Supersymmetric Sigma Models And Their Indices

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Abstract: Supersymmetric indices for σ-models are known to compute topological invariants of the target space on which the σ-model is built. In the case where the target space is a $K3$ surface, the worldsheet of the σ-model enjoys an $\mathcal{N} = 4$ superconformal symmetry. A supersymmetric index known as the elliptic genus can be constructed for this theory and decomposed into a sum of massless and massive characters of the $\mathcal{N} = 4$ superconformal algebra governing the symmetries. This index exhibits a phenomenon known as Mathieu moonshine, in which the coefficients of the massive characters in that decomposition are dimensions of representations of the sporadic group Mathieu 24. In this thesis, motivated by this moonshine phenomenon for theories with $\mathcal{N} = 4$ superconformal symmetries, we consider σ-models which exhibit a larger $\mathcal{N} = 4$ superconformal symmetry on the worldsheet, and discuss two supersymmetric indices which could be applied to such σ-models in search of a new moonshine. We discuss the states which contribute to these indices and calculate one of them for some specific theories.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

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Chapter 1

Introduction

The world is both relativistic and quantum mechanical. The Standard Model of particle physics is a quantum field theory which reconciles the principles of special relativity and quantum mechanics and is, simultaneously, a gauge theory that unifies three of the four fundamental forces of Nature. It is perhaps the most well-tested theory of modern physics. Its success tells us that any fundamental theory should look like a quantum field theory at sufficiently low energies. However, incorporating gravity within this framework has proven impossible so far, as quantum corrections to general relativity diverge very badly. The inability to reconcile quantum mechanics and general relativity suggests that the quantum theory which will unify the four fundamental forces may not even be a field theory, although it must encompass the Standard Model as an effective low energy theory.

String theory is one such quantum theory. By nature, a relativistic quantum string theory is a theory of general relativity that contains gauge interactions and avoids the ultraviolet divergences that plague quantum theories of relativistic particles. This is because its fundamental objects, the one-dimensional strings, do not interact at spacetime points. String theory has been an active area of research since the 1960's and many good introductory texts on the subject exist, such as [Ton09; BBS06; Pol98; GSW87; Kir11; BLT12; Zwi04] among others.

In string theory, the string sweeps out a two-dimensional surface known as the string worldsheet as it evolves. Two-dimensional non-linear sigma models (σ-models) therefore arise naturally in the context of string theory, where the fields are interpreted as spacetime coordinates, i.e. they are maps from the string worldsheet into the 'target space', which we call spacetime. The Polyakov action, that is the classical free bosonic string action, is invariant under global spacetime Poincaré transformations, local changes of scale (Weyl transformations) and two-dimensional reparameterisations, which include local conformal transformations. The quantisation of the Polyakov action may be achieved through the path integral formalism, and requires the elimination of as much of the redundancy encoded in the local symmetries of the Polyakov action as possible, through a process called 'gauge-fixing'. After gauge-fixing, there remains enough symmetry to fix the two-dimensional intrinsic metric through combining a local conformal transformation and a Weyl rescaling. This residual symmetry allows a portion of the Polyakov action to be considered as an action where the intrinsic two-dimensional metric is fixed (also called a non-linear σ-model), which therefore corresponds to a two-dimensional conformal field theory (CFT). Since the bosonic string theory propagating on D-dimensional Minkowski space is consistent only if $D = 26$, one might consider substituting 22 of the string spacetime coordinates with a CFT through the use of the residual symmetry described above, and interpret this as a compactification from 26 to 4 spacetime dimensions. The bosonic string, however, suffers from the presence of tachyons and the absence of fermions, two facts that are in stark contradiction with observations.

It is remarkable that the introduction of fermions in string theory naturally leads to the concept of supersymmetry, which in turn eliminates tachyons from the theory. The basic idea is to introduce one fermionic partner for each bosonic string coordinate in the form of a two-dimensional spinor in the Polyakov action, and to impose a two-dimensional (or worldsheet) supersymmetry that transforms bosonic and fermionic degrees of freedom into each other. This leads to a superstring theory which is consistent only in 10 dimensions. As in the case of bosonic string theory, the superstring action retains some residual symmetry after gauge-fixing. It is again possible to replace a portion of the (super) Polyakov action by the action of a (super) CFT, a supersymmetric non-linear σ-model, and use this mechanism to compactify down from 10 spacetime dimensions in an attempt to model Nature realistically. The choice of compactified space (target manifold) determines the spectrum of the theory after compactification. In particular, Calabi-Yau 3-folds have received a lot of attention as their geometry allows for spacetime supersymmetry, which guarantees a tachyon-free theory. Furthermore, the existence of covariantly constant complex structures on the target manifold is intimately linked to the presence of extended supersymmetry on the worldsheet, and therefore Calabi-Yau 3-folds allow for $N = 1$ spacetime supersymmetry and $\mathcal{N} = 2$ extended worldsheet supersymmetry. This may sound phenomenologically promising, but the number of non-diffeomorphic Calabi-Yau 3-folds is unknown and the problem of knowing how to choose one is known as the string landscape problem.

A classification of supersymmetric σ-models was provided in [AF81], and the authors argued that $\mathcal{N} = 4$ was the largest amount of worldsheet supersymmetry one could obtain for a σ-model. In particular, they showed that $\mathcal{N} = 4$ extended supersymmetry occurs when the target space is hyperkähler. In two complex dimensions, such a space is either a 2-tori or a $K3$ surface, which is a simply connected compact Kähler manifold of complex dimension two admitting a Ricci-flat metric. All $K3$ surfaces are diffeomorphic.

$K3$ theories, on the other hand, are $\mathcal{N} = (2, 2)$ superconformal field theories at central charges $c = 6$, $\bar{c} = 6$ with spacetime supersymmetry, integral left and right-moving $\mathfrak{u}(1)$ charges and elliptic genus (discussed further on page 4) given by the elliptic genus of $K3$ [Wit88; HBJL92; Gri00; Wen15]. In other words, a $K3$ theory is an $\mathcal{N} = (4, 4)$ superconformal field theory at central charges $c = 6, \bar{c} = 6$ and elliptic genus given by the (geometric) elliptic genus of $K3$. Although a proof that every $K3$ theory allows a non-linear $\sigma-$model interpretation on a $K3$ surface does not exist to date, compelling arguments put forward in [Wen15; NW01; Wen00] strengthen the expectation that the statement is correct. It is in this string-related context that the Mathieu moonshine phenomenon, which we elaborate on below, was observed.

The underlying algebraic structure of $K3$ theories is a left and a right $\mathcal{N} = 4$ superconformal algebra (SCA) at central charge $c = 6$. The modular properties of the $\mathcal{N} = 4$, $c = 6$ characters, which are the building blocks of the worldsheet partition function, are such that it has been impossible so far to write a generic modular invariant partition function for the $c = 6, \bar{c} = 6$, $\mathcal{N} = (4, 4)$ SCFT (i.e. for a generic point in the moduli space of SCFTs describing strings compactified on $K3$). However we know how to do so at specific points in that moduli space. In particular, one way of constructing such an $\mathcal{N} = (4, 4)$ partition function is through the use of Gepner models [Gep87]. Their construction involves taking the tensor product of minimal $\mathcal{N} = 2$ theories in order to construct a theory with $c = 6$, and augmenting the algebra generated by the $\mathcal{N} = 2$ SCA of each factor by the operator of two-fold spectral flow [EOTY89]. This gives a method of constructing modular invariant partition functions for $\mathcal{N} = (4, 4)$ theories based on the known modular properties of the minimal $\mathcal{N} = 2$ characters. Such a partition function depends on variables q, $\bar{q}, z, \bar{z}$, and can be written as a power series in q, $\bar{q}$ with a typical term being of the form $c(m, n, j, \bar{j})\, q^m \bar{q}^n\, z^j \bar{z}^{\bar{j}}$ where $c(m, n, j, \bar{j})$ is the number of states with conformal weights (m, n) and $\mathfrak{u}(1)$ charges $(j, \bar{j})$. Here $\mathfrak{u}(1)$ is the zero mode subalgebra of the $\widehat{\mathfrak{u}(1)}$ Kac-Moody subalgebra of $\mathcal{N} = 2$ (or of the $\widehat{\mathfrak{su}(2)}$ Kac-Moody subalgebra of $\mathcal{N} = 4$). The partition functions of these theories clearly depend on the combination of minimal $\mathcal{N} = 2$ theories which are tensored together. However, one can construct a moduli-independent quantity known as the (conformal field theoretic) elliptic genus, a quantity first introduced in the context of field theories by Witten [Wit87]. For an $\mathcal{N} = 2$ or $\mathcal{N} = 4$ theory the elliptic genus may be defined as the restriction of the partition function to the $\tilde{R}$ sector (the sector where fermions are periodic in both torus periods) and evaluated at the point $\bar{z} = 1$. This has the effect of projecting onto only right-moving ground states, and hence this quantity counts $\frac{1}{4}$-BPS states.

The elliptic genus is an example of a supersymmetric index, that is a quantity which is invariant under small perturbations of the relevant moduli. The first example of such a quantity is the Witten index $\mathrm{Tr}(-1)^F$, introduced by Witten in order to study supersymmetry breaking [Wit82]. The Witten index counts all bosonic states with a $+1$, and all fermionic states with a -1. In a theory with spontaneously broken supersymmetry, where the ground state has positive energy, all bosonic states have fermionic partners, and hence a suitably regularised version of this sum is guaranteed to be zero. In an unbroken theory, this quantity gives the difference between the number of bosonic and fermionic ground states. Witten showed that for a one-dimensional non-linear σ-model with target space M, $\mathrm{Tr}(-1)^F$ is equal to the Euler characteristic $\chi(M)$. As an index, the elliptic genus of a two-dimensional σ-model is also moduli space invariant and can be shown to be related to other topological invariants of the target space. As a specialisation of the modular invariant partition function, the elliptic genus also has well defined modular properties and can be shown to be a weak Jacobi form of weight zero and index one. The space of such forms is one-dimensional and hence the elliptic genus can easily be written in terms of Jacobi theta functions [EOT11].

