The Non-Abelian Wilson Loop as a Theory of Strings with Contact Interaction

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Abstract

We investigate a reformulation of Yang-Mills theory at the level of the expectation of the non-abelian Wilson loop using a string theory with non-standard interaction that forms a generalisation of the model formulated in [26]. We find that the path-ordering of the Wilson loop can be generated either from considering a worldsheet generalisation of the field theory found in [35] or by introducing a gauge field onto the worldsheet. Only the gauge theory has the sufficient structure to accommodate the three gluon vertex of Yang-Mills theory in the string model. Supersymmetric analogues of these two models are also investigated which, specifically in the gauge theory model, can be made the basis of a realistic string model formulation of Yang-Mills theory coupled to spinors.

Declaration

The work in this thesis is based on research carried out at the CPT, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. Aspects of this thesis were carried out in collaboration with Prof. Paul Mansfield. This will be clearly referenced within the text by citation or other appropriate acknowledgement.

Acknowledgements

First and foremost I would like to thank my supervisor, Professor Paul Mansfield, for the help and tutoring he has given me during my studies in Durham. He is incredibly knowledgeable in all areas I quizzed him on and I can't remember a time he didn't know the answer to a question given to him. I actually opted to change supervisor in the first year of my studies to him because of the scale of the project and my interest in his previous papers. I wish him every success in the future.

I would also like to thank my parents and brother for their continuing support throughout my studies. Our fairly close proximity meant that we could see each other every couple of months despite the protracted road works on the A1. Our holiday in France stands out as a brilliant two weeks we spent together just after the MSc exams that we were required to sit in the first year. It provided a nice break before the research began once I returned.

My life in Durham began at Ustinov College where I could usually be found in the bar watching sport or playing pool in the games room on an evening. I joined a pool team at Ustinov at the request of a friend who needed more players, being slightly cautious about joining due to me having only a few trips to Riley's sports bar in my undergraduate worth of experience. I am extremely happy that I did as I met, who would within two and a half years be my fiancée, an exceptional pool player called Lauren. Since then we have enjoyed many trips away together and moved in to a beautiful house in Darlington. Many of these trips I must thank her parents for as they have certainly maximised the amount of enjoyment one can have from leisure time. I would also like to thank them for my resurgence in playing golf, at one point taking me to play at Gleneagles in Scotland.

In the second year at Durham I moved to Neville's Cross with four chemists who would spend many hours into the evening in the labs, leaving me with the run of the house needing only a pen and paper to carry out my research. I can safely say that we all enjoyed the large kitchen-living area of the house used to stage multiple cheese and wine nights, BBQs and roast dinners. Special thanks to Carl who brought an outdoor dart board and loungers (and even a marquee in the third year!).

In the department, four of us moved from the first year PhD area to a large ex-teaching room that gave me and Mike the perfect area to play bin basketball with the paper from the recycling bin. My favourite feature though were the three ancient blackboards we used to challenge each other with problems or write out our most recent findings.

In the second year I overcame an injury sustained in undergraduate to re-take up playing football for Ustinov College with my house mate, Eddy. Wednesday nights then became training and champions league night at the college, followed by the long walk back to Neville's Cross.

It was during my third year that I moved to Darlington with Lauren who worked at the Stockton campus of Durham University. I would then go to Stockton three times a week and work from there in a computer room reducing the amount of time I spent in Durham itself. Stockton campus had the advantage of being next to the river Tees which proved to be the perfect place to unscramble my circuits after spending hours on the same calculation. Lauren and I would often spend our lunch hours walking along here and then going for a Waterside sandwich. I would like to thank her for the support and joy she has given me over the past three years.

In the April of my third year, or about 2 years into my research, I attended a quantum field theory conference in York where I met Elizabeth Winstanley who had interviewed me for a PhD position 4 years earlier. I also attended the Young Theorists Forum hosted in Durham each January which allows PhD students to present their research so far to one another.

I have thoroughly enjoyed my three and a half years in Durham and I wish all of my fellow PhD candidates and the people who I have had the pleasure of meeting here, all the best for the future.

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Chapter 1

Introduction

The standard model is to date the most complete and well tested theory we have of particle physics at the most fundamental level. At its core is the existence of local internal symmetries in the Lagrangians of the various field theories that make it up. These symmetries, known as gauge symmetries, have been intensely studied in the context of particle physics yielding one of the most precise agreements between theory and experiment we have yet seen in the measurement of the electron magnetic moment [1] [2]. This stunning agreement was reached by using the quantum theory of relativistic fields (QFT) which stands as our primary means of attacking problems in particle physics and it is QFT in which the standard model is most commonly formulated. There do, however, exist other methods that allow us to solve problems that are difficult or sometimes not apparent in QFT. In this thesis we will explore an alternative to field theory, based on a theory of strings in 4 dimensions with a non-standard contact interaction.

With all the successes of field theory, one may wish to ask, why consider alternative formulations? Well, I pose two reasons for doing so. The first and fundamentally most important reason would be that there still exist many unsolved problems in fundamental particle physics. Mostly these occur within the framework of quantum chromodynamics (QCD), the theory of the strong force, as perturbation theory has limited use at low energies due to the phenomena of asymptotic freedom. We are therefore forced to use non-perturbative methods to ask important questions in this regime. This is fine of course except for the fact that these methods are extremely difficult. The area of lattice gauge theory is of use here, though this method requires complex numerical computations requiring significant computing power. This method has another downfall in the form of the numerical sign problem [3] whereby Boltzmann factors, interpreted as probabilities, come out with the wrong sign or are even complex. A new approach to computing amplitudes would therefore be highly useful to phenomenologists.

The second reason we may wish to seek alternative formulations would be interest; is it not interesting that there exists a perfectly good alternative to field theory that reproduces the same results but reaches these conclusions from a different perspective? Philosophically it may tell us more about the structure of gauge field theory and therefore about the structure of the standard model.

The model we investigate consists of a field strength supported on a surface bounded by two interacting particles moving along their respective worldlines. When inserted into the standard Maxwell action this model actually describes a string theory with a non-standard interaction that is only non-zero when the string intersects itself. We will show that this theory produces results equivalent to the expectation of the Wilson loop computed using standard Yang-Mills theory. This formulation of gauge theory where the degrees of freedom are strings is reminiscent of Faraday's lines of force [4].

We will begin with a discussion of gauge theories and how they are usually formulated. We will then review how to formulate the most simple Yang-Mills theory, electromagnetism, using string theory. The aim of this thesis will then be to generalise this string model to include non-abelian gauge groups such as those at the heart of the standard model. We will use a result from the worldline formalism to motivate introducing new fields onto our worldsheet theory whose dynamics will give rise to the additional features of Yang-Mills theory.

1.1 Gauge theory

Field theories with local internal Lie group symmetries have proved incredibly useful in particle physics since they were first introduced, unwittingly, in the theory of electromagnetism (EM). Extending these symmetries to the matter sector, and thereby introducing interactions between bosons and leptons and quarks, is the cornerstone of the standard model. To understand these symmetries, which we call gauge symmetries, we will look at the simplest case, that of EM or the U(1) invariant gauge theory. The physical fields of the theory are the electric field, $\mathbf{E}$, and magnetic field, $\mathbf{B}$. The equations governing their dynamics are Maxwell's equations [5], which in differential form and S.I. units are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.1.1}$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{1.1.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.1.3}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \dot{\mathbf{E}}). \tag{1.1.4}$$

where ρ and $\mathbf{J}$ are the charge density and current density respectively and $\dot{\mathbf{E}} \equiv d\mathbf{E}/dt$. The two curl equations are satisfied by introducing the scalar potential, φ, and vector potential, $\mathbf{A}$, such that

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{1.1.5}$$

The electric and magnetic fields are not completely determined by a single choice of potentials. These solutions are, in fact, invariant under the transformations

$$\varphi \to \varphi' = \varphi + \dot{\Lambda}, \qquad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\Lambda \tag{1.1.6}$$

where $\Lambda = \Lambda(t; \mathbf{x})$ is a scalar space-time function. These are the gauge transformations of EM. Since we can alter the potentials in this way without affecting the physics, φ and $\mathbf{A}$ are unphysical fields. This is why the electric and magnetic fields are used more commonly. To see why EM is also known as U(1) gauge theory, it will be useful to move to a more covariant form of Maxwell's equations. This is done by placing the two types of potentials into components of a single potential four vector, A^μ, such that $A^\mu = (\varphi, \mathbf{A})$. The charge and current densities are similarly packaged as the four current, $j^\mu = (\rho, \mathbf{J})$. The covariant form of Maxwell's equations are then written in terms of the field strength tensor, defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and take the form

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{1.1.7}$$

$$\epsilon_{\mu\nu\rho\sigma}\partial^\rho F^{\mu\nu} = 0 \tag{1.1.8}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the 4 dimensional Levi-Civita symbol. The gauge transformations are then neatly written as

$$A^\mu \to A^{'\mu} = A^\mu - \partial^\mu \Lambda. \tag{1.1.9}$$

The field strength itself is important as it is invariant under gauge transformations. The first of the covariant Maxwell's equations can be obtained by minimising the action

$$S_{EM} = \int d^4x \left(- \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu \right). \tag{1.1.10}$$

Note, there is a more natural reason for choosing this action to describe the dynamics of the field theory. It is the only functional that is gauge invariant, Lorentz invariant, parity invariant and time-reversal invariant. The gauge invariance of the second term follows from the continuity equation, $\partial_\nu j^\nu = 0$, which follows from differentiating (1.1.7) with respect to x^ν. This action then describes pure EM, i.e. EM without matter. To couple matter to EM, we simply add the matter action to S_{EM} in a gauge invariant way. For simplicity we can consider coupling electromagnetism to a complex free massless scalar field, ϕ, with free action

$$S_\phi = \int d^4x \; \partial^\mu \phi^* \partial_\mu \phi. \tag{1.1.11}$$

To work out how to add matter to EM in a gauge invariant way requires us to understand how ϕ changes under a gauge transformation. To start with note that this action is invariant under the global transformation

$$\phi \to \phi' = e^{iqa}\phi \qquad \phi^\dagger \to \phi^{\dagger'} = \phi^\dagger e^{-iqa} \tag{1.1.12}$$

where q is a coupling constant and for now a is constant. The gauge transformation (1.1.9) is local, however. The action for ϕ is not invariant if we promote a to a general spacetime function $\Lambda(x)$. It can, however, be made invariant if we replace the partial derivatives with the covariant derivatives, D_μ, defined as

$$D_\mu = \partial_\mu + iqA_\mu \tag{1.1.13}$$

so that the combination $D_\mu\phi$ transforms as $(D_\mu\phi)' = e^{iq\Lambda}D_\mu\phi$ iff A_μ transforms like the four potential (1.1.9). The gauge invariant form of the kinetic action for ϕ is then

$$S'_\phi = \int d^4x \; D^\mu\phi^\dagger D_\mu\phi. \tag{1.1.14}$$

The physical effect of this replacement is to introduce interactions between ϕ and the four potential, A_μ. The full field theory is then the sum of the matter action S'_ϕ and the free Maxwell action, S_{EM}. Without the Maxwell action, the action would describe the dynamics of a complex field coupled to a background four potential. When a is replaced by a local field, (1.1.12) suggests that ϕ transforms in the fundamental representation of the Lie group U(1) i.e. it gets rotated by a factor of $U = e^{iq\Lambda}$. Similarly ϕ^* transforms in the anti-fundamental representation. The covariant derivative is so called because it covaries with the field ϕ. i.e. it is defined to transform as $D_\mu\phi \to UD_\mu\phi = (UD_\mu U^{-1})(U\phi)$. This suggests that D_μ transforms in the adjoint representation of the Lie group. Finally, this is achieved iff the four potential, in general called the gauge field, transforms as

$$A_\mu \to A'_\mu = UA_\mu U^{-1} - \frac{i}{q}(\partial_\mu U)U^{-1}. \tag{1.1.15}$$

Explicitly inserting the definition of U into this reduces the transformation to (1.1.9). We can be more general and consider a phase factor of the form

$$U = e^{iq\Gamma(x)} \tag{1.1.16}$$

where $\Gamma(x)$ belongs to a general Lie algebra and so can be expanded in terms of the group generators, τ^a, as $\Gamma = \Gamma^a(x)\tau^a$. In this way, we see that the function Λ as just a spacetime function is an element of the Lie algebra of U(1). In general Γ won't commute introducing non-linearities into the Maxwell action. We will come back to the more general gauge theory where the Lie group is left arbitrary later after a discussion of string theory and its connection to EM.

1.1.1 String theory

String theory and field theories on the 2 dimensional worldsheet will form a major part of this thesis and so we give a brief discussion of them here. String theory has aroused significant interest as a possible theory of quantum gravity ever since the quantisation of the string and the discovery of the graviton in its spectrum. It first appeared as the dual resonance theory which was an S-matrix approach to the dynamics of hadrons, a major result of which was the Veneziano amplitude [6]. Ultimately, it was shown to be an unsuccessful theory when applied to hadrons, with QCD proving to be the correct theory of the strong interaction. Nambu [7], Nielsen [8] and Susskind [9]- [11], however, were able to show that the theory was equivalent to a theory of bosonic strings. Since then string theory, particularly superstring theory, has had many successes such as the derivation of the Einstein field equations and Hawking's black hole entropy formula [12]. It has even found some applications to field theory such as the discovery of the Bern-Kosower formula [13] which computes one loop N-gluon amplitudes. More modern aspects of string theory include the ADS/CFT correspondence [14] and the discovery of the connection to monstrous moonshine [15]. There are, however, some problems such as the prediction of extra dimensions and the requirement of spacetime supersymmetry. The string landscape of superstring theory is also an issue that draws into question whether or not string theory is even a theory of science.

The study of string theory as a potential quantum theory of gravity is irrelevant to us in this thesis. Instead, we will use the machinery that has been built to study string theory over the past 40 years to formulate a string theory in 4 dimensions that can reproduce field theory results. The problems associated with working in a non-critical dimension are addressed in [26] and correspond to the appearance of additional Liouville and super-Liouville degrees of freedom.

A quantum theory of strings is most simply obtained via first quantisation. Just as a point particle sweeps out a worldline in spacetime, a 1 dimensional extended object sweeps out a 2 dimensional worldsheet; and just as the point particle's worldline minimises its proper time, a string's worldsheet minimises its proper area which leads naturally to the Nambu-Goto action describing the dynamics of the string

$$S_{NG} = \int d^2\sigma \, \sqrt{h} \tag{1.1.17}$$

with h the determinant of the induced metric of the worldsheet. It is the pull-back of the flat metric on Minkowski space so that

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{1.1.18}$$

The square root in the action makes quantisation difficult. Brink, Di Vecchia and Howe showed that introducing an additional auxiliary field onto the worldsheet allows one to obtain a classically equivalent action now known as the Polyakov action [16], given by

$$S_p[X, g] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\xi \sqrt{g} \; g^{ab} \partial_a X^\mu(\xi) \partial_b X_\mu(\xi) \tag{1.1.19}$$

so named because Polyakov was the first to show how to quantise the string via the functional integral using this action to weight the random surfaces [17]. The extra field, g, is interpreted as the two dimensional metric on the worldsheet, just as the einbein introduced in the worldline action is the one dimensional metric on the worldline. We will see that summing a classical solution to one of Maxwell's equations over all possible (genus 0) surfaces weighted by e^{-S_p} leads to the full classical solution satisfying all field equations. Functional integration over surfaces is rather non-trivial compared to its path integral sibling. The surfaces encountered in string theory are diffeomorphism and Weyl invariant and so lead to a large overcounting of possible surfaces in the partition function $Z = \int \mathcal{D}[X, g] \; e^{-S_p[X,g]}$. It is, therefore, ill defined without a proper treatment of the symmetries. To become well defined we must divide out the gauge equivalent configurations via the method of Faddeev-

Popov which introduces ghosts into the theory [18].
The ground state of the bosonic string is tachyonic and poses a serious threat to the validity of the theory as a realistic physical model. It turns out that a string theory with fermions on the worldsheet lacks this unphysical mode in its spectrum. In the same way that there exists a worldline supersymmetry between worldline fermions and bosons, there exist worldsheet supersymmetries between the worldsheet fermions and bosons. There are five distinct superstring theories with different numbers of supersymmetries and gauge fields. Of particular interest in this thesis will be bosonic and fermionic strings whose worldsheets are closed surfaces in spacetime.

1.1.2 The classical electrostatic field between two point charges

Before deriving the full electromagnetic field produced by two moving charges, we look at the simpler case of the electric field produced by two, fixed, equal and opposite point charges in D dimensions. This will introduce the lines of force method that will be used throughout this thesis. We will also find this particular case useful when we come to consider how to generalise the string theory to accommodate general gauge groups.
Consider particle 1 with position vector $\mathbf{a}$ and charge $+q$ and particle 2 with position vector $\mathbf{b}$ and charge $-q$. The physics of the system is described by Gauss' law (1.1.1)

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}(\delta^D(\mathbf{x}-\mathbf{a}) - \delta^D(\mathbf{x}-\mathbf{b})). \tag{1.1.20}$$

By inspection, a solution is

$$\mathbf{E}_c(\mathbf{x}) = \frac{q}{\epsilon_0}\int_C \delta^D(\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{1.1.21}$$

The curve C is any curve joining $\mathbf{a}$ and $\mathbf{b}$. The proof of this solution is straight forward

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}\int_C \nabla_{\mathbf{x}}\delta^D(\mathbf{x}-\mathbf{y})\cdot d\mathbf{y} = -\frac{q}{\epsilon_0}\int_C \nabla_{\mathbf{y}}\delta^D(\mathbf{x}-\mathbf{y})\cdot d\mathbf{y}$$

$$= \frac{q}{\epsilon_0}(\delta^D(\mathbf{x}-\mathbf{a}) - \delta^D(\mathbf{x}-\mathbf{b})). \tag{1.1.22}$$

This form of solution describes a single string of field connecting the two particles. The trouble is, this solution doesn't satisfy Faraday's law in D dimensions, which is from (1.1.8)

$$\partial^j E^i - \partial^i E^j = 0. \tag{1.1.23}$$

We know, however, that the unique solution of Gauss' law and Faraday's law that decays at infinity is

$$\mathbf{E}(\mathbf{x}) = \frac{q}{2\epsilon_0 \pi^{D/2}} \Gamma\left(D/2\right) \left[\frac{(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^D} - \frac{(\mathbf{x}-\mathbf{b})}{|\mathbf{x}-\mathbf{b}|^D}\right]. \tag{1.1.24}$$

We will now see that a statistical sum of our string solution over all possible curve configurations reproduces this electric field solution as in [21]. To do this we split the string solution into N strings, each with associated charge $q_0 = q/N$. We then perform a path integration between $\mathbf{a}$ and $\mathbf{b}$ weighted by a suitable Boltzmann factor, i.e. we require a weight, βH, such that

$$\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^D} - \frac{\mathbf{x}-\mathbf{b}}{|\mathbf{x}-\mathbf{b}|^D} = \frac{1}{Z}\int \mathcal{D}\mathbf{y} \int_C \delta^D(\mathbf{x}-\mathbf{y})\; d\mathbf{y}\; e^{-\beta H} \tag{1.1.25}$$

where Z is a suitable normalisation given by

$$Z = \int \mathcal{D}\mathbf{y}\; e^{-\beta H}. \tag{1.1.26}$$

There is a natural weight that arises in the study of Brownian motion and thermal conduction, the heat kernel, which takes the form

$$\langle \mathbf{b}| e^{-H_0 T} |\mathbf{a}\rangle = \int \mathcal{D}\mathbf{y}\; e^{-\int_0^T dt \frac{\dot{\mathbf{y}}^2}{2}} = \frac{e^{-\frac{|\mathbf{a}-\mathbf{b}|^2}{2T}}}{(2\pi T)^{D/2}} \tag{1.1.27}$$

where H_0 is the Hamiltonian for a free scalar bosonic particle. The heat kernel is related by a Wick rotation to the quantum expectation for a particle to travel from $\mathbf{a}$ to $\mathbf{b}$. We have parametrised the curve by t such that $\mathbf{y}(0) = \mathbf{a}$ and $\mathbf{y}(T) = \mathbf{b}$. The total electric field should then correspond to the expectation value of the string solution (1.1.21). For a general observable, Ω, we have as usual

$$\langle \Omega \rangle = \frac{1}{Z} \int \mathcal{D}\mathbf{y} \; \Omega \; e^{-\beta H}. \tag{1.1.28}$$

We can then obtain the total electric field $\mathbf{E} = \langle \mathbf{E}_c \rangle$ by introducing a source function, $\mathbf{A}$, so that

$$\langle \mathbf{E}_c(\mathbf{x}) \rangle = \frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \frac{1}{Z} \int \mathcal{D}\mathbf{y} \; e^{-\int_0^T dt \frac{\dot{\mathbf{y}}^2}{2} - \frac{q}{\epsilon_0} \int_a^b \mathbf{A} \cdot d\mathbf{y}} \Big|_{\mathbf{A}=0} \tag{1.1.29}$$

where we have used the usual functional differentiation identity

$$\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \int \mathcal{D}\mathbf{y} \; e^{-\frac{q}{\epsilon_0} \int_a^b \mathbf{A}(\mathbf{y}) \cdot d\mathbf{y}} = \int \mathcal{D}\mathbf{y} \left(-\frac{q}{\epsilon_0} \int_a^b \; \delta^D(\mathbf{x} - \mathbf{y}) \; d\mathbf{y} \right) e^{-\frac{q}{\epsilon_0} \int_a^b \mathbf{A}(\mathbf{y}) \cdot d\mathbf{y}}. \tag{1.1.30}$$

The exponent in (1.1.29) is the action of a point particle coupled to a background gauge field. From non-relativistic quantum mechanics, we know that this is equivalent to introducing a potential into the Hamiltonian of (1.1.27) so that $H_0 \to H = \frac{(p+iA)^2}{2}$ as is done when passing from the classical Lagrangian formulation of electrodynamics of a point particle to the Hamiltonian formulation via a Legendre transform. Considering the kinetic part of the action as the Wick rotated quantum action means we need to Wick rotate the source term, leading to the factor of i here. We now run into an operator ordering ambiguity which is familiar from the path integral formulation of non-relativistic quantum mechanics. To avoid this problem we interpret the Hamiltonian as the Laplacian minimally coupled to the vector potential that acts on scalars so that

$$\hat{H} = -\frac{1}{2}(\nabla - \mathbf{A})^2. \tag{1.1.31}$$

We, therefore, find

$$2 \frac{\delta \hat{H}}{\delta \mathbf{A}(\mathbf{x})} \Big|_{\mathbf{A}=0} = \nabla \delta^D(\hat{\mathbf{q}} - \mathbf{x}) + \delta^D(\hat{\mathbf{q}} - \mathbf{x}) \nabla. \tag{1.1.32}$$

We can use the completeness relation of position eigenstates, $\int d^D\mathbf{c}\,|\mathbf{c}\rangle\langle\mathbf{c}| = \mathbb{I}$, to simplify this to

$$2\frac{\delta\hat{H}}{\delta\mathbf{A}}\bigg|_{\mathbf{A}=0} = -\int d^D\mathbf{c}\,\overrightarrow{\nabla}\,|\mathbf{c}\rangle\langle\mathbf{c}|\,\delta^D(\mathbf{c}-\mathbf{x}) - |\mathbf{c}\rangle\,\overleftarrow{\nabla}\,\langle\mathbf{c}|\,\delta^D(\mathbf{c}-\mathbf{x}) = |\mathbf{x}\rangle\,\overleftrightarrow{\nabla}\,\langle\mathbf{x}|\,. \quad (1.1.33)$$

Carrying out the functional integration on the amplitude gives

$$\begin{aligned}\frac{\delta}{\delta\mathbf{A}}\langle\mathbf{b}|e^{-\hat{H}T}|\mathbf{a}\rangle\bigg|_{\mathbf{A}=0} &= -\int_0^T dt\,\langle\mathbf{b}|e^{-\hat{H}_0(T-t)}\,\frac{\delta\hat{H}}{\delta\mathbf{A}}\bigg|_{\mathbf{A}=0} e^{-\hat{H}_0 t}|\mathbf{a}\rangle\\ &= -\frac{1}{2}\int_0^T dt\,\langle\mathbf{b}|e^{-\hat{H}_0(T-t)}\,|\mathbf{x}\rangle\,\overleftrightarrow{\nabla}\,\langle\mathbf{x}|\,e^{-\hat{H}_0 t}|\mathbf{a}\rangle\,. \end{aligned} \quad (1.1.34)$$

We now have two amplitudes which we recognise as heat kernels (1.1.27), hence, we have

$$\frac{1}{Z}\int\mathcal{D}\mathbf{y}\int_C \delta^D(\mathbf{x}-\mathbf{y})\;d\mathbf{y}\;e^{-\beta H} = -\frac{q(2\pi T)^{\frac{D}{2}}}{2\epsilon_0 e^{-\frac{|a-b|^2}{2T}}}\int_0^T dt\frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{(T-t)}}}{(2\pi(T-t))^{D/2}}\overleftrightarrow{\nabla}\frac{e^{-\frac{|\mathbf{a}-\mathbf{x}|^2}{2t}}}{(2\pi t)^{D/2}}. \quad (1.1.35)$$

In the large time limit, $T\to\infty$, the integral is only non-negligible at $t\approx 0$ and $t\approx T$ and, therefore, it splits into two integrals. Firstly, we note that the exponential factor outside of the integral becomes unity in the high temperature limit. In the $t\approx 0$ limit (1.1.35) becomes

$$\approx -\frac{1}{2}\int_0^\infty dt\;e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2T}}\overleftrightarrow{\nabla}\frac{e^{-\frac{|\mathbf{a}-\mathbf{x}|^2}{2t}}}{(2\pi t)^{D/2}}. \quad (1.1.36)$$

But, the first exponential goes to unity as $T\to\infty$, so this reduces to

$$-\nabla\int_0^\infty dt\;\frac{e^{-\frac{|\mathbf{a}-\mathbf{x}|^2}{2t}}}{2(2\pi t)^{D/2}}. \quad (1.1.37)$$

When $t\approx T$ we find that (1.1.35) simplifies to

$$\approx -\frac{1}{2}\int_0^\infty dt\;\frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2(T-t)}}}{(2\pi T)^{D/2}\left(1-\frac{Dt}{2T}\right)}\overleftrightarrow{\nabla}e^{-\frac{|\mathbf{a}-\mathbf{x}|^2}{2t}} \quad (1.1.38)$$

setting $T \approx t$ and letting $T \to \infty$, which means that we are only considering long curves, we find

$$\approx \nabla \int_0^\infty dt \, \frac{e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2t}}}{2(2\pi t)^{D/2}} \tag{1.1.39}$$

hence,

$$\frac{1}{Z}\int \mathcal{D}y \int_C \delta^D(\mathbf{x}-\mathbf{y})\, d\mathbf{y}\; e^{-\beta H} \underset{T\to\infty}{=} \nabla \int_0^\infty dt \, \frac{1}{2(2\pi t)^{D/2}} \left(e^{-\frac{|\mathbf{x}-\mathbf{b}|^2}{2t}} - e^{-\frac{|\mathbf{a}-\mathbf{x}|^2}{2t}} \right). \tag{1.1.40}$$

The calculation of the electrostatic field then comes down to solving the integral

$$\mathbf{I} \equiv \int_0^\infty dt \, \nabla \frac{e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2t}}}{t^{D/2}}. \tag{1.1.41}$$

Calculating the gradient first, we find

$$\mathbf{I} = -(\mathbf{x}-\mathbf{a}) \int_0^\infty dt \, \frac{e^{-\frac{|\mathbf{x}-\mathbf{a}|^2}{2t}}}{t^{D/2+1}}. \tag{1.1.42}$$

We now use the substitution $\xi \equiv \frac{|\mathbf{x}-\mathbf{a}|^2}{2t}$, so that (1.1.42) becomes

$$\mathbf{I} = -2^{D/2} \frac{(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^D} \int_0^\infty d\xi \; \xi^{D/2-1} e^{-\xi}. \tag{1.1.43}$$

This integral is just the definition of the Gamma function, $\Gamma\left(D/2\right)$, and so

$$\mathbf{I} = -2^{D/2}\Gamma\left(D/2\right) \frac{(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^D}. \tag{1.1.44}$$

Using this together with (1.1.38) we find that the D-dimensional electrostatic field is

$$\langle \mathbf{E}_c(\mathbf{x}) \rangle = \frac{q}{2\epsilon_0 \pi^{D/2}} \Gamma\left(D/2\right) \left[\frac{(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|^D} - \frac{(\mathbf{x}-\mathbf{b})}{|\mathbf{x}-\mathbf{b}|^D} \right] = \mathbf{E}(\mathbf{x}). \tag{1.1.45}$$

An interesting feature of the above derivation is the relation between the heat kernel and the volume of the (D-1)-sphere. Consider (1.1.44) again but by placing $\mathbf{a}$ at the origin so that we have

$$\frac{\mathrm{I(r)}}{2(2\pi)^{D/2}} = -\frac{\Gamma\left(D/2\right)}{2\pi^{D/2}\; r^{D-1}} \tag{1.1.46}$$

where $r = |\mathbf{x}|$ and $I = |\mathbf{I}|$. Now the volume of the (D-1)-sphere is

$$\mathrm{Vol}(S_{D-1}(r)) = \frac{2\pi^{D/2}\ r^{D-1}}{\Gamma\left(D/2\right)} \tag{1.1.47}$$

so that

$$\frac{1}{2}\nabla \int_0^\infty K(t,r)\ dt = -\frac{1}{\mathrm{Vol}(S_{D-1})}. \tag{1.1.48}$$

The relationship between the electrostatic field of the point particle in D dimensions and the volume of the (D-1)-sphere comes from the spherically symmetric nature of the solution to Gauss's law. Here the relationship comes from the relation of the heat kernel to the volume of the (D-1)-sphere. This calculation has been studied in the finite T regime in 3 dimensions in [19] where a deviation from the inverse square law is observed. We will come back to this calculation in chapter 3 in which we focus solely on the 2 dimensional case and consider averaging the electric field line of force solution (1.1.21) over a curved surface.

1.2 Time dependent electromagnetic fields

We now let the two charges move with respect to each other. This generates time dependent electric and magnetic fields with dynamics determined by the full set of Maxwell's equations. We will see that there is once again a string like solution to one of the equations in covariant form that upon averaging becomes the unique solution that vanishes at infinity that satisfies the other equations of motion. This result is important and directly leads to the string theory that this thesis is based on. We shall, therefore, review the derivation of the full field solution from the string like solution. Maxwell's equations in covariant form are (1.1.7) and (1.1.8). The four-current $j(x)$ arising from two moving charges at 4-positions a^μ and b^μ with charges $+q$ and $-q$ respectively is

$$j(x) = q\int_{-\infty}^{\infty} dt\ \left(\delta^4(x-a)\dot{a^\mu} - \delta^4(x-b)\dot{b}^\mu\right). \tag{1.2.49}$$

We, again, seek a string like solution to (1.1.7) with this four-current. The appropriate solution is

$$F_{\mu\nu}(x) = -q \int_{\Sigma} \delta^4(x-y) d\Sigma_{\mu\nu}(y) \tag{1.2.50}$$

and was first considered by Dirac while studying the electrodynamics of magnetic monopoles [20]. This solution requires explanation. The two interacting particles trace out worldlines C_1 and C_2 respectively. We may define any surface, Σ, bounded by the two worldlines that will be open if the two scattering particles go off to (and/or come in from) infinity or closed if they are spontaneously created and annihilated. Either way, this field strength is supported on this surface just as the electric field was supported along the curve, C, in the electrostatic field case. We parametrise Σ by the two "worldsheet" coordinates ξ^a. $d\Sigma_{\mu\nu}$ is then an infinitesimal element of area on Σ.

We now prove that $F_{\mu\nu}$ does indeed solve (1.1.7). Differentiating we find

$$\partial_x^\mu F_{\mu\nu}(x) = -q \int_{\Sigma} \partial_x^\mu \delta^4(x-y)\ d\Sigma_{\mu\nu}(y) = q \int_{\Sigma} \partial_y^\mu \delta^4(x-y)\ d\Sigma_{\mu\nu}(y). \tag{1.2.51}$$

Now $d\Sigma_{\mu\nu}(y)$ is the usual area element on a surface given by

$$d\Sigma_{\mu\nu}(y) = \frac{\epsilon^{ab}}{2} \partial_a y_\mu\ \partial_b y_\nu\ d^2\xi \tag{1.2.52}$$

with here $\partial_a \equiv \frac{\partial}{\partial \xi^a}$ is a worldsheet derivative. After expanding the ϵ sum this becomes

$$\begin{aligned} \partial^\mu F_{\mu\nu} = \frac{q}{2} \int_{\Sigma} d^2\xi\ \partial_y^\mu \delta^4(x-y)\ (\partial_1 y_\mu \partial_2 y_\nu - \partial_2 y_\mu \partial_1 y_\nu) = \\ \frac{q}{2} \int_{\Sigma} d^2\xi\ \big(\partial_1 \delta^4(x-y) \partial_2 y_\nu - \partial_2 \delta^4(x-y) \partial_1 y_\nu\big) \end{aligned} \tag{1.2.53}$$

where in the last line we used the chain rule. Green's theorem for two functions $M(x,y)$ and $L(x,y)$ is

$$\int_{\Sigma} \partial_x M\ \partial_y L - \partial_y M\ \partial_x L\ dxdy = \int_{\partial\Sigma} M\ \partial_y L\ dy + M\ \partial_x L\ dx. \tag{1.2.54}$$

We can use this to write (1.2.53) as

$$\partial^\mu F_{\mu\nu} = \frac{q}{2} \int \delta^4(x-y) \left(\partial_2 y_\nu d\xi^2 + \partial_1 y_\nu d\xi^1\right)\Big|_{\partial\Sigma} = q \int \delta^4(x-y)\ dy_\nu \Big|_{\partial\Sigma}. \quad (1.2.55)$$

Inserting the boundary values (worldlines of the interacting particles) confirms that $F^{\mu\nu}$ does, indeed, solve Gauss's law. The field strength defined in this way on a surface is already reminiscent of string theory. This relationship is enhanced when considering the expectation of the field strength over all possible surfaces bounded by $\partial\Sigma$. This kind of functional integration is exactly what is done to quantise the string. Indeed, averaging the field strength over all surfaces, where each surface is weighted by the Polyakov action yields the full field solution satisfying all field equations [21]

$$\langle F_{\mu\nu}(x)\rangle_\Sigma = \frac{q}{4\pi^2} \left(\partial_\mu \int_{\partial\Sigma} \frac{dy_\nu}{||y-x||^2} - \partial_\nu \int_{\partial\Sigma} \frac{dy_\mu}{||y-x||^2} \right) \quad (1.2.56)$$

where we define the average of some quantity, Ω, over all surfaces, Σ, spanning $\partial\Sigma$ as

$$\langle\Omega\rangle_\Sigma = \frac{1}{Z} \int \mathcal{D}g \mathcal{D}_g x\ \Omega\ e^{-S_p[x,g]} \quad (1.2.57)$$

and $S_p[x,g]$ is the Polyakov action (1.1.19). g is an intrinsic metric on Σ which must be integrated over. Note this result assumes a Euclidean worldsheet and target spacetime so that $||y-x||$ is the Euclidean distance between y and x and $1/||y-x||^2$ is the Euclidean Green's function of the Laplacian. The 3+1 dimensional result is found by Wick rotating back to Minkowski spacetime. The normalisation constant is $Z = \int \mathcal{D}g\ e^{-F}$ where F is the sum of $S_p[x,g]$ minimised with respect to x and gives rise to the Liouville theory associated with doing string theory in a non-critical dimension.

1.2.1 The action and relation to string theory

We can now draw the connection to string theory closer by using the field strength defined above to formulate a theory of strings with non-standard interaction. The

Lagrangian of pure electrodynamics without sources is

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.2.58}$$

as was found in our discussion on gauge theory. Above, we found a solution for the field strength that satisfied half of Maxwell's equations. Upon averaging over all configurations this solution lead to the full physical field strength that satisfies all of Maxwell's equations. Simply inserting the line of force solution (1.2.50) into the action gives

$$\begin{aligned} S = \int d^4X \ \mathcal{L} &= \frac{q^2}{4} \int d^4X \ \int_\Sigma \int_\Sigma \delta^4(X - Y(\xi))\delta^4(X - Y(\tilde{\xi}))d\Sigma(\xi)d\Sigma(\tilde{\xi}) \\ &= \frac{q^2}{4} \int_\Sigma \int_\Sigma \ d\Sigma(\xi)\delta^4(Y(\xi) - Y(\tilde{\xi}))d\Sigma(\tilde{\xi}). \end{aligned} \tag{1.2.59}$$

The action is only non-zero when the argument of the delta function is zero. This splits the action into the sum of two pieces; one in which $\xi = \tilde{\xi}$ and the other in which $Y(\xi) = Y(\tilde{\xi})$ when $\xi \neq \tilde{\xi}$. These two contributions reduce the action to

$$S = \frac{q^2}{4}\delta^2(0)\mathrm{Area}(\Sigma) + \frac{q^2}{4} \int_\Sigma \int_\Sigma \ d\Sigma(\xi)\delta^4(Y(\xi) - Y(\tilde{\xi}))d\Sigma(\tilde{\xi})\bigg|_{\xi \neq \tilde{\xi}}. \tag{1.2.60}$$

The first term is just the Nambu-Goto action albeit multiplied by a divergent constant corresponding to the free part of the string action. The interesting piece is the second term that corresponds to a contact interaction that occurs when the worldsheet self intersects. This kind of interaction has been used in a formulation of non-linear electrodynamics by Nielsen and Olsen [22] to form a field theory describing the dual string. Its dual has been used to describe the effective field theory for a Dirac string linking two magnetic monopoles [23] [24].

The bosonic theory obtained from (1.2.60) was shown to contain unwanted divergences potentially ruining the equivalence between this model and bosonic QED. It was conjectured in [26] that quantising a suitable supersymmetric analogue of this action lead to an equivalent formulation of QED without these unwanted divergences.

In the quantum theory the contact interaction can then be considered as a small perturbation to the free string. Indeed, the partition function

$$Z \equiv \int \mathcal{D}[X]\; e^{-S} =$$

$$\int \mathcal{D}[X]\; e^{-S_{NG}} \exp\bigg(-\frac{q^2}{4} \int_{\Sigma}\int_{\Sigma} d\Sigma(\xi)\delta^4(X(\xi) - X(\tilde{\xi}))d\Sigma(\tilde{\xi})\bigg|_{\xi\neq\tilde{\xi}} \bigg) \tag{1.2.61}$$

describes the average of the contact interaction over all worldsheets. We have already mentioned that the Nambu-Goto action is difficult to work with and so we may go ahead and replace this with the classically equivalent Polyakov action, being sure to integrate over the worldsheet metric, g. In fact we will show that the partition function corresponds to the expectation of products of pairs of vertex operators. For the bosonic case above, it was shown that the correct kind of dynamics are produced, namely the perturbative expansion lead to the insertion of propagators onto the boundary of the worldsheet. This result is equivalent to the perturbative expansion of the Wilson loop. This equivalence with the Wilson loop follows from considering the expectation of the contact interaction over genus 0 worldsheets, essentially because the Wilson loop of QED is evaluated as a closed curve in spacetime, which contains no holes.

Problems arose, however, when the insertions approached each other near the boundary of the worldsheet leading to divergences that ruin the validity of the theory. It was shown that when supersymmetry was included on the worldsheet, these divergences were removed by the additional structure and so we have a starting point of a reformulation of QED. These results will be repeated and streamlined in chapter 2 where we will also discuss a way to generalise the model to non-abelian gauge theory. A full reformulation of QED will necessarily require us to not only quantise the gauge fields, but also quantise the charged particles that form the boundary of the worldsheet. The most natural way to do this is to use the worldline formulation of QFT [27].

1.3 The worldline formalism of QFT

We will give a brief review of how one can go from field theory to amplitudes via worldline quantisation beginning with the simplest case of scalar quantum electrodynamics [27] [28] [29].

1.3.1 Scalar QED

We first consider quantising a charged spin-zero scalar field coupled to an external gauge field, A_μ. In the standard field theory the coupling of the field to the gauge field is achieved through the gauge covariant derivative, D_μ, so that for a massless field, the action is (1.1.14). Working now in Euclidean space, where functional integrals are better behaved, we have

$$S = -\int d^Dx \; D\phi^\dagger \cdot D\phi = \int d^Dx \; \phi^\dagger D^2 \phi. \tag{1.3.62}$$

We have performed an integration by parts and dropped the total derivative requiring the fields to vanish at infinity. The effective action is then

$$\Gamma[A] = \log Z = \log \int \mathcal{D}\phi \; e^{-\int d^4x \mathcal{L}} = -\log(\det(-D^2)) = -\mathrm{Tr}(\log(-D^2)) \tag{1.3.63}$$

where we have used the standard result for the integral of a Gaussian operator. We now introduce a Schwinger time parameter, noting that we should introduce a regulator to properly define the resulting integral

$$\Gamma[A] = \mathrm{Tr}\left(\int_0^\infty \frac{dT}{T} \; e^{-T(-D^2)}\right) \tag{1.3.64}$$

where $D^2 = i^2(p + qA)^2$. Formally, this representation is known as the Mellin transform of the effective action.

We can now perform the functional trace over momentum states as

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \int \frac{d^Dp}{(2\pi)^D} \; \langle p| \, e^{-\frac{\sqrt{\hbar}}{2}T(p+qA)^2} \, |p\rangle \tag{1.3.65}$$

where we have included the arbitrary constant, $\sqrt{h}$, which can be associated with an intrinsic metric on the worldline (not to be confused with the intrinsic metric on the worldsheet appearing in the Nambu-Goto action). Inserting two factors of the completeness relation, $1 = \int d^D x \, |x\rangle \langle x|$, and carrying out the p integral produces a delta function, $\delta^D(x - x')$, so the effective action becomes

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \int d^D x d^D x' \; \delta^D(x - x') \, \langle x| \, e^{-\frac{\sqrt{h}}{2} T (p+qA)^2} \, |x'\rangle \, . \tag{1.3.66}$$

We recognise the integrand as the amplitude for a bosonic particle coupled to an external gauge field to travel from x' to x. This can be written as a phase space path integral by computing the inverse Legendre transform of the Hamiltonian. The delta function enforces periodic boundary conditions on the amplitude so that the path integral sums all paths forming a closed loop. The resulting effective action is

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \; e^{-Tm^2} \int_{x(T)=x(0)} \mathcal{D}x \mathcal{D}p \; e^{-\int_0^T d\tau \;\, ip\cdot\dot{x} + \frac{\sqrt{h}}{2}(p+qA)^2} . \tag{1.3.67}$$

Thus, we have succeeded in writing a field theory effective action as a point particle path integral, where the path can be interpreted as the particles worldline. Now, completing the square in p allows us to write the effective action as

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \; e^{-Tm^2} \; \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x \; e^{-\int_0^T d\tau \; (\frac{\dot{x}^2}{2\sqrt{h}} - iq\dot{x}\cdot A)} . \tag{1.3.68}$$

$\mathcal{N}$ is a normalisation factor that contains the p^2 dependence that came from completing the square. At this point, we realise that by separating off the last term in the integral and using the chain rule, the effective action can be written as

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} \; e^{-Tm^2} \; \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x \; e^{-\int_0^T d\tau \; (\frac{\dot{x}^2}{2\sqrt{h}})} \exp\left(iq \oint dx \cdot A(x) \right) . \tag{1.3.69}$$

We see that the effective action is the expectation value of a Wilson loop of the background field.

Going back to (1.3.68), dropping the normalisation constant which won't be relevant to the following discussion, and setting $\sqrt{h} = 2$ so that we obtain a similar form to free particle action, we have

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-Tm^2} \int_{x(T)=x(0)} \mathcal{D}x\; e^{-\int_0^T d\tau\, \left(\frac{\dot{x}^2}{4} - iq\dot{x}\cdot A\right)}. \tag{1.3.70}$$

We proceed as in [42] by expanding the background gauge field as a sum of N plane waves

$$A^\mu = \sum_{i=1}^N \epsilon_i^\mu e^{ik_i\cdot x} \tag{1.3.71}$$

where ϵ^μ is the polarisation four vector associated with each constituent plane wave. Using this decomposition, we can write the interaction part of the exponential as

$$e^{\int_0^T d\tau\;\, iq\dot{x}\cdot A} = \exp\left(\int_0^T d\tau\; iq \sum_{i=1}^N \epsilon_i\cdot \dot{x} e^{ik_i\cdot x}\right) \tag{1.3.72}$$

$$= \sum_{n=1}^\infty \frac{1}{n!}\,\left(\int_0^T d\tau\; iq \sum_{i=1}^N \epsilon_i\cdot \dot{x} e^{ik_i\cdot x}\right)^n. \tag{1.3.73}$$

Consider the $n = N$ term in this expansion which corresponds to the N vertex loop amplitude. This is

$$\frac{1}{N!}\,\left(\int_0^T d\tau\; iq \sum_{n=1}^N \epsilon_i\cdot \dot{x} e^{ik_i\cdot x}\right)^N. \tag{1.3.74}$$

Now, we expand this, only keeping terms with different polarisations, of which there are $N!$ such terms so that

$$\frac{1}{N!}\,\left(\int_0^T d\tau\; iq \sum_{n=1}^N \epsilon_i\cdot \dot{x} e^{ik_i\cdot x}\right)^N\Bigg|_{\epsilon_1\epsilon_2\ldots\epsilon_N\;\text{ term}} = (iq)^N \prod_{i=1}^N \left(\int_0^T d\tau_i\; \epsilon_i\cdot \dot{x}_i e^{ik_i\cdot x_i}\right). \tag{1.3.75}$$

On the right hand side we have defined $x_i \equiv x(\tau_i)$, where the subscript on τ labels each integral in the product. This can then be written as

$$(iq)^N \prod_{i=1}^N V[k_i, \epsilon_i] \tag{1.3.76}$$

where we have defined the photon vertex operator for QED as

$$V[k,\epsilon] \equiv \int_0^T d\tau \ \epsilon \cdot \dot{x} \ e^{ik\cdot x}. \tag{1.3.77}$$

The amplitude is then the expectation of products of vertex operators which is reminiscent of how amplitudes are obtained in string theory. We now wish to get the integrand in exponential form. To do this we rewrite the linear polarisation term in the vertex operator as the linear term in the expansion of an exponential, i.e.

$$\epsilon_i \cdot \dot{x}_i = e^{\epsilon_i \cdot \dot{x}_i}\Big|_{\mathrm{lin}(\epsilon_i)}. \tag{1.3.78}$$

Now that we have separated off the interaction part of the exponential, we can separate the position as $x = x_0 + q$, where x_0 is the loop centre of mass defined as

$$x_0 \equiv \frac{1}{T}\int_0^T d\tau \ x^\mu(\tau) \tag{1.3.79}$$

and q is a quantum fluctuation. The path integration measure then factorises as $\int \mathcal{D}x = \int d^D x_0 \int \mathcal{D}q$. Integrating the position over the entire time period gives the extra condition on the fluctuation

$$\int_0^T q^\mu(\tau) \ d\tau = 0. \tag{1.3.80}$$

The amplitude then becomes

$$\Gamma_N[k_1, \epsilon_1, ..., k_N, \epsilon_N] =$$

$$(iq)^N \int_0^\infty \frac{dT}{T} \int d^D x_0 \int \mathcal{D}q \prod_{i=1}^N \int_0^T d\tau_i \ e^{\epsilon_i \cdot \dot{q}_i}\Big|_{\mathrm{lin}(\epsilon_i)} e^{ik_i\cdot(x_0+q)} e^{-\int_0^T d\tau \ \frac{1}{4}q - \frac{d^2}{d\tau^2}q}. \tag{1.3.81}$$

We can now carry out the x_0 integral, which simply implements momentum conservation as $\int d^D x_0 \ e^{\sum_i ik_i\cdot x_0} = (2\pi)^D \delta^D(\sum_i k_i)$. The process of including the loop centre of mass coordinate has removed the zero mode from the path integration. We can now invert the operator appearing in the path integral and use the linear algebra result

$$\frac{\int d^Dx\ e^{\frac{-1}{4}x\cdot Mx+x\cdot j}}{\int d^Dx\ e^{\frac{-1}{4}x\cdot Mx}} = e^{j\cdot M^{-1}\cdot j} \tag{1.3.82}$$

hence, we need to find an appropriate source term, $j(\tau)$, in (1.3.81). It is simply

$$j(\tau) \equiv \sum_{i=1}^{N}(i\delta(\tau-\tau_i)k_i - \delta'(\tau-\tau_i)\epsilon_i) \tag{1.3.83}$$

so that

$$\int_0^T d\tau\ j(\tau)\cdot q(\tau) = \sum_{i=1}^{N}\left(ik_i\cdot\int_0^T d\tau\ \delta(\tau-\tau_i)q(\tau) - \epsilon_i\int_0^T d\tau\delta'(\tau-\tau_i)\cdot q(\tau)\right) \tag{1.3.84}$$

$$= \sum_{i=1}^{N}(ik_i\cdot q_i + \epsilon_i\cdot\dot{q}_i). \tag{1.3.85}$$

The path integral can then be rewritten

$$\frac{\int \mathcal{D}q\ e^{-\int_0^T d\tau\ \frac{1}{4}q-\frac{d^2}{d\tau^2}q}e^{\sum_{i=1}^{N}(ik_i\cdot q_i+\epsilon_i\cdot\dot{q}_i)}}{e^{-\int_0^T d\tau\ \frac{1}{4}q-\frac{d^2}{d\tau^2}q}} = \frac{\int \mathcal{D}q\ e^{-\int_0^T d\tau\ \frac{1}{4}q-\frac{d^2}{d\tau^2}q-j\cdot q}}{e^{-\int_0^T d\tau\ \frac{1}{4}q-\frac{d^2}{d\tau^2}q}} = e^{-\int\ j\cdot G\cdot j} \tag{1.3.86}$$

where G is the Green's function of $-\frac{d^2}{d\tau^2}$ in this reduced space. This is easily computed as

$$G(\tau,\tau') = |\tau-\tau'| - \frac{(\tau-\tau')^2}{T} \tag{1.3.87}$$

so that

$$e^{-\int\ j\cdot G\cdot j} = \exp\left(-\frac{1}{2}\int_0^T d\tau\int_0^T d\tau'\ j(\tau)G(\tau,\tau')j(\tau')\right). \tag{1.3.88}$$

Note, this Green's function contains an extra term to what one might expect. This comes from the fact that we were trying to solve Poisson's equation on a circle which requires us to add a constant external field [27]. After expanding the js and using the derivative property of the delta function, we find this can be rewritten as

$$\exp\left(\sum_{i,j=1}^{N}\frac{1}{2}G_{ij}k_i\cdot k_j - i\epsilon_i\cdot k_j\dot{G}_{ij} + \frac{1}{2}\epsilon_i\cdot\epsilon_j\ddot{G}_{ij}\right) \tag{1.3.89}$$

where $G_{i,j} \equiv G(\tau_i, \tau_j)$ and $\dot{G_{i,j}} = \frac{\partial G}{\partial \tau_i}$. Then the effective action can be written as

$$\Gamma(A) = (iq)^N (2\pi)^D \delta^D \left(\sum_{i=1}^{N} k_i\right) \int_0^\infty \frac{dT}{T} (4\pi T)^{D/2} e^{-m^2 T} \prod_{i=1}^{N} \int_0^T d\tau_i \times$$

$$\exp\left(\sum_{i,j=1}^{N} \frac{1}{2} G_{ij} k_i \cdot k_j - i\epsilon_i \cdot k_j \dot{G}_{ij} + \frac{1}{2}\epsilon_i \cdot \epsilon_j \ddot{G}_{ij}\right)\Bigg|_{\text{lin } \epsilon_{i,j}} . \tag{1.3.90}$$

This is the Bern-Kosower formula for the one loop, N-photon amplitude in scalar QED, originally discovered from the particle limit of string theory, here derived using the worldline formalism of QFT [13]. This is of use when quantising the bosonic particles whose worldlines form the boundary of the worldsheet in the abelian model.

