π_1 contains some fundamental stratum $[\mathfrak{A}, 1, 0, a]$ with $e(\mathfrak{A}) = 2$. If π_1 contains some stratum, then it contains all of its G-conjugates. Therefore we can assume π_1 contains a stratum $[\mathfrak{I}, 1, 0, a]$ for certain a. We want to show that this stratum is simple. By [10, 13.1 Proposition 1], we have $a\mathfrak{I} = \mathfrak{P}_{\mathfrak{I}}^{-1}$. In particular, $a \in \mathfrak{K}(\mathfrak{I})$, $\nu_{\mathfrak{I}}(a) = -1$ so $a = \Pi_{\mathfrak{I}} B$ for some $B \in U_{\mathfrak{I}}$. By Proposition 2.2.16, the stratum $[\mathfrak{I}, 1, 0, a]$ is simple. By Lemma 2.2.21 and Lemma 2.2.22, the representation $\pi_1 \cong \text{c-Ind}_J^G \Lambda$ for certain J and Λ such that $\Lambda\mid_{U_{\mathfrak{I}}^2} = 1_{U_{\mathfrak{I}}^2}$. By the unicity of types $\rho \cong \mathrm{Ind}_{J\cap K_2}^{K_2}(\Lambda\mid_{J\cap K_2})$ and by Mackey's formula it contains $\Lambda\mid_{J\cap K_2}$. The group K_2 is compact so $\rho\mid_{U_{\mathfrak{I}}^2}$ is trivial. Since $U_{\mathfrak{M}}^2 \subseteq U_{\mathfrak{I}}^2$ this means $\rho\mid_{U_{\mathfrak{M}}^2}$ is trivial. The conductor of ρ is 3 so this is impossible.
Assume now that $l(\pi_1) = 1$ and for every χ character of $F^\times$ we have $l(\pi_1) \leqslant l(\chi\pi_1)$. By ([10],14.5 Exhaustion theory), π_1 contains a simple stratum. Since the level is 1 it has to be of the form $[\mathfrak{A}, e(\mathfrak{A}), e(\mathfrak{A}) - 1, a]$. As before we can assume $\mathfrak{A} = \mathfrak{M}$ or $\mathfrak{I}$. We have $U_{\mathfrak{M}}^2 \subseteq U_{\mathfrak{I}}^3$ so we can assume π_1 contains a stratum of the form $[\mathfrak{M}, 1, 0, a]$ for certain a. In similar way as before, by the unicity of types and by Lemma 2.2.22 we deduce $\rho\mid_{U_{\mathfrak{M}}^2}$ is trivial which again is impossible.
Assume now that there exists χ a character of $F^\times$ such that $l(\chi\pi_1) = 0$ or $\frac{1}{2}$. Applying analogous arguments as before but for $\chi\rho$ and $\chi\pi_1$ we deduce that $\chi\rho\mid_{U_{\mathfrak{I}}^2}$ is trivial. In particular, $\chi\rho\mid_{U_{\mathfrak{M}}^2}$ is trivial. Therefore $\chi \circ \det\mid_{U_{\mathfrak{M}}^3}$ is trivial. By the following lemma it is impossible.

Lemma 2.3.11. *Let ρ be an irreducible smooth representation of K with conductor $r > 1$ and an orbit containing a matrix whose characteristic polynomial is Eisenstein. Let χ be a character of $F^\times$ such that $\chi \circ \det\mid_{U_{\mathfrak{M}}^r}$ is trivial. Then the conductor of $\chi\rho$ is bigger than or equal to r.*

Proof. Let $\{\beta_i\}_{i\in I}$ be an orbit of ρ. Without loss of generality we can assume that the characteristic polynomial of β_1 is Eisenstein.

By Lemma 2.2.20 $\chi \circ \det\mid_{U_{\mathfrak{M}}^{r-1}}$ is of the form ψ_α for some $\alpha \in \mathfrak{P}_{\mathfrak{M}}^{-r+1}$ with α scalar. By the definitions of ψ_{β_i} and ψ_α

$$\chi\rho\mid_{U_{\mathfrak{M}}^{r-1}} = \bigoplus_{i\in I} \psi_{\varpi_F^{-r+1}\beta_i + \alpha}\mid_{U_{\mathfrak{M}}^{r-1}}.$$

The restriction $\chi\rho\mid_{U_{\mathfrak{M}}^{r-1}}$ is trivial if and only if every $\varpi_F^{-r+1}\beta_i + \alpha \in \mathfrak{P}_{\mathfrak{M}}^{-r+2}$. If it holds then in particular $\beta_1 + \varpi_F^{r-1}\alpha \in \mathfrak{P}_{\mathfrak{M}}$ which is impossible.

$\square$

Now we prove the converse: if there exists an irreducible smooth representation of G which contains ρ and whose normalized level is strictly greater than $r-2$ then ρ is a cuspidal type. Therefore assume that there exists π an irreducible representation of G which contains ρ and such that $l(\pi) > r-2$. The representation ρ has an orbit containing a matrix of the form $\alpha_0 = \Pi_{\mathfrak{J}} B$ with $B \in U_{\mathfrak{J}}$. Let $\alpha = \varpi_F^{-r+1}\alpha_0$. In particular, ρ contains the character ψ_α of $U_{\mathfrak{M}}^{\lfloor \frac{r+1}{2} \rfloor} = U_{\mathfrak{M}}^{r-1}$. We want to show that ψ_α has an extension to $U_{\mathfrak{J}}^{2r-3}$ which is trivial on $U_{\mathfrak{J}}^{2r-2}$ and which is contained in ρ. Since $\alpha U_{\mathfrak{J}}^{2r-2} \subseteq U_{\mathfrak{J}}^{1}$, ψ_α is trivial not only on $U_{\mathfrak{M}}^{r}$ but also on $U_{\mathfrak{J}}^{2r-2}$. Therefore ψ_α can be seen as a character of $U_{\mathfrak{M}}^{r-1}/U_{\mathfrak{J}}^{2r-2}$. We have $U_{\mathfrak{M}}^{r-1}/U_{\mathfrak{J}}^{2r-2} \subseteq U_{\mathfrak{J}}^{2r-3}/U_{\mathfrak{J}}^{2r-2}$ and the last one is abelian so ψ_α has a one-dimensional extension to $U_{\mathfrak{J}}^{2r-3}/U_{\mathfrak{J}}^{2r-2}$ which is contained in π. This extension is of the form ψ_β for $\beta \in \mathfrak{P}_{\mathfrak{J}}^{-2r+3}$. This means the stratum $[\mathfrak{J}, 2r-3, 2r-4, \beta]$ is contained in π and ψ_β is contained in ρ. Since the normalized level of π is strictly greater than $r-2$, it is equal to $\frac{2r-3}{2}$. By ([10], 12.9 Theorem) the stratum $[\mathfrak{J}, 2r-3, 2r-4, \beta]$ is fundamental which as before implies it is simple. Similarly as in the proof of Theorem 2.3.1 this means that ρ is a cuspidal type. This ends the proof of the first step.

By Remark 2.2.24, the representation ρ is isomorphic to $\mathrm{Ind}_{\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})}^{K_2}\theta$ for some $\bar{\beta} \in \mathrm{M}_2(\mathcal{O}_F)$ and θ an irreducible representation of $\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})$.

Step 2 Assume now that ρ has conductor $r = 2$ or 3. To finish the proof it is enough to show the following statement:

The representation θ contains the trivial character of the subgroup $\begin{pmatrix} 1 & \mathfrak{p}_F^{r-2} \\ 0 & 1 \end{pmatrix}$ if and only if every irreducible smooth representation of G containing ρ has normalized level less than or equal to $r-2$.

Denote by ψ_β the lift of $\bar{\psi}_{\bar{\beta}}$ to the character of $U_{\mathfrak{M}}^{r-1}$ which is trivial on $U_{\mathfrak{M}}^{r}$. Since $U_{\mathfrak{M}}^{r-1}$ is a normal subgroup of $\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})$ by Clifford theory we can assume $\theta\mid_{U_{\mathfrak{M}}^{r-1}}$ is a multiple of ψ_β. Set $\beta_0 := \varpi_F^{r-1}\beta$. Up to conjugation $\bar{\beta}_0 \in \mathrm{M}_2(k_F)$ is of the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore we can assume that the character ψ_β is trivial when restricted to $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-1} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$ and θ is trivial when restricted to this subgroup. Since $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-1} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$ and the group $\begin{pmatrix} 1 & \mathfrak{p}_F^{r-2} \\ 0 & 1 \end{pmatrix}$ generate the subgroup $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$ we can replace in the statement the subgroup $\begin{pmatrix} 1 & \mathfrak{p}_F^{r-2} \\ 0 & 1 \end{pmatrix}$ by $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$. The subgroups $U_{\mathfrak{M}}^r$ and $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$ generate the group $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ \mathfrak{p}_F^{r} & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$ which is conjugate by the matrix $\begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$ to $U_{\mathfrak{M}}^{r-1}$. Therefore if θ contains the trivial character of $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$ then every irreducible smooth representation of G containing ρ contains the trivial character of $U_{\mathfrak{M}}^{r-1}$.

Now assume that every irreducible smooth representation of G which contains ρ has normalized level less than or equal to $r-2$. Since $\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})$ is a closed subgroup of the Iwahori subgroup $\begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F^\times \end{pmatrix}$ which is compact it is a compact subgroup. By ([10, 11.1 Proposition 1]), there exists $g \in G$ such that

$$\mathrm{Hom}_{(U_{\mathfrak{M}}^{r-1})^g \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})}(1_{(U_{\mathfrak{M}}^{r-1})^g \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})}, \theta) \neq 0.$$

Therefore θ contains the trivial character of $(U_{\mathfrak{M}}^{r-1})^g \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})$. This property depends only on the double coset $N_G(U_{\mathfrak{M}}^{r-1})g\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})$ where N_G denotes the normalizer of a subgroup in G.

Firstly we will show to what kind of elements of G we can restrict our considerations. By [33], section 2, $\mathrm{Stab}_{\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p}_F^r)}(\bar{\psi}_{\bar{\beta}}) = (\mathcal{O}_F/\mathfrak{p}_F^r)[\widehat{\beta}]^\times K_2^1$ where $\widehat{\beta}$ is a lift of $\bar{\beta}$ to a matrix in $\mathrm{M}_2(\mathcal{O}_F/\mathfrak{p}_F^r)$ and $K_2^1 = K^1$ with $p=2$. Therefore

$$\mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}}) = \bigcup_{a \in \mathcal{O}_F^\times} \begin{pmatrix} a+\mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & a+\mathfrak{p}_F \end{pmatrix}.$$

By [10, 12.3] $N_G(U_{\mathfrak{M}}^{r-1}) = K_2 Z$. Since $\begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F^\times \end{pmatrix} = \bigcup_{c \in \mathcal{O}_F^\times} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}})$ and $\varpi_F^m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K_2 Z$ for any $m \in \mathbb{Z}$ by [10, 17.1 Proposition] we can assume that g is of the form either $g_{1,n,c} = \begin{pmatrix} 1 & 0 \\ 0 & c\varpi_F^n \end{pmatrix}$ or $g_{2,n,c} = \begin{pmatrix} c\varpi_F^n & 0 \\ 0 & 1 \end{pmatrix}$ for some $c \in \mathcal{O}_F^\times$ and $n \in \mathbb{N}$.

Compute

$$(U_{\mathfrak{M}}^{r-1})^{g_{1,n,c}} \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}}) = \begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-1+n} \\ \mathfrak{p}_F^{r-1-n} \cap \mathfrak{p}_F & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$$

$$(U_{\mathfrak{M}}^{r-1})^{g_{2,n,c}} \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}}) = \begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-1-n} \cap \mathcal{O}_F \\ \mathfrak{p}_F^{r-1+n} & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}.$$

In particular, θ contains the trivial character of at least one of the following:

1. $\bigcap_{n \geqslant 0, c \in \mathcal{O}_F^\times} (U_{\mathfrak{M}}^{r-1})^{g_{1,n,c}} \cap \bigcap_{c \in \mathcal{O}_F^\times} (U_{\mathfrak{M}}^{r-1})^{g_{2,0,c}} \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}}) = \begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & 0 \\ \mathfrak{p}_F^{r-1} & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$

2. $\bigcap_{n>0, c\in\mathcal{O}_F^\times}(U_{\mathfrak{M}}^{r-1})^{g_{2,n,c}} \cap \mathrm{Stab}_{K_2}(\bar{\psi}_{\bar{\beta}}) = \begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$.

The restriction $\theta\mid_{U_{\mathfrak{M}}^{r-1}}$ is a multiple of ψ_β. In particular, since $\psi_\beta\left(\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & 0 \\ \mathfrak{p}_F^{r-1} & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}\right) = \psi(\mathcal{O}_F)$ is non-trivial θ does not contain the trivial character of $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & 0 \\ \mathfrak{p}_F^{r-1} & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$.
Therefore θ contains the trivial character of $\begin{pmatrix} 1+\mathfrak{p}_F^{r-1} & \mathfrak{p}_F^{r-2} \\ 0 & 1+\mathfrak{p}_F^{r-1} \end{pmatrix}$. □

Corollary 2.3.12. *Every cuspidal type on K_2 is a regular representation. However not every regular representation of K_2 is a cuspidal type on K_2.*

2.3.3 Regularity of cuspidal types

In this subsection we determine which cuspidal types are regular. More precisely we determine which matrices from Theorem 2.3.1 (2) and (3) are regular. Of course matrices whose characteristic polynomial is irreducible modulo $\mathfrak{p}_F$ are regular. In Proposition 2.3.14we prove that matrices of the form $\Pi_{\mathfrak{I}}B$ for $B\in U_{\mathfrak{I}}$ are regular. In Proposition 2.3.15 we prove that matrices of the form $\Pi_{\mathfrak{I}}^j B$ for $j\in\mathbb{N}$, $1<j<p$ and for $B\in U_{\mathfrak{I}}$ are not regular. In our consideration we do not have to restrict our consideration to the prime dimension of V. We work with matrices of arbitrary dimension n.

Fix n a natural number bigger than or equal to 2. Define $\mathfrak{I}$, $\Pi_{\mathfrak{I}}$ analogously as in subsection 2.2.2 but for the dimension n. First we prove a lemma useful for the proof of Proposition 2.3.14.

Lemma 2.3.13. *Let $M\in\mathrm{GL}_n(\mathcal{O}_F)$ be a matrix whose characteristic polynomial is Eisenstein. Then any $g\in\mathrm{GL}_n(F)$ such that $g^{-1}Mg\in\mathrm{M}_n(O)$ is of the form $g\in Z_{\mathrm{GL}_n(F)}(M)\mathrm{GL}_n(\mathcal{O}_F)$ where $Z_{\mathrm{GL}_n(F)}(M)$ denotes the centralizer of M in $\mathrm{GL}_n(F)$. In particular, there exists an element $h\in\mathrm{GL}_n(\mathcal{O}_F)$ such that $h^{-1}Mh = g^{-1}Mg$.*

Proof. Take the lattice $\Lambda=\mathcal{O}_F^n$. We want to show that for any matrix $Q\in\mathrm{GL}_n(F)$

$$Q\Lambda=\Lambda \quad \text{if and only if} \quad Q\in\mathrm{GL}_n(\mathcal{O}_F). \tag{2.3.12}$$

Indeed, if for any matrix $Q_1\in\mathrm{GL}_n(F)$ then $Q_1\Lambda\subseteq\Lambda$ if and only if $Q_1\in\mathrm{M}_n(\mathcal{O}_F)$. Therefore if $Q\Lambda=\Lambda$ then $Q\in\mathrm{M}_n(\mathcal{O}_F)$ and also $\Lambda=Q^{-1}\Lambda$ so $Q^{-1}\in\mathrm{M}_n(\mathcal{O}_F)$. Hence $Q\in\mathrm{GL}_n(\mathcal{O}_F)$. Conversely if $Q\in\mathrm{GL}_n(\mathcal{O}_F)$ then $Q\Lambda\subseteq\Lambda$ and $Q^{-1}\Lambda\subseteq\Lambda$ so $\Lambda=Q\Lambda$. By the condition (2.3.12) to prove the lemma it is enough to show that $g\Lambda=z\Lambda$ for some $z\in Z_{\mathrm{GL}_n(F)}(M)$.

Let $E=F[M]$. Since the characteristic polynomial of M is irreducible E is a field. The action of M on F^n makes it a one-dimensional E-vector space. Fix a non-zero element

$v \in F^n$. The following map is an isomorphism of E-vector spaces $i : E \ni x \mapsto xv \in F^n$. By definition it is E-linear homomorphism. Since both E and F^n are finitely dimensional over E to check that i is a bijection it is enough to check that i is injective. Assume it is not. Then take $x, y \in E$ such that $x \neq y$ and $xv = yv$. Then $\det(x - y) = 0$ but this is impossible because $x - y$ is invertible.

Below we will show that Λ and $g\Lambda$ are O_E-modules. Assume for now that it is true. Recall that fractional ideals of E are finitely generated O_E-submodules of E. Since i is E-linear there exist fractional ideals I_1 and I_2 of E such that $i(I_1) = \Lambda$ and $i(I_2) = g\Lambda$. Since fractional ideals of E are generated by powers of M there exists an integer number j such that $g\Lambda = M^j\Lambda$. Of course $M^j \in Z_{\mathrm{GL}_n(F)}(M)$. Therefore to finish the proof it is enough to show that $\mathcal{O}_E\Lambda \subseteq \Lambda$ and $\mathcal{O}_E g\Lambda \subseteq g\Lambda$.

We want to show that $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F M + \ldots + \mathcal{O}_F M^{n-1}$. Since the residue field of E and F are the same it we have $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_E M$. Therefore $\mathcal{O}_E = \sum_{j=0}^{\infty} \mathcal{O}_F M^j$. We want to show that for $j \geqslant n$ we have $\mathcal{O}_F M^j \subseteq \mathcal{O}_F + \mathcal{O}_F M + \ldots + \mathcal{O}_F M^{n-1}$. We do it by the induction. If $j = n$ then since the characteristic polynomial of M is equal to the minimal polynomial it is true. Assume that we have the inclusion for j. Then $\mathcal{O}_F M^{j+1} \subseteq \mathcal{O}_F M + \mathcal{O}_F M^2 + \ldots + \mathcal{O}_F M^n \subseteq \mathcal{O}_F + \mathcal{O}_F M + \ldots + \mathcal{O}_F M^{n-1}$. Therefore $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F M + \ldots + \mathcal{O}_F M^{n-1}$.

Of course $\mathcal{O}_F\Lambda \subseteq \Lambda$ and $\mathcal{O}_F g\Lambda \subseteq g\Lambda$. Therefore it is enough to check that $M\Lambda \subseteq \Lambda$ and $Mg\Lambda \subseteq g\Lambda$. Since both M and $g^{-1}Mg$ are matrices in $\mathrm{M}_n(\mathcal{O}_F)$, these are true. □

Proposition 2.3.14. *Let $M \in \mathrm{M}_n(\mathcal{O}_F)$ be a matrix whose characteristic polynomial is Eisenstein and denote the characteristic polynomial by f. Then M is $\mathrm{GL}_n(\mathcal{O}_F)$-conjugate to a companion matrix of f which is regular. In particular, M is regular.*

Proof. Denote by C the companion matrix of f. We prove that since f is irreducible the matrices M and C are $\mathrm{GL}_n(F)$-conjugate. Define the following maps: for any polynomial b with coefficients in F

$$h_1 : F[M] \ni b(M) \mapsto b(M) \in \mathrm{M}_n(F) \quad \text{and} \quad h_2 : F[M] \ni b(M) \mapsto b(C) \in \mathrm{M}_n(F).$$

Of course h_1 is well defined. To check that h_2 is well defined it is enough to check that $h_2(0) = 0$. The polynomial f is irreducible and therefore the minimal polynomial of both M and C is equal f. Therefore $h_2(0) = h_2(f(M)) = f(C) = 0$. By definition h_1 and h_2 are F-algebra homomorphisms. Since f is irreducible $F[M]$ is a field. By the Skolem–Noether theorem there exists $B \in \mathrm{GL}_n(F)$ such that $h_1 = Bh_2B^{-1}$ so in particular $M = BCB^{-1}$. Therefore by Lemma 2.3.13 the matrices M and C are $\mathrm{GL}_n(\mathcal{O}_F)$-conjugate. Since C is regular M is also regular.

□

Observe that $f(x)$ the characteristic polynomial of $\Pi_{\mathfrak{J}}B$, with $B \in U_{\mathfrak{J}}$, is equal to x^p

modulo $\mathfrak{p}_F$ and $f(0) = \det(\Pi_{\mathfrak{J}}B) = u\varpi_F$ for some $u \in \mathcal{O}_F^\times$. Therefore $f(x)$ is Eisenstein and Proposition 2.3.14 proves that matrices of the form $\Pi_{\mathfrak{J}}B$ are regular.

Proposition 2.3.15. *Matrices of the form $\Pi_{\mathfrak{J}}^j B$ for $j \in \mathbb{N}$, $1 < j < p$ and $B \in U_{\mathfrak{J}}$ are not regular.*

Proof. Denote by D the image of $\Pi_{\mathfrak{J}}^j B$ modulo $\mathfrak{p}_F$. Assume the contrary, that D is regular. Denote by $\bar{\Pi}_{\mathfrak{J}}$ the reduction of $\Pi_{\mathfrak{J}}$ modulo $\mathfrak{p}_F$. Every matrix of the form D is nilpotent. By [15, 14.11 Proposition], D is $\mathrm{GL}_p(k_F)$-conjugate to $\bar{\Pi}_{\mathfrak{J}}$. This is impossible because $D^{p-1} \neq 0$ and $\left(\bar{\Pi}_{\mathfrak{J}}^j\right)^{p-1} = 0$. This gives us a contradiction and we proved that D is not regular. $\square$

2.4 Example

In this section we give an example of two representations of $K_2 = \mathrm{GL}_2(\mathcal{O}_F)$ with the same orbits and the same conductor but one of them will be a cuspidal type and the second will not. This illustrates the fact that it is not always enough to determine orbits to determine if a given representation is a cuspidal type.

Let $S = \bigcup_{a \in \mathcal{O}^\times} \begin{pmatrix} a + \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & a + \mathfrak{p}_F \end{pmatrix}$, $\beta_1 = \varpi_F^{-1} \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$ and $\beta_2 = \varpi_F^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Define $\theta_1, \theta_2 : S \to \mathbb{C}$ as follows:

$$\theta_1(a\mathrm{Id}_p + x) = \psi(\mathrm{tr}(a^{-1}\beta_1 x)) \quad \text{and} \quad \theta_2(a\mathrm{Id}_p + x) = \psi(\mathrm{tr}(a^{-1}\beta_2 x))$$

where $a \in \mathcal{O}^\times$ and $x \in \begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$. Denote

$$\rho_1 := \mathrm{Ind}_S^{K_2}\theta_1 \quad \text{and} \quad \rho_2 := \mathrm{Ind}_S^{K_2}\theta_2.$$

Proposition 2.4.1. *The maps θ_1 and θ_2 are well-defined homomorphisms. Both representations ρ_1 and ρ_2 have conductor 2 and contain the matrix $\beta_0 := \varpi_F\beta_1$ in their orbits. The representation ρ_1 is a cuspidal type but ρ_2 is not.*

Proof. First we show that θ_1 and θ_2 are well-defined homomorphisms. Take $a, b \in \mathcal{O}_F^\times$ and $x, y \in \begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$ such that $a\mathrm{Id}_2 + x = b\mathrm{Id}_2 + y$. Then $(b^{-1} - a^{-1})\mathrm{Id}_2 = a^{-1}b^{-1}(y - x) \in \begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$ and $(b^{-1} - a^{-1}) \in \mathfrak{p}\mathrm{Id}_2$. Therefore $a^{-1}x - b^{-1}y = a^{-1}x - b^{-1}(x + (a-b)\mathrm{Id}_2)) = (a^{-1} - b^{-1})x - (ab^{-1} - 1)\mathrm{Id}_2 \in \begin{pmatrix} \mathcal{O}_F & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & \mathcal{O}_F \end{pmatrix}$. Compute

$$\begin{aligned}\theta_1(a\mathrm{Id}_2+x)\theta_1(b\mathrm{Id}_2+y)^{-1} =&\psi(\mathrm{tr}(\beta_1(a^{-1}x-b^{-1}y)))\\ \subseteq&\psi\left(\mathrm{tr}\left(\varpi_F^{-1}\begin{pmatrix}0 & 1\\ \varpi_F & 0\end{pmatrix}\begin{pmatrix}\mathcal{O}_F & \mathfrak{p}_F\\ \mathfrak{p}_F^2 & \mathcal{O}_F\end{pmatrix}\right)\right)=1\end{aligned}$$

and similarly

$$\theta_2(a\mathrm{Id}_2+x)\theta_2(b\mathrm{Id}_2+y)^{-1}\subseteq\psi\left(\mathrm{tr}\left(\varpi_F^{-1}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathcal{O}_F & \mathfrak{p}_F\\ \mathfrak{p}_F^2 & \mathcal{O}_F\end{pmatrix}\right)\right)=1.$$

This shows that θ_1 and θ_2 are well defined. For any $c\in\mathcal{O}_F^\times$, $x,y\in\begin{pmatrix}\mathfrak{p}_F & \mathcal{O}_F\\ \mathfrak{p}_F & \mathfrak{p}_F\end{pmatrix}$ we have

$$\mathrm{tr}(c\beta_1xy)\in\mathfrak{p}_F\quad\text{and}\quad\mathrm{tr}(c\beta_2xy)\in\mathfrak{p}_F.$$

Therefore θ_1 and θ_2 are homomorphisms.

Both $\rho_1|_{U^1_{\mathfrak{M}}}$ and $\rho_2|_{U^1_{\mathfrak{M}}}$ contain ψ_{β_1} so ρ_1 and ρ_2 have conductor 2 and contain $\varpi_F\beta_1$ in their orbits.

First we prove that ρ_2 is not a cuspidal type. This of course can be deduced from Theorem 2.3.9 but to give an explicit example we give a specific proof. For contradiction assume that ρ_2 is a cuspidal type for an irreducible cuspidal representation π of $\mathrm{GL}_2(F)$. The representation ρ_2 has conductor 2 so $l(\pi)>0$. On the other hand $\theta_2|_{\begin{pmatrix}1+\mathfrak{p}_F & \mathcal{O}_F\\ 0 & 1+\mathfrak{p}_F\end{pmatrix}}=1$ and $\theta_2|_{U^2_{\mathfrak{M}}}=1$. The groups $\begin{pmatrix}1+\mathfrak{p}_F & \mathcal{O}_F\\ 0 & 1+\mathfrak{p}_F\end{pmatrix}$ and $U^2_{\mathfrak{M}}$ generate $\begin{pmatrix}1+\mathfrak{p}_F & \mathcal{O}_F\\ \mathfrak{p}_F^2 & 1+\mathfrak{p}_F\end{pmatrix}$. The latter is $\mathrm{GL}_2(F)$-conjugate to $U^1_{\mathfrak{M}}$. This means that $l(\pi)=0$ and we get the contradiction.

By Theorem 2.3.9, ρ_1 is a cuspidal type.

□

Part II

On the optimal rate of equidistribution in number fields.

Chapter 3

On the optimal rate of equidistribution in number fields.

This chapter is joint work with Mikołaj Frączyk and it comes from our preprint paper [18].

3.1 Introduction

3.1.1 Optimal rate of equidistribution in number fields.

In this chapter we study the optimal rate of "local" equidistribution in the rings of integers of number fields. First we will make precise what kind of equidistribution we mean. For any ring A we may map it into the profinite completion $\widehat{A} = \lim_{\leftarrow} A/I$ where I runs over all cofinite ideals in A. The additive group of $\widehat{A}$ is a compact topological group so it is equipped with a unique Haar probability measure m. We say that a sequence of finite subsets $E_n \subset A$ **equidistributes** in A if the sequence of probability measures on $\widehat{A}$

$$\mu_n := \frac{1}{|E_n|} \sum_{x \in E_n} \delta_x$$

converges weakly-* to the Haar measure m. If k is a number field and $A = \mathcal{O}_k$ is its ring of integers this means that $(E_n)_{n\in\mathbb{N}}$ equidistributes in $\widehat{\mathcal{O}_k} = \prod_{\mathfrak{p}} \mathcal{O}_{k_\mathfrak{p}}$ where $\mathfrak{p}$ runs over prime ideals of $\mathcal{O}_k$ and $\mathcal{O}_{k_\mathfrak{p}}$ is the ring of integers in the completion $k_\mathfrak{p}$. In practice, for example when E_n are given by some arithmetic construction, it is often easier to prove that the equidistribution holds in $\mathcal{O}_{k_\mathfrak{p}}$ for each prime $\mathfrak{p}$ than that it holds in the product $\prod_\mathfrak{p} \mathcal{O}_{k_\mathfrak{p}}$. This is why we focus on the weaker notion of **local equidistribution** in $\mathcal{O}_k$. We say that $(E_n)_{n\in\mathbb{N}}$ locally equidistributes in $\mathcal{O}_k$ if for every prime ideal $\mathfrak{p}$ the sequence of probability measures μ_n (defined as above) converges weakly to the unique Haar probability measure on $\mathcal{O}_{k_\mathfrak{p}}$. We can measure the rate of equidistribution in $\mathcal{O}_{k_\mathfrak{p}}$ by looking at the $\mathfrak{p}$-adic valuation of the product of differences $\prod_{s\neq s'\in E_n}(s-s')$. Using the pigeon principle, one can show that $\nu_{\mathfrak{p}}\left(\prod_{s\neq s'\in E_n}(s-s')\right) \geq \sum_{m=1}^{|E_n|-1}\sum_{i=1}^{\infty}\lfloor\frac{m}{q^i}\rfloor$, where q is the size of the residue field of $k_{\mathfrak{p}}$. When the equality is achieved for each n we say that $(E_n)_{n\in\mathbb{N}}$ **equidistributes optimally** in $\mathcal{O}_{k_{\mathfrak{p}}}$. It happens, for example, when E_n are sets of the first n elements of a sequence $(a_i)_{i\in\mathbb{N}}$ which is a $\mathfrak{p}$-ordering (Definition 3.1.3). The $\mathfrak{p}$-orderings were introduced by Manjul Bhargava in [5] in order to generalize the notion of the factorial to any Dedekind domain (or even subsets of Dedekind domains) and to extend the classical results of Pólya on integer valued polynomials in $\mathbb{Q}[t]$ to arbitrary Dedekind domains [5, Theorem 14]. While it is easy to see that for a fixed finite set P of primes $\mathfrak{p}$ one can find a sequence E_n that equidistributes optimally in $\mathcal{O}_{k_{\mathfrak{p}}}$ for all $\mathfrak{p}\in P$ it is not clear if there exists a sequence of sets E_n that equidistributes optimally for all primes $\mathfrak{p}$ at the same time. It is certainly possible in $\mathbb{Z}$ because we we can take $E_n=\{1,2,\ldots,n\}$. As the main result of this chapter we prove that $k=\mathbb{Q}$ is the **only** number field for which $\mathcal{O}_k$ enjoys this property. As a corollary we answer the question of Bhargava [5, Question 3] for rings of integers in number fields. Bhargava asked which Dedekind domains admit simultaneous $\mathfrak{p}$-orderings. Our main result implies that $\mathbb{Z}$ is the only ring of integers where this is possible.

