A Geometric Theta Correspondence for Picard Modular Surfaces

Robert J. Little

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Department of Mathematical Sciences
Durham University
United Kingdom

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Abstract: This thesis aims to generalise the work of Kudla-Millson on intersection numbers of special cycles on Picard modular surfaces. We will use geometric and arithmetic techniques to generalise these results to the case of odd integral weight modular forms, and then use a compactification to extend the new special cycles to the boundary components.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

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"Lenin walks around the world.
Frontiers cannot bar him.
Neither barracks nor barricades impede.
Nor does barbed wire scar him.

Lenin walks around the world.
Black, brown, and white receive him.
Language is no barrier.
The strangest tongues believe him.

Lenin walks around the world.
The sun sets like a scar.
Between the darkness and the dawn
There rises a red star."

— Langston Hughes

*This thesis is dedicated
to my parents, who have
given me everything*

Contents

Chapter 1

Introduction

If one had to pick out a particular word in the title of this thesis that gave an indication to the work herein, it would be "geometric". The reader may be assumed to have some knowledge of the theory of modular forms - though for the benefit of those in need of a recap, we shall give a brief one.

Definition 1.0.1. Let $\Gamma(M) \subset \mathrm{SL}_2(\mathbb{Z})$ be given by

$$\Gamma(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod M \right\};$$

we say $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup if it contains some $\Gamma(M)$. Let $k \geq 1$ be an integer. Then a modular form for Γ is a holomorphic function on the upper-half plane $f : \mathbb{H} \to \mathbb{C}$ such that

(i) $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

(ii) $f(\tau)$ is holomorphic at the cusps $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ of $\Gamma\backslash\mathbb{H}$; namely, it has a non-negative Fourier expansion $f(\tau) = \sum_{n\geq 0} a(n)e^{2\pi i n\tau}$.

We say $f(\tau)$ is a cusp form if it is zero at all the cusps (more precisely, if its Fourier expansion in $q = e^{2\pi i\tau}$ at each cusp has no terms in q^k for $k \leq 0$); we say it has level M where M is the smallest integer such that $\Gamma(M) \subset \Gamma$.

We may now immediately deal with another important pair of words in the title: theta series. Once one has had a modular form defined, one of the first examples of such an object will be the following:

Theorem 1.0.2 (Hecke, Schoeneberg)**.** *For a lattice $L \subset \mathbb{Q}^N$ with a positive-definite and even inner product $(\ ,\ )$, and a harmonic homogeneous polynomial*

$p(\boldsymbol{x}) \in \mathbb{Q}[x_1, \ldots, x_N]$:

$$\theta(\tau, L, p) = \sum_{\boldsymbol{x} \in L} p(\boldsymbol{x}) e^{2\pi i (\boldsymbol{x},\boldsymbol{x})\tau} \in M_{\frac{N}{2}+deg(p)}\left(\Gamma(M)\right).$$

is a modular form of weight $N/2 + deg(p)$ and level $M = \mathrm{disc}(L)$; if $deg(p) \neq 0$, $\theta(\tau, L, p)$ is a cusp form.

For example, from here, one may classify for a given weight what the non-cuspidal theta series are, and then use linear algebra to express the coefficients of the Eisenstein series in terms of the simpler coefficients of θ.

If we wish, we may see the subsequent developments in this area - to be described forthwith - as a way of replicating the result of Theorem 1.0.2 through numerous different geometric avenues; in particular, what we shall focus on is analogies to Theorem 1.0.2 coming from the geometry of locally symmetric spaces. Because of $p(\mathbf{x})e^{2\pi i(\mathbf{x},\mathbf{x})\tau}$ having a modular transformation law under orthogonal transformations (analogously to Definition 1.0.1), one may see this as a correspondence between automorphic forms for $\mathrm{O}(N)$ and automorphic forms for $\mathrm{GL}_2(\mathbb{R})$. The principles we shall explore are the following:

(i) What are the properties of $\theta(\tau, L, p)$ when L is not positive-definite?

(ii) Can we use theta series to give us correspondences between automorphic forms, for more general finite-dimensional reductive Lie groups replacing $\mathrm{SO}(N)$?

The reader will immediately see that if we allow L to be non-positive-definite, then several parts of Theorem 1.0.2 will immediately disappear - for example, it will no longer have a positive q-expansion, as it will generically have non-zero terms of the form $e^{2\pi i n\tau}$ for negative n. Moreover, it is not clear that it will even remain a holomorphic function of τ.

Our work shall be based on answering (i) and (ii) at the same time. To be more specific, we shall examine theta series corresponding to the unitary group $U(2,1)$. This will require translating the above into the universe of differential forms using the generalised cohomological machinery of Kudla and Millson. We shall then look at two consecutive extensions of this theory. The first will be representation-theoretic, which will generalise the theta series to be in a cohomology group with coefficients in a vector bundle; this will allow the theta series to be of arbitrary odd integer weight.

The second extension will be in taking these theta series to the boundary components. This will give us a new class of modular forms of odd integral weight, and give a programme for the extension of general unitary theta series of split signature.

1.1 A Brief History

In the author's opinion, a good starting point for introducing this area is the work of Goro Shimura, Takuro Shintani and Shinji Niwa in a series of papers - of especial interest is [Shi73], [Shi75] and [Niw74] - on a correspondence between integral and half-integral weight modular forms, that has subsequently become known as the Shimura-Shintani correspondence. We refer the reader also to e.g. [Kob93, §4] for a good summary of the theory of half-integer weight modular forms and their Hecke operators; one may use the rubric of Definition 1.0.1 for k a half-integer, where one requires a finite-degree character ψ on Γ accompanying the automorphy factor in part (i) of the above definition.

The details of this correspondence are unnecessary, but the broad idea may still be elucidated: namely, in [Shi75], the author integrates a modular form G of weight $2k+2$ and level N against a theta kernel (more specifically, the sum over the positive-discriminant parts of the 3-dimensional lattice L_N of quadratic forms in two variables):

$$\theta(z, L_N) = \sum_{\substack{Y\in\Gamma_0(N)\backslash L_N \\ \Delta(Y)\geq 0}} \exp\left(\frac{2\pi i z\Delta(Y)}{N}\right)\int_{C_Y} Y(1,-\tau)G(\tau)\mathrm{d}\tau \qquad (1.1.1)$$

Here the $\{C_Y\}$ are special 1-cycles on the modular curve $\Gamma_0(N)\backslash\mathbb{H}$, parameterised by this set of positive-length vectors in the lattice. This produces a modular form of weight $k+3/2$.

This relationship is inverted in Niwa's paper [Niw74] - namely, an almost identical theta series is found that takes half-integral- to integral-weight modular forms. This generalises the relationship of Shimura in [Shi73], which produces this lift for half-integral Hecke eigenforms by splitting its Dirichlet L-function into two Euler products. We hence have the following foundational result:

Theorem 1.1.1 (Shimura-Shintani-Niwa). *Let N be a positive integer. Then for $k\geq 1$ another positive integer, there are maps*

$$S_{k+3/2}(\Gamma_0(4N)) \xrightarrow{\Phi_{Shim}} S_{2k+2}(\Gamma_0(N)), \quad S_{2k+2}(\Gamma_0(N)) \xrightarrow{\Phi_{Shint}} S_{k+3/2}(\Gamma_0(4N))$$

which both come from integrals against theta kernels (as detailed in (1.1.1)*), are arithmetic - they preserve the action of Hecke operators - and are adjoint to each other: namely, for f an integral and g a half-integral cusp form*

$$(f, \Phi_{Shim}g) = (\Phi_{Shint}f, g)$$

with respect to the Petersson inner product (see e.g. [DS05, p.182]).

We note here that this result, though in some senses elementary compared to the work that currently exists, contains many of the components that we shall consider. Namely, it shows that relationships between spaces of modular forms for SL_2 may be found using theta series, which use the geometry of the symmetric space $\mathbb{H}$ and special cycles on its quotients to construct the coefficients.

A general approach to non-vanishing (using p-adic methods) may be found in e.g. [Pra09], which shows that the coefficients of the modular form in the image of the Shintani correspondence are proportional to special values of the Hecke L-functions of the original form, which we know to be generically non-zero.

It is at this point instructive to consider the relationship between modular forms and cohomology. There are many ways this may be interpreted; for our purposes, the simplest formulation is in the Eichler-Shimura isomorphism, which expresses the modular forms as a cohomology group with coefficients in $\mathcal{L}_k$, the k'th power of the standard representation of $\mathrm{SL}_2(\mathbb{R})$:

$$S_{2k+2}(\Gamma) \oplus \overline{S_{2k+2}(\Gamma)} \simeq H^1_! \left(\Gamma\backslash\mathbb{H}, \mathcal{L}_k\right), \quad f(\tau) \to \omega_f = f(\tau)\mathrm{d}\tau \otimes (\tau v_1 + v_2)^{2k} \quad (1.1.2)$$

Because of this relation, we may interpret both Φ_{Shim} and Φ_{Shint} from Theorem 1.1.1 as having domains in the compactly-supported & rapidly decreasing cohomology of a modular curve.

We now move on to two more examples from Hirzebruch-Zagier and Cogdell - this will illuminate the dialectic of compactness that is central to our approach for the rest of this thesis. The results given are almost identical, but illuminate two special generalisations of the locally symmetric manifold replacing the modular curve, and give us links to classical number theory; the idea of these results giving maps on homology and cohomology is now central.

In [HZ76], the authors consider a Hilbert modular surface which replaces one of the modular curves $\Gamma\backslash\mathbb{H}$ in the Shimura-Shintani correspondence in Theorem 1.1.1. They fix an odd prime $p \equiv 1 \bmod 4$ and the real quadratic field $K = \mathbb{Q}(\sqrt{p})$, with Hilbert modular group $\mathrm{SL}_2(K)$. This acts on $\mathbb{H}$ via the two real places of K, which embed $\mathrm{SL}_2(K) \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ - see [BvdGHZ08, §2, (1.3)] - and hence, for the fixed arithmetic subgroup $\Gamma = SL_2(\mathcal{O}_K)$, they use the non-compact Hilbert modular surface $X = \Gamma\backslash(\mathbb{H} \times \mathbb{H})$.

For all positive integers N, a closed 2-cycle (more specifically, a modular curve) T_N is constructed on the Hilbert modular surface X. It is here that the problem of compactness manifests: namely, from its definitions, they find:

$$T_N = \begin{cases} \text{bounded} & \text{if } N \text{ is } \textit{not} \text{ a norm of } \mathcal{O}_K \\ \textit{not} \text{ bounded} & \text{if } N \text{ is a norm of } \mathcal{O}_K. \end{cases}$$

Hence, the cycles T_N will generically have non-trivial intersection with the cusps of X. This problem is not a new one - indeed, what we are dealing with is the exact analogy of the problem of domains in Theorem 1.1.1. Namely, in the Shimura-Shintani correspondence, the solution is to let the domains of each of the maps only be in the cusp forms, which - by our demanding they be 0 at all cusps - decay fast enough so that the integrals still converge. The innovation in this paper is to modify the special cycles to allow for calculation of the intersection numbers on X with all of the T_N. They compactify $\iota : X \hookrightarrow \overline{X}$ by taking the minimal resolution at each cusp; this decomposes the homology as follows:

$$H_2(\overline{X}) = \iota_* H_2(X) \oplus H_2(X_\infty) \tag{1.1.3}$$

where X_∞ is the subspace generated by all the homology cycles in the boundary. They hence consider T_N^c as the image of $\overline{T_N}$ (the closure of T_N in $\overline{X}$) in $\iota_* H_2(X)$ under the decomposition in (1.1.3); by explicitly finding what the cap of T_N looks like, they find the following two arithmetic results:

$$(T_1, T_N)_X = H_p(N) = \sum_{\substack{x\in\mathbb{Z} \\ x^2 \leq 4N \\ p|(4N-x^2)}} H\left(\frac{4N - x^2}{p}\right)$$

where $H(c)$ is the classical number of equivalence classes of positive-definite binary quadratic forms of discriminant $-c$, and

$$I(T_1^c, T_N^c)_{X^\infty} = I_p(N) = \frac{1}{\sqrt{p}} \sum_{\substack{\lambda\in\mathcal{O}_K \\ \lambda \text{ strictly positive} \\ \lambda|N}} \min\left(\lambda, N/\lambda\right).$$

They hence show the following:

Theorem 1.1.2. *[HZ76, §3, Theorem 1] Let $p \equiv 1 \bmod 4$ be an odd prime. Then the generating series*

$$\frac{1}{2}\text{vol}(T_1) + \sum_{N=1}^{\infty} I(T_1^c, T_N^c)_{\overline{X}} = -\frac{1}{12} + \sum_{N=1}^{\infty} \left(H_p(N) + I_p(N)\right) q^N \in M_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right).$$

is a classical modular form of weight 2 and level p with Nebentypus (character). Moreover, it is the sum of the holomorphic parts of non-holomorphic modular forms:

$$-\frac{1}{12} + \sum_{N=1}^{\infty} H_p(N) q^N, \quad \sum_{N=1}^{\infty} I_p(N) q^N$$

are both the positive q-series of weight 2, non-holomorphic weak Maass forms on $\mathbb{H}$.

More generally, for any class $T \in H_2(\overline{X}, \mathbb{Q})$*, the sum*

$$\frac{1}{2}\mathrm{vol}(T) + \sum_{N=1}^{\infty} I(T, T_N^c)_{\overline{X}} \tag{1.1.4}$$

is a modular form of weight 2*, level* p*.*

We note that the general statement (1.1.4) in Theorem 1.1.2 is equivalent to the existence of a map on homology:

$$\Phi_{HZ} : H_2(\overline{X}, \mathbb{Q}) \supset \mathrm{Span}_{\mathbb{Q}}\,[T_N^c] \to M_2\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right).$$

The reader is now invited to compare this to the result of Theorem 1.1.1 - crucially, the difference in the domains used. In the latter, one is restricted - by the presence of infinite geodesics - to only use Φ_{Shim} or Φ_{Shint} on cusp forms, as the integrals considered would not converge on Eisenstein series. This work of Hirzebruch-Zagier hence offers one solution to the problem of non-compactness.

We now offer a brief word on the non-vanishing of the Hirzebruch-Zagier mapping; indeed, there are explicit results on this contained in [HZ76]. In [HZ76, §3, Theorem 1], they show that the map $H_2(\overline{X}) \to M_2(\Gamma_0(p), \cdot)$ is injective. In particular the second Chern form $c_2(\overline{X})$ (which defines a cohomology class) on $\overline{X}$ satisfies

$$\int_{\overline{X}} c_2(\overline{X}) = 2\zeta_k(-1)$$

(see e.g. [HZ76, §3, (1)]). Hence, taking the Poincaré dual of this second Chern form, we see that when this zeta value is non-zero, there are non-zero modular forms in the image of the Hirzebruch-Zagier map.

In the work of Kudla in [Kud78], the ideas of Hirzebruch-Zagier are generalised to compact quotients of the r-fold Cartesian product of the 2-disc $\mathbb{D}$; in particular, in the middle homology $H_r(\Gamma\backslash\mathbb{D}^r)$, non-trivial cycles C_N may be constructed whose generating series is a Hilbert modular form of weight 3. By specifying work in compact locally symmetric domains from the start, this indicates a solution to the problems indicated above, as well as generalising the Hirzebruch-Zagier results to all totally real fields $k/\mathbb{Q}$.

Of particular interest on the topic of special unitary theta lifts is the work of Cogdell in [Cog85]. As we shall see, both the general motivation and techniques used are very similar to those used for Theorem 1.1.2, but it is much more directly applicable to our context of Picard modular surfaces. In this paper, Cogdell considers a split hermitian vector space V of signature $(2,1)$, an imaginary quadratic field $k/\mathbb{Q}$ and the real split Lie group $\mathrm{SU}(V) \simeq \mathrm{SU}(2,1)$. The associated locally symmetric space is a 4-manifold, and more specifically a quotient of the 4-disc by some arithmetic subgroup. As in the work of Hirzebruch-Zagier, one may naturally construct special cycles $C_N \hookrightarrow X = \Gamma\backslash\mathbb{D}$ indexed by $N \in \mathbb{N}$. Cogdell is hence able to replicate the Hirzebruch-Zagier result exactly, and finds:

Theorem 1.1.3. *[Cog85, §4, Theorem] Fix a special cycle $C_M \hookrightarrow X$: using the Hirzebruch-Zagier method of Theorem 1.1.2, this may be compactified as*

$$C_M \to C_M^c \in \iota_* H_2(X),$$

so that the series

$$\begin{aligned} \frac{1}{2}\mathrm{vol}\left(C_M\right) &+ \sum_{N=1}^{\infty} I\left(C_M^c, C_N^c\right)_{\overline{X}} q^N \\ &= \frac{1}{2}\mathrm{vol}\left(C_M\right) + \sum_{N=1}^{\infty}\left[I(C_M, C_N)_X - \sum_{[\ell]} I\left(C_M, C_{N,\ell}\right)_{\overline{X}}\right] q^N \qquad (1.1.5) \end{aligned}$$

is a holomorphic modular form of weight 3 *and level $D = \mathrm{disc}(k)$; as in Theorem 1.1.2, the global and local parts of* (1.1.5) *are the holomorphic parts of non-holomorphic modular forms on $\mathbb{H}$ with the same non-holomorphic parts.*

As with the Hirzebruch-Zagier results, we may see the existing of these modular generating series as, equivalently, the existence of a map from homology to a space of modular forms:

$$\Phi_{Cog} : H_2\left(\overline{X}, \mathbb{Q}\right) \supset \mathrm{Span}_{\mathbb{Q}}\left[C_M^c\right] \to M_3\left(\Gamma_0(D), \chi_D\right).$$

We now reach a decisive moment in the history of this theory. It is conjectured in many papers - included those cited above - that the existence of all of these results, all of which appear to have the same form, is not an accident (indeed, even what we shall try to transcribe of this generalisation is really only one part of the spectrum of conjectures made, as we are focusing on the holomorphic side of the theory). In a series of papers by both Stephen Kudla and John Millson throughout the 1970s and '80s - see e.g. [Kud78], [Kud79], [KM81], [Mil81], [Mil85] - there is an enormous amount of work done, both computational and theoretical, on creating a uniform theory for theta correspondences between spaces of archimedean automorphic forms, of the form

$$\{\text{Geometry on } \Gamma\backslash G/K\} \xrightarrow[\text{Integration}]{\text{Theta}} \{\text{Modular Forms on } \mathbb{H}\}.$$

We shall state this result as a correspondence only between split, reductive and finite dimensional Lie groups and the special linear group $SL_2(\mathbb{R})$ - however, it must be stressed, this is but a very special case of this theory, as what we have in full generality is a correspondence between split finite dimensional Lie groups and symplectic groups $Sp_n(\mathbb{R})$.

Theorem 1.1.4. *[KM86, KM87] Let G be either a special orthogonal group or a special unitary group of a split vector space V of signature (p,q), defined respectively over $\mathbb{R}$ or $\mathbb{C}$. We let $r = p+q$ or $2(p+q)$ respectively be the real dimension of V. For some maximal compact subgroup $K \subset G$, let $\mathbb{D} = G/K$ be the associated symmetric space, of real dimension $m = m_G = pq$ or $2pq$ respectively. For φ_0 the Gaussian on V, we let $\mathcal{S}(V) \subset S(V)$ be the space of polynomial Schwartz forms $p(\boldsymbol{x})\varphi_0(\boldsymbol{x})$ on V. For each fixed integer $1 \le a \le p$, and $k = aq$ or $2aq$ respectively, there is a non-trivial Schwartz form φ such that*

$$\varphi \in \left[\mathcal{S}(V^{\otimes a}) \otimes \Omega^k(\mathbb{D})\right]^G \tag{1.1.6}$$

which is closed with respect to the differential in this complex.

The complex in (1.1.6) is acted upon by the Howe dual pair $G \times G'$ through the Weil representation of a symplectic group $Sp(V \otimes V')$, where $G' = SL_2(\mathbb{R})$. Fixing any arithmetic subgroup Γ of G, and a Γ-invariant lattice $L \subset V$, φ will in particular be Γ-invariant. Taking $g = g_z$ the element of G taking the basepoint of $\mathbb{D}$ to $z \in \mathbb{D}$, and $g' = g'_\tau$ an element of G' taking i to $\tau \in \mathbb{H}$, we may form a theta series as follows:

$$\theta_L(\varphi, z, \tau) = \sum_{\boldsymbol{x} \in L} \omega(g'_\tau)\varphi(g_z^{-1}\boldsymbol{x}).$$

With $X = \Gamma\backslash\mathbb{D}$ the locally symmetric space, this will define a closed cohomology class $[\theta_L(\varphi, z, \tau)] \in H^k(X)$, which is modular of weight $r/2$ in τ. This may be integrated against closed and compactly supported $m-k$-forms $\eta \in H_c^{m-k}(X)$ to give holomorphic modular forms of level M and weight $r/2$:

$$\int_X \eta \wedge \theta_L(\varphi, z, \tau) \in M_{\frac{r}{2}}(\Gamma(M))$$

More specifically, we may see what the coefficients of these modular forms are: there exist special cycles $C_n \subset \Gamma\backslash\mathbb{D}$ such that

$$\int_X \eta \wedge \theta_L(\varphi, z, \tau) = c \int_X \eta \wedge \Omega_X + \sum_{n>0} \left[\int_{C_n} \eta\right] q^n$$

for some geometric constant $c \in \mathbb{C}$ and Ω_X a certain G-invariant k-form on X.

The proof method offers the key to why the work in this generality may be proven - namely, it uses the Weil representation ω of $\mathfrak{sp}(V \otimes V')$ to construct Howe operators ∇ in a universal enveloping algebra; the Schwartz forms are then given by $\varphi = \nabla\varphi_0$, and using the algebraic properties of the Weil representation the necessary properties are proven.

We note that the solution found here to the holomorphicity problem for non-positive-definite lattices is very elegant - the negative length parts of the theta series are still included, but these components are all exact, and so by an easy application of Stokes' theorem the integrals are zero. Hence, another way to write the results of Theorem 1.1.4 is that on the level of differential forms:

$$\theta_L(\varphi, z, \tau) \in \Omega^k(X) \otimes M^{\text{NonHol}}_{\frac{n}{2}}(\Gamma(M));$$

and more specifically, in cohomology we may say

$$[\theta_L(\varphi, z, \tau)] \in H^k(X) \otimes M_{\frac{n}{2}}(\Gamma(M)).$$

We now observe some of the developments of this relevant to our work. In [FM06], the authors aim to develop the results of Theorem 1.1.4 where $G = SO(p,q)$, but where the theta series (and special cycles) are in more general cohomology groups; this allows a complete generalisation of the weight of the resulting modular forms.

In summary: they consider the weight-indexed irreducible representations $\mathbb{S}_{[\lambda]}(V)$ of $SO(p,q)$ (whose construction is given in full generality in [FH04]), and construct Schwartz forms in complexes with more generic coefficients - and hence more generic modular weight. From here, the proof structure is broadly similar to that of Theorem 1.1.4 - namely, they use the Weil representation of the symplectic group to prove that the Schwartz form is closed, holomorphic, dual to the appropriate special cycles and so on. Their result is hence:

Theorem 1.1.5 (Funke & Millson, 2006)**.** *Let $G = SO(p,q)$, and let K, $\mathbb{D}$ and X be as in Theorem 1.1.4, so $r = p+q$ and $\mathbb{D}$ is of real dimension $k = pq$. We keep $a = 1$. Fix the trivial partition $\lambda = l$ of l, and let $\mathcal{H}^l(V)$ be the corresponding irreducible representation. Then there are non-trivial, closed Schwartz functions in the following complex:*

$$\varphi_{[l]} \in \left[\mathcal{S}(V) \otimes \Omega^q(\mathbb{D}) \otimes \mathcal{H}^l(V)\right]^G$$

and so for L a lattice of level M, we may form a cohomological theta series as in Theorem 1.1.4:

$$\left[\theta_{L,[l]}(\varphi, z, \tau)\right] \in H^q\left(X, \widetilde{\mathcal{H}^l(V)}\right) \otimes M_{\frac{p+q}{2}+l}(\Gamma(M)).$$

There are closed cycles $C_{n,[\lambda]}$ defining homology classes:

$$C_{n,[\lambda]} \in H_{q(p-1)}\left(X, \partial X, \widetilde{\mathcal{H}^l(V)}\right);$$

hence, for a closed and rapidly decreasing $\mathcal{H}^l(V)$-valued smooth differential $q(p-1)$-

form η, the generating series

$$\int_X \eta \wedge \theta_{L,[\lambda]}(\varphi, z, \tau) = \delta_{l=0} \int_X \eta \wedge \Omega^q_{X,[l]} + \sum_{n>0} \left[\int_{C_{n,[l]}} \eta \right] q^n \in M_{\frac{p+q}{2}+l}(\Gamma(M))$$

is a holomorphic modular form, which is cuspidal for $l \geq 1$.

In particular, if one wishes to go back to our very first theta result - in Theorem 1.0.2 - we may see this as a generalisation of the work of Hecke and Schoeneberg for non-positive-definite lattices!

This result will be what we attempt to recreate in the setting of $G = SU(p,q)$. There are two other Funke-Millson papers that were also used as key references for the writing of this thesis. The first, [FM13], which may be viewed as a sequel to [FM06], is on the extension of the vector-valued theta series from Theorem 1.1.5 to the boundary components of the Borel-Serre compactification $\overline{X}^{BS}$ of X. This is a homotopy-invariant compactification whose boundary components $e(P)$ are in 1-1 correspondence with the rational parabolic subgroups P of G. They show that for each such component, the restriction of the Funke-Millson orthogonal theta series is a convergent differential form which is also a theta series:

$$\left[\iota_P^* \left(\theta_{L,[\lambda]}(\varphi, z, \tau) \right) \right] = \left[\theta_{W_P \cap L, [\tilde{\lambda}]}(\varphi_P, \tilde{z}, \tau) \right]. \tag{1.1.7}$$

This is proved using a mixture of techniques - almost all of which will feature herein - including geometric analysis of the cohomology, the mixed model of the Weil representation, Fourier analysis, representation theory, and more. We now look at a particular example of this work which has motivated a lot of ours.

In their paper [FM11], the authors consider the case of split orthogonal groups of signature $(2,1)$. As the reader may note, because $\mathrm{SL}_2 \simeq \mathrm{SO}(2,1)$, this is a case we have observed already - namely, the setting of the Shintani-Shimura correspondence!

Using the machinery outlined in [FM06], they create a vector-valued Schwartz form $\varphi_{\mathcal{H}} \in \left[\mathcal{S}(V) \otimes \Omega^1(\mathbb{D}) \otimes \mathcal{H}^k \right]^G$ - here (see e.g. [FH04, §11]) the representations used will be the harmonic subspaces $\mathcal{H}^k$ of $\mathrm{Sym}^k(\mathbb{C}^2)$ - and for the usual choices of L and Γ, may make a theta series $\theta_{L,\mathcal{H}}(\varphi, z, \tau)$ which is a 1-form on X with coefficients in the representation $\mathcal{H}^k$. For η a closed and rapidly decreasing 1-form with coefficients in $\mathcal{H}^k$, the results of Theorem 1.1.5 give us:

$$\int_X \eta \wedge \theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau) \in M_{k+3/2}(\Gamma(M)). \tag{1.1.8}$$

The cusps of V are parameterised by the isotropic lines ℓ; the result of (1.1.7) give that θ_L restricts to a theta series on the boundary, and they further show that the boundary Schwartz form is *exact*: $\varphi_\ell = \mathrm{d}\phi_\ell$. This allows them to build a non-trivial, *compactly-supported* cohomology class in $H^1_c(X)$:

$$\theta_{L,\mathcal{H}}(\varphi, z, \tau) - \sum_{[\ell]} \mathrm{d}\iota^*_\ell \left(\theta_{W_\ell \cap L,\mathcal{H}}(\phi_\ell, z, \tau)\right) \in H^1_c(X).$$

This allows them to integrate against the non-compact cohomology in $H^1(X)$. We note here that, examining the form of the Eichler-Shimura isomorphism in (1.1.2), this result formally extends this relationship to the non-compact cohomology - and hence, equivalently, to Eisenstein series! Indeed, one often refers to the non-compact part of $H^1(X)$ as the Eisenstein cohomology, and this approach (which we shall follow) offers an approach to analyse the arithmetic of this subspace.

1.2 Our Results

We have now done enough work to contextualise our own! It is appropriate that we finished on the paper [FM11], as this offers the most appropriate context for our own work of anything in the existing literature. We shall start our work with two preliminary chapters. The first will largely deal with the geometry of the locally symmetric spaces X of the form

$$X = \Gamma \backslash SU(2,1)/S(U(2) \times U(1)),$$

and consider their compactifications, homology and cohomology; in particular, the geometry of the boundary components of these compactifications. The second such chapter is on the subject of irreducible representations for $SU(2,1)$; this will allow us to create the coefficient systems which give homology and cohomology objects generalising the Kudla-Millson forms.

For the remainder of this section, we fix a hermitian vector space $\underline{V}/k$ over an imaginary quadratic field of signature $(2,1)$, with complex points V and special unitary group $G = \mathrm{SU}(V)$.

What we shall show first is the extension of the homological side of Theorem 1.1.4 for $G = SU(2,1)$. Starting from the cycles C_n from Theorem 1.1.4, we first extend these to cycles with coefficients in an irreducible representation $\mathcal{H}^{l,l}(V)$. We then create caps in the homology group $H_2(\overline{X}, \widetilde{\mathcal{H}^{l,l}(V)})$, allowing us to create the compactified cycles C^c_n. We record this as a first theorem.

Theorem 1.2.1. *Let $X = \Gamma\backslash\mathbb{D}$ be the locally symmetric space corresponding to some arithmetic subgroup $\Gamma \subset SU(2,1)$. For all positive integers n, we define $C_n \subset X$ as in Kudla-Millson. Then for all symmetric, finite-dimensional irreducible* *representations $\mathcal{H}^{l,l}$ of $SU(2,1)$, we may define closed classes $C_{n,[l,l]}$ as follows:*

$$C_{n,[l,l]} = \sum_{\substack{(\boldsymbol{x},\boldsymbol{x})=2n \\ mod\ \Gamma}} C_{\boldsymbol{x}} \otimes \pi_{\mathcal{H}} \left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l \right)$$

These classes define generically non-trivial classes in homology:

$$\left[C_{n,[l,l]}\right] \in H_2 \left(X, \partial X, \widetilde{\mathcal{H}^{l,l}} \right)$$

and the integrals $\int_{C_{n,[l,l]}} \eta$ converge for all $\mathcal{H}^{l,l}(V)$-valued, rapidly decreasing and compactly supported smooth differential forms η on X.

For L an integral and even lattice of level M and $\mathcal{L} \in L'/L$ a lattice coset, the $[C_{n,[l,l]}]$ are the Fourier coefficients of a weight $2l+3$ holomorphic modular form with values in $\mathcal{H}^{l,l}$:

$$\frac{1}{2\pi} \delta_{l=0} \left[\Omega_X\right]^{PD} + \sum_{\substack{n>0 \\ n\ an\ \mathcal{L}-norm}} \left[C_{n,[l,l]}\right] q^n \in H_2 \left(X, \partial X, \widetilde{\mathcal{H}^{l,l}} \right) \otimes M_{2l+3}(\Gamma(M)).$$

It will be a cusp form for $l \geq 1$.

The Borel-Serre compactification $\overline{X}^{BS}$ has finitely many boundary components $e(P_\ell)$, corresponding to the classes of rational isotropic lines $\Gamma \backslash Iso(\underline{V})$. We may cap these cycles with coefficients at each cusp with closed cycles $A^\ell_{n,[l,l]} \subset e(P_\ell)$ such that for all positive rational numbers n:

$$C^c_{n,[l,l]} = C_{n,[l,l]} - \sum_{[\ell]} A^\ell_{n,[l,l]}$$

defines a closed and bounded - hence compact - class $[C^c_{n,[l,l]}] \in H_2(X, \widetilde{\mathcal{H}^{l,l}})$. This may be convergently integrated against the full cohomology group $H^2(X, \widetilde{\mathcal{H}^{l,l}})$, and in particular the sum of the capped special cycles will also define a modular form with coefficients in the homology group, which will be cuspidal for $l \geq 0$:

$$\frac{1}{2} \delta_{l=0} \left[\Omega_X\right]^{PD} + \sum_{\substack{n>0 \\ n\ an\ \mathcal{L}-norm}} \left[C^c_{n,[l,l]}\right] q^n \in H_2 \left(X, \widetilde{\mathcal{H}^{l,l}} \right) \otimes M_{2l+3}(\Gamma(M)).$$

The rest of the paper is hence dedicated to the cohomological picture, and then to conclude, the duality between these constructions. The first work in this direction shall be to construct Schwartz forms with coefficients for $SU(2,1)$. This will require a chapter on the Weil representation of dual pairs $\mathfrak{su}(p,q) \times \mathfrak{su}(1,1)$, which will give us the algebraic properties required to work in the complexes with coefficients. Indeed, we shall dedicate the entirety of the proceeding chapter §6 to constructing the appropriate vector-valued Schwartz forms, and proving - largely with abstract algebraic techniques - that they satisfy the correct properties that will generalise the

Kudla-Millson result. We now state the dual to Theorem 1.2.1:

Theorem 1.2.2. *Fix a positive integer $l \geq 1$, and an irreducible representation $\mathcal{H}^{l,l}(V)$ of $G = \mathrm{SU}(2,1)$; the geometric constructions will be the same as in Theorem 1.2.1. Then there are non-trivial, closed Schwartz functions in the complex:*

$$\varphi_{l,l}^{\mathcal{H}} \in \left[\mathcal{S}(V) \otimes \Omega^2(\mathbb{D}) \otimes \mathcal{H}^{l,l}(V)\right]^G.$$

For L a lattice of level M and $\mathcal{L} \in L'/L$, we may use the non-trivial & closed Schwartz forms φ to form a theta series on $X = \Gamma\backslash\mathbb{D}$:

$$\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau) = \sum_{\boldsymbol{x}\in\mathcal{L}} \varphi_{l,l}^{\mathcal{H}}(\boldsymbol{x}, z, \tau)$$

This is a closed differential form, and its cohomology class defines a holomorphic cusp form of weight $2l+3$:

$$[\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)] \in H^2\left(X, \widetilde{\mathcal{H}^{l,l}(V)}\right) \otimes S_{3+2l}(\Gamma(M)).$$

This is dual to the special cycle generating series from Theorem 1.2.1; hence, for some closed and rapidly decreasing $\mathcal{H}^{l,l}(V)$-valued smooth differential 2-form η on X, the generating series

$$\int_X \eta \wedge \theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau) = i \sum_{\substack{n>0 \\ n \text{ an } \mathcal{L}-norm}} \left[\int_{C_{n,[l,l]}} \eta\right] q^n \in S_{3+2l}(\Gamma(M))$$

is a cuspidal, holomorphic modular form, with coefficients given by the integrals against the $C_{n,[l,l]}$. Equivalently, in cohomology we may write:

$$[\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)] = i \sum_{\substack{n>0 \\ n \text{ an } \mathcal{L}-norm}} \left[C_{n,[l,l]}\right]^{PD} q^n \in H^2\left(X, \mathcal{H}^{l,l}(V)\right) \otimes S_{3+2l}(\Gamma(M))$$

So, we may now move onto the crux of this thesis - namely, the geometric work on the restriction of these objects to the boundary components of the Borel-Serre boundary components. This work will use the mixed model of the Weil representation for the Witt decomposition at each cusp, and then use geometric arguments in the boundary complex to show that the restriction of the theta series is a convergent differential form, given by a 1-dimensional theta series.

Theorem 1.2.3. *Fix a rational isotropic line $[\ell]$ of $\underline{V}$, with associated Witt splitting of $\underline{V}$ given by*

$$V = k\ell \oplus \underline{W}_\ell \oplus k\ell',$$

where ℓ' is the complementary cusp and $\underline{W}_\ell = \ell^\perp \cap \ell'^\perp$ is a positive-definite vector space spanned by some arbitrary rational vector w_ℓ. Let N_ℓ be the nilpotent part of *the parabolic subgroup fixing $[\ell]$, and $\Gamma_\ell = N_\ell \cap \Gamma$, so that the boundary component of $\overline{X}^{BS}$ at ℓ is written $e(P_\ell) = \Gamma_\ell \backslash N_\ell$.*

Then we may define a new Schwartz function in the boundary complex

$$\varphi_{l,l}^{e(P_\ell)} \in [\mathcal{S}(W_\ell) \otimes \Omega^2(N_\ell) \otimes \mathcal{H}^{l,l}(V)]^{N_\ell}$$

such that the theta series extends to a convergent differential form on the boundary components, and the natural restriction of the theta series to this boundary component may be written.

$$\iota_\ell^* \left(\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)\right) = \theta_{W_\ell \cap \mathcal{L}} \left(\varphi_{l,l}^{e(P_\ell)}\right) = \sum_{\boldsymbol{x} \in W_\ell \cap \mathcal{L}} \varphi_{l,l}^{e(P_\ell)}(\boldsymbol{x}, \tilde{z}, \tau)$$

Further, this boundary form is exact: namely, there is a primitive

$$\phi_{l,l}^{e(P_\ell)} \in [\mathcal{S}(W_\ell) \otimes \Omega^1(N_\ell) \otimes \mathcal{H}^{l,l}(V)]^{G}$$

such that $\mathrm{d}\phi_{l,l}^{e(P_\ell)} = \varphi_{l,l}^{e(P_\ell)}$. Hence, we may form a non-trivial class in the cone cohomology group:

$$\left[\theta_{\mathcal{L},\mathcal{H}}(\varphi, \tau), \sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}} \left(\phi_{l,l}^{e(P_\ell)}, \tau\right)\right] \in H^2_{cone}\left(\overline{X}^{BS}, \partial \overline{X}^{BS}\right),$$

which gives us a compactly supported cohomology class on X:

$$\theta_{\mathcal{L},\mathcal{H}}(\varphi, \tau) - \sum_{[\ell]} \mathrm{d}\iota_\ell^* \left(\theta_{W_\ell \cap \mathcal{L}} \left(\phi_{l,l}^{e(P_\ell)}, \tau\right)\right).$$

This is a holomorphic modular form of weight $2l+3$ in τ, and by its compact support we may integrate this against the non-compact *cohomology on X.*

We now have one more piece of work left to state: namely, the duality between the constructions in Theorems 1.2.1 and 1.2.3.

Theorem 1.2.4. *For $l = 0$, the boundary constructions in Theorems 1.2.1 and 1.2.3 are dual; that is, the Fourier coefficients of the capped theta series are given by the capped special cycles:*

$$\left[\theta_{\mathcal{L}}(\varphi, \tau), \sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}} \left(\phi^\ell, \tau\right)\right] = \frac{1}{2\pi}[\Omega_X] + \sum_{n>0} [C_n^c]^{PD} \in H^2_{cone}\left(\overline{X}^{BS}, \partial \overline{X}^{BS}\right)$$

We may conclude by finding the work of Cogdell in Theorem 1.1.3 as an immediate corollary. Indeed, in our work in Theorem 1.2.1, we saw a recreation of this for a different (but related) boundary component. By using the homotopy equivalence of $\overline{X}^{BS}$ and X, we may see the C_n^c as classes in $H_2(X)$, and show that the inclusion map $\iota_{TOR} : X \to \overline{X}^{TOR}$ to the toroidal compactification of X (considered by Cogdell) maps our C_n^c to his compactified class $\tilde{C_n}^c$ for all n:

$$(\iota_{TOR})_* (C_n^c) = \tilde{C_n}^c.$$

This yields Cogdell's pairing on homology as a composition of my Borel-Serre pairing and $(\iota_{TOR})_*$, and hence gives us the main result of [Cog85] as a corollary of Theorem 1.2.1.

1.3 Outlook

I also wish to make a few comments on the future direction of this work.

The first piece of outstanding work is the duality for the case of general coefficients. This is completed in the case of trivial coefficients, and I believe should be achievable with the right work on the cohomology groups with coefficients in the vector bundle $\mathcal{H}^{l,\tilde{l}}(V)$.

One of our primary motivations for undertaking this work is the completion of the unitary analogy of [FM11, §9] - namely, that the denominators of the Eisenstein cohomology in $H^2_!(X, \widetilde{\mathcal{H}^{l,l}}(V))$ give quadratic zeta values, corresponding to Hecke characters on k. There are more than enough indications in our work that this should work - indeed, the capping process gives exactly the right structure of result that mirrors [FM11], and there is no reason to suggest that such calculations should not give the right structure of results.

Another generalisation which I believe will be worthwhile to examine is the case of lattice characters. In [ANS16], a theory is laid out for "twisting" the finite Weil representation on L'/L by a Dirichlet character modulo M; then, using the theory of vector-valued modular forms, they show that the resulting object is modular with respect to $\Gamma_0(M)$, not just $\Gamma(M)$. In particular, a simplified version of this is used in [FM11] to find the lift of vector-valued Eisenstein series. This process should be recreatable in our setting, using a Hecke character χ and a twisted Weil representation ω_χ of $\mathfrak{sp}(W)$.

The reader may also have noticed that much of the work in §6 was not particularly specialised to the case of signature $(2, 1)$. Indeed, as in the orthogonal case - considered in full generality in [FM06] - it should be fairly harmless to extend the work in this chapter to the consideration of the case of general signature (p, q).

The main generalisation outstanding is therefore of the cuspidal behaviour. As a first example, can this be recreated for the case of V of hermitian signature $(p, 1)$, for $p \geq 3$?

In this setting, the cusps are still stabilisers of 1-dimensional complex isotropic lines, so the results of §7.1 should intuitively go through more or less the same. Indeed, we will be able to work with more general representations $\mathbb{S}_{[\lambda]}(V)$, and we will have p different types of positive Howe operators, so the combinatorial calculations will be more involved, but I see absolutely no reason that the same result should not be obtained - namely, that the generalised Kudla-Millson forms $\varphi_{[\lambda]}^{p,1}$ will restrict to special forms

$$\varphi_{[\lambda]}^{p,1,e(P_\ell)} \in \left[\mathcal{S}(W_\ell) \otimes \wedge^{2p}\mathfrak{n}^* \otimes \mathbb{S}_{[\lambda]}(V)\right]$$

In this respect, at least, the results should nicely mirror the generalised orthogonal results found in [FM13] - moreover, there is no reason to suspect that the restriction arguments on the special cycles shouldn't succeed. Where the symmetry with our results will likely fall down is in the capping procedures.

The problem here is that in our case, the torus form $\Omega_\ell \wedge \overline{\Omega_\ell} \in \wedge^2\mathfrak{n}^*$ had a natural primitive, given by the 1-form κ_ℓ in the corner component. However, this process will naturally fall down at this point - there is no obvious primitive in the Lie algebra to a general torus element

$$\Omega_{\ell,1} \wedge \overline{\Omega_{\ell,1}} \wedge \Omega_{\ell,2} \wedge \overline{\Omega_{\ell,2}} \wedge \ldots \wedge \Omega_{\ell,p-1} \wedge \overline{\Omega_{\ell,p-1}},$$

as we will now require a non-trivial $2p-1$ form. Because of this, there is no reason to suspect that the process with the cone complex - which allows us to lift the non-compact cohomology - will carry over. However, we have reason to believe that there are solutions to be found in more generic cohomology groups - for example, L^2.

Related to this, I believe, will be an investigation of the non-holomorphic parts of this theory. Namely, because of our focusing only on the cohomological lifting, the negative coefficients disappear by exactness. Implicit, however, in our work in chapter 8, was that there is a non-holomorphic part of this theory existing when we drop the requirement for the pairing form η to be closed - in particular, this means that we will leave behind the perspective of this being a pairing on cohomology. As is indicated in e.g. [FM14] (wherein an analogous type of boundary component is considered in the Hilbert modular case) or [Cog85] (where the same capping procedure is considered on the toroidal boundary components), the modular forms resulting from pairing weakly converging differential forms on X with the global and local forms $\varphi_{l,l}$, $\varphi_{l,l}^{e(P_\ell)}$ should result in non-holomorphic modular forms of generic odd weight. Again, this is an area for future exploration.

Chapter 2

Picard Modular Surfaces & Their Geometry

In this chapter we will give the necessary elucidation of the geometric structure of the locally symmetric spaces under consideration. This treatment will largely follow that done in e.g. [Cog85] or [Kud79]. We shall introduce some of the natural compactifications of these spaces, mostly from a practical perspective - indeed, the abstract construction of such objects will be largely unnecessary - and then look at the geometry of the boundary components of these spaces; for the work on the Borel-Serre compactification, we follow the theoretical work in [BJ06], and then specialise this to our context. We will present the Lie algebras at both a global and a local level.

2.1 The Symmetric Space $\mathbb{D}$ and its Models

We let $d < 0$ be a square free integer, and $k = \mathbb{Q}(\sqrt{d})$ an imaginary quadratic field with discriminant $D_k < 0$. We fix:

$$\omega_k = \begin{cases} \sqrt{d} & \text{if } d \not\equiv 1 \bmod 4, \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \bmod 4 \end{cases} \tag{2.1.1}$$

and $\delta_k = \sqrt{D_k} \in i\mathbb{R}_{>0}$, so that k has ring of integers $\mathfrak{o}_k = \mathbb{Z}[\omega_k]$ and different $\mathfrak{d}_k = \delta_k \mathfrak{o}_k \subset \mathfrak{o}_k$. From Galois theory, we have two algebraic embeddings $k \hookrightarrow \mathbb{C}$: the identity and the conjugate identity. We fix the identity embedding once and for all, using this to define the trace $\mathrm{Tr}(\alpha) = \alpha + \overline{\alpha}$ and norm $\mathrm{N}(\alpha) = \alpha\overline{\alpha}$ in k; we will often write $N(\alpha) = |\alpha|^2$.

Let $\underline{V}/k$ be a 3-dimensional vector space over k with a non-degenerate Hermitian form

$$(\, , \,) : \underline{V} \times \underline{V} \mapsto k, \tag{2.1.2}$$

which is anti-linear in the first variable and linear in the second, of complex signature $(2, 1)$ at the place given by the identity embedding above. Using our embedding of k in $\mathbb{C}$, this implies that there is an orthonormal basis $\{v_1, v_2, v_3\}$ of $V := \underline{V} \otimes_k \mathbb{C}$, fixed hereafter, such that for $v = z_1v_1 + z_2v_2 + z_3v_3, v' = z'_1v_1 + z'_2v_2 + z'_3v_3 \in V$, we may extend our inner product linearly to find

$$(v, v') = \overline{z_1}z'_1 + \overline{z_2}z'_2 - \overline{z_3}z'_3. \tag{2.1.3}$$

We note that generically in what follows, we shall use the underlined notation to refer to rational objects, and non-underlined to refer to the real (or occasionally complex) points of said object, as we have done with for e.g. $\underline{V}$ and V.

Definition 2.1.1. We let

$$\underline{G}/\mathbb{Q} = \mathrm{SU}(\underline{V}) = \{g \in \mathrm{SL}(\underline{V}) \mid (gu, gv) = (u, v) \text{ for all } u, v \in \underline{V}\} \tag{2.1.4}$$

be the special unitary group of the pair $(\underline{V}, (\, , \,))$. The group of real points of $\underline{G}$ is denoted $G = \underline{G}(\mathbb{R}) \simeq \mathrm{SU}(2, 1)$.

Using the natural embedding $\mathbb{Q} \hookrightarrow k$, we may consider $\underline{V}$ as a $\mathbb{Q}$-vector space, and then let $V_{\mathbb{R}} = \underline{V} \otimes_{\mathbb{Q}} \mathbb{R}$. This is a real orthogonal vector space of signature $(4, 2)$, which has a complex structure given by $v \mapsto iv$, $iv \mapsto -v$. In this way, we have

$$\underline{G} \subset \mathrm{SO}(\underline{V}), \quad G \subset \mathrm{SO}(4, 2).$$

Definition 2.1.2. For any k-vector space $\underline{W}$, an $\mathfrak{o}_k$-lattice is a projective $\mathfrak{o}_k$-module $L \subset \underline{W}$ such that $L \otimes_{\mathfrak{o}_k} k = \underline{W}$.

Such a lattice is *integral* if $(v, v') \in \mathfrak{d}_k^{-1}$ for all $v, v' \in L$, and is *even* if $(v, v) \in \mathbb{Z}$ for all $v \in L$. As in [BHY15], we define the $\mathbb{Z}$- and $\mathfrak{o}_k$-*dual lattices*:

$$L' = L'_{\mathbb{Z}} = \{w \in \underline{W} \mid (w, v)_{\mathbb{Q}} = \mathrm{Tr}(w, v) \in \mathbb{Z} \text{ for all } v \in L\} \tag{2.1.5}$$

$$L'_{\mathfrak{o}_k} = \{w \in \underline{W} \mid (w, v) \in \mathfrak{o}_k \text{ for all } v \in L\}. \tag{2.1.6}$$

Throughout the paper, we shall be using the integral $\mathbb{Z}$-dual, largely because it gives the following desirable properties:

(i) $L'_{\mathfrak{o}_k} = \mathfrak{d}_k L'$;

(ii) For an even integral lattice L, $L \subset L'$, and L'/L is a finite $\mathfrak{o}_k$-module.

We now fix such an even and integral lattice L in the vector space $\underline{V}$ taken above. We assume initially:

(i) There exists a *primitive* isotropic vector $\ell \in L$; namely, $k\ell \cap L = \mathfrak{o}_k \ell$ and $(\ell, \ell) = 0$.

(ii) There exists another *primitive* isotropic vector $\ell' \in L'$ such that $(\ell, \ell') \neq 0$.

Using these vectors, we have a *Witt splitting* of our vector space $\underline{V}$:

$$\underline{V} = k\ell \oplus \underline{W}_\ell \oplus k\ell', \tag{2.1.7}$$

where $\underline{W}_\ell := \ell^\perp \cap \ell'^\perp$ is a positive definite 1-dimensional subspace of $\underline{V}$, of complex signature $(1, 0)$.

We now describe the symmetric space $\mathbb{D}$ that we will study. We first define this space as a set of cosets and analogously as a subset of the projective space of V; both of these models exist in the orthonormal picture. We also introduce the Siegel model of $\mathbb{D}$, which uses the Witt co-ordinates; this will lead to the parabolic model, which will be our primary geometric model for the analysis of the cuspidal behaviour.

We first introduce some vector space notation:

Definition 2.1.3. (i) Let $\epsilon_k : \underline{V}\backslash\{0\} \mapsto \mathbb{P}_k\underline{V}$ be the standard projection map from $\underline{V}$ to the rational projective space, and $\epsilon : V\backslash\{0\} \mapsto \mathbb{P}_{\mathbb{C}}V$ the equivalent map to the complex projective space. For a vector $v \in V$, we denote $\epsilon_k(v) = kv$ or $\epsilon(v) = \mathbb{C}v$ - it should be clear from the context which is being used. The notation $\epsilon(v) = [v]$ will also be used.

(ii) In the vector space $\underline{V}$, we denote by $\underline{V}_+$ the subset of positive vectors, by $\underline{V}_0 = \mathrm{Iso}(\underline{V})$ the subset of isotropic vectors and by $\underline{V}_-$ the subset of negative vectors; we do exactly the same for V. We note that these three subsets are not vector subspaces.

(iii) We denote ${}_+V$ and ${}_-V$ for the maximal positive and negative vector subspaces of V, spanned respectively over $\mathbb{C}$ by $\{v_1, v_2\}$ and $\{v_3\}$.

We may now define our symmetric space.

Definition 2.1.4. (i) Let K be the stabiliser in G of the negative line $\epsilon(v_3)$, so that $K \simeq S(U(2) \times U(1))$. This is a maximally compact subgroup, and we let $\mathbb{D} := G/K$.

(ii) We let Γ_L be the arithmetic subgroup of $\underline{G}(\mathbb{Q})$ acting trivially on the discriminant group $\mathcal{G}$ of L; this is referred to as the *discriminant kernel.* For a fixed arithmetic subgroup $\Gamma \subset \Gamma_L$, we let

$$X = \Gamma\backslash\mathbb{D} = \Gamma\backslash G/K \tag{2.1.8}$$

be the space of double cosets of G with respect to the natural matrix actions. X is often referred to as a *Picard modular surface.*

We note first that in choosing the K in Definition 2.1.4 we were being fairly arbitrary - given any other fixed negative length vector in V, the stabiliser of the associated line in $\mathbb{P}V_-$ would give us a maximally compact subgroup of G.

We also note that in assuming that V is isotropic - in other words, that cusps exist - we have equivalently assumed that $\mathbb{D}$ and X are non-compact manifolds. As we shall see throughout this thesis, this non-compactness is one of the central problems attempted to be solved both here and in many other authors' work. The cusps of $\mathbb{D}$ are parameterised by the rational isotropic lines

$$\mathrm{Iso}(\underline{V}) = \{[\ell] \in \mathbb{P}\underline{V} \mid (\ell, \ell) = 0\} = \epsilon_k(\underline{V}_0).$$

We note here that while $\mathbb{D}$ does not, in a geometric sense, have cusps, it makes sense to talk about "the cusps of $\mathbb{D}$" as there will be a natural relationship between this set and the set of *geometric* cusps of $X = \Gamma\backslash\mathbb{D}$, corresponding to the Γ action.

Hence, because of our interest in X rather than $\mathbb{D}$, we must define the cusps of X. By the G-invariance of the inner product (2.1.2), Γ acts on $\mathrm{Iso}(\underline{V})$ by matrix translation on the lines; hence, the cusps of X are parameterised by the set

$$\Gamma\backslash\mathrm{Iso}\,(\underline{V}_0)$$

which is a finite set by [BJ06, Proposition III.2.16]; if we fix L to be the $\mathfrak{o}_k$-span of the Witt basis and $\Gamma = \Gamma_L$ as in [Hol98, Theorem 2.2], then this is the class number of the ring $\mathfrak{o}_k$. Throughout the following, we generally denote the fixed set of classes in $\Gamma\backslash\mathrm{Iso}(\underline{V}_0)$ by $\{[\ell]\}$; so when we refer to "a fixed cusp of X", we mean to choose one of these finitely many cusps.

2.1.1 The projective model

Our first model of $\mathbb{D}$ is as a subset of the Grassmanian $\mathbb{P}V$.

Lemma 2.1.5. $\mathbb{D} \simeq \mathbb{P}V_-$.

Proof. G acts on $\mathbb{P}V$ by $g \cdot [v] = [gv]$; because the inner product (2.1.2) is G-invariant, this restricts to an action on $\mathbb{P}V_-$. We define the isomorphism as:

$$G/K \mapsto \mathbb{P}V_-, \quad gK \mapsto g[v_3] = [gv_3]. \tag{2.1.9}$$

One may check that it is well-defined: indeed, if $gK = g'K$, then $g^{-1}g' \in K$, so by definition $[g^{-1}g'v_3] = [v_3] \implies [gv_3] = [g'v_3]$.

It is an injective map, because if $gK \mapsto [v_3]$, by definition $[gv_3] = [v_3]$, so $g \in K$ and gK is the trivial coset.

To show surjectivity, let v be a negative vector in V; by scaling, and without loss of generality (because this representation is both possible and unique) we may assume that v may be written $v = av_1 + bv_2 + v_3$. The space $v^\perp$ is positive-definite, so we may choose an appropriate orthonormal basis t, u of $v^\perp$ (with respect to $(\ ,\ )$) so that the matrix g_v defined by

$$g_v v_1 = t, \quad g_v v_2 = u, \quad g_v v_3 = v$$

is in G. Hence, $g_v K \mapsto [v]$, and this map is an isomorphism. □

We note that this proof did not use the signature in any way - indeed, for any unitary group $SU(p,1)$ and a fixed maximally compact subgroup K, $SU(p,1)/K \simeq \mathbb{P}V_-$ for the vector space V of signature $(p,1)$ (and more generally, for signature (p,q), it will be isomorphic to the negative part of the Grassmannian).

In this model, $\underline{G}$ (and hence, in particular, Γ) acts continuously on $\mathbb{P}V_-$ from the right as

$$[v] \times \underline{g} \mapsto \left[{}^t\left({}^t v\underline{g}\right)\right] = \left[\left({}^t g\right) v\right].$$

In Lemma 2.1.5, we explicitly parameterised $\mathbb{P}V_-$ as

$$\mathbb{P}V_- = \left\{[av_1 + bv_2 + v_3] \mid (a,b) \in \mathbb{C}^2,\ |a|^2 + |b|^2 < 1\right\} \tag{2.1.10}$$

which gives a more explicit domain model as a complex 2-disc:

$$\mathbb{P}V_- \simeq D^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}, \tag{2.1.11}$$

where $D^2_{\mathbb{C}}$ has the hyperbolic metric - indeed, this above map is a metric isomorphism.

2.1.2 The Siegel Domain

We now introduce the Siegel model for the upper half plane. Similar to the disc from (2.1.11), this is an affine model for G/K; analogously to how the disc model is using an affine projection of the orthonormal co-ordinates, the Siegel model uses the affine projection of the co-ordinates for the Witt splitting from (2.1.7). In this way, we may think of the Siegel model as a change of basis from the disc model $\mathbb{D}^2_{\mathbb{C}} \simeq \mathbb{P}V_-$.

With respect to each Witt basis $\{\ell, w_\ell, \ell'\}$ of $\underline{V}$ (with $w_\ell \in \underline{W}_\ell$ some arbitrarily chosen rational basis vector), the inner product by definition is of the form

$$\begin{aligned}(\mathbf{v}, \mathbf{v}') &= \overline{\begin{pmatrix} a & b & c \end{pmatrix}} \begin{pmatrix} & & (\ell, \ell') \\ & ||w_\ell|| & \\ (\ell', \ell) & & \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \\ &= |b|^2 ||w_\ell||^2 + 2\mathcal{R}\left(\overline{a}c(\ell, \ell')\right) \end{aligned} \tag{2.1.12}$$

for $\mathbf{v} = a\ell + bw_\ell + c\ell'$, $\mathbf{v}' = a'\ell + b'w_\ell + c'\ell'$.

Definition 2.1.6. With our isotropic lines ℓ and ℓ' from the Witt splitting (2.1.7), we let $W_\ell = \underline{W}_\ell \otimes \mathbb{C}$ be the complex points of the positive-definite space given by $\underline{W}_\ell = \ell^\perp \cap \ell'^\perp$. We define:

$$\mathcal{H}_{\ell,\ell'} = \{(\tau, \sigma) \in \mathbb{C} \times W_\ell \mid 2\mathcal{I}(\tau)|\delta_k||(\ell, \ell')|^2 > (\sigma, \sigma)\} \tag{2.1.13}$$

Here $\mathcal{I}$ refers to the imaginary part (similarly, $\mathcal{R}$ refers to the real part); this notation is continued throughout.

The topology of $\mathcal{H}_{\ell,\ell'}$ is the subspace topology coming from $\mathbb{C}^2$.

We may easily form a bijection between these two spaces: for $(\tau, \sigma) \in \mathcal{H}_{\ell,\ell'}$, we associate

$$z(\tau, \sigma) = \ell' + \sigma + \delta_k(\ell, \ell')\tau\ell \in V. \tag{2.1.14}$$

This is clearly a bijection; indeed, we first note that we may assume that the ℓ' co-ordinate is non-zero (and hence, by scaling, is 1) because

$$(\sigma + \tau\ell, \sigma + \tau\ell) = (\sigma, \sigma) > 0.$$

We have included the unnatural-looking factor of $\delta_k(\ell, \ell')$ because at certain points later on, without any loss of generality, we shall assume that this equals 1.

Lemma 2.1.7. *Write $\sigma = \alpha w_\ell$ for $\alpha \in \mathbb{C}$, and $w_\ell \in \underline{W}_\ell$ some basis vector as above. We may now specifically write the Siegel domain as:*

$$\mathcal{H}_{\ell,w_\ell,\ell'} = \{(\tau, \alpha) \in \mathbb{C} \times \mathbb{C} \mid 2\mathcal{I}(\tau)|\delta_k||(\ell, \ell')|^2 > |\alpha|^2||w_\ell||\}.$$

A bijection between $\mathcal{H}_{\ell,w_\ell,\ell'}$ and the disc $D^2_{\mathbb{C}}$ is given by:

$$\rho_\ell : \mathcal{H}_{\ell,w_\ell,\ell'} \mapsto D^2_{\mathbb{C}},$$

$$\rho_\ell(\tau,\alpha) = \left(\frac{2\delta_k\tau+1}{2\delta_k\tau-1}, \frac{2\alpha||w_\ell||}{(\ell,\ell')(2\delta_k\tau-1)}\right), \tag{2.1.15}$$

with inverse given by

$$(\rho_\ell)^{-1}(z_1,z_2) = \left(\frac{z_1+1}{2\delta_k(z_1-1)}, \frac{(\ell,\ell')z_2}{||w_\ell||(z_1-1)}\right). \tag{2.1.16}$$

Proof. This is the result of three maps: for $(\tau,\alpha) \in \mathcal{H}_{\ell,w_\ell,\ell'}$, we use (2.1.14) to map this to $\ell' + \alpha w_\ell + \delta_k(\ell,\ell')\tau\ell \in V'$. An easy matrix calculation gives that a good change of basis matrix between $\{v_1, v_2, v_3\}$ & $\{\ell, w_\ell, \ell'\}$ is

$$\begin{pmatrix} 1 & & \frac{(\ell,\ell')}{2} \\ & ||w_\ell|| & \\ 1 & & \frac{-(\ell,\ell')}{2} \end{pmatrix}. \tag{2.1.17}$$

Hence, applying G to this vector, we divide through by the v_3 component so that the v_1, v_2 components are our disc co-ordinates. The inverse is calculated in exactly the same way. □

2.2 Parabolic Decompositions

We here elucidate some more of the theory of the parabolic subgroups of $\underline{G}$. We start with the theoretical viewpoint given in e.g. [BJ06] - which applies to any finite rank Lie group - then work out the details for our specific special unitary group G. This will give us another model of the symmetric space - the *horospherical model*.