1.3.2 Spinor QED

We now turn to the coupling of a spinning particle to the external gauge field. The Dirac action describing the dynamics of a spinor coupled to an external gauge field is

$$S = \int d^4x \; \bar{\psi}(i\not{D} - m)\psi. \tag{1.3.91}$$

The effective action as a function of the gauge field, A_μ, is then

$$\Gamma[A] = \log\left[\det(i\not{D} - m)\right]. \tag{1.3.92}$$

We carry out a similar procedure to [43] in which a first order action is transformed to a second order action so that the effective action can be rewritten as

$$\Gamma[A] = \frac{1}{2}\log\left[\det(i\not{D} - m)\det(-i\not{D} - m)\right]. \tag{1.3.93}$$

Then, using $\det(A)\det(B) = \det(AB)$ and expanding we find

$$\det(i\not{D} - m)\det(-i\not{D} - m) = \det(\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2). \tag{1.3.94}$$

Now,

$$\gamma^\mu\gamma^\nu D_\mu D_\nu = (2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu)(-igF_{\mu\nu} + D_\nu D_\mu) = 2D^2 - iq\gamma^\mu\gamma^\nu F_{\mu\nu} - \gamma^\nu\gamma^\mu D_\nu D_\mu \tag{1.3.95}$$

where we have used $[D_\mu, D_\nu] = -iqF_{\mu\nu}$ and the Clifford algebra relation of the gamma matrices. Rearranging this we find that

$$\gamma^\mu\gamma^\nu D_\mu D_\nu = D^2 - \frac{iq}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} = D^2 - \frac{iq}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} \tag{1.3.96}$$

where we have used $AB = \frac{1}{2}([A,B] + \{A,B\})$ and $\{\gamma^\mu, \gamma^\nu\} F_{\mu\nu} = 0$ so that the effective action is

$$\Gamma[A] = \frac{1}{2}\log\left[\det(D^2 - \frac{iq}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} + m^2)\right]. \tag{1.3.97}$$

This effective action corresponds to the second order action given in [27]. We would like to carry out a similar procedure to the scalar QED case. To do this we introduce the Grassmann odd fields, $\psi^\mu(\tau)$, as partners of the $x^\mu(\tau)$ fields. The fermionic commutation relations are

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu} \tag{1.3.98}$$

implying that $\psi^\mu = \sqrt{\frac{1}{2}}\gamma^\mu$. We can now turn the effective action into a path integral over the x and ψ fields as we did in the previous subsection. For the massless case we have

$$\Gamma[A] = \frac{1}{2}\text{Tr}\ \log\left[D^2 - iq\psi^\mu\psi^\nu F_{\mu\nu}\right] =$$

$$-\frac{1}{2}\int_0^\infty \frac{dT}{T}\sum_\alpha \int \frac{d^4p}{(2\pi)^4}\,\langle\alpha, p|\exp\left[-\frac{1}{2}\sqrt{h}T\left((p+qA)^2 + iqF_{\mu\nu}\psi^\mu\psi^\nu\right)\right]|\alpha, p\rangle =$$

$$-\frac{1}{2}\int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}[x,\psi]\text{Tr}\ \exp\left[-\int_0^T d\tau\left(\frac{1}{2\sqrt{h}}\dot{x}^2 + \frac{1}{2}\psi\cdot\dot{\psi}\right.\right. \tag{1.3.99}$$

$$\left.\left.-iqA_\mu\dot{x}^\mu + \frac{iq\sqrt{h}}{2}F_{\mu\nu}\psi^\mu\psi^\nu\right)\right]. \tag{1.3.100}$$

The exponent has the form of the action of a worldline spinor. When the path integral is over a closed loop the last two terms make up the exponent of the super-

Wilson loop, so called because the action is invariant under the supersymmetry transformation

$$\delta_\eta x^\mu = -\sqrt{h}\ \eta\psi^\mu \qquad \delta_\eta\psi^\mu = \eta\dot{x}^\mu \tag{1.3.101}$$

with η an arbitrary Grassmann function of τ. This will be useful in the proceeding chapters when we come to consider a string theory that reproduces the properties of the loop.

We expand the exponential of the gauge terms and write the field as a sum of plane waves as we did before. Note, the only difference from the scalar case is the coupling of the fermionic field to the gauge field through the field strength, which after the plane wave expansion is

$$F_{\mu\nu} = \sum_{i=1}^{N} \left(i\epsilon_{i\nu}k_{i\mu} - i\epsilon_\mu k_{i\nu}\right) e^{ik_i\cdot x} \tag{1.3.102}$$

so that the effective action at order q^N is

$$\Gamma_N[A] = -\frac{(iq)^N}{2}\int_0^\infty \frac{dT}{T}\mathcal{N}\int \mathcal{D}x\mathcal{D}\psi\ \exp\left[-\int_0^T d\tau\left(\frac{1}{2\sqrt{h}}\dot{x}^2 + \frac{1}{2}\psi\cdot\dot{\psi}\right)\right]$$

$$\mathrm{Tr}\prod_{i=1}^{N}\int_0^T dt_i T^{a_i}[\epsilon_i\cdot\partial_i x(t_i) + i\sqrt{h}\epsilon_i\cdot\psi(t_i)k_i\cdot\psi(t_i)]e^{ik_i\cdot x(t_i)}. \tag{1.3.103}$$

We now have our supersymmetric generalisation of the bosonic vertex operator we had in scalar QED. From here we would proceed as before by introducing source terms and Green's functions to obtain the Bern-Kosower formula for spinor QED.

The key point of this section is that we have a way to quantise single particles along their worldlines that is equivalent to the usual field theory method. This will be useful as point particles worldline's will form the boundary of our worldsheet. After computing worldsheet expectations with the boundary fixed, we can then quantise the boundary itself using the above results, though we would have to generalise the method to non-abelian gauge fields.

1.4 Non-abelian gauge theory

We have seen how to compute amplitudes from the worldlines of particles coupled to an external gauge field. We can now turn to a discussion of the dynamics of the gauge field itself, in particular a non-abelian gauge field. A non-abelian gauge field theory is one that is invariant under the action of some representation of a general Lie group. In particular the transformation described earlier becomes a matrix which can still be written in exponential form as

$$U = \exp(q\Gamma) \tag{1.4.104}$$

where we have absorbed the factor of i into the coupling q. The exponential of a matrix is understood by its Taylor expansion so that

$$U = \mathbb{I} + q\Gamma + \frac{q^2}{2}\Gamma^2 + \ldots\ . \tag{1.4.105}$$

$\mathbb{I}$ is the identity matrix of dimension equal to the dimension of the representation. Γ can be expanded as $\Gamma = \Gamma^A \tau^A$, where Γ^A constitutes a set of linearly independent, real parameters and τ^A are the generators of the Lie group by which the representation is defined, which in general are non-commutative. To form a representation of the group, G, U must satisfy the associated group axioms. We can use the Taylor expansion of U to determine its properties from these axioms. Closure of the group demands that the generators close under commutation, i.e.

$$[\tau^A, \tau^B] = f^{ABC}\tau^C \tag{1.4.106}$$

where f^{ABC} are the structure constants.

Yang-Mills theory is the non-abelian gauge theory dealing with the $SU(N)$ group. In this case the generators are anti-Hermitian, traceless, $N \times N$ matrices. There exist $N^2 - 1$ linearly independent anti-Hermitian, traceless matrices, so that there are $(N^2 - 1)$ Γ^A coefficients.

The gauge field itself is now Lie algebra valued and can be decomposed in terms of the Lie algebra generators as

$$A_\mu(x) = A^A_\mu(x)\tau^A. \tag{1.4.107}$$

The covariant derivative in this case is $D_\mu = \partial_\mu + q[A_\mu, \cdot]$. The field strength defined as the curvature of the covariant derivatives gains an extra term not present in the abelian case:

$$\begin{aligned} F^A_{\mu\nu} \equiv \frac{1}{q}[D_\mu, D_\nu]^A &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + q[A_\mu, A_\nu]^A \\ &= \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + qf^{ABC}A^B_\mu A^C_\nu. \end{aligned} \tag{1.4.108}$$

This commutator term is responsible for self interactions of the gauge field. The action is a straight forward generalisation of the abelian case, where now we must include a trace over the Lie algebra generators to ensure gauge invariance

$$S^{YM} = \frac{1}{2}\int d^4x \ \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \tag{1.4.109}$$

where throughout this work we shall use the convention

$$\mathrm{Tr}(\tau^A\tau^B) = -\frac{1}{2}\delta^{AB}. \tag{1.4.110}$$

The quantum theory is obtained from the Euclidean partition function

$$Z[j] = \int \mathcal{D}A \ e^{-S^{YM} + \int d^4x \ j^\mu A_\mu}, \tag{1.4.111}$$

and expectations are then obtained by taking functional derivatives

$$\left\langle A^A_\mu(x_1)A^B_\nu(x_2)...A^M_\sigma(x_n)\right\rangle = \frac{\delta}{\delta j^A_\mu(x_1)}\frac{\delta}{\delta j^B_\nu(x_2)}\cdots\frac{\delta}{\delta j^M_\sigma(x_n)} \ Z[j]|_{j=0}. \tag{1.4.112}$$

There is a problem, however, when we come to define the propagator. Focussing just on the quadratic part of the action we have

$$Z[j] = \int \mathcal{D}A \ e^{-\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \int d^4x \ j^\mu A_\mu + ...}$$

$$= \int \mathcal{D}A \exp\bigg(\frac{1}{2} \int d^4x d^4y \; A^A_\mu(x) [\delta^{AB} \delta^4(x-y)(\partial^2 \delta^{\mu\nu} - \partial^\mu \partial^\nu)] A^B_\nu(y) + \int d^4x j^\mu A_\mu \bigg)$$

$$= (\mathrm{Det(M)})^{-1/2} \exp\bigg(\frac{1}{2} \int d^4x d^4y \; j^A_\mu(x) (M^{-1})^{AB\mu\nu}(x-y) j^B_\nu(y) \bigg) \qquad (1.4.113)$$

where in going from the second to the third line we have carried out the Gaussian integral in A, with $M^{\mu\nu}$ the operator sandwiched between the gauge fields in the second line. M^{-1} is then the propagator for the gauge field. This is where the problem with the partition function (1.4.111) lies; the operator M is not invertible as it has eigenvectors with zero eigenvalues of the form $v^0_\nu = \partial_\nu f$ since

$$(\partial^2 \delta^{\mu\nu} - \partial^\mu \partial^\nu) \partial_\nu f = 0. \qquad (1.4.114)$$

These zero modes arise by performing gauge transformations of $A_\mu = 0$. The partition function thus sums over gauge equivalent configurations of the fields.

To remedy this situation, the method of Faddeev and Popov imposes a gauge condition into the functional integral which is designed to cut each gauge orbit once so that the partition function sums only one representative from each gauge orbit[1]. The effect of this is to introduce unphysical fields with the wrong spin statistics called ghost fields into the functional integral. The gauge fixed partition function is, thus,

$$Z[j] = \int \mathcal{D}[A, c, b] \exp\bigg(- S^{YM} + \int d^4x \, \Big(j^\mu A_\mu - \partial^\mu \bar{c}^A D^{AB}_\mu c^B + \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \Big) \bigg) \qquad (1.4.115)$$

where $\bar{c}$ and c are the Grassmann odd ghost fields. The quadratic part of the action can be manipulated as before and the cubic and quartic terms can be treated as perturbations to the free action so that the full generating functional is

$$Z_{tot}[j, c] = \tilde{Z}[\varepsilon] Z[j] \qquad (1.4.116)$$

[1] Gribov showed that for certain gauge choices like the Landau gauge condition, $\partial_\mu A^\mu = 0$, there remain gauge equivalent configurations in the functional integral.

where

$$Z[j] = \exp(-S_I)\exp\bigg(\frac{1}{2}\int d^4x d^4y\ j_\mu^A(x) D^{AB\mu\nu}(x-y) j_\nu^B(y)\bigg) \tag{1.4.117}$$

is the gauge field part of the partition function and S_I is the term generating the three and four gluon vertices respectively given by

$$S_I = q\int d^4x\ f^{ABC}\partial_\mu\bigg(\frac{\delta}{j_\nu^A(x)}\bigg)\frac{\delta}{j_\mu^B(x)}\frac{\delta}{j^{\nu C}(x)}$$

$$+\frac{q^2}{4}\int d^4x\ f^{ABC}f^{ARS}\frac{\delta}{j_\mu^B(x)}\frac{\delta}{j_\nu^C(x)}\frac{\delta}{j^{\mu R}(x)}\frac{\delta}{j^{\nu S}(x)}. \tag{1.4.118}$$

Each three gluon vertex introduces a factor of q while each four gluon vertex introduces a factor of q^2. $D^{AB\mu\nu}(x-y)$ is the propagator for non-abelian gauge fields in ξ gauge

$$D^{AB\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\ \delta^{AB}\bigg(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\bigg)\frac{e^{ik\cdot(x-y)}}{k^2}. \tag{1.4.119}$$

The extra quadratic term produced by the Faddeev-Popov method does enough to make the operator sandwiching the gauge fields invertible.
The ghost piece of the partition function is

$$\tilde{Z}[\varepsilon] = \exp(-S^I_{ghost})\exp\bigg(-\int d^4x d^4y\ \bar{\varepsilon}^A(x) C^{AB}(x-y)\varepsilon^B(y)\bigg) \tag{1.4.120}$$

with the ghost propagator defined as

$$C^{AB}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\ \frac{\delta^{AB}}{k^2}\ e^{ik\cdot(x-y)} \tag{1.4.121}$$

and S^I_{ghost} describes the gluon-ghost-ghost interaction

$$S^I_{ghost} = q\int d^4x\ f^{ABC}\partial_\mu\bigg(\frac{\delta}{\bar{\varepsilon}^A(x)}\bigg)\frac{\delta}{j_\mu^B(x)}\frac{\delta}{\varepsilon^C(x)}. \tag{1.4.122}$$

Gauge invariant amplitudes are obtained by taking functional derivatives of the partition function, (1.4.116).

1.4.1 The Wilson loop

A particularly useful observable in the worldline formalism is the Wilson loop [30] defined as the path-ordered exponential of the gauge field transported along a closed loop in spacetime

$$W = \mathrm{Tr}\Big[\mathcal{P}\,\exp\Big(-q\oint dw^{\mu}\;A_{\mu}\Big)\Big]. \tag{1.4.123}$$

$\mathcal{P}$ denotes path ordering, something not present in abelian gauge theory. The trace once again is needed for gauge invariance and we will use w to parametrise the closed curve. They are observable in an analogous way to the Aharonov-Bohm effect of EM [31]. The path-ordering procedure for a product of N operators is defined as

$$\mathcal{P}\big(O_1(\xi_1)O_2(\xi_2)...O_N(\xi_N)\big) \equiv O_{l_1}(\xi_{l_1})O_{l_2}(\xi_{l_2})...O_{l_N}(\xi_{l_N}) \tag{1.4.124}$$

where on the right hand side the operators are ordered by the position of their arguments, i.e. $\xi_{l_1} \geq \xi_{l_2} \geq ... \geq \xi_{l_N}$. The path-ordering keeps track of the position of the matrix valued integrands as we expand the exponential.

The usefulness of the Wilson loop comes from the fact that any local operator can be written in terms of it. They are also used to differentiate between the confinement phase and asymptotically free phase of Yang-Mills theory. They have even been used as a first step towards a quantum theory of gravity, where Wilson loops, written in terms of a certain set of variables, have been shown to solve the Wheeler De-Witt equation of quantum gravity [25]. This lead to the study of Loop Quantum Gravity, a competing theory to string theory.

In the present case of Yang-Mills theory, we can integrate the Wilson loop over the gauge field and expand it as a Taylor series in powers of q so that

$$\langle W\rangle = \mathrm{Tr}\Big[\mathcal{P}\,\sum_{n=0}^{\infty}\frac{(-q)^n}{n!}\Big\langle\prod_{i=0}^{n}\oint dw_i^{\mu_i}A_{\mu_i}\Big\rangle\Big]. \tag{1.4.125}$$

The evaluation of the expectation of the Wilson loop therefore requires the calculation of $\langle A^n\rangle$. This is how amplitudes are computed in field theory.

The first non-trivial contribution to $\langle W\rangle$ is shown in Figure 1.1. Analytically, this

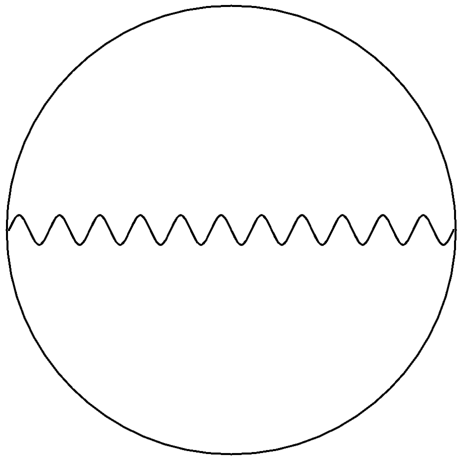


Figure 1.1: The first non-trivial diagram in the perturbative expansion of the Wilson loop describing the propagation of a gluon joining two distinct points on the boundary.

is

$$\frac{q^2}{2}\mathrm{Tr}\left[\mathcal{P}\oint\oint dw_1^\mu dw_2^\nu\ \left\langle A_\mu^A(w_1)A_\nu^B(w_2)\right\rangle\ \tau^A\tau^B\right]. \tag{1.4.126}$$

From the partition function, we see that the first non-trivial term of the expectation of the expansion of the Wilson loop is therefore

$$\frac{q^2}{2}\mathrm{Tr}\left[\mathcal{P}\oint\oint dw_1^\mu dw_2^\nu\ \frac{d^4k}{(2\pi)^4}\left(\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{e^{ik\cdot(w_1-w_2)}}{k^2}\ \tau^A\tau^A\right]. \tag{1.4.127}$$

This result continues to an arbitrary number of pairs of points on the boundary joined by propagators. Omitting the effects of self interactions, it is clear from the form of the generating functional that the expectation of the Wilson loop requires the computation of $\langle A^{2n}\rangle$. This will be

$$\left\langle A^{2n}\right\rangle = D^{A_1A_2\mu_1\mu_2}(x_1-x_2)D^{A_3A_4\mu_3\mu_4}(x_3-x_4)...+\text{permutations}. \tag{1.4.128}$$

There are $(2n-1)!!$ different ways of joining pairs of points, where !! is the double factorial defined as $k!! = k(k-2)(k-4)...$. In the Wilson loop the symmetries are such that each term in the above sum is equal which can be seen by interchanging the subscripts of the points on the boundary. Each term in the Taylor expansion of the Wilson loop is weighted by a factor of $1/(2n)!$ and so overall for each term we have

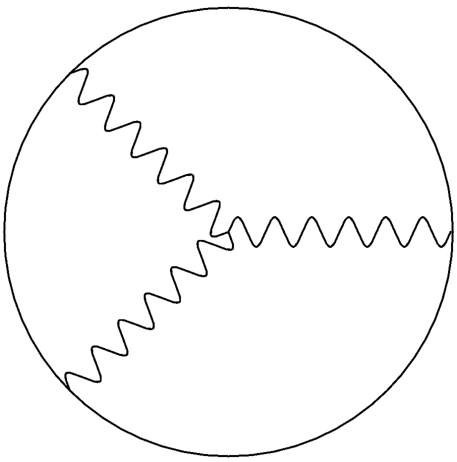
$$\frac{(2n-1)!!}{(2n)!} = \frac{(2n)!}{2^n(2n)!n!} = \frac{1}{2^n n!}. \tag{1.4.129}$$

The expectation of the Wilson loop, neglecting self interactions, is thus neatly evaluated as

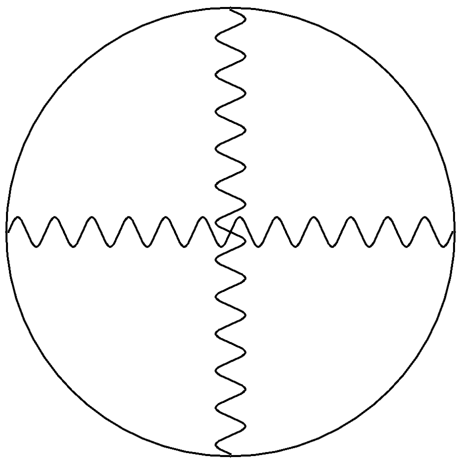
$$\langle W\rangle = \mathrm{Tr}\bigg(\mathcal{P}\exp\bigg(\frac{q^2}{2}\oint\oint dw_1^\mu dw_2^\nu \frac{d^4k}{(2\pi)^4}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\frac{e^{ik\cdot(w_1-w_2)}}{k^2}\,\tau^A\tau^A\bigg)\bigg). \tag{1.4.130}$$

For the abelian $U(1)$ gauge theory the generators are just 1 and so the trace and path-ordering are trivial. This result, with $\xi = 0$, is reproduced in [26] using the bosonic and fermionic string theory with contact interaction. Producing the path-ordered result above will be the first step towards a non-abelian generalisation of this model.

The next non-trivial difference between the abelian and non-abelian theories is the existence of self interactions. These are the three and four gluon vertices which appear at $O(q^3)$ and $O(q^4)$ of the expansion of the Wilson loop. Figures 1.2a and 1.2b show the lowest order in q in which the interactions appear. We can explicitly



(a) The first appearance of the three gluon vertex in the expectation of the Wilson loop at q^4.



(b) The first appearance of the four gluon vertex in the expectation of the Wilson loop at q^6.

Figure 1.2: The self interactions of Yang-Mills theory appearing in the perturbative expansion of the Wilson loop.

calculate their contribution to the expectation of the Wilson loop from the generating functional. In Landau gauge ($\xi = 0$) the three gluon vertex is

$$
\begin{aligned}
\mathrm{Tr}\Bigg(\mathcal{P}\ \frac{q^4}{2} f^{ABC} \tau^A \tau^B \tau^C \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4}\ \oint \oint \oint\ \frac{1}{k_1^2\ k_2^2\ (k_1+k_2)^2} \\
\times \left(dw_1^\mu - \frac{k_1^\mu k_1 \cdot dw_1}{k_1^2} \right) \left(dw_{2\mu} - \frac{k_{2\mu} k_2 \cdot dw_2}{k_2^2} \right) e^{-ik_1\cdot w_1 - ik_2 \cdot w_2} \\
\times i k_1^\mu \left(dw_{3\mu} - \frac{(k_1+k_2)_\mu (k_1+k_2)\cdot dx_3}{(k_1+k_2)^2} \right) e^{i(k_1+k_2)\cdot w_3} \Bigg)
\end{aligned}
\tag{1.4.131}
$$

while the four gluon vertex is

$$
\begin{aligned}
\mathrm{Tr}\Bigg(\mathcal{P}\ \frac{q^6}{4}\ f^{EAB} f^{ECD} \tau^A \tau^B \tau^C \tau^D \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} d^4k_4\ \frac{\delta^4(k_1+k_2+k_3+k_4)}{k_1^2\ k_2^2\ k_3^2\ k_4^2} \\
\times \left(dw_1^\mu - \frac{k_1^\mu k_1 \cdot dw_1}{k_1^2} \right) \left(dw_{2\mu} - \frac{k_{2\mu} k_2 \cdot dw_2}{k_2^2} \right) \left(dw_3^\nu - \frac{k_3^\nu k_3 \cdot dw_3}{k_3^2} \right) \left(dw_{4\nu} - \frac{k_{4\nu} k_4 \cdot dw_4}{k_4^2} \right) \\
\times\ e^{-ik_1\cdot w_1 - ik_2\cdot w_2 - ik_3 \cdot w_3 - ik_4 \cdot w_4} \Bigg).
\end{aligned}
\tag{1.4.132}
$$

The momentum conserving delta function has been left explicit here for clarity of the result. The final basic building block for all other diagrams is the 1 ghost loop shown in Fig. 1.3. The amplitude for this diagram is

$$
\begin{aligned}
\frac{q^2}{2} \mathrm{Tr}\Bigg(\mathcal{P}\ f^{ACD} f^{BCD} \tau^A \tau^B\ \int \frac{d^4k}{(2\pi)^4} \frac{d^4k_1}{(2\pi)^4}\ \frac{1}{k^2\ (k+k_1)^2} \\
\times\ \frac{ik^\mu}{k_1^2} \left(dw_1^\mu - \frac{k_1^\mu k_1 \cdot dw_1}{k_1^2} \right)\ \frac{ik^\mu}{k_1^2} \left(dw_2^\mu - \frac{k_1^\mu k_1 \cdot dw_2}{k_1^2} \right) e^{-ik_1 \cdot (w_1 - w_2)} \Bigg).
\end{aligned}
\tag{1.4.133}
$$

We will find that it is the expectation of the non-abelian Wilson loop that is reproduced in the string model. (1.4.127) is the simplest result to reproduce as it differs from the abelian result by the path-ordering of the Lie algebra generators. The additional interactions in the theory will prove more difficult to reproduce and, in fact, we will only be able to obtain the three gluon vertex in the string theory, though we believe the other interactions do exist within the model. We will comment on this later.

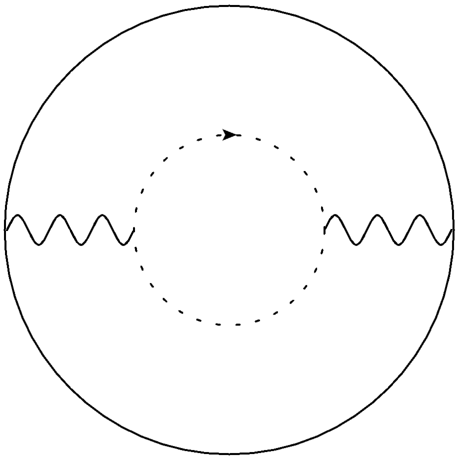


Figure 1.3: The first appearance of the ghosts in the expectation of the Wilson loop at q^4.

1.4.2 The super-Wilson loop

There exists an analogous way to describe the dynamics of gauge fields coupled to spinors. The action (1.3.100) gives us the Wilson loop for (non-supersymmetric) gauge fields coupled to particles with spin degrees of freedom. We call this the super-Wilson loop and define it as

$$W_s = \mathrm{Tr}\bigg(\mathcal{P}\,\exp\bigg(-q\oint dt\,\left(\dot{x}^\mu A_\mu - \frac{\sqrt{h}}{2}F_{\mu\nu}\psi^\mu\psi^\nu\right)\bigg)\bigg). \tag{1.4.134}$$

It is "super" because of the existence of the previously mentioned worldline supersymmetry.

The first non-trivial term in the expansion of the expectation of the super-Wilson loop is analogous to the bosonic result. To show this we need to use the fact that

$$\begin{aligned}\left\langle A^A_\sigma(x)F^B_{\mu\nu}(x')\right\rangle \psi^\mu(x')\psi^\nu(x') &= 2\partial'_\mu \left\langle A^A_\sigma(x)A^B_\nu(x')\right\rangle \psi^\mu(x')\psi^\nu(x') \\ &= 2\psi^\mu(x')\psi^\nu(x')\partial'_\mu D^{AB}_{\sigma\nu}(x-x').\end{aligned} \tag{1.4.135}$$

The commutator term in the field strength will lead to $O(q^3)$ terms which won't contribute to the propagator and so we omit these here. With this the $O(q^2)$ expectation of W_S is

$$\langle W_s \rangle \ni \frac{q^2}{2}\mathrm{Tr}\Big[\mathcal{P} \oint \oint dtdt' \, (dx^\mu + \sqrt{h}\psi^\mu\psi^\nu\partial_\nu)(dx'^\alpha + \sqrt{h'}\psi'^\alpha\psi'^\beta\partial'_\beta)D^{AB}_{\mu\alpha}(x-x')\tau^A\tau^B\Big]. \tag{1.4.136}$$

We will calculate this in Landau gauge when we come to look at the string theory. We will need to extend the worldline supersymmetry along the loop into the interior which obviously hints at use of the superstring. Higher order diagrams are generated in the same manner as in the bosonic case, just replacing bosonic propagators with the supersymmetric worldline structure. We will see that two different methods give rise to the path-ordering of the generators but only one method that we study will contain the structure needed to produce the self interactions.

1.5 Notation

We will use $\langle \Omega[A, B, ...] \rangle_{A,B...}$ to denote the functional integral

$$\int \mathcal{D}[A, B, ...] \; \Omega[A, B, ...] \; e^{-S[A,B,...]}. \tag{1.5.137}$$

The subscripts on $\langle \cdot \rangle$ will denote the variable being integrated over when it is not trivial, for multiple integrations etc.

We will use the terminology average to mean thermal average as this is not quantum mechanical. We will use expectation to mean the functional integral used to calculate quantum mechanical expectations such as the path integral or Polyakov type surface integral.

Chapter 2

A String Model of Gauge Theory

It has been shown that abelian gauge theory can be reformulated as a string theory in which a line of flux joining two oppositely charged particles is treated as the degrees of freedom of the gauge field [26]. In the fermionic case, quantising the string theory as well as the worldlines of the interacting particles was shown to be equivalent to QED in the tensionless limit [45]. Our aim here is to generalise this prescription to non-abelian gauge theory. The way in which this will be done is by introducing Lie algebra valued worldsheet variables, J^A, into the vertex operator, generalising the boundary field theory of [35]. In the following chapters we will look at two particular field theories that can be used to describe the dynamics of J^A. We will begin with a review of the bosonic abelian case and introduce a streamlined calculation of the perturbative expansion of the interacting action. We will then generalise this to the fermionic abelian case where we will deal with the realistic case of worldline fermions interacting with the gauge bosons. Building on these results, we will show how to generalise these models to reproduce the expectation of the non-abelian Wilson loop computed in Yang-Mills theory. This will give certain requirements for J^A that the rest of the thesis will be devoted to fulfilling.

2.1 A review of scalar abelian gauge theory

Let us briefly recapitulate on the argument for a string theory formulation of gauge theory. The covariant form of Gauss' law is

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{2.1.1}$$

This is usually solved by introducing a gauge field and formulating an action whose equations of motion reduce to this. The quantum dynamics of the system is then obtained by quantising this field theory. We will take a different approach. Firstly, consider two free equal but opposite charges moving with respect to each other. The system is described by the four-current

$$j^\mu(x) = q\int_B \delta^4(x-w)dw^\mu \tag{2.1.2}$$

i.e. the charge density exists on the worldlines of the two interacting particles, denoted by B. The solution to (2.1.1) with this four-current is

$$F^{\mu\nu}(x) = -q\int_\Sigma \delta^4(x-X)d\Sigma^{\mu\nu}(X) \tag{2.1.3}$$

where $d\Sigma^{\mu\nu}$ is an infinitesimal element of area on the surface, Σ. B therefore constitutes the boundary of this surface. Inserting this solution into the Maxwell action we get

$$S_{EM} = -\frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} = \frac{q^2}{4}\int_\Sigma d\Sigma_{\mu\nu}(X(\xi))\delta^4(X(\xi)-X(\xi'))d\Sigma^{\mu\nu}(X(\xi')). \tag{2.1.4}$$

The integrand is only non-zero when either $\xi = \xi'$ or the worldsheet self intersects i.e. $X(\xi) = X(\xi')$ with $\xi \neq \xi'$. This gives two contributions to the action

$$S_{EM} = \frac{q^2}{4}\delta^2(0)\mathrm{Area}(\Sigma) + \frac{q^2}{4}\int_\Sigma d\Sigma_{\mu\nu}(X(\xi))\delta^4(X(\xi)-X(\xi'))d\Sigma^{\mu\nu}(X(\xi'))|_{\xi\neq\xi'}. \tag{2.1.5}$$

The first term is proportional to the Nambu-Goto action (and therefore classically equivalent to the Polyakov action) and the second term is a contact interaction.

Scalar QED can then be formulated by considering the partition function for this action

$$Z = \frac{1}{Z_0} \int \mathcal{D}(X,g) e^{-S_p(X,g) - \frac{q^2}{4} S_I(X)} \tag{2.1.6}$$

where $S_p(X,g)$ is the Polyakov action and S_I is the contact interaction. Treating the gauge coupling, q, as a small parameter allows one to Taylor expand the interaction so that the partition function can be written as a perturbative series of expectations of the contact interaction. The object of interest in this theory is then the expectation, $q^n \langle (S_I)^n \rangle$, over worldsheets, Σ, spanning B. The first order interaction of the bosonic theory is thus

$$\frac{q^2}{4Z_0} \int \mathcal{D}(X,g)\ e^{-S_p} \int_\Sigma d\Sigma^{\mu\nu} \delta^4(X - X') d\Sigma'_{\mu\nu\ k}. \tag{2.1.7}$$

A brief word on notation here; $X' \equiv X(\xi')$ and $d\Sigma_k^{'\mu\nu} \equiv d\Sigma^{\mu\nu}(X_k(\xi'))$. We can, in fact, write the contact interaction as an integral of vertex operator insertions at ξ and ξ' respectively. To show this, we can Fourier decompose the delta function so that the contact interaction can be written as

$$S_I = \int \frac{d^4 k}{(2\pi)^4} \int_\Sigma\ d\Sigma^{\mu\nu}\ e^{ik\cdot(X-X')}\ d\Sigma'_{\mu\nu}. \tag{2.1.8}$$

The infinitesimal element of area is as usual

$$d\Sigma^{\mu\nu}(X(\xi)) = \frac{1}{2}\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu d^2\xi. \tag{2.1.9}$$

Roman letters here represent worldsheet indices and Greek letters represent target space indices. Inserting the surface element into (2.1.8) allows us to then write the contact interaction as

$$S_I = \int \frac{d^4 k}{(2\pi)^4}\ V_k^{\mu\nu} V'_{\mu\nu\ -k} \tag{2.1.10}$$

with the vertex operator, V, defined as

$$V_k^{\mu\nu} \equiv \frac{1}{2} \int d^2\xi\ \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X}. \tag{2.1.11}$$

$\langle S_I \rangle$ was calculated using Wick's theorem previously [26], thus requiring a determination of the entire contraction algebra. Here, we will obtain the same results using a simpler method that involves projecting X along the direction of the four momentum k. This reduces the vertex operator to a "projected" vertex plus a total derivative. The term in S_I that consists of the product of total derivatives leads to the propagator of a scalar boson joining two points on the boundary of the worldsheet. This is then equivalent to the expectation of the abelian Wilson loop squared. We begin by defining the projection operator, $\mathbb{P}_k$, that acts on four vectors, v^μ, as

$$\mathbb{P}_k(v)^\mu = v^\mu - k^\mu \frac{k \cdot v}{k^2}. \tag{2.1.12}$$

This is defined so that $k_\mu \mathbb{P}_k(v)^\mu = 0$. Using the projection operator to project X along the direction of k allows us to write the vertex operator as

$$V_k^{\mu\nu} = \frac{1}{2}\int d^2\xi \; \varepsilon^{ab} \; \partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu e^{ik\cdot X} - \int d^2\xi \; \partial_a \left(\frac{i\varepsilon^{ab}}{k^2} k^{[\mu} \partial_b \mathbb{P}_k^{\nu]} e^{ik\cdot X} \right). \tag{2.1.13}$$

One can then define the projected vertex operator as

$$\mathbb{V}_k^{\mu\nu} \equiv \frac{1}{2}\int d^2\xi \; \varepsilon^{ab} \; \partial_a \mathbb{P}_k(X)^\mu \partial_b \mathbb{P}_k(X)^\nu e^{ik\cdot X}. \tag{2.1.14}$$

The vertex operator has been decomposed into a projected vertex plus a total derivative. The contact interaction can now be written as

$$\begin{aligned} &\int \frac{d^4k}{(2\pi)^4} \left[\mathbb{V}_k^{\mu\nu} - d^2\xi \; \partial_a \left(\frac{i\varepsilon^{ab}}{k^2} k^{[\mu} \partial_b \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X} \right) \right] \times \\ &\left[\mathbb{V}'_{\mu\nu -k} + d^2\xi' \; \partial'_c \left(\frac{i\varepsilon^{cd}}{k^2} k_{[\mu} \partial'_d \mathbb{P}_k(X')_{\nu]} e^{-ik\cdot X'} \right) \right]. \end{aligned} \tag{2.1.15}$$

Expanding, and using the fact that $\mathbb{V}_k^{\mu\nu} k_\mu = \mathbb{V}_k^{\mu\nu} k_\nu = 0$ leaves just two terms

$$\begin{aligned} S_I = &\int \frac{d^4k}{(2\pi)^4} \; \mathbb{V}_k^{\mu\nu} \mathbb{V}'_{\mu\nu -k} \\ - &\int \frac{d^4k}{(2\pi)^4} d^2\xi d^2\xi' \; \partial_a \left(\frac{i\varepsilon^{ab}}{k^2} k^{[\mu} \partial_b \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X} \right) \partial'_c \left(\frac{i\varepsilon^{cd}}{k^2} k_{[\mu} \partial'_d \mathbb{P}_k(X')_{\nu]} e^{-ik\cdot X'} \right). \end{aligned} \tag{2.1.16}$$

Focussing firstly on the second term, we recognise this as simply a product of total derivatives that can be readily calculated. We will show this for the first term in the product. Define the integral

$$I \equiv \int d^2\xi \; \partial_a \bigg(\frac{i\varepsilon^{ab}}{k^2} k^{[\mu} \partial_b \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X} \bigg). \tag{2.1.17}$$

In complex worldsheet coordinates this is[1]

$$I = \frac{i}{k^2} \int d^2 z \; \Big(\partial(k^{[\mu} \bar{\partial} \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X}) - \bar{\partial}(k^{[\mu} \partial \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X}) \Big) \tag{2.1.18}$$

where $z = x + iy$ and $\bar{z} = x - iy$ so that $d^2 z = 2dxdy$. We can use Stoke's theorem to take these integrals to the boundary

$$\begin{aligned} I &= \frac{i}{k^2} \bigg(\int d\bar{z} \; k^{[\mu} \bar{\partial} \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X} + \int dz \; k^{[\mu} \partial \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X} \bigg) \\ &= \frac{i}{k^2} \int_B k^{[\mu} d\mathbb{P}_k(w)^{\nu]} e^{ik\cdot w} \end{aligned} \tag{2.1.19}$$

where w is the boundary value of X. For the case of two particles created in the vacuum and then annihilating each other shortly after, the boundary is a closed curve. This is the case that was proven to reproduce the expectation value of the Wilson loop to $O(q^2)$ and so we shall consider it here.

Inserting the boundary integral into the second term of (2.1.16) gives

$$\begin{aligned} &\int \frac{d^4k}{(2\pi)^4} \oint_B \oint_B \frac{1}{k^2} k^{[\mu} d\mathbb{P}_k(w)^{\nu]} k_{[\mu} d\mathbb{P}_k(w')_{\nu]} e^{ik\cdot(w-w')} \\ &= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \oint_B \oint_B \bigg(\frac{d\mathbb{P}_k(w)\cdot d\mathbb{P}_k(w')}{k^2} - \frac{k\cdot d\mathbb{P}_k(w) \; k\cdot d\mathbb{P}_k(w')}{k^4} \bigg) e^{ik\cdot(w-w')} \end{aligned} \tag{2.1.20}$$

The second term vanishes because we have the inner product of the projection operator and its associated momentum leaving us with just the first term. The

[1] It is not necessary to use complex coordinates here, however, when we come to the supersymmetric analogue of this integral we will use Stoke's theorem in superspace, which is naturally written in complex coordinates, hence, using them here will be useful when comparing the two cases.

contact interaction can then be written as

$$S^I = \int \frac{d^4k}{(2\pi)^4}\, \mathbb{V}_k^{\mu\nu}\mathbb{V}'_{\mu\nu -k} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \oint_B \oint_B \frac{d\mathbb{P}_k(w)\cdot d\mathbb{P}_k(w')}{k^2} e^{ik\cdot(w-w')}. \tag{2.1.21}$$

We can now consider functionally integrating this over all worldsheets spanning B. The second term depends only on the boundary and so averaging, holding B fixed, will have no effect. To consider why the first term vanishes we need to consider the possible contractions between terms via Wick's theorem. The full set of contractions can be found in [26], we will just list the main results. The simplest of which is

$$\langle X_\mu(\xi)X_\nu(\xi')\rangle_\Sigma =: X_\mu(\xi)X_\nu(\xi') : + \underbracket{X^\mu(\xi)X}^\nu(\xi'). \tag{2.1.22}$$

The colons indicate normal ordering, meaning all contractions have been carried out in the functional integral. The generating functional for the string theory was obtained by expanding around the classical field, X_c, and so

$$: X_\mu(\xi)X_\nu(\xi') := X_c^\mu X_c^\nu. \tag{2.1.23}$$

The main contraction is

$$\underbracket{X^\mu(\xi)X}^\nu(\xi') = \alpha'\delta^{\mu\nu}G(\xi,\xi'). \tag{2.1.24}$$

α' is the string scale and $G(\xi,\xi')$ is the Green's function for the worldsheet Laplacian. For a general term of the form $A^\mu e^{ik\cdot B}$ we have

$$ik_\nu \underbracket{A^\mu B}^\nu \exp\Big(-\frac{1}{2}k_\nu k_\sigma \underbracket{B^\nu B}^\sigma\Big) : e^{ik\cdot B} : + \exp\Big(-\frac{1}{2}k_\nu k_\sigma \underbracket{B^\nu B}^\sigma\Big) : A^\mu e^{ik\cdot B} : \tag{2.1.25}$$

and then together with

$$\langle \partial_1 X_\mu(\xi)\partial'_1 X'_\nu(\xi')\rangle_\Sigma = 4\pi\alpha'\delta_{\mu\nu}\partial_1\partial'_1 G(\xi,\xi') + \partial_1 X_{c\mu}(\xi)\partial'_1 X_{c\nu}(\xi') \tag{2.1.26}$$

we can evaluate the functional integral of $\mathbb{V}$ over worldsheets spanning B. It is easy to convince oneself that the anti-symmetry of the vertex plus the symmetry of (2.1.26) means the only contraction that will contribute to the result will be the self-contractions of the exponentials, $e^{\pm ik\cdot X}$, which from (2.1.25) are

$$e^{\pm ik\cdot X} =: e^{\pm ik\cdot X} : e^{-\pi\alpha' k^2 G(\xi,\xi)}. \tag{2.1.27}$$

Therefore, inside the functional integral we can replace the "projected" vertex operator with

$$\mathbb{V}_{\mu\nu}(k,\xi) =: \mathbb{V}_{\mu\nu}(k,\xi) : e^{-\pi\alpha' k^2 G(\xi,\xi)}. \tag{2.1.28}$$

The Green's function at coincident points diverges and should be regulated with a short-distance cut-off, ϵ. We replace it with the regulated heat kernel

$$G^\epsilon(\xi,\xi') = \int_\epsilon^\infty d\tau\ \mathcal{G}(\xi,\xi';\tau) \tag{2.1.29}$$

satisfying

$$(\partial_\tau + \Delta)\mathcal{G} = 0, \quad \mathcal{G}(\xi,\xi';0) = \frac{\delta^2(\xi-\xi')}{\sqrt{g}}. \tag{2.1.30}$$

The heat kernel has the spectral decomposition in terms of the eigenfunctions of the Laplacian, u_n,

$$\mathcal{G}(\xi,\xi';\tau) = \sum_n u_n(\xi)u_n(\xi')e^{-\tau\lambda_n}. \tag{2.1.31}$$

The short distance divergence of the Green's function is then associated with the short-time behaviour of the heat kernel. We can use the Seeley-DeWitt expansion for the heat kernel at short times [32], modified to take into account the effect of the boundary [33] [34] so that

$$\begin{aligned} G^\epsilon(\xi,\xi) \equiv \psi(\xi) &\sim \int_\epsilon^\infty \frac{d\tau}{4\pi\tau}\Big(1-\exp\Big(-\frac{\sigma}{4\tau}\Big)\Big)\Big(1+\frac{\tau}{6}R(\xi)\Big) \\ &= \begin{cases} \frac{\sigma}{16\pi\epsilon} - \frac{\sigma\ln(\epsilon R)}{96\pi} & \sigma \ll \epsilon \\ \frac{1}{4\pi}\ln\left(\frac{\sigma}{4\epsilon}\right) - \frac{\epsilon R}{24\pi} & \sigma \gg \epsilon \end{cases} \end{aligned} \tag{2.1.32}$$

where σ is the square of the distance of the shortest path between ξ and itself via a reflection from the boundary. In complex coordinates and conformal gauge the line element is $ds^2 = e^\phi dzd\bar{z}$, therefore $\sigma(z,z') = e^\phi|z-z'|^2$. The heat kernel can then be written as

$$\mathcal{G}(\xi,\xi;\tau) = \frac{1}{4\pi\tau}\left(\exp\left(-\frac{e^{\phi}|z-z'|^2}{4\tau}\right) - \exp\left(-\frac{e^{\phi}|z-\bar{z}'|^2}{4\tau}\right)\right). \tag{2.1.33}$$

At leading order in the cut-off it is sufficient to work with $\phi = 0$. The Green's function can then be written as

$$G^{\epsilon}(z,z') = -f\left(\frac{|z-z'|}{2\sqrt{\epsilon}}\right) + f\left(\frac{|z-\bar{z}'|}{2\sqrt{\epsilon}}\right) \tag{2.1.34}$$

where

$$f(s) = \int_1^{\infty} \frac{d\tau}{4\pi\tau}\left(1 - \exp\left(-\frac{s^2}{\tau}\right)\right). \tag{2.1.35}$$

This means that $f(s) \sim \frac{1}{4\pi}\ln s^2$ when $s \gg 1$. For the case above, (2.1.28), where we have the Green's function evaluated at coincident points in the interior of the worldsheet, we have $G(\xi,\xi) \sim \frac{1}{2\pi}\ln(y/\sqrt{\epsilon})$. In the Wick-rotated theory, $k^2 > 0$ and so $e^{-\pi\alpha' k^2 G(\xi,\xi)/2}$ is suppressed in the interior of the worldsheet for Fourier modes for which $\alpha' k^2$ is finite as the cut-off is removed. The tensionless limit corresponds to taking $\alpha'/L^2 \to \infty$ where L is a length scale characterising B, enhancing the suppression. The only remaining term in the expectation of the contact interaction is then

$$\langle S_I \rangle_{\Sigma} = \frac{1}{2}\oint_B \oint_B d\mathbb{P}_k(w)^{\mu}\; G_B(w,w')\; d\mathbb{P}_k(w')_{\mu} \tag{2.1.36}$$

where

$$G_B(w,w') = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(w-w')}}{k^2} \tag{2.1.37}$$

is the propagator of a scalar boson. This verifies the claim that at first order, the string theory that follows from the line of force solution to Gauss' law, reproduces the expectation value of the Wilson loop, expanded to order q^2, where the propagator is in Landau gauge ($\xi = 0$) (1.4.127). Note, in this gauge the photon propagator can be written as

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{e^{ik\cdot(x-y)}}{k^2} = \int \frac{d^4k}{(2\pi)^4}\;\mathbb{P}_{k,\mu\nu}\frac{e^{ik\cdot(x-y)}}{k^2} \tag{2.1.38}$$

with

$$\mathbb{P}_{k,\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \tag{2.1.39}$$

$$dx_1^\mu \mathbb{P}_{k,\mu\nu} dx_2^\nu = \mathbb{P}_k(dx_1) \cdot dx_2 = dx_1 \cdot \mathbb{P}_k(dx_2) = \mathbb{P}_k(dx_1) \cdot \mathbb{P}_k(dx_2) \tag{2.1.40}$$

i.e. we see the appearance of the projection operator and why we have equivalence in this particular gauge. This suggests that the expectation value of the Wilson loop to all orders might be expressed as the worldsheet average of the exponential of S_I. However, divergences appear when the exponential is expanded in powers of S_I that potentially spoil the suppression of unwanted terms. Vertex operators placed at points close to each other and near the boundary lead to divergences that are not necessarily suppressed in the tensionless limit.

It has been shown in [26] that these extra terms are not produced in the supersymmetric generalisation of this model. The supersymmetric case is also more realistic as it naturally incorporates the coupling of fermions to the gauge bosons. It is this case that we will study in the next section. The problem of extra unwanted terms will arise again when we look at a non-abelian generalisation of the bosonic string theory for the same reasons. Even though the bosonic theory is simpler, and the computations of expectation values are similar, it will ultimately be the supersymmetric theory that we wish to obtain.

2.2 Fermionic abelian gauge theory (QED)

In this section we consider replacing the bosonic point particles that live on the boundary of the worldsheet by fermionic point particles. Doing this gives the boundary extra spin degrees of freedom, characterised by the worldline spinors, ψ (1.3.2). These spin degrees of freedom will then naturally exist within the interior of the worldsheet. The free string theory that describes the dynamics of the string will then be that of the spinning string given by a straightforward generalisation of the Polyakov action. The underlying structure of the spinning particle and string is an $N = 1$ worldline and worldsheet supersymmetry respectively. These are 1 and 2 dimensional symmetries relating the bosonic and spin degrees of freedom. Before defining the action and contact interaction in the supersymmetric case we will briefly describe the superspace formulation of 2 dimensional supersymmetric field theories as this will be used extensively throughout.

2.2.1 Superspace

An efficient method of incorporating supersymmetry into a theory is to introduce the anti-commuting superpartners, θ and $\bar{\theta}$, for the coordinates z and $\bar{z}$ respectively. By superpartner we mean that the coordinates will be related to the new variables by a supersymmetry transformation. This changes the original 2 dimensional surface into a so called "super" surface. The supersymmetry transformations by which the two sets of coordinates are related are parametrised by η and $\bar{\eta}$ and given by

$$\delta z = -\eta\theta, \qquad \delta\bar{z} = -\bar{\eta}\bar{\theta}, \qquad \delta\theta = \eta, \qquad \delta\bar{\theta} = \bar{\eta} \tag{2.2.41}$$

with θ, $\bar{\theta}$, η and $\bar{\eta}$ Grassmann-odd. This can be compared with the worldline supersymmetry (1.3.101). The fact that the new variables anti-commute allows one to Taylor expand any "super"-function, $f(z, \bar{z}, \theta, \bar{\theta})$, into its component functions as

$$f = f_0(z, \bar{z}) + \theta f_1(z, \bar{z}) + \bar{\theta}\bar{f}_1(z, \bar{z}) + \theta\bar{\theta} f_2(z, \bar{z}). \tag{2.2.42}$$

The supersymmetry transformation of a superfunction can then be written as

$$\delta f = \eta(\partial_\theta - \theta\partial + \partial_{\bar{\theta}} - \bar{\theta}\bar{\partial})f = \eta(Q + \bar{Q})f \tag{2.2.43}$$

where $Q = \partial_\theta - \theta\partial$ and $\bar{Q} = \partial_{\bar{\theta}} - \bar{\theta}\bar{\partial}$ are the generators of the supersymmetry. We define the covariant derivatives as

$$\eta D \equiv \delta\theta\ \partial_\theta - \delta z\ \partial = \eta(\partial_\theta + \theta\partial) \tag{2.2.44}$$

$$\bar{\eta}\bar{D} \equiv \delta\bar{\theta}\ \partial_{\bar{\theta}} - \delta\bar{z}\ \bar{\partial} = \bar{\eta}(\partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}). \tag{2.2.45}$$

One defines left differentiation as

$$\partial_\theta\ (\theta f_1) = -\partial_\theta\ (f_1\theta) = f_1 \tag{2.2.46}$$

for f_1 Grassmann-odd. Integration is defined by

$$\int d^2\theta\ 1 = \int d^2\theta\ \theta = \int d^2\theta\ \bar{\theta} = 0, \qquad \int d^2\theta\ \bar{\theta}\theta = 1\ \rightarrow\ d^2\theta = d\theta d\bar{\theta} \tag{2.2.47}$$

i.e. integration picks out the $\bar{\theta}\theta$ term of a superfunction and is equivalent to differentiation by $\partial_\theta\partial_{\bar{\theta}}$. The supersymmetric generalisation of Stokes' theorem is

$$\int d^2zd^2\theta\ (Df+\bar{D}g) = \int d^2zd^2\theta\ (\theta\partial f+\bar{\theta}\bar{\partial}g) = \oint d\bar{z}d^2\theta\ \theta f| - \oint dzd^2\theta\ \bar{\theta}g| \tag{2.2.48}$$

where $|$ denotes the functions f and g are evaluated on the boundary. There is a nice way of making any functional supersymmetric that we utilise over and over again. This makes the additional boundary term in the action for an open spinning string less mysterious. Consider a general functional, W, such that

$$W = \int_\Sigma d^2zd^2\theta\ \Psi(z,\theta). \tag{2.2.49}$$

We take Σ to be closed and so choose the unit disk for simplicity. A conformal transformation can take the disk to the upper half plane. The boundary of the surface is, thus, the x axis. The variation under a supersymmetry transformation is

$$\delta W = \int d^2zd^2\theta\ \epsilon(Q+Q)\Psi = -\int d^2zd^2\theta\ \epsilon(\theta\partial+\bar{\theta}\bar{\partial})\Psi = -\int dxd^2\theta\ \epsilon(\theta-\bar{\theta})\Psi|. \tag{2.2.50}$$

$\Psi|$ denotes the boundary value of Ψ. We can expand Ψ in terms of its component fields as $\Psi = \Psi_0 + \theta\Psi_1 + \bar{\theta}\Psi_2 + \theta\bar{\theta}\Psi_3$ so that this variation can be written as

$$\delta W = \int dx\ \epsilon(\Psi_2+\Psi_1)|. \tag{2.2.51}$$

Note, that $\delta\Psi_0 = \epsilon(\Psi_2 + \Psi_1)$ and so the total variation of the functional under a supersymmetry transformation is

$$\delta W = \int dx\ \delta(\Psi_0). \tag{2.2.52}$$

Therefore, adding the boundary term $-\int dx\ \Psi_0$ to W will make it supersymmetric. There is a nice way of writing this boundary term by noting that $\Psi_0 = \int d^2\theta\ \bar{\theta}\theta\ \Psi_0$, then the supersymmetric functional W' is

$$W' = \int d^2z d^2\theta\ \Psi\big(1 - \bar{\theta}\theta\delta(y)\big). \tag{2.2.53}$$

The action describing the dynamics of the spinning string turns out to be the superspace generalisation of the Polyakov action. To motivate this we start with the gauge fixed Polyakov action for the bosonic string

$$S_{bos} = \frac{1}{4\pi\alpha'}\int d^2z\ \bar{\partial}X^\mu\partial X_\mu. \tag{2.2.54}$$

Now replace X^μ by the superfield $\mathbf{X}^\mu$ which is expanded as

$$\mathbf{X}^\mu = X^\mu + \theta\Psi^\mu + \bar{\theta}\bar{\Psi}^\mu + \bar{\theta}\theta B^\mu. \tag{2.2.55}$$

On the boundary this is $X^\mu| + \theta(\Psi + \bar{\Psi})^\mu|$. We impose Dirichlet boundary conditions at $y = 0$ that relate $\mathbf{X}$ to the worldline variables as

$$X| = w, \qquad (\Psi + \bar{\Psi})| = h^{1/4}\psi \tag{2.2.56}$$

where the factor of $h^{1/4}$ is required as ψ is a worldline scalar. Ψ and $\bar{\Psi}$ form a spinor on the worldsheet and a vector in spacetime. The worldsheet scalar B is an auxiliary field that plays no role in the action. These boundary fields play a role if we consider the abelian super-Wilson loop in terms of the boundary superfield, which is

$$W_s[C] = \mathrm{Tr}\Big[\exp\Big(-\oint dt d\theta\ D\mathbf{X}\cdot A(\mathbf{X})\Big)\Big] \tag{2.2.57}$$

where $\mathbf{X} = w + i\theta(h)^{1/4}\psi$. The first term appears as a superfield generalisation of the bosonic Wilson loop. There exists a non-abelian generalisation of this result, but it requires an understanding of path-ordering in superspace. We will look at this in chapter 4 when considering the so-called loop equations.