3.1.2 $\mathfrak{p}$-orderings and equidistribution

Let A be a ring and let I be an ideal of A. We say that a finite subset $S\subseteq A$ is **almost uniformly distributed modulo** I if for any $a,b\in A$ we have

$$|\{s\in S \mid s-a\in I\}| - |\{s\in S \mid s-b\in I\}| \in \{-1,0,1\}. \tag{3.1.1}$$

If A/I is finite the condition (3.1.1) is equivalent to the following

$$\left| |\{s\in S \mid s-a\in I\}| - \frac{|S|}{|A/I|}\right| < 1. \tag{3.1.2}$$

Let k be a number field and let $\mathcal{O}_k$ be its ring of integers.

Definition 3.1.1. *We call a finite subset $S\subseteq\mathcal{O}_k$ n-**optimal** if $|S|=n+1$ and S is almost uniformly equidistributed modulo every power $\mathfrak{p}^l, l\geq 1$ for every prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$.*

The n-optimal sets are in a sense locally as uniformly equidistributed as possible. The sequences of n-optimal sets are precisely the ones that equidistribute optimally in $\mathcal{O}_{k_{\mathfrak{p}}}$ for all primes $\mathfrak{p}$ at the same time. The main result of this chapter determines the numbers fields k where the rings of integers $\mathcal{O}_k$ admits arbitrarily large n-optimal sets.

Theorem 3.1.2. *Let k be a number field different than $\mathbb{Q}$. Then there is a natural number n_0 such that there are no n-optimal sets for $n\geq n_0$.*

In particular, unless $k=\mathbb{Q}$ there are no sequences of finite subsets that equidistrubute optimally modulo all prime powers. Motivation for considering n-optimal subsets comes from the theory of integer valued polynomials and from the study of $\mathfrak{p}$-orderings. We recall the definition of a $\mathfrak{p}$-ordering in a subset of $\mathcal{O}_k$, following [5].

Definition 3.1.3. *Let $S \subset \mathcal{O}_k$ and let $\mathfrak{p}$ be a non-zero proper prime ideal. A sequence $(a_i)_{i\in\mathbb{N}} \subset S$ is a $\mathfrak{p}$-**ordering in** S if for every $n \in \mathbb{N}$ we have*

$$v_S(\mathfrak{p}, n) := v_{\mathfrak{p}}\left(\prod_{i=0}^{n-1}(a_i - a_n)\right) = \min_{s\in S} v_{\mathfrak{p}}\left(\prod_{i=0}^{n-1}(a_i - s)\right),$$

where $v_{\mathfrak{p}}$ stands for the additive $\mathfrak{p}$-adic valuation on k. The value $v_S(\mathfrak{p}, n)$ does not depend on the choice of a $\mathfrak{p}$-ordering ([5]).

Bhargava defines the **generalized factorial** as the ideal $n!_S = \prod_{\mathfrak{p}} \mathfrak{p}^{v_S(\mathfrak{p},n)}$ where $\mathfrak{p}$ runs over primes in $\mathcal{O}_k$. A sequence $(a_i)_{i\in\mathbb{N}} \subset S$ is called a **simultaneous $\mathfrak{p}$-ordering** in S if it is a $\mathfrak{p}$-ordering in S for every prime ideal $\mathfrak{p}$. Simultaneous $\mathfrak{p}$-orderings are also called Newton sequences [12, 13]. A sequence $(a_i)_{i\in\mathbb{N}} \subset \mathcal{O}_k$ is a simultaneous $\mathfrak{p}$-ordering in $\mathcal{O}_k$ if and only if the set $\{a_0, a_1, \ldots, a_n\}$ is n-optimal for every $n \in \mathbb{N}$ (see [11, Proposition 2.6]) . In [5, 6] Bhargava asks what are the subsets $S \subset \mathcal{O}_k$ (or more general Dedekind domains) admitting simultaneous $\mathfrak{p}$-orderings and in particular for which k the ring $\mathcal{O}_k$ admits a simultaneous $\mathfrak{p}$-ordering. The last question was addressed by Melanie Wood in [36] where she proved that there are no simultaneous $\mathfrak{p}$-orderings in $\mathcal{O}_k$ if k is an imaginary quadratic field. This result was extended in [1, Theorem 16] to all real quadratic number fields $\mathbb{Q}(\sqrt{d})$ except possibly for $d = 2, 3, 5$ and $d \equiv 1 \mod 8$. Existence of a simultaneous $\mathfrak{p}$-ordering implies that there are n-optimal sets in $\mathcal{O}_k$ for all n. As a corollary of Theorem 3.1.2 we get:

Corollary 3.1.4. *$\mathbb{Q}$ is the unique number field whose ring of integers admits a simultaneous $\mathfrak{p}$-ordering.*

This answers [5, Question 3] for rings of integers in number fields. Note that having an upper bound on n such that there exists an n-optimal set is *a priori* stronger than non-existence of simultaneous $\mathfrak{p}$-orderings because not every n-optimal set can be ordered into an initial fragment of a simultaneous $\mathfrak{p}$-ordering. We do not know any example of a Dedekind domain that has arbitrarily large n-optimal sets but no simultaneous $\mathfrak{p}$-orderings. We remark that the ring $\mathbb{F}_q[t]$ admits a simultaneous $\mathfrak{p}$-ordering [5, p. 125]. It would be interesting to know which finite extensions F of $\mathbb{F}_q[t]$ have the property that $\mathcal{O}_F$ admits a simultaneous $\mathfrak{p}$-ordering.

3.1.3 Test sets for integer valued polynomials.

The notions of $\mathfrak{p}$-orderings and n-optimal sets are connected to the theory of integer valued polynomials. Let $P \in k[X]$ be a polynomial. We say that P is **integer valued** on $S \subset \mathcal{O}_k$

if $P(S) \subset \mathcal{O}_k$. Following [12] we denote the module of integer valued polynomials of degree at most n by

$$I_n(S, \mathcal{O}_k) = \{P \in k[X] |\deg P \leq n, P(S) \subset \mathcal{O}_k\}.$$

We call a subset $E \subset \mathcal{O}_k$ an n-**universal** set if the following holds. A polynomial $P \in k[X]$ is integer valued (on $\mathcal{O}_k$) if and only if $P(E) \subset \mathcal{O}_k$. It is easy to prove, using Lagrange interpolation, that $|S| \geq n+1$ for any n-universal set S. It was shown in [11, 35] that if $|S| = n+1$ then S is n-universal if and only if it is almost uniformly distributed modulo all powers of all prime ideals. In our notation the latter is equivalent to S being n-optimal. It is proved in [11] that for every $n \in \mathbb{N}$ there exists an n-universal set of size $n+2$, so it is interesting to ask whether there are n-universal sets of cardinality $n+1$ (i.e. n-optimal sets). For k quadratic imaginary number field it was proven in [11] that there is an upper bound on n such that there exists an n-optimal set. This generalizes the analogous result for $k = \mathbb{Q}(\sqrt{-1})$ from [35]. For general quadratic number fields Cahen and Chabert [13] proved that there are no 2-optimal sets, except possibly in $\mathbb{Q}(\sqrt{d}), d = -3, -1, 2, 3, 5$ and $d \equiv 1 \mod 8$. From our main result and [11, Theorem 4.1.] we deduce the following.

Corollary 3.1.5. *Let $k \neq \mathbb{Q}$ be a number field. Then for $n \in \mathbb{N}$ sufficiently large the minimal cardinality of an n-universal set in $\mathcal{O}_k$ is $n+2$.*

3.1.4 Average number of solutions of a unit equation

One of our key technical ingredients in the proof of Theorem 3.1.2 is the following bound, which can be interpreted as a bound on the average number of solutions of the unit equation [37]. To shorten notation we will write $\|x\| = |N_{k/\mathbb{Q}}(x)|$ for $x \in k$.

Theorem 3.1.6. *Let k be a number field of degree N with d Archimedean places and let $B \in \mathbb{R}$. There are constants $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ dependent only on k and B such that for every $a \in \mathcal{O}_k$, $0 < X \leq \|a\|e^B$ and $\kappa = \min\left\{\frac{1}{2N(N-1)}, \frac{1}{4N-1}\right\}$ we have*

$$|\left\{x \in \mathcal{O}_k | \ \|x(a-x))\| \leq X^2\right\}| \leq \Theta_1 X^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log X)^{2d-2} + \Theta_3 \log\log\log\log \|a\| + \Theta_4.$$

The traditional form of the unit equation is

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 = 1 \text{ where } \alpha_1, \alpha_2 \in k^\times$$

and the indeterminates λ_1, λ_2 are the units of $\mathcal{O}_k$. We may consider an equivalent form of the unit equation

$$\alpha_1\lambda_1 + \alpha_2\lambda_2 = \alpha_3 \text{ where } \alpha_1, \alpha_2, \alpha_3 \in \mathcal{O}_k. \tag{3.1.3}$$

It is clear that the number of solutions depends only on the class of $(\alpha_1, \alpha_2, \alpha_3)$ in the quotient of the projective space $\mathbb{P}^2(k)/(\mathcal{O}_k^\times)^3$. Let $\nu(\alpha_1, \alpha_2, \alpha_3)$ be the number of solutions of (3.1.3). It was known since Siegel [32] that $\nu(\alpha_1,\alpha_2,\alpha_3)$ is finite and Evertse [16] found an upper bound independent of $\alpha_1,\alpha_2,\alpha_3$

$$\nu(\alpha_1,\alpha_2,\alpha_3) \leq 3\times 7^N.$$

In fact, Evertse, Győry, Stewart and Tijdeman [17] showed that except for finitely many points $[\alpha_1,\alpha_2,\alpha_3]\in\mathbb{P}^2(k)/(\mathcal{O}_k)^3$ the equation (3.1.3) has at most two solutions. Theorem 3.1.6 gives a quantitative control on the "average" number of solutions of (3.1.3) as $\alpha_1,\alpha_2\in\mathcal{O}_k/\mathcal{O}_k^\times, \|\alpha_1\alpha_2\|\leq X^2$ and $\|\alpha_3\|$ is fixed with $\|\alpha_3\|$ not much smaller than X.

Theorem 3.1.7. *Let k be a number field of degree N with d Archimedean places, let $B\in\mathbb{R}$ and put $\kappa=\min\left\{\frac{1}{2N(N-1)},\frac{1}{4N-1}\right\}$. There exist constants $\Theta_1,\Theta_2,\Theta_3,\Theta_4$ dependent only on k and B such that for every $\alpha_3\in\mathcal{O}_k, 0<X\leq\|\alpha_3\|e^B$ we have*

$$\sum_{\substack{\alpha_1,\alpha_2\in\mathcal{O}_k/\mathcal{O}_k^\times\\ \|\alpha_1\alpha_2\|\leq X^2}}\nu(\alpha_1,\alpha_2,\alpha_3)\leq\Theta_1X^{1+\kappa}\|\alpha_3\|^{-\kappa}+\Theta_2(\log X)^{2d-2}+\Theta_3\log\log\log\log\|\alpha_3\|+\Theta_4.$$

The number of terms in the sum is of order $X^2\log X$ so Theorem 3.1.7 shows that the average value of $\nu(\alpha_1,\alpha_2,\alpha_3)$ is

$$O(X^{\kappa-1}\|\alpha_3\|^{-\kappa}(\log X)^{-1}+(\log X)^{2N-1}X^{-2}+(\log\log\log\log\|\alpha_3\|)^NX^{-2}(\log X)^{-1}).$$

Unless $\|\alpha_3\|\gg e^{e^{e^{X^2\log X}}}$ this improves (on average) on the pointwise bound of Evertse, Győry, Stewart and Tijdeman [17].

3.1.5 Outline of the proof

To prove Theorem 3.1.2 we argue by contradition. We assume that there exists a sequence $\mathcal{S}_{n_i}$ of n_i-optimal subsets where n_i tend to infinity. Let $V:=k\otimes_{\mathbb{Q}}\mathbb{R}\simeq\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$. First we show (Theorem 3.3.1) that for each n_i there exists a cylinder (see Definition 3.2.18) $\mathcal{C}_{n_i}\subseteq V$ of volume $O(n_i)$ containing S_{n_i}. This fact was implicit in the proofs of Theorem 3.1.2 for $k=\mathbb{Q}(\sqrt{-1})$ in [35] and for k quadratic imaginary in [11]. The argument in [11, 35] relied on a technique called "discrete collapsing" [1] which crucially uses the fact that the norm $N_{k/\mathbb{Q}}$ is convex for any quadratic imaginary number field k. Finding a way to prove Theorem 3.3.1 for a general number field k is one of the main contributions of this chapter. A key number-theoretical input is provided by Proposition 3.2.5 which counts the number of $x\in\mathcal{O}_k$ such that $|N_{k/\mathbb{Q}}(x(a-x))|\leq X^2$ for some $X>0$ and $a\in\mathcal{O}_k$ subject to the condition $|N_{k/\mathbb{Q}}(a)|\geq Xe^{-B}$ where B is a fixed real number. The proof of Proposition

[1] In [11] it was called simply "collapsing". We add the adjective discrete to distinguish it from the collapsing for measures used in the present work.

3.2.5 combines a variant of Ikehara's Tauberian theorem, counting points of $\mathcal{O}_k$ in thin cylinders and the Baker–Wüstholz's theorem on linear forms in logarithms.

Let Δ_k be the discriminant of k. From Theorem 3.3.1 we deduce (Corollary 3.3.2) that there exists a compact set Ω and sequences $(s_{n_i})_{i\in\mathbb{N}}, (t_{n_i})_{i\in\mathbb{N}} \subset V$ with $\|s_{n_i}\| = n_i|\Delta_k|^{1/2}$ such that the rescaled sets $s_{n_i}^{-1}(\mathcal{S}_{n_i} - t_{n_i})$ are all contained in Ω. Thus, it makes sense to look at subsequential weak-* limits of measures

$$\mu_{n_i} := \frac{1}{n_i}\sum_{x\in\mathcal{S}_{n_i}} \delta_{s_{n_i}^{-1}(x-t_{n_i})}.$$

Any such limit will be called a **limit measure**. It is always a probability measure supported on Ω, absolutely continuous with respect to the Lebesgue measure and of density[2] at most one[3] (see Lemma 3.5.2). By passing to a subsequence if necessary we can assume that μ_{n_i} converges to a limit measure μ. The measure μ contains the information about the asymptotic geometry of the sets $\mathcal{S}_{n_i}$. Our strategy is to exploit the properties of n-optimal sets to show that no such limit measure can exist. We introduce a notion of energy of probability measures on V (see Definition 3.5.3). For any compactly supported probability measure ν on V, absolutely continuous with respect to the Lebesgue measure and of bounded density we define

$$I(\nu) := \int_V \int_V \log\|x-y\| d\nu(x)d\nu(y),$$

where $\|\cdot\| : V \to \mathbb{R}$ extends the norm $|N_{k/Q}|$ from k to $V = k\otimes_{\mathbb{Q}}\mathbb{R}$. The volume formula for n-optimal sets (see [11, Corollary 5.2]) allows us to prove (Proposition 3.5.4) that for any limit measure μ we have

$$I(\mu) = -\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}},$$

where $\gamma_k, \gamma_{\mathbb{Q}}$ are the Euler–Kronecker constants of k and $\mathbb{Q}$ respectively (c.f. [22]). We know that the norm of the product of differences in an n-optimal set must be minimal among the norms of products of differences in all subsets of $\mathcal{O}_k$ of cardinality $n+1$ ([11, Corollary 5.2]). In other words the volume of an n-optimal set is minimal among volumes of subsets of $\mathcal{O}_k$ of cardinality $n+1$. This is used to show that μ minimizes the energy $I(\mu)$ among all probability measures of density bounded by one (Lemma 3.5.5). The last property forces strong geometric constraints on μ. In Proposition 3.5.6 we show that any such energy-minimizing measure must be of the form $\mu(A) = \mathrm{Leb}(A\cap U)$ where Leb is the Lebesgue measure on V and U is an open set of measure 1 whose boundary satisfies certain regularity conditions. This part of the argument uses the collapsing procedure

[2] By density we mean the Radon–Nikodym derivative with respect to the Lebesgue measure.

[3] The reason why we introduced the factor $|\Delta_k|^{1/2}$ in the formula $s_{n_i} = n_i|\Delta_k|^{1/2}$ is to ensure that the limits have density at most 1.

for measures (Definition 3.4.1) which is analogous to the discrete collapsing from [11] and similar to the Steiner symmetrization. We remark that if the field k is not imaginary quadratic then there is no reasonable discrete collapsing procedure for subsets of $\mathcal{O}_k$. The passage from subsets of $\mathcal{O}_k$ to measures on V seems crucial for this part of the argument.

At this point we have established that μ_{n_i} converges weakly-* to $\mu = \mathrm{Leb}|_U$ for some open subset U of V with sufficiently regular boundary. This is equivalent to saying that $\mathcal{S}_{n_i} = (\mathcal{O}_k \cap (s_{n_i}U + t_{n_i})) \sqcup R_{n_i}$ where the remainder satisfies $|R_{n_i}| = o(n_i)$. The idea for the last part of the proof is to show that for n_i sufficiently large, there is a prime ideal $\mathfrak{p}_{n_i}$ such that $\mathcal{S}_{n_i}$ fails to be almost uniformly equidistributed modulo $\mathfrak{p}_{n_i}$. This part is analogous to the proofs in [11,35] but slightly harder since we do not know the shape of U explicitly. This problem is solved by relating the almost uniform distribution of $\mathcal{S}_{n_i}$ with the lattice point discrepancy of U (see 3.6.1). If $\mathcal{S}_{n_i}$ were almost uniformly distributed modulo all prime ideals then the maximal discrepancy of U would be strictly less than 1 (Lemma 3.6.3). On the other hand, we show (Lemma 3.6.4) that once $\dim_{\mathbb{R}} V \geq 2$ and ∂U is smooth enough the maximal discrepancy of U must be strictly greater than 1. This is the only place in the proof where we use the assumption that $k \neq \mathbb{Q}$. We deduce that there must be a prime $\mathfrak{p}_{n_i}$ such that $\mathcal{S}_{n_i}$ in not uniformly equidistributed modulo $\mathfrak{p}_{n_i}$. This contradicts the fact that $\mathcal{S}_{n_i}$ is n_i-optimal and concludes the proof.

3.1.6 Notation

Let k be a number field of degree N and let $\mathcal{O}_k$ be the ring of integers of k. Numbers r_1, r_2 are respectively the number of real and complex places of k. Put $d = r_1 + r_2$. The field k is fixed throughout the chapter and so are the numbers N, r_1, r_2, d. We identify V with $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. For $v = (v_1, \ldots, v_d) \in V$ define $\|v\| = \prod_{i=1}^{r_1} |v_i| \prod_{i=r_1+1}^{r_1+r_2} |v_i|^2$. We will write $V^\times = \{v \in V | \, \|v\| \neq 0\}$ and $\mathcal{O}_k^\times$ for the unit group of $\mathcal{O}_k$. Let $N_{k/\mathbb{Q}} \colon k \to \mathbb{Q}$ be the norm of the extension $k/\mathbb{Q}$. The field k embeds in V and $\|x\| = |N_{k/\mathbb{Q}}(x)|$ for every $x \in k$. We write Δ_k for the discriminant of k. We use standard big-O and little-o notation. The base of all logarithms is e. We will write $\mathbb{A}, \mathbb{A}_\infty, \mathbb{A}_f$ for the rings of adeles, infinite adeles and finite adeles respectively[4]. We will write Leb for the Lebesgue measure on V, which is the product of Lebesgue measures on the real and complex factors. For any measure μ and measurable sets E, F we will write $\mu|_E(F) = \mu(E \cap F)$. We write $B_{\mathbb{R}}(x, R)$ ($B_{\mathbb{C}}(x, R)$) for the ball of radius R around $x \in \mathbb{R}$ ($x \in \mathbb{C}$). We will write $\mathcal{M}^1(V)$ (resp. $\mathcal{P}^1(V)$) for the set of finite (resp. probability) measures ν on V which are absolutely continuous with respect to the Lebesgue measure and such that the Radon–Nikodym derivative satisfies $d\nu(v)/d\mathrm{Leb}(v) \leq 1$ for almost every $v \in V$. For any real number t we will write $[t] = \max\{z \in \mathbb{Z} | \, z \leq t\}$. If G is a group we will write $\widehat{G}$ for the group of unitary characters of G.

[4] The adeles and ideles are present only in the last part of the Appendix, in the proof of Lemma 3.7.2.

3.1.7 Structure of the chapter

In Section 3.2 we develop estimates on the number of lattice points in $\mathcal{O}_k$ satisfying certain norm inequalities. The goal is to prove Proposition 3.2.5 and deduce Theorem 3.2.1. These inequalities control the number of points $x \in \mathcal{O}_k$ for which the product of norms $N_{k/\mathbb{Q}}((x-y)(x-z))$ is bounded where y, z are two far-away points in $\mathcal{O}_k$. In 3.2.1 we recall the Aramaki–Ikehara tauberian theorem. In 3.2.2 we prove Proposition 3.2.5 modulo some lemmas relying on diophantine approximation techniques. The sub-section 3.2.3 completes the missing part of the proof using Baker–Wüstholz inequalities on linear forms in logarithms. In 3.2.4 we explain briefly why Theorem 3.1.7 follows from Theorem 3.2.1.

Section 3.3 is devoted to the proof of Theorem 3.3.1. This is the technical heart of the chapter where we prove that n-optimal sets can be suitably renormalized. In 3.3.1 we gather some basic observations on norms of differences in an n-optimal set and in 3.3.2 couple them with the results of Section 3.2 to get the Theorem 3.3.1.

In Section 3.5 we define and study the properties of limit measures. In 3.5.1 we prove that they have density bounded by 1. In 3.5.2 we define the notion of energy for measures on V and prove that the limit measures must minimize the energy in the class of probability measures of density at most 1. Next in 3.5.3 we study the geometric properties of such energy minimizing measures and describe them in Proposition 3.5.6. The proof of Proposition 3.5.6 crucially uses the collapsing procedure which is described in detail in Section 3.4.

In Section 3.6 we show that limit measures cannot exist. In the first part 3.6.1 we recall the notion of lattice point discrepancy and relate it to limit measures. In 3.6.2 we prove Theorem 3.1.2.

The Section 3.4 is devoted to the collapsing procedure for measures on V. We mainly study its effect on the energy.

Finally in the Appendix we provide a proof of a folklore result on density of measures and likely well known variant of the prime number theorem for number fields.

3.2 Counting problem

The main result of this section is Proposition 3.2.5. It is a key ingredient in the proof of Theorem 3.3.1 on the shape of n-optimal sets in $\mathcal{O}_k$. As a corollary of Proposition 3.2.5 we get the following counting result that may be of independent interest.

Theorem 3.2.1. *Let k be a number field of degree N with d Archimedean places, let $B \in \mathbb{R}$ and put $\kappa = \min\left\{\frac{1}{2N(N-1)}, \frac{1}{4N-1}\right\}$. There exist constants $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ dependent only on k and B such that for every $X > 0$ and $a \in \mathcal{O}_k$ such that $\|a\| \geq Xe^{-B}$ we have*

$$|\left\{x \in \mathcal{O}_k | \ \|x(a-x)\| \leq X^2\right\}| \leq \Theta_1 X^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log X)^{2d-2} + \Theta_3 \log\log\log\log \|a\| + \Theta_4.$$

To state Proposition 3.2.5 we need to introduce some notations and auxiliary objects. For $v \in V$ we will write $|v|_i$ for the absolute value of i-th coordinate.

Definition 3.2.2. *A* ***good fundamental domain*** *of $\mathcal{O}_k^\times$ in $V^\times$ is a set $\mathcal{F}$ which is a finite union of convex closed cones in $V^\times$ such that $\mathcal{F}/\mathbb{R}^\times$ is compact in the projective space $\mathbb{P}(V)$, $\mathrm{int}\mathcal{F} \cap \lambda(\mathrm{int}\mathcal{F}) = \emptyset$ for every $\lambda \in \mathcal{O}_k^\times, \lambda \neq 1$ and $V^\times = \bigcup_{\lambda \in \mathcal{O}_k^\times} \lambda\mathcal{F}$. For technical reasons we will also require that the boundary $\partial\mathcal{F}$ does not contain any points of $\mathcal{O}_k$.*

We have the following elementary observation.

Lemma 3.2.3. *Let $\mathcal{F}$ be a good fundamental domain of $\mathcal{O}_k^\times$ in $V^\times$. Then there exists a constant $C_0 > 0$ such that every $v \in \mathcal{F}$ satisfies $C_0^{-1}\|v\|^{1/N} \leq |v|_i \leq C_0\|v\|^{1/N}$ for $i = 1, \ldots, d$.*

Proof. The set $\mathcal{F}/\mathbb{R}^\times$ is a compact subset of $V^\times/\mathbb{R}^\times$. The functions $v\mathbb{R}^\times \mapsto |v|_i/\|v\|^{1/N}$ and $v\mathbb{R}^\times \mapsto \|v\|^{1/N}/|v|_i$ are continuous so they are bounded and admit maxima on $\mathcal{F}/\mathbb{R}^\times$. We can take C_0 to be the biggest of the two maxima. □

We will often use this lemma in the latter part of the proof and sometimes we shall do so without additional comment. Let W_k be the torsion subgroup of $\mathcal{O}_k^\times$ and let $\xi_1, \ldots, \xi_{d-1}$ be a basis of a maximal torsion free subgroup[5] of $\mathcal{O}_k^\times$. Every element $\lambda \in \mathcal{O}_k^\times$ is uniquely expressed as a product $\lambda = w\xi_1^{b_1} \ldots \xi_{d-1}^{b_{d-1}}$ with $w \in W_k$ and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, d-1$. We define an l^∞ norm on $\mathcal{O}_k^\times$ by $\|\lambda\|_\infty := \max_{i=1,\ldots,d-1} |b_i|$. From now on we fix the basis $\xi_1, \ldots, \xi_{d-1}$ as well as the associated norm $\|\cdot\|_\infty$.

Lemma 3.2.4. *There exists a constant $\alpha > 0$ such that $\max_{i=1,\ldots,d} \log|\lambda|_i \geq \alpha\|\lambda\|_\infty$ for every $\lambda \in \mathcal{O}_k^\times$.*

Proof. Put $\|\lambda\|_0 := \max_{i=1,\ldots,d} \log|\lambda|_i$. Both $\|\cdot\|_0, \|\cdot\|_\infty$ extend uniquely to norms on $\mathcal{O}_k^\times \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^{d-1}$. Since any two norms on $\mathbb{R}^{d-1}$ are comparable, there exists a constant $\alpha > 0$ such that $\alpha\|\lambda\|_\infty \leq \|\lambda\|_0 \leq \alpha^{-1}\|\lambda\|_\infty$ for every $\lambda \in \mathcal{O}_k^\times$. □

By definition if we are given a good fundamental domain $\mathcal{F}$ then every element $y \in \mathcal{O}_k$ except 0 decomposes uniquely as $y = x\lambda$ for $x \in \mathcal{F} \cap \mathcal{O}_k, \lambda \in \mathcal{O}_k^\times$. Let us fix a good fundamental domain $\mathcal{F}$. For $a \in \mathcal{O}_k, a \neq 0$ and $X > 0$ we define the set

$$S(a, X) = \left\{(x, \lambda) \in (\mathcal{F} \cap \mathcal{O}_k) \times \mathcal{O}_k^\times | \, \|x(a - x\lambda^{-1})\| \leq X^2, \|x\| \leq X\right\}.$$

Proposition 3.2.5. *Let k be a number field of degree N with d Archimedean places, let $B \in \mathbb{R}$ and put $\kappa = \min\left\{\frac{1}{2N(N-1)}, \frac{1}{4N-1}\right\}$. Choose a good fundamental domain $\mathcal{F}$. There exist constants $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ dependent only on $k, \mathcal{F}$ and B such that for every $X > 0$ and $a \in \mathcal{O}_k$ such that $\|a\| \geq Xe^{-B}$ we have*

[5] We recall that the rank of $\mathcal{O}_k^\times$ is $d-1$ by Dirichlet's unit theorem.