Definition 2.2.1. (a) For $g, h \in G$, we write ${}^h g := hgh^{-1}$ as the conjugation map.

(b) We let $\underline{P} \subset \underline{G}$ be any rational parabolic subgroup - namely, a closed subgroup such that $\underline{G}(\mathbb{R})/\underline{P}(\mathbb{R})$ is a projective variety. We then notate:

  (i) $\underline{N_P}$, the unipotent radical of $\underline{P}$ (namely, the subgroup of Rad($\underline{P}$) of elements with all eigenvalues equalling 1),

  (ii) $\underline{L_P} = \underline{N_P}\backslash\underline{P}$, the Levi quotient of $\underline{P}$,

  (iii) $X(\underline{L_P}) = \{\chi : \underline{L_P} \mapsto \underline{G_m}\}$, the algebraic maps from $\underline{L_P}$ to the multiplicative group $\underline{G_m}$,

  (iv) $\underline{M_P} \subset \underline{L_P}$ the subgroup given by

$$\underline{M_P} = \cap_{\chi \in X(\underline{L_P})} \mathrm{Ker}(\chi^2), \tag{2.2.1}$$

  (v) $\underline{S_P}$, the split centre of $\underline{L_P}$ over $\mathbb{Q}$ - namely, the maximal $\mathbb{Q}$-split component in the centre of $\underline{L_P}$ - and

(vi) A_P, the connected component of 1 in $S_P = \underline{S_P}(\mathbb{R})$.

Similarly to how we have notated $G = \underline{G}(\mathbb{R})$, we duplicate this for all other rational groups and write the real points without underlining. The real Langlands decomposition (see [BJ06, I.1.10]) is given by:

$$P = N_P A_P M_P = M_P A_P N_P \simeq N_P \times A_P \times M_P \tag{2.2.2}$$

where the isomorphism, given by multiplication in the group, is an analytic diffeomorphism of manifolds. This induces a *horospherical decomposition* of $\mathbb{D}$:

$$\mathbb{D} = G/K \simeq N_P \times A_P \times \mathbb{D}_P, \tag{2.2.3}$$

where $\mathbb{D}_P = M_P/(K \cap M_P)$.

We now specify how this construction may be understood for our specific case of $\underline{G}$.

Because $\underline{V}$ has negative signature 1, we know that the parabolic subgroups of $\underline{G}$ will be in 1-1 correspondence with the isotropic k-subspaces of $\underline{V}$; more specifically, they will be the stabilisers of lines $[\ell] \in \mathrm{Iso}(\underline{V})$. We denote by $\underline{P_\ell}$ the rational parabolic group attached to the line $[\ell]$, and without loss of generality from here on we may switch between the two notations when the context is clear, e.g. $N_P \equiv N_{P_\ell} \equiv N_\ell$. In particular, this means that the components $\mathbb{D}_P$ are trivial for all parabolic subgroups, so we may remove this from the horospherical decomposition in (2.2.3).

We know from (2.1.7) that for our particular isotropic line $[\ell]$ used in the Witt splitting, we may form a flag of $\underline{V}$ by:

$$F_\ell = \Big[\{0\} \subset k\ell \subset (k\ell)^\perp = k\ell \oplus \underline{W}_\ell \subset \underline{V}\Big].$$

The quotients of this flag are hence the three spaces:

$$k\ell/\{0\} = k\ell, \quad (k\ell)^\perp/k\ell \simeq \underline{W}_\ell, \quad \underline{V}/(k\ell)^\perp \simeq k\ell';$$

and so we may write $\underline{L}$ as a subgroup - rather than a quotient group - of $\underline{G}$, as the stabiliser in $\underline{G}$ of the quotients of the flag:

$$\begin{aligned}\underline{L} &= \left\{ \begin{pmatrix} x & & \\ & y & \\ & & \overline{x}^{-1} \end{pmatrix} \;\middle|\; x,\, y \in k^*, N(y) = 1, x\overline{x}^{-1}y = 1 \right\} \\ &= \left\{ l(x) = \begin{pmatrix} x & & \\ & \overline{x}x^{-1} & \\ & & \overline{x}^{-1} \end{pmatrix} \;\middle|\; x \in k^* \right\}.\end{aligned}$$

Because all characters $\chi : \underline{L} \mapsto \underline{G_m}$ are of the form $\chi((l(x)) = N(x)^m$, (with $N : k \to \mathbb{Q}$ the norm in the field) for some $m \in \mathbb{Z}$, then we know that $\underline{M}$ is given by

$$\underline{M} = \{l(x) \mid x \in k^*,\, N(x) = 1\}, \tag{2.2.4}$$

so that $M = \underline{M}(\mathbb{R}) \simeq U(1)$. Hence, using the definition of all the above groups, we see that

$$A = \{l(t) \mid t \in \mathbb{R}\}.$$

So, we are left with the unipotent group $\underline{N_\ell}$. As we shall see, this is a Heisenberg group isomorphic to $\underline{W_\ell} \ltimes \mathbb{R}$.

Definition 2.2.2. For $r \in \mathbb{Q}$, $w \in \underline{W_\ell}$, we may define elements of $\underline{G} = \mathrm{GL}(\underline{V})$:

$$n(w,0) : v \mapsto v + (\ell,v)w - (w,v)\ell - \frac{1}{2}(\ell,v)(w,w)\ell \tag{2.2.5}$$

$$n(0,r) : v \mapsto v - (\ell,v)r\delta_k\ell. \tag{2.2.6}$$

We then let $n(w,r) = n(w,0) \circ n(0,r)$, and let

$$\underline{N_\ell} = \{n(w,r) \mid w \in \underline{W_\ell},\, r \in \mathbb{Q}\} \subset \underline{G}$$

be the rational subgroup generated by all such translations. We call this the Heisenberg group attached to the cusp $[\ell]$.

One may easily check (by computing the action on ℓ') that this is a group with respect to the natural composition and inverse laws:

$$n(w,r) \cdot n(w',r') = n\left(w + w', r + r' + \frac{\mathcal{I}(w,w')}{|\delta_k|}\right) \tag{2.2.7}$$

$$n(w,r)^{-1} = n(-w,-r) \tag{2.2.8}$$

Moreover, we may check that $(n(w,r)v, n(w,r)v) = (v,v)$, so that these maps are in G. These are all unipotent elements, and so by a dimension calculation we know that this is the full unipotent subgroup that we were looking for.

We summarise all of the above in the following:

Proposition 2.2.3. *With respect to the integral cusp ℓ and the associated Witt splitting $\underline{V} = [\ell] \oplus [w_\ell] \oplus [\ell']$, we may write the rational groups in Definition* (2.2.1) *as follows:*

$$\underline{N} = \left\{ n(sw_l, r) = \begin{pmatrix} 1 & -\overline{s}(w_\ell,w_\ell) & -(\ell,\ell')\left(r\delta_k + \frac{1}{2}|s|^2(w_\ell,w_\ell)\right) \\ & 1 & (\ell,\ell')s \\ & & 1 \end{pmatrix} \,\middle|\, s \in k,\, r \in \mathbb{Q} \right\}$$

$$\underline{L} = \left\{ l(x) = \begin{pmatrix} x & & \\ & \overline{x}x^{-1} & \\ & & \overline{x}^{-1} \end{pmatrix} \;\middle|\; x \in k \right\}$$

$$\underline{M} = \{ l(x) \mid x \in k,\ N(x) = 1 \}$$

and hence write the groups in the Langlands decomposition (2.2.2) *as:*

$$N = \{ n(s,r) \mid s \in \mathbb{C},\ r \in \mathbb{R} \}$$

$$A = \left\{ a(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} \;\middle|\; t \in \mathbb{R}_{>0} \right\}$$

$$M = \left\{ m(\theta) = \begin{pmatrix} e^{i\theta} & & \\ & e^{-2i\theta} & \\ & & e^{i\theta} \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}.$$

We note that because $W_\ell = \mathbb{C}w_\ell$, we may harmlessly choose a co-ordinate w_ℓ on W_ℓ and write $n(s,r) = n(sw_\ell, r)$. The co-ordinate version is often preferred in later chapters as we may assume $w_\ell = v_2$ or similar. We now have all the tools for the Iwasawa decomposition of G.

Lemma 2.2.4. *We let $N = \underline{N}(\mathbb{R})$ be the real points of our unipotent group introduced in Definition 2.2.2 and K the maximally compact subgroup fixing $[v_3]$ introduced in Definition 2.1.4. We then have the Iwasawa decomposition:*

$$N \times A \times K \simeq NAK = G, \tag{2.2.9}$$

where this is an analytic diffeomorphism of manifolds, given by multiplication in G.

Proof. This is computed explicitly in [Saw16], while the Iwasawa decomposition is discussed in full generality in [Bum04] and [BJ06]. □

Corollary 2.2.5. *We may realise $\mathbb{D} = G/K$ as a space of left matrix cosets in the most direct way possible*

$$\mathbb{D} \simeq NA \simeq N \times A.$$

Hence, the horospherical decomposition of $\mathbb{D}$ from (2.2.3) *may be written as*

$$\mathbb{D} \simeq \{ [n(s,r), t] \mid s \in \mathbb{C},\ r \in \mathbb{R},\ t \in \mathbb{R}_+ \}.$$

We now describe the necessary isomorphism between this model of $\mathbb{D}$ in Corollary 2.2.5 and the Siegel model of Lemma 2.1.7. To find this, we first give a simple Lemma.

Lemma 2.2.6. *The Heisenberg group acts on $\mathcal{H}_{\ell,\ell'}$ as follows:*

$$n(w,0) : z(\tau,\sigma) \mapsto z\left(\tau - \frac{(w,\sigma)}{\delta_k(\ell,\ell')} - \frac{(w,w)}{2\delta_k}, \sigma + (\ell,\ell')w\right) \tag{2.2.10}$$

$$n(0,r) : z(\tau,\sigma) \mapsto z(\tau - r, \sigma) \tag{2.2.11}$$

Proof. This is just a simple application of (2.2.5) and (2.2.6) along with (2.1.14). □

We may find a diffeomorphism between the horospherical decomposition and the Siegel domain as follows. For any $(\tau,\sigma) \in \mathcal{H}_{\ell,\ell'}$, this has length

$$2|\delta_k||(\ell,\ell')|^2\mathcal{I}(\tau) - (\sigma,\sigma) \coloneqq L(\tau,\sigma) \in \mathbb{R}_{>0} \tag{2.2.12}$$

by definition. We consider the natural isomorphism between A_ℓ and $\mathbb{R}_{>0}$. Then we wish to find co-ordinates $(\zeta, \upsilon, L) \in W'_\ell \times \mathbb{R} \times \{L\}$ such that for

$$\mathcal{H}^L_{\ell,\ell'} \coloneqq \{(\tau,\sigma) \in \mathbb{C} \times W_\ell \mid L(\tau,\sigma) = L\},$$

and for some isomorphism

$$\rho'_L : \mathcal{H}^L_{\ell,\ell'} \mapsto W_\ell \times \mathbb{R},$$

$(\zeta,\upsilon) = \rho'_L(\tau,\sigma)$ is acted on via $(\rho'_L)^{-1}$ as

$$n(w,r)\cdot(\zeta,\upsilon) = \left(w + \zeta, r + \upsilon + \frac{\mathcal{I}(w,\zeta)}{|\delta_k|}, \epsilon\right), \tag{2.2.13}$$

and so that $(\rho'_L)^{-1}(n(w,r)\cdot(\zeta,\upsilon))$ has length L in the sense of (2.2.12).

Indeed, one may check that the map ρ'_L required is of the form:

$$\rho'_L(\tau,\sigma) = \left(\frac{\sigma}{(\ell,\ell')}, -\mathcal{R}(\tau)\right),$$

$$(\rho'_L)^{-1}(\zeta,\upsilon) = \left(-\upsilon + \frac{i}{2|\delta_k||(\ell,\ell')|^2}\left(L + |(\ell,\ell')|^2(\zeta,\zeta)\right), (\ell,\ell')\zeta\right).$$

Hence, gluing the maps ρ'_L together for all L, we get a diffeomorphism:

$$\rho' : \mathcal{H}_{\ell,\ell'} \xrightarrow{\sim} N_\ell \times A_\ell,$$

$$\rho' : (\tau,\sigma) \mapsto \left(n\left(\frac{\sigma}{(\ell,\ell')}, -\mathcal{R}(\tau)\right), 2\mathcal{I}(\tau)|\delta_k||(\ell,\ell')|^2 - (\sigma,\sigma)\right) \tag{2.2.14}$$

2.3 Enlargements of $\mathbb{D}$ and Compactifications of X

In this section we introduce some relevant enlargements and compactifications of the models introduced in subsections 2.1 and 2.2. As referred to above, we know that the cusps of $\mathbb{D}$ are parameterised by the isotropic lines $\mathrm{Iso}(\underline{V})$ of $\underline{V}$; we shall use $\{[\ell]\}$ to notate representatives of the finite set of cusps of $X = \Gamma\backslash\mathbb{D}$.

2.3.1 The Baily-Borel Compactification

To illustrate this compactification, we shall use the projective disc model of $\mathbb{D}$ from §2.1.1; here, the cusps of $\mathbb{D}$ correspond to $\mathrm{Iso}(\underline{V})$ in a very literal way - we attach the set $\epsilon(\underline{V}_0)$, which is in bijection with $\mathrm{Iso}(\underline{V})$. Formally, the set $\epsilon(\underline{V}_0)$ is a subset of $\mathbb{P}V$, and is by definition the set of *complex* isotropic lines which contain a *rational* vector; by associating $\mathbb{C}\ell \to k\ell$, it is naturally isomorphic to $\mathrm{Iso}(\underline{V})$. So, we make our first definition:

Definition 2.3.1. The Baily-Borel enlargement of $\mathbb{D}$ is given by

$$\overline{\mathbb{D}}^{BB} = \mathbb{D} \cup \epsilon(\underline{V}_0), \tag{2.3.1}$$

with a topology called the *Satake topology*, described in full in [BJ06, §III.3], the Satake topology is given by the enlargement of the subspace topology on $\mathbb{P}V_-$ by adding in a system of neighborhoods of $\epsilon(\ell)$ given by:

$$U_M = \left\{[z] \in \mathbb{P}V_- \mid \frac{(z,z)|(\ell,\ell')|^2}{|(z,\ell)|^2} < -M\right\} \cup \epsilon(\ell), \qquad M \in \mathbb{R}_{>0}. \tag{2.3.2}$$

As each cusp $\epsilon(\ell)$ must have non-zero v_3 component also, we may represent it in $\epsilon(\underline{V}_0)$ as $\epsilon(\ell) = [av_1 + bv_2 + v_3]$ for some $a,\, b \in k$, $a\overline{a} + b\overline{b} = 1$, and hence in the enlargement $\overline{\mathbb{D}}^{BB}$ by the point $(a,b) = (a,b)_\ell \in \mathbb{C}^2$.

For the compactification of $X = \Gamma\backslash\mathbb{D}$, we need to define the action of Γ on $\mathbb{D}$. As each of the lines $[\ell]$ in $\epsilon(\underline{V}_0)$ is represented *uniquely* by a point $(a,b)_\ell$, the quotient by Γ - giving the cusps of X - is given by the finite set of points representing the lines in $\Gamma\backslash\epsilon(\underline{V}_0)$. Hence we have:

Definition 2.3.2. The Baily-Borel compactification of $X \simeq \Gamma\backslash D^2_{\mathbb{C}}$ is given by

$$\overline{X}^{BB} = \Gamma\backslash\mathbb{D} \cup \bigcup_{\underline{P}\in\Gamma\backslash\underline{G}(\mathbb{Q})/\underline{P}(\mathbb{Q})} \Gamma\backslash\{\star\} \simeq \Gamma\backslash\mathbb{D} \cup \bigcup_{[\ell]\in\Gamma\backslash\epsilon_k(\underline{V}_0)} (a,b)_\ell \tag{2.3.3}$$

The topology on $\overline{X}^{BB}$ is then the quotient topology. The complex structure on this is defined by pullbacks to the complex structure on $\mathbb{D}$ at each cusp; this gives a normal complex space, but with singularities at the cusps - see [BJ06, Proposition III.3.14] for proof. For this reason, while this is in a way the easiest compactification to understand (one could even draw the real points), it is not ideal for our purposes.

2.3.2 The Toroidal Compactification

Because of these singularities associated with the Baily-Borel compactification, it makes sense to introduce the toroidal compactification instead. The work in this section is largely based on the description in [Hof16, §1], which treats the general case of $SU(1, n)$; the case of all finite dimensional locally symmetric manifolds may be found in [BJ06, §III.7]. Topologically, this is a blowing up of the Baily-Borel compactification in the Siegel model $\mathcal{H}_{\ell,\ell'}$ from Definition 2.1.6.

In the subgroup $\underline{P}$ stabilising the fixed cusp ℓ, we have the full rational subgroup $\underline{N} \subset \underline{P}$ from Definition 2.2.2. Because we have assumed our group Γ to be torsion-free, one may easily check that $\Gamma \cap \underline{P}(\mathbb{Q}) = \Gamma \cap \underline{N}(\mathbb{Q})$. We denote this subgroup by Γ_ℓ to emphasise its dependence on the cusp ℓ. One may easily calculate that the centre of $\underline{N}(\mathbb{Q})$ is

$$C(\underline{N}(\mathbb{Q})) = \{n(0, r) \mid r \in \mathbb{Q}\}, \tag{2.3.4}$$

and so there exists a rational number $C_{\ell,\Gamma} \in \mathbb{Q}$ such that

$$C(\Gamma_\ell) = C(\Gamma \cap \underline{N}(\mathbb{Q})) = \{n(0, r) \mid r \in C_{\ell,\Gamma}\mathbb{Z}\}, \tag{2.3.5}$$

where $\Gamma_\ell = \{\gamma \in \Gamma \mid \gamma[\ell] = [\ell]\}$ is the stabiliser of the cusp. In [Cog85], where a special case of the lattice L and subgroup Γ is considered, the rational number is calculated in terms of the depth of Γ and the basis of L. In the $\mathcal{H}_{\ell,\ell'}$ model of X, $[\ell]$ is the "cusp at infinity", so the neighborhoods U_M from (2.3.2) may now be written as:

$$U_M = \{(\tau, \sigma) \in \mathcal{H}_{\ell,\ell'} \mid 2\mathcal{I}(\tau)|\delta_k||(\ell, \ell')|^2 > (\sigma, \sigma) + M\}. \tag{2.3.6}$$

So, in X, a basis for the neighbourhoods of the cusp are here given by

$$\tilde{U}_M \coloneqq \Gamma_\ell \backslash U_M. \tag{2.3.7}$$

We know from (2.2.6) that an element $n(0, r)$ in the centre acts as $\tau \mapsto \tau - r$, $\sigma \mapsto \sigma$, so taking exponentials and letting $q = \exp(2\pi i\tau / C_{\ell,\Gamma})$, we have

$$C(\Gamma_\ell)\backslash U_M \simeq \{(q, \sigma) \mid 0 < |q| < \exp\left(-\frac{\pi(\sigma, \sigma) + M}{|\delta_k|^2|(\ell, \ell')|^2}\right)\}, \tag{2.3.8}$$

which we may recognize as a punctured disc bundle over W_ℓ. We may then put in the central point $(0,0)$ to this, on which Γ_ℓ acts trivially. Denoting this space $C(\Gamma_\ell)\backslash U_M^*$, we then have an inclusion

$$\tilde{U}_M \hookrightarrow \Big[\Gamma_\ell/C(\Gamma_\ell)\Big]\backslash\Big[C(\Gamma_\ell)\backslash U_M^*\Big] = \tilde{\tilde{U}}_M \tag{2.3.9}$$

where the right hand space is a torus bundle. We define the topology "at ∞" around a point $(0,\sigma)$ by adding in the open sets

$$B_\epsilon = \{(q',\sigma') \mid ||\sigma' - \sigma||^2 < \epsilon, |q'| < \epsilon\} \tag{2.3.10}$$

for any $\epsilon > 0$.

Definition 2.3.3. For each cusp class $[\ell]$, we may glue the spaces $\tilde{\tilde{U}}_M$ to X for each $M \in \mathbb{R}_{>0}$; these stratify, and the resulting manifold is the *toroidal compactification* of X, denoted $\overline{X}^{TOR}$.

We write the topological inclusion as $\iota_{TOR} : X \hookrightarrow \overline{X}^{TOR}$. By results in [BJ06, §III.7], $\overline{X}^{TOR}$ is a compact Hausdorff space without singularities, and the identity map $\iota : X \mapsto X$ extends naturally to a surjective map $\overline{\iota} : \overline{X}^{TOR} \mapsto \overline{X}^{BB}$.

We note that the natural interpretation of this space (which we shall use later on) is as a compact 4-manifold whose boundary $\partial\overline{X}^{TOR}$ is a union of elliptic curves.

2.3.3 The Borel-Serre Compactification

We now describe the compactification we shall be using predominately throughout this paper. We recall the horospherical decomposition $\mathbb{D} \simeq N_\ell \times A_\ell$ at each cusp ℓ from Corollary 2.2.5.

We have shown in Proposition 2.2.3 that we may identify A_ℓ with $\mathbb{R}_+ = \mathbb{R}_{>0}$. This can be compactified as $\overline{A}_\ell \simeq \overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{\infty\}$ and so for the subgroup $\underline{P}$ fixing $[\ell]$, we define the associated *corner* as:

$$\mathbb{D}(\underline{P}) = N_\ell \times \overline{A}_\ell, \tag{2.3.11}$$

which is a real analytic manifold with corners. For any parabolic subgroup $\underline{P}$, we let $\iota_{\underline{P}} : \mathbb{D}(\underline{G}) = \mathbb{D} \hookrightarrow \mathbb{D}(\underline{P})$ be the natural inclusion. We now define our Borel-Serre enlargement:

Definition 2.3.4. For any point $x \in \mathbb{D}(\underline{G}) = \mathbb{D}$, we let $\iota_{\underline{P}}(x) \sim \iota_{\underline{Q}}(x)$. Then we set:

$$\overline{\mathbb{D}}^{BS} = \mathbb{D} \cup \bigcup_{\underline{P}\text{ parabolic}} \mathbb{D}(\underline{P})/\sim \tag{2.3.12}$$

$$= \mathbb{D} \cup \bigcup_{\underline{P} \text{ parabolic}} N_P. \tag{2.3.13}$$

To define the Borel-Serre compactification of $X = \Gamma\backslash\mathbb{D}$, we must define the action of G on $\mathbb{D}$.

Using the horospherical decomposition of $\mathbb{D}$ in (2.2.3) and the Langlands decomposition of P in (2.2.2), we may write $x = (n, a) \in N_\ell \times A_\ell$, so that for $p = n'a'm' \in NAM$, P acts on $\mathbb{D}$ as:

$$p \cdot x := \left({}^{a'm'}(n'n), a'a \right). \tag{2.3.14}$$

Theorem 2.3.5. *The action of $\underline{G}$ on $\mathbb{D}$ extends to $\overline{\mathbb{D}}^{BS}$ - namely, it naturally extends continuously to each face $\mathbb{D}(\underline{P})$. It permutes the faces by $g \cdot N_P = N_{{}^gP}$, meaning that $Stab_\Gamma(\underline{P}) = \Gamma \cap P = \Gamma_P$.*

Proof. See [BJ06, §III.5.13] for details. There it is proved that if one writes $g = km'a'n'$, $k \in K$, then $N_{{}^gP} = N_{{}^kP}$, and it is from there a simple calculation that it acts as required and is an analytic diffeomorphism. □

Corollary 2.3.6. *Topologically, we may write the Borel-Serre compactification $\overline{X}^{BS}$ of X as*

$$\overline{X}^{BS} := \Gamma\backslash\overline{\mathbb{D}}^{BS} = \Gamma\backslash\mathbb{D} \cup \bigcup_{\underline{P} \in \Gamma\backslash\underline{G}/\underline{Q}} e(\underline{P}), \tag{2.3.15}$$

where $e(\underline{P}) := \Gamma_P\backslash N_P$ and $\underline{Q}$ is any proper parabolic subgroup (so that $\Gamma\backslash\underline{G}/\underline{Q}$ gives the Γ-conjugacy classes of proper parabolic subgroups). We often write $e(\underline{P})$ as $e(P)$ for simplicity of notation, but this will always refer to the underlying rational *parabolic subgroup.*

Taking the isotropic line parameterisation of the cusps of X, we will largely write

$$e(P_\ell) = \Gamma_\ell\backslash N_\ell,$$

and $\iota_\ell : e(P_\ell) \to \overline{X}^{BS}$ for the natural inclusion of the boundary component for each cusp $[\ell]$.

The topology of $\overline{X}^{BS}$ is defined analogously to (2.3.10). For any $T > 0$, we define the open neighbourhoods

$$V_T := N_\ell \times (T, \infty] \subset \overline{\mathbb{D}}^{BS}$$

and

$$\tilde{V_T} := \Gamma_\ell\backslash V_T = e(P_\ell) \times (T, \infty] \subset \overline{X}^{BS} \tag{2.3.16}$$

of the ℓ cusp in $\overline{\mathbb{D}}^{BS}$ and $\overline{X}^{BS}$ respectively. The latter provide a basis for all open neighbourhoods of the given cusp, so in particular given *any* open neighbourhood U of ℓ in $\overline{X}^{BS}$, we must have $U \supset \tilde{V_T}$ for some large enough T. Hence, to define the topology for $\overline{X}^{BS}$ it suffices to choose a topological basis for X and then choose in addition the $\tilde{V_T}$ for all $[\ell]$ and all $T > 0$.

Lemma 2.3.7 (Borel, Serre)**.** *The Borel-Serre compactification $\overline{X}^{BS}$ is homotopy-equivalent to X.*

Proof. See e.g. [BS73, §9]. □

2.4 The Lie Algebra of SU(2, 1)

We here discuss the Lie algebra of G. This will use standard constructions, and is mostly of use in defining notation for later use.

Definition 2.4.1. Let G be as in Definition 2.1.4, and J the matrix defining the group:

$$J = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Let $X^\dagger$ be the conjugate transpose of the matrix X. By differentiating the relation $X^\dagger J X = J$, we find that the real Lie Algebra $\mathfrak{g}_0$ of G is given by

$$\mathfrak{g}_0 \coloneqq \{X \in M_3(\mathbb{C}) \mid XJ + JX^\dagger = 0\}$$

A 1-line calculation gives the following algebraic realisation:

Lemma 2.4.2. $\mathfrak{g}_0$ *is an* 8*-dimensional real Lie Algebra, parameterised as:*

$$\mathfrak{g}_0 = \left\{ X \in M_3(\mathbb{C}), \quad X = \begin{pmatrix} ia & d & e \\ -\overline{d} & ib & f \\ \overline{e} & \overline{f} & ic \end{pmatrix} \mid a,\, b,\, c \in \mathbb{R},\, e,\, f \in \mathbb{C},\, a+b+c = 0 \right\}.$$

We reiterate that this is a real Lie Algebra, not a complex one. We write

$$\mathfrak{g} \coloneqq \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$$

for the complexification of $\mathfrak{g}_0$, which is taken to be a right $\mathbb{C}$-vector space. This notation is replicated throughout the paper.

With $\mathfrak{k}_0$ the real Lie algebra of the maximal compact subgroup $K = \mathrm{Stab}(v_3) \simeq S(U(2) \times U(1))$ from Definition 2.1.4(i), we may write $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ where

$$\mathfrak{k}_0 = \{ \begin{pmatrix} ia & d & 0 \\ -\overline{d} & ib & 0 \\ 0 & 0 & ic \end{pmatrix} \mid a+b+c=0\}$$

and

$$\mathfrak{p}_0 = \{ \begin{pmatrix} 0 & 0 & e \\ -0 & 0 & f \\ \overline{e} & \overline{f} & 0 \end{pmatrix} \}.$$

Note that $\mathfrak{k}_0$ is a Lie subalgebra of $\mathfrak{g}_0$, but $\mathfrak{p}_0$ is *not* - one may calculate that $[\mathfrak{p}_0, \mathfrak{p}_0] \subset \mathfrak{k}_0$. The adjoint representation of $\mathfrak{k}_0$ is naturally derived from this relation, as $[\mathfrak{k}_0, \mathfrak{p}_0] \subset \mathfrak{p}_0$ - this will be what we use to build representations of G in §3.1.

Definition 2.4.3. (i) We define the $\mathbb{R}$-linear map

$$\phi_V : \wedge^2_{\mathbb{R}} V \to \mathfrak{u}(V), \quad \phi_V(v \wedge v')(z) = (v, z)v' - (v', z)v$$

and hence define the elements

$$\alpha_{r,s} := \phi_V(v_r \wedge v_s), \quad \beta_{r,s} := \phi_V(iv_r \wedge v_s) \in \mathfrak{u}(V).$$

We hence have that $\mathfrak{k}_0 = \mathrm{span}_{\mathbb{R}}\{\beta_{1,1} - \beta_{2,2}, \beta_{2,2} - \beta_{3,3}, \alpha_{1,2}, \beta_{1,2}\}$ and $\mathfrak{p}_0 = \mathrm{span}_{\mathbb{R}}\{\alpha_{1,3}, \beta_{1,3}, \alpha_{2,3}, \beta_{2,3}\}$

(ii) We hence define the following dual forms:

$$\omega_{r,3} = \alpha^*_{r,3}, \quad \omega'_{r,3} = \beta^*_{r,3}$$

and

$$\xi_r := \frac{1}{2}\left(\omega_{r,3} + \omega'_{r,3} i\right), \quad \overline{\xi_r} := \frac{1}{2}\left(\omega_{r,3} - \omega'_{r,3} i\right)$$

so that ξ_1, $\overline{\xi_1}$, ξ_2, $\overline{\xi_2} \in \mathfrak{p}^*$.

We now fix a cusp ℓ as in our Witt splitting (2.1.7), and analyse the Lie algebra of the associated nilpotent subgroup $N = N_\ell$ of SU(2, 1), written explicitly in Definition 2.2.1.

Definition 2.4.4. As in §2.4, we denote the real Lie algebra of N by $\mathfrak{n}_0$, and the right complexification by $\mathfrak{n} = \mathfrak{n}_0 \otimes_{\mathbb{R}} \mathbb{C}$. With respect to the Witt basis (2.1.7), we may write

$$\mathfrak{n}_0 = \{m(s,r) = \begin{pmatrix} 0 & -\overline{s}||w_\ell||^2 & -(\ell,\ell')r\delta_k \\ & 0 & (\ell,\ell')s \\ & & 0 \end{pmatrix} \mid s \in \mathbb{C},\, r \in \mathbb{R}\},$$

as one may easily check that $\exp(m(s,r)) = n(s,r)$.

The bracket relation in this Lie Algebra is:

$$[m(s,r), m(s',r')] = m\left(0, \frac{2||w_\ell||^2}{|\delta_k|}\mathcal{I}(\overline{s}s')\right). \tag{2.4.1}$$

As in the global case, we wish to define duals forms on each nilpotent subalgebra.

Definition 2.4.5. We define the dual forms $\tilde{\omega}_\ell = m(1,0)^*$ and $\tilde{\omega}'_\ell = m(i,0)^*$ as the duals of $m(1,0)$ and $m(i,0)$ respectively, and hence let

$$\Omega = \frac{1}{2}\left(\tilde{\omega}_\ell + \tilde{\omega}'_\ell i\right), \quad \overline{\Omega} = \frac{1}{2}\left(\tilde{\omega}_\ell - \tilde{\omega}'_\ell i\right)$$

We then define κ as the dual of $m(0,1) \in \mathfrak{n}$; hence Ω, $\overline{\Omega}$, $\kappa \in \mathfrak{n}^*$.

Lemma 2.4.6. *Let Ω, $\overline{\Omega}$ and $\kappa \in \mathfrak{n}^*_\ell$ be as in Definition 2.4.5. Then they satisfy the following relation:*

$$\mathrm{d}\kappa = -\frac{4||w_\ell||^2}{\delta_k}\Omega \wedge \overline{\Omega} \in \wedge^2\mathfrak{p}^*$$

Proof. Because the elements $m(0,r)$ and $m(0,r')$ commute for any r, $r' \in \mathbb{R}$, we use (2.4.1) to find

$$\begin{aligned} \mathrm{d}\kappa(m(s',r'), m(s'',r'')) &= \kappa\left(m\left(0, \frac{2||w_\ell||^2\mathcal{I}(s''\overline{s'})}{|\delta_k|}\right)\right) \\ &= \frac{2||w_\ell||^2\mathcal{I}(s''\overline{s'})}{|\delta_k|}. \end{aligned} \tag{2.4.2}$$

Similarly, by definition of the wedge product, the wedge product acts on $\wedge^2\mathfrak{n}^*$ as:

$$\Omega \wedge \overline{\Omega}(m(s',r'), m(s'',r'')) = \frac{i}{2}\mathcal{I}(s'\overline{s''}).$$

This completes the proof. □

2.5 The Geometry of the Heisenberg Group

We here expand some of the geometry of the Heisenberg group, to assist with our analysis of the Borel-Serre boundary components. Throughout this chapter we assume a fixed cusp ℓ of $\mathbb{D}$, its associated rational parabolic group $\underline{P}_\ell$, the nilpotent Heisenberg group $\underline{N}_\ell \subset \underline{P}_\ell$, and its real points N_ℓ. First we analyse the bundle structure arising from the quotient group $\Gamma_\ell \backslash N_\ell$.

Recall from (2.3.5) that we have a number $C_{\ell,\Gamma} \in \mathbb{Q}$ such that

$$C(\Gamma \cap N_\ell) = C(\Gamma_\ell) = \{n(0,r) \mid r \in C_{\ell,\Gamma}\mathbb{Z}\}.$$

Hence, we may write:

$$\Gamma_\ell = N_\ell \cap \Gamma = \{n(s,r) \mid s \in \mathfrak{q}, r \in C_{\ell,\Gamma}\mathbb{Z}\} =: N(\mathfrak{q}, C_{\ell,\Gamma}) \tag{2.5.1}$$

for some ideal $\mathfrak{q} \subset k$ such that for all q, q' in $\mathfrak{q}$:

$$\mathcal{I}(\overline{q}q') \in \frac{|\delta_k| C_{\ell,\Gamma}}{||w_\ell||^2}\mathbb{Z}. \tag{2.5.2}$$

Hence, we see that we may express $e(P_\ell)$ as a fibre bundle over a base torus:

$$S^1 \to e(P_\ell) \xrightarrow{\pi_\ell} T_\ell^2 := W_\ell/\mathfrak{q}, \tag{2.5.3}$$

where the second map is projection onto the torus:

$$\pi_\ell : e(P_\ell) \mapsto T_\ell^2, \quad \pi_\ell\left(\Gamma_\ell n(s,r)\right) = s + \mathfrak{q},$$

and S^1 is the fibre circle above $s \in W_\ell/\mathfrak{q}$. We now specialise:

Definition 2.5.1. For any $s \in W_\ell$, we define

$$c_{s,\ell} \equiv c_s \subset e(P_\ell)$$

to be the fibre above s with respect to the bundle (2.5.3).

Note that we often write the base fibre circle c_0 as S^1. Throughout the paper, it will often be useful to consider $e(P_\ell)$ as N_ℓ with an equivalence relation on it, so an examination of the product rule in N_ℓ gives that for any $\lambda \in \mathfrak{q}$ and $\delta \in C_{\ell,\Gamma}\mathbb{Z}$, we may write

$$e(P_\ell) = (\mathbb{C}/\mathfrak{q}) \times \mathbb{R}/\sim, \quad (\lambda + s, r + \langle \lambda, s\rangle) \sim (s,r) \sim (s, r+\delta). \tag{2.5.4}$$

Here the symplectic product $\langle\, , \rangle$ is defined by

$$\langle s, s'\rangle \equiv \langle s, s'\rangle_\ell := \frac{||w_\ell||^2}{|\delta_k|}\mathcal{I}(\overline{s}s'). \tag{2.5.5}$$

We pick a positively oriented integral basis λ_1, λ_2 of $\mathfrak{q}$, so that

$$\mathcal{I}\left(\overline{\lambda_1}\lambda_2\right) = \mathrm{vol}\left(\mathbb{C}/\mathfrak{q}\right). \tag{2.5.6}$$

In particular, this quantity does not depend on the choice of λ_1, λ_2 (so long as we retain the condition that they are positively oriented).

Definition 2.5.2. We define the *depth* of the group Γ at the cusp $[\ell]$ as the quantity

$$d(\Gamma, \ell) := \frac{\langle \lambda_1, \lambda_2 \rangle_\ell}{C_{\ell,\Gamma}},$$

which we have assumed in (2.5.2) to be an integer.

We now wish to discuss the integral homology of the boundary components $e(P_\ell) = \Gamma_\ell \backslash N_\ell$, for $\Gamma_\ell \equiv N(\mathfrak{q}, C_{\ell,\Gamma})$ as in (2.5.1) (this is fixed for the rest of this discussion).

Proposition 2.5.3. *$e(P_\ell)$ is a manifold with Poincaré duality, and the 0th, 1st and 3rd integral homology groups of $e(P_\ell)$ are given by*

$$H_0(e(P_\ell), \mathbb{Z}) = \mathbb{Z}, \quad H_1(e(P_\ell), \mathbb{Z}) = \mathfrak{q} \oplus \frac{\mathbb{Z}}{2d(\Gamma, \ell)\mathbb{Z}}, \quad H_3(e(P_\ell), \mathbb{Z}) = \mathbb{Z}$$

Proof. We know that N_ℓ is a Lie group, so this is an orientable manifold, and every element of the discrete subgroup Γ_ℓ will act on N_ℓ as an orientation-preserving diffeomorphism: hence, $e(P_\ell) = \Gamma_\ell \backslash N_\ell$ will also be orientable. It is clearly path-connected and hence connected, and the given bundle structure in (2.5.3) should convince us that $e(P_\ell)$ is compact. Hence, by definition we see that Poincaré duality does indeed apply.

The result for H_0 is simple enough: we know that $e(P_\ell)$ is a connected topological space, so by elementary algebraic topology, it has 0'th integral homology $\simeq \mathbb{Z}$, spanned by the class of any fixed point in $e(P_\ell)$.

By Hurewicz' theorem (see eg [BT95, Theorem 17.20]) we know that as $e(P_\ell)$ is path-connected, $H_1(e(P_\ell), \mathbb{Z})$ is the abelianisation of the first fundamental group $\pi_1(e(P_\ell))$. In [Sco83, pp.470] it is calculated that $\pi_1(e(P_\ell))$ is isomorphic to Γ_ℓ, so with the simple calculation that

$$[n(s, r), n(s', r')] = n(0, 2\langle s, s' \rangle)$$

we see that

$$[\Gamma_\ell, \Gamma_\ell] = \{n(0, r) \mid r \in 2\langle \lambda_1, \lambda_2 \rangle \mathbb{Z}\}.$$

Hence, by the above two cited results, we have found $H_1(e(P_\ell), \mathbb{Z})$. The result for H_3 follows again because it is an orientable, connected & compact manifold, so we know that $H_3(e(P_\ell), \mathbb{Z}) = \mathbb{Z}[e(P_\ell)] \simeq \mathbb{Z}$ is generated by the fundamental class. □

Furthermore, we may use our above analysis of the bundle structure of $e(P_\ell)$ to say which geometric elements represent the basis elements in e.g. simplicial homology:

consulting (2.5.3) and the proof of Proposition 2.5.3, it is clear that the basis of $H_1(e(P_\ell), \mathbb{Z})$ isomorphic to $\mathfrak{q}$ is just given by the homology basis of the base torus T^2_ℓ, and the finite abelian group corresponds to the circle fibre S^1, which may be wrapped around $2d(\Gamma, \ell)$ times before it becomes trivial.

Chapter 3

Coefficients and Representation Theory

In this chapter, we give a brief exposition of the representation theory of $SU(2,1)$, and a similar treatment of homology and cohomology with coefficients in a vector bundle. Our aim throughout will be giving a necessary theoretical exposition of the vector bundles derived from G-representations, which will allow us to work with generalised homological and cohomological objects in §4 and §6. The work on representation theory is largely based on the work of [FH04] and on the exposition given on harmonic operators in [FM06, §3]; the work on vector bundles and coefficient systems comes similarly from [BT95] and [FM06, §2].

3.1 Finite Dimensional Irreducible Representations of $\mathrm{SU}(2,1)$ and their Weights

We start by letting V be an arbitrary hermitian vector space (not necessarily positive-definite), and hence let $G = \mathrm{SU}(V)$. The representation used will, as always, be the standard representation of V, whereby G acts as a matrix on the column vectors of V:

$$G \times V \to V, \quad (g, v) \to g \cdot v, \tag{3.1.1}$$

and we let V^* be the dual representation of V; we note that because V is here a unitary representation, $\rho^*(g) = \overline{\rho(g)}$.

We may hence define

$$T^{l,l'}(V) := V^{\otimes l} \otimes (V^*)^{\otimes l'} \tag{3.1.2}$$

with $l, l' \in \mathbb{N}_0$ non-negative integers; this will be a representation of G as a vector product of representations. This space has an inner product on it given by extending

$$\left(\left(\mathbf{x}_i \otimes \tilde{\mathbf{x}}_j^*\right), \left(\mathbf{y}_i \otimes \tilde{\mathbf{y}}_j^*\right)\right) = \prod_{i,j} \left(\mathbf{x}_i, \mathbf{y}_i\right)\left(\tilde{\mathbf{y}}_j, \tilde{\mathbf{x}}_j\right)$$

For every pair of integers $I = (i,j) \in [l] \times [l']$, we define maps

$$P_I \equiv P_{i,j} : T^{l,l'}(V) \to T^{l-1,l'-1}(V),$$

which remove the (i,j)'th place, so that

$$\begin{aligned} &P_{i,j}\Big((w_1 \otimes \ldots \otimes w_l) \otimes (w_1^* \otimes \ldots \otimes w_{l'}^*) \Big) \\ &\quad = w_j^*(w_i)\,(w_1 \otimes \ldots \otimes \hat{w}_i \otimes \ldots \otimes w_l) \otimes (w_1^* \otimes \ldots \otimes \hat{w}_j{}^* \otimes \ldots \otimes w_{l'}^*)\,. \end{aligned}$$

We note that if l or l' is 0 then we may still allow this map; if e.g. $l' = 0$ then as we have defined $[0] = \emptyset$, we let $P_{i,0} : T^{l,0}(V) \to V^{\otimes(l-1)}$.

Definition 3.1.1. (i) For any non-negative integers $l, l' \in \mathbb{N}$, we define

$$V^{[l,l']} := \cap_{I \subset [l]\times[l']} \mathrm{Ker}(P_I)$$

We let $\mathcal{H} : T^{l,l'}(V) \mapsto V^{[l,l']}$ be the vector space projection map.

(ii) We may write the symmetric powers $S^{l,l'}(V)$ as a subspace of $T^{l,l'}(V)$; we may hence define the harmonic subspace as the image of a projection map from the symmetric powers:

$$\mathcal{H}^{l,l'}(V) := \pi_{\mathcal{H}}\left(\mathrm{Sym}^l(V) \otimes \mathrm{Sym}^{l'}(V^*)\right),$$

We emphasise here the difference between $\mathcal{H}$ and $\pi_{\mathcal{H}}$ (namely, that they have different domains); the latter shall be of primary interest.

Theorem 3.1.2. *For all finite-dimensional hermitian vector spaces V, the $SU(V)$-module $\mathcal{H}^{l,l'}(V)$ is irreducible with highest weight (l, l').*

Proof. Let $\mathfrak{g}_0$ refer to the real Lie algebra of the Lie group $SU(V)$, with right complexification $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$; one may check that $\mathfrak{g} \simeq \mathfrak{sl}_{p+q}(\mathbb{C})$. Further, one may check in e.g. [FH04, §15] that $\mathcal{H}^{l,l'}(V)$ is an irreducible representation of $\mathfrak{sl}_{p+q}(\mathbb{C})$ - these are classified by the exact weight structures that we have constructed, and so give us irreducible representations by the general representation theory of special linear groups, [FH04, Proposition 15.15]. □

We now wish to interpret the harmonic space in Theorem 3.1.2 using a representation space of polynomials.; we herein fix V to be our signature $(2,1)$ hermitian vector space from §2. It is well known that the symmetric powers are isomorphic to the space of homogeneous polynomials of degree l in the variables z_i:

$$\mathrm{Sym}^l(V) \simeq \mathbb{C}[z_i]_l,$$

and by identifying V^* with $\overline{V}$, we may similarly write

$$\mathrm{Sym}^{l'}(V) \simeq \mathbb{C}[\overline{z}_i]_{l'}.$$

We define the Laplacian operator for V as

$$\Delta := \sum_{\alpha=1}^{2} \frac{\partial^2}{\partial z_\alpha \partial \overline{z_\alpha}} - \frac{\partial^2}{\partial z_3 \partial \overline{z_3}}, \tag{3.1.3}$$

so that

$$\mathcal{H}^{l,l}(V) \simeq \mathrm{Ker}\{\Delta : \mathbb{C}[z_i,\, \overline{z_i}]_{l,l} \to \mathbb{C}[z_i,\, \overline{z_i}]_{l-1,l-1}\}.$$

Example 3.1.3. We start with the first non-trivial example: namely, with V as above, we look at the case $l = l' = 1$. Then $S^{1,1}(V) = V \otimes V^*$, and an easy starting subrepresentation is given by the metric:

$$\varepsilon = \{\mathbb{C}\,(v_1 \otimes v_1^* + v_2 \otimes v_2^* - v_3 \otimes v_3^*)\};$$

indeed, by definition of the representation $V \otimes V^*$, $G = SU(V)$ acts trivially on ε, so $\varepsilon \simeq \mathbb{1}_G$. ∇ here is a surjective map

$$\nabla : \mathbb{C}\left[z_i, \overline{z_i}\right]_{1,1} \to \mathbb{C},$$

so by the rank-nullity theorem, we have $V \otimes V^* \simeq \mathbb{1}_G \oplus \mathcal{H}^{l,l}(V)$.

We now move onto a more general analysis of the structure of the weight spaces of $\mathcal{H}^{l,l}(V)$, for V our fixed Hermitian vector space from §2. As ever in our analysis of the weight spaces of a finite dimensional Lie algebra, what we start with is the eigenspaces of the centre of the Lie algebra under the adjoint action.

We may write a basis of $\mathfrak{su}(2,1)$ with respect to the orthonormal basis $\{v_1,\, v_2,\, v_3\}$ as follows:

$$\lambda_1 = \begin{pmatrix} i & & \\ & -i & \\ & & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & & \\ & i & \\ & & -i \end{pmatrix}, \quad \lambda_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & & \\ 0 & & \end{pmatrix}, \quad \lambda_4 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 \\ i & & \\ 0 & & \end{pmatrix}$$

$$\lambda_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & & \\ 1 & & \end{pmatrix}, \quad \lambda_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & & \\ -i & & \end{pmatrix}, \lambda_7 = \frac{1}{2}\begin{pmatrix} & & 0 \\ & & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{2}\begin{pmatrix} & & 0 \\ & & i \\ 0 & -i & 0 \end{pmatrix}.$$

The central subalgebra may be seen as $\mathfrak{k}_0 = \mathrm{span}_{\mathbb{R}}\{\lambda_1,\ \lambda_2\}$; hence, for the principal eigenvalues, we find the following lemma:

Lemma 3.1.4. *The action of* $\mathfrak{k}_0$ *through the adjoint representation on* $\mathfrak{su}(2,1)$ *is as follows:*

$$[\lambda_1,\lambda_3] = -2\lambda_4,\ [\lambda_1,\lambda_4] = 2\lambda_3,\ [\lambda_1,\lambda_5] = \lambda_6,\ [\lambda_1,\lambda_6] = -\lambda_5,\ [\lambda_1,\lambda_7] = -\lambda_8,\ [\lambda_1,\lambda_8] = \lambda_7$$
$$[\lambda_2,\lambda_3] = \lambda_4,\ [\lambda_2,\lambda_4] = -\lambda_3,\ [\lambda_2,\lambda_5] = \lambda_6,\ [\lambda_2,\lambda_6] = -\lambda_5,\ [\lambda_2,\lambda_7] = 2\lambda_8,\ [\lambda_2,\lambda_8] = -2\lambda_7.$$

Hence, we may write down the following eigenbases of $\mathfrak{g}$ *with respect to the adjoint action of* $\mathfrak{k}$*:*

$$\begin{aligned}
[\lambda_1, \lambda_3 \pm i\lambda_4] &= (\pm 2i)\,(\lambda_3 \pm i\lambda_4)\,,\\
[\lambda_1, \lambda_5 \pm i\lambda_6] &= (\mp i)\,(\lambda_5 \pm i\lambda_6)\,,\\
[\lambda_1, \lambda_7 \pm i\lambda_8] &= (\pm i)\,(\lambda_7 \pm i\lambda_8)\,,
\end{aligned}$$

and

$$\begin{aligned}
[\lambda_2, \lambda_3 \pm i\lambda_4] &= (\pm i)\,(\lambda_3 \pm i\lambda_4)\,,\\
[\lambda_2, \lambda_5 \pm i\lambda_6] &= (\mp i)\,(\lambda_5 \pm i\lambda_6)\,,\\
[\lambda_2, \lambda_7 \pm i\lambda_8] &= (\pm 2i)\,(\lambda_7 \pm i\lambda_8)\,.
\end{aligned}$$

Proof. The action of $\mathfrak{k}_0$ is simple matrix calculations; from there, the existence of the eigenbases follows immediately. □

Our objective here is hence the following: in the irreducible representation $\mathcal{H}^{l,l}(V)$, to write down a highest weight vector and compute the weight changing operators (analogous to the raising and lowering operators for $\mathfrak{sl}_2(\mathbb{C})$). Throughout, we refer to the operators $\lambda_j + i\lambda_{j+1}$, $j \in \{3,\ 5,\ 7\}$ as being in the $-$ space and $\lambda_j - i\lambda_{j+1}$, $j \in \{3,\ 5,\ 7\}$ being "in the $+$ space"; this is because the former includes the anti-holomorphic part $\mathfrak{p}^-$ of $\mathfrak{p}$, and the latter includes the holomorphic part $\mathfrak{p}^+$.

Calculations in linear algebra give us the following:

$$[\lambda_3+i\lambda_4, \lambda_3-i\lambda_4] = i\lambda_1, \quad [\lambda_5+i\lambda_6, \lambda_5-i\lambda_6] = i(\lambda_1-\lambda_2), \quad [\lambda_7+i\lambda_8, \lambda_7-i\lambda_8] = i\lambda_2,$$

so that $\lambda_j + i\lambda_{j+1}$ is the weight inverse of $\lambda_j - i\lambda_{j+1}$. We further may find the following relations in the $-$ and $+$ space:

$$[\lambda_3 - i\lambda_4, \lambda_5 - i\lambda_6] = \lambda_7 - i\lambda_8, \quad [\lambda_3 + i\lambda_4, \lambda_5 + i\lambda_6] = \lambda_7 + i\lambda_8.$$

So, in this setting, we do the following - which borrows largely from the analysis of $\mathfrak{sl}_3(\mathbb{C})$ representations in [FH04, §12].

Definition 3.1.5. Let W be any representation of $\mathfrak{g}$. We say that a vector w in W is a highest weight vector if it satisfies the two following conditions:

(i) It is an eigenvector for the action of the central algebra $\mathfrak{k}_0$.

(ii) It is annihilated by the + space matrices $\lambda_3 - i\lambda_4$, $\lambda_5 - i\lambda_6$, $\lambda_7 - i\lambda_8$:

$$w \in \mathrm{Ker}(\lambda_3 - i\lambda_4) \cap \mathrm{Ker}(\lambda_5 - i\lambda_6) \cap \mathrm{Ker}(\lambda_7 - i\lambda_8)$$

Of course, this is not necessarily unique - but again, we have some ideas for how to construct one. We start by finding a weight 0 vector v_0: indeed, a bit of calculation gives the following:

$$v_0 := \pi_{\mathcal{H}} \left(v_2^l \otimes (v_2^*)^l \right)$$

as one may check that $\lambda_1(v_0) = \lambda_2(v_0) = 0$, using the additive action of the lie algebra on vector products. Using the isomorphism between $\mathfrak{g}$ and $\mathfrak{sl}_3(\mathbb{C})$, we have immediately the following:

Proposition 3.1.6. *[FH04, Claim 12.10] Let W be some finite-dimensional irreducible representation of $\mathfrak{g}$ and $w \in W$ some highest weight vector, as in Definition 3.1.5. Then W is generated by the image of w under the action of $\lambda_3 + i\lambda_4$, $\lambda_5 + i\lambda_6$ and $\lambda_7 + i\lambda_8$.*

So, by the standard rubric of weight diagrams, we wish to find a highest weight vector by applying the + space to v_0. We may quickly check that the + space acts trivially on V, and $(\lambda_3 - i\lambda_4)(v_2^*) = -v_1^*$ and $(\lambda_7 - i\lambda_8)(v_2^*) = v_3^*$ are the only non-trivial actions on the basis of V. We hence make the following educated guess for a highest weight vector:

$$v_H = (\lambda_7 - i\lambda_8)^l (v_0) = \pi_{\mathcal{H}} \left((\lambda_7 - i\lambda_8)^l (v_2^l \otimes (v_2^*)^l) \right)$$

Indeed, using the above calculations, we may find:

$$v_H = l! \pi_{\mathcal{H}} \left(v_2^l \otimes (v_3^*)^l \right).$$

Proposition 3.1.7. *v_H is a highest weight vector in $\mathcal{H}^{l,l}(V)$.*

Proof. By our calculations in Lemma 3.1.4, v_H is a weight vector:

$$\begin{aligned}
\lambda_1(v_H) &= \pi_{\mathcal{H}} \left[\lambda_1 \left((\lambda_7 - i\lambda_8)^l (v_0) \right) \right] \\
&= \pi_{\mathcal{H}} \left[\left([\lambda_1, \lambda_7 - i\lambda_8] + (\lambda_7 - i\lambda_8)\lambda_1 \right) (\lambda_7 - i\lambda_8)^{l-1} (v_0) \right]
\end{aligned}$$

$$= \pi_{\mathcal{H}} \left[\left(-i(\lambda_7 - i\lambda_8) + (\lambda_7 - i\lambda_8)\lambda_1 \right) (\lambda_7 - i\lambda_8)^{l-1} (v_0) \right]$$
$$= -iv_H + \pi_{\mathcal{H}} \left[(\lambda_7 - i\lambda_8)\lambda_1 (\lambda_7 - i\lambda_8)^{l-1} (v_0) \right].$$

Iterating this calculation - namely, using the bracket relation from Lemma 3.1.4 to interchange λ_1 and $\lambda_7 - i\lambda_8$ - a further $l-1$ times, we find that $\lambda_1(v_H) = -ilv_H$.

By an identical calculation with λ_2, we see that

$$\lambda_2(v_H) = -2ilv_H,$$

so that v_H is a weight vector for the action of $\mathfrak{k}$. It is now a fairly trivial calculation to check that the + space acts trivially on v_H: indeed, we may check that the vectors v_2 and v_3^* are mapped to 0 by the action of all the matrices in the − space, so by definition of the action of the Lie algebra on vector products, v_H is in the kernel of these maps. □

The astute reader may have noticed that this is hardly the only choice of highest weight vector we could have picked - indeed, using a hexagonal weight diagram as in e.g. [FH04, §12], anything in the top right edge will work equally well.

Though this is not all we could say about the representation theory of $SU(2,1)$, this is more or less sufficient for our purposes - namely, it will allow us to move around the weight diagram for each representation, and to find primitives for v_0 with respect to all of the raising operators in the + space.

3.2 Homology and Cohomology with Coefficients

In this section we revisit the theory of simplicial homology and cohomology with coefficients in a flat vector bundle E. In principle, it would not be fundamentally harder to develop the theory of singular (co)homology, but as all the objects under consideration in future chapters will be simplicial complexes, this would be redundant.

Let X_0 be a simplicial complex such that its topological space X is a finite dimensional manifold, and let $E \to X$ be a flat vector bundle. For $p \in \mathbb{N}_0$, the abelian group of E-valued p-chains is written:

$$Z_p(X,E) = \{\sum_{j=1}^{n} \sigma_j \otimes s_j \mid n \in \mathbb{N}, \sigma_j \text{ an oriented } p\text{−simplex and } s_j \text{ a flat section of } E \text{ over } \sigma_j\}.$$

For convenience, we define $Z_p(X,E) = \{0\}$ for $p \leq -1$. For σ a fixed oriented p-simplex, we let $\Gamma(\sigma, E)$ be the group of sections of E over σ.

For any face $\tilde{\sigma_j}$ of σ_j, a flat section $\tilde{s}_j$ of E over $\tilde{\sigma_j}$ may be uniquely extended to a section s_j of E over σ_j; we write this section $e_{\sigma_j,\tilde{\sigma_j}}(\tilde{s_j})$.

Definition 3.2.1. (i) Writing a p-simplex σ as $\sigma = (v_0, \ldots, v_p)$, we define the j'th face of σ as $\sigma_j = (v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_p)$.

(ii) The boundary operator is hence defined as:

$$\partial : Z_p(X, E) \to Z_{p-1}(X, E), \quad \partial(\sigma \otimes s) = \sum_{j=0}^{p} (-1)^j \sigma_j \otimes r_{\sigma_j,\sigma}(s),$$

where $r_{\sigma_j,\sigma}(s)$ is the natural restriction of s to the j'th face.

One may show as always that $\partial^2 : Z_p(X, E) \to Z_{p-2}(X, E)$ is trivial, so this defines a boundary operator. We let $C_p(X, E) = \mathrm{Ker}(\partial)$ be the cycles in $Z_p(X, E)$, and $B_p(X, E) \subset C_p(X, E)$ as the boundaries and so we have a homology theory, with the group written $H_p(X, E)$.

We now treat the cohomological theory. The abelian group of E-valued p-cochains is written:

$$Z^p(X, E) = \{f : Z_p(X) \to \Gamma(\cdot, E),\ f(\sigma) \in \Gamma(\sigma, E)\}$$

where $Z_p(X)$ is the abelian group of p-simplices.

Definition 3.2.2. The coboundary operator is defined:

$$\delta : Z^p(X, E) \to Z^{p+1}(X, E), \quad \delta(f)(\sigma) = \sum_{j=0}^{p} (-1)^j e_{\sigma,\sigma_j}(f(\sigma_j)).$$

This satisfies $\delta^2 = 0$ and so this defines a cohomology theory, with the cohomology groups written $H^p(X, E)$.

Remark 3.2.3. We may define relative homology (resp. cohomology) groups for any simplicial subspace Y of X, written $H_p(X, Y, E)$ (resp. $H^p(X, Y, E)$), and defined as the homology of the complex of E-valued p-chains whose boundary is a non-zero $p - 1$ chain only on Y (similarly for the cohomology).

We now discuss the usual pairings and products in the homology and cohomology theories; for this section we fix flat bundles E, F, G over X and a parallel section μ of $\mathrm{Hom}(E \otimes F, G)$ (ie a bundle map $\mu : E \otimes F \to G$).

Definition 3.2.4. (i) Let $f \in C^p(X, E)$ and $\sigma \otimes s \in C_p(X, E)$. Then the Kronecker pairing of f and $\sigma \otimes s$ is given by:

$$\langle f, \sigma \otimes s\rangle := \mu\left(f(\sigma) \otimes s\right)$$

(ii) In exactly the same way as in the case of trivial coefficients - see eg [BT95, p.192] - we may introduce the cup and cap products

$$\cup : H^p(X,E)\otimes H^{p'}(X,F)\to H^{p+p'}(X,G)$$
$$\cap : H^p(X,E)\otimes H_{p'}(X,F)\to H_{p-p'}(X,G)$$

where for the latter we require $p \geq p'$ (implicitly, this will use our specified choice of μ as above).