Since the elliptic genus is constant across connected components of the moduli space of $K3$ compactifications, the Gepner models give a simple way to calculate the elliptic genus of any $K3$ compactification [EOTY89]. Eguchi, Ooguri and Tachikawa [EOT11] observed that when this elliptic genus was expanded in terms of the characters of the underlying $\mathcal{N} = 4$ SCA, a mock-modular form $\Sigma(\tau)$, multiplied by the massive $\mathcal{N} = 4$ characters at threshold was obtained. Furthermore, the first few coefficients of $\Sigma(\tau)$ as a q-series were calculated, and were all observed to be dimensions of representations of the sporadic group Mathieu 24 (M_{24}). This suggests that a graded module of M_{24} with $\Sigma(\tau)$ as its graded dimension exists.

This phenomenon has become known as 'Mathieu moonshine' due to its similarities with a phenomenon called 'monstrous moonshine'. In monstrous moonshine the coefficients in the q-expansion of a particularly important modular function known as Klein's j-invariant (or simply, the j function) were noticed to coincide with dimensions of representations of the largest sporadic group, the Fischer-Griess Monster (M) [CN79]. It was therefore conjectured that there existed a Monster module which had $j(\tau)$ as its graded dimension. A proof of this follows from bosonic string theory compactified on a $\mathbb{Z}_2$-orbifold of a real 24-dimensional torus known as the Leech torus $T^{\Lambda_{24}}$ [DGH88]. This torus is formed by quotienting $\mathbb{R}^{24}$ by the unique even unimodular rank-24 lattice without roots, the famous Leech lattice Λ_{24} [CS13]. The chiral part of the CFT describing the worldsheet theory has an action of M and partition function given by $j(\tau)$. Given the similarities between the monstrous and Mathieu moonshines, one should consider moonshine to be the study of surprising connections between the representation theory of sporadic groups and modular (as well as mock-modular) forms.

Gannon has proved that in $\Sigma(\tau) = q^{-1/8}(-2 + \sum_{n\in\mathbb{N}} A_n q^n)$, the coefficients A_n are all characters of representations of M_{24} [Gan16] and hence proved the existence of the conjectured Mathieu moonshine module. However, the graded module of M_{24} has not been explicitly constructed and the origin of M_{24} symmetry in $K3$ σ-models is still poorly understood.

Subsequently, Mathieu moonshine has been incorporated into a larger theory of moonshine known as 'umbral moonshine' [CDH14a; CDH14b]. In umbral moonshine, the Niemeier lattices, the remaining 23 even unimodular rank-24 lattices (with roots), are also connected to a moonshine. These lattices are uniquely determined by their root systems, which admit an ADE classification. The Niemeier lattices are therefore referred to as $(X)^+$, where X is an ADE root system whose components all have the same Coxeter number. Specifically, there exists a process for constructing a (vector-valued) mock modular form known as the umbral form, for each lattice $(X)^+$. For each form the coefficients in the q-expansion are observed to be dimensions of representations of a group known as the umbral group G^X (which is defined for each lattice $(X)^+$). When one takes the Niemeier lattice $(A_1^{24})^+$, the umbral group can be shown to be M_{24}, and the (single component, vector valued) umbral form is $\Sigma(\tau)$. In this way Mathieu moonshine may be viewed as one component of umbral moonshine.

However, based on the above definition of moonshine, it should be noted that not all of the umbral groups are sporadic, and so the definition of moonshine should perhaps be weakened to include finite non-sporadic groups. Umbral moonshine has also been connected to the elliptic genus of $K3$ σ-models through the ADE classification of du Val singular points that a $K3$ surface may possess [CH15]. In particular, a way to split the elliptic genus into a 'singularity' term and a term dependent on the (vector valued) umbral forms was described. As for the other examples of moonshine, for each of the umbral groups a graded module is conjectured to exist whose graded dimension gives the umbral forms. That such a module exists in each case has been proved [DGO15], though as for Mathieu moonshine, in all but one case no construction of the module exists [DH14]. There also exist moonshine conjectures for other sporadic groups including the pariahs [GM16; DMO17a; DMO17b] though a discussion of these is beyond the scope of this thesis.

In the Mathieu and umbral moonshines (viewed separately), the $\mathcal{N} = 4$ symmetry plays a key role. In Mathieu moonshine, the importance of $\mathcal{N} = 4$ came from decomposing the elliptic genus into $\mathcal{N} = 4$ characters and identifying the function multiplying the massive character at threshold. In the umbral moonshine case, the $\mathcal{N} = 4$ characters are used to construct the umbral forms. Moreover, the splitting of the elliptic genus of $K3$ in [CH15] is also defined in terms of $\mathcal{N} = 4$ characters. However, as shown in [SSTV88a; STVS88] if Wess-Zumino terms are added to the σ-model then, on non-abelian group manifolds, a larger SCA than the usual 'small' $\mathcal{N} = 4$ SCA discussed above can be obtained, namely the A_γ SCA we now introduce.

Besides Calabi-Yau manifolds, other types of target manifolds include orbifolds and group manifolds. This thesis will be primarily concerned with SCFTs with $\mathcal{N} = 4$ extended worldsheet supersymmetry, known as 'large' $\mathcal{N} = 4$ theories, or again as A_γ theories, where γ is a real parameter. These were first studied in [SSTV88a; STVS88] and are related there to compactifications on group manifolds. The A_γ SCA also provides a unifying viewpoint in the context of $\mathcal{N} = 4$ Liouville theory, as for two specific values of the A_γ central charge, corresponding to two different dilaton background charges, the theory reduces to the Coulomb branch ('short string' sector) and the Higgs branch ('long string' sector) of a string theory in an NS5-NS1 background [ES16; CHS91]. As we shall see, the algebra A_γ contains a greater number of operators than the 'small' $\mathcal{N} = 4$ algebra associated with $K3$ compactifications, which we shall just refer to as the $\mathcal{N} = 4$ SCA. The ultimate motivation of this thesis has been, in analogy with Mathieu Moonshine, to identify a moonshine phenomenon in the context of certain theories exhibiting A_γ symmetry; that is, to discover a number theoretic function (possibly a mock modular form) whose q-series expansion exhibits coefficients that are the dimensions of representations of a finite group. In light of the previous discussion of $\mathcal{N} = 4$ theories, a natural question is then whether there exists an index for A_γ theories which could be used to track a new moonshine phenomenon. Although one can show that the trivial extension of the definition of the elliptic genus to A_γ theories is identically zero, an alternative index which we call I_1 has been proposed for A_γ theories in [GMMS04]. Furthermore, a coset method exists to construct a class of partition functions exhibiting $\tilde{A}_\gamma$ symmetry, with $\tilde{A}_\gamma$ an algebra closely related to A_γ [OPT92; PT93].

The work presented here provides an understanding of the A_γ representation theory and of the I_1 index generalising the Witten index, offering an original description of the states it counts in terms of representations of the zero mode subalgebra of A_γ (which is shown to be equivalent to $\mathfrak{su}(2|2)$ in the Ramond sector). Young supertableaux [BB81] are utilised to consider the branching of Ramond representations of A_γ into its zero mode subalgebra $\mathfrak{su}(2|2)$ [Fea18]. This thesis also aims to construct a modular invariant partition function for a theory with A_γ symmetry and calculate the index I_1 of this theory. This requires us to understand the character sum rules, derived from the knowledge that realisations of $\tilde{A}_\gamma$ on certain group cosets together with a number of free fermions exist. In particular, in order to capture a potential new moonshine phenomenon, one must understand better the contributions to the sum rules from the massive representations of A_γ within the sum rules. We present here a relatively simple example of partition function, as part of a wider project with collaborators [FTT18].

The structure of this thesis is as follows. In Chapter 2, we give an introduction to two-dimensional superconformal algebras. After briefly recapping the notion of a conformal algebra, we review the results of [AF81] and [SSTV88a; STVS88], showing how a σ-model on a non-abelian group manifold can possess the 'large' $\mathcal{N} = 4$ superconformal algebra known as A_γ. In particular, we explicitly construct an almost-quaternionic structure on the $SU(3)$ group manifold, since this example will be relevant later in the thesis.

We then discuss the representation theory of A_γ in Chapter 3, developed in [GPTV89]. In particular, we discuss the existence of an isomorphism known as spectral flow for A_γ [DST88], and how this implies that there is no unique highest weight state for a Ramond representation of A_γ. This introduces some subtleties in the representation theory of A_γ, such as the representation being labelled by charges which no state actually possesses. We also discuss the relation between A_γ and the non-linear algebra $\tilde{A}_\gamma$. We then show how one can construct the character formulae for irreducible representations of A_γ [PT90a; PT90b].

In Chapter 4 we introduce the supersymmetric indices which play a role in the Mathieu moonshine story, namely the Witten index [Wit82] and the elliptic genus [Wit87] of a field theory. We show how for a 1d σ-model, the Witten index and the signature of a target space manifold M may be defined as the analytical index of a supercharge, and how this demonstrates the results of the Atiyah-Singer index theorem from the perspective of the σ-model [Alv83]. The elliptic genus is then seen to be a generalisation of the Witten index to the case of the two-dimensional σ-model and is the analytical index of an operator on the loop space of M [Wit88]. Furthermore, the elliptic genus is known independently in the mathematical literature as a homomorphism from the cobordism ring into a ring of modular functions [Och09] and we therefore discuss the relation of the two definitons.