1. $|S(a,X)| \leq \Theta_1 X^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log X)^{2d-2} + \Theta_3 \log\log\log\log\|a\| + \Theta_4$.

2. *Suppose that $a \in \mathcal{F}$. For every $\varepsilon > 0$ there exists M such that*

$$|\{(x,\lambda) \in S(a,X) \mid \|\lambda\|_\infty \geq M\}| \leq \\ \varepsilon X^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log X)^{2d-2} + \Theta_3 \log\log\log\log\|a\| + \Theta_4.$$

The proof consists of dividing the set $S(a,X)$ in two parts S_1, S_2 where S_1 consists of pairs (x,λ) where $\|\lambda\|_\infty$ is "not too big" compared to $\log\|a\| - \log\|x\|$ and S_2 is the complement of S_1. To estimate the size of S_1 we will use the Aramaki–Ikehara Tauberian theorem (Section 3.2.1) and to control S_2 we rely on Baker–Wüsholz's theorem on linear forms in logarithms and counting integer points in cylinders (Section 3.2.3). Theorem 3.2.1 is an easy consequence of Proposition 3.2.5.

Proof of Theorem 3.2.1. It is enough to show that $|\left\{x \in \mathcal{O}_k \mid \|x(a-x)\| \leq X^2\right\}| \leq 2|S(a,X)| + 2$. Note that the set $\left\{x \in \mathcal{O}_k \mid \|x(a-x)\| \leq X^2\right\}$ is invariant under the map $x \mapsto a - x$. The inequality $\|x(a-x)\| \leq X^2$ implies that either $\|x\| \leq X$ or $\|a - x\| \leq X$. For any such x different than 0 and a there exists a pair $(y,\lambda) \in S(a,X)$ such that $\lambda^{-1}y = x$ or $\lambda^{-1}y = a - x$. This proves that $|\left\{x \in \mathcal{O}_k \mid \|x(a-x)\| \leq X^2\right\}| \leq 2|S(a,X)| + 2$. Theorem 3.2.1 now follows[6] from Proposition 3.2.5 (1). □

3.2.1 Aramaki–Ikehara theorem

We will need an extension of the classical Tauberian theorem of Wiener and Ikehara due to Aramaki [3]. Our goal is Lemma 3.2.8 and it is the only result form this section that we will be using later.

Theorem 3.2.6. *(Aramaki [3]) Let $Z(s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s}$ be a Dirichlet series convergent for* $\mathrm{Re}(s)$ *sufficiently large. Assume that $Z(s)$ satisfies the following conditions:*

1. *$Z(s)$ has a meromorphic extension to $\mathbb{C}$ with poles on the real line.*

2. *$Z(s)$ has the first singularity at $s = a > 0$ and $A_j \in \mathbb{C}$ for $j = 1, \ldots, p$ are such that*

$$Z(s) - \sum_{j=1}^{p} \frac{A_j}{(j-1)!}\left(-\frac{d}{ds}\right)^{j-1}\frac{1}{s-a}$$

is holomorphic in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > a - \delta\}$ for some $\delta > 0$.

3. *$Z(s)$ is of polynomial order of growth with respect to* $\mathrm{Im}(s)$ *in all vertical strips, excluding neighborhoods of the poles.*

[6] With roughly 2 times bigger constants.

Then, there exists $\delta_0 > 0$ such that for all $X \geq 1$

$$\sum_{n \leq X} a_n = \sum_{j=1}^{p} \frac{A_j}{(j-1)!} \left(\frac{d}{ds}\right)^{j-1} \left(\frac{X^s}{s}\right)\bigg|_{s=a} + O(X^{a-\delta_0}).$$

Corollary 3.2.7. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that the Dirichlet series $Z(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ satisfies the hypotheses of Theorem 3.2.6. Then for every integer $m \geq 0$ and $X \geq 1$ we have*

1. *there exists $\delta_m > 0$ such that*

$$\sum_{n \leq X} a_n (\log n)^m = \sum_{j=1}^{p} \frac{A_j}{(j-1)!} \left(\frac{d}{ds}\right)^{m+j-1} \left(\frac{X^s}{s}\right)\bigg|_{s=a} + O(X^{a-\delta_m}).$$

2. *If $Z(s)$ has a simple pole at 1 with residue ρ then there exists $\delta > 0$ such that*

$$\sum_{n \leq X} a_n (\log X - \log n)^m = m!\rho X + O(X^{1-\delta}).$$

Proof. 1. Note that $\sum_{n=1}^{\infty} \frac{a_n (\log n)^m}{n^s} = \left(-\frac{d}{ds}\right)^m Z(s)$. The derivative $\left(-\frac{d}{ds}\right)^m Z(s)$ is meromorphic on $\mathbb{C}$ with poles on the real line. Cauchy's integral formula implies that $\left(-\frac{d}{ds}\right)^m Z(s)$ is of polynomial order of growth with respect to $\mathrm{Im}(s)$ on vertical strips away from the poles. The desired formula follows from Aramaki theorem applied to $\left(-\frac{d}{ds}\right)^m Z(s)$.

2. By the previous point we have $\sum_{n\leq X} a_n (\log n)^m = \rho \left(\frac{d}{ds}\right)^m \left(\frac{X^s}{s}\right)\Big|_{s=1} + O(X^{1-\delta_m})$. We use this identity in the following computation:

$$\begin{aligned}
&\sum_{n \leq X} a_n (\log X - \log n)^m = \\
&\quad = \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} (\log X)^l \sum_{n \leq X} a_n (\log n)^{m-l} \\
&\quad = \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \left(\frac{d}{ds}\right)^l X^{s-1}\bigg|_{s=1} \rho \left(\frac{d}{ds}\right)^{m-l} \left(\frac{X^s}{s}\right)\bigg|_{s=1} + O(X^{1-\delta}) \\
&\quad = \rho \sum_{l=0}^{m} \binom{m}{l} \left(-\frac{d}{ds}\right)^l X^{1-s}\bigg|_{s=1} \left(-\frac{d}{ds}\right)^{m-l} \left(\frac{X^s}{s}\right)\bigg|_{s=1} + O(X^{1-\delta}) \\
&\quad = \rho \left(-\frac{d}{ds}\right)^m \left(X^{1-s}\frac{X^s}{s}\right)\bigg|_{s=1} + O(X^{1-\delta}) \\
&\quad = m!\rho X + O(X^{1-\delta})
\end{aligned}$$

where $\delta = \min\{\delta_0, \ldots, \delta_m\}$.

□

The following lemma is a key ingredient in the proof of Proposition 3.2.5.

Lemma 3.2.8. *Let ρ_k be the residue of the Dedekind zeta function $\zeta_k(s)$ at $s = 1$, let h_k be the class number of k and let w_k be the size of the torsion subgroup of $\mathcal{O}_k^\times$. For every $m \geq 0$ there exists $\delta_0 > 0$ such that for every $X \geq 1$ we have*

1. $$\sum_{\substack{a \in \mathcal{O}_k/\mathcal{O}_k^\times \\ 0 < N(a) \leq X}} \log N(a)^m = \frac{\rho_k}{h_k} X (\log X)^m + O(X^{1-\delta_0})$$

   and

2. $$\sum_{\substack{a \in \mathcal{O}_k/\mathcal{O}_k^\times \\ 0 < N(a) \leq X}} (\log X - \log N(a))^m = m! \frac{\rho_k}{h_k} X + O(X^{1-\delta_0}).$$

Proof. Let $\chi_1, \ldots, \chi_{h_k}$ be the characters of the class group of k, with $\chi_1 = 1$. The L-functions $L(s, k, \chi) = \sum_{\mathfrak{a}} \frac{\chi_i(\mathfrak{a})}{(N\mathfrak{a})^s}$ are entire for $i \geq 2$ and $L(s, k, 1)$ is the Dedekind zeta function of k with unique simple pole at $s = 1$ with residue ρ_k. All of them are of polynomial growth on vertical strips. Consider the Dirichlet series

$$G(s) = \sum_{\substack{a \in \mathcal{O}_k/\mathcal{O}_k^\times \\ 0 < N(a)}} \frac{1}{N(a)^s} = \sum_{\substack{\mathfrak{a} \\ \text{principal}}} \frac{1}{(N\mathfrak{a})^s} = \frac{1}{h_k} \sum_{i=1}^{h_k} L(s, k, \chi_i).$$

It has non-negative coefficients and extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue $\frac{\rho_k}{h_k}$. Equalities (1),(2) follow from Corollary 3.2.7 applied to $G(s)$. □

3.2.2 Proof of Proposition 3.2.5

We adopt the following convention. The constants C_i, B_i appearing in the inequalities successively throughout the proof are dependent on k and B alone. This is usually not a straightforward observation, but the proof is structured so that it is clear that C_i, B_i depend only on k, B and the constants C_j, B_j for $j < i$. As we want to keep the proof reasonably short we omit the computations of exactly how big C_i, B_i should be in terms of k and B.

Proof. (1) The problem is invariant under multiplying a by $\mathcal{O}_k^\times$ so we may assume, without loss on generality, that $a \in \mathcal{F}$. Recall that $\|a\| \geq Xe^{-B}$ and

$$\begin{aligned} S := S(a,X) &= \big\{(x,\lambda) \in (\mathcal{F} \cap \mathcal{O}_k) \times \mathcal{O}_k^\times | \|x(a - x\lambda^{-1})\| \leq X^2, \|x\| \leq X\big\} \\ &= \left\{(x,\lambda) | \log\left\|\lambda - \frac{x}{a}\right\| \leq 2\log X - \log\|a\| - \log\|x\|, \|x\| \leq X\right\}. \end{aligned}$$

Let α be the constant from Lemma 3.2.4. We define

$$S_1 := \left\{(x,\lambda) \in S | \|\lambda\|_\infty \leq \frac{2}{\alpha}\left(2\log X - \left(2 - \frac{1}{2N}\right)\log\|x\| - \frac{1}{2N}\log\|a\|\right)\right\}$$

and $S_2 := S \setminus S_1$. We start be estimating the size of S_1. We will use the fact that for non-negative R the number of $\lambda \in \mathcal{O}_k^\times$ with $\|\lambda\|_\infty \leq R$ is at most $O(R^{d-1}) + |W_k|$.

$$\begin{aligned} |S_1| &\leq \sum_{\substack{x \in \mathcal{F} \cap \mathcal{O}_k \\ \|x\| \leq X}} \left|\left\{\lambda \in \mathcal{O}_k^\times | \|\lambda\|_\infty \leq \frac{2}{\alpha}\left(2\log X - \left(2 - \frac{1}{2N}\right)\log\|x\| - \frac{1}{2N}\log\|a\|\right)\right\}\right| \\ &= \sum_{\substack{x \in \mathcal{F} \cap \mathcal{O}_k \\ \|x\| \leq X}} \left|\left\{\lambda \in \mathcal{O}_k^\times | \|\lambda\|_\infty \leq \frac{(4N-1)}{N\alpha}\left(\frac{4N}{4N-1}\log X - \frac{1}{4N-1}\log\|a\| - \log\|x\|\right)\right\}\right| \end{aligned}$$

Put $\log Y = \frac{4N}{4N-1}\log X - \frac{1}{4N-1}\log\|a\|$. The summands in the last formula vanish unless $\|x\| \leq Y$ so we get

$$\begin{aligned} |S_1| &\leq \sum_{\substack{x \in \mathcal{F} \cap \mathcal{O}_k \\ \|x\| \leq Y}} \left|\left\{\lambda \in \mathcal{O}_k^\times | \|\lambda\|_\infty \leq \frac{(4N-1)}{N\alpha}(\log Y - \log\|x\|)\right\}\right| \\ &\leq \sum_{\substack{x \in \mathcal{F} \cap \mathcal{O}_k \\ \|x\| \leq Y}} \left(C_1(\log Y - \log\|x\|)^{d-1} + C_2\right) \\ &\leq C_3 Y = C_3 X^{1+\frac{1}{4N-1}}\|a\|^{-\frac{1}{4N-1}}. \end{aligned}$$

The last passage uses Lemma 3.2.8. It remains to bound the size of $|S_2|$.

Lemma 3.2.9. *Put $B_1 := \alpha^{-1}((\log X - \log\|a\|)N^{-1} + 2\log C_0 + \log 2)$ where C_0 is as in Lemma 3.2.3. Let $(x,\lambda) \in S_2$. Then either $\|\lambda\|_\infty < B_1$ or there exists $i \in \{1,\ldots,d\}$ such that*

$$\log\left|\frac{x}{a} - \lambda\right|_i \leq -\frac{\alpha\|\lambda\|_\infty}{2N-2} - \left(\frac{1}{N} + \frac{1}{2N(N-1)}\right)(\log\|a\| - \log\|x\|) + \log 2. \qquad (3.2.1)$$

Proof. Assume that $\|\lambda\|_\infty \geq B_1$ and that $(x,\lambda) \in S_2$. Let $j \in \{1,\ldots,d\}$ be such that $|\lambda|_j$ is maximal. By Lemma 3.2.4 we have $\log|\lambda|_j \geq \alpha\|\lambda\|_\infty$. Since $\|\lambda\|_\infty \geq B_1$ we have $\log|\lambda|_j \geq (\log X - \log\|a\|)N^{-1} + 2\log C_0 + \log 2$. By Lemma 3.2.3 $\log\left|\frac{x}{a}\right|_j \leq (\log\|x\| - \log\|a\|)N^{-1} + 2\log C_0 \leq (\log X - \log\|a\|)N^{-1} + 2\log C_0$. It follows that $|\lambda|_j \geq 2|\frac{x}{a}|_j$ so we have $\log|\frac{x}{a} - \lambda|_j \geq \log|\lambda|_j - \log 2$. From this and the fact that $(x,\lambda) \in S_2$ we deduce that

$$\log|\lambda|_j \geq \alpha\|\lambda\|_\infty \geq \frac{\alpha}{2}\|\lambda\|_\infty + \left(2\log X - \frac{4N-1}{2N}\log\|x\| - \frac{1}{2N}\log\|a\|\right)$$

and

$$\log\left|\frac{x}{a} - \lambda\right|_j \geq \frac{\alpha}{2}\|\lambda\|_\infty + \left(2\log X - \frac{4N-1}{2N}\log\|x\| - \frac{1}{2N}\log\|a\|\right) - \log 2. \tag{3.2.2}$$

At the same time $|\lambda|_j \geq 1$ because $\|\lambda\| = 1$ so we also have $\log\left|\frac{x}{a} - \lambda\right| \geq -\log 2$. This observation is valid even if $B_1 < 0$. By definition of S we have

$$\log\left\|\frac{x}{a} - \lambda\right\| \leq 2\log X - \log\|a\| - \log\|x\|.$$

Let $f = 1$ if $j > r_1$ and $f = 0$ otherwise. Substracting (3.2.2) we get

$$\sum_{i=1, i\neq j}^{r_1} \log\left|\frac{x}{a} - \lambda\right|_i + 2\sum_{i=r_1+1, i\neq j}^{d} \log\left|\frac{x}{a} - \lambda\right|_i + f\log\left|\frac{x}{a} - \lambda\right|_j \leq$$
$$-\frac{\alpha}{2}\|\lambda\|_\infty - \frac{2N-1}{2N}(\log\|a\| - \log\|x\|) + \log 2.$$

At least one term in the sum must be smaller or equal to the average. Therefore, for some i we have

$$\log\left|\frac{x}{a} - \lambda\right|_i \leq -\frac{\alpha\|\lambda\|_\infty}{2N-2} - \left(\frac{1}{N} + \frac{1}{2N(N-1)}\right)(\log\|a\| - \log\|x\|) + \frac{\log 2}{N-1}. \tag{3.2.3}$$

This is slightly better than what we needed to prove. □

Put $S_2^0 := \{(x,\lambda) \in S_2 | \|\lambda\|_\infty \leq B_1\}$ and for $i = 1,\ldots,d$ let

$$S_2^i := \{(x,\lambda) \in S_2 | \text{ inequality (3.2.1) holds }\}. \tag{3.2.4}$$

Lemma 3.2.10. *There is a constant C_5 dependent only on k, B such that*

$$|S_2^0| \leq C_5 X^{1+\kappa}\|a\|^{-\kappa}.$$

Proof. The number of λ satisfying $\|\lambda\|_\infty \leq B_1$, where B_1 is as in Lemma 3.2.9, is at most

$O(1+B_1)^{d-1} \leq O(X^{\kappa}\|a\|^{-\kappa})$ so there is a constant C_4 such that

$$|S_2^0| \leq C_4 X^{\kappa}\|a\|^{-\kappa} \sum_{\substack{x\in\mathcal{F}\cap\mathcal{O}_k\\ \|x\|\leq X}} 1 \leq C_5 X^{1+\kappa}\|a\|^{-\kappa}.$$

The last inequality uses Lemma 3.2.8. □

We have the following estimate on $|S_2^i|$ for $i=1,\ldots,d$.

Lemma 3.2.11. *Let $\kappa' = \frac{1}{2N(N-1)}$. There are constants C_6, C_7, C_8, C_9 dependent on k, B alone such that for $i=1,\ldots,d$ we have*

$$|S_2^i| \leq C_6 X^{1+\kappa'}\|a\|^{-\kappa'} + C_7(\log X)^{2(d-1)} + C_8 \log\log\log\log\|a\| + C_9.$$

The proof of the Lemma 3.2.11 relies on Baker–Wüstholz's bounds on linear forms in logarithms. We postpone it to the next section. By Lemma 3.2.9 we have $S_2 = \bigcup_{i=0}^{d} S_2^i$ so

$$\begin{aligned}
|S| \leq& |S_1| + \sum_{i=0}^{d} |S_2^i| && (3.2.5)\\
\leq& C_3 X^{1+\frac{1}{4N-1}}\|a\|^{-\frac{1}{4N-1}} + C_5 X^{1+\kappa}\|a\|^{-\kappa} + dC_6 X^{1+\kappa'}\|a\|^{-\kappa'} && (3.2.6)\\
&+ dC_7(\log X)^{2(d-1)} + dC_8\log\log\log\log\|a\| + dC_9. && (3.2.7)
\end{aligned}$$

As $\kappa = \min\left\{\frac{1}{4N-1}, \kappa'\right\}$ and $X\|a\|^{-1} \leq e^B$ we can deduce that

$$|S| \leq \Theta_1 X^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log X)^{2(d-1)} + \Theta_3 \log\log\log\log\|a\| + \Theta_4, \qquad (3.2.8)$$

where $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ depend only on k, B. This proves the first part of Proposition 3.2.5.

(2) Let $M > 0$. Put $S[M] := \{(x,\lambda) \in S \mid \|\lambda\|_\infty \geq M\}$ and $S_1[M] = S_1 \cap S[M], S_2[M] = S_2 \cap S[M], S_2^i[M] = S_2^i \cap S[M]$. The proof of this case is reduced to the following lemmas.

Lemma 3.2.12. *For every $\delta > 0$ there exists M_1 such that for every $M \geq M_1$*

$$S_1[M] \leq \delta X^{1+\kappa}\|a\|^{-\kappa}.$$

Proof.

$$\begin{aligned}
&|S_1[M]| \leq \\
&\sum_{\substack{x\in\mathcal{F}\cap\mathcal{O}_k\\ \|x\|\leq X}} \left|\left\{\lambda \in \mathcal{O}_k^\times \middle| M \leq \|\lambda\|_\infty \leq \frac{2}{\alpha}\left(2\log X - \left(2 - \frac{1}{2N}\right)\log\|x\| - \frac{1}{2N}\log\|a\|\right)\right\}\right| \\
=& \sum_{\substack{x\in\mathcal{F}\cap\mathcal{O}_k\\ \|x\|\leq X}} \left|\left\{\lambda \in \mathcal{O}_k^\times \middle| M \leq \|\lambda\|_\infty \leq \frac{4N-1}{N\alpha}\left(\frac{4N}{4N-1}\log X - \frac{1}{4N-1}\log\|a\| - \log\|x\|\right)\right\}\right|
\end{aligned}$$

The summands in the last formula vanish unless $M \leq \frac{(4N-1)}{N\alpha}(\frac{4N}{4N-1}\log X - \frac{1}{4N-1}\log\|a\| - \log\|x\|)$ i.e. $\log\|x\| \leq \frac{4N}{4N-1}\log X - \frac{1}{4N-1}\log\|a\| - \frac{N\alpha}{(4N-1)}M$. Put $\log Y_M = \frac{4N}{4N-1}\log X - \frac{1}{4N-1}\log\|a\| - \frac{MN\alpha}{(4N-1)}$. We get

$$\begin{aligned}
|S_1[M]| \leq & \sum_{\substack{x\in\mathcal{F}\cap\mathcal{O}_k \\ \|x\|\leq Y_M}} \left|\left\{\lambda \in \mathcal{O}_k^\times \middle| \|\lambda\|_\infty \leq \frac{4N-1}{N\alpha}\left(\log Y_M - \log\|x\|\right)\right\}\right| \\
\leq & \sum_{\substack{x\in\mathcal{F}\cap\mathcal{O}_k \\ \|x\|\leq Y_M}} \left(C_1\left(\log Y_M - \log\|x\|\right)^{d-1} + C_2\right) \\
\leq & C_3 Y_M = C_3 X^{1+\frac{1}{4N-1}}\|a\|^{-\frac{1}{4N-1}} e^{-\frac{MN\alpha}{4N-1}}.
\end{aligned}$$

For the last inequality we have used Lemma 3.2.8. As $\|a\| \geq Xe^{-B}$ we have $|S_1[M]| \leq C_{21}X^{1+\kappa}\|a\|^{-\kappa}e^{-\frac{MN\alpha}{4N-1}}$. Clearly for $M \geq M_1$ sufficiently large we have $C_{21}e^{-\frac{MN\alpha}{4N-1}} \leq \delta$. The Lemma is proven.

□

We have the following analogue of Lemma 3.2.11.

Lemma 3.2.13. *Let $\kappa' = \frac{1}{2N(N-1)}$. For every $\delta > 0$ and $i = 1, \ldots, d$ there exists M_2 such that for every $M \geq M_2$*

$$S_2^i[M] \leq \delta X^{1+\kappa'}\|a\|^{-\kappa'} + C_7(\log X)^{2(d-1)} + C_8\log\log\log\log\|a\| + C_9.$$

The proof is postponed to the next section. We will also need the following trivial observation.

Lemma 3.2.14. *There exists M_3 such that for every $M \geq M_3$ the set $S_2^0[M]$ is empty.*

We are ready to prove Proposition 3.2.5 (2). Choose M such that $S_2^0[M]$ is empty, Lemma 3.2.12 and Lemma 3.2.13 hold with $\delta = \frac{\varepsilon}{e^B(d+1)}$. By Lemma 3.2.9 we have:

$$S[M] = S_1[M] \cup S_2[M] = S_1[M] \cup S_2^0[M] \cup \bigcup_{i=1}^{d} S_2^i[M]$$

$$|S[M]| \leq \varepsilon X^{1+\kappa}\|a\|^{-\kappa} + dC_7(\log X)^{2(d-1)} + dC_8\log\log\log\log\|a\| + dC_9.$$

This concludes the proof of Proposition 3.2.5. □

3.2.3 Linear forms in logarithms and bound on $|S_2^i|$

The aim of this section is to show Lemma 3.2.11 i.e. an upper bound on $|S_2^i|$ where S_2^i is the set defined by (3.2.4). Next we apply more or less the same argument to prove Lemma 3.2.13. Our main tool is the Baker–Wüstholz inequality on linear forms in logarithms [4, Theorem 7.1]. We recall the definition of the logarithmic Weil height of an algebraic number. Let K be a finite extension of $\mathbb{Q}$ and let $\omega \in K$. Write Σ for the set of valuations of K.

Definition 3.2.15. *The logarithmic Weil height of ω is defined as*

$$h(\omega) = \frac{1}{[K:\mathbb{Q}]} \sum_{\nu \in \Sigma} a_\nu \max\{0, \log|\omega|_\nu\},$$

where $a_\nu = 2$ if ν is a complex Archimedean place and $a_\nu = 1$ otherwise. The value of $h(\omega)$ does not depend on the choice of K.

The height enjoys the following sub-additivity property $h(xy) \leq h(x) + h(x)$ and $h(x/y) \leq h(x) + h(y)$. For later use we define $h'(\omega) = \max\{h(\omega), 1\}$. This definition agrees with the one from [4, 7.2] up to a constant depending only on $[\mathbb{Q}(\omega) : \mathbb{Q}]$.

Theorem 3.2.16 ([4, Theorem 7.1])**.** *Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ and let $\log \alpha_i$ be the value of the main branch of logarithm for $i = 1, \ldots, n$. Let $D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]$. For every $b_1, \ldots, b_n \in \mathbb{Z}$ such that*

$$\Lambda := b_1 \log \alpha_1 + \ldots b_n \log \alpha_n \neq 0$$

we have

$$\log|\Lambda| \geq -C_{n,D} h'(\alpha_1) \ldots h'(\alpha_n) \max\left\{1, \log \max_{i=1,\ldots,n} |b_i|\right\},$$

where $C_{n,D}$ is a constant depending only on n and D.

We will apply this Theorem with α_i being equal to the absolute values of units in $\mathcal{O}_k^\times$ or elements of $\mathcal{F} \cap \mathcal{O}_k$. Recall that $\xi_1, \ldots, \xi_{d-1}$ form a basis of a maximal torsion free subgroup of $\mathcal{O}_k^\times$ so that every element $\lambda \in \mathcal{O}_k^\times$ can be written as $\lambda = w\xi_1^{b_1} \ldots \xi_{d-1}^{b_{d-1}}$ with w torsion and $\|\lambda\|_\infty := \max_{i=1,\ldots,d-1} |b_i|$.

Corollary 3.2.17. *Let $x, y \in \mathcal{F} \cap \mathcal{O}_k$, let $i \in \{1, \ldots, d\}$ and let $\lambda \in \mathcal{O}_k^\times$. Then*

$$\log\left|\log\left|\frac{x}{y}\lambda\right|_i\right| \geq -C_{10}(1 + \log\|x\| + \log\|y\|) \max\{1, \log\|\lambda\|_\infty\},$$

where C_{10} depends only on k and the choice of $\xi_1, \ldots, \xi_{d-1}$.

Proof. As $x, y \in \mathcal{F} \cap \mathcal{O}_k$ the definition of Weil height with $K = k$ and Lemma 3.2.3 imply that $h(x) \leq \frac{1}{N}\log\|x\| + \log C_0$ and similarly for y. We have $h'(\frac{x}{y}) \leq 1 + h(\frac{x}{y}) \leq$

$1 + h(x) + h(y) = O(1 + \log\|x\| + \log\|y\|)$. Write $\lambda = w\xi_1^{b_1}\xi_2^{b_2}\dots\xi_{d-1}^{b_{d-1}}$ with w being a torsion element. Theorem 3.2.16 yields

$$\log\left|\log\left|\frac{x}{y}\lambda\right|_i\right| \geq -C_{d,N!}h'(\frac{x}{y})h'(\xi_1)\dots h'(\xi_{d-1})\max\left\{1, \log\max_{j=1,\dots,d-1}|b_j|\right\}.$$

Since $\max_{j=1,\dots,d-1}|b_j| = \|\lambda\|_\infty$ the Corollary follows. □

Definition 3.2.18. *A cylinder in $V \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ is a set $\mathcal{C}$ which is a coordinate-wise product of closed balls*

$$\mathcal{C} = \prod_{i=1}^{r_1} B_{\mathbb{R}}(t_i, R_i) \times \prod_{i=r_1+1}^{d} B_{\mathbb{C}}(t_i, R_i),$$

with $t_i \in \mathbb{R}$ for $i = 1, \dots, r_1$, $t_i \in \mathbb{C}$ for $i = r_1+1, \dots, d$ and $R_i \in \mathbb{R}_{\geq 0}$ for $i = 1, \dots, d$.

Lemma 3.2.19. *Let $\mathcal{C}$ be a cylinder. Then $|\mathcal{C} \cap \mathcal{O}_k| \leq 1 + C_{11}\mathrm{Leb}(\mathcal{C})$ where C_{11} is a constant depending only on k.*

Proof. First we prove that any cylinder $\mathcal{C}'$ of volume strictly below $\pi^{r_2}4^{-r_2}$ cannot contain more than one point of $\mathcal{O}_k$. Write

$$\mathcal{C}' = \prod_{i=1}^{r_1} B_{\mathbb{R}}(t_i', R_i') \times \prod_{i=r_1+1}^{d} B_{\mathbb{C}}(t_i', R_i').$$

If $x, y \in \mathcal{C}'$ then $|x-y|_i \leq 2R_i$ for every $i = 1, \dots, d$. We deduce that

$$\|x-y\| \leq \prod_{i=1}^{r_1} 2R_1 \prod_{i=r_1+1}^{d} 4R_2^2 = 4^{r_2}\pi^{-r_2}\mathrm{Leb}\mathcal{C}' < 1.$$

On the other hand if $x, y \in \mathcal{O}_k$ are distinct then $\|x-y\| = |N_{k/\mathbb{Q}}(x-y)| \geq 1$. Hence $\mathcal{C}'$ can contain at most one point from $\mathcal{O}_k$. The lemma follows since we can cover $\mathcal{C}$ with at most $1 + C_{11}\mathrm{Leb}(\mathcal{C})$ cylinders of volume $\pi^{r_2}4^{-r_2}$. □

Lemma 3.2.20. *For every $z \in \mathbb{C}$ with $|1-z| \leq \frac{1}{2}$ we have* $\log|\log|z|| \leq \log|1-z| + \log 2$.