In the case of the string, we replace partial derivatives with the superderivatives and integrate over the θ coordinates. Then, from the above discussion to make the resulting action supersymmetric we multiply the integrand by $(1-\bar{\theta}\theta\delta(y))$. This leads us to the action for the spinning string

$$S_{spin} = \frac{1}{4\pi\alpha'}\int d^2z d^2\theta\ \bar{D}\mathbf{X}\cdot D\mathbf{X}(1-\bar{\theta}\theta\delta(y)) \tag{2.2.58}$$

which can be expanded into its more familiar form [44]

$$S_{spin} = \frac{1}{4\pi\alpha'}\bigg(\int d^2z d^2\theta\ \bar{D}\mathbf{X}\cdot D\mathbf{X} - \int_{y=0} dx\ \bar{\Psi}\cdot\Psi\bigg). \tag{2.2.59}$$

Inserting the expansion of the superfield and integrating over the θ coordinates reduces this to the more familiar form of the spinning string action which is the Polyakov action plus the action for a worldsheet spinor.

The same thing can be done with the infinitesimal area element which now takes the form

$$d\tilde{\Sigma}^{\mu\nu} = \int d^2z d^2\theta\ \bar{D}\mathbf{X}^{[\mu}D\mathbf{X}^{\nu]}(1-\bar{\theta}\theta\delta(y)) = \int d^2z d^2\theta\ \big(\bar{D}\mathbf{X}^{[\mu}D\mathbf{X}^{\nu]} - \theta\bar{\theta}\delta(y)\bar{\Psi}^{[\mu}\Psi^{\nu]}\big). \tag{2.2.60}$$

Using this we can form the supersymmetric generalisation of the contact interaction as

$$\tilde{S}_I = q^2\int_{\tilde{\Sigma}\tilde{\Sigma}'} d\tilde{\Sigma}^{\mu\nu}(z,\theta)\ \delta(\mathbf{X}(z,\theta)-\mathbf{X}(z',\theta'))\ d\tilde{\Sigma}_{\mu\nu}(z',\theta')|_{z\neq z',\theta\neq\theta'}. \tag{2.2.61}$$

Fourier decomposing as before allows one to write the contact interaction as the product of two vertex operators so that

$$\tilde{S}_I = q^2\int \frac{d^4k}{(2\pi)^4}\ \tilde{V}_k^{\mu\nu}\tilde{V}'_{\mu\nu\ -k} \tag{2.2.62}$$

where the supersymmetric vertex operators are now

$$\tilde{V}_k^{\mu\nu} = \int d^2z d^2\theta\ \bar{D}\mathbf{X}^{[\mu} D\mathbf{X}^{\nu]}\big(1 - \theta\bar{\theta}\delta(y)\big)e^{ik\cdot\mathbf{X}}. \tag{2.2.63}$$

The method of projecting along k works for the supersymmetric case too, and we will show that the vertex operator can be split into a piece on the boundary plus a "projected" vertex. Projecting $\mathbf{X}$ along k so that $\mathbf{X}^\mu = \mathbb{P}(\mathbf{X})^\mu + \frac{k^\mu k\cdot\mathbf{X}}{k^2}$ allows the vertex operator to be written as

$$\tilde{V}_k^{\mu\nu} = \int d^2z d^2\theta\ \bar{D}\bigg(\mathbb{P}_k(\mathbf{X}) + k\frac{k\cdot\mathbf{X}}{k^2}\bigg)^{[\mu} D\bigg(\mathbb{P}_k(\mathbf{X}) + k\frac{k\cdot\mathbf{X}}{k^2}\bigg)^{\nu]} e^{ik\cdot\mathbf{X}}(1 - \theta\bar{\theta}\delta(y))$$

$$= \tilde{\mathbb{V}}_k^{\mu\nu} + \frac{k^{[\mu}}{k^2}\int d^2z d^2\theta\bigg(k\cdot\bar{D}\mathbb{P}_k(\mathbf{X})D\mathbb{P}_k(\mathbf{X})^{\nu]} + k\cdot D\mathbb{P}_k(\mathbf{X})\bar{D}\mathbb{P}_k(\mathbf{X})^{\nu]}\bigg)e^{ik\cdot\mathbf{X}}(1-\bar{\theta}\theta\delta(y)) \tag{2.2.64}$$

where again we have defined a "projected" vertex operator. Rewriting the second term as the sum of total derivatives requires subtracting off the contributions from the derivatives of the boundary piece. The result is

$$\tilde{V}_k^{\mu\nu} = \tilde{\mathbb{V}}_k^{\mu\nu} - \frac{ik^{[\mu}}{k^2}\int d^2z d^2\theta\ \bar{D}\bigg(D\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}(1 - \bar{\theta}\theta\delta(y))\bigg)$$

$$-\frac{ik^{[\mu}}{k^2}\int d^2z d^2\theta\ D\bigg(\bar{D}\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}(1 - \bar{\theta}\theta\delta(y))\bigg)$$

$$-\frac{ik^{[\mu}}{k^2}\int dx d^2\theta\ \theta\bigg(D\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\bigg) + \frac{ik^{[\mu}}{k^2}\int dx d^2\theta\ \bar{\theta}\bigg(\bar{D}\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\bigg). \tag{2.2.65}$$

Now, note that applying Stokes' theorem in reverse allows us to write the last two terms as

$$\frac{ik^{[\mu}}{k^2}\int d^2z d^2\theta\ \bigg[D\bigg(D\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\bigg) + \bar{D}\bigg(\bar{D}\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\bigg)\bigg]. \tag{2.2.66}$$

To complete the derivation we can show that the first two integrals simplify. Using Stokes' theorem on the total derivatives produces a single θ and $\bar{\theta}$ cancelling the

$\delta(y)$ terms and so we drop them. This leaves us with the conclusion that

$$\tilde{V}_k^{\mu\nu} = \tilde{\mathbb{V}}_k^{\mu\nu} - \frac{ik^{[\mu}}{k^2} \int d^2z d^2\theta \; (D + \bar{D}) \Big((D + \bar{D}) \mathbb{P}_k(\mathbf{X})^{\nu]} e^{ik\cdot \mathbf{X}} \Big). \tag{2.2.67}$$

We can now apply Stokes' theorem to the total derivatives and expand the integrand in terms of the component fields. The result of this is

$$\tilde{V}_k^{\mu\nu} = \tilde{\mathbb{V}}_k^{\mu\nu} + \int_{y=0} \frac{i}{k^2} k^{[\mu} \mathbb{P}_k(dX)^{\nu]} e^{ik\cdot X} + \frac{k^{[\mu}}{k^2} \int_{y=0} dx \; \mathbb{P}_k(\Psi + \bar{\Psi})^{\nu]} (\Psi + \bar{\Psi}) \cdot k \; e^{ik\cdot X} \tag{2.2.68}$$

which is the supersymmetric generalisation of (2.1.13). Using the boundary values (2.2.56) allows us to write the vertex operator as

$$\tilde{V}_k^{\mu\nu} = \tilde{\mathbb{V}}_k^{\mu\nu} + \frac{i}{k^2} k^{[\mu} \int_B dx \left(\frac{\mathbb{P}_k(dw)^{\nu]}}{dx} - \sqrt{h} \; \mathbb{P}_k(\psi)^{\nu]} \; \psi \cdot ik \right) e^{ik\cdot w}. \tag{2.2.69}$$

The interaction term in the action (2.2.62) then becomes

$$\begin{aligned} \tilde{S}_I = \int \frac{d^4k}{(2\pi)^4} \; \tilde{\mathbb{V}}_k^{\mu\nu} \tilde{\mathbb{V}}_{\mu\nu-k} \\ + \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \int_B \int_B dx dx' \left(\frac{\mathbb{P}_k(dw)}{dx} - \sqrt{h} \; \mathbb{P}_k(\psi) \; \psi \cdot ik \right)^{\mu} \\ \times \left(\frac{\mathbb{P}_k(dw')}{dx'} + \sqrt{h'} \; \mathbb{P}_k(\psi') \; \psi' \cdot ik \right)_{\mu} e^{ik\cdot(w-w')}. \end{aligned} \tag{2.2.70}$$

Now, consider the expectation of the contact interaction, computed by functionally integrating over all worldsheets bounded by B, which is held fixed. This has no effect on the second term as before. The first term is suppressed in the same way as in the bosonic model. The only possible remaining contractions are the self-contractions of the exponential which are

$$e^{\pm ik\cdot \mathbf{X}} =: e^{\pm ik\cdot \mathbf{X}} : e^{-\pi\alpha' k^2 G_F(z,z)}. \tag{2.2.71}$$

$G_F(z, \theta; z', \theta')$ is the Green's function of the super-Laplacian, $-4\bar{D}D$, subject to the boundary conditions $G_F = 0$ when $z_i = \bar{z}_i$ and $\theta_i = \bar{\theta}_i$. The required solution is

$$G_F = -\frac{1}{4\pi}\log(z_{12}\bar{z}_{12}) + \frac{1}{4\pi}\log(z_{12}^R \bar{z}_{12}^R) \tag{2.2.72}$$

where

$$z_{12} = z_1 - z_2 - \theta_1\theta_2, \quad \bar{z}_{12} = \bar{z}_1 - \bar{z}_2 - \bar{\theta}_1\bar{\theta}_2, \quad z_{12}^R = z_1 - \bar{z}_2 - \theta_1\bar{\theta}_2, \quad \bar{z}_{12}^R = \bar{z}_1 - z_2 - \bar{\theta}_1\theta_2. \tag{2.2.73}$$

At coincident points on the worldsheet, G_F diverges and so must be regulated. Introducing a short distance cut-off as in the bosonic case allows us to rewrite the G_F as

$$G_F^\epsilon = -f\left(\sqrt{\frac{z_{12}\bar{z}_{12}}{\epsilon}}\right) + f\left(\sqrt{\frac{z_{12}^R \bar{z}_{12}^R}{\epsilon}}\right) \tag{2.2.74}$$

where

$$f(s) = \int_1^\infty \frac{d\tau}{4\pi\tau}\left(1 - \exp\left(-\frac{s^2}{\tau}\right)\right). \tag{2.2.75}$$

Expanding the exponential term in (2.2.71) in powers of θ gives

$$e^{-\pi\alpha' k^2 G_F^\epsilon(0)} = \left(1 + \frac{i}{2}\theta\bar{\theta}\frac{\partial}{\partial y}\right) e^{-\pi\alpha' k^2 f(\frac{2y}{\sqrt{\epsilon}})}. \tag{2.2.76}$$

When s is large $f(s) \sim (\log s)/2\pi$ as in the bosonic case so, for $k^2 > 0$ and taking the tensionless limit, this exponential suppresses (2.2.71) at all points in the interior of Σ when the cut-off is removed. The "projected" vertex, therefore, doesn't contribute to the expectation of the contact interaction.

We are left with the conclusion that

$$\left\langle \tilde{S}_I \right\rangle = \frac{q^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_B\int_B dx dx' \left(\frac{\mathbb{P}_k(dw)}{dx} - \sqrt{h}\,\mathbb{P}_k(\psi)\,\psi\cdot ik\right)^\mu \times \left(\frac{\mathbb{P}_k(dw')}{dx'} + \sqrt{h'}\,\mathbb{P}_k(\psi')\,\psi'\cdot ik\right)_\mu e^{ik\cdot(w-w')} \tag{2.2.77}$$

which is equal the expectation of the super-Wilson loop to order q^2 (1.4.136). At higher orders the unwanted divergences that arise in the bosonic theory do not appear [45], and so we can make the conclusion that this result can be exponentiated and we can make the equivalence between the abelian string theory and the expectation of the super-Wilson loop. The way in which we will generalise this model to incorporate non-abelian gauge theories will not affect this result and so at least at this level we will be able to state the equivalence between the non-abelian string theory and the non-abelian super-Wilson loop. The downside of course is that the bosonic theory will still contain the unwanted divergences and will therefore not be equivalent to the expectation of the non-abelian Wilson loop.

2.3 Non-abelian gauge theory

The main aim of this thesis is to formulate a non-abelian generalisation of the above theory, i.e. find a string theory with contact interaction that can reproduce the expectation value of the super-Wilson loop computed in Yang Mills theory. In this section we detail how one can reproduce the results from perturbatively expanding the super-Wilson loop by introducing additional fields onto the worldsheet, at this point leaving out the details of the fields which will be filled in in the proceeding chapters. The one detail we do specify is the propagator the fields must have.
To motivate how we may wish to generalise the string theory to include non-abelian gauge groups we look at the simpler case of generalising the bosonic non-abelian Wilson loop. In Euclidean spacetime it is

$$W[C] = \mathrm{Tr}\Big[\,\mathcal{P}\,\exp\Big(-q\oint_B\,A_\mu dw^\mu\Big)\Big] \tag{2.3.78}$$

where the gauge field, A, can be expanded in terms of the anti-Hermitian Lie algebra generators, τ^A, as $A = \tau^A A^A$. Taylor expanding the exponential gives a power series weighted by q^n:

$$W[C] = \mathrm{Tr}\Big[\,\mathcal{P}\,\sum_{n=0}^{\infty}\frac{(-q)^n}{n!}\Big(\oint_B\,A_\mu dw^\mu\Big)^n\Big]. \tag{2.3.79}$$

To obtain the expectation of the Wilson loop to $O(q^n)$ in field theory requires us to calculate the expectation of $\langle A^n\rangle$. We have already calculated the expectation of the Wilson loop neglecting self interactions (1.4.127). This differs from the abelian result by the path-ordering of the Lie algebra generators. This path-ordering can be replaced by a functional integral over an anti-commuting field, ψ, on the boundary, B, [35] as

$$\int \mathcal{D}(\psi^\dagger, \psi)\, \psi^\dagger(1)\, \psi(0)\, \exp\Bigg(-\int_0^1 \psi^\dagger \dot{\psi}\, dt +$$

$$\frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} \oint_B \oint_B \left(\psi^\dagger \tau^A \psi\, \mathbb{P}_k(dw)^\mu\right)|_\xi \left(\frac{e^{ik\cdot(w-w')}}{k^2}\right) \left(\psi^\dagger \tau^A \psi\, \mathbb{P}_k(dw)_\mu\right)|_{\xi'} \Bigg). \quad (2.3.80)$$

The step functions following from the free worldline fermion action mean that the integrand is only non-zero when the generators are in the correct order as we traverse the boundary between 0 and 2π, which is the definition of path-ordering (1.4.124). Apart from the kinetic term for ψ, this differs from the abelian case by the inclusion of the Lie algebra terms $J^A \equiv \psi^\dagger \tau^A \psi$. This suggests a natural extension of the string model where we let the boundary field, ψ, extend into the interior of the worldsheet. This will be the first method we study. In the second method, we obtain the same relation from a 2 dimensional gauge theory where J^A will be restricted to the boundary.

(2.3.80) suggests a generalisation to the bosonic contact interaction of the form

$$S_I^{YM} = q^2 \int J^A(\xi) d\Sigma_{\mu\nu} \delta^4(X(\xi) - X(\xi')) J^A(\xi') d\Sigma'^{\mu\nu}. \quad (2.3.81)$$

This essentially modifies the vertex operator to

$$V_k^{\mu\nu A} = \frac{1}{2} \int d^2\xi\; J^A \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu e^{ik\cdot X}. \quad (2.3.82)$$

We can again use the projection of X along the direction of k and to rewrite the vertex operator as

$$V_k^{\mu\nu A} = \mathbb{V}_k^{\mu\nu A} + \int d^2\xi\; \partial_a \left(\frac{i\varepsilon^{ab}}{k^2} k^{[\mu} \partial_b \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X} \right) J^A. \quad (2.3.83)$$

For simplicity we will define

$$L_k^{a\mu\nu} \equiv \frac{i\varepsilon^{ab}}{k^2} k^{[\mu} \partial_b \mathbb{P}_k(X)^{\nu]} e^{ik\cdot X}. \tag{2.3.84}$$

The contact interaction is then

$$S_I^{YM} = \int \frac{d^4k}{(2\pi)^4} \left(\mathbb{V}_k^{\mu\nu A} \mathbb{V}_{\mu\nu - k}^{A} + d^2\xi d^2\xi' \; \partial_a L_k^{a\mu\nu} J^A \; \partial'_c L'^{c}_{\mu\nu - k} J'^{A} \right). \tag{2.3.85}$$

We will look at each product of the second term separately by defining

$$I_2 \equiv \int d^2\xi \; \partial_a L_k^{a\mu\nu} J^A. \tag{2.3.86}$$

Integrating by parts we find

$$I_2 = \int d^2\xi \; \left[\partial_a (L_k^{a\mu\nu} J^A) - L_k^{a\mu\nu} \partial_a (J^A) \right]. \tag{2.3.87}$$

Now, the first term is a total derivative and similar to what we had in the abelian case but the second term here is new. The first integral is computed using Stokes' theorem as before to give

$$\int_\Sigma d^2\xi \; \partial_a (L_k^{a\mu\nu} J^A) = \frac{i}{k^2} \int_B k^{[\mu} \mathbb{P}_k (dw)^{\nu]} e^{ik\cdot w} J^A \equiv B_k^{\mu\nu A}. \tag{2.3.88}$$

We have denoted this term $B_k^{\mu\nu A}$ as it is an integral around the boundary that will lead to propagators in the functional integral as in the abelian case. We will define the new term in (2.3.87) as $C_k^{\mu\nu A} \equiv \int d^2\xi L_k^{a\mu\nu} \partial_a J^A$ as this is the term that will lead to the self interactions in the Wilson loop via contractions of the derivatives of J^A. The expectation value of the contact interaction is then

$$\left\langle S_I^{YM} \right\rangle_{\Sigma,J} = q^2 \left\langle \int \frac{d^4k}{(2\pi)^4} \; \mathbb{V}_k^{\mu\nu A} \mathbb{V}_{\mu\nu - k}^{A} \right\rangle_{\Sigma,J}$$

$$+ q^2 \left\langle \int \frac{d^4k}{(2\pi)^4} (B - C)_k^{\mu\nu A} (B' - C')_{\mu\nu \; -k}^{A} \right\rangle_{\Sigma,J} \tag{2.3.89}$$

from which we obtain three new terms. At this order the $B \cdot C$ cross terms and $C \cdot C$ terms do not contribute to the expectation of the contact interaction as there is always at least one factor of $e^{\pm ik\cdot X}$ in the interior of the worldsheet. As discussed in the abelian model, these terms will be suppressed in the tensionless limit and so do not contribute to the expectation of the contact interaction. The same argument can be applied to the first term of (2.3.89). We will thus require the expectations $\left\langle J^A \partial_b J^B \right\rangle|_J$ and $\left\langle \partial_a J^A \partial_b J^A \right\rangle|_J$ to not produce anything that will lead to additional terms that will not be suppressed in the string functional integral. The second condition will follow from the three gluon vertex condition we will find later. The first condition must be satisfied within the particular model describing J^A.

The only term contributing to the expectation of the contact interaction is then

$$\left\langle \int \frac{d^4k}{(2\pi)^4} B_k^{\mu\nu A} B'^{A}_{\mu\nu\ -k} \right\rangle_{B,J^A}$$

$$= \left\langle \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \oint_B \oint_B \frac{\mathbb{P}_k(dw) \cdot \mathbb{P}_k(dw')}{k^2} e^{ik\cdot(w-w')} (J^A J'^A) \right\rangle_{B,J} \tag{2.3.90}$$

$$= \frac{1}{2} \left\langle \int \frac{d^4k}{(2\pi)^4} \oint_B \oint_B \frac{\mathbb{P}_k(dw) \cdot \mathbb{P}_k(dw')}{k^2} e^{ik\cdot(w-w')} (J^A J'^A) \right\rangle_J \tag{2.3.91}$$

where we have carried out the functional integration over Σ holding B constant. If $J^A = \psi^\dagger \tau^A \psi$ then we have reproduced the first order expansion term of (2.3.80). In any case, for this to be equivalent to the expectation of the non-abelian Wilson loop we require a contraction of the J^As that produces path-ordering of Lie algebra generators, i.e. a contraction such that we can make the replacement

$$J^A|_B J'^A|_B \sim \mathcal{P}(\tau^A \tau^A). \tag{2.3.92}$$

It will be the subject of the following chapters to explain how this comes about. By considering this property of J^A to arise by averaging the number of intersections of curves we will essentially find a way to continue path-ordering into the interior of the worldsheet. This reproduces the propagator of the worldline theory (2.3.80) and so can be considered a worldsheet generalisation of this theory. The other model we consider is similar in the sense that we will use a path integral over ψ and $\psi^\dagger$ to obtain (2.3.92). In this model, the new variables are restricted to the boundary and arise as the source for a new gauge field on the worldsheet.

This path-ordering result naturally generalises to all orders and reproduces (1.4.130). The expectation of the contact interaction to the n'th power contains n factors of $B\cdot B$ which leads to n propagators joining pairs of points on the boundary as

$$\frac{\left\langle(-S_I^{YM})^n\right\rangle_{\Sigma,J}}{n!}=$$

$$\mathcal{P}\,\frac{(q^2)^n}{2^n n!}\int\prod_{i=1}^{n}\left\langle\frac{d^4k_i}{(2\pi)^4}\oint_B\oint_B\frac{\mathbb{P}_k(dw_i)\cdot\mathbb{P}_k(dw_i')}{k_i^2}e^{ik_i\cdot(w_i-w_i')}\big(J^{A_i}J'^{A_i}\big)\right\rangle_J. \quad (2.3.93)$$

The condition (2.3.92) then generalises for $2n$ insertions on the boundary to

$$\left\langle\prod_{i=1}^{n}J^{A_i}(\xi_i)|_B J^{A_i}(\xi_i')|_B\right\rangle=\mathcal{P}\big(\prod_i^n\tau^{A_i}\tau^{A_i}\big). \quad (2.3.94)$$

If this is satisfied then $\left\langle e^{-S_I^{YM}}\right\rangle_{\Sigma,J}$ contains (1.4.130), but, we know from the abelian case that worldsheet supersymmetry is required to eliminate extra divergences when there are products of interactions. The contractions of the J and X are independent from each other and so the abelian result still stands. We will look at the supersymmetric case after looking at how the three gluon vertex arises in the bosonic model.

2.3.1 The three gluon vertex

So, we must find a field theory that can implement path-ordering along the boundary of the worldsheet. This is the first step to generalising the string model to incorporate non-abelian gauge symmetries. The next step is to include the self interactions of the gauge fields. The three gluon vertex comes from contractions of the $C_k^{\mu\nu A}$ terms in the functional integral. This interaction is of $O(q)$ and first appears in the $O(q^3)$ expansion of the expectation of the Wilson loop and so the diagram is itself of order $O(q^4)$. We, therefore, expect this interaction to appear in the expectation of $(S_I^{YM})^2$. Omitting the projected vertex terms this is

$$\left\langle (S_I^{YM})^2 \right\rangle_{\Sigma,J} \ni q^4 \left\langle \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}(B_1 - C_1)^{\mu\nu A}_{k}(B_2 - C_2)^{A}_{\mu\nu\ -k} \right.$$

$$\left. \times\ (B_3 - C_3)^{\lambda\sigma B}_{k'}(B_4 - C_4)^{B}_{\lambda\sigma\ -k'} \right\rangle_{\Sigma,J}. \tag{2.3.95}$$

From the previous section, we can immediately identify the term B^4 that leads to two gauge bosons joining two pairs of vertices on the boundary of the worldsheet. The three point vertex comes from the contraction of the derivatives of the J^As in the Cs in each term of the form

$$\left\langle (S_I^{YM})^2 \right\rangle_{\Sigma,J} \ni q^4 \left\langle \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\ B^{\mu\nu A}_{1k}C^{A}_{2\mu\nu\ -k}C^{\lambda\sigma B}_{3k'}B^{B}_{4\lambda\sigma\ -k'} \right\rangle_{\Sigma,J}. \tag{2.3.96}$$

To produce exactly the three point vertex we require

$$\partial_a \underbracket{J^A \partial_b J^B} \sim \varepsilon_{ab} f^{ABC} J^C \delta^2(\xi_1 - \xi_2). \tag{2.3.97}$$

where $\sim$ denotes that this is satisfied within the functional integral. There could (and will) be extra terms on the right hand side, but these must be suppressed upon integration. With this ansatz (2.3.96) becomes

$$q^4 \left\langle \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} f^{ABC} B^{\mu\nu A}_{1k} \right.$$

$$\left. \times \int d^2\xi\ \frac{\varepsilon^{ab}}{k^2k'^2} k_{[\mu}\partial_a \mathbb{P}_k(X_2)_{\nu]} k'^{[\lambda}\partial_b \mathbb{P}_{k'}(X_2)^{\sigma]} J^C(X_2) e^{i(k'-k)\cdot X_2} B^{B}_{4\lambda\sigma\ -k'} \right\rangle_{\Sigma,J}. \tag{2.3.98}$$

Expanding the boundary integrals and carrying out the (tensor) contractions we find

$$\frac{q^4}{2} \left\langle \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4} f^{ABC} \oint_B \oint_B \frac{\mathbb{P}_k(dw_1)^\mu}{k^2} J^A(w_1) e^{ik\cdot w_1} \right.$$

$$\left. \int d^2\xi\ \varepsilon^{ab}\partial_a X_{2\mu}\partial_b X_{2\nu} J^C(X_2) e^{i(k'-k)\cdot X_2}\ \frac{\mathbb{P}_{k'}(dw_4)^\nu}{k'^2} J^B(w_4) e^{-ik'\cdot w_4} \right\rangle_{\Sigma,J}. \tag{2.3.99}$$

We recognise here the emergence of a new vertex operator insertion. We can then project X_2 along $(k'-k)$ to produce the third leg connecting the interacting vertex to the boundary. Finally, we find

$$\left\langle e^{-S_I^{YM}} \right\rangle_{\Sigma,J} \ni \frac{q^4}{2} \left\langle \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} f^{ABC} \oint_B \oint_B \oint_B \frac{\mathbb{P}_k(dw_1)^\mu}{k^2} J^A(w_1) e^{ik\cdot w_1} \right.$$

$$\left. \frac{i(k'-k)_{[\mu}\mathbb{P}_{(k'-k)}(dw_2)_{\nu]}}{(k'-k)^2} J^C(w_2) e^{i(k'-k)\cdot w_2} \; \frac{\mathbb{P}_{k'}(dw_4)^\nu}{k'^2} J^B(w_4) e^{-ik'\cdot w_4} \right\rangle_J . \qquad (2.3.100)$$

Expanding the inner products and relabelling the dummy indices on the momenta and positions so that the momentum contracted with the propagator is the momentum associated with the propagator, we reproduce (1.4.131). Note, the antisymmetry on μ and ν introduces a factor of a half and two terms. One can then replace the factors of J with the path-ordered product of the Lie algebra generators to verify the equivalence of these results.

2.3.2 Worldsheet field theory

Thus, we are searching for a field theory on the worldsheet that allows one to replace products of the field on the boundary with the path-ordered product of the Lie algebra generators placed at these points. The first model we consider is a two dimensional generalisation for the boundary field $\psi(w)$. The kinetic action for the boundary theory is

$$S_\psi = \int_0^1 \psi^\dagger \dot{\psi} \, dt \qquad (2.3.101)$$

which leads to the propagator

$$\left\langle \psi^\dagger_\alpha(\xi_1)\psi_\beta(\xi_2) \right\rangle = \delta_{\alpha\beta} \, \mathrm{sign}(t_1 - t_2). \qquad (2.3.102)$$

This naturally generalises to the propagator for the 2 dimensional worldsheet field theory

$$\left\langle \psi^\dagger_\alpha(z_1)\psi_\beta(z_2) \right\rangle = \delta_{\alpha\beta} \, \mathrm{sign}(z_1 - z_2) \qquad (2.3.103)$$

where a Euclidean worldsheet is understood and we are using complex coordinates. On the boundary, this propagator reduces to the boundary theory and so can be used to represent the path-ordering of Lie algebra generators. In this form, $\mathrm{sign}(z_1 - z_2)$, has the form of an angle between z_1 and z_2 in the complex plane since $\mathrm{sign}(z_1 - z_2) =$

$e^{i\arg(z_1-z_2)}$. This relation will lead to the field theory we propose that generalises ψ into the interior of the worldsheet.

This field theory will be studied in the next section. We note that in this theory we require the condition

$$\partial_a\psi^\dagger_\alpha(z_1)\partial_b\psi_\beta(z_2) \sim \delta_{\alpha\beta}\ \frac{1}{2}\varepsilon_{ab}\delta^2_c(z_1-z_2) \tag{2.3.104}$$

to produce the three gluon vertex. We will, in fact, find that we cannot satisfy this relation in the worldsheet ψ theory and so we will be unable to make the equivalence between the string theory supplemented with this worldsheet theory and the expectation of the Wilson loop.

The next theory we will investigate is similar in the sense that it is introduced onto the worldsheet via the vertex operator in the same way as the ψ theory. The guiding principle behind this theory is a gauge symmetry of the contact interaction with the Lie algebra valued fields, J^A. We can use this to introduce a gauge field theory onto the worldsheet and we will show that this produces the correct contractions required to implement the above properties of the string theory.

2.3.3 The four gluon vertex and ghost-ghost-gluon vertices

The string theory described above, with contact interaction and additional worldsheet field(s), reproduces the path-ordering of the Lie algebra generators and the three gluon vertex in the expectation of the Wilson loop. We have yet to mention how the four gluon vertex, Fig. 1.2b, and ghost-ghost-gluon vertex, Fig. 1.3, are generated. In fact, this will be beyond the scope of this thesis. This has been studied in the context of this string theory independently from me by Prof. Mansfield and will be detailed in an upcoming paper. Although these interactions do not seem to appear in the work above, one can show that this is due to using a heat kernel regulator for the worldsheet Laplacian Green's function as we will use throughout, and that by switching to dimensional regularisation, one can show that these missing interactions do appear.

The terms that give rise to these extra interactions are suppressed in our work and so we do not see their effect in the work presented here. The same goes for the supersymmetric model discussed in the next section, although, it is not clear how dimensional regularisation of the super-Laplacian Green's function will work at this point.

2.4 Supersymmetric model

The above extension to non-abelian gauge groups generalises straight forwardly to incorporate the supersymmetry of the worldsheet. J^A gets promoted to a superfield, $\mathbf{J}^A$, with expansion $J_0^A + \theta J_1^A + \bar{\theta}\bar{J}_1^A + \theta\bar{\theta}J_2^A$. If we were to include a factor of $\mathbf{J}^A$ into the integrand of (2.2.63) we would not get the expected results, specifically the the third leg of the three point vertex, produced by the contraction of the Cs, gains an extra term that prevents this new boundary piece from being supersymmetric. Rather, we should include the factor in (2.2.67). The non-abelian generalisation of the supersymmetric vertex operator is then

$$\tilde{V}_k^{\mu\nu A} = \tilde{\mathbb{V}}_k^{\mu\nu A} - \frac{ik^{[\mu}}{k^2}\int d^2zd^2\theta\ \mathbf{J}^A(D+\bar{D})\Big((D+\bar{D})\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\Big) \tag{2.4.105}$$

where

$$\tilde{\mathbb{V}}_k^{\mu\nu A} = \int d^2zd^2\theta\ \mathbf{J}^A\bar{D}\mathbb{P}_k(\mathbf{X})^{[\mu}D\mathbb{P}_k(\mathbf{X})^{\nu]}\big(1-\theta\bar{\theta}\delta(y)\big)e^{ik\cdot\mathbf{X}} \tag{2.4.106}$$

is the supersymmetric "projected" vertex operator. Integrating the vertex by parts again gives two terms which we associate with the supersymmetric analogues of B and C from the bosonic theory

$$\tilde{V}_k^{\mu\nu A} = \tilde{\mathbb{V}}_k^{\mu\nu A} - \frac{ik^{[\mu}}{k^2}\int d^2zd^2\theta\ (D+\bar{D})\Big(\mathbf{J}^A(D+\bar{D})\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\Big)$$

$$+\frac{ik^{[\mu}}{k^2}\int d^2zd^2\theta\ (D+\bar{D})\mathbf{J}^A(D+\bar{D})\mathbb{P}_k(\mathbf{X})^{\nu]}e^{ik\cdot\mathbf{X}}\Big) \equiv \tilde{\mathbb{V}}_k^{\mu\nu A} - (\tilde{B}-\tilde{C})_k^{\mu\nu A}. \tag{2.4.107}$$

The contact interaction is then

$$\tilde{S}_I^{YM} = q^2 \int \frac{d^4k}{(2\pi)^4} \left(\tilde{\mathbb{V}}_k^{\mu\nu A} \tilde{\mathbb{V}}'^{A}_{\mu\nu - k} + \tilde{B}_k^{\mu\nu A} \tilde{B}'^{A}_{\mu\nu - k} \right) + \text{cross terms.} \tag{2.4.108}$$

Integrating over worldsheets spanning the fixed boundary results in

$$\left\langle \tilde{S}_I^{YM} \right\rangle_{\Sigma, J} = q^2 \int \frac{d^4k}{(2\pi)^4} \left\langle \tilde{B}_k^{\mu\nu A} \tilde{B}'^{A}_{\mu\nu - k} \right\rangle_{\mathbf{J}}. \tag{2.4.109}$$

where again the $B \cdot C$ cross terms and $C \cdot C$ term do not contribute as in the bosonic case. The "projected" vertex term is also suppressed in the functional integral because of the self-interactions of the exponentials. For this to reproduce the expected result note that we must have $J_1^A| = \bar{J}_1^A| = 0$. This result will arise naturally in the second model we study.

The integral $\tilde{B}$ can then be evaluated using Stokes' theorem

$$\tilde{B}_k^{\mu\nu A} = \frac{ik^{[\mu}}{k^2} \oint \int dx d^2\theta \; (\theta - \bar{\theta}) \Big(\mathbf{J}^A (D + \bar{D}) \mathbb{P}_k(\mathbf{X})^{\nu]} e^{ik\cdot \mathbf{X}} \Big)$$

$$= -\frac{ik^{[\mu}}{k^2} \oint dx \; J_0^A \left(\frac{d\mathbb{P}_k(w)^{\nu]}}{dx} - \sqrt{h} \; \mathbb{P}_k(\psi^{\nu]}) \; \psi \cdot ik \right) e^{ik\cdot w} \equiv -\frac{ik^{[\mu}}{k^2} \oint db_k^{\nu]A} \tag{2.4.110}$$

where we have defined a boundary element

$$db_k^{\nu A} = dx \; J_0^A \left(\frac{d\mathbb{P}_k(w)^{\nu}}{dx} - \sqrt{h} \; \mathbb{P}_k(\psi^{\nu}) \; \psi \cdot ik \right) e^{ik\cdot w}. \tag{2.4.111}$$

Defining this will reduce clutter when considering higher orders in the contact interaction as db_k will remain unaltered by the functional integrals as it lies on the boundary which remains fixed and contains no derivatives of J^A. The expectation of the contact interaction is then

$$\left\langle \tilde{S}_I^{YM} \right\rangle_{\Sigma, \mathbf{J}} = \frac{q^2}{2} \int \frac{d^4k}{(2\pi)^4} dx dx' \; \Bigg\langle J_0^A \left(\frac{\mathbb{P}_k(dw)}{dx} - \sqrt{h} \; \mathbb{P}_k(\psi) \; \psi \cdot ik \right)^{\mu}$$

$$\times \; J_0'^A \left(\frac{\mathbb{P}_k(dw')}{dx'} + \sqrt{h'} \; \mathbb{P}_k(\psi') \; \psi' \cdot ik \right)_{\mu} e^{ik\cdot(w - w')} \Bigg\rangle_{\mathbf{J}}. \tag{2.4.112}$$

The particular model for $\mathbf{J}^A$ will tell us how to proceed from here, however, we will essentially require the bosonic condition (2.3.92). This result can then be compared with the expectation of the non-abelian super-Wilson loop. Being able to replace J^A on the boundary with the Lie algebra generator τ^A will allow this result to coincide with the first order expectation of the super-Wilson loop. Exponentiating this result will then reproduce the full expectation of the super-Wilson loop neglecting self interactions of the gauge field. In the supersymmetric model we no longer produce unwanted divergences that ruin this equivalence [45].

2.4.1 Three gluon vertex in the supersymmetric theory

The three gluon vertex comes again from the particular term in the contact interaction squared

$$\left\langle (\tilde{S}_I^{YM})^2 \right\rangle_{\Sigma,\mathbf{J}} \ni q^4 \left\langle \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\ \tilde{B}^{\mu\nu A}_{1k} \tilde{C}^{A}_{2\mu\nu\ -k} \tilde{C}^{\lambda\sigma B}_{3k'} \tilde{B}^{B}_{4\lambda\sigma\ -k'} \right\rangle_{\Sigma,\mathbf{J}} . \tag{2.4.113}$$

We now require the contractions

$$\bar{D}\mathbf{J}_1^A D\mathbf{J}_2^B \sim f^{ABC}\mathbf{J}^C \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) \tag{2.4.114}$$

$$\bar{D}\mathbf{J}_1^A \bar{D}\mathbf{J}_2^B \sim D\mathbf{J}_1^A D\mathbf{J}_2^B \sim 0 \tag{2.4.115}$$

for this term to reproduce one contribution to the three gluon vertex in the super-Wilson loop. Using these contractions, the expectation of the contact interaction squared contains

$$2q^4 \left\langle \int \frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}\ \tilde{B}^{\mu\nu A}_{1k} f^{ABC} \int d^2z_2 d^2\theta_2\ \frac{k_{[\mu}}{k^2}\frac{k'_{[\lambda}}{k'^2}\mathbf{J}^C (D+\bar{D})\mathbb{P}_k(\mathbf{X}_2)_{\nu]} \right.$$

$$\left. (D+\bar{D})\mathbb{P}_{k'}(\mathbf{X}_2)_{\sigma]} e^{-i(k-k')\cdot \mathbf{X}_2}\ \tilde{B}^{B}_{4\lambda\sigma\ -k'} \right\rangle_{\Sigma,\mathbf{J}} . \tag{2.4.116}$$

Carrying out the (tensor) contractions of the relevant terms we find

$$\left\langle (\tilde{S}_I^{YM})^2 \right\rangle_{\Sigma,\mathbf{J}} \ni q^4 \left\langle \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \oint \frac{db_k^{\mu A}}{k^2} f^{ABC} \int d^2z_2 d^2\theta_2 \; \mathbf{J}^C (D+\bar{D})\mathbf{X}_{2\mu} \right.$$

$$\left. (D+\bar{D})\mathbf{X}_{2\nu} e^{-i(k-k')\cdot\mathbf{X}_2} \oint \frac{db_{k'}^{\nu B}}{k_2^2} \right\rangle_{\Sigma,\mathbf{J}}. \tag{2.4.117}$$

The integral produced is not quite a new vertex, we must do a little work to get the form we require. Again, we use the projection of $\mathbf{X}_2$ along $(k_1 - k_2)$. Focussing just on the worldsheet integral we have

$$\int d^2z_2 d^2\theta_2 \; \mathbf{J}^C (D+\bar{D})\mathbf{X}_{2\mu} (D+\bar{D})\mathbf{X}_{2\nu} e^{-i(k-k')\cdot\mathbf{X}_2}$$

$$= \int d^2z_2 d^2\theta_2 \; \mathbf{J}^C (D+\bar{D})\mathbb{P}(\mathbf{X})_{2\mu} (D+\bar{D})\mathbb{P}(\mathbf{X})_{2\nu} e^{-i(k-k')\cdot\mathbf{X}_2}$$

$$+2i\frac{(k-k')_{[\mu}}{(k-k')^2} \int d^2z_2 d^2\theta_2 \; \mathbf{J}^C (D+\bar{D})\big((D+\bar{D})\mathbb{P}_{(k-k')}(\mathbf{X}_2) e^{-i(k-k')\cdot\mathbf{X}_2}\big). \tag{2.4.118}$$

In the functional integral the first term is suppressed again and so we will drop it. The second term now looks like a vertex operator. To obtain the boundary integral we carry out an integration by parts. This will produce another term with a factor of $D\mathbf{J}^C$, but this will be suppressed by the self-contractions of $e^{-i(k-k')\cdot\mathbf{X}_2}$. With this we can effectively drop the $D\mathbf{J}^C$ terms leaving us with just the boundary integral

$$2i\frac{(k-k')_{[\mu}}{(k-k')^2} \int d^2z_2 d^2\theta_2 \; (D+\bar{D})\big(\mathbf{J}^C (D+\bar{D})\mathbb{P}_{(k-k')}(\mathbf{X}_2) e^{-i(k-k')\cdot\mathbf{X}_2}\big) = 2\tilde{B}^C_{2\mu\nu-(k-k')}. \tag{2.4.119}$$

Inserting this into the expectation of $(S_I^{YM})^2$, we find

$$\left\langle (S_I^{YM})^2 \right\rangle_{\Sigma,\mathbf{J}} \ni$$

$$q^4 \left\langle \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} f^{ABC} \oint_B \oint_B \oint_B \frac{db_k^{\mu A}}{k^2} \frac{i(k-k')_{[\mu} db^C_{-(k-k')\nu]}}{(k-k')^2} \frac{db_{k'}^{\nu B}}{k'^2} \right\rangle_{\mathbf{J}}. \tag{2.4.120}$$

Again replacing $\mathbf{J}$ on the boundary by the path-ordered product of the Lie algebra generators reduces this to the first contribution to the three point function in the expectation of the super-Wilson loop.

We will see that there exists a natural analogue of the bosonic ψ theory. Again, this theory won't contain the required relation to produce the three gluon vertex as described above. To obtain a theory that does have this structure, we consider the supersymmetric analogue of the worldsheet gauge theory model. There are some subtle differences coming from the supersymmetry requirement but this theory will form a correct generalisation of the bosonic theory, up to path-ordering and the three gluon vertex. We look at this in chapter 5.

Chapter 3

The Intersection of Random Curves and Path-Ordering of the Wilson Loop

In this section we attempt to find a field theory that satisfies our requirements to allow a way of introducing path-ordering into the string model and possibly include the extra self interactions of non-abelian gauge theory. The method of this section will generalise the worldline theory of [35] where the path-ordering of the Wilson loop is produced by an additional worldline field on the loop. The field theory we obtain provides a way of continuing the path-ordering into the interior of the loop. We will find that there exists both a bosonic and fermionic field theory that one can use to generalise path-ordering into the interior of the worldsheet. This method doesn't contain the correct structure to form the three gluon vertex however, though we can show how one might be able to use this model to obtain it. In chapter 5 we will introduce another field theory that does include the necessary ingredients to allow both path-ordering and self interactions in the string model. We will begin this section with a discussion of one of the simplest 2 dimensional field theories, 2 dimensional electrostatics, a particular case of (1.1.45). We will then show how one can use this to produce an appropriate field theory for J in the string model.

3.1 2 dimensional electrostatics

Of particular interest in this thesis is the case when $D = 2$ in (1.1.45) since we are working on the 2 dimensional worldsheet. The existence of Weyl symmetric quantities in 2 dimensions will allow us to generalise the calculation of the average of the electric field to curved spaces. This will be of use when adding fields to the worldsheet of our string model as we will when considering the generalisation of the string model to include non-abelian gauge fields. The electrostatic field produced by two oppositely charged particles is, from (1.1.45)

$$\mathbf{E}(\mathbf{x}) = \frac{q}{2\epsilon_0 \pi}\left(\frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^2} - \frac{\mathbf{x}-\mathbf{b}}{|\mathbf{x}-\mathbf{b}|^2}\right). \tag{3.1.1}$$

As this is an important case we will study it in some detail and eventually generalise the discussion to curved space. It is useful in 2 dimensions to use complex coordinates in which $z = x + iy$ and $\bar{z} = x - iy$ and we denote the contravariant components of the electrostatic field as $E^z = E$ and $E^{\bar{z}} = \bar{E}$. In flat space the metric is $ds^2 = dzd\bar{z}$. It will be the geometric properties of the fields that will be important to us so we will drop the factor of q/ϵ_0 in the discussion. The appropriate field equations are then Gauss' law in 2 dimensions

$$\nabla \cdot \mathbf{E} = \partial E + \bar{\partial}\bar{E} = 2\left(\delta_c^2(z-a) - \delta_c^2(z-b)\right) \tag{3.1.2}$$

and Faraday's law

$$\partial E - \bar{\partial}\bar{E} = 0. \tag{3.1.3}$$

The factor of 2 in Gauss' law comes from the fact that the invariant delta function takes the form

$$\frac{\delta^2(x_1 - a)}{\sqrt{g(x_1)}}. \tag{3.1.4}$$

This is due to the fact that the invariant volume element is $d^2x_1\ \sqrt{g(x_1)}$ and so

$$\int d^2x_1\ \sqrt{g(x_1)}\ \frac{\delta^2(x_1 - a)}{\sqrt{g(x_1)}} = 1. \tag{3.1.5}$$

For flat space and complex coordinates we have $\sqrt{|g|} = \frac{1}{2}$. It is then easy to see that the solution to (3.1.2) and (3.1.3) is

$$E = \frac{1}{2\pi}\left(\frac{1}{(\bar{z}-\bar{a})} - \frac{1}{(\bar{z}-\bar{b})}\right), \quad \bar{E} = \frac{1}{2\pi}\left(\frac{1}{(z-a)} - \frac{1}{(z-b)}\right) \tag{3.1.6}$$

which agrees with (3.1.1), noting that we have

$$\partial\bar{\partial}\log(z\bar{z}) = 2\pi\ \delta^2(z). \tag{3.1.7}$$

The single line of force solutions are

$$E_c(z) = \int_C \delta_c^2(z-w)dw, \qquad \bar{E}_c(z) = \int_C \delta_c^2(z-w)d\bar{w} \tag{3.1.8}$$

with $w(0) = a$ and $w(1) = b$. The important results we need are then obtained from (1.1.45)

$$\langle E_c\rangle = \left\langle \int_C \delta_c^2(z-w)dw \right\rangle = \frac{1}{2\pi}\left(\frac{1}{(\bar{z}-\bar{a})} - \frac{1}{(\bar{z}-\bar{b})}\right), \tag{3.1.9}$$

$$\langle \bar{E}_c\rangle = \left\langle \int_C \delta_c^2(z-w)d\bar{w} \right\rangle = \frac{1}{2\pi}\left(\frac{1}{(z-a)} - \frac{1}{(z-b)}\right) \tag{3.1.10}$$

which is just a restatement of the previous results in complex coordinates. Consider now integrating over a second curve, C', with coordinates z and end points A and B. Using the above results, observe that the combination

$$\langle\tilde{n}\rangle_w = \int_{C'} d\bar{z}\ \left\langle \int_C \delta_c^2(z-w)dw \right\rangle_w - \int_{C'} dz\ \left\langle \int_C \delta_c^2(z-w)d\bar{w} \right\rangle_w \tag{3.1.11}$$

naively satisfies

$$\langle\tilde{n}\rangle_w = \frac{1}{2\pi}\log\left(\frac{(\bar{B}-\bar{a})}{(B-a)}\ \frac{(A-a)}{(\bar{A}-\bar{a})}\ \frac{(\bar{B}-\bar{b})}{(B-b)}\ \frac{(A-a)}{(\bar{A}-\bar{a})}\right) \tag{3.1.12}$$

which is the sum of angles between the endpoints of the two curves, which is easily seen by writing the displacements in polar form. We have already shown that the propagator for the worldsheet field, ψ, goes like the angle between two points in the upper half plane, and so this gives us a hint that the quantity $\tilde{n}$ is the one that will lead to the field theory representing ψ in the interior of the worldsheet. Note also that it appears to satisfy

$$\bar{\partial}_B \partial_a \langle \tilde{n} \rangle_w = \delta_c^2(B-a) \tag{3.1.13}$$

and

$$\partial_B \bar{\partial}_a \langle \tilde{n} \rangle_w = -\delta_c^2(B-a), \tag{3.1.14}$$

the condition we require to reproduce the three point vertex of yang-Mills theory (see (2.3.97) and section 3.3.3). The designation of the symbol $\tilde{n}$ for this quantity will become apparent after a discussion of the supersymmetric version of this.

After removing the averages in (3.1.11), $\tilde{n}$ can be written as

$$\tilde{n} = \int_{c_z, c_w} (d\bar{z}dw - dzd\bar{w})\ \delta_c^2(z-w) = i \int_{c_z, c_w} \delta_c^2(z-w)\ \epsilon_{\mu\nu} dz^\mu dw^\nu. \tag{3.1.15}$$

We will define the same quantity on a curved surface and investigate its properties in the next section.