We then show that the obvious extension of the definition of the elliptic genus for $\mathcal{N} = 4$ theories is identically zero for any theory with A_γ symmetry at the start of Chapter 5. Since the elliptic genus is of no use for these theories, we therefore introduce the index I_1 [GMMS04]. This index counts the spectral flow orbits of the massless A_γ highest weight states appearing in the theory [Sau05] and hence counts states throughout the massless representations which satisfy the masslessness conditions [GMMS04]. This index therefore obtains contributions of theta functions from massless representations of A_γ [GMMS04]. We show how the factorisation of A_γ into the non-linear $\tilde{A}_\gamma$ and the algebra A_{QU} may be used to interpret the contributions to the index. Next, we show how one can describe the contributions of a representation of A_γ using Young supertableaux. We show that the zero mode subalgebra of Ramond representations of A_γ is the Lie superalgebra $\mathfrak{su}(2|2)$ and explicitly construct a basis for $\mathfrak{su}(2|2)$ which satisfies the algebra of A_γ. We then introduce the representation theory of $\mathfrak{su}(2|2)$ and describe how such representations may be classified by supertableaux [BB81]. We introduce the supertableaux method for the branching of $\mathfrak{su}(2|2)$ into $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ [BB82]. This can then be applied to A_γ and we show how supertableaux can be used to branch A_γ into $\mathfrak{su}(2|2)$. We compare this with earlier results to identify the representations of $\mathfrak{su}(2|2)$ containing the states of A_γ which are counted by I_1.

In Chapter 6, we then apply the results of the previous chapter to explicitly calculate the index I_1 for a class of theories with A_γ symmetry. We introduce the character sum rules for $\tilde{A}_\gamma$, relating characters of $\tilde{A}_\gamma$ to those of $\widehat{\mathfrak{su}(3)}$ [OPT92; PT93]. We investigate the massive $\tilde{A}_\gamma$ contributions to the sum rules and obtain results for $\tilde{k^+} \in \{2, 3, 4, 5\}$. We then use the sum rules to construct modular invariant diagonal A_γ theories and show how one may calculate their I_1 index.

Finally, we conclude the thesis in Chapter 7, summarising the main points and suggesting avenues for future research.

Chapter 2

2d Superconformal Algebras

The aim of this chapter is to remind the reader of the structure of the 2d Conformal charge algebra (the Virasoro algebra) and to introduce its superconformal extensions, the 'Small' $\mathcal{N} = 4$ SCA and the 'Large' $\mathcal{N} = 4$ SCA. We briefly discuss the representation theory of the Virasoro algebra before discussing the representation theory of A_γ in Chapter 3. We shall assume the reader has some familiarity with 2d Conformal Field Theories (CFTs); there are many excellent texts on CFTs, readers who would like to familiarise themselves with anything not covered in details here are referred to [DMS97; Ton09; Sch96; Sch08; Gin88]. Section 2.1 is very standard and similar discussions will appear in many introductory CFT texts. We include it here in order to introduce some basic terminology and definitions which we will use later in a less standard context. In section 2.2 we introduce the notion of the σ-model and the Wess-Zumino-Novikov-Witten model (WZW model). Following [AF81; SSTV88a; STVS88], we then discuss the possibility for extended supersymmetry on σ-models and WZW models. In particular, we introduce an $\mathcal{N} = 4$ SCA known as A_γ and show that a WZW model on $SU(3)$ has this algebra for its charge algebra.

2.1 The Conformal Charge Algebra

2.1.1 The Stress-Energy Tensor for a 2d CFT

In this work we are primarily interested in 2d conformal field theories. We define a 2d conformal transformation to be an invertible change of coordinates which fixes the metric up to a scale,

$$x^\mu \to x'^\mu, \qquad g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x), \quad \mu, \nu \in \{0, 1\}. \tag{2.1.1}$$

For an infinitesimal transformation $x'^\mu = x^\mu + \epsilon^\mu$ and $\Lambda(x) = 1 - 2\lambda(x)$, and for the Euclidean metric $\delta_{\mu\nu}$, eq. (2.1.1) yields the 2d Cartan-Killing equation

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta_{\mu\nu} \partial_\rho \epsilon^\rho. \tag{2.1.2}$$

A quantum field theory is said to be conformal if both the action and the measure are invariant under such transformations.

In two dimensions, it is convenient to introduce complex coordinates on the plane, given in terms of the cartesian coordinates as

$$z := x^0 + ix^1, \qquad \bar{z} := x^0 - ix^1, \tag{2.1.3}$$

along with the Wirtinger derivatives,

$$\partial := \partial_z \equiv \frac{1}{2}(\partial_0 - i\partial_1), \qquad \bar{\partial} := \partial_{\bar{z}} \equiv \frac{1}{2}(\partial_0 + i\partial_1). \tag{2.1.4}$$

It is usual to extend the domain of the cartesian coordinates x^μ to $\mathbb{C}$ such that eq. (2.1.3) defines a change of coordinates on $\mathbb{C}^2$ and $z, \bar{z}$ are are then viewed as independent complex variables. We must then remember that physically relevant answers lie in the real subspace $\mathbb{R}^2 \subset \mathbb{C}^2$ defined by $\bar{z} = z^*$, where z^* now defines the complex conjugate of z.

Under the change of coordinates eq. (2.1.3), and defining $\epsilon(z, \bar{z}) := \epsilon^0 + i\epsilon^1$, $\bar{\epsilon}(z, \bar{z}) :=$

$\epsilon^0 - i\epsilon^1$, requirement eq. (2.1.2) that the metric change only by a scale factor implies,

$$\partial_{\bar{z}}\epsilon(z,\bar{z}) = 0 \quad \text{and} \quad \partial_z\bar{\epsilon}(z,\bar{z}) = 0, \tag{2.1.5}$$

which is equivalent to the Cauchy-Riemann equations for a holomorphic (antiholomorphic) function. Any holomorphic function $\epsilon(z)$ (respectively antiholomorphic function $\bar{\epsilon}(\bar{z})$) satifies the first (respectively second) equation in eq. (2.1.5), so that any infinitesimal transformation

$$z \to z + \epsilon(z) \qquad (\text{respectively } \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})) \tag{2.1.6}$$

is conformal. It follows that finite 2d conformal transformations are coordinate transformations given by

$$z \to \omega(z) \qquad \text{and} \qquad \bar{z} \to \bar{\omega}(\bar{z}) \tag{2.1.7}$$

for $\omega(z)$ (respectively $\bar{\omega}(\bar{z})$) an arbitrary holomorphic (respectively antiholomorphic) function. We refer to z and $\bar{z}$ as the holomorphic and antiholomorphic variables respectively.

Whenever we have a continuous symmetry, parameterised by a set of infinitesimals ω_α, Noether's theorem tells us that we have a classically conserved current j^μ_α and an associated conserved charge $Q_\alpha = \int \mathrm{d}^{d-1}x \; j^0_\alpha$. At the quantum level we will be interested in statements about correlation functions, where consistency conditions due to symmetry are known as *Ward identities*. The conserved charge Q_α can then be seen to be the generator of the symmetry on the operators of the theory. The conserved current associated with translation symmetry, a simple example of a conformal symmetry, is known as the energy-momentum tensor, whose tracelessness is a key feature of a classically conformally invariant system.

Example 2.1.1. Let us consider some of the features of conformal field theories mentioned above in the simple example of a free scalar field theory. We begin by writing the action for a free scalar field on the plane in cartesian coordinates with the standard flat metric,

$$S = \int \mathrm{d}^2x \; \partial_\mu\phi \, \partial^\mu\phi. \tag{2.1.8}$$

If we let the new coordinates x'^μ be given by

$$(x'^0, x'^1) = (z, \bar{z}) := (\, (f^{-1})^0(x^0, x^1), (f^{-1})^1(x^0, x^1) \,), \tag{2.1.9}$$

where, using eq. (2.1.3),

$$x^0 = \frac{z + \bar{z}}{2} =: f^0(z, \bar{z}), \qquad x^1 = \frac{z - \bar{z}}{2i} =: f^1(z, \bar{z}), \tag{2.1.10}$$

then the action transforms as

$$S \to S' = \int \mathrm{d}^2x' g'^{\mu\nu} \frac{\partial}{\partial x'^\mu}\phi'(x') \frac{\partial}{\partial x'^\nu}\phi'(x'), \tag{2.1.11}$$

where $g'_{\mu\nu}$ is the transformed metric tensor.

The measure transforms as

$$\mathrm{d}x^0 \, \mathrm{d}x^1 \to \frac{1}{2} \mathrm{d}z \, \mathrm{d}\bar{z}. \tag{2.1.12}$$

Note that this is the standard volume form

$$\omega = \sqrt{g'} \, \mathrm{d}z \, \mathrm{d}\bar{z}, \tag{2.1.13}$$

where g' is the determinant of the matrix (g') whose components are those of the metric tensor, i.e. $(g')_{ab} = g'_{ab}$, for $a, b \in \{z, \bar{z}\}$ in that order. The matrix (g') and its inverse are given by,

$$(g') = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad (g')^{-1} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{2.1.14}$$

Finally, the derivatives of the fields transform as

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\partial}{\partial x'^\mu}\phi'(x') = \frac{\partial}{\partial x'^\mu}\phi(f(x')), \tag{2.1.15}$$

so we have

$$\begin{aligned} S' &= \int (\frac{1}{2}\,\mathrm{d}z\,\mathrm{d}\bar{z}) g'^{ab} \frac{\partial}{\partial x'^a}\phi(f(x'))\frac{\partial}{\partial x'^b}\phi(f(x')) \\ &= 2\int \mathrm{d}z\,\mathrm{d}\bar{z}\ \partial\phi\ \bar{\partial}\phi. \end{aligned} \tag{2.1.16}$$

We now consider the effect of an infinitesimal translation

$$x''^\mu = x'^\mu + \omega^\mu, \qquad \phi''(x'') = \phi(x'). \tag{2.1.17}$$

Clearly this is a symmetry of the action and so we have an associated Noether current, the *energy-momentum tensor* (sometimes called the stress-energy tensor) given by

$$T^\mu_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta^\mu_\nu \mathcal{L}, \tag{2.1.18}$$

where $\mathcal{L}$ is the Lagrangian density. Applying this to eq. (2.1.16), we therefore have

$$\begin{aligned} T^z_z &= 2(\bar{\partial}\phi\,\partial\phi - \partial\phi\,\bar{\partial}\phi) = 0, \qquad & T^z_{\bar{z}} &= 2\bar{\partial}\phi\,\bar{\partial}\phi, \\ T^{\bar{z}}_{\bar{z}} &= 2(\partial\phi\,\bar{\partial}\phi - \partial\phi\,\bar{\partial}\phi) = 0, \qquad & T^{\bar{z}}_z &= 2\partial\phi\,\partial\phi. \end{aligned} \tag{2.1.19}$$

We can therefore see that the energy-momentum tensor is traceless

$$T^\mu_\mu = 0, \quad \mu \in \{z, \bar{z}\}. \tag{2.1.20}$$

In fact, this is a feature of conformal invariance at the classical level. A proof of this can be found in, for example, [DMS97].

