

# On interpolations from SUSY to non-SUSY strings and their properties

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This thesis is submitted to the University of Durham, in fulfilment of  
the requirements for the degree of

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## **Abstract:**

In an effort to obtain phenomenologically acceptable models, string theories can be formulated without spacetime supersymmetry at their fundamental energy scale. However, results pertaining to the relations between such non-supersymmetric models and their stable and UV finite supersymmetric counterparts, are few in number. In this thesis, the interpolation from supersymmetric to non-supersymmetric heterotic string theories is studied, via the Scherk-Schwarz compactification of supersymmetric  $6D$  theories to  $4D$ . A general modular-invariant Scherk-Schwarz deformation is deduced from the properties of the  $6D$  theories at the endpoints of the interpolation, which significantly extends previously known examples of such compactifications. This wider class of non-supersymmetric  $4D$  theories opens up new possibilities for model building. The full one-loop cosmological constant of such theories is studied as a function of compactification radius for a number of cases, and the following interpolating configurations are found: two supersymmetric  $6D$  theories related by a T-duality transformation, with intermediate  $4D$  maximum or minimum at the string scale; a non-supersymmetric  $6D$  theory interpolating to a supersymmetric  $6D$  theory, with the  $4D$  theory possibly having an AdS minimum; and a “metastable” non-supersymmetric  $6D$  theory interpolating via a  $4D$  theory to a supersymmetric  $6D$  theory. The replication of the arguments relating  $6D$  and  $4D$  theories by interpolation to  $4D$  and  $2D$  theories, is suggested.



# Declaration

The work in this thesis is based on research carried out in the Institute for Particle Physics Phenomenology at Durham University. No material presented in this thesis has previously been submitted by myself in whole or in part for consideration for any other degree or qualification at this or any other University. The research described in this thesis has been carried out in collaboration with my supervisor, Professor Steven A. Abel. Parts of this thesis therefore have been published in the following collaborative works:

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Open all the boxes.  
Open all the boxes.  
Open *all* the boxes.  
Open all the boxes.

— from *Be Safe* by Lee Renato



# Contents

<b>Abstract</b>	<b>iii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Motivation . . . . .	1
1.1.1 Motivation for studying string theory . . . . .	1
1.1.2 Motivation for studying non-supersymmetric strings . . . . .	3
1.1.3 Motivation for studying interpolating models . . . . .	4
1.2 Overview . . . . .	7
<b>2 String theory background</b>	<b>9</b>
2.1 String theory basics and bosonic strings . . . . .	9
2.1.1 Classical strings and symmetries . . . . .	9
2.1.2 Quantizing the classical theory . . . . .	15
2.1.3 Light-cone gauge . . . . .	17
2.1.4 Spectrum of the bosonic string. . . . .	20
2.1.5 Ghost fields . . . . .	22
2.2 Fermionic strings . . . . .	24
2.2.1 Fermionic fields . . . . .	24
2.2.2 Mode expansions . . . . .	27
2.2.3 Quantizing the RNS superstring . . . . .	28
2.2.4 Light-cone gauge . . . . .	30
2.2.5 Superstring spectra . . . . .	34
2.2.6 Constructing the closed superstring theory . . . . .	39
2.3 One-loop string partition functions . . . . .	43
2.3.1 Virasoro algebra from conformal field theory . . . . .	44
2.3.2 Defining a partition function . . . . .	47
2.3.3 Generating functions . . . . .	53
2.3.4 Bosonic partition functions . . . . .	54
2.3.5 Fermionic and superstring partition functions . . . . .	56
2.4 Compactification of the background space time . . . . .	58
2.5 Heterotic string theory . . . . .	63

2.5.1	Fermionic construction . . . . .	70
2.6	Free fermionic formulation . . . . .	71
2.7	Non-supersymmetric strings . . . . .	76
2.7.1	Gravitons, gravitinos and their protos . . . . .	76
2.7.2	Misaligned supersymmetry . . . . .	78
2.7.3	Stability . . . . .	81
2.7.4	Interpolating Models . . . . .	82
2.7.5	Suppression of the cosmological constant . . . . .	84
<b>3</b>	<b>The cosmological constant &amp; generalized Scherk-Schwarz construction</b>	<b>89</b>
3.1	Overview . . . . .	89
3.1.1	Coordinate Dependent Compactification . . . . .	89
3.1.2	Summary: Procedure for Identifying the Massless Spectrum of a Free Fermionic Model . . . . .	92
3.1.3	Model construction . . . . .	93
3.2	CDC-Modified Virasoro Operators . . . . .	94
3.3	Details of Cosmological Constant Calculation . . . . .	98
3.3.1	CDC modified partition function . . . . .	100
3.4	The zero radius theory and a more general formulation of Scherk-Schwarz . . . . .	102
<b>4</b>	<b>On SUSY Restoration</b>	<b>105</b>
4.1	Gravitinos . . . . .	105
4.2	Energy Scales . . . . .	107
4.3	Is the theory at small radius supersymmetric? . . . . .	107
4.3.1	SUSY restoration when the CDC vector has zero left-moving entries . . . . .	110
4.3.2	Example of a CDC vector with non-zero left-moving entries . . . . .	112
4.3.3	Formula for $N_b = N_f$ ? . . . . .	113
<b>5</b>	<b>Surveying the interpolation landscape</b>	<b>115</b>
5.1	Interpolation Between Two Supersymmetric Theories . . . . .	116
5.1.1	$N_{b(0)} > N_{f(0)}$ . . . . .	116
5.1.2	$N_{b(0)} < N_{f(0)}$ . . . . .	117
5.2	Interpolation from a Non-supersymmetric to a Supersymmetric Theory	118
5.2.1	$N_{b(0)} = N_{f(0)}$ . . . . .	118
5.2.2	$N_{b(0)} > N_{f(0)}$ . . . . .	119
5.2.3	$N_{b(0)} < N_{f(0)}$ . . . . .	119

6	Conclusions	123
A	Conformal transformations	129
B	Notation and conventions for partition functions	131
C	Modular transformations of the partition function	135
D	Structure of the calculation	137
E	Group Representations	139
F	Algebraic Infeasibility	141
	Bibliography	149



# Chapter 1

## Introduction

### 1.1 Motivation

#### 1.1.1 Motivation for studying string theory

The standard model (SM), a quantum field theory which encompasses the electromagnetic, weak and strong forces, is renormalizable and self-consistent. However, in addition to the fact that the SM fails to incorporate the gravitational force, a number of problematic factors preclude the SM from acquiring the status of a theory of everything. Of initial concern is the fact that the model requires a seemingly *ad hoc* number of arbitrary numerical constants. It is hoped that a theory which encompasses all four of the fundamental forces would not have to make recourse to such otherwise arbitrary parameters. Furthermore, there exists a drastic hierarchy between the Higgs mass squared parameter and the Planck mass, reflecting a huge difference between the scales at which effects of the weak force and of gravitation appear. As quantum corrections are typically power-law divergent, in order to understand how cancellations between the fundamental quantity (the non-zero value for the Higgs mass-squared parameter) and the relevant quantum corrections might lead to the aforementioned hierarchy, a theory must be obtained which describes physics at energies higher than those so far probed.

Supersymmetry has been proposed as a solution to the Hierarchy problem [2, 3]. The process of renormalization relates the fundamental value of a physical parameter to its effective value via the application of quantum corrections. Fine tuning must be applied to prevent quantum corrections, arising from loop diagrams, from making large contributions to the square of the Higgs mass. This problem can be alleviated if there exist superpartners to those particles making large contributions to the Higgs mass squared parameter, with opposite spin statistics, such that the overall contribution arising from quantum corrections cancels (as shown in Figure 1.1).

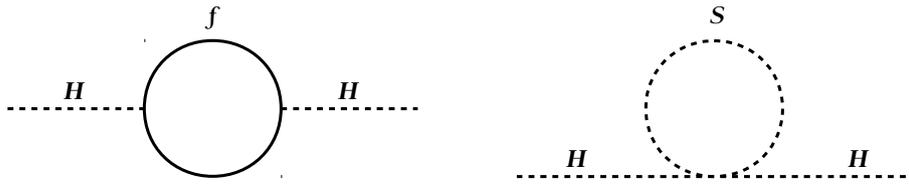


Figure 1.1: The one-loop quantum corrections to  $m_H^2$ , the Higgs mass squared parameter, arising from a Dirac fermion  $f$  (e.g. a top quark) and a scalar  $S$  (e.g. a scalar stop squark). Supersymmetry guarantees overall cancellation between the fermionic and bosonic contributions to all loop orders.

However, given the lack of experimental observation of super partners of the SM particles, if such a supersymmetry is a symmetry of nature, it must be broken at a scale above any so far experimentally accessed.

The problem of incorporating gravity into a theory of everything exists regardless of the problematic hierarchy of scales, yet a solution to the former may also provide an explanation of the latter. The fourth fundamental force, classically described by Einstein's theory of general relativity, is not perturbatively renormalizable when attempts at quantization, similar to those used for the other three fundamental forces of the standard model, are made [4–6]. Given the need to eradicate unnatural hierarchies, it is natural to attempt to accommodate the above described supersymmetry with general relativity, into a theory of supergravity [7, 8]. However, such a theory remains infinite and non-renormalizable at the quantum level [5].

Instead of starting with the classical theory and attempting to find an ultraviolet completion for Einstein's gravity, working from first principles of the quantum theory, string theory represents a unification of the four fundamental forces of nature [5, 9, 10]. String theory provides a consistent theory of quantum gravity, without ultraviolet divergences. When supersymmetry is incorporated, superstring theory can also reproduce the standard model at low energies [11]. Instead of the point particles of quantum field theories, the fundamental objects of string theories are extended, one dimensional stringy objects. Thus, string theories are parametrised by a single, fundamental quantity, the string length,  $l_s = \frac{1}{M_s}$ , which is set by the Planck scale. At low energies, different vibrational modes of the fundamental object, the string, are manifested as point particles with different properties (quantum numbers). In particular, the graviton emerges as part of the string spectrum. Thus, in contrast to the inconsistency when attempts are made to describe gravity in terms of the quantum field-theoretic methods used to define the standard model, gravity is a *necessary* component of string theories [9]. (Note that supergravity can be understood as the effective field theory which emerges from the string theory at low energies.)

### 1.1.2 Motivation for studying non-supersymmetric strings

String phenomenology represents an attempt to connect string theories, motivated mathematically and constructed from the top down, with phenomenologically realistic models [12, 13]. As experiments at the LHC have probed higher and higher energies, the minimum possible energy range of supersymmetric particles has been raised. Thus, it is of interest to string phenomenologists to either break the supersymmetry of superstring models in such a way as to produce a phenomenologically acceptable difference between the energies of standard model particles and of their superpartners, or to construct string models that are non-supersymmetric by design; that is, models which are non-supersymmetric at all energy scales.

A great deal of effort has been devoted to the former; those frameworks in which supersymmetry is broken non-perturbatively in the supersymmetric effective field theory. Much less effort has been devoted to the latter option, which involves the study of string theories that are non-supersymmetric by construction. Non-supersymmetric string models, which generically give rise to non-zero tadpole diagrams, are at risk of being fatally unstable. However, as will be seen, it can be argued that as long as the SUSY breaking is spontaneous and parametrically smaller than the string scale, the associated instability is under perturbative control [14].

Non-supersymmetric string models exhibit a ‘misaligned supersymmetry’ [15–17], which controls the extent to which supersymmetry is broken, thus preserving a degree of the finiteness associated with supersymmetric models. Misaligned SUSY renders generic non-SUSY string models finite at one-loop. This study will focus on a particular subset of non-SUSY models; those which have undergone a Scherk-Schwarz compactification on a manifold parameterised by a length scale  $R$ . In such models, the cosmological constant  $\Lambda$ , which, as will be described, is proportional to the tadpole diagram and is hence intricately related to the stability of a given model, is found to be proportional to  $\frac{1}{R^d}$ . Thus, these to be described interpolating models provide a tuneable parameter with which the cosmological constant can effectively be suppressed at large values of this parameter. Furthermore, in models containing an equal number of fermionic and bosonic states at the massless level, it is possible to generate exponential suppression in  $\Lambda$ , which is found to be  $\sim e^{-2\pi RM_s}$ .

It is important to emphasize that such non-supersymmetric models possess no supersymmetry at their fundamental energy scales. It is not the case that such models experience supersymmetry breaking at an energy scale higher than scales typically discussed in the context of breaking the  $\mathcal{N} = 1$  supersymmetry of the Minimal Supersymmetric Standard Model. Rather, no residual supersymmetry exists in such non-supersymmetric models. Such models exist within the string theory landscape, alongside their supersymmetric counterparts, as equally viable candidate theories of

the physical world.

It is further important to stress that, like any string models, the non-supersymmetric heterotic models presented in this study generically contain unfixed moduli, whose vacuum expectation values would need to be determined in order to fully specify the theory. Thus, these models face the same hurdles as their supersymmetric counterparts in their attempt to provide a fundamental theory of nature. However, the presented suppression of those additional instabilities which arise due to the lack of spacetime supersymmetry, endows interpolating models with an degree of finiteness which would otherwise be absent from non-supersymmetric models. Thus, these models should be placed on a level footing with supersymmetric models as regards their validity as phenomenologically predictive theories.

### 1.1.3 Motivation for studying interpolating models

The danger of studying such non-supersymmetric models in isolation is that they can end up lying disconnected from the wider web of string theories. Furthermore, the order one tadpole common to non-SUSY models typically renders them unstable. In order to maintain a degree of control over these instabilities, it would be desirable to provide a generic procedure relating these non-supersymmetric models to a corresponding set of supersymmetric counterparts, when certain parameters within the theory sit within certain ranges. The aim of the study is to link non-supersymmetric, 4-dimensional ( $4D$ ) string theories with stable, supersymmetric, higher dimensional, tachyon free models.

Parametric control over SUSY breaking requires a generic method for passing from a non-supersymmetric theory to a supersymmetric counterpart, under certain limiting conditions. The method that was studied in ref.[14] is interpolation via compactification to lower dimensions, with SUSY broken by the Scherk-Schwarz mechanism [18]. It will be shown in the following section how to interpolate between particular supersymmetric and non-supersymmetric  $6D$  theories via compactifications to  $4D$  theories, and the exact nature of the theories that are found at the endpoints of the interpolation will be defined. The special feature of interpolating models is that the radii of compactification, by which they are defined, represent parameters that link the lower dimensional non-supersymmetric models with the higher dimensional supersymmetric models, the latter emerging in the limit that the radii vanish or go to infinity. The process of varying the value of  $R$  establishes a continuous connection between a class of supersymmetric theories and a related class of non-supersymmetric counterparts, as in Figure 1.2.

Thus it can be understood how the desirable features of the supersymmetric models emerge under limiting conditions [19]. Two particular advantages associated

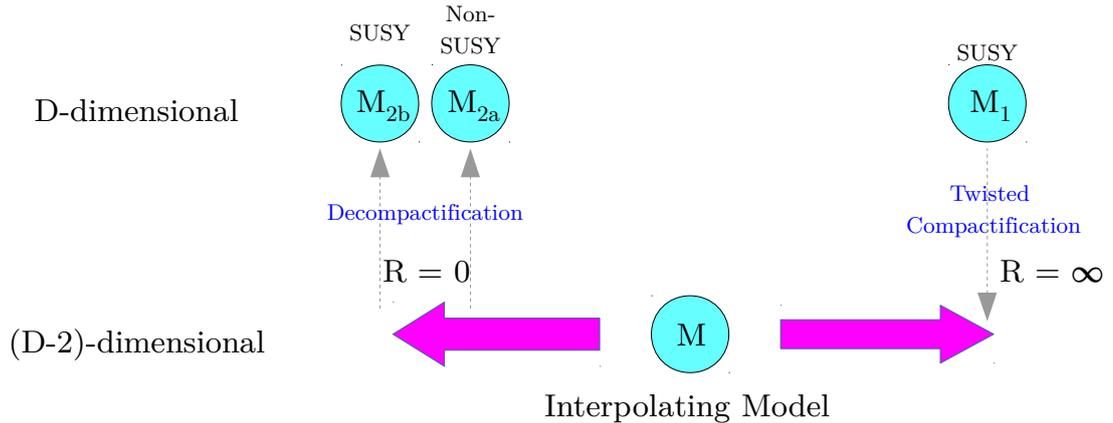


Figure 1.2: The interpolation map between a  $6D$  supersymmetric theory at infinite radius,  $\mathcal{M}_1$  and its supersymmetric and non-supersymmetric, interpolated duals,  $\mathcal{M}_{2a}$  and  $\mathcal{M}_{2b}$ , which are defined in the vanishing radius limit. Whether or not SUSY is restored in the  $r = 0$  model is determined by the choice of basis vectors and structure constants.

specifically with interpolating models are as detailed in [14]. First, the compactification volumes of interpolating models can be tuned to make the cosmological constant  $\Lambda$  arbitrarily small. And second, some of these models exhibit enhanced stability due to a one-loop cosmological constant that is exponentially suppressed with respect to the generic SUSY breaking scale [14]. They can be viewed as natural and phenomenologically interesting extensions of the original observation in refs.[20, 21] that the  $10D$  tachyon-free non-supersymmetric  $SO(16) \times SO(16)$  model interpolates to the supersymmetric heterotic  $E_8 \times E_8$  model, via a Scherk-Schwarz compactification to  $9D$ .

Before this study, the general properties under interpolation of theories broken by the Scherk-Schwarz mechanism were not well understood. For example, there did not exist any generic procedure for identifying whether or not the zero-radius endpoint theory is supersymmetric. This study focusses on the properties of  $4D$  theories that interpolate between stable,  $6D$  tachyon-free models, whose SUSY properties are to be determined. Three main results are presented.

- First, the general form of the  $6D$  endpoint theories are derived and studied, and their modular invariance properties are shown to derive directly from the Scherk-Schwarz deformation. This provides a generalisation of the construction of modular invariant Scherk-Schwarz deformed theories by beginning with the  $6D$  endpoint theory.
- Second, a simple criterion for whether a SUSY theory, broken by Scherk-Schwarz, will interpolate to a SUSY or a non-SUSY one at zero radius, is

determined: the zero radius theory is non-supersymmetric, if and only if the Scherk-Schwarz acts on the gauge group as well as the spacetime side.

- Third, a preliminary survey (in the sense that the studied models only have orthogonal gauge groups) of some representative models that confirm these two properties, involving an examination of their potentials and spectra, is undertaken.

The general framework for the interpolations are as shown in Figure 1.2. Beginning with a supersymmetric  $6D$  theory generically referred to as  $\mathcal{M}_1$ , the theory is compactified to a non-supersymmetric  $4D$  theory  $\mathcal{M}$  by adapting the Coordinate Dependent Compactification (CDC) technique first presented in refs.[22–25]. This is the string version of the Scherk-Schwarz mechanism, which spontaneously breaks  $\mathcal{N} = 1$  supersymmetry in the  $4D$  theory by lifting the masses of some of the states within the initial theory, including the gravitino, and splitting the spectrum at a scale  $\mathcal{O}(\frac{1}{2r_i})$ , where  $r_i$  is the largest radius of compactification carrying a Scherk-Schwarz twist. (“CDC” and “Scherk-Schwarz” will be used interchangeably.) As usual, this gravitino mass of  $\mathcal{O}(\frac{1}{2r_i})$  represents the order parameter for SUSY breaking; it can be continuously dialled to zero at large radius where SUSY is restored and  $\mathcal{M}_1$  regained.

One of the main properties that will be addressed is the nature of the theory as the radii of compactification are taken to zero. This depends upon the precise details of the Scherk-Schwarz compactification, and indeed it will be found that the presence or absence of SUSY at zero radius depends on the choice of basis vectors and structure constants defining the model. It is possible that the  $4D$  theory interpolates to either a supersymmetric or a non-supersymmetric model ( $\mathcal{M}_{2a}$  or  $\mathcal{M}_{2b}$  respectively). Models of the latter kind correspond to a  $6D$  theory in which SUSY is broken by discrete torsion [14].

The radius of compactification acts as the order parameter which controls the degree to which SUSY is broken at intermediate radius. In the case in which SUSY does not appear in the  $R=0$  endpoint theory, the extremely heavy gravitinos remain projected out of the spectrum. Thus, while it is appropriate to describe supersymmetry as being broken spontaneously in the  $4D$  model to order  $\mathcal{O}(\frac{1}{2r_i})$ , the zero radius endpoint theory is non-supersymmetric at its fundamental energy scale, in the same sense that the  $10D$   $SO(16) \times SO(16)$  theory of [20, 21] is non-supersymmetric.

In summary, interpolating models thus provide a link between non-supersymmetric models and supersymmetric models that lie nearby in the moduli space. This study focuses on how the finiteness and stability properties of interpolating models represent an improvement upon generic non-supersymmetric models.

## 1.2 Overview

Some requisite background string theory, in particular, the derivation of the one-loop partition function for a string, is presented in §2. The free fermionic formulation is described in §2.6. A review of the basic formalism for interpolation is provided in §3.1. In §3.2, the construction of  $4D$  non-supersymmetric models as compactifications of  $6D$  supersymmetric ones is presented. The modification of the massless spectra in the decompactification and  $r_i \rightarrow 0$  limits (with the latter corresponding to the decompactification limit of a  $6D$  T-dual theory) is analysed, in order to determine the nature of the theories at the small and large radii endpoints. The technique for rendering the cosmological constant in an interpolating form, allowing it to be calculated across a regime of small and large radii, is discussed in §3.3. The modification of the projection conditions and massless spectrum by the choice of basis vectors and structure constants is made explicit. Based on these observations, in particular how the CDC correlates with the modified GSO projections in the  $6D$  endpoint theories, the general form of deformation within this framework, extending previous constructions, is derived in §3.4. This more general formulation may prove to be useful for future model building.

The conditions under which SUSY is preserved or broken at the endpoints of the interpolation are discussed in §4. Particular focus is given to the constraints on the appearance of light gravitino winding modes in the zero radius limit. It is found that models in which the CDC acts only on the spacetime side, are inevitably supersymmetric at zero radius, while models within which the CDC vector is non-trivial on the gauge side generically yield a non-supersymmetric model in the same limit. This analysis paves the way for a presentation in §5 of explicit interpolations (in terms of their cosmological constants) in particular models that display various different behaviours. Namely, examples are found of:

- interpolation between two supersymmetric  $6D$  theories via  $4D$  theories with negative or positive cosmological constant;
- interpolation between a non-supersymmetric  $6D$  theory and a supersymmetric one, with or without an intermediate  $4D$  AdS minima;
- “metastable” non-supersymmetric  $6D$  theories (by which it is meant theories that have a positive cosmological constant with an energy barrier) that can decay to supersymmetric ones.

As mentioned, this investigation follows on from a reasonably large body of work on non-supersymmetric strings that is nonetheless much smaller than the work on supersymmetric theories. Following on from the original studies of the ten-dimensional

$SO(16) \times SO(16)$  heterotic string [26, 27], there were further studies of the one-loop cosmological constants [15–17, 19–21, 28–49], their finiteness properties [15–17, 36, 37, 50], their relations to strong/weak coupling duality symmetries [51–56], and string landscape ideas [57, 58]. The relationship to finite temperature strings was explored in refs. [25, 59–75]). Further development of the Scherk-Schwarz mechanism in the string context was made in refs.[76–80]. Progress towards phenomenology within this class has been made in refs. [29, 30, 56, 81–90]. Related aspects concerning solutions to the large-volume “decompactification problem” were discussed in refs.[91–95]. Non-supersymmetric string models have also been explored in a wide variety of other configurations [96–113], including studies of the relations between scales in various schemes [114–120]. Some aspects of this study are particularly relevant to the recent work in refs.[121].

Note that in the following work, properties of the non-supersymmetric  $4D$  theory at radii of order the string length shall be elaborated upon. As will be seen, and as found in ref.[14], often there is a minimum in the cosmological constant at this point which suggests some kind of enhancement of symmetry at a special radius. (Indeed often it is possible to identify gauge boson winding modes that become massless at the minimum.) There is therefore the possibility of establishing connections to yet more non-supersymmetric  $4D$  theories. Conversely one can ask if *every* non-supersymmetric tachyon-free  $4D$  theory can be interpolated to a supersymmetric higher dimensional theory. Comments on this and other prospects are made in the §6.

# Chapter 2

## String theory background

### 2.1 String theory basics and bosonic strings

#### 2.1.1 Classical strings and symmetries

The dynamics of any physical system are encoded within an action of the form

$$S = \int_{t_1}^{t_2} L dt. \quad (2.1.1)$$

A relativistic point particle traces out a worldline in spacetime. Demanding that the action yield Lorentz invariant equations of motion means that observers in all frames agree that the worldline which makes the action stationary is that which satisfies said equations of motion. Dimensionally analysing eq.(2.1.1) allows the action to be defined as the integral over the invariant interval  $ds$  corresponding to the path taken by the particle, multiplied by the mass of the particle  $m$ :

$$S = -m \int_{\text{path}} ds = -m \int_{\tau_0}^{\tau_1} d\tau \left[ -\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \eta_{\mu\nu} \right]^{1/2}, \quad (2.1.2)$$

with Minkowski signature  $\eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1)$ .  $\tau$  is an arbitrary parametrisation along the particle's worldline. The real functions  $x^\mu(\tau)$ , with  $\mu = 0, \dots, D - 1$ , provide a map via which the worldline is embedded in  $D$ -dimensional Minkowski spacetime. Critically, this action is reparametrization invariant, meaning that it is independent of the choice of parameters used in its definition. This invariance corresponds to a gauge symmetry, meaning that there is a redundancy in the description. The freedom associated with this redundancy can be used to ensure that the theory is consistent.

As the *length* of a point particle's worldline represents a reparametrization invariant action, so does the *area* of the worldsheet for a string (as depicted in Figure 2.1) [6, 122, 123]. A string worldsheet can be parametrized by a timelike and a spacelike

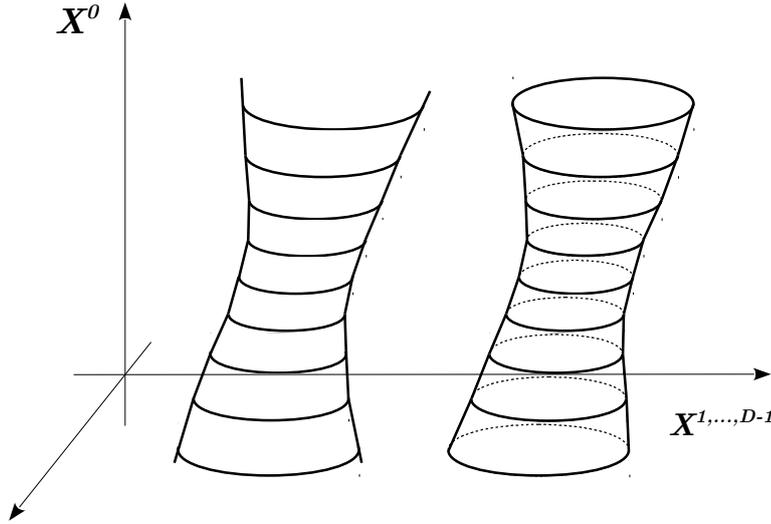


Figure 2.1: Open and closed string worldsheets. This study will focus solely upon heterotic strings, and hence will exclusively require the formalism for closed strings.

coordinate,  $\tau$  and  $\sigma$  respectively, packaged as  $\sigma^\alpha = (\tau, \sigma)$ ,  $\alpha = 0, 1$ . The worldsheet is embedded in the  $D$ -dimensional target space or spacetime, such that the image of the parameter space is a physical surface, described in spacetime in terms of the functions  $X^\mu(\tau, \sigma)$ , with  $\mu = 0, \dots, D - 1$ .

An infinitesimal area element of the parameter space, defined by  $d\tau$  and  $d\sigma$ , gives rise to a quadrilateral area element in the target space, with sides, defined as  $dv_1^\mu$ ,  $dv_2^\mu$ , rotated relative to each other by an angle  $\theta$ . The line elements in the target space are related to those in the parameter space as

$$dv_1^\mu = \frac{\partial X^\mu}{\partial \tau} d\tau \quad , \quad dv_2^\mu = \frac{\partial X^\mu}{\partial \sigma} d\sigma . \quad (2.1.3)$$

Consider first the target space area element  $dA$  of a parameterised *spatial* surface, which takes the form of the area of a parallelogram

$$\begin{aligned} dA &= |dv_1| |dv_2| |\sin(\theta)| = \sqrt{|dv_1|^2 |dv_2|^2 (1 - \cos^2 \theta)} \\ &= \sqrt{(dv_1 \cdot dv_1)(dv_2 \cdot dv_2) - (dv_1 \cdot dv_2)^2} . \end{aligned} \quad (2.1.4)$$

The appropriate area functional for target *spacetime* surfaces requires that the opposite sign be taken for the object under the square root in eq.(2.1.4)<sup>1</sup>. Ensuring

<sup>1</sup>For a full treatment, see [6]. In order that eq.(2.1.4) describe physical motion, both timelike and spacelike directions must exist at any regular point on the worldsheet. This is guaranteed by the given choice of signs.

the correct signs, the target space area element  $dA$  is thus given by

$$dA = d\tau d\sigma \sqrt{\left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \tau}\right) \left(\frac{\partial X^\nu}{\partial \sigma} \frac{\partial X_\nu}{\partial \sigma}\right)}. \quad (2.1.5)$$

Notationally,  $\frac{\partial X^\mu}{\partial \tau} = \dot{X}^\mu$ ,  $\frac{\partial X_\mu}{\partial \sigma} = X'^\mu$ . The reparametrization invariant Nambu-Goto action is proportional to this worldsheet area:

$$S_{\text{NG}} = -T \int_{\tau_i}^{\tau_f} \int_0^{\sigma_1} d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}. \quad (2.1.6)$$

The string tension  $T$  is the constant of proportionality, and is related to the string length  $l_s$  by  $T = 1/2\pi l_s^2 = 1/2\pi\alpha'$ . The integral is taken over the initial and final values of  $\tau$ , and between 0 and some constant  $\sigma_1$  value of the spacelike coordinate.

The Nambu-Goto action can also be expressed in terms of the induced metric on the worldsheet,  $\gamma_{\alpha\beta}$ , which is the pull-back of the flat metric on Minkowski space:

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} = \begin{pmatrix} \dot{X}^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X'^2 \end{pmatrix}. \quad (2.1.7)$$

Thus,

$$S_{\text{NG}} = -T \int d^2\sigma \sqrt{-\det\gamma}. \quad (2.1.8)$$

By introducing a new, independent variable, a dynamical metric on the worldsheet,  $h_{\alpha\beta}$ , the Polyakov action avoids the square root found in the Nambu-Goto action:

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \quad (2.1.9)$$

where  $h \equiv \det(h_{\alpha\beta})$  and  $h^{\alpha\beta}$  is the inverse of the worldsheet metric  $h_{\alpha\beta}(\tau, \sigma)$ . By comparing the equations of motion for the bosonic fields,  $X^\mu$ , to which both actions give rise, the Polyakov and Nambu-Goto actions can be seen to be classically equivalent<sup>2</sup>

$$\partial_\alpha (\sqrt{-h} h^{\alpha\beta} \partial_\beta X^\mu) = 0. \quad (2.1.10)$$

It is also instructive to relate the worldsheet and the pull-back metrics by varying the Polyakov action

$$\delta S_{\text{P}} = -\frac{T}{2} \int d^2\sigma \delta h^{\alpha\beta} \left( \sqrt{h} \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \sqrt{-h} h_{\alpha\beta} h^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X_\mu \right) = 0, \quad (2.1.11)$$

such that

$$h^{\alpha\beta} = f^{-1}(\sigma) \gamma^{\alpha\beta}, \quad (2.1.12)$$

with the pull-back metric as defined in eq.(2.1.7). With this expression for the world-

<sup>2</sup>Using the variation of the determinant  $\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}$ .

sheet metric, the Polyakov action reduces to the Nambu-Goto action, demonstrating their coincidence.

Armed with the fundamental object that is the string action, the dynamics of strings can be investigated. The symmetries of a string theory described by the Polyakov action, must be identified. The Polyakov string action remains invariant under one global (from the point of view of the worldsheet) and two local sets of transformations.

- The action is invariant under  $X^\mu \rightarrow \Lambda_\nu^\mu X^\nu + c^\mu$ , (where  $c^\mu$  is a constant spacetime vector), as  $\Lambda_\nu^\mu$  satisfies  $\Lambda_\rho^\mu \eta_{\mu\nu} \Lambda_\sigma^\nu = \eta_{\rho\sigma}$ . This invariance of the  $D$ -dimensional spacetime corresponds to the full symmetry of special relativity, Poincaré invariance. This global symmetry of the worldsheet theory leaves the worldsheet metric invariant,  $h'_{\alpha\beta} = h_{\alpha\beta}$ .
- Under  $(\tau, \sigma) \rightarrow (\tilde{\tau}, \tilde{\sigma})$ , the spacetime fields transform as worldsheet scalars:

$$X^\mu(\tau, \sigma) \rightarrow \tilde{X}^\mu(\tilde{\tau}, \tilde{\sigma}) = X^\mu(\tau, \sigma). \quad (2.1.13)$$

The worldsheet metric transforms as a two index tensor:

$$h_{\alpha\beta} \rightarrow \tilde{h}_{\alpha\beta}(\tilde{\tau}, \tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha} \frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta} h_{\gamma\delta}(\tau, \sigma). \quad (2.1.14)$$

This worldsheet coordinate reparametrization invariance, or diffeomorphisms, corresponds to a gauge symmetry on the world sheet. An equivalent expression is obtained for the action when it is formulated in terms of  $(\tau, \sigma)$  or in terms of  $(\tilde{\tau}, \tilde{\sigma})$ .

- Under a Weyl transformation, which leaves the fields invariant,  $X^\mu(\tau, \sigma) \rightarrow X^\mu(\tau, \sigma)$ , the worldsheet metric transforms up to a scale:

$$h_{\alpha\beta}(\tau, \sigma) \rightarrow \tilde{h}(\tau, \sigma) = \Omega^2(\tau, \sigma) h_{\alpha\beta}(\tau, \sigma). \quad (2.1.15)$$

Invariance under local rescalings of the worldsheet metric, novel to the Polyakov action, corresponds to a gauge symmetry, in that metrics related by a Weyl transformation give rise to physically equivalent states.

These symmetries can be exploited in order to identify coordinates in which the string dynamics, described by the system's equations of motion, can be more easily studied. Reparametrization invariance allows the conformal gauge, in which two of the three independent degrees of freedom contained within the metric  $h_{\alpha\beta}$  are eliminated, to be chosen. Weyl invariance allows a transformation to be performed that eliminates the remaining degree of freedom. Thus, the worldsheet metric can be

taken to be the flat Minkowski metric  $\eta_{\alpha\beta}$ , and the Polyakov action in the conformal gauge simplifies to

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu. \quad (2.1.16)$$

Strings described by the gauge simplified Polyakov action are constrained by the equations of motion for both the fields  $X^\mu$  and the metric  $h_{\alpha\beta}$ . In the conformal gauge, the former, eq.(2.1.10), simplify to an expression for a free wave,  $\partial_\alpha \partial^\alpha X^\mu = 0$ . In terms of the worldsheet light-cone coordinates,  $\sigma^\pm = \tau \pm \sigma$ , and the corresponding derivatives,  $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ , the  $X^\mu$  equations of motion take the form  $\partial_+ \partial_- X^\mu = 0$ . The most general solution to this 2-dimensional wave equation involves a superposition of left- and a right-moving (defined as functions of  $(\tau + \sigma)$  and  $(\tau - \sigma)$  respectively) travelling waves:

$$X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma). \quad (2.1.17)$$

Closed strings, upon which this study will exclusively focus, are constrained by the periodicity condition  $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$ . Thus, the most general expansions of  $X_L$ ,  $X_R$  in terms of Fourier modes, with period  $2\pi$ , take the form

$$\begin{aligned} X_L^\mu(\tau + \sigma) &= \frac{1}{2}x_{L(0)}^\mu + \sqrt{\frac{\alpha'}{2}}\tilde{\alpha}_0^\mu(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau + \sigma)}, \\ X_R^\mu(\tau - \sigma) &= \frac{1}{2}x_{R(0)}^\mu + \sqrt{\frac{\alpha'}{2}}\alpha_0^\mu(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau - \sigma)}. \end{aligned} \quad (2.1.18)$$

The relation between the periodicities of  $X_L^\mu$ ,  $X_R^\mu$  means that  $\tilde{\alpha}_0^\mu = \alpha_0^\mu$ . By evaluating the canonical momentum, the zero modes can be seen to be proportional to the momentum of the centre of mass of the string:  $\tilde{\alpha}_0^\mu = \alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}p^\mu$ . The coordinate zero modes are similarly equal to each other, and correspond to the centre of mass position:  $x_{L(0)}^\mu = x_{R(0)}^\mu = x^\mu$ . Reality of  $X^\mu$  requires that the positive and negative Fourier modes are conjugate to each other:

$$\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^* \quad \text{and} \quad \alpha_{-n}^\mu = (\alpha_n^\mu)^*. \quad (2.1.19)$$

Thus, the classical solutions to the wave equation can be neatly expressed as

$$X^\mu(\tau, \sigma) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0} \frac{1}{n} \left[ \tilde{\alpha}_n^\mu e^{-in(\tau + \sigma)} + \alpha_n^\mu e^{-in(\tau - \sigma)} \right]. \quad (2.1.20)$$

The  $h_{\alpha\beta}$  equations of motion, can also be phrased as the requirement that the energy-momentum or stress-energy tensor,  $T_{\alpha\beta}$ , vanishes:

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S}{\delta h_{\alpha\beta}} = \frac{1}{\alpha'} \left( \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu - \partial_\alpha X^\mu \partial_\beta X_\mu \right) = 0, \quad (2.1.21)$$

or in components,

$$0 = \partial_\tau X^\mu \partial_\tau X_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu \quad \text{and} \quad 0 = \partial_\tau X^\mu \partial_\sigma X_\mu. \quad (2.1.22)$$

The requirement that  $T_{\alpha\beta}$  vanishes places constraints upon the above solutions, in particular, upon the Fourier modes  $\tilde{\alpha}_n^\mu, \alpha_n^\mu$ . In terms of worldsheet light-cone coordinates, these constraints read:

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X_\mu = 0, \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu = 0. \end{aligned} \quad (2.1.23)$$

Imposing these constraints upon the classical solution, eq.(2.1.20), yields

$$\begin{aligned} T_{++} &= (\partial_+ X)^2 = 0 \\ &= \frac{\alpha'}{2} \sum_{p,m} \tilde{\alpha}_p \cdot \tilde{\alpha}_m e^{-i(p+m)(\tau+\sigma)} = \frac{\alpha'}{2} \sum_{m,n} \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m e^{-in(\tau+\sigma)} = \alpha' \sum_n \tilde{L}_n e^{-in(\tau+\sigma)}, \\ T_{--} &= (\partial_- X)^2 = 0 \\ &= \frac{\alpha'}{2} \sum_{p,m} \alpha_p \cdot \alpha_m e^{-i(p+m)(\tau-\sigma)} = \frac{\alpha'}{2} \sum_{m,n} \alpha_{n-m} \cdot \alpha_m e^{-in(\tau-\sigma)} = \alpha' \sum_n L_n e^{-in(\tau-\sigma)}, \end{aligned} \quad (2.1.24)$$

where the left- and right-moving Virasoro operators, the two objects upon which large portions of this study will focus, are defined as

$$\tilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m, \quad L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \cdot \alpha_m. \quad (2.1.25)$$

Note that the contraction runs over the Lorentz indices  $\mu = 0, \dots, D-1$ . Also note that the expressions in eq.(2.1.24) can be inverted, yielding expressions for the Virasoro operators in terms of the components of the energy momentum tensor. Thus  $\tilde{L}_n, L_n$  are defined as the conserved charges

$$\tilde{L}_n = \int d^2\sigma e^{in(\tau+\sigma)} T_{++}, \quad L_n = \int d^2\sigma e^{in(\tau-\sigma)} T_{--}. \quad (2.1.26)$$

Thus, classically the constraints eq.(2.1.24) imply that

$$\tilde{L}_n = L_n = 0, \quad \forall n \in \mathbb{Z}. \quad (2.1.27)$$

This allows the relationship between  $\tilde{\alpha}_0^\mu, \alpha_0^\mu$  and  $p^\mu$ , defined below eq.(2.1.18), to be used to express the square of the rest mass of a string in terms of the non-zero oscillator modes,

$$M^2 = -p^\mu p_\mu = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \frac{4}{\alpha'} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n. \quad (2.1.28)$$

The contributions from the  $\tilde{\alpha}_n$  and  $\alpha_n$  oscillators can be considered to constitute left- and right-moving contributions to the mass squared:

$$M^2 = \frac{2}{\alpha'} \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \alpha_{-n} \cdot \alpha_n) = m_L^2 + m_R^2. \quad (2.1.29)$$

Level matching refers to the equivalence  $m_L^2 = m_R^2$ . The worldsheet Hamiltonian, in terms of the canonical momentum,  $\Pi^\mu = \partial L / \partial \dot{X}_\mu$ , is

$$\begin{aligned} H &= \int_0^{2\pi} d\sigma (\dot{X}^\mu \Pi_\mu - L) = \frac{T}{2} \int_0^{2\pi} d\sigma (\dot{X}^2 + X'^2) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \alpha_{-n} \cdot \alpha_n) \\ &= (\tilde{L}_0 + L_0). \end{aligned} \quad (2.1.30)$$

### 2.1.2 Quantizing the classical theory

So far the classical theory has been considered. In the quantum theory, the Fourier expansion coefficients of the classical theory eq.(2.1.18) obey commutation relations [11, 122]. Of immediate interest is the relation

$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = [\alpha_m^\mu, \alpha_n^\nu] = m \eta^{\mu\nu} \delta_{m+n,0}. \quad (2.1.31)$$

Thus, there exists an ambiguity between the order of those operators which do not commute. It is necessary to assign a prescription for the ordering of the oscillators within the Virasoro operators. The choice is made to define  $\tilde{L}_n, L_n$  by their normal ordered expressions, with the lowering operators,  $\tilde{\alpha}_n, \alpha_n; n > 0$ , appearing on the right-hand side of the raising operators,  $\tilde{\alpha}_{-n}, \alpha_{-n}; n > 0$ :

$$\tilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m : , \quad L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m} \cdot \alpha_m : . \quad (2.1.32)$$

Of concern is the fact that  $\tilde{L}_0, L_0$  contain pairs of oscillator modes of equal and opposite sign, which, thanks to the  $\delta_{m+n,0}$  factor in eq.(2.1.31), do not commute. A tactic is to define  $\tilde{L}_0, L_0$  by the normal ordered expression

$$\begin{aligned} \tilde{L}_0 &= \sum_{i=1}^{D-2} \left( \sum_{m=1}^{\infty} \tilde{\alpha}_{-m} \cdot \tilde{\alpha}_m + \frac{1}{2} (\tilde{\alpha}_0)^2 \right), \\ L_0 &= \sum_{i=1}^{D-2} \left( \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \frac{1}{2} (\alpha_0)^2 \right), \end{aligned} \quad (2.1.33)$$

and to introduce the to be determined ordering constants  $\tilde{a}, a$ , when the operators  $\tilde{L}_0, L_0$  are applied to physical states  $|\phi\rangle$  (which, according to eq.(2.1.27), they annihilate):

$$(\tilde{L}_0 - \tilde{a}) |\phi\rangle = (L_0 - a) |\phi\rangle = 0. \quad (2.1.34)$$

In order that the operator corresponding to  $\tilde{L}_0 - L_0$  generate  $\sigma$  translations (see §2.3.1), it is necessary that  $(\tilde{L}_0 - L_0)|\phi\rangle = 0$ , where  $|\phi\rangle$  represent physical states, which constitutes the level-matching condition [11].

Taking into account the normal ordering prescription, the operators in eq.(2.1.32) are found to obey the commutation relation [11]

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0} , \quad (2.1.35)$$

which defines a centrally extended Virasoro algebra. From a physical point of view, the central charge  $c$  corresponds to the vacuum energy of the system. For  $D$  non-interacting free scalar fields,  $\tilde{c} = c = D$ , the spacetime dimension; that is, the central charges count the number of degrees of freedom of the system [122].

There exist multiple methods with which to proceed with quantizing the theory. The method of covariant quantization involves defining a quantum theory, before subsequently imposing the gauge fixing constraints. While this method has the advantage of preserving manifest Lorentz invariance, it is necessary to find a means with which to deal with the states of negative norm which arise within the Fock space once the above commutation relations between the Fourier modes have been defined. In particular, states that are comprised of an odd number of excited timelike oscillators, such as  $|\chi\rangle = \alpha_{-n}^0 |0; p\rangle^3$ , yield

$$\langle\chi|\chi\rangle = \langle 0; p | \alpha_n^0 \alpha_{-n}^0 | 0; p \rangle = \langle 0; p | [\alpha_n^0, \alpha_{-n}^0] | 0; p \rangle = - \langle 0; p | 0; p \rangle < 0 . \quad (2.1.36)$$

According to eq.(2.1.27), the constraints, eq.(2.1.23), imply that classically all  $\tilde{L}_n, L_n$  vanish. In the quantum theory, the absence of negative norm states can be ensured by demanding that  $\tilde{L}_n, L_n$  have vanishing matrix elements when their inner product is taken with physical states:

$$\langle\phi' | \tilde{L}_n | \phi \rangle = \langle\phi' | L_n | \phi \rangle = 0 \quad n \neq 0 . \quad (2.1.37)$$

However, it is inconsistent to demand that all physical states are annihilated by  $\tilde{L}_n, L_n$  [6, 11]. This can be seen by using the commutation relationship eq.(2.1.35)

$$\langle\phi | [L_n, L_{-n}] | \phi \rangle = \langle\phi | 2nL_0 | \phi \rangle + \frac{c}{12}n(n^2 - 1) \langle\phi | \phi \rangle . \quad (2.1.38)$$

If  $\tilde{L}_n |\phi\rangle = L_n |\phi\rangle = 0$  for all  $n$ , the only non-trivially zero term is the anomalous term generated on the right-hand side, which only vanishes if all  $|\phi\rangle = 0$ , rendering the theory trivial. Rather, it is demanded that the *positive* modes annihilate physical states:  $\tilde{L}_n |\phi\rangle = L_n |\phi\rangle = 0$  for  $n > 0$ . Noting that  $\tilde{L}_n^\dagger = \tilde{L}_{-n}$  and  $L_n^\dagger = L_{-n}$ , the constraints on the negative modes are effectively imposed, as eq.(2.1.37) is satisfied

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<sup>3</sup>Where the vacuum state  $|0; p\rangle$  is defined in §2.1.4

for  $n$  positive *and* negative. Thus, combining this constraint for the positive modes with that for the zero modes, eq.(2.1.34), the physical quantum states are defined, in terms of the degenerate ordering constants  $\tilde{a} = a$ , by

$$(\tilde{L}_n - \tilde{a}\delta_{n,0})|\phi\rangle = (L_n - a\delta_{n,0})|\phi\rangle = 0, \quad n \geq 0. \quad (2.1.39)$$

Note that, while beyond the scope of this study, there exists a no-ghost theorem which states that the states of negative norm decouple in  $D = 26$  [122] [11]. This defines the critical dimension, which in turn fixes the ordering constants for the bosonic theory. As will be seen in the following subsection discussing alternate means of quantization, demanding that the theory be free from conformal anomalies will provide more concrete means of deriving the critical dimension.

### 2.1.3 Light-cone gauge

Alternate to the method of covariant quantization, light-cone quantization involves quantizing the classical solutions to the constraint equations. Solutions to the wave equation of the form eq.(2.1.17) can more readily be found in light-cone gauge, in which  $X^+$  is set proportional to  $\tau$ . Making a particular choice for the time direction violates manifest Lorentz invariance. This approach will still prove to be instructive, but to see that Lorentz invariance is preserved would require a more complete treatment. However, the fact (which will be discussed) that the theory contains a massless graviton with an appropriate choice of  $\tilde{a}, a$ , indicates that Lorentz invariance is indeed preserved. For concreteness,  $D$ -dimensional spacetime can be described by the light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad (2.1.40)$$

with  $X^i$  for the remaining  $i = 2, \dots, D-1$  coordinates. Thus, the Minkowski metric is

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} dX^i dX^i. \quad (2.1.41)$$

As the Fock space is constructed by acting on the vacuum state with the creation operators corresponding only to the transverse modes,  $\tilde{\alpha}_{-n}^i, \alpha_{-n}^i$ , only +1 terms are selected from  $\eta^{\mu\nu}$  in eq.(2.1.31). Thus, the Hilbert space is positive definite; it contains no ghosts.