Proof. Let $z = 1-t$. Then $|t| \leq \frac{1}{2}$ and $\log|z| = -\sum_{n=1}^{\infty}\frac{t^n}{n}$. Hence $|\log|z|| \leq 2|t|$ and consequently $\log|\log|z|| \leq \log|1-z| + \log 2$. □

We can are ready to prove Lemma 3.2.11.

Proof of Lemma 3.2.11. Recall that $\kappa' := \frac{1}{2N(N-1)}$. For $\lambda \in \mathcal{O}_k^\times$ define

$$S_2^i(\lambda) := \{x \in \mathcal{F} \cap \mathcal{O}_k | (x, \lambda) \in S_2^i\}$$
$$T^i := \{\lambda \in \mathcal{O}_k^\times | S_2^i(\lambda) \neq \emptyset\}.$$

Put $\beta = \frac{\alpha}{2N-2}$. By definition, for every $x \in S_2^i(\lambda)$ we have

$$\begin{aligned}\log\left|\frac{x}{a}-\lambda\right|_i \leq& -\beta\|\lambda\|_\infty - \left(\frac{1}{N}+\kappa'\right)(\log\|a\| - \log\|x\|) + \log 2.\\ \log|x-a\lambda|_i \leq& -\beta\|\lambda\|_\infty + \log|a|_i - \left(\frac{1}{N}+\kappa'\right)(\log\|a\| - \log\|x\|) + \log 2\\ \leq& -\beta\|\lambda\|_\infty - \kappa'(\log\|a\| - \log X) + \frac{1}{N}\log X + \log 2 + \log C_0.\end{aligned}$$

By Lemma 3.2.3 the set $\{x \in \mathcal{F}\cap\mathcal{O}_k \,|\, \|x\| \leq X\}$ is contained in the cylinder

$$\prod_{j=1}^{r_1} B_{\mathbb{R}}(0, C_0X^{1/N}) \times \prod_{j=r_1+1}^{d} B_{\mathbb{C}}(0, C_0X^{1/N}).$$

Hence, $S_2^i(\lambda)$ is contained in the cylinder

$$\begin{aligned}\mathcal{C}^i(\lambda) = \prod_{j=1,j\neq i}^{r_1} B_{\mathbb{R}}(0, C_0X^{1/N}) \times &\prod_{j=r_1+1,j\neq i}^{d} B_{\mathbb{C}}(0, C_0X^{1/N}) \times\\ &B_{\mathbb{K}}(a_i\lambda_i, 2C_0X^{1/N+\kappa'}\|a\|^{-\kappa'}e^{-\beta\|\lambda\|_\infty}),\end{aligned}$$

where $\mathbb{K} = \mathbb{R}$ if $i = 1, \ldots, r_1$ and $\mathbb{K} = \mathbb{C}$ otherwise. We have

$$\mathrm{Leb}(\mathcal{C}^i(\lambda)) \leq C_{12}X^{1+\kappa'}\|a\|^{-\kappa'}e^{-\beta\|\lambda\|_\infty} \quad \text{if} \quad i = 1, \ldots, r_1$$

and

$$\mathrm{Leb}(\mathcal{C}^i(\lambda)) \leq C_{12}X^{1+2\kappa'}\|a\|^{-2\kappa'}e^{-2\beta\|\lambda\|_\infty} \quad \text{if} \quad i = r_1+1, \ldots, r_2.$$

We work under assumption that $X \leq \|a\|e^B$ so in the second case we have

$$C_{12}X^{1+2\kappa'}\|a\|^{-2\kappa'}e^{-2\beta\|\lambda\|_\infty} \leq e^{B\kappa'}C_{12}X^{1+\kappa'}\|a\|^{-\kappa'}e^{-\beta\|\lambda\|_\infty}.$$

By Lemma 3.2.19 we get

$$|S_2^i(\lambda)| \leq 1 + C_{11}\mathrm{Leb}(\mathcal{C}^i(\lambda)) \leq 1 + C_{13}X^{1+\kappa'}\|a\|^{-\kappa'}e^{-\beta\|\lambda\|_\infty}.$$

Hence

$$\begin{aligned}|S_2^i| \leq \sum_{\lambda\in T^i}|S_2^i(\lambda)| \leq& |T^i| + C_{13}X^{1+\kappa'}\|a\|^{-\kappa'}\sum_{\lambda\in\mathcal{O}_k^\times} e^{-\beta\|\lambda\|_\infty} \qquad (3.2.9)\\ \leq& |T^i| + C_{14}X^{1+\kappa'}\|a\|^{-\kappa'}. \qquad (3.2.10)\end{aligned}$$

It remains to bound $|T^i|$. First we show that for every $\lambda \in T^i$ we have

$$\|\lambda\|_\infty \leq C_{15} \log \|a\| \log\log \|a\| + C_{16}$$

(equation (3.2.13). We have

$$\begin{aligned}
\log \left|\frac{x}{a} - \lambda\right|_i \leq & -\beta\|\lambda\|_\infty - \left(\frac{1}{N} + \kappa'\right)(\log\|a\| - \log\|x\|) + \log 2. \\
\log \left|1 - \frac{a}{x}\lambda\right|_i \leq & -\beta\|\lambda\|_\infty + \log|a|_i - \log|x|_i - \left(\frac{1}{N} + \kappa'\right)(\log\|a\| - \log\|x\|) + \log 2 \\
\leq & -\beta\|\lambda\|_\infty - \kappa'(\log\|a\| - \log X) + \log 2 + 2\log C_0 \\
\leq & -\beta\|\lambda\|_\infty + \kappa' B + \log 2 + 2\log C_0 =: -\beta\|\lambda\|_\infty + B_2.
\end{aligned}$$

Here we define the constant $B_2 = \kappa' B + \log 2 + 2\log C_0$ to lighten the notation. It follows that for $\|\lambda\|_\infty \geq \frac{B_2 + \log 2}{\beta}$ we will have $\left|1 - \frac{a}{x}\lambda\right|_i \leq \frac{1}{2}$ and by Lemma 3.2.20

$$\log|\log\left|\frac{a}{x}\lambda\right|_i| \leq -\beta\|\lambda\|_\infty + B_2 + \log 2. \tag{3.2.11}$$

Put $B_3 = \max\left\{\frac{B_2+\log 2}{\beta}, 3\right\}$. For $\|\lambda\|_\infty \geq B_3$ Corollary 3.2.17 yields

$$-C_{10}(1 + \log\|x\| + \log\|a\|)\log\|\lambda\|_\infty \leq -\beta\|\lambda\|_\infty + B_2 + \log 2. \tag{3.2.12}$$

The only thing we used is that $\|\lambda\|_\infty \geq 3$ so $\max\{1, \log\|\lambda\|_\infty\} = \log\|\lambda\|_\infty$. Using inequalities $\log\|x\| \leq \log X \leq \log\|a\| + B$ we get

$$-C_{10}(1 + B + 2\log\|a\|)\log\|\lambda\|_\infty \leq -\beta\|\lambda\|_\infty + B_2 + \log 2.$$

For $\|\lambda_\infty\| \geq B_3 \geq \frac{B_2+\log 2}{\beta}$ we deduce that

$$\|\lambda\|_\infty \leq C_{15}\log\|a\|\log\log\|a\| + C_{16}. \tag{3.2.13}$$

We proved this inequality under the assumption that $\|\lambda\|_\infty \geq B_3$ but by making C_{16} bigger if necessary this inequality is also valid if $\|\lambda\|_\infty \leq B_3$. Inequality 3.2.13 already implies a non-trivial upper bound of form $|T^i| = O((\log\|a\|\log\log\|a\|)^{d-1}) + O(1)$. This is too weak for our purposes when $\|a\|$ is large. To get the desired bound we need to consider the relations between pairs $\lambda, \lambda' \in T^i$ with $B_3 \leq \|\lambda\|_\infty \leq \|\lambda'\|_\infty$.

Lemma 3.2.21. *Let $\lambda, \lambda' \in T^i$ with $B_3 \leq \|\lambda\|_\infty \leq \|\lambda'\|_\infty$. Then*

$$\beta\|\lambda\|_\infty - B_2 - 2\log 2 \leq C_{10}(1 + 2\log X)\log 2\|\lambda'\|_\infty.$$

Proof. Let $x, x' \in \mathcal{F} \cap \mathcal{O}_k$ be such that $(x,\lambda), (x',\lambda') \in S_2^i$. Inequality (3.2.11) yields

$|\log|\frac{a}{x}\lambda|_i| \leq 2e^{-\beta\|\lambda\|_\infty + B_2}$ and the same for λ'. Taking the difference we get

$$\left|\log\left|\frac{x'}{x}\lambda\lambda'^{-1}\right|_i\right| = \left|\log\left|\frac{a}{x}\lambda\right|_i - \log\left|\frac{a}{x'}\lambda'\right|_i\right| \leq 2e^{B_2}(e^{-\beta\|\lambda\|_\infty} + e^{-\beta\|\lambda'\|_\infty}) \leq 4e^{-\beta\|\lambda\|_\infty + B_2}.$$

Using Corollary 3.2.17 we get

$$\begin{aligned} -C_{10}(1 + 2\log X)\log 2\|\lambda'\|_\infty \leq & -C_{10}(1 + \log\|x\| + \log\|x'\|)\max\left\{1, \log\|\lambda\lambda'^{-1}\|_\infty\right\} \\ \leq & -\beta\|\lambda\|_\infty + B_2 + 2\log 2. \end{aligned}$$

Therefore $\beta\|\lambda\|_\infty - B_2 - 2\log 2 \leq C_{10}(1 + 2\log X)\log 2\|\lambda'\|_\infty$. □

Let $B_4 \geq \max\{9370, B_3\}$ be a constant dependent only on C_{10}, B_3 and B_2 such that whenever $\|\lambda\|_\infty \geq B_4(\log X)^2 + B_4$ we have $\beta\|\lambda\|_\infty - B_2 - 2\log 2 \geq C_{10}(1 + 2\log X)\|\lambda\|_\infty^{1/2}$. We divide the set T^i into two parts: a "tame" part $T_t^i := \left\{\lambda \in T^i | \|\lambda\|_\infty \leq B_4(\log X)^2 + B_4\right\}$ and a "wild" part $T_w^i := T^i \setminus T_t^i$. We have a simple estimate for $|T_t^i|$

$$|T_t^i| \leq C_{18}(\log X)^{2(d-1)} + C_{18,5}. \tag{3.2.14}$$

Let us list the elements of T_w^i as $\lambda_1, \ldots, \lambda_L$ in such a way that $\|\lambda_l\|_\infty \leq \|\lambda_{l+1}\|_\infty$ for $l = 1, \ldots, L-1$. Note that $L = |T_w^i|$. By Lemma 3.2.21 and choice of B_4 we have

$$C_{10}(1 + 2\log X)\|\lambda_l\|_\infty^{1/2} \leq \beta\|\lambda_l\|_\infty - B_2 - 2\log 2 \leq C_{10}(1 + 2\log X)\log 2\|\lambda_{l+1}\|_\infty.$$

Therefore $\|\lambda_l\|_\infty^{1/2} \leq \log 2\|\lambda_{l+1}\|_\infty$ for $l = 1, \ldots, L-1$. Since $\|\lambda_l\|_\infty \geq 9370$ we have $(\log 2\|\lambda_l\|_\infty)^2 \leq \|\lambda_l\|_\infty^{1/2}$ so

$$(\log 2\|\lambda_l\|_\infty)^2 \leq \log 2\|\lambda_{l+1}\|_\infty.$$

Now an elementary induction shows that $\log 2\|\lambda_L\|_\infty \geq (\log 2 \times 9370)^{2^{L-1}} > e^{2^L}$. Together with (3.2.13) this yields

$$e^{e^{2^L}} = e^{e^{2^{|T_w^i|}}} \leq 2\|\lambda_L\|_\infty \leq 2C_{15}\log\|a\|\log\log\|a\| + 2C_{16}$$

$$|T_w^i| \leq C_{19}\log\log\log\log\|a\| + C_{19,5}. \tag{3.2.15}$$

By (3.2.14) and (3.2.15) we get

$$|T^i| \leq C_{18}(\log X)^{2(d-1)} + C_{19}\log\log\log\log\|a\| + C_{20} \tag{3.2.16}$$

where[7] $C_{20} = C_{18,5} + C_{19,5}$. Together with (3.2.9) this gives Lemma 3.2.11.

[7] The non-integer indexes are a result of a correction of the proof that required introduction of additional constants.

□

The proof of Lemma 3.2.13 is very similar.

Proof of Lemma 3.2.13. We adopt notation from the proof of Lemma 3.2.11. By the same reasoning as in the proof of Lemma 3.2.11 we get

$$|S_2^i[M]| \leq \sum_{\substack{\lambda\in T^i \\ \|\lambda\|_\infty \geq M}} |S_2^i(\lambda)| \leq |T^i| + C_{13}X^{1+\kappa'}\|a\|^{-\kappa'} \sum_{\substack{\lambda\in \mathcal{O}_k \\ \|\lambda\|_\infty \geq M}} e^{-\beta\|\lambda\|_\infty}. \tag{3.2.17}$$

For M_2 sufficiently large we have

$$\sum_{\substack{\lambda\in \mathcal{O}_k \\ \|\lambda\|_\infty \geq M_2}} e^{-\beta\|\lambda\|_\infty} \leq \delta C_{13}^{-1}$$

so (3.2.17) yields $|S_2^i[M_2]| \leq \delta X^{1+\kappa'}\|a\|^{-\kappa'} + |T^i|$. By inequality (3.2.16) we get

$$|S_2^i[M_2]| \leq \delta X^{1+\kappa'}\|a\|^{-\kappa'} + C_{18}(\log X)^{2(d-1)} + C_{19}\log\log\log\log\|a\| + C_{20}.$$

The Lemma is proven. □

3.2.4 Average number of solutions of unit equations

For completeness we explain how Theorem 3.1.7 follows from Theorem 3.2.1.

Proof of Theorem 3.1.7. Let $a = \alpha_3$. Assume that $\alpha_1\lambda_1 + \alpha_2\lambda_2 = \alpha_3$ for some $\lambda_1, \lambda_2 \in \mathcal{O}_k^\times$ and $\alpha_1, \alpha_2 \in \mathcal{O}_k$. If we put $x = \alpha_1\lambda_1$ then $\alpha_2\lambda_2 = a - x$ and $\|x(a-x)\| = \|\alpha_1\alpha_2\|$. Hence, the sum

$$\sum_{\substack{\alpha_1,\alpha_2\in \mathcal{O}^k/\mathcal{O}_k^\times \\ \|\alpha_1\alpha_2\|\leq X^2}} \nu(\alpha_1,\alpha_2,\alpha_3)$$

counts the number of $x \in \mathcal{O}_k$ such that $\|x(a-x)\| \leq X^2$. This is the same quantity we bound in Theorem 3.2.1. □

3.3 Geometry of n-optimal sets.

As before let k be a number field of degree N and let d be the number of Archimedean places of k. Recall that $V \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ where r_1, r_2 are the numbers of real and complex Archimedean places of k. The aim of this Section is to show the following theorem.

Theorem 3.3.1. *There exists a positive constant Θ_5 dependent only on k with the following property. For every n-optimal set $\mathcal{S} \subset \mathcal{O}_k$ there exists a cylinder (see Definition 3.2.18) $\mathcal{C}$ of volume $\Theta_5 n$ such that $\mathcal{S} \subset \mathcal{C}$.*

We prove it in Section 3.3.2. As an easy consequence we get:

Corollary 3.3.2. *There exist a positive constant A_3 depending only on k such that the cylinder $\Omega = B_{\mathbb{R}}(0, A_3)^{r_1} \times B_{\mathbb{C}}(0, A_3)^{r_2}$ has the following property. Let $\mathcal{S}$ be an n-optimal set in $\mathcal{O}_k$. Then there exists $t, s \in V$ with $\|s\| = n|\Delta_k|^{1/2}$ such that $s^{-1}(\mathcal{S} - t) \subset \Omega$.*

Proof. Let $\mathcal{C}$ the cylinder from Theorem 3.3.1. Let t be the center of $\mathcal{C}$. We have

$$\mathcal{S} - t \subset \mathcal{C} - t = \prod_{i=1}^{r_1} B_{\mathbb{R}}(0, v_i) \times \prod_{i=r_1+1}^{d} B_{\mathbb{C}}(0, v_i),$$

with $\prod_{i=1}^{r_1}(2v_i)\prod_{i=r_1+1}^{d}(\pi v_i^2) = \mathrm{Leb}(\mathcal{C}) \leq \Theta_5 n$. Let $A_3 = (\Theta_5|\Delta_k|^{-1/2}2^{-r_1}\pi^{-r_2/2})^{1/N}$. Put $s = (s_1, \ldots, s_d)$ where $s_i = v_i(n|\Delta_k|^{1/2}2^{r_1}\pi^{r_2/2}\mathrm{Leb}(\mathcal{C})^{-1})^{1/N}$. Then $\|s\| = n|\Delta_k|^{1/2}$ and $s^{-1}(\mathcal{C} - t) \subset \Omega$ because $|s_i^{-1}v_i| = (\mathrm{Leb}(\mathcal{C})n^{-1}|\Delta_k|^{-1/2}2^{-r_1}\pi^{-r_2/2})^{1/N} \leq A_3$. □

3.3.1 Generalities on n-optimal sets

Recall that for a finite subset $F \subset \mathcal{O}_k$ we define $\mathrm{Vol}(F) = \prod_{x \neq y \in F}(x - y)$. For $m \in \mathbb{N}$, $m \geqslant 0$ let $m!_k := m!_{\mathcal{O}_k}$ be the generalized factorial in $\mathcal{O}_k$ in the sense of Bhargava [5] (see subsection 3.1.2) . We remark that $m!_k$ is an ideal of $\mathcal{O}_k$, not a number. We have shown in [11, Proposition 2.6] that a set $\mathcal{S} \subset \mathcal{O}_k$ of size $n + 1$ is n-optimal if and only if

$$\mathcal{O}_k \mathrm{Vol}(\mathcal{S}) = \prod_{m=0}^{n} m!_k^2. \tag{3.3.1}$$

Also by [11, Proposition 2.6] for every subset $F \subset \mathcal{O}_k$ of size $n + 1$ we have

$$\|\mathrm{Vol}(F))\| = N_{k/\mathbb{Q}}(\mathcal{O}_k\mathrm{Vol}(F)) \geq \prod_{m=0}^{n} |N_{k/\mathbb{Q}}(m!_k^2)|. \tag{3.3.2}$$

Lemma 3.3.3. *Let $\mathcal{S} \subset \mathcal{O}_k$ be an n-optimal set. Then for every $x \in \mathcal{S}$ we have*

$$\sum_{y \in \mathcal{S} \setminus \{x\}} \log\|x - y\| \leq \log N_{k/\mathbb{Q}}(n!_k) \leq n \log n + A_1 n,$$

where $A_1 \geq 1$ is a [8] constant depending only on k.

Proof. By (3.3.2) we have

$$\log\|\mathrm{Vol}(\mathcal{S})\| - 2\sum_{y \in \mathcal{S} \setminus \{x\}} \log\|x - y\| = \log\|\mathrm{Vol}(\mathcal{S} \setminus \{x\})\| \geq \sum_{m=0}^{n-1} \log N_{k/\mathbb{Q}}(m!_k)^2.$$

[8] We require $A_1 \geq 1$ only for technical reasons that will become apparent in the proof of Lemma 3.3.7.

Using formula (3.3.1) we get $\sum_{y\in\mathcal{S}\setminus\{x\}}\log\|x-y\|\leq\log N_{k/\mathbb{Q}}(n!_k)$. Second equality in the lemma is [28, Theorem 1.2.4]. $\square$

We immediately get:

Corollary 3.3.4. *Let $\mathcal{S}$ be an n-optimal set. Then for every $x\neq y\in\mathcal{S}$ we have* $\log\|x-y\|\leq n\log n+A_1n$.

Remark 3.3.5. *A posteriori we know that the bound in the above Corollary is very far off but it will be used in the proofs to ensure that the quadruple-logarithmic error term from Proposition 3.2.5 is negligible.*

3.3.2 Proof of Theorem 3.3.1

As before, write $N=[k:\mathbb{Q}]$, d for the number of Archimedean places of k and $\kappa=\min\left\{\frac{1}{2N(N-1)},\frac{1}{4N-1}\right\}$. Our first goal is to give an upper bound on the norms of differences of pairs of elements in hypothetical n-optimal sets. We start with the following lemma, giving a non-trivial lower bound on the product of norms of elements in two translates $F-x,F-y$ of a set $F\subset\mathcal{O}_k$.

Lemma 3.3.6. *Let $B\in\mathbb{R}$, let F be a finite subset of $\mathcal{O}_k$ and let $x,y\in F$ be such that $\log|F|\leq\log\|x-y\|+B$. Then for every $0<\log X\leq\log\|x-y\|+B$ we have*

$$\sum_{z\in F\setminus\{x,y\}}(\log\|(x-z)(y-z)\|)\geq 2|F|\log X-\frac{2\Theta_1\|x-y\|^{-\kappa}}{1+\kappa}(X^{1+\kappa}-1)$$
$$-\frac{2\Theta_2}{2d-1}(\log X)^{2d-1}-2\Theta_3\log\log\log\log\|x-y\|\log X-2\Theta_4\log X.$$

The constants Θ_i depend only on k and B.

Proof. By translating F if necessary we can assume that $x=0$. Put $a=y$. Then the leftmost sum takes the form

$$\sum_{z\in F\setminus\{0,a\}}\log\|z(a-z)\|.$$

For $t\geq 1$ let $E(a,t)=\left\{z\in\mathcal{O}_k\setminus\{0,a\}\,|\,\|z(a-z)\|\leq t^2\right\}$. We obviously have

$$\sum_{z\in F\setminus\{0,a\}}\log\|z(a-z)\|\geq\sum_{z\in F\setminus\{0,a\}}\min\left\{\log\|z(a-z)\|,\log X^2\right\}.$$

Hence

$$\begin{aligned}\sum_{z\in F\setminus\{0,a\}} \min\left\{\log\|z(a-z)\|, \log X^2\right\} =& \sum_{z\in F\setminus\{0,a\}}\left(2\log X - \int_{\|z(a-z)\|}^{X^2}\frac{dt}{t}\right)\\ \geq& 2|F|\log X - \sum_{z\in F\setminus\{0,a\}}\int_1^{X^2}\mathbf{1}_{E(a,t^{1/2})}(z)\frac{dt}{t}\\ =& 2|F|\log X - 2\int_1^X |E(a,t)\cap F|\frac{dt}{t}\\ \geq& 2|F|\log X - 2\int_1^X |E(a,t)|\frac{dt}{t}\\ \geq& 2|F|\log X - 2\int_1^X \Big(\Theta_1 t^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log t)^{2d-2}\\ &+\Theta_3\log\log\log\log\|a\| + \Theta_4\Big)\frac{dt}{t}.\end{aligned}$$

The last inequality is an application of Theorem 3.2.1. Integrating the last expression we get the desired inequality. □

Lemma 3.3.7. *There exists a constant Θ_7, dependent only on k, such that for every n sufficiently large and every n-optimal set $\mathcal{S}$ we have $\log\|x-y\| \leq \log n + \Theta_7$ for every $x \neq y \in \mathcal{S}$.*

Proof. Let A_1 be the constant from Lemma 3.3.3, we recall that $A_1 \geq 1$ and it depends only on k. Let $x \neq y \in \mathcal{S}$. Either $\log\|x-y\| \leq \log n + A_1$ or we can we can apply Lemma 3.3.6 with $F = \mathcal{S}$, $\log X = \log n + 2A_1$ and $B = A_1$. In the latter case we get

$$\begin{aligned}\sum_{z\in\mathcal{S}\setminus\{x,y\}}(\log\|z-x\| + \log\|z-y\|) \geq 2(n+1)(\log n + 2A_1) - \frac{2\Theta_1\|x-y\|^{-\kappa}}{1+\kappa}(X^{\kappa+1}-1)\\ -\frac{2\Theta_2}{2d-1}(\log X)^{2d-1} - 2\Theta_3\log\log\log\log\|x-y\|\log X - 2\Theta_4\log X,\end{aligned}$$

where the constants depend only on k. By Corollary 3.3.4 we have $\log\|x-y\| \leq n\log n + A_1 n$ so $2\Theta_3\log\log\log\log\|x-y\|\log X = o(n)$. The same holds for other error terms. Hence, for n sufficiently large we have

$$\begin{aligned}\sum_{z\in\mathcal{S}\setminus\{x,y\}}(\log\|z-x\| + \log\|z-y\|) \geq& 2n(\log n + 2A_1) - \frac{2\Theta_1\|x-y\|^{-\kappa}}{1+\kappa}(ne^{2A_1})^{\kappa+1} - o(n)\\ \geq& 2n\log n + 3nA_1 - \frac{2\Theta_1\|x-y\|^{-\kappa}}{1+\kappa}(ne^{2A_1})^{\kappa+1}.\end{aligned}$$

By Lemma 3.3.3 we get

$$2n\log n + 2A_1 n \geq \sum_{z\in\mathcal{S}\setminus\{x,y\}}(\log\|z-x\| + \log\|z-y\|) + 2\log\|x-y\|.$$

Of course $\log\|x-y\| \geq 0$ so we deduce that

$$\begin{aligned}
2n\log n + 2A_1 n \geq& 2n\log n + 3nA_1 - \frac{2\Theta_1\|x-y\|^{-\kappa}}{1+\kappa}(ne^{2A_1})^{\kappa+1}\\
\frac{2\Theta_1\|x-y\|^{-\kappa}}{1+\kappa}(ne^{2A_1})^{\kappa+1} \geq& A_1 n\\
\Theta_6 := \frac{2\Theta_1 e^{2A_1+2\kappa A_1}}{A_1(1+\kappa)} \geq& \|x-y\|^{\kappa} n^{-\kappa}\\
\log\Theta_6 \geq& \kappa(\log\|x-y\| - \log n).
\end{aligned}$$

Hence, for n sufficiently large $\log\|x-y\| \leq \log n + \kappa^{-1}\log\Theta_6$ where Θ_6 depends only on k. Lemma holds with $\Theta_7 = \max\left\{\kappa^{-1}\log\Theta_6, A_1\right\}$. The constant Θ_7 depends only on k. $\square$

The second ingredient in the proof of Theorem 3.3.1 is the following weaker version of Theorem 3.3.1.