3.1.1 Electrostatic field in curved space

The worldsheet is a 2 dimensional curved surface embedded in a target spacetime. Therefore, we would like to generalise the computation above for a curved surface. Gauss' law for two equal oppositely charged particles becomes

$$\nabla \cdot \mathbf{E} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} E^\mu) = \frac{\delta^2(x-a)}{\sqrt{g}} - \frac{\delta^2(x-b)}{\sqrt{g}}. \tag{3.1.16}$$

In 2 dimensions the metric has three independent components. The worldsheet has 2 dimensional diffeomorphism invariance that can be used to reduce the number of independent components to just one. The computations are simplest in conformal gauge and complex coordinates in which the line element takes the form $ds^2 = e^\phi dz d\bar{z}$. Gauss' law becomes

$$\partial(e^\phi E) + \bar{\partial}(e^\phi \bar{E}) = 2\ \left(\delta_c^2(z-a) - \delta_c^2(z-b)\right). \tag{3.1.17}$$

This is easily solved by converting to the covariant components of the field which are

$$E_z = g_{z\bar{z}}\bar{E} = \frac{e^{\phi}\bar{E}}{2}, \qquad E_{\bar{z}} = g_{\bar{z}z}E = \frac{e^{\phi}E}{2}. \tag{3.1.18}$$

Gauss' law is then

$$\partial E_{\bar{z}} + \bar{\partial} E_z = \ \delta_c^2(z-a) - \delta_c^2(z-b) \tag{3.1.19}$$

so that the conformal factor no longer appears in the field equation. This is because in 2 dimensions the combination $\sqrt{g}E^{\mu} = \sqrt{g}g^{\mu\nu}E_{\nu}$ is Weyl invariant, i.e. invariant under metric rescalings of the form $g_{\mu\nu} \to \Omega\ g_{\mu\nu}$. The conformal gauge is obtained by making a Weyl transformation, $\Omega = e^{\phi}$, of the flat metric and so writing $g_{\mu\nu} = \frac{1}{2}e^{\phi}\sigma_{\mu\nu}$, where $\sigma_{\mu\nu}$ has elements $\sigma_{11} = \sigma_{22} = 0$ and $\sigma_{12} = \sigma_{21} = 1$, then $\sqrt{g}g^{\mu\nu} = \sigma^{\mu\nu}$. Because of this, we can just use the results above by interchanging $E \to 2E_{\bar{z}}$ etc. The solution is then

$$E_{\bar{z}} = \frac{1}{4\pi}\left(\frac{1}{(\bar{z}-\bar{a})} - \frac{1}{(\bar{z}-\bar{b})}\right), \quad E_z = \frac{1}{4\pi}\left(\frac{1}{(z-a)} - \frac{1}{(z-b)}\right). \tag{3.1.20}$$

Faraday's law for the covariant components of the field is now $\partial E_{\bar{z}} - \bar{\partial} E_z = 0$ and so is still satisfied. The simplicity of (3.1.19) allows us to, again, use the flat space results for the line of force analysis. This time we have

$$E_{\bar{z}} = \frac{1}{2}\int \delta_c^2(z-w)dw, \qquad E_z = \frac{1}{2}\int \delta_c^2(z-w)d\bar{w}. \tag{3.1.21}$$

We cannot use the flat space results to calculate the averages over these quantities however. The functional average over some quantity, Ω, is still given by the functional integral

$$\langle\Omega\rangle = \frac{1}{Z}\ \int \mathcal{D}[x,h]\left(\int_0^1 d\xi\sqrt{h(\xi)} - T\right)\Omega\ e^{-S[x,\sqrt{h}]}, \tag{3.1.22}$$

but now the action is that of a free particle in curved space

$$S[x,\sqrt{h}] = \frac{1}{2}\int_0^T dt\ \frac{g_{\mu\nu}(x)}{\sqrt{h}}\dot{x}^{\mu}\dot{x}^{\nu}. \tag{3.1.23}$$

So the modification of the calculation involves dealing with the conformal factor, e^{ϕ}, in the metric [50]. Note, the normalisation constant, Z, fixes $\langle 1\rangle = 1$. To obtain a scale invariant weight we will take the large T limit corresponding to averaging over long curves in terms of the einbein, $\sqrt{h}$. The action and average are invariant under reparametrisations and so we can choose a gauge in which $\sqrt{h}$ is constant [17], explaining the appearance of the delta function insertion.

We will be general in our calculation by considering the average of the quantity $\int_{C_1} \delta^2(x_1 - x_2) dx_1^{\mu}$. We will also this time choose to work on a surface bounded by a closed curve. There are a few reasons for this, the first and most important is that this is the case that we will eventually need in our string model where the worldsheet has boundaries. We will also find that quantities on a closed curved space will be more manageable due to the existence of a normalisable zero mode of the Laplacian. Working on a closed surface requires us to consider what happens when the lines of force reach the boundary. The simplest possibility is to require curves that reach the boundary be specularly reflected so that the angle of incidence between the curve and the normal to the boundary is equal to the angle of reflection between the reflected curve and the normal. This is the most natural choice for the calculation of the electric field.

For simplicity, we will choose to work on the unit disk as it is conformally equivalent to the upper half plane. The boundary is now the x axis and we identify $x = -\infty$ and $x = +\infty$ as the same point. Working in the upper half plane allows for a simple use of the method of images to deal with the effects of the boundary.

Fourier decomposing the delta function allows us to write the average as

$$\left\langle \int_{C_1} \delta^2(x_1 - x_2) dx_1^{\mu} \right\rangle_{C_1} = \frac{1}{Z} \int \mathcal{D}[x] \left(\int_{C_1} \delta^2(x_1 - x_2) dx_1^{\mu} \right) e^{-S[x_1]}$$

$$= \frac{\delta}{\delta \mathcal{A}_{\mu}(x_2)} \frac{1}{Z} \int \mathcal{D}[x] \ e^{-\int_0^T dt \ \left(\frac{1}{2} g_{\mu\nu}(x_1) \dot{x}_1^{\mu} \dot{x}_1^{\nu} - \mathcal{A}\cdot\dot{x}_1\right)} \Big|_{\mathcal{A}=0} . \tag{3.1.24}$$

The functional integral is the curved space generalisation of (1.1.29) in 2 dimensions. It is interpreted as the amplitude of a bosonic particle coupled to a vector potential, $i\mathcal{A}$, in the presence of a curved background, to travel from $x(0)$ to $x(T)$. The factor of i here is again due to the Wick rotation of the action. From this we can straight forwardly obtain the classical Hamiltonian by a Legendre transformation, $H = \frac{1}{2}g_{\mu\nu}(p + i\mathcal{A})^\mu(p + i\mathcal{A})^\nu$. Again, in the quantum theory there is an ordering ambiguity which is resolved by interpreting the Hamiltonian as the Laplacian minimally coupled to the vector potential, acting on scalars.

Taking the coordinates, x_μ, to be points in the upper half-plane, Σ, then we can use points in the lower-half-plane to parametrise a surface Σ_R attached along the boundary. Σ_R is the reflection of Σ in the sense that the value of the metric at a point in the lower half-plane is taken to be the value of the metric at the point in the upper half-plane that is its reflection. Any curve C_1 from a_1 to b_1 that is restricted to Σ but is reflected once has the same Boltzmann factor as a curve that crosses the boundary between Σ and Σ_R but either starts at a_1^R, the reflection of a_1, or ends at b_1^R, the reflection of b_1 . Curves that are reflected an even number of times have the same weight as curves from a_1 to b_1 (or from a_1^R to b_1^R) that are not restricted to Σ and curves that are reflected an odd number of times have the same weight as curves from a_1 to b_1^R (or from a_1^R to b_1) that are not restricted to Σ. These two cases are shown in Figures 3.1-3.3. So, by including reflected curves we are effectively working on the full plane parametrising $\Sigma \cup \Sigma_R$ but including curves with ends that are the reflections of one of the original end-points and so we can identify

$$\int \mathcal{D}[x] \quad e^{-\int_0^T dt \ (\frac{1}{2}g_{\mu\nu}(x_1)\dot{x}_1^\mu \dot{x}_1^\nu - \mathcal{A}\cdot\dot{x}_1)} = \langle b_1|e^{-\hat{H}T}|a_1\rangle + \langle b_1|e^{-\hat{H}T}|a_1^R\rangle \tag{3.1.25}$$

$$= \langle b_1|e^{-\hat{H}T}|a_1\rangle + \langle b_1^R|e^{-\hat{H}T}|a_1\rangle = \mathcal{G}_T(b_1, a_1). \tag{3.1.26}$$

The average of the line of force is then computed by taking a functional derivative of the heat kernel

$$\left\langle \int_{C_1} \delta^2(x_1 - x_2) dx_1^\mu \right\rangle_{C_1} = \frac{\delta}{\delta \mathcal{A}_\mu(x_2)} \ \mathcal{G}_T(b_1, a_1)\bigg|_{\mathcal{A}=0}$$

$$= -\frac{1}{Z}\int_0^T dt \int_{\Sigma\cup\Sigma_R} d^2x d^2y \sqrt{g(x)}\sqrt{g(y)} \ \langle b_1|\, e^{-t\hat{H}_0}\,|x\rangle \, \langle x| \ \frac{\delta \hat{H}}{\delta \mathcal{A}_\mu(x_2)}\bigg|_{\mathcal{A}=0} |y\rangle$$

$$\times \ \langle y|\, e^{(t-T)\hat{H}_0}(|a_1\rangle + |a_1^R\rangle) \tag{3.1.27}$$

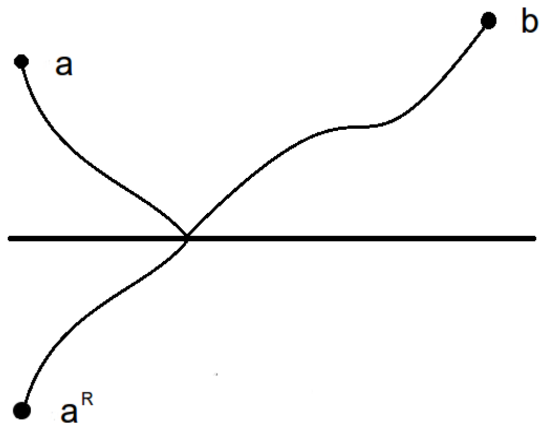


Figure 3.1: An example of a curve with one reflection off the boundary with Σ above the central line and Σ^R below it. The curve has a Boltzmann factor equivalent to that of the curve that joins a^R and b, shown below the boundary.

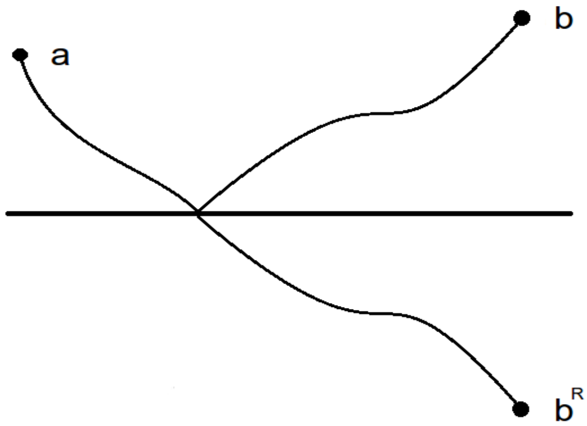


Figure 3.2: An example of a curve with one reflection off the boundary. The curve has a Boltzmann factor equivalent to that of the curve that joins a and b^R. This Boltzmann factor is also equal to the Boltzmann factor of the curve in Figure 3.1.

where $\mathcal{G}_T$ is the heat kernel defined above and we have used the short hand for the completeness relation over the whole plane,

$$\int_{\Sigma\cup\Sigma_R} d^2x\sqrt{g(x)}\,|x\rangle\langle x| \equiv \int_{\Sigma} d^2x\sqrt{g(x)}\,|x\rangle\langle x| + \int_{\Sigma_R} d^2x\sqrt{g(x)}\,|x\rangle\langle x| = 1. \quad (3.1.28)$$

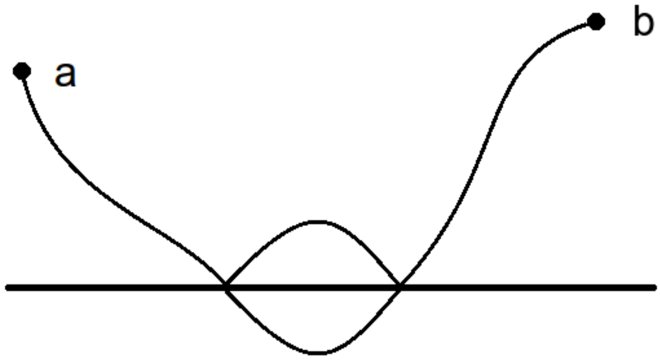


Figure 3.3: An example of a curve with two reflections off the boundary. In this case the Boltzmann factor of the curve restricted to the upper half plane is equal to the corresponding curve that is allowed to pass through the boundary.

We recognise the appearance of the heat kernel of the Laplacian at $\mathcal{A} = 0$, denoted $\mathcal{G}^0$, so (3.1.27) can be written

$$= -\frac{1}{Z}\int_0^T dt \int_{\Sigma\cup\Sigma_R} d^2x d^2y \sqrt{g(x)}\sqrt{g(y)}\; \mathcal{G}_t^0(b_1, x)\,\langle x|\; \frac{\delta \hat{H}}{\delta \mathcal{A}_\mu(x_2)}\bigg|_{\mathcal{A}=0} |y\rangle\, \mathcal{G}_{T-t}^0(y, a_1). \tag{3.1.29}$$

Now, we have the matrix elements

$$\langle\, x \,|\, \hat{H} \,|\, y \,\rangle = -\frac{1}{2\sqrt{g(x)}}\,(\partial - \mathcal{A})_\alpha \left(\sqrt{g}(x)\, g^{\alpha\nu}(x)(\partial - \mathcal{A})_\nu\, \frac{\delta^2(x-y)}{\sqrt{g(y)}}\right). \tag{3.1.30}$$

and so

$$\langle x|\; \frac{\delta \hat{H}}{\delta \mathcal{A}_\mu(x_2)}\bigg|_{\mathcal{A}=0} |y\rangle = \frac{1}{2\sqrt{g(x)}}\delta^2(x-x_2)\sqrt{g(x)}g^{\mu\nu}(x)\partial_\nu\left(\frac{\delta^2(x-y)}{\sqrt{g(y)}}\right)$$
$$+\frac{1}{2\sqrt{g(x)}}\partial_\nu\left(\sqrt{g(x)}g^{\mu\nu}(x)\delta^2(x-x_2)\frac{\delta^2(x-y)}{\sqrt{g(y)}}\right). \tag{3.1.31}$$

Substituting this into (3.1.29) gives

$$-\frac{1}{2Z}\int_0^T dt \int_{\Sigma\cup\Sigma_R} d^2x d^2y \left(\mathcal{G}_t^0(b_1,x)\delta^2(x-x_2)\sqrt{g(x)}g^{\mu\nu}(x)\partial_\nu^x\big(\delta^2(x-y)\big)\mathcal{G}_{T-t}^0(y,a_1)\right.$$
$$\left.+\mathcal{G}_t^0(b_1,x)\partial_\nu^x\big(\sqrt{g(x)}g^{\mu\nu}(x)\delta^2(x-x_2)\delta^2(x-y)\big)\mathcal{G}_{T-t}^0(y,a_1)\right). \tag{3.1.32}$$

Firstly, computing the y integrals gives

$$-\frac{1}{2Z}\int_0^T dt \int_{\Sigma\cup\Sigma_R} d^2x \,\Big(\mathcal{G}_t^0(b_1,x)\delta^2(x-x_2)\sqrt{g(x)}g^{\mu\nu}(x)\partial_\nu^x\mathcal{G}_{T-t}^0(x,a_1)$$

$$+\mathcal{G}_t^0(b_1,x)\partial_\nu\big(\sqrt{g(x)}g^{\mu\nu}(x)\delta^2(x-x_2)\mathcal{G}_{T-t}^0(x,a_1)\big)\Big) \tag{3.1.33}$$

and carrying out an integration by parts on the second term and computing the x integral we find

$$\left\langle\int_{C_1}\delta^2(x_1-x_2)dx_1^\mu\right\rangle_{C_1} = -\frac{1}{2Z}\int_0^T dt\,\Big(\mathcal{G}_t^0(b_1,x_2)\sqrt{g(x_2)}g^{\mu\nu}(x_2)\partial_\nu\mathcal{G}_{T-t}^0(x_2,a_1)$$

$$-\partial_\nu\big(\mathcal{G}_t^0(b_1,x_2)\big)\sqrt{g(x_2)}g^{\mu\nu}(x_2)\mathcal{G}_{T-t}^0(x_2,a_1)\Big). \tag{3.1.34}$$

In the $T\to\infty$ limit the integral splits into two pieces. One where t is close to 0 and the other where t is close to T. We will briefly discuss the spectral decomposition of the heat kernel as this will allows us to evaluate these cases.
We will start with the eigenfunction equation for the Laplacian

$$-\Delta u_\lambda = \lambda u_\lambda. \tag{3.1.35}$$

As we are working on a compact space we have a normalisable zero mode that satisfies $\Delta u_0 = 0$. To find this zero mode it is most useful to work in complex coordinates where the zero mode now satisfies $\bar\partial\partial u_0(z,\bar z) = 0$ which has a solution composed of a holomorphic and anti-holomorphic function, $u_0 = f_+(z) + f_-(\bar z)$. We may expand $f_\pm$ as a power series in z^n with corresponding coefficients a_n. We note, however, that all coefficients with $n>0$ must be zero as otherwise f would blow up as $z\to\pm\infty$. This leaves us with the conclusion that the zero mode is a constant. We can normalise so that

$$\int |u_0|^2\sqrt{g(x)}d^2x = |u_0|^2 A \tag{3.1.36}$$

where $A = \int \sqrt{g(x)} d^2x$ is the area of the surface. The normalised zero mode is then $u_0 = 1/\sqrt{A}$. As the eigenfunctions are mutually orthogonal we then find

$$\int d^2x \ \sqrt{g(x)} \ u_0 u_\lambda = \frac{1}{\sqrt{A}} \int d^2x \ \sqrt{g(x)} u_\lambda = 0 \tag{3.1.37}$$

i.e. the integral of u_λ with $\lambda \neq 0$ over the surface is zero. The heat kernel has the decomposition

$$\mathcal{G}_t^0(x_1, x_2) = \sum_\lambda e^{-t\lambda} u_\lambda(x_1) u_\lambda(x_2). \tag{3.1.38}$$

Taking the limit of the spectral decomposition of the heat kernel picks out the zero mode contribution and so $\lim_{T\to\infty} \mathcal{G}_T^0 = 1/A$. The spectral decomposition of the heat kernel allows us to set $\mathcal{G}_{T-t}^0(x_2, a_1) = 1/A$ when t is close to 0 and $\mathcal{G}_t^0(b_1, x_2) = 1/A$ when $t \sim T$. Taking the $T \to \infty$ limit and using these values then gives

$$\lim_{T\to\infty} \left\langle \int_{C_1} \delta^2(x_1 - x_2) dx_1^\mu \right\rangle_{C_1} =$$

$$\frac{1}{2AZ} \int_0^\infty dt \left(\sqrt{g(x_2)} g^{\mu\nu}(x_2) \partial_\nu \mathcal{G}_t^0(b_1, x_2) - \sqrt{g(x_2)} g^{\mu\nu}(x_2) \partial_\nu \mathcal{G}_{T-t}^0(x_2, a_1) \right). \tag{3.1.39}$$

The normalisation constant is obtained from

$$\langle 1 \rangle = \lim_{T\to\infty} \left(\frac{1}{Z} \int \mathcal{D}x_1 \ e^{-S[x_1]} \right) = \frac{1}{Z} \lim_{T\to\infty} \mathcal{G}_T^0(x_1, a_1) = \frac{1}{ZA} \tag{3.1.40}$$

Now, (3.1.39) contains the t integral over the heat kernel which is related to the Green's function satisfying Neumann boundary conditions

$$\int_0^\infty dt \left(\mathcal{G}_t^0(x_1, x_2) - \frac{1}{A} \right) = \sum_\lambda u_\lambda(x_1) \frac{1}{\lambda} u_\lambda(x_2) = 2G(x_1, x_2) \tag{3.1.41}$$

where the Green's function, G, solves

$$-\frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu G(x_1, x_2) \right) = \frac{\delta^2(x_1 - x_2)}{\sqrt{g}} - \frac{1}{A} \tag{3.1.42}$$

where the Laplacian acts at either x_1 or x_2. The appearance of the $1/A$ terms is due to the existence of the zero mode. Working in the upper half plane allows us to use the method of images to solve for the Green's function by placing a point charge at the point in the lower half plane that corresponds to the reflection of the original charge. In complex coordinates, the Green's function equation is

$$-4e^{-\phi}\bar{\partial}\partial G(z_1,z_2) = 2e^{-\phi}\delta^2(z_1-z_2) - \frac{1}{A} \tag{3.1.43}$$

which has solution

$$G(z_1,z_2) = -\frac{1}{2\pi}\log(|z_1-z_2|) - \frac{1}{2\pi}\log(|z_1-\bar{z}_2|) - \Psi(z_1,z_2). \tag{3.1.44}$$

The second term is due to the image charge; note that $\Delta\log(|z_1-\bar{z}_2|) = 4\pi\delta^2(z_1-\bar{z}_2) = 0$ since we are working on the upper half plane. The last term satisfies

$$4e^{-\phi(z_1)}\bar{\partial}_1\partial_1\Psi = 4e^{-\phi(z_2)}\bar{\partial}_2\partial_2\Psi = -\frac{1}{A} \tag{3.1.45}$$

and Neumann boundary conditions and is again a consequence of the zero mode. Note that (3.1.45) suggests $\Psi(z_1,z_2) = \psi(z_1)+\psi(z_2)$. From (3.1.39) we find

$$\begin{aligned}\left\langle \int_{C_1}\delta^2(z_1-z_2)dz_1\right\rangle_{C_1} &= \bar{\partial}_2 G(b_1,z_2) - \bar{\partial}_2 G(z_2,a_1)\\ &= \frac{1}{4\pi}\left(\frac{1}{\bar{b}_1-\bar{z}_2}+\frac{1}{b_1-\bar{z}_2}-\frac{1}{\bar{a}_1-\bar{z}_2}-\frac{1}{a_1-\bar{z}_2}\right)\end{aligned} \tag{3.1.46}$$

so the zero mode contribution drops out. We then see that the average of the line of force reproduces the full electric field (3.1.20) modified by the effect of the boundary. Similarly we find the conjugate result

$$\begin{aligned}\left\langle \int_{C_1}\delta^2(z_1-z_2)d\bar{z}_1\right\rangle_{C_1} &= \partial_2 G(b_1,z_2) - \partial_2 G(z_2,a_1)\\ &= \frac{1}{4\pi}\left(\frac{1}{b_1-z_2}+\frac{1}{\bar{b}_1-z_2}-\frac{1}{a_1-z_2}-\frac{1}{\bar{a}_1-z_2}\right).\end{aligned} \tag{3.1.47}$$

Inserting these into the Gauss' law verifies that these are the solutions, as the terms corresponding to the reflection vanish since $\delta^2(a-\bar{z}) = 0$ etc. on the upper half plane. We can see that the conformal factor has dropped out of the calculation and so we have a result similar to the flat space case. We can form the functional $\tilde{n}$ again that naively looks as if it satisfies the criteria for producing the three point function. We shall do this in detail after a study of the supersymmetric electrostatic field.

3.2 Supersymmetric electrostatics

We now turn to the supersymmetric generalisation of the electrostatic field. To do this we enlarge the parameter space by introducing the Grassmann odd variables θ and $\bar{\theta}$, being the superpartners of z and $\bar{z}$ respectively. The electric field becomes a superfield, $\mathbb{E}$, and the partial derivatives become superderivatives. We will again consider the electrostatic field produced by two equal opposite charges placed at (a, η_a) and (b, η_b). We will use the covariant components of the electrostatic field, $\mathbb{E}_\mu$, as we have shown above that Gauss' law takes the same form in flat and curved space. This is a unique feature of 2 dimensions where the combination $\sqrt{g}g^{\mu\nu}$ is Weyl invariant. We will show in here that just as in the bosonic case, the problem in curved 2 dimensional superspace involves dealing with a superconformal factor which will eventually drop out. In flat superspace, Gauss' law takes the form[1]

$$D\bar{\mathbb{E}} + \bar{D}\mathbb{E} = \delta^2(\theta - \eta_a)\delta^2(z-a) - \delta^2(\theta - \eta_b)\delta^2(z-b). \tag{3.2.48}$$

The solution can be obtained from the Green's function for the super-Laplacian

$$-2\bar{D}DG_F = \bar{D}(-DG_F) + D(\bar{D}G_F) = \delta^2(\theta_1 - \theta_2)\delta^2(z_1 - z_2) \tag{3.2.49}$$

and so $\bar{\mathbb{E}} = \bar{D}G_F$ and $\mathbb{E} = -DG_F$. These can be evaluated using the results from Appendix A so that

$$\bar{\mathbb{E}} = -\frac{1}{4\pi}\frac{(\bar{\theta} - \bar{\eta}_a)}{(\bar{z} - \bar{a})} + \frac{1}{2}(\bar{\theta} - \bar{\eta}_a)\theta\eta_a\delta^2(z-a)$$

[1] $\delta^2(\theta - \theta') = (\bar{\theta} - \bar{\theta}')(\theta - \theta')$ and we denote $\mathbb{E}_{z,\theta} = \mathbb{E}$ and $\bar{\mathbb{E}}_{\bar{z},\bar{\theta}} = \bar{\mathbb{E}}$.

$$+\frac{1}{4\pi}\frac{(\bar{\theta}-\bar{\eta}_b)}{(\bar{z}-\bar{b})}+\frac{1}{2}(\bar{\theta}-\bar{\eta}_b)\theta\eta_b\delta^2(z-b), \tag{3.2.50}$$

$$\mathbb{E}=\frac{1}{4\pi}\frac{(\theta-\eta_a)}{(z-a)}-\frac{1}{2}\bar{\theta}\bar{\eta}_a(\theta-\eta_a)\delta^2(z-a)$$

$$-\frac{1}{4\pi}\frac{(\theta-\eta_b)}{(z-b)}-\frac{1}{2}\bar{\theta}\bar{\eta}_b(\theta-\eta_b)\delta^2(z-b). \tag{3.2.51}$$

As in the bosonic case, there exists a supersymmetric line of force solution

$$\bar{\mathbb{E}}_c(z,\theta)=-\int_C dt\ \delta^2(z-w-\theta\eta)(\bar{\theta}-\bar{\eta})(\dot{w}+\theta\dot{\eta}) \tag{3.2.52}$$

$$\mathbb{E}_c(z,\theta)=\int_C dt\ \delta^2(z-w-\theta\eta)(\theta-\eta)(\dot{\bar{w}}+\bar{\theta}\dot{\bar{\eta}}) \tag{3.2.53}$$

where $w=w(t)$ and $\eta=\eta(t)$ are the bosonic and fermionic coordinates of the curve, C. The proof is more involved than in the bosonic case. The superderivatives of the field components are

$$D\bar{\mathbb{E}}_c=\int_C dt\ \Big(\eta\partial_z\delta^2(z-w)(\bar{\theta}-\bar{\eta})\dot{w}+\delta^2(z-w)(\bar{\theta}-\bar{\eta})\dot{\eta}-\theta\partial_z\delta^2(z-w)(\bar{\theta}-\bar{\eta})\dot{w}\Big) \tag{3.2.54}$$

which can be written as

$$=-\int_C dt\ \left((\bar{\theta}-\bar{\eta})(\theta-\eta)\dot{w}\partial_w\delta^2(z-w)+(\bar{\theta}-\bar{\eta})\left(\frac{d}{dt}(\theta-\eta)\right)\delta^2(z-w)\right). \tag{3.2.55}$$

Similarly, one finds

$$\bar{D}\mathbb{E}_c=-\int_C dt\ \left((\bar{\theta}-\bar{\eta})(\theta-\eta)\dot{\bar{w}}\partial_{\bar{w}}\delta^2(z-w)+\left(\frac{d}{dt}(\bar{\theta}-\bar{\eta})\right)(\theta-\eta)\delta^2(z-w)\right). \tag{3.2.56}$$

Combining these results in Gauss' law we get

$$D\bar{\mathbb{E}}_c+\bar{D}\mathbb{E}_c=-\int_C dt\ \frac{d}{dt}\Big((\bar{\theta}-\bar{\eta})(\theta-\eta)\delta^2(z-w)\Big)$$

$$=\delta^2(\theta-\eta_a)\delta^2(z-a)-\delta^2(\theta-\eta_b)\delta^2(z-b) \tag{3.2.57}$$

as desired. The form of $\mathbb{E}_c$ and $\bar{\mathbb{E}}_c$ can be simplified by defining the super-displacement, $l = z - w - \theta\eta$, so that

$$\bar{\mathbb{E}}_c = \int_C dt\ \delta^2(l)\bar{D}\bar{l}\ \dot{l}, \qquad \mathbb{E}_c = \int_C dt\ \delta^2(l)Dl\ \dot{\bar{l}} \tag{3.2.58}$$

where the superderivative acts with respect to w and η on C. Note, we can write the bosonic field in a similar way by defining $s = z - w$. With this, the bosonic lines of force can then be written as

$$\bar{E}_{c\ z} = \int_C dt\ \delta^2(s)\bar{\partial}\bar{s}\ \dot{s}, \qquad E_{c\ z} = \int_C dt\ \delta^2(s)\partial s\ \dot{\bar{s}}. \tag{3.2.59}$$

It may then be expected from our previous results that summing over all curves weighted by a suitable action functional reproduces the full solution, (3.2.50) and (3.2.51). The suitable action in question must be the supersymmetric generalisation of the massless, free bosonic particle's action. In [45], the sums over surfaces were carried out by weighting each curve with the gauge fixed superstring action in the appropriate number of dimensions. Here, then, we need the action describing the worldline of a superparticle with a 2 dimensional target space. There are a few choices of possible actions. As we would like this 2 dimensional theory to live on the worldsheet of our original string model it is useful to use a superparticle action with manifest spacetime supersymmetry. This naturally leads us to use the Green-Schwarz superparticle.

3.2.1 The Green-Schwarz superparticle

Let us recapitulate what we have at this point. In the bosonic case, averaging over lines of force corresponds to summing over all possible curves joining two fixed end points with each curve weighted by a factor of e^{-S}. S is naturally interpreted as the worldline action of a free particle on the surface upon which we are working so that the curves we are averaging over are like particle worldlines embedded on the surface. The delta function appearing in the line of force solution can be written as a functional derivative of an exponential of the form

$$\int d^D x_1 \; \delta^D(x_1 - x_2) = \frac{\delta}{\delta A(x_2)} e^{\int dt \; \dot{x}_1 \cdot A(x_1)} \tag{3.2.60}$$

The exponent of (3.2.60) describes a Wilson line and when combined with the action, S, describes the motion of a particle moving under the influence of an external gauge field, $A(x)$. In the present case, our target space is the worldsheet of a superstring. Spacetime quantities are obtained from the superstring model by integrating out the Grassmann odd coordinates, θ, and so field theories on the superstring worldsheet should be explicitly dependent on all worldsheet coordinates, including the θ coordinates. The worldsheet has an $N = 1$ global supersymmetry between bosonic and fermionic coordinates given by

$$\delta z = -\eta\theta, \quad \delta\theta = \eta. \tag{3.2.61}$$

These two arguments rule out using the traditional superparticle action of Brink, Howe and Di Vecchia [46] as our weight functional, S. In superstring theory, there exists a formulation which has manifest target space supersymmetry known as the Green-Schwarz (GS) superstring [56]. The GS superstring action has an important additional symmetry not existing in the RNS formalism known as κ invariance. This is because as it stands the GS action has twice as many degrees of freedom as the RNS action. The kappa symmetry can then be used to remove these extra degrees of freedom.

There exists a superparticle analogue of the GS superstring that also has target space supersymmetry and kappa invariance. We will, therefore, use the action of the GS superparticle as our weighting functional. We will find that on a 2 dimensional target space, the κ invariance can be used to reduce the action to that of the free bosonic particle, making calculations much simpler. The form of the Lagrangian we use is similar to that studied in [48] except that we are working in a Euclidean signature and have introduced an extra mass parameter, μ. The Lagrangian of the

GS superparticle in flat 2 dimensional superspace is then

$$L_0 = \frac{\pi\bar{\pi}}{\sqrt{h}} - \mu(\theta\dot{\bar{\theta}} + \bar{\theta}\dot{\theta}) + \mu^2\sqrt{h} \tag{3.2.62}$$

where $\pi \equiv \dot{z}+\theta\dot{\theta}$ and $\bar{\pi} \equiv \dot{\bar{z}}+\bar{\theta}\dot{\bar{\theta}}$ are the globally supersymmetric generalisations of $\dot{z}$ and $\dot{\bar{z}}$ respectively. The Lagrangian is invariant under reparametrisations, $t \to f(t)$ and the global supersymmetric variations (3.2.61). The action also possesses an additional worldline symmetry known as kappa invariance which takes the form

$$\delta_\kappa z = -\theta\delta_\kappa\theta, \quad \delta_\kappa\theta = -\left(\kappa + \frac{\bar{\kappa}\pi}{\mu\sqrt{h}}\right)$$

$$\delta_\kappa \bar{z} = -\bar{\theta}\delta_\kappa\bar{\theta}, \quad \delta_\kappa\bar{\theta} = -\left(\bar{\kappa} + \frac{\kappa\bar{\pi}}{\mu\sqrt{h}}\right)$$

$$\delta_\kappa\sqrt{h} = \frac{2}{\mu}(\dot{\bar{\theta}}\kappa + \dot{\theta}\bar{\kappa}) \tag{3.2.63}$$

where $\kappa = \kappa(t)$ and $\bar{\kappa} = \bar{\kappa}(t)$ are Grassmann-odd worldline functions. We can use this symmetry to choose a gauge in which $\dot{\theta} = \dot{\bar{\theta}} = 0$. We can also use the reparametrisation invariance to fix $\sqrt{h} = T$. In this gauge the θ dependence completely drops out of (3.2.62) and so it reduces to the Lagrangian of the massive free bosonic particle. In the bosonic model we used the massless particle action, so setting $\mu = 0$ gives the action

$$S = \frac{1}{2}\int_0^1 dt\ \frac{\dot{z}\dot{\bar{z}}}{T} = \frac{1}{2}\int_0^T dt'\ \dot{z}\dot{\bar{z}} \tag{3.2.64}$$

with $t' = tT$. This then is the action for the free massless particle. Provided that a functional, Ω, is κ invariant, then it's average (in flat space) can be computed as $\langle\Omega\rangle = \langle\Omega'\rangle_B$ where Ω' is the functional after gauge fixing and the subscript B denotes the functional integral computed with the bosonic weight.

We would like to generalise this result to a curved target space which is equivalent to coupling the superparticle to supergravity. This coupling is best described in terms of forms and so we first write (3.2.62) as

$$L_0 = \frac{\dot{e}^z \dot{e}^{\bar{z}}}{\sqrt{h}} + 2\mu \dot{e}^A \Gamma_A + \mu^2 \sqrt{h}. \tag{3.2.65}$$

$\dot{e}^A = \dot{z}^M e_M^A$ is the supervielbein and Γ_A are gauge fields. In particular we have $\dot{e}^z = \pi$, $\dot{e}^{\bar{z}} = \bar{\pi}$, $\dot{e}^\theta = \dot{\theta}$, $\dot{e}^{\bar{\theta}} = \dot{\bar{\theta}}$, $\Gamma_\theta = \bar{\theta}/2$ and $\Gamma_{\bar{\theta}} = \theta/2$.

3.2.2 Lorentz superparticle

We are now in a position to consider the dynamics of the GS superparticle on a curved supermanifold. This is most naturally done in terms of forms, hence why we wrote the flat space Lagrangian as (3.2.65). We will begin with a discussion of the superparticle on a supermanifold in Lorentz signature as given in [48] as there are a few steps which seem unclear or even incorrect. After this, we will turn to the case that we require of the superparticle on a supermanifold with Euclidean signature. The difference between each model comes from the supergravity constraints, basically coming down to factors of i.

The Lagrangian of the superparticle on a curved supermanifold background takes the same form as (3.2.65) where we promote the flat vielbein, e, to the curved vielbein, $\mathcal{E}$, and promote the gauge field, Γ, to a general superfield so that the Lagrangian becomes

$$L = \frac{\dot{\mathcal{E}}^z \dot{\mathcal{E}}^{\bar{z}}}{\sqrt{h}} + 2\mu \dot{\mathcal{E}}^A \Gamma_A + \mu^2 \sqrt{h} \tag{3.2.66}$$

where $\dot{\mathcal{E}}^z \equiv \dot{z}^M {\mathcal{E}_M}^z$ etc. Now, we have to satisfy the Bianchi identities and constraints on the connection that allow us to simplify the above Lagrangian. The full covariant derivative containing the effect of spatial curvature and the gauge fields is

$$\hat{\nabla}_A = \nabla_A + \Gamma_A = \mathcal{E}_A^M \partial_M + \Omega_A + \Gamma_A \tag{3.2.67}$$

where $\Omega_A = \omega_A M$ is the spin connection. Note, we have $M\Gamma_\theta = \frac{1}{2}\Gamma_\theta$ and $M\Gamma_{\bar{\theta}} = -\frac{1}{2}\Gamma_{\bar{\theta}}$. The (1,1) supergravity constraints in Lorentz signature are

$$\{\hat{\nabla}_\theta, \hat{\nabla}_\theta\} = 2i\hat{\nabla}_z, \quad \{\hat{\nabla}_{\bar{\theta}}, \hat{\nabla}_{\bar{\theta}}\} = 2i\hat{\nabla}_{\bar{z}} \tag{3.2.68}$$

$$F_{\theta\bar{\theta}} = F_{\bar{\theta}\theta} = \nabla_{\bar{\theta}}\Gamma_\theta + \nabla_\theta\Gamma_{\bar{\theta}} = i. \tag{3.2.69}$$

Substituting (3.2.67) into (3.2.68) and noting that $\{\nabla_\theta, \Gamma_\theta\} = \nabla_\theta\Gamma_\theta$ and $\{\Gamma_\theta, \Gamma_\theta\} = 0$, allows us to solve for Γ_z in terms of Γ_θ and an equation relating the covariant derivatives of z and θ, with similar results for $\bar{\theta}$ and $\bar{z}$:

$$\Gamma_z = -i\nabla_\theta\Gamma_\theta, \quad \Gamma_{\bar{z}} = -i\nabla_{\bar{\theta}}\Gamma_{\bar{\theta}} \tag{3.2.70}$$

$$\{\nabla_\theta, \nabla_\theta\} = 2i\nabla_z, \quad \{\nabla_{\bar{\theta}}, \nabla_{\bar{\theta}}\} = 2i\nabla_{\bar{z}}. \tag{3.2.71}$$

The constraints on the covariant derivatives are then solved in conformal gauge in terms of a compensator superfield, S, as

$$\nabla_\theta = e^S\big(D + 2(DS)M\big), \quad \nabla_{\bar{\theta}} = e^S\big(\bar{D} - 2(\bar{D}S)M\big) \tag{3.2.72}$$

where $D \equiv \partial_\theta + i\theta\partial$ and $\bar{D} \equiv \partial_{\bar{\theta}} + i\bar{\theta}\bar{\partial}$ are the Lorentzian superderivatives. (3.2.70) gives us

$$\Gamma_z = -ie^S[D\Gamma_\theta + (DS)\Gamma_\theta], \quad \Gamma_{\bar{z}} = -ie^S[\bar{D}\Gamma_{\bar{\theta}} + (\bar{D}S)\Gamma_{\bar{\theta}}] \tag{3.2.73}$$

and (3.2.71) gives us

$$\nabla_z = e^{2S}\big(\partial - 2i(DS)D + 2(\partial S)M\big), \quad \nabla_{\bar{z}} = e^{2S}\big(\bar{\partial} - 2i(\bar{D}S)\bar{D} - 2(\bar{\partial}S)M\big). \tag{3.2.74}$$

From these we can read off the elements of the inverse supervielbein, $\mathcal{E}_A{}^M$,

$$\mathcal{E}_A{}^M = \begin{pmatrix} e^{2S}\big(1 + 2(DS)\theta\big) & 0 & -2ie^{2S}DS & 0 \\ 0 & e^{2S}\big(1 + 2(\bar{D}S)\bar{\theta}\big) & 0 & -2ie^{2S}\bar{D}S \\ ie^S\theta & 0 & e^S & 0 \\ 0 & ie^S\bar{\theta} & 0 & e^S \end{pmatrix}$$

One can then invert this to obtain the supervielbein. To invert we need the formula for the inverse of a supermatrix. For a general even supermatrix,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A and D composed of commutative elements and B and C composed of anti-commutative elements then

$$M^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

We can use this to compute the supervielbein:

$$\mathcal{E}_M^{\ A} = \begin{pmatrix} e^{-2S} & 0 & 2ie^{-S}DS & 0 \\ 0 & e^{-2S} & 0 & 2ie^{-S}\bar{D}S \\ -ie^{-2S}\theta & 0 & e^{-S}[1 - 2(DS)\theta] & 0 \\ 0 & -ie^{-2S}\bar{\theta} & 0 & e^{-S}[1 - 2(\bar{D}S)\bar{\theta}] \end{pmatrix}$$

From this, we obtain $\mathrm{sdet}(\mathcal{E}_M^{\ A}) = e^{-2S}$ [49]. We now have everything we need to write the Lagrangian in terms of the superspace coordinates (z, θ) as

$$L = e^{-4S}\frac{\pi\bar{\pi}}{\sqrt{h}} - 2i\mu e^{-S}\Big(\pi(D\Gamma_\theta - (DS)\Gamma_\theta) + \bar{\pi}(\bar{D}\Gamma_{\bar{\theta}} - (\bar{D}S)\Gamma_{\bar{\theta}})\Big)$$

$$-2\mu e^{-S}(\Gamma_\theta\dot{\theta} + \Gamma_{\bar{\theta}}\dot{\bar{\theta}}) + \mu^2\sqrt{h}. \tag{3.2.75}$$

Introducing $G \equiv e^{-S}\Gamma_\theta$ and $\bar{G} \equiv e^{-S}\Gamma_{\bar{\theta}}$ allows us to simplify the Lagrangian to

$$L = \frac{e^{-4S}\pi\bar{\pi}}{\sqrt{h}} - 2i\mu[\pi DG + \bar{\pi}\bar{D}\bar{G}] - 2\mu(G\dot{\theta} + \bar{G}\dot{\bar{\theta}}) + \mu^2\sqrt{h}. \tag{3.2.76}$$

Comparing this Lagrangian with the Lagrangian for the bosonic particle in a curved space and in conformal gauge, we see that e^{-4S} represents a superconformal factor. When expanded into its component fields, one can obtain the various components of the supergravity multiplet as is done in [48].

The constraint (3.2.69) becomes

$$D\bar{G} + \bar{D}G = ie^{-2S}. \tag{3.2.77}$$

One can go ahead and solve for G and so obtain the full Lorentzian Lagrangian for the superparticle coupled to a curved background. With this, one can then use this Lagrangian as the starting point for a path integral formalism. This will require a treatment of the various symmetries of the superparticle. We will do this in the next section for the case of the Euclidean superparticle.

3.2.3 Euclidean superparticle

We now turn to the dynamics of the superparticle on a curved supermanifold with Euclidean signature. We begin with (3.2.66) as the Lagrangian. The Euclidean supergravity constraints are

$$\{\hat{\nabla}_\theta, \hat{\nabla}_\theta\} = 2\hat{\nabla}_z, \quad \{\hat{\nabla}_{\bar{\theta}}, \hat{\nabla}_{\bar{\theta}}\} = 2\hat{\nabla}_{\bar{z}} \tag{3.2.78}$$

$$F_{\theta\bar{\theta}} = F_{\bar{\theta}\theta} = \nabla_{\bar{\theta}}\Gamma_\theta + \nabla_\theta\Gamma_{\bar{\theta}} = 1. \tag{3.2.79}$$

Substituting in the covariant derivatives, (3.2.67), gives this time

$$\Gamma_z = \nabla_\theta\Gamma_\theta, \quad \Gamma_{\bar{z}} = \nabla_{\bar{\theta}}\Gamma_{\bar{\theta}} \tag{3.2.80}$$

$$\{\nabla_\theta, \nabla_\theta\} = 2\nabla_z, \quad \{\nabla_{\bar{\theta}}, \nabla_{\bar{\theta}}\} = 2\nabla_{\bar{z}}. \tag{3.2.81}$$

We again solve the constraints on the covariant derivatives by

$$\nabla_\theta = e^S\big(D + 2(DS)M\big), \quad \nabla_{\bar{\theta}} = e^S\big(\bar{D} - 2(\bar{D}S)M\big) \tag{3.2.82}$$

but now with the Euclidean superderivatives $D = \partial_\theta + \theta\partial$ and $\bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}$, so that

$$\Gamma_z = e^S\big(D\Gamma_\theta + (DS)\Gamma_\theta\big), \quad \Gamma_{\bar{z}} = e^S\big(\bar{D}\Gamma_{\bar{\theta}} + (\bar{D}S)\Gamma_{\bar{\theta}}\big). \tag{3.2.83}$$

Substituting (3.2.82) into (3.2.81) then gives

$$\nabla_z = e^S[\partial + 2(DS)D + 2(\partial S)M], \quad \nabla_{\bar{z}} = e^S[\bar{\partial} + 2(\bar{D}S)\bar{D} - 2(\bar{\partial}S)M]. \quad (3.2.84)$$

From these we can write down the inverse supervielbein:

$$\mathcal{E}_A{}^M = \begin{pmatrix} e^{2S}\big(1+2(DS)\theta\big) & 0 & 2e^{2S}DS & 0 \\ 0 & e^{2S}\big(1+2(\bar{D}S)\bar{\theta}\big) & 0 & 2e^{2S}\bar{D}S \\ e^S\theta & 0 & e^S & 0 \\ 0 & e^S\bar{\theta} & 0 & e^S \end{pmatrix}$$

and invert to obtain the supervielbein

$$\mathcal{E}_M{}^A = \begin{pmatrix} e^{-2S} & 0 & 2e^{-S}DS & 0 \\ 0 & e^{-2S} & 0 & 2e^{-S}\bar{D}S \\ -e^{-2S}\theta & 0 & e^{-S}[1-2(DS)\theta] & 0 \\ 0 & -e^{-2S}\bar{\theta} & 0 & e^{-S}[1-2(\bar{D}S)\bar{\theta}] \end{pmatrix}$$

With this, we can write the Lagrangian in superspace coordinates as

$$L = e^{-4S}\frac{\pi\bar{\pi}}{\sqrt{h}} + 2\mu e^{-S}\Big(\pi(D\Gamma_\theta - (DS)\Gamma_\theta) + \bar{\pi}(\bar{D}\Gamma_{\bar{\theta}} - (\bar{D}S)\Gamma_{\bar{\theta}})\Big)$$
$$-2\mu e^{-S}(\Gamma_\theta\dot{\theta} + \Gamma_{\bar{\theta}}\dot{\bar{\theta}}) + \mu^2\sqrt{h}. \quad (3.2.85)$$

Defining G and $\bar{G}$ as before we find

$$L = e^{-4S}\frac{\pi\bar{\pi}}{\sqrt{h}} + 2\mu(\pi DG + \bar{\pi}\bar{D}\bar{G}) - 2\mu(G\dot{\theta} + \bar{G}\dot{\bar{\theta}}) + \mu^2\sqrt{h} \quad (3.2.86)$$

and the supergravity constraint (3.2.79) becomes

$$D\bar{G} + \bar{D}G = e^{-2S}. \quad (3.2.87)$$

The change of the Lagrangian under a general variation of the form $\delta z = -\theta\delta\theta$ is

$$\delta L = 2e^{-4S}\frac{\pi\bar{\pi}}{\sqrt{h}}\left(\delta\bar{\theta}\bar{D}(-2S) + \delta\theta D(-2S)\right) - \frac{2e^{-4S}}{\sqrt{h}}\left(\pi\dot{\bar{\theta}}\delta\bar{\theta} + \bar{\pi}\dot{\theta}\delta\theta\right)$$

$$-2\mu e^{-2S}(\delta\bar{\theta}\dot{\theta} + \delta\theta\dot{\bar{\theta}}) - 2\mu e^{-2S}\left(\pi\delta\bar{\theta}D(-2S) + \bar{\pi}\delta\theta\bar{D}(-2S)\right)$$

$$-\delta\sqrt{h}\left(e^{-4S}\frac{\pi\bar{\pi}}{h} - \mu^2\right) \tag{3.2.88}$$

where we have used (3.2.87). The Lagrangian is then kappa invariant iff

$$\delta_\kappa\theta = -\left(\kappa e^{2S} + \frac{\bar{\kappa}\pi}{\mu\sqrt{h}}\right), \qquad \delta_\kappa\bar{\theta} = -\left(\frac{\kappa\bar{\pi}}{\mu\sqrt{h}} + \bar{\kappa}e^{2S}\right)$$

$$\delta_\kappa\sqrt{h} = \frac{2\dot{\bar{\theta}}\kappa}{\mu} - \frac{2\kappa\bar{\pi}\bar{D}(-2S)}{\mu} + \frac{2\dot{\theta}\bar{\kappa}}{\mu} - \frac{2\bar{\kappa}\pi D(-2S)}{\mu}. \tag{3.2.89}$$

As in the bosonic model we wish to consider the massless limit of the superparticle. Taking $\mu = 0$ gives the Lagrangian

$$L = e^{-4S}\frac{\pi\bar{\pi}}{\sqrt{h}}. \tag{3.2.90}$$

The massless kappa transformations are obtained by setting $\kappa' \equiv \kappa/\mu$ and then letting $\mu \to 0$.

$$\delta_\kappa\theta = -\frac{\bar{\kappa}\pi}{\sqrt{h}}, \quad \delta_\kappa\bar{\theta} = -\frac{\kappa\bar{\pi}}{\sqrt{h}}, \quad \delta_\kappa z = -\theta\delta_\kappa\theta, \quad \delta_\kappa\bar{z} = -\bar{\theta}\delta_\kappa\bar{\theta}$$

$$\delta_\kappa\sqrt{h} = 2\dot{\bar{\theta}}\kappa - 2\kappa\bar{\pi}\bar{D}(-2S) + 2\dot{\theta}\bar{\kappa} - 2\bar{\kappa}\pi D(-2S). \tag{3.2.91}$$

The importance of this is that one can use the kappa transformations to pick a gauge in which $\dot{\theta} = \dot{\bar{\theta}} = 0$ so that $\theta(t) = \theta(a)$ and $\bar{\theta}(t) = \bar{\theta}(a)$ for all t. Gauge fixing θ_i and denoting the gauge fixed form of $-4S(z, \theta(b))$ as $\tilde{\phi}(z)$ we find that the Lagrangian reduces to

$$L_2' = \frac{1}{\sqrt{h}}e^{\tilde{\phi}}\dot{z}\dot{\bar{z}} \tag{3.2.92}$$

which is the Lagrangian of the free massless bosonic particle in conformal gauge. We cannot gauge fix the einbein in the same way as in the bosonic case as $\int dt\sqrt{h} - T$ is no longer kappa invariant. We will return to this point when evaluating functional integrals.

In the bosonic case we could consider calculating the expectation of $\delta(z_1 - z_2)$. The problem here is that $\mathbb{E}$ defined in (3.2.58) isn't kappa invariant so we cannot use gauge fixing to reduce the average to a bosonic calculation. We can, however, build kappa invariants from $\mathbb{E}$. Recall the functional $\tilde{n}$ defined by (3.1.15). Rewriting in terms of the variable $s = z - w$ we have

$$\tilde{n} = \int_{C_z, C_w} dt dt' \, (\dot{s}\bar{s}' - \dot{\bar{s}}s') \, \delta^2(s) \tag{3.2.93}$$

where $s = z(t) - w(t')$ and $\dot{s} \equiv \frac{d}{dt}s$ and $s' \equiv \frac{d}{dt'}s$. Note, in this form we can quite easily generalise $\tilde{n}$ to be supersymmetric by choosing a suitable s as discussed earlier. Under a general variation of C_z we have

$$\delta\tilde{n} = \int dt dt' \left(\big(\delta\dot{s}\ \bar{s}' - \delta\dot{\bar{s}}\ s'\big)\delta^2(s) + \big(\dot{s}\bar{s}' - \dot{\bar{s}}s'\big)\big(\delta s\ \partial + \delta\bar{s}\ \bar{\partial}\big)\delta^2(s) \right). \tag{3.2.94}$$

Integrating by parts on the first term gives

$$\delta\tilde{n} = \int dt dt' \left(\frac{d}{dt}\big(\delta s\ \bar{s}'\delta^2(s) - \delta\bar{s}\ s'\delta^2(s)\big) - \big(\delta s\ \dot{\bar{s}}' - \delta\bar{s}\ \dot{s}'\big)\delta^2(s) \right.$$
$$\left. - \frac{d}{dt'}\big(\delta s\ \dot{\bar{s}}\ \delta^2(s) - \delta\bar{s}\ \dot{s}\ \delta^2(s)\big) \right) \tag{3.2.95}$$

for $s = z - w$ the middle term vanishes and the last term is a total derivative. The variation of this quantity then depends only on the end points of the two curves. In the next section we will show the reason for this. For now, if we instead use $l = z_1 - z_2 - \theta_1\theta_2$ in (3.1.15), then the middle term doesn't vanish and the last term is not a total derivative. The variation of $\tilde{n}$ in this case is therefore not so simple. Also, inserting the κ transformations shows that $\tilde{n}$ is not a kappa invariant. If instead we use $l' = z_1 - z_2 + \theta_1^0\theta_1 - \theta_2^0\theta_2 - \theta_1^0\theta_2^0$ in $\tilde{n}$, where θ_i^0 are the end point values of θ_i, then we see that again the variation depends only on the endpoints and as $\delta_\kappa l' = 0$, we see that it is also kappa invariant. Importantly, l' is also supersymmetric. This is then the functional we wish to consider.