Worldsheet coordinate transformations of the form  $\sigma^\pm \rightarrow \xi^\pm(\sigma^\pm)$ , which change the metric up to a scale, can be undone by a Weyl transformation, eq.(2.1.15). These transformations represent residual gauge freedom, which can be fixed by making a choice for  $X^+$ . The transformed worldsheet coordinate  $\tilde{\tau} = 1/2 [\xi^+(\sigma^+) + \xi^-(\sigma^-)]$

satisfies the two-dimensional wave equation  $\partial_\alpha \partial^\alpha \tilde{\tau}$ . Thus, in analogy with the mode expansions eq.(2.1.20), light-cone gauge involves making the choice

$$X^+(\tilde{\tau}, \tilde{\sigma}) = x^+ + \alpha' p^+ \tilde{\tau}, \quad (2.1.42)$$

which can be split into left- and right-moving components (setting  $(\tilde{\tau}, \tilde{\sigma}) = (\tau, \sigma)$ ),

$$X_L^+ = \frac{x^+}{2} + \frac{\alpha'}{2} p^+ (\tau + \sigma), \quad X_R^+ = \frac{x^+}{2} + \frac{\alpha'}{2} p^+ (\tau - \sigma). \quad (2.1.43)$$

That is, the only non-vanishing oscillator mode is the zero mode  $\tilde{\alpha}_0^+ = \alpha_0^+ = \sqrt{\alpha'/2} p^+$ . As in the classical theory, the equation of motion for  $X^-$ ,  $\partial_+ \partial_- X^-$  implies that  $X^- = X_L^-(\tau + \sigma) + X_R^-(\tau - \sigma)$ , such that  $X^-$  can be expanded into its modes as:

$$\begin{aligned} X_L^-(\tau + \sigma) &= \frac{x^-}{2} + \frac{\alpha' p^-}{2} (\tau + \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^-}{n} e^{-in(\tau + \sigma)}, \\ X_R^-(\tau - \sigma) &= \frac{x^-}{2} + \frac{\alpha' p^-}{2} (\tau - \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^-}{n} e^{-in(\tau - \sigma)}, \end{aligned} \quad (2.1.44)$$

which can be summed to give

$$X^-(\tau, \sigma) = x^- + \alpha' p^- \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} [\tilde{\alpha}_n^- e^{-in(\tau + \sigma)} + \alpha_n^- e^{-in(\tau - \sigma)}]. \quad (2.1.45)$$

The constraints, eq.(2.1.23), allow the oscillator modes of  $X^-$  to be completely determined by  $p^+$  and the transverse oscillator modes  $\tilde{\alpha}_n^i, \alpha_n^i$ . That is, all the dynamics of the system are encoded within the transverse coordinates  $X^i$ . Using eq.(2.1.41), the constraints yield

$$0 = -2\partial_+ X^- \partial_+ X^+ + \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i. \quad (2.1.46)$$

Inserting the above coordinate expansions for  $X^+, X^-$ , along with the expansions for  $X^i$ , which are identical to those for  $X^\mu$  in the classical theory, eq.(2.1.18), determines the negative  $\alpha_n^-$  modes in terms of the transverse oscillators (with  $\tilde{a}, a$  left generic):

$$\begin{aligned} \tilde{\alpha}_n^- &= \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} : \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i : = \frac{1}{\sqrt{2\alpha'}} \frac{2}{p^+} (\tilde{L}_n - \tilde{a} \delta_n), \\ \alpha_n^- &= \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{D-2} : \alpha_{n-m}^i \alpha_m^i : = \frac{1}{\sqrt{2\alpha'}} \frac{2}{p^+} (L_n - a \delta_n), \end{aligned} \quad (2.1.47)$$

where the Lorentz contraction between the oscillators in the Virasoro operators in eq.(2.1.32) is replaced by the sum over the non light-cone coordinates,  $i = 1, \dots, D-2$

4.

$$\tilde{L}_n = \frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} : \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i : , \quad L_n = \frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : . \quad (2.1.48)$$

Having chosen an ordering for the oscillators within the Virasoro operators, the sums over the positive modes appearing in the light-cone versions of the operators in eq.(2.1.33) can be denoted as the light-cone oscillation number operators, while the equal zero modes (using the relationship, defined below eq.(2.1.18), that  $\tilde{\alpha}_0^i = \alpha_0^i$ ), are expressed as the transverse momentum components:

$$\begin{aligned} \tilde{L}_0 &= \sum_{i=1}^{D-2} \left( \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i + \frac{1}{2} (\tilde{\alpha}_0^i)^2 \right) = \tilde{N}_{l.c.} + \frac{1}{2} \sum_{i=1}^{D-2} (\tilde{\alpha}_0^i)^2 = \tilde{N}_{l.c.} + \frac{\alpha'}{4} \sum_{i=1}^{D-2} (p^i)^2, \\ L_0 &= \sum_{i=1}^{D-2} \left( \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i + \frac{1}{2} (\alpha_0^i)^2 \right) = N_{l.c.} + \frac{1}{2} \sum_{i=1}^{D-2} (\alpha_0^i)^2 = N_{l.c.} + \frac{\alpha'}{4} \sum_{i=1}^{D-2} (p^i)^2. \end{aligned} \quad (2.1.49)$$

The normal ordering constants cancel when the difference between the two expressions in eq.(2.1.34) is taken. Thus,

$$(\tilde{L}_0 - L_0) |\phi\rangle = 0. \quad (2.1.50)$$

Inserting eq.(2.1.49) yields the level matching constraint,  $\tilde{N}_{l.c.} = N_{l.c.}$ .

The effect of the ordering ambiguity in eq.(2.1.34) is manifested in, for example, the shift in the mass squared of the spectrum of states within the theory. Inserting  $n = 0$  into eq.(2.1.47) yields an expression relating  $\tilde{L}_0$ ,  $L_0$  to the zero modes  $\tilde{\alpha}_0^-$ ,  $\alpha_0^-$ . Thus, using the relationship defined below eq.(2.1.18),  $\tilde{\alpha}_0^- = \alpha_0^- = \sqrt{\alpha'/2} p^-$ , eq.(2.1.47) can be rearranged to give<sup>5</sup>

$$\frac{\alpha'}{2} p^+ p^- = \tilde{L}_0 - \tilde{a} = L_0 - a. \quad (2.1.51)$$

It is thus possible to relate the mass squared to the transverse oscillators

$$\begin{aligned} M^2 &= -p^\mu p_\mu = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{2}{\alpha'} [(\tilde{L}_0 - \tilde{a}) + (L_0 - a)] - (p^i)^2 \\ &= \frac{2}{\alpha'} \left[ \left( \sum_{i=1}^{D-2} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i - \tilde{a} \right) + \left( \sum_{i=1}^{D-2} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i - a \right) \right] \\ &= \frac{2}{\alpha'} [(\tilde{N}_{l.c.} - \tilde{a}) + (N_{l.c.} - a)] = m_L^2 + m_R^2. \end{aligned} \quad (2.1.52)$$

Level matching requires that  $m_L^2 = m_R^2$ . The light-cone Hamiltonian is calculated

<sup>4</sup>That is,  $\tilde{L}_n$ ,  $L_n$  in eq.(2.1.47) are the light-cone Virasoro operators,  $(\tilde{L}_n)_{l.c.}$ ,  $(L_n)_{l.c.}$ , but the subscript 'l.c.' will be dropped for convenience of notation.

<sup>5</sup>Note that the level-matching condition can also be seen from this relationship.

as in eq.(2.1.30), and consequently contains the zero point energies,  $\tilde{E}_0$ ,  $E_0$ , which correspond to the degenerate normal ordering constants in eq.(2.1.34), and which are hence equal to each other <sup>6</sup>:

$$H_{l.c.} = \sum_{i=1}^{D-2} \frac{\alpha'}{2} p^i p^i + \left[ \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \alpha_{-n}^i \alpha_n^i) + \tilde{E}_0 + E_0 \right] = \tilde{L}_0 + L_0 + \tilde{E}_0 + E_0. \quad (2.1.54)$$

It is clear from the commutation relationship between the oscillators, eq.(2.1.31), that normal ordering the contracted pairs  $\tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^j$ ,  $\alpha_{n-m}^i \alpha_m^j$  in eq.(2.1.48) yields factors of

$$-\tilde{a} = -a = \sum_{i=1}^{D-2} \frac{1}{2} \sum_{n=1}^{\infty} n = \frac{D-2}{2} \sum_{n=1}^{\infty} n. \quad (2.1.55)$$

The analogous construction of the left- and right-moving sectors of the closed bosonic string means that ordering both the  $\tilde{\alpha}_n^\mu$  and the  $\alpha_n^\mu$  operators gives rise to identical constants  $\tilde{a}$ ,  $a$ . Thus,  $\tilde{a} = a$ .<sup>7</sup> (This logic represents an alternative argument to that presented beneath eq.(2.1.34) for the degeneracy of the left- and right-moving ordering constants.) Employing, for example, zeta function regularization, allows the infinite sum, which can be interpreted as an infinite sum of zero point energies [122], to be evaluated to  $-\frac{1}{12}$ . Thus,

$$-\tilde{a}/\tilde{E}_0 = -a/E_0 = -\frac{(D-2)}{24}. \quad (2.1.56)$$

Thus, expressions involving the zero-point energies can be expressed in terms of arbitrary  $D$ . For example:

$$M^2 = \frac{2}{\alpha'} (\tilde{N}_{l.c.} + N_{l.c.} + \tilde{E}_0 + E_0) = \frac{2}{\alpha'} \left( \tilde{N}_{l.c.} + N_{l.c.} - \frac{(D-2)}{12} \right). \quad (2.1.57)$$

### 2.1.4 Spectrum of the bosonic string.

Of greatest interest are the lowest lying, lightest states. The vacuum state  $|0\rangle$  of a single string is defined as the state annihilated by the lowering operators:  $\tilde{\alpha}_n^\mu |0\rangle = \alpha_n^\mu |0\rangle = 0$ , for all positive  $n$ . The true ground state of the string must also reflect the vacuum state of the spacetime. Thus, the vacuum state will be denoted  $|0; p\rangle$ , where  $|p\rangle$  reflects the state whose eigenvalue is the spacetime momentum  $p^\mu$ ;

<sup>6</sup>Note that

$$H_{l.c.} = \frac{\alpha'}{2} (M^2 + p^i p^i). \quad (2.1.53)$$

<sup>7</sup>Importantly, this is not the case for the superstring, where the choice of periodicity conditions, which are independently assigned to the left- and the right-moving sectors, dictates the form taken by the respective normal ordering constants.

that is,

$$\tilde{\alpha}_0^\mu |0; p\rangle = \alpha_0^\mu |0; p\rangle = \sqrt{\frac{2}{\alpha'}} p^\mu |0; p\rangle . \quad (2.1.58)$$

The spectrum can be constructed by exciting this ground state,  $|0; p\rangle$  and calculating the corresponding mass squared. The excited states of the theory are defined by the corresponding excitation numbers  $\tilde{N}_{l.c.}$ ,  $N_{l.c.}$ . The ground state, with zero excitations, gives rise to a scalar field, whose quanta have negative mass squared. It is helpful to interpret these quanta as arising from the expansion around a maximum (that is, an unstable point) of the potential of a tachyonic field  $T(x)$ . Tachyonic instabilities are fortunately absent from *superstring* theories.

In light-cone gauge, a generic state takes the form

$$|\tilde{\lambda}, \lambda\rangle = \left[ \prod_{i=2}^{25} \prod_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i)^{\tilde{\lambda}_{n,i}} \times \prod_{j=2}^{25} \prod_{m=1}^{\infty} (\alpha_{-m}^j)^{\lambda_{m,j}} \right] |0; p\rangle , \quad (2.1.59)$$

where generic non-negative integers  $\tilde{\lambda}_{n,i}$ ,  $\lambda_{m,j}$  denote the number of creation operators  $\tilde{\alpha}_{-n}^i$ ,  $\alpha_{-m}^j$ , which comprise the state. The requirement that the left and right sectors are level matched,  $\tilde{N}_{l.c.} = N_{l.c.}$ , means that the first excited states of the bosonic string are comprised of both an  $\tilde{\alpha}_{-1}^i$  and an  $\alpha_{-1}^j$  excitation. The states  $\tilde{\alpha}_{-1}^i \alpha_{-1}^j |0; p\rangle$ , labelled by the indices of the transverse directions,  $i, j$ , comprise a two-index tensor. From eq.(2.1.57), these states have mass squared

$$\frac{\alpha'}{2} M^2 = \tilde{N}_{l.c.} + N_{l.c.} - \frac{(D-2)}{12} = 2 - \frac{(D-2)}{12} . \quad (2.1.60)$$

In light-cone gauge, manifest Lorentz invariance is preserved only under an  $SO(D-2)$  subgroup of the Lorentz group. Full Lorentz invariance can be recovered for a specific value of the critical dimension [5].

Alternately, physical polarization states belong to representations of the subgroup of the Lorentz group which preserves a particle's momentum in the full  $D$ -dimensional spacetime. Massive particles transform under the subgroup  $SO(D-1)$ , while the subgroup for massless particles is  $SO(D-2)$  [11]. Meanwhile, with  $i, j$  running over  $1, \dots, D-2$ , the first excited states fill out a representation of  $SO(D-2)$ . Thus these states must be massless. Thus, in order that the quantum theory preserve full Lorentz invariance under  $SO(1, D-1)$ , bosonic strings must propagate in a spacetime of critical dimension  $D = 26$ , such that eq.(2.1.60) vanishes [5, 122]. Thus for the critical dimension  $D = 26$ , massless particles transform under the  $SO(24)$  subgroup.

The massless states at the first excited level can be decomposed into irreducible representations of  $SO(24)$ . The states within the spectrum can be thought of as quanta of spacetime field.  $G_{\mu\nu}(X)$  is a symmetric, traceless field, corresponding to

the  $26d$  massless, spin-two graviton.  $B_{[\mu\nu]}(X)$  is an antisymmetric 2-form, commonly referred to as the Kalb-Ramond field. The trace term  $\phi(X)$  is a massless scalar, the dilaton field, which determines the string coupling constant. The low lying states of the bosonic theory are recorded in Table 2.1.

### 2.1.5 Ghost fields

While both instructive, neither presented method of quantization has been fully satisfactory. Covariant quantization preserves the Lorentz invariance of the action, yet is plagued by the presence of those states with negative norm. Meanwhile, it has been fairly straightforward to extract the physical spectrum using the method of light-cone gauge quantization, but the choice of a particular time direction violates manifest Lorentz invariance. Indeed, Lorentz invariance is only manifest when working in light-cone gauge with a particular choice of the spacetime dimension.

As in the covariant formulation, the method of BRST quantization exhibits manifest Lorentz invariance. While the details are beyond the scope of this study, the procedure is the string theory equivalent of the Fadeev-Popov method of quantizing the quantum field theory path integral. Anticommuting ghost fields  $b$  and  $c$  are introduced in order to fix the gauge symmetries of the theory, and ultimately to ensure that the overall theory, containing both the matter and the ghost fields, be anomaly free. The ghost fields cancel the unphysical gauge degrees of freedom in a Lorentz invariant manner, leaving only the  $D - 2$  transverse modes of  $X^\mu$  in the theory. The action is modified to

$$S = S[X] + S_{\text{ghost}}[b, c], \quad (2.1.61)$$

where, in conformal gauge, [11, 123, 124]

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z \left( 2\partial X^\mu \bar{\partial} X_\mu + b\bar{\partial}c + \tilde{b}\partial\tilde{c} \right). \quad (2.1.62)$$

Following the same procedure in the previous subsections as for the bosonic fields, mode expansions solving the equations of motion can be found for the ghost fields. A pair of Virasoro operators analogous to eq.(2.1.32) can be defined for the ghost fields<sup>8</sup>

$$L_m = \sum_{n=-\infty}^{\infty} (m-n) : b_{m+n} c_{-n} :, \quad \tilde{L}_m = \sum_{n=-\infty}^{\infty} (m-n) : \tilde{b}_{m+n} \tilde{c}_{-n} :. \quad (2.1.63)$$

---

<sup>8</sup>The normal ordering  $::$  requires that the positive modes  $b_n, c_n$  with  $n > 0$ , are moved to the right.

Level	State	$\alpha' M^2$	Little group	Representation contents	26d field
$\tilde{N}_{l.c.} = N_{l.c.} = 0$	$ 0; p\rangle$	-4	$SO(25)$	$\mathbf{1}$	$T(X)$
$\tilde{N}_{l.c.} = N_{l.c.} = 1$	$\tilde{\alpha}_{-1}^i \alpha_{-1}^j  0; p\rangle$	0	$SO(24)$	$\square\square + \square + \mathbf{1}$	$\phi(X), B_{[\mu\nu]}(X), G_{\mu\nu}(X)$

Table 2.1: The Table records the mass squared for the low-lying states at the first two levels of the bosonic theory (labelled by  $\tilde{N}_{l.c.}$ ),  $N_{l.c.}$ , along with their decomposition into irreducible representations of the respective subgroups ( $SO(24)$  for the massless states,  $SO(25)$  for the massive states), and the 26-dimensional fields to which they give rise [5].

Using the anticommutation relation

$$\{b_m, c_n\} = \delta_{m+n}, \quad (2.1.64)$$

a corresponding centrally extended Witt algebra is defined for the ghost fields by the relation

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{1}{12}(-26m^3 + 2m) \delta_{m+n,0}, \quad (2.1.65)$$

which defines the central charge of the ghost system to be  $c = -26$ . Unless the overall central charge vanishes, the Weyl symmetry is anomalous. By requiring the cancellation of the central terms in the combined Virasoro algebra due to the Fadeev-Popov ghosts and the spacetime fields,

$$L_m^{\text{total}} = L_m^X + L_m^{\text{ghost}}, \quad (2.1.66)$$

BRST quantization provides a means by which to derive the critical dimension of the string in question. Combining the central extension term due to the ghost fields with that from the bosonic fields, eq.(2.1.35)<sup>9</sup>

$$c^{\text{total}} = c^{[X]} + c^{[b],[c]} = \frac{D}{12}(m^3 - m) + \frac{1}{12}(-26m^3 + 2m), \quad (2.1.67)$$

which fixes the critical dimension of the bosonic string to be  $D = 26$ . Consistency of the bosonic theory thus also fixes the normal ordering constants,  $\tilde{a} = a = -1$ .

## 2.2 Fermionic strings

### 2.2.1 Fermionic fields

The consistency conditions (anomaly freedom) just outlined are completely generic; they can be satisfied by fields of either integer or half-integer spin. In order that the theory be Weyl invariant, the string must be described by a CFT with central charge  $c = 26$ . In addition to the bosonic string, one is naturally motivated to find other consistent string theories, not least because nature is observed to contain particles of a fundamentally different nature to the integer spin bosons discussed in the previous section. Any theory hoping to describe the observed universe must also contain half-integer spin fermions.

In the Ramond-Neveu-Schwarz (RNS) formalism, the addition of fermionic modes to the worldsheet, in a manner appropriate to match the degrees of freedom of their

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<sup>9</sup>The notation  $c^{[X]}$  denotes the total contribution from the fields  $[X]$  to the central charge, as opposed to referring to  $c$  of, for example, a single field  $X^\mu$ .

bosonic counterparts, produces a theory with worldsheet supersymmetry, namely superstring theory. Furthermore, as will be demonstrated, the unphysical tachyonic modes seen in the bosonic theory are absent from the spectra of superstring theories. As for the bosonic string, the critical dimension of the superstring is determined by the requirement that the theory be anomaly free.

Fermions can be introduced by modifying the field content of the  $2D$  worldsheet theory [125, 126]. The modification alters the conformal anomaly, such that superstrings consistently propagate in 10 rather than 26 dimensions. The theory can be constructed with an appropriate projection operator to ensure that the spectrum of the superstring is tachyon free. Given the supersymmetric relationship between the bosonic and fermionic fields on the worldsheet,  $X^i(\tau, \sigma)$  and  $\psi^i(\tau, \sigma)$  ( $i = 1, \dots, D - 2$ ), much of the formalism of the bosonic string extends to the construction of the superstring.

Working in  $D$  dimensions, an appropriate worldsheet action is required to describe the fermionic fields, which are two-component Majorana<sup>10</sup> spinors on the worldsheet,  $\psi_\alpha^\mu(\tau, \sigma)$ ,  $\alpha = 1, 2$ . The two anticommuting components  $\psi_1^\mu$ ,  $\psi_2^\mu$  represent the necessary variables to describe worldsheet fermions. The complete action for the RNS superstring, which combines the Polyakov action for the bosonic string, eq.(2.1.9), with analogous terms for fermionic fields, is [127–129]

$$\begin{aligned} S_P &= S_{\text{boson}} + S_{\text{fermion}} = S_X + S_\psi = \\ &= -\frac{1}{8\pi} \int d^2\sigma \sqrt{-h} \left[ \frac{2}{\alpha'} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right. \\ &\quad \left. - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left( \sqrt{\frac{2}{\alpha}} \partial_\beta X_\mu - \frac{i}{4} \bar{\chi}_{\beta\mu} \psi_\mu \right) \right]. \end{aligned} \quad (2.2.1)$$

$\chi_\alpha$  is the 2-dimensional gravitino.  $\rho^\alpha$ ,  $\alpha = 0, 1$  are the 2-dimensional Dirac matrices, satisfying the anticommutation relation  $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$ . The action is invariant under the following local worldsheet symmetries: supersymmetry, Weyl and super-Weyl transformations,  $2d$  Lorentz transformations and reparametrizations [11]. The Dirac conjugate of a spinor is defined as  $\bar{\psi} = \psi^\dagger \rho^0$ .<sup>11</sup> The fermionic fields take the form of two component spinors  $\psi_A^\mu$ ,  $A = \pm$ ,

$$\psi^\mu = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix}.$$

<sup>10</sup>Majorana spinors satisfy the reality condition  $\psi_\mu = \psi_\mu^*$ .

<sup>11</sup>A convenient basis for the  $\rho^\alpha$ , in terms of the Pauli matrices, is

$$\rho^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^1 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In analogy with the conformal gauge in the bosonic theory, going to superconformal gauge involves using local supersymmetry, reparametrizations and Lorentz transformations to remove unphysical degrees of freedom from the general system. Further employing Weyl and super-Weyl transformations<sup>12</sup> leaves  $h_{\alpha\beta} = \eta_{\alpha\beta}$ ,  $\chi_\alpha = 0$ , such that the action simplifies to

$$S = -\frac{1}{4\pi} \int d^2\sigma \left( \frac{1}{\alpha'} \partial_\alpha X^\mu \partial^\alpha X_\mu - i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right). \quad (2.2.2)$$

The action eq.(2.2.2) is invariant under supersymmetry transformations, parametrised by the infinitesimal, constant Majorana spinor  $\epsilon$ ,

$$\begin{aligned} \delta_\epsilon X^\mu &= i \bar{\epsilon} \psi^\mu, \\ \delta_\epsilon \psi^\mu &= \frac{1}{2} \rho^\alpha \partial_\alpha X^\mu \epsilon, \end{aligned} \quad (2.2.3)$$

These transformations parametrise the  $\mathcal{N} = 1$  supersymmetry associated with the left-moving fermionic fields  $\psi^\mu$ . An analogous symmetry exists for the right-moving  $\bar{\psi}^\mu$  fields. The action gives rise to the equations of motion

$$\begin{aligned} \partial_\alpha \partial^\alpha X^\mu &= 0, \\ \rho^\alpha \partial_\alpha \psi^\mu &= 0. \end{aligned} \quad (2.2.4)$$

For the fermionic fields, these equations constitute the 2-dimensional massless Dirac equation. In analogy with eq.(2.1.21), the energy-momentum tensor of the RNS superstring, which is required to vanish, takes the form

$$\begin{aligned} T_{\alpha\beta} &= \frac{4\pi}{\sqrt{-h}} \frac{\partial S}{\partial h^{\alpha\beta}} \\ &= \frac{1}{\alpha'} \left( \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu - \partial_\alpha X^\mu \partial_\beta X_\mu \right) + \frac{i}{4} \left( \bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu - \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu \right) = 0. \end{aligned} \quad (2.2.5)$$

As the energy-momentum tensor encodes the conserved current associated with the global translation symmetry of the action, so the worldsheet supercurrent  $J_A^\alpha$ ,  $A = \pm$ , encodes a conserved current which is associated with the global worldsheet supersymmetry of the RNS superstring [124]:

$$J_A^\alpha = \frac{4\pi\alpha'}{\sqrt{-h}} \left( \frac{\partial S}{i\partial\bar{\chi}_\alpha} \right)_A = -\frac{1}{4} \sqrt{\frac{2}{\alpha'}} (\rho^\beta \rho^\alpha \psi^\mu)_A \partial_\beta X_\mu = 0. \quad (2.2.6)$$

Together with the equations of motion eq.(2.2.4), the vanishing of  $T_{\alpha\beta}$  and  $J_A^\alpha$  implies

<sup>12</sup>Weyl and super-Weyl rescalings, which hold in the classical theory, are broken in quantum theories defined in any number of dimensions other than the critical number.

that the energy-momentum and supercurrent are conserved

$$\begin{aligned}\partial^\alpha T_{\alpha\beta} &= 0, \\ \partial_\alpha J_A^\alpha &= 0.\end{aligned}\tag{2.2.7}$$

As with the bosonic string, the action can be re-expressed in worldsheet light-cone coordinates,

$$S = \frac{1}{2\pi} \int d^2\sigma \left( \frac{2}{\alpha'} \partial_+ X^\mu \partial_- X_\mu + i (\psi_+^\mu \partial_- \psi_{+\mu} + \psi_-^\mu \partial_+ \psi_{-\mu}) \right).\tag{2.2.8}$$

The equations of motion take the form

$$\begin{aligned}0 &= \partial_+ \partial_- X^\mu, \\ 0 &= \partial_+ \psi_-^\mu = \partial_- \psi_+^\mu.\end{aligned}\tag{2.2.9}$$

Thus, as for the bosonic fields, the two components of the fermionic fields can be identified as left- and right-movers,  $\psi_+^\mu = \psi_+^\mu(\tau + \sigma)$  and  $\psi_-^\mu = \psi_-^\mu(\tau - \sigma)$ . Using  $\epsilon = (\epsilon_-, \epsilon_+)$ , supersymmetry transformations take the form

$$\begin{aligned}\delta_\epsilon X^\mu &= i(\epsilon_+ \psi_-^\mu - \epsilon_- \psi_+^\mu), \\ \delta_\epsilon \psi_-^\mu &= (-2\partial_- X^\mu) \epsilon_+, \\ \delta_\epsilon \psi_+^\mu &= (2\partial_+ X^\mu) \epsilon_-.\end{aligned}\tag{2.2.10}$$

The non-trivial components of the energy-momentum tensor, which are required to vanish by the constraints, can be expressed as

$$\begin{aligned}T_{++} &= \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2} \psi_+^\mu \partial_+ \psi_{+\mu}, \\ T_{--} &= \partial_- X_\mu \partial_- X^\mu + \frac{i}{2} \psi_-^\mu \partial_- \psi_{-\mu}.\end{aligned}\tag{2.2.11}$$

Equally, the to be constrained supercurrent takes the form

$$J_+ = \psi_+^\mu \partial_+ X_\mu \quad \text{and} \quad J_- = \psi_-^\mu \partial_- X_\mu.\tag{2.2.12}$$

Application of the equations of motion yields conservation equations for the energy-momentum tensor and supercurrent, the light-cone equivalents of eq.(2.2.7):

$$\partial_- T_{++} = \partial_+ T_{--} = 0 \quad \text{and} \quad \partial_- J_+ = \partial_+ J_- = 0.\tag{2.2.13}$$

### 2.2.2 Mode expansions

In closed string theories, upon which this study is focused, there exist decoupled left- and right-moving fermionic sectors. Varying the fermionic action in eq.(2.2.8)

produces a boundary term,

$$\delta S = \frac{i}{2\pi} \int_{\tau_0}^{\tau_1} d\tau (\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-) \Big|_{\sigma=0}^{\sigma=l=2\pi}, \quad (2.2.14)$$

whose vanishing imposes upon the string endpoints, located at, for example,  $\sigma = 0$  and  $\sigma = l = 2\pi$ , the condition

$$(\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-)(\sigma) = (\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-)(\sigma + 2\pi). \quad (2.2.15)$$

Thus, the left- and right-moving fermionic modes must independently satisfy periodic (Ramond) or antiperiodic (Neveu-Schwarz) boundary conditions:

$$\psi_{\pm}^{\mu}(\sigma) = \pm \psi_{\pm}^{\mu}(\sigma + 2\pi) \quad \text{and} \quad \psi_{\pm}^{\mu}(\sigma) = \pm \psi_{\pm}^{\mu}(\sigma + 2\pi), \quad \text{with } \pm = \text{R/NS}, \quad (2.2.16)$$

which can be denoted

$$\psi_{\pm}(\sigma + 2\pi) = e^{2\pi i \phi} \psi_{\pm}(\sigma), \quad \text{with } \phi = 0/\frac{1}{2} \text{ in the R/NS sector.} \quad (2.2.17)$$

Denoting separately the periodicity conditions for the two spinor components, there exist four distinct closed-string sectors, (NS, NS), (R, R), (NS, R) and (R, NS). Depending upon the type of string theory in question, different states arise in the different sectors.

As for the bosonic fields, the most general solutions (in terms of general string endpoint displacement  $l$ ) to the equations of motion for the fermionic fields, eq.(2.2.9), with periodic (R) and antiperiodic (NS) boundary conditions, take the form of left- and right-moving mode expansions, where the periodicities, which are used to label the sectors and which give rise to respectively integer and half-integer mode numbers, are specified by  $\phi = 0/\frac{1}{2}$  for R/NS:

$$\begin{aligned} \psi_{+}^{\mu}(\tau, \sigma) &= \sqrt{\frac{2\pi}{l}} \sum_{r \in \mathbb{Z} + \phi} \bar{b}_r^{\mu} e^{-2\pi i r(\tau + \sigma)/l}, \\ \psi_{-}^{\mu}(\tau, \sigma) &= \sqrt{\frac{2\pi}{l}} \sum_{r \in \mathbb{Z} + \phi} b_r^{\mu} e^{-2\pi i r(\tau - \sigma)/l}. \end{aligned} \quad (2.2.18)$$

### 2.2.3 Quantizing the RNS superstring

Just as the modes of the bosonic fields satisfy *commutation* relations, §2.1.2, so the modes of the fermionic fields, which are governed by the free Dirac equation on the world sheet, obey *anticommutation* relations. As is appropriate for objects exhibiting fermionic statistics, Pauli's exclusion principle is automatically satisfied by anticommuting variables, since states containing two particles with identical momentum and spin immediately vanish. As for the bosonic string,  $\tilde{\alpha}_m^{\mu}$ ,  $\alpha_m^{\mu}$  obey

eq.(2.1.31), while the fermionic modes satisfy the anticommutation relationship

$$\{\tilde{b}_r^\mu, \tilde{b}_s^\nu\} = \{b_r^\mu, b_s^\nu\} = \eta^{\mu\nu} \delta_{r+s,0}. \quad (2.2.19)$$

The constraint equations eq.(2.2.11), together with the conservation of the energy-momentum tensor (a consequence of the diffeomorphism invariance of the Polyakov action [11]), eq.(2.2.13), imply that there exist an infinite number of conserved charges associated with the modes of the energy-momentum tensor,  $T_{\alpha\beta}$ , and the supercurrent  $J_A^\alpha$ . Thus, the generators of the conformal and superconformal transformations can be decomposed into modes. Reminiscent of eq.(2.1.26), the bosonic and fermionic contributions to the former are given by:

$$\begin{aligned} \tilde{L}_n &= -\frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{in\sigma} T_{++}(\sigma) = \tilde{L}_n^{(\text{Boson})} + \tilde{L}_n^{(\text{Fermion})}, \\ L_n &= -\frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{-in\sigma} T_{--}(\sigma) = L_n^{(\text{B})} + L_n^{(\text{F})}. \end{aligned} \quad (2.2.20)$$

The contribution from the bosonic fields to the super-Virasoro generators is found by applying the constraints eq.(2.1.23) as in eq.(2.1.24). Equally, the fermionic mode expansions eq.(2.2.18) are inserted into the constraint equations eq.(2.2.11). Focussing on the right-moving sector (there exists an identical copy of the algebra for the left-movers, in terms of  $\tilde{\alpha}_m^\mu, \tilde{b}_r^\mu$ ), the normal ordered mode expansions, for  $n \in \mathbb{Z}$ , take the form

$$\begin{aligned} L_n^{(\text{B})} &= \frac{1}{2} \sum_{m \in \mathbb{Z}} : \alpha_{n-m} \cdot \alpha_m : , \\ L_n^{(\text{F})} &= \frac{1}{2} \sum_{r \in \mathbb{Z} + \phi} \left( r - \frac{n}{2} \right) : b_{n-r} \cdot b_r , \end{aligned} \quad (2.2.21)$$

where the  $r$  summation indices are determined by the sector, as in eq.(2.2.18). As for the purely bosonic string, a normal ordering constant must be included when  $\tilde{L}_0, L_0$  act on physical states.

Novel to the superstring, the supercurrent gives rise to the generator

$$G_r = -\frac{1}{\pi} \int_0^{2\pi} d\sigma e^{-ir\sigma} T_{F-}(\sigma), \quad (2.2.22)$$

with, using eq.(2.2.12), modes given by (with  $r$  again taking values as in eq.(2.2.18))

$$G_r = \sum_{m \in \mathbb{Z}} \alpha_{-m} \cdot b_{r+m}. \quad (2.2.23)$$

In analogy with eq.(2.1.33), the generators of the super-Virasoro algebra obey

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{D}{8}m(m^2 - 2\phi)\delta_{m+n},$$

$$\begin{aligned}
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2} \left(r^2 - \frac{\phi}{2}\right) \delta_{r+s}.
\end{aligned}
\tag{2.2.24}$$

Similarly, in analogy with eq.(2.1.39), the constraints on the states of the theory, that is, the vanishing of the energy-momentum and supercurrent components in eqs.(2.2.11) and (2.2.12), are imposed by demanding that physical states  $|\phi\rangle$  are annihilated by the super-Virasoro operators in the R and NS sectors thus:

$$\begin{aligned}
L_m |\phi\rangle &= 0, \quad m > 0, \\
(L_0 - a_{\text{R}}) |\phi\rangle &= 0, \\
G_r |\phi\rangle &= 0, \quad r \geq 0,
\end{aligned}
\tag{R}
\tag{2.2.25}$$

and

$$\begin{aligned}
L_m |\phi\rangle &= 0, \quad m > 0, \\
(L_0 - a_{\text{NS}}) |\phi\rangle &= 0, \\
G_r |\phi\rangle &= 0, \quad r > 0.
\end{aligned}
\tag{NS}
\tag{2.2.26}$$

As for the purely bosonic string, the level-matching condition for the closed string requires that  $(\tilde{L}_0 - L_0) |\phi\rangle = 0$ .

## 2.2.4 Light-cone gauge

As was the case when discussing the bosonic string, it is instructive to solve the Virasoro constraints in light-cone gauge, such that only the physical degrees of freedom remain in the theory. The same choice as in eq.(2.1.42) is made for the  $X^+$  coordinate, which fixes the gauge freedom associated with reparametrization invariance. The freedom associated with local supersymmetry transformations, which has not thus far been fixed by going to super-conformal gauge, allows  $\psi^+$  (where the superscript represents a light-cone component,  $\psi^\pm = \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^1)$ ) to be fixed to

$$\psi^+ = 0. \tag{2.2.27}$$

As in the bosonic theory, upon going to light-cone gauge, Lorentz contractions representing sums over all  $\mu = 0, \dots, D-1$  Minkowski spacetime dimensions, are replaced by sums over the transverse, or light-cone directions  $i = 1, \dots, D-2$ .

Following the same steps as those taken in §2.1.3, the constraints associated with the vanishing of the energy momentum tensor,  $T_{\pm\pm} = 0$ , detailed in eq.(2.2.11),

become

$$\partial_{\pm} X^{-} = \frac{1}{2p^{+}} \left( \frac{2}{\alpha'} \partial_{\pm} X^i \partial_{\pm} X^i + i \psi_{\pm}^i \partial_{\pm} \psi_{\pm}^i \right). \quad (2.2.28)$$

Equally, those in eq.(2.2.12) associated with the vanishing of the supercurrent,  $J_{\pm} = 0$ , read

$$\psi_{\pm}^{-} = \frac{2}{\alpha' p^{+}} \psi_{\pm}^i \partial_{\pm} X^i. \quad (2.2.29)$$

These equations can be used to solve for the negative bosonic and fermionic oscillator modes, which, as for the oscillators in §2.1.3, can be expressed in terms of the *light-cone* super-Virasoro operators,  $\tilde{L}_n^{(B)}$ ,  $\tilde{L}_n^{(F)}$ ,  $L_n^{(B)}$ ,  $L_n^{(F)}$ , (the latter two of which were defined in eq.(2.2.21)), but with the contraction running over only the transverse coordinates

$$\begin{aligned} \tilde{\alpha}_n^{-} &= \frac{1}{\sqrt{2\alpha'} p^{+}} \frac{1}{\sum_{i=1}^{D-2}} \left[ \sum_{m \in \mathbb{Z}} : \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i : + \sum_{r \in \mathbb{Z} + \phi} \left( r - \frac{n}{2} \right) : \tilde{b}_{n-r}^i \tilde{b}_r^i : - \tilde{a} \delta_m \right] \\ &= \frac{1}{\sqrt{2\alpha'} p^{+}} (\tilde{L}_n^{(B)} + \tilde{L}_n^{(F)} - \tilde{a} \delta_m), \\ \alpha_n^{-} &= \frac{1}{\sqrt{2\alpha'} p^{+}} \frac{1}{\sum_{i=1}^{D-2}} \left[ \sum_{m \in \mathbb{Z}} : \alpha_{n-m}^i \alpha_m^i : + \sum_{r \in \mathbb{Z} + \phi} \left( r - \frac{n}{2} \right) : b_{n-r}^i b_r^i : - a \delta_m \right] \\ &= \frac{1}{\sqrt{2\alpha'} p^{+}} (L_n^{(B)} + L_n^{(F)} - a \delta_m), \end{aligned} \quad (2.2.30)$$

and

$$\tilde{b}_r^{-} = \sqrt{\frac{2}{\alpha' p^{+}}} \sum_q \tilde{\alpha}_{r-q}^i \tilde{b}_q^i, \quad b_r^{-} = \sqrt{\frac{2}{\alpha' p^{+}}} \sum_q \alpha_{r-q}^i b_q^i. \quad (2.2.31)$$

Following eq.(2.1.33),  $\tilde{L}_0$ ,  $L_0$  are defined by their normal ordered expressions, while an ordering constant is introduced whenever the operators act on physical states. Thus,

$$\begin{aligned} \tilde{L}_0^{(B)} + \tilde{L}_0^{(F)} &= \sum_{i=1}^{D-2} \left( \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i + \frac{1}{2} (\tilde{\alpha}_0^i)^2 + r \tilde{b}_{-r}^i \tilde{b}_r^i \right) = \tilde{N}_{l.c.} + \frac{\alpha'}{4} \sum_{i=1}^{D-2} (p^i)^2, \\ L_0^{(B)} + L_0^{(F)} &= \sum_{i=1}^{D-2} \left( \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i + \frac{1}{2} (\alpha_0^i)^2 + r b_{-r}^i b_r^i \right) = N_{l.c.} + \frac{\alpha'}{4} \sum_{i=1}^{D-2} (p^i)^2, \end{aligned} \quad (2.2.32)$$

where the superstring (light-cone) number operators appear as the sums over the positive modes, split into separate contributions from the bosonic and fermionic modes,  $\tilde{N}_{l.c.} = \tilde{N}_{l.c.}^{(B)} + \tilde{N}_{l.c.}^{(F)}$ ,  $N_{l.c.} = N_{l.c.}^{(B)} + N_{l.c.}^{(F)}$ :

$$\begin{aligned} \tilde{N}_{l.c.}^{(B)} &= \sum_{i=1}^{D-2} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i, & N_{l.c.}^{(B)} &= \sum_{i=1}^{D-2} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i, \\ \tilde{N}_{l.c.}^{(F)} &= \sum_{i=1}^{D-2} \sum_{r \in \mathbb{Z} + \phi \geq 0} r \tilde{b}_{-r}^i \tilde{b}_r^i, & N_{l.c.}^{(F)} &= \sum_{i=1}^{D-2} \sum_{r \in \mathbb{Z} + \phi \geq 0} r b_{-r}^i b_r^i. \end{aligned} \quad (2.2.33)$$

As in eq.(2.1.55), the superstring normal ordering constants arise when ordering the oscillators in the light-cone expressions for  $\tilde{L}_0$ ,  $L_0$ , eq.(2.2.30), in order to obtain the ordered expressions in eq.(2.2.32). In the NS sector (again employing  $\zeta$ -function regularization)<sup>13</sup>,

$$\begin{aligned} -\tilde{a}_{\text{NS}} = -a_{\text{NS}} &= \sum_{i=1}^{D-2} \frac{1}{2} \left( \sum_{n=1}^{\infty} n - \sum_{r=\frac{1}{2}}^{\infty} r \right) \\ &= \frac{D-2}{2} \left( \sum_{n=1}^{\infty} n - \sum_{r=\frac{1}{2}}^{\infty} r \right) = \frac{D-2}{2} \left( -\frac{1}{12} - \frac{1}{24} \right) = -\frac{(D-2)}{16}. \end{aligned} \quad (2.2.34)$$

Following similar arguments to those proposed for the critical dimension of the bosonic string, negative norm states are found to be absent from the superstring theory in the critical dimension  $D = 10$  [6, 11, 122, 124]. Thus, in the critical dimension,  $\tilde{a}_{\text{NS}} = a_{\text{NS}} = 1/2$ . Conversely, the sum over the integer fermionic modes in the R sector is equal and opposite to the sum over bosonic modes, such that  $\tilde{a}_{\text{R}} = a_{\text{R}} = 0$ .

As in the bosonic theory, the critical dimension for the string can also be obtained by demanding that there be no conformal anomaly. As for the bosonic theory §2.1.5, BRST quantization of the superstring involves introducing a pair of fermionic ghost fields,  $b$  and  $c$ , and a pair of bosonic superghost fields,  $\beta$  and  $\gamma$ ; that is, the superstring theory, with  $X^\mu$ ,  $\psi^\mu$ , is augmented by an anticommuting  $bc$  theory and a commuting  $\beta\gamma$  system (c.f. eq.(2.1.62):

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z (b\bar{\partial}c + \bar{b}\partial\bar{c} + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}). \quad (2.2.35)$$

As for the ghost fields, the superghost fields give rise to a conformal anomaly, such that the total ghost central charge is found to be [11, 124, 130]

$$c^{\text{ghost}} = c^{[b],[c]} + c^{[\beta],[\gamma]} = -26 + 11 = -15. \quad (2.2.36)$$

Just as each scalar field gives rise to a central charge  $c = 1$ , each fermionic field contributes  $c = \frac{1}{2}$ . Thus, in order that the  $D$ -dimensional superstring theory be free from conformal anomalies, the total central charge

$$c^{\text{total}} = c^{[X]} + c^{[\psi]} + c^{[b],[c]} + c^{[\beta],[\gamma]} = D\left(1 + \frac{1}{2}\right) - 15 = \frac{3}{2}D - 15, \quad (2.2.37)$$

<sup>13</sup>Note the relative minus sign coupled to the sum over the fermionic mode numbers. This sign arises from the fact that the integer  $r$  is present in the fermionic component of the super-Virasoro operators,  $L_n^{(F)}$  in eq.(2.2.21), and absent from the anticommutator, eq.(2.2.19). For the bosonic modes, the situation is reversed; the integer is absent in the operator,  $L_n^{(B)}$  eq.(2.2.21), and present in the commutator, eq.(2.1.31). The additional ‘-’ sign appears upon reordering the terms in the infinite sum for the expressions for  $\tilde{L}_0$ ,  $L_0$ .

must vanish, which fixes the critical dimension of the superstring to  $D = 10$ .

Inserting  $n = 0$  into eq.(2.2.30) yields

$$\frac{\alpha'}{2} p^+ p^- = \left( \tilde{L}_0^{(B)} + \tilde{L}_0^{(F)} - \tilde{a} \right) = \left( L_0^{(B)} + L_0^{(F)} - a \right), \quad (2.2.38)$$

such that, as in eq.(2.1.52),

$$\begin{aligned} M^2 &= -p^\mu p_\mu = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i \\ &= \frac{2}{\alpha'} \left[ \left( \tilde{L}_0^{(B)} + \tilde{L}_0^{(F)} - \tilde{a} \right) + \left( L_0^{(B)} + L_0^{(F)} - a \right) \right] - (p^i)^2 \\ &= \frac{2}{\alpha'} \left[ \left( \tilde{N}_{l.c.}^{(B)} + \tilde{N}_{l.c.}^{(F)} - \tilde{a} \right) + \left( N_{l.c.}^{(B)} + N_{l.c.}^{(F)} - a \right) \right] = m_L^2 + m_R^2. \end{aligned} \quad (2.2.39)$$

The physical state condition

$$\left( \tilde{L}_0 - \tilde{a} \right) |\phi\rangle = \left( L_0 - a \right) |\phi\rangle = 0 \quad (2.2.40)$$

yields the level matching constraint,

$$\tilde{N}_{l.c.}^{(B)} + \tilde{N}_{l.c.}^{(F)} - \tilde{a} = N_{l.c.}^{(B)} + N_{l.c.}^{(F)} - a. \quad (2.2.41)$$

Alternatively expressed, as stated in the context of the bosonic string, level matching requires that the left- and the right-moving sectors contribute equally to the mass squared, such that  $m_L^2 = m_R^2$ .

The light-cone Hamiltonian is as defined for the bosonic string in eq.(2.1.54), but now contains the super-Virasoro versions of the zero mode operators defined in eq.(2.1.49). That is, the light-cone number operators are now given by eq.(2.2.33), such that the total operator, in terms of the sector-dependent values for the zero point energies, is

$$\begin{aligned} H_{l.c.} &= \sum_{i=1}^{D-2} \frac{\alpha'}{2} p^i p^i + \sum_{i=1}^{D-2} \left[ \sum_{n=1}^{\infty} \left( \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \alpha_{-n}^i \alpha_n^i \right) + \sum_{r \in \mathbb{Z} + \phi \geq 0} r \left( \tilde{b}_{-r} \cdot \tilde{b}_r + b_{-r} \cdot b_r \right) \right] \\ &\quad + \tilde{E}_0 + E_0 \\ &= \sum_{i=1}^{D-2} \frac{\alpha'}{2} p^i p^i + \left[ \tilde{N}_{l.c.}^{(B)} + N_{l.c.}^{(B)} + \tilde{N}_{l.c.}^{(F)} + N_{l.c.}^{(F)} \right] + \tilde{E}_0 + E_0 \\ &= \tilde{L}_0 + L_0 + \tilde{E}_0 + E_0. \end{aligned} \quad (2.2.42)$$

By virtue of the contributions due to the zero point energies, the values of  $\tilde{a}_{\text{NS}}$ ,  $a_{\text{NS}}$ ,  $\tilde{a}_{\text{R}}$ ,  $a_{\text{R}}$  for the 10-dimensional critical superstring theory yield sector-dependent expressions for the mass squared and the Hamiltonian.

### 2.2.5 Superstring spectra

In order to obtain the lightest states of closed superstring spectra, it will be clearest to write down a set of independent expressions for the left- and the right-moving sectors, each of which is labelled by the NS or R boundary conditions which it satisfies. Other than by the level matching condition which constrains the physical states, the sectors are decoupled from each other, such that the full spectra of closed superstring theories are comprised of the product of a left- and a right-moving set of states. (In an effort to emphasize the fact that the following presentation describes a right-moving copy of the spectrum, the normal text, free-standing subscript R refers to Ramond, while the italicised  $R$  subscript, denotes right-moving states.)

Consider the right-moving states; as for the superconformal algebra above, there exists a copy of the below set of states for the left-movers. There exists a single ground state in the Neveu-Schwarz sector, annihilated by the positive modes. Thus, this (right-moving) state, which can be assigned a unique label  $|0; p\rangle_{\text{NS}}$ , satisfies

$$\begin{aligned} \alpha_m^\mu |0; p\rangle_{(\text{NS}),R} &= 0, \quad m = 1, 2, \dots, \\ b_r^\mu |0; p\rangle_{(\text{NS}),R} &= 0, \quad r \in \mathbb{Z} + \left(\phi = \frac{1}{2}, \frac{3}{2}, \dots\right), \end{aligned} \quad (2.2.43)$$

and

$$\alpha_0^\mu |0; p\rangle_{(\text{NS}),R} = \sqrt{\frac{2}{\alpha'}} p^\mu |0; p\rangle_{(\text{NS}),R}. \quad (2.2.44)$$

Meanwhile, the action of the operator  $b_0^\mu$  (and equally  $\tilde{b}_0^\mu$ ) leaves the mass of the Ramond ground state unchanged. That is, the states  $|0; p\rangle_{(\text{R}),R}$  and  $b_0^\mu |0; p\rangle_{(\text{R}),R}$  are degenerate in mass. As is appropriate for states which are composed of operators which, obeying the anticommutator eq.(2.2.19), satisfy a Clifford algebra, these degenerate R sector ground states, labelled  $|R_A; p\rangle_R$ , transform as a spinor under  $SO(D-1, 1)$ . Explicitly, fermionic raising and lowering operators can be formed from linear combinations of the eight light-cone zero modes  $b_0^i$  [5, 6, 11]

$$b_i^\pm = \frac{1}{\sqrt{2}}(b_0^{2i} \pm ib_0^{2i+1}), \quad i = 1, \dots, 4, \quad (2.2.45)$$

such that these operators define a Clifford algebra

$$\{b_i^+, b_j^-\} = \delta_{ij}. \quad (2.2.46)$$

Having defined a highest weight state corresponding to a unique R sector vacuum,  $|0; p\rangle_{(\text{R}),R}$ , for  $D$ -even, application of the lowering operators generates a  $2^{\frac{d}{2}}$ -dimensional spinor representations of  $SO(D-1, 1)$ . The ground states can be denoted  $|R_A; p\rangle_R$ ,  $A = 1, \dots, 2^{\frac{d}{2}}$ . Dropping the momentum label,  $|R_a\rangle_R$ ,  $|R_{\bar{a}}\rangle_R$ , with  $a, \bar{a} = 1, \dots, 2^{\frac{d-2}{2}}$ , are used to denote those states formed from an even, respectively

odd, number of lowering operators:

$$|R_a\rangle_R = \begin{cases} |0; p\rangle_{(R),R} \\ b_{i_1}^- b_{i_2}^- |0; p\rangle_{(R),R} \\ b_{i_1}^- b_{i_2}^- b_{i_3}^- b_{i_4}^- |0; p\rangle_{(R),R} \end{cases}, \quad |R_{\bar{a}}\rangle_R = \begin{cases} b_{i_1}^- |0; p\rangle_{(R),R} \\ b_{i_1}^- b_{i_2}^- b_{i_3}^- |0; p\rangle_{(R),R} \end{cases}. \quad (2.2.47)$$

$|R_a\rangle_R, |R_{\bar{a}}\rangle_R$  thus denote the two possible chiralities of the R sector ground state. Summarizing, the ground states in the R sector can be denoted as

$$\begin{aligned} \alpha_m^\mu |R_A; p\rangle_R &= 0, \quad m = 1, 2, \dots, \\ b_r^\mu |R_A; p\rangle_R &= 0, \quad r \in \mathbb{Z} + (\phi = 1, 2, \dots). \end{aligned} \quad (2.2.48)$$

In particular, in light-cone gauge there exist  $2^{\frac{d}{2}} = 16$  degenerate R sector ground states, denoted  $|R_A\rangle_R, A = 1, \dots, 16$ , which transform in a 16-component spinor representation of  $SO(8)$ . Thus, there exist 8 ground states of each chirality,  $|R_a\rangle_R, a = 1, \dots, 8$  and  $|R_{\bar{a}}\rangle_R, \bar{a} = 1, \dots, 8$  (later denoted  $\mathbf{8}_S$  and  $\mathbf{8}_C$ ).