Lemma 3.3.8. *For every $\delta > 0$ there exists a constant $\Theta_8 = \Theta_8(\delta)$ such that for every n sufficiently large and every n-optimal set $\mathcal{S}$ there exists a cylinder $\mathcal{C}_1$ of volume at most $n\Theta_8$ such that $|\mathcal{S}\cap\mathcal{C}_1| \geq (1-\delta)n$.*

Proof. We shall crucially use Proposition 3.2.5 (2) together with Lemma 3.3.7. In order to use Proposition 3.2.5 (2) we fix a good fundamental domain $\mathcal{F}$ of $\mathcal{O}_k^\times$ in $V^\times$, a basis $\xi_1,\ldots,\xi_{d-1}$ of a maximal torsion free subgroup of $\mathcal{O}_k^\times$ and the associated norm $\|\cdot\|_\infty$ on $\mathcal{O}_k^\times$. Put $A_2 = \gamma_k - \gamma_{\mathbb{Q}} - 2$. First note that by the volume formula [11, Corollary 5.2] for large enough n we have

$$\log(\mathrm{Vol}(\mathcal{S})) = \sum_{x\neq y\in\mathcal{S}} \log\|x-y\| = n^2\log n + n^2(\gamma_k - \gamma_{\mathbb{Q}} - \frac{3}{2}) + o(n^2) \geq (n^2+n)(\log n + A_2).$$

Together with Lemma 3.3.7 this implies that there exists at least one pair $x, y \in \mathcal{S}$ such that $\log n + A_2 \leq \log\|x-y\| \leq \log n + \Theta_7$. Let us fix a pair x_0, y_0 with $\|x_0 - y_0\|$ maximal among all pairs in $\mathcal{S}$. By translating $\mathcal{S}$ if necessary we may assume that $x_0 = 0$ and put $a = y_0$. Let $X = \|a\|$. Then $\log n + A_2 \leq \log X \leq \log n + \Theta_7$. The question is invariant under multiplying $\mathcal{S}$ by elements of $\mathcal{O}_k^\times$ so we may assume without loss of generality that $a \in \mathcal{F}$. For every $z \in \mathcal{S}$ we have $\|z\| \leq X$ and $\|a-z\| \leq X$ so $\|z(a-z)\| \leq X^2$. Therefore, with notation from Proposition 3.2.5 we have

$$\mathcal{S}\setminus\{0\} \subset \left\{x\lambda^{-1} | (x,\lambda) \in S(a,X)\right\}. \tag{3.3.3}$$

Let $M > 0$ be such that Proposition 3.2.5 (2) holds with $\varepsilon = \frac{\delta}{2}e^{-\Theta_7}$ and $B = 0$. The constant M depends only on k and δ. For n sufficiently large we have

$$\begin{aligned}|S(a,X)[M]| \leq & \frac{\delta e^{-\Theta_7}}{2} X^{1+\kappa}\|a\|^{-\kappa} + \Theta_2(\log X)^{2d-2} + \Theta_3 \log\log\log\log \|a\| + \Theta_4 \\ \leq & \frac{\delta}{2} n + o(n) \leq \delta n.\end{aligned}$$

Let $\mathcal{S}' := \mathcal{S} \setminus \left\{(x\lambda^{-1} | (x,\lambda) \in S(a,X)[M]\right\}$. By the inequality above $\mathcal{S}'$ contains at least $(1-\delta)n$ elements. To prove the lemma it is enough to show that $\mathcal{S}'$ is contained in a cylinder of volume at most $n\Theta_8$. By (3.3.3) we have $\mathcal{S}' \subset \left\{(x\lambda^{-1} | (x,\lambda) \in S(a,X) \setminus S(a,X)[M]\right\} \cup \{0\}$.

$$S(a,x) \setminus S(a,X)[M] \subset \{(x,\lambda) | x \in \mathcal{F} \cap \mathcal{O}_k, \|x\| \leq X, \|\lambda\|_\infty \leq M\}.$$

By Lemma 3.2.3 we have a constant $C_0 > 0$ such that $C_0^{-1}\|x\|^{1/N} \leq |x|_i \leq C_0\|x\|^{1/N}$ for every $x \in \mathcal{F}$ an every $i = 1, \ldots, d$. Let $C_{21} = \max_{\|\lambda\|_\infty \leq M} \max_{i=1,\ldots,d} |\lambda^{-1}|_i$. Therefore, for every $(x,\lambda) \in S(a,x) \setminus S(a,X)[M]$ and $i = 1, \ldots, d$ we have $|x\lambda^{-1}|_i \leq C_{21}C_0\|x\|^{1/N} \leq C_{21}C_0X^{1/N} \leq C_{21}C_0e^{\Theta_7/N}n^{1/N}$. It follows that $\mathcal{S}'$ is contained in the cylinder

$$\mathcal{C}_1 = B_{\mathbb{R}}(0, C_{21}C_0e^{\Theta_7/N}n^{1/N})^{r_1} \times B_{\mathbb{C}}(0, C_{21}C_0e^{\Theta_7/N}n^{1/N})^{r_2}.$$

The volume of $\mathcal{C}_1$ is $n2^{r_1}\pi^{r_2}e^{\Theta_7}C_0^N C_{21}^N =: n\Theta_8$ where Θ_8 depends only on k and δ. $\square$

We are ready to prove Theorem 3.3.1

Proof of Theorem 3.3.1. Assume that n is sufficiently large so that Lemma 3.3.7 holds and Lemma 3.3.8 holds with $\delta = 1/100$ and $\Theta_8 = \Theta_8(1/100)$. Also for technical reasons we require $n \geq 4d, \Theta_8 \geq 1$ and the constant C_{11} from Lemma satisfies $C_{11} \geq 1$. This is not a problem since they can be always replaced by a bigger constants as long as these constants depend only on k. Let $\mathcal{S}$ be an n-optimal set and let $\mathcal{C}_1$ be a cylinder of volume $n\Theta_8$ containing at least $\frac{99n}{100}$ points of $\mathcal{S}$. Write

$$\mathcal{C}_1 = \prod_{i=1}^{r_1} B_{\mathbb{R}}(t_i, R_i) \times \prod_{i=r_1+1}^{d} B_{\mathbb{C}}(t_i, R_i)$$

with $t = (t_1, \ldots, t_d) \in V$. Note that $2^{r_1}\pi^{r_2}\prod_{i=1}^{r_1} R_i \prod_{i=r_1+1}^{d} R_i^2 = n\Theta_8$. For a positive constant $A > 0$ (how big will be precised later) we put $\mathcal{C}_1^A = \prod_{i=1}^{r_1} B_{\mathbb{R}}(t_i, AR_i) \times \prod_{i=r_1+1}^{d} B_{\mathbb{C}}(t_i, AR_i)$. The idea of the proof is to show that for large A (how large depends only on k) and every $y \notin \mathcal{C}_1^A$ the intersection $\mathcal{C}_1 \cap \left\{x \in V | \|x-y\| \leq ne^{\Theta_7}\right\}$ is too small to contain 99% of $\mathcal{S}$. Then from Lemma 3.3.7 and Lemma 3.3.8 we can deduce that $y \notin \mathcal{S}$ and consequently that $\mathcal{S} \subset \mathcal{C}_1^A$.

Let C_{11} be the constant from Lemma 3.2.19 and put

$$A = \max\left\{2, e^{\Theta_7}(2N)^N 2^{r_1}\pi^{r_2}\Theta_8^{N-1}C_{11}^N + 1\right\}.$$

Suppose that $y \in \mathcal{S} \setminus \mathcal{C}_1^A$. Since $y \notin \mathcal{C}_1^A$ for every $x \in \mathcal{C}_1$ there exists a coordinate $i \in \{1, \ldots, d\}$ such that $|x - y|_i \geq (A-1)R_i$. Put $\iota = 1$ if $i \in \{1, \ldots, r_1\}$ and $\iota = 2$ otherwise. If additionally $\|x - y\| \leq ne^{\Theta_7}$, then we have

$$\prod_{j=1, j\neq i}^{r_1} |x-y|_j \prod_{j=r_1+1, j\neq i}^{d} |x-y|_j^2 \leq ne^{\Theta_7} R_i^{-\iota}(A-1)^{-1} \tag{3.3.4}$$

$$\leq (A-1)^{-1}\frac{e^{\Theta_7}2^{r_1}\pi^{r_2}}{\Theta_8} \prod_{j=1,j\neq i}^{r_1} R_j \prod_{j=r_1+1, j\neq i}^{d} R_j^2 \tag{3.3.5}$$

$$\leq \prod_{j=1,j\neq i}^{r_1} \frac{R_j}{2N\Theta_8 C_{11}} \prod_{j=r_1+1,j\neq i}^{d} \left(\frac{R_j}{2N\Theta_8 C_{11}}\right)^2. \tag{3.3.6}$$

Hence, there exists $j \neq i$ such that $|x - y|_j \leq \frac{R_j}{2N\Theta_8 C_{11}}$. Define

$$\mathcal{C}_1(j) = \prod_{l=1, l\neq j}^{r_1} B_{\mathbb{R}}(t_l, R_l) \times \prod_{l=r_1+1, l\neq j}^{d} B_{\mathbb{C}}(t_l, R_l) \times B_{k_{\nu_j}}\left(t_j, \frac{R_j}{2N\Theta_8 C_{11}}\right)$$

and note that $\mathrm{Leb}(\mathcal{C}_1(j)) = \frac{n\Theta_8}{(2N\Theta_8 C_{11})^{[k_{\nu_j}:\mathbb{R}]}} \leq \frac{n}{2NC_{11}}$. From inequalities (3.3.4-3.3.6) we deduce that

$$\left\{x \in \mathcal{C}_1 |\, \|x-y\| \leq ne^{\Theta_7}\right\} \subset \bigcup_{l=1}^{d} \mathcal{C}_1(l). \tag{3.3.7}$$

By Lemma 3.2.19 we get

$$|\left\{x \in \mathcal{C}_1 \cap \mathcal{O}_k |\, \|x-y\| \leq ne^{\Theta_7}\right\}| \leq d + C_{11}\frac{dn}{2NC_{11}} \leq d + \frac{n}{2} \leq \frac{3n}{4}. \tag{3.3.8}$$

Lemma 3.3.7 yields $\mathcal{S} \subset \left\{x \in \mathcal{O}_k |\, \|x-y\| \leq ne^{\Theta_7}\right\}$ so we have

$$\mathcal{S} \cap \mathcal{C}_1 \subset \left\{x \in \mathcal{C}_1 \cap \mathcal{O}_k | \|x-y\| \leq ne^{\Theta_7}\right\}.$$

In particular $|\left\{x \in \mathcal{C}_1 \cap \mathcal{O}_k |\, \|x-y\| \leq ne^{\Theta_7}\right\}| \geq |\mathcal{S} \cap \mathcal{C}_1| \geq \frac{99}{100}n$. This contradicts (3.3.8). Thereby we showed that $\mathcal{S} \setminus \mathcal{C}_1^A$ is empty, that is $\mathcal{S} \subset \mathcal{C}_1^A$. As A depends only on k, the volume of $\mathcal{C}_1^A$ is $n\Theta_5$ where $\Theta_5 = \Theta_8 A^N$ depends only on k. Theorem 3.3.1 is proven. □

3.4 Collapsing of measures

Write $\mathcal{M}^1(V), (\mathcal{P}^1(V))$ for the set of finite measures (resp. probability measures) ν on V which are absolutely continuous with respect to the Lebesgue measure such that the density $d\nu/d\mathrm{Leb}$ is almost everywhere less or equal to 1. For $i \in \{1, \ldots, d\}$ and $v_i \in \mathbb{R}$ or $\mathbb{C}$ (depending on whether i corresponds to real or complex place) we will define an operation called collapsing $c_{i,v_i} : \mathcal{P}^1(V) \to \mathcal{P}^1(V)$ that has the following property: either

$I(c_{i,v_i}(\nu)) < I(\nu)$ or ν is of a very specific form. It is a version of the Steiner symmetrization ([26]), but for measures in $\mathcal{M}^1(V)$ instead of subsets of V. We shall make it precise in a moment. The operation of collapsing is the continuous analogue of the collapsing operation on subsets of $\mathcal{O}_k$ used in [35] and [11] where it was defined for k quadratic imaginary. We remark that for number fields k other than quadratic imaginary ones there is no reasonable discrete collapsing procedure for subsets of $\mathcal{O}_k$. In this section we study the effect of collapsing on the energy of measures. Our goal is Corollary 3.4.8 which says that the measures ν in $\mathcal{P}^1(V)$ that minimize the energy $I(\nu)$ are, up to translation, invariant under all collapsing operations.

Definition 3.4.1. *Let $i \in \{1, \ldots, d\}$ and $v_i \in \mathbb{R}$ if $i \in \{1, \ldots, r_1\}$ or $v_i \in \mathbb{C}$ otherwise. Let $\nu \in \mathcal{M}^1(V)$ be a measure with density $f \in L^1(V)$. For $x = (x_1, \ldots, x_d) \in V$ define*

$$F_i(x) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}} f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)dt & \text{if } i \in \{1, \ldots, r_1\} \\ \frac{1}{\sqrt{\pi}}\left(\int_{\mathbb{C}} f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d)dt\right)^{1/2} & \text{if } i \in \{r_1+1, \ldots, d\}. \end{cases} \tag{3.4.1}$$

Let $h : V \to \mathbb{R}_{\geq 0}$ be given by

$$h_i = \begin{cases} \int \cdots \int \mathbf{1}_{B_{\mathbb{R}}(v_i, F_i(t_1, \ldots, v_i, \ldots, t_d))} dt_1 \ldots \widehat{dt_i} \ldots dt_d & \text{if } i \in \{1, \ldots, r_1\} \\ \int \cdots \int \mathbf{1}_{B_{\mathbb{C}}(v_i, F_i(t_1, \ldots, v_i, \ldots, t_d))} dt_1 \ldots \widehat{dt_i} \ldots dt_d & \text{if } i \in \{r_1+1, \ldots, d\}. \end{cases} \tag{3.4.2}$$

The collapsed measure $c_{i,v_i}(\nu)$ is given by the density h_i. By construction $c_{i,v_i}(\nu)$ is symmetric with respect to the subspace $V^i := \{v = (v_1, \ldots, v_d) \in V | v_i = 0\}$.

Collapsing is closely related the Steiner symmetrization in the following way. If $V = \mathbb{R}^d$, then for any measurable subset $E \subset V$ we have $c_{i,0}(\mathrm{Leb}|_E) = \mathrm{Leb}|_{\mathrm{St}_i(E)}$ where $\mathrm{St}_i(E)$ is the Steiner symmetrization of E with respect to the hyperplane V^i (c.f. [26]). For further use we introduce a symmetric bilinear form on $\mathcal{M}^1(V) \times \mathcal{M}^1(V)$

$$\langle \nu, \nu' \rangle = \int_V \int_V \log \|x - y\| d\nu(x) d\nu'(y). \tag{3.4.3}$$

The integral converges as soon as ν, ν' are finite signed measures with bounded density. The energy can be expressed as $I(\nu) = \langle \nu, \nu \rangle$. We will also need a modified version of the bilinear form $\langle \cdot, \cdot \rangle$ defined as

$$\langle \nu_1, \nu_2 \rangle_\delta := \int_V \int_V \frac{1}{2} \log(\|x - y\|^2 + \delta^2) d\nu_1(x) d\nu_2(y) \text{ for } \delta \geq 0.$$

Note that $\langle \nu_1, \nu_2 \rangle_0 = \langle \nu_1, \nu_2 \rangle$. The following result is intuitively obvious but the proof is quite involved.

Lemma 3.4.2. *Let $\nu_1, \nu_2 \in \mathcal{M}^1(\mathbb{R})$ and $x \in \mathbb{R}$ and let $\delta \geq 0$. Then* $\langle c_{1,x}(\nu_1), c_{1,x}(\nu_2) \rangle_\delta \leq$

$\langle \nu_1, \nu_2 \rangle_\delta$ and equality holds if and only if there exists $y \in \mathbb{R}$ such that ν_1, ν_2 are restrictions of the Lebesgue measure to some intervals centered in y.

Proof. Let $m_1 = \nu_1(\mathbb{R}), m_2 = \nu_2(\mathbb{R})$. Throughout the proof we will write $E_i = [-\frac{m_i}{2}, \frac{m_i}{2}]$. We will prove that the minimum of $\langle \mu_1, \mu_2 \rangle_\delta$ with $\mu_i \in \mathcal{M}^1(\mathbb{R})$ subject to condition $\mu_i(\mathbb{R}) = m_i$ for $i = 1, 2$ is realized if and only if μ_1, μ_2 are Lebesgue measures restricted to translates $E_1 + y, E_2 + y$ for some $y \in \mathbb{R}$. This is clearly equivalent to the lemma. The proof is actually easier for $\delta > 0$ and we will first prove it for $\delta > 0$ and then deduce the general statement. By abuse of notation for every pair of measurable sets $I_1, I_2 \subset \mathbb{R}$ we will write $\langle I_1, I_2 \rangle_\delta := \langle \mathrm{Leb}|_{I_1}, \mathrm{Leb}|_{I_2} \rangle_\delta$.

Step 1. We will introduce a shifting/gluing operation G on finite sums of closed intervals I_1, I_2 that strictly reduces the value of $\langle I_1, I_2 \rangle_\delta$, preserves the measures of I_1, I_2 and can be applied until I_1, I_2 are two intervals centered in the same point. Write $(I_1', I_2') = G(I_1, I_2)$ for the result of the operation G. We will show that for every $\delta \geq 0$

$$\langle I_1, I_2 \rangle - \langle I_1', I_2' \rangle \geq \langle I_1, I_2 \rangle_\delta - \langle I_1', I_2' \rangle_\delta > 0. \tag{3.4.4}$$

We will show also that after finitely many applications G produces two concentric intervals. This step proves the lemma for pairs Lebesgue measures restricted to finite unions of intervals.

Before defining G we need to set up some notation. Let $I_i = C_i^1 \sqcup \ldots \sqcup C_i^{n_i}$ be the decomposition of I_i into connected components for $i = 1, 2$. We assume that $C_i^1, \ldots, C_i^{n_i}$ are listed from the leftmost to the rightmost connected component. Let $C_i^j = [a_i^j, b_i^j]$ and put $c_i^j = (a_i^j + b_i^j)/2$. First we look at the rightmost components $C_1^{n_1}, C_2^{n_2}$. Choose $i \in 1, 2$ such that $c_i^{n_i} = \max\{c_1^{n_1}, c_2^{n_2}\}$. Consider two cases: first when $c_1^{n_1} \neq c_2^{n_2}$ and the second when $c_1^{n_1} = c_2^{n_2}$ and $n_1 > 1$ or $n_2 > 1$. **Case 1.** In the first case operation G replaces $C_i^{n_i}$ with the translate $C_i^{n_i} - \kappa$ where $\kappa = \min\left\{a_i^{n_i} - b_i^{n_i - 1}, |c_1^{n_1} - c_2^{n_2}|\right\}$. In this case either G reduces the total number of connected components by 1 or makes $C_1^{n_1}, C_2^{n_2}$ concentric. We estimate $\langle I_1, I_2 \rangle_\delta - \langle I_1', I_2' \rangle_\delta$. Without loss on generality assume $i = 2$. We have

$$\Delta_{1,\delta} := \langle I_1, I_2 \rangle_\delta - \langle I_1', I_2' \rangle_\delta \tag{3.4.5}$$
$$= \frac{1}{2} \sum_{l=1}^{n_1} \int_{C_1^l} \left(\int_{C_2^{n_2}} \log(|x - y|^2 + \delta^2) - \log(|x - \kappa - y|^2 + \delta^2) dx \right) dy. \tag{3.4.6}$$

We know that $0 < \kappa \leq c_2^{n_2} - c_1^{n_1} \leq c_2^{n_2} - c_1^l$ for $l = 1, \ldots, n_1$. By Lemma 3.4.3 $\Delta_{1,0} \geq \Delta_{1,\delta} > 0$. **Case 2.** Put $\kappa = \min\left\{a_1^{n_1} - b_1^{n_1 - 1}, a_2^{n_2} - b_2^{n_2 - 1}\right\}$ with the convention that $b_i^{-1} = -\infty$. Number κ is finite because we assume that $n_1 > 1$ or $n_2 > 1$. Operation G replaces $C_1^{n_1}, C_1^{n_2}$ with the translates $C_1^{n_1} - \kappa, C_2^{n_2} - \kappa$ respectively. In this case the operation G reduces the total number of connected components by at least 1. We have

$$\begin{aligned}\Delta_{2,\delta} :=& \langle I_1, I_2\rangle_\delta - \langle I_1', I_2'\rangle_\delta \\ =& \frac{1}{2}\sum_{l=1}^{n_1-1}\int_{C_1^l}\left(\int_{C_2^{n_2}} \log(|x-y|^2+\delta^2) - \log(|x-\kappa-y|^2+\delta^2)dx\right)dy \\ +& \frac{1}{2}\sum_{l=1}^{n_2-1}\int_{C_2^l}\left(\int_{C_1^{n_1}} \log(|x-y|^2+\delta^2) - \log(|x-\kappa-y|^2+\delta^2)dx\right)dy.\end{aligned}$$

By Lemma 3.4.3 $\Delta_{2,0} \geq \Delta_{2,\delta} > 0$. We have shown that G reduces the value of $\langle I_1, I_2\rangle$ and that the reduction is the highest if $\delta = 0$. We can apply G unless $n_1 = n_2 = 1$ and I_1 and I_2 are concentric. If a single application of G does not reduce the total number of connected components then we were in the first case and the rightmost connected components of I_1', I_2' are concentric. This means that if we apply G to I_1', I_2' we will be in the second case so this iteration of G will reduce the total number of connected components by at least 1. This proves that G stops after at most $2(n_1+n_2)$ iterations and then we are left with two concentric intervals.

Step 2. We show that there exist bounded measurable sets J_i with $\mathrm{Leb}(J_i) = m_i$ for $i = 1, 2$ such that $\langle J_1, J_2\rangle_\delta \leq \langle \nu_1, \nu_2\rangle_\delta$ and the following holds: if we have equality, then either $\nu_1 = \mathrm{Leb}|_{J_1}$ and $\nu_2 = \mathrm{Leb}|_{J_2}$ or one of J_1 or J_2 is disconnected[9]. Moreover if $\delta > 0$ then J_1, J_2 are finite unions of closed intervals and equality holds if and only if $\nu_1 = \mathrm{Leb}|_{J_1}$ and $\nu_2 = \mathrm{Leb}|_{J_2}$.

Let $P_\delta(x) := \frac{1}{2}\int_{\mathbb{R}} \log((x-y)^2+\delta^2)d\nu_1(y)$. Since $\nu_1 \in \mathcal{M}^1(\mathbb{R})$ this function is continuous and bounded from below so there exists an α such that $\mathrm{Leb}(P_\delta^{-1}((-\infty,\alpha))) \leq m_2 \leq \mathrm{Leb}(P_\delta^{-1}((-\infty,\alpha]))$. Let J_2 be any measurable subset of measure m_2 such that

$$S_1 := P_\delta^{-1}((-\infty,\alpha)) \subset J_2 \subset P_\delta^{-1}((-\infty,\alpha]) := S_2.$$

If $\delta > 0$ then P_δ is analytic so $\mathrm{Leb}(P_\delta^{-1}((-\infty,\alpha)) = \mathrm{Leb}(P_\delta^{-1}((-\infty,\alpha])$. In that case we choose $J_2 = P_\delta^{-1}((-\infty,\alpha])$. It is a finite sum of closed intervals because P_σ is analytic. We go back to the general case and argue that $\langle \nu_1, \mathrm{Leb}|_{J_2}\rangle \leq \langle \nu_1, \nu_2\rangle$ with an equality if and only if $\mathrm{supp}\nu_2 \subset P_\delta^{-1}((-\infty,\alpha])$ and $\nu_2|_{S_1} = \mathrm{Leb}|_{S_1}$. Indeed

$$\begin{aligned}\langle \nu_1, \nu_2\rangle_\delta =& \int_{S_1} P_\delta(x)d\nu_2(x) + \int_{S_2\setminus S_1} P_\delta(x)d\nu_2(x) + \int_{\mathbb{R}\setminus S_2} P_\delta(x)d\nu_2(x) \\ =& \int_{S_1} P_\delta(x)d\nu_2(x) + \alpha\nu_2(S_2\setminus S_1) + \int_{\mathbb{R}\setminus S_2} P_\delta(x)d\nu_2(x) \\ \leq& \int_{S_1} P_\delta(x)dx + \alpha(m_2 - \mathrm{Leb}(S_1)) = \langle \nu_1, \mathrm{Leb}|_{J_2}\rangle_\delta.\end{aligned}$$

[9]This technical dychotomy is needed to prove the "if and only if" part of the lemma.

The equality holds if and only if $\nu_2(\mathbb{R}\setminus S_2)=0$ and the mass of ν_2 is as concentrated on S_1 as possible i.e. $\nu_2|_{S_1}=\mathrm{Leb}|_{S_1}$. If the equality holds and ν_2 is not a restriction of Lebesgue measure to S_1 or S_2 then we can choose J_2 to be disconnected. Thus we can replace ν_2 with $\nu_2'=\mathrm{Leb}|_{J_2}$ in such a way that either $\langle\nu_1,\mathrm{Leb}|_{J_2}\rangle<\langle\nu_1,\nu_2\rangle$, or $\nu_2=\mathrm{Leb}|_{J_2}$ for some closed interval J_2, or $\langle\nu_1,\mathrm{Leb}|_{J_2}\rangle=\langle\nu_1,\nu_2\rangle$ and J_2 is disconnected. Next, we perform the same trick for ν_2' to replace ν_1 with $\mathrm{Leb}|_{J_1}$ for some measurable set J_1 of measure m_1. If $\nu_2=\mathrm{Leb}|_{J_2}$ then the symmetric argument provides J_1 such that $\langle\mathrm{Leb}|_{J_1},\mathrm{Leb}|_{J_2}\rangle<\langle\nu_1,\nu_2\rangle$ or $\langle\mathrm{Leb}|_{J_1},\mathrm{Leb}|_{J_2}\rangle=\langle\nu_1,\nu_2\rangle$ and J_1 is disconnected or $\nu_1=\mathrm{Leb}|_{J_1}$. If $\delta>0$ we chose J_1,J_2 as finite sums of closed intervals and the equality $\langle\nu_1,\nu_2\rangle_\delta=\langle J_1,J_2\rangle_\delta$ holds if and only if $\nu_i=\mathrm{Leb}|_{J_i}$ for $i=1,2$. This proves Step 2.

Step 3. We prove the lemma for $\delta>0$.

Let $E_i:=[-\frac{m_i}{2},\frac{m_i}{2}]$. If $\delta>0$ then by Step 2 there are finite unions of closed intervals J_1,J_2 such that $\langle\nu_1,\nu_2\rangle_\delta\geq\langle I_1,I_2\rangle_\delta$ with an equality if and only if $\nu_i=\mathrm{Leb}|_{J_i}$ for $i=1,2$. By Step 1 $\langle J_1,J_2\rangle_\delta\geq\langle E_1,E_2\rangle_\delta$ with an equality if and only if J_1,J_2 are concentric intervals. Those two observations put together prove the lemma in the case $\delta>0$.

Step 4. Let E_i be as in Step 3, let $\delta>0$ and let I_1,I_2 be finite unions of intervals of total lengths m_1,m_2 respectively. We show that $\langle I_1,I_2\rangle-\langle E_1,E_2\rangle\geq\langle I_1,I_2\rangle_\delta-\langle E_1,E_2\rangle_\delta$.

Let m be the number of times we can apply operation G to I_1,I_2. Write $I_1^{(j)},I_2^{(j)}$ for the result of j-th iteration of G. Since G can be applied until we get two concentric intervals we have up to translation $I_1^{(m)}=E_1,I_2^{(m)}=E_2$. By inequality (3.4.4) we get

$$\langle I_1^{(j)},I_2^{(j)}\rangle-\langle I_1^{(j+1)},I_2^{(j+1)}\rangle\geq\langle I_1^{(j)},I_2^{(j)}\rangle_\delta-\langle I_1^{(j+1)},I_2^{(j+1)}\rangle_\delta.$$

Taking the sum from $j=0$ to $m-1$ we get $\langle I_1,I_2\rangle-\langle E_1,E_2\rangle\geq\langle I_1,I_2\rangle_\delta-\langle E_1,E_2\rangle_\delta$.

Step 5. We prove the Lemma for compactly supported ν_1,ν_2.

Let Σ be an interval containing the supports of ν_1,ν_2. Let E_i be as in Step 3 and fix $\delta>0$. We argue that $\langle\nu_1,\nu_2\rangle-\langle E_1,E_2\rangle\geq\langle\nu_1,\nu_2\rangle_\delta-\langle E_1,E_2\rangle_\delta$. By Lemma 3.4.4 the map

$$(\nu_1,\nu_2)\mapsto\langle\nu_1,\nu_2\rangle-\langle E_1,E_2\rangle-\langle\nu_1,\nu_2\rangle_\delta+\langle E_1,E_2\rangle_\delta$$

is weakly-* continuous on $\mathcal{M}^1(\Sigma)\times\mathcal{M}^1(\Sigma)$. The set measures of form Leb_I where I is a finite union of intervals is dense in $\mathcal{M}^1(\Sigma)$ so by Step 4 we deduce that $\langle\nu_1,\nu_2\rangle-\langle E_1,E_2\rangle-\langle\nu_1,\nu_2\rangle_\delta+\langle E_1,E_2\rangle_\delta\geq 0$. From Step 3 it follows now that $\langle\nu_1,\nu_2\rangle\geq\langle E_1,E_2\rangle$. We can have an equality only if $\langle\nu_1,\nu_2\rangle_\delta=\langle E_1,E_2\rangle_\delta$. In that case, again by Step 3, ν_1,ν_2 restrictions of Lebesgue measure to concentric intervals of lengths m_1,m_2 respectively.

Step 6. We prove the general case. By Step 2 either we can find bounded measurable sets I_1,I_2 of measures m_1,m_2 respectively such that $\langle\nu_1,\nu_2\rangle>\langle J_1,J_2\rangle$ in which case Step 5 finishes the proof, or $\nu_i=\mathrm{Leb}|_{J_i}, i=1,2$ in which case Step 5 again finishes the proof or $\langle\nu_1,\nu_2\rangle=\langle J_1,J_2\rangle$ and and J_1 or J_2 is disconnected. In the last case Step 5 yields $\langle J_1,J_2\rangle>\langle E_1,E_2\rangle$. The lemma is proven. □

Lemma 3.4.3. *Let $\delta \geq 0$, let $a_i < b_i, i = 1,2$ be real numbers and put $c_i = (a_i + b_i)/2$. Assume that $c_2 > c_1$. For every $0 < \kappa \leq c_2 - c_1$ we have*

$$\Delta_\delta := \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \log(|x-y|^2 + \delta^2) - \log(|x-y-\kappa|^2 + \delta^2) dx \right) dy > 0.$$

Moreover $\Delta_0 \geq \Delta_\delta$ for every $\delta \geq 0$.

Proof. For $0 < \kappa < c_2 - c_1$ we have

$$\begin{aligned}
&\frac{d}{d\kappa} \int_{a_1}^{b_1} \left(\int_{a_2}^{b_2} \log(|x-y|^2 + \delta^2) - \log(|x-y-\kappa|^2 + \delta^2) dx \right) dy \\
&= \int_{a_1}^{b_1} \left(-\log(|a_2 - y - \kappa|^2 + \delta^2) + \log(|b_2 - y - \kappa|^2 + \delta^2) \right) dy \\
&= \int_{b_2 - b_1 - \kappa}^{b_2 - a_1 - \kappa} \log(x^2 + \delta^2) dx - \int_{a_2 - b_1 - \kappa}^{a_2 - a_1 - \kappa} \log(x^2 + \delta^2) dx > 0.
\end{aligned}$$

For the last inequality observe that $[b_2 - b_1 - \kappa, b_2 - a_1 - \kappa]$ and $[a_2 - b_1 - \kappa, a_2 - a_1 - \kappa]$ are both intervals of length $b_1 - a_1$. The center of the first one is $b_2 - \kappa - c_1$ and the center of the second is $a_2 - \kappa - c_1$. We always have $|b_2 - \kappa - c_1| > |a_2 - \kappa - c_1|$ so the first integral is bigger because $\log(x^2 + \delta^2)$ is strictly increasing in $|x|$.