3.2.4 Supersymmetric average

We can now compute the average of the supersymmetric "line of force" on a curved supermanifold [47]. To this we should add Faddeev-Popov terms associated with the fixing of the reparametrisation invariance and kappa symmetry. The former are the same as in the bosonic case, whilst for the kappa symmetry they take the form

$$\bar{\lambda}\left(\theta(b)-\theta\right)+\lambda\left(\bar{\theta}(b)-\bar{\theta}\right)+\bar{B}\frac{C\bar{\pi}}{\sqrt{g}}+B\frac{\bar{C}\pi}{\sqrt{g}}\,. \tag{3.2.96}$$

λ acts is a Lagrangian multiplier imposing the gauge condition and the ghosts B and C generate the Faddeev-Popov determinant of a local quantity (as opposed to a differential operator) which can be ignored.

The observables that we work with should be BRST invariant. For example, in the bosonic case we focus on long curves by inserting $\delta(\int_0^1\sqrt{g}\,dt-T)$ into the functional integral. This is reparametrisation invariant, which is sufficient in the bosonic case, but it is not κ invariant which we also need in the supersymmetric case. However, the kappa variation of

$$\sqrt{h}\left(1-(\bar{\theta}-\bar{\theta}^0)\bar{D}(-2S)-(\theta-\theta^0)D(-2S)\right)\equiv\zeta \tag{3.2.97}$$

is zero when we impose the gauge conditions so we can use this to make a BRST invariant insertion. Actually, imposing the kappa gauge conditions reduces this to the bosonic condition and so imposing all of the gauge conditions will reduce the reparametrisation fixing delta function in the functional integral to the bosonic delta function. The average of some supersymmetric and κ invariant functional, $\Omega[C_1]$, over C_1 is found by computing the functional integral

$$\langle\Omega\rangle_{C_1}\equiv\frac{1}{Z}\int\mathcal{D}g\mathcal{D}z_1\mathcal{D}\theta_1\mathcal{D}\lambda\mathcal{D}B\mathcal{D}C\;\delta\left(\int_0^1\zeta dt-T\right)\,\Omega[C_1]\;e^{-S_{FP}[\sqrt{h},z_1,\theta_1,\lambda,B,C]}\,, \tag{3.2.98}$$

where S_{FP} is the gauge fixing action including the superparticle action and Faddeev-Popov terms. The gauge conditions reduce the functional integral to

$$\langle\Omega\rangle_{C_1}=\frac{1}{Z}\int\mathcal{D}z_1\;\;\Omega'[C_1]\;e^{-S'[z_1]} \tag{3.2.99}$$

with

$$S'[z] = \frac{1}{2}\int_0^T dt\ e^{\tilde{\phi}}\dot{z}\dot{\bar{z}} \tag{3.2.100}$$

and Ω' is the gauge fixed form of Ω. (3.2.99) is then equivalent to the bosonic functional integral considered before. There is a slight problem with our symmetric l'. Inserting it into (3.2.58), we see that the electric field no longer satisfies Gauss' law. This is not too much of a problem as the quantity of interest will be $\tilde{n}$.

The average we calculate will then be the supersymmetric and kappa invariant line of force

$$\left\langle \int_{C_1} dl_1'\ \delta^2(l') \right\rangle_{C_1} = \frac{1}{Z}\int \mathcal{D}[z_1] \left(\int_{C_1} dz_1\ \delta^2(z_1 - z_2 - \theta_1^0\theta_2^0) \right) e^{-\frac{1}{2}\int_0^T dt\ e^{\tilde{\phi}}\dot{z}_1\dot{\bar{z}}_1}. \tag{3.2.101}$$

This represents a supersymmetric generalisation of the bosonic "lines of force" rather than a generalisation of the electrostatic field.

The Faddeev-Popov terms in the action do not take part in the integral and so can be absorbed into the normalisation constant. Now, if we define $z_2' \equiv z_2 + \theta_1^0\theta_2^0$ then this average just reduces to the bosonic average. We shall, therefore, revert back to general coordinates, x^μ, to mirror the bosonic derivation, pausing to explain a few important points. The average to compute now is then

$$\left\langle \int_{C_1} \delta^2(x_1 - x_2')dx_1^\mu \right\rangle_{C_1} =$$

$$\left\{ \frac{\delta}{\delta\mathcal{A}_\mu(x_2')}\frac{1}{Z}\int \mathcal{D}x_1\ e^{-\int_0^T \left(\frac{1}{2}g_{\mu\nu}(x_1)\,\dot{x}_1^\mu\,\dot{x}_1^\nu - \mathcal{A}_\mu\dot{x}_1^\mu\right)dt} \right\}\Big|_{\mathcal{A}=0}. \tag{3.2.102}$$

By analogy with the bosonic case this becomes

$$= \frac{1}{2}\int_0^\infty dt\left(-\sqrt{g}g^{\mu\nu}\partial_\nu\mathcal{G}_t'^0(b_1, x_2') + \sqrt{g}g^{\mu\nu}\partial_\nu\mathcal{G}_{T-t}'^0(x_2', a_1)) \right) \tag{3.2.103}$$

after fixing the normalisation constant and taking the $T \to \infty$ limit. Now, $\mathcal{G}_t'^0(x_1, x_2')$ is not equivalent to the bosonic heat kernel used in the previous section due to subtleties involving the θ coordinates. In this case, we have

$$\mathcal{G}_T'^0(x_1, x_2') = \langle\, x_1 |\, e^{-T\hat{H}_0}\,|\, x_2'\,\rangle + \langle\, x_1\,|\, e^{-T\hat{H}_0}\,|\, x_2'^R\,\rangle. \tag{3.2.104}$$

We are still considering specular reflections of the curves when they reach the boundary but in this case the reflected coordinate of the i'th curve is (x_i^R, θ_i^R) and so $x_2'^R = x_2^R + \theta_1^0\theta_2^{0R}$. The integral of $\mathcal{G}_T'^0$ over t results in a generalisation of the Green's function discussed earlier, denoted by G':

$$\int_0^\infty dt \left(\mathcal{G}_T'(z_1, z_2') - \frac{1}{A}\right) = 2G'(z_1, z_2'). \tag{3.2.105}$$

It satisfies

$$-4e^{-\tilde{\phi}}\bar{\partial}\partial G' = 2e^{-\tilde{\phi}}\delta^2(z_1 - z_2 - \theta_1^0\theta_2^0) - \frac{1}{A} \tag{3.2.106}$$

and modified Neumann conditions

$$(\partial_i - \bar{\partial}_i)G'(z_i, z_j, \theta_i, \theta_j)|_{z_i=\bar{z}_i, \theta_i=\bar{\theta}_i} = 0. \tag{3.2.107}$$

The solution to (3.2.106) satisfying (3.2.107) is then

$$G' = -\frac{1}{2\pi}\log(|z_1 - z_2 - \theta_1^0\theta_2^0|) - \frac{1}{2\pi}\log(|z_1 - \bar{z}_2 - \theta_1^0\bar{\theta}_2^0|) - \Psi(z_1, z_2, \theta_1^0, \theta_2^0) \tag{3.2.108}$$

where Ψ solves

$$-4e^{\tilde{\phi}(z_1,\theta_1^0)}\bar{\partial}_1\partial_1\Psi = -4e^{\tilde{\phi}(z_2,\theta_2^0)}\bar{\partial}_2\partial_2\Psi = \frac{1}{A} \tag{3.2.109}$$

and the modified Neumann conditions. G' actually solves the Green's function equation for the super-Laplacian, $\Delta_F \equiv -4\bar{D}D$. When θ_1 and θ_2 are dynamical we believed the spectral decomposition of the supersymmetric heat kernel was needed to compute the functional integral. With the gauge fixing described above, the problem reduced to the bosonic calculation and so there was no need for this. Appendix A gives the derivation of the spectral decomposition of the supersymmetric heat kernel. Using the results above we find

$$\left\langle \int_{C_1} \delta^2(z_1 - z_2')dz_1 \right\rangle_{C_1} = \frac{\partial G'(b_1, z_2')}{\partial \bar{z}_2} - \frac{\partial G'(z_2', a_1)}{\partial \bar{z}_2}. \tag{3.2.110}$$

From (3.2.109) it is clear that Ψ can be decomposed as $\Psi(z_1, z_2, \theta_1, \theta_2) = B(z_1, \theta_1) + B(z_2, \theta_2)$. Because of this, (3.2.110) is independent of the zero mode contribution to the Green function. Using (3.2.108) we find

$$\left\langle \int_{C_1} \delta^2(z_1 - z_2 - \theta_1^0\theta_2^0)\ dz_1 \right\rangle_{C_1} =$$

$$\frac{1}{4\pi}\left(\frac{1}{\bar{b}_1 - \bar{z}_2 - \bar{\theta}_1^0\bar{\theta}_2^0} + \frac{1}{b_1 - \bar{z}_2 - \theta_1^0\theta_2^0}\right) - \frac{1}{4\pi}\left(\frac{1}{\bar{a}_1 - \bar{z}_2 - \bar{\theta}_1^0\bar{\theta}_2^0} + \frac{1}{a_1 - \bar{z}_2 - \theta_1^0\bar{\theta}_2^0}\right). \tag{3.2.111}$$

Similarly, we find

$$\left\langle \int_{C_1} \delta^2(z_1 - z_2 - \theta_1^0\theta_2^0)\ d\bar{z}_1 \right\rangle_{C_1} =$$

$$\frac{1}{4\pi}\left(\frac{1}{b_1 - z_2 - \theta_1^0\theta_2^0} + \frac{1}{\bar{b}_1 - z_2 - \bar{\theta}_1^0\bar{\theta}_2^0}\right) - \frac{1}{4\pi}\left(\frac{1}{a_1 - z_2 - \theta_1^0\theta_2^0} + \frac{1}{\bar{a}_1 - z_2 - \bar{\theta}_1^0\bar{\theta}_2^0}\right). \tag{3.2.112}$$

This turns out to be a straight forward analogue of the bosonic result. This is a good job as the supersymmetric ψ theory used to implement path-ordering has propagator $\left\langle \psi^\dagger(z_1, \theta_1)\psi(z_2, \theta_2)\right\rangle \sim \mathrm{sign}(z_1 - z_2 - \theta_1\theta_2)$. If we can obtain the bosonic propagator from the quantity n then we should similarly be able to obtain the fermionic propagator from the supersymmetric generalisation.

We are now finally in a position to obtain this result for the bosonic and fermionic theories.

3.3 The intersection of random curves

Our aim is find a field theory with the required dynamics for J^A. We have seen hints of the kinds of properties we need. Let us return to the bosonic result (3.1.46):

$$\left\langle \int_{C_1} \delta^2(z_1 - z_2) dz_1 \right\rangle_{C_1} = \frac{1}{4\pi}\left(\frac{1}{\bar{b}_1 - \bar{z}_2} + \frac{1}{b_1 - \bar{z}_2} - \frac{1}{\bar{a}_1 - \bar{z}_2} - \frac{1}{a_1 - \bar{z}_2}\right) \tag{3.3.113}$$

along with the conjugate result (3.1.47)

$$\left\langle \int_{C_1} \delta^2(z_1 - z_2) d\bar{z}_1 \right\rangle_{C_1} = \frac{1}{4\pi}\left(\frac{1}{b_1 - z_2} + \frac{1}{\bar{b}_1 - z_2} - \frac{1}{a_1 - z_2} - \frac{1}{\bar{a}_1 - z_2}\right). \tag{3.3.114}$$

Consider associating z_2 to the coordinates of a second curve, C_2. Integrating the first result over $\bar{z}_2$ and the second result over z_2 in the combination

$$\int_{C_2} \left\langle \int_{C_1} \delta^2(z_1 - z_2) dz_1 \right\rangle_{C_1} d\bar{z}_2 - \int_{C_2} \left\langle \int_{C_1} \delta^2(z_1 - z_2) d\bar{z}_1 \right\rangle_{C_1} dz_2$$

$$= \frac{1}{4\pi} \left(\log_{C_2} \left(\frac{b_1 - b_2}{\bar{b}_1 - \bar{b}_2} \right) - \log_{C_2} \left(\frac{b_1 - a_2}{\bar{b}_1 - \bar{a}_2} \right) + ... \right). \tag{3.3.115}$$

These integrations give a complex logarithm. The complex logarithm is multivalued and so requires a branch cut in the complex plane to be well defined. On the worldsheet we have no preferred direction; the only directional option we have after averaging over the first curve is the second curve C_2. We, thus, choose to cut the logarithm along the second curve, which explains the subscript in (3.3.115). We will return to this point when we consider averaging over the second curve.

Notice that this result has the form of an angle that C_2 makes between the points b_1 and b_2 and b_1 and a_2 respectively since

$$\log\left(\frac{z}{|z|}\right) = 2i \ \arg(z). \tag{3.3.116}$$

This is the right kind of thing we require for the propagator of the Lie algebra variables in the contact interaction. We, therefore, propose that the object and its averages that we should study is:

$$n[C_1, C_2] = -i \int_{C_2} \int_{C_1} \ \delta^2(z_1 - z_2) \ (dz_1 d\bar{z}_2 - d\bar{z}_1 dz_2). \tag{3.3.117}$$

This functional actually counts the number of intersections of the two curves C_1 and C_2. It was originally found when we looked for κ invariant functionals in the supersymmetric theory. Its antisymmetry is emphasised if we use tensor notation

$$n[C_1, C_2] = \int_{C_2} \int_{C_1} \ \delta^2(x_1 - x_2) \ \epsilon_{ab} dx_1^a dx_2^b. \tag{3.3.118}$$

In contrast to the previous section, we now have a double integral to worry about. For naturalness, it therefore makes sense to average over both curves. Taking into account the skew symmetry of $n[C_1,\, C_2]$ under interchange of C_1 and C_2 its expectation value will be shown to depend on b_1 and b_2 as

$$\langle n[C_1,\, C_2]\rangle_{C_1,C_2} = k\frac{(b_1 - b_2)}{|b_1 - b_2|} = k\ \mathrm{sign}(b_1 - b_2) \qquad (3.3.119)$$

with constant k. This function can be used to implement path-ordering along the boundary, but also by taking the ends of the curve to move into the interior of the world-sheet we would obtain an extension of path-ordering into the body of the world-sheet. Above, we have essentially calculated the expectation value over one curve. We will now show how one can obtain (3.3.119) by integrating over the second.

We mentioned earlier that the only directional object on the worldsheet after averaging over the first curve is the second curve and so chose to cut the logarithms along C_2. The trouble with this is that we want to average over C_2. To avoid issues with averaging over functions cut along the curve we are averaging over, we can instead express these as integrals cut along a fixed reference curve, C_2^*, from a_2 to b_2 plus $2\pi i$ multiples of the winding number about the the points b_1 and a of the closed curve made up of C_2 and C_2^* reversed. The winding numbers can then be

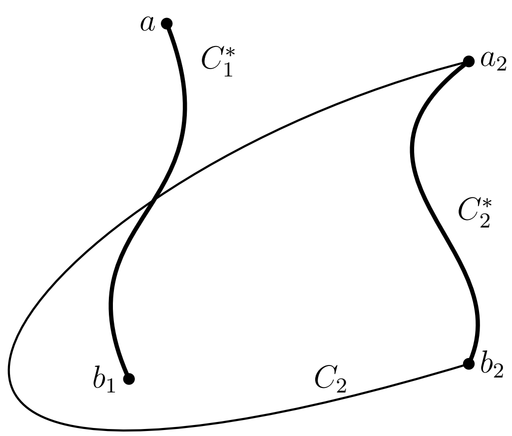


Figure 3.4: A possible configuration of the curves C_1^*, C_2 and C_2^* illustrating (3.3.120). This diagram was produced by Prof. Mansfield and is found in [50].

written in terms of the number of intersections of C_2 and C_2^* with a reference curve

C_1^* from b_1 to a, so

$$\langle\, n[C_1,\, C_2]\,\rangle_{C_1} =$$
$$-\frac{1}{2\pi}\Im\log_{C_2^*}\left(\frac{(b_1-b_2)(\bar{b}_1-b_2)}{(b_1-a_2)(\bar{b}_1-a_2)}\right)+\frac{1}{2}\Im\log_{C_2^*}\left(\frac{(a-b_2)(\bar{a}-b_2)}{(a-a_2)(\bar{a}-a_2)}\right)$$
$$-\left(n[C_2,C_1^*]-n[C_2^*,C_1^*]\right). \tag{3.3.120}$$

$\Im\log$ denotes the imaginary part of the logarithm and the subscript denotes that the logarithms, viewed as functions of b_1, a and their complex conjugates are cut along C_2^*. The only dependence on C_2 is via $n[C_2, C_1^*]$ and a_2 so if we now average over C_2 using

$$\langle\, n[C_2,\, C_1^*]\,\rangle_{C_2} = \tag{3.3.121}$$
$$-\frac{1}{2\pi}\Im\log_{C_1^*}\left(\frac{(b_2-b_1)(\bar{b}_2-b_1)}{(b_2-a)(\bar{b}_2-a)}\right)+\Im\log_{C_1^*}\left(\frac{(a_2-b_1)(\bar{a}_2-b_1)}{(a_2-a)(\bar{a}_2-a)}\right) \tag{3.3.122}$$

where now the subscript denotes that the logarithms, viewed as functions of b_2, a_2 and their complex conjugates, are cut along C_1^*.

Observe that the following difference in logarithms cut along C_1^* and C_2^* is proportional to the number of times C_1^* and C_2^* intersect:

$$\log_{C_2^*}\left(\frac{(b_1-b_2)(a-a_2)}{(b_1-a_2)(a-b_2)}\right)-\log_{C_1^*}\left(\frac{(b_2-b_1)(a_2-a)}{(a_2-b_1)(b_2-a)}\right)=2\pi i\; n[C_2^*,C_1^*]\,. \tag{3.3.123}$$

This is illustrated in Figure 3.5. The angle swept out by the line from z to b_1 as

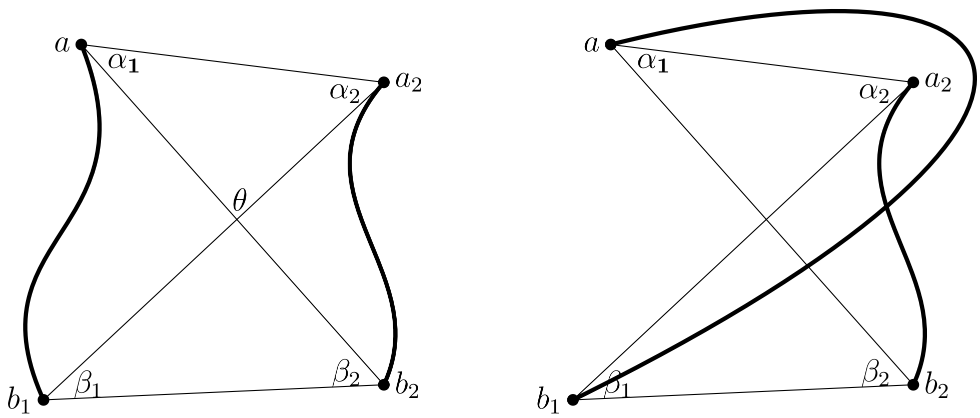


Figure 3.5: Two configurations of the curves C_1^* and C_2^*. This diagram was produced by Prof. Mansfield and is found in [50].

z moves along C_2^* from a_2 to b_2 is the imaginary part of $\log_{C_2^*}((b_1-b_2)/(b_1-a_2))$ which is $-\beta_1$ in both figures. Similarly the angle swept out by the line from z to a is the imaginary part of $\log_{C_2^*}((a-b_2)/(a-a_2))$ which is $-\alpha_1$ in both figures. The imaginary part of $\log_{C_1^*}((b_2-b_1)(a_2-a)/((b_2-a)(a_2-b_1)))$ is the difference in the angles swept out by the lines from z to b_2 and from z to a_2 as z moves along C_1^* from a to b_1. For the left hand figure, in which the curves C_1^* and C_2^* do not intersect, this is $\beta_2-\alpha_2$. In the right hand figure the line from z to a_2 sweeps out $-(2\pi-\alpha_2)$ so the difference in the two angles is $\beta_2+(2\pi-\alpha_2)$. Also, in the right hand figure the curves C_1^* and C_2^* intersect with $n[C_2^*, C_1^*]=-1$, so for the two figures (3.3.123) is

$$\alpha_1-\beta_1-(\beta_2-\alpha_2)=0, \quad \text{and} \quad \alpha_1-\beta_1-(\beta_2+2\pi-\alpha_2)=-2\pi\,, \tag{3.3.124}$$

both of which hold because α_1, α_2 and θ are the angles of the top triangle in the figure and β_1, β_2 and θ are the angles in the lower triangle. I must thank Prof. Mansfield for help with computing this second average.

Using (3.3.123), we are just left with

$$\langle\,\langle\, n[C_1,\, C_2]\,\rangle_{C_1}\,\rangle_{C_2} = -\Im\left(\log_{C_2^*}\frac{(\bar b_1-b_2)(\bar a-a_2)}{(\bar a-b_2)(\bar b_1-a_2)} - \log_{C_1^*}\frac{(\bar b_2-b_1)(\bar a_2-a)}{(\bar b_2-a)(\bar a_2-b_1)}\right). \tag{3.3.125}$$

To obtain (3.3.119) we must integrate over one end point of each curve. Integrating over a and a_2 (to obtain a result in terms of b_1 and b_2) we find

$$\int d^2a d^2a_2\;\frac{e^{\phi(a)+\phi(a_2)}}{8\pi A^2}\;\langle\,\langle\, n[C_1,\, C_2]\,\rangle_{C_1}\,\rangle_{C_2} =$$

$$-\Im\int_\Sigma\left(\log_{C_2^*}\frac{(\bar b_1-b_2)(\bar a-a_2)}{(\bar a-b_2)(\bar b_1-a_2)} - \log_{C_1^*}\frac{(\bar b_2-b_1)(\bar a_2-a)}{(\bar b_2-a)(\bar a_2-b_1)}\right)\frac{e^{\phi(a)+\phi(a_2)}}{8\pi A^2}\,d^2a\,d^2a_2\,. \tag{3.3.126}$$

We can now interpret this expression in the light of the comments relating to path-ordering along the boundary using (3.3.119). Let b_1 and b_2 approach the real axis so $b_1=x_1+i\epsilon_1$ and $b_2=x_2+i\epsilon_2$ with x_1 and x_2 real, and denote by $G(x_1,x_2)$ the resulting value of $\langle\, \langle\, n[C_1,\, C_2]\, \rangle_{C_1}\, \rangle_{C_2}$ then

$$\log_{C_2^*}(\bar{b}_1 - b_2) - \log_{C_1^*}(\bar{b}_2 - b_1) = i\pi \frac{x_1 - x_2}{|x_1 - x_2|}\,. \tag{3.3.127}$$

As this is independent of a and a_2, the area integrals in (3.3.126) can be done to give

$$G(x_1, x_2) = -\frac{x_1 - x_2}{2|x_1 - x_2|} + F(x_1) - F(x_2)\,, \tag{3.3.128}$$

which is (3.3.119) apart from the function F. To interpret F, differentiate with respect to x_1

$$\frac{\partial}{\partial x_1} G(x_1, x_2) = -\delta(x_1 - x_2) + F'(x_1). \tag{3.3.129}$$

The real axis parametrises the boundary of Σ which has finite length and the co-ordinates $x = \pm\infty$ describe the same point on this boundary so for consistency we should have

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial x_1} \langle\, \langle\, n[C_1,\, C_2]\, \rangle_{C_1}\, \rangle_{C_2}\, dx_1 = -1 + \int_{-\infty}^{\infty} F'(x_1)\, dx_1\,, \tag{3.3.130}$$

but from (3.3.126)

$$F'(x_1) = \Im \int_\Sigma \left(\frac{1}{x_1 - a_2} - \frac{1}{x_1 - \bar{a}_2} \right) \frac{e^{\phi(a_2)}}{4\pi A}\, d^2 a_2 \tag{3.3.131}$$

which does indeed integrate to $+1$. Now (3.3.129) is a Green's function equation for $\partial/\partial x$ on a closed loop, i.e. the propagator for a one-dimensional field ψ with action $\int dx\ \psi^\dagger \psi'$.

3.3.1 Path-ordering of the Wilson loop

The field theory, ψ, with propagator $\left\langle \psi_1^\dagger \psi_2 \right\rangle \sim \mathrm{sign}(x_1 - x_2)$ has been used to represent path-ordering around the loop in [35]- [40]. Since $G(x_1, x_2)$ is just the boundary value of the average of the intersection number we have a natural way of extending path-ordering into the interior of Σ. This extension coincides with the propagator of the topological field theory constructed in [41] for just this purpose. To see this connection note that in Broda's model the boundary field ψ is assumed to be the boundary value of a bulk field and the extension into the bulk can be done arbitrarily giving rise to a topological field theory with invariance $\delta\psi = \theta$ with θ being any function vanishing on the boundary. This invariance is gauge-fixed by requiring ψ to be harmonic. Just as in the topological theory the average intersection number satisfies Laplace's equation in the bulk because it is non-singular as b_1 approaches b_2 in the interior.

In the string theory, after averaging over worldsheets spanning the fixed boundary, B, we are left with (2.3.94). We can then use the function (3.3.128) defined on the boundary as the propagator for ψ

$$\psi^\dagger_\alpha(b_1)\psi_\beta(b_2) = \delta_{\alpha\beta}\int d^2a d^2a_2\,\frac{e^{\phi(a)+\phi(a_2)}}{8\pi A^2}\langle\,\langle\, n[C_1,\,C_2]\,\rangle_{C_1}\,\rangle_{C_2} = \delta_{\alpha\beta}G(b_1,b_2). \tag{3.3.132}$$

With this, the string expectation becomes (2.3.80) and so provides a way to introduce path-ordering into the string theory. We note that this model lacks the necessary singularities required to produce the three gluon vertex. We will come back to this point after calculating the average number of intersections of curves on a supermanifold.

3.3.2 Intersection of curves on a supermanifold

We will now turn to the supersymmetric generalisation of the calculation given above. As we have shown when looking at supersymmetric electrostatics, we must modify the displacement such that it is both supersymmetric and kappa invariant. This leads us to the number of intersections of supersymmetric curves taking the form

$$n_F[C_1,C_2] = i\int_{C_1,C_2} (\pi_1^0\bar{\pi}_2^0 - \bar{\pi}_1^0\pi_2^0)\;\delta^2(l')\;d\xi_1 d\xi_2 \tag{3.3.133}$$

with $\pi_i^0 \equiv \dot{z}_i + \theta_i^0\dot{\theta}_i$. To begin with, we will consider averaging this functional over both curves, keeping all four end points fixed. We will then consider averaging over one of the end points of each curve using a gauge fixed volume element. This will lead to the supersymmetric analogue of the bosonic result above and so can be used to implement path-ordering into the supersymmetric string theory.

Using (3.2.111), (3.2.112) and (3.3.120) we can conclude that the number of intersections of two curves, C_1 and C_2, on a supermanifold, averaged over C_1 is

$$\langle n_F[C_1,C_2]\rangle_{C_1} = -\frac{1}{2\pi}\Im\mathrm{log}_{2^*}\left(\frac{(b_1-b_2-\theta_1^0\theta_2^0)(\bar b_1-b_2-\bar\theta_1^0\theta_2^0)}{(b_1-a_2-\theta_1^0\theta_2^0)(\bar b_1-a_2-\bar\theta_1^0\theta_2^0)}\right)$$

$$+\frac{1}{2\pi}\Im\mathrm{log}_{2^*}\left(\frac{(a_1-b_2-\theta_1^0\theta_2^0)(\bar a_1-b_2-\bar\theta_1^0\theta_2^0)}{(a_1-a_2-\theta_1^0\theta_2^0)(\bar a_1-a_2-\bar\theta_1^0\theta_2^0)}\right)-\big(n[C_2,C_1^*]-n[C_2^*,C_1^*]\big). \quad (3.3.134)$$

The number of intersections of C_2^* and C_1^* is a straight forward generalisation of the bosonic case and can be obtained by shifting x_2 as before, we have

$$n[C_2^*,C_1^*] = \frac{1}{2\pi}\Im\mathrm{log}_{2*}\left(\frac{(b_1-b_2-\theta_1^0\theta_2^0)(a-a_2-\theta_1^0\theta_2^0)}{(b_1-a_2-\theta_1^0\theta_2^0)(a-b_2-\bar\theta_1^0\theta_2^0)}\right)$$

$$-\frac{1}{2\pi}\Im\mathrm{log}_{1*}\left(\frac{(b_1-b_2-\theta_1^0\theta_2^0)(a-a_2-\theta_1^0\theta_2^0)}{(b_1-a_2-\theta_1^0\theta_2^0)(a-b_2-\theta_1^0\theta_2^0)}\right). \quad (3.3.135)$$

We also have

$$\langle n_F[C_2,C_1^*]\rangle_{C_2} = -\frac{1}{2\pi}\Im\mathrm{log}_{1*}\left(\frac{(b_1-b_2-\theta_1^0\theta_2^0)(b_1-\bar b_2-\theta_1^0\bar\theta_2^0)}{(a-b_2-\theta_1^0\theta_2^0)(a-\bar b_2-\theta_1^0\bar\theta_2^0)}\right)$$

$$+\frac{1}{2\pi}\Im\mathrm{log}_{1*}\left(\frac{(b_1-a_2-\theta_1^0\theta_2^0)(b_1-\bar a_2-\theta_1^0\bar\theta_2^0)}{(a-a_2-\theta_1^0\theta_2^0)(a-\bar a_2-\theta_1^0\bar\theta_2^0)}\right). \quad (3.3.136)$$

Using these results we find

$$\big\langle\langle n_F[C_1,C_2]\rangle_{C_1}\big\rangle_{C_2} = \frac{1}{2\pi}\Im\mathrm{log}_{1*}\left(\frac{(b_1-\bar b_2-\theta_1^0\bar\theta_2^0)(a_1-\bar a_2-\theta_1^0\bar\theta_2^0)}{(a_1-\bar b_2-\theta_1^0\bar\theta_2^0)(b_1-\bar a_2-\theta_1^0\bar\theta_2^0)}\right)$$

$$-\frac{1}{2\pi}\Im\mathrm{log}_{2*}\left(\frac{(\bar b_1-b_2-\bar\theta_1^0\theta_2^0)(\bar a_1-a_2-\bar\theta_1^0\theta_2^0)}{(\bar b_1-a_2-\bar\theta_1^0\theta_2^0)(\bar a_1-b_2-\bar\theta_1^0\theta_2^0)}\right). \quad (3.3.137)$$

We can now consider integrating over one of the end points. This is not as straight forward as in the bosonic case as the invariant volume element on the supermanifold is $\mathrm{sdet}(\mathcal{E})\ d^2ad^2\theta_a$. There are a couple of problems with this. The first is that we mentioned earlier that we would like the result to be dependent on the θ coordinates, but, integrating with this measure will remove all this dependence since we have gauge fixed θ to take the same value along the curves. The second problem is that $\mathrm{sdet}(\mathcal{E}) = e^{-2S} = e^{\tilde\phi/2}$ with an extra factor of $1/2$ appearing compared to the bosonic case. This is because we are taking into account the θ coordinates. What we really need is the reduced volume element corresponding to just the bosonic coordinates. We can obtain this by writing down the line element and imposing the gauge conditions

$$ds^2 = \eta_{AB}\mathcal{E}_M^{\ A}dz^M\mathcal{E}_N^{\ B}dz^N = \mathcal{E}_z^{\ \bar{z}}\mathcal{E}_{\bar{z}}^{\ z}dzd\bar{z} + \frac{1}{2}\{\mathcal{E}_z^{\ \theta}, \mathcal{E}_{\bar{z}}^{\ \bar{\theta}}\}dzd\bar{z}$$
$$= e^{\tilde{\phi}}dzd\bar{z}\,, \qquad (3.3.138)$$

therefore, the invariant volume element for the i'th end point is $\sqrt{g}\ d^2a_i = e^{\tilde{\phi}}d^2a_i$. We can now integrate over a_1 and a_2 and let (b_1, θ_1) and (b_2, θ_2) approach the boundary. Call the result of these actions $G_F(x_1, \theta_1; x_2, \theta_2)$ so that

$$G_F = -\frac{(x_1 - x_2 - \theta_1\theta_2)}{2|x_1 - x_2 - \theta_1\theta_2|} + \tilde{F}(x_1, x_2, \theta_1\theta_2). \qquad (3.3.139)$$

The first term generalises the step function to superspace. Differentiating with respect to (x_1, θ_1) gives

$$D_1G_F = -(\theta_1 - \theta_2)\delta(x_1 - x_2) + D_1\tilde{F}. \qquad (3.3.140)$$

Integrating along the boundary requires

$$0 = \int_{-\infty}^{+\infty} dx_1 \int d\theta_1\ D_1G_F = -1 + \int_{-\infty}^{+\infty} dx_1 \int d\theta_1\ \theta_1 \frac{\partial}{\partial x_1}\tilde{F}. \qquad (3.3.141)$$

The integral on the RHS is exactly the same as in the bosonic case after integrating out θ_1 and so this does hold. (3.3.139) is then the Green's function equation for $D = \partial/\partial\theta + \theta\partial/\partial x$ on a closed loop. This is a suitable supersymmetric generalisation of the bosonic case that one can use to introduce path-ordering into the interior of the spinning string model.

3.3.3 Three point vertex?

As alluded to earlier, this model on the worldsheet does not include the necessary structure required to produce the three gluon vertex. The reason for this is that we are forced to cut the logarithms along C_2 which we then average over. To do this calculation we replaced the average over C_2 with the average over a reference curve, C_2^*, plus a winding number. Integrating over this winding number cancelled out the terms that would give rise to the singularities we seek. Had our theory had some notion of direction built in we would not have to average over the second branch cut and would find that the required structure does in fact appear. We will show this in the bosonic model, though it similarly exists in the supersymmetric model. Imagine that we have some notion of direction already in our model, say perpendicular to the x axis for simplicity. Then the average number of intersections between two curves over C_1 is

$$\langle\, n[C_1,\, C_2]\,\rangle_{C_1} = -\frac{1}{2\pi}\Im\log_y\left(\frac{(b_1-b_2)(\bar{b}_1-b_2)}{(b_1-a_2)(\bar{b}_1-a_2)}\right)+\frac{1}{2}\Im\log_y\left(\frac{(a-b_2)(\bar{a}-b_2)}{(a-a_2)(\bar{a}-a_2)}\right). \tag{3.3.142}$$

Noting that

$$\Im\,\log(b_1-b_2)=\arg(b_1-b_2)=\frac{1}{2i}\log\left(\frac{b_1-b_2}{\bar{b}_1-\bar{b}_2}\right), \tag{3.3.143}$$

then differentiating this with respect to $\bar{b}_2$ we find

$$\frac{\partial}{\partial\bar{b}_2}\langle\, n[C_1,\, C_2]\,\rangle_{C_1}=\frac{i}{4\pi}\left(\frac{1}{\bar{b}_1-\bar{b}_2}+\frac{1}{b_1-\bar{b}_2}-\frac{1}{\bar{a}_1-\bar{b}_2}-\frac{1}{a_1-\bar{b}_2}\right). \tag{3.3.144}$$

Averaging this result over C_2 will then have no effect when the end points are held fixed. Differentiating with respect to b_1 gives

$$\frac{\partial}{\partial b_1}\left\langle\frac{\partial}{\partial\bar{b}_2}\,\langle n[C_1,C_2]\rangle_{C_1}\right\rangle_{C_2}=\frac{i}{2}\delta_c(b_1-b_2)-\frac{i}{4\pi}\frac{1}{(b_1-\bar{b}_2)^2}. \tag{3.3.145}$$

Similarly, one finds

$$\frac{\partial}{\partial\bar{b}_1}\langle\,\frac{\partial}{\partial b_2}\langle\, n[C_1,C_2]\,\rangle_{C_2}\,\rangle_{C_1}=-\frac{i}{2}\delta_c(b_1-b_2)+\frac{i}{4\pi}\frac{1}{(\bar{b}_1-b_2)^2}. \tag{3.3.146}$$

The difference in sign of the first term on the right hand side of these two equations is exactly what we require to produce the three gluon vertex (2.3.97). To show this we need to look at the contraction terms, C, in the string theory. The relevant term is

$$C^{A}_{1\mu\nu\ k}C^{\lambda\sigma B}_{2k'} =$$

$$\int d^2\xi_1 d^2\xi_2\ \partial_a J^A(X_1)\partial_c J^B(X_2)\frac{i\varepsilon^{ab}}{k^2}k_{[\mu}\partial_b\mathbb{P}_k(X_1)_{\nu]}\frac{i\varepsilon^{cd}}{k^2}k^{[\lambda}\partial_d\mathbb{P}_k(X_2)^{\sigma]}e^{ik\cdot X_1+ik'\cdot X_2}. \tag{3.3.147}$$

where $J^A = \psi^\dagger\tau^A\psi$. The contractions we need are

$$\underbrace{\partial_a J^A\partial_c}J^B = \psi^\dagger\tau^A\underbrace{\partial_a\psi\ \partial_c}\psi^\dagger\tau^B\psi + \psi^\dagger\tau^B\underbrace{\partial_c\psi\ \partial_a}\psi^\dagger\tau^A\psi. \tag{3.3.148}$$

These contractions can then be evaluated using the above

$$\underbrace{\partial_a\psi\ \partial_c}\psi^\dagger = \partial_a\partial_c\ \left\langle\langle n[C_1,C_2]\rangle_{C_2}\right\rangle_{C_1} = \frac{1}{2}\varepsilon_{ac}\delta^2(\xi_1-\xi_2)+... \tag{3.3.149}$$

where the dots represent the extra terms appearing in (3.3.145) and (3.3.146). A little algebra leads finally to

$$\underbrace{\partial_a J^A\partial_c}J^B = \varepsilon_{ab}f^{ABC}\delta^2(\xi_1-\xi_2)\ J^C \tag{3.3.150}$$

which is precisely (2.3.97). It would remain for us to show that the extra terms (3.3.145) and (3.3.146) do not contribute to the string functional integral. We will not do this here as we have a different method of generalising the string theory to include non-abelian gauge theories that includes path-ordering and self interactions of the gauge bosons that is applicable to our worldsheet model. Without a natural notion of direction on the surface, so that we do not have to integrate over a reference curve, we believe that it may be impossible to find a function, f, that satisfies

$$\partial_a\partial_b f(z_1,z_2) = \epsilon_{ab}\delta^2_c(z_1-z_2). \tag{3.3.151}$$

The next model we look at satisfies such a relation at the level of the quantum expectation, $\left\langle\partial_a J^A\partial_b J^B\right\rangle = \epsilon_{ab}f^{ABC}\ J^C\ \delta^2_c(z_1-z_2)$.
We note that there exists an analogous result within the supersymmetric theory. In

this case we find

$$D_1^a \left\langle D_2^b \left\langle n_F[C_1, C_2] \right\rangle_{C_1} \right\rangle_{C_2} = \frac{i}{2} \sigma^{ab} (\bar{\theta}_1^0 - \bar{\theta}_2^0)(\theta_1^0 - \theta_2^0) \delta_c(b_1 - b_2) + ... \qquad (3.3.152)$$

Again, we won't pursue this line of enquiry any further, since the next model we consider will contain everything we require.

Chapter 4

Loop Dynamics of the ψ Theory

4.1 Non-abelian loop dynamics

Work in the 1980s on loop dynamics lead to an interesting result relating the Wilson loop in the $N \to \infty$ limit to the planar diagrams to all orders of $SU(N)$ gauge theory [51] [52] [53]. In this section we will show that there exists an equivalent construction for the ψ theory whose dynamics were investigated above.

We will begin with the result in standard Yang-Mills theory and then generalise this to the ψ theory. We will show that the loop equation for the ψ theory is equivalent to the Mandelstam formula [54] but where the path-ordering of the Wilson loop is achieved by the path integral of the ψ field. The Mandelstam formula leads to the Migdal-Makeenko equation from which the planar diagrams are obtained, and so this is an important result to reproduce.

We begin with the standard non-abelian loop variable

$$L[C] \equiv \mathcal{P}\,\exp\left(\oint_C A_\mu dX^\mu\right) \tag{4.1.1}$$

where C is a closed curve in spacetime parametrised by $X^\mu = X^\mu(t)$, $0 \leq t \leq 2\pi$. Consider now functionally varying the loop, that is considering the effect of shifting the path of the loop by δX. We need the following relations for path-ordered exponentials:

$$\frac{d}{dt}\Big(\mathcal{P}\ \exp\int_0^t M(\tau)d\tau\Big) = \Big(\mathcal{P}\ \exp\int_0^t M(\tau)d\tau\Big)M(t) \tag{4.1.2}$$

$$\delta\Big(\mathcal{P}\exp\int_0^{2\pi} M(\tau)d\tau\Big) = \int_0^{2\pi} dt\Big(\mathcal{P}\ \exp\int_0^t M(\tau_1)d\tau_1\Big)\delta M(t)\Big(\mathcal{P}\ \exp\int_t^{2\pi} M(\tau_2)d\tau_2\Big)$$
$$\equiv \int_0^{2\pi} dt\ \mathcal{P}\ \Big(\delta M(t)\exp\int_0^{2\pi} M(\tau)d\tau\Big). \tag{4.1.3}$$

Using these results we find that the variation of (4.1.1) is

$$\delta L[C] = \int_0^{2\pi} dt\Big(\mathcal{P}\ \exp\Big(\int_0^t A_\mu dX^\mu\Big)\Big)\delta(A_\mu \dot{X}^\mu)\Big(\mathcal{P}\ \exp\Big(\int_t^{2\pi} A_\mu dX^\mu\Big)\Big). \tag{4.1.4}$$

Expanding the variation in the integrand and performing an integration by parts gives, using the definition given in (4.1.3),

$$\delta L[C] = \int_0^{2\pi} dt\ \mathcal{P}\Big(F_{\mu\nu}\ \exp\Big(\oint_C A\cdot dX\Big)\Big)\dot{X}^\nu\delta X^\mu. \tag{4.1.5}$$

The only gauge invariant quantity we can build from the loop is $W[C] = \mathrm{Tr}\ L[C]$, hence, the quantities that make sense in the quantum theory will be expectations of the form $\langle W[C]\rangle$, $\langle W[C]W[C']\rangle$ etc. We will consider the first of these expectations and define

$$\Psi[C] \equiv \langle W[C]\rangle = \Big\langle \mathrm{Tr}\Big(\mathcal{P}\ \exp\Big(\oint_C A_\mu dX^\mu\Big)\Big)\Big\rangle \tag{4.1.6}$$

where

$$\langle\Omega\rangle = \int \mathcal{D}[A,X]\ e^{-S_{YM}}. \tag{4.1.7}$$

Using (4.1.5) we find

$$\delta\Psi[C] = \Big\langle \mathrm{Tr}\int_0^{2\pi} dt\ \mathcal{P}\Big(F_{\mu\nu}\ \exp\Big(\oint_C A\cdot dX\Big)\dot{X}^\nu\delta X^\mu\Big)\Big\rangle. \tag{4.1.8}$$

We now make a second variation of X. We obtain two sets of terms, one coming from the variation of $F_{\mu\nu}\dot{X}^\nu\delta X^\mu$ and the other coming from the loop exponent. The result is

$$\delta_2\delta_1\Psi[C] = \left\langle \mathrm{Tr} \int_0^{2\pi} dt \; \mathcal{P}\left(D_\alpha F_{\mu\nu} \; \exp\left(\oint_C A \cdot dX \right) \dot{X}^\nu \delta_1 X^\mu \delta_2 X^\alpha \right) \right\rangle$$

$$+ \left\langle \mathrm{Tr} \int_0^{2\pi} dt_1 \int_0^{2\pi} dt_2 \; \mathcal{P}\left(F_{\mu\nu} F_{\alpha\beta} \; \exp\left(\oint_C A \cdot dX \right) \dot{X}^\nu \dot{X}^\beta \delta_1 X^\mu \delta_2 X^\alpha \right) \right\rangle \quad (4.1.9)$$

where $D_\alpha F_{\mu\nu} \equiv \partial_\alpha F_{\mu\nu} - [F_{\mu\nu}, A_\alpha]$ is the non-abelian gauge covariant derivative of the field strength. Dividing through by $\delta X^\mu(t_1)\delta X_\mu(t_2)$ we find

$$\frac{\delta^2}{\delta X^\mu(t_1)\delta X_\mu(t_2)} \Psi[C] = \left\langle \delta(t_1 - t_2) \mathrm{Tr} \; \mathcal{P}\left(D^\mu F_{\mu\nu} \; \exp\left(\oint_C A \cdot dX \right) \dot{X}^\nu \right) \right\rangle$$

$$+ \left\langle \mathrm{Tr} \; \mathcal{P}\left(F_{\mu\nu} \dot{X}^\nu |_{t_1} F^\mu{}_\alpha \dot{X}^\alpha |_{t_2} \; \exp\left(\oint_C A \cdot dX \right) \right) \right\rangle . \quad (4.1.10)$$

We introduce a local derivative

$$\Delta \equiv \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dt' \; \frac{\delta^2}{\delta X^\mu(t + t'/2) \delta X_\mu(t - t'/2)} \quad (4.1.11)$$

also known as the area derivative, which, when applied to $W[C]$ picks out only the first term of (4.1.10) containing the delta function. It is like a Laplacian on the space of loops. The area derivative of the Wilson loop is then

$$\Delta\Psi[C] = \left\langle \mathrm{Tr} \; \mathcal{P}\left(D^\mu F_{\mu\nu} \; \dot{X}^\nu \; \exp\left(\oint_C A \cdot dX \right) \right) \right\rangle . \quad (4.1.12)$$

Note, $D^\mu F_{\mu\nu}$ is proportional to the variation of the Yang-Mills action. In fact we can integrate out the gauge field by considering the variation of the Yang-Mills part of the functional integral and relating it to the variation to the integrand of (4.1.12). Varying the Yang-Mills partition function we find

$$\delta_A Z_{YM} = \delta_A \int \mathcal{D}[A] \; \exp\left(-\frac{1}{4q^2} \int d^4X \; F^{\mu\nu A} F^A_{\mu\nu} \right) \mathrm{Tr}\left(\mathcal{P} \; e^{\oint A \cdot dX} \right) = 0. \quad (4.1.13)$$

This leads to the relation

$$\int \mathcal{D}[A] \left(\frac{1}{q^2} \int d^4X \; \delta A^A_\nu D_\mu F^{\mu\nu A} \right) e^{-S_{YM}} \; \mathrm{Tr}\left(\mathcal{P} \; e^{\oint A \cdot dX} \right)$$

$$= - \int \mathcal{D}[A] \; e^{-S_{YM}} \mathrm{Tr}\Big(\mathcal{P} \; \oint \delta A_\nu^A \tau^A dX^\nu \; e^{\oint A \cdot dX} \Big). \tag{4.1.14}$$

Choosing $\delta A_\nu^A = q^2 \tau^A \dot{X}_\nu \delta^4(X - X')$ so that we can compare the first term with the integrand of (4.1.12) we find

$$\Delta\Psi[C] = -q^2 \left\langle \mathrm{Tr}\Big(\mathcal{P} \; \oint \; \tau^A dX_\nu \delta^4(X - X') \tau^A dX'^\nu \; e^{\oint A \cdot dX} \Big) \right\rangle. \tag{4.1.15}$$

Migdal was able to show how one can get from this result to the sum of all planar diagrams of $SU(N)$ theory in the $N \to \infty$ limit [52]. This is the relation we aim to reproduce in the ψ theory. We expect that the path-ordering will be replaced by a path integral over ψ, as in the result of the expectation of the Wilson loop (2.3.80).

4.2 Loop dynamics of the ψ theory

We would like to carry out a similar calculation for the boundary ψ theory. In this theory the path-ordering is achieved by a functional integral over ψ which has a step function propagator. This simplifies the problem in some sense as the relations (4.1.2) and (4.1.3) do not need to be used.