Proposition 3.2.5. *(i) Let X be a compact oriented manifold of dimension k with boundary ∂X. Then the pairing with the fundamental class $[X,\partial X]$ gives the isomorphism*

$$\mathcal{P} : H^p(X,E)\to H_{k-p}(X,\partial X,E), \quad \mathcal{P}(f) = f\cap [X,\partial X]$$

(ii) More generally, let X be a not-necessarily compact manifold with boundary ∂X; then the Poincaré duality is as follows:

$$\mathcal{P} : H^p_c(X,E)\to H_{k-p}(X,\partial X,E)$$

where without loss of generality we denote the map with the same letter. Cohomology with compact support here formally means that the representative of the class is compactly supported in each of the fibres of $E\to X$.

We now quote one more result which will be important in our consideration of the duality of our elements. We may also give a de Rham theory of (co)homology with coefficients in E; one may find the details in eg [BT95, §6], but given a vector bundle E with a connection ∇, a differential p-form is a section of

$$\wedge^p T^*(X)\otimes E \simeq \mathrm{Hom}\left(\wedge^p T(X), E\right)$$

which is closed w.r.t. the differential d_∇ given by the equation:

$$\mathrm{d}_\nabla(\omega)(X_1,\ldots,X_{p+1}) = \sum_{j=1}^{p}(-1)^{j-1}\nabla_{X_i}\left(\omega\left(X_1,\ldots,\hat{X}_j,\ldots,X_{p+1}\right)\right)$$
$$+\sum_{i<j}(-1)^{i+j}\omega\left([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots\hat{X}_j,\ldots X_{p+1}\right).$$

The group of such de Rham cochains, with the differential d_∇, is denoted $C^p_{dR}(X,E)$; for σ any p-simplex and U some open neighbourhood of σ in X, we may write $\sigma = \sum_j \omega_j\otimes s_j$ for ω_j ordinary de Rham forms and s_j sections of $E \mid U$. The natural integration map between $C^p_{dR}(X,E)$ and $C^p(X,E)$ is hence given by

$$\iota_{dR}(\omega)(\sigma) = \sum_j \left(\int_\sigma \omega_j \right) s_j \tag{3.2.1}$$

Proposition 3.2.6. *(i) The map ι_{dR} is trivial on cochains and descends to a map $\iota_{dR} : H^\bullet_{dR}(X,E) \to H^\bullet(X,E)$ on cohomology which is an isomorphism.*

(ii) Hence, we have Poincaré duality on flat vector bundles $E \to X$ of the form

$$\mathcal{P} : H^p_{c,dR}(X,E) \to H_{k-p}(X, \partial X, E).$$

Proof. Part (i) is proven in e.g. [BT95, Theorem 12.15]; part (ii) is hence an immediate corollary using the Poincaré duality of Proposition 3.2.5(ii). □

Chapter 4

Special Cycles on Picard Modular Surfaces

In this chapter, we introduce the homological side of the Kudla-Millson theory. First, we define the special cycles $C_{\mathbf{x}}$ on our Picard modular surface X for all positive vectors $\mathbf{x}$, which will allow us to formulate the first version of the main theorem of Kudla & Millson from [KM86] and [KM87]; namely, that the generating series of these special cycles is a modular form of weight 3. Our first step will be to look at their natural extensions to the chain complexes constructed in §3.2. This will allow us to state the first extension of the theorem of Kudla-Millson - namely, that the generating series of the special cycles $C_{\mathbf{x},[l,l]}$ with coefficients in $\mathcal{H}^{l,l}(V)$ is modular of weight $2l+3$. This theorem will not be proven at this stage - it makes more sense to wait until the cohomological statement can be proven, and then use duality - so we shall have to wait until the end of §6 for this.

Following this, our main focus in this chapter will be to look at the interaction between the generalised special cycles $C_{\mathbf{x},[l,l]}$ and each of the boundary components $e(P_\ell)$. Using a similar argument to other work on modular cycles on Borel-Serre compactifications, we shall create chains $A_{\mathbf{x},[l,l]} \in Z_2(\partial\overline{X}^{BS}, \mathcal{H}^{l,l}(V))$ such that $\partial A_{\mathbf{x},[l,l]} = \partial\overline{C_{\mathbf{x},[l,l]}}$. This will allow us to define capped cycles on $\overline{X}^{BS}$ which are closed and whose generating series will be modular; analogously, this will be proven in §8.

4.1 Special Cycles on X

We now return to the geometry of §2, namely where $\mathbb{D} = G/K$ is the symmetric space of a unitary group of signature $(2,1)$, corresponding to a hermitian vector space $\underline{V}/k$.

Definition 4.1.1. Let $\mathbf{x} \in \underline{V}$ be a positive, rational vector. In the projective model, we may define

$$\mathbb{D}_{\mathbf{x}} := \{z \in \mathbb{D} \mid z \perp \mathbf{x}\}$$

and for $\Gamma_{\mathbf{x}}$ the stabiliser of $\mathbf{x}$ in Γ, we let $C_{\mathbf{x}} := \Gamma_{\mathbf{x}} \backslash \mathbb{D}_{\mathbf{x}}$. For $n \in \mathbb{Q}_{>0}$, $h \in L'/L$ and $\mathcal{L} = L + h$ as before, we define the (finite) sum of $C_{\mathbf{x}}$'s:

$$C_n \equiv C_{n,h} := \sum_{\substack{\mathbf{x} \in \mathcal{L}, (\mathbf{x},\mathbf{x})=2n \\ \bmod \Gamma}} C_{\mathbf{x}} \tag{4.1.1}$$

$\mathbb{D}_{\mathbf{x}}$ is embedded in $\mathbb{D}$ as a topological subspace; indeed, by its definition, we may realise it as a subset of the projective lines in $\mathbb{P}V_{-}$. For $\Gamma_{\mathbf{x}}[v] \in C_{\mathbf{x}}$, the natural map from $C_{\mathbf{x}} \to X$ given by

$$\Gamma_{\mathbf{x}}[v] \to \Gamma[v] \tag{4.1.2}$$

is well-defined.

We say an element $\gamma \in G$ is *neat* if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of γ is torsion-free; we hence say that the arithmetic subgroup Γ is neat if all $\gamma \in \Gamma$ are neat elements. In [FM06, Proposition 4.4], it is shown that for all $\mathbf{x} \in \underline{V}$, there exists a neat subgroup $\Gamma(\mathbf{x}) \subset \Gamma$ such that $C_{\mathbf{x}}$ injects into $\Gamma(\mathbf{x})\backslash\mathbb{D}$.

In general, for any chosen Γ, however, the $C_{\mathbf{x}}$ will not inject into X; henceforth, we will identify $C_{\mathbf{x}}$ with its natural image in $X = \Gamma\backslash\mathbb{D}$, and so write it as a chain on X. As in §2, we shall herein assume that Γ is indeed torsion-free.

So, we are now in a position to state the homological part of the Kudla-Millson theorem.

Theorem 4.1.2 (Kudla-Millson, '86)**.** *Let η be a closed, compactly supported and rapidly decreasing differential form on X, representing a class $[\eta] \in H^2_c(X)$. Let L be an even, integral lattice in $\underline{V}$ of level M, and let $\mathcal{L}$ be some coset of L'/L.*

Using the map from (4.1.2)*, we may consider $C_{\boldsymbol{x}}$ as a chain on X: this represents a relative homology class on X:*

$$[C_{\boldsymbol{x}}] \in H_1(X, \partial X, \mathbb{Z}).$$

This class is generically non-compact, and the integrals given by the Kronecker pairing

$$\int_{C_{\boldsymbol{x}}} \eta$$

all converge. We let Ω_X be the Kähler form on X, so that $c_1(X) = \frac{i}{2\pi}\Omega_X$ is the Chern form on X; then the sum

$$\frac{1}{2\pi}\int_X (\eta \wedge \Omega_X) + \sum_{n>0} \left[\int_{C_n} \eta\right] e^{2\pi i n\tau} \in M_3(\Gamma(M))$$

is a holomorphic modular form of weight 3 *and level* M*.*

This theorem is proven in the papers [KM86] and [KM87] in complete generality; namely, when V is any real, complex or quaternionic split vector space - and hence when G is any finite dimensional orthogonal, unitary or symplectic Lie group.

We here note that when we sum over "positive n" in e.g. Theorem 4.1.2 and the sum in consideration is over some collection of special cycles, what we mean is to sum over all non-trivial norms n of elements in $\mathcal{L}$, so that this is really a sum over a well-ordered and discrete set as usual.

For the rest of this section, we shall attempt to recreate Theorem 4.1.2 for generic odd weight $2l+3$; we recall our work on G-representations from §3.1. For any G-representation E, and any $\mathbf{x} \in \underline{V}$ of positive length, the bundle we will be working with is the natural one given locally as a projection:

$$C_{\mathbf{x}} \times_{\Gamma_{\mathbf{x}}} E \to C_{\mathbf{x}}.$$

For any $\Gamma_{\mathbf{x}}$-invariant vector w in E, we may write sections s_w of the bundle as

$$s_w(z) = (z, w);$$

for simplicity we write $C_{\mathbf{x}} \otimes w \equiv C_{\mathbf{x}} \otimes s_w$. Hence, fixing $E = S^{l,l'}(V)$, the naturally chosen w here is given by $\mathbf{x}^l \otimes (\mathbf{x}^*)^{l'}$.

This is a constant and thus parallel section; we may now write down the special cycle with coefficients in the relevant representations!

Proposition 4.1.3. *Fix integers $l, l' \in \mathbb{N}_0$, and any positive vector $\boldsymbol{x} \in \underline{V}$. We then define the special cycle with coefficients in $S^{l,l'}(V)$ as follows:*

$$C_{\boldsymbol{x},l,l'} \coloneqq C_{\boldsymbol{x}} \otimes \boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^{l'}.$$

Similarly, we then define the special cycle with coefficients for the representation $\mathcal{H}^{l,l'}(V)$ from Definition 3.1.1 as

$$C_{\boldsymbol{x},[l,l']} = C_{\boldsymbol{x}} \otimes \pi_{\mathcal{H}} \left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^{l'} \right).$$

These are cycles - namely, they are closed - and so in particular represent classes in homology:

$$[C_{\boldsymbol{x},l,l'}] \in H_2 \left(X, \partial X, \widetilde{S^{l,l'}(V)} \right), \qquad \left[C_{\boldsymbol{x},[l,l']} \right] \in H_2 \left(X, \partial X, \widetilde{\mathcal{H}^{l,l'}(V)} \right).$$

Proof. We shall prove this in the complex with symmetric coefficients; it should be clear that as the second complex is a restriction of the first, closure in the first implies closure in the second. Indeed, examining the boundary operator from Definition 3.2.1, and picking some simplicial decomposition of $C_{\mathbf{x}}$, it is clear that for all $C_{\mathbf{x}} \otimes v \in C_2(X, S^{l,l'}(V))$, we will have

$$\partial\left(C_{\mathbf{x}} \otimes v\right) = \partial_0\left(C_{\mathbf{x}}\right) \otimes v$$

where the boundary operator without coefficients is denoted ∂_0. Hence, the closure of $C_{\mathbf{x},l,l'}$ follows immediately from $\partial_0(C_{\mathbf{x}}) = 0$ from the equivalent statement for closure without coefficients in Theorem 4.1.2. □

Remark 4.1.4. We note here that although, *a priori*, we cannot say that the harmonic projection $\pi_{\mathcal{H}}(\mathbf{x}^l \otimes (\mathbf{x}^*)^{l'})$ is non-zero, it should not be hard to imagine why it is generically so. For example, in [FM11, §4], they are able to describe the integrals of the analogous orthogonal special cycles $C_{\mathbf{x},[k]}$ against differential forms to give weighted periods of $f(z)$ over the cycle $C_{\mathbf{x}}$ - which in particular are described by Kohnen and Zagier in [KZ84].

More specifically, one may e.g. look at $l = l' = 1$ to get an idea of why these vectors are generically non-zero; indeed, in this case, $V \otimes V^* = \mathbb{1} \oplus \mathcal{H}^{1,1}(V)$, where $\mathbb{1}$ is a 1-dimensional representation spanned by the metric. In particular, this tells us that *all* vectors $\mathbf{x} \otimes \mathbf{x}^*$ will not project to 0 in $\mathcal{H}^{1,1}(V)$, as there is no vector $\mathbf{x} \in V$ such that $\mathbf{x} \otimes \mathbf{x}^*$ is proportional to the metric.

As an example of this, see Example 8.2.5 to see why the lift of the capped theta class is non-trivial - and in particular why these vectors $\pi_{\mathcal{H}}(\mathbf{x}^l \otimes (\mathbf{x}^*)^l)$ are generically non-zero.

We may hence state the main theorem analogising Theorem 4.1.2; the proof of the most important part will be deferred until the following chapter (where the cohomological side will be treated), but it makes sense to state it before we move onto the boundary behaviour.

Theorem 4.1.5. *Fix a non-negative integer $l \geq 0$. Let $\eta \in H^2_c(X, \widetilde{\mathcal{H}^{l,l}(V)})$ be a compactly supported and rapidly decreasing differential form with coefficients in the irreducible representation $\mathcal{H}^{l,l}(V)$, and let $\mathcal{L}$ be some coset of L'/L, where L is of level M. Considering $C_{\boldsymbol{x}} \to X$, and letting $\eta \in H^2_c(X, \widetilde{\mathcal{H}^{l,l}(V)})$ be some rapidly decreasing and compactly supported differential form, then the integrals*

$$\int_{C_{\boldsymbol{x},[l,l]}} \eta := \int_{C_{\boldsymbol{x}}} \left(\eta, \pi_{\mathcal{H}}\left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l\right)\right)$$

all converge - we here take this integral as the scalar integral resulting in pairing the *coefficients in each fibre. We let Ω_X be the Kähler form on X, and η as above. Then*

$$\frac{1}{2\pi}\delta_{l=0}\int_X(\eta\wedge\Omega_X)+\sum_{n>0}\left[\int_{C_{n,[l,l]}}\eta\right]e^{2\pi in\tau}\in M_{3+2l}(\Gamma(M))$$

is a holomorphic modular form of weight $3+2l$ and level M.

Proof. The convergence of all the integrals $\int_{C_{\mathbf{x},[l,l]}}\eta$ is an immediate consequence of our requirement that η be compactly supported and rapidly decreasing on each fibre. The proof of the weight and modularity will be at the end of the chapter on the construction of the Schwartz forms - in Corollary 6.4.2 to Theorem 6.4.1 - but it should be intuitively clear to the reader at this point why we believe it to be true. □

4.2 Restriction and Capping of Special Cycles

In this section, we shall look at the restriction of the special cycles from §4.1 to the boundary components. What we shall see is that at each boundary component $e(P_\ell)$ of $\overline{X}^{BS}$, each cycle $C_{\mathbf{x},[l,l]}$ has boundary a finite collection of 1-cycles. These 1-cycles are themselves boundaries in $Z_1(e(P_\ell),\mathcal{H}^{l,l}(V))$, and so we may create modified 2-cycles $C^c_{\mathbf{x},[l,l]}$ such that $\partial C^c_{\mathbf{x},[l,l]}=0$ in $\overline{X}^{BS}$; crucially, these new cycles will be integrable against non-compact cohomology, and so we will be able to expand the results of Theorem 4.1.5 to drop the condition on η being compactly supported and rapidly decreasing.

A quadratic space is said to be *split* if there is a subspace that is equal to its own orthogonal complement. In [Fun02, Lemma 3.6], in the analogous real case, it is proven that

$$C_{\mathbf{x}}\text{ is an infinite geodesic at }[\ell]\iff\mathbf{x}^\perp\text{ is split}\iff q(\mathbf{x})=||w_\ell||^2N(\alpha)$$

for some $\alpha\in k$; the proof for the complex vector space is identical. As we assume $\mathbf{x}$ is of positive length, then $\mathbf{x}^\perp$ is a hyperbolic space of complex signature $(1,1)$; in particular, if $C_{\mathbf{x}}$ is infinite at $[\ell]$ (namely, for all neighborhoods U of $[\ell]$ in X, $U\cap C_{\mathbf{x}}\neq 0$), $\mathbf{x}=\beta\ell+\alpha w_\ell$. We focus on this case for the moment.

We start by investigating the interaction of

$$\mathbb{D}_{\mathbf{x},[l,l]}\coloneqq\mathbb{D}_{\mathbf{x}}\otimes\pi_{\mathcal{H}}\left(\mathbf{x}^l\otimes(\mathbf{x}^*)^l\right)\in C_2\left(\mathbb{D},\mathcal{H}^{l,l}(V)\right)$$

with N_ℓ, the enlargement of $\mathbb{D}$ at the cusp ℓ; it is clear (because $\Gamma_{\mathbf{x}}$ acts trivially on the vector components of $\mathbb{D}_{\mathbf{x},[l,l]}$) that $C_{\mathbf{x},[l,l]}=\Gamma_{\mathbf{x}}\backslash\mathbb{D}_{\mathbf{x},[l,l]}$.

We now recall the horospherical decomposition $N_\ell \times A_\ell$ of $\mathbb{D}$ given in Corollary 2.2.5, and the map $\psi : \mathcal{H}_{\ell,w_\ell,\ell'} \to N_\ell \times A_\ell$ from (2.2.14); by definition of the space $\mathbb{D}_{\mathbf{x}}$ as the space of vectors perpendicular to $\mathbf{x}$, we may write:

$$\begin{aligned}\mathbb{D}_{\mathbf{x},[l,l]} &\simeq \{[n(s,r),t] \otimes \pi_{\mathcal{H}}\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right) \mid \left(\psi^{-1}[n(s,r),t],\mathbf{x}\right) = 0\} \\ &= \{[n(s,r),t] \otimes \pi_{\mathcal{H}}\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right) \mid \beta + \alpha||w_\ell||^2\overline{s} = 0\} \\ &= \left\{\left[n\left(s(\mathbf{x}),r\right),t\right] \otimes \pi_{\mathcal{H}}\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right) \mid r \in \mathbb{R}, t \in \mathbb{R}_+\right\}.\end{aligned}$$

where we have defined $s(\mathbf{x}) = -\overline{\beta}/(\overline{\alpha}||w_\ell||^2)$.

Lemma 4.2.1. *Fix a rational isotropic line ℓ of $\underline{V}$. Let $\boldsymbol{x} = \beta\ell + \alpha w_\ell \in \underline{V}$ be a positive-length vector split at $[\ell]$, so that $\mathbb{D}_{\boldsymbol{x},[l,l]}$ intersects non-trivially with the boundary component N_ℓ of the Borel-Serre enlargement $\overline{\mathbb{D}}^{BS}$. For such an $\boldsymbol{x}$, we let $s(\boldsymbol{x}) := -\overline{\beta}/(\overline{\alpha}(w_\ell, w_\ell))$ as above. Then considering the Borel-Serre enlargement, the intersection at the cusp corresponding to ℓ is given by:*

$$\overline{\mathbb{D}_{\boldsymbol{x},[l,l]}} \cap N_\ell = \left\{n\left(s(\boldsymbol{x}),r\right) \otimes \pi_{\mathcal{H}}\left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l\right) \mid r \in \mathbb{R}\right\}.$$

Now, using this, we may characterise what the equivalent picture is on the quotient $X = \Gamma\backslash\mathbb{D}$ and its Borel-Serre compactification.

Lemma 4.2.2. *We now fix a cusp class $[\ell]$ of X; for any positive $n \in \mathbb{Q}$, we may introduce the following subset of the lattice coset $\mathcal{L}$:*

$$\mathcal{L}_{n,\ell} := \{\boldsymbol{x} \in \mathcal{L} \text{ split} \mid (\boldsymbol{x},\boldsymbol{x}) = 2n,\ \boldsymbol{x} \perp \ell\};$$

by our work above, we know that for all $\boldsymbol{x} \in \mathcal{L}_{n,\ell}$, $\overline{C_{\boldsymbol{x},[l,l]}}$ will intersect non-trivially with the boundary component $e(P_\ell)$ at $[\ell]$.

For $\boldsymbol{y} \in \mathcal{L}_{n,\ell}$, we let $c_{\boldsymbol{y},\ell} = c_{s(\boldsymbol{y}),\ell} \subset e(P_\ell)$ be the fibre circle above $s(\boldsymbol{y}) \in T^2_\ell$. Then:

$$\partial_\ell C_{\boldsymbol{x},[l,l]} := \overline{C_{\boldsymbol{x},[l,l]}} \cap e(P_\ell) = \coprod_{\substack{\boldsymbol{y}\in\Gamma_\ell\backslash\mathcal{L}_{n,\ell}\\ \boldsymbol{y}=\gamma\boldsymbol{x},\,\gamma\in\Gamma}} c_{\boldsymbol{y},\ell} \otimes \pi_{\mathcal{H}}\left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l\right),$$

and for all $\boldsymbol{x} \notin \mathcal{L}_{n,\ell}$, $\partial_\ell C_{\boldsymbol{x},[l,l]} = 0$.

Proof. We fix elements $d = [n(s(\mathbf{x},r),t]$ and $d' = [n(s(\mathbf{x}),r'),t']$ in $\mathbb{D}_{\mathbf{x}}$, and let d_ℓ, d'_ℓ be the images in the boundary component $e(P_\ell)$. Then by definition $d \equiv d'$ in X if and only if $d = d'\gamma$ for some $\gamma \in \Gamma$; analogously, $d_\ell \equiv d'_\ell$ if and only if $d_\ell = d'_\ell\gamma_\ell$ for some $\gamma_\ell \in \Gamma_\ell$. Hence, by definition, the closure of $C_{\mathbf{x}}$ includes all the Γ translates of $c_{\mathbf{x}}$, and so the statement is proven for trivial coefficients.

Moreover, it is clear that for any $\Gamma_{\mathbf{x}}$-invariant vector $v \in S^{l,l}(V)$,

$$\partial_\ell(C_{\mathbf{x}} \otimes v) = \partial_{0,\ell}(C_{\mathbf{x}}) \otimes v;.$$

hence, the coefficients fit harmlessly onto the end - in particular, the identical result holds in the complex $C_2(X, \mathcal{H}^{l,l}(V))$. $\square$

So, for any given $\mathbf{x} \in \underline{V}$ of positive length, we wish to find a two-cycle $A^{\ell}_{\mathbf{x},[l,l]} \subset e(P_\ell)$ such that $\partial_\ell A^{\ell}_{\mathbf{x},[l,l]} = \partial_\ell C_{\mathbf{x},[l,l]}$.

We first introduce some generic algebraic objects which will allow us to deal with the denominators of these objects. Formally, the cycles $c_{\mathbf{x}}$ are cycles with coefficients in the trivial representation $\mathbb{Q}$ of $\underline{G}$; in this way, the natural integral structure came from $\mathbb{Z}$, the ring of integers of $\mathbb{Q}$. When we now work in $\mathcal{H}^{l,l}(V)$, we wish to define a $\mathbb{Z}$-submodule which replicates the natural integral structure; this will follow the work of Harder on this subject.

Definition 4.2.3. Let $S^{l,l'}(V)$ be the vector space of symmetric powers as above, and let $\pi_{\mathcal{H}} : S^{l,l'}(V) \to \mathcal{H}^{l,l'}(V)$ be the projection into the harmonic subspace.

We let $S^{l,l'}(\underline{V})$ be the symmetric powers over k of $\underline{V}^l \otimes_k (\underline{V}^*)^{l'}$, and $\mathcal{H}^{l,l'}(\underline{V})$ the harmonic vectors; we may understand these objects as either a $\mathbb{Q}$ or a k-vector space.

For $\mathcal{L} \in L'/L$ some coset of an even and integral lattice L, we may define the $\mathbb{Z}$-module $S^{l,l'}(\mathcal{L})$ as follows:

$$S^{l,l'}(\mathcal{L}) = \{\mathbf{x} \in S^{l,l'}(V) \mid \text{all components of } \mathbf{x} \text{ in } \mathcal{L}\},$$

and hence define $\mathcal{H}^{l,l'}(\mathcal{L}) = \pi_{\mathcal{H}}(S^{l,l'}(\mathcal{L}))$. For R a $\mathbb{Z}$-module, we may extend the coefficients on these modules by letting $\mathcal{H}^{l,l'}(\mathcal{L})(R) = \mathcal{H}^{l,l'}(\mathcal{L}) \otimes_{\mathbb{Z}} R$.

In particular, we shall be interested in these module constructions when R' is a subring of $\mathbb{Q}$ given by the inversion of some integers in $\mathbb{Z}$.

We start our analysis of the capping procedure with the immediate question: why should such a cap exist? In the existing literature, such things are common - see for example [FM14] - but there are also examples where they explicitly cannot exist - see e.g. [FM11], where the caps exist only when the coefficient system is non-trivial. We shall start by looking at the case $l = 0$ - where the coefficients are in $\mathbb{Q}$ - and then try to adapt these methods to the case of coefficients in $\mathcal{H}^{l,l}(\underline{V})$ for $l \geq 1$. We let $\mathrm{d}w$ and $\mathrm{d}\overline{w}$ be the toroidal 1-forms on W_ℓ, given by the image in the evaluation map of the forms Ω_ℓ and $\overline{\Omega_\ell}$, written in Definition 2.4.5:

$$\mathrm{d}w = \pi_\ell\left(\Omega_\ell\right), \quad \mathrm{d}\overline{w} = \pi_\ell\left(\overline{\Omega_\ell}\right)$$

In the case $l = 0$, we may see as an immediate corollary of Proposition 2.5.3 that $H^1_{dR}(e(P_\ell))$ is spanned by the projections into $e(P_\ell)$ of $\mathrm{d}w$ and $\mathrm{d}\overline{w}$, so that in particular

$$\int_{c_{\mathbf{x},\ell}} \eta = \langle c_{\mathbf{x}}, \eta \rangle = 0 \tag{4.2.1}$$

for all $\eta \in H^1_{dR}(e(P_\ell))$. This tells us that this 1-cycle is exact, and so in this case the cap must exist.

Lemma 4.2.4. *Let all the $\boldsymbol{y}$ which parameterise the boundary fibre circles of $C_{\boldsymbol{x}}$ in $e(P_\ell)$ be denoted*

$$\{\boldsymbol{x} \mid \ell\} = \{\boldsymbol{y} \in \Gamma_\ell \backslash \mathcal{L}_{n,\ell} \mid \boldsymbol{y} = \gamma \boldsymbol{x}\}$$

For $\boldsymbol{x} \in \mathcal{L} \cap W_\ell$ and $\boldsymbol{y} \in \{\boldsymbol{x} \mid \ell\}$, the special cycles $c_{\boldsymbol{x}} \otimes \boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l$ and $c_{\boldsymbol{y}} \otimes \boldsymbol{y}^l \otimes (\boldsymbol{y}^*)^l$ are homologous.

Proof. More specifically, we may say that they are parallel: we consider the action of N on the complex $\Gamma(e(P_\ell), \widetilde{\mathcal{H}^{l,l}(V)})$ of sections on the bundle generated by the harmonic representation. Let $\mathbf{x} = \alpha w_\ell$ and $\mathbf{y} = \beta\ell + \alpha w_\ell$. Then the nilpotent matrix $n(s(\mathbf{x}), 0)$ acts on the fibre circle as:

$$n(s(\mathbf{x}), 0) \cdot n(0, r) = n(s(\mathbf{x}), r),$$

and on the vector components as:

$$n(s(\mathbf{x}), 0)\mathbf{x} = -\overline{s(\mathbf{x})}\alpha\ell + \alpha w_\ell = \left(\frac{\beta}{\alpha}\right)\alpha\ell + \alpha w_\ell = \mathbf{y},$$

and exactly analogously for the action on $\mathbf{x}^*$. □

In particular, this lemma tells us that, heuristically, all the fibre circles of the same norm related by Γ-maps are equivalent, so once we know a property up to homology for one of them, we know it for them all.

We now look at the bounding of the cycles with trivial coefficients. By the results of Lemma 4.2.4, we know here that all the distinct fibre circles in $\partial_\ell C_{\mathbf{x}}$ are homologous.

We recall here the equivalence relation formulation from (2.5.4), and in particular the chosen integral basis λ_1, λ_2 of the ideal $\mathfrak{q} \subset k$. Let $\mathbf{x} = \beta\ell + \alpha w_\ell \in \mathcal{L}_{n,\ell}$ be an arbitrary vector, with associated constant $s(\mathbf{x}) = -\overline{\beta}/\overline{\alpha} \in k$. Using the inclusion of $\mathfrak{q}$ in k, we write

$$s = s(\mathbf{x}) = x\lambda_1 + y\lambda_2$$

for some rational numbers x and y (because we know that $s(\mathbf{x}) \in k$). We define the following 2-cycle in $e(P_\ell)$:

$$\chi_s : [0,1]^2 \mapsto e(P_\ell), \quad \chi_s(a,b) = n\Big[(a+x)\lambda_1 + (b+y)\lambda_2, -(a+x)(b+y)\langle\lambda_1, \lambda_2\rangle_\ell\Big].$$

In the following, we use the algebraic notation of singular homology. The boundary of χ_s is hence given by:

$$\begin{aligned}\partial\chi_s =& n\Big[(a+x)\lambda_1 + y\lambda_2, -(a+x)y\langle\lambda_1,\lambda_2\rangle_\ell\Big]\\ &+ n\Big[(1+x)\lambda_1 + (b+y)\lambda_2, -(1+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\\ &- n\Big[(\tilde{a}+x)\lambda_1 + (1+y)\lambda_2, -(\tilde{a}+x)(1+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\\ &- n\Big[x\lambda_1 + (\tilde{b}+y)\lambda_2, x(\tilde{b}+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big],\end{aligned}$$

where a, b, $\tilde{a}$, $\tilde{b} \in [0,1]$. For the rest of the calculations we drop the specifications of where the variables lie. Using the equivalence relations for the Γ_ℓ element $n(-\lambda_2, 0)$, we find that:

$$\begin{aligned}&n\Big[(\tilde{a}+x)\lambda_1 + (1+y)\lambda_2, -(\tilde{a}+x)(1+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\\ &\quad\sim n\Big[-\lambda_2 + (\tilde{a}+x)\lambda_1 + (1+y)\lambda_2, \langle -\lambda_2, (\tilde{a}+x)\lambda_1 + (1+y)\lambda_2\rangle_\ell - (\tilde{a}+x)(1+y)\Big]\\ &\quad= n\Big[(a+x)\lambda_1 + y\lambda_2, -(a+x)y\langle\lambda_1,\lambda_2\rangle_\ell\Big],\end{aligned}$$

where the equality follows from $\langle\lambda_2,\lambda_1\rangle_\ell = -\langle\lambda_1,\lambda_2\rangle_\ell$. Similarly:

$$\begin{aligned}&n\Big[(1+x)\lambda_1 + (b+y)\lambda_2, -(1+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\\ &\quad\sim n\Big[-\lambda_1 + (1+x)\lambda_1 + (b+y)\lambda_2, \langle -\lambda_1, (1+x)\lambda_1 + (b+y)\lambda_2\rangle_\ell\\ &\qquad\qquad - (1+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\\ &\quad= n\Big[x\lambda_1 + (b+y)\lambda_2, -(2+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big].\end{aligned}$$

Hence, we may write:

$$\partial\chi_s = n\Big[x\lambda_1 + (b+y)\lambda_2, -(2+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big] - n\Big[x\lambda_1 + (\tilde{b}+y)\lambda_2, x(\tilde{b}+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big], \tag{4.2.2}$$

where b, $\tilde{b} \in [0,1]$. One may easily see that

$$\begin{aligned}&\partial\chi_s - n(x\lambda_1 + y\lambda_2, -2(\tilde{\tilde{b}}+y)\langle\lambda_1,\lambda_2\rangle_\ell)\\ &\quad= n(b\lambda_2, -2b\langle\lambda_1,\lambda_2\rangle_\ell) - n(\tilde{b}\lambda_2, 0) - n(s(\mathbf{x}), -2(\tilde{\tilde{b}}+y)\langle\lambda_1,\lambda_2\rangle_\ell) \qquad (4.2.3)\\ &\quad= \partial T_{s,\ell}\end{aligned}$$

is an oriented 1-cycle which bounds a singular 2-cycle $T_{s,\ell} \in Z_2(e(P_\ell), \mathbb{Q})$ - indeed, we may define $T_{s,\ell}$ to be the 2-chain defined by the closure of the interior of the 1-cycle $\partial T_{s,\ell}$. By definition, we may write the cycle $c_{s(\mathbf{x}),\ell}$ as

$$c_{s(\mathbf{x}),\ell} = \{n(s(\mathbf{x}), r) \mid r \in [0, C_{\ell,\Gamma}]\},$$

where we again use (2.5.4) to join $n(0,0)$ with $n(0, C_{\ell,\Gamma})$. We may harmlessly rotate it round to remove the constant y in the r component in (4.2.3), and hence, in the group of rational chains $Z_1(e(P_\ell),\mathbb{Q})$ we have:

$$\partial\chi_s = \frac{-2\langle\lambda_1,\lambda_2\rangle_\ell}{C_{\ell,\Gamma}}c_{s,\ell} + \partial T_{s,\ell} = -2d(\Gamma,\ell)c_{s,\ell} + \partial T_{s,\ell}. \tag{4.2.4}$$

Hence, we have capped the fibre circle c_{w_ℓ} with a rational 2-chain contained entirely in the boundary component $e(P_\ell)$.

We hence may define the capped cycles for the trivial coefficients $l = 0$.

Definition 4.2.5. For a fixed cusp $[\ell]$ and positive vector $\mathbf{x}$ split at ℓ, we define $A^\ell_{\mathbf{x}}$ to be the two-cycle defined by

$$A^\ell_{\mathbf{x}} \coloneqq \frac{1}{2d(\Gamma,\ell)}\left(T_{s(\mathbf{x}),\ell} - \chi_{s(\mathbf{x}),\ell}\right).$$

For all other $\mathbf{x}$ and $[\ell]$, we let $A^\ell_{\mathbf{x}} \equiv 0$. Using the calculations of Lemma 4.2.2 we define the compactified two-cycle in the Borel-Serre compactification $\overline{X}^{BS}$ of X by

$$C^c_{\mathbf{x}} \coloneqq \overline{C_{\mathbf{x}}} - \sum_{[\ell]}\sum_{\substack{\mathbf{y}\in\Gamma_\ell\backslash\mathcal{L}_{n,\ell}\\ \mathbf{y}=\gamma\mathbf{x},\,\gamma\in\Gamma}} A^\ell_{\mathbf{y}}$$

and hence for $n\in\mathbb{Q}$ positive, similarly define C^c_n as

$$C^c_n = \overline{C_n} - \sum_{[\ell]}\sum_{\mathbf{y}\in\Gamma_\ell\backslash\mathcal{L}_{n,\ell}} A^\ell_{\mathbf{y}}$$

We shall now record all of the above in a theorem, which generalises Theorem 4.1.2 for the case of trivial coefficients. We are stating this theorem now for several reasons: firstly, because the general case of $l \geq 1$ requires one more result to express the integrality; secondly, it also will still contain the Kähler form, which we will see in later chapters to disappear for $l \geq 1$. Finally, at the end of the thesis we will relate our work to that of Cogdell, who worked in this same setting of $l = 0$, and so separating it out seems sensible.

Theorem 4.2.6. *For all cusp classes $[\ell]$ the cycle $c_{\mathbf{x},\ell}$ is a* rational *boundary in $e(P_\ell)$, and so is trivial in the rational homology group $H_1(e(P_\ell),\mathbb{Q})$.*

For all positive $n\in\mathbb{Q}$, the compactified special cycles C^c_n define homology classes

$$[C^c_n] \in H_2\left(\overline{X}^{BS},\mathbb{Z}\left[\frac{1}{d_\Gamma}\right]\right) \simeq H_2\left(X,\mathbb{Z}\left[\frac{1}{d_\Gamma}\right]\right)$$

which are generically non-exact, and have denominator dividing the even integer $d_\Gamma \coloneqq lcm_{[\ell]}2d(\Gamma,\ell)$. These classes may be convergently integrated against non-compactly-supported cohomology classes $\eta \in H^2_{dR}(X)$, and when we specify $\eta \in$

$H^2(X,\mathbb{Z})$, the resulting integrals satisfy

$$\int_{C_n^c} \eta \in \frac{1}{3d_\Gamma}\mathbb{Z}.$$

For any fixed $\eta \in H^2_{dR}(X)$, the generating series

$$\frac{1}{2\pi}\delta_{\mathcal{L}=L}\int_X (\eta \wedge \Omega_X) + \sum_{n>0}\left[\int_{C_n^c}\eta\right] e^{2\pi i n\tau} \in M_3(\Gamma(M))$$

is a holomorphic modular form of weight 3 and level M.

Proof. The exactness of the $c_{\mathbf{x},\ell}$ was shown above; by their definition and the results of Lemma 4.2.2, it follows that all the $C^c_{\mathbf{x}}$ are closed with respect to the boundary operator ∂ in $Z_2(\overline{X}^{BS})$. By Lemma 2.3.7, $\overline{X}^{BS}$ is a compact space homotopy equivalent to X, so we know that

$$H_2(\overline{X}^{BS}, R) \simeq H_2(X,R)$$

for any $\mathbb{Z}$-module R. Hence, for all $\mathbf{x}$, $[C^c_{\mathbf{x}}]$ defines a compact class in $H_2(X)$; this compactness tells us that

$$\int_{C^c_{\mathbf{x}}} \eta$$

converges for all $\mathbf{x}$ and all choices of η as in the statement of the theorem. Further, because the Kronecker pairing

$$\Big\langle\, ,\Big\rangle : H_2(\overline{X}^{BS}, R) \otimes H^2(\overline{X}^{BS}, R) \to R$$

is perfect for all $\mathbb{Z}[1/6]$-modules R, then the fractional integrality holds up to the extra factor of 3 in the denominator (we know that 2 will always divide d_Γ, so we only need 3-divisibility).

So, we are left with the modularity: this, again, shall be proven using geometric arguments in §8. □

We now treat the general case of the capping of the cycles $C_{\mathbf{x},[l,l]}$ in $\overline{X}^{BS}$. We shall naively try to use the same objects as in Theorem 4.2.6, and show that these objects give us the right capping properties.

We will now attempt to adapt the above machinery to the general case of coefficients our irreducible representations of $\underline{G} = \mathrm{SU}(\underline{V})$, constructed in §3.1. Without loss of generality, we shall consider the rational part of this - namely, $\mathcal{H}^{l,l}(\underline{V})$, which we may consider as an irreducible representation of $\underline{G}$.

Lemma 4.2.7. *Let ℓ be an arbitrary isotropic vector, n be an arbitrary positive rational number, l a positive integer and $\boldsymbol{x} \in \mathcal{L}_{n,\ell}$ a vector of length n, split at the* *cusp represented by $[\ell]$. Let $\lambda \in k$ be some arbitrary element of the field. Then in the irreducible representation $\mathcal{H}^{l,l}(\underline{V})$, we have the following equivalence:*

$$\left[n(\lambda,0)\left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l\right)\right] = \left[\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l\right]$$

Proof. This is a fairly simple exercise in arithmetic: indeed, one may easily calculate that if we write $\mathbf{x} = \beta\ell + \alpha w_\ell$ as usual, then:

$$n(\lambda,0)\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right) = \left(\mathbf{x} + \overline{\lambda}\alpha\ell\right)^l \otimes (\mathbf{x}^* + \lambda\overline{\alpha}\ell^*)^l. \tag{4.2.5}$$

It is a simple calculation to check that with respect to the Witt basis, the Laplacian operator ∇ from (3.1.3) may be written:

$$\nabla = \frac{\partial^2}{\partial w_\ell \partial w_\ell^*} + \delta_k\left(\frac{\partial^2}{\partial \ell' \partial \ell^*} - \frac{\partial^2}{\partial \ell \partial (\ell')^*}\right).$$

As there are no non-zero terms with ℓ' or $(\ell')^*$ in (4.2.5), then we only need to consider the central terms - namely, the coefficients of w_ℓ and w_ℓ^*. It is hence a trivial exercise in linear algebra to check that:

$$\nabla\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right) = l^2|\alpha|^2\left(\mathbf{x}^{l-1} \otimes (\mathbf{x}^*)^{l-1}\right) \tag{4.2.6}$$

and

$$\nabla\left(n(\lambda,0)\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right)\right) = l^2|\alpha|^2\left(\left(\mathbf{x} + \overline{\lambda}\alpha\ell\right)^{l-1} \otimes (\mathbf{x}^* + \lambda\overline{\alpha}\ell^*)^{l-1}\right) \tag{4.2.7}$$

Hence, as are assuming that $l \geq 1$ and α must be non-zero, then the difference between (4.2.6) and (4.2.7) must be non-zero: indeed, it may be written

$$l^2|\alpha|^2 \sum_{\substack{j,j' \in \{0,1,\dots,l-1\}^2 \\ \text{not both } l-1}} (-1)^{j+j'-1}\binom{l-1}{j}\binom{l-1}{j'}\mathbf{x}^j\left(\overline{\lambda}\alpha\ell\right)^{l-1-j} \otimes (\mathbf{x}^*)^{j'}(\lambda\overline{\alpha}\ell^*)^{l-1-j'}$$

which for all chosen parameters will be a non-zero vector, and so their difference will project to zero in harmonic coefficients. □

This now allows us to cap our cycles over $\mathbb{Q}$.

Proposition 4.2.8. *Let ℓ, n and l be as in Lemma 4.2.7. Let $\boldsymbol{x} \in \mathcal{L}_{n,\ell}$, and let $\boldsymbol{y} \in \{\boldsymbol{x} \mid \ell\}$. Then there exists a collection of chains $A^\ell_{\boldsymbol{x},l,l} \subset e(P_\ell)$ such that $\partial_\ell A^\ell_{\boldsymbol{x},l,l} = \partial_\ell C_{\boldsymbol{x},[l,l]}$.*

Proof. We start by noting that by Lemma 4.2.7, the boundary of $C_{\mathbf{x},[l,l]}$ in $e(P_\ell)$ from Lemma 4.2.2 can now be written

$$\partial_\ell C_{\mathbf{x},[l,l]} = \coprod_{\mathbf{y} \in \{\mathbf{x}|\ell\}} c_{\mathbf{y},\ell} \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)$$

by acting with the element $\gamma \in \Gamma$ taking $\mathbf{x}$ to each $\mathbf{y}$ - such a γ exists by definition of the set $\{\mathbf{x}, \ell\}$. Because we have assumed Γ to preserve each lattice coset, this action will preserve integrality. We write $s(\mathbf{y}) = x\lambda_1 + y\lambda_2$ as in Definition 4.2.5, so the following holds:

$$\partial\Big[\chi_{s(\mathbf{y})} \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)\Big] = \Big[n\Big[(a+x)\lambda_1 + y\lambda_2, -(a+x)y\langle\lambda_1,\lambda_2\rangle_\ell\Big] \tag{4.2.8}$$

$$+ n\Big[(1+x)\lambda_1 + (b+y)\lambda_2, -(1+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big] \tag{4.2.9}$$

$$- n\Big[(\tilde{a}+x)\lambda_1 + (1+y)\lambda_2, -(\tilde{a}+x)(1+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big] \tag{4.2.10}$$

$$- n\Big[x\lambda_1 + (\tilde{b}+y)\lambda_2, x(\tilde{b}+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\Big] \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right). \tag{4.2.11}$$

We now see that our argument follows identically as before - namely, by acting with $n(-\lambda_2, 0)$ on (4.2.10) we find

$$n(-\lambda_2,0)\left(n\Big[(\tilde{a}+x)\lambda_1 + (1+y)\lambda_2, -(\tilde{a}+x)(1+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big] \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)\right)$$
$$\sim n\Big[(a+x)\lambda_1 + y\lambda_2, -(a+x)y\langle\lambda_1,\lambda_2\rangle_\ell\Big] \otimes \pi_{\mathcal{H}}\left(n(-\lambda_2,0)\circ\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)\right)$$

so by applying Lemma 4.2.7 again, this is equal to (4.2.8); similarly, acting with $n(-\lambda_1, 0)$ on (4.2.11), we have

$$n(-\lambda_1,0)\left(n\Big[(1+x)\lambda_1 + (b+y)\lambda_2, -(1+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big] \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)\right)$$
$$\sim n\Big[x\lambda_1 + (b+y)\lambda_2, -(2+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big] \otimes \pi_{\mathcal{H}}\left(n(-\lambda_1,0)\circ\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)\right)$$

and identically to above, we may apply Lemma 4.2.7 so that with the coefficient system,

$$\partial\Big[\chi_s \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)\Big] = \Bigg[n\Big[x\lambda_1 + (b+y)\lambda_2, -(2+x)(b+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]$$
$$- n\Big[x\lambda_1 + (\tilde{b}+y)\lambda_2, x(\tilde{b}+y)\langle\lambda_1,\lambda_2\rangle_\ell\Big]\Bigg] \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)$$

and so defining $T^\ell_{s(\mathbf{y}),[l,l]} = T_{s(\mathbf{y}),\ell} \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right)$, we see by an identical argument to the case of $l = 0$ that the cycle is indeed capped, so that

$$\partial\left(T^\ell_{\mathbf{y},[l,l]} - \chi^\ell_{s(\mathbf{y}),[l,l]}\right) = 2d(\Gamma,\ell)c_{\mathbf{y}} \otimes \pi_{\mathcal{H}}\left(\mathbf{y}^l \otimes (\mathbf{y}^*)^l\right).$$

Hence, taking $A^\ell_{\mathbf{x},l,l}$ to be the collection of these 2-cycles over all $\mathbf{y} \in \{\mathbf{x} \mid \ell\}$, the proof is complete. □

We note that because the matrices $n(-\lambda_i, 0)$ takes $\mathbf{x}$ to an integral vector, the cap $A^{\ell}_{\mathbf{x},[l,l]}$ will have the same denominator as in the trivial coefficients case.

So, we now may state our full theorem.

Theorem 4.2.9. *Fix an integer $l \geq 1$. For all cusp classes $[\ell]$ the cycle $c_{\boldsymbol{x},\ell} \otimes \boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l$ is a* rational *boundary in $e(P_\ell)$, and so is trivial in the rational homology group $H_1(e(P_\ell), \widetilde{\mathcal{H}^{l,l}(\underline{V})})$ with coefficients in the rational part of the harmonic vectors.*

For $\boldsymbol{x}$ split at ℓ, we define $A^{\ell}_{\boldsymbol{x},[l,l]}$ to be the two-chain defined by

$$A^{\ell}_{\boldsymbol{x},[l,l]} := \frac{1}{2d(\Gamma,\ell)} \left(T_{s(\boldsymbol{x}),\ell} - \chi_{s(\boldsymbol{x}),\ell}\right) \otimes \pi_{\mathcal{H}} \left(\boldsymbol{x}^l \otimes (\boldsymbol{x}^*)^l\right);$$

for all other $\boldsymbol{x}$ and $[\ell]$, we let $A^{\ell}_{\boldsymbol{x},[l,l]} \equiv 0$. Using the calculations of Lemma 4.2.2 we define the compactified two-cycle in the Borel-Serre compactification $\overline{X}^{BS}$ of X by

$$C^c_{\boldsymbol{x},[l,l]} := \overline{C_{\boldsymbol{x},[l,l]}} - \sum_{[\ell]} \sum_{\substack{\boldsymbol{y} \in \Gamma_\ell \backslash \mathcal{L}_{n,\ell} \\ \boldsymbol{y} = \gamma \boldsymbol{x},\, \gamma \in \Gamma}} A^{\ell}_{\boldsymbol{y},[l,l]}$$

and hence for $n \in \mathbb{Q}$ positive, similarly define C^c_n as

$$C^c_{n,[l,l]} = \overline{C_{n,[l,l]}} - \sum_{[\ell]} \sum_{\boldsymbol{y} \in \Gamma_\ell \backslash \mathcal{L}_{n,\ell}} A_{\boldsymbol{y},\ell}$$

For all positive $n \in \mathbb{Q}$, the compactified special cycles $C^c_{n,[l,l]}$ define homology classes

$$[C^c_n] \in H_2\left(\overline{X}^{BS}, \widetilde{\mathcal{H}^{l,l}(\mathcal{L})\left(\mathbb{Z}\left[\frac{1}{d_\Gamma}\right]\right)}\right) \simeq H_2\left(X, \widetilde{\mathcal{H}^{l,l}(\mathcal{L})\left(\mathbb{Z}\left[\frac{1}{d_\Gamma}\right]\right)}\right)$$

which are generically non-exact, and have denominator dividing the even integer $d_\Gamma := lcm_{[\ell]} 2d(\Gamma,\ell)$. These classes may be convergently integrated against non-compact cohomology classes $\eta \in H^2_{dR}(X, \widetilde{\mathcal{H}^{l,l}(V)})$, and when we specify $\eta \in H^2(X, \widetilde{\mathcal{H}^{l,l}(L)})$, the resulting integrals satisfy

$$\int_{C^c_{n,[l,l]}} \eta \in \frac{1}{3d_\Gamma}\mathbb{Z}.$$

Finally, for any fixed $\eta \in H^2_{dR}(X, \widetilde{\mathcal{H}^{l,l}(V)})$, the generating series

$$\sum_{n>0} \left[\int_{C^c_n} \eta\right] e^{2\pi i n \tau} \in M_{3+2l}(\Gamma(M))$$

is a holomorphic modular form of weight $3+2l$ and level M.

Proof. The proof of this (or indeed, the lack thereof, given our need to wait until we have further machinery to prove modularity) is identical to that of Theorem 4.2.6, except that we must note one more thing about the denominator of the pairing: we have defined L to be integral, and hence in particular the natural pairing between the homological and cohomological coefficients - using the inner product $(\ ,\ )$ on V - will produce a product of integers if we input lattice vectors. The factor of 3 appears for the same reason as stated in the proof of Theorem 4.2.6.

Hence, because we may write

$$3d_\Gamma C^c_{n,[l,l]} \in H_2\left(X, \widetilde{\mathcal{H}^{l,l}(\mathcal{L})}\right),$$

the pairing between this class and $\eta \in H^2(X, \widetilde{\mathcal{H}^{l,l}(\mathcal{L})})$ will be an integer. □

Chapter 5

The Weil Representation for Unitary Groups

In this section, we shall write in detail the action of the Weil representation of the dual pair $U(p,q) \times U(1,1)$. This will be the foundation stone for our construction of generalised Kudla-Millson forms in §6. In particular, we shall show how the Weil representation acts in both the Fock and Schrödinger models; in line with the existing literature, the former will be the model we perform most of our computation in - because it is homogeneous - whereas the latter model will give us the structure necessary to interpret the resulting objects as differential forms. This work will follow that existing in many places in the existing literature - to see an equivalent setup of the unitary case, one may examine e.g. [FH19, Appendix B]. We shall largely omit the computation in this section, as the results are only of interest to support our main arguments in §6.

5.1 The Fock Model of the Weil Representation

We start here by giving an abstract treatment of the Fock model of the Weil representation.

Let $\mathcal{W}$ be a real vector space of positive even dimension $2M$, equipped with a non-degenerate symplectic inner product

$$\langle\ ,\ \rangle : \mathcal{W} \otimes \mathcal{W} \to \mathbb{R}$$

and a positive-definite complex structure $J : \mathcal{W} \to \mathcal{W}$. We let $\{e_1, \ldots e_M,\ f_1, \ldots f_M\}$ be the symplectic basis of $\mathcal{W}$ with respect to $\langle\ ,\ \rangle$ and J such that:

(i) $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ and $\langle e_i, f_j \rangle = \delta_{ij}$.

(ii) $Je_i = f_i$ and $Jf_i = -e_i$.

In particular, condition (ii) tells us that J acts with eigenvalues $\pm i$ on $\mathcal{W}$. We now define the *right* vector space $\mathcal{W}_{\mathbb{C}} := \mathcal{W} \otimes \mathbb{C}$; this decomposes into the two complex symplectic eigenspaces

$$\mathcal{W}_{\mathbb{C}} = \mathcal{W}' \oplus \mathcal{W}'' \tag{5.1.1}$$

under the induced action of J (formally acting as $J \otimes 1$ on $\mathcal{W} \otimes \mathbb{C}$). One may check that appropriate symplectic bases over $\mathbb{C}$ are given by $\{g'_j = e_j - f_j i\}_{j=1}^{M}$ for the $-i$ eigenspace $\mathcal{W}'$, and $\{g''_j = e_j + f_j i\}_{j=1}^{M}$ for the $+i$ eigenspace $\mathcal{W}''$.

We let $\lambda \in \mathbb{C}\backslash\{0\}$ be some constant, and we hence define an action of $\mathcal{W}$ on $\mathcal{P}_{\mathbb{C}}^{M} := \mathbb{C}[z_1, \ldots, z_M]$ as follows

$$\rho_\lambda(g'_j) = 2i\lambda \frac{\partial}{\partial z_j}, \quad \rho_\lambda(g''_j) = z_j. \tag{5.1.2}$$

We wish to find an action of the associated symplectic Lie algebra $\mathfrak{sp}(\mathcal{W}_{\mathbb{C}})$ using ρ_λ. The symmetric vector product $\mathrm{Sym}^2(\mathcal{W})$ is by definition the quotient of $\mathcal{W} \otimes \mathcal{W}$ given by:

$$\mathrm{Sym}^2(\mathcal{W}) = \mathcal{W} \otimes \mathcal{W} / \langle a \otimes b - b \otimes a \rangle;$$

as is standard, we write

$$x \circ y = \frac{1}{2}(x \otimes y + y \otimes x) \in \mathrm{Sym}^2(\mathcal{W}). \tag{5.1.3}$$

One may check that the algebras $\mathrm{Sym}^2(\mathcal{W})$ and $\mathfrak{sp}(\mathcal{W})$ are isomorphic by writing

$$(x \circ y) \in \mathfrak{sp}(\mathcal{W}), \quad (x \circ y)(z) = \langle x, z \rangle y + \langle y, z \rangle x. \tag{5.1.4}$$

Using (5.1.3) and (5.1.4), we may write down the Weil representation.

Definition 5.1.1. The Weil representation of $\mathfrak{sp}(\mathcal{W}) \otimes \mathbb{C} = \mathfrak{sp}(\mathcal{W}_{\mathbb{C}})$ on $\mathcal{P}_{\mathbb{C}}^{M}$ with central character $\lambda \in \mathbb{C}\backslash\{0\}$ is written:

$$\omega_\lambda(x \circ y) = \frac{1}{2\lambda}\left(\rho_\lambda(x)\rho_\lambda(y) + \rho_\lambda(y)\rho_\lambda(x)\right) \tag{5.1.5}$$

This is the Fock model of the Weil representation, with character λ; we also write this as $\omega_\lambda^{\mathcal{F}}$ later on to refer to the Fock model.

5.2 The Fock Model of the Weil Representation for Unitary Dual Pairs

We now give the explicit action of the Weil representation when the symplectic space $\mathcal{W}$ is specified as a space representing the unitary dual pair $\mathfrak{u}(p,q)\times\mathfrak{u}(1,1)$. We shall write down the Lie algebras separately, form an action ρ_λ on the product of their relevant vector spaces, and hence use this to form a symplectic Weil representation which $\mathfrak{u}(p,q)$ and $\mathfrak{u}(1,1)$ act via inclusions.

We now invite the reader to temporarily amend the notation given in §2; namely, in that chapter, we used $\underline{V}$ to denote a vector space over k of signature $(2,1)$ with an inner product $(\, , \,)$, antilinear in the first place and linear in the second. We now let these objects denote a k-vector space of split signature (p,q) for p, $q>0$. We pick a basis of $V=\underline{V}\otimes_k\mathbb{C}$ given by orthonormal $\{v_\alpha\}_{\alpha=1}^{p}$ and $\{v_\mu\}_{\mu=p+1}^{p+q}$ such that

$$(v_\alpha, v_\alpha') = \delta_{\alpha\alpha'}, \quad (v_\mu, v_{\mu'}) = -\delta_{\mu\mu'}, \quad (v_\alpha, v_\mu) = 0. \tag{5.2.1}$$

We let $G=U(V)$, $\mathfrak{g}_0=\mathfrak{u}(V)$, and $\mathfrak{g}=\mathfrak{g}_0\otimes\mathbb{C}$. We may construct an $\mathbb{R}$-linear map $\phi_{V,\mathbb{R}}:\wedge^2_{\mathbb{R}}V\to\mathfrak{g}_0$ - which naturally generalises that in Definition 2.4.3 - by

$$\phi_{V,\mathbb{R}}(v_1\wedge v_2)(z) = (v_1,z)v_2-(v_2,z)v_1. \tag{5.2.2}$$

Indeed, one may check that $\phi_{V,\mathbb{R}}(v_1\wedge v_2)$ satisfies

$$(\phi_{V,\mathbb{R}}(v_1\wedge v_2)(z_1), z_2) = (z_1, \phi_{V,\mathbb{R}}(v_1\wedge v_2)^\dagger(z_2)) \tag{5.2.3}$$

for all vectors v_1, v_2, z_1, $z_2\in V$, so that this is a well-defined map. It is surjective, so we may without loss of generality write a generic element of $\mathfrak{g}_0$ as $v_1\wedge v_2$.

Definition 5.2.1. We write $\alpha_{r,s}\coloneqq v_r\wedge v_s$ and $\beta_{r,s}\coloneqq iv_r\wedge v_s\in\mathfrak{g}_0$; the Lie algebra $\mathfrak{g}_0$ decomposes as $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$, where:

$$\mathfrak{k}_0 = \{\alpha_{r,s},\ \beta_{r,s} \mid 1\le r,s\le p \text{ or } p+1\le r,s\le p+q\}$$
$$\mathfrak{p}_0 = \{\alpha_{r,s},\ \beta_{r,s} \mid 1\le r\le p,\ p+1\le s\le p+q\}$$

For a basis of $\mathfrak{g}$, we write $Z'_{r,s}=(\alpha_{r,s}-\beta_{r,s}i)/2$ and $Z''_{r,s}=(\alpha_{r,s}+\beta_{r,s}i)/2$, so that $\mathfrak{g}=\mathfrak{g}_0\otimes\mathbb{C}$ is spanned by all the $Z'_{r,s}$ and $Z''_{r,s}$ for all possible entries of r,s.

We let $\underline{W}$ be a 2-dimensional vector space over k of signature $(1,1)$, assuming as with $\underline{V}$ that the inner product is antilinear in the first variable and linear in the second. We give $W=\underline{W}\otimes_k\mathbb{C}$ the quasi-orthonormal basis $\{e_1,e_2\}$ such that

$$(e_1,e_1)=i, \quad (e_2,e_2)=-i, \quad (e_1,e_2)=0.$$

We let $\underline{G}'$ be the unitary group of $\underline{W}$, with real points G'. We hence may define the real vector space $\mathcal{W} := (V \otimes_{\mathbb{C}} W)_{\mathbb{R}}$, which has a symplectic form given by:

$$\langle v \otimes w, v' \otimes w' \rangle = \mathrm{Re}\left((v, v')_V (w, w')_W\right). \tag{5.2.4}$$

Analogously to (5.1.1), we split the complexification of W as follows: we let J_W be the natural complex structure on W such that $J_W(e_1) = -ie_1$ and $J_W(e_2) = ie_2$; then the right $\mathbb{C}$-vector space

$$W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C} = W' \oplus W''$$

splits into $\pm i$ eigenspaces. We may calculate, for example, that

$$J_W(w_1') = J_W(e_1) + J(ie_1)i = -ie_1 + e_1 i = (ie_1 i + e_1)i = w_1' i$$

and hence split into eigenbases:

$$W' = \{w_1' = e_1 + ie_1 i,\, w_2' = e_2 - ie_2 i\}, \quad W'' = \{w_1'' = e_1 - ie_1 i,\, w_2'' = e_2 + ie_2 i\} \tag{5.2.5}$$

We define the $\mathbb{R}$-linear map $\phi_{W,\mathbb{R}} : \mathrm{Sym}^2_{\mathbb{R}} W \to \mathfrak{g}_0'$:

$$\phi_{W,\mathbb{R}}(w_1 \circ w_2)(z) = (w_1, z)w_2 + (w_2, z)w_1$$

Entirely analogously to (5.2.3), one may check that this map is well-defined and surjective; hence, we may without loss of generality refer to elements of $\mathfrak{u}(W) = \mathfrak{g}_0'$ as $w_1 \circ w_2$.