As they transform in a spinor representation of the spacetime rotation group  $SO(D-1, 1)$  [11], the R sector ground states are fermionic. Conversely, the unique ground state of the NS sector engenders a spacetime spin zero, bosonic state. The action of the spacetime vector oscillators cannot change the spin-statistics upon exciting the ground states. Thus, the Ramond sector gives rise to spacetime fermions, while the Neveu-Schwarz sector yields spacetime bosonic string states.

From eq.(2.2.39), the lowest lying states can be identified in terms of the excitation numbers. For the right-movers, in the NS and R sectors,

$$\frac{\alpha'}{2} m_{(\text{NS}),R}^2 = (N_{l.c.}^B + N_{l.c.}^F - \frac{1}{2}), \quad \frac{\alpha'}{2} m_{(R),R}^2 = (N_{l.c.}^B + N_{l.c.}^F), \quad (2.2.49)$$

where the summation indices in the NS and R versions of  $N_{l.c.}^F$  take respectively half-integer and integer values. The action of a bosonic creation oscillator,  $\tilde{\alpha}_n^\mu, \alpha_n^\mu$ , increases the energy of any given state by an amount  $n$  (in units of  $\frac{1}{\alpha'}$ ). However, the fermionic oscillators can produce half-integer separations. The presence of the mode number  $r$  in the expressions for the fermionic number operators, eq.(2.2.33), indicates that in the NS / R sector, states are separated by respectively half-integer / integer units of mass. That is, in the NS sector, fermionic excitations yield states which sit at levels between those produced by the bosonic oscillators. In particular, the first excited NS state involves a single fermionic excitation.

Consider first the NS sector. In light-cone gauge, a generic right-moving state

takes the form

$$|\lambda\rangle_{(\text{NS}),R} = \left[ \prod_{i=2}^9 \prod_{m=1}^{\infty} (\alpha_{-m}^i)^{\lambda_{m,i}} \right] \times \left[ \prod_{j=2}^9 \prod_{r=\frac{1}{2}}^{\infty} (b_{-r}^j)^{\rho_{r,j}} \right] |0; p\rangle_{(\text{NS}),R} , \quad (2.2.50)$$

where, as in eq.(2.1.59), generic non-negative integers  $\lambda_{m,i}$ ,  $\rho_{r,j}$  denote the number of creation operators  $\alpha_{-m}^i$ ,  $b_{-r}^j$  which comprise the state. The anticommuting behaviour of the fermionic oscillator modes  $b_{-r}^j$  means that  $\rho_{r,j}$  are either 0 or 1. The ground state, which corresponds to zero excited modes,  $N_{l.c.}^B = N_{l.c.}^F = 0$ , is, as in the purely bosonic theory, tachyonic, with right-moving mass squared given by  $\alpha' m_{(\text{NS}),R}^2 = -2a_{\text{NS}}$ . As explained in the previous paragraph, in the NS sector, a single fermionic excitation represents a half-unit energy shift compared to a bosonic excitation. Thus, at the first excited level, there exists the state  $b_{-1/2}^i |0; p\rangle_{(\text{NS}),R}$   $i = 1, \dots, D-2$ , with  $\alpha' m_{(\text{NS}),R}^2 = 2(\frac{1}{2} - a_{\text{NS}})$ . Following the argument made in §2.1.4, in order that this vector state, which transforms under  $SO(D-2)$ , be massless,  $a_{\text{NS}}$  must equal  $-1/2$ . Thus, eq.(2.2.34) confirms that the critical dimension of the superstring is 10. The low-lying NS sector states are recorded in Table 2.2.

Bosonic and fermionic states can be distinguished by whether the value for their corresponding worldsheet fermion number  $F$  is even or odd. By convention, the value of the operator  $(-1)^F$ , is  $+1$  for bosonic states, and  $-1$  for fermionic states. The NS sector ground state is defined to be fermionic, with  $(-1)^F |0; p\rangle_{(\text{NS}),R} = -|0; p\rangle_{(\text{NS}),R}$ . Taking  $\{(-1)^F, b_r^i\} = 0$ , the action of  $(-1)^F$  on generic NS sector states  $|\lambda\rangle_{(\text{NS}),R}$ , comprised of an even / odd number of fermionic oscillators, yields

$$(-1)^F |\lambda\rangle_{(\text{NS}),R} = -(-1)^{\left(\sum_{r \in (\mathbb{Z}_+ + 1/2)} b_{-r}^i, b_r^i\right)} |\lambda\rangle_{(\text{NS}),R} , \quad (2.2.51)$$

which defines the worldsheet fermion number  $F$ . Thus states with an even number of fermionic oscillators preserve the fermionic statistics of the NS ground state, and are thus themselves fermionic on the worldsheet. These states are all those with integer values for  $N_{l.c.}$ , (which thus take odd integer values of the mass squared under the normalisation of eq.(2.2.49)). Conversely, states with an odd number of fermionic oscillators, such as those at the first excited level, with half-integer values for  $N_{l.c.}$  and even integer values of the mass squared, are worldsheet bosons. The final column of the Table records the value of  $(-1)^F$  on the worldsheet.

The only change for a generic R sector state is in the integer rather than half-integer mode numbers of the fermionic oscillators:

$$|\lambda\rangle_{(\text{R}),R} = \left[ \prod_{i=2}^9 \prod_{m=1}^{\infty} (\alpha_{-m}^i)^{\lambda_{m,i}} \right] \times \left[ \prod_{j=2}^9 \prod_{r=1}^{\infty} (b_{-r}^j)^{\rho_{r,j}} \right] |R_A; p\rangle_R . \quad (2.2.52)$$

The vanishing of the ordering constant  $a_R$  corresponds to the fact that in the R

Sector	State	$\alpha' m_{(\text{NS}),R}^2$	Little group	Representation contents	$(-1)^F$
$N_{l.c.} = 0$	$ 0; p\rangle_{(\text{NS}),R}$	-1	$SO(9)$	<b>1</b>	-1
$N_{l.c.} = N_{l.c.}^F = \frac{1}{2}$	$b_{-1/2}^i  0; p\rangle_{(\text{NS}),R}$	0	$SO(8)$	<b>8<sub>V</sub></b>	+1

Table 2.2: The mass squared for the low-lying (that is, up to massless), NS sector right-moving states, along with their decomposition into irreducible representations of the respective subgroups, and the eigenvalue of the operator  $(-1)^F$  [5]. **8<sub>V</sub>** denotes the vector representation of  $SO(8)$ .

Sector	State	$\alpha' m_{(\text{R}),R}^2$	Little group	Representation contents	$(-1)^F$
$N_{l.c.} = 0$	$ R_a; p\rangle_R$	0	$SO(8)$	<b>8<sub>S</sub></b>	+1
$N_{l.c.} = 0$	$ R_{\bar{a}}; p\rangle_R$	0	$SO(8)$	<b>8<sub>C</sub></b>	-1

Table 2.3: The massless R sector right-moving states, along with their decomposition into irreducible representations of the respective subgroups, and the eigenvalue of the operator  $(-1)^F$  [5]. **8<sub>S</sub>**, **8<sub>C</sub>** denote the two opposite chirality representations of the 16-component spinor representation of  $SO(8)$ .

sector, all ground states are massless. Thus, the lowest mass states, contained in Table 2.3, are precisely the two sets of opposite chirality components of the spinor of eq.(2.2.48):  $|R_a\rangle_R$  and  $|R_{\bar{a}}\rangle_R$ .

In the R sector, the worldsheet statistics of the states can be determined by the action of the operator

$$(-1)^F |R_A; p\rangle_R = \gamma (-1)^{\sum_{r \in \mathbb{Z}_+} b_{-r}^i b_r^i} |R_A; p\rangle_R, \quad (2.2.53)$$

where the 8-dimensional chirality operator  $\gamma$ , corresponding to the transverse dimensions, is comprised of the product of the 8 zero mode oscillators,  $\gamma = 16b_0^2 \cdots b_0^9$  [11].  $\sum_{r \in \mathbb{Z}_+} b_{-r}^i b_r^i$  defines the worldsheet fermion number operator. The eigenvalue of the highest weight R sector ground state is declared to be positive  $(-1)^F |0; p\rangle_R = |0; p\rangle_R$ . As in the NS sector,  $\{(-1)^F, b_r^i\} = 0$ , such that the eigenvalues of the different chirality ground states,  $|R_a\rangle_R, |R_{\bar{a}}\rangle_R$ , built by the successive action of the lowering operators, are  $\pm 1$ , corresponding to the even / odd number of fermionic oscillators contained therein. Thus, with  $(-1)^F = +1$ ,  $|R_a\rangle_R$  have bosonic statistics on the worldsheet, while, with  $(-1)^F = -1$ ,  $|R_{\bar{a}}\rangle_R$  are fermionic.

From a brief inspection of Tables 2.2 & 2.3, it is evident that spacetime supersymmetry (which guarantees the vanishing of the one-loop partition function, to be discussed in the following subsection) cannot be generated while the spectrum contains the NS sector tachyon, nor while there exist twice as many states at the massless level in the R sector than in the NS sector. To form consistent superstring theories, it is necessary to restrict the spacetime spectrum. This restriction can be imposed upon the states based on their eigenvalue under the action of the operator  $(-1)^F$ . Concretely, this truncation of the spectrum is described by the Gliozzi-Scherk-Olive (GSO) projection [131].

Imposing the GSO projection in the NS sector involves keeping those states with positive eigenvalue under the action of the operator  $(-1)^F$ , while discarding those whose eigenvalue is negative. Thus, the preserved NS+ sector states are comprised of an odd number of  $b_r^i$  oscillators. In this way, the NS- sector tachyon, for which  $(-1)^F |0; p\rangle_{(NS),R} = -|0; p\rangle_{(NS),R}$ , is eliminated by the spacetime spectrum forming projection. Thus the massless vector boson at the first excited level,  $b_{-1/2}^i |0; p\rangle_{(NS),R}$ , forms the NS+ sector ground state. In order that the theory generate spacetime supersymmetry, the number of bosonic and fermionic degrees of freedom at each mass level must match. The R+ and R- sector ground states correspond to the opposite chirality states  $|R_a\rangle_R, |R_{\bar{a}}\rangle_R$ . Since these two 8-component spinors in the R sector engender twice as many real fermionic degrees of freedom as are found at the massless level in the bosonic NS+ sector, the GSO projection must select only those ground states of one chirality. The following subsection details the theories which

arise from both choices.

### 2.2.6 Constructing the closed superstring theory

The full spectrum of a closed string theory is formed by multiplicatively combining copies of the above right- and (the corresponding) left-moving sectors. The independence of the sectors means that in the RNS formalism, there exist four closed string sectors, differentiated by the choice of periodicity conditions, eq.(2.2.16), for the (left, right)-moving sectors: (NS, NS), (R, R), (NS, R) and (R, NS). States are formed from left- and right-moving creation operators acting respectively on left- and right-moving versions of the NS and R ground states, eqs.(2.2.43) & (2.2.48). (The NS and R subscript labels are dropped from the above states, in order to allow left ( $L$ ) and right ( $R$ ) labels to be clearly assigned.<sup>14</sup>) Following eqs.(2.2.50) & (2.2.52), in the (NS, NS) sector, schematically a generic state takes the form

$$|\tilde{\lambda}, \lambda\rangle = \left[ \prod_{i=2}^9 \prod_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i)^{\tilde{\lambda}_{n,i}} \right] \times \left[ \prod_{k=2}^9 \prod_{p=\frac{1}{2}}^{\infty} (\tilde{b}_{-p}^k)^{\tilde{\rho}_{p,k}} \right] |0; p\rangle_L \\ \otimes \left[ \prod_{j=2}^9 \prod_{m=1}^{\infty} (\alpha_{-m}^j)^{\lambda_{m,j}} \right] \times \left[ \prod_{l=2}^9 \prod_{q=\frac{1}{2}}^{\infty} (b_{-q}^l)^{\rho_{q,l}} \right] |0; p\rangle_R, \quad (2.2.54)$$

while in the (R, R) sector,

$$|\tilde{\lambda}, \lambda\rangle = \left[ \prod_{i=2}^9 \prod_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i)^{\tilde{\lambda}_{n,i}} \right] \times \left[ \prod_{k=2}^9 \prod_{p=1}^{\infty} (\tilde{b}_{-p}^k)^{\tilde{\rho}_{p,k}} \right] |R_A; p\rangle_L \\ \otimes \left[ \prod_{j=2}^9 \prod_{m=1}^{\infty} (\alpha_{-m}^j)^{\lambda_{m,j}} \right] \times \left[ \prod_{l=2}^9 \prod_{q=1}^{\infty} (b_{-q}^l)^{\rho_{q,l}} \right] |R_A; p\rangle_R, \quad (2.2.55)$$

where the notation for the generic non-negative integer exponents is extended from eq.(2.2.50). States in the (NS, R) and (R, NS) sectors involve combinations of the left- and right-moving components of these (NS, NS) and (R, R) states.

States are constrained by the superstring level matching condition, which requires that the left- and the right-moving contributions to the mass squared, eq.(2.2.49), of each of these states, must be equal:

$$\frac{\alpha'}{2} m_L^2 = \frac{\alpha'}{2} m_R^2 \implies (\tilde{N}_{l.c.}^B + \tilde{N}_{l.c.}^F - \tilde{a}) = (N_{l.c.}^B + N_{l.c.}^F - a). \quad (2.2.56)$$

In the (NS, NS) and (R, R) sectors, the normal ordering constants cancel, such that allowed states require equality between the number operators. Note that equality

<sup>14</sup>Following the afore defined notation, it should be clear that  $|0; p\rangle_{L/R}$  denote Neveu-Schwarz ground states, and  $|R_A; p\rangle_{L/R}$  their Ramond counterparts.

can be satisfied by states comprised of different numbers of left- and right-moving *bosonic* and *fermionic* operators; that is, eq.(2.2.56) can be satisfied without there being left-right equality separately for the bosonic and fermionic number operators. For example, the state  $\tilde{\alpha}_{-1}^i |R_a; p\rangle_L \otimes b_{-1}^j |R_b; p\rangle_R$  has  $\tilde{N}_{l.c.}^B = N_{l.c.}^F = 1$ . Conversely, in the (NS, R) and (R, NS) sectors, the disparity between the normal ordering constants must be off-set by there being different numbers of left- and right-moving oscillators. For example, the state  $|R_a; p\rangle_L \otimes b_{-1/2}^i |0; p\rangle_{(NS),R}$  has  $\tilde{N}_{l.c.} = 0$ ,  $N_{l.c.}^F = \frac{1}{2}$ , such that

$$\begin{aligned} (\tilde{N}_{l.c.}^B + \tilde{N}_{l.c.}^F - \tilde{a}_R) &= (N_{l.c.}^B + N_{l.c.}^F - a_{NS}), \\ (0 - 0) &= (0 + \frac{1}{2} - \frac{1}{2}). \end{aligned} \quad (2.2.57)$$

Table 2.4 contains the possible low-lying (tachyonic plus massless) closed string states, differentiated by sector. The values of  $(-1)^{\tilde{F}}$ ,  $(-1)^F$  respectively correspond to the projection operators for the left- and the right-movers. Combining two bosonic NS sectors yields spacetime bosons. Furthermore, spacetime bosons arise in the (R, R) sector, which is ‘doubly fermionic’ [6]. Conversely, the fermionic statistics within the R sector are preserved when combined with the NS sector, such that spacetime fermions arise in the (NS, R) and (R, NS) sectors. Massless states in the different sectors are formed from combinations of the eight-component R sector ground state spinors  $|R_a; p\rangle_{L/R}$ ,  $|R_{\bar{a}}; p\rangle_{L/R}$  and the first excited NS sector states,  $b_{-1/2}^i |0; p\rangle_{(NS)L/R}$ . Thus, in each sector, there exist  $8 \times 8 = 64$  massless states. The 10-dimensional fields in each sector can be found by decomposing the states with respect to their respective little group. These fields are given in the final column of Table 2.4.

As discussed in the previous subsection, it is necessary to perform the GSO projection on the states contained within Table 2.4, in order to produce a consistent superstring spectrum; namely, in order to eliminate the tachyonic ground state which exists in the NS– sector, and to ensure that the spectrum is spacetime supersymmetric. The GSO projection can be performed separately for left- and right-movers. The (NS, NS) sector projection is fixed by the need to remove the tachyon. Thus the states in the left- and the right-moving sectors must satisfy  $(-1)^{\tilde{F}} = +1$  and  $(-1)^F = +1$ . That is, the NS sectors for both the left- and the right-moving states are truncated to NS+. The (NS+, NS+) sector is tachyon free. In the R sectors for the left- and right-movers, there remains the freedom to select equal or opposite values of  $(-1)^{\tilde{F}}$  and  $(-1)^F$ . The so called ‘type IIB/A’ theories correspond to choosing both the left- and the right-movers to have R sector states of equal/opposite chirality [131, 132]. For the IIB theory, an arbitrary, but equal, choice of sign (positive) is made for the R sector states; that is,  $(-1)^{\tilde{F}} = +1$  and  $(-1)^F = +1$  (by arbitrary choice), such that the R sectors for both the left- and

$\alpha' M^2$	State	$SO(8)$ contents	Little group	$(-1)^{\bar{F}}$	$(-1)^F$	Repr. contents	10d field
(NS, NS)							
-2	$ 0; p\rangle_L \otimes  0; p\rangle_R$	$\mathbf{1} \otimes \mathbf{1}$	$SO(9)$	-1	-1	$\mathbf{1}$	$T(X)$
0	$\tilde{b}_{-1/2}^i  0; p\rangle_L \otimes b_{-1/2}^j  0; p\rangle_R$	$\mathbf{8}_V \otimes \mathbf{8}_V$	$SO(8)$	+1	+1	$\mathbf{1} + \mathbf{28} + \mathbf{35}_V$	$\phi(X), B_{[\mu\nu]}(X), G_{\mu\nu}(X)$
(R, R)							
	$ R_a; p\rangle_L \otimes  R_b; p\rangle_R$	$\mathbf{8}_S \otimes \mathbf{8}_S$		+1	+1	$\mathbf{1} + \mathbf{28} + \mathbf{35}_S$	$a, C_{\mu\nu}, C_{\mu\nu\rho\sigma}$
	$ R_{\bar{a}}; p\rangle_L \otimes  R_{\bar{b}}; p\rangle_R$	$\mathbf{8}_C \otimes \mathbf{8}_C$		-1	-1	$\mathbf{1} + \mathbf{28} + \mathbf{35}_C$	$a, C_{\mu\nu}, C_{\mu\nu\rho\sigma}$
0	$ R_{\bar{a}}; p\rangle_L \otimes  R_b; p\rangle_R$	$\mathbf{8}_C \otimes \mathbf{8}_S$	$SO(8)$	-1	+1	$\mathbf{8}_V + \mathbf{56}_V$	$C_\mu, C_{\mu\nu\rho}$
	$ R_a; p\rangle_L \otimes  R_{\bar{b}}; p\rangle_R$	$\mathbf{8}_S \otimes \mathbf{8}_C$		+1	-1	$\mathbf{8}_V + \mathbf{56}_V$	$C_\mu, C_{\mu\nu\rho}$
(R, NS)							
	$ R_a; p\rangle_L \otimes b_{-1/2}^i  0; p\rangle_R$	$\mathbf{8}_S \otimes \mathbf{8}_V$		+1	+1	$\mathbf{8}_C + \mathbf{56}_C$	$\lambda_\alpha^1, \psi_{\mu\alpha}^1$
0	$ R_{\bar{a}}; p\rangle_L \otimes b_{-1/2}^i  0; p\rangle_R$	$\mathbf{8}_C \otimes \mathbf{8}_V$	$SO(8)$	-1	+1	$\mathbf{8}_S + \mathbf{56}_S$	$\lambda_{\bar{\alpha}}^1, \psi_{\mu\bar{\alpha}}^1$
(NS, R)							
	$\tilde{b}_{-1/2}^i  0; p\rangle_L \otimes  R_a; p\rangle_R$	$\mathbf{8}_V \otimes \mathbf{8}_S$		+1	+1	$\mathbf{8}_C + \mathbf{56}_C$	$\lambda_\alpha^2, \psi_{\mu\alpha}^2$
0	$\tilde{b}_{-1/2}^i  0; p\rangle_L \otimes  R_{\bar{a}}; p\rangle_R$	$\mathbf{8}_V \otimes \mathbf{8}_C$	$SO(8)$	+1	-1	$\mathbf{8}_S + \mathbf{56}_S$	$\lambda_{\bar{\alpha}}^2, \psi_{\mu\bar{\alpha}}^2$

Table 2.4: The low-lying (tachyonic and massless) states of the closed superstring theory. The states are decomposed into irreducible representations of the respective subgroups. Also recorded are the eigenvalues of the operators  $(-1)^{\bar{F}}$ ,  $(-1)^F$  [11], and the 10-dimensional fields to which the decomposed states give rise. The yellow, respectively blue, rows are selected by the GSO projection for the spectrum of the IIB, respectively IIA, theory. The green states in the (NS, NS) and (R, R) sectors are common to both theories.

the right-movers are truncated to  $R_+$ . Conversely, in the IIA theory, the left- and right-moving  $R$  sectors have opposite parity; that is,  $(-1)^{\bar{F}} = -1$  and  $(-1)^F = +1$ , such that the left- and right-moving sectors are respectively are truncated to  $R_-L$  and  $R_+R$ . For example, the  $(R, R)$  sector becomes  $(R_-, R_+)$ .

The action of the GSO projection on the possible closed string states is indicated by the color-coding in Table 2.4. The massless spectrum of the ‘type IIB’ string theory, which corresponds to the (shared) green and (unique) yellow entries of the Table, is recorded in Table 2.5. As explained, each sector contains 64 states. The gravity multiplet, which arises in the  $(NS_+, NS_+)$  sector, is as described for the bosonic string; the scalar dilaton  $\phi$  (a single state), a two-index antisymmetric tensor  $B_{[\mu\nu]}$  (28 states), and a 2-index traceless symmetric tensor  $G_{\mu\nu}$  (the graviton) (35 states). The remaining bosonic degrees of freedom arise in the  $(R_+, R_+)$  sector; a scalar (one state), the axion  $a$ , and two- and four-index antisymmetric tensors,  $C_{\mu\nu}$  (28 states) and  $C_{\mu\nu\rho\sigma}$  (35 states). The fermionic  $(R_+, NS_+)$  and  $(NS_+, R_+)$  sectors give rise to two Rarita-Schwinger fields, the gravitinos  $\psi_{\mu\alpha}^{1,2}$  (56 states each), and two spinors, the dilatinos  $\lambda_{\alpha}^{1,2}$  (8 states each), where  $\alpha$  is a  $10D$  chiral spinor index [5]. Thus, the counting of states explicitly demonstrates that, at the massless level, the GSO projection yields a spectrum in which there exist an equal number of bosonic and fermionic degrees of freedom. Although beyond the scope of this study, the Green-Schwarz formalism provides a proof that the GSO projection guarantees equality across all mass levels, thus ensuring that the theory is spacetime supersymmetric [124]. Specifically, the equal chirality of the gravitino states indicate that the theory gives rise to an  $\mathcal{N} = (2, 0)$  supersymmetry.

Sector	$ \rangle_L \otimes  \rangle_R$	$(-1)^{\bar{F}}$	$(-1)^F$	$SO(8)$ contents	10d field
$(NS_+, NS_+)$	$\mathbf{8}_V \otimes \mathbf{8}_V$	+1	+1	$\mathbf{1} + \mathbf{28} + \mathbf{35}_V$	$\phi(X), B_{[\mu\nu]}(X), G_{\mu\nu}(X)$
$(R_+, R_+)$	$\mathbf{8}_S \otimes \mathbf{8}_S$	+1	+1	$\mathbf{1} + \mathbf{28} + \mathbf{35}_S$	$a, C_{\mu\nu}, C_{\mu\nu\rho\sigma}$
$(R_+, NS_+)$	$\mathbf{8}_C \otimes \mathbf{8}_V$	+1	+1	$\mathbf{8}_C + \mathbf{56}_C$	$\lambda_{\alpha}^1, \psi_{\mu\alpha}^1$
$(NS_+, R_+)$	$\mathbf{8}_V \otimes \mathbf{8}_C$	+1	+1	$\mathbf{8}_C + \mathbf{56}_C$	$\lambda_{\alpha}^2, \psi_{\mu\alpha}^2$

Table 2.5: The massless states of type IIB superstring theory, in the notation of [5], with colours coordinated with those in Table 2.4.

Analogously, Table 2.6 contains the massless spacetime spectrum of the ‘IIA’ theory, which corresponds to the green and blue entries of Table 2.4. The  $(NS_+, NS_+)$  sector is identical to that of the IIB theory. The  $(R_-, R_+)$  sector bosonic fields are one- and three-index antisymmetric tensors,  $C_{\mu}$  (8 states) and  $C_{\mu\nu\rho}$  (56 states). The

fermions in the (R−, NS+) and (NS+, R+) sectors come in pairs of opposite chirality; the gravitinos  $\psi_{\mu\dot{\alpha}}^1, \psi_{\mu\alpha}^2$  and the dilatinos  $\lambda_{\dot{\alpha}}^1, \lambda_{\alpha}^2$ , where the spinor indices  $\alpha, \dot{\alpha}$  denote spinors of opposite chirality. The opposite chirality of the gravitinos corresponds to the fact that the IIA theory has  $\mathcal{N} = (1, 1)$  supersymmetry, again evidenced by the equal number of massless bosonic and fermionic states.

Sector	$ \rangle_L \otimes  \rangle_R$	$(-1)^{\bar{F}}$	$(-1)^F$	$SO(8)$ contents	$10d$ field
(NS+, NS+)	$\mathbf{8}_V \otimes \mathbf{8}_V$	+1	+1	$\mathbf{1} + \mathbf{28} + \mathbf{35}_V$	$\phi(X), B_{[\mu\nu]}(X), G_{\mu\nu}(X)$
(R−, R+)	$\mathbf{8}_C \otimes \mathbf{8}_S$	−1	+1	$\mathbf{8}_V + \mathbf{56}_V$	$C_{\mu}, C_{\mu\nu\rho}$
(R−, NS+)	$\mathbf{8}_C \otimes \mathbf{8}_V$	−1	+1	$\mathbf{8}_S + \mathbf{56}_S$	$\lambda_{\dot{\alpha}}^1, \psi_{\mu\dot{\alpha}}^1$
(NS+, R+)	$\mathbf{8}_V \otimes \mathbf{8}_S$	+1	+1	$\mathbf{8}_C + \mathbf{56}_C$	$\lambda_{\alpha}^2, \psi_{\mu\alpha}^2$

Table 2.6: The massless states of type IIA superstring theory, in the notation of [5], with colours coordinated with those in Table 2.4. The ‘+’ or ‘−’ labels on the NS and R sectors indicate the eigenvalue of the GSO projection operators  $(-1)^{\bar{F}}, (-1)^F$ . Note in particular the opposite chirality of the fermions, which describe the  $\mathcal{N} = (1, 1)$  supersymmetry of the theory.

Before discussing the remaining two types of (purely) closed string theories, the so called heterotic theories, which are the objects of principle interest in this study, it will be instructive to introduce string partition functions and necessary to introduce the formalism of compactifications.

## 2.3 One-loop string partition functions

The fundamental object of interest for the study of stable theories is the one-loop partition function for the closed string;

$$\mathcal{Z}(\tau, \bar{\tau}) = \text{Tr} (-1)^F \bar{q}^{H_L} q^{H_R}. \quad (2.3.1)$$

$q$  is the nome  $q = e^{2\pi i\tau}$  (as usual, the real and imaginary parts of  $\tau$  are defined by  $\tau = \tau_1 + i\tau_2$ ). The eigenvalues of the left- and right-moving worldsheet Hamiltonians,  $(H_R, H_L)$ , are the left- and right-moving worldsheet energies  $(E_R, E_L)$ .  $F$  denotes the spacetime fermion number. It is worth going into some detail in order to understand the origin of this expression, which can also be interpreted as a generating functional, before outlining how it can be used to study, among other features, the spectrum of a given (of interest in this study, heterotic) theory.

It should be emphasized that the following sections treat the *one*-loop partition function. [133] constitutes a preliminary investigation in to how the treatment might be extended to higher loop order. It is expected that higher order corrections will be suppressed compared to the one-loop contribution to  $\mathcal{Z}$ .

### 2.3.1 Virasoro algebra from conformal field theory

Conformal field theories (CFTs), whose properties will now be described, represent a special subset of quantum field theories, which have no preferred scale. In the context of string theory, a CFT arises as the 2-dimensional field theory which lives on the worldsheet traced out by a string propagating through spacetime [134]. Indeed, the Polyakov action is Weyl invariant. The machinery of CFT provides a powerful tool with which to obtain solutions for the dynamics of such strings. The relevant formalism will now be provided.

A differentiable map  $\phi : U \rightarrow V$ , where  $U \subset M$  and  $V \subset M'$  are open subsets, represents a conformal transformation if under such a transformation, the metric tensor  $g_{\mu\nu}$  transforms up to a scale factor [122, 134, 135]:

$$g'_{\rho\sigma}(x') \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x) , \quad (2.3.2)$$

where  $x' = \phi(x)$  and  $x \in U$ . Restricting to flat space, the condition simplifies to:

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) \eta_{\mu\nu} . \quad (2.3.3)$$

Conformal field theories are those which remain invariant under such, locally angle preserving, conformal transformations. CFTs look the same at all length scales. Consider infinitesimal coordinate transformations to first order in the parameter  $\epsilon(x) \ll 1$ ;

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2) . \quad (2.3.4)$$

By requiring that such infinitesimal coordinate transformations satisfy eq.(2.3.2), conformally invariant transformations are determined to be those which satisfy (see Appendix A for the full calculation):

$$\partial_{\nu} \epsilon_{\mu} + \partial_{\mu} \epsilon_{\nu} = \frac{2}{d} (d \cdot \epsilon) \eta_{\mu\nu} . \quad (2.3.5)$$

Given the extended nature of the string worldsheets depicted in Figure 2.1, the conformal group in two dimensions is of particular interest, and will now be described.

Complex coordinates and their derivatives can be formed as<sup>15</sup>

$$\begin{aligned} z &= x^0 + ix^1, & \partial_z &= \frac{1}{2}(\partial_0 - i\partial_1), \\ \bar{z} &= x^0 - ix^1, & \partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + i\partial_1), \end{aligned} \quad (2.3.6)$$

with similar, light-cone like infinitesimal parameters formed as

$$\begin{aligned} \epsilon &= \epsilon^0 + i\epsilon^1, \\ \bar{\epsilon} &= \epsilon^0 - i\epsilon^1. \end{aligned} \quad (2.3.7)$$

Calculations are simplified if done on a Euclidean worldsheet, with coordinates  $(x^0, x^1)$ , but the results could equally be obtained in Minkowski space. With  $\eta_{00} = \eta_{11} = +1$ , eq.(2.3.5) yields the pair of constraints,

$$\partial_0\epsilon_0 = \partial_1\epsilon_1, \quad \partial_0\epsilon_1 = -\partial_1\epsilon_0, \quad (2.3.8)$$

which restrict  $\epsilon(z)$  to be a homomorphic function in some open set. Thus a Laurent expansion of the function about  $z = 0$  can be performed, yielding, in terms of the infinitesimal constants  $\epsilon_n, \bar{\epsilon}_n$ :

$$\begin{aligned} z' &= z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n (-z^{n+1}), \\ \bar{z}' &= \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n (-\bar{z}^{n+1}). \end{aligned} \quad (2.3.9)$$

The operators

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}, \quad (2.3.10)$$

which generate these infinitesimal conformal transformations, form two, commuting, copies of a Witt algebra, defined by the relations:

$$[l_m, l_n] = (m - n) l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m - n) \bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0. \quad (2.3.11)$$

To avoid the ambiguity at  $z = 0$ , the operators are considered on the Riemann sphere  $S^2 \simeq \mathbb{C} \cup \{\infty\}$  rather than working on the Euclidean plane  $\mathbb{R}^2 \simeq \mathbb{C}$ . It is still necessary to restrict the generators in eq.(2.3.10), which are non-singular at  $z = 0$  for  $n \geq -1$ , and, as can be seen by setting  $z = -\frac{1}{w}$ , non-singular at  $z = \infty$  for  $n \leq +1$ . Thus conformal transformations on the Riemann sphere are generated by the set of operators  $\{l_{-1}, l_0, l_1\}$ .

Consider the action of the three generators eq.(2.3.10).

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<sup>15</sup>Bars will be used to denote complex conjugation. Thus, following the convention in the literature, the operators  $\bar{L}_0, L_0$  will be used analogously to  $\tilde{L}_0, L_0$  of the previous subsections.

- $l_{-1} = -\partial_z$  generates constant shifts or translations;  $z \mapsto z + b$
- $l_0 = -z\partial_z$  generates scale transformations;  $z \mapsto az$
- $l_1 = -z^2\partial_z$  generates Special Conformal Transformations;  $z \mapsto \frac{z}{cz+d}$

A general conformal transformation generated by the operators  $\{l_{-1}, l_0, l_1\}$  thus takes the form

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}. \quad (2.3.12)$$

With  $ad - bc = 1$  to ensure invertibility, and noting that the same transformation is generated under the exchange  $(a, b, c, d) \mapsto (-a, -b, -c, -d)$ , the group which generates conformal transformations on the Riemann sphere of this form is identified as the Möbius group  $SL(2, \mathbb{C})/\mathbb{Z}_2$ .

Of particular interest are those transformations generated by  $l_0, \bar{l}_0$ . The complex coordinates eq.(2.3.6) can be rephrased in polar form as  $z = re^{i\phi}$ ,  $\bar{z} = re^{-i\phi}$ . Thus the operators can be re-expressed as:

$$l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi, \quad \bar{l}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi, \quad (2.3.13)$$

and combined as:

$$l_0 + \bar{l}_0 = -r\partial_r \quad \text{and} \quad i(l_0 - \bar{l}_0) = -\partial_\phi. \quad (2.3.14)$$

Thus the operator  $l_0 + \bar{l}_0$  can be interpreted geometrically as generating infinitesimal scale transformations or dilations, while  $l_0 - \bar{l}_0$  corresponds to infinitesimal rotations.

The elements of the central extension of the Witt algebra of infinitesimal conformal transformations, denoted  $L_n$ ,  $n \in \mathbb{Z}$ , obey commutation relations modified by a mapping  $p$  coupled to a central charge  $c$ :

$$[L_m, L_n] = (m - n) L_{m+n} + c p(m, n), \quad (2.3.15)$$

By noting the anti-symmetry of the Lie bracket, using the Jacobi identity and applying the conventional normalisation, the Virasoro algebra, which defines the central extension of the Witt algebra, is found to be defined by the commutation relations:

$$[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0}, \quad (2.3.16)$$

(This expression has already been seen in eq.(2.1.35).) Thus, there is no central extension for  $L_0$  and  $\bar{L}_0$ , meaning that their geometric interpretations are equivalent to those for  $l_0$  and  $\bar{l}_0$  (eq.(2.3.14)).

If the 2-dimensional worldsheet theory is to be conformally invariant, the conformal symmetry of the classical theory must be preserved at the quantum level; the

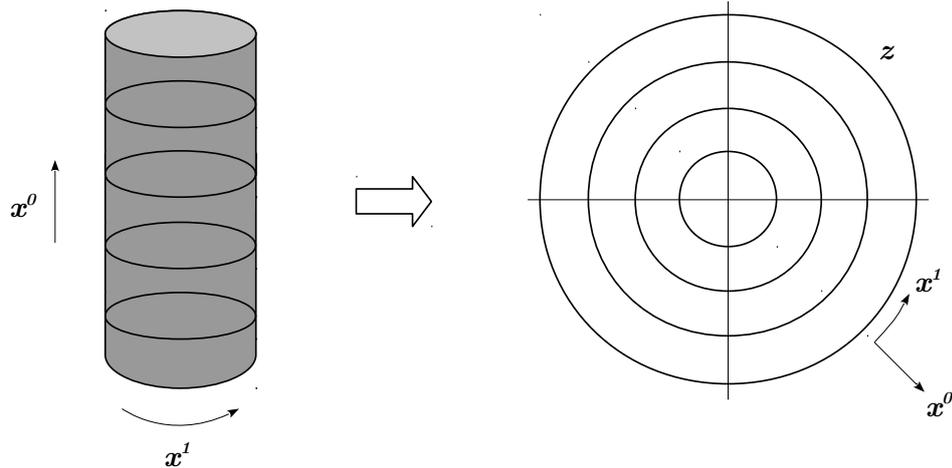


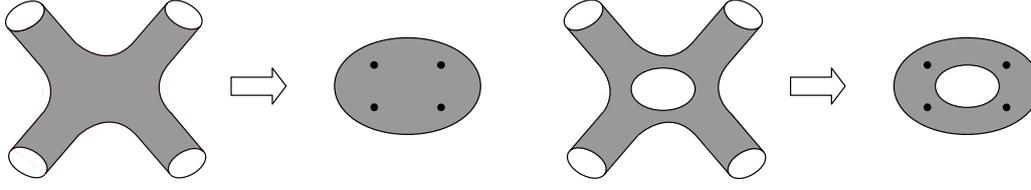
Figure 2.2: Mapping the two dimensional surface of a cylinder, which is infinitely long in the  $x^0$  direction, to the complex plane, under the relationship  $z = e^w = e^{x^0} \cdot x^{ix^1}$ .

quantum theory must be anomaly free. The potential conformal anomaly, defined in terms of the central charge, can be cancelled by a specific choice of the number of fields; namely 26 in the bosonic theory. The field content of the theory corresponds to the number of spacetime dimensions in which the string propagates. Thus, in agreement with the result found in §2.1.4, the critical dimension, or critical central charge, of the bosonic string theory, is  $D = 26$ . Similarly, the superstring is anomaly free in  $D = 10$ .

### 2.3.2 Defining a partition function

CFTs defined on the Riemann sphere correspond to tree level perturbations. That is, a tree level amplitude corresponds to a world sheet which takes the form of a Riemann sphere. As all possible states can propagate in loops, loop diagrams reveal the particle content of a theory. Loop diagrams correspond to world sheets of higher genus than the genus zero Riemann sphere. The first order diagram corresponds to a 2-torus,  $T^2$ , world sheet. It will be seen that the vacuum-to-vacuum amplitude *one-loop* diagram depicted in Figure2.3, the order to which this study will be restricted, can be interpreted as a one-loop partition function, a weighted sum over all states in the theory. Thus, given that the surface of a 2-torus is defined by the two parameters  $\tau_1, \tau_2$ , the preceding treatment of 2-dimensional CFTs can be employed.

As a stepping stone from flat space to the 2-torus, the spatial dimension (for concreteness  $x^1$ ) of the Euclidean plane can be compactified on to a circle of radius  $R$ , to form a cylinder of infinite length. Points on the surface of the cylinder are defined by the coordinate  $w = x^0 + ix^1$ , such that the imaginary component of  $w$  is periodic,  $w \sim w + 2\pi i$ . To make contact with the preceding study of conformal



(a) String tree-level amplitude.

(b) String one-loop amplitude.

Figure 2.3: The tree-level and one-loop string diagrams, to the left of the arrows, with four closed strings stretching to infinity, correspond to the genus  $g = 0, 1$  Riemann surfaces, the sphere and the torus, with four vertex operators inserted [134].

theories on flat space parametrised by a complex coordinate  $z$  (eq.(2.3.6)), the two dimensional Euclidean conformal field theory defined on the cylinder can be mapped to the complex plane using the mapping  $z = e^w$ , as shown in Figure 2.2. Critically, temporal and spatial translations on the cylinder are respectively mapped to dilations and rotations on the complex plane:

$$\begin{aligned} x^0 \mapsto x^0 + a & \quad \text{becomes} \quad z \mapsto e^a z, \\ x^1 \mapsto x^1 + b & \quad \text{becomes} \quad z \mapsto e^{ib} z. \end{aligned} \quad (2.3.17)$$

Using the relations which define dilations and rotations for two dimensional conformal field theories parametrised by a complex coordinate, eq.(2.3.14), the generators of time and space translations, the Hamiltonian and momentum operators, can be expressed in terms of the Virasoro operators:

$$H_{\text{cyl}} = (L_{\text{cyl}})_0 + (\bar{L}_{\text{cyl}})_0, \quad P_{\text{cyl}} = i \left[ (L_{\text{cyl}})_0 - (\bar{L}_{\text{cyl}})_0 \right]. \quad (2.3.18)$$

A torus can be built by taking a segment of the cylinder and performing a second identification, in addition to that described above to build the cylinder in the first place, such that the both the spatial and temporal coordinates are periodic. Formally constructing the torus from the complex plane  $\mathbb{C}$  involves constructing a fundamental domain, containing the set of points  $w$ . By identifying the parallel edges of the fundamental domain, a lattice is constructed. Points  $w$  within the fundamental domain are identified with points separated by integer units of the lattice vectors,  $w \equiv w + m\alpha_1 + n\alpha_2$ , with  $\alpha_1, \alpha_2 \in \mathbb{C}$ ,  $m, n \in \mathbb{Z}$ , as in Figure 2.4. The modular parameter is defined as the ratio of the two lattice vectors:  $\tau = \frac{\alpha_2}{\alpha_1} = \tau_1 + i\tau_2$ . Multiplying by  $\frac{2\pi}{m\alpha_1}$ , the identification can alternatively be stated

$$w' = w' + 2\pi + m'2\pi\tau. \quad (2.3.19)$$

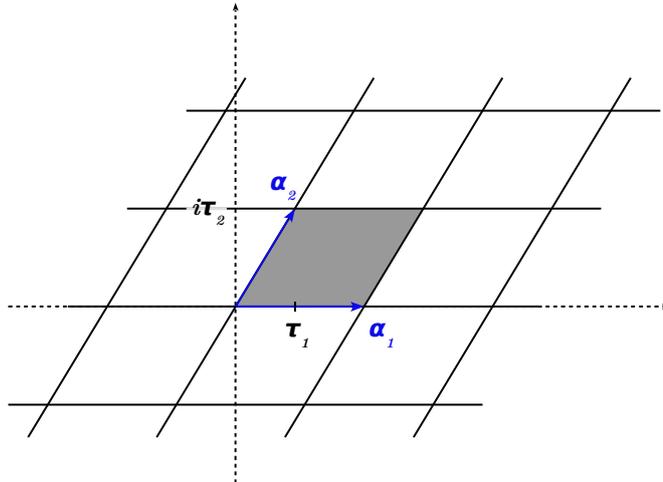


Figure 2.4: The fundamental domain of the torus, shaded, is defined as the smallest cell of the lattice spanned by the lattice vectors  $(\alpha_1, \alpha_2)$ , where the torus itself is generated by the identification of the opposite, parallel edges of said fundamental domain.

In terms of the real coordinates  $w = x^0 + ix^1$ ,

$$(x^0, x^1) \equiv (x^0 + 2\pi, x^1) \equiv (x^0 + 2\pi\tau_1, x^1 + 2\pi\tau_2), \quad (2.3.20)$$

which aids with a geometrical interpretation; the torus can be constructed by connecting the ends of a cylinder, of circumference  $2\pi$  and with length  $2\pi\tau_2$ , which have been rotated relative to each other by an angle  $2\pi\tau_1$ .

It turns out that the exhaustive set of transformations, termed modular transformations, that preserve the structure of any given torus, take the form

$$\mathcal{T} : \tau \rightarrow \tau + 1, \quad \mathcal{S} : \tau \rightarrow -\frac{1}{\tau}. \quad (2.3.21)$$

Figure 2.5 depicts the action of the  $\mathcal{T}$  and  $\mathcal{S}$  transformations, both in terms of the lattice and in terms of the non-contractible cycles around the torus. Under a  $\mathcal{T}$  transformation, which corresponds to a shift in the real component of the lattice vector, the torus is cut, and its ends are only identified once one of its ends has been twisted through  $2\pi$ . An  $\mathcal{S}$  transformation, which corresponds a scaling of the modulus, corresponds to the switching of the two cycles.

Combinations of  $\mathcal{T}$  and  $\mathcal{S}$  transformations give rise to all general modular transformations. The modular group of the torus is an isometry group acting on the modular parameter as [134]

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2. \quad (2.3.22)$$

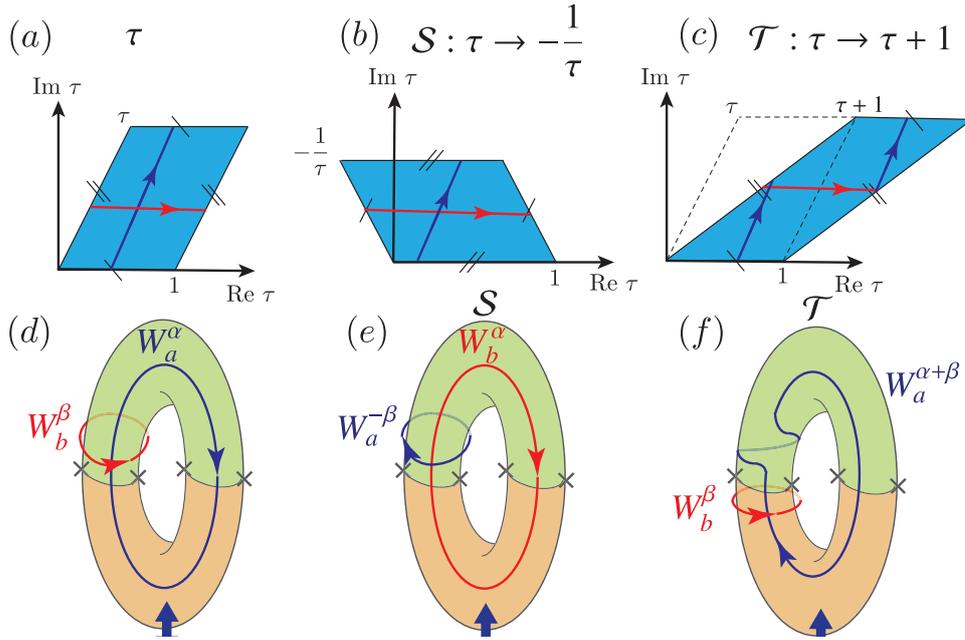


Figure 2.5: The  $\mathcal{T}$  and  $\mathcal{S}$  modular transformations for the torus, in terms of the shift in lattice defined by the parameter  $\tau$ , and in terms of the transformations of the two non-contractible cycles,  $\alpha, \beta$ , which loop around the torus. In the notation of [136], from which the Figure is taken,  $W_a^\alpha, W_b^\beta$  denote the Wilson loops (which will not be discussed in detail here) around the  $\alpha$  and  $\beta$  cycles respectively. In this notation,  $\mathcal{T}: (\alpha, \beta) \rightarrow (\alpha + \beta, \alpha)$ , and  $\mathcal{S}: (\alpha, \beta) \rightarrow (-\beta, \alpha)$ .

These transformations constitute the action of the group  $SL(2, \mathbb{Z})$ . Transformations between pairs of lattice vectors are generated by  $SL(2, \mathbb{Z})/\mathbb{Z}_2$  matrices. Modular transformations are the transformations under which the lattice which defines a given torus (and hence the metric of the theory which lives on the torus) transform only up to a scale (the conformal scale). Note that  $SL(2, \mathbb{C})/\mathbb{Z}_2$ , which acts on the complex coordinate  $z$ , generates conformal transformations on the Riemann sphere, while  $SL(2, \mathbb{Z})/\mathbb{Z}_2$ , which acts on  $\tau$ , the modular parameter, produces modular transformations.

Specifically, the  $\mathcal{T}$  transformations can be repeatedly employed to shift  $\tau$  in to the region  $\tau_1 \in [-\frac{1}{2}, \frac{1}{2}]$ . Furthermore,  $\mathcal{S}$  transformations, which send  $|\tau| \rightarrow \frac{1}{|\tau|}$ , can be used to map points inside the circle defined by  $|\tau| < 1$  to those outside of it. Thus, it is possible to define a fundamental domain,  $\mathcal{F}$ , shown in Figure 2.6, into which any value of  $\tau$  in the upper half-plane can be mapped by successive modular transformations:

$$\mathcal{F} = \left\{ \tau \in \mathbb{H} : |\tau_1| \leq \frac{1}{2}, |\tau| \geq 1 \right\}. \quad (2.3.23)$$

Any torus defined by a value of  $\tau$  which sits outside of the fundamental domain can be transformed in to a conformally equivalent torus lying inside  $\mathcal{F}$  using modular transformations.