If we lower the indices using the metric we get

$$\begin{aligned} T_{z\bar{z}} &= 0, \qquad & T_{\bar{z}\bar{z}} &= \bar{\partial}\phi\bar{\partial}\phi, \\ T_{\bar{z}z} &= 0, \qquad & T_{zz} &= \partial\phi\partial\phi. \end{aligned} \tag{2.1.21}$$

The classical conservation equation $\partial_\mu T^{\mu\nu} = 0$ becomes

$$\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0, \tag{2.1.22}$$

and so we see that $T_{zz}, T_{\bar{z}\bar{z}}$ must be holomorphic and antiholomorphic respectively.

This motivates the definitions

$$\begin{aligned} T(z) &:= -2\pi T_{zz} &&= -2\pi\partial\phi\partial\phi, \\ \bar{T}(\bar{z}) &:= -2\pi T_{\bar{z}\bar{z}} &&= -2\pi\bar{\partial}\phi\bar{\partial}\phi, \end{aligned} \tag{2.1.23}$$

where the normalisation will turn out to be convenient later. It might not be clear that the expressions for $T_{zz}, T_{\bar{z}\bar{z}}$ in eq. (2.1.21) are holomorphic and antiholomorphic respectively, but the conservation equation $\partial_\mu T^{\mu\nu} = 0$ holds for solutions obeying the equations of motion which for this example are

$$\partial\bar{\partial}\phi = 0, \tag{2.1.24}$$

with the general solution

$$\phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z}), \tag{2.1.25}$$

and hence $T(z)$ and $\bar{T}(\bar{z})$ are clearly holomorphic and antiholomorphic respectively.

△

2.1.2 The Witt Algebra

We now consider the generators of local 2d conformal transformations. As discussed in the previous section, the finite 2d conformal transformations are given by eq. (2.1.7), or infinitesimally by eq. (2.1.6) where we can expand the holomorphic function $\epsilon(z)$ as a Laurent series

$$\epsilon(z) = \sum_{-\infty}^{\infty} a_n z^{n+1}. \tag{2.1.26}$$

While we are living in the extended coordinate space of $\mathbb{C}^2$, where z and $\bar{z}$ are independent, then we should view $\bar{\epsilon}(\bar{z})$ as an independent function. From here on, we shall only write down statements about the holomorphic coordinate z and the effects of the infinitesimal conformal transformation given by ϵ, and shall take it for granted that all such statements have an antiholomorphic counterpart.

The change of coordinates is generated by

$$l_n := -z^{n+1}\partial, \tag{2.1.27}$$

by which we mean that spinless, dimensionless fields transform as

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_{-\infty}^{\infty}(a_n l_n\phi + \bar{a}_n\bar{l}_n\phi). \tag{2.1.28}$$

The labelling of the Laurent modes may appear odd, but with this choice the generators satisfy the *Witt algebra*,

$$[l_m, l_n] = (m-n)l_{m+n}. \tag{2.1.29}$$

The antiholomorphic $\bar{l}_n$ generate another copy of the Witt algebra

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \tag{2.1.30}$$

which commutes with the holomorphic copy, i.e. $[l_m, \bar{l}_n] = 0$.

We should note that the infinitesimal transformations given in eq. (2.1.6) do not necessarily exponentiate to globally defined invertible transformations. In fact the Witt algebra has a finite subalgebra generated by l_{-1}, l_0 and l_1 isomorphic to $sl(2,\mathbb{R})$, and the direct product of the finite subalgebras of the holomorphic and antiholomorphic Witt algebras, $sl(2,\mathbb{R}) \times sl(2,\mathbb{R}) \cong sl(2,\mathbb{C}) \cong so(3,1)$ is the Lie algebra of globally defined conformal transformations.

Up to this point we have not been concerned with quantising our theories, that is we have really been discussing 2d classical conformal field theory. When we quantise the theory we expect to obtain a Hilbert space of states which could be described by a projective representation of the classical symmetry algebra. We therefore should be interested in the projective representations of the Witt algebra. However, we can instead lift projective representations of the Witt algebra to true (non-projective) representations of the unique central extension of the Witt algebra, the Virasoro algebra. We will not discuss the Virasoro algebra in this context further, but instead refer the interested reader to [Sch08].

2.1.3 Operator Product Expansions and Ward Identities

The preceding sections have focussed on 2d CFT; we now turn our attention to the quantised version of a classical CFT theory with action $S[\phi]$, bearing in mind that a Wick rotation might be necessary. In the quantum theory we will mainly be interested in the correlation functions of some operators $\mathcal{O}_i(\mathbf{x}_i)$, each at position $\mathbf{x}_i$. The correlation function of n operators is defined as

$$\begin{aligned}\langle \mathcal{O}_1(\mathbf{x}_1)\ldots\mathcal{O}_n(\mathbf{x}_n)\rangle &:= \langle 0|\,\mathcal{T}(\mathcal{O}_1(\mathbf{x}_1)\ldots\mathcal{O}_n(\mathbf{x}_n))\,|0\rangle \\ &= \frac{1}{Z}\int \mathcal{D}\phi\; e^{-S[\phi]}\mathcal{O}_1(\mathbf{x}_1)\ldots\mathcal{O}_n(\mathbf{x}_n),\end{aligned} \tag{2.1.31}$$

where $\mathcal{T}$ is the time-ordering operator, defined by

$$\mathcal{T}(\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2)) = \begin{cases}\mathcal{O}_1(\mathbf{x}_1)\mathcal{O}_2(\mathbf{x}_2) & \iff\; x_1^0 \geq x_2^0, \\ \mathcal{O}_2(\mathbf{x}_2)\mathcal{O}_1(\mathbf{x}_1) & \iff\; x_2^0 > x_1^0,\end{cases} \tag{2.1.32}$$

and

$$Z = \int \mathcal{D}\phi\; e^{-S[\phi]}, \tag{2.1.33}$$

is known as the *partition function.*

The *operator product expansion* (*OPE*) states that inside correlation functions, the product of two operators at nearby points can be approximated by a sum of operators at one of the points, with radius of convergence given by the location of the nearest other operator in the product. That is, two fields $\mathcal{O}_1(z), \mathcal{O}_2(w)$ have an operator product expansion of the form

$$\langle \mathcal{O}_1(z)\mathcal{O}_2(w)X\rangle = \langle(\sum_{-\infty}^{n}\frac{\tilde{\mathcal{O}}_i(w)}{(z-w)^i})X\rangle, \tag{2.1.34}$$

where X denotes the product of other operators

$$X = \mathcal{O}_3(x_3^{\mu_3})\ldots\mathcal{O}_k(x_k^{\mu_k}).$$

The radius of convergence for this statement around w would be given by $|w - x_j|$ if x_j is the nearest other operator in the insertion. This is shown in fig. 2.1, where the dashed line shows the radius of convergence of the OPE.

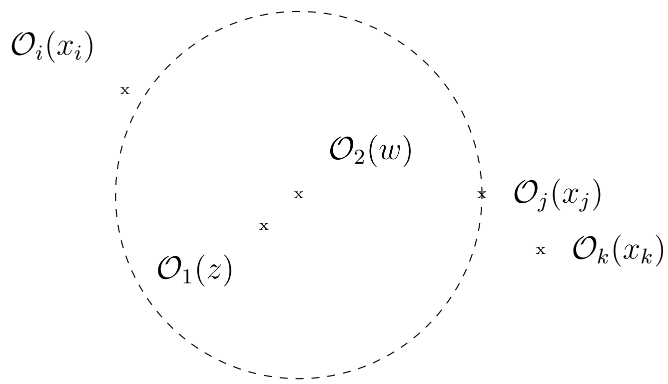


Figure 2.1: The Operator Product Expansion

The OPE is usually written without the explicit brackets for the correlator, $\langle\rangle$, so we would typically write eq. (2.1.34) as

$$\mathcal{O}_1(z)\mathcal{O}_2(w) = \sum_{-\infty}^{n} \frac{\tilde{\mathcal{O}}_i(w)}{(z-w)^i}, \tag{2.1.35}$$

where the product is implicitly assumed to belong to a time-ordered correlator.

In the classical theory, Noether's theorem told us that for every continuous symmetry we should have an associated current; the energy-momentum tensor was defined as such a current, associated to translation invariance. In the quantum theory, the effect of a symmetry (defined to leave the partition function invariant) is expressed through the *Ward identities*. For brevity we shall not give the derivation as this is standard and may be found for example in [DMS97]. Given an infinitesimal transformation defined by

$$\mathcal{O}'(\mathbf{x}) = \mathcal{O}(\mathbf{x}) - i\omega_a G_a \mathcal{O}(\mathbf{x}), \tag{2.1.36}$$

such that

$$\delta\mathcal{O} = -i\omega_a G_a \mathcal{O} \tag{2.1.37}$$

for a set of infinitesimals ω_a, the Ward identity for the current j_a^μ is

$$\partial_\mu \langle j_a^\mu(\mathbf{x})\mathcal{O}_1(\mathbf{x}_1)\ldots\mathcal{O}_n(\mathbf{x}_n)\rangle = -i\sum_{i=1}^{n}\delta(\mathbf{x}-\mathbf{x}_i)\langle\mathcal{O}_1\ldots G_a\mathcal{O}_i(\mathbf{x}_i)\ldots\mathcal{O}_n(\mathbf{x}_n). \tag{2.1.38}$$

G_a is then defined to be the *generator* of the transformation. Equation (2.1.38) can then be integrated to identify the conserved charges

$$Q_a := \int \mathrm{d}^{d-1}x \; j_a^0(\mathbf{x}), \tag{2.1.39}$$

as the generators of the transformation,

$$[Q_a, \mathcal{O}] = -iG_a\mathcal{O}. \tag{2.1.40}$$

Taking the transformation to be an infinitesimal conformal transformation given by eq. (2.1.6) leads to the *conformal Ward identity*

$$\delta_{\epsilon,\bar{\epsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C \mathrm{d}z \; \epsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C \mathrm{d}\bar{z} \; \bar{\epsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle, \tag{2.1.41}$$

where X denotes a product of local fields at positions $\mathbf{x}_i$, and C is taken to be a contour containing the positions of all the fields in X. Since the conformal current $\epsilon(z)T(z)$ is holomorphic, we can compute the (holomorphic) integral appearing in eq. (2.1.41) using the residue theorem. We can therefore use this conformal Ward identity to calculate the OPEs of fields with the energy-momentum tensor if we know how the field transforms under a conformal transformation. Since this information is encapsulated by the singular terms of the OPE, these will be the only terms of the OPE that we are interested in; we therefore suppress non-singular terms when writing down OPEs in the following and use $\sim$ to indicate equivalence up to non-singular terms.