Definition 5.2.2. The complexified Lie algebra $\mathfrak{g}'$ decomposes as $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ where

$$\mathfrak{k}' = \mathrm{span}_{\mathbb{C}}\{e_1 \circ e_1 + ie_1 \circ e_1 i,\, e_2 \circ e_2 + ie_2 \circ e_2 i\}$$
$$\mathfrak{p}' = \mathrm{span}_{\mathbb{C}}\{e_1 \circ e_2 - ie_1 \circ e_2 i,\, e_1 \circ e_2 + ie_1 \circ e_2 i\}$$

The isomorphism $\mathfrak{su}(W) \simeq \mathfrak{sl}_2(\mathbb{R})$ may be realised by changing basis in W to $\{e_1 + e_2,\, -ie_1 + ie_2\}$. This Lie algebra splits into

$$\mathfrak{su}(W) = \mathfrak{k}' \cap \{\mathrm{tr}(X) = 0\} + \mathfrak{p}',$$

and we may split $\mathfrak{p}'$ into $\mathfrak{p}' = \mathfrak{p}'^{+} + \mathfrak{p}'^{-}$ spanned by operators giving rise to the classical Maass raising and lowering operators respectively:

$$\mathfrak{p}'^{+} = \mathrm{span}_{\mathbb{C}}\left\{\frac{-i}{2}(e_1 \circ e_2 - ie_1 \circ e_2 i)\right\} = \mathrm{span}_{\mathbb{C}}\left\{R := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\right\}$$
$$\mathfrak{p}'^{-} = \mathrm{span}_{\mathbb{C}}\left\{\frac{-i}{2}(e_1 \circ e_2 + ie_1 \circ e_2 i)\right\} = \mathrm{span}_{\mathbb{C}}\left\{L := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\right\}$$

The groups G and G' embed in $\mathrm{Sp}(\mathcal{W})$, forming a *dual reductive pair* as in [How77]; equivalently, $\mathfrak{g}$ and $\mathfrak{g}' = \mathfrak{su}(W)\otimes\mathbb{C}$ form a dual pair in $\mathfrak{sp}(\mathcal{W}_{\mathbb{C}})$. We wish to construct a Weil representation of this algebra; in order to do so, we must find a symplectic basis of the complex space $\mathcal{W}_{\mathbb{C}} = V \otimes W \otimes \mathbb{C}$. Using the inner product (5.2.4) of $\mathcal{W}$, we have a naturally occuring Lagrangian basis:

$$\begin{aligned}&\{v_\alpha \otimes e_1, \quad v_\alpha \otimes e_2,\ v_\mu \otimes e_1,\ v_\mu \otimes e_2\}\\ &\{v_\alpha \otimes -ie_1,\ v_\alpha \otimes ie_2,\ v_\mu \otimes ie_1,\ v_\mu \otimes -ie_2\},\end{aligned} \tag{5.2.6}$$

where we have notated (5.2.6) to emphasise the pairing as with the $\{e_j,\ f_j\}$ in §5.1. With κ the Cartan involution on V induced by the maximally compact subgroup $U(p)\times U(q)$ of G, written

$$\kappa(v_\alpha) = v_\alpha, \quad \kappa(v_\mu) = -v_\mu,$$

then $\mathcal{W}$ has an inherited complex structure of the form $J = \kappa \otimes J_W$. we may split $\mathcal{W}_{\mathbb{C}} = \mathcal{W} \otimes_{\mathbb{R}} \mathbb{C}$ into $\pm i$ eigenspaces for J using the complex basis for $W_{\mathbb{C}}$ in (5.2.5) in the natural way:

$$\mathcal{W}' = \mathrm{span}_{\mathbb{C}}\{v_\alpha \otimes w_1',\ v_\alpha \otimes w_2',\ v_\mu \otimes w_1'',\ v_\mu \otimes w_2''\} \tag{5.2.7}$$

$$\mathcal{W}'' = \mathrm{span}_{\mathbb{C}}\{v_\alpha \otimes w_1'',\ v_\alpha \otimes w_2'',\ v_\mu \otimes w_1',\ v_\mu \otimes w_2'\}. \tag{5.2.8}$$

Hence, because of our construction of a proper symplectic basis in (5.2.7) and (5.2.8), we may write down a Weil representation of $\mathfrak{sp}(\mathcal{W})\otimes\mathbb{C}$. In the Fock model, we know that $\mathfrak{sp}(\mathcal{W})\otimes\mathbb{C}$ will act on $\mathrm{Sym}^\bullet(\mathcal{W}'')$. We denote the variables in our polynomial space by $\{z_{rs}',\ z_{rs}''\}_{1\le r\le p+q,\,1\le s\le 2}$. We may naturally identify $\mathrm{Sym}^\bullet(\mathcal{W}'')$ with a space of complex polynomials in $2(p+q)$ variables as follows:

$$\begin{aligned}v_\alpha \otimes w_1'' &\leftrightarrow z_\alpha'',\\ v_\alpha \otimes w_2'' &\leftrightarrow z_\alpha',\\ v_\mu \otimes w_1' &\leftrightarrow z_\mu',\\ v_\mu \otimes w_2' &\leftrightarrow z_\mu''.\end{aligned} \tag{5.2.9}$$

From this, we can use (5.1.2) to write down the action of $\mathcal{W}$:

$$\begin{aligned}\rho_\lambda(v_\alpha \otimes w_1') &= 2i\lambda\frac{\partial}{\partial z_\alpha''}, &\qquad \rho_\lambda(v_\mu \otimes w_1') &= z_\mu',\\ \rho_\lambda(v_\alpha \otimes w_1'') &= z_\alpha'', & \rho_\lambda(v_\mu \otimes w_1'') &= 2i\lambda\frac{\partial}{\partial z_\mu'},\\ \rho_\lambda(v_\alpha \otimes w_2') &= 2i\lambda\frac{\partial}{\partial z_\alpha'}, & \rho_\lambda(v_\mu \otimes w_2') &= z_\mu'',\end{aligned}$$

$$\rho_\lambda(v_\alpha \otimes w_2'') = z_\alpha', \qquad \rho_\lambda(v_\mu \otimes w_2'') = 2i\lambda \frac{\partial}{\partial z_\mu''} \tag{5.2.10}$$

We may now write down the inclusions of the subalgebras $\mathfrak{g}_0$ and $\mathfrak{g}_0'$ into $\mathfrak{sp}(\mathcal{W})$. By definition of the wedge product $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1$, we may write the inclusion $j_V : \mathfrak{g}_0 \to \mathfrak{sp}(\mathcal{W})$ as

$$\begin{aligned} j_V(v_1 \wedge v_2) = &[(v_1 \otimes iw_1) \circ (v_2 \otimes w_1) - (v_1 \otimes w_1) \circ (v_2 \otimes iw_1)] \\ &- [(v_1 \otimes iw_2) \circ (v_2 \otimes w_2) - (v_1 \otimes w_2) \circ (v_2 \otimes iw_2)] \,. \end{aligned}$$

Similarly, using the form of the symmetric product, we may write the inclusion j_W of $\mathfrak{g}_0'$ as:

$$\begin{aligned} j_W(w_1 \circ w_2) = &\sum_{\alpha=1}^{p} [(v_\alpha \otimes w_1) \circ (v_\alpha \otimes w_2) + (iv_\alpha \otimes w_1) \circ (iv_\alpha \otimes w_2)] \\ &- \sum_{\mu=p+1}^{p+q} [(v_\mu \otimes w_1) \circ (v_\mu \otimes w_2) + (iv_\mu \otimes w_1) \circ (iv_\mu \otimes w_2)] \,. \end{aligned}$$

We hence may extend these maps to inclusions $j_{V,\mathbb{C}} : \mathfrak{g} \to \mathfrak{sp}(\mathcal{W}_\mathbb{C})$ and $j_{W,\mathbb{C}} : \mathfrak{g}' \to \mathfrak{sp}(\mathcal{W}_\mathbb{C})$ as follows:

$$\begin{aligned} j_{V,\mathbb{C}}(v_1 \wedge v_2 + (iv_1 \wedge v_2)i) &= -i(v_1 \otimes w_1') \circ (v_2 \otimes w_1'') + i(v_1 \otimes w_2'') \circ (v_2 \otimes w_2'), \\ j_{V,\mathbb{C}}(v_1 \wedge v_2 - (iv_1 \wedge v_2)i) &= i(v_1 \otimes w_1'') \circ (v_2 \otimes w_1') - i(v_1 \otimes w_2') \circ (v_2 \otimes w_2''), \end{aligned}$$

and

$$\begin{aligned} j_{W,\mathbb{C}}(w_1 \circ w_2 + (iw_1 \circ w_2)i) = &\sum_{\alpha=1}^{p} [(v_\alpha \otimes (w_1 + iw_1 i)) \circ (v_\alpha \otimes (w_2 - iw_2 i))] \\ &- \sum_{\mu=p+1}^{p+q} [(v_\mu \otimes (w_1 + iw_1 i)) \circ (v_\mu \otimes (w_2 - iw_2 i))] \,, \\ j_{W,\mathbb{C}}(w_1 \circ w_2 - (iw_1 \circ w_2)i) = &\sum_{\alpha=1}^{p} [(v_\alpha \otimes (w_1 - iw_1 i)) \circ (v_\alpha \otimes (w_2 + iw_2 i))] \\ &- \sum_{\mu=p+1}^{p+q} [(v_\mu \otimes (w_1 - iw_1 i)) \circ (v_\mu \otimes (w_2 + iw_2 i))] \,. \end{aligned}$$

So, we are now in a position where we may write down the action of the subalgebras $\mathfrak{g}$, $\mathfrak{g}'$ through the Fock model of the Weil representation; we shall write for e.g. $\omega_\lambda(Z_{r,s}') \equiv \omega_\lambda(j_{V,\mathbb{C}}(Z_{r,s}'))$. When we write $\mathcal{P}_\mathbb{C}^{2(p+q)}$ in the following lemma, we mean - in the notation of (5.2.9) - the complex polynomials in the variables z_α'', z_μ', z_α' and z_μ'' for $1 \le \alpha \le p$ and $p+1 \le \mu \le p+q$.

Lemma 5.2.3. *For the remainder of this lemma, we let numbers between* 1 *and* p *be represented by the indices* α *and* β*, and numbers between* $p+1$ *and* $p+q$ *be* *represented by* μ *and* ν.

The basis elements $Z'_{\alpha,\beta}$ *and* $Z''_{\alpha,\beta}$ *of* $\mathfrak{u}(V) \subset \mathfrak{k}$ *(defined in Definition 5.2.1) act on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *as follows:*

$$\omega_\lambda(Z'_{\alpha,\beta}) = -z''_\alpha \frac{\partial}{\partial z''_\beta} + z'_\beta \frac{\partial}{\partial z'_\alpha} = -\omega_\lambda(Z''_{\alpha,\beta});$$

further, the basis elements $Z'_{\mu,\nu}$ *and* $Z''_{\mu,\nu}$ *of* $\mathfrak{u}(W) \subset \mathfrak{k}$ *act on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *as follows:*

$$\omega_\lambda(Z'_{\mu,\nu}) = -z'_\nu \frac{\partial}{\partial z'_\mu} + z''_\mu \frac{\partial}{\partial z''_\nu} = -\omega_\lambda(Z''_{\mu,\nu}).$$

The basis elements $Z'_{\alpha,\mu}$ *of* $\mathfrak{p}^+$ *act on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *as*

$$\omega_\lambda(Z'_{\alpha,\mu}) = -\frac{1}{2i\lambda} z''_\alpha z'_\mu + 2i\lambda \frac{\partial^2}{\partial z'_\alpha \partial z''_\mu}$$

and the basis elements $Z''_{\alpha,\mu}$ *of* $\mathfrak{p}^-$ *act on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *as*

$$\omega_\lambda(Z''_{\alpha,\mu}) = -\frac{1}{2i\lambda} z'_\alpha z''_\mu + 2i\lambda \frac{\partial^2}{\partial z''_\alpha \partial z'_\mu}.$$

Using the same strategy, we may write down the action of the bases of $\mathfrak{k}'$ *and* $\mathfrak{p}'$*: the action of* $e_r \circ e_r + ie_r \circ e_r i$ *for* $r = 1, 2$ *on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *is given by*

$$\omega_\lambda(e_1 \circ e_1 + ie_1 \circ e_1 i) = 2i \left[\sum_{\alpha=1}^{p} z''_\alpha \frac{\partial}{\partial z''_\alpha} - \sum_{\mu=p+1}^{p+q} z'_\mu \frac{\partial}{\partial z'_\mu} \right] + i(p-q),$$

and

$$\omega_\lambda(e_2 \circ e_2 + ie_2 \circ e_2 i) = 2i \left[\sum_{\alpha=1}^{p} z'_\alpha \frac{\partial}{\partial z'_\alpha} - \sum_{\mu=p+1}^{p+q} z''_\mu \frac{\partial}{\partial z''_\mu} \right] + i(p-q).$$

Finally, the action of $\mathfrak{p}'^+$ *on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *is given by*

$$\omega_\lambda(e_1 \circ e_2 - ie_1 \circ e_2 i) = \frac{1}{\lambda} \sum_{\alpha=1}^{p} z''_{\alpha 1} z'_{\alpha 2} + 4\lambda \sum_{\mu=p+1}^{p+q} \frac{\partial^2}{\partial z'_{\mu 1} \partial z''_{\mu 2}},$$

and the action of $\mathfrak{p}'^-$ *on* $\mathcal{P}_{\mathbb{C}}^{2(p+q)}$ *is given by*

$$\omega_\lambda(e_1 \circ e_2 + ie_1 \circ e_2 i) = -4\lambda \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z''_{\alpha 1} \partial z'_{\alpha 2}} - \frac{1}{\lambda} \sum_{\mu=p+1}^{p+q} z'_{\mu 1} z''_{\mu 2}.$$

Proof. The proof of all of these identities uses the formulae for the inclusions $j_{V,\mathbb{C}}$ and $j_{W,\mathbb{C}}$, the definition of the Weil representation in the Fock model in (5.1.5) and then the formulae in (5.2.10). For example, for the action of $Z'_{\alpha,\mu}$, we may write

$$\begin{aligned} j_{V,\mathbb{C}}(Z'_{\alpha,\mu}) &= \frac{1}{2} j_{V,\mathbb{C}}(v_\alpha \wedge v_\mu - (iv_\alpha \wedge v_\mu)i) \\ &= \frac{i}{2}\left((v_\alpha \otimes w''_1) \circ (v_\mu \otimes w'_1) - (v_\alpha \otimes w'_2) \circ (v_\mu \otimes w''_2)\right). \end{aligned}$$

Hence, we may write the action of the Weil representation as

$$\begin{aligned} \omega_\lambda(Z'_{\alpha,\mu}) &= \frac{i}{4\lambda}\big[\rho_\lambda(v_\alpha \otimes w''_1)\rho_\lambda(v_\mu \otimes w'_1) + \rho_\lambda(v_\mu \otimes w'_1)\rho_\lambda(v_\alpha \otimes w''_1) \\ &\quad - \rho_\lambda(v_\alpha \otimes w'_2)\rho_\lambda(v_\mu \otimes w''_2) - \rho_\lambda(v_\mu \otimes w''_2)\rho_\lambda(v_\alpha \otimes w'_2)\big] \\ &= \frac{i}{4\lambda}\left[z''_\alpha z'_\mu + z'_\mu z''_\alpha - \left(2i\lambda\frac{\partial}{\partial z'_\alpha}\right)\left(2i\lambda\frac{\partial}{\partial z''_\mu}\right) - \left(2i\lambda\frac{\partial}{\partial z''_\mu}\right)\left(2i\lambda\frac{\partial}{\partial z'_\alpha}\right)\right] \\ &= -\frac{1}{2i\lambda} z''_\alpha z'_\mu + 2i\lambda\frac{\partial^2}{\partial z'_\alpha \partial z''_\mu} \end{aligned}$$

and the reader may check that this is exactly the prescribed formula in the statement of the theorem. The rest are proven in exactly the same way. □

5.3 The Schrödinger Model, Intertwiners and the Mixed Model

We now give a summary of the Schrödinger model of the Weil representation. For notational convenience, we let $\mathcal{F} \equiv \mathcal{F}^{p+q} = \mathbb{C}[z'_\alpha, z''_\alpha, z'_\mu, z''_\mu]$ be the space of polynomials used in the Fock model in §5.2.

Definition 5.3.1. (i) We write vectors in V with respect to the orthonormal co-ordinates (5.2.1) as

$$\mathbf{x} = \sum_\alpha z_\alpha v_\alpha + \sum_\mu z_\mu v_\mu. \tag{5.3.1}$$

(ii) We let $\mathbb{S}(V)$ be the space of Schwartz functions on V:

$$\mathbb{S}(V) = \{f : V \mapsto \mathbb{C} \mid \forall \text{ multi-indices } \beta_1, \beta_2,\ \sup_{\mathbf{x}}|\partial_{\beta_1}\mathbf{x}^{\beta_2} f(\mathbf{x})| \text{ and } \sup_{\mathbf{x}}|\overline{\partial_{\beta_1}}\mathbf{x}^{\beta_2} f(\mathbf{x})| < \infty\};$$

we may think of this as the collection of functions that decay faster than any power of the monomials z_i, $\overline{z}_i$. We define the principal majorant for V as follows:

$$(\mathbf{x},\mathbf{x})_0 = \begin{cases} (\mathbf{x},\mathbf{x}) & \text{if } \mathbf{x} \notin \mathrm{Span}_\mathbb{C}\{v_\mu\}_{\mu=p+1}^{p+q}, \\ -(\mathbf{x},\mathbf{x}) & \text{if } \mathbf{x} \in \mathrm{Span}_\mathbb{C}\{v_\mu\}_{\mu=p+1}^{p+q} \end{cases} \tag{5.3.2}$$

This defines a positive-definite inner product on V - we will see more on such objects in §6.1. With respect to the basis from (5.3.1) , we have

$$(\mathbf{x},\mathbf{x})_0 = \sum_\alpha |z_\alpha|^2 + \sum_\mu |z_\mu|^2.$$

We let $\varphi_0(\mathbf{x}) \coloneqq \exp(-\pi(\mathbf{x},\mathbf{x})_0)$ be the standard Gaussian on V, and hence let

$$\mathcal{S}(V) = \{f(\mathbf{x}) = p(\mathbf{x})\varphi_0(\mathbf{x}) \mid p \in \mathbb{C}[z_i, \overline{z_i}]\} \subset S(V)$$

be the subset of the Schwartz space spanned by products of the Gaussian with complex polynomials in the coefficients.

We first note that it is clear that $\mathcal{S}(V) \subset \mathbb{S}(V)$; indeed, by elementary analysis, we know that $\exp(-x^2)$ dominates any polynomial in x, so all elements of $\mathcal{S}(V)$ have a normed supremum, and hence are elements of $\mathbb{S}(V)$.

Secondly, we note that $\mathcal{F}$ is algebraically isomorphic to $\mathcal{S}(V)$. Analogously to (5.2.9), we may write down an action of $\mathcal{W}$ on $\mathcal{S}(V)$ which will lead to a Weil representation. We now specify to the central character $\lambda = 2\pi i$, and remove this from the notation - we instead use a subscript $\mathcal{S}$ to specify the Schrödinger model, so that $\rho_{\mathcal{S}} \equiv \rho_{2\pi i,\mathcal{S}}$.

$$\begin{aligned}
\rho_{\mathcal{S}}(v_\alpha \otimes w_1') &= \sqrt{2}\pi i\left(z_\alpha + \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right), & \rho_{\mathcal{S}}(v_\mu \otimes w_1') &= -\sqrt{2}\pi i\left(z_\mu - \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\mu}\right),\\
\rho_{\mathcal{S}}(v_\alpha \otimes w_1'') &= \sqrt{2}\pi i\left(\overline{z}_\alpha - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right), & \rho_{\mathcal{S}}(v_\mu \otimes w_1'') &= -\sqrt{2}\pi i\left(\overline{z}_\mu + \frac{1}{\pi}\frac{\partial}{\partial z_\mu}\right),\\
\rho_{\mathcal{S}}(v_\alpha \otimes w_2') &= \sqrt{2}\pi i\left(z_\alpha + \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right), & \rho_{\mathcal{S}}(v_\mu \otimes w_2') &= -\sqrt{2}\pi i\left(\overline{z}_\mu - \frac{1}{\pi}\frac{\partial}{\partial z_\mu}\right),\\
\rho_{\mathcal{S}}(v_\alpha \otimes w_2'') &= \sqrt{2}\pi i\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right), & \rho_{\mathcal{S}}(v_\mu \otimes w_2'') &= -\sqrt{2}\pi i\left(z_\mu + \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\mu}\right).
\end{aligned} \tag{5.3.3}$$

By examining how each of these act on the Gaussian φ_0, we may construct an intertwiner between $\mathcal{F}$ and $\mathcal{S}(V)$; for example, $\overline{z}_\alpha - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}(\varphi_0) = 2\overline{z}_\alpha\varphi_0$, so we wish to relate this operator to the one resulting from $v_\alpha \otimes w_1''$ in the Fock model - this is z_α''. Hence, following this logic, we may write down the intertwiner:

Lemma 5.3.2. *There is a unique $\mathfrak{sp}(\mathcal{W}_\mathbb{C})$-intertwiner $\mathcal{J} : \mathcal{F} \to \mathcal{S}(V)$ satisfying $\mathcal{J}(1) = \varphi_0$, and the intertwiner satisfies*

$$\begin{aligned}
\mathcal{J} z_\alpha' \mathcal{J}^{-1} &= \sqrt{2}\pi i\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right), & \mathcal{J}\frac{\partial}{\partial z_\alpha''}\mathcal{J}^{-1} &= \frac{1}{2\sqrt{2}i}\left(z_\alpha + \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right)\\
\mathcal{J} z_\alpha'' \mathcal{J}^{-1} &= \sqrt{2}\pi i\left(\overline{z}_\alpha - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right), & \mathcal{J}\frac{\partial}{\partial z_\alpha'}\mathcal{J}^{-1} &= \frac{1}{2\sqrt{2}i}\left(\overline{z}_\alpha + \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right)\\
\mathcal{J} z_\mu' \mathcal{J}^{-1} &= -\sqrt{2}\pi i\left(z_\mu - \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\mu}\right), & \mathcal{J}\frac{\partial}{\partial z_\mu''}\mathcal{J}^{-1} &= \frac{-1}{2\sqrt{2}i}\left(z_\mu + \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\mu}\right)
\end{aligned}$$

$$\mathcal{J} z''_\mu \mathcal{J}^{-1} = -\sqrt{2}\pi i \left(\overline{z}_\mu - \frac{1}{\pi}\frac{\partial}{\partial z_\mu}\right), \; \mathcal{J}\frac{\partial}{\partial z'_\mu}\mathcal{J}^{-1} = \frac{-1}{2\sqrt{2}i}\left(\overline{z}_\mu + \frac{1}{\pi}\frac{\partial}{\partial z_\mu}\right)$$

Hence, herein we shall always refer to $\mathcal{S}(V)$ for the representation $\omega_{\mathcal{S}}$ in the Schrödinger model for $\lambda = 2\pi i$.

Proof. The proof that the intertwiner exists and is unique is proven in [KM86, §2]; there it is formulated in terms of the quantum algebra generated by $\mathcal{W}$, but for our purposes the statement above suffices. For the calculation of the intertwiner, one only needs to use the uniqueness and then compare how the elements acts on the elements 1 and φ_0 in $\mathcal{F}$ and $\mathcal{S}(V)$ respectively. □

In §6, we shall use the Weil representations to create operators ∇ such that $\nabla\varphi_0$ is a non-trivial form in complexes $[\mathcal{S}(V) \otimes \wedge^\bullet \mathfrak{p}^* \otimes E]^K$ and $\nabla\varphi_0$ is a closed form!

We now introduce one more model for the Weil representation - the mixed model of the Weil representation. For V here still assumed to be of arbitrary signature (p, q), we fix $E \subset V$ some non-trivial, totally isotropic and maximal vector subspace (so that $(\mathbf{x}, \mathbf{x}) = 0$ for all $\mathbf{x} \in E$). Our reason for prescribing maximality is that the parabolic subgroups of $SU(V)$ are classified by flags

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_k$$

of complex parabolic subspaces; hence, choosing a maximal isotropic subspace is equivalent to choosing a conjugacy class of maximal parabolic subgroups. Then we may define a conjugate isotropic subspace, denoted E', as follows:

$$E' = \{v \in V \mid (v, e)_0 = 0 \text{ for all } e \in E^\perp\} = \left(E^\perp\right)^{\perp_0},$$

where $\perp$ is the perpendicular space with respect to $(\,,\,)$ and $\perp_0$ the perpendicular space with respect to $(\,,\,)_0$. We note that we may identify E' naturally with the dual of E - namely, they are of the same real dimension, and we may pick bases $\{e_i\}_{i=1}^r$ of E and $\{e'_j\}_{j=1}^r$ such that $(e_i, e'_j) = \delta_{ij}$. Then we may define $W_E = E^\perp/E$ as a non-degenerate space of signature $(p - \dim(E), q - \dim(E))$, and a *Witt splitting*:

$$V = E \oplus W_E \oplus E'.$$

Definition 5.3.3. The mixed model of the Weil representation is on the space

$$\mathcal{S}(E^*) \otimes \mathcal{S}(W_E) \otimes \mathcal{S}(E') \simeq \mathcal{S}(E') \otimes \mathcal{S}(W_E) \otimes \mathcal{S}(E')$$

where we have initially used the isomorphism between E' and E^*. We may derive it as an isomorphism from $\mathcal{S}(V)$ as follows:

$$\begin{aligned} \mathcal{S}(V) &\to \mathcal{S}\left(E'\right) \otimes \mathcal{S}\left(W_E\right) \otimes \mathcal{S}\left(E'\right), \\ \phi &\to \widehat{\phi} \end{aligned}$$

where $\hat{\phi}$ is the Fourier transform, given by

$$\widehat{\phi}(u_1', w, u_2') = \int_E \phi(u, w, u_2')\exp(-2\pi i(u, u_1'))\mathrm{d}u$$

for u_1', u_2' co-ordinates on E' and w a co-ordinate on W_E.

For the sake of brevity, we denote the mixed model with respect to a subspace E as $\mathcal{S}(V)_E^{MM}$; here we understand the complementary subspace E' to be fixed.

We first note that it is well known that the Fourier transform takes Schwartz forms to Schwartz forms, and in particular will take polynomial Schwartz forms to polynomial Schwartz forms (this is implicitly proven later on for our specific choice of V of signature $(2,1)$, in e.g. Lemma 7.1.3). One may easily check that this is a G-equivariant map, so it follows that this is indeed an intertwiner. We will not need to do this at this point, but one may e.g. write down the action of G on $\mathcal{S}\left(E'\right) \otimes \mathcal{S}\left(W_E\right) \otimes \mathcal{S}\left(E'\right)$ - see e.g. [FM13, §4.2].

Chapter 6

A Generalisation of Kudla-Millson's Schwartz Form To Complex Harmonic Coefficients

In this chapter, we shall look to fully generalise the construction of G-invariant Schwartz forms for G a special unitary group of signature $(2,1)$. More specifically, for $\mathcal{H}^{l,l}(V)$ an irreducible representation of $SU(2,1)$ as we constructed in §3.1, we will use this chapter to show that one may construct a Schwartz form

$$\varphi_{l,l} \in [\mathcal{S}(V) \otimes \Omega^2(\mathbb{D}) \otimes \mathcal{H}^{l,l}(V)]^G$$

which is closed and of weight $2l+3$.

Further, it will give us a theta series $\theta_{\mathcal{L}}(\varphi_{l,l}, \tau)$ which is holomorphic as a cohomology class, whose Fourier coefficients are given by the duals of the special cycle $C_{\mathbf{x},[l,l]}$ from Proposition 4.1.3.

The work in this chapter will draw on the algebraic arguments in [FM06], and in particular will make heavy use of the Fock model calculations from §5.2.

6.1 The Kudla-Millson Schwartz Form

We start here by giving some detail on the construction of the Schwartz forms by Kudla and Millson in their papers [KM86] and [KM87]; these shall be referred to hereon as "Kudla-Millson Schwartz forms", and notated as φ_{KM}.

We start with a definition of the complexes that we shall work in. We recall the polynomial Schwartz space $\mathcal{S}(V)$ and the polynomial Fock space $\mathcal{F}(V)$ from Definition 5.3.1; in particular, we shall stress that we are now working with our particular

V of signature $(2,1)$, so that the notation of §5 may be carried over unchanged with this specialisation.

Definition 6.1.1. Let $G \simeq SU(2,1)$ be the real points of the special unitary group of V; let $\mathfrak{g}$ be the complexification of its Lie algebra $\mathfrak{g}_0$ as in Definition 5.2.1, which decomposes into $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathcal{A}$ be either of the spaces $\mathcal{S}(V)$ or $\mathcal{F}$ with associated Weil representation $\omega_{\mathcal{A}}$, and hence define the two complexes we shall work in:

$$[\mathcal{A} \otimes \wedge^{\bullet}\mathfrak{p}^*]^K, \qquad [\mathcal{A} \otimes \Omega^{\bullet}(\mathbb{D})]^G.$$

Here the group K (resp. G) acts on $\mathcal{A}$ via the Weil representation and on the wedge product (resp. the differential forms on $\mathbb{D}$) via the canonical left matrix actions respectively. Hence the notation $]^K$ (resp. $]^G$) refers to the set of K-invariants (resp. G-invariants) in these sets.

We now cite a result which we shall use throughout the thesis.

Proposition 6.1.2. *For all the objects defined in Definition 6.1.1, and E any finite-dimensional representation of the group G, we have:*

$$\pi : [\mathcal{S}(V) \otimes \wedge^{\bullet}\mathfrak{p}^* \otimes E]^K \xrightarrow{\simeq} [\mathcal{S}(V) \otimes \Omega^{\bullet}(\mathbb{D}) \otimes E]^G$$

where the isomorphism is canonically given by evaluating the right-hand side at the basepoint of $\mathbb{D}$.

Proof. See the discussion in [KM87, §3]. □

We note that this fact is generically true for all finite-dimensional split Lie groups G - indeed, this idea is central to the construction of the Kudla-Millson forms in full generality, as it allows the authors to work only with the Lie algebra differentials. We may hence write down the differentials in these complexes:

Definition 6.1.3. For $\lambda^* \in \mathfrak{p}^*$, let $A(\lambda^*) : \wedge^{\bullet}\mathfrak{p}^* \to \wedge^{\bullet+1}\mathfrak{p}^*$ be the wedge product on the left with λ^*. We fix $\{\lambda\}$ to be some complex basis of $\mathfrak{p}$.

Let $\mathcal{B}$ be some representation of G acting with ν_B, and as in [BW00, §1], we define the differential in the complex $[\mathcal{B} \otimes \wedge^{\bullet}\mathfrak{p}^*]^K$ as

$$\mathrm{d}_{\mathcal{B}} := \sum_{\lambda} \nu_{\mathcal{B}}(\lambda) \otimes A(\lambda^*).$$

Lemma 6.1.4. *For any such $\mathcal{B}$, the differential satisfies $\mathrm{d}_{\mathcal{B}}^2 = 0$ and $\mathrm{d}_{\mathcal{B}}$ preserves the K-invariance.*

So, we now may write down the Kudla-Millson Schwartz form.

Definition 6.1.5. With the orthonormal co-ordinates $\{z_1, z_2, z_3\}$ corresponding to the basis $\{v_1, v_2, v_3\}$, we define the following operators:

$$\nabla^{\mathcal{S}} = \frac{1}{2}\sum_{\alpha=1}^{2}\left[\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right)\otimes A\left(\overline{\xi_\alpha}\right)\right] : [\mathcal{S}(V)\otimes \wedge^\bullet \mathfrak{p}^*]^K \to \left[\mathcal{S}(V)\otimes \wedge^{\bullet+1}\mathfrak{p}^*\right]^K,$$
$$\overline{\nabla}^{\mathcal{S}} = \frac{1}{2}\sum_{\alpha=1}^{2}\left[\left(\overline{z}_\alpha - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right)\otimes A(\xi_\alpha)\right] : [\mathcal{S}(V)\otimes \wedge^\bullet \mathfrak{p}^*]^K \to \left[\mathcal{S}(V)\otimes \wedge^{\bullet+1}\mathfrak{p}^*\right]^K;$$

in the literature - see e.g. [KM86] - these are known as *Howe operators*, so we shall follow this tradition. Let $\varphi_0 \in \mathcal{S}(V)$ be the standard Gaussian on V from Definition 5.3.1:

$$\varphi_0(\mathbf{x}) = e^{-\pi(\mathbf{x},\mathbf{x})_0}.$$

Then the Kudla-Millson Schwartz form $\varphi_{KM}(\mathbf{x})$ is defined by

$$\varphi_{KM}(\mathbf{x}) = \left(\overline{\nabla}^{\mathcal{S}}\circ \nabla^{\mathcal{S}}\circ\varphi_0\right)(\mathbf{x}) \in \left[\mathcal{S}(V)\otimes\wedge^2\mathfrak{p}^*\right]^K.$$

We shall start with the properties necessary for showing the main parts of the Kudla-Millson result; we then shall examine how this may be developed into a suitable cohomological object. Using the notation of Definition 5.2.1, we recall the chosen basis of $\mathfrak{p}$:

$$Z'_r \equiv Z'_{r,3} = \frac{1}{2}\left(\alpha_{r,3} - \beta_{r,3}i\right), \quad Z''_r \equiv Z''_{r,3} = \frac{1}{2}\left(\alpha_{r,3} + \beta_{r,3}i\right) \tag{6.1.1}$$

for $r = 1, 2$, and similarly the basis $\mathfrak{p}^*$:

$$\xi_\alpha = \left(Z'_{\alpha,3}\right)^*, \quad \overline{\xi_\alpha} = \left(Z''_{\alpha,3}\right)^*.$$

Proposition 6.1.6 (Kudla & Millson, 1987)**.** *$\varphi_{KM}(\boldsymbol{x})$ is closed, and has weight 3 with respect to the action of $\mathfrak{k}'$.*

Proof. Throughout, we shall work in the Fock complex. Using the intertwiner from Lemma 5.3.2, we may rewrite the ∇ operators in the Fock model as:

$$\nabla^{\mathcal{F}} = \frac{-i}{2\sqrt{2}\pi}\sum_{\beta=1}^{2} z'_\beta \otimes A\left(\overline{\xi_\beta}\right),$$
$$\overline{\nabla}^{\mathcal{F}} = \frac{-i}{2\sqrt{2}\pi}\sum_{\beta=1}^{2} z''_\beta \otimes A\left(\xi_\beta\right).$$

Hence, utilising the isomorphism between the Fock and Schrödinger models, we will equivalently prove that the form

$$\varphi_{KM}^{\mathcal{F}} = \left(\frac{-i}{2\sqrt{2}\pi}\right)^2 \sum_{\beta,\beta'=1}^{2} z'_\beta z''_{\beta'} \otimes \xi_{\beta'} \wedge \overline{\xi_\beta} \in \left[\mathcal{F} \otimes \wedge^2 \mathfrak{p}^*\right]^K$$

is closed. Using the definition of the action of $\mathfrak{p}$ in the Fock model from Lemma 5.2.3, we may use the definition of the differential in Definition 6.1.3, use the basis $\{Z'_\alpha,\, Z''_\alpha\}_{\alpha=1}^2$ of $\mathfrak{p}$, and hence write the differential $\mathrm{d}_{\mathcal{F}} = \mathrm{d}_{\mathcal{F}}^+ + \mathrm{d}_{\mathcal{F}}^-$ for the complex $[\mathcal{F} \otimes \wedge^\bullet \mathfrak{p}^*]^K$:

$$\begin{aligned} \mathrm{d}_{\mathcal{F}}^+ &= \frac{1}{4\pi} \sum_{\alpha=1}^{2} \left(z''_\alpha z'_3 \otimes A(\xi_\alpha) + z'_\alpha z''_3 \otimes A\left(\overline{\xi_\alpha}\right)\right), \\ \mathrm{d}_{\mathcal{F}}^- &= -4\pi \sum_{\alpha=1}^{2} \left(\frac{\partial^2}{\partial z'_\alpha \partial z''_3} \otimes A(\xi_\alpha) + \frac{\partial^2}{\partial z''_\alpha \partial z'_3} \otimes A\left(\overline{\xi_\alpha}\right)\right) \end{aligned} \tag{6.1.2}$$

Because $\varphi_{KM}^{\mathcal{F}}$ has no terms with z'_3 or z''_3 in, we see immediately that $\mathrm{d}_{\mathcal{F}}^- \varphi_{KM}^{\mathcal{F}} = 0$, so we only need prove that $\mathrm{d}_{\mathcal{F}}^+ \varphi_{KM}^{\mathcal{F}} = 0$. Because the polynomial terms act symmetrically and $\mathfrak{p}$ acts antisymmetrically, one immediately finds the formulae

$$\sum_{\alpha=1}^{2} \left(z''_\alpha \otimes A(\xi_\alpha)\right) \varphi^{\mathcal{F}} = \sum_{\alpha=1}^{2} \left(z'_\alpha \otimes A\left(\overline{\xi_\alpha}\right)\right) \varphi^{\mathcal{F}} = 0; \tag{6.1.3}$$

Hence, $\mathrm{d}_{\mathcal{F}}^+ \varphi^{\mathcal{F}} = 0$ follows directly, and so the proof of closedness is complete.

For the weight statement, we again may read off from Lemma 5.2 that the basis for $\mathfrak{k}'$ acts through $\omega_{\mathcal{F}}$ as

$$\begin{aligned} \omega_{\mathcal{F}}(e_1 \odot e_1 + ie_1 \odot e_1 i) &= 2i\left[\sum_{\alpha=1}^{2} z''_\alpha \frac{\partial}{\partial z''_\alpha} - z'_3 \frac{\partial}{\partial z'_3}\right] + i \\ \omega_{\mathcal{F}}(e_2 \odot e_2 + ie_2 \odot e_2 i) &= 2i\left[\sum_{\alpha=1}^{2} z'_\alpha \frac{\partial}{\partial z'_\alpha} - z''_3 \frac{\partial}{\partial z''_3}\right] + i. \end{aligned}$$

One may now calculate that $\omega_{\mathcal{F}}(e_r \odot e_r + ie_r \odot e_r i)\left(\varphi_{KM}^{\mathcal{F}}\right) = 3i\varphi_{KM}^{\mathcal{F}}$ for $r = 1$ and 2, and so the Schwartz form has weight 3. □

Definition 6.1.7. Let $\mathfrak{p}^+$ be the subspace of $\mathfrak{p}$ spanned by the Z'_α and $\mathfrak{p}^-$ the subspace spanned by the Z''_α; hence let

$$\wedge^{1,1}\mathfrak{p}^* = \left(\mathfrak{p}^+\right)^* \wedge \left(\mathfrak{p}^-\right)^* \subset \wedge^2 \mathfrak{p}^*$$

We define the interior multiplication maps A_γ^*, $\overline{A_\gamma^*}$ on $\wedge^{1,1}\mathfrak{p}^*$ by:

$$A_\gamma^*\left(\xi_\alpha \wedge \overline{\xi_{\alpha'}}\right) = \delta_{\gamma\alpha}\overline{\xi_{\alpha'}}, \quad \overline{A_\gamma^*}\left(\xi_\alpha \wedge \overline{\xi_{\alpha'}}\right) = -\delta_{\gamma\alpha'}\xi_\alpha$$

which act as inverses to $A(\xi_\gamma)$, $A(\overline{\xi_\gamma})$; indeed, one may see that the action of these operators (which we defined to be an action by the wedge on the left) recovers the original form. We hence define the homotopy operators h, $\overline{h}$:

$$h,\, \overline{h} : \left[\mathcal{F} \otimes \wedge^{1,1}\mathfrak{p}^*\right]^K \to \left[\mathcal{F} \otimes \mathfrak{p}^*\right]^K$$

by:

$$h = \frac{1}{4}\sum_{\gamma=1}^{2} z_3' \frac{\partial}{\partial z_\gamma'} \otimes \overline{A_\gamma^*}, \quad \overline{h} = \frac{1}{4}\sum_{\gamma=1}^{2} z_3'' \frac{\partial}{\partial z_\gamma''} \otimes A_\gamma^*$$

We define (in the Fock model) the second Schwartz form $\psi^{\mathcal{F}}$ by:

$$\psi^{\mathcal{F}} := \left(h + \overline{h}\right)\left(\varphi^{\mathcal{F}}\right)$$

Lemma 6.1.8. *Let $L \in \mathfrak{p}'^-$ be the basis element that gives the lowering operator. Then the following equation holds in the Fock model:*

$$\omega_{\mathcal{F}}(L)\left(\varphi^{\mathcal{F}}\right) = \mathrm{d}_{\mathcal{F}}\psi^{\mathcal{F}}$$

Proof. Using the calculations in Lemma 5.2.3, and fixing $\lambda = 2\pi i$, we may write

$$\omega_{\mathcal{F}}(L) = -4\pi \sum_{\gamma=1}^{2} \frac{\partial^2}{\partial z_\gamma'' \partial z_\gamma'} + \frac{1}{4\pi} z_3'' z_3'. \tag{6.1.4}$$

An easy calculation then gives us:

$$\omega_{\mathcal{F}}(L)(\varphi^{\mathcal{F}}) = \frac{1}{4\pi} z_3' z_3'' \varphi^{\mathcal{F}} + \frac{1}{2\pi}\left(\xi_1 \wedge \overline{\xi_1} + \xi_2 \wedge \overline{\xi_2}\right).$$

We may calculate that in the Fock model, $\psi^{\mathcal{F}}$ may be written:

$$\begin{aligned}
\psi^{\mathcal{F}} &= \frac{1}{4}\sum_{\gamma=1}^{2}\left(z_3' \frac{\partial}{\partial z_\gamma'} \otimes \overline{A_\gamma^*} + z_3'' \frac{\partial}{\partial z_\gamma''} \otimes A_\gamma^*\right)\left(\varphi^{\mathcal{F}}\right) \\
&= \frac{1}{4}\left(\frac{-i}{2\sqrt{2}\pi}\right)^2 \sum_{\alpha=1}^{2}\left(z_3' \cdot 2z_\alpha'' \otimes (-\xi_\alpha) + z_3'' \cdot 2z_\alpha' \otimes \left(\overline{\xi_\alpha}\right)\right) \\
&= \frac{-1}{16\pi^2}\sum_{\alpha=1}^{2}\left(-z_3' z_\alpha'' \otimes \xi_\alpha + z_3'' z_\alpha' \otimes \overline{\xi_\alpha}\right).
\end{aligned} \tag{6.1.5}$$

Recalling the definition of the differential $\mathrm{d}_{\mathcal{F}} = \mathrm{d}_{\mathcal{F}}^+ + \mathrm{d}_{\mathcal{F}}^-$ in (6.1.2), we calculate:

$$\begin{aligned}
\mathrm{d}_{\mathcal{F}}^+\psi^{\mathcal{F}} &= \frac{-1}{64\pi^3}\sum_{\gamma=1}^{2}\left(z_3' z_\gamma'' \otimes A\left(\xi_\gamma\right) + z_3'' z_\gamma' \otimes A\left(\overline{\xi_\gamma}\right)\right) \cdot \sum_{\alpha=1}^{2}\left(-z_3' z_\alpha'' \otimes \xi_\alpha + z_3'' z_\alpha' \otimes \overline{\xi_\alpha}\right) \\
&= \frac{-1}{64\pi^3}\sum_{\alpha,\gamma=1}^{2}\Big(-(z_3')^2 z_\alpha'' z_\gamma'' \otimes \xi_\gamma \wedge \xi_\alpha - z_3' z_3'' z_\gamma' z_\alpha'' \otimes \overline{\xi_\gamma} \wedge \xi_\alpha \qquad (6.1.6) \\
&\qquad + z_3' z_3'' z_\alpha' z_\gamma'' \otimes \xi_\gamma \wedge \overline{\xi_\alpha} + (z_3'')^2 z_\gamma' z_\alpha' \otimes \overline{\xi_\gamma} \wedge \overline{\xi_\alpha}\Big) \\
&= \frac{1}{8\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^2 z_3' z_3'' \sum_{\alpha,\gamma=1}^{2}\left(z_\gamma' z_\alpha'' \otimes \xi_\alpha \wedge \overline{\xi_\gamma} + z_\alpha' z_\gamma'' \otimes \xi_\gamma \wedge \overline{\xi_\alpha}\right), \qquad (6.1.7)
\end{aligned}$$

where the last equality in (6.1.7) holds because all of the terms of the form $\xi_\gamma \wedge \xi_\alpha$, $\overline{\xi_\gamma} \wedge \overline{\xi_\alpha}$ disappear - this is because of the anti-symmetric properties of these wedge products. Hence, as what remains is recognisable as two copies of the original $\varphi^{\mathcal{F}}$, we hence have:

$$\mathrm{d}^{+}_{\mathcal{F}}\psi^{\mathcal{F}} = \frac{1}{4\pi} z'_3 z''_3 \varphi^{\mathcal{F}}.$$

Similarly, we calculate:

$$\begin{aligned}
\mathrm{d}^{-}_{\mathcal{F}}\psi^{\mathcal{F}} &= -4\pi \sum_{\gamma=1}^{2} \left(\frac{\partial^2}{\partial z_{\gamma'} \partial z''_3} \otimes A\left(\xi_\gamma\right) + \frac{\partial^2}{\partial z_{\gamma''} \partial z'_3} \otimes A(\overline{\xi_\gamma}) \right) \cdot \frac{-1}{16\pi^2} \sum_{\alpha=1}^{2} \left(-z'_3 z''_\alpha \otimes \xi_\alpha + z''_3 z'_\alpha \otimes \overline{\xi_\alpha} \right) \\
&= \frac{1}{4\pi} \sum_{\alpha,\gamma=1}^{2} \left(\frac{\partial^2}{\partial z_{\gamma'} \partial z''_3} (z''_3 z_{\alpha'}) \otimes \xi_\gamma \wedge \overline{\xi_\alpha} + \frac{\partial^2}{\partial z_{\gamma''} \partial z'_3} (z'_3 z''_\alpha) \otimes \xi_\alpha \wedge \overline{\xi_\gamma} \right) \\
&= \frac{1}{4\pi} \sum_{\alpha,\gamma=1}^{2} \left(\delta_{\alpha\gamma} \otimes \xi_\gamma \wedge \overline{\xi_\alpha} + \delta_{\alpha\gamma} \otimes \xi_\alpha \wedge \overline{\xi_\gamma} \right) \\
&= \frac{1}{2\pi} \left(\xi_1 \wedge \overline{\xi_1} + \xi_2 \wedge \overline{\xi_2} \right)
\end{aligned}$$

and so the proof is completed. □

Given all of the above - which essentially treated the Schwartz form as an abstract algebraic object - we now examine how we may view this Schwartz form on the right hand side of the isomorphism in Proposition 6.1.2; in particular, we wish to understand φ_{KM} as a differential form on $\mathbb{D}$.

Definition 6.1.9. For $z \in \mathbb{P}V_- \simeq \mathbb{D}$ and $\mathbf{x} \in V$, we may construct the majorant attached to z as follows:

$$(\mathbf{x}, \mathbf{x})_z := \begin{cases} (\mathbf{x}, \mathbf{x}) & \text{if } \mathbf{x} \in z \\ -(\mathbf{x}, \mathbf{x}) & \text{if } \mathbf{x} \in z^\perp; \end{cases} \tag{6.1.8}$$

one may recognise this as a generalisation of the majorant for the line spanned by v_3, constructed in Definition 5.3.1. Indeed, as an alternate definition of the majorant, one may take $(\ , \ )_0$ from Definition 5.3.1 and the property

$$(\mathbf{x}, \mathbf{x})_z = (g_z^{-1}\mathbf{x}, g_z^{-1}\mathbf{x})_B, \tag{6.1.9}$$

where $g_z \in G$ such that $g_z z_B = z$. We hence form

$$\varphi_0(\mathbf{x}, z) = \varphi_0\left(g_z^{-1}(\mathbf{x})\right) = \exp(-\pi(\mathbf{x}, \mathbf{x})_z).$$

This allows us a further redefinition of $\mathbb{D}$: as the space of majorants for the hermitian space V. We also note that all choices of majorants $(\ , \ )_z$ give us a *positive-definite*, and hence signature $(3, 0)$, inner product on V; in particular, this will allow us to address convergence properties.

We may now realise the isomorphism π from Proposition 6.1.2 on the given co-ordinates.

Definition 6.1.10. We define the canonical G-invariant differential forms Ξ_α, $\overline{\Xi_\alpha}$ on $\mathbb{D}$ as $\Xi_\alpha = \pi(\xi_\alpha)$, $\overline{\Xi_\alpha} = \pi(\overline{\xi_\alpha})$. By the general theory of Maurer-Cartan forms, we may write these as follows:

$$\Xi_\alpha = \frac{\mathrm{d}z_\alpha}{1-|z_1|^2-|z_2|^2}, \quad \overline{\Xi_\alpha} = \frac{\mathrm{d}\overline{z_\alpha}}{1-|z_1|^2-|z_2|^2}$$

We may now write down the Kudla-Millson Schwartz forms in the complex of differential forms. For $\alpha = 1, 2$, we let $\mathcal{D}_\alpha$ and $\overline{\mathcal{D}_\alpha}$ be the operators on $\mathcal{S}(V)$ given by

$$\mathcal{D}_\alpha = \frac{1}{2}\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right), \qquad \overline{\mathcal{D}_\alpha} = \frac{1}{2}\left(\overline{z_\alpha} - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right) \tag{6.1.10}$$

so that $\nabla^{\mathcal{S}} = \sum_\alpha \mathcal{D}_\alpha \otimes \Xi_\alpha$ and $\overline{\nabla}^{\mathcal{S}} = \sum_\alpha \overline{\mathcal{D}}_\alpha \otimes \overline{\Xi_\alpha}$.

$$\varphi_{KM}(\mathbf{x}, g_z) = \sum_{\alpha,\alpha'=1}^{2} \left(\mathcal{D}_\alpha \circ \overline{\mathcal{D}_{\alpha'}}(\varphi_0)\right)\left(g_z^{-1}\mathbf{x}\right) \otimes \Xi_{\alpha'} \wedge \Xi_\alpha \tag{6.1.11}$$

Because we know that $G/K = \mathbb{D}$ parameterises the set of negative length lines z, we may view this as a function of z rather than g_z, and write this as $\varphi_{KM}(\mathbf{x}, z)$.

Further, we may write down the polynomials resulting from the ∇ operators: it is a simple calculation that

$$\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right) \circ \left(\overline{z_\alpha} - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right) = \left(\overline{z_\alpha} - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right) \circ \left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right)$$

as operators on $\mathcal{S}(V)$, and so

$$\left(\mathcal{D}_\alpha \circ \overline{\mathcal{D}_\alpha}(\varphi_0)\right)(\mathbf{x}) = \left(|z_\alpha|^2 - \frac{1}{2\pi}\right)\varphi_0(\mathbf{x}),$$

and for $\{\alpha, \alpha'\} = \{1, 2\}$:

$$\left(\mathcal{D}_\alpha \circ \overline{\mathcal{D}_{\alpha'}}(\varphi_0)\right)(\mathbf{x}) = \left(z_{\alpha'}\overline{z_\alpha}\right)\varphi_0(\mathbf{x})$$

Throughout this work, we shall use φ^0 to refer to the polynomial part - namely, $\varphi^0(\mathbf{x}) = \varphi(\mathbf{x})e^{\pi(\mathbf{x},\mathbf{x})}$. We now illustrate how we may insert the $SL_2(\mathbb{R}) \simeq SU(W)$ variable; we first must calculate how the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ acts, using the matrix exponential:

$$\exp\begin{pmatrix} \frac{1}{2}\log v & \frac{u\log v}{v-1} \\ & -\frac{1}{2}\log v \end{pmatrix} = \begin{pmatrix} \sqrt{v} & \frac{u}{\sqrt{v}} \\ & \frac{1}{\sqrt{v}} \end{pmatrix} =: g'_\tau$$

where $\tau = u + iv$ is some point in the upper-half plane $\mathbb{H}$.

Lemma 6.1.11. *Let* $g = \begin{pmatrix} a & b \\ & d \end{pmatrix}$ *be a generic upper-triangular matrix in* $SL_2(\mathbb{R})$*. Then this acts via the Weil representation on a polynomial Schwartz form* $f(\boldsymbol{x}) \in \mathcal{S}(V)$ *as follows:*

$$((exp(\omega)(g))\,(f))\,(\boldsymbol{x}) = |a|^3 exp\left(\pi i \mathcal{R}\left[\overline{a}b(\boldsymbol{x},\boldsymbol{x})\right]\right) f(a\boldsymbol{x})$$

Proof. This is just a change of basis from the unitary basis e_1, e_2 of W to the necessary symplectic basis (which gives $\mathfrak{su}(W) \simeq \mathfrak{sl}_2(\mathbb{R})$): this gives

$$\begin{pmatrix} \frac{bi}{2} & a - \frac{bi}{2} \\ a + \frac{bi}{2} & -\frac{bi}{2} \end{pmatrix} \in \mathfrak{su}(W)$$

and then an application of the Weil representation actions of $\mathfrak{k}'$ and $\mathfrak{g}'$ in Lemma 5.2.3. □

Definition 6.1.12. For an arbitrary $\tau \in \mathbb{H}$, we define $\varphi(\mathbf{x},\tau)$ as a function of τ as follows:

$$\begin{aligned}\varphi(\mathbf{x},\tau) &:= j(g'_\tau, i)^3(\exp(\omega)(g'_\tau)\varphi)(\mathbf{x}) \\ &= v^{-3/2}\left(\sqrt{v}^3 \exp(\pi i u(\mathbf{x},\mathbf{x}))\varphi_{KM}(\sqrt{v}\mathbf{x})\right) \\ &= \sum_{\alpha,\alpha'=1}^{2} \varphi^0_{\alpha,\alpha'}(\sqrt{v}\mathbf{x})e^{\pi i(\mathbf{x},\mathbf{x})\tau} \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha}\end{aligned} \tag{6.1.12}$$

where $\varphi^0_{\alpha,\alpha'}(\mathbf{x}) = \varphi_{\alpha,\alpha'}(\mathbf{x})e^{\pi(\mathbf{x},\mathbf{x})}$ is the polynomial part of φ_{KM} in the $\xi_{\alpha'} \wedge \overline{\xi_\alpha}$ component.

Since all the actions will commute, one may check that this is closed in the complex $[\mathcal{S}(V) \otimes \wedge^\bullet \mathfrak{p}^*]^K$ for all τ, so in particular when inserting z as in (6.1.11), we acquire a closed form $\varphi(\mathbf{x}, z, \tau)$ given by

$$\varphi(\mathbf{x}, z,\tau) = \sum_{\alpha,\alpha'=1}^{2} \varphi^0_{\alpha,\alpha'}\left(g_z^{-1}\left(\sqrt{v}\mathbf{x}\right)\right) e^{\pi i(\mathbf{x},\mathbf{x})_z\tau} \otimes \Xi_{\alpha'} \wedge \overline{\Xi_\alpha} \in \left[\mathcal{S}(V) \otimes \Omega^2(\mathbb{D})\right]^G.$$

We are now ready to introduce the theta series.

Theorem 6.1.13 (Kudla-Millson, 1986)**.** *Let* $\boldsymbol{x} \in \underline{V}$ *be a positive length vector. Up to a constant, the special cycle* $C_{\boldsymbol{x}}$ *is a Poincaré dual of* $\varphi_{KM}(\boldsymbol{x}, z, \tau)$*; namely, for* η *a closed, G-invariant and rapidly decreasing closed differential* 2*-form on* $\Gamma_{\boldsymbol{x}}\backslash\mathbb{D}$*:*

$$\int_{\Gamma_{\boldsymbol{x}}\backslash\mathbb{D}} \eta \wedge \varphi_{KM}(\boldsymbol{x}, z, \tau) = ie^{\pi i(\boldsymbol{x},\boldsymbol{x})\tau}\int_{C_{\boldsymbol{x}}} \eta \tag{6.1.13}$$

For $\boldsymbol{x} \in \underline{V}$ *of negative length,* $\varphi_{KM}(\boldsymbol{x}, z, \tau)$ *is exact, and so by Stokes' theorem in particular the integral on the left hand side of* (6.1.13) *is* 0*.*

Let $L \subset \underline{V}$ be an even, integral lattice of level $M \in \mathbb{N}$ such that all of its cosets $\mathcal{L} \subset L'/L$ are fixed by some arithmetic group $\Gamma \subset \underline{G}$. We may hence define the Kudla-Millson theta series:

$$\theta_{\mathcal{L}}(\varphi_{KM}, \tau) = \sum_{\boldsymbol{x} \in \mathcal{L}} \varphi_{KM}(\boldsymbol{x}, z, \tau) \in \Omega^2(X) \otimes M_3^{nonhol}(\Gamma(M));$$

This defines a differential form on X which is uniformly convergent on compact subsets of X, and as a function in τ is a non-holomorphic modular form of level M. We may treat this as a cohomology class in $H^2_{dR}(X)$; this class is non-trivial, and defines a holomorphic *modular form in τ:*

$$[\theta_{\mathcal{L}}(\varphi_{KM})] = \sum_{\boldsymbol{x} \in \mathcal{L}} [\varphi(\boldsymbol{x}, z, \tau)] \in H^2(X) \otimes M_3(\Gamma(M)).$$

Further, by duality, we may write what the Fourier coefficients are: with

$$\Omega_X = \sum_{j=1}^{2} \frac{\mathrm{d}z_j \wedge \mathrm{d}\overline{z_j}}{(1 - |z_1|^2 - |z_2|^2)^2}$$

the Kähler form on X - so that $c_1(X) = \frac{i}{2\pi}\Omega$ is the first Chern form on X - we may write:

$$[\theta_{\mathcal{L}}(\varphi_{KM})] = [c_1(X)] + \sum_{n>0} [C_n]^{PD} q^n$$

Hence, for any rapidly decreasing form $\eta \in H^2_c(X)$, we have

$$\int_X \eta \wedge \theta_{\mathcal{L}}\left(\varphi_{KM}\right) = i \left[\frac{1}{2\pi} \delta_{\mathcal{L}=L} \int_X \eta \wedge \Omega_X + \sum_{n>0} e^{2\pi i n \tau} \left(\int_{C_n} \eta\right)\right] \in M_3(\Gamma(M))$$

Proof. The full proof of this is contained entirely (not to mention in much broader generality) in the papers [KM86] and [KM87]; we note only that once the duality equation is proven (which we need not attempt - see e.g. [KM87, Proposition 6.3]), the rest of the theorem largely follows from this and our earlier work.

In particular, the fact that $\theta_{\mathcal{L}}(\varphi_{KM}, \tau)$ defines a differential form on X follows immediately from the Γ-invariance of $\mathcal{L}$. The modularity of weight 3 comes from the weight calculation in Proposition 6.1.6; similarly, the holomorphy in cohomology comes from Lemma 6.1.8 - which, when translated to the cohomological language, means that $[L_\tau \varphi_{KM}(\mathbf{x}, z, \tau)] = [0]$ in cohomology.

The fact that it is a specifically holomorphic modular form - i.e., that for negative length $\mathbf{x}$, the integral in (6.1.13) is 0 - follows from a standard construction of Kudla and Funke in [KF17, §3.3]. Namely, given the relationship $\omega(L)\varphi_{KM} = \mathrm{d}\psi_{KM}$, one forms an auxiliary form $\tilde{\psi}$ by

$$\tilde{\psi}(\mathbf{x}) = -\int_1^\infty \psi^0_{KM}\left(\sqrt{r}\mathbf{x}\right) \frac{\mathrm{d}r}{r} e^{-\pi(\mathbf{x},\mathbf{x})} = -\frac{1}{2\pi|z_3|^2} \psi_{KM}(\mathbf{x}).$$

Inserting z in the usual way, we see that this is a smooth function exactly for $z \notin \mathbb{D}_{\mathbf{x}}$ - indeed, for z at the basepoint, it is defined for z such that $|z_3| \neq 0$, or rather $\mathbf{x}$ not perpendicular to $[v_3]$. We may hence write

$$\mathrm{d}\tilde{\psi}^0(\mathbf{x}, z) = -\int_1^{\infty} \mathrm{d}\left(\psi^0_{KM}\left(\sqrt{r}\mathbf{x}, z\right)\right)\frac{\mathrm{d}r}{r} = -\int_1^{\infty} \frac{\partial}{\partial r}\left(\varphi^0_{KM}(\sqrt{r}\mathbf{x}, z)\right)\mathrm{d}r = \varphi^0_{KM}(\mathbf{x}, z), \tag{6.1.14}$$

where the 2nd equality in (6.1.14) is a rewriting of Lemma 6.1.8 for $v = r$. The rest of the theorem then follows from the duality statement. □

6.2 Working towards a Generalised Schwartz Form

We now dedicate some time to motivating and then writing down the generalised Schwartz form with coefficients in our chosen irreducible representations of G. Namely, the problem naturally arises: how does one actually write down a K-invariant Schwartz function? We shall hope to motivate this, using the Howe operators of Kudla and Millson as inspiration. After all, at this point (with only the original Kudla-Millson form to work with) we could, a priori, do a quite large number of things in order to write down a vector-valued Schwartz form.