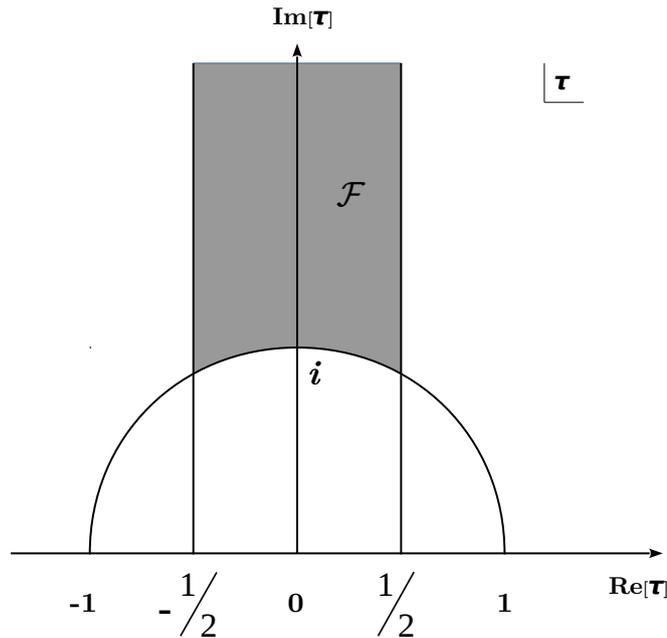


Figure 2.6: The fundamental domain for the identifications  $\tau \sim \tau + 1$  and  $\tau \sim -\frac{1}{\tau}$  of  $\mathbb{H}$ , which define the moduli space of tori.

Conformal transformations were defined in eq.(2.3.2) as the set of transformations of a manifold with metric, which fixes the metric up to an overall scale. The conformal group represents the group of conformal transformations. All tori can be endowed with a flat metric, but not all flat metrics on the torus are equivalent; those which are equivalent are those related by the modular transformations. Thus, the conformal group of the torus is the modular group.

As a consequence of the conformal symmetry preserved by CFTs, physical quantities (such as amplitudes) relating to such theories must be invariant under conformal transformations. In the context of string theory, the CFT on the torus must be invariant under modular transformations. The physical quantity of interest to this study, the string theory one-loop vacuum-to-vacuum amplitude, which, as stated, corresponds to the one-loop string partition function, should therefore be modular invariant, such that it is preserved under modular (conformal) transformations. Note that modular invariance is not a property of a general CFT; it must be imposed with respect to a particular application. Modular invariance arises for the one-loop amplitude precisely because it involves a CFT on a torus.

The fundamental domain  $\mathcal{F}$  defines the space of conformally inequivalent tori. In order to generate the one-loop amplitude, it is necessary to integrate over all inequivalent tori (corresponding to a sum over all metrics); that is, all tori defined by distinct values of  $\tau$  that lie within the fundamental domain of the modular group. The power of modular invariance is that it is only necessary to treat tori defined by values of  $\tau$  within this restricted range; all other tori can be mapped into  $\mathcal{F}$  by

modular transformations defined by eq.(2.3.22). Using the  $SL(2, \mathbb{Z})/\mathbb{Z}_2$  invariant measure, the partition function is given by the integral over an  $SL(2, \mathbb{Z})/\mathbb{Z}_2$  invariant integrand:

$$Z = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \mathcal{Z}(\tau). \quad (2.3.24)$$

Critically, the potentially UV divergent  $\tau \rightarrow 0$  region is excluded from this region. In this way, modular invariance regulates the potential one-loop divergences, bestowing a degree of finiteness upon these theories. Ultimately, modular invariance is responsible for controlling the degree to which spacetime SUSY can be broken in any non-SUSY string model; a ‘misaligned supersymmetry’, to be discussed, is preserved by the spectrum [15–17].

It is useful to think about the physical interpretation of the partition function.  $Z(\tau)$  for a conformal field theory on a torus, alternatively thought of as the one-loop vacuum-to-vacuum amplitude, can be built by taking a field theory on a circular line (with unit radius), evolving for a Euclidean time  $2\pi\tau_2$ , followed by performing a translation in  $x^0$  by  $2\pi\tau_1$ , and identifying the ends [122, 123, 134]. That is, performing a periodic temporal displacement of length  $\tau_2$ , as in Figure 2.4, results in a spatial displacement of length  $\tau_1$ . Thus any expression for  $Z(\tau)$  will involve a term coupling the generator for temporal translations,  $H$ , to the temporal component of the modular parameter,  $\tau_2$ , and a corresponding term coupling the generator for spatial translations,  $P$ , to its spatial component,  $\tau_1$ <sup>16</sup>

$$\mathcal{Z}(\tau_1, \tau_2) = \sum_{|\psi\rangle \in \mathcal{H}_{\text{cl.}}} \langle \psi | e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} | \psi \rangle. \quad (2.3.25)$$

The trace over all states in the theory makes clear how this expression connects with classical interpretations of a partition functions. Using the expressions for  $H$  and  $P$  in terms of the zero modes of the Laurent expansion on the cylinder, eq.(2.3.18), the partition function can be expressed as

$$\begin{aligned} \mathcal{Z}(\tau_1, \tau_2) &= \text{Tr} \left( e^{-2\pi\tau_2 H_{\text{cyl}}} e^{2\pi\tau_1 P_{\text{cyl}}} \right) \\ &= \text{Tr} \left( e^{-2\pi\tau_2 [(L_{\text{cyl}})_0 + (\bar{L}_{\text{cyl}})_0]} e^{2\pi i\tau_1 [(L_{\text{cyl}})_0 - (\bar{L}_{\text{cyl}})_0]} \right). \end{aligned} \quad (2.3.26)$$

The cylinder engenders a negative Casimir energy relative to the CFT on the plane studied above [122]. Thus the zero mode for the theory on the cylinder is shifted from that for the theory on the plane:

$$(L_{\text{cylinder}})_0 = L_0 - \frac{c}{24}, \quad (\bar{L}_{\text{cylinder}})_0 = \bar{L}_0 - \frac{\bar{c}}{24}. \quad (2.3.27)$$

---

<sup>16</sup> $|\psi\rangle$  are used to denote generic states. They contain as a subset the physical states  $|\phi\rangle$ . This will be an important distinction for later sections, as even unphysical states make a contribution to the partition function.

Using the nome  $q = e^{2\pi i\tau}$ ,  $\bar{q} = e^{-2\pi i\bar{\tau}}$ , terms can be collected, giving the expression for the partition function for a conformal field theory defined on a torus, which will be used as the fundamental item of study in the following sections:

$$\mathcal{Z}(\tau_1, \tau_2) = \text{Tr}_{\mathcal{H}_{\text{closed}}} \left( \bar{q}^{\left(\bar{L}_0 - \frac{\bar{c}}{24}\right)} q^{\left(L_0 - \frac{c}{24}\right)} \right) = (\bar{q}^{\bar{c}} q^c)^{\frac{-1}{24}} \text{Tr}_{\mathcal{H}_{\text{cl}}} \left( \bar{q}^{\bar{L}_0} q^{L_0} \right). \quad (2.3.28)$$

Instead of tracing over the Hilbert space of closed strings constrained by the level matching condition, eq.(2.4.17), the trace can be taken over an extended Hilbert space of states of unconstrained oscillator structure, in the knowledge that the  $\tau_1$  integral within the partition function itself imposes the level matching condition, acting to project out non-physical states as desired [5]. In particular, the integral over  $\tau_1$  dependent terms in  $\bar{q}^{\bar{L}_0} q^{L_0}$  takes the form

$$\int d\tau_1 \bar{q}^{\bar{L}_0} q^{L_0} \rightarrow \int d\tau_1 e^{2\pi i\tau_1(L_0 - \bar{L}_0)} \sim \delta_{\bar{N}, N}. \quad (2.3.29)$$

### 2.3.3 Generating functions

The partition function encodes information about the number of states within a theory at a given energy level. The infinite tower of string states is encoded within the partition function as an infinite sequence of numbers. This sequence can be encoded in a generating function, which treats the individual terms in the partition function as the coefficients of a power series.

Consider explicitly constructing the bosonic string state space using integer moded operators,  $a_m^\dagger$  corresponding to creation operators. Similar methods are applied when constructing partition functions for heterotic strings, the strings which will ultimately be of most interest in this study. A set of states constructed from  $0, 1, 2, 3, \dots$  powers of such operators, can be expressed as a polynomial  $f(q)^{17}$ , each term of which enumerates the number of states constructed from a given combination of operators (and hence, with a given energy). For example, those states constructed from the repeated action of a creation operator of mode 1,  $a_1^\dagger$ , could be expressed as

$$f(q) = \left[ 1 + (q^1) + (q^1)^2 + (q^1)^3 + \dots \right] = \frac{1}{(1 - q^1)}, \quad (2.3.30)$$

with 1 corresponding to the state  $|0\rangle$ ,  $(q^1)$  to  $a_1^\dagger|0\rangle$ ,  $(q^1)^2$  to  $(a_1^\dagger)^2|0\rangle$ , and so on. Oscillators with mode number  $k$  give rise to the set of states

$$f(q) = \left[ 1 + (q^k) + (q^k)^2 + (q^k)^3 + \dots \right] = \frac{1}{(1 - q^k)}. \quad (2.3.31)$$

---

<sup>17</sup>Much of the formalism of the theory of modular forms, upon which string theory relies so heavily, is expressed in terms of the nome  $q = e^{2\pi i\tau}$ . Thus it is natural to construct the partition function  $f(q, \bar{q})$  as an infinite sequence of powers of  $q, \bar{q}$ .

Taking the product of the infinite set of such polynomials, each set corresponding to those states generated by operators of differing mode numbers (or a different combination of oscillators), yields the generating function

$$\begin{aligned} f(q) &= \left[1 + (q^1) + (q^1)^2 + (q^1)^3 + \dots\right] \left[1 + (q^2) + (q^2)^2 + (q^2)^3 + \dots\right] \times \dots \\ &\quad \dots \times \left[1 + (q^m) + (q^m)^2 + (q^m)^3 + \dots\right] \\ &= \prod_{m=1}^{\infty} (1 - q^m)^{-1}. \end{aligned} \quad (2.3.32)$$

For closed strings, the  $\tilde{a}_n^\dagger$  operators create an equivalent set of states, encoded in  $\bar{q}$  polynomials. Thus the state space, encoded by the function  $f(q, \bar{q})$ , is comprised of combinations of  $\bar{q}^n$ ,  $q^m$  terms. The spacetime mass squared of the states at each level is given by  $m + n$ . Concretely, the closed string version of eq.(2.3.32) can be neatly expressed in terms of the Dedekind  $\eta$ -function eq.(B.0.1):

$$f(q, \bar{q}) = \prod_{n=1}^{\infty} (1 - \bar{q}^n)^{-1} \prod_{m=1}^{\infty} (1 - q^m)^{-1} = \frac{(\bar{q}q)^{1/24}}{\bar{\eta}(\bar{q})\eta(q)}, \quad (2.3.33)$$

### 2.3.4 Bosonic partition functions

It is possible to make contact with the CFT expression for the partition function, eq.(2.3.28), whose trace represents a sum over the states in the theory. In the bosonic theory, each state takes the form given by eq.(2.1.59). The components of the closed string Hilbert space for the left- and the right-movers corresponding to the centre of mass and to the oscillators, decouple [5]

$$\mathcal{H}_{\text{cl.}} = \mathcal{H}_{\text{c.o.m.}} \otimes \left(\otimes_{i=1}^{D-2} \otimes_{n=1}^{\infty} \bar{\mathcal{H}}_{\text{osc.}}\right)_L \otimes \left(\otimes_{i=1}^{D-2} \otimes_{n=1}^{\infty} \mathcal{H}_{\text{osc.}}\right)_R. \quad (2.3.34)$$

Thus, the trace can be taken separately for the zero and the non-zero mode contributions. Using the expressions for the Virasoro operators split in to the zero and the positive modes, eq.(2.1.49), and with  $c = \bar{c} = 1$  for a free scalar field,

$$\mathcal{Z}(\tau_1, \tau_2) = (\bar{q}q)^{-\frac{1}{24}} \left[ \text{Tr}_{\mathcal{H}_{\text{c.o.m.}}} e^{-\tau_2 \pi \alpha' p^2} \right] \left[ \text{Tr}_{\mathcal{H}_{\text{osc.}}} \bar{q}^{\bar{N}} q^N \right]. \quad (2.3.35)$$

Acting with the operator  $\alpha_{-n}$  produces an excited state with energy  $n$ . Acting  $K$  times creates a state with energy  $Kn$ . Thus, for a single oscillator, the trace over the positive modes yields

$$\sum_{K=0}^{\infty} \langle 0 | \left(\alpha_n^i\right)^K q^N \left(\alpha_{-n}^i\right)^K | 0 \rangle = \sum_{K=0}^{\infty} q^{nK} = \frac{1}{1 - q^n}. \quad (2.3.36)$$

The Fock space for a single scalar field is built by the repeated application of such operators, (and the equivalent tilde operators for the left-movers), contained within

$L_0$  ( $\bar{L}_0$ ), each bearing different mode numbers,  $n \in \mathbb{Z}^+$ .

$$\mathcal{Z}_{\text{osc.}} = \text{Tr}_{\mathcal{H}_{\text{osc.}}} \bar{q}^{\bar{N}} q^N = \prod_{n=1}^{\infty} \frac{1}{1 - \bar{q}^n} \prod_{m=1}^{\infty} \frac{1}{1 - q^m} = \frac{(\bar{q}q)^{1/24}}{\bar{\eta}(\tau)\eta(\tau)}. \quad (2.3.37)$$

Thus, the contribution from the left- and right-moving oscillators is encoded in a generating function precisely of the form eq.(2.3.33).

The zero mode  $p$  of a scalar field yields [5]

$$\mathcal{Z}_{\text{c.o.m.}}(\tau) = \int \frac{dp}{2\pi} \langle p | e^{-\pi\alpha'\tau_2 p^2} | p \rangle = \int \frac{dp}{2\pi} e^{-\pi\alpha'\tau_2 p^2} = \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}}. \quad (2.3.38)$$

Thus, inserting the zero mode and oscillator contributions into eq.(2.3.35), the constant factors of  $\bar{q}q$  cancel, such that the partition function integrand for a single free scalar takes the form

$$\mathcal{Z}_{\text{scalar}}(\tau, \bar{\tau}) = \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \frac{1}{\bar{\eta}(\bar{q})} \frac{1}{\eta(q)}. \quad (2.3.39)$$

Working in lightcone gauge, there are contributions from the 24 oscillator modes, which are given by the  $\eta$ -functions, yet all 26 zero modes contribute a term equal to the prefactor in eq.(2.3.39). It is necessary to divide by the volume of the conformal Killing group,  $4\pi^2\tau_2$ , and to sum over all inequivalent tori, equivalent to integrating over the fundamental domain of the modular group, eq.(2.3.24). Extracting two factors of  $\tau_2$  allows the integral to be written with the  $SL(2, \mathbb{Z})/\mathbb{Z}_2$  modular invariant measure. Thus, in terms of  $\eta(q)$ ,  $\bar{\eta}(\bar{q})$ , the modular invariant one-loop partition function for the bosonic string can be expressed as [122]

$$Z_{\text{string}}(\tau, \bar{\tau}) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left( \frac{1}{\sqrt{4\pi^2\alpha'\tau_2}} \frac{1}{\bar{\eta}(\bar{q})} \frac{1}{\eta(q)} \right)^{24}. \quad (2.3.40)$$

The partition function can easily be seen to be modular invariant by using the modular transformations of the  $\eta$  function,

$$\eta(\tau + 1) = e^{2\pi i/24} \eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \quad (2.3.41)$$

Thus, the integrand in eq.(2.3.40) transforms as (ignoring constant factors)

$$\begin{aligned} \mathcal{Z}(\tau + 1) &= \frac{1}{e^{-2\pi i} e^{2\pi i}} \left( \frac{1}{\sqrt{\tau_2}} \frac{1}{\bar{\eta}(\bar{q})} \frac{1}{\eta(q)} \right)^{24} = \mathcal{Z}(\tau), \\ \mathcal{Z}(-1/\tau) &= \left( \frac{\sqrt{\bar{\tau}\tau}}{\sqrt{\tau_2}} \frac{1}{\sqrt{i\bar{\tau}}\bar{\eta}(\bar{q})} \frac{1}{\sqrt{-i\tau}\eta(q)} \right)^{24} = \mathcal{Z}(\tau). \end{aligned} \quad (2.3.42)$$

### 2.3.5 Fermionic and superstring partition functions

The worldsheet of the closed string takes the form of a genus  $g$  Riemann surface,  $\Sigma_g$ , which is defined by its number of holes. There exist two non-contractible loops associated with each of the  $g$  holes. Spinor fields which live on  $\Sigma_g$  surfaces can be assigned either periodic or anti-periodic boundary conditions around the  $2g$  cycles. The set of boundary condition assignments defines the spin structure for spinor fields [11]. Riemann surfaces admit of more than one spin structure. Modular transformations permute the different spin structures amongst themselves. Thus, when calculating the fermionic contribution to, for example, the partition function, it is necessary to sum over the different sets of boundary conditions which give rise to the different spin structures on the Riemann surface. Modular invariance fixes the relative phases between the different contributions.

In particular, the two cycles  $\sigma_1, \sigma_2$  which correspond to the genus one torus worldsheet are shown in Figure 2.5. The possible boundary conditions give rise to four distinct spin structures.

$$\begin{aligned}\psi(\sigma_1 + 2\pi, \sigma_2) &= \pm\psi(\sigma_1, \sigma_2), \\ \psi(\sigma_1, \sigma_2 + 2\pi) &= \pm\psi(\sigma_1, \sigma_2).\end{aligned}\tag{2.3.43}$$

Periodic and anti-periodic boundary conditions in  $\sigma_1$  correspond respectively to the Ramond and Neveu-Schwarz sectors. The boundary conditions for the two cycles are recorded in the form  $(\pm, \pm)$ .

The closed string partition function factorises in to traces over the left- and the right-moving Hilbert spaces:

$$\begin{aligned}\mathcal{Z}(\tau) &= (4\pi^2\alpha'\tau_2)^{-4} \text{Tr}_{\overline{\mathcal{H}}_L} \bar{q}^{H_L} \text{Tr}_{\mathcal{H}_R} q^{H_R} \\ &= (4\pi^2\alpha'\tau_2)^{-4} \text{Tr}_{\overline{\mathcal{H}}_L} \bar{q}^{\tilde{N}_{l.c.} + \tilde{E}_0} \text{Tr}_{\mathcal{H}_R} q^{N_{l.c.} + E_0}.\end{aligned}\tag{2.3.44}$$

It is thus possible to treat both the left- and the right-moving contributions independently. Consider the right-movers. Following the convention of [11] and denoting by  $\chi_{(\pm, \pm)}^F(\tau)$  the contribution to the partition function from the right-moving fermions with the different spin structures, states in the R and NS sectors contribute [11, 130]<sup>18</sup>

$$\begin{aligned}\chi_{(+, +)}^F(\tau) &\simeq \text{Tr} q^{H_R} (-1)^F, \\ \chi_{(+, -)}^F(\tau) &\simeq \text{Tr} q^{H_R}, \\ \chi_{(-, -)}^F(\tau) &\simeq \text{Tr} q^{H_{NS}},\end{aligned}$$

<sup>18</sup>As clarified in §2.2.5, the regular script R is used to refer to the Ramond sector, while the italicised  $R$  denotes a right-moving state.

$$\chi_{(-,+)}^{\text{F}}(\tau) \simeq \text{Tr} q^{H_{\text{NS}}} (-1)^F, \quad (2.3.45)$$

where the indeterminacy corresponds to unfixed phases which can be determined by the requirement that the expressions be modular invariant. Expressions for the R and NS Hamiltonians with their explicit vacuum energies can be extracted from eq.(2.2.42)

$$\begin{aligned} H_{\text{l.c.}}^{(\text{NS})} &= H_{\text{l.c.,L}}^{(\text{NS})} + H_{\text{l.c.,R}}^{(\text{NS})} = \sum_{i=1}^{D-2} \sum_{r=\frac{1}{2}}^{\infty} r \left( \tilde{b}_{-r} \cdot \tilde{b}_r + b_{-r} \cdot b_r \right) - \frac{1}{6} - \frac{1}{6}, \\ H_{\text{l.c.}}^{(\text{R})} &= H_{\text{l.c.,L}}^{(\text{R})} + H_{\text{l.c.,R}}^{(\text{R})} = \sum_{i=1}^{D-2} \sum_{r=1}^{\infty} r \left( \tilde{b}_{-r} \cdot \tilde{b}_r + b_{-r} \cdot b_r \right) + \frac{1}{3} + \frac{1}{3}, \end{aligned} \quad (2.3.46)$$

where the normal fermionic normal ordering constants have been extracted from eq.(2.2.34) as

$$\begin{aligned} -\tilde{a}_{\text{NS}}^{\text{F}} &= -a_{\text{NS}}^{\text{F}} = \sum_{i=1}^{D-2} \frac{1}{2} \left( -\sum_{r=\frac{1}{2}}^{\infty} r \right) = \frac{D-2}{2} \left( -\frac{1}{24} \right) = -\frac{(D-2)}{48} \stackrel{D=10}{=} -\frac{1}{6}, \\ -\tilde{a}_{\text{R}}^{\text{F}} &= -a_{\text{R}}^{\text{F}} = \sum_{i=1}^{D-2} \frac{1}{2} \left( -\sum_{r=1}^{\infty} r \right) = \frac{D-2}{2} \left( \frac{1}{12} \right) = \frac{(D-2)}{12} \stackrel{D=10}{=} \frac{1}{3}. \end{aligned} \quad (2.3.47)$$

The Hilbert space of a single right-moving fermionic oscillator,  $\psi_r^i$ , is comprised of the vacuum  $|0\rangle$ , and the excited state  $\psi_{-r}^i |0\rangle$ . Thus,

$$\text{Tr}_{\mathcal{H}_R} q^{N_{\text{l.c.}} + E_0} = q^{E_0} (1 + q^r). \quad (2.3.48)$$

The Fock space is built by taking the product over all mode numbers  $r$ . In light-cone gauge, 8 fermions contribute to each sector. Thus, for example, using the definitions for  $\vartheta$ -functions with general characteristics defined in eq.(B.0.3), the  $\chi_{(-,-)}^{\text{F}}$  term, which involves a trace over the NS Hamiltonian, contributes

$$\chi_{(-,-)}^{\text{F}}(\tau) = q^{-\frac{1}{6}} \prod_r (1 + q^r)^8 = \frac{\vartheta_3^4(0, \tau)}{\eta^4(\tau)}. \quad (2.3.49)$$

The conventions for the Jacobi  $\vartheta$ -functions, which represent a powerful tool in the manipulation of string theory partition functions, can be found in Appendix B.

The fermion boundary conditions in eq.(2.3.43) can be expressed in terms of the general boundary conditions  $\alpha, \beta \in [0, \frac{1}{2}]$ :

$$\begin{aligned} \psi(\sigma_1 + 2\pi, \sigma_2) &= -e^{2\pi i \alpha} \psi(\sigma_1, \sigma_2) = \pm \psi(\sigma_1, \sigma_2), \\ \psi(\sigma_1, \sigma_2 + 2\pi) &= -e^{2\pi i \beta} \psi(\sigma_1, \sigma_2) = \pm \psi(\sigma_1, \sigma_2). \end{aligned} \quad (2.3.50)$$

Table 2.7 records the different spin structures in terms of the phases  $\alpha, \beta$ , and in terms of the short-hand  $\vartheta$ -function notation defined in Appendix B. Modular

(R)	(+, +)	$\alpha = \frac{1}{2}$	$\beta = \frac{1}{2}$	$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vartheta_1$
	(+, -)	$\alpha = \frac{1}{2}$	$\beta = 0$	$\vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \vartheta_2$
(NS)	(-, -)	$\alpha = 0$	$\beta = 0$	$\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vartheta_3$
	(-, +)	$\alpha = 0$	$\beta = \frac{1}{2}$	$\vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} = \vartheta_4$

Table 2.7: The different spin structures in the different sectors of the superstring, and the corresponding  $\vartheta$ -function encoding of the trace over the states in the individual sectors.

invariance fixes the phases in the various sectors, such that the total right-moving fermionic contribution to the partition function is given by

$$\begin{aligned} \chi^F(\tau) &= \text{Tr} \left( q^{H_{1.c.}^{(\text{NS})}} \cdot \frac{(1 - (-1)^F)}{2} \right) - \text{Tr} \left( q^{H_{1.c.}^{(\text{R})}} \cdot \frac{(1 - \eta_{++}(-1)^F)}{2} \right) \\ &= \frac{1}{2\eta^4(\tau)} \left[ \vartheta_3^4(\tau) - \vartheta_4^4(\tau) - \vartheta_2^4(\tau) + \eta_{++} \vartheta_1^4(\tau) \right], \end{aligned} \quad (2.3.51)$$

where  $\eta_{++}$  is a to be determined phase. Note that  $\vartheta_1$  vanishes.

In the type II 10D superstring theory, the 8 + 8 left- and right-moving transverse bosonic operators contribute, using eq.(2.3.39),

$$\mathcal{Z}(\tau, \bar{\tau}) = \frac{1}{(4\pi^2 \alpha' \tau_2)^4} \frac{1}{|\eta(\tau)|^{16}}. \quad (2.3.52)$$

Coupled with the contribution from the left- and the right-moving fermions, the total partition function for the type II theory is given by

$$\mathcal{Z}(\tau, \bar{\tau}) = \frac{1}{(4\pi^2 \alpha' \tau_2)^4} \frac{1}{|\eta(\tau)|^{24}} |\vartheta_3^4(\tau) - \vartheta_4^4(\tau) - \vartheta_2^4(\tau)|^2. \quad (2.3.53)$$

## 2.4 Compactification of the background space time

By way of introduction, consider compactifying a single *spacetime* dimension of a generic  $D$ -dimensional field theory, with the compact dimension forming a circle of radius  $R$ , such that the background takes the form  $M_d \times \mathbf{S}^1$ , with  $D = d + 1$ . Thus,

$$\phi(x^0, \dots, x^{d-1}, x^{D-1}) = \phi(x^0, \dots, x^{d-1}, x^{D-1} + 2\pi R). \quad (2.4.1)$$

The  $x^{D-1}$  dependence of a scalar field  $\phi$  can be expanded in Fourier modes around the circle of circumference  $L = 2\pi R$  as

$$\phi(x^0, \dots, x^{d-1}, x^{D-1}) = \sum_{k \in \mathbb{Z}} e^{2\pi i k x^{D-1}/L} \phi_k(x^0, \dots, x^{d-1}). \quad (2.4.2)$$

The kinetic terms for the scalar are (where  $M = 0, \dots, D-1$ ,  $\mu = 0, \dots, d-1$ )

$$\begin{aligned} \int_{M_d \times \mathbf{S}^1} d^D x \partial_M \phi \partial^M \phi^* &= \int d^D x \left( \partial_\mu \phi \partial^\mu \phi^* + |\partial_{(D-1)} \phi|^2 \right) \\ &= 2\pi R \int d^d x \sum_{n=-\infty}^{\infty} \left( \partial_\mu \phi_k \partial^\mu \phi_k^* + \frac{k^2}{R^2} |\phi_k|^2 \right). \end{aligned} \quad (2.4.3)$$

The infinite tower of so called Kaluza-Klein (KK) states on  $M_{d-1} \times \mathbf{S}^1$  have mass

$$M_{KK}^2 = \frac{k^2}{R^2}. \quad (2.4.4)$$

Now specifically consider 26-dimensional bosonic string theory defined in the spacetime  $\mathbf{R}^{1,24} \times \mathbf{S}^1$ ; that is, consider the KK reduction on the string worldsheet [122, 137, 138].<sup>19</sup> Explicitly, the set of 26 coordinates  $X^0, \dots, X^{25}$  are split up in to light-cone coordinates  $X^+$ ,  $X^-$ , transverse coordinates  $X^I$ ,  $I = 1, \dots, D-3$  (that is, one fewer than in the non-compact case), and the compact coordinate  $X^{25}$ :

$$X^+, X^-, \{X^I\}, X^{25} \quad I = 1, \dots, D-3. \quad (2.4.5)$$

The periodic nature of  $X^{25}$  results in two novel types of quantized momenta for strings propagating in this background. First, the string wavefunction includes a factor of  $e^{ip \cdot X}$ , which must be single valued under a translation  $X^{25} \rightarrow X^{25} + 2\pi R$  around the periodic dimension. Thus, in terms of the KK number  $m$ ,

$$p^{25} = \frac{m}{R}, \quad m \in \mathbb{Z}. \quad (2.4.6)$$

The KK momentum is inversely proportional to the radius of compactification,  $R$ ; decreasing the wavelength of a KK mode results in a more energetic state. The infinite range of values of  $m$  generates a KK tower of momentum states. Second, closed strings can wrap around the compact dimension, such that the periodicity boundary condition in the compact dimension is

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi n R, \quad n \in \mathbb{Z}. \quad (2.4.7)$$

An energy, which is directly proportional to the radius of compactification  $R$ , is associated with the tension of the wound string; a string which is wound round a

<sup>19</sup>Following the literature, the light-cone coordinates will be denoted using general  $D$ , though concretely,  $D = 26$  in this example, such that the compact coordinate takes the superscript 25.

greater distance (larger  $R$ ) engenders a more energetic state.

Following the logic of §2.1, the mode expansions for the left- and right-moving components of the periodic coordinate,  $X_L^{25}(\sigma^+)$ ,  $X_R^{25}(\sigma^-)$ , take the same form as eq.(2.1.18). As in the non-compact case, the momentum corresponds to the zero modes

$$\tilde{\alpha}_0^{25} + \alpha_0^{25} = \sqrt{2\alpha'} p^{25}. \quad (2.4.8)$$

However, eq.(2.4.7) means that the zero modes, being coupled to  $\sigma^+$ ,  $\sigma^-$ , are no longer equal, but are related by

$$\tilde{\alpha}_0^{25} - \alpha_0^{25} = \sqrt{\frac{2}{\alpha'}} nR. \quad (2.4.9)$$

Thus

$$\tilde{\alpha}_0^{25} = \sqrt{\frac{\alpha'}{2}} \left( p^{25} + \frac{nR}{\alpha'} \right), \quad \alpha_0^{25} = \sqrt{\frac{\alpha'}{2}} \left( p^{25} - \frac{nR}{\alpha'} \right). \quad (2.4.10)$$

The similarity in structure of these equations suggests that it is natural to interpret the winding as a momentum, such that both the KK terms and the winding contribute to the momenta in the  $X^{25}$  dimension. Thus, using  $\tilde{\alpha}_0^{25} = \alpha_0^{25} \equiv \sqrt{\frac{\alpha'}{2}} p^{25}$ , eq.(2.4.10) defines the left- and right-moving quantized momenta in the compact dimension:

$$p_{L/R}^{25} = \left( \frac{m}{R} + / - \frac{nR}{\alpha'} \right), \quad (2.4.11)$$

$p_{L/R}^{25}$  will henceforth be denoted as  $p_{L/R}$  for convenience of notation. Thus the mode expansions differ from their non-periodic counterparts, eq.(2.1.18), only in the form of the momentum terms  $p_{L/R}^{25}$  coupled to the light-cone worldsheet coordinates,  $\sigma^\pm$ ,

$$\begin{aligned} X_L^{25}(\tau + \sigma) &= \frac{1}{2} x_{L(0)}^{25} + \frac{\alpha'}{2} p_L(\tau + \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^{25}}{n} e^{-in(\tau + \sigma)}, \\ X_R^{25}(\tau - \sigma) &= \frac{1}{2} x_{R(0)}^{25} + \frac{\alpha'}{2} p_R(\tau - \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_n^{25}}{n} e^{-in(\tau - \sigma)}, \end{aligned} \quad (2.4.12)$$

Combining these expressions yields the full coordinate for the compact field

$$X^{25}(\tau, \sigma) = x_{(0)}^{25} + \frac{\alpha' m}{R} \tau + nR\sigma + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left[ \tilde{\alpha}_n^{25} e^{-in(\tau + \sigma)} + \alpha_n^{25} e^{-in(\tau - \sigma)} \right]. \quad (2.4.13)$$

Solving the constraint equations as for the non-compact background yields expressions for the compact Virasoro operators  $\bar{L}_0$ ,  $L_0$ , which, using the mode expansions eq.(2.4.12), can be seen to differ from those in the non-compact theory, eq.(2.1.33),

by factors of  $p_R^2, p_L^2$ <sup>20</sup>:

$$\begin{aligned}\bar{L}_0 &= \sum_{i=1}^{D-2} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i + \sum_{I=1}^{D-3} \frac{1}{2} (\tilde{\alpha}_0^I)^2 + \frac{\alpha'}{4} p_L^2, \\ L_0 &= \sum_{i=1}^{D-2} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i + \sum_{I=1}^{D-3} \frac{1}{2} (\alpha_0^I)^2 + \frac{\alpha'}{4} p_R^2.\end{aligned}\quad (2.4.14)$$

Note that the sum over the non-zero modes is over both the compact and non-compact transverse coordinates, denoted by  $i = 1, \dots, D-2$ , whereas the zero mode contributions are split in to those from the non-compact ( $I = 1, \dots, D-3$ ) and the compact ( $X^{25}$ ) coordinates. As before, the zero modes (of the non-compact dimensions) are related to the spacetime momentum  $\tilde{\alpha}_0^I = \alpha_0^I = \sqrt{\frac{\alpha'}{2}} p^I$ . In analogy with eq.(2.1.52), the expression for the mass squared in the 25-dimensional spacetime, modified by the Kaluza-Klein and winding terms, thus takes the form

$$\begin{aligned}M^2 &= \frac{2}{\alpha'} (\bar{L}_0 + L_0 - 2) - (p^I)^2 = \frac{1}{2} (p_L^2 + p_R^2) + \frac{2}{\alpha'} (\bar{N}_{l.c.} + N_{l.c.} - 2) \\ &= \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} (\bar{N}_{l.c.} + N_{l.c.} - 2),\end{aligned}\quad (2.4.15)$$

where the summation in the light-cone number operators  $\bar{N}_{l.c.}, N_{l.c.}$  is over both the compact ( $X^{25}$ ) and non-compact ( $X^I$ ) dimensions (that is, the full set of transverse dimensions), as in eq.(2.1.49). In the interests of clarity, the terms in the final bracket of eq.(2.4.15) will henceforth be referred to as ‘oscillator contributions’. Splitting the mass into left- and right-moving components

$$M_L^2 = \frac{p_L^2}{2} + \frac{2}{\alpha'} (\bar{N}_{l.c.} - 1), \quad M_R^2 = \frac{p_R^2}{2} + \frac{2}{\alpha'} (N_{l.c.} - 1). \quad (2.4.16)$$

Level matching, which requires that  $M_L^2 = M_R^2$ , implies that

$$N_{l.c.} - \bar{N}_{l.c.} = nm. \quad (2.4.17)$$

The Hamiltonian for a compact model differs from its non-compact counterpart only by the winding and KK terms in eq.(2.4.15). The non-compact Hamiltonian in

<sup>20</sup>In order to make contact with the literature (e.g. [14]), the compact operators (which are the sole form of the Virasoro operators to which will be referred from now on), will henceforth be denoted by the barred and unbarred operators  $\bar{L}_0, L_0$ , in contrast to the non-compact operators,  $\tilde{L}_0, L_0$ , presented in §2.1. Evidently, there exists no difference in notation for the right-movers between the compact and the non-compact cases. However, in all subsequent expressions,  $\bar{L}_0$  and  $L_0$  will always appear in conjunction, making clear the fact that the compact versions of the operators are those to which are being referred. Note also that, again following the convention in the literature, the operators  $\bar{L}_0, L_0$  appeared in the context of the discussion of conformal field theory, §2.3, despite the fact that the presentation in no way concerns compactifications.

eq.(2.1.30) can also be expressed as

$$H_{1.c.} = \frac{1}{2} \int_0^{2\pi} d\sigma \left[ 2\pi\alpha' \Pi^i \Pi^i + \frac{1}{2\pi\alpha'} \partial_\sigma X^i \partial_\sigma X^i \right], \quad (2.4.18)$$

where  $i = 1, \dots, 24$  encompasses all transverse (compact and non-compact) coordinates. Thus, it is clear that the compact mode expansion eq.(2.4.13) introduces two new terms in to  $H_{1.c.}$ , corresponding to the terms linear in  $\tau$  and  $\sigma$ . Inserting the mode expansions  $X^{25}$  and their conjugate momenta  $\Pi^{25}$ , in addition to the non-compact coordinates  $X^I$  and their conjugate momenta  $\Pi^I$ ,

$$H_{1.c.} = \frac{\alpha'}{2} \left( \sum_{I=1}^{23} p^I p^I + \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} \right) + \bar{N}_{l.c.} + N_{l.c.} - 2 = \bar{L}_0 + L_0 - 2. \quad (2.4.19)$$

As in eq.(2.3.1), the left- and right-moving light-cone Hamiltonians, related by  $H_{1.c.} = H_L + H_R$ , are useful when constructing the partition function:

$$\begin{aligned} H_L &= \frac{\alpha'}{4} \left( \sum_{I=1}^{23} p^I p^I + p_L^2 \right) + \bar{N}_{l.c.} - 1 = \bar{L}_0 + \tilde{E}_0, \\ H_R &= \frac{\alpha'}{4} \left( \sum_{I=1}^{23} p^I p^I + p_R^2 \right) + N_{l.c.} - 1 = L_0 + E_0. \end{aligned} \quad (2.4.20)$$

Similarly, the momentum, defined by

$$P_{1.c.} = \int_0^{2\pi} d\sigma \Pi_i \partial_\sigma X^i, \quad (2.4.21)$$

takes the form

$$P_{1.c.} = L_0 - \bar{L}_0 + mn. \quad (2.4.22)$$

Thus, in analogy with eq.(2.3.26), the partition function for the compact theory is modified to

$$\mathcal{Z}(\tau_1, \tau_2) = \text{Tr} \left( e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} \right) = \text{Tr} \left( e^{-2\pi\tau_2 [L_0 + \bar{L}_0]} e^{2\pi i \tau_1 [L_0 - \bar{L}_0]} \right), \quad (2.4.23)$$

where the modification is that the compact  $\bar{L}_0, L_0$  take the form given in eq.(2.4.14).

Taking the expression for the partition function, eq.(2.3.1), written in terms of the left- and right-moving Hamiltonians, and inserting the expressions in eq.(2.4.20), the compact partition function modifies eq.(2.3.40) to

$$\mathcal{Z} = \text{Tr}_{\mathcal{H}_L} (\bar{q}^{H_L}) \text{Tr}_{\mathcal{H}_R} (q^{H_R}) = (2\pi\alpha'\tau_2)^{-\frac{23}{2}} |\eta(\tau)|^{-48} \sum_{p_L, p_R} \bar{q}^{\frac{p_L^2}{2}} q^{\frac{p_R^2}{2}}, \quad (2.4.24)$$

where the sum is taken over the compact momenta  $p_L, p_R$ , as defined in eq.(2.4.11). As in the non-compact case, the first term corresponds to the  $25D$  centre of mass momentum, while the  $\eta$ -functions encode the oscillator trace. Note that generally,

the partition function of the bosonic string compactified on a torus  $T^d$  has a centre of mass prefactor, corresponding to the transverse non-compact momenta, proportional to  $\tau_2^{-(24-d)/2}$ . The novel feature in the compact theory is the trace over the internal KK and winding degrees of freedom. Explicitly,<sup>21</sup>

$$\mathcal{Z} = (2\pi\alpha'\tau_2)^{-\frac{23}{2}} |\eta(\tau)|^{-48} \sum_{m,n=-\infty}^{\infty} \exp \left[ -\pi\tau_2 \left( \frac{m^2}{R^2} + \frac{R^2 n^2}{\alpha'^2} \right) - 2\pi i \tau_1 \frac{mn}{\alpha'} \right], \quad (2.4.25)$$

Consider redefining

$$R \rightarrow \tilde{R} = \alpha'/R. \quad (2.4.26)$$

Eq.(2.4.15) is invariant under this exchange so long as  $m$  and  $n$  are also exchanged. Thus, the spectrum for the string moving in a circularly compactified geometry is equivalent to that for a string moving on a circle of dual radius  $\alpha'/R$ , with the winding and KK number reinterpreted as each other. A complete analysis reveals that the theories defined on the dual radii are entirely equivalent at the level full conformal field theory [139]. The so called ‘T-duality’ represents an exact quantum symmetry of perturbative closed strings.

## 2.5 Heterotic string theory

The set of consistent string theories is summarized in Table 2.8. There exist five consistent superstring theories, each exhibiting some overlapping and some differing properties. The two type II theories were discussed in §2.2.5. Having introduced the one-loop string partition function and having reviewed the theory behind compactification, it is now possible to introduce the remaining two of the four theories of oriented closed strings, the heterotic string models, which, albeit not uniquely among the superstring theories detailed in Table 2.8, give rise to chiral fermions and non-abelian vector gauge bosons.

Other than the level-matching constraint which must be satisfied by physical states, the left- and right-moving sectors of any closed string theory are completely decoupled. It is thus possible to construct theories which contain fermions only in the right-moving string sectors. The left-moving sector of these so called heterotic theories, introduced in [142–144], constitutes a  $D = 26$  bosonic string theory, while the right-moving states constitute superstrings propagating in 10 dimensions. Heterotic theories exhibit a 10-dimensional  $\mathcal{N} = (1, 0)$  supersymmetry associated with gravitinos of a given chirality, generated by 16 supercharges, which is manifest in the

<sup>21</sup>Note that the sign of the complex term in the exponential depends upon the convention taken for the left- and the right-moving sectors. Had the  $\tilde{\alpha}_L^\mu$  operators been associated with the right-moving rather than the left-moving sector, as has been the case throughout this study, the complex term in the exponential would have been positive, as per [5].

Type	$D$	SUSY generators	Chiral	Open strings	Gauge group
Bosonic (closed)	26	$\mathcal{N} = 0$	No	No	None
Bosonic (open)	26	$\mathcal{N} = 0$	No	Yes	$U(1)$
I	10	$\mathcal{N} = (1, 0)$	Yes	Yes	$SO(32)$
IIA	10	$\mathcal{N} = (1, 1)$	No	No	$U(1)$
IIB	10	$\mathcal{N} = (2, 0)$	Yes	No	None
HO	10	$\mathcal{N} = (1, 0)$	Yes	No	$SO(32)$
HE	10	$\mathcal{N} = (1, 0)$	Yes	No	$E_8 \times E_8$
M-theory	11	$\mathcal{N} = 1$	No	No	None

Table 2.8: A summary of the consistent string theories. Purely bosonic theories were discussed in §2.1. The  $\mathcal{N} = 1$  supersymmetry of the ‘type I’ string is generated by 16 supercharges. This is the only theory of both open and closed strings. The ‘type II’ strings, defined in §2.2.6, are differentiated by whether the two fermionic generators, which give rise to the 32 supercharges, are of the same, or opposite, chirality. ‘Type IIA/B’ strings correspond to the handedness of the gravitinos being equal / opposite. Finally, the two heterotic theories, to be introduced presently, are comprised of a left-moving bosonic string and right-moving superstring, and are distinguished by their 10-dimensional gauge groups. Each of these five string theories can be considered to arise as weak coupling limits of the more fundamental M-theory [123, 140, 141].

supersymmetric right-moving spectrum<sup>22</sup>. In order that the left- and right-moving sectors consistently propagate in 10-dimensional spacetime, (that is, in order that the theory be modular invariant), 16 of the left-moving dimensions must be compactified.

The spacetime fields in the  $10D$  theory,

$$X^\mu[(\tau + \sigma), (\tau - \sigma)], \quad \text{and} \quad \psi_R^\mu(\tau - \sigma), \quad (2.5.1)$$

give rise to a total central charge  $(c_L^{[X]}, c_R^{[X], [\psi]}) = (10, 15)$ . Given that there exist bosonic degrees of freedom in both sectors of the theory,  $b$  and  $c$  ghosts are introduced for both left- and right-movers. Conversely, it is only necessary to introduce  $\beta$  and  $\gamma$  ghosts for the spacetime right-moving fermions. The ghost and superghost

<sup>22</sup>Note that a single supersymmetry  $\mathcal{N} = (1, 0)$  is often denoted  $\mathcal{N} = 1$ .

central charges for the bosonic left- and the supersymmetric right-moving theories sum to  $(c_L^{[b],[c]}, c_R^{[b],[c],[\beta],[\gamma]}) = (-26, -15)$ . The choice of the remaining left-moving matter fields is thus constrained by the requirement that the rest of the theory have  $(c_L, c_R) = (16, 0)$  which ensures the cancellation of the conformal anomaly.

Consider first the bosonic construction of the heterotic string. For concreteness, in light-cone gauge, the fields in the right-moving sector comprise 8 bosons,  $X_R^i(\tau - \sigma)$ , and 8 superpartner fermions,  $\psi_R^i(\tau - \sigma)$ ,  $i = 1, \dots, 8$ . It is convenient to split the 24 non-light-cone bosons of the left-moving sector into two sets; 8 dynamic coordinates,  $X_L^i(\tau + \sigma)$ , and 16 coordinates  $X_L^I(\tau + \sigma)$ ,  $I = 1, \dots, 16$ , which must be compactified on a 16-dimensional torus. These 16 bosonic coordinates satisfy the requirement that  $c_L^{\text{total}} = 0$ . In summary, the fields comprising the 10 physical spacetime dimensions are (in light-cone gauge)

$$\left. \begin{aligned} X^i(\tau, \sigma) &= X_L^i(\tau + \sigma) + X_R^i(\tau - \sigma), \\ \psi^i(\tau, \sigma) &= \psi_R^i(\tau - \sigma), \end{aligned} \right\} i = 1, \dots, 8. \quad (2.5.2)$$

The mode expansion for the compact left-moving fields is given by the left-moving expression in eq.(2.4.12). The compactification on a 16-dimensional torus yields a discrete momentum spectrum for the left-moving bosons  $X_L^I(\tau + \sigma)$ . Being discrete, the momenta of these bosons,  $p_L^I$ , live on a 16-dimensional lattice, denoted  $\Lambda_{16}$ , which is spanned by the basis vectors  $e_i^I$  [11, 124]<sup>23</sup>

$$\mathbf{p}_L \in \Lambda_{16}, \quad p_L^I = \sum_{j=1}^{16} p_j e_j^I, \quad p_j \in \mathbb{Z}. \quad (2.5.3)$$

Recalling eq.(2.3.1), for the heterotic theory

$$\mathcal{Z}(\tau, \bar{\tau})_{\text{het}} = \text{Tr}_{\mathcal{H}_{\text{het}}} (-1)^F \bar{q}^{H_L} q^{H_R}. \quad (2.5.4)$$

Owing to the 16 compact dimensions of the left-moving theory, the expression for the left-moving Hamiltonian contains exponents of the quantized momenta as in eq.(2.4.20). Meanwhile, the Hamiltonian for the supersymmetric right-movers contains the fermionic number operators, which are dependent upon the periodicity conditions. Thus, the left- and right-moving light-cone Hamiltonians are [11]

$$\begin{aligned} H_L &= \frac{1}{2}(p^i)^2 + \tilde{N}_{L(\text{l.c.})}^{(\text{B})} + \frac{1}{2}(p_L^I)^2 + \tilde{E}_0, \\ H_R &= \frac{1}{2}(p^i)^2 + N_{R(\text{l.c.})}^{(\text{B})} + N_{R(\text{l.c.})}^{(\text{F})} + E_0, \end{aligned} \quad (2.5.5)$$

where as usual,  $p^i$ ,  $i = 1, \dots, D - 2$ , represent the transverse components of the

<sup>23</sup> $\mathbf{p}_L$  denotes the *set* of 16 internal left-moving momenta [11].

spacetime momentum.

The partition function for the heterotic string factorises into separate terms corresponding to the traces over the different states (bosonic and fermionic) in the different sectors (left and right) of the theory [5]:

$$Z(\tau, \bar{\tau}) = \frac{1}{(4\pi\alpha'\tau_2)^4} \frac{1}{|\eta(\tau)|^{16}} Z_\psi(\tau) \bar{Z}_{X^I}(\bar{\tau}). \quad (2.5.6)$$

Recalling the treatment surrounding eq.(2.3.39), the  $\tau_2$  prefactor arises from the zero modes of the uncompactified transverse bosonic coordinates, which correspond to the centre of mass momentum. 8 factors each of  $\eta, \bar{\eta}$  arise from the trace over the oscillators of the left- and right-moving dynamic bosons  $X_L^i, X_R^i$ . In terms of the sum over the fermionic spin structures, the trace over the worldsheet fermionic oscillators in  $\psi_R^i$  takes the form given in eq.(2.3.51):

$$\begin{aligned} Z_\psi &= \frac{1}{\eta^4} \sum_{a,b=0}^1 e^{i\pi(a+b+ab)} \vartheta^4 \begin{bmatrix} a/2 \\ b/2 \end{bmatrix} = \frac{1}{\eta^4} \left( \vartheta^4 \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \vartheta^4 \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} - \vartheta^4 \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \vartheta^4 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \right) \\ &= \frac{1}{\eta^4} \left( \vartheta_3^4(\tau) - \vartheta_4^4(\tau) - \vartheta_2^4(\tau) \right). \end{aligned} \quad (2.5.7)$$

Jacobi's abstruse identity, (see Appendix B), guarantees that the one-loop partition vanishes for theories that exhibit spacetime supersymmetry.