To be able to make use of the conformal Ward identity to calculate OPEs, we therefore need to know how fields transform under conformal transformations. Under a local infinitesimal conformal transformation $z \to z + \epsilon(z)$, a *primary* field ϕ is one which transforms as

$$\delta_\epsilon\phi = -(h\partial\epsilon\phi + \epsilon\partial\phi), \qquad h = \frac{\Delta + s}{2}, \tag{2.1.42}$$

for Δ the scaling dimension and s the spin of ϕ. h is called the *weight* of the field, sometimes known as the *conformal dimension*. Primary fields also have an antiholomorphic weight $\bar{h}$ defined similarly. From this transformation, we can read off the $T(z)\phi(\omega)$ OPE as

$$T(z)\phi(\omega) \sim \frac{h\phi(\omega)}{(z-\omega)^2} + \frac{\partial\phi(\omega)}{z-\omega}. \tag{2.1.43}$$

In fact, this gives an alternate definition of a primary field of weight h as one whose OPE with $T(z)$ is of the form in eq. (2.1.43).

Example 2.1.2. Let us return to the example of the free boson. Our starting point is the propagator for ϕ,

$$\langle\phi(z)\phi(\omega)\rangle = -\frac{1}{4\pi}\ln(z-\omega). \tag{2.1.44}$$

We will not prove this here for brevity, but as the Green's function for the operator $(-\partial^2)$ it may be calculated using standard methods.

Due to the logarithmic behaviour of the propagator of ϕ we will be more interested in the behaviour of $\partial\phi$ whose OPE with itself is

$$\partial\phi(z)\partial\phi(\omega) \sim -\frac{1}{4\pi(z-\omega)^2}. \tag{2.1.45}$$

This OPE with one singular term is characteristic of a free field; this is crucial in order for us to define the energy-momentum tensor. We already calculated the form of $T(z)$ for the free scalar field in example 2.1.1 for the classical case as

$$T(z) = -2\pi\partial\phi\partial\phi.$$

Now we are interested in the quantum theory, and the product of operators at the same point is badly defined, so we should normal order. For a free field, whose OPE with itself contains only one singular term, we can normal order by subtracting the propagator which ensures the vanishing of the vacuum expectation value;

$$T(z) = -2\pi :\partial\phi(z)\partial\phi(z):\ \ := -2\pi\lim_{\omega\to z}(\partial\phi(z)\partial\phi(\omega) - \langle\partial\phi(z)\partial\phi(\omega)\rangle). \tag{2.1.46}$$

We want to calculate the $T\partial\phi$ OPE as a valid statement inside a time-ordered correlator. Wick's theorem tells us that the time-ordered product is equal to the normal-ordered product plus the sum of all possible contractions. We therefore calculate the time ordered OPE as

$$\begin{aligned} T(z)\partial\phi(\omega) &= -2\pi : \partial\phi(z)\partial\phi(z) : \partial\phi(\omega) = \; -4\pi : \partial\phi(z)\partial\phi(z) : \partial\phi(\omega) \\ &\sim \frac{\partial\phi(z)}{(z-\omega)^2} \sim \frac{\partial\phi(\omega)}{(z-\omega^2)} + \frac{\partial_\omega^2\phi(\omega)}{z-\omega}, \end{aligned} \tag{2.1.47}$$

where we have expanded $\partial\phi(z)$ around ω to put the OPE in the form of eq. (2.1.35). As explained previously, it is the poles of the OPE that contain the information we are interested in, and so we have suppressed all regular terms. Comparing this to eq. (2.1.43), we see that the field $\partial\phi$ is a primary field of weight one. This is what we should expect, since in two dimensions the free scalar ϕ has scaling dimension 0, hence $\partial\phi$ is spin 1 and dimension 1, giving a conformal weight of one using eq. (2.1.42).

Finally we calculate the TT OPE in a similar manner to above,

$$\begin{aligned} T(z)T(\omega) &= 4\pi^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(\omega)\partial\phi(\omega) : \\ &= 8\pi^2 : \partial\phi(z)\partial\phi(z) :: \partial\phi(\omega)\partial\phi(\omega) : + 16\pi^2 : \partial\phi(z)\partial\phi(\omega) :: \partial\phi(z)\partial\phi(\omega) : \\ &\sim \frac{1/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}. \end{aligned} \tag{2.1.48}$$

The energy-momentum tensor is therefore not a primary field. △

As the previous example showed, $T(z)$ is not a primary field, since its OPE with itself contained a $(z-\omega)^{-4}$ term. $T(z)$ is an example of a *quasi-primary* field of weight 2, meaning it has an OPE of the form

$$T(z)\mathcal{O}(\omega) \sim \sum_{n\geq 4} \frac{\mathcal{O}_n(\omega)}{(z-\omega)^n} + \frac{h\mathcal{O}(\omega)}{(z-\omega)^2} + \frac{\partial\mathcal{O}(\omega)}{z-\omega}, \tag{2.1.49}$$

for $n \in \mathbb{N}$ and $\mathcal{O}_n$ of dimension $4-n$, or alternatively that $T(z)$ transforms according to eq. (2.1.42) only for global conformal transformations. The most singular term a quasi-primary operator of weight 2 can have in a unitary CFT, is a term c-proportional to $(z-\omega)^{-4}$ and hence the most general form for the TT OPE is

$$T(z)T(\omega) \sim \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}, \tag{2.1.50}$$

where c is known as the *central charge* of the algebra.

2.1.4 The Virasoro Algebra

Being able to relate OPEs to (anti)commutation relations is a useful skill as it the symmetry information encoded in the OPEs to be expressed in operator language. The method for this is described in standard CFT books, such as [DMS97]. For brevity, we only give a few pointers here that can be used to relate the OPE eq. (2.1.50) to the Virasoro algebra.

1. The radial quantisation of 2d CFTs is particularly helpful in this context as the time ordering within correlation functions becomes radial ordering. Consequently, the left hand side of OPEs must also be radially ordered.

   One way to think of the process of radial quantisation is from the point of view of closed string theory, where the Euclidean worldsheet CFT is naturally defined on an infinite cylinder, parameterised by the complex coordinate $\omega = i\sigma + \tau$, for $\sigma \in [0, 2\pi)$ and $\tau \in \mathbb{R}$, indicating that we take the cylinder to be of radius 1 here. We can then make the conformal transformation

   $$z = e^{\omega}, \tag{2.1.51}$$

   which maps the infinite cylinder to the punctured plane as shown in fig. 2.2. Spatial slices of the cylinder therefore get mapped to circles in the plane, where a later time slice gets mapped to a circle of greater radius than a earlier time slice; this is shown with the dashed lines in fig. 2.2. One does not have to start with a string picture in order to consider radial quantisation however; on the Euclidean plane, there is no preferred direction to be chosen as the time-like direction, and so one may still freely choose to identify spatial slices as circles around the origin.

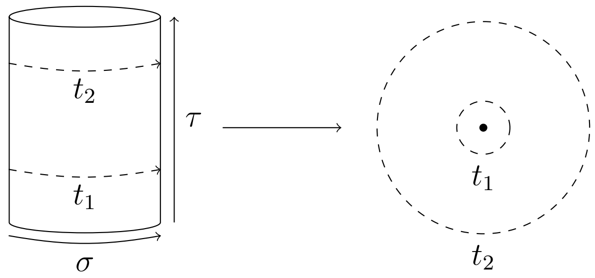


Figure 2.2: A map from the cylinder to the plane

2. The quantisation of a field $\phi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ proceeds from Laurent expanding it as,

$$\phi(z,\bar{z}) = \sum_{n,\bar{n}\in\mathbb{Z}} z^{-n-h}\,\bar{z}^{-\bar{n}-\bar{h}}\,\phi_{n,\bar{n}}, \tag{2.1.52}$$

and promoting the modes $\phi_{n,\bar{n}}$ to operators. This is consistent with first considering the theory on a cylinder, Fourier expanding $\phi(\sigma,\tau)$, promoting the Fourier coefficients to operators via quantisation, and then mapping to the plane using eq. (2.1.51).

We note here that, if one drops the antiholomorphic dependence of the field ϕ, the Laurent expansion takes the form

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n, \tag{2.1.53}$$

where

$$\phi_n = \frac{1}{2\pi i}\oint \mathrm{d}z\; z^{n+h-1}\phi(z). \tag{2.1.54}$$

3. Radial quantisation suggests that well-defined asymptotic states for Euclidean time $\tau \to -\infty$ should be defined as

$$|\phi_{\text{in}}\rangle \equiv |h,\bar{h}\rangle := \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})\,|0\rangle = \phi_{-h,-\bar{h}}\,|0\rangle\,, \tag{2.1.55}$$

given eq. (2.1.51) and assuming that a vacuum state $|0\rangle$ exists and that the Hilbert space for the theory is built by acting on it with creation operators.