Definition 6.2.1. For two arbitary non-negative integers l, l', let $T^{l,l'}(V) = V^{\otimes l} \otimes (V^*)^{\otimes l'}$ be the vector product space, and $S^{l,l'}(V) \subset T^{l,l'}(V)$ be the symmetric powers. As in earlier chapters, let A denote the insertion on the left of a vector - so that $A(v) : V^{\otimes l} \to V^{\otimes l+1}$, and similarly for the dual. Then in the Schrödinger model, we define the following Howe operators:

$$\nabla^{\mathcal{S}}_V = \frac{1}{2}\sum_{\alpha=1}^{2}\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right) \otimes 1 \otimes A(v_\alpha) \otimes 1,$$
$$\overline{\nabla}^{\mathcal{S}}_V = \frac{1}{2}\sum_{\alpha=1}^{2}\left(\overline{z_\alpha} - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right) \otimes 1 \otimes 1 \otimes A(v^*_\alpha),$$

which both act as endormophisms on $[\mathcal{S}(V) \otimes \wedge^\bullet \mathfrak{p}^* \otimes T^\star(V)]^K$.

One may see the logic to this as follows: recall from Definition 2.4.3 that we have defined the Lie algebra elements as

$$\xi_j = \frac{1}{2}\left(\alpha^*_j + \beta^*_j i\right) = \frac{1}{2}\left((v_j \wedge v_3)^* - (iv_j \wedge v_3)^* i\right),$$
$$\overline{\xi_j} = \frac{1}{2}\left(\alpha^*_j - \beta^*_j i\right) = \frac{1}{2}\left((v_j \wedge v_3)^* + (iv_j \wedge v_3)^* i\right)$$

V has a left action by $\mathbb{C}$, and we may give $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ a right action as well, which decomposes $V_{\mathbb{C}}$ as

$$V_{\mathbb{C}} = V_{\mathbb{C},+i} \oplus_{\mathbb{C}} V_{\mathbb{C},-i}$$

where $V_{\mathbb{C},\pm i} = \{v \mp ivi \mid v \in V\}$ is the $\pm i$ eigenspace for the complex structure extended to $V_{\mathbb{C}}$. We may check that as left vector-spaces $V \simeq V_{\mathbb{C},+i}$ and $V^* \simeq V_{\mathbb{C},-i}$; hence, we may write

$$\xi_j = \frac{1}{2}\left((v_j + iv_ji)^* \wedge v_3^*\right) \in \mathfrak{p}^{*,-}, \quad \overline{\xi_j} = \frac{1}{2}\left((v_j - iv_ji)^* \wedge v_3^*\right) \in \mathfrak{p}^{*,+}. \tag{6.2.1}$$

Indeed, using the forms from (6.2.1), we may write:

$$\nabla^{\mathcal{S}} = \frac{1}{2}\sum_{\alpha=1}^{2}\left[\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z}_\alpha}\right) \otimes A\left(\frac{1}{2}\left((v_\alpha - iv_\alpha i)^* \wedge v_3^*\right)\right)\right], \tag{6.2.2}$$

$$\overline{\nabla}^{\mathcal{S}} = \frac{1}{2}\sum_{\alpha=1}^{2}\left[\left(\overline{z}_\alpha - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right) \otimes A\left(\frac{1}{2}\left((v_\alpha + iv_\alpha i)^* \wedge v_3^*\right)\right)\right]. \tag{6.2.3}$$

So, exploiting all the above isomorphisms, we may write the vector-valued Howe operators ∇_V, $\overline{\nabla}_V$ as:

$$\nabla_V^{\mathcal{S}} = \frac{1}{2}\sum_{\alpha=1}^{2}\left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right) \otimes 1 \otimes A\left(v_\alpha - iv_\alpha i\right) \otimes 1$$

$$\overline{\nabla}_V^{\mathcal{S}} = \frac{1}{2}\sum_{\alpha=1}^{2}\left(\overline{z_\alpha} - \frac{1}{\pi}\frac{\partial}{\partial z_\alpha}\right) \otimes 1 \otimes 1 \otimes A\left(v_\alpha + iv_\alpha i\right).$$

The correspondence between these two sets of operators may be seen as follows (we treat the correspondence $\nabla^{\mathcal{S}} \leftrightarrow \nabla_V^{\mathcal{S}}$, the other is identical). Both of them are sums over the positive parts of V, and the terms in the $\mathcal{S}(V)$ component are identical; further, each has one term acting on $\wedge^\bullet \mathfrak{p}^*$ or $T^l(V)$, indexed by the negative signature of V - hence, this correspondence may be reduced to the study of these terms.

Indeed, using the form of $\nabla^{\mathcal{S}}$ given in (6.2.2), we see that the map

$$_+V \to (_+V \wedge_- V)^*, \; v \to v^* \wedge v_3^*$$

is an isomorphism - indeed, this is by definition of the wedge product map in (5.2.2). Hence, using all of the above, we may see the new operators ∇^V and $\overline{\nabla}^V$ to be the obvious choice of operators with coefficients.

We also note (because we know, from Kudla-Millson, that the operators $\nabla^{\mathcal{S}}$ and $\overline{\nabla}^{\mathcal{S}}$ are K-invariant) the K-invariance of the $\nabla_V^{\mathcal{S}}$ and $\overline{\nabla}_V^{\mathcal{S}}$ follows immediately from these observations.

We may now write down our Schwartz forms in full generality.

Definition 6.2.2. Let l, l' be two non-negative integers. Throughout, we use the superscript $\mathcal{S}$ to refer to objects in the Schrödinger model, and $\mathcal{F}$ to objects in the Fock model.

(i) In the Schrödinger model, we define the Schwartz form with coefficients as follows:

$$\varphi^{\mathcal{S}}_{l,l'} = \left(\nabla^{\mathcal{S}}_V\right)^l \cdot \left(\overline{\nabla}^{\mathcal{S}}_V\right)^{l'} \left(\varphi^{\mathcal{S}}\right) \in \left[\mathcal{S}(V) \otimes \wedge^2\mathfrak{p}^* \otimes T^{l,l'}(V)\right]^K$$

(ii) Similarly, in the Fock model, we may use the intertwiners between $\mathcal{S}$ and $\mathcal{F}$ to write

$$\nabla^{\mathcal{F}}_V = \frac{-i}{2\sqrt{2}\pi}\sum_{\alpha=1}^{2} z'_\alpha \otimes 1 \otimes A(v_\alpha) \otimes 1,$$
$$\overline{\nabla}^{\mathcal{F}}_V = \frac{-i}{2\sqrt{2}\pi}\sum_{\alpha=1}^{2} z''_\alpha \otimes 1 \otimes 1 \otimes A(v^*_\alpha).$$

We hence define the relevant Schwartz form as:

$$\varphi^{\mathcal{F}}_{l,l'} = \left(\nabla^{\mathcal{F}}_V\right)^l \cdot \left(\overline{\nabla}^{\mathcal{F}}_V\right)^{l'} \left(\varphi^{\mathcal{F}}\right) \in \left[\mathcal{F} \otimes \wedge^2\mathfrak{p}^* \otimes T^{l,l'}(V)\right]^K$$

(iii) As noted in §3.1, the irreducible representations will be subspaces of $T^{l,l'}(V)$; hence, for any such $\mathcal{B} \subset T^{l,l'}(V)$, with projection map $\pi_{\mathcal{B}} : T^{l,l'}(V) \to \mathcal{B}$, we write the form with coefficients in $\mathcal{B}$ as

$$\varphi^{\mathcal{S},\mathcal{B}}_{l,l'} = (1 \otimes 1 \otimes \pi_{\mathcal{B}})\left(\varphi^{\mathcal{S}}_{l,l'}\right),$$

and similarly for the Fock model. Of particular interest to us is the subspace $S^{l,l'}(V)$ of symmetric vectors, which has corresponding irrep. $\mathcal{H}^{l,l'}(V)$, discussed in §3.1.

We here mention that the advantage of working in the complex $[\mathcal{S}(V) \otimes \wedge^\bullet\mathfrak{p}^* \otimes S^{\bullet,\bullet}(V)]^K$ is that all of the Howe operators fully commute. We also mention here that our primary focus is on the case $l = l'$ - indeed, we shall see in the next section that this is the only case giving us holomorphic theta series - but because of the need for some auxiliary forms later on, we will stay in the non-specialised case as often as possible early on.

We now record the full form of the vector-valued Schwartz functions in a lemma; it is at this point that the reader will see the usefulness of the Fock model, because in the Schrödinger model the expression involved is significantly more opaque.

Lemma 6.2.3. *Let $\boldsymbol{x} = z_1v_1 + z_2v_2 + z_3v_3$ be the orthonormal co-ordinates of V. In* *the Schrödinger model we may explicitly write the form* $\varphi_{l,l'}^{\mathcal{S}}$ *as*

$$\varphi_{l,l'}^{\mathcal{S}}(\boldsymbol{x}) = \frac{1}{2^{l+l'+2}} \sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta'}}} \left(z_\alpha - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\right)\left(\overline{z_{\alpha'}} - \frac{1}{\pi}\frac{\partial}{\partial z_{\alpha'}}\right) \prod_{\underline{\beta},\underline{\beta'}} \left(z_\beta - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\beta}}\right)\left(\overline{z_{\beta'}} - \frac{1}{\pi}\frac{\partial}{\partial z_{\beta'}}\right)(\varphi_0)$$
$$\otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \underline{v_\beta} \otimes \underline{v^*}_{\underline{\beta}}$$

where

$$\underline{\beta} = (\beta_1, \ldots, \beta_l)\,,\ \underline{\beta'} = (\beta'_1, \ldots, \beta'_{l'})\,,$$

all α, α', β_i, β'_i *run from* 1 *to* 2 *and we use the usual notation of* $\underline{v_\beta} = v_{\beta_1} \otimes \ldots \otimes v_{\beta_l}$ *and* $\underline{v^*_{\beta'}} = v^*_{\beta'_1} \otimes \ldots \otimes v^*_{\beta_{l'}}$.

Similarly, we may write for the Fock model

$$\varphi_{l,l'}^{\mathcal{F}} = \left(\frac{-i}{2\sqrt{2}\pi}\right)^{l+l'+2} \sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta'}}} z'_\alpha z''_{\alpha'} \underline{z'_\beta} \underline{z''_{\beta'}} \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \underline{v_\beta} \otimes \underline{v^*}_{\underline{\beta'}}.$$

6.3 The Many Properties of $\varphi_{l,l}^{\mathcal{F},\mathcal{H}}$

So, as promised a while ago - we look through the mists of time, back through all the way to §4.1 and in particular Theorem 4.1.5 - we shall soon be able to use these new Schwartz functions to prove our highly generalised statement about modular generating series of odd integral weight, stated originally in the introductory Theorem 1.2.2. For the rest of the section we write, unless otherwise necessary:

$$\varphi_{l,l'} = \varphi_{l,l'}^{\mathcal{F},T^{l,l'}(V)}.$$

We now prove the following properties of the Schwartz function $\varphi_{l,l}$, when $l = l'$:

(i) With respect to the action of the central subalgebra $\mathfrak{k}' \subset \mathfrak{g}'$ under the Weil representation, $\varphi_{l,l}$ has weight $2l + 3$ [Lemma 6.3.2].

(ii) $\varphi_{l,l}$ is closed in the differential for the complex $[\mathcal{F} \otimes \wedge^2 \mathfrak{p}^* \otimes T^{l,l}(V)]^K$ - and hence in $[\mathcal{F} \otimes \wedge^2 \mathfrak{p}^* \otimes E]^K$ for any subrepresentation $E \subset T^{l,l}(V)$ [Proposition 6.3.5].

(iii) The lowering operator L spanning $\mathfrak{p}'^-$ from Definition 5.2.2 acts trivially in cohomology with coefficients: namely, $\omega_{\mathcal{F}}(L)(\varphi_{l,l})$ defines a Schwartz form that satisfies

$$\omega_{\mathcal{F}}(L)\,(\varphi_{l,l}) = \mathrm{d}\,(\beta_{l,l}) + \Delta_{l,l}$$

where $\beta_{l,l}$ is some Schwartz form and $\Delta_{l,l}$ is a Schwartz form whose vector component is proportional to the metric in $V \otimes V^*$, so that in particular

$(1 \otimes 1 \otimes \pi_{\mathcal{H}}(\Delta_{l,l})) = 0$. Hence, by using the Schrödinger model with harmonic coefficients, it follows that

$$\left[\omega_{\mathcal{S}}(L)\varphi_{l,l}^{\mathcal{S},\mathcal{H}}\right] = [0]$$

in $H^2\left(\mathbb{D}, \widetilde{\mathcal{H}^{l,l}(V)}\right)$ [Theorem 6.3.8].

(iv) In the Schrödinger model, $\varphi_{l,l}^{\mathcal{S}}(\mathbf{x})$ is cohomologous to a "geometric" Schwartz form: namely, we may define

$$\begin{aligned}\varphi_{l,l,G}(\mathbf{x}) &= [1 \otimes 1 \otimes A(\mathbf{x}) \otimes 1]^l \circ [1 \otimes 1 \otimes 1 \otimes A(\mathbf{x})]^l \varphi_{KM}(\mathbf{x}) \\ &= \varphi_{KM}(\mathbf{x}) \otimes \mathbf{x}^l \otimes (\mathbf{x}^*)^l\end{aligned}$$

so that $\varphi_{l,l,G}(\mathbf{x})$ is the result of replacing all the vectors v_β with $\mathbf{x}$ (and similarly in the dual) in the Howe operators from Definition 6.2.1 - or rather, we "shift" $\mathbf{x}$ from the $\mathcal{S}(V)$ part of the operator to the vector product part. Then this form is cohomologous to $\varphi_{l,l}^{\mathcal{S}}$ when we work with coefficients in the irrep $\mathcal{H}^{l,l}(V)$ [Theorem 6.3.9].

Before we start, a couple of notes: firstly, I have listed these results here because, despite my most ardent optimism, I think it is fair to assume that these results are themselves a good deal more interesting than their proofs, which are largely algebraic. Of course, these are necessary, and the insights derived from these results are at the core of our work, but one need not understand the proofs to understand the remainder of this thesis.

A second thing worth noting is exactly why part (*iv*) exists. The reader will hopefully see the utility in parts (*i*)-(*iii*) - these results collectively will give us that the associated theta series will define a modular class in cohomology. The most obvious motivation is in our definition of the special cycles with coefficients in Proposition 4.1.3; indeed, once one sees this, it is suddenly very clear what this result will give us - duality!

Definition 6.3.1. For any multi-index $\underline{\beta} = (\beta_1, \ldots, \beta_l)$ of length l, with all $\beta_i \in \{1, 2\}$, we let $r(\underline{\beta})$ be the number of indices equalling 1, so that $l - r(\underline{\beta})$ is the number of indices equalling 2.

We start with a nice introductory result.

Lemma 6.3.2. *$\varphi_{l,l}$ is an eigenvector of weight $(2l+3)$ for the action of $\mathfrak{k}'$ under the Weil representation.*

Proof. This proof is virtually identical to that of Proposition 6.1.6. We recall from our work on the Weil representation in Lemma 5.2.3 that the basis $\{e_r \odot e_r + ie_r \odot e_r i\}_{r=1}^2$ acts in the Fock model as

$$\omega_{\mathcal{F}}(e_1 \odot e_1 + ie_1 \odot e_1 i) = 2i\Big[\sum_{\gamma=1}^{2} z''_\gamma \frac{\partial}{\partial z''_\gamma} - z'_3 \frac{\partial}{\partial z'_3}\Big] + i$$
$$\omega_{\mathcal{F}}(e_2 \odot e_2 + ie_2 \odot e_2 i) = 2i\Big[\sum_{\gamma=1}^{2} z'_\gamma \frac{\partial}{\partial z'_\gamma} - z''_3 \frac{\partial}{\partial z''_3}\Big] + i$$

We prove the statement for the first basis element - the proof for the other is completely identical.

$$\begin{aligned}
\omega_{\mathcal{F}}(e_1 \odot e_1 + ie_1 \odot e_1 i)\,(\varphi_{l,l}) &= 2i \sum_{\gamma=1}^{2} z''_\gamma \frac{\partial}{\partial z''_\gamma}\,(\varphi_{l,l}) + i\varphi_{l,l} \\
&= 2i\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2} \sum_{\substack{\gamma,\alpha,\alpha' \\ \underline{\beta},\underline{\beta'}}} \left(z''_\gamma \frac{\partial}{\partial z''_\gamma}\left(z'_\alpha z''_{\alpha'} \underline{z}'_{\underline{\beta}} \underline{z}''_{\underline{\beta'}}\right) \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta'}} \right) + i\varphi_{l,l} \\
&= 2i\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2} \sum_{\substack{\gamma,\alpha,\alpha' \\ \underline{\beta},\underline{\beta'}}} \left(z''_\gamma z'_\alpha \underline{z}'_{\underline{\beta}} \frac{\partial}{\partial z''_\gamma}\left(z''_{\alpha'} \underline{z}''_{\underline{\beta'}}\right) \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta'}} \right) + i\varphi_{l,l}.
\end{aligned}$$

We may hence write:

$$\begin{aligned}
\sum_{\gamma=1}^{2} z''_\gamma \frac{\partial}{\partial z''_\gamma}\left(z''_{\alpha'} \underline{z}''_{\underline{\beta'}}\right) &= \sum_{\gamma=1}^{2} z''_\gamma \left(\delta_{\gamma\alpha'} \underline{z}''_{\underline{\beta'}} + z''_{\alpha'} \frac{\partial}{\partial z''_\gamma}\left(\underline{z}''_{\underline{\beta'}}\right)\right) \\
&= \sum_{\gamma=1}^{2} z''_\gamma \Big(\delta_{\gamma\alpha'} \underline{z}''_{\underline{\beta'}} + z_{\alpha'} \Big[\delta_{\gamma 1}(r(\underline{\beta'})(z''_1)^{r(\underline{\beta'})-1}(z''_2)^{l-r(\underline{\beta'})} \\
&\qquad + \delta_{\gamma 2}\left(l - r(\underline{\beta'})\right)(z''_1)^{r(\underline{\beta'})}(z''_2)^{l-r(\underline{\beta'})-1}\Big]\Big) \\
&= z''_{\alpha'} \underline{z}''_{\underline{\beta'}} + z''_{\alpha'}\Big[r(\underline{\beta'})\underline{z}''_{\underline{\beta'}} + (l - r(\underline{\beta'}))\underline{z}''_{\underline{\beta'}}\Big] \\
&= (l+1) z''_{\alpha'} \underline{z}''_{\underline{\beta'}}
\end{aligned}$$

and so we immediately have that

$$\omega_{\mathcal{F}}(e_1 \odot e_1 + ie_1 \odot e_1 i)\,(\varphi_{l,l}) = 2i(l+1)\varphi_l + i\varphi_{l,l} = i(2l+3)\varphi_{l,l}.$$

□

We note here that implicit to this proof is that the element $\varphi_{l,l'}$ for $l \neq l'$ will *not* give a theta series which is a holomorphic modular form in τ! This is because the two basis elements (for $r = 1$ and $r = 2$) will generate different weights - $2l + 3$ and $2l' + 3$ respectively.

We may now start with a full definition of the differentials in the complexes used;

this was something slightly fudged in Definition 6.1.3, where (because of our only needing to work in coefficient-free complexes) we gave only one half of the picture.

Definition 6.3.3. We recall the complex basis $\{Z'_r, Z''_r\}$ of $\mathfrak{p}$ from (6.1.1). Then we define the vector-valued differentials by:

$$\begin{aligned} \mathrm{d}_V^+ &= \sum_{\mathfrak{p}-\text{basis}} 1 \otimes A(\lambda)^* \otimes \rho_V(\lambda) \otimes 1 \\ &= \sum_{\alpha=1}^{2} [1 \otimes A(\xi_\alpha) \otimes \rho_V(Z'_\alpha) \otimes 1] + \sum_{\alpha=1}^{2} \left[1 \otimes A\left(\overline{\xi_\alpha}\right) \otimes \rho_V(Z''_\alpha) \otimes 1\right] \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}_V^- &= \sum_{\mathfrak{p}-\text{basis}} 1 \otimes A(\lambda^*) \otimes 1 \otimes \rho_V^*(\lambda) \\ &= \sum_{\alpha=1}^{2} [1 \otimes A(\xi_\alpha) \otimes 1 \otimes \rho_V^*(Z'_\alpha)] + \left[1 \otimes A\left(\overline{\xi_\alpha}\right) \otimes 1 \otimes \rho_V^*(Z''_\alpha)\right]. \end{aligned}$$

Here ρ_V, ρ_V^* are, respectively, the derived actions (from the standard representation) of the Lie algebra on the symmetric powers $V^{\otimes l}$ and $(V^*)^{\otimes l'}$. The $+$ and $-$ superscripts are to signify (as in the discussion in §6.2) how V and V^* are respectively isomorphic to the $+i$ and $-i$ eigenspaces of $V_{\mathbb{C}}$. We hence define the differential

$$\mathrm{d}_V := \mathrm{d}_V^+ + \mathrm{d}_V^-.$$

We here recall the restricted differential $\mathrm{d}_{\mathcal{F}}$ in $[\mathcal{F} \otimes \wedge^\bullet \mathfrak{p}^*]^K$ from (6.1.2); hence, in the Fock model (and, identically in the Schrödinger model), we may define the differential

$$\mathrm{d} := \mathrm{d}_{\mathcal{F}} + \mathrm{d}_V,$$

on the complex $[\mathcal{F} \otimes \wedge^\bullet \mathfrak{p}^* \otimes T^{l,l'}(V)]^K$. We note also that for any subrepresentation $\mathcal{B}$ of $T^{l,l'}(V)$, we may use the exact same differential, just restricted using the map $\pi_{\mathcal{B}}$ in exactly the same way as we may restrict the Schwartz functions from Definition 6.2.2.

The reader may notice where the full differential $\mathrm{d} = \mathrm{d}_{\mathcal{F}} + \mathrm{d}_V$ is derived from. Indeed, for E any G-representation, in [BW00] one associates a differential as outlined here to $[E \otimes \wedge^\bullet \mathfrak{p}^*]^K$; so in particular, this is the case of $E = \mathcal{F} \otimes T^{l,l'}(V)$, and in the case of trivial coefficients, d_V will just act trivially.

Lemma 6.3.4. *We here record some small results which we will need to refer back to throughout the key proofs.*

(i) For γ, $\gamma' \in \{1, 2\}$, the action of the basis elements Z'_γ, Z''_γ on the relevant basis

elements of V and V^* is given by:

$$\begin{aligned} &\rho_V(Z'_\gamma)(v_{\gamma'}) = 0, && \rho_V(Z''_\gamma)(v_{\gamma'}) = \delta_{\gamma\gamma'} v_3,\\ &\rho_V(Z'_\gamma)(v_3) = v_\gamma, && \rho_V(Z''_\gamma)(v_3) = 0\\ &\rho^*_V(Z'_\gamma)(v^*_{\gamma'}) = -\delta_{\gamma\gamma'} v^*_3, && \rho^*_V(Z''_\gamma)(v^*_{\gamma'}) = 0\\ &\rho^*_V(Z'_\gamma)(v^*_3) = 0, && \rho^*_V(Z''_\gamma)(v^*_3) = -v^*_\gamma. \end{aligned}$$

(ii) For l, l' non-negative integers, we define an auxiliary Schwartz form as follows:

$$\varphi^{\mathcal{F}}_{0,l,l'} := \sum_{\underline{\beta},\underline{\beta'}} \underline{z'_\beta}\,\underline{z''_{\beta'}} \otimes 1 \otimes \underline{v_\beta} \otimes \underline{v^*_{\beta'}} \tag{6.3.1}$$

so that $\varphi^{\mathcal{F}}_{l,l'} = \varphi^{\mathcal{F}}_{KM} \cdot \varphi^{\mathcal{F}}_{0,l,l'}$. We then have:

$$\begin{aligned} &\Big(1 \otimes 1 \otimes \rho_V(Z'_\gamma) \otimes 1\Big)\, \varphi_{0,l,l'} = 0\\ &\Big(1 \otimes 1 \otimes \rho_V(Z''_\gamma) \otimes 1\Big)\, \varphi_{0,l,l'} = \frac{-i}{2\sqrt{2}\pi} \sum_{j=1}^{l} \Big(z'_\gamma \otimes 1 \otimes A_j(v_3) \otimes 1\Big) \cdot \varphi_{0,l-1,l'}\\ &\Big(1 \otimes 1 \otimes 1 \otimes \rho^*_V(Z'_\gamma)\Big)\, \varphi_{0,l,l'} = \frac{i}{2\sqrt{2}\pi} \sum_{j=1}^{l} \Big(z''_\gamma \otimes 1 \otimes 1 \otimes A_j(v^*_3)\Big) \cdot \varphi_{0,l,l'-1}\\ &\Big(1 \otimes 1 \otimes 1 \otimes \rho^*_V(Z''_\gamma)\Big)\, \varphi_{0,l,l'} = 0, \end{aligned}$$

where $A_j(v) : V^{l-1} \to V^l$ is the insertion of v in the j'th place (and similarly for the dual vector space):

$$A_j(v)\,(v_1 \otimes \ldots \otimes v_{l-1}) = (v_1 \otimes \ldots v_{j-1} \otimes v \otimes v_{j+1} \otimes \ldots \otimes v_{l-1})\,.$$

(iii) Similarly, we have:

$$\begin{aligned} &\left(\frac{\partial}{\partial z'_\gamma} \otimes 1 \otimes 1 \otimes 1\right) \varphi_{0,l,l'} = \frac{-i}{2\sqrt{2}\pi} \sum_{j=1}^{l} (1 \otimes 1 \otimes A_j(v_\gamma) \otimes 1)\, \varphi_{0,l-1,l'}\\ &\left(\frac{\partial}{\partial z''_\gamma} \otimes 1 \otimes 1 \otimes 1\right) \varphi_{0,l,l} = \frac{-i}{2\sqrt{2}\pi} \sum_{j=1}^{l'} \Big(1 \otimes 1 \otimes 1 \otimes A_j(v^*_\gamma)\Big)\, \varphi_{0,l,l'-1} \end{aligned}$$

Proof. Part (i) is simple linear algebra, based on the definition of how ρ_V and ρ^*_V act and the matrix forms of the basis elements. Parts (ii) and (iii) are then just applications of part (i). □

Proposition 6.3.5. *$\varphi^{\mathcal{F}}_{l,l'} \in \Big[\mathcal{F} \otimes \wedge^\bullet \mathfrak{p}^* \otimes T^{l,l'}(V)\Big]^K$ is a closed form.*

Proof. We start by showing $\mathrm{d}_{\mathcal{F}} \varphi_{l,l'} = 0$. We recall that we may write $\mathrm{d}_{\mathcal{F}} = \mathrm{d}^+_{\mathcal{F}} + \mathrm{d}^-_{\mathcal{F}}$, where

$$\mathrm{d}^+_{\mathcal{F}} = \frac{1}{4\pi}\sum_{\alpha=1}^{2}\left(z''_\alpha z'_3 \otimes A\left(\xi_\alpha\right) + z'_\alpha z''_3 \otimes A\left(\overline{\xi_\alpha}\right)\right),$$
$$\mathrm{d}^-_{\mathcal{F}} = -4\pi\sum_{\alpha=1}^{2}\left(\frac{\partial^2}{\partial z'_\alpha \partial z''_3} \otimes A\left(\xi_\alpha\right) + \frac{\partial^2}{\partial z''_\alpha \partial z'_3} \otimes A\left(\overline{\xi_\alpha}\right)\right).$$

Because of the lack of z'_3, z''_3 variables in $\varphi_{l,l'}$, we may immediately write $\mathrm{d}^-_{\mathcal{F}}\varphi_{l,l'} = 0$. Because $\mathrm{d}^+_{\mathcal{F}}$ acts multiplicatively, we may write

$$\mathrm{d}^+_{\mathcal{F}}\varphi_{l,l'} = \left(\mathrm{d}^+_{\mathcal{F}}\varphi_{KM}\right)\cdot\varphi_{0,l,l'} = 0\cdot\varphi_{0,l,l'} = 0$$

as we already showed that $\left(\mathrm{d}^+_{\mathcal{F}}\varphi_{KM}\right) = 0$ in Proposition 6.1.6.

We now calculate the d^+_V action. We write:

$$\begin{aligned}
\mathrm{d}^+_V\varphi_{l,l'} &= \sum_{\gamma=1}^{2}\left(1\otimes A(\xi_\gamma)\otimes\rho_V(Z'_\gamma)\otimes 1 + 1\otimes A(\overline{\xi_\gamma})\otimes\rho_V(Z''_\gamma)\otimes 1\right)\cdot\left(\varphi_{KM}\cdot\varphi_{0,l,l'}\right)\\
&= \sum_{\gamma=1}^{2}\Big[\left(1\otimes A(\xi_\gamma)\otimes 1\otimes 1\right)(\varphi_{KM})\cdot\left(1\otimes 1\otimes\rho_V\left(Z'_\gamma\right)\otimes 1\right)(\varphi_{0,l,l'})\\
&\qquad + \left(1\otimes A(\overline{\xi_\gamma})\otimes 1\otimes 1\right)(\varphi_{KM})\cdot\left(1\otimes 1\otimes\rho_V\left(Z''_\gamma\right)\otimes 1\right)(\varphi_{0,l,l'})\Big]\\
&= \sum_{\gamma=1}^{2}\Big[0 + \left(1\otimes A(\overline{\xi_\gamma})\otimes 1\otimes 1\right)(\varphi_{KM})\cdot\frac{-i}{2\sqrt{2}\pi}\sum_{j=1}^{l}\left(z'_\gamma\otimes 1\otimes A_j(v_3)\otimes 1\right)(\varphi_{0,l-1,l'})\Big]\\
&= \sum_{\gamma=1}^{2}\frac{-i}{2\sqrt{2}\pi}\left(z'_\gamma\otimes A(\overline{\xi_\gamma})\otimes 1\otimes 1\right)(\varphi_{KM})\cdot\sum_{j=1}^{l}\left(1\otimes 1\otimes A_j(v_3)\otimes 1\right)(\varphi_{0,l-1,l'}).
\end{aligned} \tag{6.3.2}$$

The 2nd equality in (6.3.2) follows from the fact that φ_{KM} has no vector components and $\varphi_{0,l,l'}$ has no $\wedge^2\mathfrak{p}^*$ components, and the third equality is the first statement in Lemma 6.3.4(*ii*). From (6.1.3) (in the proof of Proposition 6.1.6), we already have that

$$\sum_{\gamma=1}^{2}\left(z'_\gamma\otimes A(\overline{\xi_\gamma})\otimes 1\otimes 1\right)(\varphi_{KM}) = 0$$

and so $\mathrm{d}^+_V\varphi_{l,l'} = 0$. The proof of $\mathrm{d}^-_V\varphi_{l,l'} = 0$ is exactly identical. □

We now move onto the first of our two sizeable proofs. To improve readability, we shall put as much of the necessary notation and auxiliary algebraic objects as possible before the statement, as most of these objects are only needed specifically for this chapter.

Definition 6.3.6. (i) We recall the Schwartz form $\psi^{\mathcal{F}}_{KM} \equiv \psi^{\mathcal{F}}$ from Definition 6.1.7, and the form $\varphi_{0,l,l'}$ from (6.3.1). Then for any positive integers l, l', we define the form

$$\psi_{l,l'}^{\mathcal{F}} := \psi^{\mathcal{F}} \cdot \varphi_{0,l,l'}$$

(ii) For $1 \leq j \leq l$, we define the Schwartz forms as follows:

$$\begin{aligned}
A_j &= \frac{i}{2\sqrt{2}\pi}\left(z_3' \otimes 1 \otimes A_j(v_3) \otimes 1\right)\varphi_{l-1,l} \\
B_j &= i\sqrt{2}\sum_\alpha \left(\frac{\partial}{\partial z_\alpha''} \otimes 1 \otimes 1 \otimes 1\right)\varphi_{KM} \cdot \left(1 \otimes 1 \otimes A_j(v_\alpha) \otimes 1\right)\varphi_{0,l-1,l} \\
C_j^+ &= \frac{1}{2\pi}\varphi_{KM} \cdot \sum_\alpha \sum_{k=1}^{l}\left(1 \otimes 1 \otimes A_j(v_\alpha) \otimes A_k(v_\alpha^*)\right)\varphi_{0,l-1,l-1} \\
C_j^- &= \frac{1}{2\pi}\varphi_{KM} \cdot \sum_{k=1}^{l}\left(1 \otimes 1 \otimes A_j(v_3) \otimes A_k(v_3^*)\right)\varphi_{0,l-1,l-1}
\end{aligned}$$

and

$$\begin{aligned}
\overline{A_j} &= \frac{i}{2\sqrt{2}\pi}\left(z_3'' \otimes 1 \otimes 1 \otimes A_j(v_3^*)\right)\varphi_{l,l-1} \\
\overline{B_j} &= i\sqrt{2}\sum_\alpha \left(\frac{\partial}{\partial z_\alpha'} \otimes 1 \otimes 1 \otimes 1\right)\varphi_{KM} \cdot \left(1 \otimes 1 \otimes 1 \otimes A_j(v_\alpha^*)\right)\varphi_{0,l,l-1} \\
\overline{C_j}^+ &= \frac{1}{2\pi}\varphi_{KM} \cdot \sum_\alpha \sum_{k=1}^{l}\left(1 \otimes 1 \otimes A_k(v_\alpha) \otimes A_j(v_\alpha^*)\right)\varphi_{0,l-1,l-1} \\
\overline{C_j}^- &= \frac{1}{2\pi}\varphi_{KM} \sum_{k=1}^{l}\left(1 \otimes 1 \otimes A_j(v_3) \otimes A_k(v_3^*)\right)\varphi_{0,l-1,l-1}.
\end{aligned}$$

(iii) For any $1 \leq j \leq l$, we also define the following homotopy operators

$$h_j = \sum_{\gamma=1}^{2}\left(\frac{\partial}{\partial z_\gamma''} \otimes A^*\left(\xi_\gamma\right) \otimes A_j(v_3) \otimes 1\right), \quad \overline{h_j} = \sum_{\gamma=1}^{2}\left(\frac{\partial}{\partial z_\gamma'} \otimes A^*\left(\overline{\xi_\gamma}\right) \otimes 1 \otimes A_j(v_3^*)\right)$$

and forms:

$$\Lambda_{j,l} := \frac{i\sqrt{2}}{l+2}h_j\varphi_{l-1,l}^{\mathcal{F}}, \quad \overline{\Lambda_{j,l}} := \frac{i\sqrt{2}}{l+2}\overline{h_j}\varphi_{l,l-1}^{\mathcal{F}}.$$

The next proposition shall give us a lot of control over these objects, and in particular indicates how one may construct the necessary primitives of $\omega(L)\varphi_{l,l}$. The rubric shall be that the differential $\mathrm{d} = \mathrm{d}_{\mathcal{F}}^+ + \mathrm{d}_{\mathcal{F}}^- + \mathrm{d}_V^+ + \mathrm{d}_V^-$ acts with each of its constituent parts on $\Lambda_{j,l}$ to give one of the A_j, B_j, C_j^-.

Proposition 6.3.7. *For* d *the differential in the Fock complex with coefficients, we have*

$$\mathrm{d}\Lambda_{j,l} = -A_j - B_j - C_j^-, \quad \mathrm{d}\overline{\Lambda_{j,l}} = -\overline{A_j} - \overline{B_j} - \overline{C_j}^-. \tag{6.3.3}$$

More specifically, we have the following 8 *algebraic relations:*

$$\mathrm{d}_{\mathcal{F}}^+\Lambda_{j,l} = -A_j, \quad \mathrm{d}_{\mathcal{F}}^-\Lambda_{j,l} = 0, \quad \mathrm{d}_V^+\Lambda_{j,l} = -B_j, \quad \mathrm{d}_V^-\Lambda_{j,l} = -C_j^- \tag{6.3.4}$$

$$\mathrm{d}_{\mathcal{F}}^{+}\overline{\Lambda_{j,l}} = -\overline{A_j}, \quad \mathrm{d}_{\mathcal{F}}^{-}\overline{\Lambda_{j,l}} = 0, \quad \mathrm{d}_{V}^{+}\overline{\Lambda_{j,l}} = -\overline{B_j}, \quad \mathrm{d}_{V}^{-}\overline{\Lambda_{j,l}} = -\overline{C_j^{-}} \tag{6.3.5}$$

Proof. I hope to convince the reader of the need only to show one half of this; namely, because of the symmetry between the definitions of e.g. A_j and $\overline{A_j}$, h_j and $\overline{h_j}$ etc, the algebra for the proofs of (6.3.4) and (6.3.5) will be identical. Assuming that this is satisfactory, we start by showing that $\mathrm{d}_{\mathcal{F}}\Lambda_{j,l} = -A_j$. Re-arranging the constants, this is equivalent to:

$$\mathrm{d}_{\mathcal{F}}\left(h_j\varphi_{l-1,l}\right) = \frac{-i(l+2)}{\sqrt{2}}A_j. \tag{6.3.6}$$

Firstly, it is clear that $\mathrm{d}_{\mathcal{F}}^{-}\left(h_j\varphi_{l-1,l}\right) = 0$, because by examination none of the terms in $h_j\varphi_{l-1,l}$ contain either of the variables z_3', z_3''. Hence, we have reduced (6.3.6) to:

$$\mathrm{d}_{\mathcal{F}}^{+}\left(h_j\varphi_{l-1,l}\right) = \frac{-i(l+2)}{\sqrt{2}}A_j.$$

Putting the two operators together, we have

$$\mathrm{d}_{\mathcal{F}}^{+}h_j = \frac{1}{4\pi}\sum_{\gamma,\gamma'=1}^{2}\Bigg(z_\gamma'' z_3'\frac{\partial}{\partial z_{\gamma'}''}\otimes A(\xi_\gamma)A^*(\xi_{\gamma'})\otimes A_j(v_3)\otimes 1 \\ + z_\gamma' z_3''\frac{\partial}{\partial z_{\gamma'}''}\otimes A(\overline{\xi_\gamma})A^*(\xi_{\gamma'})\otimes A_j(v_3)\otimes 1\Bigg). \tag{6.3.7}$$

The differential operators act on $\wedge^2\mathfrak{p}^*$ as:

$$A(\overline{\xi_\gamma})A^*(\xi_{\gamma'})(\xi_{\alpha'}\wedge\overline{\xi_\alpha}) = -\delta_{\alpha\gamma'}\overline{\xi_\gamma}\wedge\overline{\xi_{\alpha'}}; \tag{6.3.8}$$

here we see that exchanging α' and γ acts as -1, whereas exchanging α and γ' is invariant. Hence the second term in (6.3.7) acts trivially on $\varphi_{l-1,l}$:

$$\frac{1}{4\pi}\sum_{\gamma,\gamma'=1}^{2}\left(z_\gamma' z_3''\frac{\partial}{\partial z_{\gamma'}''}\otimes A(\overline{\xi_\gamma})A^*(\xi_{\gamma'})\otimes A_j(v_3)\otimes 1\right)\varphi_{l-1,l} = 0.$$

We now treat the other half. We see very similarly that

$$A(\xi_\gamma)A^*(\xi_{\gamma'})(\xi_{\alpha'}\wedge\overline{\xi_\alpha}) = \delta_{\alpha\gamma'}\xi_\gamma\wedge\overline{\xi_\alpha}$$

and

$$\frac{\partial}{\partial z_{\gamma'}''}\left(z_\alpha' z_{\alpha'}'' \underline{z}_{\underline{\beta}}' \underline{z}_{\underline{\beta}'}''\right) = z_\alpha'\underline{z}_{\underline{\beta}}'\Big[\delta_{\alpha'\gamma'}\underline{z}_{\underline{\beta}'}'' + z_{\alpha'}''\Big(\delta_{\gamma'1}r(\underline{\beta}')(z_1'')^{r(\underline{\beta}')-1}(z_2'')^{l-r(\underline{\beta}')} \\ + \delta_{\gamma'2}(l-r(\underline{\beta}'))(z_1'')^{r(\underline{\beta}')}(z_2'')^{l-r(\underline{\beta}')-1}\Big)\Big].$$

Hence, splitting into $\gamma'=1$ and $\gamma'=2$, we may write

$$\mathrm{d}_{\mathcal{F}}^{+}h_j\varphi_{l-1,l} = \frac{1}{4\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1}\sum_{\gamma,\alpha=1}^{2}\sum_{\underline{\beta},\underline{\beta}'}\Big[z_3' z_\gamma'' z_\alpha' \underline{z}_{\underline{\beta}}'\Big((\underline{z}_{\underline{\beta}'}'' + r(\underline{\beta}')\underline{z}_{\underline{\beta}'}'')\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes A_j(v_3)\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^*\Big)$$

$$+ z'_3 z''_\gamma z'_\alpha \underline{z'_\beta}\Big((\underline{z''_{\beta'}} + (l - r(\underline{\beta'}))\underline{z''_{\beta'}}) \otimes \xi_\gamma \wedge \overline{\xi_\alpha} \otimes A_j(v_3)\underline{v_\beta} \otimes \underline{v^*_{\beta'}}\Big)\Big]$$
$$= \frac{1}{4\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \sum_{\gamma,\alpha=1}^{2} \sum_{\underline{\beta},\underline{\beta'}} \left[z'_3 z''_\gamma z'_\alpha \underline{z'_\beta}\left((l+2)\underline{z''_{\beta'}}\right) \otimes \xi_\gamma \wedge \overline{\xi_\alpha} \otimes A_j(v_3)\underline{v_\beta} \otimes \underline{v^*_{\beta'}}\right]$$
$$= \frac{-i(l+2)}{\sqrt{2}} A_j.$$

Next, we wish to prove that $\mathrm{d}^+_V h_j \varphi_{l-1,l} = \frac{-i(l+2)}{\sqrt{2}} B_j$. As before, we write:

$$\mathrm{d}^+_V h_j = \sum_{\gamma,\gamma'=1}^{2} \left(\frac{\partial}{\partial z_{\gamma''}} \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes \rho_V\left(Z'_{\gamma'3}\right) \circ A_j(v_3) \otimes 1\right)$$
$$+ \left(\frac{\partial}{\partial z_{\gamma''}} \otimes A(\overline{\xi_{\gamma'}})A^*(\xi_\gamma) \otimes \rho_V\left(Z''_{\gamma'3}\right) \circ A_j(v_3) \otimes 1\right) \qquad (6.3.9)$$

As in the proof of the A_j statement, because of the action on $\wedge^2\mathfrak{p}^*$ from (6.3.8) we may see quite easily that the second term in (6.3.9) acts trivially on $\varphi_{l-1,l}$, for exactly the same reasons. Hence, we only need to look at the first term.

By Lemma 6.3.4(i), we know that $\rho_V\left(Z'_{\gamma'3}\right)(v_3) = v_{\gamma'}$ and that $\rho_V\left(Z'_{\gamma'3}\right)(v_\alpha) = 0$, so that

$$\left(\rho_V\left(Z'_{\gamma'3}\right) \circ A_j(v_3)\right)\underline{v_\beta} = A_j(v_{\gamma'})\underline{v_\beta}. \qquad (6.3.10)$$

Using the product rule and (6.3.10), we may apply (6.3.9) to $\varphi_{l-1,l}$ and write:

$$\mathrm{d}^+_V h_j \varphi_{l-1,l} = \sum_{\gamma,\gamma'=1}^{2} \left(\frac{\partial}{\partial z''_\gamma} \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes 1 \otimes 1\right)(\varphi_{KM}) \cdot (1 \otimes 1 \otimes A_j(v_{\gamma'}) \otimes 1)\,\varphi_{0,l-1,l}$$
$$+ \sum_{\gamma,\gamma'=1}^{2} (1 \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes 1 \otimes 1)\,(\varphi_{KM}) \cdot \left(\frac{\partial}{\partial z''_\gamma} \otimes 1 \otimes A_j(v_{\gamma'}) \otimes 1\right)\varphi_{0,l-1,l}.$$

We may calculate that

$$\frac{\partial}{\partial z''_\gamma} \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes 1 \otimes 1\,(\varphi_{KM}) = \left(\frac{-i}{2\sqrt{2}\pi}\right)^2 \sum_{\alpha=1}^{2} z'_\alpha \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha}$$

and so splitting the second sum into $\gamma = 1,\, 2$, we may write

$$\mathrm{d}^+_V h_j \varphi_{l-1,l} = \left(\frac{-i}{2\sqrt{2}\pi}\right)^2 \sum_{\alpha,\gamma'=1}^{2} \left(2z'_\alpha \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha}\right) \cdot (1 \otimes 1 \otimes A_j(v_{\gamma'}) \otimes 1)\,\varphi_{0,l-1,l}$$
$$+ \sum_{\alpha,\gamma'=1}^{2} \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \left(z_\alpha z''_1 \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha}\right) \cdot \sum_{\underline{\beta},\underline{\beta'}} r(\underline{\beta'})\underline{z'_\beta}(z''_1)^{r(\underline{\beta'})-1}(z''_2)^{l-r(\underline{\beta'})}$$
$$\otimes 1 \otimes A_j(v_{\gamma'})\underline{v_\beta} \otimes \underline{v^*_{\beta'}}$$
$$+ \sum_{\alpha,\gamma'=1}^{2} \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \left(z_\alpha z''_2 \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha}\right) \cdot \sum_{\underline{\beta},\underline{\beta'}} (l - r(\underline{\beta'}))\underline{z'_\beta}(z''_1)^{r(\underline{\beta'})}(z''_2)^{l-r(\underline{\beta'})-1}$$

$$\otimes 1 \otimes A_j(v_{\gamma'})\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}$$

$$= (l+2)\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \sum_{\alpha,\gamma'=1}^{2} \left(z'_\alpha \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha}\right) \sum_{\underline{\beta},\underline{\beta}'} \underline{z}'_{\underline{\beta}} \underline{z}''_{\underline{\beta}'} \otimes 1 \otimes A_j(v_{\gamma'})\underline{v}_{\beta} \otimes \underline{v}^*_{\underline{\beta}'}$$

$$= \frac{-i(l+2)}{\sqrt{2}} B_j$$

Finally, we show that $\mathrm{d}'^{-}_V h_j \varphi_{l-1,l} = \frac{-i(l+2)}{\sqrt{2}} C_j^-$. As before, we write the operator as

$$\mathrm{d}^-_V h_j = \sum_{\gamma,\gamma'=1}^{2} \left(\frac{\partial}{\partial z''_\gamma} \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes A_j(v_3) \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\right) + \sum_{\gamma,\gamma'=1}^{2} \left(\frac{\partial}{\partial z''_\gamma} \otimes A(\overline{\xi_{\gamma'}})A^*(\xi_\gamma) \otimes A_j(v_3) \otimes \rho_V^*\left(Z''_{\gamma' 3}\right)\right) \tag{6.3.11}$$

As in the previous parts, the second term in (6.3.11) will act trivially on $\varphi_{l-1,l}$ because the action on $\wedge^2\mathfrak{p}^*$ is as in (6.3.8). So:

$$\begin{aligned}
\mathrm{d}^-_V h_j \varphi_{l-1,l} &= \sum_{\gamma,\gamma'=1}^{2} \left(\frac{\partial}{\partial z''_\gamma} \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes 1 \otimes 1\right)(\varphi_{KM}) \\
&\qquad \cdot \left(1 \otimes 1 \otimes A_j(v_3) \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\right)\varphi_{0,l-1,l} \\
&+ \sum_{\gamma,\gamma'=1}^{2} \left(1 \otimes A(\xi_{\gamma'})A^*(\xi_\gamma) \otimes 1 \otimes 1\right)(\varphi_{KM}) \cdot \left(\frac{\partial}{\partial z''_\gamma} \otimes 1 \otimes A_j(v_3) \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\right)\varphi_{0,l-1,l} \\
&= \left(\frac{-i}{2\sqrt{2}\pi}\right)^2 \sum_{\alpha,\gamma'=1}^{2} \left(2z'_\alpha \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha}\right) \cdot \left(1 \otimes 1 \otimes A_j(v_3) \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\right)\varphi_{0,l-1,l} \\
&+ \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \sum_{\substack{\alpha,\gamma' \\ \underline{\beta},\underline{\beta}'}} z'_\alpha z''_1 r(\underline{\beta}') \underline{z}'_{\underline{\beta}} (z''_1)^{r(\underline{\beta}')-1} (z''_2)^{l-r(\underline{\beta}')} \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha} \otimes A_j(v_3)\underline{v}_{\underline{\beta}} \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\underline{v}^*_{\underline{\beta}'} \\
&+ \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \sum_{\substack{\alpha,\gamma' \\ \underline{\beta},\underline{\beta}'}} z'_\alpha z''_2 (l - r(\underline{\beta}')) \underline{z}'_{\underline{\beta}} (z''_1)^{r(\underline{\beta}')} (z''_2)^{l-r(\underline{\beta}')-1} \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha} \otimes A_j(v_3)\underline{v}_{\underline{\beta}} \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\underline{v}^*_{\underline{\beta}'} \\
&= \frac{-i(l+2)}{\sqrt{2}} \frac{1}{2\pi} \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l} \sum_{\substack{\alpha,\gamma' \\ \underline{\beta},\underline{\beta}'}} z'_\alpha \underline{z}'_{\underline{\beta}} \underline{z}''_{\underline{\beta}'} \otimes \xi_{\gamma'} \wedge \overline{\xi_\alpha} \otimes A_j(v_3)\underline{v}_{\underline{\beta}} \otimes \rho_V^*\left(Z'_{\gamma' 3}\right)\underline{v}^*_{\underline{\beta}'}.
\end{aligned}$$

Using Lemma 6.3.4(i) again to find the action of ρ_V^*, we see that this last expression is equal to $\frac{-i(l+2)}{\sqrt{2}} C_j^-$. □

Using the above, we may state a central result: namely, that the form $\omega_{\mathcal{F}}(L)(\varphi_{l,l})$ is exact with coefficients in the harmonic representation $\mathcal{H}^{l,l}$. We shall discuss afterwards the consequences that may be drawn from this - as well as its relation to the coefficients-free result equivalent to it in Lemma 6.1.8.

Theorem 6.3.8. *Let $l \geq 1$ be a positive integer. The lowering element*

$$L = \frac{-i}{2}\left(e_1 \odot e_2 + ie_1 \odot e_2 i\right) \in \mathfrak{p}'^{-}$$

acts through the Weil representation in the Fock model on $\varphi_{l,l}$ as follows:

$$\omega_{\mathcal{F}}(L)\left(\varphi_{l,l}^{\mathcal{F}}\right) = \mathrm{d}\left(\psi_{l,l} - \frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right) + \sum_{j=1}^{l}\left(C_j^{+} - C_j^{-}\right)$$

Proof. We first note that we've already calculated what quite a lot of this equation is: indeed, from Proposition 6.3.7, we know that

$$\mathrm{d}\left(\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right) = -\sum_{j=1}^{l}\left(A_j + \overline{A_j} + B_j + \overline{B_j} + C_j^{-} + \overline{C_j}^{-}\right),$$

so what we shall show is the following:

(i) $\omega_{\mathcal{F}}(L)\left(\varphi_{l,l}^{\mathcal{F}}\right) = \left(\omega_{\mathcal{F}}(L)\left(\varphi_{KM}^{\mathcal{F}}\right)\right)\varphi_{0,l,l} + \sum_{j=1}^{l}\left(B_j + \overline{B_j} + C_j^{+}\right)$

(ii) $\mathrm{d}_{\mathcal{F}}\psi_{l,l} = \left(\mathrm{d}_{\mathcal{F}}\psi^{\mathcal{F}}\right)\varphi_{0,l,l} + \frac{1}{2}\sum_{j=1}^{l}\left(B_j + \overline{B_j}\right)$

(iii) $\mathrm{d}_V^{+}\psi_{l,l} = -\frac{1}{2}\sum_{j=1}^{l} A_j$ and $\mathrm{d}_V^{-}\psi_{l,l} = -\frac{1}{2}\sum_{j=1}^{l}\overline{A_j}$.

We first note that $\sum_j C_j^{\pm} = \sum_j \overline{C_j}^{\pm}$, which simplifies some of the above equations. We also note that the above implicitly uses the results of Lemma 6.1.8 - that $\mathrm{d}_{\mathcal{F}}\psi^{\mathcal{F}} = \omega_{\mathcal{F}}(L)\varphi_{KM}^{\mathcal{F}}$.

We start by showing part (i), the action of the lowering operator. We know from Lemma 5.2.3 that L acts in the Fock model as:

$$\omega_{\mathcal{F}}(L) = -4\pi\sum_{\gamma=1}^{2}\frac{\partial^2}{\partial z_\gamma''\partial z_\gamma'} + \frac{1}{4\pi}z_3''z_3'$$

The $\frac{1}{4\pi}z_3''z_3'$ acts purely linearly, so it makes sense to focus on the action of the derivatives - our main tool here will be the product rule.

$$\begin{aligned}
\frac{\partial^2}{\partial z_\gamma''\partial z_\gamma'}\varphi_{l,l} &= \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}\frac{\partial}{\partial z_\gamma'}\left(z_\alpha'\underline{z}_{\underline{\beta}}'\right)\frac{\partial}{\partial z_\gamma''}\left(z_{\alpha'}''\underline{z}_{\underline{\beta}'}''\right)\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^{*}\\
&= \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\left[\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}\delta_{\gamma\alpha}\underline{z}_{\underline{\beta}}'\delta_{\gamma\alpha'}\underline{z}_{\underline{\beta}'}''\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^{*}\right.\\
&\quad+\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}\frac{\partial}{\partial z_\gamma'}(z_\alpha')\underline{z}_{\underline{\beta}}'z_{\alpha'}''\frac{\partial}{\partial z_\gamma''}\left(\underline{z}_{\underline{\beta}'}''\right)\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^{*}
\end{aligned}$$

$$+\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}} z_{\alpha'}\frac{\partial}{\partial z'_\gamma}\left(\underline{z'_{\beta}}\right)\frac{\partial}{\partial z''_\gamma}(z''_{\alpha'})\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}$$

$$\left.+\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}} z_{\alpha'}\frac{\partial}{\partial z'_\gamma}\left(\underline{z'_{\beta}}\right)z''_{\alpha'}\frac{\partial}{\partial z''_\gamma}\left(\underline{z''_{\beta'}}\right)\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}\right].$$

We may use the identities in Lemma 6.3.4(iii) to write the above as:

$$\frac{\partial^2}{\partial z''_\gamma\partial z'_\gamma}=\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\left[\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}\delta_{\gamma\alpha}\underline{z'_\beta}\delta_{\gamma\alpha'}\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}\right.$$

$$+\frac{-i}{2\sqrt{2}\pi}\sum_{j=1}^{l}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}\frac{\partial}{\partial z'_\gamma}(z'_\alpha)\underline{z'_\beta}z''_{\alpha'}\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v_\beta}\otimes A_j(v^*_\gamma)\underline{v^*_{\beta'}} \tag{6.3.12}$$

$$+\frac{-i}{2\sqrt{2}\pi}\sum_{j=1}^{l}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}z'_\alpha\underline{z'_\beta}\frac{\partial}{\partial z''_\gamma}(z''_{\alpha'})\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes A_j(v_\gamma)\underline{v_\beta}\otimes\underline{v^*_{\beta'}} \tag{6.3.13}$$

$$\left.+\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2}\sum_{j,k=1}^{l}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}z'_\alpha\underline{z'_\beta}z''_{\alpha'}\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes A_j(v_\gamma)\underline{v_\beta}\otimes A_k(v^*_\gamma)\underline{v^*_{\beta'}}\right], \tag{6.3.14}$$

where the $\underline{\beta}$, $\underline{\beta}'$ sums in (6.3.12), (6.3.13) and (6.3.14) are respectively over $\{1,2\}^l$ & $\{1,2\}^{l-1}$, $\{1,2\}^{l-1}$ & $\{1,2\}^l$ and $\{1,2\}^{l-1}$ & $\{1,2\}^{l-1}$. Hence, summing over γ, and re-arranging the sums (which is harmless, as they are all over finite sets), we find:

$$-4\pi\sum_{\gamma=1}^{2}\frac{\partial^2}{\partial z''_\gamma\partial z'_\gamma}\varphi_{l,l}=\frac{1}{2\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l}\sum_{\underline{\beta},\underline{\beta}'}\underline{z'_\beta}\underline{z''_{\beta'}}\otimes\left(\xi_1\wedge\overline{\xi_1}+\xi_2\wedge\overline{\xi_2}\right)\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}$$

$$+\underbrace{\sum_{j=1}^{l}i\sqrt{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\gamma=1}^{2}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}\frac{\partial}{\partial z'_\gamma}(z'_\alpha)\underline{z'_\beta}z''_{\alpha'}\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes\underline{v_\beta}\otimes A_j(v^*_\gamma)\underline{v^*_{\beta'}}}_{=\overline{B_j}}$$

$$+\underbrace{\sum_{j=1}^{l}i\sqrt{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\gamma=1}^{2}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}z'_\alpha\underline{z'_\beta}\frac{\partial}{\partial z''_\gamma}(z''_{\alpha'})\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes A_j(v_\gamma)\underline{v_\beta}\otimes\underline{v^*_{\beta'}}}_{=B_j}$$

$$+\underbrace{\sum_{j=1}^{l}\frac{1}{2\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\gamma=1}^{2}\sum_{k=1}^{l}\sum_{\substack{\alpha,\alpha'\\ \underline{\beta},\underline{\beta}'}}z'_\alpha\underline{z'_\beta}z''_{\alpha'}\underline{z''_{\beta'}}\otimes\xi_{\alpha'}\wedge\overline{\xi_\alpha}\otimes A_j(v_\gamma)\underline{v_\beta}\otimes A_k(v^*_\gamma)\underline{v^*_{\beta'}}.}_{=C_j^+}$$

Hence, we may write

$$\begin{aligned}\omega_{\mathcal{F}}(L)\left(\varphi_{l,l}\right) &= \frac{1}{4\pi}z_3'z_3''\varphi_{l,l} + \frac{1}{2\pi}\left(1\otimes\left(\sum_{\gamma=1}^{2}\xi_\gamma\wedge\overline{\xi_\gamma}\right)\otimes 1\otimes 1\right)\cdot\varphi_{0,l,l}\\ &\quad+\sum_{j=1}^{l}\overline{B_j}+\sum_{j=1}^{l}B_j+\sum_{j=1}^{l}C_j^+\\ &= \left(\omega_{\mathcal{F}}(L)\left(\varphi_{KM}\right)\right)\varphi_{0,l,l}+\sum_{j=1}^{l}\left(B_j+\overline{B_j}+C_j^+\right),\end{aligned}$$

which completes the proof of (i). Next, we show part (ii), the action of $\mathrm{d}_{\mathcal{F}}$. From (6.1.5), we have the explicit algebraic form given by

$$\psi^{\mathcal{F}} = \frac{1}{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^2\sum_{\alpha=1}^{2}\left(-z_3'z_\alpha''\otimes\xi_\alpha+z_3''z_\alpha'\otimes\overline{\xi_\alpha}\right),$$

so we write:

$$\psi_{l,l}^{\mathcal{F}} = \frac{1}{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\alpha,\underline{\beta},\underline{\beta}'}\left(-z_3'z_\alpha''\underline{z}_{\underline{\beta}}'\underline{z}_{\underline{\beta}'}''\otimes\xi_\alpha\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^*+z_3''z_\alpha'\underline{z}_{\underline{\beta}}'\underline{z}_{\underline{\beta}'}''\otimes\overline{\xi_\alpha}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^*\right) \tag{6.3.15}$$

We recall the forms of $\mathrm{d}_{\mathcal{F}} = \mathrm{d}_{\mathcal{F}}^+ + \mathrm{d}_{\mathcal{F}}^-$ from (6.1.2). The action of $\mathrm{d}_{\mathcal{F}}^+$ is completely multiplicative, so we may write

$$\mathrm{d}_{\mathcal{F}}^+\left(\psi_{l,l}\right) = \left(\mathrm{d}_{\mathcal{F}}^+\left(\psi^{\mathcal{F}}\right)\right)\cdot\varphi_{0,l,l},$$

and hence we may focus on the action of $\mathrm{d}_{\mathcal{F}}^2$. By the usual argument - see e.g. (6.3.8) - for the differential operators in $\wedge^\bullet\mathfrak{p}^*$, we may discard terms of the form $\xi_\gamma\wedge\xi_\alpha$, $\overline{\xi_\gamma}\wedge\xi_\alpha$ in $\mathrm{d}_{\mathcal{F}}^-\psi_{l,l}^{\mathcal{F}}$. Hence, we may write:

$$\begin{aligned}\mathrm{d}_{\mathcal{F}}^-\psi_{l,l}^{\mathcal{F}} = -2\pi\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}&\Bigg[\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta}'}\frac{\partial^2}{\partial z_\gamma'\partial z_3''}\left(z_3''z_\alpha'\underline{z}_{\underline{\beta}}'\underline{z}_{\underline{\beta}'}''\right)\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^*\\ &+\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta}'}\frac{\partial^2}{\partial z_\gamma''\partial z_3'}\left(z_3'z_\alpha''\underline{z}_{\underline{\beta}}'\underline{z}_{\underline{\beta}'}''\right)\otimes\xi_\alpha\wedge\overline{\xi_\gamma}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^*\Bigg].\end{aligned} \tag{6.3.16}$$

Our first observation is that the first term (resp. the second term) in (6.3.16) has exactly one z_3'' and one $\partial/\partial z_3''$ (resp. one z_3' and one $\partial/\partial z_3'$) term, so these may be moved to the front of the equation - though we keep them in some of the terms to help our correspondence with the action of $\mathrm{d}_{\mathcal{F}}^-$. We may hence split this equation using the chain rule:

$$\mathrm{d}_{\mathcal{F}}^-\psi_{l,l}^{\mathcal{F}} = -2\pi\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\Bigg[\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta}'}\underline{z}_{\underline{\beta}}'\underline{z}_{\underline{\beta}'}''\frac{\partial^2}{\partial z_\gamma'\partial z_3''}\left(z_3''z_\alpha'\right)\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes\underline{v}_{\underline{\beta}}\otimes\underline{v}_{\underline{\beta}'}^*$$

$$+\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} \underline{z'_{\beta}}\,\underline{z''_{\beta'}}\frac{\partial^2}{\partial z''_{\gamma}\partial z'_3}\left(z'_3 z''_{\alpha}\right)\otimes\xi_\alpha\wedge\overline{\xi_\gamma}\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}\Bigg]$$

$$-2\pi\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\Bigg[\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z'_\alpha \underline{z''_{\beta'}}\frac{\partial}{\partial z'_\gamma}\left(\underline{z'_{\beta}}\right)\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}$$

$$+\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z''_\alpha \underline{z'_{\beta}}\frac{\partial}{\partial z''_\gamma}\left(\underline{z''_{\beta'}}\right)\otimes\xi_\alpha\wedge\overline{\xi_\gamma}\otimes\underline{v_\beta}\otimes\underline{v^*_{\beta'}}\Bigg]. \tag{6.3.17}$$

The first two lines of (6.3.17) are easily recognisable as equal to $(\mathrm{d}^-_{\mathcal{F}}(\psi^{\mathcal{F}}))\cdot\varphi_{0,l,l}$; we may use the relations of Lemma 6.3.4(iii) to rewrite the remaining two lines as follows:

$$\begin{aligned}
\mathrm{d}^-_{\mathcal{F}}\psi^{\mathcal{F}}_{l,l} &= \left(\mathrm{d}^-_{\mathcal{F}}\left(\psi^{\mathcal{F}}\right)\right)\cdot\varphi_{0,l,l} - 2\pi\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+3}\sum_{j=1}^{l}\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z'_\alpha\underline{z''_{\beta'}}\,\underline{z'_\beta}\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes A_j(v_\gamma)\underline{v_\beta}\otimes\underline{v^*_{\beta'}}\\
&\quad - 2\pi\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+3}\sum_{j=1}^{l}\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z''_\alpha\underline{z'_{\beta}}\,\underline{z''_{\beta'}}\otimes\xi_\alpha\wedge\overline{\xi_\gamma}\otimes \underline{v_\beta}\otimes A_j(v^*_\gamma)\underline{v^*_{\beta'}}\\
&= \left(\mathrm{d}^-_{\mathcal{F}}\left(\psi^{\mathcal{F}}\right)\right)\cdot\varphi_{0,l,l} + \frac{1}{2}\sum_{j=1}^{l}\underbrace{i\sqrt{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z'_\alpha\underline{z''_{\beta'}}\,\underline{z'_\beta}\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes A_j(v_\gamma)\underline{v_\beta}\otimes\underline{v^*_{\beta'}}}_{=B_j}\\
&\quad + \frac{1}{2}\sum_{j=1}^{l}\underbrace{i\sqrt{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z''_\alpha\underline{z'_{\beta}}\,\underline{z''_{\beta'}}\otimes\xi_\alpha\wedge\overline{\xi_\gamma}\otimes \underline{v_\beta}\otimes A_j(v^*_\gamma)\underline{v^*_{\beta'}}}_{=\overline{B_j}}\\
&= \left(\mathrm{d}^-_{\mathcal{F}}\left(\psi^{\mathcal{F}}\right)\right)\cdot\varphi_{0,l,l} + \frac{1}{2}\sum_{j=1}^{l}B_j + \frac{1}{2}\sum_{j=1}^{l}\overline{B_j};
\end{aligned}$$

hence we have shown part (ii). We finish by showing part (iii), the action of d_V; by the symmetry of the operators, we only show the d^+_V action, as the d^-_V action is essentially identical.