Reminiscent of the structure of the partition function in eq.(2.3.1), the trace over the oscillators and left-momentum for the 16  $X_L^I$  contains a lattice sum [11]

$$\bar{Z}_{X^I}(\bar{\tau}) = \frac{1}{\bar{\eta}(\bar{\tau})^{16}} \sum_{\mathbf{p}_L \in \Lambda_{16}} \bar{q}^{\frac{1}{2} \mathbf{p}_L^2}. \quad (2.5.8)$$

One-loop modular invariance of the partition function, which is ultimately responsible for ensuring that the theory remain anomaly free, tightly constrains the type of 16-dimensional lattice upon which heterotic theories can be compactified. First, invariance under T transformations requires that  $\mathbf{p}_L^2 \in 2\mathbb{Z}, \forall \mathbf{p}_L \in \Lambda_{16}$ , constraining  $\Lambda_{16}$  to be even. Second, using the Poisson resummation formula (see Appendix C), invariance under S transformations requires that  $\Lambda_{16}$  be self-dual ( $\Lambda_{16}^* = \Lambda_{16}$ ) [5, 11]. In 8-dimensions,  $\Gamma_8$ , the root lattice of the Lie group  $E_8$ , is the unique even self-dual lattice. Thus, a 16-dimensional lattice can be formed by taking the product  $\Gamma_8 \times \Gamma_8$ . Alternatively,  $\Gamma_{16}$  denotes the weight lattice of  $Spin(32)/\mathbb{Z}_2$ , which contains the root lattice of  $SO(32)$ . The root lattice of  $E_8 \times E_8$  and the weight lattice of  $Spin(32)/\mathbb{Z}_2$  are the roots of  $E_8 \times E_8$  and  $SO(32)$  respectively [11]. The Lie algebras of the two groups are realized in spacetime as gauge symmetries in the form of non-abelian spacetime gauge fields, the gauge bosons. The two even, self-dual lattices, unique to the 16 dimensions corresponding to the compactified

space, differ by their gauge groups in 10 dimensions [5, 11]. In order that the theory be anomaly free, the dimension of the gauge group must be 496. The two consistent choices are  $SO(32)$  and  $E_8 \times E_8$ . These distinct groups define the two types of heterotic theories, sometimes labelled HO and HE, as in Table 2.8. The heterotic theory thus contains a vector multiplet of the gauge group; that is, gauge bosons transforming under the adjoint of either  $SO(32)$  or  $E_8 \times E_8$ . Using Table E.1, it can be seen that for a vector multiplet transforming in the adjoint representation of, for example,  $SO(32)$ , there exist  $32 \times \frac{31}{2} = 496$  states.

Returning to the expressions for  $H_L, H_R$ ; while both the 8 transverse spacetime and the 16 internal left-moving bosonic oscillators contribute to  $\tilde{N}_{L(\text{l.c.})}^{(\text{B})}$ ,  $N_{R(\text{l.c.})}^{(\text{B})}$  is comprised of only the 8 right-moving spacetime bosonic oscillators. The light-cone bosonic number operators take the form of eq.(2.2.33), with the following coordinate summations:

$$\tilde{N}_{L(\text{l.c.})}^{(\text{B})} = \sum_{i=1}^8 \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i + \sum_{I=1}^{16} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^I \tilde{\alpha}_m^I, \quad N_{R(\text{l.c.})}^{(\text{B})} = \sum_{i=1}^8 \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i \quad (2.5.9)$$

For the right-moving fermions

$$N_{R(\text{l.c.})}^{(\text{F})} = \sum_{i=1}^8 \sum_{r \in \mathbb{Z} + \phi \geq 0} r b_{-r}^i b_r^i, \quad (2.5.10)$$

where as usual, the boundary conditions are specified by the value of  $\phi$ . The transverse bosonic fields contribute a factor of  $\tilde{E}_0^{(\text{B})} = E_0^{(\text{B})} = -(D-2)/24$  to the normal ordering constant, while the internal bosons contribute  $\tilde{E}_0^{(\text{B})} = -D/24$ . Thus, the total ordering constant in the left-moving sector is  $-(10-2)/24 - 16/24 = -1$ . The values of  $a_{(\text{NS})} = 1/2$ ,  $a_{(\text{R})} = 0$  for the fermionic fields with different boundary conditions, were found in eq.(2.2.34). The left-moving fermionic ordering constants can thus be denoted as  $E_0^{(\text{F})} = -2\phi(1-\phi)$ , with  $\phi = 0, 1/2$  for the R, NS sectors. As for the superstring, the spectrum is obtained by coupling the left- and the right-moving states, with the physical states being those which satisfy  $m_L^2 = m_R^2$ :

$$\begin{aligned} \frac{\alpha' m_L^2}{2} &= \tilde{N}_{L(\text{l.c.})}^{(\text{B})} + \frac{1}{2}(\mathbf{p}_L^I)^2 - 1, \\ \frac{\alpha' m_R^2}{2} &= N_{R(\text{l.c.})}^{(\text{B})} + N_{R(\text{l.c.})}^{(\text{F})} - 2\phi(1-\phi). \end{aligned} \quad (2.5.11)$$

Permitted states satisfy the level matching constraint

$$\tilde{N}_{L(\text{l.c.})}^{(\text{B})} + \frac{1}{2}(\mathbf{p}_L^I)^2 - 1 = \begin{cases} N_{R(\text{l.c.})}^{(\text{B})} + N_{L(\text{l.c.})}^{(\text{F})}, & \text{R sector.} \\ N_{R(\text{l.c.})}^{(\text{B})} + N_{L(\text{l.c.})}^{(\text{F})} - \frac{1}{2}, & \text{NS sector,} \end{cases} \quad (2.5.12)$$

The right-moving massless states are shown in Tables 2.2 & 2.3. To recap, massless

states in the right-moving sector are generated by the action on the NS ground state of a single half-integer excitation,  $b_{-\frac{1}{2}}^\mu$ , or by the spinor  $|S^\alpha\rangle$  in the R sector. The GSO projection in the NS sector projects out the tachyon, and selects the state  $\mathbf{8}_V$ . Taking  $(-1)^F = +1$  in the R sector selects  $\mathbf{8}_S$ .

A single oscillator excitation acting upon the left-moving ground state yields a zero contribution to the mass squared of a state. The dynamic oscillators,  $\bar{\alpha}_{-1}^\mu |0\rangle$ , transform as spacetime vectors under the gauge group. Conversely, the internal oscillators,  $\bar{\alpha}_{-1}^I |0\rangle$ , transform as an abelian gauge boson, which is a singlet with respect to the subgroup of the full Lorentz group for massless states,  $SO(8)$ . States with non-trivial internal momenta  $\mathbf{p}_L$ ,  $N_L = 0$  also give rise to massless states. The states  $|\mathbf{p}_L^2 = 2\rangle$  generate non-abelian gauge bosons. The tachyonic and massless left-moving states are recorded in Table 2.9.

$\tilde{N}_{L(\text{l.c.})}^{(B)}; \mathbf{p}_L^2$	$ \rangle_L$	$\alpha' m_L^2$	$SO(8)$
$\tilde{N}_{L(\text{l.c.})}^{(B)} = 0; \mathbf{p}_L^2 = 0$	$ 0\rangle_L$	-2	$\mathbf{1}$
$\tilde{N}_{L(\text{l.c.})}^{(B)} = 1; \mathbf{p}_L^2 = 0$	$\tilde{\alpha}_{-1}^i  0\rangle_L$	0	$\mathbf{8}_V$
$\tilde{N}_{L(\text{l.c.})}^{(B)} = 1; \mathbf{p}_L^2 = 0$	$\tilde{\alpha}_{-1}^I  0\rangle_L$	0	$\mathbf{1}$
$\tilde{N}_{L(\text{l.c.})}^{(B)} = 0; \mathbf{p}_L^2 = 2$	$ \mathbf{p}_L^2 = 2\rangle_L$	0	$\mathbf{1}$

Table 2.9: The Table records the tachyonic and massless left-moving states of the heterotic theory, along with their decomposition into irreducible representations of  $SO(8)$  [5]. The states with  $\tilde{N}_{L(\text{l.c.})}^{(B)} = 0$ ,  $|\mathbf{p}_L^2 = 2\rangle_L$ , correspond to a  $(\text{length})^2 = 2$  root vector of either  $E_8 \times E_8$  or  $SP(32)$ , and give rise to non-abelian gauge bosons of these groups.

As stated, the full massless spectrum for the heterotic string is formed by coupling the left-moving states in Table 2.9 with the right moving massless states,  $\mathbf{8}_V$ ,  $\mathbf{8}_S$ . As the left-moving tachyon cannot be level matched with any right-moving state, it is not possible to form a spacetime tachyon in the final spectrum of the  $10D$  theory.

The right-moving ground state labels, (NS) and (R), identify whether the state arises in a bosonic or fermionic sector. The spectrum is comprised of:

- (NS) The 10-dimensional gravity multiplet, comprised of the scalar dilaton  $\phi$ , the antisymmetric tensor  $B_{[\mu\nu]}$  and the graviton  $G_{\mu\nu}$ :

$$\tilde{\alpha}_{-1}^i |0; p\rangle \otimes b_{-\frac{1}{2}}^i |0; p\rangle_{\text{NS}} . \quad (2.5.13)$$

Sector	$ \rangle_L \times  \rangle_R$	$SO(8)$	10d field
NS	$\mathbf{8}_V \times \mathbf{8}_V$	$\mathbf{1} + \mathbf{28}_V + \mathbf{35}_V$	$\phi, B_{[\mu\nu]}, G_{\mu\nu}$
R	$\mathbf{8}_V \times \mathbf{8}_S$	$\mathbf{8}_C + \mathbf{56}_C$	$\lambda_\alpha, \psi_{M\alpha}$
NS	$\tilde{\alpha}_{-1}^I  0\rangle \times \mathbf{8}_V$	$\mathbf{8}_V$	$A_\mu^{(I)}$
NS	$ \mathbf{p}_L^2 = 2\rangle \times \mathbf{8}_V$	$\mathbf{8}_V$	$A_\mu^{(\mathbf{p}_L^2)}$
R	$\tilde{\alpha}_{-1}^I  0\rangle \times \mathbf{8}_S$	$\mathbf{8}_S$	$\lambda_\alpha^{(I)}$
R	$ \mathbf{p}_L^2 = 2\rangle \times \mathbf{8}_S$	$\mathbf{8}_S$	$\lambda_\alpha^{(\mathbf{p}_L^2)}$

Table 2.10: The Table records the massless states in the heterotic theory, along with their decomposition into irreducible representations of  $SO(8)$ , and the 10-dimensional fields to which they give rise [5].

- (R) The superpartners of the gravity multiplet, the gravitino  $\psi_{M\alpha}$  and the dilatino  $\lambda_\alpha$ :

$$\tilde{\alpha}_{-1}^i |0; p\rangle \otimes |S^\alpha\rangle_R . \quad (2.5.14)$$

- (NS) The fields  $A_\mu$  correspond to the afore-described 496 gauge bosons which transform under a spacetime non-abelian gauge symmetry.  $A_\mu^{(I)}$  are gauge bosons of a  $U(1)^{16}$  Cartan subalgebra corresponding to the KK mechanism which arises from the compactification.  $A_\mu^{(\mathbf{p}_L^2)}$  are charged under  $U(1)_I$ . The gauge bosons realise the Lie algebra which corresponds to the vector of momenta  $\mathbf{p}_L$ .

$$\begin{aligned} & \bar{\alpha}_{-1}^I |0; p\rangle \otimes b_{-\frac{1}{2}}^i |0; p\rangle_{\text{NS}} , \\ & |\mathbf{p}_L^2 = 2\rangle \otimes b_{-\frac{1}{2}}^i |0; p\rangle_{\text{NS}} . \end{aligned} \quad (2.5.15)$$

- (R) The 496 superpartners of the gauge bosons, the gauginos  $\lambda_\alpha$ :

$$\begin{aligned} & \bar{\alpha}_{-1}^I |0; p\rangle \otimes |S^\alpha\rangle_R , \\ & |\mathbf{p}_L^2 = 2\rangle \otimes |S^\alpha\rangle_R . \end{aligned} \quad (2.5.16)$$

The lowest lying state in Table 2.9 represents a bosonic tachyon, with  $\frac{\alpha' m_L^2}{2} = -1$ , which is associated with the left-moving sector of the heterotic string. Equally, from Table 2.2 it is clear that the right-moving sector also contains a tachyon, with

$\frac{\alpha' m_R^2}{2} = -\frac{1}{2}$ . Thus for heterotic strings, the first term in the expansion of the partition function  $\bar{q}^n q^m$  has  $m = -\frac{1}{2}$ ,  $n = -1$ . These unphysical states do not constitute part of the string spectrum, but, as off-shell states, they can contribute to loop amplitudes. They will therefore be necessary when calculating the full one-loop partition function.

### 2.5.1 Fermionic construction

In contrast to the afore-discussed bosonic construction, it is also possible to construct the heterotic string using fermionic fields to describe the internal left-moving degrees of freedom. This allows contact to be made with the following subsection, in which the method of fermionization is employed in and extended to the context of free fermionic models.

The spacetime fields in the fermionic construction match those in the bosonic construction, eq.(2.5.2), but now, rather than 16 left-moving bosonic fields  $X_L^I$ , the theory contains 32 Majorana fermions,  $\lambda^A(\tau + \sigma)$ ,  $A = 1, \dots, 32$ . These fermionic fields, which transform under the internal component of the symmetry of the world-sheet theory,  $SO(9, 1) \times SO(32)$ , satisfy the requirement that the total left-moving central charge of the heterotic theory, which corresponds to the bosonic theory, be

$$c_{1.m}^{\text{total}} = c^{[X_L]} + c^{[\lambda]} = 10 \cdot (1) + 32 \cdot \left(\frac{1}{2}\right) = 26, \quad (2.5.17)$$

as required for anomaly cancellation. Thus, the fermionic construction of the heterotic string involves describing the left-moving degrees of freedom by 10 bosonic and 32 fermionic fields [124, 130]. Imposing the respective light-cone derivatives on the right- ( $\psi^\mu$ ) and left-moving ( $\lambda^A$ ) fermions, the action for the heterotic string in the fermion formulation is

$$S = \frac{1}{\pi} \int d^2\sigma \left( \frac{2}{\alpha'} \partial_+ X_\mu \partial_- X^\mu + i \psi^\mu \partial_+ \psi_\mu + i \sum_{A=1}^{32} \lambda^A \partial_- \lambda^A \right). \quad (2.5.18)$$

$\mu = 0, \dots, 9$ , while  $\lambda^A$  are singlets with respect to the Lorentz group. Both  $\psi^\mu$  and  $\lambda^A$  are Majorana-Weyl fermions.

The manifest  $SO(32)$  symmetry under which the  $\lambda^A$  transform, which is a global symmetry of the worldsheet, gives rise to a local gauge symmetry in spacetime [124]. This symmetry is precisely that which gives rise to the gauge bosons described in the bosonic formulation, and which defines the HO theory. Ultimately, the choice of GSO projections for  $\lambda^A$  chooses between  $SO(32)$  and  $E_8 \times E_8$ . Without providing a full presentation of the Virasoro and super-Virasoro algebra which govern the left- and

right-moving sectors<sup>24</sup>, which is analogous to that used to obtain the full bosonic and superstring theories in the previous sections, it is stated that the fermionic construction gives rise to the same massless spectrum for the heterotic string as for the bosonic construction.

## 2.6 Free fermionic formulation

The free-fermionic construction in the KLST formalism of refs. [145–148] (and equivalently ref.[149]), is used in the formulation of all models studied in this work. Ref.[5] provides a concise contemporary summary of the procedure. As the methodology provides a direct manner in which to compactify  $10D$  superstring theories, the free fermionic formulation is of particular phenomenological interest.

Following the principles of the fermionic construction of the heterotic string presented in the previous subsection, free fermionic models involve fermionising all internal worldsheet degrees of freedom. That is, upon compactifying  $d$  of the initial  $D$  dimensions of the original theory, the  $d$  left- and  $d$  right-moving bosonic modes, eq.(2.5.2), are expressed in terms of complex worldsheet fermionic degrees of freedom, by employing the transformation

$$\partial_\alpha X_\mu \partial^\alpha X^\mu \rightarrow i\psi^{*\mu} \partial_\alpha \psi_\mu + i\bar{\psi}^{*\mu} \partial_\alpha \bar{\psi}_\mu - h\psi^* \psi \bar{\psi}^* \bar{\psi}. \quad (2.6.1)$$

By performing the fermionization procedure at a particular point in the moduli space, namely the self-dual radius, the so called Thirring coupling  $h$  goes to zero, such that the action describes free fermions  $\psi, \bar{\psi}$ . Each of the  $d$  compactified bosonic coordinates gives rise to  $2d$  left- and  $2d$  right-moving real free fermions. As it is constructed by fermionizing the heterotic string, the free fermionic construction is entirely equivalent to its (compactified) parent theory.

Following the discussion in §2.1.1, §2.2.1 & §2.3.1 of the symmetries of the  $2D$  worldsheet action, anomaly freedom of the conformal field theory on the worldsheet can be guaranteed by an appropriate choice of worldsheet fields. Equally, in the free-fermionic construction of the heterotic string, all worldsheet conformal anomalies are cancelled through the introduction of free worldsheet fermions. In a  $(10 - d)$ -dimensional model in which  $d$  of the original 10 dimensions of the heterotic theory are compactified using the transformation eq.(2.6.1), the appropriate number of complex fermionic degrees of freedom in the left- and right-moving sectors are  $N_L = \frac{1}{2}(32+2d)$ ,  $N_R = \frac{1}{2}(8+2d)$ , where 32 and 8 enumerate the real left- and right-moving worldsheet fermions of the original 10-dimensional heterotic theory. The complex free fermions

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<sup>24</sup>See [124] for further details.

can be recorded as the elements of  $(N_L + N_R) = (20 + 2d)$ -component vectors:

$$f \equiv \{f_R; f_L\} \equiv \{f_{i_R}; f_{i_L}\}, \text{ where } i_{R/L} = 1, \dots, N_{R/L}. \quad (2.6.2)$$

For heterotic strings in six uncompactified spacetime dimensions, which will appear in all examples throughout this study, there are  $N_R = 8$ ,  $N_L = 20$  complex Weyl fermions on the worldsheet.

Recalling eq.(2.5.2), the  $6D$  spacetime fields (in light-cone gauge) can be obtained by splitting the  $10D$  heterotic fields, from which the  $6D$  model is obtained, in to compact and non-compact sets:

$10D$ fields	Transverse Non-compact $6D$	Compact $4D$
$i = 1, \dots, 8$	$j = 1, \dots, 4$	$k = 5, \dots, 8$
$X_L^i(\tau + \sigma) + X_R^i(\tau - \sigma)$	$\rightarrow X_L^j(\tau + \sigma) + X_R^j(\tau - \sigma)$	$X_L^k(\tau + \sigma) + X_R^k(\tau - \sigma)$
$\psi_R^i(\tau - \sigma)$	$\rightarrow \psi_R^j(\tau - \sigma)$	$\psi_R^k(\tau - \sigma)$

In conjunction with the 32 real left-moving fermions  $\lambda^A(\tau + \sigma)$ ,  $A = 1, \dots, 32$ , the [8|20] complex fermions in the  $6D$  theory are labelled in the conventional manner [14, 150–152]:

Right-movers (total of 8 complex degrees of freedom):

- $\psi^{34}, \psi^{56}$  : 2 complex spacetime fermions  $\Leftrightarrow$  the transverse modes of  $\psi^\mu$ ,  $\mu = 1, \dots, 6$ ;
- $\chi^{34}, \chi^{56}$  : 2 complex internal fermions  $\Leftrightarrow$  originate from the 10D heterotic string model;
- $y^{34}, y^{56}, \omega^{34}, \omega^{56}$  : 4 complex, internal fermions  $\Leftrightarrow$  obtained from the fermionization of each compactified bosonic coordinate in the 6D theory.

Left-movers (total of 20 complex degrees of freedom):

- $\bar{\psi}^{1, \dots, 5}, \bar{\eta}^{1, \dots, 3}, \bar{\phi}^{1, \dots, 8}$  : 16 complex left-moving fermions  $\Leftrightarrow$  originate from the 10D heterotic theory;
- $\bar{y}^{34}, \bar{y}^{56}, \bar{\omega}^{34}, \bar{\omega}^{56}$  : 4 complex, internal fermions  $\Leftrightarrow$  correspond to the internal right-moving fermions obtained from the fermionization procedure.

The free fermionic formulation involves assigning periodic or antiperiodic boundary conditions to worldsheet fermions along specific directions of the worldsheet of a given model. Models are defined by the phases acquired under parallel transport

around non-contractible cycles of the one-loop worldsheet. The torus worldsheet, discussed in detail in §2.3.2, is described by the two cycles  $\sigma_i$ ,  $i = 1, 2$ , depicted in Figure 2.5, which correspond to the  $\mathcal{T}$  and  $\mathcal{S}$  modular transformations, defined in eq.(2.3.21).

The phases for the individual complex left-moving fermions around the  $\sigma_1, \sigma_2$  loops can be recorded in the form

$$\begin{aligned} \mathbf{1} : f_{i_{R/L}}(\sigma_1, \sigma_2) &\rightarrow f_{i_{R/L}}(\sigma_1 + 2\pi, \sigma_2) = -e^{-2\pi i v_{i_{R/L}}} f_{i_{R/L}}(\sigma_1, \sigma_2), \\ \tau : f_{i_{R/L}}(\sigma_1, \sigma_2) &\rightarrow f_{i_{R/L}}(\sigma_1, \sigma_2 + 2\pi) = -e^{-2\pi i u_{i_{R/L}}} f_{i_{R/L}}(\sigma_1, \sigma_2), \end{aligned} \quad (2.6.3)$$

where, for the heterotic string in  $6D$ ,  $i_R = 1, \dots, 8$  and  $i_L = 1, \dots, 20$ . These phases can be collected into two sets of vectors, one for each non-contractible cycle of the torus, and written as

$$\begin{aligned} v &\equiv \{v_R; v_L\} \equiv \{v_{i_R}; v_{i_L}\}, \\ u &\equiv \{u_R; u_L\} \equiv \{u_{i_R}; u_{i_L}\}, \end{aligned} \quad (2.6.4)$$

where modular invariance constrains the phases to lie within the range  $v_{i_{R/L}}, u_{i_{R/L}} \in [-\frac{1}{2}, \frac{1}{2})$ . Modular invariance requires that the sets  $\{v\}$  and  $\{u\}$  be equivalent. The spin structure, introduced in §2.3.5 and conventionally recorded in a set of basis vectors  $\{V_i\}$ , encodes the allowed set of phases [148].

The boundary conditions for the left-moving fermions determine the gauge symmetry that is preserved by an allowed set of basis vectors,  $\{V_i\}$ . The massless states in the theory are characterized by their transformation properties under this symmetry. Conversely, the spacetime supersymmetry (which can be considered to be a global symmetry) is specified by the right-moving complex fermions. For a heterotic theory in  $D$  dimensions,  $G_{\text{global}} = SO(14 - D)$ , while  $G_{\text{gauge}} = SO(26 - D)$ . To be clear, the convention is that the boundary conditions for the right- and left-moving fermions are recorded in the left- and right-hand sides of the basis vectors as

$$V_i = [V_R|V_L]_i. \quad (2.6.5)$$

Thus in a 6-dimensional model<sup>25</sup>, the basis vectors describe the boundary conditions for  $V_i = [(\ )^8 | (\ )^{20}]$  complex fermions. From the point of view of the  $\tilde{D} = (D - d)$ -dimensional compactified theory,  $\{V_i\}$  are a set of  $(14 - \tilde{D}) + (26 - \tilde{D})$ -dimensional basis vectors. For the  $\tilde{D} = 6$ -dimensional mode, this is equally seen to require 28-dimensional vectors.

<sup>25</sup>As will be explained in the following section, the 4-dimensional models of interest to this study are derived via a lift to a  $D = 6$  model and a subsequent twisted compactification on circles with radii  $r_i, i = 1, 2$  [14].

In terms of the complex fermionic worldsheet fields, and the remaining bosonic degrees of freedom, the worldsheet supercurrent is given by [152]

$$T_F(z) = \psi^\mu(z) \partial_z X_\mu(z) + \sum_{I=3}^6 \chi^I y^I \omega^I, \quad (2.6.6)$$

where  $\mu = 1, \dots, 6$  and  $I$  label the different real fermions, which are related to the complex fermions listed at the beginning of this subsection, by

$$\begin{aligned} \chi_c^{(1)} &\equiv \chi^{34} = \frac{1}{\sqrt{2}} (\chi^3 + i\chi^4), & \chi_c^{(2)} &\equiv \chi^{56} = \frac{1}{\sqrt{2}} (\chi^5 + i\chi^6), \\ w_c^{(1)} &\equiv w^{34} = \frac{1}{\sqrt{2}} (w^3 + iw^4), & w_c^{(2)} &\equiv w^{56} = \frac{1}{\sqrt{2}} (w^5 + iw^6), \\ y_c^{(1)} &\equiv y^{34} = \frac{1}{\sqrt{2}} (y^3 + iy^4), & y_c^{(2)} &\equiv y^{56} = \frac{1}{\sqrt{2}} (y^5 + iy^6). \end{aligned} \quad (2.6.7)$$

In addition to the requirement that models be conformally invariant and that they preserve worldsheet SUSY defined by the supercurrent, in order to define consistent models, the assignment of boundary conditions must respect modular invariance. The complete set of constraints that must be imposed upon all vectors that can be added to the set  $\{V_i\}$  is as follows.  $m_i$  is defined to be the order of  $V_i$ , the smallest possible integer required to generate  $\tilde{V}_i = m_i V_i$  (index  $i$  not summed) such that all elements of  $\tilde{V}_i$  are integer valued. The otherwise arbitrary structure constants  $k_{ij}$  that completely specify the theory are typically recorded as

$$\begin{pmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,j} \\ k_{2,1} & k_{2,2} & \cdots & k_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ k_{i,1} & k_{i,2} & \cdots & k_{i,j} \end{pmatrix}. \quad (2.6.8)$$

The structure constants are constrained by:

$$m_j k_{ij} = 0 \pmod{1}. \quad (2.6.9)$$

For vectors with elements  $V_i \in \{0, \frac{1}{2}\}$ ,  $m_i = 2$ , and thus the elements of the matrix  $k_{ij}$  can only take values of 0 or  $\frac{1}{2}$ . Following eq.(2.2.17), integer and half-integer phases correspond respectively to NS and R states. Many choices of additional vectors  $V_i$  that would require values of  $k_{ij}$  outside of this set, are ruled out by the condition relating the elements of  $k_{ij}$  to  $V_i \cdot V_j$ :

$$k_{ij} + k_{ji} = V_i \cdot V_j \pmod{1}. \quad (2.6.10)$$

Finally,  $V_i$  are constrained by

$$k_{ii} + k_{i0} + s_i = \frac{1}{2} V_i \cdot V_i \pmod{1}, \quad (2.6.11)$$

where  $s_i$ , which denote the spin statistics of the  $i$ th sector vector  $V_i$ , are determined by the boundary condition of the first right-moving fermion, such that  $s_i \equiv V_i^1$ .

The basis vectors span a finite additive group  $G = \sum_k \alpha_k V_k$  where  $\alpha_k \in \{0, \dots, m-1\}$ , each element of which describes the boundary conditions associated with a different individual sector of the theory. The individual sectors of the theory are given by the set of  $\overline{\alpha V} \equiv \alpha_i V_i + \Delta$  where  $\Delta \in \mathbb{Z}$  so that  $\overline{\alpha V} \in [-\frac{1}{2}, \frac{1}{2})$ . The usual convention, that  $\alpha_i$  denotes the sum over spin structures on the  $\alpha$  cycle, is followed. The spectrum of the theory at generic radius in any sector is determined by imposing the GSO projection conditions, which are expressed in terms of the vectors  $\{V_i\}$  and a set of structure constants  $k_{ij}$ . Within each sector  $\overline{\alpha V}$ , the physical states are those which are level-matched and whose fermion-number operators  $N_{\overline{\alpha V}}$  satisfy the generalized GSO projections

$$V_i \cdot N_{\overline{\alpha V}} = \sum_j k_{ij} \alpha_j + s_i - V_i \cdot \overline{\alpha V} \pmod{1} \quad \text{for all } i. \quad (2.6.12)$$

The worldsheet energies associated with such states are given by

$$M_{L,R}^2 = \sum_{\ell} \left\{ E_{\overline{\alpha V}^{\ell}} + \sum_{q=1}^{\infty} \left[ (q - \overline{\alpha V}^{\ell}) \bar{n}_q^{\ell} + (q + \overline{\alpha V}^{\ell} - 1) n_q^{\ell} \right] \right\} - \frac{(D-2)}{24} + \sum_{i=2}^D \sum_{q=1}^{\infty} q M_q^i, \quad (2.6.13)$$

where  $\ell$  sums over left- or right worldsheet fermions, where  $n_q, \bar{n}_q$  are the occupation numbers for complex fermions, where  $M_q$  are the occupation numbers for complex bosons, and where  $E_{\overline{\alpha V}^{\ell}}$  is the vacuum-energy contribution of the  $\ell^{\text{th}}$  complex worldsheet fermion:

$$E_{\overline{\alpha V}^{\ell}} = \frac{1}{2} \left[ (\overline{\alpha V}^{\ell})^2 - \frac{1}{12} \right]. \quad (2.6.14)$$

A vector of  $U(1)$  charges for each complex worldsheet fermion can be defined by

$$\mathbf{Q} = N_{\overline{\alpha V}} + \overline{\alpha V}, \quad (2.6.15)$$

where the elements of the sector vector  $\overline{\alpha V}$  are 0 and  $-\frac{1}{2}$  respectively for a Neveu-Schwarz and a Ramond boundary condition. The charge vector will be of critical importance to the evaluation of the masses of the states in, and hence the supersymmetry properties of, the models to be investigated in the following sections.

Following eq.(2.3.51), each complex fermion degree of freedom makes a contribution to the partition function which depends upon its world sheet boundary

conditions,  $v \equiv \overline{\alpha V}_i$  and  $u \equiv \beta V_i$ , as

$$\begin{aligned} \mathcal{Z}_u^v &= \text{Tr} \left[ q^{\hat{H}_v} e^{-2\pi i u \hat{N}_v} \right] \\ &= q^{\frac{1}{2}(v^2 - \frac{1}{12})} \prod_{n=1}^{\infty} (1 + e^{2\pi i(v\tau - u)} q^{n - \frac{1}{2}}) (1 + e^{-2\pi i(v\tau - u)} q^{n - \frac{1}{2}}) \\ &= e^{2\pi i u v} \frac{1}{\eta(\tau)} \vartheta \left[ \begin{matrix} v \\ -u \end{matrix} \right] (0, \tau). \end{aligned} \quad (2.6.16)$$

For complex left-moving fermions, with boundary conditions twisted by  $e^{2\pi i v}$ ,  $\hat{H}_v$  and  $\hat{N}_v$  denote the Hamiltonian and the fermion number operator respectively. In order to obtain the contribution to the one-loop partition function which arises from the entire set of free fermions in the  $6D$  theory, it is necessary to collect their contributions, while summing over their possible boundary conditions. That is,

$$\mathcal{Z}_f = \sum_{\{\alpha, \beta\}} C_\beta^\alpha Z_{\beta \mathbf{V}}^{\overline{\alpha \mathbf{V}}}, \quad (2.6.17)$$

where

$$Z_{\beta \mathbf{V}}^{\overline{\alpha \mathbf{V}}} = \frac{1}{\eta^8 \bar{\eta}^{20}} \prod_{i_R} \vartheta \left[ \begin{matrix} \overline{\alpha \mathbf{V}_i} \\ -\beta \mathbf{V}_i \end{matrix} \right] \prod_{j_L} \vartheta \left[ \begin{matrix} \overline{\alpha \mathbf{V}_j} \\ -\beta \mathbf{V}_j \end{matrix} \right], \quad (2.6.18)$$

and the GSO phases, expressed in terms of the structure constants  $k_{ij}$ , and spin-statistic  $s_i = V_i^1$ , as in the original literature, are given by

$$C_\beta^\alpha = \exp [2\pi i (\alpha s + \beta s + \beta_i k_{ij} \alpha_j)]. \quad (2.6.19)$$

For the  $6D$  theory, in which the fermions are recorded as [8|20],  $i_R = 1, \dots, 8$ , and  $i_L = 1, \dots, 20$ .

## 2.7 Non-supersymmetric strings

### 2.7.1 Gravitons, gravitinos and their protos

When formulated in  $D$  spacetime dimensions, the integrand of the one-loop partition function  $\mathcal{Z}(\tau)$  for any string theory, including those heterotic models upon which this study will focus, eq.(2.3.24), takes the form of a double power-series, whose coefficients,  $a_{mn}$  denote the net number of spacetime bosonic minus spacetime fermionic string states (net Bose-Fermi number) with worldsheet energies  $(E_R, E_L) = (m, n)$ . Expanding  $\mathcal{Z}(\tau, \bar{\tau})$  in eq.(2.3.28), in powers of  $\bar{q}$ ,  $q$ , one finds:

$$\mathcal{Z}(\tau, \bar{\tau}) = \tau_2^{1-D/2} \sum_{m,n} a_{mn} \bar{q}^n q^m, \quad (2.7.1)$$

where  $m$  and  $n$ , which correspond to  $m_R^2$  and  $m_L^2$  of eq.(2.2.39), define the spacetime mass of any given state<sup>26</sup>. States with  $m + n < 0$  are tachyonic. Physical states are those which are level-matched, with  $m = n$ . Non-level matched or off-shell states are unphysical, but can make a contribution to calculated quantities via their presence in loop diagrams. Modular invariance requires that  $(m - n) \in \mathbb{Z}$  [78]. While the vanishing of  $a_{mn}$  at a particular mass level signifies a degeneracy between the number of fermionic and bosonic states at that particular energy, the coefficients vanish at all mass levels, by definition, only in a completely supersymmetric theory. Thus, for  $\mathcal{Z}(\tau, \bar{\tau})$ , it is necessary that  $a_{mn} = 0$  for all  $(m, n)$ . Thus, calculating whether or not the partition function vanishes acts as a means of verifying whether or not any given theory is supersymmetric.

As explained in detail in [14], it can be shown that ‘every non-supersymmetric string model necessarily contains off-shell tachyonic states with  $(m, n) = (0, -1)$ ’. The argument is as follows. As suggested by its presence in all of the string models thus described, every string model contains a completely NS/NS sector from which the gravity multiplet arises. In the heterotic theory, the state (given in eq.(2.5.13))

$$\text{graviton} \subset \psi_{-1/2}^\mu |0\rangle_R \otimes \tilde{\alpha}_{-1}^\nu |0\rangle_L, \quad (2.7.2)$$

describes the gravity multiplet, which contains the graviton. No self-consistent GSO projection which eliminates the NS sector tachyon can possibly eliminate this gravity multiplet from the string spectrum. However, as long as the graviton is present in the string spectrum, so must there exist a state for which the left-moving coordinate oscillator is *not* excited:

$$\text{proto-graviton:} \quad \psi_{-1/2}^\mu |0\rangle_R \otimes |0\rangle_L. \quad (2.7.3)$$

This state, dubbed the ‘‘proto-graviton’’, corresponds to the first line of Table 2.9, and can be seen to be tachyonic by virtue of its world sheet energies,  $(E_R, E_L) = (m, n) = (0, -1)$ . There exists a contribution to the partition function of the form  $\sim \bar{q}^{-1}$ . However, just as the graviton cannot be projected out of the theory, neither can the proto-graviton; the fate of the graviton and its proto state are inextricably woven together.

In the supersymmetric theory, there also exists a gravitino, given by eq.(2.5.14), formed from the Ramond zero-mode combinations acting on the right-moving vacuum

$$\text{gravitino} \subset |S^\alpha\rangle_R \otimes \tilde{\alpha}_{-1}^\nu |0\rangle_L, \quad (2.7.4)$$

---

<sup>26</sup>In this context, the values of  $m$  and  $n$  (which later may take non-integer values), are distinct from the integer KK and winding numbers  $m$  and  $n$ , defined in §2.4.

with superpartner, the “proto-gravitino”:

$$\text{proto-gravitino:} \quad |S^\alpha\rangle_R \otimes |0\rangle_L, \quad (2.7.5)$$

The relationship between the proto-gravitino and the gravitino is identical to that between the proto-graviton and the graviton. Therefore, in a supersymmetric theory, just as the contributions to the partition function from the graviton and the gravitino cancel, so do the contributions from the proto-graviton and the proto-gravitino. The potential  $\sim \bar{q}^{-1}$  term vanishes, along with all other terms in the partition function, as is necessary for supersymmetry. Flipping the logic, it is clear that any GSO projection which eliminates the gravitino from the string spectrum, producing a non-supersymmetric string, will also correspondingly eliminate the proto-gravitino state. The contribution to the partition function from the proto-graviton state will remain uncanceled, such that the term with  $a_{0,-1} > 0$  will render the resulting partition function non-supersymmetric.

The Lorentz index associated with the proto-graviton states in eq.(2.7.3) indicates that, like the graviton, they transform as vectors under the transverse spacetime Lorentz symmetry  $SO(D-2)$ . Thus, any non-supersymmetric string theory in  $D$  spacetime dimensions must have a partition function which begins with the contribution<sup>27</sup>

$$Z(\tau) = \frac{D-2}{\bar{q}} + \dots \quad (2.7.7)$$

Much can be learnt about string theories by consulting their one-loop partition function. The presence or absence of this off shell contribution acts as an immediate indicator of the supersymmetry properties of any given theory.

## 2.7.2 Misaligned supersymmetry

The cosmological constant  $\Lambda$  is defined as the integral of the partition function over the fundamental domain of the modular group (the relationship will be formally stated in eq.(3.3.1)). Thus, supersymmetric theories give rise to a vanishing cosmological constant. This cancellation is indicative of the fact that supersymmetric theories exhibit a great number of finite characteristics. Although an in depth analysis is beyond the scope of this study, some results relating to the degree of this finiteness will presently be stated and discussed. Supertraces represent sums over the

<sup>27</sup>Note that this contrasts with the term  $\sim q^{-1}$  in [14], in which the powers of  $(m, n)$  are flipped by the alternate choice (c.f. eq.(2.3.1))

$$\mathcal{Z}(\tau, \bar{\tau}) = \text{Tr}(-1)^F \bar{q}^{H_R} q^{H_L}. \quad (2.7.6)$$

The opposite choice is made in this study in order to be consistent with the choice made in eq.(2.1.18) that the  $\tilde{\alpha}_n^\mu$  operators correspond to the left-movers.

spectrum of any given theory, weighted by the statistics of the fields therein [14]

$$\text{Str } \mathcal{M}^{2\beta} \equiv \sum_{\text{states } i} (-1)^{F_i} (M_i)^{2\beta} . \quad (2.7.8)$$

By virtue of the fact that they have equal numbers of bosons and fermions throughout the spectrum, all supertraces vanish in supersymmetric theories:

$$\text{Str } \mathcal{M}^{2\beta} = 0 \quad \text{for all } \beta \geq 0 . \quad (2.7.9)$$

It can be shown that supertraces are intrinsically related to the quantum mechanical sensitivities of light energy scales, such as the Higgs mass squared parameter  $m_H$  or the cosmological constant  $\Lambda$  to heavy mass scales, such as a cutoff  $\lambda$  [14]:

$$\begin{aligned} \delta m_H^2 &\sim (\text{Str } \mathcal{M}^0) \lambda^2 + (\text{Str } \mathcal{M}^2) \log \lambda + \dots \\ \Lambda &\sim (\text{Str } \mathcal{M}^0) \lambda^4 + (\text{Str } \mathcal{M}^2) \lambda^2 + (\text{Str } \mathcal{M}^4) \log \lambda + \dots \end{aligned} \quad (2.7.10)$$

By virtue of setting  $\delta m_H^2$  and  $\Lambda$  to zero, the vanishing of the supertraces solves the hierarchy problems associated with these parameters.

While the finite behaviour associated with supersymmetric strings is inherent to their formulation, it must be ascertained to what extent non-supersymmetric strings exhibit any similar features. Loop diagrams for the closed string, such as the vacuum-to-vacuum amplitude and other diagrams at higher order, constitute closed surfaces. There exist powerful symmetries governing those theories which exist on such surfaces. Of particular relevance to this study is the fact that the one-loop amplitude for the closed string corresponds to the torus, such that the worldsheet CFT must be modular invariant. It was shown in §2.3.2 that modular transformations can be employed to remove the ultraviolet  $\tau \rightarrow 0$  region from the fundamental domain of all modular integrals. It turns out that modular invariance is responsible for bestowing upon, even non-supersymmetric strings, a high degree of finiteness.

From the point of view of the string spectrum, modular invariance manifests itself by preserving a residual, so-called ‘misaligned supersymmetry’ between the fermionic and the bosonic states [15–17]. Misaligned supersymmetry governs the degree to which supersymmetry can be broken without destroying the finiteness of string amplitudes. In non-supersymmetric theories, cancellations which occur level by level in supersymmetric theories instead occur through conspiracies between  $N_f$  and  $N_b$  across *all* mass levels.

In a supersymmetric theory,  $N_b = N_f$  at any given mass level. In non-supersymmetric theories, it is the ‘sector-averaged’ state degeneracies, rather than the state degeneracies  $a_{nn}$ , which experience cancellation [15]. While the number of fermions or

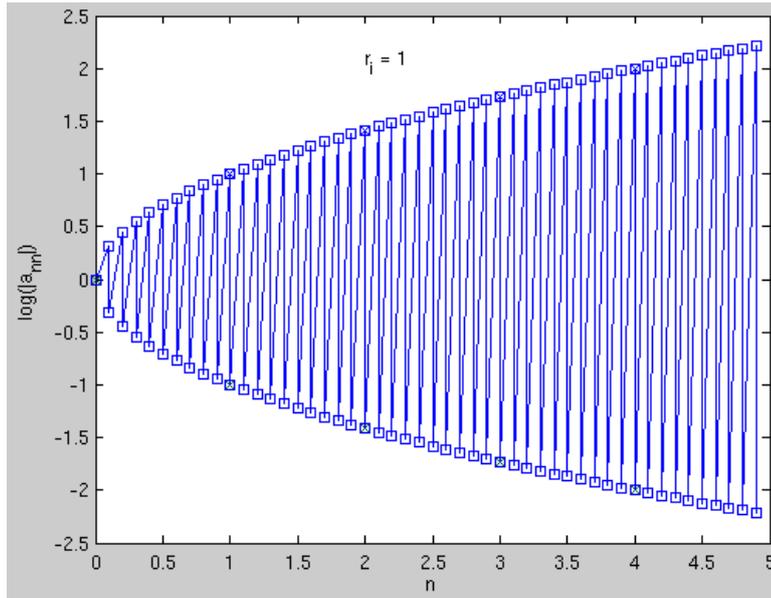


Figure 2.7: A plot of (the logarithm of) the net number of spacetime bosonic minus spacetime fermionic string states,  $a_{nn}$ , at each energy level, versus the energy level  $n$  itself. The spectrum exhibits an alternating surplus of bosonic and fermionic states within an exponentially growing envelope function. There exists a misaligned supersymmetry within the spectrum.

bosons are not equal at a particular level in a non-supersymmetric theory, the inequality is offset in the opposite direction at the subsequent mass level. The surplus of states of either spacetime spin statistics at one mass level are offset by a greater surplus at the next. The values of  $a_{nn}$  are found to exhibit oscillatory growth within an exponential envelope,  $\Phi(n) \sim |a_{nn}| \sim e^{c\sqrt{n}}$ , as depicted in Figure 2.7. If one type of state is found to outnumber the other by  $\Phi(n_i)$  at a given level, the states of opposite statistics ‘numerically retaliate’ with  $\Phi(n_i + \Delta n)$  at the subsequent level, before the spectrum exhibits a surplus of  $\Phi(n_i + 2\Delta n)$  of the original type of state, and so on and so forth. In this way, non-supersymmetric models exhibit a ‘misaligned supersymmetry’ in their spectrum. The relative number of fermionic and bosonic states oscillates with increasing energy level. The states are arranged in such a way as to preserve a residual degree of finiteness even without exhibiting spacetime supersymmetry in their spectra.

Note, that this analysis describes strings which are non-supersymmetric by construction. It is not the case that supermultiplets have been split. The fact that the surpluses of states at subsequent mass levels grow exponentially makes it clear that the spectrum is fundamentally non-supersymmetric. Instead, “it is only through a conspiracy between the physics at all mass levels across the entire string spectrum that finiteness is achieved” [14].

In analogy with the supersymmetric expression eq.(2.7.8), a regulated supertrace can be defined for non-supersymmetric theories:

$$\text{Str } M^{2\beta} \equiv \lim_{y \rightarrow 0} \sum_{\text{states}} (-1)^F M^{2\beta} e^{-y\alpha' M^2} . \quad (2.7.11)$$

The regulator  $y$  leads to a convergent sum over states and is then removed once the sum is evaluated. It is found that [14]

$$\text{Str } M^0 , \dots , \text{Str } M^{D-4} = 0 , \quad \text{and} \quad \text{Str } M^{D-2} \propto \Lambda . \quad (2.7.12)$$

The supertraces for  $0, \dots, D-4$  vanish because of the nature of the spectrum as a whole, rather than as a result of level by level cancellations. The fact that  $\text{Str } M^{D-2}$  is proportional to the value of  $\Lambda$  suggests that suppression of the cosmological constant corresponds to the stabilisation of those non-supersymmetric models which might at first appear to be inherently *unstable* owing to their lack of SUSY. By virtue of the relationships in eq.(2.7.10), it is evident that the solutions offered by supersymmetric models to those problems described in §1.1.1 can still be provided by those non-supersymmetric models in which  $\Lambda$  is suppressed. In its enhancement of the theory's finiteness properties and in its resolution of field-theoretic hierarchy problems, misaligned supersymmetry can be understood to take over where supersymmetry left off.

In summary, misaligned supersymmetry constrains supersymmetry-breaking scenarios to those in which the bosonic and fermionic states are at most misaligned from one another within the spectrum of a given theory. In order that supersymmetry be broken in such a way that physical tachyons are not introduced and modular invariance is maintained, the mismatch between the bosonic and fermionic states at a given level is constrained by the requirement that the theory as a whole can at most compensate for whatever surplus of states of opposite statistics exists at the previous level. Furthermore, since modular invariance and the freedom from physical tachyons ensure that string theory amplitudes remain finite, one can interpret the misaligned nature of the states within the non-supersymmetric theory as being the responsible for the ultimate finiteness of such theories.

### 2.7.3 Stability

As mentioned in the introduction, typically, all string theories contain flat directions which correspond to massless moduli. Attempts to fix these moduli largely result in runaway potentials. One of the moduli which would need to be fixed, is the dilaton  $\phi$ . A runaway potential for the dilaton leads to unacceptable values for the string coupling, whose value is set by the vacuum expectation value of  $\phi$ ,  $g_{\text{string}} = e^{\langle \phi \rangle}$ .

This instability is felt keenly in non-supersymmetric string theories. The dilaton tadpole diagram for such closed strings is found to be directly proportional to the cosmological constant [14]. Thus in any model in which  $\Lambda$  does not vanish there will also exist a non-vanishing dilaton tadpole diagram. The presence of such a diagram corresponds to the existence of a linear term  $\phi$  in the potential for the dilaton,  $V(\phi)$ , which acts to destabilise the vacuum.

There exists no standard procedure with which to evade the problems posed by dilaton tadpoles. Lack of a full understanding of the moduli space makes it impossible to know whether there exist proximal stable vacua. Furthermore, attempts to construct non-supersymmetric strings with vanishing values for  $\Lambda$  have also proved inconclusive [14]. Therefore, the main guiding principle when model building, and indeed the principle which has directed this investigation, is to find ways in which to formulate non-supersymmetric strings with the smallest possible values for the cosmological constant.

### 2.7.4 Interpolating Models

Generically, individual string states are defined by the mass scale  $M_{\text{string}} \equiv 1/\sqrt{\alpha'}$ , where  $\alpha'$  is related to the string tension  $T$  as defined below eq.(2.1.6). However, compactified background geometries can give rise to their own mass scales. In any non-supersymmetric theory,  $\Lambda$  will typically be of the order of the fundamental mass scales associated with those states whose one-loop vacuum amplitudes it describes, namely  $M_{\text{string}}$  and the compactification volume  $M_c$  [14]. Even with the most natural choice,  $M_{\text{string}} \sim M_c$ ,  $\Lambda$  will typically inherit the scale  $M_{\text{string}}$ . This relationship between scales is clearly not conducive to obtaining models in which  $\Lambda$  is suppressed.

With the intention of gaining a degree of control over the value of  $\Lambda$ , models can be constructed in which  $M_c$  is a free parameter. Consider for example compactifying a given, problematic (in the sense of its instability)  $D$ -dimensional non-supersymmetric theory, labelled  $\mathcal{M}_1$ , on a  $d$ -dimensional manifold such as a  $d$ -torus, with radii of compactification  $R_i$ ,  $i = 1, \dots, d$ . As the compactification is turned off, and the volume of compactification  $V_d$  is taken to infinity, the original  $D$ -dimensional model is restored. When the radii of compactification are sent to zero, closed string T-duality, described around eq.(2.4.26), relates  $\mathcal{M}_1$  to another  $D$ -dimensional model  $\mathcal{M}_2$ . As depicted in Figure 1.2, the  $(D - d)$ -dimensional model which exists at intermediate radius ‘*interpolates*’ between the pair of higher dimensional models.

If the model  $\mathcal{M}_1$  at the  $V_d \rightarrow \infty$  endpoint of the interpolation is supersymmetric, the value of the radii of compactification can be used as a parameter with which to tune the value of the cosmological constant. The setup has the further advantage that, if  $\mathcal{M}_2$  is non-supersymmetric, spacetime supersymmetry is likely to be broken

as long as  $V_d$  is finite. Lest it be worried that the setup involves a form of SUSY breaking, whose scale is being arbitrarily shifted, recall that, as is the case for all non-supersymmetric theories, the spectra of interpolating models exhibit a misaligned symmetry. As discussed in the previous subsection, misaligned supersymmetry is the string spectrum manifestation of modular invariance, and it is this symmetry which can be credited with ensuring that the theory be UV finite.

The setup has the further advantage that the cosmological constant scale and the effective scale of supersymmetry breaking need not necessarily be related to one another. Interpolating models give rise to the possibility of separating the two scales, such that methods of suppressing the cosmological constant can be employed without producing a phenomenologically unacceptable SUSY breaking scale.