We show that Δ_δ is strictly decreasing in $\delta \geq 0$. We have

$$\begin{aligned}
&\frac{d}{d\delta} \int_{b_2 - b_1 - \kappa}^{b_2 - a_1 - \kappa} \log(x^2 + \delta^2) dx - \int_{a_2 - b_1 - \kappa}^{a_2 - a_1 - \kappa} \log(x^2 + \delta^2) dx \\
&= \int_{b_2 - b_1 - \kappa}^{b_2 - a_1 - \kappa} \frac{2\delta}{x^2 + \delta^2} dx - \int_{a_2 - b_1 - \kappa}^{a_2 - a_1 - \kappa} \frac{2\delta}{x^2 + \delta^2} dx < 0.
\end{aligned}$$

For the last inequality observe that the function $\frac{2\delta}{x^2+\delta^2}$ is decreasing in $|x|$, both integrals are over intervals of length $b_1 - a_1$ and the first one is further from 0 than the second. We deduce that $\frac{d}{d\kappa}\Delta_\delta$ is decreasing in δ. Hence $\Delta_0 \geq \Delta_\delta$. □

Lemma 3.4.4. *Let Σ be a compact subset of $\mathbb{R}$ or $\mathbb{C}$. The map $\mathcal{M}^1(\Sigma) \times \mathcal{M}^1(\Sigma) \ni \nu_1, \nu_2 \to \langle \nu_1, \nu_2 \rangle \in \mathbb{R}$ is continuous with respect to weak-* topology.*

Proof. Let f_1, f_2 be the densities of ν_1, ν_2 respectively. We have

$$\langle \nu_1, \nu_2 \rangle = \int_\Sigma \int_\Sigma f_1(x) f_2(y) \log \|x - y\| dx dy.$$

The map $(\nu_1, \nu_2) \mapsto f_1 \times f_2 \in L^2(\Sigma \times \Sigma)$ is weakly-* continuous on $\mathcal{M}^1(\Sigma) \times \mathcal{M}^1(\Sigma)$ with the weak topology on $L^2(\Sigma \times \Sigma)$. The function $(x,y) \mapsto \log \|x - y\|$ is in $L^2(\Sigma \times \Sigma)$ so the map $(\nu_1, \nu_2) \mapsto \langle \nu_1, \nu_2 \rangle$ is weakly-* continuous on $\mathcal{M}^1(\Sigma) \times \mathcal{M}^1(\Sigma)$. □

Lemma 3.4.5. *Let $\nu_1, \nu_2 \in \mathcal{M}^1(\mathbb{R})$ and $x \in \mathbb{C}$. Then $\langle c_{1,x}(\nu_1), c_{1,x}(\nu_2)\rangle \le \langle \nu_1, \nu_2\rangle$ and equality holds if and only if there is an $y \in \mathbb{C}$ such that ν_1, ν_2 are the restrictions of the Lebesgue measure to balls centered in y.*

Proof. **Step 1.** We define collapsing along a line ℓ in $\mathbb{C}$. First let us assume that ℓ is the real line $\mathbb{R} \subset \mathbb{C}$. Let ν be finite a measure on $\mathbb{C}$ of bounded density $f \in L^1(\mathbb{C})$. For $x \in \mathbb{R}$ let $F(x) = \int_{-\infty}^{+\infty} f(x+it)dt$. We define $h \in L^1(\mathbb{C})$ as

$$h(x+iy) = \begin{cases} 1 & \text{if } |y| \le F(x)/2 \\ 0 & \text{otherwise.} \end{cases} .$$

We write $c_{\mathbb{R}}(\nu)$ for the measure $h(x+iy)dxdy$. Let $\nu_1, \nu_2 \in \mathcal{M}^1(\mathbb{C})$, we argue that $\langle \nu_1, \nu_2\rangle \ge \langle c_{\mathbb{R}}(\nu_1), c_{\mathbb{R}}(\nu_2)\rangle$ with an equality if and only if there exists $t \in \mathbb{R}$ such that ν_1, ν_2 are translates of $c_{\mathbb{R}}(\nu_1), c_{\mathbb{R}}(\nu_2)$ by it. Let f_1, f_2 be the densities of ν_1, ν_2 respectively. For $x \in \mathbb{R}$ define $\nu_i^x \in \mathcal{M}^1(\mathbb{R})$ by $d\nu_i^x(y) = f_i(x+iy)dy$. We have

$$\begin{aligned}
\langle \nu_1, \nu_2\rangle =& \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \log((x_1-x_2)^2 + (y_1-y_2)^2) f_1(x_1+iy_1) f_2(x_2+iy_2) dx_1dy_1dx_2dy_2 \\
=& \int_{\mathbb{R}}\int_{\mathbb{R}} \langle \nu_1^{x_1}, \nu_2^{x_2}\rangle_{|x_1-x_2|} dx_1dx_2 \le \int_{\mathbb{R}}\int_{\mathbb{R}} \langle c_0(\nu_1^{x_1}), c_0(\nu_2^{x_2})\rangle_{|x_1-x_2|} dx_1dx_2 \\
=& \int_{\mathbb{R}}\int_{-F(x_1)}^{F(x_1)}\int_{\mathbb{R}}\int_{-F(x_2)}^{F(x_2)} \log((x_1-x_2)^2 + (y_1-y_2)^2) f_1(x_1+iy_1) dx_1dy_1dx_2dy_2 \\
=& \langle c_{\mathbb{R}}(\nu_1), c_{\mathbb{R}}(\nu_2)\rangle.
\end{aligned}$$

The inequality in the second line holds by Lemma 3.4.2 with equality if and only if $\nu_1^{x_1}, \nu_2^{x_2}$ are Lebesgue measures restricted to concentric intervals for every $x_1, x_2 \in \mathbb{R}$. Call t the common center of these intervals. Then ν_1, ν_2 are translates of $c_{\mathbb{R}}(\nu_1), c_{\mathbb{R}}(\nu_2)$ by it.

For $\ell \ne \mathbb{R}$ we choose any isometry ι of $\mathbb{C}$ such that $\iota(\ell) = \mathbb{R}$ and put $c_\ell(\nu) = \iota^{-1}(c_{\mathbb{R}}(\iota^*\nu))$. Like before we have that $\langle \nu_1, \nu_2\rangle \ge \langle c_\ell(\nu_1), c_\ell(\nu_2)\rangle$ with an equality if and only if there exists $z \in \ell^\perp$ such that ν_1, ν_2 are translates of $c_{\mathbb{R}}(\nu_1), c_{\mathbb{R}}(\nu_2)$ by z. Equivalently we have $\langle \nu_1, \nu_2\rangle = \langle c_\ell(\nu_1), c_\ell(\nu_2)\rangle$ if and only if there exists a line ℓ' parallel to ℓ such that $\nu_i = c_{\ell'}(\nu_i)$ for $i = 1, 2$.

Step 2. Let $m_i = \nu_i(\mathbb{C})$ and let B_1, B_2 be closed balls of volumes m_1, m_2 respectively, centered at 0. We show that for every $\nu_1, \nu_2 \in \mathcal{M}^1(\mathbb{C})$ compactly supported we have either $\langle \nu_1, \nu_2\rangle > \langle B_1, B_2\rangle$ or ν_1, ν_2 are the Lebesgue measure restricted to concentric balls.

Let $R > 0$ be such that $\operatorname{supp} \nu_i \subset B_{\mathbb{C}}(0, R)$ for $i = 1, 2$. By Lemma 3.4.4 there exists a pair of measures $\nu_1', \nu_2' \in \mathcal{M}^1(\mathbb{C})$ supported on $B_{\mathbb{C}}(0, R)$ with $\nu_1'(\mathbb{C}) = m_1, \nu_2'(\mathbb{C}) = m_2$ such that

$$\langle \nu_1', \nu_2'\rangle = \min\left\{\langle \mu_1, \mu_2\rangle | \mu_1, \mu_2 \in \mathcal{M}^1(B_{\mathbb{C}}(0,R)), \mu_1(\mathbb{C}) = m_1, \mu_2(\mathbb{C}) = m_2\right\}.$$

We either have $\langle \nu_1, \nu_2 \rangle > \langle \nu_1', \nu_2' \rangle$ or we can assume that $\nu_i = \nu_i'$ for $i = 1, 2$. Choose $z, w \in \mathbb{C}$ such that $\arg(z) - \arg(w) \notin \pi\mathbb{Q}$. By Step 1 and choice of ν_1', ν_2' we have $\langle c_{z\mathbb{R}}(\nu_1'), c_{z\mathbb{R}}(\nu_2') \rangle = \langle c_{w\mathbb{R}}(\nu_1'), c_{w\mathbb{R}}(\nu_2') \rangle = \langle \nu_1', \nu_2' \rangle$. Hence, by Step 1 there exist lines ℓ_1, ℓ_2 parallel to $z\mathbb{R}, w\mathbb{R}$ respectively such that $\nu_i' = c_{\ell_j}(\nu_i')$ for $i = 1, 2$ and $j = 1, 2$. By translating ν_1', ν_2' if necessary we may assume that $\ell_1 = z\mathbb{R}, \ell_2 = w\mathbb{R}$. Being collapsed implies that densities of ν_1, ν_2 are characteristic functions of measurable sets, so we have $\nu_i = \mathrm{Leb}|_{I_i}$ for some bounded measurable sets I_i. Let s_i be the orthogonal reflection in ℓ_i for $i = 1, 2$. Since ν_1', ν_2' are collapsed along ℓ_1, ℓ_2 they are invariant under the group S of isometries generated by s_1, s_2. Since $\arg(z) - \arg(w) \notin \pi\mathbb{Q}$ the group S is dense in $\mathrm{O}(2)$ (the orthogonal group group of $\mathbb{C}$ seen as $\mathbb{R}^2$). We deduce that I_1, I_2 must be (up to a measure 0 set) closed balls B_1, B_2 respectively. This proves Step 2.

Step 3. We prove the lemma. Without loss of generality we can assume $x = 0$. Let B_1, B_2 be as in Step 2. We need to show that $\langle B_1, B_2 \rangle \leq \langle \nu_1, \nu_2 \rangle$ with equality if and only if ν_1, ν_2 are Lebesgue measures restricted to concentric balls. The method is similar to Step 2 from the proof of Lemma 3.4.2. Consider $P(z) = 2\int_{\mathbb{C}} \log|x - z| d\nu_1(x)$. Then P is a continuous function on $\mathbb{C}$ such that $|P(z)|$ tends to ∞ as $|z| \to \infty$. There exists $\alpha \in \mathbb{R}$ such that $\mathrm{Leb}(P^{-1}((-\infty, \alpha))) \leq m_2 \leq \mathrm{Leb}(P^{-1}((-\infty, \alpha]))$. Choose a bounded measurable set I_2 of measure m_2 such that

$$S_1 := P^{-1}((-\infty, \alpha)) \subset I_2 \subset P^{-1}((-\infty, \alpha]) := S_2.$$

Like in the Step 2 from the proof of Lemma 3.4.2 we have $\langle \nu_1, \mathrm{Leb}|_{I_2} \rangle \leq \langle \nu_1, \nu_2 \rangle$ with an equality if and only if $\nu_2|_{S_1} = \mathrm{Leb}|_{S_1}$ and $\operatorname{supp} \nu_2 \subset S_2$. If the inequality is strict we replace ν_2 by $\mathrm{Leb}|_{I_2}$ and apply the same reasoning to find I_1 of measure m_1 such that $\langle I_1, I_2 \rangle \leq \langle \nu_1, \mathrm{Leb}|_{I_2} \rangle < \langle \nu_1, \nu_2 \rangle$. In the second case we deduce that $\operatorname{supp} \nu_2 \subset S_2$ so ν_2 is compactly supported. By the symmetry of the problem this is enough to deduce that either ν_1, ν_2 are compactly supported or we can find bounded measurable sets I_1, I_2 with measures m_1, m_2 such that $\langle I_1, I_2 \rangle < \langle \nu_1, \nu_2 \rangle$. In the first case Step 2 finishes the proof and in the second case again by Step 2 we have $\langle B_1, B_2 \rangle \leq \langle I_1, I_2 \rangle < \langle \nu_1, \nu_2 \rangle$.

□

As an easy consequence of Lemma 3.4.2 and Lemma 3.4.5 we get

Lemma 3.4.6. *Let $V = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Let $\nu_1, \nu_2 \in \mathcal{M}^1(V)$ and $v = (v_1, \ldots, v_d) \in V$. Then $\langle c_{i,v_i}(\nu_1), c_{i,v_i}(\nu_2) \rangle \leq \langle \nu_1, \nu_2 \rangle$ and equality holds if and only if there is an $w = (w_1, \ldots, w_d) \in V$ such that $\nu_1 = c_{i,w_i}(\nu_1)$ and $\nu_2 = c_{i,w_i}(\nu_2)$ for $i = 1, \ldots, d$.*

Proof. Assume without loss of generality that $v = 0$. We will first treat the case where i corresponds to a real place. Write $V^i = \{v \in V | v_i = 0\}$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in V$ where the unique non-zero entry is placed in the i-th coordinate. Let f_1, f_2 be the densities of ν_1, ν_2. For $x \in V^i$ and $j = 1, 2$ define the measure ν_j^x on $\mathbb{R}$ as $d\nu_j^x(t) = f_j(x + te_i)dt$.

Note that for every $f \in L^1(V)$ we have

$$\int_V f(v) d\nu_j(v) = \int_{V^i} \int_{\mathbb{R}} f(x + te_i) d\nu_j^x(t) dx.$$

By Lemma 3.4.2 we get

$$\begin{aligned}\langle \nu_1, \nu_2 \rangle =& \int_{V^i} \int_{V^i} \langle \nu_1^x, \nu_1^y \rangle dx dy \\ \leq& \int_{V^i} \int_{V^i} \langle c_{v_i}(\nu_1^x), c_{v_i}(\nu_2^y) \rangle dx dy \\ =& \langle c_{i,v_i}(\nu_1), c_{i,v_i}(\nu_2) \rangle.\end{aligned}$$

By Lemma 3.4.2 the equality holds if and only if there exists $w_i \in \mathbb{R}$ such that for all $x, y \in V^i$ the measures ν_1^x, ν_2^y are the Lebesgue measure restricted to intervals centered in $w_i \in \mathbb{R}$. In that case we also have $\nu_1 = c_{i,w_i}(\nu_1)$ and $\nu_2 = c_{i,w_i}(\nu_2)$. If i corresponds to a complex case the proof is identical but we use Lemma 3.4.5 in place of Lemma 3.4.2. □

Lemma 3.4.7. *Let $\nu \in \mathcal{P}^1(V)$ and let $i \in \{1, \ldots, d\}$, $v_i \in \mathbb{R}$ if $i \in \{1, \ldots, r_1\}$ or $v_i \in \mathbb{C}$ otherwise. Then either $I(c_{i,v_i}(\nu)) < I(\nu)$ or $I(c_{i,v_i}(\nu)) = I(\nu)$ and there exists v_i' such that $\nu = c_{i,v_i'}(\nu)$.*

Proof. Use Lemma 3.4.6 for $\nu_1 = \nu_2 = \nu$. □

As a consequence of Lemma 3.4.7 we get:

Corollary 3.4.8. *Let $\nu \in \mathcal{P}^1(V)$ be a measure minimizing the energy $I(\nu)$ on $\mathcal{P}^1(V)$. Then there exists $v = (v_1, \ldots, v_d) \in V$ such that $c_{i,v_i}(\nu) = \nu$ for every $i = 1, \ldots, d$.*

3.5 Limit measures and energy

Let $(n_i)_{i=1}^\infty$ be an increasing sequence of natural numbers. Let k be a number field and assume that $(\mathcal{S}_{n_i})_{i \in \mathbb{N}}$ is a sequence of n_i-optimal sets in $\mathcal{O}_k$. By Corollary 3.3.2 there are sequences $(t_{n_i})_{i \in \mathbb{N}} \subset V, (s_{n_i})_{i \in \mathbb{N}} \subset V$ such that $\|s_{n_i}\| = n_i |\Delta_k|^{1/2}$ and $s_{n_i}^{-1}(\mathcal{S}_{n_i} - t_{n_i}) \subset \Omega$. Define a sequence of measures

$$\mu_{n_i} := \frac{1}{n_i} \sum_{x \in \mathcal{S}} \delta_{s_{n_i}^{-1}(x - t_{n_i})}. \tag{3.5.1}$$

Since Ω is compact we can assume, passing to a subsequence if necessary, that μ_{n_i} converges to a limit probability[10] measure μ. This observation uses crucially Corollary 3.3.2 and is the key step in the proof of Theorem 3.1.2. Such limit measures are the central object of study in this section.

[10]While μ_{n_i} are not probability measures because $\mu_{n_i}(V) = 1 + \frac{1}{n_i}$ any weak-* limit point will be a probability measure.

Definition 3.5.1. *A probability measure μ on V is called a* ***limit measure*** *if it is a weak-* limit of a sequence of measures μ_{n_i} constructed as above.*

3.5.1 Density of limit measures

Let ν be a probability measure on V, absolutely continuous with respect to the Lebesgue measure on V. The density of ν is the unique non-negative function $f \in L^1(V)$ such that $d\nu = f(t)dt$ where dt is the Lebesgue measure. We say that ν is of density at most D if $f(t) \leq D$ for Lebesgue-almost all $t \in V$.

Lemma 3.5.2. *Any limit measure μ on V is of density at most* 1.

Proof. Let $(n_i)_{i\in\mathbb{N}}$ and let $(\mu_{n_i})_{i\in\mathbb{N}}$ be a sequence of measures defined as in (3.5.1) such that μ is the weak limit of μ_{n_i} as $i \to \infty$. By Lebesgue density theorem it is enough to verify that $\mu(\mathcal{C}) \leq \mathrm{Leb}(\mathcal{C})$ for every bounded cylinder $\mathcal{C} \subset V$. We have

$$\mu_{n_i}(\mathcal{C}) = \frac{1}{n_i}|\mathcal{S}_{n_i} \cap (s_{n_i}\mathcal{C} + t_{n_i})| \leq \frac{1}{n_i}|\mathcal{O}_k \cap (s_{n_i}\mathcal{C} + t_{n_i})|.$$

Put $\mathcal{C}_i = s_{n_i}\mathcal{C} + t_{n_i}$. Since $\|s_{n_i}\| = |\Delta_k|^{1/2}n_i$ the cylinder $\mathcal{C}_i$ has volume $|\Delta_k|^{1/2}n_i\mathrm{Leb}(\mathcal{C})$. As $\mathcal{O}_k$ is a lattice of covolume $|\Delta_k|^{1/2}$ we get[11] $|\mathcal{O}_k \cap \mathcal{C}_i| = |\Delta_k|^{-1/2}\mathrm{Leb}(C_i) + o(\mathrm{Leb}(C_i))$. Hence

$$\mu(\mathcal{C}) = \lim_{i\to\infty} \mu_{n_i}(\mathcal{C}) \leq \lim_{i\to\infty} \frac{1}{n_i}\left(n_i\mathrm{Leb}(\mathcal{C}) + o(n_i\mathrm{Leb}(\mathcal{C}))\right) = \mathrm{Leb}(\mathcal{C}).$$

□

3.5.2 Energy of limit measures

We start by defining the two notions of energy for probability measures on V.

Definition 3.5.3. *Let ν be a probability measure on V and write $\Delta(V) = \{(v,v)| \, v \in V\} \subset V \times V$. We define energies $I(\nu), I'(\nu)$ as*

$$I(\nu) = \int_{V\times V} \log\|x-y\|d\nu(x)d\nu(y)$$
$$I'(\nu) = \int_{V\times V\setminus\Delta(V)} \log\|x-y\|d\nu(x)d\nu(y)$$

provided that the integrals converge.

We will refer to I as energy and to I' as discretized energy since it is designed to handle finitely supported atomic measures. The integral defining the energy converges for

[11] This does not work for a general lattice $\Lambda \subset V$. However, we know that $\mathcal{O}_k$ is invariant under multiplication by $\mathcal{O}_k^\times$ so we can multiply $\mathcal{C}_i$ by an element of $\mathcal{O}_k^\times$ so that it becomes "thick" in every direction.

all compactly supported measures of bounded density. The main goal of this section is to establish:

Proposition 3.5.4. *Let k be a number field, let $V = k \otimes_{\mathbb{Q}} \mathbb{R}$ and suppose that μ is a limit measure on V. Then $I(\mu) = -\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}}$ where $\gamma_k, \gamma_{\mathbb{Q}}$ are Euler–Kronecker constants of $k, \mathbb{Q}$ respectively.*

Proof. Let us fix a sequence $(\mu_{n_i})_{i\in\mathbb{N}}$ of measure defined as in (3.5.1) such that μ is the weak-* limit of μ_{n_i} as $i \to \infty$. Observe that by the volume formula [11, Corollary 5.2]

$$\sum_{x\neq y\in\mathcal{S}_{n_1}} \log\|x-y\| = n_i^2\log n_i + n_i(-\frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}}) + o(n_i^2)$$

we have

$$\begin{aligned} I'(\mu_{n_i}) =& \frac{1}{n_i^2}\sum_{x\neq y\in\mathcal{S}_{n_i}} \log\|s_{n_i}^{-1}(x-y)\| \\ =& -\frac{n_i+1}{n_i}\log\|s_{n_i}\| + \frac{1}{n_i^2}(n_i^2\log n_i + n_i^2(-\frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}}) + o(n_i^2)) \\ =& -\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}} + o(1). \end{aligned}$$

Our task is reduced to proving that $\lim_{i\to\infty} I'(\mu_{n_i}) = I(\mu)$. This doesn't simply follow from the weak-* convergence because the logarithm is not continuous in the neighborhood of 0. We remedy that by approximating the logarithm by a well chosen family of continuous functions.

Let $T > 0$. For $x > 0$ put $\log^T x := \max\{-T, \log x\}$ and let $\log^T 0 := -T$. For any compactly supported probability measure ν on V put:

$$I_T(\nu) = \int_{V\times V} \log^T\|x-y\| d\nu(x)d\nu(y). \tag{3.5.2}$$

Note that we integrate over the diagonal as well. The function $\log^T$ is continuous so we get $\lim_{i\to\infty} I_T(\mu_{n_i}) = I_T(\mu)$. On the other hand, by Lebesgue dominated convergence theorem we have $\lim_{T\to\infty} I_T(\mu) = I(\mu)$ so $I(\mu) = \lim_{T\to\infty}\lim_{i\to\infty} I_T(\mu_{n_i})$. We estimate the difference $I_T(\mu_{n_i}) - I'(\mu_{n_i})$.

$$I'(\mu_{n_i}) - I_T(\mu_{n_i}) = \frac{T(n_i+1)}{n_i^2} + \frac{1}{n_i^2}\sum_{\substack{x\neq y\in\mathcal{S}_{n_i}\\ \|s_n^{-1}(x-y)\|\leq e^{-T}}} \left(\log\|s_n^{-1}(x-y)\| + T\right) \tag{3.5.3}$$

$$= \frac{T(n_i+1)}{n_i^2} + \frac{1}{n_i^2}\sum_{\substack{x\neq y\in\mathcal{S}_{n_i}\\ \|x-y\|\leq\|s_{n_i}\|e^{-T}}} (\log\|x-y\| - \log\|s_{n_i}\| + T) \tag{3.5.4}$$

Hence

$$-\frac{1}{n_i^2}\sum_{\substack{x\neq y\in\mathcal{S}_{n_i}\\ \|x-y\|\le\|s_{n_i}\|e^{-T}}}(\log\|s_{n_i}\|-\log\|x-y\|-T)\le I'(\mu_{n_i})-I_T(\mu_{n_i})\le\frac{T(n_i+1)}{n_i^2}. \quad (3.5.5)$$

We proceed to estimate the left hand side. Note that by Corollary 3.3.2 and our choice of s_{n_i}, t_{n_i} there is a compact cylinder $\Omega = B_{\mathbb{R}}(0,A)^{r_1}\times B_{\mathbb{C}}(0,A)^{r_2}$ such that $\mathcal{S}_{n_i}\subset s_{n_i}(\Omega - t_{n_i})$. Let $\Omega' = B_{\mathbb{R}}(0,2A)^{r_1}\times B_{\mathbb{C}}(0,2A)^{r_2}$. Then for every $x,y\in\mathcal{S}_{n_i}$ we have $x-y\in s_{n_i}\Omega'$. Hence

$$\sum_{\substack{x\neq y\in\mathcal{S}_{n_i}\\ \|x-y\|\le\|s_{n_i}\|e^{-T}}}(\log\|s_{n_i}\|-\log\|x-y\|-T)\le\sum_{x\in S_{n_i}}\sum_{\substack{z\in s_{n_i}\Omega'\\ \|z\|\le\|s_{n_i}\|e^{-T}}}(\log\|s_{n_i}\|-\log\|z\|-T) \quad (3.5.6)$$

$$=(n_i+1)\sum_{\substack{z\in s_{n_i}\Omega'\\ \|z\|\le\|s_{n_i}\|e^{-T}}}(\log\|s_{n_i}\|-\log\|z\|-T). \quad (3.5.7)$$

Let us fix a good fundamental domain of $\mathcal{O}_k^\times$ acting on $V^\times$ (see Definition 3.2.2) and a basis $\xi_1,\ldots,\xi_{d-1}$ of a maximal torsion free subgroup of $\mathcal{O}_k^\times$ together with the associated norm $\|\cdot\|_\infty$ on $\mathcal{O}_k^\times$. We can write $s_{n_i}=v\lambda_0$ with $\lambda_0\in\mathcal{O}_k^\times, v\in\mathcal{F}$ and $\|v\|=\|s_{n_i}\|=n_i|\Delta_k|^{1/2}$. Put $A_4 := 2|\Delta_k|^{1/2N}C_0A$. By Lemma 3.2.3 we have

$$\lambda_0^{-1}s_{n_i}\Omega' = v\Omega'\subset\Omega'' := B_{\mathbb{R}}(0,n_i^{1/N}A_4)^{r_1}\times B_{\mathbb{C}}(0,n_i^{1/N}A_4)^{r_2}.$$

By Lemmas 3.2.4 and 3.2.3 for every $x\in\mathcal{F}$ and $\lambda\in\mathcal{O}_k^\times$ such that $x\lambda\in\Omega''$ we have

$$\|\lambda\|_\infty\le\alpha^{-1}(\log(n_i^{1/N}\|x\|^{-1/N}A_4C_0)) =: C_{22}(\log n_i-\log\|x\|)+C_{23}.$$

We can estimate the sum in (3.5.7) by

$$\sum_{\substack{z\in s_{n_i}\Omega'\\ \|z\|\le\|s_{n_i}\|e^{-T}}}(\log\|s_{n_i}\|-\log\|z\|-T)=\sum_{\substack{z\in\lambda_0^{-1}s_{n_i}\Omega'\\ \|z\|\le\|s_{n_i}\|e^{-T}}}(\log\|s_{n_i}\|-\log\|z\|-T)$$

$$\le\sum_{\substack{x\in\mathcal{F}\\ \|x\|\le n_i|\Delta_k|^{1/2}e^{-T}}}(\log\|s_{n_i}\|-\log\|x\|-T)|\left\{\lambda\in\mathcal{O}_k^\times\,|\,\|\lambda\|_\infty\le C_{22}(\log n-\log\|x\|)+C_{23}\right\}|.$$

Once T is sufficiently large we will have $C_{22}(\log n_i-\log\|x\|)+C_{23}\le 2C_{22}(\log n_i-\log\|x\|)$ for every x satisfying $\|x\|\le n_i|\Delta_k|^{1/2}e^{-T}$. Therefore, for T sufficiently large we can bound the last expression by

$$\begin{aligned}
\leq & \sum_{\substack{x\in\mathcal{F}\\ \|x\|\leq n_i|\Delta_k|^{1/2}e^{-T}}} (\log\|s_{n_i}\| - \log\|x\| - T)\left|\left\{\lambda\in\mathcal{O}_k^\times \mid \|\lambda\|_\infty \leq 2C_{22}(\log n_i - \log\|x\|)\right\}\right| \\
\leq & C_{24} \sum_{\substack{x\in\mathcal{F}\\ \|x\|\leq n_i|\Delta_k|^{1/2}e^{-T}}} (\log\|s_{n_i}\| - \log\|x\| - T)(\log n_i - \log\|x\|)^{d-1} \\
= & C_{24} \sum_{\substack{x\in\mathcal{F}\\ \|x\|\leq n_i|\Delta_k|^{1/2}e^{-T}}} (\log n_i - \log\|x\| + \frac{1}{2}\log|\Delta_k| - T)(\log n_i - \log\|x\|)^{d-1}.
\end{aligned}$$

Put $Y = n_i|\Delta_k|^{1/2}e^{-T}$. The last expression becomes

$$\begin{aligned}
& C_{24}\sum_{\substack{x\in\mathcal{F}\\ \|x\|\leq Y}} (\log Y - \log\|x\|)(\log Y - \log\|x\| + (T - \frac{1}{2}\log|\Delta_k|))^{d-1} \\
= & C_{24}\sum_{i=1}^{d-1}\binom{d-1}{i}(T - \frac{1}{2}\log|\Delta_k|)^i \sum_{\substack{x\in\mathcal{F}\\ \|x\|\leq Y}} (\log Y - \log\|x\|)^{d-i} \\
\leq & C_{25}T^{d-1}e^{-T}n_i.
\end{aligned}$$

For the last inequality we have used Lemma 3.2.8. The constant C_{25} depends only on k. We wrap inequalities together to get

$$-\frac{n_i+1}{n_i}C_{25}T^{d-1}e^{-T} \leq I'(\mu_{n_i}) - I_T(\mu_{n_i}) \leq \frac{T(n_i+1)}{n_i^2} \tag{3.5.8}$$

$$|I'(\mu_{n_i}) - I_T(\mu_{n_i})| \leq \frac{T(n_i+1)}{n_i^2} + \frac{n_i+1}{n_i}C_{25}T^{d-1}e^{-T}. \tag{3.5.9}$$

It follows that for any T sufficiently large we have $\limsup_{i\to\infty}|I'(\mu_{n_i}) - I_T(\mu_{n_i})| \leq C_{25}T^{d-1}e^{-T}$. Consequently

$$I(\mu) = \lim_{T\to\infty}\lim_{i\to\infty} I_T(\mu_{n_i}) = \lim_{i\to\infty} I'(\mu_{n_i}) = -\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}}.$$

The proposition is proved. □

3.5.3 Measures of minimal energy

In this section we show that the limit measures, provided that they exist, realize the minimal energy among all probability measures of density at most 1. Next we study the properties of energy minimizing measures.