By the same logic as in the proof of parts (i) and (ii), using e.g. (6.3.8), when applying d_V to $\psi^{\mathcal{F}}_{l,l}$ we may discard all terms of the form $\xi_\gamma\wedge\xi_\alpha$, $\overline{\xi_\gamma}\wedge\overline{\xi_\alpha}$. doing this, we find

$$\begin{aligned}
\mathrm{d}^+_V\psi^{\mathcal{F}}_{l,l} =& \frac{1}{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2}\Bigg[\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z''_3 z'_\alpha\underline{z'_\beta}\,\underline{z''_{\beta'}}\otimes\xi_\gamma\wedge\overline{\xi_\alpha}\otimes\rho_V(Z'_\gamma)(\underline{v_\beta})\otimes\underline{v^*_{\beta'}}\\
&+\sum_{\gamma,\alpha,\underline{\beta},\underline{\beta'}} z'_3 z''_\alpha\underline{z'_\beta}\,\underline{z''_{\beta'}}\otimes\xi_\alpha\wedge\overline{\xi_\gamma}\otimes\rho_V(Z''_\gamma)(\underline{v_\beta})\otimes\underline{v^*_{\beta'}}\Bigg].
\end{aligned}$$

We know from our calculations in Lemma 6.3.4(i) how ρ_V will act on the symmetric products $\underline{v_\beta}$. For any multi-index $\underline{\beta}$, we let $\hat{\underline{\beta}}_j = (\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_l)$ be the multi-index with β_j excluded - we may write the above as

$$\mathrm{d}_V^+\psi_{l,l}^{\mathcal{F}} = \frac{1}{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2} \sum_{\gamma,\alpha,\underline{\beta},\underline{\beta}'} z_3' z_\alpha'' \underline{z}_{\underline{\beta}}' \underline{z}_{\underline{\beta}'}'' \otimes \xi_\alpha \wedge \overline{\xi_\gamma} \otimes \left(\sum_{j=1}^{l} \delta_{\beta_j\gamma} A_j(v_3) \underline{v}_{\underline{\hat{\beta}}_j}\right) \otimes \underline{v}_{\underline{\beta}'}^*,$$

Keeping the indices $\underline{\beta}$, $\underline{\beta}' \in \{1,2\}^l$, and for any given $1 \le j \le l$:

$$\begin{aligned}
\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} & \sum_{\alpha,\gamma,\underline{\beta},\underline{\beta}'} \delta_{\beta_j\gamma} z_\alpha'' \underline{z}_{\underline{\beta}}' \underline{z}_{\underline{\beta}'}'' \otimes \xi_\alpha \wedge \overline{\xi_\gamma} \otimes \underline{v}_{\underline{\hat{\beta}}_j} \otimes \underline{v}_{\underline{\beta}'}^* \\
&= \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \sum_{\alpha,\gamma,\underline{\beta},\underline{\beta}'} \delta_{\beta_j\gamma} z_\alpha'' z_{\beta_j}' \underline{z}_{\underline{\hat{\beta}}_j}' \underline{z}_{\underline{\beta}'}'' \otimes \xi_\alpha \wedge \overline{\xi_\gamma} \otimes \underline{v}_{\underline{\hat{\beta}}_j} \otimes \underline{v}_{\underline{\beta}'}^* \\
&= \left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+1} \sum_{\alpha,\gamma,\underline{\hat{\beta}}_j,\underline{\beta}'} z_\gamma' z_\alpha'' \underline{z}_{\underline{\hat{\beta}}_j}' \underline{z}_{\underline{\beta}'}'' \otimes \xi_\alpha \wedge \overline{\xi_\gamma} \otimes \underline{v}_{\underline{\hat{\beta}}_j} \otimes \underline{v}_{\underline{\beta}'}^* \\
&= \varphi_{l-1,l}^{\mathcal{F}}.
\end{aligned}$$

Hence we have:

$$\begin{aligned}
\mathrm{d}_V^+\psi_{l,l}^{\mathcal{F}} &= \frac{-i}{4\sqrt{2}\pi} \sum_{j=1}^{l} \left(z_3' \otimes 1 \otimes A_j(v_3) \otimes 1\right) \varphi_{l-1,l}^{\mathcal{F}} \\
&= -\frac{1}{2}\sum_{j=1}^{l} A_j.
\end{aligned}$$

The proof of the d_V^- action is identical, so we shall skip it. □

We have already given a small amount of motivation, at the beginning of this section, for the final theorem we shall prove; we invite the reader to revisit this, as the duality problem is one that should be kept at the back of the mind throughout; indeed, the relevant duality calculation (which, *a priori*, looks very daunting) is only a few lines' work once the following result is shown.

In the Schrödinger model, we define operators

$$\begin{aligned}
G_{j,\mathcal{S}} &= \sum_{\alpha=1}^{3} \left(z_\alpha \otimes 1 \otimes A_j(v_\alpha) \otimes 1\right) = 1 \otimes 1 \otimes A_j\left(\sum_{\alpha=1}^{3} z_\alpha v_\alpha\right) \otimes 1 \\
\overline{G_{j,\mathcal{S}}} &= \sum_{\alpha=1}^{3} \left(\overline{z_\alpha} \otimes 1 \otimes 1 \otimes A_j(v_\alpha^*)\right) = 1 \otimes 1 \otimes 1 \otimes A_j\left(\sum_{\alpha=1}^{3} (z_\alpha v_\alpha)^*\right)
\end{aligned}$$

where the second equality comes from the linearity of the vector products. As one may see, by inspection, these operators correspond to the insertion of the vector $z_1v_1 + z_2v_2 + z_3v_3$ in the j'th slot of the vector product and its dual respectively.

Theorem 6.3.9. *For any $1 \leq j \leq l$, the following holds:*

$$\begin{aligned}\varphi_{l,l} &= G_j\left(\varphi_{l-1,l}\right) + \mathrm{d}\Lambda_{j,l} - \left(C_j^+ - C_j^-\right)\\ &= \overline{G_j}\left(\varphi_{l,l-1}\right) + \mathrm{d}\overline{\Lambda_{j,l}} - \left(\overline{C_j}^+ - \overline{C_j}^-\right)\end{aligned}$$

Proof. Because of the convenience we have found in working in the Fock model, it makes sense to start by find these operators in the Fock complex: for this, we use the intertwiners in Lemma 5.3.2. We may use the intertwiner notation from Lemma 5.3.2) to write $G_{j,\mathcal{F}} = \mathcal{J}^{-1}G_{j,\mathcal{S}}\mathcal{J}$ as

$$\begin{aligned}G_j =& i\sqrt{2}\left(\sum_{\alpha=1}^{2}\left(\left(\frac{\partial}{\partial z''_\alpha} - \frac{1}{4\pi}z'_\alpha\right)\otimes 1\otimes A_j(v_\alpha)\otimes 1\right) - \left(\frac{\partial}{\partial z''_3} - \frac{1}{4\pi}z'_3\right)\otimes 1\otimes A_j(v_3)\otimes 1\right)\\ =& \frac{-i}{2\sqrt{2}\pi}\left(\sum_{\alpha=1}^{2}\left(z'_\alpha\otimes 1\otimes A_j(v_\alpha)\otimes 1\right) - z'_3\otimes 1\otimes A_j(v_3)\otimes 1\right) \qquad (6.3.18)\\ &+ i\sqrt{2}\left(\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z''_\alpha}\otimes 1\otimes A(v_\alpha)\otimes 1\right) - \frac{\partial}{\partial z''_3}\otimes 1\otimes A_j(v_3)\otimes 1\right). \qquad (6.3.19)\end{aligned}$$

Herein, we write the expression in (6.3.18) as G'_j, and that in (6.3.19) as G''_j, so that $G_j = G'_j + G''_j$. The operator $\overline{G_{j,\mathcal{S}}}$ may similarly be written in the Fock model as

$$\begin{aligned}\overline{G_j} =& i\sqrt{2}\left(\sum_{\alpha=1}^{2}\left(\left(\frac{\partial}{\partial z'_\alpha} - \frac{1}{4\pi}z''_\alpha\right)\otimes 1\otimes 1\otimes A_j(v^*_\alpha)\right) - \left(\frac{\partial}{\partial z'_3} - \frac{1}{4\pi}z''_3\right)\otimes 1\otimes 1\otimes A_j(v_3*)\right)\\ =& \frac{-i}{2\sqrt{2}\pi}\left(\sum_{\alpha=1}^{2}\left(z''_\alpha\otimes 1\otimes 1\otimes A_j(v^*_\alpha)\right) - z''_3\otimes 1\otimes 1\otimes A_j(v^*_3)\right) \qquad (6.3.20)\\ &+ i\sqrt{2}\left(\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z'_\alpha}\otimes 1\otimes 1\otimes A(v^*_\alpha)\right) - \frac{\partial}{\partial z'_3}\otimes 1\otimes 1\otimes A_j(v^*_3)\right); \qquad (6.3.21)\end{aligned}$$

analogously we write the expression in (6.3.20) as $\overline{G_j}'$ and the expression in (6.3.21) as $\overline{G_j}''$, so that $\overline{G_j} = \overline{G_j}' + \overline{G_j}''$.

Applying all of these operators to $\varphi_{l-1,l}$ and $\varphi_{l,l-1}$ respectively, we have:

$$G'_j\left(\varphi_{l-1,l}\right) = \varphi_{l,l} + \underbrace{\frac{i}{2\sqrt{2}\pi}\left(z'_3\otimes 1\otimes A_j(v_3)\otimes 1\right)\varphi_{l-1,l}}_{=A_j}$$

and

$$\begin{aligned}G''_j\left(\varphi_{l-1,l}\right) &= i\sqrt{2}\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z''_\alpha}\otimes 1\otimes A_j(v_\alpha)\otimes 1\right)\varphi_{l-1,l}\\ &= i\sqrt{2}\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z''_\alpha}\otimes 1\otimes 1\otimes 1\right)\varphi_{KM}\cdot\left(1\otimes 1\otimes A_j(v_\alpha)\otimes 1\right)\varphi_{0,l-1,l}\\ &\quad + i\sqrt{2}\varphi_{KM}\cdot\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z''_\alpha}\otimes 1\otimes A_j(v_\alpha)\otimes 1\right)\varphi_{0,l-1,l}\end{aligned}$$

$$= \underbrace{i\sqrt{2}\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z''_\alpha}\otimes 1\otimes 1\otimes 1\right)\varphi_{KM}\cdot(1\otimes 1\otimes A_j(v_\alpha)\otimes 1)\,\varphi_{0,l-1,l}}_{=B_j}$$
$$+\underbrace{\frac{1}{2\pi}\varphi_{KM}\cdot\sum_{\alpha=1}^{2}\sum_{k=1}^{l}(1\otimes 1\otimes A_j(v_\alpha)\otimes A_k(v^*_\alpha))\,\varphi_{0,l-1,l-1}}_{=C_j^+},$$

so that $G_j\varphi_{l-1,l} = \varphi_{l,l} + A_j + B_j + C_j^+$. Similarly, we have

$$\overline{G_j}'\,(\varphi_{l,l-1}) = \varphi_{l,l} + \underbrace{\frac{i}{2\sqrt{2}\pi}\,(z''_3\otimes 1\otimes 1\otimes A_j(v^*_3))\,\varphi_{l,l-1}}_{=\overline{A_j}}$$

and

$$\begin{aligned}\overline{G_j}''\,(\varphi_{l,l-1}) &= i\sqrt{2}\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z'_\alpha}\otimes 1\otimes 1\otimes A_j(v^*_\alpha)\right)\varphi_{l,l-1}\\
&= i\sqrt{2}\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z'_\alpha}\otimes 1\otimes 1\otimes 1\right)\varphi_{KM}\cdot(1\otimes 1\otimes 1\otimes A_j(v^*_\alpha))\,\varphi_{0,l,l-1}\\
&\quad + i\sqrt{2}\varphi_{KM}\cdot\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z'_\alpha}\otimes 1\otimes 1\otimes A_j(v^*_\alpha)\right)\varphi_{0,l,l-1}\\
&= \underbrace{i\sqrt{2}\sum_{\alpha=1}^{2}\left(\frac{\partial}{\partial z'_\alpha}\otimes 1\otimes 1\otimes 1\right)\varphi_{KM}\cdot(1\otimes 1\otimes 1\otimes A_j(v^*_\alpha))\,\varphi_{0,l,l-1}}_{=\overline{B_j}}\\
&\quad \underbrace{+\frac{1}{2\pi}\varphi_{KM}\cdot\sum_{\alpha=1}^{2}\sum_{k=1}^{l}(1\otimes 1\otimes A_k(v_\alpha)\otimes A_j(v^*_\alpha))\,\varphi_{0,l-1,l-1}}_{=\overline{C_j}^+}.\end{aligned}$$

so that $\overline{G_j}\varphi_{l,l-1} = \varphi_{l,l} + \overline{A_j} + \overline{B_j} + \overline{C_j}^+$.

All we need use now is the calculations on $\mathrm{d}\Lambda_{j,l}$ and $\mathrm{d}\overline{\Lambda_{j,l}}$ from Proposition 6.3.7; this completes the proof. □

6.4 The Extension of the Kudla-Millson Result to Higher Weights

We have now reached the point where the proof of our first main vector-valued theorem is possible; namely, about the modularity of the theta series attached to the Schwartz form $\varphi_{l,l}^{\mathcal{S},\mathcal{H}}$. On top of this, because of the duality result in Theorem 6.3.9, we will also be able to prove Theorem 4.1.5.

The reader has hopefully been illuminated as to why the delay between the statement of Theorem 4.1.5 and the proof of its main result exists. In §1, we saw several preliminary approaches to proofs about modular forms coming from special cycles; however, these largely relied on the specific arithmetic of the group Γ, and were in particular quite specialised proofs, using Hirzebruch-Zagier methods. In this work, we have made no assumptions on Γ, beyond it being small enough to be torsion-free (in equivalent work - see e.g. [FM11, §10] - even this condition may be relaxed), and hence we have needed the constructions using the Weil representation. It is here that this generalised approach shows its power.

Theorem 6.4.1. *Let $\Gamma \subset \underline{G}$ be a torsion-free arithmetic subgroup of the group $\underline{G}$, and let $L \subset \underline{V}$ be a full and integral level M lattice in the k-vector space $\underline{V}$ such that Γ acts trivially on all the cosets $\mathcal{L} = L + h$ of L'/L.*

We fix a positive integer $l \geq 1$, and work in the Schrödinger model, with all our forms understood to have coefficients in the harmonic subspace $\mathcal{H}^{l,l}(V) \subset S^{l,l}(V)$. Then there exists a closed, non-trivial Schwartz form

$$\varphi_{l,l}^{\mathcal{S},\mathcal{H}} \in \left[\mathcal{S}(V) \otimes \Omega^2\left(\mathbb{D}\right) \otimes \mathcal{H}^{l,l}(V)\right]^G.$$

We may form a theta series $\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)$ which defines a closed differential form on X with coefficients in $\mathcal{H}^{l,l}(V)$; this theta series converges uniformly on compact subsets of X. Moreover, it is a non-holomorphic modular form of weight $2l + 3$:

$$\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau) := \sum_{\boldsymbol{x} \in \mathcal{L}} \varphi_{l,l}^{\mathcal{S},\mathcal{H}}(\boldsymbol{x}, z, \tau) \in \Omega^2(X) \otimes \mathcal{H}^{l,l}(V) \otimes M_{3+2l}^{NonHol}(\Gamma(M)). \tag{6.4.1}$$

All the φ are closed, and so taking it as a cohomology class, it defines a cuspidal holomorphic modular form in τ:

$$[\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)] \in H^2\left(X, \widetilde{\mathcal{H}^{l,l}(V)}\right) \otimes S_{3+2l}(\Gamma(M)).$$

Moreover, the coefficients of $q^n = e^{2\pi i n \tau}$ in this modular form are given by duals of the special cycles $C_{n,[l,l]}$ defined in Proposition 4.1.3; in cohomology, we may write:

$$[\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)] = i \sum_{\substack{n>0 \\ n \text{ an } \mathcal{L}-norm}} \left[C_{n,[l,l]}\right]^{PD} q^n.$$

Hence, for some closed and rapidly decreasing $\mathcal{H}^{l,l}(V)$-valued smooth differential 2-form η on X, the generating series

$$\int_X \theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau) \wedge \eta = i \sum_{\substack{n>0 \\ n \text{ an } \mathcal{L}-norm}} \left[\int_{C_{n,[l,l]}} \eta\right] q^n \in S_{3+2l}(\Gamma(M)) \tag{6.4.2}$$

is a holomorphic modular form, with coefficients given by the integrals against the $C_{n,[l,l]}$.

Proof. Throughout this proof we will be using the G-isomorphism between the complexes

$$\pi : [\mathcal{S}(V) \otimes \wedge^\bullet \mathfrak{p}^* \otimes E]^K \xrightarrow{\cong} [\mathcal{S}(V) \otimes \Omega^\bullet(\mathbb{D}) \otimes E]^G$$

from Proposition 6.1.2. The existence of $\varphi_{l,l}$ for $l = 0$ was proven by Kudla and Millson - see Theorem 6.1.13 - and the form $\varphi_{l,l}$ with coefficients that we constructed in Definition 6.2.2 was proven in Proposition 6.3.5 to be closed.

$\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)$ defines a differential form on X (and not just on $\mathbb{D}$) because we have shown that $\varphi_{l,l}^{\mathcal{S},\mathcal{H}}$ is G-invariant; hence it is in particular Γ-invariant. Further, we have assumed that $\mathcal{L}$ is Γ-invariant, so that the entirety of the sum is Γ-invariant - hence, as a function of the co-ordinates $z \in \mathbb{D}$, it is Γ-invariant, so that it defines a differential form on $\Gamma \backslash \mathbb{D} = X$.

As it is a closed form - by Proposition 6.3.5 - we may take this theta series as a cohomology class with coefficients in $\mathcal{H}^{l,l}(V)$:

$$[\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)] = \left[\sum_{\mathbf{x} \in \mathcal{L}} \varphi_{l,l}^{\mathcal{S},\mathcal{H}}(\mathbf{x}, z, \tau)\right] \in H^2\left(X, \widetilde{\mathcal{H}^{l,l}(V)}\right).$$

The element $\sum_j (C_j^+ - C_j^-)$ projects to 0 in the coefficient system $\mathcal{H}^{l,l}(V)$; hence using the results of Theorem 6.3.8, the cohomology element $[\omega_{\mathcal{F}}(L)(\varphi_{l,l}^{\mathcal{F}})]$ equals $[0]$ in this cohomology group, as it is exact. Immediately, this gives us that this is a *holomorphic* function of $\tau \in \mathbb{H}$, as $\omega(L) = -2iv\partial_{\overline{\tau}}$ once τ is inserted.

We now address modularity. Using the isomorphism from Proposition 6.1.2, we may take $\varphi_{l,l}$ as a differential form on $\mathbb{D}$ with coefficients in the representation. From Lemma 6.3.2, we know that it is closed, so defines a cohomology class in $H^2(\mathbb{D}, \widetilde{\mathcal{H}^{l,l}(V)})$. From the general theory of the Weil representation, it follows from the above and Proposition 6.3.5 - which gives us that it is an eigenvector of weight $2l + 3$ under the action of the maximally compact subgroup $K' \subset G'$ - that the associated theta series defined in (6.4.1) has a modular transformation law of weight $2l + 3$ with respect to the correct congruence subgroup. For further reading on this from a theoretical standpoint, see e.g. [KM87, §4 & 5], and in particular Theorem 5.2 from this work.

The results of Theorem 6.3.9 tell us that in the cohomology with harmonic coefficients, $[\varphi_{l,l}] = [G_j \varphi_{l-1,l}] = [\overline{G_j} \varphi_{l,l-1}]$ for all j. Hence, we may define a new Schwartz form in the Schrödinger model as follows:

$$\varphi_{l,l,G}(\mathbf{x}) = [1 \otimes 1 \otimes A(\mathbf{x}) \otimes 1]^l \circ [1 \otimes 1 \otimes 1 \otimes A(\mathbf{x})]^l (\varphi_{KM}(\mathbf{x}))$$

$$= \varphi_{KM}(\mathbf{x}) \otimes \mathbf{x}^l \otimes (\mathbf{x}^*)^l$$

So, using Theorem 6.3.9 repeatedly, it follows that

$$\begin{aligned}\varphi^{\mathcal{H}}_{l,l,G}(\mathbf{x}) &= (1 \otimes 1 \otimes \pi_{\mathcal{H}})\left(\varphi_{l,l,G}(\mathbf{x})\right) \\ &= (1 \otimes 1 \otimes \pi_{\mathcal{H}})\left(\prod_{j=1}^{l}\left(G_j \circ \overline{G_j}\right)\left(\varphi^{\mathcal{S}}_{l,l}(\mathbf{x})\right)\right)\end{aligned}$$

satisfies $\varphi^{\mathcal{H}}_{l,l,G}(\mathbf{x}) = \varphi^{\mathcal{H}}_{l,l}(\mathbf{x}) + \mathrm{d}\beta$ for all $\mathbf{x}$ and for β some differential form on X.

Let $\mathbf{x} \in \underline{V}$ be any non-negative vector. By Stokes' theorem, integrals of exact forms are 0, so for η any compactly supported and rapidly decreasing differential form, we have

$$\int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}} \varphi^{\mathcal{H}}_{l,l}(\mathbf{x}, z, \tau) \wedge \eta = \int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}} \varphi^{\mathcal{H}}_{l,l,G}(\mathbf{x}, z, \tau) \wedge \eta.$$

Locally, we may write (without loss of generality) $\eta = \omega \otimes v$. Hence, using the inner product on $\mathcal{H}^{l,l}(V)$ given by extending (,) to the symmetric product, as well as the corresponding Kudla-Millson result from (6.1.13) in Theorem 6.1.13, we find:

$$\begin{aligned}\int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}} \varphi^{\mathcal{H}}_{l,l}(\mathbf{x}, z, \tau) \wedge \eta &= \int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}} \varphi_{KM}(\mathbf{x}, z, \tau) \wedge \omega \cdot \left(v, \pi_{\mathcal{H}}\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right)\right) \\ &= \left[ie^{\pi i(\mathbf{x},\mathbf{x})\tau} \int_{C_{\mathbf{x}}} \omega\right] \cdot \left(v, \pi_{\mathcal{H}}\left(\mathbf{x}^l \otimes (\mathbf{x}^*)^l\right)\right). \qquad (6.4.3)\end{aligned}$$

However, we know that the integral of a section of $\wedge^k T^* \otimes E$ over a k-cycle with coefficients in E will also be given by taking the pairing in the fibre; hence, we may write (6.4.3) as:

$$\int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}} \varphi^{\mathcal{H}}_{l,l}(\mathbf{x}, z, \tau) \wedge \eta = ie^{\pi i(\mathbf{x},\mathbf{x})\tau} \int_{C_{\mathbf{x},[l,l]}} \eta$$

and so duality is established for $\mathbf{x}$ of non-negative length. A completely identical argument - namely, using that

$$\int_{\Gamma_{\mathbf{x}}\backslash\mathbb{D}} \varphi_{KM}(\mathbf{x}, z, \tau) \wedge \omega = 0$$

for all negative-length $\mathbf{x}$ and closed scalar differential forms ω, shown in Theorem 6.1.13 - gives us the exactness of $\varphi^{\mathcal{H}}_{l,l}(\mathbf{x})$ for all such $\mathbf{x}$.

We also comment that the constant coefficient also vanishes here, unlike in the case $l = 0$; indeed, in that case, it is given by integration against the Chern form $c_1(X)$. However, in the case $l = 0$, another application of the above homotopy argument shows the constant coefficient (which, by definition, will be the parts of $\varphi_{l,l}$ given by $\mathbf{x} = 0$) to be given by integration against

$$c_1(X) \otimes 0^l \otimes (0^*)^l = 0$$

and hence will be identically zero; in particular, we may say here that these forms will not only be modular, but will be cuspidal.

Combining these two results gives us the result for the form of the Fourier coefficients, by summing over all $\mathbf{x} \in \mathcal{L}$. □

Corollary 6.4.2. *Theorem 4.1.5 is proven; namely, the special cycles sum with coefficients is modular of weight* $2l + 3$.

Proof. This is a restatement of (6.4.2) in Theorem 6.4.1. □

Chapter 7

Restriction to Boundary Components

Given our work in §4 on special cycles and their restriction, and in §6 on the creation of new Schwartz forms and theta series, which give us geometric modular forms of generic odd weight ≥ 3, one of the outstanding problems is the restriction of these theta series to the boundary components of $\overline{X}^{BS}$. In this section, we will aim to prove Theorem 1.2.3 from the introduction, using geometric techniques, the mixed model of the Weil representation, Fourier transforms and Poisson summation. This will draw heavily on equivalent work done by Funke and Millson, primarily in the orthogonal setting - for example, [FM11], [FM13] and [FM14].

There will, however, also be some arithmetic near the end! What we may turn the proof into (once most of the geometry has been sorted) is a series of combinatorial proofs about the vanishing of coefficients of $\widehat{\varphi_{l,l}}$, which is quite interesting in its own right as a result in combinatorics.

7.1 Fourier Transforms of Laguerre Polynomials

We start with a redefinition of the Schwartz forms from Lemma 6.2.3.

Definition 7.1.1. Fix an integer l; then in the Schrödinger model, we may write the Schwartz form with coefficients in $\mathcal{H}^{l,l}(V) = \pi_{\mathcal{H}}\Big(\mathrm{Sym}^l(V) \otimes \mathrm{Sym}^l(V^*)\Big)$ as

$$\varphi_{l,l}(\mathbf{x}) = \frac{1}{2^{2l+2}} \sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta}'}} \left[\underline{\mathcal{D}}_{\underline{\beta}} \circ \overline{\underline{\mathcal{D}}}_{\underline{\beta}'} \circ \mathcal{D}_{\alpha} \circ \overline{\mathcal{D}}_{\alpha'}\right] (\varphi_0)(\mathbf{x}) \otimes \xi_{\alpha'} \wedge \overline{\xi_{\alpha}} \otimes \pi_{\mathcal{H}}\left(\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}\right) \tag{7.1.1}$$

$$\in \left[\mathcal{S}(V) \otimes \wedge^{1,1}\mathfrak{p} \otimes \mathcal{H}^{l,l}(V)\right]^K,$$

where $\mathcal{D}_\gamma = z_\gamma - \frac{1}{\pi}\frac{\partial}{\partial \overline{z_\gamma}}$ and $\overline{\mathcal{D}_\gamma} = \overline{z_\gamma} - \frac{1}{\pi}\frac{\partial}{\partial z_\gamma}$ for $\gamma = 1,\, 2$. By using the isomorphism detailed in Definition 6.1.10, we may insert $z \in \mathbb{D}$ by acting with g_z on all the terms, and find the differential form on $\mathbb{D}$ written as:

$$\begin{aligned}\varphi_{l,l}(\mathbf{x}, z) &= \frac{1}{2^{2l+2}} \sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta}'}} \left[\underline{\mathcal{D}}_{\underline{\beta}} \circ \underline{\overline{\mathcal{D}}}_{\underline{\beta}'} \circ \mathcal{D}_\alpha \circ \overline{\mathcal{D}}_{\alpha'}\right](\varphi_0)\left(g_z^{-1}\mathbf{x}\right) \otimes \Xi_{\alpha'} \wedge \overline{\Xi_\alpha} \otimes \pi_{\mathcal{H}}\left(g_z\left(\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}\right)\right) \\ &\in \left[\mathcal{S}(V) \otimes \Omega^{1,1}(\mathbb{D}) \otimes \mathcal{H}^{l,l}(V)\right]^G,\end{aligned}$$

We note here that this notation $\underline{v}_{\underline{\beta}}$ is used repeatedly throughout this chapter, to mean a vector product $v_{\beta_1} \otimes v_{\beta_2} \otimes \ldots \otimes v_{\beta_l}$ - we similarly do this for polynomials in the Fock model, operators $\underline{\mathcal{D}}$, etc.

We now fix an isotropic vector ℓ, which we will assume without any loss of generality is one of our finite representatives of the cusps on $\overline{X}^{BS}$; hence, our relevant boundary component will be written $\iota_\ell : e(P_\ell) \hookrightarrow \overline{X}^{BS}$. We now fix a cusp ℓ of $\mathbb{D}$, and - as in the definition of the mixed model in §5.3 - fix a Witt splitting of $\underline{V}$ as

$$\underline{V} = k\ell \oplus W_\ell \oplus k\ell'.$$

Without any real loss of generality - as we may rescale the inner product to achieve this - we may assume the following

(i) $\underline{W_\ell}$ is spanned by a rational w_ℓ such that $||w_\ell||^2 = 1$;

(ii) $(\ell, \ell') = \delta_k^{-1}$

Indeed, what we shall see throughout this chapter is that these constants are not particularly important - the rubric shall be that the coefficients of ℓ and ℓ' will vanish at the boundary, and hence that the behaviour in $\underline{W_\ell}$ (which is orthogonal to the other co-ordinates) will be what survives.

Examining the form of the inner product in (2.1.12), we see that the inner product may hence be written (with respect to the above basis) as:

$$\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}\right) = |b|^2 - \frac{2\mathcal{I}\left(\overline{a}c\right)}{|\delta_k|}.$$

We recall from (2.2.2) that we have decomposed the real points of the parabolic subgroup $P = \underline{P}(\mathbb{R})$ fixing $k\ell$ as $P = N_P A_P M_P$; hence, on the level of Lie algebras, we immediately have the following direct sum:

$$\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m}_P. \tag{7.1.2}$$

We recall also from Definition 2.3.4 that for our group G, the parabolic part $\mathfrak{m}_P$ of $\mathfrak{m}$ will be trivial and the boundary component is realised by compactifying $A \to \overline{A} = \mathbb{R}_{>0} \cup \{\infty\}$, so that $e(P_\ell) = \Gamma_\ell \backslash N_\ell$. Hence, in our case, the maximally compact subgroup of P_ℓ may be denoted K_P.

The natural restriction to the boundary for Lie algebras will be to project $\mathfrak{p} \to \mathfrak{n}$ in (7.1.2). We recall our work on the mixed model of $\mathcal{S}(V)$ at each cusp ℓ, from §5.3. Throughout this chapter, we shall denote $\widehat{f}$ as the Fourier transform in the ℓ variable.

So, for any fixed cusp ℓ of $\mathbb{D}$, and for any G-representation E, we hence define the restriction map r_ℓ as follows:

$$r_\ell : [\mathcal{S}(V) \otimes \wedge^\bullet \mathfrak{p}^* \otimes E]^K \to [\mathcal{S}(W_\ell) \otimes \wedge^\bullet \mathfrak{n}^* \otimes E] \tag{7.1.3}$$

by

$$r_\ell \left(f(\mathbf{x}) \otimes \wedge^i \omega_{j_i}^* \otimes w \right) = \widehat{f} \left(\mathbf{x} \left|_{W_\ell} \right.\right) \otimes \wedge^i \left(\omega_{j_i} \left|_{\mathfrak{n}} \right.\right)^* \otimes w. \tag{7.1.4}$$

We shall now give a brief remark on the reason for introducing this map. We first comment that calling it a "restriction map" is itself a bit of a fudge; indeed, a fuller understanding of it is given as following. It is the composition of two maps: $r_\ell = \widetilde{r}_\ell \circ f_\ell$, where

$$f_\ell : [\mathcal{S}(V) \otimes \wedge^\bullet \mathfrak{p}^* \otimes E]^K \to \left[\mathcal{S}(V)_\ell^{MM} \otimes \wedge^\bullet \mathfrak{p}^* \otimes E \right]^K$$

is the Fourier transform map, acting as the identity on $\wedge^\bullet \mathfrak{p}^* \otimes E$ and acting as the Fourier transform in the ℓ variable (as in Definition 5.3.3); $\widetilde{r}_\ell$ is hence a "true" restriction map, given by

$$\widetilde{r}_\ell : \left[\mathcal{S}(V)_\ell^{MM} \otimes \wedge^\bullet \mathfrak{p}^* \otimes E \right]^K \to [\mathcal{S}(W_\ell) \otimes \wedge^\bullet \mathfrak{n}^* \otimes E],$$

acting as the restriction to W_ℓ on the Schwartz component and as restriction to $\mathfrak{n}^* \subset \mathfrak{p}^*$ in the Lie algebra. Despite this, it makes sense to describe r_ℓ as a restriction map, as we know that f_ℓ is an isomorphism.

Remark 7.1.2. So, why have we used this map in the first place? As hinted in the title of this chapter, our ultimate aim herein is to extend the differential form $\theta_{\mathcal{L},\mathcal{H}}(\varphi, \tau)$ to the boundary of $\overline{X}^{BS}$, and to find what the restriction is on each component. One may examine the form of the scalar parts of φ in e.g. (7.1.7) to see that the critical problem in finding this restriction (which will crudely be given by taking $t \to \infty$ outside the sum over the lattice) is that there is a factor of $1/t^2$ accompanying the $|a|^2$ in the exponential factor.

In particular, this should convince the reader that individually, each of the $\varphi(\mathbf{x}, z, \tau)$

do *not* converge near the boundary components. We shall hence be using Poisson summation on the sum over $a \in k\ell \cap \mathcal{L}$; of course, *a priori*, this solves nothing, but what we shall see is that under the map $f_\ell : \varphi \to \widehat{\varphi}$, this shifts the t^2 factor to the numerator, and hence renders the term inside the exponential polynomial in t! In particular, this explains why we use the map f_ℓ - namely, that on the level of the complex of differential forms, it gives a Schwartz form with satisfactory convergence properties.

Similarly, we may now explain the reason for the restriction map $\widetilde{r_\ell}$. Because of the action of f_ℓ as explained above, we see that the geometric restriction of $\theta_{\mathcal{L},\mathcal{H}}(\varphi, \tau)$ may be taken termwise; in particular, in the sum over $\mathcal{L}$, we shall see that all the terms $\varphi_{l,l}(\mathbf{x}, z, \tau)$ not lying in $W_\ell \cap \mathcal{L}$ will go to zero under the restriction map!

In particular, what this tells us is that in some sense, the Fourier transform is unnecessary - namely, following the rubric of the above, the only part of $\widehat{\varphi_{l,l}}$ that will survive is the origin, where the new variables $\underline{\phi} = 0$ - hence, this is really the trivial part of the Fourier transform. For those familiar with the literature, this is also recognisable as the 0'th coefficient of the Fourier-Jacobi expansion of the theta series.

We shall now spend the rest of this section showing that the previewed properties of this restriction map do indeed hold. We shall hence focus on finding the image under f_ℓ of the scalar parts of $\varphi_{l,l}$, given in the Schrödinger model in Definition 7.1.1.

As previously, we let $\underline{\beta}$, $\underline{\beta}'$ be two collections of indices in $\{1, 2\}^l$, and hence define two counting functions: let $0 \leq r(\alpha, \underline{\beta}) \leq l + 1$ be the number of indices in $\{\alpha\} \cup \underline{\beta}$ which are 1, and $0 \leq r(\alpha', \underline{\beta}') \leq l + 1$ the number of indices in $\{\alpha'\} \cup \underline{\beta}'$ which are 1. Hence, for fixed indices $0 \leq r, r' \leq l + 1$, we define the scalar Schwartz form $\varphi_{l,l,r,r'}(\mathbf{x})$ and the polynomial $g_{r,r'}(\mathbf{x})$ by:

$$\varphi_{l,l,r,r'}(\mathbf{x}) := g_{r,r'}(\mathbf{x})\varphi_0(\mathbf{x}) = \frac{1}{2^{2l+2}} \left[\mathcal{D}_{\underline{\beta}} \circ \overline{\mathcal{D}}_{\underline{\beta}'} \circ \mathcal{D}_\alpha \circ \overline{\mathcal{D}}_{\alpha'}\right] (\varphi_0)(\mathbf{x}) \tag{7.1.5}$$

The change of variables between the orthonormal basis $\{v_1, v_2, v_3\}$ and the Witt basis $\{\ell, w_\ell, \ell'\}$ may be assumed to be given by the co-ordinate change

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \to \begin{pmatrix} a + \frac{c}{2\delta_k} \\ b \\ a - \frac{c}{2\delta_k} \end{pmatrix},$$

where as in the rest of the paper we write vectors $\mathbf{x} = a\ell + bw_\ell + c\ell' \in V$. The insertion of τ into $\varphi_{l,l}(\mathbf{x})$ is analogous to that given in Definition 6.1.12; by Lemma 6.2.3, $\varphi_{l,l}$ has weight $2l + 3$ under the ω-action of the Lie algebra $\mathfrak{k}'$, so in order to make the new form SL_2-invariant we define

$$\begin{aligned}\varphi_{l,l}(\mathbf{x},\tau) &= j(g'_\tau,i)^{-(2l+3)}\exp\left(\pi i u(\mathbf{x},\mathbf{x})\right)\left(\sqrt{v}\right)^3\varphi(\sqrt{v}\mathbf{x})\\ &= v^{-l}\exp\left(\pi i u(\mathbf{x},\mathbf{x})\right)\varphi(\sqrt{v}\mathbf{x}).\end{aligned}$$

In order to incorporate the action of A on the Schwartz form in the mixed model, we act by definition as $a(t)^{-1} = a(t^{-1})$ on the column vector $\mathbf{x}$. Hence the resulting scalar term for the above fixed α, α', $\underline{\beta}$, $\underline{\beta}'$ is given by:

$$\begin{aligned}\varphi_{l,l,r,r'}(a,b,c,a(t),\tau) &:= \frac{1}{2^{2l+2}}\left[\mathcal{D}_{\underline{\beta}}\circ\overline{\mathcal{D}}_{\underline{\beta}'}\circ\mathcal{D}_\alpha\circ\overline{\mathcal{D}}_{\alpha'}\right](\varphi_0)(a,b,c,a(t),\tau)\\ &= v^{-l}g_{r,r'}\left(a(t)^{-1}\left(\sqrt{v}\mathbf{x}\right)\right)\exp\left(\pi i u\left(|b|^2-\frac{2\mathcal{I}(a\bar{c})}{|\delta_k|}\right)\right) \qquad (7.1.6)\\ &\quad\times\exp\left(-\pi v\left(\frac{2}{t^2}|a|^2+|b|^2+\frac{t^2}{|\delta_k|^2}|c|^2\right)\right)\\ &= v^{-l}g_{r,r'}\left(a(t)^{-1}\left(\sqrt{v}\mathbf{x}\right)\right)\exp\left(-\frac{2\pi v}{t^2}|a|^2+\frac{2\pi i u}{|\delta_k|}\left(\mathcal{I}(c)\mathcal{R}(a)-\mathcal{R}(c)\mathcal{I}(a)\right)\right)\\ &\quad\times\exp\left(\pi i\tau|b|^2-\frac{\pi v t^2}{2|\delta_k|^2}|c|^2\right). \qquad (7.1.7)\end{aligned}$$

As above, we write $\mathcal{R}(a) = X$ and $\mathcal{I}(a) = Y$. Letting the polynomial $f_{r,r'}(X,Y)$ be defined by

$$f_{r,r'}(X,Y) = v^{-l}g_{r,r'}\left(\frac{\sqrt{v}}{t}X+\frac{i\sqrt{v}}{t}Y+\frac{\sqrt{v}c}{2\delta_k},\sqrt{v}b,\sqrt{v}tX+i\sqrt{v}tY-\frac{\sqrt{v}c}{2\delta_k}\right), \qquad (7.1.8)$$

our objective is to Fourier transform (at the origin) the following function of two real variables X and Y:

$$f_{r,r'}(X,Y)\exp\left(\frac{-2\pi v}{t^2}\left[X^2+Y^2-\frac{iut^2\mathcal{I}(c)}{v}X+\frac{iut^2\mathcal{R}(c)}{v}Y\right]\right) \qquad (7.1.9)$$

Indeed, one may see that this is a scalar part of $\varphi_{l,l,r,r'}(\mathbf{x},a(t),\tau)$, and hence in finding the Fourier transform in the $a = X+iY$ variable will give us the first part of the restriction map.

By definition of the real Fourier transform, our calculation will be the following:

$$\begin{aligned}&\iint_{\mathbb{R}^2}\left[f_{r,r'}(X,Y)\exp\left(\frac{-2\pi v}{t^2}\left[X^2+Y^2-\frac{iut^2\mathcal{I}(c)}{v}X+\frac{iut^2\mathcal{R}(c)}{v}Y\right]\right)\right.\\ &\qquad\left.\times\exp\left(2\pi i(X\phi_1+Y\phi_2)\right)\right]\mathrm{d}X\mathrm{d}Y\\ &= \iint_{\mathbb{R}^2}f_{r,r'}(X,Y)\exp\left(-\frac{2\pi v}{t^2}\left[X^2+Y^2+\beta_1X+\beta_2Y\right]\right)\mathrm{d}X\mathrm{d}Y \qquad (7.1.10)\end{aligned}$$

where the constants β_1, β_2 are given by

$$\beta_1 = -\frac{iut^2}{v}\mathcal{I}(c) - \frac{it^2}{v}\phi_1 = -\frac{it^2}{v}\widetilde{\beta_1}, \qquad \beta_2 = \frac{iut^2}{v}\mathcal{R}(c) - \frac{it^2}{v}\phi_2 = -\frac{it^2}{v}\widetilde{\beta_2}.$$

We now give the two main results that will allow us to calculate this integral.

Lemma 7.1.3. *Let $A \in \mathbb{R}_{>0}$ be a strictly positive constant, and $f(X,Y) = \sum_{m,n} b_{m,n}X^mY^n$ a finite degree complex polynomial function of two variables X and Y. Then:*

$$\iint_{\mathbb{R}^2} f(X,Y)\exp(-A(X^2+Y^2))\mathrm{d}X\mathrm{d}Y = \frac{\pi}{A}\sum_{n\geq 0}\left(\frac{1}{4A}\right)^n\frac{1}{n!}\sum_{k=0}^{n} b_{2k,2n-2k}(2k)!(2n-2k)!\binom{n}{k}$$

Proof. We start with two results from elementary calculus, where k is a positive integer and A a positive real number.

$$\int_{-\infty}^{\infty} X^{2k}e^{-AX^2}\mathrm{d}X = \frac{(2k)!}{(4A)^k k!}\sqrt{\frac{\pi}{A}}, \qquad \int_{-\infty}^{\infty} X^{2k+1}e^{-AX^2}\mathrm{d}X = 0 \tag{7.1.11}$$

These may be proved respectively by differentiating by A and replacement of variables for negative X. Hence, we may only focus on the purely even coefficients. We write:

$$\begin{aligned}&\iint_{\mathbb{R}^2} f(X,Y)\exp\left(-A(X^2+Y^2)\right)\mathrm{d}X\mathrm{d}Y \\ &\quad= \sum_{n\geq 0}\sum_{k=0}^{n} b_{2k,2n-2k}\iint_{\mathbb{R}^2} X^{2k}Y^{2n-2k}\exp\left(-A(X^2+Y^2)\right)\mathrm{d}X\mathrm{d}Y \\ &\quad= \sum_{n\geq 0}\sum_{k=0}^{n} b_{2k,2n-2k}\left[\int_{\mathbb{R}} X^{2k}\exp(-AX^2)\mathrm{d}X\right]\left[\int_{\mathbb{R}} Y^{2n-2k}\exp(-AY^2)\mathrm{d}Y\right].\end{aligned}$$

The result is hence a simple application of the first formula in (7.1.11). □

Lemma 7.1.4. *Let $j \geq 0$ be a non-negative integer, $a \in \mathbb{R}$ a real number and $\alpha \in \mathbb{R}_{>0}$ a positive real number. Then*

$$\int_{\mathbb{R}+ia} X^j\exp\left(-\alpha X^2\right)\mathrm{d}X = \int_{\mathbb{R}} X^j\exp\left(-\alpha X^2\right)\mathrm{d}X \tag{7.1.12}$$

Proof. We start with the case of $j=0$, so this is just a classical Gaussian integral. We may switch variables to $Y = X - ia$, and write the left hand side of (7.1.12) as

$$\int_{\mathbb{R}+ia} \exp\left(-\alpha X^2\right)\mathrm{d}X = \int_{\mathbb{R}} \exp\left(-\alpha\left(Y+ia\right)^2\right)\mathrm{d}Y; \tag{7.1.13}$$

hence, as a function of $\alpha \in \mathbb{R}$, the right-hand side of (7.1.13) is continuously differentiable. Because of this, we may differentiate under the integral by a and find:

$$\begin{aligned}\frac{\partial}{\partial a}\left[\int_{\mathbb{R}} \exp\left(-\alpha\left(Y+ia\right)^2\right)\mathrm{d}Y\right] &= \int_{\mathbb{R}} \frac{\partial}{\partial a}\left[\exp\left(-\alpha\left(Y+ia\right)^2\right)\right]\mathrm{d}Y \\ &= \int_{\mathbb{R}} \left(-i\alpha(Y+ia)\exp\left(-\alpha\left(Y+ia\right)^2\right)\right)\mathrm{d}Y\end{aligned}$$

$$= i\left[\exp\left(-\alpha\left(Y+ia\right)^2\right)\right]_{Y=-\infty}^{Y=\infty}$$
$$= 0.$$

The value of this integral does not depend on a, and so we may conclude that (7.1.12) holds for $j=0$.

We next treat the case of $j=2k$ an even, non-negative integer. Indeed, because the function $X^{2k}\exp(-\alpha X^2)$ is $\mathcal{C}^\infty$ and rapidly decreasing as a function of α, the integral gives a continuously differentiable function of α. Hence, in particular, we may differentiate inside the integral and write:

$$\begin{aligned}\int_{\mathbb{R}+ia} X^{2k}\exp\left(-\alpha X^2\right)\mathrm{d}X &= \int_{\mathbb{R}+ia}(-1)^k\frac{\partial^k}{\partial\alpha^k}\left(\exp\left(-\alpha X^2\right)\right)\mathrm{d}X\\ &= (-1)^k\frac{\partial^k}{\partial\alpha^k}\int_{\mathbb{R}+ia}\left(\exp\left(-\alpha X^2\right)\right)\mathrm{d}X.\end{aligned}$$

Hence, (7.1.12) for even integers $j=2k$ follows immediately from the case $j=k=0$, which we have proven above.

For the case of $j\geq 1$ odd, we use both the proof method and the result of the even case. Indeed, we may again differentiate under the integral, change variables as above, and hence write:

$$\begin{aligned}&\frac{\partial}{\partial a}\left[\int_{\mathbb{R}+ia}X^j\exp\left(-\alpha X^2\right)\mathrm{d}X\right] = \frac{\partial}{\partial a}\left[\int_{\mathbb{R}}(Y+ia)^j\exp\left(-\alpha(Y+ia)^2\right)\mathrm{d}X\right]\\ &\quad= \int_{\mathbb{R}}\left[ij(Y+ia)^{j-1}\exp\left(-\alpha(Y+ia)^2\right) - 2i\alpha(Y+ia)^{j+1}\exp\left(-\alpha(Y+ia)^2\right)\right]\mathrm{d}Y.\end{aligned} \tag{7.1.14}$$

We see now that both terms in (7.1.14) are shifted integrals of *even* powers; hence, we may apply the result of this lemma for even powers and write this as:

$$ij\int_{\mathbb{R}}Y^{j-1}\exp\left(-\alpha Y^2\right)\mathrm{d}Y - 2i\alpha\int_{\mathbb{R}}Y^{j+1}\exp\left(-\alpha Y^2\right)\mathrm{d}Y.$$

We may hence apply (7.1.11) - namely, the numerical evaluations of these integrals - and see that this equals 0. Hence, we have shown:

$$\frac{\partial}{\partial a}\left[\int_{\mathbb{R}+ia}X^j\exp\left(-\alpha X^2\right)\mathrm{d}X\right] = 0,$$

and so this again is independent of a. This completes the proof. □

In particular, this shows a very neat result for Gaussian integrals: namely, that all integrals of the form $\int_{\mathbb{R}}X^j\exp(-\alpha X^2)\mathrm{d}X$ are invariant under linear transformations $X\to X+z$ for *all* $z\in\mathbb{C}$!

Using Lemmas 7.1.3 and 7.1.4, we may prove the following result:

Proposition 7.1.5. *(a) We let the coefficient of* X^mY^n *in* $f_{r,r'}(X,Y)$ *be notated* $a_{m,n} = a_{m,n,r,r'}$*, and hence for each* m*,* n *define the finite sum*

$$\widetilde{a_{m,n}} = \sum_{i,j\geq 0} a_{i,j}\binom{i}{m}\binom{j}{n}\left(\frac{-\beta_1}{2}\right)^{i-m}\left(\frac{-\beta_2}{2}\right)^{j-n}$$

Then the integral in (7.1.10) *is given by*

$$\begin{aligned}&\iint_{\mathbb{R}^2} f_{r,r'}(X,Y)\exp\left(-\frac{2\pi v}{t^2}\left[X^2+Y^2+\beta_1X+\beta_2Y\right]\right)\mathrm{d}X\mathrm{d}Y \qquad (7.1.15)\\ &= \frac{t^2}{2v}\exp\left(-\frac{\pi t^2}{2v}(\widetilde{\beta_1}^2+\widetilde{\beta_2}^2)\right)\sum_{n\geq 0}\left(\frac{t^2}{8\pi v}\right)^n\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(2k)!(2n-2k)!\widetilde{a_{2k,2n-2k}}\end{aligned}$$

(b) The real Fourier transform of $\varphi_{l,l,r,r'}(a,b,c,a(t),\tau)$ *with respect to the* $a = X+iY$ *variable is given by:*

$$\begin{aligned}\widehat{\varphi_{l,l,r,r'}}(\phi_1,\phi_2,b,c,a(t),\tau) &= \frac{t^2}{2v}\exp\left(\pi i\tau|b|^2 - \frac{\pi v t^2}{2|\delta_k|^2}|c|^2 - \frac{\pi t^2}{2v}(\widetilde{\beta_1}^2+\widetilde{\beta_2}^2)\right)\\ &\times\sum_{n\geq 0}\left(\frac{t^2}{8\pi v}\right)^n\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(2k)!(2n-2k)!\widetilde{a_{2k,2n-2k}}\end{aligned}$$

Proof. The proof of the first statement is an application of Lemma 7.1.3: we change variables to $X \to X+\beta_1/2$, $Y \to Y+\beta_2/2$, and may hence write (7.1.15) as

$$\exp\left(-\frac{\pi t^2}{2v}(\widetilde{\beta_1}^2+\widetilde{\beta_2}^2)\right)\int_{\mathbb{R}+\beta_2}\int_{\mathbb{R}+\beta_1}\widetilde{f_{r,r'}}(X,Y)\exp\left(-\frac{2\pi v}{t^2}\left[X^2+Y^2\right]\right)\mathrm{d}X\mathrm{d}Y, \qquad (7.1.16)$$

where $\widetilde{f_{r,r'}}(X,Y) = f(X-\beta_1/2, Y-\beta_2/2)$.

Directly from the results of Lemma 7.1.4, we see that the β_i factors in the integrals in (7.1.16) are trivial, so we may take these integrals to be over the real line $\mathbb{R}$.

One may easily calculate that the X^mY^n coefficient in $\widetilde{f_{r,r'}}(X,Y)$ is given by the $\widetilde{a_{m,n}}$ defined above. Hence, part (i) is given by the application of Lemma 7.1.3 to the equation in (7.1.16), and so part (ii) is an immediate corollary of part (i) by multiplying by the extra term in (7.1.7). □

With all of the above, we are now ready to explicitly find the image of $\varphi_{l,l}$ under r_ℓ. We have one more piece of business to attend to before finding the image $r_\ell\varphi_{l,l}$; namely, we must find the restriction of the Lie algebra elements $\xi_{\alpha'} \wedge \overline{\xi_\alpha}$.

A good change of basis matrix was found in (2.1.17); we may use this to write $m(s,r)$ with respect to the orthonormal basis as:

$$m(s,r) = \begin{pmatrix} r\delta_k & -\overline{s} & r\delta_k \\ s & 0 & -s \\ -r\delta_k & -\overline{s} & -r\delta_k \end{pmatrix}. \qquad (7.1.17)$$

Hence, from this, we may immediately read off how the elements restrict. We know $\mathfrak{n}_\ell$ is a real Lie algebra of dimension 3, so $\mathfrak{n}_\ell^*$ will have 3 basis elements - indeed, in Definition 2.4.5, we have written down 3 non-trivial elements of this space, which by their definition are linearly independent. Using the construction of the ξ_α, $\overline{\xi_\alpha}$ from Definition 2.4.3, we find the following relations:

Lemma 7.1.6. *The action of $\mathfrak{p}^*$ on the basis of $\mathfrak{n}_\ell \subset \mathfrak{g}$ is given by:*

$$\xi_1(m(s,r)) = \frac{\delta_k}{2}r, \quad \overline{\xi_1}(m(s,r)) = -\frac{\delta_k}{2}r, \quad \xi_2(m(s,r)) = -\frac{1}{2}s, \quad \overline{\xi_2}(m(s,r)) = -\frac{1}{2}\overline{s}.$$

Hence, the restriction of the forms in $\mathfrak{p}^*$ under the map r_ℓ is given by

$$r_\ell \mid_{\mathfrak{p}^*}(\xi_1) = -\frac{\delta_k}{2}\kappa_\ell, \quad r_\ell \mid_{\mathfrak{p}^*}\left(\overline{\xi_1}\right) = \frac{\delta_k}{2}\kappa_\ell, \quad r_\ell \mid_{\mathfrak{p}^*}(\xi_2) = -\Omega_\ell, \quad r_\ell \mid_{\mathfrak{p}^*}\left(\overline{\xi_2}\right) = -\overline{\Omega_\ell}.$$

Proof. This is essentially by definition of the forms ξ_α, $\overline{\xi_\alpha}$; one only needs check their action on the matrix in (7.1.17). □

So, as we have now obtained the restriction of all the components of $\varphi_{l,l}$, we are ready to find the restriction of the theta series!

Theorem 7.1.7. *Let $q \geq 0$ be a non-negative integer. For a complex variable w, we define the q'th Laguerre polynomial $L_q(w)$ by*

$$L_q(w) = \frac{e^w}{q!}\frac{d^q}{dw^q}\left(e^{-w}w^q\right) = \sum_{r=0}^{q}\gamma_{q,r}w^r.$$

Let $\varphi_{l,l}^{\mathcal{H},\ell}$ be the Schwartz form in the boundary complex $\left[\mathcal{S}(W_\ell) \otimes \wedge^\bullet \mathfrak{n}^* \otimes \mathcal{H}^{l,l}(V)\right]$ given by:

$$\varphi_{l,l}^{\mathcal{H},\ell}(\boldsymbol{x},\tau) = \frac{(-1)^{l+1}(l+1)!}{2(2\pi v)^{l+1}}L_{l+1}\left(2\pi v||\boldsymbol{x}||\right)\exp\left(\pi i\tau||\boldsymbol{x}||\right) \otimes \Omega_\ell \wedge \overline{\Omega_\ell} \otimes \pi_{\mathcal{H}}\left(v_2^l \otimes (v_2^*)^l\right)$$

Then this form is invariant with respect to the maximally compact subgroup K_P, and for $\boldsymbol{x} \in \underline{V}$:

$$r_\ell\left(\varphi_{l,l}^{\mathcal{H}}(\boldsymbol{x},\tau)\right) = \varphi_{l,l}^{\mathcal{H},\ell}(bw_\ell,\tau).$$

Proof. We start by noting that it is almost trivial that this form is K_P-invariant; indeed, we may check that the relevant subgroup is given by the circle group $S^1 \simeq M \subset P$. As $\varphi_{l,l}^{\mathcal{H},\ell}$ is in reality only a function of the norm of $\mathbf{x}$, it is clear that the scalar parts are invariant (because the action of S^1 leaves the norm invariant). Similarly, the action on $\Omega_\ell \wedge \overline{\Omega_\ell}$ will be as

$$S^1 \times \wedge^{1,1}\mathfrak{n}_\ell^* \to \wedge^{1,1}\mathfrak{n}_\ell^*, \quad (e^{i\theta}, \Omega_\ell \wedge \overline{\Omega_\ell}) \to \left(e^{i\theta}\Omega_\ell\right) \wedge \left(e^{-i\theta}\overline{\Omega_\ell}\right) = \Omega_\ell \wedge \overline{\Omega_\ell},$$

and identically for the powers of v_2. Hence, the invariance follows.