In order to construct an interpolating string model,

1. First, a  $D$ -dimensional heterotic model is selected to constitute one of the models at the endpoint of the interpolation.
2. Second, the  $D$ -dimensional model is compactified on a circle of arbitrary radius  $R$ . T-duality relates the  $D$ -dimensional model at the  $R \rightarrow \infty$  to a dual model at  $R \rightarrow 0$ .
3. The interpolation is made non-trivial by introducing a twist into the  $(D - 1)$ -dimensional compactified model. The twist facilitates the breaking of spacetime SUSY within the interpolation.

First consider an untwisted circle compactification. As in the afore derived expression for the contribution to the partition function from a compact bosonic coordinate, eq.(2.4.24), the  $10D$  heterotic fields compactified on a circle give rise to contributions to the partition function of the form of the final term in eq.(2.4.25)

$$\mathcal{Z}_{\text{circ}}(\tau, \bar{\tau}, R) = \sqrt{\tau_2} \sum_{m,n \in \mathbb{Z}} \bar{q}^{p_L^2} q^{p_R^2} = \sqrt{\tau_2} \sum_{m,n \in \mathbb{Z}} \bar{q}^{(m/R+nR)^2} q^{(m/R-nR)^2}, \quad (2.7.13)$$

and the total partition function of the  $(D - 1)$ -dimensional theory representing the untwisted compactification, (where  $\mathcal{Z}(\tau)$  reproduces the original  $D$ -dimensional partition function), is

$$\mathcal{Z}(R) = \mathcal{Z}(\tau) \mathcal{Z}_{\text{circ}}(R). \quad (2.7.14)$$

Consider now performing the compactification of the  $D$ -dimensional model on a  $\mathbb{Z}_2$  twisted circle to give a  $(D - 1)$ -dimensional model. The procedure is detailed in [14]. There it is derived that the only allowable twists are those which yield valid  $D$ -dimensional endpoint theories at both ends of the interpolation. The  $(D - 1)$ -dimensional twisted string model must interpolate between a  $D$ -dimensional model at  $R \rightarrow \infty$  ( $\mathcal{M}_1$ ) and the T-dual  $R \rightarrow 0$  theory ( $\mathcal{M}_2$ ). The  $(D - 1)$ -dimensional

model can be considered as a twisted compactification of  $\mathcal{M}_1$ , where the allowable twists correspond to the consistent zero radius endpoint models. In detail, the steps involved in the construction of the model are as follows.

1. Begin with the  $D$ -dimensional model  $\mathcal{M}_1$ .
2. Identify another  $D$ -dimensional model  $\mathcal{M}_2$  to which  $\mathcal{M}_1$  is related through the action of a particular  $\mathbb{Z}_2$  orbifold twist.
3. Compactify  $\mathcal{M}_2$  on a circle of radius  $R$ , and orbifold the resulting  $(D - 1)$ -dimensional theory by the twist  $\mathcal{TQ}$ , where the orbifold twist  $\mathcal{Q}$  acts on the internal component of the string, and the  $\mathbb{Z}_2$  shift  $\mathcal{T}$  acts on the compactified circle.
4. The  $(D - 1)$  model which is defined at intermediate radius interpolates between  $\mathcal{M}_1$  at  $R \rightarrow \infty$  and  $\mathcal{M}_2$  at  $R \rightarrow 0$ .

The determination of the partition function for the  $(D - 1)$  theory compactified on twisted circles is given in [14]. However, ultimately the determination of the cosmological constant in the main body of this study will not require the contribution from the twisted sectors, as they are supersymmetric, so their contribution to the partition function will not be included here.

Analysis in [14] of particular models provides concrete examples of misaligned spectra. As a consequence of the misaligned symmetry, the supertrace relations obeyed by the models depend upon  $\Lambda$ . If  $\Lambda$  can be suppressed, as a result of conspiracies across all string energy levels, the supertraces are smaller than they would have been if they had been evaluated supermultiplet by supermultiplet.

### 2.7.5 Suppression of the cosmological constant

It has been explained that the resolution of the issues relating to finiteness, stability and hierarchy are all related to the magnitude of the cosmological constant. Thus, it would be preferable to determine the means by which interpolating models can be used to suppress the cosmological constant. It is instructive to consider which states make the leading contributions to  $\Lambda$ .

Using the power series expression for  $\mathcal{Z}$  given in 2.7.1, the cosmological constant receives contributions from the sum of terms within the following integral:

$$\Lambda \sim \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left[ \tau_2^{1-D/2} \sum_{m,n} a_{mn} \bar{q}^n q^m \right]. \quad (2.7.15)$$

For large  $n$ ,  $\Lambda$  can be shown to take the form  $\sim e^{-4\pi n}$  [14]. Thus it is to be expected that massless physical states (those with  $m = n = 0$ ) make the largest contributions

to the cosmological constant. However, in the non-compact theory, the largest contributions to  $\Lambda$  in fact originate from the off-shell (unphysical) tachyonic states, with  $(E_R, E_L) = (0, -1)$ , described in §2.7.1 [14].

By definition, these tachyonic states are absent from the supersymmetric  $\mathcal{M}_1$  theory that sits at the infinite radius endpoint. Of interest is the leading correction to the value of  $\Lambda(R)$  in the compactified theory at large but finite radius. It turns out that, for large enough  $R$ , the contributions to the total partition function  $\mathcal{Z}(R) = \mathcal{Z}(\tau)\mathcal{Z}_{\text{circ}}(R)$  from the massless physical states exceed those which arise from the above tachyonic proto-gravitons. Thus, for large but finite  $R$ , the leading contribution to the cosmological contribution is given by<sup>28</sup>

$$\Lambda \sim \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left[ \tau_2^{1-D/2} (N_{b(0)} - N_{f(0)}) (\bar{q}q)^0 + \dots \right] \mathcal{Z}_{\text{circ}}(R). \quad (2.7.16)$$

It can be shown that the leading behaviour as  $R \rightarrow \infty$  is given by the  $m = \pm 1$  (where  $m$  is the KK number) terms, such that [14]

$$\Lambda \sim (N_{b(0)} - N_{f(0)}) \frac{1}{R^{D-1}} + \dots \quad (2.7.17)$$

Thus, for models in which  $N_{b(0)} \neq N_{f(0)}$ , that is, those in which there does not exist degeneracy between the number of massless fermions and bosons, the cosmological constant takes the form  $\Lambda \sim \frac{1}{R^{(D-1)}}$ .

The dominant sub-leading contributions to  $\Lambda$ , which arise from massive states, are found to be [14]

$$\Lambda \sim \sum_{m>0} a_{mm} \left( \frac{\sqrt{m}}{R} \right)^{(D-1)/2} e^{-4\pi\sqrt{m}R}. \quad (2.7.18)$$

Thus, for models in which there are an equal number of massless bosonic and fermionic states,  $N_{b(0)} = N_{f(0)}$ , it is the lightest massive states which make the leading contributions to  $\Lambda$ . In this case,  $\Lambda$  is exponentially suppressed at large radius. Thus, whether or not  $N_{b(0)} = N_{f(0)}$ , as previously argued,  $\Lambda$  can be dialled to an arbitrarily small value by taking  $R \rightarrow$  large. In this way, interpolating models provide a tunable parameter (or parameters) with which the supersymmetric limit can be recovered.

Conversely, the theory at small  $R$  is generically non-supersymmetric. Note that the partition function of a  $(D-d)$ -dimensional string theory,  $Z^{(D-d)}$ , which is obtained by compactifying a  $D$ -dimensional theory on a  $d$ -dimensional volume  $V_d$ ,

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<sup>28</sup> $N_{b(0)}, N_{f(0)}$  denote the number of bosonic and fermionic states at the massless level after SUSY breaking has occurred [14].

is related to the partition function  $Z^{(D)}$  of its parent theory as

$$\mathcal{Z}^{(D)} = \lim_{V_d \rightarrow \infty} \left[ \frac{1}{\mathcal{M}^d V_d} \mathcal{Z}^{(D-d)} \right] \quad (2.7.19)$$

where the reduced string scale is defined as  $\mathcal{M} \equiv M_{\text{string}}/(2\pi) = 1/(2\pi\sqrt{\alpha'})$ . T-duality guarantees the existence of an identical relationship (with  $\tilde{V}_d$  replacing  $V_d$ ) in the  $V_d \rightarrow 0$  limit. Applying this relationship to the compact partition function in eq.(2.7.14), it can be seen that in the zero radius limit, the  $D$ -dimensional partition function,  $\mathcal{Z}(\tau)$ , is recovered. The leading contribution in this (generically non-supersymmetric) limit is given by the unphysical tachyons, corresponding to the  $\frac{1}{q}$  term given in eq.(2.7.7). This term yields a constant contribution, such that the overall term is simply proportional to  $N_{\text{proto}}$ , the number of proto-graviton states in the model [14]. Therefore, at the zero radius endpoint, the models to be surveyed in §5 can be expected to exhibit a constant value (which can be zero) of  $\Lambda \propto (N_{b(0,-1)} - N_{f(0,-1)}) = N_{\text{proto}}$ .

To generalise the above construction, in which the  $D$ -dimensional theory is a compactification on a single circle, the cosmological constant for a theory which is obtained by compactifying a  $D$ -dimensional supersymmetric theory down to a  $d$ -dimensional model takes the form

$$\Lambda_d \sim (N_{b(0)} - N_{f(0)}) \frac{1}{R^d} + \dots \quad (2.7.20)$$

Of particular interest in this study are  $D$ -dimensional supersymmetric theories which are compactified on  $S^1 \times S^1$  manifolds, where each circle  $S^1$  is of radius  $r_i$  and is subject to its own supersymmetry-breaking  $\mathbb{Z}_2$  orbifold twist. The partition function takes the form

$$\Lambda \sim r_1 r_2 \left[ (N_{b(0)} - N_{f(0)}) \frac{1}{r_1^D} + (\tilde{N}_{b(0)} - \tilde{N}_{f(0)}) \frac{1}{r_2^D} \right] + \dots \quad \text{as } r_1, r_2 \rightarrow \infty, \quad (2.7.21)$$

where  $(N_{b(0)} - N_{f(0)})$  and  $(\tilde{N}_{b(0)} - \tilde{N}_{f(0)})$  denote the net numbers of physical massless states which are invariant under the first and second twists, respectively. Note that the leading factor  $\sim r_1 r_2$  describes the two-torus volume factor. Furthermore, taking either  $r_1$  or  $r_2 \rightarrow \infty$  in eq.(2.7.21) reproduces the result in eq.(2.7.17) for a single twisted circle. As for the  $(D - 1)$ -dimensional model, the subleading corrections are found to be exponentially suppressed [14].

To reiterate, an equal number of massless bosonic and fermionic states does not require supersymmetry. Rather than states arising from the same supermultiplet, it is only necessary that there be numerical equality between the fermionic and the bosonic degrees of freedom. Furthermore, numerical equality need only be guaranteed by the sum over both the observable and the hidden sectors. Any disparity between

$N_{b(0)}$  and  $N_{f(0)}$  in one sector can be offset by an equal and opposite disparity in the other [14].

Since  $N_b = N_f$  by definition in supersymmetric models, such models have no dilaton tadpole. However, models exhibiting the phenomenologically favourable features associated with a vanishing tadpole need not be supersymmetric [149, 153]; it is possible to find non-supersymmetric models, for which the condition  $N_b = N_f$  still holds at the massless level; that is,  $N_{b(0)} = N_{f(0)}$ . Under a coordinate dependent compactification (introduced in §3.1.1), certain sets of particles in the original  $6D$   $\mathcal{N} = 1$  model, can be projected out of the spectrum, while their superpartners are not, leaving a non-supersymmetric theory. Critically though, the massless spectrum of the  $\mathcal{N} = 0$  theory can preserve  $N_{b(0)} = N_{f(0)}$ .

Having outlined in some detail the formalism behind the main different types superstring theory, in particular, heterotic strings in the free fermionic construction, it will now be possible to attempt to obtain phenomenologically preferable, lower dimensional, non-supersymmetric models. The interpolating formalism will be employed to build heterotic string models for which the degree of instability associated with the dilaton tadpole can be exponentially suppressed compared with the degree which might otherwise be expected.



# Chapter 3

## The cosmological constant & generalized Scherk-Schwarz construction

### 3.1 Overview

The motivation for this investigation is to study phenomenologically interesting  $4D$  non-supersymmetric string models. String theories are consistent in a higher number of dimensions (10 with and 26 without spacetime SUSY) than are macroscopically observed. In order to make contact with low energy models, it is necessary to reduce the number of dimensions upon which the parameters of the theory depend. Compactifying a  $D$ -dimensional string theory on a  $d$ -dimensional manifold, yields a model in which the parameters depend only upon the remaining  $(D - d)$ -dimensions. The nature of this compactification determines the properties of the resulting lower dimensional model.

In this section, the calculation of the cosmological constant in theories that have been compactified using the Scherk-Schwarz mechanism is elucidated, and in particular, a means of formulating the partition functions of interpolating models, which is useful for the analysis that is undertaken in the following section, is derived. The discussion is a natural generalisation of the ‘compactification-on-a-circle’ treatment of ref.[14], and as will be seen, it ultimately leads to an improved and more general construction for this class of theory.

#### 3.1.1 Coordinate Dependent Compactification

The fundamental principles of compactification were introduced in §2.4. It will be helpful to briefly summarize the implementation of the Scherk-Schwarz mechanism

[25], in which  $d$  of the  $D$ -dimensions, in which a theory is initially defined, are compactified. As already mentioned, this is incorporated using a Coordinate Dependent Compactification (CDC) [25] of, in the models of interest in this study, an initially  $\mathcal{N} = 1$ , tachyon-free supersymmetric  $6D$  theory, namely the  $\mathcal{M}_1$  model of Figure 1.2. For the purposes of the current presentation, it is useful to define the model in the fermionic formulation, described in the previous section, §2.6, although any construction method would be applicable.

The model is further compactified down to  $4D$  on a  $T_2/\mathbb{Z}_2$  orbifold. In the absence of any CDC the result would simply be an  $\mathcal{N} = 1$  model resulting from an overall  $(K_3 \times T_2)/\mathbb{Z}_2$  compactification. The  $K_3$  in question corresponds to the  $6D$   $\mathcal{N} = 1$  theory in the fermionic construction in the examples presented in this study. In theories of the type discussed in [14], in which the orbifold twist preserves SUSY, the twisted sectors have a supersymmetric spectrum, and therefore do not contribute to the cosmological constant, and thus the nature of the orbifold is unimportant. As will be explained explicitly in §4.3, the CDC is implemented by introducing a deformation, described by another vector  $\mathbf{e}$ , of shifts in the charge lattice that depend on the radii  $r_{i=1,2}$  of the  $T_2$ . As detailed in eq.(2.6.12), there exist a set of GSO projections associated with the  $\{V_i\}$  from which the initial  $K_3$  is constructed. Under the CDC, the Virasoro generators of the theory are modified, yielding an extra effective projection condition, which is governed by  $\mathbf{e}$ , on the states constituting the massless spectrum of the  $4D$  theory. The remaining massless states are characterized by their charges under the  $U(1)$  symmetry associated with  $\mathbf{e}$ . To qualify as a Scherk-Schwarz mechanism, this  $U(1)$  symmetry has to include some component of the  $R$ -symmetry such that bosons are distinguished from fermions. In this way, the gravitinos can be projected out, such that spacetime SUSY is broken.

As can be seen from the CDC modified Virasoro operators which defined the mass squared of the states in the theory, eq.(3.2.4), in the strict  $r \rightarrow \infty$  limit, the Kaluza-Klein (KK) spectrum becomes continuous. The effect of the CDC disappears and the  $6D$  endpoint model  $\mathcal{M}_1$  is recovered. On the other hand, as will be seen in §4.3, in the  $r_i \rightarrow 0$  limit, where states either remain massless or become infinitely massive, the CDC turns into another GSO projection vector.

The ‘initial’  $(D - d)$ -dimensional theory (in the particular examples in this study,  $(D - d) = (6 - 2) = 4$ ) refers to the model which is derived via a twisted compactification of the  $D$ -dimensional theory  $\mathcal{M}_1$  on circles of radius  $r_i$ ,  $i = 1, \dots, d$ , (or a higher-dimensional internal manifold consisting of the product of various circles, each of radius  $r_i$ ). Upon T-dualising (recall eq.(2.4.26) with  $\alpha'$  set to 1.),

$$r_i \rightarrow \tilde{r}_i = 1/r_i, \tag{3.1.1}$$

the  $\tilde{r}_i \rightarrow \infty$  model becomes the non-compact theory  $\mathcal{M}_2$ , whose properties depend precisely on the form of  $\mathbf{e}$ . Under the replacement eq.(2.4.26),  $m_i$  and  $n_i$ , which denote the winding and Kaluza-Klein numbers,  $n_i$  and  $m_i$  respectively, in the original expression eq.(2.4.11), are reinterpreted as the winding and KK modes in the ‘dual’ theory. The T-dual  $(D - d)$ -dimensional theory is defined on a new background with dual compactification radii  $\tilde{r}_i$ . Just as the  $D$ -dimensional theory is recovered in the  $r_i \rightarrow \infty$  decompactification limit, taking the radius of compactification of the original  $(D - d)$  theory to zero,  $r_i \rightarrow 0$ , corresponds to the decompactification limit of the dual theory,  $\tilde{r}_i \rightarrow \infty$ . In passing from one limit to the other, the  $(D - d)$ -dimensional theory interpolates between the two  $D$ -dimensional theories that sit at the endpoints of the radii ranges (see Figure 1.2).

As will be described in the following sections, the effect of the CDC on spectrum of the theory is radius dependent. The presence of the CDC can act to modify the gauge symmetry of the models at either end of the interpolation. In the large and small radii limits, the modification of the gauge symmetry of the model as a consequence of the CDC corresponds to a modification of massless spectrum. A different gauge symmetry can arise from the same set of basis vectors  $\{V_i\}$ , via a different decomposition of the parent symmetry, brought about by the CDC vector. Thus, the theories at the two endpoints can contain a different number of massless states, charged under their respective gauge symmetries. The appearance of new massless states corresponds to an enhancement of the symmetries of the original  $4D$  theory. Because  $\mathbf{e}$  can overlap the gauge degrees of freedom,  $\mathcal{M}_2$  will generically have a gauge symmetry that differs from that of  $\mathcal{M}_1$ , and possibly no SUSY. As will be seen in §4.3, the two are in fact linked: the gauge group of  $\mathcal{M}_2$  is the same as that of the supersymmetric  $\mathcal{M}_1$  theory if and only if  $\mathcal{M}_2$  is supersymmetric.

To summarize, the absence or presence of any coordinate dependence in the compactification of a higher dimensional theory determines whether or not the lower dimensional theory preserves any SUSY, as depicted in Figure 3.1. At large radius there exists a supersymmetric  $6D$  theory. Performing a twisted, and coordinate dependent, compactification (a generalization of Scherk-Schwarz SUSY breaking [14]) yields a non-supersymmetric  $4D$  model with SUSY spontaneously broken at order  $\frac{1}{2r_i}$ . As the radius of compactification is taken to zero, the theory decompactifies back to a  $6D$  theory. The presence of the two models  $\mathcal{M}_{2a}$ ,  $\mathcal{M}_{2b}$  in Figure 1.2 reflects the fact that it is possible either that SUSY remains broken, or that it is restored, in the zero radius endpoint theory. Thus the model interpolates between two  $6D$  endpoint theories via a non-supersymmetric  $4D$  theory.

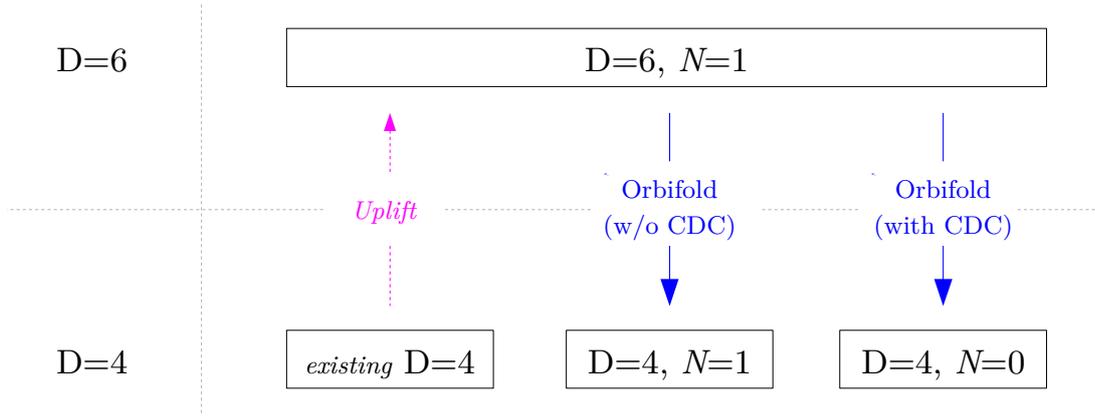


Figure 3.1: *The roadmap for constructing both SUSY and non-SUSY 4D models, the latter of which will be used extensively in this study, depending upon the choice of compactification. As detailed in [14], for technical reasons, it is advantageous to begin with a 6D model which is derived by ‘lifting’ to 6D a semi-realistic 4D model. The aim of this Figure is to demonstrate the role played by the CDC in breaking SUSY; for a clear picture of the interpolation, see Figure 1.2.*

### 3.1.2 Summary: Procedure for Identifying the Massless Spectrum of a Free Fermionic Model

For reference, the procedure for identifying the massless spectrum of a model subject to a coordinate dependent compactification, which calls upon the formalism and uses the conventions of §2.6, can be summarized as follows.

1. Choose a set of basis vectors  $\{V_i\}$ , populated with periodic  $(-\frac{1}{2})$ , anti-periodic  $(0)$ , or non-periodic  $(\neq 0, -\frac{1}{2})$  values, and corresponding structure constants  $k_{ij}$ , which satisfy the above described conditions, eq.(2.6.9,2.6.10,2.6.11).
2. Identify all possible sectors,  $\overline{\alpha V}$ , as described in §2.6, that can be formed by combining different sets of basis vectors.
3. Identify the vacuum energies associated with each sector, denoted  $[\epsilon_R, \epsilon_L]$ , using eq.(2.6.13), and consider only those which can potentially contribute massless states (that is, sectors in which  $\epsilon_R, \epsilon_L \leq 0$ ). Identify whether each sector is bosonic or fermionic in its spacetime statistics (determined by the boundary conditions for the spacetime fermions, recorded in the first entry of the sector vector right movers,  $\overline{\alpha V}^1$ ).
4. Identify the fermion number operator vector,  $N_{\overline{\alpha V}}$ , for each state within the individual sectors.

5. Perform the GSO projections associated with the basis vectors  $\{V_i\}$ , on each sector, eq.(2.6.12).
6. Calculate  $\mathbf{e} \cdot \mathbf{Q}$ , where  $\mathbf{Q}$ , defined in eq.(2.6.15),  $\mathbf{Q} = N_{\overline{\alpha V}} + \overline{\alpha V}$  gives the vector of  $U(1)$  charges for each complex world sheet fermion, and  $\mathbf{e}$  is the coordinate dependent compactification (CDC) vector, constrained by  $\mathbf{e} \cdot \mathbf{e} = 1$ .
7. Having removed those states forbidden by the GSO projection conditions, further remove those states which become massive via the CDC (as presented below eq.(3.2.8), the non-winding states for which  $\mathbf{e} \cdot \mathbf{Q} \neq 0$ ). Consequently identify and enumerate the spectrum of allowed states.
8. Combine the total set of massless states from both the bosonic and fermionic sectors to identify a value for  $(N_{b(0)} - N_{f(0)})$ .

### 3.1.3 Model construction

As an example, consider a specific model containing the vectors  $\{V_0, V_1, V_2, V_4\}$  and a CDC vector  $\mathbf{e}$ . The set of vectors, referred to as the spin structure, and structure constants  $k_{ij}$  are given by:

$$\begin{aligned}
V_0 &= -\frac{1}{2}[ 11 111 111 | 1111 11111 111 11111111 ] \\
V_1 &= -\frac{1}{2}[ 00 011 011 | 1111 11111 111 11111111 ] \\
V_2 &= -\frac{1}{2}[ 00 101 101 | 0101 00000 011 11111111 ] \\
V_4 &= -\frac{1}{2}[ 00 101 101 | 0101 00000 011 00000000 ] \\
\mathbf{e} &= \frac{1}{2}[ 00 101 101 | 1011 00000 000 00011111 ] . \tag{3.1.2}
\end{aligned}$$

The pre-factor of  $\frac{1}{2}$  means that the ‘1’ entries denote Ramond ground states, while ‘0’ entries are states with Neveu-Schwarz periodicity. The spacetime fields on the left-hand side of the above vectors, are the right-moving fermions, while the left-moving fields which define the gauge symmetry, sit to the right. This matches the notation of 2.6, in which  $V_i$  was denoted  $[V_R|V_L]$ . For clarity, sets of elements of the basis vectors which are often common to pairs of vectors are grouped together. This complicates overlaying the sector fermions, in the notation of §2.6, which emulates [14]. However, it is still useful for reference to specify which columns of the above vectors correspond to which of the free fermions of the  $\mathcal{N} = 1$   $6D$  theory. For the  $6D$  theory, in which the vectors take the form [8|20], the complex fermions defined at the end of §2.6, are located in the following columns:

$$\text{Right-moving Sector} = [\psi^{34}\psi^{56}\chi^{34}y^{34}\omega^{34}\chi^{56}y^{56}\omega^{56} | \dots ,$$

$$\text{Left-moving Sector} = \dots [y^{34} \omega^{34} \bar{y}^{56} \bar{\omega}^{56} \bar{\psi}^1 \bar{\psi}^2 \bar{\psi}^3 \bar{\psi}^4 \bar{\psi}^5 \bar{\eta}^1 \bar{\eta}^2 \bar{\eta}^3 \bar{\phi}^1 \bar{\phi}^2 \bar{\phi}^3 \bar{\phi}^4 \bar{\phi}^5 \bar{\phi}^6 \bar{\phi}^7 \bar{\phi}^8]. \quad (3.1.3)$$

The dot product of vectors and the  $k_{ij}$  are respectively given by

$$V_i \cdot V_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \text{ mod}(2); \quad k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Following §2.6, written in terms of the representations corresponding to real fermions,  $G_{\text{global}} \otimes G_{\text{gauge}} = SO(16) \otimes SO(40)$ . The conventional choice of  $V_0, V_1$ , as is made here, preserves the full gauge symmetry, while breaking the global symmetry from  $SO(16)$  to  $SO(8) \otimes SO(8)$ . The inclusion within the basis set of an additional vector  $V_2$  further breaks the global symmetry to  $SO(4) \otimes SO(4) \otimes SO(4) \otimes SO(4)$ . Equivalently, the  $\mathcal{N} = 4$  symmetry preserved by the vectors  $V_0, V_1$  is broken to  $\mathcal{N} = 2$ . In the most basic models, containing an additional basis vector  $V_4$  and a CDC vector  $\mathbf{e}$ , the symmetry breaking patterns depend upon the number of zero elements on the gauge side of  $V_4$ , and the size of the zero overlap between the two vectors  $V_4$  and  $\mathbf{e}$ .

A full calculation of the massless spectrum, including an identification of the transformation properties of the states under the elements of the group and global symmetry factors, allows a calculation of the number of states using Table E.1). Following steps 2 through 8 above,  $N_{b(0)}, N_{f(0)}$  are found to be equal. Critically, this condition holds despite the fact that the theory is non-supersymmetric (as can be seen by the absence of any massless gravitinos when the full spectrum is calculated).

## 3.2 CDC-Modified Virasoro Operators

The above description will now be elaborated upon. The standard supersymmetric construction in the entirely complex fermionic formalism is employed. The conventions for the fermionic construction are as in refs.[145–148] and for the CDC are as outlined in ref.[14], and have been summarised in §2.6.

Extending the analysis of §2.4 to describe strings propagating in a space with two, circularly compact directions, denoted  $r_{i=1,2}$ , as above, the contributions to the mass squared of the string from KK and winding modes, denoted  $m_i$  and  $n_i$ , respectively decrease and grow, with increasing radius  $r_i$ . That is, following eq.(2.4.14), the

unmodified Virasoro operators for the left- and right-moving sectors are defined as

$$\bar{L}_0/L_0 = \frac{1}{2}\alpha' p_{L/R}^2 + (\text{zero mode} + \text{oscillator contributions}), \quad (3.2.1)$$

where, in analogy with eq.(2.4.11), in terms of the winding and KK numbers,  $n_i$  and  $m_i$  respectively<sup>1</sup>, the left- and right-moving momenta for a theory compactified on two circles of radii  $r_{i=1,2}$  take the form (dropping factors of  $\alpha'$ )

$$p_{L/R}^2 \sim \sum_{i=1,2} \left( \frac{m_i}{r_i} + /- n_i r_i \right)^2. \quad (3.2.2)$$

Ultimately, the aim in this and related studies is to derive the largest possible class of deformations to the Virasoro operators which preserve modular invariance. This represents a more general set of models than that considered in refs.[24, 25]. In order to achieve this, the most general possible modification of the Virasoro operators, defined in eq.(3.2.1), under the Scherk-Schwarz action, with two bosonic coordinates compactified with radii  $r_{i=1,2}$ , takes the form:

$$\begin{aligned} \bar{L}'_0 &= \frac{1}{2} [\mathbf{Q}_L - \mathbf{e}_L(n_1 + n_2)]^2 + \frac{1}{4} \left[ \frac{m_1 + m_e}{r_1} + n_1 r_1 \right]^2 \\ &\quad + \frac{1}{4} \left[ \frac{m_2 + m_e}{r_2} + n_2 r_2 \right]^2 - 1 + \text{other oscillator contributions}, \\ L'_0 &= \frac{1}{2} [\mathbf{Q}_R - \mathbf{e}_R(n_1 + n_2)]^2 + \frac{1}{4} \left[ \frac{m_1 + m_e}{r_1} - n_1 r_1 \right]^2 \\ &\quad + \frac{1}{4} \left[ \frac{m_2 + m_e}{r_2} - n_2 r_2 \right]^2 - \frac{1}{2}/0 + \text{other oscillator contributions}. \end{aligned} \quad (3.2.3)$$

which can be written compactly as

$$\begin{aligned} \bar{L}'_0/L'_0 &= \frac{1}{2} [\mathbf{Q}_{L/R} - \mathbf{e}_{L/R}(n_1 + n_2)]^2 + \frac{1}{4} \sum_{i=1,2} \left[ \frac{m_i + m_e}{r_i} + /- n_i r_i \right]^2 \\ &\quad - 1/\left[\frac{1}{2}/0\right] + \text{other oscillator contributions}. \end{aligned} \quad (3.2.4)$$

The other oscillator contributions can be deduced from eq.(2.6.13). The vacuum energies are  $-1$  for the bosonic string in the left-moving sector (as in eq.(2.1.56) and the below evaluation for  $D = 26$ ), and  $a_{\text{NS}} = \frac{1}{2}$ ,  $a_{\text{R}} = 0$  for the right-moving superstring (as in eq.(2.2.34)).  $\mathbf{Q}$  are the vectors of Cartan gauge and  $R$ -charges ( $\mathbf{Q}$  is the local generator associated with the parent  $U(1)$  world sheet SUSY), defined by  $\mathbf{Q} = \mathbf{N}_{\alpha\bar{V}} + \overline{\alpha\bar{V}}$ .  $\mathbf{e}_{L,R}$  and  $\mathbf{Q}_{L,R}$  denote the left- and right-moving elements of

<sup>1</sup>To reiterate, the integers  $n_i$  and  $m_i$  are distinct from the left and right-moving space time masses, as in eq.(2.7.1).

the CDC vector and of  $\mathbf{Q}$  respectively. The dot products are Lorentzian, with there being a relative minus sign between the left- and the right-moving components of these vectors;  $\mathbf{e} \cdot \mathbf{Q} = (\mathbf{e}_L \cdot \mathbf{Q}_L - \mathbf{e}_R \cdot \mathbf{Q}_R)$ . The constraint  $\mathbf{e} \cdot \mathbf{e} = 1$ , necessary for modular invariance, requires the difference between the number of non-zero left- and right-moving entries of  $\mathbf{e}$  to be 0 modulo(4). The free parameter  $m_e$  will ultimately be fixed by imposing modular invariance.

Modular invariance will now indeed be used to determine  $m_e$ . Following §2.3, the one-loop partition function of the CDC modified theory can be expressed in the “charge-lattice” formalism [14], where  $\mathbf{g}$  is the generalized GSO fermion-number projection operator, and  $F$  denotes the spacetime fermion number:

$$\mathcal{Z}(\tau) = \text{Tr} \sum_{m_{1,2}, n_{1,2}} \mathbf{g} \bar{q}^{\bar{L}'_0} q^{L'_0} = \text{Tr}(-1)^F \bar{q}^{H_L} q^{H_R}. \quad (3.2.5)$$

Modular invariance associated with closed strings requires that the difference between the exponents of  $\bar{q}$  and  $q$  in each  $\bar{q}^n q^m$  term within the partition function is integer valued; that is,  $\bar{L}'_0 - L'_0 \in \mathbb{Z}$  (see Appendix C). Given that the initial theory is modular invariant, that is,  $L_0 - \bar{L}_0 \in \mathbb{Z}$  (where  $\bar{L}_0, L_0$  are the Virasoro operators with  $\mathbf{e} = \mathbf{0}$ ),  $m_e$  can be consistently determined by requiring that  $L'_0 - \bar{L}'_0 = L_0 - \bar{L}_0$ , such that the action of the CDC preserves modular invariance. Concretely,

$$\begin{aligned} \bar{L}'_0 - L'_0 &= (m_1 n_1 + m_2 n_2) + \frac{1}{2} [\mathbf{Q}_L^2 - \mathbf{Q}_R^2] \\ &\quad + (n_1 + n_2) m_e - \mathbf{e} \cdot \mathbf{Q} (n_1 + n_2) + \mathbf{e} \cdot \mathbf{e} \frac{(n_1 + n_2)^2}{2} \\ &= L_0 - \bar{L}_0 + (n_1 + n_2) m_e - (n_1 + n_2) \mathbf{e} \cdot \left[ \mathbf{Q} - \mathbf{e} \frac{(n_1 + n_2)}{2} \right]. \end{aligned} \quad (3.2.6)$$

Thus a KK shift of

$$m_e = \mathbf{e} \cdot \mathbf{Q} - \frac{1}{2} (n_1 + n_2) \mathbf{e} \cdot \mathbf{e}, \quad (3.2.7)$$

is *sufficient* to maintain modular invariance in the deformed theory. This matches the result of ref. [25]. Note that  $L'_0 - \bar{L}'_0 = L_0 - \bar{L}_0$  (rather than  $L'_0 - \bar{L}'_0 = \mathbb{Z}$ ) ensures that the CDC cannot violate the level matching condition of the initial theory ( $(L_0 - \bar{L}_0)|phys\rangle = 0$ ). Thus it is the *mass spectrum*, rather than the *number* of degrees of freedom contained within the theory, that is modified, as required for a spontaneous breaking of SUSY [22–25].

For completeness, the mass squared in the CDC-deformed theory is

$$\begin{aligned} \bar{L}'_0 + L'_0 &= \bar{L}_0 + L_0 + \frac{1}{2} \left[ \mathbf{e} \cdot \mathbf{Q} - \frac{(n_1 + n_2)}{2} \mathbf{e}^2 \right]^2 \left( \frac{1}{r_1^2} + \frac{1}{r_2^2} \right) \\ &\quad - \frac{1}{2} (n_1 + n_2) (\mathbf{e}_L \cdot \mathbf{Q}_L + \mathbf{e}_R \cdot \mathbf{Q}_R) + \frac{1}{2} (n_1 + n_2)^2 (\mathbf{e}_L^2 + \mathbf{e}_R^2) \end{aligned}$$

$$+ \left( \frac{m_1}{r_1^2} + \frac{m_2}{r_2^2} \right) \left[ \mathbf{e} \cdot \mathbf{Q} - \frac{(n_1 + n_2)}{2} \mathbf{e}^2 \right]. \quad (3.2.8)$$

It is clear that the masses of the states in the CDC-deformed theory depend upon the radii of compactification,  $r_i$ , via the terms that couple the winding / KK numbers to the compactification radii. Note for reference that setting the CDC vector  $\mathbf{e}$  in eq.(3.2.8) to zero yields the expression for the original supersymmetric model in four dimensions (that is, purely the sum of the unprimed Virasoro operators). The remaining terms take the form (summing over the number of compactified dimensions):

$$\begin{aligned} \bar{L}_0 + L_0 &= \frac{1}{2}(p_L^2 + p_R^2) - 1 - \frac{1}{2}/0 + \text{osc.} \\ &= \frac{1}{2} \sum_{i=1,2} \left[ \left( \frac{m_i}{r_i} + n_i r_i \right)^2 + \left( \frac{m_i}{r_i} - n_i r_i \right)^2 \right] - 1 - \frac{1}{2}/0 + \text{osc.} \\ &= \sum_{i=1,2} \left[ \left( \frac{m_i}{r_i} \right)^2 + (n_i r_i)^2 \right] - 1 - \frac{1}{2}/0 + \text{osc.} . \end{aligned} \quad (3.2.9)$$

It is clear from eqs.(3.2.4) and (3.2.8) that the vector  $\mathbf{e}$  lifts the masses of states according to their charges under the linear combination  $q_e = \mathbf{e} \cdot \mathbf{Q}$ . For zero winding modes ( $n_i = 0$ ), states for which  $q_e = \mathbf{e} \cdot \mathbf{Q} \neq 0 \pmod{1}$  become massive under the action of the CDC. Conversely, all zero winding states which are chargeless with respect to  $\mathbf{e}$  remain unshifted by the CDC. Thus the CDC imposes conditions on the possible light spectrum of the lower dimensional theory. The CDC has the potential to project states out of the higher dimensional theory. Upon the transition to a lower dimensional theory, the CDC can break the spacetime SUSY associated with the higher dimensional model by projecting out the gravitinos. Section 4 establishes the conditions under which theories contain massless or massive gravitinos, depending on whether the effective projection  $\mathbf{e} \cdot \mathbf{Q}$  is aligned with the GSO projections which define the matter content of the theory.

Restricting the discussion to half-integer mass-shifts imposes the constraint  $\mathbf{e} \cdot \mathbf{e} = 1 \pmod{2}$ . In the following subsection, the partition function will be reorganised into sums over different values of  $4m_e = 0 \dots 3$  (as the study is restricted to  $\frac{1}{2}$  phases in all examples, fractions of at most  $\frac{1}{4}$  can arise in the GSO projections via odd numbers of overlapping  $\frac{1}{2}$ 's). So far these deformations are precisely those of refs.[22–25]; a consideration of the interpolation to the 6D theories will make clear how these deformations can be made general.

### 3.3 Details of Cosmological Constant Calculation

As described in §2.7.3, the finiteness of string models is directly tied to the cancellation of the cosmological constant. Thus, the remainder of this study will involve the calculation of  $\Lambda$ , and of its dependence upon the radii of compactification in Scherk-Schwarz deformed models. In particular, given that in exactly supersymmetric theories the partition function, and thus  $\Lambda$ , by definition vanish, the SUSY properties of interpolating models can be determined by an analysis of the value of the cosmological constant at the different stages of the interpolation.

To evaluate the cosmological constant, at given radii  $r_1 = r_2 = r$ , one must integrate each  $\bar{q}^n q^m$  term (weighted by its coefficient  $a_{mn}$ ) in the total one-loop partition function over the fundamental domain  $\mathcal{F}$  of the modular group:

$$\Lambda^{(D)} \equiv -\frac{1}{2} \mathcal{M}^{(D)} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \mathcal{Z}_{\text{total}}(\tau), \quad (3.3.1)$$

where  $D$  is the number of uncompactified spacetime dimensions (equal to 4 at all intermediate radii between the small and large radius  $6D$  endpoint theories, along which the cosmological constant will be evaluated), and  $\mathcal{M} \equiv M_{\text{string}}/(2\pi) = 1/(2\pi\sqrt{\alpha'})$  is the reduced string scale. Henceforth  $\mathcal{M}$  is set to 1; it can be reinserted by dimensional analysis at the end of the calculation if desired.

Recall the fundamental domain of the modular group, defined in eq.(2.3.23):

$$\mathcal{F} \equiv \left\{ \tau : |\text{Re } \tau| \leq \frac{1}{2}, \text{Im } \tau > 0, |\tau| \geq 1 \right\}. \quad (3.3.2)$$

The integral splits into upper ( $\tau_2 \geq 1$ ) and lower ( $\tau_2 < 1$ ) regions of the fundamental domain. Only terms for which  $m = n$ , (where  $m, n$  represent the left- and right-moving components of the mass squared, as defined in eq.(2.7.1)), which correspond to the physical states, can receive contributions from both regions. Conversely, when  $m \neq n$  which corresponds to unphysical states, the  $\tau_1$  integral yields zero in the upper region, enforcing level matching in the infra-red (but allowing contributions from unphysical proto-graviton modes in the ultra-violet as described in ref.[14]).

The  $4D$  model is treated as a  $6D$  model with a final orbifold compactification, as depicted in Figure 3.1. The identification of the cosmological constant involves calculating the partition function as an interpolation, in the sense, outlined in §2.7.4, that the treatment is formulated in terms of the interpolation parameters, the compactification radii  $r_i$  of the twisted two-torus. The detailed structure of the code which was written to automate the computation, the development of which constituted a large part of the preparatory work for this study, is summarized in

Appendix D.

At generic radii, the evaluation of the cosmological constant is complicated immensely by the fact that  $m, n$  vary with  $r_i$ . In order to provide a tractable means of evaluation, it is necessary to rearrange the total partition function,  $\mathcal{Z}_{\text{total}}(\tau)$ , into separate bosonic and fermionic factors as follows. It is convenient to define  $n = (n_1 + n_2)$  and  $\ell = (\ell_1 + \ell_2)$ . Twisted sectors do not need to be considered in this implementation as, being supersymmetric, they do not contribute to the cosmological constant. In other words, the cosmological constant calculated without the orbifolding, is the same up to a factor of two, as the actual cosmological constant, as explained in detail in [14, 22–25]. However, further comments on twisted sectors will be made when the construction is generalised.

The expression corresponding to the compactification of 26-dimensional bosonic string, eq.(2.4.25), can be Poisson-resummed, as explained in Appendix C. An expression for the complex boson compactified on *two* cycles  $r_1, r_2$  can be obtained as a straightforward generalization (see Appendix B). Thus, in the untwisted sector, the modular-invariant partition function for the two compact bosonic degrees of freedom is given by

$$Z_{\mathbf{B}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) = \sum_{\vec{\ell}, \vec{n}} Z_{\vec{\ell}, \vec{n}}, \quad (3.3.3)$$

where

$$\mathcal{Z}_{\vec{\ell}, \vec{n}} = \frac{r_1 r_2}{\tau_2 \eta^2 \bar{\eta}^2} \sum_{\vec{\ell}, \vec{n}} \exp \left\{ -\frac{\pi}{\tau_2} \left[ r_1^2 |\ell_1 - n_1 \tau|^2 + r_2^2 |\ell_2 - n_2 \tau|^2 \right] \right\}. \quad (3.3.4)$$

Equally, following eqs.(2.6.17) & (2.6.18), the fermionic contribution is given by the Jacobi  $\vartheta$ -function products. Thus, collecting the bosonic and the fermionic contributions, the complete one-loop partition function  $Z(\tau)$  for the  $\mathcal{N} = 1$ , 4D model is [14]

$$Z(\tau) = \frac{\mathcal{M}^2}{\tau_2 |\eta|^4} \frac{1}{[\eta(\tau)]^8 [\bar{\eta}(\bar{\tau})]^{20}} \sum_{\{\alpha, \beta\}} C_{\beta}^{\alpha} Z_{\mathbf{B}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) \prod_{i_R} \vartheta \begin{bmatrix} \frac{\alpha \mathbf{v}_i}{-} \\ -\beta \mathbf{v}_i \end{bmatrix}(\tau) \prod_{i_L} \bar{\vartheta} \begin{bmatrix} \frac{\alpha \mathbf{v}_i}{-} \\ -\beta \mathbf{v}_i \end{bmatrix}(\bar{\tau}), \quad (3.3.5)$$

where  $C_{\beta}^{\alpha}$  are as in the original literature and eq.(2.6.19),

$$C_{\beta}^{\alpha} = \exp [2\pi i (\alpha s + \beta s + \beta_i k_{ij} \alpha_j)]. \quad (3.3.6)$$

### 3.3.1 CDC modified partition function

The modification to this expression owing to the CDC will now be determined. The partition function can more compactly be expressed as

$$\mathcal{Z}(\tau) = \frac{1}{\tau_2 \eta^8 \bar{\eta}^{20}} \sum_{\vec{\ell}, \vec{n}} \mathcal{Z}_{\vec{\ell}, \vec{n}} \sum_{\alpha, \beta} \Omega_{\ell, n} \left[ \begin{matrix} \alpha \\ \beta \end{matrix} \right], \quad (3.3.7)$$

where each Jacobi  $\vartheta$ -function within the product has characteristic defined by the sectors  $\alpha, \beta$ , with their respective CDC shifts:

$$\Omega_{\ell, n} \left[ \begin{matrix} \alpha \\ \beta \end{matrix} \right] = \tilde{C}_{\beta, -\ell}^{\alpha, -n} \prod_{i_R} \vartheta \left[ \begin{matrix} \overline{\alpha \mathbf{V}_i - n \mathbf{e}_i} \\ -\beta \mathbf{V}_i + \ell \mathbf{e}_i \end{matrix} \right] (\tau) \prod_{j_L} \bar{\vartheta} \left[ \begin{matrix} \overline{\alpha \mathbf{V}_j - n \mathbf{e}_j} \\ -\beta \mathbf{V}_j + \ell \mathbf{e}_j \end{matrix} \right] (\bar{\tau}). \quad (3.3.8)$$

The conventions for  $\vartheta$ -functions, including those with characteristics, can be found in Appendix B. The coefficients of the  $\vartheta$ -function products, which correspond to the fermions in the left- and right-moving sectors of the theory, are given by

$$\tilde{C}_{\beta, -\ell}^{\alpha, -n} = \exp \left\{ -2\pi i \left[ n \mathbf{e} \cdot \beta \mathbf{V} - \frac{1}{2} n \ell \mathbf{e}^2 \right] \right\} C_{\beta}^{\alpha}, \quad (3.3.9)$$

and  $C_{\beta}^{\alpha}$  are defined above as the coefficients of the original theory before the CDC.

It is convenient to use the resummed version of this expression; certainly for the  $q$ -expansion, this is the preferred method, as it makes explicit the modular invariance. Using the resummed version removes the  $r_1 r_2$  pre-factor and adds a pre-factor of  $\tau_2$ . The bosonic factor in the partition function, denoted above as  $\mathcal{Z}_{\vec{\ell}, \vec{n}}$ , depends upon the radii of compactification, the winding and resummed KK numbers and the CDC induced shift in the KK levels,  $m_e$ , as follows (appropriately reassigning the subscripts):

$$\begin{aligned} \mathcal{Z}_{\vec{m}, \vec{n}, m_e} &= \frac{1}{\eta^2 \bar{\eta}^2} \sum_{\vec{m}, n_1, k} q^{\frac{1}{4} \left( \frac{m_1 + m_e}{r_1} + n_1 r_1 \right)^2 + \frac{1}{4} \left( \frac{m_2 + m_e}{r_2} + (n - n_1 + 4k) r_2 \right)^2} \\ &\quad \times q^{\frac{1}{4} \left( \frac{m_1 + m_e}{r_1} - n_1 r_1 \right)^2 + \frac{1}{4} \left( \frac{m_2 + m_e}{r_2} - (n - n_1 + 4k) r_2 \right)^2}. \end{aligned} \quad (3.3.10)$$

The effective shift in the KK number, given by the requisite  $m_e \equiv \mathbf{e} \cdot (\mathbf{Q} - n \frac{\mathbf{e}}{2})$ , arises from the choice of  $\tilde{C}_{\beta, -\ell}^{\alpha, -n}$ , which contributes an overall phase  $e^{2\pi i \ell (\mathbf{e} \cdot (\mathbf{Q} - n \mathbf{e}) - n \mathbf{e}^2 / 2)}$  to the partition function. As will become evident, this shift in the KK number ultimately amounts to introducing a new vector  $V_e \equiv \mathbf{e}$ , combined with structure constants  $k_{ei} = 0$ ,  $k_{ee} = 1/2$ , to the non-compact T-dual theory at zero radius.  $\mathbf{e}$  is precisely the CDC vector discussed at the beginning of this section. Note that, as will be explained in detail in the following section, this means in the  $4D$  spectrum one may find states with  $1/4$ -charges  $\mathbf{e} \cdot \mathbf{Q} = 1/4, 3/4$ , which, since they have  $m_e \neq 0$ , become infinitely massive in the zero radius limit.