To begin with, note that (2.3.80) can be obtained from the loop variable

$$\exp\Big(- q \oint \; \psi^\dagger dX \cdot A \psi \Big). \tag{4.2.16}$$

At $O(q^2)$ the expectation of the loop is

$$-\frac{q^2}{2} \oint \oint \; \psi^\dagger \tau^A \psi|_\xi \; \frac{\mathbb{P}(dX)^\mu \; e^{ik\cdot(X-X')} \; \mathbb{P}(dX')_\mu}{k^2} \; \psi^\dagger \tau^A \psi|_{\xi'}. \tag{4.2.17}$$

The ψ path integral then reduces this to the expectation of the non-abelian Wilson loop at $O(q^2)$. This result continues to all even orders in q when neglecting self interactions of the gauge field. Therefore, we may propose that the corresponding loop variable in this theory takes the form of the path integral

$$\Psi[C] = \left\langle \int \mathcal{D}\psi \; e^{-\int dt \;\; \psi^\dagger(\dot{\psi} + \dot{X}\cdot A\psi)} \right\rangle_A \tag{4.2.18}$$

where we must include the kinetic term for ψ. We will show that this loop variable leads to (4.1.15). We begin by varying the curve as before

$$\delta_X \Psi[C] = -\left\langle \int \mathcal{D}\psi \ \delta_X S \ e^{-S} \right\rangle_A$$

$$= \left\langle \int \mathcal{D}\psi \ \int dt \ \delta X^\nu (\dot{\psi}^\dagger A_\nu \psi + \psi^\dagger A_\nu \dot{\psi} + \dot{X}^\mu \psi^\dagger (\partial_\mu A_\nu - \partial_\nu A_\mu)\psi) \ e^{-S} \right\rangle_A . \quad (4.2.19)$$

The computation is slightly easier than the previous one due to the lack of the path-ordering operator, though, the generation of the commutator terms requires a different method from above. The way in which we will do this is to make repeated use of the Schwinger-Dyson equations for $\Psi[C]$. These equations tell us that for a general functional integral of the form $\int \mathcal{D}\phi \ e^{-S[\phi]}$,

$$\int \mathcal{D}\phi \ e^{-S[\phi]} \ \frac{\delta S[\phi]}{\delta \phi_a(X)} \ \delta\phi_a(X) = 0. \quad (4.2.20)$$

Applying this to (4.2.18) for $\psi^\dagger$ and ψ gives

$$\delta_{\psi^\dagger} \Psi[C] = -\left\langle \int \mathcal{D}\psi \int dt \ \delta\psi^\dagger (\dot{\psi} + \dot{X} \cdot A\psi) \ e^{-S} \right\rangle_A = 0 \quad (4.2.21)$$

and

$$\delta_{\psi} \Psi[C] = -\left\langle \int \mathcal{D}\psi \int dt \ (-\dot{\psi}^\dagger + \psi^\dagger \dot{X} \cdot A)\delta\psi \ e^{-S} \right\rangle_A = 0. \quad (4.2.22)$$

We now choose the specific variations $\delta\psi^\dagger = \psi^\dagger \ \delta X \cdot A$ and $\delta\psi = \delta X \cdot A\psi$ so that we may use these relations to replace terms in (4.2.19). Inserting these into (4.2.21) and (4.2.22) gives

$$\left\langle \int \mathcal{D}\psi \int dt \ \delta X^\nu \psi^\dagger A_\nu \dot{\psi} \ e^{-S} \right\rangle_A = -\left\langle \int \mathcal{D}\psi \int dt \ \delta X^\nu \dot{X}^\mu \psi^\dagger A_\nu A_\mu \psi \ e^{-S} \right\rangle_A \quad (4.2.23)$$

and

$$\left\langle \int \mathcal{D}\psi \int dt \ \delta X^\nu \dot{\psi}^\dagger A_\nu \psi \ e^{-S} \right\rangle_A = \left\langle \int \mathcal{D}\psi \int dt \ \delta X^\nu \dot{X}^\mu \psi^\dagger A_\mu A_\nu \psi \ e^{-S} \right\rangle_A \quad (4.2.24)$$

respectively. Now inserting these relations into (4.2.19) gives for the variation of the loop

$$\delta_X \Psi[C] = \left\langle \int \mathcal{D}\psi \int dt\ \delta X^\nu (\dot{X}^\mu \psi^\dagger (A_\mu A_\nu - A_\nu A_\mu)\psi + \dot{X}^\mu \psi^\dagger (\partial_\mu A_\nu - \partial_\nu A_\mu)\psi)\ e^{-S} \right\rangle_A$$

$$= \left\langle \int \mathcal{D}\psi \int dt\ \delta X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu} \psi\ e^{-S} \right\rangle_A . \tag{4.2.25}$$

Which we note is a similar result to (4.1.8). Dividing out the variation, using (4.2.32), results in

$$\frac{\delta \Psi[C]}{\delta X^\mu(t_1)} = -\left\langle \int \mathcal{D}\psi \int dt\ \delta(t-t_1) \dot{X}^\nu \psi^\dagger F_{\mu\nu} \psi\ e^{-S} \right\rangle_A$$

$$= -\left\langle \int \mathcal{D}\psi\ \dot{X}^\nu \psi^\dagger F_{\mu\nu} \psi|_{t_1}\ e^{-S} \right\rangle_A . \tag{4.2.26}$$

We now vary (4.2.25) again

$$\delta_2 \delta_1 \Psi[C] = \delta_2 \left\langle \int \mathcal{D}\psi \int dt\ \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu} \psi\ e^{-S} \right\rangle_A$$

$$= \left\langle \int \mathcal{D}\psi \int dt\ \delta_1 X^\nu (\delta_2 \dot{X}^\mu \psi^\dagger F_{\mu\nu}\psi + \dot{X}^\mu \delta_2 X^\lambda \psi^\dagger \partial_\lambda F_{\mu\nu} \psi)\ e^{-S} \right\rangle_A +$$

$$\left\langle \int \mathcal{D}\psi\ e^{-S} \int dt\ \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu} \psi \int dt'\ \delta_2 X^\lambda (\dot{\psi}^\dagger A_\lambda \psi + \psi^\dagger A_\lambda \dot{\psi}\right.$$

$$\left. + \dot{X}^\sigma \psi^\dagger (\partial_\sigma A_\lambda - \partial_\lambda A_\sigma)\psi \right\rangle_A . \tag{4.2.27}$$

We can use the Schwinger-Dyson equations again on (4.2.25) to get the relations

$$\left\langle \int \mathcal{D}\psi \int dt\ \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu} \psi \int dt'\ \delta_2 X^\lambda \psi^\dagger A_\lambda \dot{\psi}\ e^{-S} \right\rangle_A$$

$$= \left\langle \int \mathcal{D}\psi \left[\int dt\ (\delta_1 X^\nu \delta_2 X^\lambda \dot{X}^\mu \psi^\dagger A_\lambda F_{\mu\nu} \psi \right.\right.$$

$$\left.\left. - \int dt\ \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu} \psi \int dt'\ \delta_2 X^\lambda \dot{X}^\sigma \psi^\dagger A_\lambda A_\sigma \psi \right]\ e^{-S} \right\rangle_A \tag{4.2.28}$$

and

$$\left\langle \int \mathcal{D}\psi \int dt\ \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu} \psi \int dt'\ \delta_2 X^\lambda \dot{\psi}^\dagger A_\lambda \psi\ e^{-S} \right\rangle_A \tag{4.2.29}$$

$$= \left\langle \int \mathcal{D}\psi \left[- \int dt \; (\delta_1 X^\nu \delta_2 X^\lambda \dot{X}^\mu \psi^\dagger F_{\mu\nu} A_\lambda \psi \right.\right.$$

$$\left.\left. + \int dt \; \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu}\psi \int dt' \; \delta_2 X^\lambda \dot{X}^\sigma \psi^\dagger A_\sigma A_\lambda \psi \right] e^{-S} \right\rangle_A . \tag{4.2.30}$$

Inserting these into (4.2.27) gives

$$\delta_2\delta_1\Psi[C] = \left\langle \int \mathcal{D}\psi \left[\int dt \; \delta_1 X^\nu \delta_2 \dot{X}^\mu \psi^\dagger F_{\mu\nu}\psi + \delta_1 X^\nu \delta_2 X^\lambda \dot{X}^\mu \psi^\dagger D_\lambda F_{\mu\nu}\psi \right] e^{-S} \right\rangle_A$$

$$+ \left\langle \int \mathcal{D}\psi \int dt \; \delta_1 X^\nu \dot{X}^\mu \psi^\dagger F_{\mu\nu}\psi \int dt' \; \delta_2 X^\lambda \dot{X}^\sigma \psi^\dagger F_{\sigma\lambda}\psi \; e^{-S} \right\rangle_A . \tag{4.2.31}$$

Now, using the standard result for functional differentiation

$$\frac{\delta X^\mu(t)}{\delta X^\nu(t')} = \delta^\mu{}_\nu \delta(t-t') \tag{4.2.32}$$

we can calculate

$$\frac{\delta^2\Psi[C]}{\delta X^\alpha(t_1)\delta X^\beta(t_2)} = \left\langle \int \mathcal{D}\psi \left[\delta'(t_1-t_2)\psi^\dagger F_{\alpha\beta}\psi + \delta(t_1-t_2)\dot{X}^\mu \psi^\dagger D_\beta F_{\mu\alpha}\psi \right] e^{-S} \right\rangle_A$$

$$+ \left\langle \int \mathcal{D}\psi \; \dot{X}^\mu \psi^\dagger F_{\mu\alpha}\psi|_{t_1} \dot{X}^\sigma \psi^\dagger F_{\sigma\beta}\psi|_{t_2} \; e^{-S} \right\rangle_A \tag{4.2.33}$$

and so

$$\frac{\delta^2\Psi[C]}{\delta X^\mu(t_1)\delta X_\mu(t_2)} = \left\langle \int \mathcal{D}\psi \left[\delta'(t_1-t_2)\psi^\dagger F^\mu_\mu \psi - \delta(t_1-t_2)\dot{X}^\nu \psi^\dagger D^\mu F_{\mu\nu}\psi \right] e^{-S} \right\rangle_A$$

$$+ \left\langle \int \mathcal{D}\psi \; \dot{X}^\nu \psi^\dagger F_{\mu\nu}\psi|_{t_1} \dot{X}^\sigma \psi^\dagger F^\mu{}_\sigma \psi|_{t_2} \; e^{-S} \right\rangle_A . \tag{4.2.34}$$

Applying the area derivative, Δ, to Ψ and using the result (4.2.34) we find

$$\Delta\Psi[C] = - \left\langle \int \mathcal{D}\psi \; \psi^\dagger D^\mu F_{\mu\nu}\psi \; \dot{X}^\nu \; e^{-S} \right\rangle_A . \tag{4.2.35}$$

This is the Mandelstam formula for the ψ theory. We are now in a position to integrate out the gauge field. The Yang-Mills part of the partition function is this time

$$Z_A = \int \mathcal{D}A \; e^{-\frac{1}{4q^2}\int d^4x \; F^A_{\mu\nu}F^{\mu\nu A}} e^{-\int_c \psi^\dagger A\psi \cdot dX}. \tag{4.2.36}$$

The Schwinger-Dyson equations tell us that

$$\delta_A Z_A = 0 = -\int \mathcal{D}A\left(\frac{1}{4q^2}\int d^4X\ \delta(F^A_{\mu\nu}F^{\mu\nu A}) + \int dX\cdot\psi^\dagger\delta A\psi\right)e^{-S}. \quad (4.2.37)$$

This gives the relation

$$\int \mathcal{D}A\ \frac{1}{q^2}\left(\int d^4X\ \delta A^{\nu A}D^\mu F^A_{\mu\nu}\right)e^{-S} = \int \mathcal{D}A\ \left(\int dX_\nu\ \psi^\dagger\delta A^\nu\psi\right)\ e^{-S}. \quad (4.2.38)$$

We now choose the variation $\delta A^{\nu A} = q^2\psi^\dagger\tau^A\psi\dot{X}^\nu\delta(X - X(t))$ so that $\delta A^\nu = \tau^A\delta A^{\nu A} = q^2\tau^A\psi^\dagger\tau^A\psi\dot{X}^\nu\delta(X - X(t))$ and so the Schwinger-Dyson equation becomes

$$\begin{aligned}&\int \mathcal{D}A\ \left(\int d^4X\ \psi^\dagger D^\mu F_{\mu\nu}\psi\dot{X}^\nu\delta(X - X(t))\right)e^{-S}\\ &= \int \mathcal{D}A\ \left(q^2\int dX_\nu\ \psi^\dagger\tau^A\psi\psi^\dagger\tau^A\psi\dot{X}^\nu\delta(X - X(t))\right)\ e^{-S} \end{aligned}\quad (4.2.39)$$

since $\tau^A F^A_{\mu\nu} = F_{\mu\nu}$. Inserting this into (4.2.35) gives

$$\Delta\Psi[C] = -q^2\int \mathcal{D}\psi\ \left(e^{-S}\int (dX_\nu\ \psi^\dagger\tau^A\psi)|_{t'}\ \delta(X(t') - X(t))\ (\psi^\dagger\tau^A\psi\dot{X}^\nu)|_t\right). \quad (4.2.40)$$

This result is a generalisation of the Migdal-Makeenko equations (4.1.12) for the non-abelian Wilson loop. In the same way that the path-ordering is achieved in the expectation of the Wilson loop, the path-ordering of the Lie algebra generators is achieved via a path integral over an anti-commuting field, ψ. We therefore see that the two results are equivalent.

4.3 Supersymmetric ψ theory loop equations

We can generalise the previous procedure to the supersymmetric ψ model. The natural extension to the bosonic model replaces the standard Wilson line term in (4.2.18) with the super-Wilson line (1.4.134). The loop variable we will now be considering is, therefore

$$\Psi_s[C] = \left\langle\int \mathcal{D}\psi\ e^{-S_1}\right\rangle_A \quad (4.3.41)$$

where

$$S_1 = \int dt\ \psi^\dagger \left(\frac{d}{dt} + \dot{X}^\mu A_\mu - \frac{\sqrt{h}}{2}\ \eta^\mu F_{\mu\nu} \eta^\nu \right) \psi. \tag{4.3.42}$$

Here, we have defined the superpartner of X as η to avoid confusion with the world-sheet field, ψ. Once again, we can carry out the gauge field integration and show that this loop variable leads to (2.4.112). There is a slight complication when considering the loop dynamics in this case due to the appearance of the field strength. The loop variable exponent is then non-linear in the gauge field making the calculation more laborious.

One can simplify the calculation by introducing the superpartners of $\psi^\dagger$ and ψ [57], denoted $\tilde{z}$ and z respectively, so that the action can be written as

$$S_2 = \int dt\ \left(\psi^\dagger \left(\frac{d}{dt} + \dot{X}^\mu A_\mu - \sqrt{h}\ \eta^\mu \partial_\mu A_\nu \eta^\nu \right) \psi + \sqrt{h}\ (\tilde{z}\eta^\mu A_\mu \psi + \psi^\dagger \eta^\mu A_\mu z + \tilde{z}z) \right). \tag{4.3.43}$$

The new fields, $\tilde{z}$ and z, do not have kinetic terms and so we may integrate them out to get back to (4.3.42). This action is now linear in the gauge field with the non-linearities being generated by the additional terms containing $\tilde{z}$ and z. The nice thing about this is that we can massage the action, S_2, into a similar form to that of the exponent of (4.3.42) by appealing to the superspace formulation. We note that if we define the boundary superfields

$$\mathbf{X} = X + ih^{1/4}\theta\eta \tag{4.3.44}$$

$$\tilde{\Gamma} = \psi^\dagger + ih^{1/4}\theta\tilde{z} \tag{4.3.45}$$

$$\Gamma = \psi + ih^{1/4}\theta z \tag{4.3.46}$$

and the superderivative

$$D \equiv \partial_\theta + \theta\partial_t, \tag{4.3.47}$$

then we can write S_2 as

$$S_2 = -\int dtd\theta\ \tilde{\Gamma}(D + D\mathbf{X} \cdot A(\mathbf{X}))\Gamma \tag{4.3.48}$$

whereby we obtain (4.3.42) by integrating out θ. The advantage of writing the action in the form (4.3.48) is that it takes a similar form to the bosonic Wilson loop, being linear in $A(\mathbf{X})$. The full loop variable in the superspace formulation is then

$$\Psi_s[C] = \left\langle \int \mathcal{D}\Gamma \ \exp\Big(\int dt d\theta \ \tilde{\Gamma}(D + D\mathbf{X}\cdot A(\mathbf{X}))\Gamma \Big) \right\rangle_A \tag{4.3.49}$$

where $\mathcal{D}\Gamma \equiv \mathcal{D}[\psi^\dagger, \psi, \tilde{z}, z]$. In this form we can vary the superfield, $\mathbf{X}$, as a whole rather than varying the X and η separately.
The variation of the loop is

$$\begin{aligned}\delta_{\mathbf{X}}\Psi_s[C] &= \left\langle \int \mathcal{D}\Gamma \int dt d\theta \ \Big(D\tilde{\Gamma} \ \delta\mathbf{X}\cdot A \ \Gamma - \tilde{\Gamma} \ \delta\mathbf{X}\cdot A \ D\Gamma \Big) \ e^{-S_2} \right\rangle_A \\ &+ \left\langle \int \mathcal{D}\Gamma \int dt d\theta \ \tilde{\Gamma} D\mathbf{X}^\nu \delta\mathbf{X}^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu)\Gamma \ e^{-S_2} \right\rangle_A \end{aligned} \tag{4.3.50}$$

where $\partial_\mu \equiv \partial/\partial\mathbf{X}^\mu$. Now, we need the superspace generalisations of integration by parts which are

$$\int dt d\theta \ AD(B) = \int dt d\theta \ D(AB) - \int dt d\theta \ D(A)B \quad \text{(commuting } A) \tag{4.3.51}$$

$$\int dt d\theta \ AD(B) = -\int dt d\theta \ D(AB) + \int dt d\theta \ D(A)B \quad \text{(anti-commuting } A). \tag{4.3.52}$$

We need to evaluate the term

$$\int dt d\theta \ D(AB) = \int dt d\theta \ \partial_\theta(AB) + \int dt d\theta \ \theta\partial_t(AB). \tag{4.3.53}$$

The integrand of the first term is independent of θ, hence the θ integral is 0. The vanishing of the second term requires the condition $\int dt \ \partial_t(AB)|_{\theta\text{less}} = 0$ which is the same condition we require in the "bosonic" integration by parts formula. Thus, effectively

$$AD(B) = \mp D(A)B \tag{4.3.54}$$

with the minus sign for commuting A and the plus sign for anti-commuting A. The Schwinger-Dyson equations for (4.3.49) for Γ and $\tilde{\Gamma}$ give

$$D\Gamma + D\mathbf{X}\cdot A\Gamma \sim 0 \tag{4.3.55}$$

$$D\tilde{\Gamma} + \tilde{\Gamma}D\mathbf{X}\cdot A \sim 0 \tag{4.3.56}$$

where these relations are understood to hold in the functional integration. These relations allow us to write

$$\begin{aligned}\delta_{\mathbf{X}}\Psi_s[C] &= \left\langle \int \mathcal{D}\Gamma \int dtd\theta\ \tilde{\Gamma}D\mathbf{X}^\nu \delta\mathbf{X}^\mu F_{\mu\nu}(\mathbf{X})\Gamma\ e^{-S_2}\right\rangle_A \\ &= \left\langle \int \mathcal{D}\Gamma \int dtd\theta\ D\mathbf{X}^\mu \delta\mathbf{X}^\nu\ \tilde{\Gamma}F_{\mu\nu}(\mathbf{X})\Gamma\ e^{-S_2}\right\rangle_A .\end{aligned} \tag{4.3.57}$$

The result (4.3.57) is analogous to (4.2.25). Varying this again, using the Schwinger-Dyson equations for (4.3.57), gives

$$\begin{aligned}\delta_2\delta_1\Psi_s[C] &= \left\langle \int \mathcal{D}\Gamma \int dtd\theta\ \delta_2 D\mathbf{X}^\mu \delta_1\mathbf{X}^\nu \tilde{\Gamma}F_{\mu\nu}(\mathbf{X})\Gamma\ e^{-S_2}\right\rangle_A \\ &+ \left\langle \int \mathcal{D}\Gamma \int dtd\theta\ D\mathbf{X}^\mu \delta_1\mathbf{X}^\nu \delta_2\mathbf{X}^\alpha \tilde{\Gamma}D_\alpha F_{\mu\nu}(\mathbf{X})\Gamma\ e^{-S_2}\right\rangle_A \\ &- \left\langle \int \mathcal{D}\Gamma \int dt_1 d\theta_1\ D\mathbf{X}^\mu \delta_1\mathbf{X}^\nu \tilde{\Gamma}F_{\mu\nu}(\mathbf{X})\Gamma \int dt_2 d\theta_2\ D\mathbf{X}^\beta \delta_2\mathbf{X}^\alpha \tilde{\Gamma}F_{\alpha\beta}(\mathbf{X})\Gamma\ e^{-S_2}\right\rangle_A .\end{aligned} \tag{4.3.58}$$

Again, it is the second term we would like to isolate. We can do this by introducing the superspace analogue of the area derivative

$$\tilde{\Delta}(t) \equiv \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dt' d\theta d\theta'\ \frac{\delta^2}{\delta\mathbf{X}^\mu((t,\theta)+(t',\theta')/2)\delta\mathbf{X}_\mu((t,\theta)-(t',\theta')/2)} . \tag{4.3.59}$$

Here, we use the superspace generalisation for functional differentiation

$$\frac{\delta\mathbf{X}^\mu(t,\theta)}{\delta\mathbf{X}^\nu(t',\theta')} = \delta^\mu{}_\nu \delta(t-t')(\theta-\theta') \tag{4.3.60}$$

so that

$$\int dt d\theta \; \frac{\delta \mathbf{X}^{\mu}(t,\theta)}{\delta \mathbf{X}^{\nu}(t',\theta')} = \delta^{\mu}_{\nu}. \tag{4.3.61}$$

To evaluate the θ delta functions we will consider the integral

$$I = \int d\eta d\theta' d\theta \; \frac{\delta_1 \mathbf{X}^{\nu}(\theta)\delta_2 \mathbf{X}^{\alpha}(\theta)}{\delta \mathbf{X}^{\mu}(\theta_1(\theta',\eta))\delta \mathbf{X}_{\mu}(\theta_2(\theta',\eta))} F(\theta) \tag{4.3.62}$$

where $F(\theta)$ represents any other terms. Using (4.3.61) we find

$$I = \delta^{\nu}{}_{\mu}\delta^{\alpha\mu} \int d\eta d\theta' d\theta \; (\theta - \theta_1)(\theta - \theta_2) F(\theta) \tag{4.3.63}$$

where we have neglected the t delta functions here. Now, if we set $\theta_1 = \eta + \theta'/2$ and $\theta_2 = \eta - \theta'/2$ we find

$$I = \delta^{\nu\alpha} \int d\eta d\theta' d\theta \; (\theta - \eta)\theta' \; F(\theta) = \delta^{\nu\alpha} \int d\eta \; F(\eta), \tag{4.3.64}$$

which is precisely what we require. Applying these results to $\tilde{\Delta}\Psi_s[C]$ we find

$$\tilde{\Delta}(t)\Psi_s[C] = \left\langle \int \mathcal{D}\Gamma \; \int d\theta \; D\mathbf{X}^{\nu} \; \tilde{\Gamma} D^{\mu} F_{\mu\nu}(\mathbf{X})\Gamma|_t \; e^{-S_2} \right\rangle_A . \tag{4.3.65}$$

Thus, we obtain the Mandelstam formula for the supersymmetric ψ theory loop variable. Integrating out Γ produces the path-ordering of the usual super-Wilson loop.

We can now use the Schwinger-Dyson equations to integrate out the gauge field. The gauge field action in this case is

$$S^{A}_{YM} = -\frac{1}{4q^2} \int d^4X' \; F'^{A}_{\mu\nu}(X) F'^{\mu\nu \; A}(X') + \int dt' d\theta' \; \tilde{\Gamma} D\mathbf{X}'^{\nu} A_{\nu}(\mathbf{X}')\Gamma. \tag{4.3.66}$$

The variation of this with respect to the gauge field is simply

$$\delta_A S^{A}_{YM} = \frac{1}{q^2} \int d^4X' \; \delta A^{A}_{\nu}(X') D_{\mu} F^{\mu\nu \; A}(X') + \int dt' d\theta' \; \tilde{\Gamma} D\mathbf{X}'^{\nu} \delta A_{\nu}(\mathbf{X}')\Gamma. \tag{4.3.67}$$

Now, we would like to use the first term to substitute an expression into (4.3.65) that is free of the gauge field. To make this first term equal to the integrand of

(4.3.65) we choose

$$\delta A^A_\nu(X') = q^2 \int d\theta \ \delta^4(X' - \mathbf{X}(t)) D\mathbf{X}_\nu \ \tilde{\Gamma}\tau^A\Gamma \tag{4.3.68}$$

i.e. we use a delta function to turn $D_\mu F^{\mu\nu}(X)$ from a function of X into a function of the superfield $\mathbf{X}$. The second term requires us to turn δA from a function of $\mathbf{X}$ into a function of X. We can do this by using the delta function again

$$\delta A^A_\nu(\mathbf{X}) = \int d^4X' \ \delta(X' - \mathbf{X}) \ \delta A^A_\nu(X). \tag{4.3.69}$$

This relation is easily verified by expanding each side

$$\begin{aligned}
\delta A^A_\nu(X) + ih^{1/4}\theta\eta^\mu\partial_\mu\delta A^A_\nu(X) = \int d^4X' \ \delta(X' - X) \ \delta A^A_\nu(X) \\
- \int d^4X' \ ih^{1/4}\theta\eta^\mu\partial_\mu\delta(X' - X) \ \delta A^A_\nu(X) \\
= \delta A^A_\nu(X) + ih^{1/4}\theta\eta^\mu\partial_\mu\delta A^A_\nu(X)
\end{aligned} \tag{4.3.70}$$

after an integration by parts. Substituting (4.3.68) into (4.3.69) gives

$$\begin{aligned}
\delta A^A_\nu(\mathbf{X}(t')) = \int d^4X(t') \ \delta^4(X(t') - \mathbf{X}(t'))\delta A^A_\nu(X(t')) \\
= q^2 \int d^4X(t')d\theta \ \delta^4(X(t') - \mathbf{X}(t'))\delta^4(X(t') - X(t))D\mathbf{X}_\nu(t)\tilde{\Gamma}\tau^A\Gamma(t) \\
= q^2 \int d\theta \ \delta^4(\mathbf{X}(t') - \mathbf{X}(t))D\mathbf{X}_\nu(t)\tilde{\Gamma}\tau^A\Gamma(t).
\end{aligned} \tag{4.3.71}$$

Finally, substituting this into (4.3.67) gives the loop equation for the supersymmetric ψ theory

$$\tilde{\Delta}\Psi_s[C] =$$

$$-q^2 \int \mathcal{D}\Gamma \int dt' d\theta' d\theta \ (\tilde{\Gamma}\tau^A \Gamma D\mathbf{X}^\nu)_{t',\theta'} \delta^4(\mathbf{X}(t') - \mathbf{X}(t))(D\mathbf{X}_\nu \tilde{\Gamma}\tau^A\Gamma)_{t,\theta} \ e^{-S_2}. \quad (4.3.72)$$

This is the supersymmetric generalisation of (4.1.15). Path-ordering is achieved by the path integral over the superfield, ψ, as it is in the expectation of the super-Wilson loop.

This result may seem less familiar than the bosonic loop equation. To show equivalence, we must obtain the loop equation for the super-Wilson loop. The loop variable in this case is obtained from (1.4.134)

$$W_s[C] \equiv \mathcal{P} \ \exp\Bigg(\oint_C dt \ \Bigg(\dot{X}^\mu A_\mu - \frac{\sqrt{h}}{2}\eta^\mu \eta^\nu F_{\mu\nu}\Bigg)\Bigg) \quad (4.3.73)$$

without the minus sign again, which is just a matter of convention. The only gauge invariant quantity in the quantum theory that can be built from $W_s[C]$ is then $\Psi_s[C] \equiv \langle \mathrm{Tr} \ W_s[C]\rangle$. We have already shown that the super-Wilson loop has an intrinsic worldline supersymmetry and seen that supersymmetric functionals have a superspace representation. This case is no different; there does exist a superspace representation of the super-Wilson loop, though the proof of its equivalence is non-trivial. Before we introduce it, we must generalise path-ordering to superspace [58]. Firstly, the superspace path-ordered exponential is defined as

$$\mathcal{P} \ e^{\int d\tilde{\phi} \ M(\tilde{\phi})} \equiv \sum_{N=0}^{\infty} \int [d\tilde{\phi}]_N \ M(\tilde{\phi}_1)...M(\tilde{\phi}_N) \quad (4.3.74)$$

where

$$[d\tilde{\phi}]_N \equiv (dt_1 d\theta_1 ... dt_N d\theta_N) \ \Theta(\tilde{\phi}_{12})\Theta(\tilde{\phi}_{23})...\Theta(\tilde{\phi}_{N-1,N}), \quad (4.3.75)$$

$$\Theta(\tilde{\phi}_{ij}) = \Theta(t_i - t_j - \theta_i\theta_j) = \Theta(t_i - t_j) - \theta_i\theta_j\delta(t_i - t_j). \quad (4.3.76)$$

With these results, we introduce the superspace representation of Ψ

$$\Psi_s[C] = \Bigg\langle \mathrm{Tr} \ \Bigg[\mathcal{P} \ \exp\Bigg(\int d\tilde{\phi} \ D\mathbf{X}^\mu A_\mu(\mathbf{X})\Bigg)\Bigg]\Bigg\rangle_A . \quad (4.3.77)$$

The exponent, therefore, matches the abelian Wilson line in superspace (2.2.57). The commutator term of $F_{\mu\nu}$ is generated by the definition of path-ordering (4.3.75) and the superspace step function (4.3.76). To see this consider expanding the path-ordered exponential so that

$$\mathcal{P}\ \exp\bigg(\int d\tilde{\phi}\ D\mathbf{X}^\mu A_\mu(\mathbf{X})\bigg) = 1 + \int dt d\theta\ D\mathbf{X}^\mu A_\mu(\mathbf{X})$$

$$+ \int\int dt_1 d\theta_1 dt_2 d\theta_2\ (D\mathbf{X}^\mu A_\mu(\mathbf{X}))_1 (D\mathbf{X}^\mu A_\mu(\mathbf{X}))_2 + \dots\ . \tag{4.3.78}$$

Integrating out the θ coordinates we get

$$\mathcal{P}\ \exp\bigg(\int d\tilde{\phi}\ D\mathbf{X}^\mu A_\mu(\mathbf{X})\bigg) = 1 + \int dt\ (\dot{X}^\mu A_\mu - \sqrt{h}\ \eta^\mu\eta^\nu) +$$

$$\int dt_1 dt_2\ \Theta(t_1 - t_2)(\dot{X}^\mu A_\mu - \sqrt{h}\ \eta^\mu\eta^\nu)_1 (\dot{X}^\mu A_\mu - \sqrt{h}\ \eta^\mu\eta^\nu)_2$$

$$- \int dt_1 dt_2\ \delta(t_1 - t_2)\sqrt{h}\ \eta^\mu\eta^\nu A_\mu A_\nu + \dots\ . \tag{4.3.79}$$

The Θ function in the first term of the second line is what we get for normal path-ordering. The third line then combines with the first line to produce $F_{\mu\nu}$. It is easy to see that this continues to all orders. The relative simplicity of (4.3.77), and its similarity to the bosonic loop allows one to easily obtain the corresponding loop equations. Making a variation of $\delta\mathbf{X}$, noting that (4.1.2) and (4.1.3) continue to hold, we find

$$\delta\Psi[C] = \bigg\langle \mathrm{Tr}\ \bigg(\mathcal{P}\ \int_0^{2\pi} dt d\theta\ D\mathbf{X}^\nu \delta\mathbf{X}^\mu F_{\mu\nu}(\mathbf{X})\ e^{\int d\tilde{\phi}\ D\mathbf{X}^\mu A_\mu(\mathbf{X})}\bigg)\bigg\rangle_A . \tag{4.3.80}$$

Computing a second variation and using the definition of the area derivative, (4.3.59), we find

$$\tilde{\Delta}(t)\Psi = \bigg\langle \mathrm{Tr}\bigg(\ \mathcal{P}\ \int d\theta\ D\mathbf{X}^\nu D^\mu F_{\mu\nu}(\mathbf{X})|_t\ e^{\int d\tilde{\phi}\ D\mathbf{X}^\mu A_\mu(\mathbf{X})}\bigg)\bigg\rangle_A . \tag{4.3.81}$$

Varying the partition function with respect to the gauge field we find

$$0 = \delta_A Z = \left\langle \frac{1}{q^2} \int d^4X \ \delta A^A_\nu D_\mu F^{\mu\nu A} \ \mathrm{Tr}\Big(\mathcal{P} \ e^{\int d\tilde{\phi} \ D\mathbf{X}^\mu A_\mu(\mathbf{X})} \Big) \right\rangle_A \tag{4.3.82}$$

$$+ \left\langle \mathrm{Tr} \left(\mathcal{P} \int dt d\theta \ D\mathbf{X}^\mu \delta A_\mu(\mathbf{X}) \ e^{\int d\tilde{\phi} \ D\mathbf{X}^\nu A_\nu(\mathbf{X})} \right) \right\rangle_A . \tag{4.3.83}$$

Choosing a similar variation to (4.3.68), this time

$$\delta A^A_\nu(X) = q^2 \int d\theta \ \delta^4(X - \mathbf{X}(t)) D\mathbf{X}_\nu \ \tau^A, \tag{4.3.84}$$

we find

$$\tilde{\Delta}\Psi =$$

$$-q^2 \ \mathrm{Tr} \left(\mathcal{P} \int dt' d\theta' d\theta \ (\tau^A D\mathbf{X}^\nu)_{t',\theta'} \delta(\mathbf{X}(t') - \mathbf{X}(t)) (\tau^A D\mathbf{X}_\nu)_{t,\theta} \ e^{\int d\tilde{\phi} \ D\mathbf{X}^\nu A_\nu(\mathbf{X})} \right). \tag{4.3.85}$$

This is a straightforward supersymmetric generalisation of (4.1.15). We may now compare this with (4.3.72) and see that, after the path integration of Γ, they are equivalent.

Chapter 5

Yang-Mills Theory on the Worldsheet

So, we have found a way to incorporate path-ordering into the interior of the Wilson loop generalising the boundary worldline field theory of [35]- [40]. However, this model lacks the possibility of generating the self interactions of the gauge bosons. Here we will describe a model that can be used to impose path-ordering and provide a way to generate the three gluon vertex of non-abelian gauge theory. This method relies on the very thing underlying most of this work up to this point: gauge theory. Turning back to the non-abelian bosonic contact interaction

$$S_I^{YM} = \mathrm{Tr}\bigg(q^2 \int J^A(\xi) d\Sigma_{\mu\nu}(\xi) \delta^4(X(\xi) - X(\xi')) J^B(\xi') d\Sigma^{\mu\nu}(\xi')|_{\xi\neq\xi'}\bigg). \tag{5.0.1}$$

We note, making the spacetime gauge transformation

$$J^A(\xi) \to g(X(\xi)) J^A(\xi) g^{-1}(X(\xi)), \tag{5.0.2}$$

with $g \in G$, leaves the action invariant. i.e. J^A transforms in the adjoint representation of the group, G. The unusual dependence on the spacetime coordinate rather than the worldsheet coordinates is required because of the insertions of J at two different points on the worldsheet. The delta function ensures invariance at the only point where the integrand is non-zero.

It is therefore natural to introduce a new gauge theory onto the worldsheet; doing so will introduce a new field we will identify with the J^As that has the correct dynamics we seek.

5.1 2 dimensional Yang-Mills theory

The standard pure Yang-Mills action in 2 dimensions is

$$S_2^{YM} = -\frac{1}{2e^2}\int d^2\xi\sqrt{g}\ \mathrm{Tr}(F^{ab}F_{ab}) \tag{5.1.3}$$

where e^2 is the gauge coupling. A simple dimensional analysis shows that in 2 dimensions the gauge coupling has mass dimension $[e^2] = 2$. This is a problem if we are to consider this gauge theory as living on the worldsheet as we expect it to be scale invariant since the worldsheet action is Weyl invariant. We would, therefore, like a 2 dimensional gauge theory with no gauge coupling. We can obtain such a theory from the action above [59]. Consider the action

$$I = -\int\ \mathrm{Tr}(\phi F) - \frac{e^2}{2}\sqrt{g}\mathrm{Tr}(\phi^2) \tag{5.1.4}$$

where we have introduced a Lie algebra valued 0-form, ϕ, that transforms in the adjoint representation. Integrating out ϕ returns us to the Yang-Mills action above. Now, consider turning off the gauge coupling so that the second term vanishes. We are left with

$$S[A,\phi] = \int d^2\xi\ \epsilon^{ab}\ \mathrm{Tr}\big(\phi F_{ab}\big) \tag{5.1.5}$$

where we have emphasised the antisymmetry of the field strength. This action defines a topological field theory. Varying ϕ gives the equation of motion for the field strength as $F = 0$, or pure gauge. The quantum theory is described by the partition function composed of the Euclidean functional integral

$$Z_0 = \int \mathcal{D}[A,\phi]\ e^{-S[A,\phi]}. \tag{5.1.6}$$

This partition function can be solved as the exponential is linear in ϕ. Therefore,

$$Z_0 = \int \mathcal{D}[A,\phi]\ e^{\frac{1}{2}\int d^2\xi\ \phi^A \epsilon^{ab} F^A_{ab}} = \int \mathcal{D}A\ \delta\left[\frac{\epsilon^{ab}}{2} F^A_{ab}\right] \tag{5.1.7}$$

where $\delta[.]$ is functional generalisation of the delta function, what we will call the delta functional. This is the quantum version of the pure gauge mentioned above. Therefore the theory defined by this partition function is uninteresting; expectations of the form $\langle f[A]\rangle_{\phi,A} = f[A_{pure}]$ are trivial.

Interesting things happen when we use this partition function to define the generating functional for the gauge field on the boundary and choose a particular form of the source. Adding a source term on the boundary to the free action allows us to define the action

$$S_\kappa[A,\phi,\kappa] = \int d^2\xi\ \epsilon^{ab}\mathrm{Tr}\big(\phi F_{ab}\big) + \oint d\xi^a A^A_a \kappa^A. \tag{5.1.8}$$

We can then define the generating functional

$$Z[\kappa] \equiv \int \mathcal{D}[A,\phi]\ e^{-S_\kappa[A,\phi,\kappa]}. \tag{5.1.9}$$

To see how introducing this field theory on the worldsheet is useful for our cause, we will make repeated use of the Schwinger-Dyson equations for the gauge field. We begin by making a variation of the gauge field. The expectation of any functional doesn't change under such a variation (as long as the functional measure, $\mathcal{D}[A,\phi]$, is invariant, which here we assume it is) and so we have

$$0 = \delta_A Z = -\left\langle \delta_A S\ \right\rangle_{A,\phi} - \left\langle \oint d\xi^a\ \delta A^A_a \kappa^A \right\rangle_{A,\phi} \tag{5.1.10}$$

where $\langle\Omega\rangle_{A,\phi} = \int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\ \Omega$. The change in the action under a variation of A is

$$\delta_A S = \int d^2\xi\ \varepsilon^{ab}\mathrm{Tr}(\phi\ \delta_A F_{ab}) =$$

$$2\int d^2\xi\ \varepsilon^{ab}\mathrm{Tr}(D_b\phi\ \delta A_a) - 2\int d^2\xi\ \partial_b(\varepsilon^{ab}\mathrm{Tr}(\phi\ \delta A_a)) \tag{5.1.11}$$

where $D_b = \partial_b + [A_b, \cdot]$ is the non-abelian covariant derivative. The second term can be written as an integral along the boundary via Stokes' theorem

$$-2\int d^2\xi\ \partial_b(\varepsilon^{ab}\mathrm{Tr}(\phi\ \delta A_a)) = 2\oint d\xi^a\ \mathrm{Tr}(\phi\ \delta A_a) = -\oint d\xi^a\ \phi^A \delta A_a^A. \qquad (5.1.12)$$

We see that the variation of the free action on the boundary has a similar form to the variation of the source term. We may then consider the variation of the gauge field along the boundary and in the interior separately.

Considering the boundary variation first, which will lead to the path-ordering condition we seek, we find the following relation

$$\left\langle \oint d\xi^a\ \phi^A \delta A_a^A \right\rangle_{A,\phi} = \left\langle \oint d\xi^a\ \delta A_a^A \kappa^A \right\rangle_{A,\phi}. \qquad (5.1.13)$$

Now, functionally differentiating and using the relation

$$\frac{\delta A_a^A(\xi)}{\delta A_b^B(\xi')} = \delta^{AB}\delta_a^b\ \delta^2(\xi - \xi') \qquad (5.1.14)$$

reduces this to

$$\left\langle\ \phi^A|_\xi \right\rangle_{A,\phi} = \left\langle\ \kappa^A|_\xi \right\rangle_{A,\phi} \qquad (5.1.15)$$

where $\phi^A|_\xi$ denotes the value of ϕ^A at the boundary point ξ. Now comes the important part; if we choose κ^A to coincide with J^A from chapter 2 so that $\kappa^A = \psi^\dagger \tau^A \psi$, and functionally integrate over ψ and $\psi^\dagger$ with the usual kinetic action, we find

$$\int \mathcal{D}[\psi^\dagger, \psi]\ \left\langle\ \phi^A| \right\rangle_{A,\phi} e^{-\oint \psi^\dagger \dot{\psi}\ d\xi} = \int \mathcal{D}[A, \phi, \psi^\dagger, \psi]\ \psi^\dagger \tau^A \psi\ e^{-\oint \psi^\dagger \dot{\psi}\ d\xi - S_\kappa}. \quad (5.1.16)$$

Dividing through by $Z[\kappa]$, which we will incorporate into the expectation when considering multiple insertions of ψ, we find

$$\int \mathcal{D}[\psi^\dagger, \psi]\ \phi^A|\ e^{-\oint \psi^\dagger \dot{\psi}\ d\xi} = \int \mathcal{D}[\psi^\dagger, \psi]\ \psi^\dagger \tau^A \psi\ e^{-\oint \psi^\dagger \dot{\psi}\ d\xi}. \qquad (5.1.17)$$

This relation is the first step towards our required result of replacing a product of J^A on the boundary with the path-ordered Lie algebra generators (2.3.94).

We can prove that the path-ordering of multiple insertions follows by induction. Consider the functional integral

$$Z_{A_1...A_rB_1...B_n}(\eta_1,...,\eta_r,\xi_1,...,\xi_n)$$

$$= \int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\ (\kappa^{A_1}(\eta_1)...\kappa^{A_r}(\eta_r))(\phi^{B_1}(\xi_1)...\phi^{B_n}(\xi_n)) \tag{5.1.18}$$

where η_i is the location of the κ^{A_i} insertion on the boundary. Then, vary the gauge field on the boundary. Using the above results we find

$$0 = \delta_{A|}Z =$$

$$\int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\left(\oint d\xi^a\phi^A\delta A_a^A\right)(\kappa^{A_1}(\eta_1)...\kappa^{A_r}(\eta_r))(\phi^{B_1}(\xi_1)...\phi^{B_n}(\xi_n))$$

$$-\int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\left(\oint d\xi^a\ \delta A_a^A\kappa^A\right)(\kappa^{A_1}(\eta_1)...\kappa^{A_r}(\eta_r))(\phi^{B_1}(\xi_1)...\phi^{B_n}(\xi_n)). \tag{5.1.19}$$

Dividing out the variation of the gauge field gives us the relation

$$Z_{A_1...A_rAB_1...B_n}(\eta_1,...,\eta_r,\eta,\xi_1,...,\xi_n) = Z_{A_1...A_rB_1...B_nA}(\eta_1,...,\eta_r,\xi_1,...,\xi_n,\eta). \tag{5.1.20}$$

Note, the position of the indices. This relation can then be used to replace all of the ϕ^{B_i} with the corresponding κ^i on the boundary. To see this we start with no factors of κ^{A_i} or ϕ^{B_i}. Then applying the above formula n times results in the relation

$$\int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\ (\phi^{A_1}(\xi_1)...\phi^{A_n}(\xi_n)) = \int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\ (\kappa^{A_1}(\xi_1)...\kappa^{A_n}(\xi_n)). \tag{5.1.21}$$

Inserting $\kappa^A = \psi^\dagger\tau^A\psi$, integrating over ψ and $\psi^\dagger$ and dividing through by $\frac{1}{Z[\kappa]}$ we find

$$\int \mathcal{D}[\psi^\dagger,\psi]\ e^{-\oint \psi^\dagger\dot{\psi}\ d\xi}\ \left\langle\phi^{A_1}(\xi_1)...\phi^{A_n}(\xi_n)\right\rangle =$$

$$\int \mathcal{D}[\psi^\dagger,\psi]\ e^{-\oint \psi^\dagger\dot{\psi}\ d\xi}(\psi^\dagger\tau^{A_1}\psi)|_{\xi_1}...(\psi^\dagger\tau^{A_n}\psi)|_{\xi_n} \tag{5.1.22}$$

where now, we have defined

$$\langle \Omega \rangle \equiv \frac{\int \mathcal{D}[A,\phi]\ e^{-S_\kappa}\ \Omega}{\int \mathcal{D}[A,\phi]\ e^{-S_\kappa}}. \tag{5.1.23}$$

We recognise the right hand side as the path integral representation of the path-ordered product of Lie algebra generators on the boundary and so we can write

$$\int \mathcal{D}[\psi^\dagger,\psi]\ e^{-\oint \psi^\dagger \dot{\psi}\ d\xi}\ \left\langle \phi^{A_1}(\xi_1)...\phi^{A_n}(\xi_n) \right\rangle = \mathrm{Tr}\big(\mathcal{P}\ \tau^{A_1}...\tau^{A_n}\big). \tag{5.1.24}$$

This tells us that any factors of ϕ on the boundary in our string theory, where we include the ψ integration, may be replaced by the trace of the path-ordered product of the Lie algebra generators inserted at the positions of the ϕs. Note, this is a different condition from the previous field theory in that path-ordering is only achieved on the boundary of the worldsheet. To reproduce the expectation of the Wilson loop we need only the path-ordering of lie algebra generators on the boundary so this is not a problem.

5.1.1 Three gluon vertex generation

Next, we can consider the variation of the gauge field in the interior of the surface. Functionally differentiating the partition function with respect to the gauge field in the interior, and using (5.1.11), we find

$$\left\langle (D_a\phi)^A \right\rangle_{A,\phi} = 0. \tag{5.1.25}$$

One can expand the covariant derivative to obtain

$$\left\langle \partial_a \phi^A \right\rangle_{A,\phi} = -\left\langle [A_a,\phi]^A \right\rangle_{A,\phi}. \tag{5.1.26}$$

This relation will be of use when we come to consider the loop equations. One can then consider a variation of the expectation of the partial derivative of ϕ

$$0 = \delta_A \left\langle \partial_a \phi^A \right\rangle_{A,\phi} \to \left\langle \partial_a \phi^A (D_b\phi)^B \right\rangle_{A,\phi} = 0. \tag{5.1.27}$$

Expanding the covariant derivative then gives the relation

$$\left\langle \partial_a \phi^A \partial_b \phi^B \right\rangle_{A,\phi} = -\left\langle \partial_a \phi^A [A_b, \phi]^B \right\rangle_{A,\phi}. \tag{5.1.28}$$

The term on the right hand side can be written in a more useful form by considering the variation of $\left\langle [A_b, \phi]^B \right\rangle_{A,\phi}$ in the interior.

$$\delta_A \left\langle [A_b, \phi]^B \right\rangle_{A,\phi} = 0$$

$$= \left\langle [\delta A_b, \phi]^B \right\rangle_{A,\phi} + \left\langle 2\left(\int d^2\xi \; \varepsilon^{ac} \mathrm{Tr}(D_c \phi \; \delta A_a) \right) [A_b, \phi]^B \right\rangle_{A,\phi}. \tag{5.1.29}$$

Functionally differentiating out the gauge field and using the relation $[A_b, \phi]^B = A_b^A \phi^C f^{ACB}$ we get

$$\delta_b^d f^{ABC} \delta^2(z - z') \; \left\langle \phi^C \right\rangle_{A,\phi} = -\left\langle \varepsilon^{dc} (D_c \phi)^A [A_b, \phi]^B \right\rangle_{A,\phi}. \tag{5.1.30}$$

Multiplying by ε_{ad} gives

$$\varepsilon_{ab} f^{ABC} \delta^2(z - z') \; \left\langle \phi^C \right\rangle_{A,\phi} = -\left\langle (D_a \phi)^A [A_b, \phi]^B \right\rangle_{A,\phi}. \tag{5.1.31}$$

Expanding the covariant derivative finally leaves us with

$$\left\langle \partial_a \phi^A [A_b, \phi]^B \right\rangle_{A,\phi} = -\varepsilon_{ab} f^{ABC} \delta^2(z - z') \; \left\langle \phi^C \right\rangle_{A,\phi} - \left\langle [A_a, \phi]^A [A_b, \phi]^B \right\rangle_{A,\phi}. \tag{5.1.32}$$

Inserting this into (5.1.28) gives

$$\left\langle \partial_a \phi^A \partial_b \phi^B \right\rangle_{A,\phi} = \varepsilon_{ab} f^{ABC} \delta^2(z - z') \; \left\langle \phi^C \right\rangle_{A,\phi} + \left\langle [A_a, \phi]^A [A_b, \phi]^B \right\rangle_{A,\phi}. \tag{5.1.33}$$

The first term on the right hand side is just what is needed to produce the three gluon vertex in the string theory (2.3.97). This result comes at the expense of introducing the second term on the right. If we can show that this term vanishes in the string theory then this is of course not a problem. We can use functional methods to evaluate this term

$$\left\langle [A_a,\phi]_1^A [A_b,\phi]_2^B \right\rangle_{A,\phi} = f^{CDA} f^{EFB} \left\langle A_{1a}^C \phi_1^D A_{2b}^E \phi_2^F \right\rangle_{A,\phi}$$

$$= f^{CDA} f^{EFB} \frac{\partial}{\partial q_1^D} \frac{\partial}{\partial q_2^F} \left\langle A_{1a}^C A_{2b}^E \; e^{q_1\phi^1 + q_2\phi^2} \right\rangle_{A,\phi} \Bigg|_{q=0}. \tag{5.1.34}$$

Subscripts on the fields here denote $A_1 \equiv A(\xi_1)$. The exponential essentially modifies the action to $S' = S - (q_1\phi^1 + q_2\phi^2)$. The effect of this modification is to introduce two sources for the gauge field, q_1 and q_2 placed at z_1 and z_2 respectively. To see this we can integrate out ϕ. This inserts a delta function of the form

$$\delta\big(F - q_1\delta^2(\xi - \xi_2) - q_2\delta^2(\xi - \xi_2)\big) \tag{5.1.35}$$

into the functional integral. F here is the single independent component of the field strength. We will find it useful to work in complex coordinates and so $F = \partial\bar{A} - \bar{\partial}A + [A,\bar{A}]$. Note, this delta function implies that the gauge theory is no longer pure gauge. To proceed, we seek a solution for the gauge field such that $F(z) = \sum_i q_i\delta^2(z - z_i)$. We will choose $A = \bar{A}$ on the boundary. This is essentially a Green's function problem. To see this we write the gauge field as a power series in q

$$A = qf_1 + \sum_{n>1} q^n f_n. \tag{5.1.36}$$

Thus, the higher order terms will contribute to the commutator term of the field strength as it goes as $O(q^2)$ and greater. These higher order terms must cancel as the field strength is linear in q from (5.1.35) and thus only the derivative piece of the field strength exists. We must then solve

$$\partial\bar{f}_1 - \bar{\partial}f_1 = \sum_i q_i\delta^2(z - z_i). \tag{5.1.37}$$

Note, the defining equation for the bosonic Laplacian is

$$-2\bar{\partial}\partial G_B(z,z') = \delta^2(z - z') \tag{5.1.38}$$

or

$$\partial(-\bar{\partial}G_B) - \bar{\partial}(\partial G_B) = \delta^2(z - z') \tag{5.1.39}$$

which we can identify as (5.1.37). Thus $\bar{f}_1 = -\bar{\partial}G_B$ and $f_1 = \partial G_B$. G_B is detailed in Appendix A and so we have

$$A(z) = -\frac{1}{4\pi}\sum_i \frac{q_i}{z - z_i} + \frac{1}{4\pi}\sum_i \frac{q_i}{z - \bar{z}_i} + a \tag{5.1.40}$$

$$\bar{A}(z) = \frac{1}{4\pi}\sum_i \frac{q_i}{\bar{z} - \bar{z}_i} - \frac{1}{4\pi}\sum_i \frac{q_i}{\bar{z} - z_i} + \bar{a}. \tag{5.1.41}$$

a and $\bar{a}$ are the higher order q terms. With this solution the field strength is thus

$$F(z) = \sum_i q_i\delta^2(z-z_i) + \partial\bar{a} - \bar{\partial}a + \frac{1}{16\pi^2}[q_1, q_2]\Bigg[\left(\frac{1}{\bar{z} - \bar{z}_1} - \frac{1}{\bar{z} - z_1}\right)\left(\frac{1}{z - z_2} - \frac{1}{z - \bar{z}_2}\right) - \left(\frac{1}{\bar{z} - \bar{z}_2} - \frac{1}{\bar{z} - z_2}\right)\left(\frac{1}{z - z_1} - \frac{1}{z - \bar{z}_1}\right)\Bigg]. \tag{5.1.42}$$

For the field strength to satisfy our requirement, all other terms must cancel meaning a and $\bar{a}$ are proportional to the commutator $[q_1, q_2]$. The exact form of each is not important here as we need only the linear piece of the gauge field as we are differentiating with respect to each q_i only once. With the linear piece of A and $\bar{A}$ we can now evaluate (5.1.34) with A now a function of the sources q_1 and q_2

$$f^{CDA}f^{EFB}\frac{\partial}{\partial q_1^D}\frac{\partial}{\partial q_2^F}\left\langle A^C_{1a}(q)A^E_{2b}(q)\ \right\rangle_A\Big|_{q=0}. \tag{5.1.43}$$

There are three terms to consider: AA, $A\bar{A}$ and $\bar{A}\bar{A}$. Note, evaluating the gauge field at z_1 and z_2 using (5.1.40) leads to divergences. Computing the derivatives of these terms leads to Kronecker deltas that will lead to repeated indices in the structure constants. We thus drop these terms so that the divergences have no effect. This can be argued by regulating the divergence. Doing this will keep the gauge field finite and upon taking the derivatives, these terms will vanish.

For the first case, with $a = b = z$, the only non-zero term we are left with is then

$$-\frac{1}{16\pi^2}f^{CDA}f^{DCB}\left(\frac{1}{z_1 - z_2} - \frac{1}{\bar{z}_1 - z_2}\right)\left(\frac{1}{z_1 - z_2} - \frac{1}{z_1 - \bar{z}_2}\right)$$

$$\equiv -\frac{1}{16\pi^2} f^{CDA} f^{DCB} f_{zz}. \tag{5.1.44}$$

For $a = z$ and $b = \bar{z}$ we have

$$\frac{1}{16\pi^2} f^{CDA} f^{DCB} \left(\frac{1}{z_1 - z_2} - \frac{1}{\bar{z}_1 - z_2} \right) \left(\frac{1}{\bar{z}_1 - \bar{z}_2} - \frac{1}{z_1 - \bar{z}_2} \right)$$

$$\equiv -\frac{1}{16\pi^2} f^{CDA} f^{DCB} f_{z\bar{z}} \tag{5.1.45}$$

and finally for $a = b = \bar{z}$ we have

$$-\frac{1}{16\pi^2} f^{CDA} f^{DCB} \left(\frac{1}{\bar{z}_1 - \bar{z}_2} - \frac{1}{\bar{z}_1 - z_2} \right) \left(\frac{1}{\bar{z}_1 - \bar{z}_2} - \frac{1}{z_1 - \bar{z}_2} \right)$$

$$\equiv -\frac{1}{16\pi^2} f^{CDA} f^{DCB} f_{\bar{z}\bar{z}}. \tag{5.1.46}$$

These additional terms will arise when considering the contraction of two C integrals

$$C^A_{1\mu\nu\ k} C^{\lambda\sigma B}_{2k'} \sim$$

$$-\frac{1}{16\pi^2} f^{CDA} f^{DCB} \int d^2z_1 d^2z_2\ f_{ac}(z_1, z_2)\ L^a_{\mu\nu k}(X(z_1)) L^{c\lambda\sigma}_{k'}(X(z_2)). \tag{5.1.47}$$

$f_{ac}(z_1, z_2)$ diverges as $z_1 \to z_2$ so the surface integrals are ill defined. But, this is multiplied by exponential terms

$$e^{ik_1\cdot X(z_1)} e^{ik_2\cdot X(z_2)} \sim : e^{ik_1\cdot X(z_1)} e^{ik_2\cdot X(z_2)} : e^{-\pi\alpha' \left(2k_1\cdot k_2 G(z_1,z_2) + k_1^2 G(z_1,z_1) + k_2^2 G(z_2,z_2)\right)}. \tag{5.1.48}$$

For $k_1 \cdot k_2 > 0$ the integral over z_1 converges for $z_1 \approx z_2$ because $2\pi G(z_1, z_2) \approx \ln|z_1 - z_2|$ and this can be used to define the integral by analytic continuation. However, the remaining terms in the exponent suppress the whole expression as the cut-off is removed and $\alpha' k^2 \to \infty$ because $2\pi G(z_1, z_1) \approx \ln(y/\sqrt{\epsilon})$ as described in chapter 2. With this overall suppression, the additional term in (5.1.33) can be ignored and so we can effectively write

$$\left\langle \partial_a \phi^A \partial_b \phi^B \right\rangle_\phi = \varepsilon_{ab} f^{ABC} \delta^2(z - z')\ \left\langle \phi^C \right\rangle_\phi \tag{5.1.49}$$

This relation is precisely what is required to produce the three gluon vertex in the string theory as the contraction of two C integrals.