Note that this requires that the operators satisfy,

$$\phi_{n,\bar{n}}\,|0\rangle = 0, \quad \text{whenever } n > -h \text{ or } \bar{n} > -\bar{h}. \tag{2.1.56}$$

We can also construct a well-defined asymptotic out state through the hermitian conjugate field,

$$\phi(z,\bar{z})^\dagger := \bar{z}^{-2h} z^{-2\bar{h}} \phi(1/\bar{z}, 1/z), \tag{2.1.57}$$

with Laurent expansion,

$$\phi(z,\bar{z})^\dagger = \sum_{n,\bar{n}\in\mathbb{Z}} \bar{z}^{n-h} z^{\bar{n}-\bar{h}} \phi_{n,\bar{n}}, \tag{2.1.58}$$

where,

$$(\phi_{n,\bar{n}})^\dagger = \phi_{-n,-\bar{n}}. \tag{2.1.59}$$

This yields, for $\xi = 1/z$ and $\bar{\xi} = 1/\bar{z}$,

$$\langle\phi_{\text{out}}| := \lim_{z,\bar{z}\to 0} \langle 0|\,\phi(z,\bar{z})^\dagger = \lim_{\xi,\bar{\xi}\to\infty} \xi^{2h}\bar{\xi}^{2\bar{h}}\,\langle 0|\,\phi(\xi,\bar{\xi}) = \langle 0|\,\phi_{h,\bar{h}}. \tag{2.1.60}$$

Given two operators $\mathcal{O}_i, i \in \{1,2\}$ written as contour integrals of holomorphic fields $o_i(z)$,

$$\mathcal{O}_i = \oint o_i(z)\,dz, \tag{2.1.61}$$

their commutator can be calculated as

$$[\mathcal{O}_1, \mathcal{O}_2] = \oint_0 dw \oint_w dz\, o_1(z) o_2(w), \tag{2.1.62}$$

where the z-integral is taken around w, and the w-integral is taken around the origin.

We demonstrate the usefulness of radial quantisation by returning to the conformal Ward identity, eq. (2.1.41). We shall consider the variation of a single field ϕ under a conformal variation $\epsilon(z)$, and we focus only on the holomorphic part of the identity,

$$\delta_\epsilon \phi(\omega) = -\frac{1}{2\pi i}\oint_C \mathrm{d}z\ \epsilon(z) T(z)\phi(\omega), \tag{2.1.63}$$

where we recall that C was to be a contour containing ω and which we therefore take to be a small circle centred on ω. Since we have radially quantised, the implicit time-ordering $\mathcal{T}$ in the correlator becomes radial ordering $\mathcal{R}$,

$$\mathcal{R}(\mathcal{O}_1(z)\mathcal{O}_2(\omega)) = \begin{cases} \mathcal{O}_1(z)\mathcal{O}_2(\omega) & \iff |z| > |\omega|, \\ \mathcal{O}_2(\omega)\mathcal{O}_1(z) & \iff |\omega| > |z|. \end{cases} \tag{2.1.64}$$

We therefore need to split the contour up to account for this radial ordering, which we can do as indicated in fig. 2.3, where the contour on the left-hand side represents the contour C around ω.

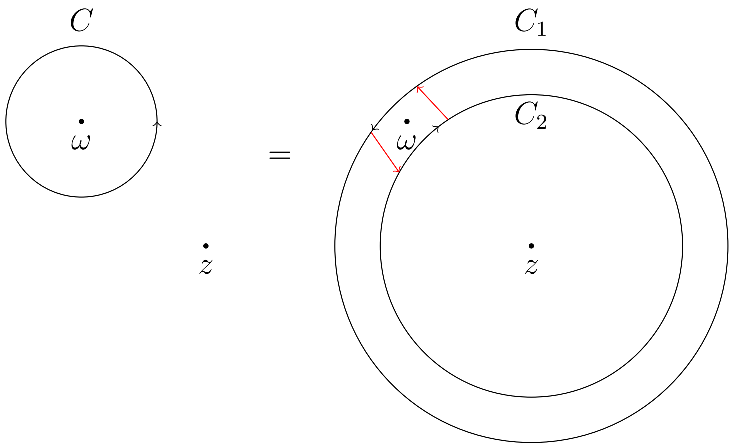


Figure 2.3: Contour for radial ordering

We see that the integral in the conformal Ward identity (eq. (2.1.63)) then becomes,

$$\begin{aligned} \delta_\epsilon \phi(\omega) &= -\frac{1}{2\pi i}\left(\oint_{C_1} \mathrm{d}z\ \epsilon(z)T(z)\phi(\omega) - \oint_{C_2} \mathrm{d}z\ \phi(\omega)\epsilon(z)T(z)\right) \\ &= -[Q_\epsilon, \phi(\omega)], \end{aligned} \tag{2.1.65}$$

where we have defined the *conformal charge*

$$Q_\epsilon := \frac{1}{2\pi i}\oint \mathrm{d}z\ \epsilon(z)T(z), \tag{2.1.66}$$

which by eq. (2.1.40) we recognise as the generator of conformal transformations.

If we expand the energy-momentum tensor into modes as in eq. (2.1.53), we get

$$T(z) = \sum_n z^{-n-2}L_n, \qquad L_n = \frac{1}{2\pi i}\oint \mathrm{d}z\ z^{n+1}T(z), \tag{2.1.67}$$

and so if we also expand

$$\epsilon(z) = \sum_{m\in\mathbb{Z}} \epsilon_m z^{m+1}, \tag{2.1.68}$$

then we can identify the conformal charge defined in eq. (2.1.66) as

$$Q_\epsilon = \sum_{m\in\mathbb{Z}} \epsilon_m L_m. \tag{2.1.69}$$

We therefore see that the modes L_n of the energy-momentum tensor are the generators of local conformal transformations in the quantum theory in the same way that the l_n of eq. (2.1.29) generated the classical local conformal transformations.

We can now calculate the charge algebra of the L_n's.

$$\begin{aligned}
[L_m, L_n] &= \frac{-1}{4\pi^2}\left(\oint \mathrm{d}z \oint \mathrm{d}\omega \; - \oint \mathrm{d}\omega \oint \mathrm{d}z\right) z^{m+1}\omega^{n+1}T(z)T(\omega) \\
&= \frac{-1}{4\pi^2}\oint \mathrm{d}\omega \oint_\omega \mathrm{d}z \; z^{m+1}\omega^{n+1}T(z)T(\omega) \\
&= \frac{1}{2\pi i}\oint \mathrm{d}\omega \;\; \mathrm{Res}_{z=\omega}\left[z^{m+1}\omega^{n+1}\left(\frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega}\right)\right] \\
&= \frac{1}{2\pi i}\oint \mathrm{d}\omega \; \omega^{n+1}(\omega^{m+1}\partial T(\omega) + 2(m+1)\omega^m T(\omega) + \frac{c}{12}m(m^2-1)\omega^{m-2}) \\
&= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0},
\end{aligned} \tag{2.1.70}$$

where in the second line, for fixed ω we recognise the radially ordered integrals as equal to integrating z in a contour around ω as in fig. 2.3.

We see that the L_n satisfy an almost identical algebra to the Witt algebra, but with a central extension $C = cI$ from which the central charge c gets its name. This algebra is known as the *Virasoro algebra* and can be shown to be the unique central extension of the Witt algebra, [Sch08]. Obviously, in all of the preceding discussion we have been ignoring the antiholomorphic parts, and hence the full algebra for the quantum theory is given by two commuting copies of the Virasoro algebra,

$$\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
&[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}, \\
&[L_m, \bar{L}_n] = 0.
\end{aligned} \tag{2.1.71}$$

Note that for $T(z)\,|0\rangle$ to be well-defined in the limit $z \to 0$ we must have

$$L_n\,|0\rangle = 0 \quad n \geq -1. \tag{2.1.72}$$

We therefore have that L_{-1}, L_0 and L_1 all annihilate the vacuum. Since these elements generate the algebra of global conformal transformations $sl(2,\mathbb{C})$, this means the vacuum is invariant under global conformal transformations.

2.1.5 Representations of the Virasoro Algebra

In the previous section we have seen that the local conformal transformation generators are the modes of the energy-momentum tensor L_n. In fact, by considering eq. (2.1.69) we see that for a dilation on the plane with $\epsilon(z) = \epsilon_0 z$ then the conformal charge is given by $Q_\epsilon = \epsilon_0 L_0$. Since we have seen that the conformal charge is the generator of the transformation, this means that L_0 is the generator of dilations on the plane. In radial quantisation however, the dilations are time translations, and hence the Hamiltonian is proportional to L_0.

The importance of primary fields is now demonstrated through the *state-operator correspondence*, which gives a bijection between states of the theory and local operators through the asymptotic 'in' states of eq. (2.1.55), now extended to all fields[1]

$$|\phi_{\text{in}}\rangle \;:= \lim_{z\to 0} \phi(z)\,|0\rangle\,.$$

For primary fields of weight h, we also refer to the state as $|h\rangle$. We now show that these states are eigenstates of the Hamiltonian L_0. We will need

$$[L_n, \phi(\omega)] = h(n+1)\omega^n \phi(\omega) + \omega^{n+1}\partial\phi(\omega), \tag{2.1.73}$$

which is easily shown using the same contour technique used in eq. (2.1.65). We

[1] Here again, we drop the antiholomorphic dependence of fields for simplicity.

therefore have

$$\begin{aligned} L_0 \left| h \right\rangle &:= \lim_{\omega \to 0} L_0 \phi(\omega) \left| 0 \right\rangle = \lim_{\omega \to 0} [L_0, \phi(\omega)] \left| 0 \right\rangle , \\ &= \lim_{\omega \to 0} h \phi(\omega) \left| 0 \right\rangle = h \left| h \right\rangle , \end{aligned} \tag{2.1.74}$$

where we have used the fact that $L_0 \left| 0 \right\rangle = 0$. Since the conformal weight h is the eigenvalue of the Hamiltonian L_0 we will also refer to h as the energy of the state $\left| h \right\rangle$. Note that since h is the eigenvalue of the Hamiltonian, it must be real for the Hamiltonian to be a Hermitian operator.