In order to reorder the sum such that it can be efficiently computed, a projection on  $\mathbf{Q}$  in the  $\mathcal{Z}_F$  is introduced in order to select the possible values of  $m_e$ . In parallel with the notation that  $\beta_i$  represents the sum over spin structures, the parameter for the projection over the vector  $\mathbf{e}$  will be denoted  $\beta_e = 0 \dots 3$ . Thus, using the results in Appendix B, one can write overall

$$\mathcal{Z}_{\text{total}}(\tau) = \frac{1}{4} \frac{1}{\tau_2 \eta^8 \bar{\eta}^{20}} \sum_{\substack{m_e=(0..3)/4 \\ \bar{m}, \bar{n}}} \mathcal{Z}_{B, \bar{m}, \bar{n}, m_e} \sum_{\alpha, \beta, \beta_e} e^{2\pi i \beta_e m_e} \Omega_n \left[ \begin{matrix} \alpha \\ \beta, \beta_e \end{matrix} \right], \quad (3.3.11)$$

where

$$\Omega_n \left[ \begin{matrix} \alpha \\ \beta, \beta_e \end{matrix} \right] = \tilde{C}_{\beta, \beta_e}^{\alpha, -n} \prod_{i_R} \vartheta \left[ \begin{matrix} \frac{\alpha V_i - n \mathbf{e}_i}{-\beta V_i - \beta_e \mathbf{e}_i} \right] \prod_{j_L} \bar{\vartheta} \left[ \begin{matrix} \frac{\alpha V_j - n \mathbf{e}_j}{-\beta V_j - \beta_e \mathbf{e}_j} \right]. \quad (3.3.12)$$

Note that the phases in  $\tilde{C}_{\beta, \beta_e}^{\alpha, -n}$  are precisely those which are needed to cancel the contribution coming from the  $\vartheta$ -functions in  $\Omega_n$ , such that overall the spectrum is merely shifted, and the GSO projections remain independent of  $\mathbf{e}$ .

The bosonic contribution to the total partition function is independent of the fermionic sectors within the theory, meaning that  $\mathcal{Z}_B$  appears as a pre-factor to the sector sum for any given  $m_e$ . Conversely, the fermionic partition function is composed of terms that depend upon the boundary conditions of the fermions within the sectors  $\alpha, \beta$ , each of which is independent of the compactification radii. The advantage of this reordering is that one can therefore collect 16 representative factors, labelled by the combinations of the pair  $n, 4m_e = 0..3 \pmod{4}$ ,

$$\mathcal{Z}_{F, n, m_e} = \frac{1}{4} \sum_{\alpha \beta \beta_e} e^{2\pi i \beta_e m_e} \Omega_n \left[ \begin{matrix} \alpha \\ \beta, \beta_e \end{matrix} \right], \quad (3.3.13)$$

which are independent of the radii, and 16 respective  $T_2/\mathbb{Z}_2$  factors ( $n, 4m_e = 0..3 \pmod{4}$ ), which being independent of the internal degrees of freedom, depend only on the  $T_2$  compactification,

$$\begin{aligned} \mathcal{Z}_{B, n, m_e} &= \frac{1}{\eta^2 \bar{\eta}^2} \sum_{\bar{m}, n_1, k} \bar{q}^{\frac{1}{4} \left( \frac{m_1 + m_e}{r_1} + n_1 r_1 \right)^2 + \frac{1}{4} \left( \frac{m_2 + m_e}{r_2} + (n - n_1 + 4k) r_2 \right)^2} \\ &\quad \times q^{\frac{1}{4} \left( \frac{m_1 + m_e}{r_1} - n_1 r_1 \right)^2 + \frac{1}{4} \left( \frac{m_2 + m_e}{r_2} - (n - n_1 + 4k) r_2 \right)^2}. \end{aligned} \quad (3.3.14)$$

The latter are radius dependent interpolating functions, analogous to the functions  $\mathcal{E}_{0,1/2}, \mathcal{O}_{0,1/2}$  in the simple circular case studied in ref.[14]. The  $\mathcal{Z}_{F, n, m_e}$  terms are referred to as ‘ $K_3$  factors’, since they involve only the *internal* degrees of freedom of the  $6D$  theory, and thus can be computed for all radii at the beginning of the calculation. The total partition function is subsequently compiled by summing over

the 16  $(n, m_e)$  sectors as

$$\mathcal{Z}(\tau) = \frac{1}{4} \frac{1}{\tau_2 \eta^8 \bar{\eta}^{20}} \sum_{n, 4m_e=0..3} \mathcal{Z}_{B,n,m_e} \mathcal{Z}_{F,n,m_e}. \quad (3.3.15)$$

To summarise, via the procedure of re-ordering the original sum eq.(3.3.7), a projection on to different consistent  $m_e$  values has been performed, such that a sum over  $m_e$  can be taken.

### 3.4 The zero radius theory and a more general formulation of Scherk-Schwarz

An interesting aspect of the above approach is that in the small radius limit, the part of the spectrum for which  $m_e \neq 0 \pmod{1}$  decouples and can be discarded, leaving the partition function of the non-compact 6D theory at  $r_i = 0$ . Indeed, Poisson resummation on  $n_1$  and  $k$  gives

$$\mathcal{Z}_{B,n,m_e} \rightarrow \sum_{\vec{m}} e^{-\left(\frac{(m_1+m_e)^2}{r_1^2} + \frac{(m_2+m_e)^2}{r_2^2}\right) \frac{\pi|\tau|^2}{\tau_2}} \frac{1}{4\tau_2 r_1 r_2} + \dots, \quad (3.4.1)$$

where the ellipsis indicate terms that are further exponentially suppressed. Thus the total untwisted partition function in the small radius limit can be expressed as

$$\mathcal{Z}(\tau) \rightarrow \frac{1}{16r_1 r_2} \frac{1}{\tau_2 \eta^8 \bar{\eta}^{20}} \sum_n \mathcal{Z}_{F,n,0}. \quad (3.4.2)$$

Note that  $1/(r_1 r_2)$  is simply the expected volume factor of the partition function in the T-dual 6D theory. In conjunction with the fermionic component of the partition function, a 6D model, with an additional basis vector  $\mathbf{e}$ , appearing in the sector definitions as  $\alpha V - n\mathbf{e}$ , is reproduced. Eq.(3.2.7) provides a new GSO projection, namely  $m_e = \mathbf{e} \cdot \mathbf{Q} - n/2 = 0 \pmod{1}$ . (The mod (1) is courtesy of the sum over  $m_i$ .)

Upon inspection therefore, eq.(3.2.7) is actually found to be the GSO projection of an additional vector  $V_e \equiv \mathbf{e}$  in the non-compact 6D theory. Beginning with the choice of  $\mathbf{e} \cdot \mathbf{e} = 1$ , one can infer that, for the examples under consideration, the 6D theory at zero radius has structure constants  $k_{ei} = 0$  and  $k_{ee} = 1/2$ , consistent with the modular invariance rules of KLST in refs.[145–148]. In fact, identifying the sectors  $\alpha V = \alpha_i V_i + \alpha_e V_e$  with the sum over the spin structures on the  $\mathbf{e}$  cycle with  $\alpha_e = -n \pmod{2}$ , the entire partition function at zero radius is that of the 6D theory with the appropriate corresponding GSO phases,

$$\tilde{C}_{\beta,\beta_e}^{\alpha,-n} = \exp \left\{ 2\pi i [\beta_e k_{ej} \alpha_j - \beta_i k_{ie} n - \beta_e k_{ee} n] \right\} C_\beta^\alpha. \quad (3.4.3)$$

Reversing the line of reasoning above finally leads to a generalisation of the construction of interpolating models based on the modular invariance of their endpoint  $6D$  theories:

- First, define a  $6D$  theory in terms of a set of basis vectors  $\{V_i\}$ , and *any* additional  $V_e \equiv \mathbf{e}$  vector that obeys the  $6D$  modular invariance rules of ref.[145–148], together with a set of consistent structure constants  $k_{ei}$  and  $k_{ee}$ . (The  $k_{ei}$  are as usual fixed by the requirement of modular invariance.)
- In theories that have an additional  $\mathbb{Z}_2$  orbifold action  $\hat{\mathbf{g}}$  upon their compactification to  $4D$ ,  $V_e \equiv \mathbf{e}$  is still constrained by the need to preserve mutually consistent GSO projections, with the condition  $\{\mathbf{e} \cdot \mathbf{Q}, \hat{\mathbf{g}}\} = 0$  (as in refs.[24, 25] and discussed in ref.[14]).
- The partition function takes the form of eqs.(3.3.11),(3.3.12) with coefficients as in eq.(3.4.3). The projection obtained by performing the  $\beta_e$  sum determines the corresponding KK shift to be

$$m_e = \mathbf{e} \cdot \mathbf{Q} + (k_{ee} - \mathbf{e}^2) n - k_{ei} \alpha_i, \quad (3.4.4)$$

generalizing eq.(3.2.7).

The final statement, namely that one may simply treat the Scherk-Schwarz action as another basis vector, leads to considerable generalisations, and is one of the main results of this study. In order to provide a proof, one may first Poisson-resum back to the original expression, but retaining  $\beta_e$ , so that entire partition function is

$$\mathcal{Z} = \frac{1}{4} \frac{1}{\tau_2 \eta^8 \bar{\eta}^{20}} \sum_{m_e=(0\dots3)/4} \sum_{\substack{\alpha, \beta, \vec{\ell}, \vec{n} \\ \beta_e}} e^{2\pi i(\ell + \beta_e)m_e} \mathcal{Z}_{\vec{\ell}, \vec{n}} \tilde{C}_{\beta, \beta_e}^{\alpha, -n} \prod_{i_R} \vartheta \left[ \begin{matrix} \overline{\alpha V_i - n \mathbf{e}_i} \\ -\beta V_i - \beta_e \mathbf{e}_i \end{matrix} \right] \prod_{j_L} \bar{\vartheta} \left[ \begin{matrix} \overline{\alpha V_j - n \mathbf{e}_j} \\ -\beta V_j - \beta_e \mathbf{e}_j \end{matrix} \right]. \quad (3.4.5)$$

Note that the sum over  $m_e$  provides a projection that equates  $\beta_e \equiv -\ell \pmod{1}$ . Using the modular transformations for  $\vartheta$ -functions detailed in Appendix B, it is straightforward to show that the partition function remains invariant under  $\tau \rightarrow \tau + 1$  provided that

$$e^{-i\pi(\overline{\alpha \mathbf{V}} - n \mathbf{e}) \cdot (2V_0 + \overline{\alpha \mathbf{V}} - n \mathbf{e})} \tilde{C}_{\beta, \beta_e}^{\alpha, -n} = \tilde{C}_{\beta - \alpha - \delta_{i_0}, \beta_e + n}^{\alpha, -n}, \quad (3.4.6)$$

and invariant under  $\tau \rightarrow -1/\tau$  provided that

$$e^{-2\pi i(\overline{\alpha \mathbf{V}} - n \mathbf{e}) \cdot (\beta \mathbf{V} + \beta_e \mathbf{e})} \tilde{C}_{\beta, \beta_e}^{\alpha, -n} = \tilde{C}_{-\alpha, n}^{\beta, \beta_e}. \quad (3.4.7)$$

This overall set of conditions is precisely that of KLST [145–148], with the original theory enlarged to include the vector  $V_e \equiv \mathbf{e}$ .  $\square$

Note that these rules are significantly more general than those of refs.[22–25], in which the choice

$$\tilde{C}_{\beta, \beta_e}^{\alpha, -n} = C_{\beta}^{\alpha} e^{-2\pi i(n\mathbf{e} \cdot \beta V + \beta_e n \frac{\mathbf{e} \cdot \mathbf{e}}{2})}, \quad (3.4.8)$$

corresponds to taking  $k_{ei} = 0$  and  $k_{ee} = 1/2$ , in eq.(3.4.3). Now, for example, the CDC vectors are no longer restricted to obey  $\mathbf{e}^2 = 1 \pmod{1}$ , and moreover the KK shifts have additional sector dependence if  $k_{ei} \neq 0$ . It is worth adding that, as well as being a generalisation, these rules simplify the construction of viable phenomenological models, because the  $\{\mathbf{e} \cdot \mathbf{Q}, \hat{\mathbf{g}}\} = 0$  condition can be implemented independently, with consistency thus guaranteed with respect to the remaining  $V_i$  vectors<sup>2</sup>. It can also be concluded that, for consistency, a theory that is compactified via the Scherk-Schwarz mechanism on an orbifold, should contain additional sectors that are twisted under the action of both the orbifold and the Scherk-Schwarz – i.e. twisted sectors that have non-zero  $\alpha_e$ . Of course  $\alpha_e$  for such sectors has no association with any windings, but it is found that those sectors (which being twisted are supersymmetric) are required for consistency (for, for example, anomaly cancellation).

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<sup>2</sup>This is a somewhat subtle point because the basis in which the orbifold action is diagonal is not the same as the basis in which the Scherk-Schwarz action is diagonal. However the two act relatively independently on the partition function. This point is discussed in explicit detail in ref.[154].

# Chapter 4

## On SUSY Restoration

### 4.1 Gravitinos

The conditions under which the endpoint theories exhibit SUSY will now be described. In all models considered, the theory at infinite radius is supersymmetric (as would be evidenced by the vanishing of the cosmological constant there). However, whether or not SUSY is restored at zero radius must be determined. This section develops arguments to address this question based on the existence or otherwise of massless gravitinos as  $r_i \rightarrow 0$ .

The pure Neveu-Schwarz (NS) sector,  $\mathbf{0}$ , has vacuum energies  $[\epsilon_R, \epsilon_L] = [-\frac{1}{2}, -1]$ . Using eq.(2.6.14), it can be seen that an NS excitation for the  $\psi^{34}$  fermion, produces a state with  $\epsilon_R = 0$ . Thus, as in eq.(2.5.13)<sup>1</sup>, the  $\mathbf{0}$  sector contains the state  $\psi_{-\frac{1}{2}}^{34} |0\rangle_R \otimes X_{-1}^{34} |0\rangle_L$ , where the superscript notation, defined in eq.(3.1.3), is as in [14]. This state gives rise to the gravity multiplet,  $G_{\mu\nu}$  (the graviton),  $B_{[\mu\nu]}$  (the two index antisymmetric tensor) and  $\phi$  (the dilaton). In the  $\mathbf{0}$  sector, the GSO projection (before the action of the CDC), eq.(4.3.7), reduces to  $V_i \cdot N_{\overline{\alpha V}} = s_i$ . In this sector, the fermion number operator vector that corresponds to an NS excitation for the  $\psi^{34}$  fermion is non-zero in only the first right-moving entry. For vectors  $V_i$  with a non-zero first right-moving entry,  $s_i = \frac{1}{2}$ , such that the single overlap with  $N_{\overline{\alpha V}}$  satisfies the projection condition. Since the CDC vector  $\mathbf{e}$  is always zero in the  $4D$  spacetime dimensions  $\psi^{34}$ , (the first right-moving column), there is no overlap between  $\mathbf{e}$  and  $N_{\overline{\alpha V}}$ , meaning that the graviton state is left unconstrained. That is, the graviton states are chargeless under  $\mathbf{e} \cdot \mathbf{Q}$  and cannot be projected out of the spectrum.

Gravitinos, which arise in an R sector in which  $[\epsilon_R, \epsilon_L] = [0, -1]$ , are formed from bosonic left-moving oscillators (integer modes); thus the left-moving elements

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<sup>1</sup>Note that, in opposition to those written in §2.5, here states are written  $|\rangle_R \otimes |\rangle_L$ , in keeping with convention and in order to match the format of the basis vectors,  $V_i = [V_R|V_L]$ .

of the gravitino states do not contribute to the determination of the fermionic charge vector  $\mathbf{Q}_\psi$ . As in the discussion introducing the massless R ground states in §2.2.5, right-moving Ramond free fermions that excite the right-moving vacuum give rise to massless gravitino states. Concretely, the excitations which give rise to the spacetime gravitinos arise from one of the fermions from the original  $10D$  right-moving superstring; that is, either  $\psi^{34}$ ,  $\psi^{56}$ ,  $\chi^{34}$  or  $\chi^{56}$ . Given the inevitable presence of the graviton, the SUSY properties of the theory are thus dictated by the presence or absence of the R-NS gravitinos in the light spectrum, as in eq.(2.5.14), namely (using the labels defined in §2.6)

$$\Psi_\alpha^\mu \equiv \left\{ \psi^{34}, \psi^{56}, \chi^{34}, \chi^{56} \right\}_\alpha |0\rangle_R \otimes X_{-1}^{34} |0\rangle_L. \quad (4.1.1)$$

The Scherk-Schwarz projections upon these states are determined purely by the Scherk-Schwarz action on the right-moving degrees of freedom.

It is necessary to consider the spectrum, found from the expressions for the modified Virasoro operators in eq.(3.2.4), in its entirety; that is, to consider the winding / KK modes together. For the non-winding gravitinos, the shifted KK momentum becomes virtually continuous in the  $r_i \rightarrow \infty$  limit and all modes of the gravitinos become massless; the full  $6D$  gravitino state is inevitably recovered there [155]. In this way, the supersymmetric  $6D$  theory is recovered as the compactification is turned off. Conversely, whenever the radii of compactification take non-zero values, the SUSY of the  $6D$  theory is broken. Indeed, the scale at which SUSY is spontaneously broken by the CDC is set by the gravitino mass  $\sim \frac{1}{2r_i}$ . Towards the opposite  $r_i \rightarrow 0$  end of the interpolation, new gravitinos, whose presence would restore supersymmetry at the zero radius endpoint, may or may not appear in the massless spectrum; for example, at this endpoint, the winding modes become light. The breaking of SUSY as the compactification radius is taken to its limits can be understood in terms of the appearance or disappearance of non-trivial winding / KK states of the gravitino at the massless level.

The argument can equally be made from the point of view of the dual theory, presented in §3.1.1, in which the mass squared of the states is expressed as

$$M^2 \sim (m_i \tilde{r}_i)^2 + \left( \frac{n_i}{\tilde{r}_i} \right)^2 + \text{oscillator contributions}. \quad (4.1.2)$$

In the limit that  $r_i \rightarrow 0$ , from eq.(3.2.9), the KK modes of the original theory become extremely heavy, while the winding modes constitute the light spectrum. Alternatively expressed, taking the limit that  $\tilde{r}_i \rightarrow \infty$ , with  $n_i, m_i$  suitably reinterpreted, the roles are reversed; the dual theory contains an infinite tower of light KK states. The projection conditions on the states in the dual theory as  $\tilde{r}_i \rightarrow 0$  will determine whether or not SUSY is restored at the opposite end of the interpolation.

## 4.2 Energy Scales

Taking the string scale as the cut-off, involves ignoring those dynamics which depend on smaller length / higher energy scales. Imposing such a restriction allows the low energy theory to be described by an effective field theory. It is interesting to identify the regimes in which there exists a low energy effective supergravity description for interpolating models.

Consider a theory whose infinite radius supersymmetry is not restored at zero radius. The scale of the spontaneous SUSY breaking in the CDC theory is set by the factor of  $\sim \frac{1}{2r_i}$  that defines the modified expression for the mass-squared of the states within the theory, given in eq.(3.2.8). Thus as  $r_i$  tends to infinity, the gravitino becomes sufficiently light to become part of the (dynamical) spectrum. It is only in the  $r_i \rightarrow \infty$  limit that SUSY is restored. Hence the appropriate description at large radius is given by an effective spontaneously broken supersymmetric field theory. In particular, the gravitinos can be incorporated within the effective superpotential of an effective supergravity theory, which is distinct from the UV string dynamics. The SUSY associated with the  $r_i \rightarrow \infty$  theory is spontaneously broken by this effective superpotential. Conversely, the gravitino mass becomes large in the small  $r_i$  limit, and exceeds the string scale. Thus the theory in this limit does not have gravitinos below the string scale. The gravitino dynamics cannot therefore be described by a light effective spontaneously broken supersymmetric theory.

In principle, the placement of the cut-off scale is a parameter that can be tuned. However, a logical choice is to take it to be of the order of the string scale, such that at the large and small radius endpoints of the interpolation, the gravitino is lighter and heavier respectively than the cut-off scale for a theory that becomes respectively supersymmetric and non-supersymmetric at these two endpoints.

## 4.3 Is the theory at small radius supersymmetric?

In order to see if gravitinos do appear in the theory at zero radius, consider how the CDC modifies the theories that sit at the endpoints of the interpolation. Let  $\mathbf{Q}_\psi^0$  denote the charge of the lightest gravitino state at large radius. SUSY is exact even in the presence of the CDC vector  $\mathbf{e}$ , with the state  $\mathbf{Q}_\psi^0$  being exactly massless, if both the first and second terms in the modified Virasoro operators of eq.(3.2.4) (with  $n = n_1 + n_2$ ), namely

$$(\mathbf{Q}_{L/R\psi}^0 - \mathbf{e}_{L/R} n)^2, \quad (4.3.1)$$

and, inserting the determined value of  $m_e$ , eq.(3.2.7), into eq.(3.2.4),

$$\frac{1}{4} \sum_{i=1,2} \left( \frac{m_i + \mathbf{e} \cdot \mathbf{Q}_\psi^0 - \frac{1}{2} n \mathbf{e}^2}{r_i} + / - n_i r_i \right)^2, \quad (4.3.2)$$

vanish. (For convenience, the original more restrictive rules of refs.[22–25] are used for this discussion; it would be trivial to extend the discussion to the more general rules of eq.(3.4.4).) With  $n_1 = n_2 = 0$ , the first term receives no extra contribution due to the CDC (compared to the theory defined by  $\bar{L}_0, L_0$ ). Furthermore, there is no winding contribution to the second term. Therefore  $m_i, n_i = 0$  gravitinos that have  $\mathbf{e} \cdot \mathbf{Q}_\psi^0 = 0$  remain massless and indicate the presence of exact SUSY. Conversely, if the only remaining gravitinos have

$$\mathbf{e} \cdot \mathbf{Q}_\psi^0 = \frac{1}{2}, \quad (4.3.3)$$

from the second term in eq.(3.2.8), which is the only term wholly decoupled from the winding and KK numbers, their mass is  $\frac{1}{2} \sqrt{\frac{1}{r_1^2} + \frac{1}{r_2^2}} = \frac{1}{2} \sqrt{\sum_{i=1,2} \frac{1}{r_i^2}}$  and SUSY is spontaneously broken.

Without loss of generality, one can consider SUSY breaking to amount to a conflict between  $\mathbf{e}$  and a single basis vector, denoted by  $V_{\text{con}}$ . That is,  $V_{\text{con}}$  constrains the gravitinos, while the remaining  $V_i$  cannot project them out of the theory. In order for the above light (but not massless) gravitino to be the one that is left un-projected, the projections due to  $\mathbf{e}$  and  $V_{\text{con}}$  must disagree. That is, the massive  $\mathbf{e} \cdot \mathbf{Q}_\psi^0 = \frac{1}{2}$  state is retained by  $V_{\text{con}}$  while the massless  $\mathbf{e} \cdot \mathbf{Q}_\psi^0 = 0$  state is projected out. Again without loss of generality, it is always possible to choose  $V_{\text{con}}$  so that the conditions are aligned; that is  $V_{\text{con}} \cdot \mathbf{Q}_\psi^0 = \frac{1}{2} \implies \mathbf{e} \cdot \mathbf{Q}_\psi^0 = \frac{1}{2}$ . These modes, with mass  $\sim \frac{1}{2r_i}$ , are preserved, while  $V_{\text{con}}$  projects the massless  $\mathbf{e} \cdot \mathbf{Q}_\psi^0 = 0$  modes out of the theory entirely.

Now consider the zero radius end of the interpolation, and denote the new would-be massless gravitino state by  $\tilde{\mathbf{Q}}_\psi$ . Although a different state, it can be related to the infinite radius gravitino  $\mathbf{Q}_\psi^0$  by a shift in the charge vector, induced by a potentially non-zero winding number;

$$\tilde{\mathbf{Q}}_\psi = \mathbf{Q}_\psi^0 - \mathbf{e} n. \quad (4.3.4)$$

As  $r_i$  vanish, the spectrum associated with the winding modes becomes continuous, while the KK states become extremely heavy. As described in the previous section, the requirement that the KK term in eq.(4.3.2) vanishes forms an effective projection that constrains the light states at zero radius, selecting the modes for which

$$\mathbf{e} \cdot \tilde{\mathbf{Q}}_\psi = \frac{n}{2} \pmod{1}, \quad (4.3.5)$$

where, as a consequence of modular invariance,  $\mathbf{e} \cdot \mathbf{e} = 1$ .

It is clear from the relation between  $\widetilde{\mathbf{Q}}_\psi$  and  $\mathbf{Q}_\psi^0$  in eq.(4.3.4) that the projection due to the CDC vector remains unchanged for any gravitino state at zero radius, since  $\mathbf{e}^2 n \in \mathbb{Z}$ ; that is

$$\mathbf{e} \cdot \widetilde{\mathbf{Q}}_\psi = \mathbf{e} \cdot \mathbf{Q}_\psi^0. \quad (4.3.6)$$

This equivalence of the vector dot product with the CDC vector for both the zero and infinite radius gravitinos means that eqs.(4.3.5) and (4.3.3) together imply that any gravitino of the spontaneously broken theory that becomes light at small radius must be an odd-winding mode:  $n \in 2\mathbb{Z} + 1$  for  $\widetilde{\mathbf{Q}}_\psi$ .

Under the shift given by eq.(4.3.4), the GSO projection that must be satisfied by the charge vector for physical states, eq.(2.6.12),  $\widetilde{Q}_{\alpha V}$ , is modified. An extra term appears in the GSO phase (where the projection holds mod(1)):

$$2\pi i\beta [V_i \cdot \widetilde{Q}_{\alpha V}] = 2\pi i\beta \left[ \sum_j k_{ij} \alpha_j + s_i - n V_i \cdot \mathbf{e} \right] = 2\pi i\beta [V_i \cdot \mathbf{Q}^0 - n V_i \cdot \mathbf{e}]. \quad (4.3.7)$$

Thus, the requirements that result in the GSO projections being non-trivially modified, are that:

- $V_i \cdot \mathbf{e} \notin \mathbb{Z}$ , and that
  1.  $(n_1 + n_2)$  be odd if  $V_i \cdot \mathbf{e} = \frac{1}{2}$ , or
  2.  $(n_1 + n_2) = 4k + 2$  or odd if  $V_i \cdot \mathbf{e} = \frac{2k'+1}{4}$ , with  $k, k' \in \mathbb{Z}$ .

In the present case, the  $V_{\text{con}}$  projection which constrains the gravitinos is

$$\begin{aligned} V_{\text{con}} \cdot \widetilde{\mathbf{Q}}_\psi &= V_{\text{con}} \cdot \mathbf{Q}_\psi^0 - n V_{\text{con}} \cdot \mathbf{e} \pmod{1} \\ &= \frac{1}{2} - n V_{\text{con}} \cdot \mathbf{e} \pmod{1}. \end{aligned} \quad (4.3.8)$$

For the effective projection in eq.(4.3.5) to agree with the modified GSO condition in eq.(4.3.8) for  $n \in 2\mathbb{Z} + 1$ , it is required that

$$V_{\text{con}} \cdot \mathbf{e} = 0 \pmod{1}. \quad (4.3.9)$$

Thus, neither condition 1. or 2. are satisfied, and the GSO projection is preserved. Eq.(4.3.9) is a necessary condition for a model with SUSY spontaneously broken by the Scherk-Schwarz mechanism to have massless gravitino states in both the infinite and zero radius limits. Note that this condition is necessary but not sufficient; it is still possible that SUSY can be broken by a CDC vector with non-zero left-moving entries, which project the gravitino states out of the theory, but which happen to be appropriately aligned such that they preserve eq.(4.3.9).

### 4.3.1 SUSY restoration when the CDC vector has zero left-moving entries

Consider the implications of the conclusion relating to eq.(4.3.9) for a specific theory. Consider the basis vector set  $\{V_0, V_1, V_2, V_4\}$ , together with a CDC vector that is empty in its left-moving elements, the standard set up outlined in [14], in which the vectors  $\{V_0, V_1, V_2\}$  project down to a 6D theory with  $\mathcal{N} = 1$  SUSY with orthogonal gauge group components:

$$\begin{aligned}
V_0 &= -\frac{1}{2}[ 11 111 111 \mid 1111 11111 111 111 11 111 ] \\
V_1 &= -\frac{1}{2}[ 00 011 011 \mid 1111 11111 111 111 11 111 ] \\
V_2 &= -\frac{1}{2}[ 00 101 101 \mid 0101 00000 011 111 11 111 ] \\
V_4 &= -\frac{1}{2}[ 00 101 101 \mid 0101 00000 011 000 00 000 ] \\
\mathbf{e} &= -\frac{1}{2}[ 00 101 101 \mid 0000 00000 000 000 00 000 ]. \tag{4.3.10}
\end{aligned}$$

The dot products between the basis vectors are:

$$V_i \cdot V_j = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \text{ mod}(1).$$

A suitable and consistent set of structure constants is

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

Consider the sector containing the gravitinos,  $\overline{V_0 + V_1} = \frac{1}{2}[11 100 100 \mid (0)^{20}]$ , for which the vacuum energies are  $[\epsilon_R, \epsilon_L] = [0, -1]$ . Gravitinos are formed from bosonic oscillators (integer modes); thus the left-moving elements of the gravitino states do not contribute to the determination of the fermionic charge vector  $\mathbf{Q}_\psi$ . Right-moving Ramond free fermions that excite the right-moving vacuum give rise to massless gravitino states. The charge operator for the non-winding gravitinos in the initial (infinite radius) theory takes the same form as the sector vector, since the

values of  $\frac{1}{2}$  in  $\overline{\alpha V}$ , corresponding to the fermions with Ramond boundary conditions, are shifted by integers by the fermion number  $N$  operator taking values of  $\pm 1$ , to other odd integer multiples of  $\frac{1}{2}$ . The gravitinos have charges determined by  $V_4$  that give the required  $\mathbf{e} \cdot \mathbf{Q} = 1/2 \pmod{1}$  for spontaneous SUSY breaking: the positive helicity states with this choice of structure constants are

$$\mathbf{Q}_\psi^0 = \frac{1}{2}[1 \ -1 \ \pm 100 \ \pm 100 \ | \ (0)^{20} \ ], \quad (4.3.11)$$

where the  $\pm$  signs on the fermions are co-dependent (meaning that the simultaneously take the same sign). It is clear from the vector overlap between  $\mathbf{Q}_\psi^0$  and  $V_4$  ( $V_4 \cdot \mathbf{Q}_\psi^0 = \frac{1}{2}$ ) that the latter is playing the role of  $V_{\text{con}}$  that constrains the gravitino states. (The structure constants have been chosen such that  $V_2$  yields identical constraints.) Whether or not any of the winding modes of the gravitinos are light at zero radius depends upon them satisfying the modified GSO projection condition of eq.(4.3.8):

$$V_4 \cdot \widetilde{\mathbf{Q}}_\psi = V_4 \cdot \mathbf{Q}_\psi^0 - V_4 \cdot \mathbf{e} (n_1 + n_2) \pmod{1}. \quad (4.3.12)$$

As explained, the projections at infinite and zero radius agree for the odd-winding modes of the  $\widetilde{\mathbf{Q}}_\psi$  states since  $V_4 \cdot \mathbf{e} = 0 \pmod{1}$ , and, using eq.(4.3.4), the charge vector for the small radius gravitino is shifted from the vector for the infinite radius theory by the CDC vector  $\mathbf{e}$ :

$$\widetilde{\mathbf{Q}}_\psi = \frac{1}{2}[1 \ -1 \ 00_{\pm 1} \ 00_{\pm 1} \ | \ (0)^{20} \ ]. \quad (4.3.13)$$

The shift in eq.(4.3.4) does not modify the numerical value of the projection eq.(4.3.12) on odd winding modes. Rather the change in the non-zero valued positions within  $\widetilde{\mathbf{Q}}_\psi$  results in a change in the particular fermions that constitute each massless state. Equivalently stated, the vector  $V_4$  does not ‘observe’ the swapping of the right-moving columns that occurs for the odd winding modes, other than in the fermions that it selects to comprise the preserved states. The boundary conditions for the  $\chi^{34}$  and  $\chi^{56}$  fermions (corresponding to columns 3 and 6 in the notation of [14]) are swapped with those of  $\omega^{34}$  and  $\omega^{56}$  (columns 5 and 8). The non-zero right-moving charges of the small radius gravitino correspond to the  $\omega^{34}$  and  $\omega^{56}$  worldsheet degrees of freedom, and thus they no longer overlap the SUSY charges of the large radius theory. Concretely, the theory contains the light gravitinos  $\omega_{-\frac{1}{2}}^{34} \omega_{-\frac{1}{2}}^{56} |0\rangle_R \otimes \alpha_{-1}^\mu |0\rangle_L$ .

The appearance of gravitino states in the light spectrum in the zero radius limit of this theory reflects a general conclusion. If the left-moving elements of the CDC vector vanish, eq.(4.3.9) is automatically satisfied. Any theory with a CDC vector acting purely on the spacetime side becomes supersymmetric at zero radius since the projection always preserves the odd-winding modes of the gravitinos. The non-

supersymmetric  $4D$  theory at generic radius is therefore an interpolation between two supersymmetric theories quite generally in these cases, which sit at the zero and infinite radius endpoints. In the following section, the supersymmetric nature of the zero radius theory will be verified by the vanishing of the cosmological constant  $\Lambda$  in the  $r_i \rightarrow 0$  limit (Figure 5.1). Note that the necessary cancellation between thousands of terms is highly non-trivial.

### 4.3.2 Example of a CDC vector with non-zero left-moving entries

Consider instead a theory composed of the same basis vector set as in eq.(4.3.10), but with a CDC vector containing non-zero left-moving entries: for example

$$\mathbf{e} = \frac{1}{2}[00\ 101\ 101\ | \ 1011\ 00000\ 000\ 100\ 01\ 111]. \quad (4.3.14)$$

Under the CDC, and for convenience of presentation dropping the  $\pm$  signs, the charge vector for the odd-winding gravitino modes is modified by eq.(4.3.4) to

$$\widetilde{\mathbf{Q}}_\psi = \frac{1}{2}[11\ 001\ 001\ | \ 1011\ 00000\ 000\ 100\ 01\ 111]. \quad (4.3.15)$$

As in the previous example the vector contains the same number of non-zero right-moving entries, but lying in different columns, so there is no contribution from eq.(4.3.1) to the mass squared on the spacetime side. However the non-zero left-moving elements now make a non-zero contribution. Under the shift,

$$(\mathbf{Q}_R^0, \mathbf{Q}_L^0)^2 \rightarrow (\widetilde{\mathbf{Q}}_R, \widetilde{\mathbf{Q}}_L)^2 = (\mathbf{Q}_R^0 + \mathbf{e}_R, \mathbf{Q}_L^0 + \mathbf{e}_L)^2, \quad (4.3.16)$$

any non-zero shift in  $\mathbf{Q}_L^0$  will inevitably produce massive gravitinos since in the R-NS<sup>2</sup> sector the charges of massless states must be zero mod (1) on the left-moving side.

In terms of the basis vectors, the additional factor in the expression for the modified charge vector (4.3.4) that arises in the zero radius limit, modifies the projection conditions (4.3.8) for those vectors  $V_i$  that overlap with  $\mathbf{e}$ . When the left-moving elements of  $\mathbf{e}$  are non-trivial, these vector overlaps can take  $\frac{1}{4}$  integer values, which can result in the impossibility of satisfying the modified GSO projection condition eq.(4.3.8). For example, it can immediately be seen that the dot product,  $V_{\text{con}} \cdot \mathbf{e} = \frac{1}{4}$  (in this case,  $V_{\text{con}} = V_4$ ), modifies (4.3.12), such that

$$V_{\text{con}} \cdot \widetilde{\mathbf{Q}}_\psi = V_{\text{con}} \cdot \mathbf{Q}_\psi^0 - \frac{1}{4}(n_1 + n_2) \pmod{1}. \quad (4.3.17)$$

---

<sup>2</sup>The gravitino is formed from the right-moving Ramond zero modes, and a coordinate oscillator acting on the left-moving vacuum (NS).

To preserve the  $\mathbf{e} \cdot \mathbf{Q}_\psi^0 = \frac{1}{2}$  gravitinos of the infinite radius theory at zero radius,  $(n_1 + n_2) = 0 \pmod{4}$ , which is incompatible with the requirement that the light states have  $(n_1 + n_2) = \text{odd}$ . These gravitino states, the only ones which could have emerged in the limit, are projected out of the spectrum, and the zero radius theory is non-supersymmetric.

Note that, as suggested in the general conclusion at the end of §4.3, had  $V_{\text{con}}$  and/or  $\mathbf{e}$  been chosen differently, it could have been the case that  $V_{\text{con}} \cdot \mathbf{e} = 0 \pmod{1}$ , even with  $\mathbf{e}_{LM}$  non-zero. In this case, the modified GSO projection would read

$$V_{\text{con}} \cdot \widetilde{\mathbf{Q}}_\psi = V_{\text{con}} \cdot \mathbf{Q}_\psi^0 - 0 \pmod{1}. \quad (4.3.18)$$

Naively, as in the previous subsection, it appears as though the projections due to  $V_{\text{con}}$  upon the zero and infinite radius states are free to align. However, the shift in  $\widetilde{\mathbf{Q}}_\psi$  owing to the  $e_{LM} \neq 0$ , eq.(4.3.16), renders massive the potential zero-radius gravitino states<sup>3</sup>. In most models studied, introducing non-zero terms to the left-hand side of  $\mathbf{e}$  will result in a non-trivial conflict with one of the basis vectors - most likely, that which constrains the gravitinos - such that supersymmetry will be broken. It is therefore instructive, but not conclusive, to consider the  $V_{\text{con}} \cdot \mathbf{e}$  projection, as above.

In conclusion, SUSY is restored at small as well as large radius if and only if the Scherk-Schwarz mechanism does not act on the gauge-side. Conversely, if SUSY is broken at zero radius, the gauge symmetry is as well. In the latter case, the non-SUSY intermediate radius model is an interpolation between a SUSY theory at the infinite radius endpoint, and a non-SUSY theory at zero radius. Under the interpolation, the CDC acts to project the gravitino of the infinite radius theory out of the theory at zero radius. This is the principal result of this study.

### 4.3.3 Formula for $N_b = N_f$ ?

Via its relationship to the one-loop partition function, eq.(2.7.1), the one-loop contribution to the cosmological constant eq.(3.3.1) is proportional to  $(N_b - N_f)$ . The net Bose-Fermi number for massless states appears as the coefficient of the constant term in the partition function  $\mathcal{Z} \supset (N_{b(0)} - N_{f(0)})q^0\bar{q}^0 + \dots$ . Thus, as explained in §2.7.5, at large  $R$ , the dominant terms in the one-loop contribution to the cosmological constant are proportional to  $(N_{b(0)} - N_{f(0)})$ , while the subleading corrections are exponentially suppressed with respect to these terms [14]. Thus non-supersymmetric models with an equal number of massless bosonic and fermionic states have an exponentially suppressed one-loop cosmological constant, and hence exhibit a greater

<sup>3</sup>As stated two paragraphs previously, non-zero values of  $e_{RM}$  simply shift the location of the non-zero entries, rather than their total number, thus leaving invariant the mass of the gravitino states.

degree of stability than those generic models in which  $N_{b(0)} \neq N_{f(0)}$ .

An interesting question is whether or not there is a consistent way of choosing the basis vectors  $\{V_i\}$ , the CDC vector  $\mathbf{e}$ , and the structure constants  $k_{ij}$ , which ensures that  $N_{b(0)} = N_{f(0)}$ . The choice of vectors  $\{V_i\}$  is constrained by the rules described in §2.6. If when choosing a set of basis vectors, a vector  $\{V_i\}$  is chosen such that  $N_{b(0)} \neq N_{f(0)}$ , it is possible that a different choice of the set of structure constants can restore the equality. The GSO projections are modified by changes in the values of the  $k_{ij}$ 's, such that certain states will be preserved within / projected out of the spectrum that is generated by the basis vector set  $\{V_i\}$  being considered. Thus, the choice of  $k_{ij}$ , in addition to the choice of the set  $\{V_i\}$ , represents a parameter that can be tuned in order to generate a set of basis vectors satisfying  $N_{b(0)} = N_{f(0)}$ .

As the number of vectors constituting the basis set increases, it rapidly becomes unfeasible to algebraically identify and describe models for which  $N_{b(0)} = N_{f(0)}$  (see Appendix F for a more detailed investigation). It appears that not only is there no principle to guarantee  $N_{b(0)} = N_{f(0)}$ , but that there is also no generic procedure for choosing the basis vectors  $\{V_i\}$ , the CDC vector  $\mathbf{e}$ , and the structure constants  $k_{ij}$ , that ensures that  $N_{b(0)} = N_{f(0)}$ . The alternative is to simply choose by hand a specific vector set, and test whether or not  $N_{b(0)} = N_{f(0)}$ . Thus, having acknowledged these difficulties, specific models that *do* exhibit  $N_{b(0)} = N_{f(0)}$ , can be introduced, and used as examples, alongside models in which  $N_{b(0)} \neq N_{f(0)}$ , to study compactifications of higher dimensional free fermionic models.

# Chapter 5

## Surveying the interpolation landscape

In order to verify the rules derived in the previous sections, in particular those that govern the supersymmetry properties of given models, the different possible interpolations are now surveyed. It should be pointed out that in order to make the exercise computationally feasible, only 0 and 1/2 phases are used, so that the theories contain only large orthogonal gauge groups. As such, no attempt is made to construct the SM, neither is the massless spectrum for each example presented. (The massless states can easily be determined using the rules in §2.6). Rather, studying the relationship between the cosmological constant  $\Lambda$  and the radii of compactification  $r_i$  exemplifies interpolation patterns between different types of model. Following the procedure outlined in Section 3.3, the total partition function,  $Z_{\text{total}}(\tau)$ , is input into the integral in eq.(3.3.1), for a range of compactification radii between either ends of the interpolation range. The  $q$ -expansion in the partition function is truncated at an order  $\mathcal{O}(q^2)$ , which is computationally manageable while displaying the qualitative behaviour. In order to facilitate the rapid processing of multiple models, the procedure detailed in §3.1.2 is implemented computationally, as described in detail in Appendix D.

The symmetries of the theories at the zero and infinite radius endpoints of the interpolation can be studied from a plot of the cosmological constant versus the radius of compactification. In the limit that the theory becomes supersymmetric, the cosmological constant vanishes. Consider models  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , as defined in Figure 1.2. A plot of the dependence of  $\Lambda$  on the radius vanishes at the infinite radius endpoint, corresponding to the supersymmetric model  $\mathcal{M}_1$ . Meanwhile, the behaviour at zero radius determines whether or not  $\mathcal{M}_2$  is supersymmetric. Thus a graphical analysis straightforwardly reveals the supersymmetry properties of the different models at different radii. Note that in the following plots, units, careful

track of which has not been kept throughout the treatment up until this point, are unimportant; rather the nature of the profiles reveal the different possible behaviours which can be exhibited by the different models.

## 5.1 Interpolation Between Two Supersymmetric Theories

### 5.1.1 $N_{b(0)} > N_{f(0)}$

Consider a theory containing  $V_0$ ,  $V_1$  and  $V_2$  as in the above basis vector set in eqs.(4.3.10), a modified  $V_4$ , an additional vector  $V_5$ , and a CDC vector that acts only on the spacetime side:

$$\begin{aligned} V_4 &= -\frac{1}{2}[ 00\ 000\ 000\ | 0101\ 00000\ 000\ 011\ 00\ 000\ ] \\ V_5 &= -\frac{1}{2}[ 00\ 000\ 011\ | 0101\ 11100\ 001\ 000\ 10\ 111\ ] \\ \mathbf{e} &= \frac{1}{2}[ 00\ 101\ 101\ | 0000\ 00000\ 000\ 000\ 00\ 000\ ]. \end{aligned} \quad (5.1.1)$$

A suitable and consistent set of structure constants  $k_{ij}$  is

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

This model can be investigated using the general method presented in the previous section.  $V_4$  plays the role of  $V_{\text{con}}$ , while  $V_5$  respects the projections of  $V_4$  on the gravitinos. As discussed, the interpolation is between two supersymmetric endpoints at both small and large radius. In between being zero at the two extremes, the cosmological constant takes a non-zero negative value at the intermediate minimum, which represents some form of stability, as displayed in Figure 5.1.

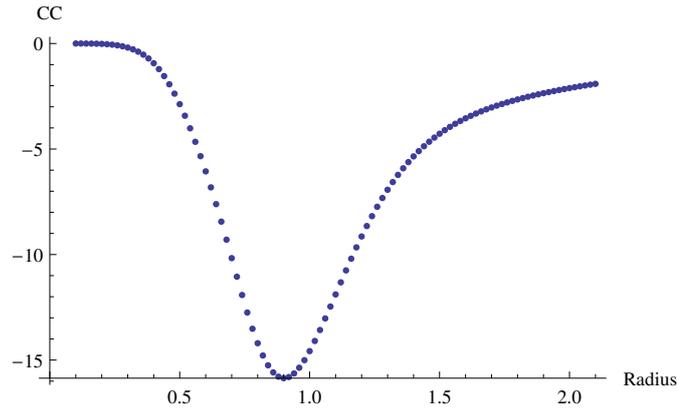


Figure 5.1: *Cosmological Constant vs. Radius*,  $r_1 = r_2 = r \in [0.1, 2.1]$  with radius increments of 0.02 for a model with  $\mathbf{e}_L = \text{trivial}$ .  $N_{b(0)} - N_{f(0)} = 28$ . This represents an example of an interpolation between two supersymmetric 6D theories via 4D theories with negative cosmological constant.

### 5.1.2 $N_{b(0)} < N_{f(0)}$

A theory in which  $N_{b(0)} < N_{f(0)}$  can be generated by performing an alternative modification to the vectors  $V_4, V_5$ ,

$$\begin{aligned}
 V_4 &= -\frac{1}{2} [ 00 \ 101 \ 101 \ | \ 0101 \ 00000 \ 011 \ 000 \ 01 \ 111 ] \\
 V_5 &= -\frac{1}{2} [ 00 \ 000 \ 011 \ | \ 0101 \ 11100 \ 010 \ 110 \ 00 \ 011 ] \\
 \mathbf{e} &= \frac{1}{2} [ 00 \ 101 \ 101 \ | \ 0000 \ 00000 \ 000 \ 000 \ 00 \ 000 ], \quad (5.1.2)
 \end{aligned}$$

with the structure constants

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly to the  $N_{b(0)} > N_{f(0)}$  model with an exclusively non-trivial right-moving CDC vector, this model interpolates between two supersymmetric endpoints at both small and large radius, but with the cosmological constant taking a non-zero positive value at intermediate radii, as displayed in Figure 5.2. This profile describes unstable runaway to decompactification at either end of the interpolation.

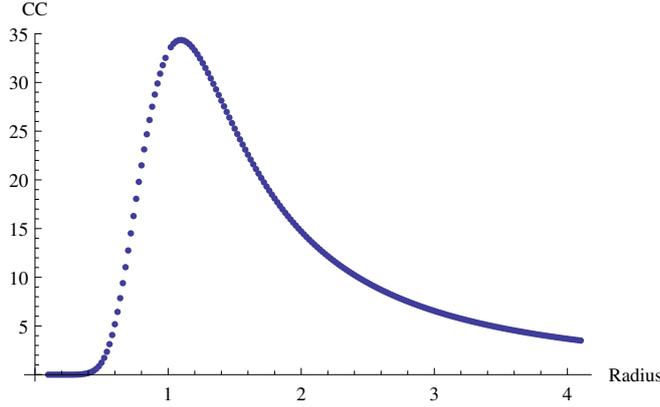


Figure 5.2: *Cosmological Constant vs. Radius*,  $r_1 = r_2 = r \in [0.1, 4.1]$  with radius increments of 0.02 for a model with  $\mathbf{e}_L = \text{trivial}$  and  $N_{b(0)} - N_{f(0)} = -228$ . This represents an example of an interpolation between two supersymmetric 6D theories via 4D theories with positive cosmological constant.

## 5.2 Interpolation from a Non-supersymmetric to a Supersymmetric Theory

### 5.2.1 $N_{b(0)} = N_{f(0)}$

A theory with Bose-Fermi degeneracy can be achieved with a theory comprised of the basis vector set in eq.(4.3.10), plus a basis vector  $V_5$  and CDC vector of the form

$$\begin{aligned} V_5 &= -\frac{1}{2} [ 00\ 000\ 011 \mid 0100\ 11100\ 000\ 111\ 10\ 011 ], \\ \mathbf{e} &= \frac{1}{2} [ 00\ 101\ 101 \mid 1011\ 00000\ 000\ 100\ 01\ 111 ], \end{aligned} \quad (5.2.1)$$

with  $k_{ij}$  given by

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

$N_{b(0)}$  and  $N_{f(0)}$  are found to be equal despite the fact that the theory is non-supersymmetric (as can be seen by the absence of any massless gravitinos when the full spectrum is calculated). For models such as this, in which the CDC vector  $\mathbf{e}$  is non-trivial in both the global (spacetime supersymmetry) *and* gauge entries,

the cosmological constant takes a non-zero value at small radius, while it vanishes exponentially quickly for large compactification scales, as displayed in Figure 5.3.

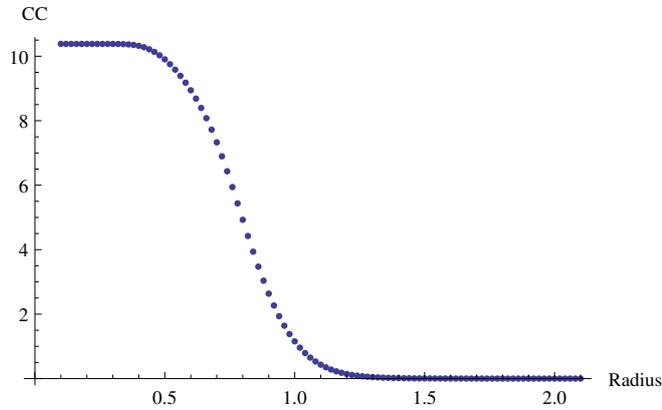


Figure 5.3: *Cosmological Constant vs. Radius,  $r_1 = r_2 = r \in [0.1, 2.1]$  with radius increments of 0.02 for a model with  $\mathbf{e}_L =$  non-trivial and  $N_{b(0)} = N_{f(0)}$ . This represents an example of an interpolation between supersymmetric and non-supersymmetric 6D theories.*

### 5.2.2 $N_{b(0)} > N_{f(0)}$

An interpolation from SUSY to non-SUSY in which  $N_{b(0)} > N_{f(0)}$ , can be achieved by taking the set of basis vectors in eqs.(5.1.1), but with a CDC vector of the form

$$\mathbf{e} = \frac{1}{2}[ 00\ 101\ 101\ | 0101\ 00000\ 000\ 110\ 11\ 011 ]. \quad (5.2.2)$$

For models in which  $N_{b(0)} > N_{f(0)}$ , the cosmological constant reduces from a constant positive value at small radius reaching a negative minimum at approximately  $r = 1.0$  in string units. As the radius increases to  $\infty$ , the cosmological constant tends to zero from negative values, consistent with the restoration of SUSY in the endpoint model, as displayed in Figure 5.4. In this particular example, the turnover appears to be at precisely 1 string unit, suggesting that a winding mode becomes massless at this point, enhancing the gauge symmetry.