Hereafter, the proof divides into two parts:

(i) For all α, α' $\underline{\beta}$, $\underline{\beta}'$ such that $r(\alpha, \underline{\beta}) = r(\alpha', \underline{\beta}') = r$, we shall prove that the restriction

$$r_\ell\left[\varphi_{l,l,r,r} \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \pi_{\mathcal{H}}\left(\underline{v}_{\beta} \otimes \underline{v}^*_{\beta'}\right)\right] = \varphi^{\mathcal{H},\ell}_{l,l}$$

if and only if $r = 0$.

(ii) For all α, α' $\underline{\beta}$, $\underline{\beta}'$ such that $r(\alpha, \underline{\beta}) = r$, $r(\alpha', \underline{\beta}') = r'$ with $r \neq r'$, we shall prove that the restriction is always trivial:

$$r_\ell\left[\varphi_{l,l,r,r'} \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \pi_{\mathcal{H}}\left(\underline{v}_{\beta} \otimes \underline{v}^*_{\beta'}\right)\right] = 0.$$

We start with part (i); we recall the operators $\mathcal{D}_\alpha$, $\overline{\mathcal{D}_\alpha}$ from (6.1.10). By the results of [KM87, (6.49), p.303] - though this may easily be proven with an induction argument - we know that

$$\frac{1}{2^{2q}}\left(\mathcal{D}_\alpha \circ \overline{\mathcal{D}_\alpha}\right)^q (\varphi_0) = (-1)^q \frac{q!}{(2\pi)^q} L_q\left(2\pi|z_\alpha|^2\right)\varphi_0 \tag{7.1.18}$$

Because all of the operators $\mathcal{D}_\alpha$, $\overline{\mathcal{D}_\alpha}$ commute, we may write

$$\begin{aligned}
\varphi_{l,l,r,r}(\mathbf{x}) &= \frac{1}{2^{2l+2}}\left(\mathcal{D}_1 \circ \overline{\mathcal{D}_1}\right)^r \circ \left(\mathcal{D}_2 \circ \overline{\mathcal{D}_2}\right)^{l+1-r}(\varphi_0)\\
&= (-1)^r \frac{r!}{(2\pi)^r} L_r\left(2\pi|z_1|^2\right)(-1)^{l+1-r}\frac{(l+1-r)!}{(2\pi)^{l+1-r}} L_{l+1-r}\left(2\pi|z_2|^2\right)\varphi_0\\
&= (-1)^{l+1}\frac{r!(l+1-r)!}{(2\pi)^{l+1}} L_r\left(2\pi|z_1|^2\right) L_{l+1-r}\left(2\pi|z_2|^2\right)\varphi_0.
\end{aligned}$$

In the co-ordinates corresponding to the Witt basis, using the natural change of basis from (7.1.17), we may write this as

$$\begin{aligned}
\varphi_{l,l,r,r}(\mathbf{x}) = (-1)^{l+1}\frac{r!(l+1-r)!}{(2\pi)^{l+1}} L_r\left(2\pi\left(|a|^2 + \frac{|c|^2}{4|\delta_k|^2} - \frac{\mathcal{I}(a\bar{c})}{|\delta_k|}\right)\right)\\
\times L_{l+1-r}\left(2\pi|b|^2\right)\exp\left(-\pi\left(2|a|^2 + |b|^2 + \frac{|c|^2}{2|\delta_k|^2}\right)\right)
\end{aligned} \tag{7.1.19}$$

Identically to earlier in this chapter, we write $a = X + iY$ and $\widehat{\varphi_{l,l,r,r}}$ as the real partial Fourier transform of $\varphi_{l,l,r,r}$ with respect to the $\{X, Y\}$ variables. We may now see that we have done a lot of the work needed to prove this already - namely, we have found the general form of such Fourier transforms in Proposition 7.1.5 - so that the main work remaining in this half of the proof is applying it to our particular polynomial. We now set $t = 1$ in this result, so that this is still a Schwartz function of $\mathbf{x}$ and τ only. Using the notation of this proposition, we have:

$$f_{r,r}(X,Y) = \frac{(-1)^{l+1} r!(l+1-r)!}{(2\pi)^{l+1} v^l} L_{l+1-r}\left(2\pi v|b|^2\right) \times L_r\left(2\pi v\left(X^2+Y^2+\frac{|c|^2}{4|\delta_k|^2} - \frac{Y\mathcal{R}(c) - X\mathcal{I}(c)}{|\delta_k|}\right)\right) \tag{7.1.20}$$

By definition of r_ℓ, we are restricting to the case $\phi_1 = \phi_2 = c = 0$. In particular, this means that $\beta_1 = \beta_2 = 0$: hence, by definition again, we have $\widetilde{a_{2k,2n-2k}} = a_{2k,2n-2k}$, so that

$$\begin{aligned} f_{r,r}(X,Y)|_{\phi_1=\phi_2=c=0} &= \frac{(-1)^{l+1} r!(l+1-r)!}{(2\pi)^{l+1} v^l} L_r\left(2\pi v(X^2+Y^2)\right) L_{l+1-r}\left(2\pi v|b|^2\right) \\ &= \frac{(-1)^{l+1-r} r!(l+1-r)!}{(2\pi)^{l+1} v^l} L_{l+1-r}\left(2\pi v|b|^2\right) \sum_{s=0}^{r} \gamma_{r,s} (2\pi v)^s \sum_{s_0=0}^{s} \binom{s}{s_0} X^{2s_0} Y^{2s-2s_0}. \end{aligned}$$

and so the $a_{2k,2n-2k}$ coefficients are of the form:

$$a_{2k,2n-2k} = \frac{(-1)^{l+1-r} r!(l+1-r)!}{(2\pi)^{l+1} v^l} L_{l+1-r}\left(2\pi v|b|^2\right) \gamma_{r,n} (2\pi v)^n \binom{n}{k}.$$

We now examine Proposition 7.1.5(b). Putting all the components together, we may now write:

$$\left(r_\ell\,|_{\mathcal{S}(V)}\right)\left(\varphi_{l,l,r,r}(\mathbf{x},\tau)\right) = \frac{1}{2v} \frac{(-1)^{l+1-r} r!(l+1-r)!}{(2\pi)^{l+1} v^l} L_{l+1-r}\left(2\pi v|b|^2\right) \times \exp\left(\pi i \tau |b|^2\right) \sum_{n\geq 0} \gamma_{r,n} \frac{1}{4^n n!} \sum_{k=0}^{n} \binom{n}{k}^2 (2k)!(2n-2k)! \tag{7.1.21}$$

We have now reduced this to arithmetic. Indeed, by two simple induction arguments, one may prove:

$$\sum_{k=0}^{n} \binom{n}{k}^2 (2k)!(2n-2k)! = 4^n (n!)^2; \qquad \gamma_{r,n} = \binom{r}{n} \frac{(-1)^{r+n}}{n!}, \tag{7.1.22}$$

for all $k \leq n \leq r$. This allows us to rewrite (7.1.21) as:

$$\left(r_\ell\,|_{\mathcal{S}(V)}\right)\left(\varphi_{l,l,r,r}(\mathbf{x},\tau)\right) = \frac{t^2}{2v} \frac{(-1)^{l+1-r} r!(l+1-r)!}{(2\pi)^{l+1} v^l} L_{l+1-r}\left(2\pi v|b|^2\right) \times \exp\left(\pi i \tau |b|^2\right) \sum_{n=0}^{r} (-1)^{r+n} \binom{r}{n}. \tag{7.1.23}$$

We may check that the last term of (7.1.23) may be expressed as:

$$\sum_{n=0}^{r} (-1)^{r+n} \binom{r}{n} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r > 0. \end{cases}$$

which gives us immediately that the restriction r_ℓ of the Schwartz function will be trivial for all $r > 0$.

We may now focus on the case $r = 0$. In this case, there is trivially only one choice of $\{\alpha,\, \alpha',\, \underline{\beta},\, \underline{\beta}'\}$ giving $r(\alpha, \underline{\beta}) = r(\alpha', \underline{\beta}') = 0$ - namely, when all the indices equal 2. So, putting (7.1.23) into the full form of r_ℓ, we have shown:

$$
\begin{aligned}
&r_\ell \left(\varphi_{l,l,0,0} \otimes \xi_2 \wedge \overline{\xi_2} \otimes \pi_{\mathcal{H}} \left(v_2^l \otimes (v_2^*)^l\right)\right) \\
&\quad = \frac{1}{2v} \frac{(-1)^{l+1}(l+1)!}{(2\pi)^{l+1} v^l} L_{l+1}\left(2\pi v |b|^2\right) \exp\left(\pi i \tau |b|^2\right) \otimes (-\Omega_\ell) \wedge \left(-\overline{\Omega_\ell}\right) \otimes \pi_{\mathcal{H}}\left(v_2^l \otimes (v_2^*)^l\right) \\
&\quad = \varphi_{l,l}^{\mathcal{H},\ell}(b w_\ell, \tau),
\end{aligned}
$$

and that

$$
r_\ell \left[\varphi_{l,l,r,r} \otimes \xi_{\alpha'} \wedge \overline{\xi_\alpha} \otimes \pi_{\mathcal{H}} \left(\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}\right)\right] = 0
$$

for all choices of α, α', $\underline{\beta}$, $\underline{\beta}'$ such that $r(\alpha, \underline{\beta}) = r(\alpha', \underline{\beta}') = r > 0$. This completes the proof of part (i).

We now move on to part (ii), the case of $r \neq r'$; this will largely follow the same structure as the proof of part (i). We start by finding the structure of the polynomials $g_{r,r'}(\mathbf{x})$ and $f_{r,r'}(\mathbf{x})$ as before. Fix an integer $q \geq 1$, and another integer $k \geq 0$, and we again work with the operators $\mathcal{D}_\alpha$ and $\overline{\mathcal{D}_\alpha}$ from Definition **??**. We claim the following:

$$
\frac{1}{2^{2q+k}} \mathcal{D}_\alpha^k \circ \left(\mathcal{D}_\alpha \circ \overline{\mathcal{D}_\alpha}\right)^q (\varphi_0) = \frac{q!}{(2\pi)^q} z_\alpha^k \sum_{j=0}^{q} \Gamma_{q,j,k} \left(2\pi |z_\alpha|^2\right)^j \varphi_0 \tag{7.1.24}
$$

and

$$
\frac{1}{2^{2q+k}} \overline{\mathcal{D}_\alpha}^k \circ \left(\mathcal{D}_\alpha \circ \overline{\mathcal{D}_\alpha}\right)^q (\varphi_0) = \frac{q!}{(2\pi)^q} \overline{z_\alpha}^k \sum_{j=0}^{q} \Gamma_{q,j,k} \left(2\pi |z_\alpha|^2\right)^j \varphi_0 \tag{7.1.25}
$$

where the coefficients $\Gamma_{q,j,k}$ are defined by $\Gamma_{q,j,0} = \gamma_{q,j}$ and:

$$
\Gamma_{q,j,k} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \gamma_{q,j+i} \prod_{l=1}^{i} (j+l) = \frac{(-1)^{j+q}}{j!} \binom{q+k}{j+k}. \tag{7.1.26}
$$

We shall refer to the $\Gamma_{q,j,k}$ as *generalised Laurent coefficients*, because of their coming from the action of the operator $\mathcal{D}_\alpha^k$ or $\overline{\mathcal{D}_\alpha}^k$ on a Laurent polynomial. Once (7.1.24) is proven, we may see that the proof of (7.1.25) will be identical, so we shall only do the first.

We know from the above-cited result (7.1.18) that this holds when $k = 0$; we shall prove (7.1.24) by induction on k. One may easily calculate that

$$\frac{1}{\pi}\frac{\partial}{\partial \overline{z_\alpha}}\left(z_\alpha^k \sum_{j=0}^{q}\Gamma_{q,j,k}\left(2\pi|z_\alpha|^2\right)(\varphi_0)\right) = -\, z_\alpha^{k+1}\left[\sum_{j=0}^{q}\Gamma_{q,j,k}\left(2\pi|z_\alpha|^2\right)^j\right]\varphi_0 \\ - 2z_\alpha^{k+1}\left[\sum_{j=0}^{q-1}(j+1)\Gamma_{q,j+1,k}\left(2\pi|z_\alpha|^2\right)^j\right]\varphi_0$$

so that

$$\frac{1}{2}\mathcal{D}_\alpha\left(z_\alpha^k \sum_{j=0}^{q}\Gamma_{q,j,k}\left(2\pi|z_\alpha|^2\right)(\varphi_0)\right) = z_\alpha^{k+1}\sum_{j=0}^{q}\left(\Gamma_{q,j,k}-(j+1)\Gamma_{q,j+1,k}\right)\left(2\pi|z_\alpha|^2\right)^j\varphi_0.$$

We may hence write (assuming nothing here about the form of $\gamma_{q,j}$)

$$\begin{aligned}\Gamma_{q,j,k}-(j+1)\Gamma_{q,j+1,k} &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\gamma_{q,j+i}\prod_{l=1}^{i}(j+l) - \sum_{i=0}^{k}(-1)^i\binom{k}{i}\gamma_{q,j+1+i}\prod_{l=1}^{i+1}(j+l)\\ &= \sum_{i=0}^{k+1}(-1)^i\gamma_{q,j+i}\prod_{l=1}^{i}(j+l)\left[\binom{k}{i}+\binom{k}{i-1}\right]\\ &= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}\gamma_{q,j+i}\prod_{l=1}^{i}(j+l)\\ &= \Gamma_{q,j,k+1};\end{aligned}$$

this completes the proof of the equality in (7.1.24). To prove (7.1.26) (the specific form of the generalised Laurent coefficients) we use the form for $\gamma_{q,j}$ given in (7.1.22). We hence find:

$$\begin{aligned}\Gamma_{q,j,k} &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left[\binom{q}{j+i}\frac{(-1)^{q+j+i}}{(j+i)!}\right]\prod_{l=1}^{i}(j+l)\\ &= \frac{(-1)^{j+q}}{j!}\sum_{i=0}^{k}\binom{k}{i}\binom{q}{j+i}. \qquad (7.1.27)\end{aligned}$$

One may proof with an inductive argument that

$$\sum_{i=0}^{k}\binom{k}{i}\binom{q}{j+i} = \binom{q+k}{j+k}$$

(it may be proven by induction on k again); hence, inserting this into (7.1.27) completes the proof of this identity.

So, we are now ready to find the coefficients of the Fourier transform at the origin. Because of the symmetry in (7.1.24) and (7.1.25), we shall focus on the case of $r > r'$, as the proof for $r' > r$ will be identical. Following the above notation, we write

$r = q + k$ and $r' = q$ for some $k > 0$, and so have

$$g_{q+k,q}(\mathbf{x}) = \frac{q!(l+1-q-k)!}{(2\pi)^{l+1-k}} \left[z_1^k \sum_{j=0}^{q} \Gamma_{q,j,k} \left(2\pi|z_1|^2\right)^j \right] \cdot \left[\overline{z_2}^k \sum_{j'=0}^{l+1-q-k} \Gamma_{l+1-q-k,j',k} \left(2\pi|z_2|^2\right)^{j'} \right]. \tag{7.1.28}$$

Using the definition of $g_{r,r'}(X,Y)$ from (7.1.8), we wish to Fourier transform $\varphi_{l,l,q+k,q}(\mathbf{x},\tau)$. So, writing (7.1.28) in terms of the Witt basis, we have

$$\begin{aligned} f_{q+k,q}(X,Y) &= v^{-l} g_{q+k,q}\left(\sqrt{v}(X+iY) + \frac{\sqrt{v}c}{2\delta_k}, \sqrt{v}b, \sqrt{v}(X+iY) - \frac{\sqrt{v}c}{2\delta_k}\right) \\ &= \frac{q!(l+1-q-k)!}{v^{l-k}(2\pi)^{l+1-k}} \overline{b}^k \sum_{j'=0}^{l+1-q-k} \Gamma_{l+1-q-k,j',k} \left(2\pi v|b|^2\right)^{j'} (X+iY)^k \times \\ &\quad \sum_{j=0}^{q} \Gamma_{q,j,k} \left(2\pi v \left(X^2 + Y^2 + \frac{|c|^2}{4|\delta_k|^2} - \frac{Y\mathcal{R}(c) - X\mathcal{I}(c)}{|\delta_k|}\right)\right)^j. \end{aligned} \tag{7.1.29}$$

For any integers α and β, we use the notation of Proposition 7.1.5 and let $a_{\alpha,\beta}$ be the coefficient of $X^\alpha Y^\beta$ in $f_{q+k,q}(X,Y)$. Firstly, by examination of the above, it is easy to see that for $n > q + k/2$, $a_{2m,2n-2m} = 0$ for all m - so in particular, this is a finite degree polynomial. Secondly, if k is odd, then this will imply the total weight (i.e. $\alpha + \beta$) of each term is odd, as the only monomials present in the expansion will be of the form $X^{2m}Y^{k+2j-2m}$ for $0 \le j \le q$. Hence, we may say immediately that for k odd, the Fourier transform of $\varphi_{l,l,q+k,q}$ - evaluated at $\phi_1 = \phi_2 = 0$ - is trivial!

Again, by definition of the map r_ℓ on $\mathcal{S}(V)_\ell^{MM}$, we wish to restrict the Fourier transform to the positive-definite space W_ℓ; by exactly the same logic as in our proof of part (i), this means we can immediately (before the Fourier transform) take $c = 0$. We write

$$C_{q,2k}(b) = \frac{q!(l+1-q-2k)!}{v^{l-2k}(2\pi)^{l+1-2k}} \overline{b}^{2k} \sum_{j'=0}^{l+1-q-2k} \Gamma_{l+1-q-2k,j',2k} \left(2\pi v|b|^2\right)^{j'},$$

which is a function of b, but we may treat as a constant as we are only interested in the behaviour related to the X, Y variables. We may hence write:

$$\begin{aligned} f_{q+2k,q}(X,Y)\,|_{c=0} &= C_{q,2k}(b) \left[\sum_{r=0}^{2k} \binom{2k}{r} X^r (iY)^{2k-r}\right] \cdot \left[\sum_{j=0}^{q} \Gamma_{q,j,2k} \left(2\pi v(X^2+Y^2)\right)^j\right] \\ &= C_{q,2k}(b) \left[\sum_{r=0}^{2k} \binom{2k}{r} X^r (iY)^{2k-r}\right] \cdot \left[\sum_{j=0}^{q} \Gamma_{q,j,2k} (2\pi v)^j \sum_{s=0}^{j} \binom{j}{s} X^{2s} Y^{2j-2s}\right] \\ &= C_{q,2k}(b) \sum_{n=k}^{q+k} \sum_{m=0}^{2n} \left[\Gamma_{q,n-k,2k} (2\pi v)^{n-k} \sum_{\substack{r,s\ge 0 \\ r+2s=m}} \binom{2k}{r} i^{2k-r} \binom{n-k}{s}\right] X^m Y^{2n-m}. \end{aligned} \tag{7.1.30}$$

We notice that the sum in (7.1.30) splits into either terms X^aY^b with a and b of the *same* parity. As we only care about the terms with both even, we discard the odd parts, replace m with $2m$ and hence write

$$f_{q+2k,q}(X,Y)_{\text{EVEN}} = C_{q,2k}(b)\sum_{n=k}^{q+k}\sum_{m=0}^{n}\Bigg[\Gamma_{q,n-k,2k}\,(2\pi v)^{n-k} \times \sum_{\substack{r,s\geq 0\\ r+s=m}}\binom{2k}{2r}(-1)^{k-r}\binom{n-k}{s}\Bigg]X^{2m}Y^{2n-2m}.$$

From this we may extract:

$$a_{2m,2n-2m} = C_{q,2k}(b)\Gamma_{q,n-k,2k}\,(2\pi v)^{n-k}\sum_{\substack{r,s\geq 0\\ r+s=m}}\binom{2k}{2r}(-1)^{k-r}\binom{n-k}{s}, \tag{7.1.31}$$

and so combining (7.1.31) with the results of Proposition 7.1.5, we may write:

$$\left(r_\ell\,|_{\mathcal{S}(V)}\right)(\varphi_{l,l,q+2k,q}) = \frac{|\delta_k|}{2v}\exp\left(\pi i\tau|b|^2\right)C_{q,2k}(b)\left(\frac{1}{2\pi v}\right)^k\sum_{n=k}^{q+k}\frac{\Gamma_{q,n-k,2k}}{4^n n!} \times\sum_{m=0}^{n}\left[\binom{n}{m}(2m)!(2n-2m)!\sum_{\substack{r,s\geq 0\\ r+s=m}}\binom{2k}{2r}(-1)^{k-r}\binom{n-k}{s}\right] \tag{7.1.32}$$

We now claim that for all k, the internal sum over m in (7.1.32), given by

$$\sum_{m=0}^{n}\left[\binom{n}{m}(2m)!(2n-2m)!\sum_{\substack{r,s\geq 0\\ r+s=m}}\binom{2k}{2r}(-1)^{k-r}\binom{n-k}{s}\right] \tag{7.1.33}$$

is trivial for all n and $k\leq n$; assuming this is true, it is clear that the restriction of all the $\varphi_{l,l,q+2k,q}$ to the boundary complex is trivial for all relevant choices.

We group the terms in this sum by $m-r=N\geq 0$ and hence define:

$$Q(n,k,N) = \sum_{r=0}^{k}(-1)^r\binom{2k}{2r}(2r+2N)!(2n-2r-2N)!\binom{n}{r+N}$$

so that (7.1.33) is given by $(-1)^k\sum_{N=0}^{n-k}Q(n,k,N)\binom{n-k}{N}$. We claim that this sum is 0 for all positive $k\leq n$. This may be shown by induction on k, but a more direct proof (shown in full for e.g. $k=1$) gives:

$$\sum_{N=0}^{n-1}\binom{n-1}{N}\left[\binom{n}{N}(2N)!(2n-2N)! - \binom{n}{N+1}(2N+2)!(2n-2N-2)!\right]$$
$$= \sum_{N=0}^{n-1}\binom{n-1}{N}\frac{n!(2N)!(2n-2N-2)!}{(N+1)!(n-N)!}\times$$

$$\times [(N+1)(2n-2N)(2n-2N-1)-(n-N)(2N+1)(2N+2)]$$
$$= \sum_{N=0}^{n-1} \binom{n-1}{N} \frac{n!(2N)!(2n-2N-2)!}{(N+1)!(n-N)!} 2(N+1)(n-N)(2n-4N-2)$$
$$= 2n \sum_{N=0}^{n-1} \binom{n-1}{N}^2 (2N)!(2n-2N-2)!(2n-4N-2). \tag{7.1.34}$$

Under the substitution $N' = n-1-N$, this sum is given by

$$\sum_{N'=0}^{n-1} \binom{n-1}{n-1-N'}^2 (2n-2-2N')!(2N')!(-2n+4N'+2)$$

which is easily seen to be the negative of the sum in (7.1.34); hence this sum is trivial.

More generally, we see that we may write:

$$Q(n,k,N) = \binom{n-k}{N} (2N)!(2n-2N-2k)! \left[\sum_{r=0}^{k} (-1)^r \binom{2k}{2r} \right.$$
$$\left. \times \prod_{i=1}^{r} (2N+2i-1) \prod_{j=1}^{k-r} (2n-2N-2j+1) \right]$$

and so one may see that the form contained in the brackets is anti-symmetric with respect to the substitution $N \to n-k-N$; this completes the proof. □

We will now look at an analogy of the work done in [FM11, §6]; namely, seeing the image of the restriction $r_\ell(\varphi_{l,l})$ as being in the image of a map from "pure" vector-valued forms. This will follow the constructions in [FM13]; in the above-cited work, there is a vast and very complex theory of restriction constructed, which we have recreated a very small part of in Theorem 7.1.7.

Definition 7.1.8. Let $l \geq 0$ be a non-negative integer, and $[\ell]$ a cusp class of X. Then we define the complex of pure Schwartz forms at the associated boundary component as

$$\left[\mathcal{S}(W_\ell) \otimes T^{l+1,l+1}(W_\ell)\right]^{K_P} \subset \left[\mathcal{S}(W_\ell) \otimes T^{l+1,l+1}(V)\right]^{K_P}$$

The associated map at the boundary is given by

$$\tau_\ell : \left[\mathcal{S}(W_\ell) \otimes T^{l+1,l+1}(W_\ell)\right]^{K_P} \to \left[\mathcal{S}(W_\ell) \otimes \wedge^{1,1}\mathfrak{n}_\ell^* \otimes T^{l,l}(W_\ell)\right]^{K_P},$$

acting as the identity on $\mathcal{S}(W_\ell)$ and mapping the first term in each power of the vector product into the dual forms on $\mathfrak{n}_\ell$:

$$\tau_\ell \left[f \otimes (w_1 \otimes w_2 \otimes \ldots \otimes w_{l+1}) \otimes (\tilde{w_1}^* \otimes \tilde{w_2}^* \otimes \ldots \otimes \tilde{w_{l+1}}^*)\right]$$

$$= f \otimes (w_1 \otimes (\ell' + i\ell' i))^* \wedge (\tilde{w_1} \otimes (\ell' - i\ell' i))^* \\ \otimes (w_2 \otimes \ldots \otimes w_{l+1}) \otimes (\tilde{w_2}^* \otimes \ldots \otimes \widetilde{w_{l+1}}^*) \,.$$

So, using this map, we may immediately write down a pre-image for the restricted Schwartz form $\varphi_{l,l}^{e(P_\ell)}$, under the map τ_ℓ.

Proposition 7.1.9. *τ_ℓ induces a map of complexes*

$$\tau_\ell : \left[\mathcal{S}(W_\ell) \otimes T^{l+1,l+1}(W_\ell)\right]^{K_P} \to \left[\mathcal{S}(W_\ell) \otimes \wedge^{1,1}\mathfrak{n}_\ell^* \otimes T^{l,l}(W_\ell)\right]^{K_P} ;$$

namely, it maps closed forms to closed forms and preserves the invariance under K_P. It is invariant under restriction of $T^{l+1,l+1}(V)$ to the symmetric powers $S^{l+1,l+1}(V)$ and hence to the harmonic vectors $\mathcal{H}^{l+1,l+1}(V)$. We may hence write down a derived map on cohomology:

$$\widetilde{\tau_\ell} : H^0\left(N_\ell, \widetilde{\mathcal{H}^{l+1,l+1}(V)}\right) \to H^2\left(N_\ell, \widetilde{\mathcal{H}^{l,l}(V)}\right)$$

Further, $\varphi_{l,l}$ has a pre-image under this map, which we may write as

$$\varphi^{\mathcal{P}}_{l+1,l+1}(\boldsymbol{x}) = \frac{(-1)^{l+1}(l+1)!}{2(2\pi v)^{l+1}} L_{l+1}\left(2\pi v||\boldsymbol{x}||\right) \exp\left(\pi i \tau ||\boldsymbol{x}||\right) \otimes \pi_{\mathcal{H}}\left(v_2^{l+1} \otimes (v_2^*)^{l+1}\right)$$

Proof. The vast majority of this - namely, everything other than the statement about the pre-image - may be found in [FM13, §6]. The pre-image statement follows immediately from the definition of the forms Ω_ℓ and $\overline{\Omega_\ell}$. □

We note that here we are really using a very specialised case of the work done therein - namely, that this is a statement about generalised theta liftings with coefficients in a vector bundle coming from a general representation of $\mathrm{SO}(p,q)$.

Our next theorem will hence give us the restriction of the theta series $\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)$. What we shall prove is that, for each boundary component $\iota_\ell : e(P_\ell) \to \overline{X}^{BS}$, the natural restriction map ι_ℓ^* on the level of differential forms on X will act - using the isomorphism π between the complexes of Lie algebra dual forms and differential forms - via the map r_ℓ.

Theorem 7.1.10. *Fix a single cusp class $[\ell]$, representing the associated Borel-Serre boundary component $e(P_\ell)$ defined in Corollary 2.3.6; let $\iota_\ell : e(P_\ell) \to \overline{X}^{BS}$ be the natural inclusion map, with pullback ι_ℓ^*.*

The complex of differential forms at this cusp is given by

$$\left[\mathcal{S}(W_\ell) \otimes \wedge^\bullet \mathfrak{n}^* \otimes \mathcal{H}^{l,l}(V)\right]^{K_P} \xrightarrow[\pi_\ell]{\simeq} \left[\mathcal{S}(W_\ell) \otimes \Omega^\bullet\left(N_\ell\right) \otimes \mathcal{H}^{l,l}(V)\right]^{N_\ell} ,$$

where the isomorphism π_ℓ is given, as with the global complexes, by evaluation at the basepoint $s = r = 0$.

Under this isomorphism we may write the boundary form $\varphi_{l,l}^{\mathcal{H},\ell}(\boldsymbol{x}, z, \tau)$ as:

$$\varphi_{l,l}^{\mathcal{H},\ell}(\boldsymbol{x}, z, \tau) = \frac{(-1)^{l+1}(l+1)!}{2(2\pi v)^{l+1}} L_{l+1}\left(2\pi v(\boldsymbol{x}, \boldsymbol{x})\right) e^{\pi i \tau(\boldsymbol{x},\boldsymbol{x})} \otimes \mathrm{d}s \wedge \mathrm{d}\overline{s} \otimes \pi_{\mathcal{H}}\left(n(s,r)\left[v_2^l \otimes (v_2^*)^l\right]\right),$$

and the theta series $\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)$ extends as a differential form to $\overline{X}^{BS}$; on each boundary component $e(P_\ell)$, it restricts to a convergent differential form with coefficients in $\mathcal{H}^{l,l}(V)$:

$$\iota_\ell^*\left(\theta_{\mathcal{L},\mathcal{H}}(\varphi)\right) = \theta_{W_\ell \cap \mathcal{L}}\left(\varphi_{l,l}^{\mathcal{H},\ell}\right) = \sum_{\boldsymbol{x} \in W_\ell \cap \mathcal{L}} \varphi_{l,l}^{\mathcal{H},\ell}(\boldsymbol{x}, z, \tau).$$

Proof. We write the lattice L in the Witt basis; keeping our assumption that L is even and integral, without loss of generality - namely, by rescaling the inner product and the basis - we may assume that it can be written as

$$L = \mathfrak{o}_k \ell \oplus \mathfrak{b} w_\ell \oplus \mathfrak{c}\ell' \tag{7.1.35}$$

for $N(\mathfrak{b})$, $N(\mathfrak{c}) \in \mathbb{Z}$. We wish to perform Poisson summation over the a variable; hence, we need a dual sublattice. Under the real inner product given by

$$(x_1 + x_2\sqrt{d}, y_1 + y_2\sqrt{d}) = x_1 y_1 + x_2 y_2),$$

the dual lattice of $\mathfrak{o}_k$ is given by

$$\mathfrak{o}'_{k,\mathbb{R}} := \begin{cases} \mathfrak{o}_k & \text{if } d \equiv 2, 3 \bmod 4 \\ 2\mathfrak{o}_k & \text{if } d \equiv 1 \bmod 4. \end{cases}$$

For our choice of L as in (7.1.35), one may easily calculate that the dual lattice is given by

$$L' = \overline{\mathfrak{b}}^{-1}\ell \oplus (\mathfrak{d}_k \mathfrak{a})^{-1} w_\ell \oplus \mathfrak{o}_k \ell'.$$

So, for some arbitrary coset $\mathcal{L}$, we write $\mathcal{L} = L + h$ for $h = h_\ell \ell + h_w w_\ell + h_{\ell'}\ell'$.

We know - from Theorem 6.4.1 - that the sum over $\mathbf{x} \in \mathcal{L}$ in the theta series defines a convergent differential form on X. So, by a standard argument with Poisson summation, we may Poisson sum over the $a = X + iY$ variables and write:

$$\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau) = \sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta}'}} \sum_{\substack{(\phi_1,\phi_2)\in \mathfrak{o}'_{k,\mathbb{R}} \\ b,c}} e^{2\pi i(\phi_1 \mathcal{R}(h_\ell) + \phi_2 \mathcal{I}(h_\ell))} \widehat{\varphi_{l,l,r(\alpha,\underline{\beta}),r(\alpha',\underline{\beta}')}}(\phi_1, \phi_2, b, c, z, \tau)$$
$$\otimes\, \Xi_{\alpha'} \wedge \overline{\Xi_\alpha} \otimes \pi_{\mathcal{H}}\left(g_z\left(\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}\right)\right). \tag{7.1.36}$$

In order to find the restriction to the boundary component, as in the case of the individual Schwartz forms in Theorem 7.1.7, we wish to insert the t variable in all necessary positions and then take $t \to \infty$. We use Proposition 7.1.5(b) to recall the form of $\widehat{\varphi}$ for all necessary parameters: namely, as a function of t, it is of the form:

$$h(t, t^{-1})\exp\left(-\frac{\pi v t^2}{|\delta_k|^2}|c|^2 - \frac{\pi t^2}{2v}\left(\widetilde{\beta_1}^2 + \widetilde{\beta_2}^2\right)\right) \tag{7.1.37}$$

for some finite degree complex polynomial h in two variables. In particular, this shows the power of the Poisson summation employed here - it removes all the terms of the form $1/t^2$ in the exponential, and hence in particular gives us a form that is rapidly decreasing in t.

Moreover, this immediately limits this theta series enormously! Because the exponential term in (7.1.37) will dominate all polynomials $h(t, t^{-1})$ whenever $|c|^2$ or $\tilde{\beta_1}^2 + \tilde{\beta_2}^2 \neq 0 \iff c$, ϕ_1 or $\phi_2 \neq 0$, we may now immediately say that under the image of the restriction map ι_ℓ^*, all terms with these variables not 0 will vanish identically. In particular, we see that $\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)$ restricts to a smooth and convergent differential form on $\overline{X}^{BS}$.

More precisely, therefore, we may say how the restriction map acts on the theta series: we are restricting to $W_\ell \subset V$! What I hope to have convinced the reader of here is that the reason the map r_ℓ was introduced was precisely for this task of finding the boundary behaviour of this theta series - namely, by using Poisson summation we have found that we need to restrict to the central component W_ℓ, and then take the limit as $t \to \infty$. Writing this in full, we may see:

$$\begin{aligned}\iota_\ell^*\left(\theta_{\mathcal{L},\mathcal{H}}(\varphi, z, \tau)\right) &= \iota_\ell^*\Bigg[\sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta}'}} \sum_{\substack{(\phi_1,\phi_2)\in\mathfrak{o}'_{k,\mathbb{R}} \\ b,c}} e^{2\pi i(\phi_1\mathcal{R}(h_\ell)+\phi_2\mathcal{I}(h_\ell))} \\ &\quad \times \widehat{\varphi_{l,l,r(\alpha,\underline{\beta}),r(\alpha',\underline{\beta}')}}(\phi_1, \phi_2, b, c, z, \tau) \otimes \Xi_{\alpha'} \wedge \overline{\Xi_\alpha} \otimes \pi_{\mathcal{H}}\left(g_z\left(\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}\right)\right)\Bigg] \\ &= \lim_{t\to\infty}\left(\sum_{\substack{\alpha,\alpha' \\ \underline{\beta},\underline{\beta}'}} \sum_b \widehat{\varphi_{l,l,r(\alpha,\underline{\beta}),r(\alpha',\underline{\beta}')}}(0, b, 0, z, \tau) \otimes \Xi_{\alpha'} \wedge \overline{\Xi_\alpha} \otimes \pi_{\mathcal{H}}\left(g_z\left(\underline{v}_{\underline{\beta}} \otimes \underline{v}^*_{\underline{\beta}'}\right)\right)\right)\end{aligned} \tag{7.1.38}$$

Inserting the t variable into the Fourier transforms, we see that we have an extra power of t^2 at the front coming from the $1/t^2$ in the exponential of φ. Using the orthogonal presentation of the nilpotent Lie algebra $\mathfrak{n}_\ell$ in (7.1.17), we may calculate, because $\mathfrak{p}$ acts on $\mathfrak{k}$ via the adjoint representation, that ξ_2 and $\overline{\xi_2}$ evaluate in the t

variable as

$$\xi_2\left(a(t)^{-1}m(s,r)a(t)\right) = -\frac{1}{t}s, \qquad \overline{\xi_2}\left(a(t)^{-1}m(s,r)a(t)\right) = -\frac{1}{t}\overline{s}.$$

Hence, evaluated in this variable, $\xi_2 \wedge \overline{\xi_2}$ acts as $(\Omega_\ell \wedge \overline{\Omega_\ell})/t^2$, and so this cancels with the t^2 from the Fourier transform. We finally note that $a(t)$ acts trivially on the vector component $v_2^l \otimes (v_2^*)^l$; hence, we may write:

$$\iota_\ell^*\left(\theta_{\mathcal{L},\mathcal{H}}(\varphi,z,\tau)\right) = \pi_\ell\left[\sum_{W_\ell\cap\mathcal{L}} r_\ell\varphi_{l,l}^{\mathcal{H}}(\mathbf{x},\tau)\right] = \pi_\ell\left[\sum_{W_\ell\cap\mathcal{L}} \varphi_{l,l}^{\mathcal{H},\ell}(\mathbf{x},\tau)\right],$$

and so applying π_ℓ to the Lie algebra forms, the proof is complete. □

We now give a small amount of context into what we have achieved. We have shown that the theta series $\theta_{\mathcal{L},\mathcal{H}}(\varphi,z,\tau)$ extends to a convergent differential form on the Borel-Serre compactification $\overline{X}^{BS}$, and that on each of the boundary components $e(P_\ell)$ of $\overline{X}^{BS}$, it restricts to a one-dimensional positive-definite theta series. Moreover, because of the results of Proposition 7.1.9, we see immediately that each of these boundary forms is N_ℓ-invariant; hence, to use the language of special cohomology classes, the restricted form is almost "special" - namely, on each boundary component $e(P_\ell)$ it is an N_ℓ-invariant form. We will see more on such forms in the next chapter, when duality is discussed.

7.2 Construction of Compactly-Supported Theta Series with Coefficients

What we wish to do in this section is to situate the work of §7.1 in the world of modular forms. The principle idea contained herein - namely, that we may use the cone cohomology group to extend the theta series - comes from the work of Funke and Millson again; we shall use much of the theoretical work on the related cochain complex from [FM11, Appendix A].

In particular, our aim in the remainder of this chapter is to use the results of Theorem §7.1.10 to construct a compactly supported differential form on X. In order to do this, we will need to construct a primitive for $\varphi_{l,l}^{\mathcal{H},\ell}(\mathbf{x})$ in the boundary complex. This will allow us to use the structure of the cone cohomology on the compact manifold $\overline{X}^{BS}$ - using the exactness at the boundary, we shall be able to use the isomorphism between the cone cohomology and the compactly supported cohomology on X to find a compactly supported class; hence, we shall be able to integrate this class against the non-compact cohomology on X.

We start with some introductory theory.

Definition 7.2.1. Let A be a smooth, finite-dimensional manifold (possibly with boundary) and B a submanifold, with inclusion $i_B : B \hookrightarrow A$, and projection $\pi_B : A \to B$. Let $E \to A$ be a flat vector bundle over A as in §3.2; we write the i-dimensional differential forms on a space C with coefficients in E as $\Omega^i(C, E)$. Then we define the mapping cone complex as:

$$C^i_{\text{cone}}(A, B, E) = \{[\omega, \eta], \quad \omega \in \Omega^i(A, E), \eta \in \Omega^{i-1}(B, E)\}$$

where the differential is given by:

$$\mathrm{d}[\omega, \eta] = [\mathrm{d}\omega, i_B^*\omega - \mathrm{d}\eta]. \tag{7.2.1}$$

One may check that this differential satisfies the necessary condition $\mathrm{d}^2 = 0$, and so the pair of the complex and this differential defines a cochain complex, with cohomology groups denoted $H^i_{\text{cone}}(A, B, E)$.

In particular, by reading off the definition, we may see immediately what a cocycle in this complex is: it is a pair $[\omega, \eta]$ such that

(i) ω is closed.

(ii) $i_B^*\omega = \mathrm{d}\eta$.

Lemma 7.2.2. *Let $\overline{X}^{BS}$ be the Borel-Serre compactification of the manifold X, and $E \to X$ a flat vector bundle on X that extends to the compactification. Then for all $i \in \mathbb{Z}$,*

$$H^i_{cone}(\overline{X}^{BS}, \partial\overline{X}^{BS}, E) \cong H^i_c(X, E),$$

where $H^i_c(X)$ denotes the cohomology on X of degree i with compact supports, defined in eg [BT95, §1].

Proof. The proof is identical to that found in [FM14, Lemma Appendix A]: indeed, the manifold considered there (corresponding to the case $G = SO(2,1)$) has the same limiting behaviour, as both have the group $A \subset P$ isomorphic to $\mathbb{R}_+$, with the boundary components given by compactifying $t = \infty$. Hence, the limiting behaviour is identical, so the proof will work identically.

The isomorphism is derived from a quasi-isomorphism on cochain complexes. Writing Z for cochain complexes, we let $k : Z^\bullet_c(X, E) \to Z^\bullet(\overline{X}^{BS}, \partial\overline{X}^{BS}, E)$ be the inclusion of the complex of compactly supported cochains into the complex of relative cochains on the Borel-Serre compactification; from [Hat09, Theorem 3.43], we know this to be a quasi-isomorphism. Defining the map j:

$$j : Z^\bullet(\overline{X}^{BS}, \partial\overline{X}^{BS}, E) \to Z^\bullet_{\text{cone}}(\overline{X}^{BS}, \partial\overline{X}^{BS}, E), \quad j(z) = [z, 0]$$

then this is a cochain map - hence well-defined - and in [FM11, Lemma A.3] it is proven that j is also a quasi-isomorphism. Hence, $j \circ k$ is our required quasi-isomorphism between $Z^\bullet_c(X, E)$ and $Z^\bullet_{\text{cone}}(\overline{X}^{BS}, \partial\overline{X}^{BS}, E)$.

In the reverse direction, there is a natural construction considered: namely, for a class $[\omega, \eta] \in Z^\bullet_{\text{cone}}(\overline{X}^{BS}, \partial\overline{X}^{BS}, E)$, one may show that there exists a closed and compactly supported class ξ and a form β, vanishing on $\partial\overline{X}^{BS}$, such that

$$\omega - \mathrm{d}\left(\pi_B^*\left(f(\eta)\right)\right) = \xi + \mathrm{d}\beta,$$

where f is some indicator function on A, non-zero only near ∞. The map

$$Z^\bullet_{\text{cone}}(\overline{X}^{BS}, \partial\overline{X}^{BS}, E) \to Z^\bullet_c(X, E)$$

is hence given by $[\omega, \eta] \to \xi$, and in [FM11, Lemma A.8] they prove that this is a quasi-isomorphism. □

The motivation for introducing the above machinery should now be clear: once we have found a primitive $\phi_{l,l}^{\mathcal{H},\ell}$ for $\varphi_{l,l}^{\mathcal{H},\ell}$, then the associated theta series will be able to take the place of η, giving us a cohomology class! Further, using all the maps in the proof of Lemma 7.2.2, we may use this class to write out what the associated compactly supported class on X will be, allowing us to integrate against the *non-compact* cohomology on X.

We now give a heuristic for the following proof. Though the case with coefficients is more complicated, because coefficients will be paired off when integrated against homology, the rubric of our argument will still apply. So, we need to look at how the differential acts in the complex $[\mathcal{S}(W_\ell) \otimes \Omega^1(N_\ell)]^{N_\ell}$.

Luckily, it turns out we have done almost all the necessary work here! Firstly, we note that N_ℓ acts trivially on $\mathcal{S}(W_\ell)$, so we only need to consider d acting on the differential form. We only have the scalar differential $\mathrm{d}_{\mathcal{S}}$ here (as our representation is trivial); in Lemma 2.4.6, we have already found a primitive for $\Omega_\ell \wedge \overline{\Omega_\ell}$: indeed, the results of this lemma are

$$\mathrm{d}\left(\frac{-\delta_k}{4}\kappa\right) = \Omega_\ell \wedge \overline{\Omega_\ell}.$$

We may check that in the case $l = 0$, the restricted Schwartz form is

$$\varphi_{KM}^{e(P_\ell)}(\mathbf{x},\tau) = \frac{-1}{4\pi v} L_1(2\pi v(\mathbf{x},\mathbf{x}))e^{\pi i\tau(\mathbf{x},\mathbf{x})} \otimes \Omega_\ell \wedge \overline{\Omega_\ell},$$

so that in the complex $[\mathcal{S}(W_\ell)\otimes \mathfrak{n}_\ell^*]^{K_P}$ with trivial coefficients, a primitive is given by

$$\phi_{KM}^{\ell}(\mathbf{x},\tau) = \frac{\delta_k}{16\pi v} L_1(2\pi v(\mathbf{x},\mathbf{x}))e^{\pi i\tau(\mathbf{x},\mathbf{x})} \otimes \kappa_\ell.$$

So, this immediately gives us a starting point for finding a generic primitive. Here, the derivative will act as a sum of the scalar differential $\mathrm{d}_{\mathcal{S}}$ and the vector-valued differential d_V, which will be summed over our chosen basis of $\mathfrak{n}_\ell$; we should also stress that the choice of primitive here will not be unique, as we will be able to add highest/lowest-weight vectors to the components of Ω_ℓ and $\overline{\Omega_\ell}$.

Proposition 7.2.3. *There is a primitive for $\varphi_{l,l}^{\mathcal{H},\ell}$ in the boundary complex*

$$\left[\mathcal{S}(W_\ell)\otimes \wedge^\bullet \mathfrak{n}^* \otimes \mathcal{H}^{l,l}(V)\right]^{K_P},$$

given by

$$\begin{aligned}\phi_{l,l}^{\mathcal{H},\ell}(\boldsymbol{x},\tau) =& \frac{(-1)^{l+1}(l+1)!}{2(2\pi v)^{l+1}} L_{l+1}\left(2\pi v(\boldsymbol{x},\boldsymbol{x})\right) e^{\pi i\tau(\boldsymbol{x},\boldsymbol{x})} \\ &\otimes \left[\frac{-\delta_k}{4}\kappa\otimes \pi_{\mathcal{H}}(v_0) + \Omega_\ell\otimes\pi_{\mathcal{H}}\left(v_{HOL}\right) + \overline{\Omega_\ell}\otimes \pi_{\mathcal{H}}\left(v_{AHOL}\right)\right] \end{aligned} \quad (7.2.2)$$

where v_0 is as usual our weight 0 vector, and v_{HOL}, v_{AHOL} are given by

$$\begin{aligned} v_{HOL} &= -\frac{l\delta_k}{2} v_2^l \otimes (\ell')^* \left(v_2^*\right)^{l-1} = -\frac{l}{4} v_2^l \otimes \left(v_1^* - v_3^*\right)\left(v_2^*\right)^{l-1}, \\ v_{AHOL} &= -\frac{l\delta_k}{2}\ell' v_2^{l-1}\otimes \left(v_2^*\right)^l = \frac{-l}{4}\left(v_1 - v_3\right) v_2^{l-1}\otimes\left(v_2^*\right)^l. \end{aligned}$$

Proof. We start by noting that the action of the scalar differential $\mathrm{d}_{\mathcal{S}}$ gives the appropriate form; indeed, we have already calculated above that the action of $\mathrm{d}_{\mathcal{S}}$ on the κ component gives $\varphi_{l,l}^{\mathcal{H},\ell}$. By our calculations of the homology in Proposition 2.5.3, we know that in the cohomology with rational coefficients, the forms Ω_ℓ and $\overline{\Omega_\ell}$ will map to closed 1-forms on $e(P_\ell)$, so that $\mathrm{d}_{\mathcal{S}}$ acts trivially on them. Hence, we have shown:

$$\mathrm{d}_{\mathcal{S}}\left(\phi_{l,l}^{\mathcal{H},\ell}(\mathbf{x},\tau)\right) = \varphi_{l,l}^{\mathcal{H},\ell}$$

Secondly, one may check that when $l = 0$, this form is identical to the one above, and in this case, $\mathrm{d}_V = 0$ anyway. Hence, we may take this case as proven, and assume herein that $l \geq 1$.

We now claim that d_V acts trivially on $\phi_{l,l}^{\mathcal{H},\ell}$. We know from the definition of the action of the differential that we shall act by the representation ρ on $\mathcal{H}^{l,l}(V)$; we write the basis of $\mathfrak{n}_\ell$ as

$$\nu_\ell = m(1,0) - im(i,0), \quad \overline{\nu_\ell} = m(1,0) + im(i,0), \quad \mu_\ell = m(0,1).$$

For notational simplicity, we may ignore the splitting of d_V into $\mathrm{d}_V = \mathrm{d}_V^+ + \mathrm{d}_V^-$ (namely, we may incorporate the action on $S^{l,l}(V)$ into one representation ρ) so that the action of d_V is

$$\mathrm{d}_V = \left(1 \otimes A\left(\Omega_\ell\right) \otimes \rho\left(\nu_\ell\right)\right) + \left(1 \otimes A\left(\overline{\Omega_\ell}\right) \otimes \rho\left(\overline{\nu_\ell}\right)\right) + \left(1 \otimes A\left(\kappa_\ell\right) \otimes \rho\left(\mu_\ell\right)\right).$$

Acting with this on (7.2.2) and collecting terms, we may write:

$$\begin{aligned} \mathrm{d}_V\left(\phi_{l,l}^{\mathcal{H},\ell}\right) =& \frac{(-1)^{l+1}(l+1)!}{2(2\pi v)^{l+1}} L_{l+1}\left(2\pi v(\mathbf{x},\mathbf{x})\right) e^{\pi i \tau(\mathbf{x},\mathbf{x})} \\ & \otimes \left[\kappa_\ell \wedge \Omega_\ell \otimes \left(\rho(\mu_\ell)(v_{HOL}) + \frac{\delta_k}{4}\rho(\nu_\ell)(v_0)\right)\right. \\ & + \kappa_\ell \wedge \overline{\Omega_\ell} \otimes \left(\rho(\mu_\ell)(v_{AHOL}) + \frac{\delta_k}{4}\rho(\overline{\nu_\ell})(v_0)\right) \\ & \left. + \Omega_\ell \wedge \overline{\Omega_\ell} \otimes \left(\rho(\nu_\ell)(v_{AHOL}) - \rho(\overline{\nu_\ell})(v_{HOL})\right)\right] \end{aligned} \tag{7.2.3}$$

So, we now check that this vector is identically 0. It is an easy calculation that

$$\rho\left(\nu_\ell\right)\left(v_0\right) = -2lv_2^l \otimes \ell^*\left(v_2^*\right)^{l-1}, \qquad \rho\left(\overline{\nu_\ell}\right)\left(v_0\right) = -2l\ell v_2^{l-1} \otimes \left(v_2^*\right)^l$$

and, writing μ_ℓ with respect to the Witt basis as in Definition 2.4.4, we have

$$\rho\left(\mu_\ell\right)\left(v_2^l \otimes (\ell')^*\left(v_2^*\right)^{l-1}\right) = -v_2^l \otimes \ell^*\left(v_2^*\right)^{l-1}, \qquad \rho\left(\mu_\ell\right)\left(\ell' v_2^{l-1} \otimes (v_2)^*\right) = -\ell v_2^{l-1} \otimes (v_2)^*$$

so our choices of v_{HOL}, v_{AHOL} ensure that the $\kappa_\ell \wedge \Omega_\ell$, $\kappa_\ell \wedge \overline{\Omega_\ell}$ terms are trivial. Using the weight operators from §3.1, we may check that

$$\nu_\ell = (\lambda_3 - i\lambda_4) + (\lambda_7 - i\lambda_8), \quad \overline{\nu_\ell} = (\lambda_3 + i\lambda_4) + (\lambda_7 + i\lambda_8), \quad \mu_\ell = |\delta_k|(\lambda_1 + \lambda_2 + 2\lambda_6),$$

so, switching to the orthogonal basis, we may use our calculations in the weight bases from §3.1 to check that that

$$\begin{aligned} \rho\left(\nu_\ell\right)\left((v_1 - v_3)v_2^{l-1} \otimes (v_2^*)^l\right) &= \left((\lambda_3 - i\lambda_4) + (\lambda_7 - i\lambda_8)\right)\left((v_1 - v_3)v_2^{l-1} \otimes (v_2^*)^l\right) \\ &= v_0 - l(v_1 - v_3)v_2^{l-1} - v_0 + l(v_1 - v_3)v_2^{l-1} \otimes v_3^*\left(v_2^*\right)^{l-1} \\ &= l(v_1 - v_3)v_2^{l-1} \otimes \left(v_3^* - v_1^*\right)\left(v_2^*\right)^{l-1} \end{aligned} \tag{7.2.4}$$

and

$$\begin{aligned}\rho\left(\overline{\nu_\ell}\right)\left(v_2^l \otimes (v_1^*-v_3^*)\,(v_2^*)^{l-1}\right) &= \left((\lambda_3+i\lambda_4)+(\lambda_7+i\lambda_8)\right) v_2^l \otimes (v_1^*-v_3^*)\,(v_2^*)^l\right) \\ &= -lv_1v_2^{l-1}\otimes (v_1^*-v_3^*)\,(v_2^*)^{l-1} + v_0 + lv_3v_2^{l-1}\otimes (v_1^*-v_3^*)\,(v_2^*)^{l-1} - \\ &= l(v_3-v_1)v_2^{l-1}\otimes (v_1^*-v_3^*)\,(v_2^*)^{l-1}\,; \qquad (7.2.5)\end{aligned}$$

we see that (7.2.5) equals (7.2.4), and so the proof is complete. □

So, we are ready now to state our main theorem. This will use much the same arguments as were contained in §7.1, but the structure of the statement will be very similar to that contained in §6.3. Namely, we will show that the natural cone form that we wish to construct is holomorphic as a function of τ, using the lowering operator. To do this, we will show that the lowering operator acts on the cone class to give an exact form, and in particular that the auxiliary forms that we used in Theorem 6.3.8 restrict to the boundary complex to give an appropriate auxiliary form here.

Theorem 7.2.4. *Let L be an even, integral lattice of level M as in our previous theorems, with $\mathcal{L}$ some coset of L fixed by Γ. For each cusp class $[\ell]$ of $\Gamma\backslash Iso(\underline{V})$, we let $\theta_{W_\ell\cap\mathcal{L}}(\phi_{l,l}^{\mathcal{H},\ell})$ be the differential form on $\partial\overline{X}^{BS}$ defined by*

$$\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell},\tau\right) = \sum_{\boldsymbol{x}\in W_\ell\cap\mathcal{L}} \phi_{l,l}^{\mathcal{H},\ell}(\boldsymbol{x},\tilde{z},\tau)$$

on the component $e(P_\ell)\subset\partial\overline{X}^{BS}$, and identically 0 on all other components.

Then the class

$$\left[\theta_{\mathcal{L},\mathcal{H}}(\varphi,\tau),\sum_{[\ell]}\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell},\tau\right)\right]$$

defines a cocycle in the cone cochain complex, and hence defines a cohomology class:

$$\left[\theta_{\mathcal{L},\mathcal{H}}(\varphi,\tau),\sum_{[\ell]}\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell},\tau\right)\right] \in H^2_{\text{cone}}\left(\overline{X}^{BS},\partial\overline{X}^{BS},\widetilde{\mathcal{H}^{l,l}(V)}\right).$$

This class is non-trivial in cohomology whenever $[\theta_{\mathcal{L},\mathcal{H}}(\varphi)]$ is non-trivial in $H^2(X,\widetilde{\mathcal{H}^{l,l}(V)})$, and is a holomorphic modular form in τ of weight $2l+3$ and level M. It is cuspidal for $l\geq 1$.

Proof. From the results of Proposition 7.2.3, Theorem 7.1.10 and Theorem 6.4.1, we may write in the cone complex:

$$\mathrm{d}\left[\theta_{\mathcal{L},\mathcal{H}}(\varphi),\sum_{[\ell]}\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right] = \left[\mathrm{d}\theta_{\mathcal{L},\mathcal{H}}(\varphi),\iota^*\theta_{\mathcal{L},\mathcal{H}}(\varphi)-\mathrm{d}\left(\sum_{[\ell]}\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right)\right]$$

$$= \left[0, \sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}} \left(\varphi_{l,l}^{\mathcal{H},\ell}\right) - \mathrm{d}\left(\sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}} \left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right)\right]$$
$$= [0,0]$$

and so it is indeed a cocycle. Because of the action of d on the first component of the cone complex, we see immediately that if $\theta_{\mathcal{L},\mathcal{H}}(\varphi)$ is not exact then the above class cannot be either.

Now we may show holomorphy. For this argument, we require a modification of our earlier restriction argument: indeed, we shall look at what form holomorphy should take, and then restrict the appropriate parts. Acting with the lowering operator on this pair, and using the results of Theorem 6.3.8, we find:

$$\begin{aligned}\omega(L)\left[\theta_{\mathcal{L},\mathcal{H}}(\varphi), \sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}} \left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right] &= \left[\omega(L)\theta_{\mathcal{L},\mathcal{H}}(\varphi), \sum_{[\ell]} \omega(L)\theta_{W_\ell \cap \mathcal{L}} \left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right] \\ &= \left[\mathrm{d}\theta_{\mathcal{L},\mathcal{H}}\left(\psi_{l,l} - \frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right), \sum_{[\ell]} \omega(L)\theta_{W_\ell \cap \mathcal{L}} \left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right] \\ &\stackrel{?}{=} \mathrm{d}\left[\theta_{\mathcal{L},\mathcal{H}}\left(\psi_{l,l} - \frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right), 0\right] \\ &= \left[\mathrm{d}\theta_{\mathcal{L},\mathcal{H}}\left(\psi_{l,l} - \frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right), \iota^*\theta_{\mathcal{L},\mathcal{H}}\left(\psi_{l,l} - \frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right)\right]\end{aligned}$$

Of course, the question mark equality is not shown - hence, we want to show the following at each cusp $[\ell]$:

$$\iota_\ell^*\theta_{\mathcal{L}}\left(\psi_{l,l} - \frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l} + \overline{\Lambda_{j,l}}\right)\right) = \theta_{\mathcal{L}}\left(\omega(L)\left(\phi_{l,l}^{\mathcal{H},\ell}\right)\right).$$

We now recall these objects; what we shall see is that a lot of our earlier work on the restriction map r_ℓ will carry over identically. In (6.3.15), we defined $\psi_{l,l}^{\mathcal{F}}$ in the complex $[\mathcal{F} \otimes \mathfrak{p}^* \otimes T^{l,l}(V)]^K$ as follows:

$$\psi_{l,l}^{\mathcal{F}} = \frac{1}{2}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l+2} \sum_{\alpha',\underline{\beta},\underline{\beta'}} \left(-z_3' z_{\alpha'}'' \underline{z}_{\underline{\beta}}' \underline{z}_{\underline{\beta'}}'' \otimes \xi_{\alpha'} \otimes \underline{v}_{\underline{\beta}} \otimes \underline{v}_{\underline{\beta'}}^* + z_3'' z_{\alpha'}' \underline{z}_{\underline{\beta}}' \underline{z}_{\underline{\beta'}}'' \otimes \overline{\xi_\alpha} \otimes \underline{v}_{\underline{\beta}} \otimes \underline{v}_{\underline{\beta'}}^*\right), \tag{7.2.6}$$

where α' is summed over $\{1,2\}$ and $\underline{\beta}$, $\underline{\beta'} \in \{1,2\}^l$. To distinguish the two terms involved here, we now define $\psi_{l,l}^{Hol}$ to be the left hand side of (7.2.6) (with the ξ_α term) and $\psi_{l,l}^{AntiHol}$ to be the right hand side (with the $\overline{\xi_\alpha}$ term). We use the intertwiner of Lemma 5.3.2 to write the above in the Schrödinger model:

$$\psi_{l,l}^{\mathcal{S}} = \frac{1}{2^{2l+3}} \sum_{\alpha',\underline{\beta},\underline{\beta'}} \left(-\mathcal{D}_3\overline{\mathcal{D}_{\alpha'}}\underline{\mathcal{D}}_{\underline{\beta}}\overline{\underline{\mathcal{D}}_{\underline{\beta'}}} \otimes \xi_{\alpha'} \otimes \underline{v}_{\underline{\beta}} \otimes \underline{v}_{\underline{\beta'}}^* + \overline{\mathcal{D}_3}\mathcal{D}_{\alpha'}\underline{\mathcal{D}}_{\underline{\beta}}\overline{\underline{\mathcal{D}}_{\underline{\beta'}}} \otimes \overline{\xi_\alpha} \otimes \underline{v}_{\underline{\beta}} \otimes \underline{v}_{\underline{\beta'}}^*\right).$$

We now recall the restriction arguments in Theorem 7.1.7. We may see that when we let $c = 0$, the scalar term in $\psi^{Hol}_{l,l}(\mathbf{x})$ will be of the form

$$-a \cdot \left(\mathcal{D}_1^r \circ \mathcal{D}_2^{l-r} \circ \overline{\mathcal{D}_1}^{r'} \overline{\mathcal{D}_2}^{l+1-r'}(\varphi_0)\right)(a, b, 0) \qquad (7.2.7)$$

Using the rubric of the proof of Theorem 7.1.7 - namely, that everything of the form of a modified Laguerre polynomial in a would Fourier transform trivially - we see that there is exactly one term here that has non-trivial Fourier transform: when $r = 0$ and $r' = 1$, we may write (7.2.7) as

$$\psi^{Hol}_{l,l,0,1} = -a\left(\mathcal{D}_2^l \circ \overline{\mathcal{D}_1}^1 \overline{\mathcal{D}_2}^l(\varphi_0)\right)(a, b, 0) = a\overline{a}(-1)^l \frac{l!}{(2\pi)^l} L_l\left(2\pi|b|^2\right)\varphi_0.$$

Inserting τ and t as usual, we hence may use Proposition 7.1.5 again to find the restriction of the Fourier transform to the W_ℓ component. With $r = 0$ and $r' = 1$, we must have all $\underline{\beta} = 2$ and exactly one of $\alpha', \underline{\beta'} = 1$ (with all the others $= 2$); there are plainly l choices of which of the β_i to equal 1 in the latter case. In Lemma 7.1.6, we found the restriction of the Lie algebra dual forms, which we will now apply here.