Using the commutation relations of the Virasoro algebra eq. (2.1.71), we can immediately see the effect of acting on a state of energy h with the generators L_n,

$$\begin{aligned} L_0 L_n \left| h \right\rangle &= ([L_0, L_n] + L_n L_0) \left| h \right\rangle , \\ &= (h - n) L_n \left| h \right\rangle . \end{aligned} \tag{2.1.75}$$

For $n > 0$ then, the operators L_n lower the energy of the state $\left| h \right\rangle$, whilst the operators L_{-n} raise the energy of the state. We therefore refer to L_n and L_{-n} as lowering and raising modes respectively. Obviously we want the energy spectrum of our theory to be bounded below and for the primary states $\left| h \right\rangle$ we see that this is exactly what happens, since for $n > 0$

$$\begin{aligned} L_n \left| h \right\rangle &= \lim_{\omega \to 0} [L_n, \phi(\omega)] \left| 0 \right\rangle , \\ &= 0, \end{aligned} \tag{2.1.76}$$

where we used eqs. (2.1.72) and (2.1.73). Taking Hermitian conjugates leads to a similar requirement for the dual states,

$$\left\langle h \right| L_{-n} = 0 \qquad \forall \quad n > 0. \tag{2.1.77}$$

Given a primary state $\left| h \right\rangle$, we therefore have a lowest-weight representation of the Virasoro algebra with lowest-weight state $\left| h \right\rangle$. We note here that although $\left| h \right\rangle$ behaves like a lowest-weight state, it is commonly referred to as a highest weight state in the literature. The basis for this representation is taken to be the *descendant* *states*,

$$L_{-n_p}L_{-n_{p-1}}\dots L_{-n_1}|h\rangle\,, \qquad 1\leq n_1\leq\dots\leq n_p, \tag{2.1.78}$$

where the commutation relations eq. (2.1.71) of the Virasoro algebra have been used to put each descendant into a standard ordering as indicated. We define the *level* of a descendant state to be,

$$N=n_1+n_2+\dots+n_p, \tag{2.1.79}$$

and the level of the primary state $|h\rangle$ to be 0. Clearly a descendant state of a primary state $|h\rangle$ of level N has energy $h+N$. States at level $n=n_1+\dots+n_k$ are orthogonal to all states at level $m=m_1+\dots+m_q$ for $m\neq n$, since the relevant inner product is of the form,

$$\langle h|\,L_{m_1}\dots L_{m_q}L_{-n_p}\dots L_{-n_1}|h\rangle\,, \tag{2.1.80}$$

where without loss of generality we assume that $m>n$. In writing this inner product we have used eq. (2.1.59) to write the hermitian conjugates of the raising operators,

$$L_{-n}^{\dagger}=L_n. \tag{2.1.81}$$

We now commute the positive modes L_{m_i} past the negative modes L_{n_j}, using the Virasoro commutation relations eq. (2.1.71). Since the positive modes L_{m_i} annihilate $|h\rangle$ for $m_i>0$, and the non-central piece of the commutator $[L_{m_i},L_{n_j}]$ is proportional to $L_{m_i+n_j}$ and by assumption $m>n$, the states are orthogonal.

As mentioned above, since local conformal transformations are generated by the modes L_n, the set of a primary, lowest-weight state and all its conformal descendants form a module for the Virasoro (Vir) algebra. Knowing the set of primary operators for a given theory, or equivalently their associated primary states, is equivalent to a full understanding of the Hilbert space of the theory. A (highest-weight) Vir-module is characterised by the weight of the lowest-weight state it is built from and the central charge of the Virasoro algebra; we therefore label such a Vir-module as $V(c,h)$.

To each Vir-module we can assign a *character*

$$\chi_{c,h}(\tau) := \mathrm{Tr}_{V(c,h)}\, q^{L_0 - c/24}, \tag{2.1.82}$$

where $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$, $\mathbb{H}$ the Poincaré upper half-plane and the trace is taken over the Vir-module in question. The character is the generating function for the degeneracy of states at each energy level,

$$\chi_{c,h}(\tau) = \sum_{n\in\mathbb{Z}_{\geq 0}} \dim(V_{n+h}) q^{n+h-c/24}, \tag{2.1.83}$$

where V_{n+h} is the linear subspace of the Vir-module of states of weight $h+n$. The highest-weight free Vir-module generated by the modes L_{-n} is known as a Verma module. The character of a Verma module is easily computed by considering eq. (2.1.79); a state at level N may be obtained from any partition of N, interpreted as the Virasoro raising modes to apply to the primary state $|h\rangle$.

Example 2.1.3. For example, beginning with a primary state $|h\rangle$, the states at level 5 are given in table 2.1. △

Partition	State
(1,1,1,1,1)	$L_{-1}^5 \|h\rangle$
(2,1,1,1)	$L_{-2}L_{-1}^3 \|h\rangle$
(2,2,1)	$L_{-2}^2 L_{-1} \|h\rangle$
(3,1,1)	$L_{-3}L_{-1}^2 \|h\rangle$
(3,2)	$L_{-3}L_{-2} \|h\rangle$
(4,1)	$L_{-4}L_{-1} \|h\rangle$
(5)	$L_{-5} \|h\rangle$

Table 2.1: The states of a Verma module at level 5.

It is therefore easy to see that the number of states at level n in a Verma module is given by $p(n)$, the number of partitions of n. The generating function for the partition numbers is easily seen to be,

$$\sum_{n\in\mathbb{Z}_{\geq 0}} p(n)q^n = \prod_{n\in\mathbb{Z}_{>0}} \frac{1}{1-q^n} =: q^{1/24}\eta^{-1}(\tau), \tag{2.1.84}$$

in terms of the Dedekind eta function, $\eta(\tau)$. Using this we can write the character for the Verma module as

$$\chi_{c,h}(\tau) = q^{h+(1-c)/24}\eta^{-1}(\tau). \tag{2.1.85}$$

Since the states of a conformal field theory must fall into irreducible representations of the Virasoro algebra, it is more interesting to study the irreducible unitary highest-weight representations of the Virasoro algebra. It is easy to see that the physical requirement of unitarity, the lack of negative-norm states, puts restrictions on the parameters of a Vir-module. If we have a primary state $|h\rangle$ of positive norm then

$$|L_{-1}\,|h\rangle\,|^2 = \langle h|\,L_1L_{-1}\,|h\rangle = \langle h|\,[L_1,L_{-1}]\,|h\rangle = \langle h|\,2L_0\,|h\rangle = 2h\,\langle h|h\rangle\,, \tag{2.1.86}$$

giving us the requirement $h \geq 0$ in order for this state to be of non-negative norm. Similarly considering the state

$$|L_{-n}\,|h\rangle\,|^2 = \langle h|\,L_nL_{-n}\,|h\rangle = \langle h|\,[L_n,L_{-n}]\,|h\rangle = \left(2nh + \frac{c}{12}n(n^2-1)\right)\langle h|h\rangle\,, \tag{2.1.87}$$

which will be negative for sufficiently large n if $c < 0$. Unitarity therefore requires $c \geq 0$ and $h \geq 0$.

Further, it is possible that in a given representation of the Virasoro algebra, some states may be *singular*, that is there may exist some state ρ other than the lowest-weight state $|h\rangle$ which satisfies

$$L_n\,|\rho\rangle = 0 \qquad \forall \quad n > 0. \tag{2.1.88}$$

This state is orthogonal to the entire Vir-module, since

$$\langle h|\,L_{n_p}\ldots L_{n_1}\,|\rho\rangle = 0. \tag{2.1.89}$$

Moreover, each descendant of $|\rho\rangle$ is also orthogonal to the whole Vir-module;

$$\langle h| \, L_{n_p} \ldots L_{n_1} L_{-m_1} \ldots L_{-m_q} \, |\rho\rangle = 0. \qquad (2.1.90)$$

Clearly if the states are of different levels then they are automatically orthogonal as described before. If not, then letting $m = \sum_j m_j$, $n = \sum_i n_i$, we must have $n > m$, since $n = m + M$ where n is the level of the descendant state and M is the level of $|\rho\rangle$. The state $|\rho\rangle$ is then itself the lowest-weight state for an entire Verma module of *null* states - states of zero-norm - since all these states are orthogonal to themselves. Such a Verma module is known as a *null submodule*. If such states exist in a Vir-module, then the module is not irreducible, since the module built from the singular state $|\rho\rangle$ gives a non-trivial submodule. In order to form irreducible highest-weight Vir-modules, one should quotient the Verma module by its maximal proper submodule, identifying states if they differ by a null state in the module. In fact, given any null state $|\omega\rangle$, we can apply raising operators until we reach a singular state $|\rho\rangle$, and hence if we work in the irreducible module obtained by quotienting out all null submodules we have removed all states of zero-norm. Since the irreducible modules are obtained through the process of quotienting out null submodules, their characters are not given by eq. (2.1.85) but instead depend on the embedding of null states in the Verma module [FF90].

2.2 Superconformal Algebras

In this section we will introduce theories with extended conformal symmetry, that is theories whose current algebra contains the Virasoro algebra as a subalgebra. As well as additional bosonic currents, it is possible to introduce fermionic currents in theories with supersymmetry. The first supersymmetric extension to the Virasoro algebra was introduced by Ramond in the context of introducing fermions into what was then known as the Dual Resonance Model [Ram71]. Shortly after, a different model was considered by Neveu and Schwarz which also attempted to add fermionic operators to the theory [NS71a; NS71b]. The modern perspective is that these algebras are the algebras of *superconformal field theories*, which are usually introduced from the point of view of string theory, where the worldsheet theory is now a 2d superconformal field theory (SCFT). The modes of the fermionic currents satisfy anticommutation relations as opposed to the commutation relations of the Virasoro algebra, eq. (2.1.71). This means that the SCAs have the structure of Lie superalgebras rather than simply the Lie algebra structure of the Virasoro algebra. Lie superalgebras are discussed for example in [Cor89] and some relevant definitions may be found in Appendix D.