### 5.2.3 $N_{b(0)} < N_{f(0)}$

Finally for a non-SUSY to SUSY interpolation with  $N_{b(0)} < N_{f(0)}$ , the model in eqs.(5.1.2) is taken, but with a CDC vector of the form

$$\mathbf{e} = \frac{1}{2}[ 00\ 101\ 101\ | 0101\ 00000\ 000\ 011\ 11\ 011 ]. \quad (5.2.3)$$

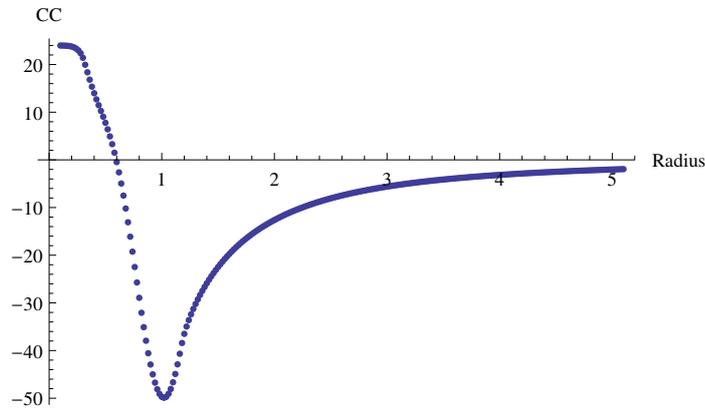


Figure 5.4: *Cosmological Constant vs. Radius,  $r_1 = r_2 = r \in [0.1, 5.1]$  with radius increments of 0.02 for a model with  $\mathbf{e}_L = \text{non-trivial}$ , and  $N_{b(0)} - N_{f(0)} = 192$ . This represents an example of an interpolation between non-supersymmetric and supersymmetric 6D theories with an Anti-de Sitter minimum.*

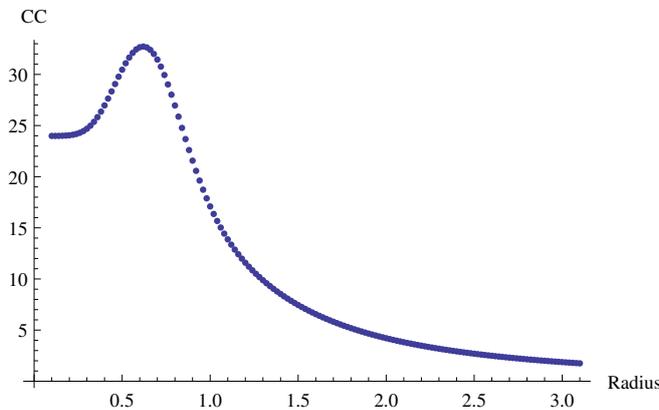


Figure 5.5: *Cosmological Constant vs. Radius,  $r_1 = r_2 = r \in [0.1, 3.1]$  with radius increments of 0.02 for a model with  $\mathbf{e}_L = \text{non-trivial}$  and  $N_{b(0)} - N_{f(0)} = -64$ . This represents an example of an interpolation between a metastable non-supersymmetric 6D theory at the zero radius endpoint and a supersymmetric 6D theory which emerges at infinite radius, with a de Sitter maximum at an intermediate radius.*

The cosmological constant of the non-SUSY  $6D$  theory at zero radius increases from a constant positive value, to a maximum (at approximately  $r = 2/3$  in string units). This time, as the radius tends to  $\infty$ , the cosmological constant tends to zero positively, defining a SUSY  $6D$  theory at infinite radius, as displayed in Figure 5.5.



# Chapter 6

## Conclusions

While it remains open as to whether or not string theories provide a more fundamental description of nature than that provided by other theories so far concocted, such theories undeniably possess many desirable features, not least that, given that the graviton is intrinsic to string spectra, they represent UV finite, anomaly free theories of quantum gravity. Furthermore, as is conjectured, all string theories emerge from the hidden underlying M-theory. This might provide an explanation for the fact that the landscape of possible string theories is more restricted than for ordinary quantum field theories, justifying the interpretation that string theories have a higher degree of naturalness. Yet for many, practical applications, in the broadest sense of the phrase, do not act as the prime motivation, since string theory provides in its own right a multitude of beautiful mathematical and physical results.

§1.1.1 provided a few motivating factors for the study of string theories, in the hope of simultaneously making contact with the wider particle physics community, while also establishing the validity of the study in its own right. Having in §2 provided the necessary background theory on bosonic and fermionic strings, partition functions at one-loop, the power of conformal symmetries in two-dimensions, compactifications, and ultimately heterotic strings in the free fermionic formulation, it has been possible to introduce and discuss in §3 the main subject of this investigation, interpolating models. Following on from ref.[14], the nature of heterotic strings in the context of Scherk-Schwarz compactification has been investigated, with particular emphasis on their properties under interpolation. It has been shown how to use Coordinate Dependent Compactifications to provide a tuneable parameter with which to interpolate between higher dimensional endpoint theories. From the starting point of supersymmetric  $6D$  theories in the infinite radius limit, Scherk-Schwarz compactification to  $4D$  yields models that have  $N_{b(0)} \{=, <, >\} N_{f(0)}$ , each possibility exhibiting different behaviours under interpolation. The discussion has been concretised with the specific models presented in §5. These theories exhibit a

radius dependent value for the cosmological constant. The behaviour of their cosmological constants has thus been studied as a function of compactification radius, and it has been found that theories can yield maximum or minimum values of the cosmological constant at intermediate values of  $r_1 = r_2 = r$ , as well as barriers with apparent metastability. The latter feature may have interesting phenomenological and/or cosmological applications. Thus, it has been shown how the value of the cosmological constant represents a means by which to evaluate the stability properties of such models.

The construction in this study relates a large class of non-supersymmetric theories to supersymmetric ones. As explained in §4, by analysing the behaviour of the gravitino under the process of interpolation, the relation between the interpolating theory at intermediate radius and the  $6D$  theories that emerge at the end-points of the interpolation has been studied. The observation that the Scherk-Schwarz action descends from an additional GSO projection in the  $6D$  zero radius endpoint theory has been made. This allows the modular invariance constraints of the  $6D$  theory to be used to derive a more general class of Scherk-Schwarz compactification. The nature of the Scherk-Schwarz action, in particular whether or not it simultaneously acts to break the gauge group, dictates whether or not SUSY emerges in the  $6D$  theory at zero radius. In the  $r_i \rightarrow 0$  or  $\infty$  limit, the overall set of projection conditions, which define those states which comprise the spectrum, are modified by this new condition. Thus the presence or absence of the would be gravitinos in the spectrum determines whether or not SUSY is preserved in a given model. This represents a remarkably straightforward procedure. This result has the potential to provide a framework upon which it would be possible to hang much of the investigation into non-supersymmetric string theories that has already been carried out.

The aim of this work has been to establish the general features of interpolating models, relating higher,  $D$ -dimensional models to  $(D - d)$ -dimensional compactified models. As depicted in Figure 1.2, having started with a supersymmetric, higher dimensional model, a description for how to interpolate to a non-supersymmetric lower dimensional model has been provided. Alternatively, from the starting point of a non-supersymmetric,  $(D - d)$ -dimensional theory, the ultimate goal would be to find a relation, by the process of interpolation, to a higher,  $D$ -dimensional model. It is conceivable that many non-supersymmetric tachyon-free  $4D$  models can be interpolated to higher dimensional supersymmetric ones in this way. By such a relation, any  $4D$  tachyon free non-supersymmetric model could be related to a  $6D$  supersymmetric model. Looking forward, it may not be possible to show that *every* non-supersymmetric theory is related to a supersymmetric counterpart via the process of interpolation. However, it seems possible that such a relation might always hold for the particular class of theory in which SUSY is broken by discrete

torsion, as in ref.[86] for example. To concretise this relation, it would be necessary to show that upon undergoing an interpolation, gravitinos, which were projected out by the CDC, reappear in the spectra of models in which SUSY is broken thus.

It is important to reiterate that the non-supersymmetric models which sit in the middle of the interpolations discussed in this study are in no sense less fundamental than the original, supersymmetric endpoint models. It is not the case that one begins with a supersymmetric theory, and at some specific energy scale, a breaking results in a non-supersymmetric theory. This would restrict the validity of the construction to a specific energy range. Rather, the final non-supersymmetric theory can be understood in terms of an original supersymmetric theory, plus some additional setup. The latter model lays just as equal a claim as the former to providing a fundamental description of nature at all energy scales.

The proposed roadmap by which one might navigate the landscape of interpolating models might take the following structure. Starting with a non-supersymmetric  $4D$  theory, (the choice of which would ultimately be made based upon the requirement that it should exhibit desirable phenomenological properties), for the specific class of non-SUSY  $4D$  models upon which this study has focussed, it is possible to interpolate back to higher dimensional,  $6D$  endpoint theories, whose supersymmetry properties are controlled by the limiting behaviour of the CDC parameters. The web which links models constructed in this way, depicted in Figure 6.1, can be traversed thus:

1. Begin with a phenomenologically appealing non-SUSY  $4D$  model.
2. Take  $d$  (in this study,  $d = 2$ ) of the dimensions which define the original compactification from  $10D$  to  $4D$ , to their limits (e.g.  $\{r_1, r_2\} \rightarrow 0$  and  $\infty$ ), to decompactify the theory to the respective  $6D$  endpoint theories, at least one of which should be supersymmetric.
3. Identify the CDC vector,  $\mathbf{e}$ , that breaks SUSY in the  $6D$  model.
4. Construct the interpolation between the endpoint theories via the non-SUSY  $4D$  model. (Note that, as described, a SUSY or non-SUSY  $4D$  model can be obtained by performing a  $\mathcal{Z}_2$  orbifold either with or without this CDC vector.)

A goal for future work would be to establish relationships, of the type found in this study, between additional lower dimensional, non-supersymmetric models, ideally of greater phenomenological appeal, and their supersymmetric counterparts. If it can be shown that non-supersymmetric models generically relate to supersymmetric theories in this way, interpolation could be used as a tool with which to relate many tachyon-free non-supersymmetric string theories to their supersymmetric siblings.

Thus it would be possible to locate non-supersymmetric models within the larger network of string theories, extending previous work in this direction.

In the present discussion, a  $6D$   $\mathcal{N} = 1$  theory is compactified on a  $T_2/\mathbb{Z}_2$  orbifold, yielding a  $4D$   $\mathcal{N} = 0$  or  $1$  model, depending on whether or not the CDC is present. In the infinite radius limit of the  $4D$  theory, the original  $6D$  theory is recovered. There exists a dual  $6D$  theory at the zero radius endpoint. The logic could potentially be repeated in fewer dimensions, to set up a framework within which the interpolation takes place between supersymmetric and non-supersymmetric  $4D$  theories, the latter obviously being of greatest phenomenological interest. A  $\mathbb{Z}_2 \times \mathbb{Z}_2$   $4D$   $\mathcal{N} = 1$  model (for example, one of the  $4D$ ,  $T_2/\mathbb{Z}_2$  orbifolds of the original  $6D$  model) could be compactified on another  $T_2/\mathbb{Z}_2$  orbifold to yield a  $2D$  theory. Thus, the decompactification limit of the dual theory could be taken in order to obtain a non-supersymmetric  $4D$  model. A similar interpolation relation would have been established in fewer dimensions, between an original  $\mathcal{N} = 1$   $4D$  model, and a new,  $\mathcal{N} = 0$ ,  $4D$ , dual model in the  $R \rightarrow 0$  limit. The difference in this set-up would be that the  $4D$  theories of phenomenological interest would be found at the endpoints of the interpolation, rather than at generic compactification radii, as is the case in the afore discussed models. In this context, interpolation could be used as a tool to pass from a non-SUSY to a SUSY  $4D$  model.

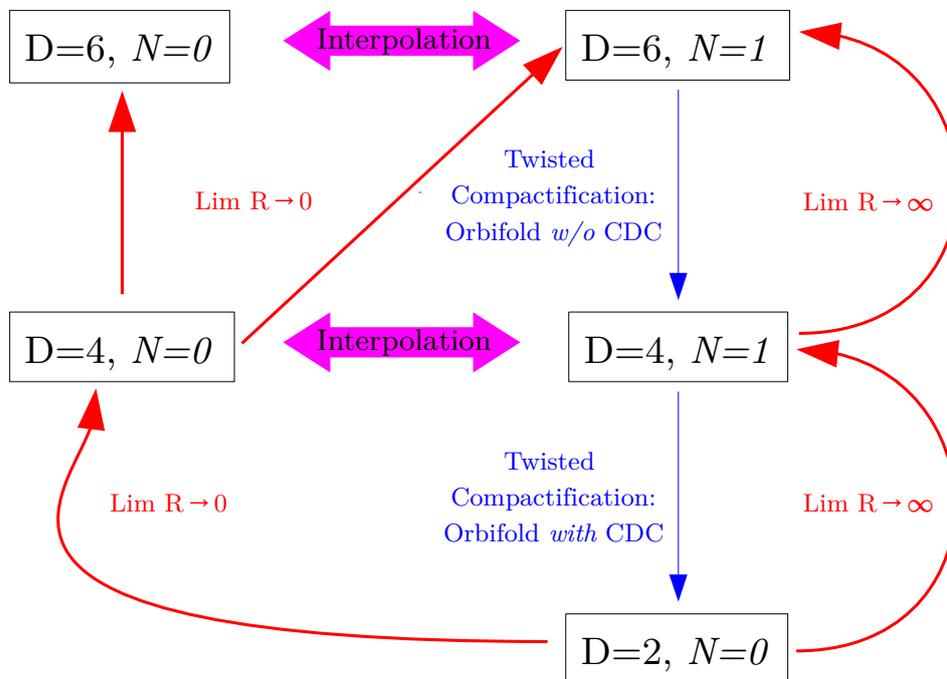


Figure 6.1: The map between  $4D$  theories in Figure (1.2) can be extended to include the interpolation between  $6D$  theories.  $\mathcal{N} = 1$  SUSY is preserved under the compactification from six to four dimensions, but broken by the CDC from four to two.

The principal challenge facing non-supersymmetric model building is that, admitting as they do non-zero dilaton tadpoles, generic non-supersymmetric strings exhibit an unfavourable degree of instability. However, it has been explained that all non-supersymmetric strings exhibit a misaligned supersymmetry, which is ultimately responsible for maintaining a degree of finiteness which naively requires spacetime supersymmetry. Furthermore, it has been shown that it is possible to construct a particular class of models in which the degree of instability can be exponentially suppressed. By implementing a Scherk-Schwarz compactification with a coordinate dependence, the generic radii of compactification from six to four dimensions have been found to represent a tunable parameter with which the cosmological constant can be controlled. The strings studied in this investigation belong to a class of models which interpolate between  $6D$  endpoint theories via a 4-dimensional theory at intermediate radii. Interpolation provides a tentative procedure by which to link unstable, non-supersymmetric models to more robust supersymmetric counterparts. More work is needed to flesh out these relations.

As it stands, there exist many hurdles to be overcome in the field of non-supersymmetric string theory and phenomenology. However, as long as the question remains open, non-supersymmetric string models cannot be ruled out as candidate theories of nature. It remains to be seen what predictive power such models may possess when applied across a broader range of applications and in a wider set of contexts.



# Appendix A

## Conformal transformations

Inserting the expression for an infinitesimal coordinate transformation,

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2) , \quad (\text{A.0.1})$$

into the left hand side of the condition that a transformation be conformal,

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) \eta_{\mu\nu} , \quad (\text{A.0.2})$$

yields

$$\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \eta_{\mu\nu} + (\partial_{\nu}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\nu}) + \mathcal{O}(\epsilon^2) . \quad (\text{A.0.3})$$

In order that the transformation be conformal, that is, that the metric transforms only up to a conformal scale, the  $\mathcal{O}(\epsilon)$  terms on the right hand side of this expression must equal some factor of the metric,

$$(\partial_{\nu}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\nu}) = f(x) \eta_{\mu\nu} , \quad (\text{A.0.4})$$

such that

$$1 + f(x) = \Lambda(x) , \quad (\text{A.0.5})$$

and the condition eq.(A.0.2) is satisfied. Taking the trace of eq.A.0.4 with  $\eta^{\mu\nu}$  yields:

$$2\partial^{\mu}\epsilon_{\mu} = f(x) d . \quad (\text{A.0.6})$$

Thus conformal invariance is guaranteed as long as:

$$\partial_{\nu}\epsilon_{\mu} + \partial_{\mu}\epsilon_{\nu} = \frac{2}{d}(d \cdot \epsilon)\eta_{\mu\nu} . \quad (\text{A.0.7})$$



# Appendix B

## Notation and conventions for partition functions

The basic Dedekind  $\eta$ -function,  $\eta$ , and Jacobi  $\vartheta$ -functions,  $\vartheta$ , are defined with the following conventions

$$\begin{aligned}
 \eta(\tau) &\equiv q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3(n-1/6)^2/2}, \\
 \vartheta_1(\tau) &\equiv -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2/2}, \\
 \vartheta_2(\tau) &\equiv 2q^{1/8} \prod_{n=1}^{\infty} (1 + q^n)^2 (1 - q^n) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2/2}, \\
 \vartheta_3(\tau) &\equiv \prod_{n=1}^{\infty} (1 + q^{n-1/2})^2 (1 - q^n) = \sum_{n=-\infty}^{\infty} q^{n^2/2}, \\
 \vartheta_4(\tau) &\equiv \prod_{n=1}^{\infty} (1 - q^{n-1/2})^2 (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2}. \quad (\text{B.0.1})
 \end{aligned}$$

$q$  is the square of the nome, *i.e.*,  $q \equiv \exp(2\pi i\tau)$ , with  $\tau_{1,2}$  respectively denoting  $\text{Re } \tau$  and  $\text{Im } \tau$ . These functions satisfy the identities  $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$  and  $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$ . Note that  $\vartheta_1(q)$  has a vanishing  $q$ -expansion and is modular invariant; its infinite-product representation has a vanishing coefficient and thus is not shown.

In order to simplify and unify the notation, several generalizations of these functions are introduced. First, the more general  $\vartheta$ -function of two arguments is defined as:

$$\begin{aligned}
 \vartheta(z, \tau) &\equiv \sum_{n=-\infty}^{\infty} \xi^n q^{n^2/2} \\
 &= q^{-1/24} \eta(\tau) \prod_{m=1}^{\infty} (1 + \xi q^{m-1/2}) (1 + \xi^{-1} q^{m-1/2}), \quad (\text{B.0.2})
 \end{aligned}$$

where  $\xi \equiv e^{2\pi iz}$ . Similarly, the  $\vartheta$ -functions with characteristics are defined as

$$\begin{aligned} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) &\equiv \sum_{n=-\infty}^{\infty} e^{2\pi i(n+a)(z+b)} q^{(n+a)^2/2} \\ &= e^{2\pi iab} \xi^a q^{a^2/2} \vartheta(z + a\tau + b, \tau). \end{aligned} \quad (\text{B.0.3})$$

These latter functions have a certain redundancy, depending on only  $z + b$  rather than  $z$  and  $b$  separately. Under shifts in their characteristics, the  $\vartheta$ -functions satisfy

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) = e^{2\pi iab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau). \quad (\text{B.0.4})$$

For  $a, b \in \{0, 1/2\}$ , there exist four permutations, which are commonly abbreviated as

$$\begin{aligned} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) &= \vartheta_3(z, \tau) \equiv \vartheta_{00}, & \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} (z, \tau) &= \vartheta_2(z, \tau) \equiv \vartheta_{10}, \\ \vartheta \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} (z, \tau) &= \vartheta_4(z, \tau) \equiv \vartheta_{01}, & \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z, \tau) &= -\vartheta_1(z, \tau) \equiv \vartheta_{11}. \end{aligned} \quad (\text{B.0.5})$$

In general, the functions in Eq. (B.0.3) have modular transformations

$$\begin{aligned} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, -1/\tau) &= \sqrt{-i\tau} e^{2\pi iab} e^{i\pi\tau z^2} \vartheta \begin{bmatrix} -b \\ a \end{bmatrix} (-z\tau, \tau), \\ \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau + 1) &= e^{-i\pi(a^2+a)} \vartheta \begin{bmatrix} a \\ a+b+1/2 \end{bmatrix} (z, \tau). \end{aligned} \quad (\text{B.0.6})$$

Similarly,

$$\eta(\tau + 1) = e^{2\pi i/24} \eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau). \quad (\text{B.0.7})$$

To evaluate the cosmological constant from the partition function in §Section 3.3, the following  $q$ -expansions, in the  $\tau_2 \gg 1$  (or  $|q| \ll 1$ ) limit, are required:

$$\begin{aligned} \eta(\tau) &\sim q^{1/24} + \dots, \\ \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (0, \tau) &\sim 1 + 2q^{1/2} + \dots, \\ \vartheta \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} (0, \tau) &\sim 1 - 2q^{1/2} + \dots, \\ \vartheta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (0, \tau) &\sim 2q^{1/8} + \dots, \\ \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (0, \tau) &= 0. \end{aligned} \quad (\text{B.0.8})$$

Regarding partition functions, the expression for the compactified bosonic com-

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ponent of the partition function is given in [14]. The expression for the untilted torus in terms of radii  $r_1, r_2$  is required. The Poisson-resummed partition function for the compactified complex boson in the untwisted sectors (denoted by  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ) is given by

$$\mathcal{Z}_{\mathbf{B}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}(\tau) = \mathcal{M}^2 \frac{r_1 r_2}{\tau_2 |\eta(\tau)|^4} \sum_{\mathbf{n}, \mathbf{m}} \exp \left\{ -\frac{\pi}{\tau_2} r_1^2 |m_1 + n_1 \tau|^2 - \frac{\pi}{\tau_2} r_2^2 |m_2 + n_2 \tau|^2 \right\}. \quad (\text{B.0.9})$$



# Appendix C

## Modular transformations of the partition function

The contribution to the partition function for a  $(D - d)$ -dimensional string theory compactified on a  $d$ -dimensional manifold, which is parametrised by compactification radii  $r_i$ , can be expressed in terms of the squared quantities  $p_R, p_L$ , defined in eq.(2.4.11). Consider, for concreteness, the theory compactified on a single circle of radius  $r_i$ :

$$\mathcal{Z}(\tau) \sim \tau_2^{1-\frac{D}{2}} \bar{q}^m q^n \sim \tau_2^{1-\frac{D}{2}} \bar{q}^{\bar{L}_0} q^{L_0} \sim \tau_2^{1-\frac{(D-1)}{2}} \bar{q}^{\bar{N}_{1.c.} + \bar{E}_0} q^{N_{1.c.} + E_0} \bar{q}^{\frac{p_L^2}{2}} q^{\frac{p_R^2}{2}}. \quad (\text{C.0.1})$$

Strictly speaking, eq.(C.0.1) defines the integrand. The partition function corresponds to an integral of the above quantity over the fundamental domain of  $SL(2, \mathbb{Z})/\mathbb{Z}_2$ . Given that the measure, included in the integral which defines the partition function, is modular invariant, eq.(C.0.1) must therefore be modular invariant in its own right. Furthermore, it was stated in eq.(2.3.42) that the non-compact contribution, which can be written

$$\mathcal{Z}_{\text{non-compact}}(\tau) \sim \tau_2^{1-\frac{D}{2}} \bar{q}^{\bar{N}_{1.c.} + \bar{E}_0} q^{N_{1.c.} + E_0}. \quad (\text{C.0.2})$$

is also modular invariant in its own right. Thus, modular invariance of the compact contribution must be shown. Expanding in terms of  $\tau$ ,

$$\mathcal{Z}_{\text{compact}}(\tau) \sim \tau_2^{\frac{1}{2}} e^{-2\pi i \bar{\tau} \frac{p_L^2}{2}} e^{2\pi i \tau \frac{p_R^2}{2}}. \quad (\text{C.0.3})$$

Under a  $\mathcal{T}$  transformation,  $\tau \rightarrow \tau + 1$ ,  $\mathcal{Z}(\tau)$  transforms as

$$\mathcal{Z}_{\text{compact}}(\tau + 1) \sim \tau_2^{\frac{1}{2}} e^{-2\pi i (\bar{\tau} + 1) \frac{p_L^2}{2}} e^{2\pi i (\tau + 1) \frac{p_R^2}{2}} = \mathcal{Z}_{\text{compact}}(\tau) e^{2\pi i \frac{p_R^2 - p_L^2}{2}}, \quad (\text{C.0.4})$$

which restricts  $(p_R^2 - p_L^2)/2$  to be integer.

Under an  $\mathcal{S}$  transformation,  $\tau \rightarrow -1/\tau$ ,

$$\mathcal{Z}_{\text{compact}}(-1/\tau) \sim \left( \frac{\tau_2}{|\tau|^2} \right)^{\frac{1}{2}} e^{\frac{2\pi i p_L^2}{\bar{\tau}} \frac{1}{2}} e^{\frac{-2\pi i p_R^2}{\tau} \frac{1}{2}}. \quad (\text{C.0.5})$$

In order to see that this expression equals  $\mathcal{Z}_{\text{compact}}(\tau)$ , it is necessary to use the Poisson resummation formula, which relates the Fourier coefficients of the periodic summation of a function to the coefficients of the periodic summation of its Fourier transform:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{t=-\infty}^{\infty} \hat{f}(t). \quad (\text{C.0.6})$$

In general

$$\sum_{n \in \mathbb{Z}} \exp \left[ -\pi A(n + \theta)^2 + 2\pi i(n + \theta)\phi \right] = A^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \exp \left[ -\pi A^{-1}(k + \phi)^2 - 2\pi i k \theta \right]. \quad (\text{C.0.7})$$

For  $f(x) = e^{-ax^2}$ , the Fourier transform  $\hat{f}(t)$  is  $\sqrt{\frac{\pi}{a}} e^{-\frac{(\pi t)^2}{a}}$ . The Poisson resummation formula allows one to take  $t = x$  in the resulting formula. Thus, one finds a factor of  $\sqrt{\bar{\tau}\tau}$  from the Poisson resummation on  $p_R, p_L$ :

$$\begin{aligned} \mathcal{Z}_{\text{compact}}(-1/\tau) &\sim \left( \frac{\tau_2}{|\tau|^2} \right)^{\frac{1}{2}} e^{-2\pi i \bar{\tau} \frac{p_L^2}{2}} e^{2\pi i \tau \frac{p_R^2}{2}} \sqrt{(i\bar{\tau})(-i\tau)} \\ &= \mathcal{Z}_{\text{compact}}(\tau) \left( \frac{1}{\sqrt{\bar{\tau}\tau}} \right) \sqrt{\bar{\tau}\tau} \\ &= \mathcal{Z}_{\text{compact}}(\tau). \end{aligned} \quad (\text{C.0.8})$$

# Appendix D

## Structure of the calculation

In order to obtain the results described in this study, it has been necessary to calculate the spectra and partition functions of a wide range of different free fermionic models. That is, models defined by sets of basis vectors  $\{V_i\}$  and structure constants  $k_{ij}$ , are subject to the procedure outlined in §3.1.2. This procedure is most easily implemented algorithmically. While conducting this investigation, scripts have been written in both Python and Mathematica, which efficiently construct models, generating their partition functions, low lying spectra and cosmological constants as a function of the compactification radii. The main building blocks of these modules are as follows. The *blue* variables, which are the principle objects defined in the programmes, use notation similar to that used in §3.3.

- **Constructing the fundamentals of the KLT [145–148] formalism.**

Define the set of basis vectors,  $\{V_i\}$ , the structure constants  $k_{ij}$ , and the form of vector sums and Lorentzian lattice contractions.

Define the  $\vartheta$  and  $\eta$  functions, and their  $q$ -expansions up to the defined order (chosen to set the order of the truncation of the massive spectrum).

- **Main routine to scan all sectors (the fermionic partition function is sector dependent).**

The set of sectors  $\alpha, \beta$  for given  $nn, \alpha_e$ , are generated.

- **Fermionic Partition Function polynomial:**  $\mathcal{Z}_F(\alpha V_{\text{in}}, \beta V_{\text{in}})$ , which corresponds to eq.(3.3.13).

Summing over the  $\alpha, \beta$  sectors,  $\Omega_{\text{vec}}(n, \alpha, \beta, \beta_e)$  stores a vector of coefficients and  $\vartheta$ -functions for the fermionic partition function polynomial for each pair of  $nn, \alpha_e$ . The vector takes the format:  $\{c_{mn}, \{\vartheta_2^a, \vartheta_3^b, \vartheta_4^c, \overline{\vartheta}_2^d, \overline{\vartheta}_3^e, \overline{\vartheta}_4^f\}\}$  (with  $c_{mn}$  representing the coefficient of each  $\vartheta$  product term, and  $\{a b c d e f\}$  representing the respective multiplicity of each of the  $\vartheta$ -functions).

Having incorporated the overall  $\eta$  factors, ultimately  $\mathcal{Z}_F(\alpha V_{\text{in}}, \beta V_{\text{in}})$  generates and stores the fermionic partition function polynomial for each of the 16  $nn$ ,  $\alpha_e$  sectors in terms of their  $q$ -expansions.

- **Bosonic Partition Function:**  $\mathcal{Z}_B(r_1, r_2, n, \alpha_e)$ , which corresponds to eq.(3.3.14).

This module produces the non-resummed, compactification radius dependent partition function factors for bosons (minus the overall Dedekind factors) for given  $nn$ ,  $\alpha_e$ .

- **Total Partition Function:**  $\mathcal{Z}_{\text{total}}(r_1, r_2)$ , which corresponds to eq.(3.3.15).

This module sums the product of the fermionic and bosonic partition function polynomials,  $\mathcal{Z}_F$  and  $\mathcal{Z}_B$ , in each of the 16  $nn$  and  $\alpha_e$  sectors. The output polynomial, in which the sum of the exponents ( $m + n$ ) of each  $q^m \bar{q}^n$  term can take values up to a defined order, is in the form:

$$\mathcal{Z}_{\text{total}} = \sum_{m,n} c_{mn} q^m \bar{q}^n. \quad (\text{D.0.1})$$

- **Cosmological Constant Calculation:**  $\Lambda(\mathcal{Z}_{\text{total}})$ , c.f. eq.(3.3.1).

The value of the cosmological constant, at given radii  $r_1 = r_2 = r$ , is given by the sum of the integrals of each  $q^m \bar{q}^n$  term in the total partition function,  $\mathcal{Z}_{\text{total}} = \mathcal{Z}_F \times \mathcal{Z}_B$ , weighted by its coefficient  $c_{mn}$ .

The integral is split up into the ‘upper’ ( $\tau_2 > 1$ ) and ‘lower’ regions of the fundamental domain. Only terms for which  $m = n \neq 0$  receive contributions from both regions; the  $\tau_1$  integral yields zero for a modular invariant partition function (in which all terms obey  $(m - n) \in \mathbb{Z}$ ) when  $m \neq n$ , and hence can be ignored.

- **Radius Dependence**

The variation of the cosmological constant  $\Lambda$  with radius  $r$ , over a defined range of values, is plotted. Note that most plots in §5 are truncated at the lower end of the range of values of  $r$ ; this is because the computation time for values of  $r \lesssim 0.1$  (in string units) is inordinate. As is clear from the plots, all of the limiting behaviours become obvious within the appropriately specified ranges.

# Appendix E

## Group Representations

Representation	Number of States
<i>Fundamental</i>	$2n$
<i>Adjoint</i>	$\frac{2n(2n-1)}{2}$
<i>(Conjugate) Spinor</i>	$\frac{1}{2}2^n = 2^{n-1}$

Table E.1: *Number of states corresponding to different representations of  $SO(2n)$ .*



# Appendix F

## Algebraic Infeasibility

### F.1 Set Theory Interpretation

Denoting <sup>1</sup>

- the number of zero elements (on the gauge side) of  $V_4$  as  $\tilde{\beta}$ ,
- the number of zero elements which are common to  $V_4$  and to  $\mathbf{e}$  as  $\tilde{\beta}_{\tilde{\gamma}}$ ,
- the number of zero elements of  $\mathbf{e}$  that do not overlap with  $V_4$  as  $\tilde{\beta}_{\tilde{\gamma}'}$ ,
- the number of zeros (on the left-moving side) of  $\mathbf{e}$  as  $\tilde{\gamma}$ ,
- the number of zero elements (on the left-moving side) of  $\mathbf{e}$  which are common to  $V_4$  as  $\tilde{\gamma}_{\tilde{\beta}}$ ,
- the number of zero elements (on the left-moving side) of  $\mathbf{e}$  which are common to  $V_4$  as  $\tilde{\gamma}_{\tilde{\beta}'}$ ,

the following relations, depicted in Figure F.1 hold:

$$\tilde{\beta} = \tilde{\beta}_{\tilde{\gamma}} + \tilde{\beta}_{\tilde{\gamma}'}, \quad (\text{F.1.1})$$

$$\tilde{\gamma} = \tilde{\gamma}_{\tilde{\beta}} + \tilde{\gamma}_{\tilde{\beta}'} = \tilde{\beta}_{\tilde{\gamma}} + \tilde{\gamma}_{\tilde{\beta}'}. \quad (\text{F.1.2})$$

Generically the gauge symmetry is broken from  $SO(2n)$  to (where the factor of 2 corresponds to the mapping between the real and the complex fermion formulation):

$$SO(2n) \rightarrow SO(2n - 2\tilde{\gamma} - 2\tilde{\beta}_{\tilde{\gamma}'}) \otimes SO(2\tilde{\beta}_{\tilde{\gamma}'}) \otimes SO(2\tilde{\gamma} - 2\tilde{\beta}_{\tilde{\gamma}}) \otimes SO(2\tilde{\beta}_{\tilde{\gamma}}). \quad (\text{F.1.3})$$

Consider a specific model containing the vectors  $\{V_0, V_1, V_2, V_4\}$  and a CDC vector

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<sup>1</sup>Considering, as in all models studied, the global side of  $V_4$  and  $\mathbf{e}$  to be equal,  $V_4^{RM} = \mathbf{e}^{RM}$

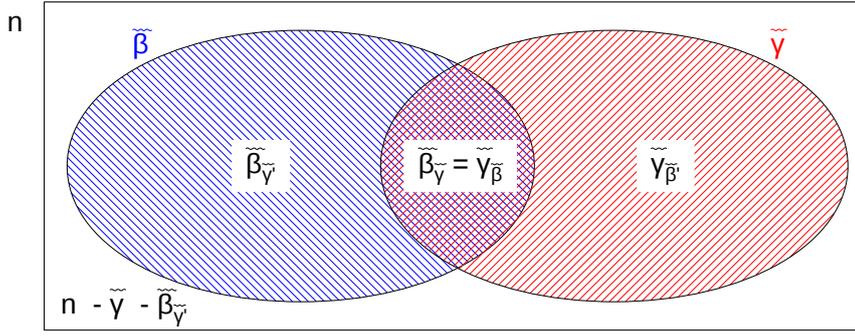


Figure F.1: The overlap between the zeros of the vectors  $V_4$  (denoted by  $\tilde{\beta}$ ) and  $\mathbf{e}$  (denoted by  $\tilde{\gamma}$ ) corresponds to the overlapping of the elements of the Venn diagram. The four distinct sets are labelled with the convention that  $\tilde{\beta}_{\tilde{\gamma}}$  denotes the number of zeros in  $V_4$  that do not overlap with the zeros in  $\mathbf{e}$ ; that is,  $\tilde{\beta} \cap \tilde{\gamma}'$  (where the prime refers to the complement of the set).

e. The vectors and structure constants  $k_{ij}$  are given by:

$$\begin{aligned}
 V_0 &= -\frac{1}{2} [ 11 \ 111 \ 111 \ | \ 1111 \ 11111 \ 111 \ 111111111 ] \\
 V_1 &= -\frac{1}{2} [ 00 \ 011 \ 011 \ | \ 1111 \ 11111 \ 111 \ 111111111 ] \\
 V_2 &= -\frac{1}{2} [ 00 \ 101 \ 101 \ | \ 0101 \ 00000 \ 011 \ 111111111 ] \\
 V_4 &= -\frac{1}{2} [ 00 \ 101 \ 101 \ | \ 0101 \ 00000 \ 011 \ 000000000 ] \\
 \mathbf{e} &= \frac{1}{2} [ 00 \ 101 \ 101 \ | \ 1011 \ 00000 \ 000 \ 000111111 ], \quad (F.1.4)
 \end{aligned}$$

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}. \quad (F.1.5)$$

For this three element basis set, there are sufficiently few different sets of states in the theory, that their overall number can be expressed by a relatively simple algebraic expression. The complexity of the expression increases rapidly as more basis vectors are added, such that the number of unknown parameters increases.  $N_{b(0)} - N_{f(0)}$  is given by (where  $\gamma_\beta$  denotes the number of zero elements of the CDC vector that overlap with  $V_4$  on the global side, and in this model,  $2n = 40$ ):

$$\begin{aligned}
 N_{b(0)} - N_{f(0)} &= 4 \{ \gamma_\beta + \frac{1}{2} \{ (2\tilde{\gamma} - 2\tilde{\beta}_{\tilde{\gamma}}) (2\tilde{\gamma} - 2\tilde{\beta}_{\tilde{\gamma}} - 1) + (2\tilde{\beta}_{\tilde{\gamma}}) (2\tilde{\beta}_{\tilde{\gamma}} - 1) \\
 &\quad + (40 - 2\tilde{\gamma} - 2\tilde{\beta}_{\tilde{\gamma}'}) (40 - 2\tilde{\gamma} - 2\tilde{\beta}_{\tilde{\gamma}'} - 1) + (2\tilde{\beta}_{\tilde{\gamma}'}) (2\tilde{\beta}_{\tilde{\gamma}'} - 1) \} \}
 \end{aligned}$$

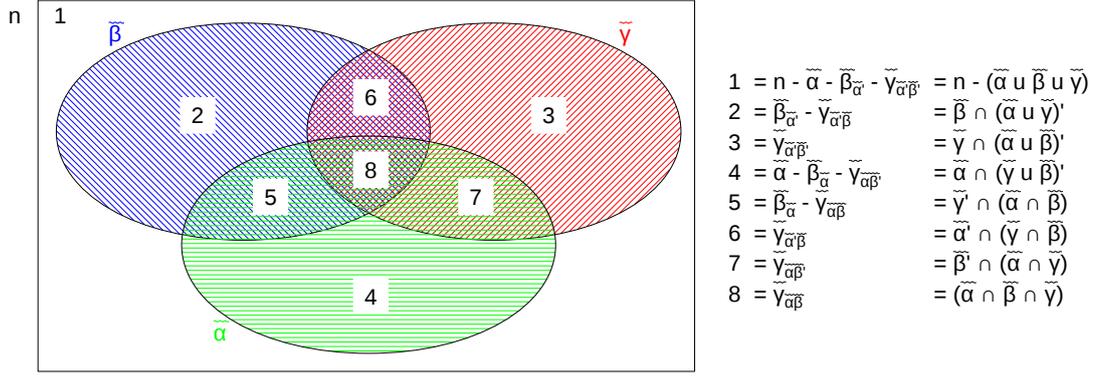


Figure F.2: The (non)-overlap between the zeros of the vectors  $V_2, V_4$  and  $\mathbf{e}$  corresponds to the overlapping of the elements of the Venn diagram. Note that the parameters  $\alpha, \beta, \gamma$  take zero, one and two subscripts respectively, rather than expressing the entire sets of overlaps with double subscripts, to reflect the incremental increase in the number of unknown parameters associated with the introduction of each new basis vector. For example,  $\tilde{\alpha} - \tilde{\beta}_{\tilde{\alpha}} - \tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}} = \tilde{\beta}_{\tilde{\alpha}'\tilde{\gamma}'}$ . Initially, a vector with  $\tilde{\alpha}$  left-moving zeroes is introduced, followed by a vector with  $\tilde{\beta}$  zeros, of which  $\tilde{\beta}_{\tilde{\alpha}}$  overlap with the zeros of the original vector. Hence, while  $\tilde{\alpha} - \tilde{\beta}_{\tilde{\alpha}} = \tilde{\alpha}_{\tilde{\beta}'}$ , the former expression makes clear the additional overlaps that result from the addition of a new vector.

$$+ 4\{(20 - \tilde{\gamma} - \tilde{\beta}_{\tilde{\gamma}'}) (2\tilde{\beta}_{\tilde{\gamma}} - \tilde{\beta}_{\tilde{\gamma}'} - \tilde{\gamma}) + (\tilde{\gamma} - \tilde{\beta}_{\tilde{\gamma}}) (\tilde{\beta}_{\tilde{\gamma}'} - \tilde{\beta}_{\tilde{\gamma}}) - \tilde{\beta}_{\tilde{\gamma}}\tilde{\beta}_{\tilde{\gamma}'}\} \}. \quad (\text{F.1.6})$$

In this case,  $\tilde{\gamma} = 12$ ,  $\tilde{\beta} = 16$ ,  $\tilde{\beta}_{\tilde{\gamma}} = 9$  and  $\tilde{\beta}_{\tilde{\gamma}'} = 7$ . These values, as well as the above expression for the number of states in the theory, have been constrained by the GSO projection conditions along with the above assignment of structure constants. Following steps 2 through 8 above,  $N_{b(0)} - N_{f(0)}$  is found to take a value of  $-64 \neq 0$ .

## F.2 Adding another vector to the set $\{V_i\}$

A second non-trivial vector  $V_2$  can be added to the basis set. Notationally, it contains  $\tilde{\alpha}$  zeros on the gauge side, of which  $\tilde{\beta}_{\tilde{\alpha}}$  overlap with the zeros in  $V_4$ ,  $\tilde{\beta}_{\tilde{\alpha}'}$  do not overlap, with the notation repeated to include the overlap between  $V_2, V_4$  and the CDC vector  $\mathbf{e}$  (for example,  $\tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}'}$  denotes the overlap between the zeros of  $\mathbf{e}$ , the zeros of  $V_2$  and the ones of  $V_4$  (alternatively the non-overlap with the zeros of  $V_4$ )). Figure F.2 provides a pictorial representation of the set-up. The gauge group is generically broken to:

$$SO(2n) \rightarrow SO(2n - 2\tilde{\alpha} - 2\tilde{\beta}_{\tilde{\alpha}'} - 2\tilde{\gamma}_{\tilde{\alpha}'\tilde{\beta}'}) \otimes SO(2\tilde{\gamma}_{\tilde{\alpha}'\tilde{\beta}'})$$

$$\begin{aligned} & \otimes SO(2\tilde{\beta}_{\tilde{\alpha}'} - 2\tilde{\gamma}_{\tilde{\alpha}'\tilde{\beta}}) \otimes SO(2\tilde{\gamma}_{\tilde{\alpha}'\tilde{\beta}}) \otimes SO(2\tilde{\alpha} - 2\tilde{\beta}_{\tilde{\alpha}} - 2\tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}'}) \\ & \otimes SO(2\tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}'}) \otimes SO(2\tilde{\beta}_{\tilde{\alpha}} - 2\tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}}) \otimes SO(2\tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}}). \end{aligned} \quad (\text{F.2.1})$$

Of interest is the form that this additional vector  $V_2$  can take, such that  $N_{b(0)} = N_{f(0)}$ . With  $V_2$  taking a form such that it does not induce any further breaking in  $G_{\text{global}}$ , it is possible to choose the structure constants  $k_{ij}$ , such that  $N_{b(0)} - N_{f(0)} = 0$ . (Note that the gauge group corresponding to the original set of basis vectors, eq.(F.1.3) can be obtained from eq.(F.2.1) by inputting the appropriate numerical values, and replacing  $SO(0)$  with  $\mathbb{I}$ .)

Having added an additional basis vector, it is infeasible to algebraically express the number of states in the theory. However, a specific case can be considered. Adding the vector  $V_2$  to the above set  $\{V_0, V_1, V_4\}$  with  $\mathbf{e}$ , and assigning the additional structure constants;

$$V_2 = -\frac{1}{2}[00\ 101\ 101\ | 0101\ 00000\ 011\ 11111111], \quad (\text{F.2.2})$$

$$k_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}. \quad (\text{F.2.3})$$

the ‘overlap parameters’ that appear in eq.(F.2.1), take the following values;

$$\begin{aligned} 2n &= 40, \\ \tilde{\alpha} &= 8, \\ \tilde{\beta} &= 16; \quad \tilde{\beta}_{\tilde{\alpha}} = 8, \quad \tilde{\beta}_{\tilde{\alpha}'} = 8, \\ \tilde{\gamma} &= 12; \quad \tilde{\gamma}_{\tilde{\alpha}'\tilde{\beta}'} = 3, \quad \tilde{\gamma}_{\tilde{\alpha}'\tilde{\beta}} = 3, \quad \tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}'} = 0, \quad \tilde{\gamma}_{\tilde{\alpha}\tilde{\beta}} = 6. \end{aligned} \quad (\text{F.2.4})$$

In this case,  $N_{b(0)}, N_{f(0)}$  are equal, both taking a value of 1312. Critically, this condition holds despite the fact that the theory is non-supersymmetric (as can be seen by the absence of any massless gravitinos in the spectrum).

### F.3 Adding additional basis vectors

Unfortunately, the number of unknown parameters, which determine the vectors that can be added to the set of basis vectors satisfying the constraint  $N_{b(0)} - N_{f(0)} = 0$ , grows drastically as the size of the basis set is increased. It rapidly becomes algebraically infeasible to solve for this number of undetermined parameters.

An alternative approach would be to generate the full set of possible additional vectors  $\{V_i\}$ , before looping over the set, calculating whether or not  $N_{b(0)} = N_{f(0)}$  for each element. It would therefore be possible to look at the set for which the condition holds, and try to deduce any patterns in the form that the ‘successful’ vectors  $V_i$  take. However, the size of the set for  $[8|20]$  vectors is  $2^{28}$  minus some amount due to constraints; that is, an impractical number. Certainly, this would represent a categorically different approach to the analytic argument presented by Ferrara, Kounnas & Porrati [22] in terms of the partition function, that  $N_{b(0)} = N_{f(0)}$  is unaffected by the form taken by any twisted sector vector .

## F.4 Vectors overlapping $\mathbf{e}$

It is possible to add vectors  $V_i$  that both do and do not overlap  $\mathbf{e}$ , but which break  $N_{b(0)} = N_{f(0)}$ . These results can be used to rule out conditions such as ‘any vector of type X can be added without violating  $N_{b(0)} = N_{f(0)}$ ’, where ‘type X’ would in this case have been ‘non-overlapping with  $\mathbf{e}$ ’. Specifically, having added to the basis set 4.3.10 supplemented by  $V_2$ , a vector that does not overlap with  $\mathbf{e}$ , an explicit calculation reveals that it is not possible to tune  $k_{ij}$  such that  $N_{b(0)} = N_{f(0)}$ . This forms a ‘non-existence proof’. It is possible to falsify, rather than verify, the statement that ‘*it is always possible to add a vector  $V_i$  that does not overlap with  $\mathbf{e}$  (or  $b_3$ ) such that a non-SUSY model with  $N_{b(0)} = N_{f(0)}$  is generated*’. It has not been proved analytically that it is false that this statement holds. Instead, a counter example to its validity has been found. Hence it is not the case that it is always possible to obtain a non-SUSY theory with  $N_{b(0)} = N_{f(0)}$  from non-overlapping vectors.

Might an analytic argument that would constrain the form of additional vectors exist? For vectors not overlapping  $\mathbf{e}$ ,  $\mathbf{e} \cdot \mathbf{Q}$  reduces to  $\mathbf{e} \cdot N_{\alpha\bar{V}}$ . Thus  $\mathbf{e} \cdot N_{\alpha\bar{V}}$  only picks out NS excitations in the new vector,  $V_i$ , since the Ramond excitations correspond to the non-zero elements of  $V_i$ , none of which overlap with  $\mathbf{e}$ . This constitutes a new constraint on the spectrum of states that has been derived from the GSO projections. But it is not clear if there exists any pattern in the ways in which the bosonic and fermionic sectors are affected respectively, or why there should be.

The problem with trying random vectors is that there is no reason to spot any rules. For example, it is possible that there might be a rule whereby ‘*any vector  $V_i$  that does not overlap the basis vector  $X$  can always be included within a model (with appropriate tuning of  $k_{ij}$ ) such that  $N_{b(0)} = N_{f(0)}$* ’. While for vectors  $V_i$  that do overlap  $X$ , this is only true some of the time. However, even if this were the case, all 4 possible combinations of results would still be encountered (overlapping & non-overlapping with  $X$ ,  $N_{b(0)} = N_{f(0)}$  &  $N_{b(0)} \neq N_{f(0)}$ ) upon the addition of any

arbitrary vector  $V_i$ . That is, before tuning  $k_{ij}$ , vectors with null overlap with  $X$  will be found that both break and preserve  $N_{b(0)} = N_{f(0)}$ , and equally, vectors which do overlap the CDC vector will be found that also yield both results, (the latter perhaps more infrequently, but still often enough for instances to be found as part of a random scan).

An alternative to the algebraic approach, is to simply choose by hand a specific vector set, and test whether or not  $N_{b(0)} = N_{f(0)}$ . Thus, having acknowledged the above difficulties, specific models that *do* exhibit  $N_{b(0)} = N_{f(0)}$ , are presented in this study.

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