5.1.2 String theory

With this result we can begin to catalogue the expectations of all possible combinations of ϕ and $\partial\phi$ when the two dimensional surface upon which the gauge theory is supported upon is the worldsheet of the string theory with contact interaction described in chapter 2. Wherever we have a factor of B in the string theory we will have a factor of ϕ on the boundary and so we can immediately replace this with the path-ordered product of Lie algebra generators described by (5.1.24). This can be seen from including insertions of ϕ and $\partial\phi$ in the interior of the worldsheet into (5.1.18). Variations of the gauge field on the boundary won't affect these insertions and so we will obtain (5.1.24) with additional insertions. With this result and the contraction of derivative terms (5.1.49) we can completely determine the expectation of the contact interaction. To first order we have

$$\left\langle S_I^{YM}\right\rangle_{X,A,\phi,\psi^\dagger,\psi} = q^2\left\langle\int\frac{d^4k}{(2\pi)^4}\Big(\mathbb{V}+(B-C)\Big)\cdot\Big(\mathbb{V}+(B-C)\Big)\right\rangle_{X,A,\phi,\psi^\dagger,\psi} \tag{5.1.50}$$

where $A\cdot B\equiv A_k^{\mu\nu A}\cdot B^A_{\mu\nu -k}$. Then, as explained earlier, any terms with the projected vertex vanish because of suppression coming from the self contractions of the exponential. Also, (5.1.49) means that $C\cdot C=0$ because of repeated indices in the structure constant. Terms of the form $B\cdot C$ require the calculation of $\left\langle\phi^A|\partial_a\phi^A\right\rangle_{A,\phi}$ which is equal to $-\left\langle\phi^A|[A_a,\phi]^A\right\rangle_{A,\phi}$. One can use functional methods similar to those above to determine that this term will be zero since one will find $f^{ABC}\delta q^B/\delta q^C = f^{ABB}=0$.

We are then left with

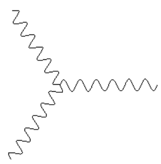
$$\left\langle S_I^{YM}\right\rangle_{X,A,\phi,\psi^\dagger,\psi} = \left\langle\int\frac{d^4k}{(2\pi)^4}B\cdot B\right\rangle_{X,A,\phi,\psi^\dagger,\psi} \tag{5.1.51}$$

which is the only term arising in the expectation of $\mathcal{P}\ \mathrm{Tr}(\int d\xi\cdot A)^2$. In fact, at higher orders, we can associate the following diagram with each factor of $B\cdot B$

$$B\cdot B\sim \tag{5.1.52}$$

joining two points on the boundary. This is the first step to obtaining the diagrams found in the introduction. The three gluon vertex arises from the contraction of two C integrals belonging to different factors of the contact interaction as

$$B\cdot \underbrace{CC}\cdot B\sim \tag{5.1.53}$$



At order q^4, the expectation of the contact interaction squared is

$$\left\langle (S_I^{YM})^2\right\rangle_{X,A,\phi,\psi^\dagger,\psi}\sim$$

$$q^4\left\langle\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}(B-C)\cdot(B-C)(B-C)\cdot(B-C)\right\rangle_{X,A,\phi,\psi^\dagger,\psi}. \tag{5.1.54}$$

We immediately identify the diagram with two independent propagators, $(B\cdot B)^2$, and the four diagrams with the three gluon vertex with external legs attached to the boundary, $(B\cdot C)^2$. From our discussion on the q^2 calculation we find that these are the only non-zero contributions to $\left\langle (S_I^{YM})^2\right\rangle$. We must mention that the ghost diagram, Fig. 1.3, fails to appear at this order. The model at this stage is therefore incomplete as a possible reformulation of Yang-Mills theory. As mentioned before, though, this missing feature may be found in the model after all if we change the regularisation scheme (see conclusion).

We can build more complicated diagrams at higher order q with these basic diagrams, such as

$$\left\langle (S_I^{YM})^3\right\rangle_{X,A,\phi,\psi^\dagger,\psi}\ni\ B\cdot\underbrace{CC}\cdot\underbrace{CC}\cdot B \tag{5.1.55}$$

which corresponds to the diagram shown in Figure 5.1. This amplitude will be calculated in the more realistic supersymmetric model.

We can continue to build the full catalogue of diagrams equivalent to those pro-

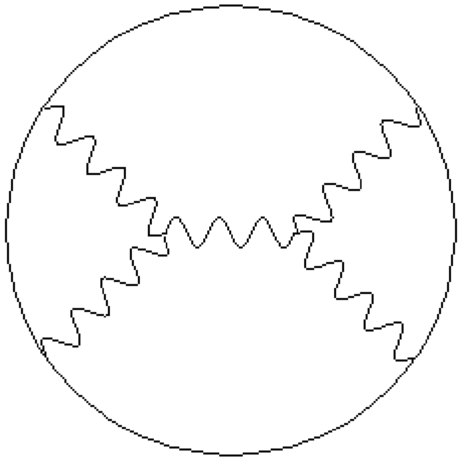


Figure 5.1: An example of a more complicated diagram produced by multiple contractions of Cs. In this case the string theory calculation is far simpler than the field theory calculation.

duced by computing the expectation of the non-abelian Wilson loop. To verify this equivalence to all orders, one can study the loop dynamics, which we do now.

5.2 Loop dynamics

The expectation of the non-abelian Wilson loop has been computed using a string theory with contact interaction and an additional gauge field on the worldsheet. This result is similarly obtainable from the loop variable (after carrying out the ψ and $\psi^\dagger$ integrals to replace ϕ^A with the path-ordered product of Lie algebra generators)

$$W_\phi = \exp\Big(- q \oint_B dt\ \phi^A \dot{X} \cdot A^A \Big) = \exp\Big(- q \oint_B dt\ \phi^A \dot{\xi}^a \partial_a X^\mu A^A_\mu \Big) \qquad (5.2.56)$$

where X is the target space coordinate and A is the 4 dimensional Yang-Mills field. On the right hand side we have written $X^\mu = X^\mu(\xi)$, where ξ are the worldsheet boundary coordinates, as we need to compare this exponent with the action of the 2 dimensional gauge theory. This result follows naturally from our discussion of the ψ theory loop dynamics. From the above results, we are lead to the following functional integral that describes the quantum dynamics of the loop variable

$$W \equiv \Big\langle \int D[\phi, a]\ e^{-\oint d\xi^a \left(\phi^A \partial_a X^\mu A^A_\mu + a^A_a (\kappa^A - \phi^A)\right)} e^{-S_{bulk}} \Big\rangle_{A,\psi^\dagger,\psi} \qquad (5.2.57)$$

where we are now using a to denote the worldsheet gauge field to avoid confusion with the spacetime gauge field. We have separated (5.1.5) into a boundary integral, $\oint d\xi^a a_a^A \phi^A$, and a bulk integral given by S_{bulk} via an integration by parts so that the total action is linear in the worldsheet gauge field as is done in (5.3.84).
Now consider varying the loop in spacetime

$$\delta_X \Psi = \left\langle \oint d\xi^a \delta X^\mu \big(\partial_a \phi^A A_\mu^A - \phi^A \partial_a X^\nu (\partial_\mu A_\nu^A - \partial_\nu A_\mu^A)\big) ... \right\rangle_{A,\phi,a,\psi^\dagger,\psi} . \tag{5.2.58}$$

We would like to be able to replace the first term with

$$\left\langle - \oint d\xi^a \ \delta X^\mu \phi^A [A_\mu, A_\nu]^A \partial_a X^\nu \right\rangle \tag{5.2.59}$$

so as to produce a similar result to that of Mandelstam. Note that using the properties of the Lie commutator, we can rewrite this as

$$\left\langle \oint d\xi^a \ \delta X^\mu [\phi, A_\nu]^A A_\mu^A \partial_a X^\nu \right\rangle = - \left\langle \oint d\xi^a \ \delta X^\mu [\phi, A_\mu]^A A_\nu^A \partial_a X^\nu \right\rangle \tag{5.2.60}$$

and so we need to essentially replace $\partial_a \phi^A$ by $[\phi, A_\nu]^A \partial_a X^\nu$ in the boundary integral. Firstly, we vary (5.2.57) with respect to ϕ so that

$$0 = \delta_\phi \Psi = \left\langle \oint d\xi^a \ \delta\phi^A \big(\partial_a X^\nu A_\nu^A - a_a^A\big) ... \right\rangle_{A,\phi,a} \tag{5.2.61}$$

which gives a relation between the spacetime gauge field and the worldsheet gauge field. The dots here represent the factors of e^{-S}. Since the variation is arbitrary, we can choose $\delta\phi^A = \delta X^\mu [\phi, A_\mu]^A$ so that we have the relation

$$\left\langle \oint d\xi^a \ \delta X^\mu [\phi, A_\mu]^A \partial_a X^\nu A_\nu^A ... \right\rangle_{A,\phi,a} = \left\langle \oint d\xi^a \ \delta X^\mu [\phi, A_\mu]^A a_a^A ... \right\rangle_{A,\phi,a} . \tag{5.2.62}$$

The left hand side is what we need and the right hand side can be shown to be the first term of (5.2.58). Again we can use the properties of the commutator:

$$[\phi, A_\mu]^A a_a^A = \phi^B A_\mu^C a_a^A f^{BCA} = -\phi^B a_a^A A_\mu^C f^{BAC} = -[\phi, a_a]^A A_\mu^A . \tag{5.2.63}$$

Now we need

$$\left\langle \oint d\xi^a \delta X^\mu \partial_a \phi^A A^A_\mu ... \right\rangle = \left\langle \oint d\xi^a \ \delta X^\mu [\phi, a_a]^A A^A_\mu ... \right\rangle \tag{5.2.64}$$

(first term of (5.2.58) equal to RHS of (5.2.62)) or

$$\left\langle \oint d\xi^a \delta X^\mu D_a \phi^A A^A_\mu ... \right\rangle = 0 \tag{5.2.65}$$

where $D_a\phi^A = \partial_a\phi^A - [\phi, a_a]^A$ is the worldsheet gauge covariant derivative. Well, we have already shown that $\left\langle D_a\phi^A \right\rangle_{a,\phi} = 0$, and so this condition appears naturally from the quantum dynamics of the system.

We can then write, going back up the steps,

$$\begin{aligned} \left\langle \oint d\xi^a \delta X^\mu \partial_a \phi^A A^A_\mu ... \right\rangle &= \left\langle \oint d\xi^a \ \delta X^\mu [\phi, a_a]^A A^A_\mu ... \right\rangle = \left\langle \oint d\xi^a \ \delta X^\mu [\phi, A_\mu]^A a^A_a ... \right\rangle \\ &= -\left\langle \oint d\xi^a \ \delta X^\mu [\phi, A_\mu]^A \partial_a X^\nu A^A_\nu ... \right\rangle = -\left\langle \oint d\xi^a \ \delta X^\mu \phi^A [A_\mu, A_\nu]^A \partial_a X^\nu ... \right\rangle \end{aligned} \tag{5.2.66}$$

which is our original requirement. Finally, then we can say

$$\delta_X \Psi = -\left\langle \left(\oint d\xi^a \ \delta X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) ... \right\rangle_{A,\phi,a,\psi^\dagger,\psi}. \tag{5.2.67}$$

A second variation yields

$$\begin{aligned} \delta_2\delta_1\Psi = &-\left\langle \left(\oint d\xi^a \ \delta_1 X^\mu \phi^A \partial_a \delta_2 X^\nu F^A_{\mu\nu} \right) ... \right\rangle - \left\langle \left(\oint d\xi^a \ \delta_1 X^\mu \phi^A \partial_a X^\nu \delta_2 X^\alpha \partial_\alpha F^A_{\mu\nu} \right) ... \right\rangle \\ &- \left\langle \left(\oint d\xi^a_1 \ \delta_1 X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi^b_2 \ \delta_2 X^\lambda \big(\partial_b \phi^B A^B_\lambda + \phi^B \partial_b X^\sigma (\partial_\sigma A^B_\lambda - \partial_\lambda A^B_\sigma) \big) \right) ... \right\rangle. \end{aligned} \tag{5.2.68}$$

Now consider varying (5.2.67) with respect to ϕ, we find

$$\begin{aligned} &\left\langle \oint d\xi^a \ \delta X^\mu \delta\phi^A \partial_a X^\nu F^A_{\mu\nu} ... \right\rangle = \\ &\left\langle \left(\oint d\xi^a_1 \ \delta X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi^b_2 \ \delta\phi^B (\partial_b X^\lambda A^B_\lambda - a^B_b) \right) ... \right\rangle. \end{aligned} \tag{5.2.69}$$

Choosing again, $\delta\phi^A = \delta_2 X^\alpha[\phi, A_\alpha]^A$ we find, after permuting the positions of the fields in the commutators

$$\left\langle \oint d\xi^a\ \delta X^\mu \delta_2 X^\alpha [\phi, A_\alpha]^A \partial_a X^\nu F^A_{\mu\nu} ... \right\rangle =$$

$$\left\langle \left(\oint d\xi_1^a\ \delta X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi_2^b\ \delta_2 X^\alpha (\phi^B \partial_b X^\lambda [A_\alpha, A_\lambda]^B + [\phi, a_b]^B A^B_\alpha) \right) ... \right\rangle . \tag{5.2.70}$$

Now, note that the first line can be written as

$$\left\langle \oint d\xi^a\ \delta X^\mu \delta_2 X^\alpha \phi^A [A_\alpha, F_{\mu\nu}]^A \partial_a X^\nu ... \right\rangle . \tag{5.2.71}$$

The second term of the bottom line can also be replaced by

$$\left\langle \left(\oint d\xi_1^a\ \delta X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi_2^b\ \delta_2 X^\alpha \partial_b \phi^B A^B_\alpha \right) ... \right\rangle \tag{5.2.72}$$

as we did for the first variation. This is simply obtained by varying (5.2.67) with respect to a and using (5.1.25). With this we can write

$$-\left\langle \left(\oint d\xi_1^a\ \delta X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi_2^b\ \delta_2 X^\alpha \partial_b \phi^B A^B_\alpha \right) ... \right\rangle =$$

$$-\left\langle \oint d\xi^a\ \delta_1 X^\mu \delta_2 X^\alpha \phi^A [A_\alpha, F_{\mu\nu}]^A \partial_a X^\nu ... \right\rangle$$

$$+\left\langle \left(\oint d\xi_1^a\ \delta X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi_2^b\ \delta_2 X^\lambda \phi^B [A_\lambda, A_\sigma]^B \partial_b X^\sigma \right) ... \right\rangle . \tag{5.2.73}$$

We can now substitute this result into (5.2.68) to find

$$\delta_2 \delta_1 \Psi = -\left\langle \left(\oint d\xi^a\ \delta_1 X^\mu \phi^A \partial_a \delta_2 X^\nu F^A_{\mu\nu} \right) ... \right\rangle$$

$$-\left\langle \left(\oint d\xi^a\ \delta_1 X^\mu \delta_2 X^\alpha \phi^A \partial_a X^\nu D_\alpha F^A_{\mu\nu} \right) \right\rangle$$

$$-\left\langle \left(\oint d\xi_1^a\ \delta_1 X^\mu \phi^A \partial_a X^\nu F^A_{\mu\nu} \right) \left(\oint d\xi_2^b\ \delta_2 X^\lambda \phi^B \partial_b X^\sigma F^B_{\sigma\lambda} \right) ... \right\rangle . \tag{5.2.74}$$

With this result we can calculate the area derivative, Δ, of the loop and so we find

$$\Delta\Psi = -\left\langle\left(\phi^A dX^\nu D^\mu F^A_{\mu\nu}\right)...\right\rangle_{A,\phi,a,\psi^\dagger,\psi}. \tag{5.2.75}$$

This is the Mandelstam formula for the gauge theory. We can now integrate out the spacetime gauge field, A. Varying Ψ with respect to A we find

$$0 = \delta_A\Psi = \left\langle \frac{1}{q^2}\int d^4X\ D^\mu F^A_{\mu\nu}\delta A^{\nu A} - \oint d\xi^a\ \phi^A\partial_a X^\mu \delta A^A_\mu\right\rangle .$$

Choosing $\delta A^{\nu A} = q^2\phi^A dX^\nu \delta^4(X(\xi)-X(\xi'))$ we find the relation

$$\left\langle\left(\phi^A dX^\nu D^\mu F^A_{\mu\nu}\right)...\right\rangle = \left\langle \oint d\xi^a\ \phi^A\partial_a X^\mu \phi^A(\xi')dX_\mu(\xi')\delta^4(X(\xi)-X(\xi'))\right\rangle \tag{5.2.76}$$

and so we have

$$\Delta\Psi = \int D[\phi,a,\psi^\dagger,\psi]\ \oint \phi^A dX^\mu|_\xi \delta^4(X(\xi)-X(\xi'))\phi^A dX_\mu|_{\xi'}\ e^{-S}. \tag{5.2.77}$$

This is the ϕ gauge field analogy of the Migdal-Makeenko equation for standard Yang-Mills theory. We can now use (5.1.21) to replace the factors of ϕ^A with the corresponding κ^A and carry out the ψ and $\psi^\dagger$ integrations to obtain the path-ordered product of Lie algebra generators.

With these results, the theory presented here may then be considered as a first step to generalising the string theory of [26] to incorporate non-abelian gauge theories. It reproduces the path-ordering of the Lie algebra generators in the expectation of the Wilson loop and contains the three gluon vertex. This can be made the basis of further study, i.e. studying the partition function

$$Z^{string}_{YM} = \int \mathcal{D}[X,g,A,\phi,\psi^\dagger,\psi]\ e^{-S_{Pol}[X,g]-S^{YM}_I[X,\phi]-S_\kappa[A,\phi,\psi^\dagger,\psi]}. \tag{5.2.78}$$

We know, however, from the abelian model that divergences arise at higher orders when vertices approach each other near the boundary, which will prevent us from identifying

$$Z^{string}_{YM} \sim \langle W\rangle_{A_4}. \tag{5.2.79}$$

We expect the supersymmetric string theory to be free of these extra unwanted divergences and so we now look at generalising this gauge theory to incorporate the supersymmetry of the superstring worldsheet.

5.3 Supersymmetric gauge theory

We have managed to show that introducing a gauge theory onto the worldsheet of the string theory can reproduce some features of the expectation of the non-abelian Wilson loop. It, therefore, seems that it may be possible to generalise the fermionic string theory to reproduce the expectation of the non-abelian super-Wilson loop by finding a suitable supersymmetric analogue of the gauge theory just described. In this section we will formulate such a generalisation.
To motivate the supersymmetric model we start with the bosonic action

$$S = \int d^2\xi \; \epsilon^{ab} \; \mathrm{Tr}\big(\phi F_{ab}\big). \tag{5.3.80}$$

In 2 dimensions F_{ab} has one independent component. In complex coordinates the action can be written as

$$S = \int d^2z \, \Big(\epsilon^{z\bar{z}} \; \mathrm{Tr}\big(\phi F_{z\bar{z}}\big) + \epsilon^{\bar{z}z} \; \mathrm{Tr}\big(\phi F_{\bar{z}z}\big)\Big) = 2\int d^2z \; \mathrm{Tr}\big(\phi F_{z\bar{z}}\big). \tag{5.3.81}$$

Now, writing the covariant components of the gauge field as $A_z \equiv A$ and $A_{\bar{z}} \equiv \bar{A}$, we have

$$F_{z\bar{z}} = \partial\bar{A} - \bar{\partial}A + A\bar{A} - \bar{A}A \equiv d\bar{A} - \bar{d}A \tag{5.3.82}$$

where we have defined the derivative $d \equiv \partial + A$ and used a lower case d to avoid confusion with the non-abelian gauge covariant derivative. The action can then be written as

$$S = 2\int d^2z \; \mathrm{Tr}\Big(\phi\big(d\bar{A} - \bar{d}A\big)\Big). \tag{5.3.83}$$

Note, it can also be written as

$$S = 2\oint dw^a \; \mathrm{Tr}\big(\phi A_a\big) - 2\int d^2z \; \mathrm{Tr}\big((d\phi)\bar{A} - (\bar{d}\phi)A\big) \tag{5.3.84}$$

where we have used the cyclic property of the trace. Written in this way it is easier to see the effect of separate boundary and interior gauge field variations, something we used in the loop equation derivation. This will also be useful for the supersymmetric model. As a first extension to (5.3.83) we can consider replacing the partial derivates by super derivatives such that d becomes

$$d_s \equiv D + \mathcal{A} \tag{5.3.85}$$

which suggests that the gauge superfield, $\mathcal{A}$, is Grassmann-odd. This then suggests that the gauge field, A, is the coefficient of the θ term in the expansion of $\mathcal{A}$. If we expand the superfields as

$$\mathcal{A} = \eta + \theta A + \bar{\theta}\lambda + \theta\bar{\theta}\sigma \tag{5.3.86}$$

$$\bar{\mathcal{A}} = \bar{\eta} + \theta\bar{\lambda} + \bar{\theta}\bar{A} - \theta\bar{\theta}\bar{\sigma} \tag{5.3.87}$$

$$\tilde{\phi} = \phi_0 + \theta\phi_1 + \bar{\theta}\bar{\phi}_1 + \theta\bar{\theta}\phi_2 \tag{5.3.88}$$

then we can propose the action

$$S' = -2\int d^2z d^2\theta \ \mathrm{Tr}\Big(\tilde{\phi}\big(d_s\bar{\mathcal{A}} + \bar{d}_s\mathcal{A}\big)\Big) \tag{5.3.89}$$

which contains the bosonic action above. The integrand then defines the superfield strength:

$$\mathcal{F} = d_s\bar{\mathcal{A}} + \bar{d}_s\mathcal{A} = D\bar{\mathcal{A}} + \bar{D}\mathcal{A} + \{\mathcal{A}, \bar{\mathcal{A}}\} \tag{5.3.90}$$

with the anti-commutator appearing this time. This is all conjecture at this point as we have merely promoted the partial derivative to the superderivative and the fields to superfields, only making sure that the bosonic result is contained within. To see that this supersymmetric field theory satisfies our requirements we first need to make sure that it is actually supersymmetric. From (2.2.53) we know that to make this action supersymmetric we must add the boundary integral

$$S_{susy} = \int d^2z d^2\theta \ \bar{\theta}\theta\delta(y)\mathrm{Tr}(\phi\mathcal{F}) = 2\int dx \ \mathrm{Tr}(\phi_0\mathcal{F}_0)$$

$$= 2\int dx\ \mathrm{Tr}\big(\phi_0(\lambda + \bar{\lambda} + \{\eta, \bar{\eta}\})\big) \tag{5.3.91}$$

to the action in order for it to be invariant with respect to the worldsheet supersymmetry (2.2.41).

We can write the supersymmetric gauge theory action in a similar way to the bosonic action (5.3.84), i.e. as a boundary piece plus a bulk term

$$S' = 2\int d^2zd^2\theta\ \mathrm{Tr}(d_s\tilde{\phi}\bar{\mathcal{A}} + \bar{d}_s\tilde{\phi}\mathcal{A}) + \int dxd^2\theta\ \Big(\bar{\theta}\mathrm{Tr}(\tilde{\phi}\mathcal{A}) - \theta\mathrm{Tr}(\tilde{\phi}\bar{\mathcal{A}})\Big). \tag{5.3.92}$$

Note, as $\mathcal{A}$ is Grassmann-odd we have the general cyclic property

$$\mathrm{Tr}(ABC) = (-)^{(C,AB)}\mathrm{Tr}(CAB) \tag{5.3.93}$$

where (C, AB) is 0 if C and AB commute or 1 if they anti-commute. This can be seen by expanding each field as $A = A^A\tau^A$. The fields can be taken outside of the trace and permuted into the desired combination picking up minus signs if the fields anti-commute while the generators can be cyclically permuted within the trace. We can then add to the boundary integral S_{susy}, so that the supersymmetric action is

$$S_{tot} = 2\int d^2zd^2\theta\ \mathrm{Tr}(d_s\tilde{\phi}\bar{\mathcal{A}} + \bar{d}_s\tilde{\phi}\mathcal{A})$$

$$+\int dx\ \mathrm{Tr}\Big(\phi_0(A + \bar{A} + \lambda + \bar{\lambda} + \{\eta, \bar{\eta}\}) + (\phi_1\eta + \bar{\phi}_1\bar{\eta})\Big). \tag{5.3.94}$$

We can, in fact, use the supersymmetry to fix some of the component gauge superfields and thus simplify the boundary piece. A pure supersymmetry transformation of the gauge superfield is

$$\delta\mathcal{A} = \epsilon(Q + \bar{Q})\mathcal{A} \tag{5.3.95}$$

which gives in terms of components gives

$$\delta\eta = \epsilon(A + \lambda) \qquad \delta A = \epsilon(\partial\eta + \sigma) \qquad \delta\lambda = \epsilon(\bar{\partial}\eta - \sigma) \qquad \delta\sigma = -\epsilon(\partial\lambda - \bar{\partial}A). \tag{5.3.96}$$

Similarly, we find that $\bar{\mathcal{A}}$ under this transformation gives in terms of component fields

$$\delta\bar{\eta} = \epsilon(\bar{A} + \bar{\lambda}) \qquad \delta\bar{A} = \epsilon(\bar{\partial}\bar{\eta} + \bar{\sigma}) \qquad \delta\bar{\lambda} = \epsilon(\partial\bar{\eta} - \bar{\sigma}) \qquad \delta\bar{\sigma} = \epsilon(\partial\bar{A} - \bar{\partial}\bar{\lambda}). \tag{5.3.97}$$

On the boundary we take the superfield $\mathcal{A}$ to be 'real' in the sense that $\mathcal{A}| = \bar{\mathcal{A}}|$ so that

$$\eta| = \bar{\eta}| \quad (A + \lambda)| = (\bar{A} + \bar{\lambda})| \tag{5.3.98}$$

and add to this $\sigma| = \bar{\sigma}|$ which is not determined by the reality condition. The reality condition is useful as (5.3.98) are invariant under the supersymmetry transformations. Using these boundary conditions we find that the boundary part of the action is

$$2\int dx\ \mathrm{Tr}\big(2\phi_0(A + \lambda + \eta\eta) + \eta(\phi_1 + \bar{\phi}_1)\big). \tag{5.3.99}$$

Setting $\phi_1 = \bar{\phi}_1 = 0$ on the boundary means that $\tilde{\phi}| = \phi_0|$ which makes sense if we want the bosonic condition. The total supersymmetric action is then

$$S_{tot} = 2\int d^2zd^2\theta\ \mathrm{Tr}(d_s\tilde{\phi}\bar{\mathcal{A}} + \bar{d}_s\tilde{\phi}\mathcal{A}) + 4\int dx\ \mathrm{Tr}\big(\phi_0(A + \lambda + \eta\eta)\big). \tag{5.3.100}$$

Under a supersymmetry transformation the combination $A+\lambda+\eta\eta$ on the boundary changes as

$$\delta(A + \lambda + \eta\eta) = \epsilon\big(\partial_x\eta + [(A + \lambda), \eta]\big) = \epsilon D_x^{(A+\lambda)}\eta. \tag{5.3.101}$$

The right hand side is the covariant derivative for the gauge field $(A + \lambda)$ along the boundary. Thus, we can define a boundary gauge field as

$$\mathcal{A}'| \equiv A + \lambda + \eta\eta. \tag{5.3.102}$$

A boundary gauge field related by a supersymmetry transformation is equivalent to boundary gauge field related by a gauge transformation. It is, therefore, supersymmetric and gauge invariant. This naturally leads to a source term for the gauge field on the boundary

$$S_{source} = 2\int dx\ (A+\lambda+\eta\eta)^A\kappa^A = 2\int dx d\theta\ \theta(D\mathcal{A}| + \mathcal{A}\mathcal{A}|)^A\kappa^A \qquad (5.3.103)$$

where on the right hand side we have written it as a boundary superspace integral. From the above, we know that this choice of source term is supersymmetric and gauge invariant and because it matches the boundary part of the action it will give us the path-ordering condition upon a variation of the boundary field $\mathcal{A}' \equiv A+\lambda+\eta\eta$. Note, here we don't require the superpartners of ψ and $\psi^\dagger$ as we did in chapter 3 to give the path-ordering condition.

With this, we now have a supersymmetric analogue of the bosonic field theory described in the last section. We can again use the Schwinger-Dyson equations to evaluate quantum expectations in this theory. We will again consider varying the interior and boundary fields separately only this time varying $\mathcal{A}$ in the interior and $\mathcal{A}'|$ on the boundary. Beginning with a variation of $\mathcal{A}'|$ we find

$$\int \mathcal{D}[\mathcal{A},\tilde{\phi}]\left(-4\int dx\ \mathrm{Tr}(\phi_0\delta\mathcal{A}')| - 2\int dx\ \mathcal{A}'^A\kappa^A|\right)e^{-S_\kappa} \qquad (5.3.104)$$

where

$$S_\kappa = S_{tot} + S_{source} \qquad (5.3.105)$$

which leads to

$$\langle \phi_0^A|\rangle_{\mathcal{A}',\phi} = \langle\kappa^A|\rangle_{\mathcal{A}',\phi} = \langle\psi^\dagger\tau^A\psi|\rangle_{\mathcal{A}',\phi} \qquad (5.3.106)$$

analogously to the bosonic result. To implement path-ordering into the supersymmetric string theory we need to consider products of $\phi_0|$ and so we define the integral analogously to the bosonic integral (5.1.18)

$$Z^s_{A_1...A_rB_1...B_n}(\eta_1,...,\eta_r,\xi_1,...,\xi_n)$$

$$\equiv \int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}\ \big(\kappa^{A_1}(\eta_1)...\kappa^{A_r}(\eta_r)\big)\big(\phi_0^{B_1}|(\xi_1)...\phi_0^{B_n}|(\xi_n)\big). \qquad (5.3.107)$$

Repeating the bosonic calculation using the above results leads to

$$\int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}\left(\kappa^{A_1}(\eta_1)...\kappa^{A_r}(\eta_r)\right)\ \left(\phi_0^A(\xi)\ \phi_0^{B_1}|(\xi_1)...\phi_0^{B_n}|(\xi_n)\right)$$

$$= \int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}\left(\kappa^A(\xi)\kappa^{A_1}(\eta_1)...\kappa^{A_r}(\eta_r)\right)\left(\phi_0^{B_1}|(\xi_1)...\phi_0^{B_n}|(\xi_n)\right) \tag{5.3.108}$$

so that

$$Z^s_{A_1...A_rAB_1...B_n}(\eta_1,...,\eta_r,\xi,\xi_1,...,\xi_n) = Z^s_{AA_1...A_rB_1...B_n}(\xi,\eta_1,...,\eta_r,\xi_1,...,\xi_n). \tag{5.3.109}$$

This relation can be used to replace each factor of $\phi_0^A(\xi)$ on the boundary by a corresponding factor of $\kappa^A(\xi)$. By starting with no factors of ϕ_0 or κ we can apply this relation n times to obtain

$$\int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}\ \phi_0^{A_1}(\xi_1)...\phi_0^{A_n}(\xi_n) = \int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}\ \kappa^{A_1}(\xi_1)...\kappa^{A_n}(\xi_n). \tag{5.3.110}$$

Now, inserting $\kappa^A = \psi^\dagger\tau^A\psi$ and integrating over ψ and $\psi^\dagger$ we find

$$\int \mathcal{D}[\psi^\dagger,\psi]\ e^{-\int \psi^\dagger\dot{\psi}\ d\xi}\ \left\langle \phi_0^{A_1}(\xi_1)...\phi_0^{A_n}(\xi_n)\right\rangle_{\mathcal{A},\tilde{\phi}} =$$

$$\int \mathcal{D}[\psi^\dagger,\psi]\ e^{-\int \psi^\dagger\dot{\psi}\ d\xi}\ (\psi^\dagger\tau^{A_1}\psi)|_{\xi_1}...(\psi^\dagger\tau^{A_n}\psi)|_{\xi_n} \tag{5.3.111}$$

where

$$\langle\Omega\rangle_{\mathcal{A},\tilde{\phi}} \equiv \frac{\int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}\ \Omega}{\int \mathcal{D}[\mathcal{A},\tilde{\phi}]\ e^{-S_\kappa}}. \tag{5.3.112}$$

The second line of (5.3.111) is the trace of the path-ordered product of Lie algebra generators, and so we can write

$$\left\langle \phi_0^{A_1}(\xi_1)...\phi_0^{A_n}(\xi_n)\right\rangle_{\mathcal{A}\tilde{\phi}} = \mathrm{Tr}\Bigg(\mathcal{P}\ (\tau^{A_1}...\tau^{A_r})\Bigg) \tag{5.3.113}$$

This is the bosonic result from the previous section, achieved with the supersymmetric analogue of the bosonic gauge theory, and so at least for the boundary variation this model reproduces the required result for path-ordering the Lie algebra generators on the boundary.

5.3.1 Three gluon vertex generation

Before we show how the three point function arises from this model we write the action in a more covariant form. To do this we introduce the metric tensor σ^{ab} with elements $\sigma^{z\bar{z}} = \sigma^{\bar{z}z} = 1$ and $\sigma^{zz} = \sigma^{\bar{z}\bar{z}} = 0$. The action (5.3.89) can then be written in a similar form to the bosonic action as

$$S' = -\int d^2zd^2\theta\ \sigma^{ab}\ \mathrm{Tr}(\tilde{\phi}\ F_{ab}) \tag{5.3.114}$$

where the superfield strength is $F_{ab} = 2(D_{(a}\mathcal{A}_{b)} +\ \mathcal{A}_{(a}\mathcal{A}_{b)})$. Now, a variation of the gauge superfield in the interior gives

$$\delta S' = 2\int d^2zd^2\theta\ \mathrm{Tr}(\mathcal{D}_a\tilde{\phi}\ \delta\mathcal{A}_b) \tag{5.3.115}$$

where $\mathcal{D}_a\tilde{\phi} \equiv D_a\tilde{\phi} + [\mathcal{A}_a, \tilde{\phi}]$ is the super gauge covariant derivative acting on a Grassmann-even field. Therefore, we have

$$\frac{\delta S'}{\delta\mathcal{A}_a^A} = -\sigma^{ab}(\mathcal{D}_b\tilde{\phi})^A. \tag{5.3.116}$$

In the interior of the worldsheet a variation of the gauge field then yields

$$0 = \frac{\delta Z}{\delta\mathcal{A}_a^A} = \left\langle -\frac{\delta S'}{\delta\mathcal{A}_a^A}\right\rangle = -\sigma^{ab}\left\langle(\mathcal{D}_b\tilde{\phi})^A\right\rangle \tag{5.3.117}$$

or

$$\left\langle(\mathcal{D}_a\tilde{\phi})^A\right\rangle = 0. \tag{5.3.118}$$

A variation of the average $\left\langle(D_a\tilde{\phi})^A\right\rangle$ must again vanish leading to

$$\left\langle(D_a\tilde{\phi})^A(\mathcal{D}_b\tilde{\phi})^B\right\rangle = 0. \tag{5.3.119}$$

This is the relation that will lead to the result needed to produce the three gluon vertex in the string theory. We therefore repeat the calculation for the supersymmetric case. Expanding this gives the correlation function relation

$$\left\langle (D_a\tilde{\phi})^A (D_b\tilde{\phi})^B \right\rangle = -\left\langle (D_a\tilde{\phi})^A [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle . \tag{5.3.120}$$

We can evaluate the right hand side by considering varying the average $\left\langle [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle$

$$0 = \delta \left\langle [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle = \left\langle [\delta\mathcal{A}_b, \tilde{\phi}]^B \right\rangle - 2\left\langle [\mathcal{A}_b, \tilde{\phi}]^B \int d^2z d^2\theta\ \sigma^{cd}\mathrm{Tr}(\mathcal{D}_c\tilde{\phi}\ \delta\mathcal{A}_d) \right\rangle . \tag{5.3.121}$$

Dividing through by $\delta\mathcal{A}_a^A$ gives

$$\left\langle \delta^a{}_b[\tau^A, \tilde{\phi}]^B \delta^2(z-z')\delta^2(\theta-\theta') \right\rangle = \left\langle \sigma^{ca}(\mathcal{D}_c\tilde{\phi})^A [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle \tag{5.3.122}$$

where we have computed the trace from the previous equation and swapped the position of the $(\mathcal{D}\tilde{\phi})^A$ picking up a minus sign. Multiplying across by σ_{ad} gives

$$\sigma_{ab}\left\langle [\tau^A, \tilde{\phi}]^B \delta^2(z-z')\delta^2(\theta-\theta') \right\rangle = \left\langle (\mathcal{D}_a\tilde{\phi})^A [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle . \tag{5.3.123}$$

Expanding the commutator and using the cyclicity and anti-commutativity of the structure constants we have

$$\left\langle (D_a\tilde{\phi})^A [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle = -\sigma_{ab}\left\langle f^{ABC}\tilde{\phi}^C \delta^2(z-z')\delta^2(\theta-\theta') \right\rangle - \left\langle [\mathcal{A}_a, \tilde{\phi}]^A [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle . \tag{5.3.124}$$

Then finally substituting this back into (5.3.120) gives

$$\left\langle (D_a\tilde{\phi})^A (D_b\tilde{\phi})^B \right\rangle = \sigma_{ab}\left\langle f^{ABC}\tilde{\phi}^C \delta^2(z-z')\delta^2(\theta-\theta') \right\rangle + \left\langle [\mathcal{A}_a, \tilde{\phi}]^A [\mathcal{A}_b, \tilde{\phi}]^B \right\rangle \tag{5.3.125}$$

giving us the supersymmetric analogue of (5.1.33) with again an additional term. For this relation to be of use in generating the three gluon vertex we require the extra term of the form

$$\left\langle [\mathcal{A}_a, \tilde{\phi}]^A [\mathcal{A}_b, \tilde{\phi}]^B\ \tilde{\phi}^A | \tilde{\phi}^B | ... \right\rangle \tag{5.3.126}$$

to vanish in the string theory. The ... denote additional insertions of $\tilde{\phi}$ placed at points non-coincident with any other insertions. We therefore repeat the bosonic calculation for the supersymmetric gauge theory constructed above. We begin by expanding the commutators

$$\left\langle ...f^{R_1S_1A}f^{R_2S_2B}\mathcal{A}_a^{R_1}(z_1)\mathcal{A}_b^{R_2}(z_2)\tilde{\phi}^{S_1}(z_1)\tilde{\phi}^{S_2}(z_2)\ \tilde{\phi}^{A}(z_1')\tilde{\phi}^{B}(z_2')...\right\rangle . \qquad (5.3.127)$$

We can replace all factors of $\tilde{\phi}$ by their corresponding generators on the boundary and so we can leave them out of the computation for simplicity. The other two factors of ϕ^{S_i} can be generated by the derivatives

$$\frac{\partial}{\partial q_1^{S_1}}\frac{\partial}{\partial q_2^{S_2}}\left\langle ...f^{R_1S_1A}f^{R_2S_2B}\mathcal{A}_a^{R_1}\mathcal{A}_b^{R_2}\ e^{\ q_1\tilde{\phi}^1+q_2\tilde{\phi}^2}...\right\rangle\Bigg|_{q=0}. \qquad (5.3.128)$$

This exponential can again be used to define a modified action in the functional integral of the form

$$S'' = S' - (q_1\tilde{\phi}^1 + q_2\tilde{\phi}^2). \qquad (5.3.129)$$

Integrating out $\tilde{\phi}$ gives a delta function insertion of the form

$$\delta(F - q_1\delta^2(z-z_1)\delta^2(\theta-\theta_1) - q_2\delta^2(z-z_2)\delta^2(\theta-\theta_2)) \qquad (5.3.130)$$

into the functional integral. The dynamics of the super gauge field are no longer pure gauge due to the sources generated by the insertions of $\tilde{\phi}$. We then solve for the super gauge field associated with this field strength. We must, therefore, solve

$$F = D\bar{\mathcal{A}} + \bar{D}\mathcal{A} + \bar{\mathcal{A}}\mathcal{A} + \mathcal{A}\bar{\mathcal{A}} = \sum_i\ q_i\ \delta^2(z-z_i)\delta^2(\theta-\theta_i) \qquad (5.3.131)$$

subject to the boundary condition $\mathcal{A}| = \bar{\mathcal{A}}|$. We can use the Green's function for the supersymmetric Laplacian to solve this. As

$$-2\bar{D}DG_F(z,\theta;z',\theta') = \delta^2(z-z')\delta^2(\theta-\theta') \qquad (5.3.132)$$

or

$$\bar{D}(-DG_F) + D(\bar{D}G_F) = \delta^2(z-z')\delta^2(\theta-\theta') \qquad (5.3.133)$$

we can therefore identify

$$\mathcal{A} = -qDG_F + a = \frac{q}{4\pi} D \log(z_{12}\bar{z}_{12} + \epsilon) + a \tag{5.3.134}$$

$$\bar{\mathcal{A}} = q\bar{D}G_F + a = \frac{q}{4\pi} D \log(z_{12}\bar{z}_{12} + \epsilon) + \bar{a} \tag{5.3.135}$$

where a and $\bar{a}$ are the higher order correction terms and $z_{12} \equiv z_1 - z_2 - \theta_1\theta_2$. Note, we have added the regulator ϵ as naively

$$-\frac{1}{2\pi}\bar{D}D\log(z_{12}\bar{z}_{12}) \neq \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2). \tag{5.3.136}$$

This behaviour is detailed further in appendix A. Calculating the derivatives, taking the $\epsilon \to 0$ limit and adding appropriate terms to satisfy the boundary condition gives

$$\mathcal{A}^{R_1}(z,\theta) = \sum_{i=1}^{2} q_i^{R_1}\left(\frac{1}{4\pi}\left(\frac{\theta-\theta_i}{z-z_i} - \frac{\theta-\bar{\theta}_i}{z-\bar{z}_i}\right) - \frac{1}{2}\bar{\theta}\bar{\theta}_i(\theta-\theta_i)\delta^2(z-z_i)\right) + a \tag{5.3.137}$$

$$\bar{\mathcal{A}}^{R_1}(z,\theta) = -\sum_{i=1}^{2} q_i^{R_1}\left(\frac{1}{4\pi}\left(\frac{\bar{\theta}-\bar{\theta}_i}{\bar{z}-\bar{z}_i} - \frac{\bar{\theta}-\theta_i}{\bar{z}-z_i}\right) - \frac{1}{2}\theta\theta_i(\bar{\theta}-\bar{\theta}_i)\delta^2(z-z_i)\right) + \bar{a}. \tag{5.3.138}$$

One can obtain the explicit form of a and $\bar{a}$ by computing the field strength. As these are higher order in q they will have no effect in the computation of (5.3.128). In analogy with the bosonic calculation we need to determine the 3 combinations $\mathcal{A}\mathcal{A}$, $\bar{\mathcal{A}}\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}\mathcal{A}$. Focussing on the first of these we have

$$\mathcal{A}^{R_1}(z_1)\mathcal{A}^{R_2}(z_2) = q_2^{R_1}\left(\frac{1}{4\pi}\left(\frac{\theta_1-\theta_2}{z_1-z_2} - \frac{\theta_1-\bar{\theta}_2}{z_1-\bar{z}_2}\right) - \frac{1}{2}\bar{\theta}_1\bar{\theta}_2(\theta_1-\theta_2)\delta^2(z_1-z_2)\right)$$

$$\times\, q_1^{R_2}\left(\frac{1}{4\pi}\left(\frac{\theta_2-\theta_1}{z_2-z_1} - \frac{\theta_2-\bar{\theta}_1}{z_2-\bar{z}_1}\right) - \frac{1}{2}\bar{\theta}_2\bar{\theta}_1(\theta_2-\theta_1)\delta^2(z_2-z_1)\right). \tag{5.3.139}$$

This is the only term that will contribute to the differentiation of the sources due to repeated indices in the structure constants. Differentiating we find

$$\frac{\partial}{\partial q^{S_1}}\frac{\partial}{\partial q^{S_2}}\mathcal{A}^{R_1}\mathcal{A}^{R_2} = \delta^{S_1R_2}\delta^{S_2R_1}\left(\frac{1}{4\pi}\left(\frac{\theta_1-\theta_2}{z_1-z_2} - \frac{\theta_1-\bar{\theta}_2}{z_1-\bar{z}_2}\right) - \frac{1}{2}\bar{\theta}_1\bar{\theta}_2(\theta_1-\theta_2)\delta^2(z_1-z_2)\right)$$

$$\times\left(\frac{1}{4\pi}\left(\frac{\theta_2-\theta_1}{z_2-z_1}-\frac{\theta_2-\bar\theta_1}{z_2-\bar z_1}\right)-\frac{1}{2}\bar\theta_2\bar\theta_1(\theta_2-\theta_1)\delta^2(z_1-z_2)\right)$$

$$=\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{16\pi^2}\left(\frac{\theta_1-\theta_2}{z_1-z_2}-\frac{\theta_1-\bar\theta_2}{z_1-\bar z_2}\right)\left(\frac{\theta_1-\theta_2}{z_1-z_2}-\frac{\bar\theta_1-\theta_2}{\bar z_1-z_2}\right)$$

$$-\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{8\pi}\bar\theta_1\bar\theta_2\theta_1\theta_2\delta^2(z_1-z_2)\left(\frac{1}{\bar z_1-z_2}-\frac{1}{\bar z_2-z_1}\right)\equiv\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{16\pi^2}f_{zz}. \quad (5.3.140)$$

Note, the delta function causes this extra term to vanish. Similarly, for $\bar{\mathcal{A}}\bar{\mathcal{A}}$ we find

$$\frac{\partial}{\partial q^{S_1}}\frac{\partial}{\partial q^{S_2}}\bar{\mathcal{A}}^{R_1}(z_1)\bar{\mathcal{A}}^{R_2}(z_2)=\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{16\pi^2}\left(\frac{\bar\theta_1-\bar\theta_2}{\bar z_1-\bar z_2}-\frac{\bar\theta_1-\theta_2}{\bar z_1-z_2}\right)\left(\frac{\bar\theta_1-\bar\theta_2}{\bar z_1-\bar z_2}-\frac{\theta_1-\bar\theta_2}{z_1-\bar z_2}\right)$$

$$-\frac{1}{8\pi}\bar\theta_1\bar\theta_2\theta_1\theta_2\delta^2(z_1-z_2)\left(\frac{1}{z_1-\bar z_2}-\frac{1}{z_2-\bar z_1}\right)\equiv\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{16\pi^2}f_{\bar z\bar z}. \quad (5.3.141)$$

and, finally, for $\bar{\mathcal{A}}\mathcal{A}$

$$\frac{\partial}{\partial q^{S_1}}\frac{\partial}{\partial q^{S_2}}\bar{\mathcal{A}}^{R_1}(z_1)\mathcal{A}^{R_2}(z_2)=$$

$$-\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{16\pi^2}\left(\frac{\bar\theta_1-\bar\theta_2}{\bar z_1-\bar z_2}-\frac{\bar\theta_1-\theta_2}{\bar z_1-z_2}\right)^2-\frac{\delta^{R_1S_2}\delta^{R_2S_1}}{4\pi}\frac{\delta^2(z_1-z_2)}{\bar z_1-z_2}\equiv\frac{\delta^{S_1R_2}\delta^{S_2R_1}}{16\pi^2}f_{z\bar z}. \quad (5.3.142)$$

In this case, the delta function term survives. In the functional integral we will see that this term does not contribute to the overall result as it cancels with the $\mathcal{A}\bar{\mathcal{A}}$ term.

All of these terms will then arise in the string theory when we consider the contractions of two factors of C coming from different vertex operators. The relevant terms will all be of the form

$$\langle CC\rangle\ni-\frac{ik_1^{[\mu}}{k_1^2}\frac{ik_2^{[\alpha}}{k_2^2}\int d^2z_1d^2\theta_1d^2z_2d^2\theta_2\ \frac{1}{16\pi^2}(f_{zz}+f_{\bar z\bar z}+f_{z\bar z}+f_{\bar zz})$$

$$(D+\bar D)\mathbb{P}_{k_1}(\mathbf{X}_1)^{\nu]}(D+\bar D)\mathbb{P}_{k_2}(\mathbf{X}_2)^{\beta]}\ e^{ik_1\cdot\mathbf{X}_1+ik_2\cdot\mathbf{X}_2}. \quad (5.3.143)$$

As one can see from above, in the functional integral we will find $f_{z\bar z}=-f_{\bar zz}$ and so the singular piece of (5.3.142) does not contribute. As in the bosonic case, f_{ab} diverges as $z_1\approx z_2$. But, once again, these terms are multiplied by the exponential $e^{ik_1\cdot\mathbf{X}_1+ik_2\cdot\mathbf{X}_2}$ which when contracted has an analogous form to the bosonic case. In the tensionless limit, then, these terms are suppressed and so will not contribute to the string functional integral. We can then conclude that

$$\left\langle (D_a\tilde{\phi})^A (D_b\tilde{\phi})^B \right\rangle \sim \sigma_{ab} \left\langle f^{ABC}\tilde{\phi}^C \delta^2(z-z')\delta^2(\theta-\theta') \right\rangle . \tag{5.3.144}$$

This means we will obtain the correct form of the three gluon vertex appearing in the perturbative expansion of the super-Wilson loop as this is precisely (2.4.114).