Lemma 3.5.5. *For every compactly supported*[12] *probability measure ν on V with density*

[12] The assumption on the support makes the proof easier but the statement should remain valid without

at most 1 *we have* $-\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}} \leq I(\nu)$.

Proof. Let ν be a compactly supported probability measure on V of density at most 1. Lemma 3.7.1 (in the appendix) affords a sequence E_n of subsets of $\mathcal{O}_k$ such that $|E_n| = n+1$ and the measures

$$\nu_{E_n,n} := \frac{1}{|E_n|}\sum_{x\in E_n} \delta_{n^{-1/N}|\Delta_k|^{-1/2N}x}$$

converge weakly-* to ν. Put $\log^* t = \log t$ of $t > 0$ and $\log^* 0 = 0$. For every measure μ on V we have $I'(\mu) = \int\int \log^* \|x-y\| d\mu(x)d\mu(y)$. The function $(x,y) \mapsto \log^* \|x-y\|$ is lower semicontinuous on $\mathbb{R}_{\geq 0}$ so

$$\limsup_{n\to\infty} I'(\nu_{E_n,n}) \leq I'(\nu) = I(\nu), \tag{3.5.10}$$

$$\limsup_{n\to\infty} \frac{1}{(n+1)^2}\sum_{x\neq y\in E_n}(\log\|x-y\| - \log n - \frac{1}{2}\log|\Delta_k|) \leq I(\nu). \tag{3.5.11}$$

By [11, Corollary 5.2] we get

$$-\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}} \leq I(\nu).$$

□

It follows that any limit measure realizes the minimal energy among all probability measures of density at most 1. We turn to the investigation of such energy minimizing measures. Our goal is to prove:

Proposition 3.5.6. *Let ν be a compactly supported probability measure on V of density at most* 1 *which is realizing the minimal energy among all such measures. Then there exists an open set U such that $\nu = \mathrm{Leb}|_U$ and moreover there exists $v \in V$ such that $(\partial U - v)\cap V^\times$ is a codimension* 1 *subvariety of class C^1 and $\lambda(\overline{U} - v) \subset U - v$ for every $0 < \lambda < 1$.*

We will write $\mathcal{M}^1(V), \mathcal{P}^1(V)$ for the sets of respectively finite measures and probability measures on V with Lebesgue density bounded by 1. One of the key tools used to prove Proposition 3.5.6 is the collapsing procedure $c_{i,v_i} : \mathcal{M}^1(V) \to \mathcal{M}^1(V)$ (see Definition 3.4.1), introduced and studied in the Section 3.4. We will also need the following identities.

Lemma 3.5.7. *1. For every $x \in \mathbb{R}$, $T > 0$ we have*

$$\frac{d}{dx}\left(\int_{-T}^{T}\log|x-t|dt\right) = \log(|T+x|) - \log(|T-x|).$$

2. Write $dxdy$ for the Lebesgue measure on $\mathbb{C}$ in coordinates $z = x+iy$. For every

it.

$se^{i\theta} \in \mathbb{C}, s > 0$ *we have*

$$\int_{B_{\mathbb{C}}(0,T)} \log |se^{i\theta} - z|^2 dxdy = \begin{cases} 2\pi T^2 \log T - \pi T^2 + \pi s^2 & \text{if } r \leq T \\ 2\pi T^2 \log s & \text{otherwise,} \end{cases}$$

$$\frac{d}{ds}\left(\int_{B_{\mathbb{C}}(0,T)} \log |se^{i\theta} - z|^2 dxdy\right) = \begin{cases} 2\pi s & \text{if } s \leq T \\ \frac{2\pi T^2}{s} & \text{otherwise.} \end{cases}$$

Proof of Proposition 3.5.6. Let $\nu \in \mathcal{P}^1(V)$ be a measure such that the energy $I(\nu)$ is minimal on $\mathcal{P}^1(V)$. By Corollary 3.4.8 there exists $v = (v_1, \ldots, v_d) \in V$ such that $c_{i,v_i}(\nu) = \nu$ for every $i = 1, \ldots, d$. Translating ν by $-v$ we may assume that $v = 0$. We will construct an open set U such that $\nu = \mathrm{Leb}|_U$, $\lambda \overline{U} \subset U$ for every $0 \leq \lambda < 1$ and $\partial U \cap V^\times$ is a C^1-submanifold of $V^\times$ of codimension 1. Let $P_V, P_i, i = 1, \ldots, d$ be functions on V defined by

$$P_i(x) = \int_V \log |x - y|_i d\nu(y) \text{ and } P_V(x) = \int_V \log \|x - y\| d\nu(y) = \sum_{i=1}^d P_i.$$

Clearly $P_i(x)$ depends only on the $i-$th coordinate of x so it makes sense to abuse the notation and write $P_i(x) = P_i(x_i)$. We will show that U can be chosen as $U = P_V^{-1}((-\infty, \alpha))$ for some $\alpha \in \mathbb{R}$. To prove that the boundary $\partial U \cap V^\times$ is a C^1-submanifold we will establish certain regularity properties of P_i for coordinates $i = 1, \ldots, d$ and use the implicit function theorem. Starting from Step 3 we assume, for the sake of the proof, that $V = \mathbb{R}^2$.

Step 1. We show that there exists a unique $\alpha \in \mathbb{R}$ such that $\mathrm{Leb}(P_V^{-1}((-\infty, \alpha))) = 1$. It is easy to see that $P_V^{-1}((-\infty, t)))$ is bounded for every $t \in \mathbb{R}$. We will consider the gradient $\nabla P_V = \left(\frac{\partial}{\partial x_i} P_i(v)\right)_{i=1,\ldots,d}$ allowing it to take value $\pm\infty$ on some coordinates. We show that for almost all $v \in V$ the gradient $\nabla P_V(v)$ is non-zero. In such case the function $t \mapsto \mathrm{Leb}(P_V^{-1}((-\infty, t)))$ is a continuous bijection from $[\operatorname{ess\,inf} P_V, +\infty)$ to $[0, +\infty)$ so we can find a unique α with $\mathrm{Leb}(P_V^{-1}((-\infty, \alpha))) = 1$.

Let F_i, h_i be the functions defined as in Definition 3.4.1. Since our measure is already collapsed with respect to all coordinates, the function $h_1 = h_2 = \ldots = h_d$ is the density of ν. Hence for every $i = 1, 2, \ldots, d$ we have

$$P_i(x_i) = \begin{cases} \int \cdots \int \left(\int_{B_{\mathbb{R}}(0, F_i(t_1, \ldots, 0, \ldots, t_d))} \log |x_i - t_i| dt_i\right) dt_1 \ldots \widehat{dt_i} \ldots dt_d & \text{if } i \in \{1, \ldots, r_1\} \\ \int \cdots \int \left(\int_{B_{\mathbb{C}}(0, F_i(t_1, \ldots, 0, \ldots, t_d))} \log |x_i - t_i|^2 dt_i\right) dt_1 \ldots \widehat{dt_i} \ldots dt_d & \text{if } i \in \{r_1 + 1, \ldots, d\} \end{cases}. \tag{3.5.12}$$

Let $x = (x_1, \ldots, x_{r_1}, s_{r_1+1} e^{i\theta_{r_1+1}}, \ldots, s_d e^{i\theta_d}) \in V$. To shorten notation we will write V^i for the subset of V defined by $v_i = 0$ and dv^i and for the Lebesgue measure on V^i. In these coordinates we have

$$P_i(x_i) = \begin{cases} \int_{V^i} \left(\int_{B_{\mathbb{R}}(0,F_i(v^i))} \log |x_i - t_i| dt_i \right) dv^i & \text{if } i \in \{1, \dots, r_1\} \\ \int_{V^i} \left(\int_{B_{\mathbb{C}}(0,F_i(v^i))} \log |x_i - t_i|^2 dt_i \right) dv^i & \text{if } i \in \{r_1 + 1, \dots, d\}. \end{cases} \tag{3.5.13}$$

By Lemma 3.5.7 for $i = 1, \dots, r_1$ we have

$$\frac{d}{dx_i} P_i(x_i) = \int_{V^i} \left(\log |F_i(v^i) + x_i| - \log |F_i(v^i) - x_i| \right) dv^i \tag{3.5.14}$$

and for $i = r_1 + 1, \dots, d$

$$\frac{d}{ds_i} P_i(s_i e^{i\theta_i}) = \int_{V^i, F_i(v^i) \leq s_i} \frac{2\pi F_i(v^i)^2}{s_i} dv^i + \int_{V^i, F_i(v^i) > s_i} 2\pi s_i dv^i. \tag{3.5.15}$$

We have

$$\frac{d}{dx_i} P_V = \frac{d}{dx_i} P_i \text{ and } \frac{d}{ds_i} P_V = \frac{d}{ds_i} P_i.$$

Note that (3.5.14), (3.5.15) are strictly positive or $+\infty$ as soon as $x_i > 0$ or $s_i > 0$ and strictly negative or $-\infty$ if $x_i < 0$. In particular the gradient $\nabla P_V(v)$ is non-zero for $v \neq 0$. This proves Step 1.

Step 2. Let $U = P_V^{-1}((-\infty, \alpha))$. We prove that $\nu = \mathrm{Leb}|_U$ and that $\lambda \overline{U} \subset U$ for every $0 \leq \lambda < 1$.

For any two measures $\mu, \mu' \in \mathcal{M}^1(V)$ we define a bilinear form

$$\langle \mu, \mu' \rangle := \int_V \int_V \log \|x - y\| d\mu(x) d\mu'(y).$$

With that definition we have $I(\mu) = \langle \mu, \mu \rangle$ for every $\mu \in \mathcal{M}^1(V)$. The function P_V is defined so that $\langle \nu, \mu \rangle = \int_V P_V(x) d\mu(x)$ for every $\mu \in \mathcal{M}^1(V)$. Let $\nu' = \mathrm{Leb}|_{U'} \in \mathcal{P}^1(V)$. By the choice of U we have $\langle \nu, \nu' \rangle \leq \langle \nu, \nu \rangle$ with equality if and only if $\nu = \mathrm{Leb}|_U$. Let $\varepsilon \geq 0$ be small and put $\nu_\varepsilon = (1 - \varepsilon)\nu + \varepsilon \nu'$. This measure is in $\mathcal{P}^1(V)$ so

$$I(\nu_\varepsilon) = (1 - \varepsilon)^2 I(\nu) + 2\varepsilon(1 - \varepsilon)\langle \nu, \nu' \rangle + \varepsilon^2 I(\nu') \geq I(\nu).$$

We deduce that

$$\left. \frac{d}{d\varepsilon} I(\nu_\varepsilon) \right|_{\varepsilon = 0} = 2(\langle \nu, \nu' \rangle - \langle \nu, \nu \rangle) \geq 0.$$

We already know that $\langle \nu, \nu' \rangle - \langle \nu, \nu \rangle \leq 0$ so $\langle \nu, \nu' \rangle = \langle \nu, \nu \rangle$ and $\nu = \mathrm{Leb}|_U$. It remains to show that $\lambda \overline{U} \subset U$ for every $0 \leq \lambda < 1$. Note that $\overline{U} \subset P_V^{-1}((-\infty, \alpha])$ so it will be enough to prove that $P_V(\lambda x) < P_V(x)$ for every $x \in V \setminus 0$. This is true because the computations from Step 1 imply that the derivative $\frac{d}{d\lambda} P_V(\lambda x)$ is strictly positive on $(0, +\infty)$ for every $v \neq 0$.

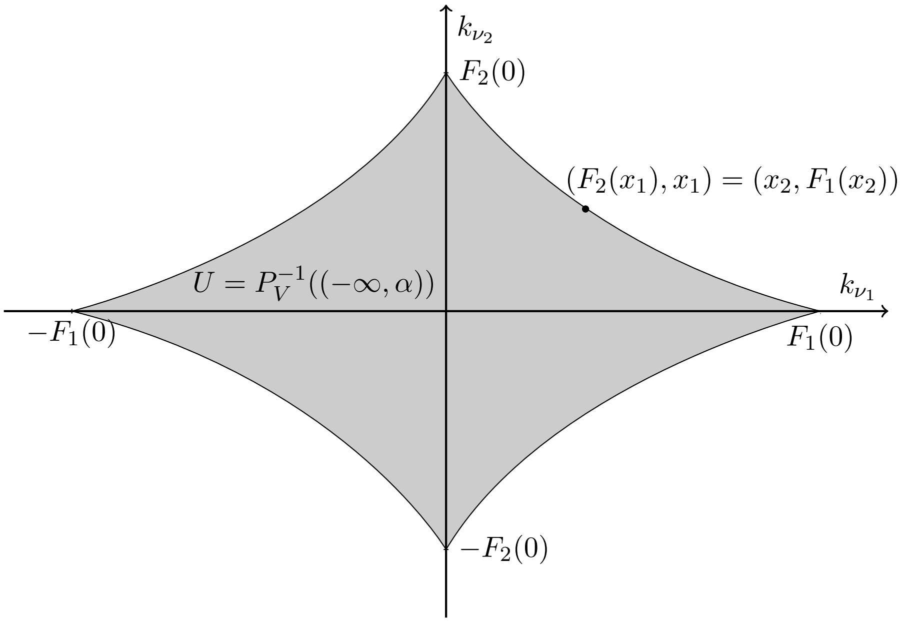

Figure 3.5.1: The set $U \subset V \simeq \mathbb{R}^2$

Step 3. From now on we assume $V = \mathbb{R}^2$. The proof of the general case follows the same outline with some parts being easier for complex coordinates[13]. By the previous steps $\nu = \mathrm{Leb}|_U$ where $U = P_V^{-1}((-\infty, \alpha))$. The set U is contained in the box $(-F_1(0), F_1(0)) \times (-F_2(0), F_2(0))$ (see Figure 3.5.1) and that is where we study the regularity properties of P_1, P_2. The functions $F_1(t), F_2(t)$ vanish outside $(-F_2(0), F_2(0)), (-F_1(0), F_1(0))$ respectively and admit their maximum at $t = 0$. We show that the derivative $\frac{d}{dx_i} P_i(x_i)$ restricted to $(-F_i(0), F_i(0))$ is in $L^2((-F_i(0), F_i(0)))$ for $i = 1, 2$. From now on we restrict P_1, P_2 to $(-F_1(0), F_1(0)), (-F_2(0), F_2(0))$ respectively.

By (3.5.14) we have

$$\begin{aligned}
\left\| \frac{d}{dx_1} P_1(x_1) \right\|_2 &= \left(\int_{-F_1(0)}^{F_1(0)} \left(\int_{\mathbb{R}} (\log |F_1(t) + x_1| - \log |F_1(t) - x_1|) \, dt \right)^2 dx_1 \right)^{1/2} \\
&\leq \int_{\mathbb{R}} \left(\int_{-F_1(0)}^{F_1(0)} (\log |F_1(t) + x_1| - \log |F_1(t) - x_1|)^2 \, dx_1 \right)^{1/2} dt \\
&\leq 2 \int_{-F_2(0)}^{F_2(0)} \left(\int_{-F_1(0)}^{F_1(0)} (\log |F_1(t) + x_1|)^2 \, dx_1 \right)^{1/2} dt \\
&\leq 2 \int_{-F_2(0)}^{F_2(0)} O(F_1(0)^{1/2}(|\log F_1(0)| + 1)) dt < +\infty.
\end{aligned}$$

Between the second and the third line we had right to restrict the outer integral from $\mathbb{R}$ to $[-F_2(0), F_2(0)]$ because outside this interval $F_1(t)$ is 0 so the inner integral vanishes. Same

[13] In the real case the derivative of P_i is locally L^2. For the complex coordinates P_i is uniformly bounded.

computations show that $\|\frac{d}{dx_2}P_2(x_2)\|_2 < +\infty$. This concludes the proof of Step 3.

Step 4. We show that for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that $\left|\frac{d}{dx_i}P_i(x_i)\right| \geq A_\varepsilon$ for $i = 1, 2$ and every $\varepsilon < |x_i| \leq F_i(0)$.

Let $\varepsilon > 0$ and let $\varepsilon < |x_1| \leq F_1(0)$. Assume that $x_1 > \varepsilon$ (i.e. x_1 is positive). Since we can always restrict to a smaller ε we will assume for technical reasons that $\frac{1}{2} - 2\varepsilon F_2(0) > 0$. We have

$$\begin{aligned}
\frac{d}{dx_1}P_1(x_1) &= \int_{-F_2(0)}^{F_2(0)} (\log|F_1(t) + x_1| - \log|F_1(t) - x_1|)dt \\
&\geq \int_{-F_2(0)}^{F_2(0)} \left(1 - \frac{|F_1(t) - x_1|}{|F_1(t) + x_1|}\right) dt = \int_{-F_2(0)}^{F_2(0)} \frac{|F_1(t) + x_1| - |F_1(t) - x_1|}{|F_1(t) + x_1|} dt \\
&= 2\int_{-F_2(0)}^{F_2(0)} \frac{\min\{F_1(t), x_1\}}{|F_1(t) + x_1|} dt \geq \frac{1}{F_1(0)} \int_{-F_2(0)}^{F_2(0)} \min\{F_1(t), x_1\}\, dt \\
&\geq \frac{1}{F_1(0)} \int_{-F_2(0)}^{F_2(0)} \min\{F_1(t), \varepsilon\}\, dt.
\end{aligned}$$

To estimate the last quantity we go back to the definition of F_1 (see Definition 3.4.1). It implies that

$$\int_{-F_2(0)}^{F_2(0)} 2F_1(t)dt = \nu(V) = 1. \tag{3.5.16}$$

Let $E_1 := \{t \in [-F_2(0), F_2(0)] \,|\, F_1(t) \geq \varepsilon\}$. For every $t \in [-F_2(0), F_2(0)]$ we have $F_1(t) \leq F_1(0)$ so (3.5.16) yields $F_1(0)\mathrm{Leb}(E_1) + \varepsilon(2F_2(0) - \mathrm{Leb}(E_1)) \geq \frac{1}{2}$. In particular $\mathrm{Leb}(E_1) \geq \frac{\frac{1}{2} - 2\varepsilon F_2(0)}{F_1(0) - \varepsilon}$. We get

$$\frac{d}{dx_1}P_1(x_1) \geq \frac{1}{F_1(0)} \int_{-F_2(0)}^{F_2(0)} \min\{F_1(t), \varepsilon\}\, dt \geq \frac{\varepsilon}{F_1(0)}\mathrm{Leb}(E_1) \geq \frac{\varepsilon(\frac{1}{2} - 2\varepsilon F_2(0))}{F_1(0)(F_1(0) - \varepsilon)} > 0.$$

The final lower bound is positive and depends only on $\varepsilon, F_1(0)$ and $F_2(0)$. The same computation gives a negative upper bound for $\frac{d}{dx_1}P_1(x_1)$ when $x_1 < 0$. The argument for $i = 2$ is identical.

Step 5. We show that $P_i(x_i), i = 1, 2$ are of class C^1 on $(-F_2(0), F_2(0))$ and $(-F_1(0), F_1(0))$ respectively.

The points $(F_1(t), t)$ for $t \in [-F_2(0), F_2(0)]$ are in the boundary ∂U. Since $U = P_V^{-1}((-\infty, \alpha))$ we have $\partial U \subset P_V^{-1}(\{\alpha\})$. As a consequence

$$P_1(F_1(t)) + P_2(t) = P_V((F_1(t), t)) = \alpha \text{ for } t \in [-F_2(0), F_2(0)].$$

The function P_1 is strictly increasing on $[0, +\infty)$ so let us write P_1^{-1} for the inverse of P_1 restricted to $[0, +\infty)$. Then, for $t \in (0, F_2(0))$ we have

$$F_1(t) = P_1^{-1}(\alpha - P_2(t)).$$

We deduce that $F_1(t)$ is strictly decreasing on $(0, F_2(0))$ and that

$$\frac{d}{dt}F_1(t) = -\frac{d}{dt}P_2(t)\left(\frac{d}{ds}P_1(s)\bigg|_{s=F_1(t)}\right)^{-1} \tag{3.5.17}$$

whenever the formula is well defined. We have an analogous equation for F_2 from which it follows that $F_2 : (0, F_1(0)) \to (0, F_2(0))$ is the inverse of F_1. Let $\varepsilon > 0$ be small. Then for every $0 \leq t < F_2(0) - \varepsilon$ we have $F_1(t) > F_1(F_2(0) - \varepsilon) > 0$. Let $\varepsilon' = F_1(F_2(0) - \varepsilon)$ and let $A = A_{\varepsilon'}$ be the constant from Step 4. Combining (3.5.17) with Step 4 we get

$$\left|\frac{d}{dt}F_1(t)\right| \leq \left|\frac{d}{dt}P_2(t)\right| A^{-1} \text{ for } 0 \leq t \leq F_2(0) - \varepsilon.$$

By Step 3 we have

$$\int_0^{F_2(0)-\varepsilon} \left|\frac{d}{dt}F_1(t)\right|^2 dt \leq \left\|\frac{d}{dt}P_2(t)\right\|_2^2 A^{-2} < +\infty. \tag{3.5.18}$$

The above will serve as an input to the Cauchy–Schwartz inequality. Put $G_x(s) := \log|s+x| - \log|s-x|$ for $s \in [0, F_2(0)]$. A simple computation shows that $G_x \in L^2([0, F_2(0)])$ and that the map $\mathbb{R} \ni x \to G_x \in L^2([0, F_2(0)])$ is continuous. We will estimate $\frac{d}{dx}P_2(x)$ for $\varepsilon \leq x \leq F_2(0) - 2\varepsilon$. By (3.5.14) we have

$$\frac{d}{dx}P_2(x) = \int_{-F_1(0)}^{F_1(0)} (\log|F_2(t) + x| - \log|F_2(t) - x|)dt$$

and we use substitution $t = F_1(s)$ to get

$$\begin{aligned}
=&2\int_0^{F_2(0)} (\log|s+x| - \log|s-x|)\left|\frac{dF_1(s)}{ds}\right| ds \\
=&2\int_0^{F_2(0)-\varepsilon} (\log|s+x| - \log|s-x|)\left|\frac{dF_1(s)}{ds}\right| ds \\
+&2\int_{F_2(0)-\varepsilon}^{F_2(0)} (\log|s+x| - \log|s-x|)\left|\frac{dF_1(s)}{ds}\right| ds.
\end{aligned}$$

We use Cauchy–Schwartz and (3.5.18) to estimate the first summand and get:

$$\begin{aligned}
\frac{d}{dx}P_2(x) \leq& 2\|G_x\|_2 \left\|\frac{dP_2(t)}{dt}\right\|_2 A^{-1} + 2\int_{F_2(0)-\varepsilon}^{F_2(0)} (\log(2F_2(0) - 2\varepsilon) - \log\varepsilon)\left|\frac{dF_1(s)}{ds}\right| ds \\
=&2A^{-1}\|G_x\|_2 \left\|\frac{dP_2(t)}{dt}\right\|_2 + 2F_1^{-1}(F_2(0) - \varepsilon)(\log|(2F_2(0) - 2\varepsilon) - \log\varepsilon) < +\infty.
\end{aligned}$$

Since $\frac{d}{dx}P_2(x) \geq 0$ for $x \geq 0$ this establishes the finiteness of $\frac{d}{dx}P_2(x)$ on $[0, F_2(0) - 2\varepsilon]$. To show continuity we choose $0 \leq x, x' < F_2(0) - 2\varepsilon$ and perform the same calculation to get

$$\left|\frac{dP_2(t)}{dt}|_{t=x} - \frac{dP_2(t)}{dt}|_{t=x'}\right| \leq 2A^{-1}\|G_x - G_{x'}\|_2 \left\|\frac{dP_2(t)}{dt}\right\|_2$$
$$+2\int_{F_2(0)-\varepsilon}^{F_2(0)} \big|\log|x+s| - \log|x'+s| + \log|x'-s| - \log|x-s|\,\big| \left|\frac{dF_1(s)}{ds}\right| ds.$$

The right hand side tends to 0 as $x' \to x$ so $\frac{d}{dx}P_2(x)$ is continuous on $(0, F_2(0) - 2\varepsilon]$. We let $\varepsilon \to 0$ and use symmetry of P_2 to deduce that P_2 is of class C^1 on $[-F_2(0), F_2(0)] \setminus \{0\}$. Same proof shows that P_1 is of class C^1 on $[-F_1(0), F_1(0)] \setminus \{0\}$.

Step 6. We will deduce that F_1, F_2 are of class C^1 on $[-F_2(0), F_2(0)] \setminus \{0\}$, $[-F_1(0), F_1(0)] \setminus \{0\}$ respectively.

By symmetry it is enough to show that F_1 is of class C^1 on $(0, F_2(0))$. For $t \in (0, F_2(0))$ we have $\alpha = P_1(F_1(t)) + P_2(t)$ so $F_1(t) = P_1^{-1}(\alpha - P_2(t))$. By Step 5 P_1 is of class C^1 so the same is true for P_1^{-1} on its domain of definition. As a composition of two C^1 functions F_1 is of class C^1.

Step 7. We have

$$\partial U \cap V^\times = \{(t, \pm F_1(t)) | t \in (-F_2(0), F_2(0)) \setminus \{0\}\}.$$

Being a finite disjoint union of graphs of functions of class C^1 the set $\partial U \cap V^\times$ is a C^1-submanifold of codimension 1. This concludes the proof. □

3.6 Non-existence of n-optimal sets.

3.6.1 Discrepancy and almost equidistribution

Let ν be any limit measure on $V = k \otimes_{\mathbb{Q}} \mathbb{R}$. In this section we study the discrepancy of the sets U such that $\nu = \mathrm{Leb}|_U$ which are provided by Proposition 3.5.6. We recall the notion of **lattice point discrepancy** (see [21]).

Definition 3.6.1. *Let $V = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and fix a bounded measurable subset U of V. For $t \in V^\times, v \in V$ let $N_t(U, v) := |\{(tU + v) \cap \mathcal{O}_k\}|$ and define the discrepancy*

$$D_t(U, v) := N_t(U, v) - |\Delta_k|^{-1/2}\mathrm{Leb}(U)\|t\|.$$

And the maximal discrepancy

$$D_t(U) := \operatorname*{ess\,sup}_{v \in V} |D_t(U, v)|.$$

We will need the following technical property of the maximal discrepancy.

Lemma 3.6.2. *Let U be a bounded measurable subset of V such that ∂U has zero Lebesgue measure. Then either $D_t(U) < 1$ for all $t \in V^\times$ or there exists $\delta > 0$ a non-empty open subset $T \subset V^\times$ and a non-empty open subset W of V such that $D_t(U, v) > 1 + \delta$ for all $t \in T, v \in W$.*

Proof. Let $E = \bigcup_{x \in \mathcal{O}_k} \bigcup_{t \in V^\times} (x - t\partial U) \times \{t\} \subset V \times V^\times$ and for every $t \in V^\times$ let $E_t = \{v \in V | (v, t) \in E\} = \bigcup_{x \in \mathcal{O}_k} (x - t\partial U)$. Because the set U is bounded, the unions defining E and E_t are locally finite. We deduce that E and E_t are closed and E_t has measure 0 for every $t \in V^\times$. The function $(v, t) \mapsto N_t(U, v)$ is locally constant on $(V \times V^\times) \setminus E$ so it is constant on the connected components of $(V \times V^\times) \setminus E$. In particular, for every $t \in V^\times$ the function $v \mapsto D_t(U, v)$ is constant on the connected components on $V \setminus E_t$.

Assume that $D_{t_0}(U) \geq 1$ for some $t_0 \in V^\times$. The set of values of $D_{t_0}(U, v)$ is discrete because $D_{t_0}(U, v) \in \mathbb{N} - |\Delta_k|^{-1/2}\mathrm{Leb}(U)\|t_0\|$. We deduce that there exists a connected component Q_{t_0} of $V \setminus E_{t_0}$ such that $D_{t_0}(U, v) \geq 1$ or $D_{t_0}(U, v) \leq -1$ for all $(v, t) \in Q_{t_0}$. Assume that the first inequality holds. Fix a point $v_0 \in Q_{t_0}$. Let Q be the unique connected component of $(V \times V^\times) \setminus E$ containing (v_0, t_0). For $\varepsilon > 0$ let $Q^\varepsilon = \{(v, t) \in Q | \|t\| < \|t_0\| - \varepsilon\}$, for ε small enough it is a non-empty open set because t_0 lies in the interior of Q. Choose open sets $T \subset V^\times, W \subset V$ such that $W \times T \subset Q^\varepsilon$. For every $(v, t) \in Q^\varepsilon$ we have

$$\begin{aligned} D_t(U, v) = N_t(U, v) - |\Delta_k|^{-1/2}\mathrm{Leb}(U)\|t\| &= N_{t_0}(U, v_0) - |\Delta_k|^{-1/2}\mathrm{Leb}(U)\|t\| \\ &> D_{t_0}(U, v_0) + \varepsilon|\Delta_k|^{-1/2}\mathrm{Leb}(U). \end{aligned}$$

We deduce that $D_t(U, v) > 1 + \delta$ with $\delta = \varepsilon\mathrm{Leb}(U)|\Delta_k|^{-1/2}$ for $t \in T$ and all $v \in W$. In the case $D_{t_0}(U, v) \leq -1$ the same argument works with $Q' = \{(v, t) \in Q | \|t\| > \|t_0\| + \varepsilon\}$. □

We show that if ν is a limit measure and U is the open set provided by Proposition 3.5.6 then U must have very low maximal discrepancy.