Using Poisson summation identically to that in the proof of Theorem 7.1.10, we may write

$$\iota_\ell^*\left(\theta_{\mathcal{L}}\left(\psi^{Hol}_{l,l}\right)\right) = \pi_\ell\Bigg(\lim_{t\to\infty}\Bigg(\sum_{b\in W_\ell\cap\mathcal{L}} \frac{-t^2(l+1)}{8\pi}(-1)^l\frac{(l+1)!}{(2\pi v)^l}L_l\left(2\pi v|b|^2\right)\exp\left(\pi i\tau|b|^2\right)$$
$$\otimes\left[-\frac{\delta_k}{2t^2}\kappa_\ell\otimes a(t)(v_0) - \frac{l}{t}\Omega_\ell\otimes a(t)\left(v_2^l\otimes(v_1^*)(v_2^*)^{l-1}\right)\right].\Bigg)\Bigg)$$

By the same logic as in the proof of Theorem 7.1.10, this limit will be exactly the terms without powers of t in. We may easily calculate how $a(t)$ acts on the symmetric powers: for example

$$a(t)\left(v_2^l\otimes(v_1^*)(v_2^*)^{l-1}\right) = \frac{t^{-1}}{2}\left(v_2^l\otimes(v_1^*-v_3^*)(v_2^*)^{l-1}\right)+\mathcal{O}(1)$$

and similarly for other terms. Hence, writing this out, we have hence shown:

$$\iota_\ell^*\left(\theta_{\mathcal{L}}\left(\psi^{Hol}_{l,l}\right)\right) = \sum_{W_\ell\cap\mathcal{L}}\Bigg(\frac{-(l+1)}{8\pi}(-1)^l\frac{(l+1)!}{(2\pi v)^l}L_l\left(2\pi v|b|^2\right)\exp\left(\pi i\tau|b|^2\right)$$
$$\otimes\left[-\frac{\delta_k}{2}\mathrm{d}r\otimes v_0 - \mathrm{d}s\otimes v_2^l\otimes\frac{l}{2}(v_1^*-v_3^*)(v_2^*)^{l-1}\right]\Bigg)$$

We may apply exactly the same argument to the anti-holomorphic part $\psi^{AntiHol}_{l,l}$ to find the same type of restriction:

$$\iota_\ell^*\left(\theta_{\mathcal{L}}\left(\psi_{l,l}^{AntiHol}\right)\right)=\sum_{W_\ell\cap\mathcal{L}}\left(\frac{l+1}{8\pi}(-1)^l\frac{(l+1)!}{(2\pi v)^l}L_l\left(2\pi v|b|^2\right)\exp\left(\pi i\tau|b|^2\right)\right.$$
$$\left.\otimes\left[\frac{\delta_k}{2}\mathrm{d}r\otimes v_0-\mathrm{d}\overline{s}\otimes\frac{l}{2}(v_3-v_1)v_2^{l-1}\otimes(v_2^*)^l\right]\right)$$

We now consider the $\Lambda_{j,l}$ and $\overline{\Lambda_{j,l}}$. Using Definition 6.3.6(iii), we may easily write down an explicit form in the Fock model:

$$\Lambda_{j,l}=\frac{1}{2\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l}\sum_{\substack{\alpha\in\{1,2\}\\ \underline{\beta}\in\{1,2\}^{l-1}\\ \underline{\beta}'\in\{1,2\}^l}}z'_\alpha\underline{z}'_{\underline{\beta}}\underline{z}''_{\underline{\beta}'}\otimes\overline{\xi_\alpha}\otimes A_j(v_3)\underline{v}_{\underline{\beta}}\otimes\underline{v}^*_{\underline{\beta}'},$$
$$\overline{\Lambda_{j,l}}=\frac{-1}{2\pi}\left(\frac{-i}{2\sqrt{2}\pi}\right)^{2l}\sum_{\substack{\alpha\in\{1,2\}\\ \underline{\beta}\in\{1,2\}^{l}\\ \underline{\beta}'\in\{1,2\}^{l-1}}}z''_\alpha\underline{z}'_{\underline{\beta}}\underline{z}''_{\underline{\beta}'}\otimes\xi_\alpha\otimes\underline{v}_{\underline{\beta}}\otimes A_j(v_3^*)\underline{v}^*_{\underline{\beta}'}.$$

We again use the restriction arguments from Theorem 7.1.7; we may use the intertwiners from Lemma 5.3.2 to write these forms in $[\mathcal{S}(V)\otimes\mathfrak{p}^*\otimes T^{l,l}(V)]^K$:

$$\Lambda_{j,l}^{\mathcal{S}}(\mathbf{x})=\frac{1}{2\pi}\sum_{\substack{\alpha\in\{1,2\}\\ \underline{\beta}\in\{1,2\}^{l-1}\\ \underline{\beta}'\in\{1,2\}^l}}\mathcal{D}_\alpha\underline{\mathcal{D}}_{\underline{\beta}}\underline{\overline{\mathcal{D}}}_{\underline{\beta}'}(\varphi_0)\otimes\overline{\xi_\alpha}\otimes A_j(v_3)\underline{v}_{\underline{\beta}}\otimes\underline{v}^*_{\underline{\beta}'},$$
$$\overline{\Lambda_{j,l}}^{\mathcal{S}}(\mathbf{x})=\frac{-1}{2\pi}\sum_{\substack{\alpha\in\{1,2\}\\ \underline{\beta}\in\{1,2\}^{l}\\ \underline{\beta}'\in\{1,2\}^{l-1}}}\overline{\mathcal{D}_\alpha}\underline{\mathcal{D}}_{\underline{\beta}}\underline{\overline{\mathcal{D}}}_{\underline{\beta}'}(\varphi_0)\otimes\xi_\alpha\otimes\underline{v}_{\underline{\beta}}\otimes A_j(v_3^*)\underline{v}^*_{\underline{\beta}'}.$$

Indeed, we see that in both $\Lambda_{j,l}^{\mathcal{S}}$ and $\overline{\Lambda_{j,l}}^{\mathcal{S}}$, we have the same number of $\mathcal{D}_i$ and $\overline{\mathcal{D}_i}$ terms for all indices, and so the pattern in the scalar terms will be identical as in the $\varphi_{l,l}$, but here we have l rather than $l+1$ of each. Hence, in order to find the restriction in the $\mathcal{S}(V)$ term, a completely identical argument to Theorem 7.1.7 may be employed, and we may say that for all indices with any of the α, α', β_i, $\beta_i'\neq 2$ the restriction of the Fourier transform will be trivial.

So, we may again project into the symmetric co-ordinates and find that the geometric restriction of the relevant theta series is given by

$$\iota_\ell^*\left(\theta_{\mathcal{L}}\left(\Lambda_{j,l}\right)\right)=\pi_\ell\left(\lim_{t\to\infty}\left(\frac{1}{4\pi}\frac{t^2(l+1)}{2v}\frac{(-1)^l(l+1)!}{(2\pi)^l v^{l-1}}L_l(2\pi v|b|^2)\exp\left(\pi i|b|^2\tau\right)\right.\right.$$
$$\left.\left.\otimes\left(\frac{-1}{t}\overline{\Omega_\ell}\right)\otimes a\left(t\right)\left(v_3v_2^{l-1}\otimes(v_2^*)^l\right)\right)\right)$$

and

$$\iota_\ell^*\left(\theta_{\mathcal{L}}\left(\overline{\Lambda_{j,l}}\right)\right) =\pi_\ell\Bigg(\lim_{t\to\infty}\Bigg(\frac{-1}{4\pi}\frac{t^2(l+1)}{2v}\frac{(-1)^l(l+1)!}{(2\pi)^l v^{l-1}}L_l(2\pi v|b|^2)\exp\left(\pi i|b|^2\tau\right) \otimes\left(\frac{-1}{t}\Omega_\ell\right)\otimes a\left(t\right)\left(v_2^l\,\otimes\,v_3^*\left(v_2^*\right)^{l-1}\right)\Bigg)\Bigg)$$

So, by an exactly identical argument to the above, we now need to find the $\mathcal{O}(t^{-1})$ parts; this is, again, identical to the above. We may combine all of these calculations to give us that:

$$\iota_\ell^*\left(\theta_{\mathcal{L}}\left(\psi_{l,l}-\frac{1}{2}\sum_{j=1}^{l}\left(\Lambda_{j,l}+\overline{\Lambda_{j,l}}\right)\right)\right)=\sum_{W_\ell\cap\mathcal{L}}\Bigg(\frac{l+1}{8\pi}\frac{(-1)^l(l+1)!}{(2\pi v)^l}L_l\left(2\pi v|b|^2\right)\exp\left(\pi i\tau|b|^2\right) \otimes\left[\delta_k \mathrm{d}r\otimes v_0+l\mathrm{d}s\otimes v_2^l\otimes\left(v_1^*-v_3^*\right)\left(v_2^*\right)^{l-1}+l\mathrm{d}\overline{s}\otimes(v_1-v_3)v_2^{l-1}\otimes\left(v_2^*\right)^l\right]\Bigg) \tag{7.2.8}$$

We now need the action of the lowering operator. For convenience, because our work so far was in the Schrödinger model, we shall continue in this vein. Here we know that the lowering operator acts as:

$$L_\tau=-2iv^2\frac{\partial}{\partial\overline{\tau}}.$$

Ignoring the constants (and the other terms in the vector product), we may calculate:

$$\frac{\partial}{\partial\overline{\tau}}\left(v^{q-(l+1)}e^{\pi i(\mathbf{x},\mathbf{x})\tau}\right)=\frac{i}{2}(q-(l+1))v^{q-(l+2)}e^{\pi i(\mathbf{x},\mathbf{x})\tau},$$

and so

$$\begin{aligned}\omega(L)\left(v^{-(l+1)}L_{l+1}(2\pi v(\mathbf{x},\mathbf{x}))e^{\pi i\tau(\mathbf{x},\mathbf{x})}\right)&=-2iv^2\sum_{q=0}^{l+1}(2\pi(\mathbf{x},\mathbf{x}))^q\gamma_{l+1,q}\frac{\partial}{\partial\overline{\tau}}\left(v^{q-(l+1)}e^{\pi i(\mathbf{x},\mathbf{x})\tau}\right)\\&=\sum_{q=0}^{l}(2\pi(\mathbf{x},\mathbf{x}))^q\gamma_{l+1,q}(q-(l+1))v^{q-l}e^{\pi i(\mathbf{x},\mathbf{x})\tau};\end{aligned} \tag{7.2.9}$$

we note that $q=l+1$ gives 0 in this sum, so we only need sum from 0 to l. It is a simple calculation using (7.1.22), the explicit form of the coefficients $\gamma_{l+1,q}$ of the Laguerre polynomials, that:

$$(q-(l+1))\gamma_{l+1,q}=(q-(l+1))\binom{l+1}{q}\frac{(-1)^{l+1+q}}{q!}=-(l+1)\gamma_{l,q},$$

and so we may write (7.2.9) as

$$\omega(L)\left(v^{-(l+1)}L_{l+1}(2\pi v(\mathbf{x},\mathbf{x}))e^{\pi i\tau(\mathbf{x},\mathbf{x})}\right) = -(l+1)v^{-l}\sum_{q=0}^{l}(2\pi v(\mathbf{x},\mathbf{x}))^{q}\,\gamma_{l,q}e^{\pi i\tau(\mathbf{x},\mathbf{x})}$$
$$= -(l+1)v^{-l}L_l(2\pi v(\mathbf{x},\mathbf{x})).$$

As $\omega(L)$ acts only non-trivially only on $\mathcal{S}(V)$, we may write:

$$\omega(L)\left(\phi_{l,l}^{S,\ell}\right) = -\,(l+1)\frac{(-1)^l(l+1)!}{2(2\pi)(2\pi v)^l}L_l(2\pi v(\mathbf{x},\mathbf{x}))\exp\left(\pi i(\mathbf{x},\mathbf{x})\tau\right)$$
$$\otimes\left[-\frac{\delta_k}{4}\kappa_\ell\otimes v_0 + \Omega_\ell\otimes v_{HOL} + \overline{\Omega_\ell}\otimes v_{AHOL}\right].$$

Inserting the forms of the vectors v_{HOL} and v_{AHOL} found in Proposition 7.2.3, we may take a factor of $-1/4$ out of this form, and hence see that this is equal to the restriction of the form found above in (7.2.8).

Finally, we may say why this class in the cone cohomology is a holomorphic modular form: indeed, we have already shown it to be a holomorphic class when taken in cohomology. We know that the global theta series $\theta_{\mathcal{L},\mathcal{H}}(\varphi,\tau)$ has a modular transformation property with respect to the subgroup $\Gamma(M)$.

Using identical arguments to those of Lemma 6.3.2, we may see that $\phi_{l,l}^{\mathcal{H},\ell}$ is also an eigenvector of weight $2l+3$ under the action of $\mathfrak{k}'$, and by the theta machinery, we may identically see that for each $[\ell]$, the boundary theta series $\theta_{W_\ell\cap\mathcal{L}}(\phi_{l,l}^{\mathcal{H},\ell},\tau)$ also has a modular transformation law of weight $2l+3$ with respect to the subgroup $\Gamma(M)$. In particular, we have shown that for all $C\in H_2(X,\widetilde{\mathcal{H}^{l,l}(V)})$, all the terms in the Kronecker pairing

$$\left\langle C,\left[\theta_{\mathcal{L},\mathcal{H}}(\varphi,\tau),\sum_{[\ell]}\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell},\tau\right)\right]\right\rangle = \int_C\theta_{\mathcal{L},\mathcal{H}}(\varphi,\tau) - \sum_{[\ell]}\int_{\partial_\ell C}\theta_{W_\ell\cap\mathcal{L}}\left(\phi_{l,l}^{\mathcal{H},\ell},\tau\right)$$

are modular of weight $2l+3$. Hence, as we know that this will be holomorphic, then we may conclude by linearity that this is a holomorphic modular form. □

Chapter 8

Duality

In this final section, we look at the last outstanding work - namely, duality. We will here be able to prove that the generating series of special cycles for $l \geq 0$ is modular, and in particular will give the relevant duality statement that will tell us the Fourier coefficients of the capped theta series. This will follow the geometric arguments of [FM14].

We will also be able to use this to interpret the work of Cogdell as a corollary of ours, using a natural map between the Borel-Serre compactification and the toroidal compactification used by Cogdell to find his geometric modular forms on similar Picard modular surfaces. We shall illustrate this with an example.

8.1 Duality

We shall start with a restatement of the global duality statement of Theorem 6.1.13. In order to do so on the manifold X, we shall define some more Schwartz forms which homogenise the indexing used previously; this exactly follows the notation of [FM14].

Definition 8.1.1. Fix $n \in \mathbb{Q}_{>0}$. For the global Schwartz form $\varphi^{\mathcal{H}}_{l,l}$, we let

$$\varphi^{\mathcal{H}}_{l,l}(n) = \sum_{\substack{\mathbf{x} \in \mathcal{L} \\ (\mathbf{x},\mathbf{x})=2n}} \varphi^{\mathcal{H}}_{l,l}(\mathbf{x}, z, \tau).$$

We may define the same object for the local differential form $\phi^{\mathcal{H},\ell}_{l,l}$ at any cusp $[\ell]$ of X:

$$\phi^{\mathcal{H},\ell}_{l,l}(n) = \sum_{\substack{\mathbf{x} \in \mathcal{L} \cap W_\ell \\ (\mathbf{x},\mathbf{x})2n}} \phi^{\mathcal{H},\ell}_{l,l}(\mathbf{x}, z, \tau).$$

At each such cusp $[\ell]$, we consider some product neighbourhood of $\tilde{V_T}$ of $e(P_\ell)$ in $\overline{X}^{BS}$ as in (2.3.16); we hence may define some smooth function $f_\ell : \tilde{V_T} \to \mathbb{R}_{>0}$ which is only a function of the t co-ordinate, and satisfies $f_\ell = 1$ near $t = \infty$ and 0 elsewhere - one may think of it as a smoothed step function. We let $\pi^{BS} : \overline{X}^{BS} \to \partial\overline{X}^{BS}$ be the topological projection into the boundary.

We hence define the compactified Schwartz function

$$\varphi_{l,l}^{\mathcal{H},c}(n) = \varphi_{l,l}^{\mathcal{H}}(n) - \sum_{[\ell]} \mathrm{d}\left[f_\ell \left(\pi^{BS}\right)^* \left(\phi_{l,l}^{\mathcal{H},\ell}(n)\right)\right].$$

In general, this notation of replacing the $\mathbf{x}$ variable with a norm n may be understood in the same way as above. So, we may now state the duality result.

It is proven in [BF04, Theorems 7.1 & 7.2] in the orthogonal case, but one may see the proof to carry over identically to the unitary setting, as in e.g. [FH19]; indeed, we here recall the form $\tilde{\psi}(\mathbf{x}, z)$ defined in the proof of Theorem 6.1.13, which is non-singular for $\mathbf{x} \not\perp z$.

Proposition 8.1.2 ([FH19]). *Let η be a compactly-supported 2-form on X. Then*

$$\int_X \eta \wedge \varphi_{KM}(n) = ie^{2\pi i n\tau} \int_{C_n} \eta - \int_X \mathrm{d}\eta \wedge \tilde{\psi}(n) \tag{8.1.1}$$

Indeed, the reader may see that when η is closed, the second integral on the right hand side of (8.1.1) is trivial, which recovers the duality expressed in Theorem . Further, an analogous form $\tilde{\psi}_{l,l}(\mathbf{x}, z)$ must exist as a primitive to $\varphi_{l,l}(\mathbf{x}, z)$ for $\mathbf{x} \not\perp z$; by an identical argument to the geometric arguments in §6.3, we may take this in harmonic coefficients as $\tilde{\psi}_{l,l}^{\mathcal{H}}(\mathbf{x}, z) = \tilde{\psi}(\mathbf{x}, z) \otimes \pi_{\mathcal{H}}(\mathbf{x}^l \otimes (\mathbf{x}^*)^l)$.

Definition 8.1.3. We say a differential form $\eta \in H^2(\overline{X}^{BS}, \widetilde{\mathcal{H}^{l,l}(V)})$ is special if

(i) For each cusp $[\ell]$ of X, there is a neighbourhood $\tilde{V_T}$ of $e(P_\ell)$ where η may be written as a pullback of a differential form η_ℓ on $e(P_\ell)$.

(ii) Under the pullback of the map $N_\ell \to \Gamma_\ell \backslash N_\ell$, η_ℓ is left N_ℓ-invariant.

These forms are closed under the action of the normal differential d from the full cochain complex, and hence give cohomology groups $H^i_{\mathrm{Sp}}(\overline{X}^{BS}, E)$. We have the following important result on special forms:

Lemma 8.1.4 ([GHM94]). *For any coefficient system E on $\overline{X}^{BS}$, the special forms on $\overline{X}^{BS}$ with coefficients in E compute the full cohomology group:*

$$H^i_{Sp}(\overline{X}^{BS}, E) \simeq H^i(\overline{X}^{BS}, E).$$

Proposition 8.1.5. *Let $\boldsymbol{x} \in \underline{W}_\ell$ be a vector of positive length in the central Witt component, and let $\boldsymbol{y} \in \{\boldsymbol{x} \mid \ell\}$ be some vector parameterising the boundary fibres of $C_{\boldsymbol{x}}$, using the notation of Lemma 4.2.4. Then for η any closed special 2-form on $\overline{X}^{BS}$, 2-forms η on $e(P_\ell)$, the boundary integrals are 0:*

$$\int_{A_{\boldsymbol{y}}^\ell} \eta = 0 = \int_{e(P_\ell)} \eta \wedge \phi_{KM}^{e(P_\ell)}(n). \tag{8.1.2}$$

Proof. We first note that the second equality in (8.1.2) is an immediate consequence of the definition of $\phi_{KM}^{e(P_\ell)}$ and of the structure of the 2nd cohomology group of $e(P_\ell)$. Indeed, using Proposition 2.5.3, we see that over $\mathbb{C}$, the de Rham cohomology group $H^2_{dR}(e(P_\ell))$ - which is naturally isomorphic to $H^2(e(P_\ell), \mathbb{C})$ - is spanned over $\mathbb{C}$ by the forms $\mathrm{d}w \wedge \mathrm{d}r$ and $\mathrm{d}\overline{w} \wedge \mathrm{d}r$. Moreover, as we assume (using Lemma 8.1.4, this is without any loss of generality) that η is special, we know it may be written as a sum of representatives of the cohomology on $e(P_\ell)$ at each cusp.

So, as we know that $\phi_{KM}^{e(P_\ell)}$ is proportional to the differential form $\mathrm{d}r$, the wedge product with this and η will be identically 0, and so this integral is 0.

We now treat the integral over the 2-chain $A_{\mathbf{y}}^\ell$; without loss of generality, it is clear we may assume that $\mathbf{y}$ satisfies $s(\mathbf{y}) = 0$, and hence that $\mathbf{y} = \mathbf{x}$. We recall this chain from Definition 4.2.5 as (proportional to) the difference of the 2-chains

$$A_{\mathbf{x}}^\ell = \frac{1}{2d(\Gamma, \ell)} \left(T_{0,\ell} - \chi_{0,\ell}\right).$$

We shall prove the integral to be trivial for the form $\mathrm{d}w \wedge \mathrm{d}r$; the reader may see that the proof for $\mathrm{d}\overline{w} \wedge \mathrm{d}r$ will be identical.

Indeed, we claim that the integral over the triangular chain $T_{0,\ell}$ is equal to the integral over $\chi_{0,\ell}$. We have defined the former to be the 2-chain bounding the triangular 1-chain

$$n(b\lambda_2, -2b\langle\lambda_1, \lambda_2\rangle_\ell) - n(\tilde{b}\lambda_2, 0) - n(0, -2\tilde{\tilde{b}}\langle\lambda_1, \lambda_2\rangle_\ell).$$

So, we now change variables in the r term: namely, we swap $r' = 2r$, so that we write $\widetilde{T_{0,\ell}}$ as the cycle bounding

$$n(b\lambda_2, -b\langle\lambda_1, \lambda_2\rangle_\ell) - n(\tilde{b}\lambda_2, 0) - n(0, -\tilde{\tilde{b}}\langle\lambda_1, \lambda_2\rangle_\ell).$$

Changing variables in the differential form as well, we have hence shown

$$\int_{T_{0,\ell}} \mathrm{d}w \wedge \mathrm{d}r = 2 \int_{\widetilde{T_{0,\ell}}} \mathrm{d}w \wedge \mathrm{d}r$$

However, we may now recognise this as a pair of chains which add together to give $\chi_{0,\ell}$. Indeed, taking the second copy of $\widetilde{T_{0,\ell}}$, we may see this as homotopic to the complementary chain $\widetilde{T_{0,\ell}}'$ bounding the 1-cycle given by

$$n(b\lambda_2, 0) - n\left(0, \tilde{b}\langle\lambda_1,\lambda_2\rangle\right) - n\left(\tilde{\tilde{b}}\lambda_2, -2\tilde{\tilde{b}}\langle\lambda_1,\lambda_2\rangle\right).$$

We hence see that the two 2-chains $\widetilde{T_{0,\ell}}$ and $\widetilde{T_{0,\ell}}'$ are the two triangular chains adding together to give $\chi_{0,\ell}$, and so we have shown

$$\int_{T_{0,\ell}} \eta = \int_{\widetilde{T_{0,\ell}}} \eta + \int_{\widetilde{T_{0,\ell}}'} \eta = \int_{\chi_{0,\ell}} \eta,$$

and so the integral of any closed η over $A_{\mathbf{x},\ell}$ will be 0. □

Theorem 8.1.6. *Let $n \in \mathbb{Q}_{>0}$. The compactified Schwartz function $\varphi^c_{KM}(n)$ is an Poincaré dual for C^c_n; namely, for η a closed 2-form on $\overline{X}^{BS}$:*

$$\int_X \eta \wedge \varphi^c_{KM}(n) = ie^{-2\pi n}\int_{C^c_n} \eta \tag{8.1.3}$$

Proof. At each cusp $[\ell]$, we let $\sigma_T \equiv \sigma_{T,\ell}$ be a smooth function on X which is 1 for $t \le T$ and 0 for $t \ge T+1$; in particular, this means we may write the left hand side of (8.1.3) as

$$\int_X \eta \wedge \varphi^c_{KM}(n) = \lim_{T\to\infty}\int_X (\sigma_T\eta)\wedge\varphi^c_{KM}(n).$$

For each T, $\sigma_T\eta$ is a compactly supported form on X, so we may apply Theorem 8.1.2: splitting $\varphi^c_{KM}(n)$ into its global and cuspidal parts, we may write the left hand side of (8.1.3) as

$$ie^{-2\pi n}\int_{C_n}\eta - \lim_{T\to\infty}\int_X \mathrm{d}\left(\sigma_T\eta\right)\wedge\tilde{\psi}(n) - \sum_{[\ell]}\lim_{T\to\infty}\int_X \sigma_T\eta\wedge\mathrm{d}\left[f_\ell\left(\pi^{BS}\right)^*\left(\phi^\ell_{KM}(n)\right)\right]. \tag{8.1.4}$$

Using the elementary equation

$$\mathrm{d}(\alpha_1\wedge\alpha_2) = \mathrm{d}\alpha_1\wedge\alpha_2 + (-1)^{\deg(\alpha_1)}\alpha_1\wedge\mathrm{d}\alpha_2,$$

and using Stokes' theorem - see e.g. [BT95, Theorem 3.5] - which tells us that $\int_X \mathrm{d}\left[\sigma_T\eta\wedge f_\ell\left(\pi^{BS}\right)^*\left(\phi^\ell_{KM}(n)\right)\right] = 0$ for all T, we may write (8.1.4) as

$$ie^{-2\pi n}\int_{C_n}\eta - \lim_{T\to\infty}\int_X \mathrm{d}\left(\sigma_T\eta\right)\wedge\tilde{\psi}(n) + \sum_{[\ell]}\lim_{T\to\infty}\int_X \mathrm{d}\left(\sigma_T\eta\right)\wedge f_\ell\left(\pi^{BS}\right)^*\left(\phi^\ell_{KM}(n)\right). \tag{8.1.5}$$

By definition of the differential on local co-ordinates, we have $[\sigma_T\eta] = \sigma'_T(t)\mathrm{d}t\wedge\eta + \sigma_T\mathrm{d}\eta$; as we have defined σ_T to be constant outside $[T, T+1]$, and $\mathrm{d}\eta$ is assumed to be 0, we may hence write (8.1.5) as

$$ie^{-2\pi n}\int_{C_n}\eta - \lim_{T\to\infty}\int_X\left[\sigma_T'(t)\mathrm{d}t\wedge\eta\wedge\left(\tilde{\psi}(n)-\sum_{[\ell]}\lim_{T\to\infty}f_\ell\left(\pi^{BS}\right)^*\left(\phi_{KM}^\ell(n)\right)\right)\right] \tag{8.1.6}$$

By Lemma 8.1.4, we may without loss of any generality assume that η is special. As each $\tilde{V_T}$ is a product of $e(P_\ell)$ with a space that is homotopically trivial, we know by the Künneth formula - see [BT95, §5, p.47] - that any special cohomology class η will be equal to η_ℓ on $\tilde{V_T}$; in particular, it will be independent of T, so we may separate the differential forms in the wedge product and write (8.1.6) as

$$ie^{-2\pi n}\int_{C_n}\eta + \lim_{T\to\infty}\int_{t=T}^{T+1}\sigma_T'(t)\mathrm{d}t\int_{e(P_\ell)}\eta\wedge\left[\sum_{[\ell]}f_\ell\left(\pi^{BS}\right)^*\left(\phi_{KM}^\ell(n)\right)-\tilde{\psi}(n)\right]. \tag{8.1.7}$$

We now claim that the form $\tilde{\psi}(n)$ restricts to a form proportional to $\mathrm{d}r$ on each boundary component $e(P_\ell)$; equivalently, looking at the restriction of the Lie algebra elements in Lemma 7.1.6, we claim that the scalar terms proportional to ξ_2 and $\overline{\xi_2}$ in $\tilde{\psi}(n)$ will restrict to 0.

By definition, the ξ_2 and $\overline{\xi_2}$ scalar terms in $\tilde{\psi}_{KM}(\mathbf{x})$ will be of the form

$$\frac{b}{t^{-1}a-\frac{tc}{2\delta_k}},\qquad \overline{\left(\frac{b}{t^{-1}a-\frac{tc}{2\delta_k}}\right)}$$

Again, putting $c=0$, we see that this is an odd function in a; by our Fourier transform calculations in Proposition 7.1.5 again, this tells us that the Fourier transform of this scalar term will be identically 0, as the restriction map only picks up the coefficients $a_{2k,2n-2k}$. Hence, this tells us that hence, this is trivially also true for $\tilde{\psi}_{l,l}^{\mathcal{H}}$.

We recall now that we have assumed without loss of generality that η is special, and hence retracts to a closed 2-form η_ℓ on each $e(P_\ell)$. In the boundary integrals, we may write $(\pi^{BS})^*\phi_{KM}^\ell(n)=\phi_{KM}^\ell(n)$, write $f_\ell\equiv 1$ for large enough T. Because η - assumed closed - will have $\mathrm{d}r$ in its wedge product in local co-ordinates, the term vanishes, and so we may split the integral in (8.1.7) and write

$$\int_X\eta\wedge\varphi_{KM}^c(n) = ie^{-2\pi n}\int_{C_n}\eta - \sum_{[\ell]}\int_{e(P_\ell)}\eta\wedge\phi_{KM}^\ell(n) \tag{8.1.8}$$

We now take the pairing in the integral on the right-hand side of (8.1.8); in Proposition 8.1.5 we showed that these integrals are all 0 for $l=0$, and in particular are hence equal to the integrals over A_n for all n. Hence, all the boundary integrals will disappear when η is closed, and we may write

$$\int_X \eta \wedge \varphi^c_{KM}(n) = ie^{-2\pi n} \int_{C^c_n} \eta.$$

□

Using the relationship between the compactly supported cohomology and the cohomology of the cone complex, we now have our main theorem.

Theorem 8.1.7. *Let L be an even integral lattice in V of level M, with $\mathcal{L} \in L'/L$ some coset. Let $\Gamma \subset \underline{G}$ be some arithmetic subgroup fixing all such $\mathcal{L}$. Then the class*

$$\left[\theta_{\mathcal{L}}(\varphi,\tau), \sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}}\left(\phi^\ell_{KM},\tau\right)\right] \in H^i_{cone}\left(\overline{X}^{BS}, \partial\overline{X}^{BS}\right)$$

defines a non-cuspidal, holomorphic modular form of weight 3 and level M, whose coefficients are given by the compactified cycles C^c_n:

$$\left[\theta_{\mathcal{L}}(\varphi,\tau), \sum_{[\ell]} \theta_{W_\ell \cap \mathcal{L}}\left(\phi^\ell_{KM},\tau\right)\right] = \frac{1}{2\pi}\delta_{\mathcal{L}=L}\left[\Omega_X\right] + \sum_{n>0} \left[C^c_n\right]^{PD} e^{2\pi i n\tau}$$

Proof. This is an immediate corollary of Theorem 7.2.4 and Theorem 8.1.6 for η a closed form, as we may notice that $\varphi^c_{KM}(n)$ is the image in $H^2_c(X)$ of the pair in the cone complex, under the map into the compactly supported cohomology constructed in Lemma 7.2.2. □

Corollary 8.1.8. *Theorem 4.2.9 is true; namely, that for all $l \geq 0$, the generating series given by*

$$\frac{1}{2\pi}\delta_{l=0}\left[\Omega_X\right]^{PD} + \sum_{n>0}\left[C^c_{n,[l,l]}\right]^{PD} q^n$$

is modular of weight $2l+3$, and is a cusp form if $l \geq 1$.

Proof. This follows immediately from the proof of Proposition 8.1.5 and from Theorem 8.1.7. Namely, given any closed η representing a cohomology class $[\eta]$ on X, the integral against C^c_n will equal the integral against C_n, as the integrals against A_n will be trivial at all the cusps. As the capping cycles $A_{n,[l,l]}$ are proportional to the A_n - namely, at each vector $\mathbf{y}$ they are equal to $A_{\mathbf{y}} \otimes \mathbf{y}^l \otimes (\mathbf{y}^*)^l$ - we see that this will also hold for general l, and hence modularity follows immediately. □

8.2 A relationship to Cogdell's modular generating series

In Cogdell's paper [Cog85], the author follows the same logic as us - namely, he recognises that on $\overline{X}^{TOR}$, there is a need to modify the natural cycles C_n in order to be able to find a pairing into the space of holomorphic modular forms.

The natural starting point here is to consider the cycles C_n embedded in $\overline{X}^{TOR}$; we hence may find the topological closure of these in $\overline{X}^{TOR}$ for all $n > 0$, and denote these cycles by $D_n \to \overline{X}^{TOR}$. By the definition of the topological closure, this will define a class (which, like the C_n in X, will be generically non-trivial) in $H_2(\overline{X}^{TOR})$. The divisors compactifying X to $\overline{X}^{TOR}$ may be explicitly written down in this case; using the notation of §2.3.2, these are the $\tilde{\tilde{U}}_M$ defined in (2.3.9). At each cusp class $[\ell] \in \Gamma\backslash\mathrm{Iso}(\underline{V})$, we let $\mathcal{D}_\ell$ be the span of all the classes given by compactifying divisors at $[\ell]$.

We may view the Borel-Serre boundary as dividing $\overline{X}^{TOR}$ into two sections:

$$\overline{X}^{TOR} = \overline{X}^{TOR}_{int} \cup \overline{X}^{TOR}_{ext}, \tag{8.2.1}$$

where $\overline{X}^{TOR}_{int} \cap \overline{X}^{TOR}_{ext} = \partial\overline{X}^{BS}$ and the interior part is just isomorphic to X. Hence, from e.g. [Cog85, p.125], we know that when $\iota_{TOR} : X \to \overline{X}^{TOR}$ is the natural inclusion map, we have a splitting of the homology of the compactified Picard modular surface as follows:

$$H_2(\overline{X}^{TOR}) = (\iota_{TOR})_* H_2(X) \oplus_{[\ell]} H_2(\mathcal{D}_\ell). \tag{8.2.2}$$

which will be orthogonal with respect to the intersection pairing.

Proposition 8.2.1. *Let D_n be the topological closure of $C_n \hookrightarrow \overline{X}^{TOR}$ as defined above, and let D_n^c be projection of D_n into $(\iota_{TOR})_* H_2(X)$ in the splitting* (8.2.2) *- this is exactly the compactified cycle considered by Cogdell in [Cog85]. For all n we have the following equation in homology:*

$$(\iota_{TOR})_* [C_n^c] = [D_n^c].$$

Proof. We largely mimic the proof given in [FM14]; as there, for simplicity, we assume that there is a single cusp $[\ell]$ of X. By the splitting of $\overline{X}^{TOR}$ given in (8.2.1), we can split along these submanifolds to get

$$D_n = \left(D_n \cap \overline{X}^{TOR}_{int}\right) + \left(D_n \cap \overline{X}^{TOR}_{ext}\right)$$

We have defined D_n so that it is closed with respect to the homological boundary operator $\partial : Z_j(\overline{X}^{TOR}) \to Z_{j-1}(\overline{X}^{TOR})$. As in e.g. §4, we let $\overline{C_n}$ be the closure of C_n in $\overline{X}^{BS}$, which - using the homotopy equivalence $\overline{X}^{BS} \simeq X$ - we may consider as a class in $H_2(X)$. As we know from e.g. [BJ06, III.15.6] that the intersection of the interior and exterior parts of the toroidal compactification are the Borel-Serre boundary $e(P_\ell)$, we have the relations

$$(\iota_{TOR})^* \overline{C_n} = D_n \cap \overline{X}^{TOR}_{int}, \quad \partial C_n = -\partial \left(D_n \cap \overline{X}^{TOR}_{ext} \right).$$

So, we can write

$$D_n = (\iota_{TOR})^* C_n^c + \left(D_n \cap \overline{X}^{TOR}_{ext} \right)^c.$$

Note that the compactification of the exterior two-chain $D_n \cap \overline{X}^{TOR}_{ext}$ is identical to the compactification of the C_n in $\overline{X}^{BS}$ - namely, we attach $-A_n$ for A_n the two chain in $\partial \overline{X}^{BS}$ from Definition 4.2.5. So, because we have now decomposed D_n into two orthogonal parts for the homological splitting given in (8.2.2), we know (by definition of the direct sum operation) that this is the unique splitting - hence we may say that

$$D_n^c = \left[(\iota_{TOR})^* C_n^c + \left(D_n \cap \overline{X}^{TOR}_{ext} \right)^c \right]^c = (\iota_{TOR})^* C_n^c$$

and so we are done. □

What is to be done? We revisit the setting of Cogdell to compare the homology therein to ours. For any positive length $\mathbf{x} \in \underline{V}$, he creates a modular form

$$\nu_{\mathbf{x}}(\tau) = \frac{1}{2} \mathrm{vol}\,(C_{\mathbf{x}}) + \sum_{n=1}^{\infty} (D_{\mathbf{x}}^c \cdot D_n^c)_X \, q^n. \tag{8.2.3}$$

Writing (8.2.3) in terms of an intersection between a homological modular form and the class $D_{\mathbf{x}}^c$, we may see this as lying in the image of a pairing on homology:

$$\nu_{\mathbf{x}}(\tau) = \left(\left([PD\,(c_1(X))] + \sum_{n>0} D_n^c \right) \cdot D_{\mathbf{x}}^c \right). \tag{8.2.4}$$

Definition 8.2.2. For any lattice coset $\mathcal{L}$, let $H_2(X)_{\mathcal{L}}$ (resp. $H_2(\overline{X}^{TOR})_{\mathcal{L}}$) be the span of all the classes $[C_{\mathbf{x}}^c]$ (resp. $[D_{\mathbf{x}}^c]$) for $\mathbf{x} \in \mathcal{L}$.

Then we may interpret the modular forms $\varphi_{\mathbf{x}}$ in (8.2.4) as specific images of the pairing:

$$\left(H_2 \left(\overline{X}^{TOR} \right) \otimes M_3(\Gamma(M)) \right) \times H_2 \left(\overline{X}^{TOR} \right)_L \to M_3(\Gamma(M))$$

It is clear that we may interpret $(\iota_{TOR})_*$ as a map from $H_2(X)_L$ to $H_2(\overline{X}^{TOR})_L$;

hence, given our knowledge of the result from Theorem 3, we have the map

$$((\iota_{TOR})_* \otimes 1) : H_2(X) \otimes M_3(\Gamma(M)) \to H_2\left(\overline{X}^{TOR}\right) \otimes M_3(\Gamma(M)). \tag{8.2.5}$$

Fixing an $\mathbf{x} \in L$ of positive length, we may interpret the results of Theorem 4.2.6 as equivalent to the existence of a map $\Xi_{BS} : H_2(X) \to M_3(\Gamma(M))$; in particular, there exists a pairing

$$\langle\, , \,\rangle_{BS} : (H_2(X) \otimes M_3(\Gamma(M))) \times H_2(X)_L \to M_3(\Gamma(M)). \tag{8.2.6}$$

Hence, the Cogdell result follows simply from (8.2.6) for the fixed class $C^c_{\mathbf{x}} \in H_2(X)_L$: we apply the map from (8.2.5) in the left-hand side and $(\iota_{TOR})_*$ in the right-hand side of the pairing.

We have hence proven the following; it is a corollary to 4.2.6, Proposition 8.2.1 and the main result of [Cog85].

Proposition 8.2.3. *The main theorem of Cogdell regarding automorphic liftings of special cycles on Picard modular surfaces, stated in [Cog85], is a corollary of Theorem 4.2.6 for the special case when η is the Poincaré dual of a Borel-Serre special cycle $C^c_{\boldsymbol{x}}$.*

Example 8.2.4. We now illustrate this with an example; in particular, we will integrate the capped theta series of weight 3 against the special cycle C_{v_2}. We will assume we work with a single cusp $[\ell]$, and the lattice $L = \mathfrak{o}_k\ell \oplus \mathfrak{o}_k w_\ell \oplus \mathfrak{o}_k\ell'$ as in [Cog85].

By definition of the mapping between $H^2_{\text{cone}}(\overline{X}^{BS}, \partial X^{BS})$ and $H^2_c(X)$ given in Lemma 7.2.2, this is given in the Kronecker pairing by

$$\langle C_{v_2}, [\theta_L(\varphi_{KM}, \tau), \theta_{W_\ell \cap L}(\phi_{KM}, \tau)]\rangle = \int_{C_{v_2}} \theta_L(\varphi_{KM}, \tau) - \int_{\partial C_{v_2}} \theta_{W_\ell \cap L}(\phi_{KM}, \tau). \tag{8.2.7}$$

As v_2 has length 1, the intersection with the cusp $[\ell]$ is given by the single fibre circle c_{v_2}. We may hence write the cuspidal integral as

$$\begin{aligned} \int_{c_{v_2}} \theta_{W_\ell \cap L}(\phi_{KM}, \tau) &= \sum_{\mathbf{x} \in W_\ell \cap L} \int_{c_{v_2}} \frac{-\delta_k}{8} \left(||\mathbf{x}|| - \frac{1}{2\pi v} \right) e^{\pi i \tau ||\mathbf{x}||} \otimes \mathrm{d}r \\ &= \sum_{\mathbf{x} \in W_\ell \cap L} -\frac{\delta_k C_{\ell,\Gamma}}{8} \left(||\mathbf{x}|| - \frac{1}{2\pi v} \right) e^{\pi i \tau ||\mathbf{x}||}. \end{aligned} \tag{8.2.8}$$

We may recognise this as a non-holomorphic modular form: we let R_1 be the raising operator from weight 1 to weight 3 modular forms, given by

$$R_1 := 2i\frac{\partial}{\partial \tau} + \frac{1}{v}.$$

Then we may easily calculate that

$$R_1\left(e^{\pi i\tau||\mathbf{x}||}\right) = -2\pi\left(||\mathbf{x}|| - \frac{1}{2\pi v}\right)e^{\pi i\tau||\mathbf{x}||}$$

and so we may recognise the sum in (8.2.8) as proportional to the action of R_1 on the weight 1 holomorphic modular form given by the theta series:

$$\sum_{\mathbf{x}\in W_\ell\cap L} e^{\pi i\tau||\mathbf{x}||};$$

in particular, this tells us that the integral in (8.2.8) is a non-holomorphic modular form of weight 3, equal to the sum of a holomorphic theta series

$$\sum_{\mathbf{x}\in W_\ell\cap L} ||\mathbf{x}||e^{\pi i\tau||\mathbf{x}||}$$

and a non-holomorphic theta series.

We may define the weight 2 Eisenstein series $E_2(\tau)$ by

$$E_2(\tau) = \frac{1}{2\zeta(2)}\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0\}}(c\tau+d)^{-2} = 1 + 24\sum_{n=1}\sigma_1(n)q^n.$$

An elementary fact from the elementary theory of modular forms is that $E_2(\tau)$ (unlike $E_{2k}(\tau)$, $k \geq 2$) is *not* modular with respect to $\mathrm{SL}_2(\mathbb{Z})$, and to retain this property we define the non-holomorphic Eisenstein series given by

$$\widehat{E_2}(\tau) = E_2(\tau) - \frac{3}{\pi v};$$

which has the correct modular property with respect to the generators of $\mathrm{SL}_2(\mathbb{Z})$. Following the example of e.g. [FM11, Lemma 5.4], we may split the global integral in (8.2.7) into a product of integrals:

$$\int_{C_{v_2}}\theta_L(\varphi_{KM},\tau) = \left(\int_{C_{v_2}}\theta_{W_\ell^\perp\cap L}(\varphi_{KM}^0,\tau)\right)\left(\sum_{\mathbf{x}\in W_\ell\cap L}e^{\pi i\tau||\mathbf{x}||}\right) \tag{8.2.9}$$

and we may calculate the first integral on the right-hand side of (8.2.9) using the work of [Sta15, §4] on unitary signature $(1,1)$ non-holomorphic liftings as follows:

$$\int_{C_{v_2}}\theta_{W_\ell^\perp\cap L}(\varphi_{KM}^0,\tau) = \frac{\delta_k C_{\ell,\Gamma}}{48}\widehat{E_2}(\tau).$$

Hence, we may conclude the following:

(i) Both the integrals on the right-hand side of (8.2.7) are modular forms of weight 3, which are explicitly not holomorphic on $\mathbb{H}$.

(ii) The non-holomorphic parts of these integrals are equal: namely, we have

$$\left[\int_{C_{v_2}} \theta_L(\varphi_{KM}, \tau)\right]_{NonHol} = \frac{\delta_k C_{\ell,\Gamma}}{16\pi v} \sum_{\mathbf{x}\in W_\ell \cap L} e^{\pi i \tau ||\mathbf{x}||} = \left[\int_{\partial C_{v_2}} \theta_{W_\ell \cap L}(\phi_{KM}, \tau)\right]_{NonHol}.$$

Hence, this replicates the work of Cogdell for the cycle C_{v_2}, so that the difference

$$\int_{C_{v_2}} \theta_{\mathcal{L}}(\varphi_{KM}, \tau) - \int_{\partial C_{v_2}} \theta_{W_\ell \cap \mathcal{L}}(\phi_{KM}, \tau)$$

is a holomorphic modular form of weight 3.

We hence in particular see the same structure of result as in [Cog85]: namely, that both the local and the global integrals give non-holomorphic modular forms of weight 3, with the same non-holomorphic parts - and hence in particular the Kronecker pairing between C_{v_2} and the capped theta series gives a *holomorphic* modular form. The advantage to this method over Cogdell's is that this vanishing property is seen as an immediate corollary of the capping procedure.

We now develop this example to give an indication of why our capped theta series from Theorem 7.2.4 are non-trivial - namely, we shall imitate the work of Example 8.2.4 for $l = 1$.

Example 8.2.5. We let $l = 1$, and keep the same initial assumptions on the lattice and number of cusps as in Example 8.2.4; in particular, we assume $\Gamma = \Gamma_L$ is the full stabiliser of the lattice. We shall integrate the capped theta class (emphasising the chosen value of l)

$$\left[\theta_{L,\mathcal{H}}\left(\varphi_{1,1}, \tau\right), \theta_{W_\ell \cap L}\left(\phi^{\ell}_{1,1}, \tau\right)\right] \in H^2_c\left(\overline{X}^{BS}, \widetilde{\mathcal{H}^{1,1}(V)}\right)$$

against the special cycle

$$C_{v_2,[1,1]} \in H_2\left(X, \widetilde{\mathcal{H}^{1,1}(V)}\right);$$

given the results of Theorem 7.2.4, we expect this to give us a holomorphic cusp form of weight 5.

We may now analogise the results of [FM11, Lemma 5.4] for the case of complex harmonic coefficients. We let $W_\ell^\perp \subset V$ be the hyperbolic subspace of signature $(1, 1)$, and let $\mathbb{H} \simeq \mathbb{D}_{W_\ell^\perp} \subset \mathbb{D}$ be the corresponding subsymmetric space. We let $\varphi_{KM}^{W_\ell}$ and $\varphi_{KM}^{W_\ell^\perp}$ be the Kudla-Millson form for W_ℓ (spanned by v_2) and $W_\ell^\perp$ respectively, so that we may immediately write

$$\varphi_{KM}^{W_\ell}(\mathbf{w}, \tau) = e^{\pi i (\mathbf{w},\mathbf{w})\tau}, \qquad \varphi_{KM}^{W_\ell^\perp}(\mathbf{x}, \tau) = \left(|z_1|^2 - \frac{1}{2\pi v}\right) e^{\pi i (\mathbf{x},\mathbf{x})\tau} \otimes \xi_1 \wedge \overline{\xi_1}.$$

The latter may have $z \in \mathbb{D}_{W_\ell^\perp}$ inserted in the usual way, as in §6. The restriction of the vector-valued operators ∇^V, $\overline{\nabla}^V$ to these subspaces gives

$$\nabla^{W_\ell} = \mathcal{D}_2 \otimes 1 \otimes A(v_2) \otimes 1, \quad \overline{\nabla}^{W_\ell} = \overline{\mathcal{D}_2} \otimes 1 \otimes 1 \otimes A(v_2^*)$$

and

$$\nabla^{W_\ell^\perp} = \mathcal{D}_1 \otimes 1 \otimes A(v_1) \otimes 1, \quad \overline{\nabla}^{W_\ell^\perp} = \overline{\mathcal{D}_1} \otimes 1 \otimes 1 \otimes A(v_1^*).$$

For l, l' arbitrary non-negative integers, we hence define the vector-valued forms in the reduced subspaces as follows:

$$\varphi_{l,l'}^{W_\ell} = \left(\nabla^{W_\ell}\right)^l \circ \left(\overline{\nabla}^{W_\ell}\right)^{l'} \left(\varphi_{KM}^{W_\ell}\right), \qquad \varphi_{l,l'}^{W_\ell^\perp} = \left(\nabla^{W_\ell^\perp}\right)^l \circ \left(\overline{\nabla}^{W_\ell^\perp}\right)^{l'} \left(\varphi_{KM}^{W_\ell^\perp}\right). \tag{8.2.10}$$

Hence, using exactly the same principles as in the discussion preceding [FM11, Lemma 5.4], we see that for $\mathbf{x} \in W_\ell^\perp$ and $\mathbf{w} \in W_\ell$, the restriction of the Schwartz form $\varphi_{l,l'}$ (defined in Lemma 6.2.3) to the subsymmetric space is given by

$$r_{\mathbb{D}_{W_\ell^\perp}}(\varphi_{l,l'}) = \sum_{\substack{0 \le j \le l \\ 0 \le j' \le l'}} \varphi_{j,j'}^{W_\ell}(\mathbf{w}, \tau) \varphi_{l-j,l'-j'}^{W_\ell^\perp}(\mathbf{x}, \tau). \tag{8.2.11}$$

The Kronecker pairing is given by

$$\left\langle C_{v_2,[l,l]}, [\theta_{L,\mathcal{H}}(\varphi_{1,1},\tau), \theta_{W_\ell \cap L,\mathcal{H}}(\phi_{1,1},\tau)]\right\rangle = \int_{C_{v_2,[1,1]}} \theta_{L,\mathcal{H}}(\varphi_{1,1},\tau) - \int_{\partial C_{v_2,[1,1]}} \theta_{W_\ell \cap L,\mathcal{H}}(\phi_{1,1},\tau). \tag{8.2.12}$$

Applying (8.2.11), we may rewrite the first integral on the right-hand side of (8.2.12) as follows:

$$\int_{C_{v_2,[1,1]}} \theta_{L,\mathcal{H}}(\varphi_{1,1},\tau) = \int_{C_{v_2,[1,1]}} \left[\sum_{j,j'=0}^{1} \theta_{W_\ell \cap L}\left(\varphi_{j,j'}^{W_\ell},\tau\right) \theta_{W_\ell^\perp \cap L}\left(\varphi_{1-j,1-j'}^{W_\ell^\perp},\tau\right)\right].$$

By definition, the integral will be given by pairing the vector $v_0 = \pi_{\mathcal{H}}(v_2 \otimes v_2^*)$ with the coefficients in the fibre; in particular, examining the form of the vector-valued forms in (8.2.10), we see that if j or $j' \neq 1$, then the integrand will have a v_1 or v_1^* term in, and hence in particular will be orthogonal to v_0. Hence, we may discard almost all the terms in this sum and take out the one-dimensional theta series in the global integral (as it does not have any differential forms involved). Further, for the cuspidal integral, we notice that this same pairing triviality occurs - namely, that the terms proportional to Ω_ℓ and $\overline{\Omega_\ell}$ in $\varphi_{1,1}$ will disappear. Hence, we may integrate at the cusp as in Example 8.2.4 and hence write the pairing between the special cycle and the capped theta series as

$$\left\langle C_{v_2,[l,l]}, [\theta_{L,\mathcal{H}}(\varphi_{1,1},\tau), \theta_{W_\ell \cap L,\mathcal{H}}(\phi_{1,1},\tau)]\right\rangle = \left[\int_{C_{v_2}} \theta_{W_\ell^\perp \cap L}\left(\varphi_{KM}^{W_\ell^\perp},\tau\right)\right] \theta_{W_\ell \cap L}\left(\mathcal{D}_1 \circ \overline{\mathcal{D}_1}\left(\varphi_{KM}^{W_\ell}\right),\tau\right)$$

$$- \frac{\delta_k C_{\ell,\Gamma}}{8} \theta_{W_\ell \cap L} \left(\left(\mathcal{D}_2 \circ \overline{\mathcal{D}_2} \right)^2 \left(\varphi_{KM}^{W_\ell} \right), \tau \right) \tag{8.2.13}$$

In particular, what we have shown here completely generalises the equivalent calculation in the Kudla-Millson case, and in particular gives a method for calculating generic integrals of this type.

We now check that the non-holomorphic parts of this lifting disappear. We know from the trivial coefficients calculations that the $1/v$ term in the integral on the right-hand side of (8.2.13) is given by $-\delta_k C_{\ell,\Gamma}/8\pi v$, and the same in the one-dimensional theta series coming from the splitting will be from the constant term of the first Laguerre polynomial:

$$\left[\theta_{W_\ell \cap L} \left(\mathcal{D}_1 \circ \overline{\mathcal{D}_1} \left(\varphi_{KM}^{W_\ell} \right), \tau \right) \right]_{\frac{1}{v}} = - \frac{1}{2\pi v} \sum_{\mathbf{w} \in L \cap W_\ell} e^{\pi i \tau ||\mathbf{w}||}.$$

(this notation of $[X]_{v^k}$ notating the v^k part of X continues throughout). Similarly, the $1/v^2$ term in the boundary integral will come from the constant term of the second Laguerre polynomial:

$$\left[\theta_{W_\ell \cap L} \left(\left(\mathcal{D}_2 \circ \overline{\mathcal{D}_2} \right)^2 \left(\varphi_{KM}^{W_\ell} \right), \tau \right) \right]_{\frac{1}{v}} = \frac{1}{2(\pi v)^2} \sum_{\mathbf{w} \in W_\ell \cap L} e^{\pi i \tau ||\mathbf{w}||}.$$

Hence, putting this together, we see that the $1/v^2$ term in (8.2.13) is given by

$$\Big[(8.2.13) \Big]_{1/v^2} = \sum_{\mathbf{w} \in W_\ell \cap L} \left[\left(\frac{-\delta_k C_{\ell,\Gamma}}{8\pi v} \right) \left(\frac{-1}{2\pi v} \right) - \frac{\delta_k C_{\ell,\Gamma}}{16(\pi v)^2} \right] e^{\pi i \tau ||\mathbf{w}||} = 0.$$

We will now use the interpretation from [Sta15] of the coefficients of the integral on the right-hand side of in (8.2.13) to be given by representation numbers in the lattice $W_\ell^\perp \cap L$ modulo the action of $\Gamma_{W_\ell^\perp}$. We write the term not proportional to $1/v$ as $G(\tau)$; hence, the full $1/v$ part of (8.2.13) is given by:

$$\begin{aligned}
&- \frac{\delta_k C_{\ell,\Gamma}}{2} G(\tau) \left[\theta_{W_\ell \cap L} \left(\mathcal{D}_1 \circ \overline{\mathcal{D}_1} \left(\varphi_{KM}^{W_\ell} \right), \tau \right) \right]_{\frac{1}{v}} + \frac{\delta_k C_{\ell,\Gamma}}{8} \theta_{W_\ell \cap L} \left(\mathcal{D}_1 \circ \overline{\mathcal{D}_1} \left(\varphi_{KM}^{W_\ell}, \tau \right) \right]_{v^0} \\
&\qquad + \frac{\delta_k C_{\ell,\Gamma}}{8} \left[\theta_{W_\ell \cap L} \left(\left(\mathcal{D}_2 \circ \overline{\mathcal{D}_2} \right)^2 \left(\varphi_{KM}^{W_\ell} \right), \tau \right) \right]_{\frac{1}{v}} \\
&= - \frac{\delta_k C_{\ell,\Gamma}}{2} G(\tau) \left(\frac{-1}{2\pi v} \right) \sum_{\mathbf{w} \in W_\ell \cap L} e^{\pi i \tau ||\mathbf{w}||} + \frac{\delta_k C_{\ell,\Gamma}}{8\pi v} \sum_{\mathbf{w} \in W_\ell \cap L} ||\mathbf{w}|| e^{\pi i \tau ||\mathbf{w}||} \\
&\qquad - \frac{\delta_k C_{\ell,\Gamma}}{4\pi v} \sum_{\mathbf{w} \in W_\ell \cap L} ||\mathbf{w}|| e^{\pi i \tau ||\mathbf{w}||} \\
&= \frac{\delta_k C_{\ell,\Gamma}}{8\pi v} \left[2 \left[\sum_n \Gamma_{W_\ell^\perp} \backslash r_{W_\ell^\perp \cap L}(n) q^n \right] \left[\sum_{\tilde{n}} \Gamma_{W_\ell} \backslash r_{W_\ell \cap L}(\tilde{n}) q^{\tilde{n}} \right] - \sum_{n'} 2n' \Gamma_{W_\ell} \backslash r_{W_\ell \cap L}(n') q^{n'} \right],
\end{aligned}$$

where as usual we have assumed that $q = e^{2\pi i \tau}$. However, by our assumptions on Γ

(namely, that it is the full lattice stabiliser), then we see that all the representation numbers are just 1, in which case we see that this $1/v$ term also disappears.

We are hence left with the constant v^0 part; we only now need check that this is non-zero. However, this is trivial: we may check that for e.g. $\mathfrak{o}_k$ the ring of Gaussian integers, the q coefficient is given by $-\delta_k C_{\ell,\Gamma}/3$, and this will replicate in general for other rings of integers.

In particular, this shows us that these constructions will not in general be trivial - indeed, when treating the case of $l = 1$, this pattern of the v^k terms disappearing for $k > 0$ should replicate, and the same non-triviality will again happen, because the holomorphic parts of the Kronecker pairing will be given by products of different weight holomorphic theta series, and hence in particular will have non-zero coefficients of q^n for some sufficiently large n.

We record all of the above in a theorem.

Theorem 8.2.6. *For $l = 1$, the cohomology class*

$$[\theta_{L,\mathcal{H}}(\varphi_{1,1},\tau), \theta_{W_\ell \cap L,\mathcal{H}}(\phi_{1,1},\tau)] \in H^2_{\text{cone}}\left(\overline{X}^{BS}, \partial\overline{X}^{BS}, \widetilde{\mathcal{H}^{l,l}(V)}\right)$$

is non-trivial; more specifically, the Kronecker pairing with the class $C_{v_2,[1,1]}$ is a holomorphic cusp form of weight 5:

$$\left\langle C_{v_2,[l,l]}, [\theta_{L,\mathcal{H}}(\varphi_{1,1},\tau), \theta_{W_\ell \cap L,\mathcal{H}}(\phi_{1,1},\tau)]\right\rangle = \sum_{n\geq 1} a(n)q^n$$

where

$$a(n) = \delta_k C_{\ell,\Gamma}\left(\frac{1}{24} n r_{W_L}(n) - \sum_{k=1}^{n-1}\left[(n-k)\sigma_1(k) r_{W_L}(n-k)\right] - \frac{1}{8} n^2 r_{W_L}(n)\right)$$

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