5.3.2 Higher order terms

We can play the same game with higher order correlation functions, i.e. expectations of the form $\left\langle (D\tilde{\phi})^n \right\rangle$ with $n > 2$ with each insertion carrying a distinct colour label correpsonding to distinct vertex operator insertions. The next order that satisfies this is S^3 which is of the form

$$S^3 \sim q^6 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \, (B-C)^{\mu\nu A}_{k_1} (B-C)^A_{\mu\nu - k_1} (B-C)^{\rho\lambda B}_{k_2} (B-C)^B_{\rho\lambda - k_2}$$
$$\times \, (B-C)^{\alpha\beta C}_{k_3} (B-C)^C_{\alpha\beta - k_3} \tag{5.3.145}$$

of which there are 6 terms like

$$S^3 \ni q^6 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \, C^{\mu\nu A}_{k_1} B^A_{\mu\nu - k_1} C^{\rho\lambda B}_{k_2} B^B_{\rho\lambda - k_2} C^{\alpha\beta C}_{k_3} B^C_{\alpha\beta - k_3} . \tag{5.3.146}$$

Evaluating these terms requires the computation of expectations of the form

$$\left\langle ... D_a\tilde{\phi}^A_1 \; D_b\tilde{\phi}^B_2 D_c\tilde{\phi}^C_3 \; \tilde{\phi}^A_{1'} \; \tilde{\phi}^B_{2'}\tilde{\phi}^C_{3'} ... \right\rangle . \tag{5.3.147}$$

Now, carrying out a similar calculation as we did for the two point correlation function we find

$$\left\langle D_a\tilde{\phi}^A_1 \; D_b\tilde{\phi}^B_2 D_c\tilde{\phi}^C_3 \right\rangle = \sigma_{ab} \left\langle [\tau^A, \tilde{\phi}_2]^B [\mathcal{A}_{c3}, \tilde{\phi}_3]^C \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) \right\rangle$$
$$-\sigma_{ac} \left\langle [\mathcal{A}_{b2}, \tilde{\phi}_2]^B [\tau^A, \tilde{\phi}_3]^C \delta^2(z_1 - z_3)\delta^2(\theta_1 - \theta_3) \right\rangle -$$
$$-\sigma_{bc} \left\langle [\mathcal{A}_{a1}, \tilde{\phi}_1]^A [\tau^B, \tilde{\phi}_3]^C \delta^2(z_2 - z_3)\delta^2(\theta_2 - \theta_3) \right\rangle$$

$$-\left\langle [\mathcal{A}_{a1}, \tilde{\phi}_1]^A [\mathcal{A}_{b2}, \tilde{\phi}_2]^B [\mathcal{A}_{c3}, \tilde{\phi}_3]^C \right\rangle \tag{5.3.148}$$

where $\mathcal{A}_{ai} \equiv \mathcal{A}_a(z_i, \theta_i)$. Focussing on the first term and expanding the commutators, we have

$$\sigma_{ab} f^{ADB} f^{EFC} \left\langle ...\tilde{\phi}_2^D \mathcal{A}_{c3}^E \tilde{\phi}_3^F \ \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2)\tilde{\phi}_{1'}^A \ \tilde{\phi}_{2'}^B \tilde{\phi}_{3'}^C ... \right\rangle . \tag{5.3.149}$$

We can replace the factors $\tilde{\phi}_2^D \tilde{\phi}_{1'}^A \ \tilde{\phi}_{2'}^B \tilde{\phi}_{3'}^C$ by the corresponding generators on the boundary using (5.3.113), leaving us to consider

$$\left\langle ...\mathcal{A}_{c3}^E \tilde{\phi}_3^F \ \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2)... \right\rangle . \tag{5.3.150}$$

Replacing this factor of $\tilde{\phi}$ with a differentiation as before leads to a solution for the gauge field, $\mathcal{A}(z_3)$

$$\mathcal{A}^E(z_3) = -\frac{q_3^E}{4\pi}\left(\frac{\theta_3 - \bar{\theta}_3}{z_3 - \bar{z}_3}\right) + a \tag{5.3.151}$$

and so (5.3.149) becomes

$$\begin{aligned} &\sim -\sigma_{ab} f^{ADB} f^{EFC} \left\langle ...\frac{\delta^{EF}}{4\pi}\left(\frac{\theta_3 - \bar{\theta}_3}{z_3 - \bar{z}_3}\right) \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) \ \mathrm{Tr}(\mathcal{P} \ \tau^D \tau^A \tau^B \tau^C)... \right\rangle \\ &\sim \sigma_{ab} f^{ADB} f^{EEC} ... = 0. \end{aligned} \tag{5.3.152}$$

The next two terms of (5.3.148) are similarly zero. The final term with three commutators will require the functional differentiation of three factors of the super gauge field. We can use the solutions (5.3.137) and (5.3.138) to determine that this term will be suppressed by self contractions of the exponential. We, therefore, determine that

$$\left\langle D_a \tilde{\phi}_1^A \ D_b \tilde{\phi}_2^B D_c \tilde{\phi}_3^C \right\rangle \sim 0. \tag{5.3.153}$$

In fact, the three point correlation function can be found by performing Wick contractions:

$$D_a \tilde{\phi}_1^A \ D_b \tilde{\phi}_2^B D_c \tilde{\phi}_3^C \sim D_a \tilde{\phi}_1^A D_b \tilde{\phi}_2^B D_c \tilde{\phi}_3^C + D_a \tilde{\phi}_1^A D_b \tilde{\phi}_2^B D_c \tilde{\phi}_3^C + ... \ . \tag{5.3.154}$$

The first term is

$$\sigma_{ab}\delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2)\ f^{ABD}\tilde{\phi}^D D_c \tilde{\phi}_3^C \tag{5.3.155}$$

but, using

$$\delta \left\langle \tilde{\phi}^D \right\rangle = 0 \quad \Rightarrow \quad \left\langle \tilde{\phi}^D D_c \tilde{\phi}_3^C \right\rangle = - \left\langle \tilde{\phi}^D [\mathcal{A}_c, \tilde{\phi}]^C \right\rangle \tag{5.3.156}$$

allows us to replace this with

$$-\sigma_{ab}\delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2)\ f^{ABD}\ \tilde{\phi}^D [\mathcal{A}_c, \tilde{\phi}]^C$$

$$= \sigma_{ab} f^{ADB} f^{EFC}\ \delta^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2)\ \tilde{\phi}^D \mathcal{A}_c^E \tilde{\phi}^F \tag{5.3.157}$$

which coincides with the first term of (5.3.148). This goes for the rest of the possible contractions. This means that the cascade of possible expectations of the form $\left\langle (D\tilde{\phi})^n \right\rangle$ terminates at the three point function. At higher orders, the expectation of factors of $D\phi$ can be obtained by applying Wick's theorem with the "propagator" (5.3.144). With this, we now look at the expectation of the exponential of the contact interaction.

5.3.3 Superstring theory

As in the bosonic theory, we can use these results to draw Feynman graphs with which we can associate expectations to all orders in q^2. The contact interaction, at n'th order, is

$$S^n = q^{2n} \prod_{i=1}^{n} \int \frac{d^4 k_i}{(2\pi)^4} \left(\mathbb{V}_i \cdot \mathbb{V}_i + (B - C) \cdot (B - C) \right) \tag{5.3.158}$$

where here $A \cdot B \equiv A_k^{\mu\nu A} B_{\mu\nu - k}^A$. Firstly, we note that there is always one such term of

$$q^{2n} \prod_{i=1}^{n} \int \frac{d^4 k_i}{(2\pi)^4}\ B \cdot B. \tag{5.3.159}$$

We have already evaluated this, it is simply

$$q^{2n}\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}\oint_B\oint_{B'}\tilde{\phi}^{A_i}\tilde{\phi}'^{A_i}\ db^{[\mu_i}_{k_i}k_i^{\nu]}\ \frac{e^{ik_i\cdot(w_i-w_i')}}{k_i^4}\ db'_{-k_i\ [\mu}k_{i\ \nu]} \tag{5.3.160}$$

$$=\frac{q^{2n}}{2^n}\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}\oint_B\oint_{B'}\tilde{\phi}^{A_i}\tilde{\phi}'^{A_i}\ db^{\mu}_{k_i}\ \frac{e^{ik_i\cdot(w_i-w_i')}}{k_i^2}\ db'_{-k_i\ \mu}. \tag{5.3.161}$$

Functionally integrating over worldsheets spanned by B has no effect on this term, and integrating over $\tilde{\phi}$ replaces each factor of $\tilde{\phi}^A$ on the boundary with the associated generator, τ^A. This term, then, describes the interaction between n pairs of points on the boundary joined by a propagator carrying momentum k_i which can be compared with (1.4.136).

Other terms which can easily be evaluated at any order are

$$\langle S^n\rangle \ni q^{2n}\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}\langle\mathbb{V}\cdot\mathbb{V}\rangle=0 \tag{5.3.162}$$

and

$$S^n \ni q^{2n}\prod_{i=1}^{n}\int\frac{d^4k_i}{(2\pi)^4}\ \langle C\cdot C\rangle=0. \tag{5.3.163}$$

The first of these is zero in the tensionless limit as self-contractions of the exponentials are suppressed as explained earlier. The second term is zero because of (5.3.144):

$$\underbrace{C^{\mu\nu A}_{k}C'^{A}_{\mu\nu}}\,_{-k}=\int d^2zd^2\theta\ d^2z'd^2\theta'\ D_a\underbrace{\tilde{\phi}^AJ^{a\mu\nu}_{k}D_b}\tilde{\phi}'^AJ'^{b}_{\mu\nu\ -k}$$

$$\sim\ -\int d^2zd^2\theta\ \sigma_{ab}f^{AAB}\tilde{\phi}^B\ J^{a\mu\nu}_{k}J'^{b}_{\mu\nu\ -k}=0. \tag{5.3.164}$$

Note, any term in the expansion of S^n with an odd number of C insertions will vanish as there will also be an extra C left over after contractions. This extra C will live in the interior of the worldsheet and there are no contractions left to do that can bring this to the boundary. Integration over the worldsheets spanning B will then lead to a suppression of this term due to self contractions of the exponential. The next non-trivial results are obtained by separating off 2 factors of $(B-C)\cdot$

$(B-C)$ as

$$S^n \sim q^{2n} \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}\ (B-C)^{A_1}_{k_1}\cdot(B-C)^{A_1}_{-k_1}(B-C)^{A_2}_{k_2}\cdot(B-C)^{A_2}_{-k_2}$$

$$\times\Big((B-C)\cdot(B-C)\Big)^{n-2}. \tag{5.3.165}$$

There are $n!/2!(n-2)!$ ways of doing this. We can now contract the Bs with the Cs in the first line producing the three point function. Additional factors of $B\cdot C$ in the second line will be distinct and will give rise to additional three point functions. For now just consider the unique term $(B\cdot B)^{n-2}$. This term will then give rise to a diagram with three points on the boundary linked by a three gluon vertex, and $n-2$ pairs of points on the boundary each joined by a single propagator. The amplitude for each i,j is then

$$\frac{q^{2n}}{2^{n-2}}\int \frac{d^4k_i}{(2\pi)^4}\frac{d^4k_j}{(2\pi)^4}\ (B^{A_i}_{k_i}\cdot C^{A_i}_{-k_i}+C^{A_i}_{k_i}\cdot B^{A_i}_{-k_i})(B^{A_j}_{k_j}\cdot C^{A_j}_{-k_j}+C^{A_j}_{k_j}\cdot B^{A_j}_{-k_j})$$

$$\times\prod_{k\neq i,j}^{n-2}\int_{k_k}\oint_B\oint_{B'}\ \tilde\phi^{A_k}\tilde\phi'^{A_k}\ db^{\mu}_{k_k}\ \frac{e^{ik_k\cdot(w_k-w'_k)}}{k_k^2}\ db'_{-k_k\ \mu}. \tag{5.3.166}$$

There are 4 ways to combine $B\cdot C$ and so we find that there are

$$2^2\ \frac{n!}{2!(n-2)!}=2n(n-1) \tag{5.3.167}$$

diagrams with one three point function and $n-2$ propagators joining $2(n-2)$ points on the boundary.

We can use Wick contractions to evaluate more complex products of vertex operators. At higher orders, there exists terms of the form

$$\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}\ (B\cdot C)(B\cdot C)(C\cdot C) \tag{5.3.168}$$

that first appear in S^3. Averaging over $\tilde\phi$ leads to contractions of the form

$$\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4k_3}{(2\pi)^4}\ (B\cdot C)(C\cdot C)(C\cdot B). \tag{5.3.169}$$

Carrying out these contractions results in

$$= \frac{i}{k_1^2} \oint db^{\mu A}_{k_1} \frac{1}{k_2^2} \int d^2z d^2\theta \; f^{ABD} J^D (D+\bar{D})\mathbf{X}_\mu (D+\bar{D})\mathbf{X}_\alpha e^{-i(k_1-k_2)\cdot \mathbf{X}}$$

$$\int d^2z' d^2\theta' \; f^{BCE} (D+\bar{D})\mathbf{X}'^{\alpha} (D+\bar{D})\mathbf{X}'^{\lambda} e^{-i(k_2-k_3)\cdot \mathbf{X}'} \frac{i}{k_3^2} \oint db^{C}_{k_3 \lambda}. \qquad (5.3.170)$$

We can project each $\mathbf{X}$ along the associated momentum to obtain two new boundary integrals. The result is

$$= f^{ABD} f^{BCE} \oint \; \tilde{\phi}^A \frac{db_{k_1 \; \mu}(w_1)}{k_1^2} e^{ik_1\cdot w_1} \frac{1}{k_2^2} \oint \tilde{\phi}^D \frac{db^{[\mu}_{-(k_1-k_2)}(w_2)}{(k_1-k_2)^2} i(k_1-k_2)^{\nu]} e^{-i(k_1-k_2)\cdot w_2}$$

$$\times \oint \tilde{\phi}^E \; \frac{db_{-(k_2-k_3) \; [\nu}(w_3)}{(k_2-k_3)^2} i(k_2-k_3)_{\rho]} e^{-i(k_2-k_3)\cdot w_3} \oint \tilde{\phi}^C \; \frac{db^{\rho}_{-k_3}(w_4)}{k_3^2} e^{-ik_3\cdot w_4}. \quad (5.3.171)$$

This gives the amplitude for the diagram in Fig. 5.1. The diagram is fairly easily computed, up to a symmetry factor, since each external leg contributes a factor of $e^{ik_i(X_i-X)}/k_i^2$. We also have an internal propagator, along with two integrals over the two positions of the three gluon vertices. These integrals produce momentum conserving delta functions that relate the momentum going into the diagram from each external leg. This explains the extra factor of $1/k_2^2$ in the amplitude computed in the string theory.

This is the only additional set of contractions that one can perform on the product of vertex operators, and it does not lead to the four gluon vertex. This exhausts the possibilities of finding the four gluon vertex in $\langle S^n \rangle$ from the quantum dynamics of ϕ^A. Again, we should mention that it is believed that the four gluon vertex, along with the ghost interaction, is indeed present in the string theory. It is obtained from contractions involving the projected vertex operators using dimensional regularisation to regulate the Green's function for the Laplacian at coincident points. These have been studied separately by Prof. Mansfield and will be presented in an upcoming paper.

5.3.4 Loop dynamics of the supersymmetric theory

The supersymmetric gauge model has so far resisted a similar treatment to that given in chapter 4 for the loop dynamics of the supersymmetric ψ theory. The reasons are that the super-Wilson line is non-linear in the gauge field leading to complications with the calculation. This was dealt with in the ψ theory by introducing two additional fields into the loop variable, $\tilde{z}$ and z respectively, that produced the quadratic terms upon integration. These extra fields naturally entered as the superpartners of the fields $\psi^\dagger$ and ψ and so we were able to use the superfield formalism to write the super-Wilson loop as linear in the gauge field $A(\mathbf{X})$.

In the present case we can introduce the extra terms linear in z and A as before except now we are unable to identify $\tilde{z}$ and z as superpartners of the Lie algebra valued field, ϕ^A. There are two reasons for this: firstly, we would need two ϕ fields to incorporate $\tilde{z}$ and z as superpartners and secondly we have shown that the superpartner of ϕ^A on the boundary vanishes, so that $\phi^A| = \phi_0^A$.

We also cannot use the standard super-Wilson loop (4.3.73) because the superspace representation specifically requires the use of path-ordering to generate the commutator terms. Trying a similar thing with this model will lead to the results lacking this extra structure. This needs resolving if we are to represent the expectation of the super-Wilson loop as a supersymmetric string theory with an additional gauge field on the worldsheet.

Chapter 6

Conclusions and further work

We have seen that introducing an additional Lie algebra valued field, J^A, onto the worldsheet of the string theory described in [26], generalising the boundary field theory of [35], allows a way of reproducing some of the features of the expectation of the non-abelian Wilson loop. Particularly, we used the boundary field theory model to introduce our field theory on the worldsheet in such a way that the path-ordering of Lie algebra generators was achieved in the same way. This lead to a study of the possible field theories that may be used to describe the dynamics of J^A. We began with a study of 2 dimensional electrostatics in curved space. An underlying Weyl invariance simplified the calculation of the average over the line of force solution to Gauss' law and generalised the two dimensional result found in [21]. This result lead to the study of the number of intersections of curves on a curved 2 dimensional surface. We found that this produces a generalisation of the boundary field theory by extending the path-ordering of Lie algebra generators into the interior of the worldsheet. Using the intersection number of curves as the field theory describing the dynamics of J^A allows an implementation of path-ordering of the Wilson loop. This result was extended to incorporate supersymmetry into the underlying surface, allowing one to obtain path ordering of the super-Wilson loop. This case is of most importance as it was shown in the abelian theory that it is free of extra divergences that would prevent one from making a formal equivalence between the string theory and the expectation of the Wilson loop.

Introducing J^A onto the worldsheet naturally leads to extra terms in the vertex operator obtained by an integration by parts so that one obtains a boundary integral, analogous to the abelian vertex operator, plus a term containing the derivatives of J^A. We determined that this additional term gives rise to the three gluon self interaction of non-abelian gauge theory when satisfying a specific relation. We, therefore, tried to find such a relation in the field theory described by the average of the number of intersections of curves but were ultimately unable to find one, though we did see some hints of where it may arise. Because of this, we investigated an alternative method for obtaining the dynamics for J^A, this time using an inherent gauge symmetry of the contact interaction to introduce a 2 dimensional Yang-Mills field onto the worldsheet. We calculated correlation functions of the gauge field finding the path-ordering condition plus the relation required for the three gluon interaction. Again, we were able to extend this result to incorporate supersymmtery by studying 2 dimensional supersymmteric Yang-Mills theory. Unfortunately, we were unable to find the four gluon vertex and ghost-ghost-gluon vertex as arising from the dynamics of the Lie algebra valued field, J^A. This, then, requires further work to see if these interactions do actually exist in the string theory described in this work.

Independent work by Prof. Mansfield has shown that these may arise from contractions of the projected vertex operator, $\mathbb{V}$. In the work presented here, we used heat kernel regularisation to regulate the divergence produced in the Green's function of the Laplacian when vertex operators were placed at coincident points on the worldsheet. With this, we found that these particular contractions are suppressed in the tensionless limit of the string theory. However, using dimensional regularisation, one finds that this contraction is not suppressed and does indeed contribute to the expectation of the contact interaction. The idea is to replace (2.1.34) with

$$G_\epsilon(x_1, x_2) = \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{e^{ik\cdot(x_1-x_2)}}{k^2} \tag{6.0.1}$$

and do the various contractions in $2+\epsilon$ dimensions, letting $\epsilon \to 0$ at the end.

If this prescription is shown to reproduce the four gluon vertex and ghost-ghost-gluon vertex (and nothing more!), then we will be one step closer to proposing this string theory as the true non-abelian generalisation of [45] that reproduces the expectation of the super-Wilson loop. By quantising the particles that make up the boundary using Strassler's worldline formulation we may obtain Yang-Mills theory coupled to spinors and so obtain a reformulation of QCD. This would be the next step in the research of this subject, followed by a study of the phenomenology predicted by this model.

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Appendix A

It was believed that we required a spectral decomposition of the supersymmetric heat kernel when investigating the number of intersections of curves on a supersurface. We, ultimately, found that one can do the calculation by gauge fixing the kappa symmetry and using the bosonic results. However, we did in fact obtain a spectral decomposition for the heat kernel, Green's function and identity operator of the super-Laplacian.

A.1 The Super-Laplacian

Consider a variation of the gauge fixed superstring action (neglecting boundary terms)

$$\begin{aligned}\delta S_{spin} &= \frac{1}{4\pi\alpha'}\int d^2z d^2\theta\,\left(\bar{D}\delta\mathbf{X}^\mu D\mathbf{X}_\mu + \bar{D}\mathbf{X}^\mu D\delta\mathbf{X}_\mu\right)\\ &= \frac{1}{4\pi\alpha'}\int d^2z d^2\theta\;\delta\mathbf{X}^\mu(-2\bar{D}D)\mathbf{X}_\mu + \ldots\quad .\end{aligned} \tag{A.1.1}$$

The equations of motion for the superfield, $\mathbf{X}$, are then

$$\bar{D}D\mathbf{X}_\mu = 0. \tag{A.1.2}$$

The operator $\bar{D}D$ is the supersymmetric analogue of the bosonic Laplacian. It is useful to define

$$\Delta_F \equiv 4\bar{D}D \tag{A.1.3}$$

where the factor of 4 is included to match the bosonic Laplacian in complex coordinates which is $\partial_x^2 + \partial_y^2 = 4\bar{\partial}\partial$. It is like the square root of the bosonic Laplacian in the sense that

$$-\Delta_F^2 = -(4)^2 \bar{D}D\bar{D}D = (4)^2 \bar{D}^2 D^2 = (4)^2 \bar{\partial}\partial = 4\Delta_B \tag{A.1.4}$$

where we have used the relation

$$D^2 = (\partial_\theta + \theta\partial)(\partial_\theta + \theta\partial) = \partial. \tag{A.1.5}$$

We are interested in the eigenfunctions and corresponding eigenvalues of Δ_F. Their defining equation is as usual

$$-\Delta_F \Psi_i = \lambda_i \Psi_i. \tag{A.1.6}$$

Applying the super-Laplacian once more and using (A.1.4) we find

$$-\Delta_B \Psi_i = \frac{\lambda^2}{4}\Psi_i. \tag{A.1.7}$$

This is actually four equations that tell us that each component eigenfunction of the super-Laplacian with eigenvalue λ_i is therefore an eigenvalue of the bosonic Laplacian with eigenvalue $\lambda^2/4$. The decomposition of Ψ can be written as

$$\Psi = f + \theta g + \bar{\theta} h + \theta\bar{\theta} k \tag{A.1.8}$$

where $f(z,\bar{z})$ and $k(z,\bar{z})$ are commuting and $g(z,\bar{z})$ and $h(z,\bar{z})$ are anti-commuting fields. Each component satisfies

$$f = -\left(\frac{4}{\lambda}\right)k \qquad g = \left(\frac{4}{\lambda}\right)\partial h \qquad h = -\left(\frac{4}{\lambda}\right)\bar{\partial} g \tag{A.1.9}$$

meaning we can write Ψ in terms of just two component fields as

$$\Psi = f(1 - \theta\bar{\theta}\frac{\lambda}{4}) + \theta g - \bar{\theta}\frac{4}{\lambda}\bar{\partial} g. \tag{A.1.10}$$

A knowledge of Ψ allows us to build the Green's function associated with Δ_F from the qualitative relation

$$G_F \sim \sum_{\lambda\neq 0} \frac{|\Psi\rangle\langle\Psi|}{\lambda}. \tag{A.1.11}$$

Applying $-\Delta_F$ to this gives the associated identity operator

$$-\Delta_F G_F = \sum_{\lambda\neq 0} |\Psi\rangle\langle\Psi| = 2\delta_c^2(z_1 - z_2)\delta_c^2(\theta_1 - \theta_2) - \sum_{\lambda=0} |\Psi\rangle\langle\Psi|. \tag{A.1.12}$$

Ignoring the zero modes, which will only be normalisable when a boundary is present, we know the functional form of the Green's function as the extension of the Laplacian Green's function

$$G_F(z_1,\theta_1;z_2,\theta_2) = -\frac{1}{4\pi}\log(z_1 - z_2 - \theta_1\theta_2)(\bar{z}_1 - \bar{z}_2 - \bar{\theta}_1\bar{\theta}_2) \tag{A.1.13}$$

neglecting any boundary effects. Applying the super-Laplacian we find that this doesn't actually satisfy (A.1.12), as applying it at z_1, θ_1 we find

$$\Delta_{F,1} G_F = 2\bar{\theta}_1(\theta_1 - \theta_2)\delta_c^2(z_1 - z_2) \tag{A.1.14}$$

i.e. it is missing some terms. We need to regulate the logarithm. The simplest way to do this is to add a small parameter, ϵ, we set to 0 as

$$G_F = -\frac{1}{4\pi}\lim_{\epsilon\to 0}\ \log(z_{12}\bar{z}_{12} + \epsilon) \tag{A.1.15}$$

where $z_{12} \equiv z_1 - z_2 - \theta_1\theta_2$. With this we can Taylor expand to find

$$G_F = -\frac{1}{4\pi}\left(\log(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) - \frac{\theta_1\theta_2}{z_1 - z_2} - \frac{\bar{\theta}_1\bar{\theta}_2}{\bar{z}_1 - \bar{z}_2} + \bar{\theta}_1\bar{\theta}_2\theta_1\theta_2 \lim_{\epsilon\to 0}\frac{\epsilon}{|z_1 - z_2|^2 + \epsilon}\right). \tag{A.1.16}$$

Taking the $\epsilon \to 0$ limit this final term becomes a delta function and so we have

$$-\frac{1}{4\pi}\left(\log(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) - \frac{\theta_1 - \theta_2}{z_1 - z_2} - \frac{\bar{\theta}_1 - \bar{\theta}_2}{\bar{z}_1 - \bar{z}_2} + 2\pi\bar{\theta}_1\bar{\theta}_2\theta_1\theta_2\delta_c^2(z_1 - z_2)\right). \tag{A.1.17}$$

This now satisfies $\Delta_F G_F = 2\delta^2(\theta_1 - \theta_2)\delta_c^2(z_1 - z_2)$.

A.2 Properties

As with any differential operator whose eigenfunctions we seek, we should test Hermiticity and therefore if the eigenvalues are real. To test the Hermiticity of the super-Laplacian we study the inner product, in direct analogy with the bosonic case,

$$\langle\Psi_n|\Delta_F\Psi_m\rangle = 4\int d^2z\ d\theta d\bar{\theta}\ \bar{\Psi}_n\bar{D}D\Psi_m$$

$$= 4\int d^2z\ d\theta d\bar{\theta}\ \Big(\bar{D}(\bar{\Psi}_n D\Psi_m) + D(\bar{D}\bar{\Psi}_n\Psi_m) + (\bar{D}D\bar{\Psi}_n)\Psi_m\Big). \tag{A.2.18}$$

The last term is

$$4\int d^2z\, d\theta d\bar{\theta}\, (\bar{D}D\bar{\Psi}_n)\Psi_m = 4\int d^2z\, d\theta d\bar{\theta}\, \overline{(D\bar{D}\Psi_n)}\Psi_m = -4\int d^2z\, d\theta d\bar{\theta}\, \overline{(\bar{D}D\Psi_n)}\Psi_m \tag{A.2.19}$$

and so we find

$$\langle\Psi_n|\Delta_F\Psi_m\rangle = -\langle\Delta_F\Psi_n|\Psi_m\rangle + 4\int d^2z\ d\theta d\bar{\theta}\ \Big(\bar{D}(\bar{\Psi}_n D\Psi_m) + D(\bar{D}\bar{\Psi}_n\Psi_m)\Big). \tag{A.2.20}$$

Now assuming the boundary terms vanish, a point we will return to, we find

$$\langle\Psi_n|\Delta_F\Psi_m\rangle = -\langle\Delta_F\Psi_n|\Psi_m\rangle \tag{A.2.21}$$

meaning the super-Laplacian is anti-hermitian. The anti-hermicity comes from the anti-commutativity of the superderivatives. Usually this would give a purely imaginary eigenvalue, but we will show that $\lambda^2 \geq 0$. We can also write the right hand side of (A.2.19) as

$$-4\int d^2z\ d\theta d\bar{\theta}\ \overline{(\bar{D}D\Psi_n)}\Psi_m = \overline{4\int d^2z\ d\theta d\bar{\theta}\ \bar{\Psi}_m(\bar{D}D\Psi_n)} \tag{A.2.22}$$

so that

$$\langle\Psi_n|\Delta_F\Psi_m\rangle = \overline{\langle\Psi_m|\Delta_F\Psi_n\rangle}. \tag{A.2.23}$$

This odd behaviour is due to the fact that our measure is anti-commutative. This leads to the conclusion

$$\lambda_m \langle\Psi_n|\Psi_m\rangle - \bar{\lambda}_n \overline{\langle\Psi_m|\Psi_n\rangle} = 0. \tag{A.2.24}$$

We can relate $\langle\Psi_m|\Psi_n\rangle$ to $\langle\Psi_n|\Psi_m\rangle$ as follows

$$\overline{\langle\Psi_m|\Psi_n\rangle} = \overline{\int d^2z d^2\theta\ \bar{\Psi}_m \Psi_n} = -\int d^2z d^2\theta\ \bar{\Psi}_n \Psi_m = -\langle\Psi_n|\Psi_m\rangle \tag{A.2.25}$$

and so (A.2.24) becomes

$$(\lambda_m + \bar{\lambda}_n)\langle\Psi_n|\Psi_m\rangle = 0. \tag{A.2.26}$$

This would usually lead to the conclusion that $\lambda = -\bar{\lambda}$ or $\lambda = i\tilde{\lambda}$ with $\tilde{\lambda} \in \mathbb{R}$. This tells us that we are doing something wrong. We will remedy this after looking at the boundary terms and the bosonic Laplacian.

A.2.1 Boundary terms

Now we turn to the boundary terms on the right hand side of (A.2.20). We can use Stoke's theorem to write them as

$$\int d^2z\ d\theta d\bar{\theta}\ \left(\bar{D}(\bar{\Psi}_n D\Psi_m) + D(\bar{D}\bar{\Psi}_n \Psi_m)\right)$$

$$= -\oint dz\ d\theta d\bar{\theta}\ \bar{\theta}\bar{\Psi}_n D\Psi_m + \oint d\bar{z}\ d\theta d\bar{\theta}\ \theta\bar{D}\bar{\Psi}_n \Psi_m. \tag{A.2.27}$$

These boundary integrals in terms of the component fields are then

$$-\oint dz\ (\bar{f}_n \partial f_m + \bar{h}_n g_m) - \oint d\bar{z}\ (\bar{\partial}\bar{f}_n f_m - \bar{g}_n h_m). \tag{A.2.28}$$

To see what these are we can consider the analogous bosonic Laplacian case

$$\int d^2z\ \bar{f}\bar{\partial}\partial f = \int d^2z\ \left(\bar{\partial}\partial\bar{f}f + \bar{\partial}(\bar{f}\partial f) - \partial(\bar{\partial}\bar{f}f)\right)$$

$$= \int d^2z\ \bar{\partial}\partial\bar{f}f - \oint(dz\bar{f}\partial f + d\bar{z}\bar{\partial}\bar{f}f). \tag{A.2.29}$$

The extra term is the exactly the f terms in (A.2.27). The remaining terms are

$$\oint (d\bar{z}\ \bar{g}_n h_m - dz\ \bar{h}_n g_m) \tag{A.2.30}$$

which we will return to. We see that f satisfies exactly the same boundary conditions as the eigenfunctions of the bosonic Laplacian. What's more the expansion of the Green's function of the super-Laplacian, G_F, is

$$G_F = -\frac{1}{2\pi}\log(|z_1 - z_2 - \theta_1\theta_2||z_1 - \bar{z}_2 - \theta_1\bar{\theta}_2|) = G_B + \theta_1\theta_2 ... \quad . \tag{A.2.31}$$

Expanding both sides in terms of their respective spectral decompositions gives a relation between f and the bosonic eigenfunctions as

$$\sum_{\lambda\neq 0} \frac{f\bar{f}}{\lambda} + \theta_1\theta_2 ... = \sum_{\lambda'\neq 0} \frac{\phi\bar{\phi}}{\lambda'} + \theta_1\theta_2 ... \tag{A.2.32}$$

where ϕ are the eigenfunctions of Δ_B with eigenvalue λ'.

A.2.2 Bosonic Laplacian

We have seen that the eigenfunctions of the super-Laplacian are also eigenfunctions of the bosonic Laplacian. We, therefore, take a brief look at the properties of the bosonic Laplacian. We begin with the eigenvalue equation

$$-4\bar{\partial}\partial\phi_n = \gamma_n\phi_n. \tag{A.2.33}$$

We can write the Green's function as a spectral decomposition of the eigenfunctions as

$$G_B = \int_{\gamma_n\neq 0} d^2p_n\ \frac{\phi_n\bar{\phi}_n}{\gamma_n} = -\frac{1}{2\pi}\log(|z_1 - z_2||z_1 - \bar{z}_2|). \tag{A.2.34}$$

Then we have

$$-\Delta_B G_B = \int_{\gamma_n\neq 0} d^2p_n\ \phi_n\bar{\phi}_n = 2\delta_c^2(z_1 - z_2). \tag{A.2.35}$$

We can define an inner product on states

$$\langle \phi_n | \phi_m \rangle \equiv \int d^2 z \ \bar{\phi}_n \phi_m. \tag{A.2.36}$$

The bosonic Laplacian is Hermitian and so we can use this to prove γ is real

$$\langle \phi_n | \Delta_B \phi_m \rangle = \langle \Delta_B \phi_n | \phi_m \rangle \tag{A.2.37}$$

which leads to

$$(\bar{\gamma}_n - \gamma_m) \langle \phi_n | \phi_m \rangle = 0 \tag{A.2.38}$$

hence $\bar{\gamma}_n = \gamma_n$ and the eigenfunctions are orthogonal. The usual convention is to choose orthonormal eigenfunctions such that

$$\langle \phi_n | \phi_m \rangle = \delta_{nm}. \tag{A.2.39}$$

We can also prove that $\gamma \geq 0$:

$$-4\gamma \int d^2 z \ \bar{\phi}\phi = \int d^2 z \ \bar{\phi}\bar{\partial}\partial\phi = -\int \bar{\partial}\bar{\phi}\bar{\partial}\phi = -\int d^2 z \ |\partial\phi|^2 \leq 0 \tag{A.2.40}$$

and therefore $\gamma \geq 0$. The heat kernel has spectral decomposition

$$K_B(t, z_1, z_2) = \int d^2 p_n \ e^{-\lambda_n t} \phi_n \bar{\phi}_n \tag{A.2.41}$$

and solves the heat equation

$$(\partial_t - \Delta_B) K_B = 0. \tag{A.2.42}$$

It also satisfies

$$K_B(0, z_1, z_2) = \int d^2 p_n \ \phi_n \bar{\phi}_n = \mathbb{I}_B. \tag{A.2.43}$$

Expanding the exponential in powers of t we find the relation

$$K_B(t, z_1, z_2) = \int d^2 p_n \ e^{-\lambda_n t} \phi_n \bar{\phi}_n = \int d^2 p_n \ \sum_m \frac{(-\lambda_n t)^m}{m!} \phi_n \bar{\phi}_n \tag{A.2.44}$$

then we can use the relation $\lambda^n \phi = (-1)^m \Delta_B^m \phi$ to write

$$K_B(t, z_1, z_2) = \sum_m \frac{t^m}{m!} \Delta_B^m \int d^2 p_n \ \phi_n \bar{\phi}_n = e^{t\Delta_B} \mathbb{I}_B. \tag{A.2.45}$$

A.2.3 Super-Laplacian

Each component eigenfunction of Ψ satisfies the bosonic Laplacian eigenvalue equation from (A.1.4), and so we can use the above information to determine that λ is real. First of all, the commuting components, f and k, satisfy

$$\langle f_n | f_m \rangle_B \equiv \int d^2 z \ \bar{f}_n f_m = \delta_{nm} \tag{A.2.46}$$

with $\lambda^2 \geq 0 \to \lambda \geq 0$ and real. The anti-commuting components, g and h, satisfy

$$\langle g_n | g_m \rangle_B = \alpha \delta_{nm} \tag{A.2.47}$$

with $\alpha = 1 = -\bar{\alpha}$. In this case we have $\lambda^2 = \bar{\lambda}^2$ so that either $\lambda \in \mathbb{I}, \Im(\lambda) \leq 0$ or $\lambda \in \mathbb{R}, \lambda \geq 0$. But g and f share eigenvalues and so we should choose $\lambda \in \mathbb{R}$. Going back to (A.2.26) we have

$$(\lambda_m + \lambda_n) \langle \Psi_n | \Psi_m \rangle = 0 \tag{A.2.48}$$

which suggests that

$$\langle \Psi_n | \Psi_m \rangle \sim \delta(\lambda_n + \lambda_m). \tag{A.2.49}$$

This means that $\langle \Psi_n | \Psi_n \rangle = 0$ and $\langle \Psi_{-n} | \Psi_n \rangle \neq 0$. This suggests that we should consider the two sets of eigenfunctions $\Psi_\pm$ with corresponding eigenvalues $\lambda_\pm$. The final piece we need is the fact that g is Grassmann-odd so it must go like $g \sim \eta e^{i\mathbf{k}\cdot\mathbf{x}}$ where $\eta_1 \eta_2 = -\eta_2 \eta_1$.

We now have enough information to determine (A.2.49). We know that the right hand side is only non-zero when $\lambda_m = -\lambda_n$ hence we shall consider the inner product $\langle \Psi_{-n} | \Psi_{+m} \rangle$. We will define it to be equal to

$$\langle \Psi_{-n} | \Psi_{+m} \rangle \equiv c_{nm} \delta_c^2 (p_n - p_m) \tag{A.2.50}$$

where $c_{nm} = c(\bar{\eta}_n, \eta_m)$ is a function to be determined. Writing this out in terms of its component fields gives

$$\langle \Psi_{-n} | \Psi_{+m} \rangle = \frac{(\lambda_n + \lambda_m)}{4} \int d^2 z \; \bar{f}_n f_m - \frac{(\lambda_n + \lambda_m)}{\lambda_n} \int d^2 z \; \bar{g}_n g_m. \tag{A.2.51}$$

Now we can take out the factors of $\bar{\eta}_n$ and η_m and rescale the fields to $f \equiv \sqrt{\frac{2}{\lambda}} F$ and $g \equiv \frac{\eta}{\sqrt{2}} G$ so that the above becomes

$$\langle \Psi_{-n} | \Psi_{+m} \rangle = \int d^2 z \; \bar{F}_n F_m - \bar{\eta}_n \eta_m \int d^2 z \; \bar{G}_n G_m \tag{A.2.52}$$

F and G are still eigenfunctions of the bosonic Laplacian with the same eigenvalues and so choosing them to be orthonormal we have

$$\langle \Psi_{-n} | \Psi_{+m} \rangle = (1 - \bar{\eta}_n \eta_m) \delta_c^2 (p_n - p_m) \tag{A.2.53}$$

so that $c_{nm} = 1 - \bar{\eta}_n \eta_m$. We can also write

$$\langle \Psi_{+n} | \Psi_{-m} \rangle = \tilde{c}_{nm} \delta_c^2 (p_n - p_m). \tag{A.2.54}$$

We can determine $\tilde{c}$ from (A.2.25). We find

$$\langle \Psi_{+n} | \Psi_{-m} \rangle = -(1 + \bar{\eta}_n \eta_m) \delta_c^2 (p_n - p_m) \tag{A.2.55}$$

and so $\tilde{c}_{nm} = -(1 + \bar{\eta}_n \eta_m)$. We can now write down the identity operator on the space of super-Laplacian eigenfunctions in terms of c and $\bar{c}$ as

$$\mathbb{I} = \int d^2 p_i d^2 \eta_i \left(\frac{\Psi_{+i} \bar{\Psi}_{-i}}{c_{ii}} - \frac{\Psi_{-i} \bar{\Psi}_{+i}}{\tilde{c}_{ii}} \right). \tag{A.2.56}$$

Which on the face of it looks like a non-trivial combination of the fields and c's. Note that this can be written as

$$\mathbb{I} = \int d^2 p_i d^2 \eta_i \left(\Psi_{+i} \bar{\Psi}_{-i} (1 + \bar{\eta}_i \eta_i) + \Psi_{-i} \bar{\Psi}_{+i} (1 - \bar{\eta}_i \eta_i) \right)$$

$$= \int d^2p_i d^2\eta_i \left((\Psi_{+i}\bar{\Psi}_{-i} + \Psi_{-i}\bar{\Psi}_{+i}) + \bar{\eta}_i\eta_i((\Psi_{+i}\bar{\Psi}_{-i} - \Psi_{-i}\bar{\Psi}_{+i}) \right). \tag{A.2.57}$$

Now due to the η integration the first term in parenthesis is only a function of G and the second term is only a function F. Applying this to a general state $\Psi = \int d^2p_j\ (a_j\Psi_{+j} + b_j\Psi_{-j})$ gives

$$\mathbb{I}\cdot\Psi = \int d^2p_i d^2\eta_i d^2p_j \left(\Psi_{+i}(1+\bar{\eta}_i\eta_i)a_j \int d^2z d^2\theta\ \bar{\Psi}_{-i}\Psi_{+j} \right.$$

$$\left. +\Psi_{-i}(1-\bar{\eta}_i\eta_i)b_j \int d^2z d^2\theta\ \bar{\Psi}_{+i}\Psi_{-j} \right)$$

$$= \int d^2p_i d^2\eta_i d^2p_j \left(a_j\Psi_{+i}(1+\bar{\eta}_i\eta_i)(1-\bar{\eta}_i\eta_j)\delta_c^2(p_i-p_j) \right.$$

$$\left. = -b_j\Psi_{-i}(1-\bar{\eta}_i\eta_i)(1+\bar{\eta}_i\eta_j)\delta_c^2(p_i-p_j) \right)$$

$$= \int d^2p_j d^2\eta_i \Big(a_i\Psi_{+i}\bar{\eta}_i(\eta_i-\eta_j) + b_i\Psi_{-i}\bar{\eta}_i(\eta_i-\eta_j) \Big). \tag{A.2.58}$$

It's straightforward to show that $\int d^2\eta_i\ \Psi(\eta_i)\bar{\eta}_i(\eta_i - \eta) = \Psi(\eta)$ since Ψ is holomorphic in η so that $\bar{\eta}_i(\eta_i - \eta_j)$ acts as the delta function. Then we are left with our general state that we started with confirming to us that $\mathbb{I}$ is indeed the identity operator. From this we obtain the Green's function

$$G_F = \int_{\lambda>0} d^2p_i d^2\eta_i \left(\frac{\Psi_{+i}\bar{\Psi}_{-i}}{\lambda_i c_{ii}} + \frac{\Psi_{-i}\bar{\Psi}_{+i}}{\lambda_i \tilde{c}_{ii}} \right) \tag{A.2.59}$$

which can be written as

$$G_F = \int_{\lambda>0} d^2p_i d^2\eta_i\ \frac{1}{\lambda_i}\Big((\Psi_{+i}\bar{\Psi}_{-i} - \Psi_{-i}\bar{\Psi}_{+i}) + \bar{\eta}_i\eta_i((\Psi_{+i}\bar{\Psi}_{-i} + \Psi_{-i}\bar{\Psi}_{+i}) \Big). \tag{A.2.60}$$

A.2.4 The heat kernel

The heat kernel in flat 2 dimensional space is

$$K_B = \frac{1}{4\pi t} e^{-\frac{(z_1-z_2)(\bar{z}_1-\bar{z}_2)}{4t}}. \tag{A.2.61}$$

It satisfies the bosonic heat equation

$$(\partial_t - \Delta_B)K_B = 0. \tag{A.2.62}$$

The straightforward generalisation to superspace is found by replacing the bosonic displacement $z_1 - z_2$ with the fermionic displacement $z_{12} \equiv z_1 - z_2 - \theta_1\theta_2$. This gives the fermionic heat kernel

$$K_F = \frac{1}{4\pi t} e^{-\frac{z_{12}\bar{z}_{12}}{4t}} \tag{A.2.63}$$

and satisfies a generalisation of the bosonic heat equation

$$(\delta^2(\theta_1 - \theta_2)\partial_t - \Delta_F)K_F = 0. \tag{A.2.64}$$

It is related to the Green's function by

$$G_F = \lim_{\epsilon \to 0} \int_\epsilon^\infty K_F \; dt. \tag{A.2.65}$$

We can expand K_F in powers of θ:

$$K_F = K_B + \theta_1\theta_2 \; \partial_2 K_B + \bar{\theta}_1\bar{\theta}_2 \; \bar{\partial}_2 K_B - \theta_1\theta_2\bar{\theta}_1\bar{\theta}_2 \; \bar{\partial}_2\partial_2 K_B. \tag{A.2.66}$$

Now we turn to the spectral decomposition of the heat kernel. The bosonic case is simply:

$$K_B = \int d^2 p_i \; e^{-\frac{\lambda^2 t}{4}} \phi_i \bar{\phi}_i \tag{A.2.67}$$

where $-\Delta_B \phi_i = \frac{\lambda_i^2}{4}\phi_i$. From (A.2.59), we can straight forwardly write the heat kernel as

$$K_F(t, z_1, z_2, \theta_1, \theta_2) = \int_{\lambda > 0} d^2 p_i d^2 \eta_i \left(e^{-\lambda_i t} \frac{\Psi_{+i}\bar{\Psi}_{-i}}{c_{ii}} - e^{\lambda_i t} \frac{\Psi_{-i}\bar{\Psi}_{+i}}{\tilde{c}_{ii}} \right) \tag{A.2.68}$$

and satisfies

$$K_F(0) = \mathbb{I} - \Psi_0 \bar{\Psi}_0. \tag{A.2.69}$$

Then

$$\int_0^\infty dt\,\left(K_F(t)+\Psi_0\bar{\Psi}_0\right)=\int_{\lambda>0}d^2p_id^2\eta_i\left(\frac{\Psi_{+i}\bar{\Psi}_{-i}}{\lambda_ic_{ii}}+\frac{\Psi_{-i}\bar{\Psi}_{+i}}{\lambda_i\tilde{c}_{ii}}\right)-\Psi_F=G_F.\quad\text{(A.2.70)}$$

Inserting c and $\tilde{c}$ allows us to write the fermionic heat kernel as

$$K_F=\int d^2p_id^2\eta\;e^{-\lambda t}\big[(\Psi_+\bar{\Psi}_--\Psi_-\bar{\Psi}_+)+\bar{\eta}\eta(\Psi_+\bar{\Psi}_-+\Psi_-\bar{\Psi}_+)\big].\qquad\text{(A.2.71)}$$

In this form it is not easy to see that K_F is equivalent to (A.2.63). To see that they are equivalent we consider the integral over t

$$\int K_F\;dt=\int dt\;d^2p_id^2\eta\;e^{-\lambda t}\big[(\Psi_+\bar{\Psi}_--\Psi_-\bar{\Psi}_+)+\bar{\eta}\eta(\Psi_+\bar{\Psi}_-+\Psi_-\bar{\Psi}_+)\big].\quad\text{(A.2.72)}$$

Now make the transformation $t\rightarrow\frac{\lambda t}{4}$ so that this becomes

$$\int K_F\;dt=\int dt\;d^2p_id^2\eta\;\frac{\lambda}{4}e^{-\frac{\lambda^2t}{4}}\big[(\Psi_+\bar{\Psi}_--\Psi_-\bar{\Psi}_+)+\bar{\eta}\eta(\Psi_+\bar{\Psi}_-+\Psi_-\bar{\Psi}_+)\big].\quad\text{(A.2.73)}$$

If we take K_F to be understood under the t integral then we can take it to be

$$K_F=\int d^2p_id^2\eta\;\frac{\lambda}{4}e^{-\frac{\lambda^2t}{4}}\big[(\Psi_+\bar{\Psi}_--\Psi_-\bar{\Psi}_+)+\bar{\eta}\eta(\Psi_+\bar{\Psi}_-+\Psi_-\bar{\Psi}_+)\big].\qquad\text{(A.2.74)}$$

We now prove that (A.2.74) is equal to (A.2.63). We can show this by expanding in powers of θ and showing they are equal to each term of (A.2.66). Firstly, we can write the eigenfunctions as

$$\Psi_\pm=\sqrt{\frac{2}{\lambda}}F_\pm\left(1\mp\frac{\lambda}{4}\theta\bar{\theta}\right)+\frac{\theta\eta}{\sqrt{2}}G_\pm\mp\frac{4\bar{\theta}\eta}{\sqrt{2}\lambda}\bar{\partial}G_\pm.\qquad\text{(A.2.75)}$$

Then consider firstly the purely bosonic term of (A.2.74)

$$K_F\ni\int d^2p_id^2\eta\;\frac{\lambda}{4}e^{-\frac{\lambda^2t}{4}}\bar{\eta}\eta\frac{4F\bar{F}}{\lambda}=\int d^2p_i\;e^{-\frac{\lambda^2t}{4}}F\bar{F}=K_B.\qquad\text{(A.2.76)}$$

So our purely bosonic terms match. Now consider the $\theta_1\theta_2$ term

$$K_F \ni \int d^2p_i d^2\eta \; \frac{\lambda}{4} e^{-\frac{\lambda^2 t}{4}} \; \theta_1\theta_2 \; \bar{\eta}\eta \; \frac{4}{\lambda} G\partial_2\bar{G}$$

$$= \theta_1\theta_2\partial_2 \int d^2p_i \; e^{-\frac{\lambda^2 t}{4}} \; G\bar{G} = \theta_1\theta_2\partial_2 K_B$$

which matches the $\theta_1\theta_2$ term of (A.2.66). Finally the $\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2$ term

$$K_F \ni \int d^2p_i d^2\eta \; \frac{\lambda}{4} e^{-\frac{\lambda^2 t}{4}} \; 2\left(\frac{\lambda}{4}\right)^3 \theta_1\bar{\theta}_1\theta_2\bar{\theta}_2 \; f\bar{f}$$

$$= -\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2\bar{\partial}_2\partial_2 \int d^2p_i d^2\eta \; e^{-\frac{\lambda^2 t}{4}} F\bar{F} = -\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2\bar{\partial}_2\partial_2 K_B \tag{A.2.77}$$

as expected. K_F satisfies

$$\int_0^\infty K_F dt = \int d^2p_i d^2\eta \; \frac{1}{\lambda}\big[(\Psi_+\bar{\Psi}_- - \Psi_-\bar{\Psi}_+) + \bar{\eta}\eta(\Psi_+\bar{\Psi}_- + \Psi_-\bar{\Psi}_+)\big] \tag{A.2.78}$$

which is the Green's function, G_F, that we found earlier. Note that

$$K_F(t=0) = \int d^2p_i d^2\eta \; \frac{\lambda}{4}\big[(\Psi_+\bar{\Psi}_- - \Psi_-\bar{\Psi}_+) + \bar{\eta}\eta(\Psi_+\bar{\Psi}_- + \Psi_-\bar{\Psi}_+)\big]$$

$$= -\frac{\Delta_F}{4}\mathbb{I} = \delta^2(z_1 - z_2 - \theta_1\theta_2). \tag{A.2.79}$$

Then

$$\delta^2(\theta_1 - \theta_2) K_F(0) = \delta^2(\theta_1 - \theta_2)\delta^2(z_1 - z_2). \tag{A.2.80}$$

So that

$$\Delta_F \int K_F \; dt = \delta^2(\theta_1 - \theta_2) K_F(0) = \delta^2(\theta_1 - \theta_2)\delta^2(z_1 - z_2). \tag{A.2.81}$$

We can expand (A.2.68) as we did in the bosonic case

$$K_F = \int_{\lambda>0} d^2p_i d^2\eta_i \; \sum_m \left(\frac{(-\lambda_i t)^m}{m!}\frac{\Psi_{+i}\bar{\Psi}_{-i}}{c_{ii}} - \frac{(\lambda_i t)^m}{m!}\frac{\Psi_{-i}\bar{\Psi}_{+i}}{\tilde{c}_{ii}}\right) \tag{A.2.82}$$

then using $\lambda^m \Psi_\pm = (\mp 1)^m \Delta_F^m \Psi_\pm$ we get

$$K_F = e^{t\Delta_F} \mathbb{I}_F \tag{A.2.83}$$

in analogy with the bosonic case. We can write the operator as $e^{t\Delta_F}$

$$e^{t\Delta_F} = \sum_{n=0}^{\infty} \frac{(t\Delta_F^n)}{n!}. \tag{A.2.84}$$

Now $\Delta_F^{2n} = (-1)^n \Delta_B^n$ and $\Delta_F^{2n+1} = (-1)^n \Delta_F \Delta_B^n$ and so we can split the sum into odd and even powered terms

$$e^{t\Delta_F} = \sum_{n=0}^{\infty} \left(\frac{(-1)^n t^{2n} \Delta_B^n}{(2n)!} + \frac{(-1)^n t^{2n+1} \Delta_F \Delta_F^n}{(2n+1)!} \right) \tag{A.2.85}$$

which gives the rather nice form

$$e^{t\Delta_F} = \cos(t\sqrt{\Delta_B}) + \frac{\Delta_F}{\sqrt{\Delta_B}} \sin(t\sqrt{\Delta_B}). \tag{A.2.86}$$

One could also write

$$e^{t\Delta_F} = \cos(t\sqrt{\Delta_B}) + t\Delta_F \ \mathrm{sinc}(t\sqrt{\Delta_B}). \tag{A.2.87}$$

A.2.5 Boundary terms, again

We can go back to (A.2.30) and complete our analysis of the boundary terms. If instead of looking at $\langle \Psi_n | \Psi_n \rangle$ we look at $\langle \Psi_{-n} | \Psi_{+n} \rangle$, then (A.2.30) becomes

$$\oint (d\bar{z} \ \bar{g}_{-n} h_{+m} - dz \ \bar{h}_{-n} g_{+n}). \tag{A.2.88}$$

The relations (A.1.9) allow us to write this as

$$-\frac{4}{\lambda} \oint (d\bar{z} \ \bar{g}_n \bar{\partial} g_n + dz \ \partial \bar{g}_n g_n). \tag{A.2.89}$$

After an integration by parts we find that this is

$$\frac{4}{\lambda}\oint \left(d\bar{z}\ \bar{\partial}\bar{g}_n g_n + dz\ \bar{g}_n \partial g_n\right) \tag{A.2.90}$$

which now has the same form as the bosonic eigenfunction boundary conditions (A.2.29), as we would expect as both f and g are eigenfunctions of the bosonic Laplacian.

A.2.6 Useful formulae

$$D_1\ \delta_c^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) = -(\bar{\theta}_1 - \bar{\theta}_2)\Big(\delta_c^2(z_1 - z_2) - \theta_1\theta_2\partial_1\delta_c^2(z_1 - z_2)\Big) \tag{A.2.91}$$

$$\bar{D}_1\delta_c^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) = (\theta_1 - \theta_2)\Big(\delta_c^2(z_1 - z_2) - \bar{\theta}_1\bar{\theta}_2\bar{\partial}_1\delta_c^2(z_1 - z_2)\Big) \tag{A.2.92}$$

$$-\bar{D}D\ \delta_c^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) = \delta^2(z_1 - z_2 - \theta_1\theta_2). \tag{A.2.93}$$

Then from the equation

$$-4\bar{D}DG_F = 2\delta_c^2(z_1 - z_2)\delta^2(\theta_1 - \theta_2) \tag{A.2.94}$$

we find

$$-\bar{D}D(-4\bar{D}DG_F) = -4\bar{D}^2D^2G_F = -\Delta_B G_F = 2\delta^2(z_1 - z_2 - \theta_1\theta_2) \tag{A.2.95}$$