Lemma 3.6.3. *Let ν be a limit measure on V and let U be an open set such that ∂U is Jordan measurable of Jordan measure 0 and $\nu = \mathrm{Leb}|_U$. Then U satisfies $D_t(U) < 1$ for all $t \in V^\times$.*

Proof. We argue by contradiction. Assume that for some $t_0 \in V^\times$ we have $D_{t_0}(U) \geq 1$. By Lemma 3.6.2 there exist open sets $T \subset V^\times, W \subset V$ and $\delta > 0$ such that $|D_t(U, v)| > 1 + \delta$ for every $t \in T, v \in W$. By making W smaller if necessary we may assume it is an open cylinder in V, similarly by taking smaller T if necessary we may assume that there exists $\kappa > 1$ such that $\kappa^{-1} \leq \|t\| \leq \kappa$ for all $t \in T$. Let $(n_i)_{i \in \mathbb{N}}, (\mathcal{S}_{n_i})_{i \in \mathbb{N}}$ be a sequence of n-optimal sets and let $(t_{n_i})_{i \in \mathbb{N}} \subset V, (s_{n_i}) \subset V^\times, \|s_{n_i}\| = n_i|\Delta_k|^{1/2}$ be such that the measures ν_{n_i} defined as in (3.5.1) converge weakly-* to ν. Translating $\mathcal{S}_{n_i}$ be appropriate elements of $\mathcal{O}_k$ we may assume that $t_{n_i} = 0$, this will simplify considerably the formulas in the proof. By [11, Corollary 2.4] the sets $\mathcal{S}_{n_i}$ are **almost uniformly distributed** modulo powers of every prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$. This means that for every prime $\mathfrak{p}$ of $\mathcal{O}_k$, $l \in \mathbb{N}$ and $a \in \mathcal{O}_k$ we have

$$\left| \left| \left\{ x \in \mathcal{S}_{n_i} | x - a \in \mathfrak{p}^l \right\} \right| - \frac{n_i + 1}{N\mathfrak{p}^l} \right| < 1.$$

In order to get a contradiction we will exhibit a prime $\mathfrak{p}_{n_i}$ for all sufficiently large n_i such that $\mathcal{S}_{n_i}$ fails to be almost uniformly equidistributed modulo $\mathfrak{p}_{n_i}$.

Let $E_{n_i} := (s_{n_i} U) \cap \mathcal{O}_k$ and put $R_{n_i} = \mathcal{S}_{n_i} \Delta E_{n_i}$. Since ν_{n_i} converges weakly-* to $\mathrm{Leb}|_U$ and the boundary ∂U has Jordan measure 0 we can deduce that $|R_{n_i}| = o(n_i)$. The set T^{-1} is open so by Corollary 3.7.3 (in the appendix) for n_i sufficiently large there exists an $\varpi_{n_i} \in s_{n_i} T^{-1} \cap \mathcal{O}_k$ such that the principal ideal $\mathfrak{p}_{n_i} := \varpi_{n_i} \mathcal{O}_k$ is prime. We argue that for every $x \in \mathcal{O}_k \cap \varpi_{n_i} W$ we have

$$|\{y \in E_{n_i} | x - y \in \mathfrak{p}_{n_i}\}| = |s_{n_i} U \cap (\varpi_{n_i} \mathcal{O}_k + x)| = N_{s_{n_i} \varpi_{n_i}^{-1}}(U, \varpi_{n_i}^{-1} x).$$

Since $s_{n_i} \varpi_{n_i}^{-1} \in T$, $\varpi_{n_i}^{-1} x \in W$ and $\|s_{n_i} \varpi_{n_i}^{-1}\| = n_i |\Delta_k|^{1/2} (N\mathfrak{p}_{n_i})^{-1}$ we get

$$\left| |\{y \in E_{n_i} | x - y \in \mathfrak{p}_{n_i}\}| - \frac{n_i}{N\mathfrak{p}_{n_i}} \right| = |D_{s_{n_i} \varpi_{n_i}^{-1}}(U, \varpi_{n_i}^{-1} x)| > 1 + \delta \text{ for all } x \in \varpi_{n_i} W. \tag{3.6.1}$$

We showed that in some sense E_{n_i} fails "badly" to be almost uniformly equidistributed modulo $\mathfrak{p}_{n_i}$. From this we need do deduce that $\mathcal{S}_{n_i}$ is not almost equidistributed modulo $\mathfrak{p}_{n_i}$. Call $x \in \varpi_{n_i} W \cap \mathcal{O}_k$ **bad** if $(x + \varpi_{n_i} \mathcal{O}_k) \cap R_{n_i} \neq \emptyset$ and **good** otherwise. For good points we have $(x + \mathfrak{p}_{n_i}) \cap E_{n_i} = (x + \mathfrak{p}_{n_i}) \cap \mathcal{S}_{n_i}$. Our next goal is to prove that for n_i sufficiently large there exists at least one [14] good element in $\varpi_{n_i} W$. Let us estimate the number of bad elements of $\varpi_{n_i} W \cap \mathcal{O}_k$. By Lemma 3.2.19 for every $r \in R_{n_i}$ we have $|(r + \varpi_{n_i} \mathcal{O}_k) \cap \varpi_{n_i} W| = |(r \varpi_{n_i}^{-1} + W) \cap \mathcal{O}_k| = O(1)$. Hence we have at most $O(|R_{n_i}|) = o(n_i)$ bad elements. On the other hand $|\varpi_{n_i} W \cap \mathcal{O}_k| = \|\varpi_{n_i}\| \mathrm{Leb}(W) |\Delta_k|^{-1/2} + o(\|\varpi_{n_i}\|)$. We chose $\varpi_{n_i} \in s_{n_i} T^{-1}$ so $\kappa^{-1} n_i |\Delta_k|^{1/2} \leq \|\varpi_{n_i}\| \leq \kappa n_i |\Delta_k|^{1/2}$ so the number of good elements is $\mathrm{Leb}(W) n_i \kappa^{-1} - o(n_i)$. We infer that for n_i sufficiently large there exists at least one good element $x \in \varpi_{n_i} W \cap \mathcal{O}_k$. Let $x \in \varpi_{n_i} W \cap \mathcal{O}_k$ be a good element. We have $E_{n_i} \cap (x + \mathfrak{p}_{n_i}) = \mathcal{S}_{n_i} \cap (x + \mathfrak{p}_{n_i})$ so by (3.6.1) we get

$$\left| |\{y \in \mathcal{S}_{n_i} | x - y \in \mathfrak{p}_{n_i}\}| - \frac{n_i + 1}{N\mathfrak{p}_{n_i}} \right| = |D_{s_{n_i} \varpi_{n_i}^{-1}}(U, \varpi_{n_i}^{-1} x) - \frac{1}{N\mathfrak{p}_{n_i}}| > 1 + \delta - \frac{1}{N\mathfrak{p}_{n_i}}. \tag{3.6.2}$$

We know that $N\mathfrak{p}_{n_i} = \|\varpi_{n_i}\| \geq \kappa^{-1} |\Delta_k|^{1/2} n_i$ so for n_i sufficiently large (3.6.2) implies that $\mathcal{S}_{n_i}$ in not almost equidistributed modulo $\mathfrak{p}_{n_i}$. This is a contradiction because n-optimal sets are almost uniformly equidistributed modulo all prime ideals of $\mathcal{O}_k$. $\square$

The last ingredient that we will need in order to show that n-optimal sets cannot exist for large n is the following lower bound on the discrepancy.

[14] In fact we will show that most of them are good.

Lemma 3.6.4. *Assume that $V = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ with $r_1 + 2r_2 > 1$. Let U be an open subset of V such that $\partial U \cap V^\times$ is a submanifold of $V^\times$ of class C^1 and $\lambda\overline{U} \subset U$ for every $0 \leq \lambda < 1$. Then there exists $t \in V^\times$ such that $D_t(U) > 1$.*

Proof. We claim that there exists $t_0 \in V^\times, v_0 \in V$ such that $(t_0\partial U + v_0) \cap \mathcal{O}_k$ contains at least 3 points[15]. Let $N = \dim_{\mathbb{R}} V$. For every $v \in V^\times$ let us identify the tangent space $T_v V^\times$ with V in the obvious way. For every point $p \in \partial U \cap V^\times$ the tangent space $T_p\partial U$ is a codimension 1 subspace of V. Call a (real) codimension 1 subspace H of V singular if we have $H \subset V \setminus V^\times$.

Step 1. We show that there is $x_0 \in \partial U \cap V^\times$ such that $T_p\partial U$ is not singular. Write $\mathrm{Gr}_{N-1}(V)$ for the space parametrizing all $(N-1)$-dimensional real vector subspaces of V. The map $\phi(p) := [T_p\partial U] \in \mathrm{Gr}_{N-1}(T_pV^\times) \simeq \mathrm{Gr}_{N-1}(V)$ is continuous on $\partial U \cap V^\times$ because the latter is a C^1-submanifold. Let M be any connected component of $\partial U \cap V^\times$. Either the exists a point $p \in M$ such that $T_p\partial U$ is nonsingular or we can assume that the image $\phi(M)$ consists solely of singular subspaces. The set of singular subspaces in $\mathrm{Gr}_{N-1}(V)$ has r_1 elements, each corresponding to a real coordinate of V. Hence, $\phi(M) = H$ for some singular subspace H. We deduce that M is contained in a hyperplane H' parallel to H. In particular the r_1 singular subspaces and H' cut out a bounded region of V. This is a contradiction because $r_1+1 \leq N+1$ and the only way $N+1$ codimension 1 hyperplanes can cut out a bounded region in N-dimensional space is when they are pairwise non-parallel.

Step 2. We construct a continuous map $\gamma\colon [0,1] \to \partial U$ such that $\gamma(0) = x_0$ and $\|\gamma(s) - \gamma(0)\| > 0$ for $0 < s \leq 1$. Choose any smooth complete Riemannian metric on $\partial U \cap V^\times$. Choose a vector $w \in T_{x_0}\partial U$ such that $\|w\| \neq 0$. Let $\gamma : [0, +\infty)$ be the unique geodesic ray such that $\gamma(0) = x_0$ and $\frac{d}{dt}\gamma(t) = w$. We have $\frac{d}{dt}\|\gamma(0) - \gamma(t)\| = \|w\| \neq 0$ so for t small enough we have $\|\gamma(s) - \gamma(0)\| > 0$ for every $s \leq t$. Up to reparametrizing γ we may assume $t = 1$.

Step 3. We show that there exists $s_1 > 0$ such that $((\gamma(s_1) - x_0)\mathcal{O}_k + x_0) \cap \partial U$ contains at least one point x_1 except $x_0, \gamma(s_1)$. First note that as s approaches 0 the norm $\|\gamma(s) - x_0\|$ tends to 0. Hence

$$\lim_{s\to 0} |(\gamma(s) - x_0)\mathcal{O}_k + x_0) \cap U| = +\infty.$$

Let $s_1 = \inf\{s > 0 | |(\gamma(s) - x_0)\mathcal{O}_k + x_0) \cap U| \leq |(\gamma(1) - x_0)\mathcal{O}_k + x_0) \cap U|\}$. The equality above ensures that $s_1 > 0$. The intersection $((\gamma(s_1) - x_0)\mathcal{O}_k + x_0) \cap \partial U$ must contain another point except x_0 and $\gamma(s_1)$ because otherwise the function $s \mapsto |(\gamma(s) - x_0)\mathcal{O}_k + x_0) \cap U|$ would be constant in an open neighborhood of s_1, contradicting the definition of s_1.

Step 4. Put $v_0 = -x_0(\gamma(s_0) - x_0)^{-1}$ and $t_0 = (\gamma(s_0) - x_0)^{-1}$. Then $(t_0\partial U + v_0) \cap \mathcal{O}_k$ contains at least 3 points p_1, p_2, p_3. Indeed, we may take $p_1 = 0, p_2 = 1$ and $p_3 = (x_1 - x_0)(\gamma(s_1) - x_0)^{-1}$ where s_1, x_1 are provided by Step 3.

[15] This is of course not true if $V = \mathbb{R}$ and U is an interval.

Step 5. We will show that for every small enough $\varepsilon > 0$ there exists open neighborhood W of v_0 such that $((1-\varepsilon)t_0U + v_1) \cap \mathcal{O}_k = (t_0U + v_0) \cap \mathcal{O}_k$ and $((1+\varepsilon)t_0U + v_1) \cap \mathcal{O}_k \supset (t_0U + v_0) \cap \mathcal{O}_k \sqcup \{p_1, p_2, p_3\}$ for every $v_1 \in W$.

Choose $\varepsilon > 0$ such that $((1-\varepsilon)t_0U+v_0)\cap\mathcal{O}_k = (t_0U+v_0)\cap\mathcal{O}_k$ and $(1+\varepsilon)t_0U+v_0)\cap\mathcal{O}_k \supset (t_0U + v_0) \cap \mathcal{O}_k \sqcup \{p_1, p_2, p_3\}$. The desired conditions are satisfied once ε is small enough because $(1-\varepsilon)t_0\overline{U} \subset t_0U$ and $t_0\overline{U} \subset (1+\varepsilon)U$. Conditions $((1-\varepsilon)t_0U + v_1) \cap \mathcal{O}_k = (t_0U + v_0) \cap \mathcal{O}_k$ and $((1+\varepsilon)t_0U + v_1) \cap \mathcal{O}_k \supset (t_0U + v_0) \cap \mathcal{O}_k \sqcup \{p_1, p_2, p_3\}$ define an open set of v_1 so they hold for all v_1 in an open neighborhood of v_0.

Step 6. We show that for small enough $\varepsilon > 0$ we have either $D_{(1-\varepsilon)t_0}(U) > 1$ or $D_{(1+\varepsilon)t_0}(U) > 1$. By Step 5 for every $v_1 \in W$ we have $N_{(1+\varepsilon)t_0}(U, v_1) - N_{(1-\varepsilon)t_0}(U, v_1) \geq 3$ so $D_{(1+\varepsilon)t_0}(U, v_1) - D_{(1-\varepsilon)t_0}(U, v_1) \geq 3 - |\Delta_k|^{-1/2}\mathrm{Leb}(U)\|t_0\|((1+\varepsilon)^N - (1-\varepsilon)^N)$. By choosing ε small enough we can ensure that $D_{(1+\varepsilon)t_0}(U, v_1) - D_{(1-\varepsilon)t_0}(U, v_1) \geq \frac{5}{2}$. Set W is open so it has positive measure. We deduce that $D_{(1-\varepsilon)t_0}(U) + D_{(1+\varepsilon)t_0}(U) \geq \frac{5}{2}$. One of them must be bigger than 1 so Step 6 and the lemma follows. □

3.6.2 Proof of the main theorem

In this section we prove the main result of this chapter.

Proof of Theorem 3.1.2. We argue by contradiction. As before $V = k \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Assume that there is a sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ with $n_i \to \infty$ such that for every $i \in \mathbb{N}$ there exists an n_i-optimal set $\mathcal{S}_{n_i}$. By Theorem 3.3.2 there exists a compact cylinder $\Omega \subset V$ and sequences $s_{n_i}, t_{n_i} \subset V$ such that $\|s_{n_i}\| = n_i|\Delta_k|^{1/2}$ and $s_{n_i}^{-1}(\mathcal{S}_{n_i} - t_{n_i}) \subset \Omega$. Put

$$\nu_{n_i} := \frac{1}{n_i} \sum_{x \in \mathcal{S}_{n_i}} \delta_{s_{n_i}(x - t_{n_i})}. \tag{3.6.3}$$

Those measures are supported in Ω. Since Ω is compact we may assume, passing to a subsequence if needed, that ν_{n_i} converges weakly-* to a probability measure ν. This a measure that we called in Section 3.5 a limit measure. By Lemma 3.5.2 the measure ν is absolutely continuous with respect to the Lebesgue measure on V and its density is bounded by 1. By Proposition 3.5.4 we have $I(\nu) = -\frac{1}{2}\log|\Delta_k| - \frac{3}{2} - \gamma_k + \gamma_{\mathbb{Q}}$ where $\gamma_k, \gamma_{\mathbb{Q}}$ are Euler–Kronecker constants of $k, \mathbb{Q}$ respectively. Recall that $\mathcal{P}^1(V)$ is the set of absolutely continuous probability measures on V of density at most 1. By Lemma 3.5.5 the measure ν realizes the minimal energy among all probability measures in $\mathcal{P}^1(V)$. Hence, by Proposition 3.5.6 there exists an open set U of measure 1 such that $\partial U \cap V^\times$ is a C^1-submanifold of $V^\times$, $\lambda\overline{U} \subset U$ for $0 < \lambda < 1$ and (up to translation) $\nu = \mathrm{Leb}|_U$. By Lemma 3.6.4 applied to U there exists $t \in V^\times$ such that $D_t(U) > 1$. On the other hand Lemma 3.6.3 yields $D_t(U) < 1$ for every $t \in V^\times$. This yields the desired contradiction. □

3.7 Appendix

3.7.1 Measure theory

Lemma 3.7.1. *Let ν be a probability measure on V of density at most 1. Then there exists a sequence of subsets $(E_n)_{n\in\mathbb{N}}$ of $\mathcal{O}_k$ such that $|E_n| = n+1$ and the sequence of measures*

$$\nu_{E_n,n} := \frac{1}{n}\sum_{x\in E_n} \delta_{n^{-1/N}|\Delta_k|^{-1/2N}x} \tag{3.7.1}$$

converges weakly- to ν.*

Proof. The proof is based on a sequence of reductions to easier problems. First note that if we manage to find a sequence of sets $E_n \subset \mathcal{O}_k$ such that the measures $\nu_{E_n,n}$ converge weakly-* to ν then $|E_n| = n + o(n)$. Removing or adding $o(n)$ points to each E_n does not affect the weak-* limit so we may easily obtain a desired sequence. The proof is reduced to finding any sequence (E_n) of finite subsets of $\mathcal{O}_k$ such that $\nu_{E_n,n}$ converges weakly-* to ν. Let $P \subset \mathcal{M}^1(V)$ be the set of finite measures for which this is possible.

Step 1. We prove that P is a closed convex subset of $\mathcal{M}^1(V)$. The fact that P is closed is immediate by definition. Thus, to prove that it is convex we only need to show that for every $\nu, \nu' \in P$ we have $\frac{1}{2}(\nu+\nu') \in P$. Fix a set $a_1, \ldots, a_{2^N}$ of representatives of $\mathcal{O}_k/2\mathcal{O}_k$. Let E_n, E'_n be sequences of subsets of $\mathcal{O}_k$ such that $\nu_{E_n,n}, \nu_{E'_n,n}$ converge weakly-* to ν, ν' respectively. Define

$$F_{2^N n} = \bigcup_{i=1}^{2^{n-1}} (a_i + 2E_n) \cup \bigcup_{i=2^{n-1}+1}^{2^n} (a_i + 2E'_n)$$

and $F_m := F_{2^N[m/2^N]}$. A simple computation shows that $\lim_{m\to\infty} \nu_{F_m,m} = \frac{1}{2}(\nu+\nu')$ so the latter belongs to P.

Step 2. Let $U \subset V$ be an open set of finite Lebesgue measure such that ∂U is Jordan measurable and has Jordan measure 0. Then the measure $\nu(A) := \mathrm{Leb}(A\cap U)$ belongs to P. Indeed it is enough to take $E_n = \mathcal{O}_k \cap (n^{1/N}, \ldots, n^{1/N})U$.

Step 3. For every measurable set $E \subset V$ of finite measure the measure $\nu_E(A) := \mathrm{Leb}(A\cap E)$ is in P. This follows from the fact that the Lebesgue measure is Radon so there exists a sequence of open sets U_n containing E such that ν_{U_n} converges weakly-* to ν_E. Removing from U_n a closed set of arbitrarily small Lebesgue measure we can assume that ∂U has Jordan measure 0 so Step 2 applies.

Step 4. The convex hull of measures ν_E from the previous step is weakly-* dense in the set of measures of density at most 1. Indeed, let ν be a finite measure with density $f \in L^1(V)$ such that $f(v) \leq 1$ almost everywhere. For every $t \in [0,1]$ let $E_t = \{v \in V | f(v) \geq t\}$. Those are measurable sets of finite measure and we have $\nu = \int_0^1 \nu_{E_t} dt$. Hence, by convexity $\nu \in P$. The lemma is proven. ☐

3.7.2 Angular distribution of prime ideals

We prove a version of prime number theorem for number fields where principal ideals are weighted with respect to their "angular" position in $V^\times/\mathcal{O}_k^\times$. This is very close to the prime number theorem for products of cylinders and sectors proved by Mitsui [29]. The version we need is a little bit different and we don't need an explicit error term. The following result is rather folklore, we include a short proof for completeness.

Lemma 3.7.2. *Let k be a number field and let $V = k \otimes_{\mathbb{Q}} \mathbb{R}$. Let $\varphi : V^\times \to \mathbb{C}$ be a continuous function such that $\varphi(t\lambda x) = \varphi(x)$ for every $x \in V^\times, \lambda \in \mathcal{O}_k^\times$ and $t \in \mathbb{R}^\times$. For a principal ideal $I = a\mathcal{O}_k$ we put $\varphi(I) := \varphi(a)$. Then*

$$\sum_{\substack{N(\mathfrak{p}^l)\leq X \\ \mathfrak{p}\ \textit{principal}}} \varphi(\mathfrak{p}) \log N\mathfrak{p} = \frac{X}{R_k h_k} \int_{\mathcal{I}/\mathcal{O}_k^\times} \varphi(t)dt + o(X),$$

where R_k, h_k are the regulator and the class number of k and $\mathcal{I} := \{v \in V \mid \|v\| = 1\}$.

Proof. Write $\mathcal{A}$ for the space of continuous functions φ satisfying the conditions in the lemma. The unitary characters $\chi : V^\times \to \mathbb{C}^\times$ such that $\chi(\lambda) = 1$ for every $\lambda \in \mathcal{O}_k^\times$ and $\chi(t) = 1$ for every $t \in \mathbb{R}^\times$ span a dense subspace of $\mathcal{A}$. As a consequence it is enough to prove the statement for $\varphi = \chi$ with χ as above.

Our first step is to associate to χ a Hecke character. Write $\mathbb{A}^\times$ for the group of ideles of k and $\mathbb{A}_\infty^\times$ and $\mathbb{A}_f^\times$ for the groups of infinite and finite ideles respectively. We distinguish the subgroup $\mathbb{A}^1$ of ideles of idelic norm 1. Let $K = \prod_{\mathfrak{p}} \mathcal{O}_{k_\mathfrak{p}}^\times$ be the maximal compact subgroup of $\mathbb{A}_f^\times$. We identify $V^\times$ with $\mathbb{A}_\infty^\times$. By abuse of notation let us write χ for the extension of χ to $\mathcal{I} \times K \subset \mathbb{A}^1$ by setting $\chi(vk) = \chi(v)$ for $v \in \mathcal{I}, k \in K$. The character χ factors through $(\mathcal{I} \times K)/\mathcal{O}_k^\times$ and the latter is a closed subgroup of $\mathbb{A}^1/k^\times$ of index h_k. Let $\widehat{\chi}$ be any extension of χ to $\mathbb{A}^1/k^\times$. There are precisely h_k such extensions and the are all of form $\psi\widehat{\chi}$ for $\psi : \mathbb{A}/\mathbb{A}_\infty^\times K k^\times =: \mathrm{Cl}_k \to \mathbb{C}^\times$ where Cl_k stands for the class group of k. Through the standard procedure $\widehat{\chi}$ gives rise to an unramified Hecke character $\widehat{\chi}$ such that for every principal prime ideal $\mathfrak{p} = a\mathcal{O}_k$ we have $\widehat{\chi}(\mathfrak{p}) = \chi(a)$. For any $\nu : \mathrm{Cl}_k \to \mathbb{C}^\times$ consider the Hecke L-function

$$L(s, \psi\widehat{\chi}) := \prod_{\mathfrak{p}} \left(1 - \frac{\psi(\mathfrak{p})\widehat{\chi}(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1}.$$

By [24, Theorem 5.34] there exists a constant $c > 0$ such that the function $L(s, \psi\widehat{\chi})$ has at most one zero in the region

$$\mathrm{Re}\, z > 1 - \frac{c}{N \log |\Delta_k|(|\mathrm{Im}\, z| + 3)^N}.$$

The exceptional zero is always real, less than 1 and can occur only when $\psi\widehat{\chi}$ is a real character. With this information at hand the standard argument used to prove the prime number theorem (see [24, Theorem 5.13]) shows that

$$\sum_{N\mathfrak{p}^l\leq X}\psi(\mathfrak{p})\widehat{\chi}(\mathfrak{p})\log N\mathfrak{p}=rX+o(X),\tag{3.7.2}$$

where $r=1$ if $L(s,\psi\widehat{\chi})$ has a simple pole at 1 and $r=0$ otherwise. In our case $r=1$ if $\psi\widehat{\chi}=1$ and $r=0$ otherwise. We take the average of (3.7.2) over all characters $\psi:\mathrm{Cl}_k\to\mathbb{C}^\times$ to get

$$\sum_{\substack{N\mathfrak{p}^l\leq X\\ \mathfrak{p}\text{ principal}}}\widehat{\chi}(\mathfrak{p})\log N\mathfrak{p}=\frac{1}{h_k}\sum_{N\mathfrak{p}^l\leq X}\sum_{\psi\in\widehat{\mathrm{Cl}_k}}\psi(\mathfrak{p})\widehat{\chi}(\mathfrak{p})\log N\mathfrak{p}=\begin{cases}\frac{X}{h_k}+o(X) & \text{if }\chi=1\\ o(X) & \text{otherwise.}\end{cases}.$$

Since $\int_{\mathcal{I}/\mathcal{O}_k^\times}1dt=R_k$ this agrees with the formula predicted by the lemma. We deduce that the lemma holds for $\phi=\chi$. By our opening remarks this concludes the proof. $\square$

Corollary 3.7.3. *Let U be a bounded open subset of $V^\times$. Then for any $t\in V^\times$ with $\|t\|$ sufficiently large there is at least one element $a\in tU\cap\mathcal{O}_k$ such that $a\mathcal{O}_k$ is a prime ideal.*

Proof. First we prove the statement for $t\in\mathbb{R}^\times\subset V^\times$. Let U' be an open subset of U such that $\overline{U}'\subset U$. Put $U''=\mathcal{O}_k^\times\mathbb{R}^\times U'$. Let $\varphi_0:V\to\mathbb{R}_{\geq0}$ be any continuous function such that $\varphi_0|_{U'}=1$ and φ_0 vanishes outside U. For $v\in V$ put $\varphi(v):=\sum_{x\in\mathcal{O}_k^\times}\int_0^\infty\varphi_0(txv)\frac{dt}{t}$. Function φ is continuous, positive on U, supported on U'' and it satisfies the assumptions of Lemma 3.7.2. As $V^\times/\mathbb{R}^\times\mathcal{O}_k^\times$ is compact the function φ is necessarily bounded. There exist $a<1<b$ such that $U''\cap\{v\in V^\times|a\leq\|v\|\leq b\}\subset\mathcal{O}_k^\times U$. By Lemma 3.7.2 there is a positive constant c such that

$$\sum_{\substack{aX\leq N(\mathfrak{p}^l)\leq bX\\ \mathfrak{p}\text{ principal}}}\varphi(\mathfrak{p})\log N\mathfrak{p}=c(b-a)X+o(bX).\tag{3.7.3}$$

The higher powers are negligible since we have $\sum_{\substack{N(\mathfrak{p}^l)\leq bX\\ l\geq2}}\log N\mathfrak{p}=o(bX)$. Equation (3.7.3) becomes

$$\sum_{\substack{aX\leq N(\mathfrak{p})\leq bX\\ \mathfrak{p}\text{ principal}}}\varphi(\mathfrak{p})\log N\mathfrak{p}=c(b-a)X+o(bX).\tag{3.7.4}$$

We deduce that for X sufficiently large there exists an element $w\in X^{1/N}\mathcal{O}_k^\times U\cap\mathcal{O}_k$ such that $w\mathcal{O}_k$ is prime. We replace w by $w\lambda$ for some $\lambda\in\mathcal{O}_k^\times$ to get an element of $X^{1/N}U\cap\mathcal{O}_k$ generating a prime ideal. This proves the statement for $t\in\mathbb{R}^\times$ because we can take $X=\|t\|$.

To get the general case choose an open set $W\subset V^\times$ and a finite set $y_1,\dots,y_m$ of elements of $V^\times$ such that for every translate $tU,t\in V^\times$ there exists an $\lambda\in\mathcal{O}_k^\times$, $t_0\in\mathbb{R}^\times$ and

$i \in \{1, \ldots, m\}$ such that $\lambda t_0 y_i W \subset tU$. This can be always arranged because $V^\times/\mathbb{R}^\times \mathcal{O}_k^\times$ is compact. The case of the corollary that we have already proved applied to the open sets $y_i W$ implies that for $\|t_0\|$ sufficiently large the sets $t_0 y_i W$ all contain a prime element. But then so do the translates $\lambda t_0 y_i W$ for every $\lambda \in \mathcal{O}_k^\times$. Since one of them is contained in tU and $\|t_0\| \to \infty$ as soon as $\|t\| \to \infty$ the corollary is proven. $\square$

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