The main goal of this section is to introduce and discuss a family of so-called 'large' $\mathcal{N}=4$ SCA, which was first discovered during an investigation into the possibility of extended supersymmetry on 2d σ-models [SSTV88a; STVS88; STV88] building on earlier work by [AF81]. In particular, the σ-models considered in [SSTV88a; STVS88; STV88] add a Wess-Zumino term to the usual σ-model action [Wit84] and take the target manifold to be an absolutely parallelisable group manifold.

2.2.1 The Wess-Zumino-Novikov-Witten Model

The Wess-Zumino-Novikov-Witten model, also known as the Wess-Zumino-Witten model and referred to hereafter as the WZW model is a 2d σ-model with the addition of a Wess-Zumino term. We briefly introduce the non-linear σ-model and discuss the WZW model and its charge algebra in this section before considering its supersymmetric generalisation in the section which follows.

Bosonic strings evolving in spacetime sweep a two-dimensional worldsheet X whose shape depends on the type of strings considered. The bosonic string quantisation proceeds from taking the Polyakov action

$$S^{\text{Polyakov}}(\phi, h) = -\frac{T}{2}\int_X \mathrm{d}^2x\,\sqrt{-h}h^{\mu\nu}\,\eta_{ab}\partial_\mu\phi^a\partial_\nu\phi^b, \tag{2.2.1}$$

where $\phi : X \to M$ is a differentiable map from the *worldsheet* space X to a d-

dimensional *target space* M, taken to be spacetime with Minkowski metric η_{ab}, and so $\phi^a, a \in \{1, \ldots d\}$ are the string coordinates. The symmetric tensor $h_{\mu\nu}, \mu, \nu \in \{0, 1\}$ is the intrinsic metric on X. In the case of closed strings, X is an infinite cyclinder parameterised by $x^0 \in \mathbb{R}$ and $x^1 \in [0, 1]$. The parameter T is the string tension. As is well-known, the Polyakov action is invariant under global spacetime Poincaré transformations, local changes of scale (Weyl transformations) and two-dimensional reparametrisations, which include local conformal transformations. The theory can then be quantised using the path integral technique. The massless spectrum of the closed string contains a rank-2 symmetric tensor (graviton g_{ab}), a rank-2 antisymmetric tensor (b-field b_{ab}) and a scalar (dilaton Φ). These zero modes are emitted and absorbed by the closed strings and, from the target space perspective, they are massless particles.

A non-linear σ-model in this context encodes the effect of these massless particles as a background in which the string evolves. The action is given by [FT85; CFMP85],

$$S := \int \mathrm{d}^2 x \left[\sqrt{-h} h^{\mu\nu} g(\phi)_{ab} \partial_\mu \phi^a \partial_\nu \phi^b + \epsilon^{\mu\nu} b(\phi)_{ab} \partial_\mu \phi^a \partial_\nu \phi^b + \sqrt{-h} \Phi(\phi) R^{(2)}(h) \right] \tag{2.2.2}$$

where $g(\phi)_{ab}, b(\phi)_{ab}$ and the dilaton $\Phi(\phi)$ are couplings chosen to ensure the resulting theory is consistent quantum mechanically. In the above, $R^{(2)}$ is the worldsheet Ricci scalar. Turning on the background fields potentially destroys the conformal invariance of the theory. Since the conformal invariance of string theory is a gauge symmetry it needs to survive quantisation for the theory to be consistent. The conditions on the background fields to ensure conformal invariance at the lowest order of approximation are a set of differential equations worked out in [CFMP85], and amount to calculating the so-called β-function for the non-linear σ-model at first loop in string perturbation theory and setting it to zero. Alongside a 3-form $h = \mathrm{d}b$, these equations involve connections and curvatures that are calculated with the metric $g_{ab}(\phi)$.

The background field equations may be obtained as the Euler-Lagrange equations of a d-dimensional action which, when the dilaton field is set to zero (we are only interested in this case here), is given by [GSW87]

$$S' := \int \mathrm{d}^d x \sqrt{-g}\, R(\tilde{\Gamma}), \tag{2.2.3}$$

where the Ricci curvature $R(\tilde{\Gamma})_{ab}$ is calculated using the connection

$$\tilde{\Gamma}^a{}_{bc} := \Gamma^a{}_{bc} - \frac{1}{2} g^{ad} h_{bcd}, \tag{2.2.4}$$

with $\Gamma^a{}_{bc}$ the Levi-Civita connection and the totally antisymmetric tensor h_{bcd} (from $h = \mathrm{d}b$) being the generator of a torsion. It turns out that the background field equations are encoded in the field equations

$$R(\tilde{\Gamma})_{ab} = 0. \tag{2.2.5}$$

One way to solve the equations eq. (2.2.5) is to select a group manifold as target space, as such manifolds are parallelisable, i.e. one can construct a connection with torsion and zero curvature on them.

The action of the WZW model that we describe next provides the correct framework to study strings propagating on a group manifold. We take the target space to be a compact simply-connected Lie group G and we let the map from the worldsheet to target space be given by,

$$\phi : S^2 \to G, \tag{2.2.6}$$

where we have extended the worldsheet space from the complex plane to the Riemann sphere. The first step is to rewrite the first term of the non-linear σ-model action eq. (2.2.2), which involves the spacetime metric $g(\phi)_{ab}$ - call it S_g - in terms of a group element

$$\phi := \exp(\phi^A T_A),\ \ A \in \{1, \ldots, \dim G\}, \tag{2.2.7}$$

with T_A the generators of the Lie algebra $\mathfrak{g}$ of G satisfying the commutation relations $[T_A, T_B] = f_{AB}{}^C T_C$.

On the group manifold the Killing form,

$$\mathcal{K}(x,y) = \mathrm{Tr}\left(\mathrm{ad}\,(x)\,\mathrm{ad}\,(y)\right), \tag{2.2.8}$$

induces a metric since all invariant bilinear forms are simply scalar multiples of the Killing form. In terms of the Lie algebra structure constants, the Killing form is given by $K_{AB} = f_{AC}{}^{D} f_{BD}{}^{C}$. Using this we can write the standard kinetic term for the σ-model on a group manifold as

$$S_\phi = C \int_{S^2} \mathrm{d}^2x \;\; \mathrm{Tr}(\partial_\mu \phi^{-1} \partial^\mu \phi), \tag{2.2.9}$$

where C is a normalisation factor, proportional to the Dynkin index if ϕ is taken in a representation other than the adjoint.

Note that the σ-model has global $G \times G$ symmetry; that is under $\phi \to a\phi(x)b$ for $a, b \in G$ we have

$$\begin{aligned} S_\phi \to C &\int_{S^2} \mathrm{d}^2x \;\; \mathrm{Tr}(\partial_\mu (b^{-1}\phi^{-1}a^{-1}) \partial^\mu (a\phi b)), \\ = C &\int_{S^2} \mathrm{d}^2x \;\; \mathrm{Tr}(b^{-1}\partial_\mu \phi^{-1} a^{-1} a \partial^\mu \phi b) = S_\phi, \end{aligned} \tag{2.2.10}$$

due to the cyclicity of the trace.

We rewrite eq. (2.2.9) as

$$S = -C \int_{S^2} \mathrm{d}^2x \;\; \mathrm{Tr}(\phi^{-1}\partial_\mu \phi\, \phi^{-1} \partial^\mu \phi) \tag{2.2.11}$$

and we take g in the adjoint representation as we wish to use the Killing form as metric. With the help of eq. (2.2.7), we get

$$\phi^{-1}\partial_\mu \phi = e^{A}{}_{a}\, \partial_\mu \phi^a\, T_A \tag{2.2.12}$$

where $e^{A}{}_{a}$ is the vielbein, and the action S_g is recovered with $g(\phi)_{ab} = \delta_{AB} e^{A}{}_{a} e^{B}{}_{b}$.

As we have just seen, the above construction is based on a metric built up from the vielbein, and therefore the curvature of the manifold would be computed with the Levi-Civita connection. In general, i.e. for groups other than tori, this would not yield a zero Ricci curvature as in eq. (2.2.5), and the conformal invariance of the theory would be lost. In order to ensure conformal invariance, a topological term containing a totally antisymmetric coupling was added to eq. (2.2.9) by Witten [Wit84]. This term is the Wess-Zumino term

$$S_{WZ} = D \int_{B^3} \mathrm{d}^3 y \; \epsilon^{ijk} \, \mathrm{Tr}(\tilde{\phi}^{-1} \partial_i \tilde{\phi} \; \tilde{\phi}^{-1} \partial_j \tilde{\phi} \; \tilde{\phi}^{-1} \partial_k \tilde{\phi}), \tag{2.2.13}$$

where again D is a normalisation factor, and $\tilde{\phi}$ represents the extension of the map $\phi : S^2 \to G$ to the 3-ball B^3, whose boundary is the original space S^2. Since the second homotopy group $\pi_2(G)$ describes the homotopic equivalence classes of maps $S^2 \to G$, and since $\pi_2(G) = 0$ for any compact, connected Lie group, any map $\tilde{\phi}$ on B^3 is homotopically equivalent to ϕ when restricted to the boundary $\partial B^3 = S^2$. The group manifold G has a 3-form defined in terms of the structure constants f^{abc}, and the Wess-Zumino term can be seen to be the pullback of this 3-form to B^3. Roughly, this can be seen by realising that the structure constants can be obtained from the Killing form evaluated on an orthonormal basis with respect to the Killing form,

$$K(T_A, [T_B, T_C]) = K(T_A, f_{BC}{}^D T_D) = f_{BC}{}^D \delta_{AD} = f_{ABC}. \tag{2.2.14}$$

If a basis of $T_x B^3$ is given by ∂_{y_i}, then $\tilde{\gamma}$ defines a map $\mathrm{d}\tilde{\gamma}_x : T_x B^3 \to T_{\tilde{\gamma}(x)} G$ known as the *pushforward*. Since we can translate on a Lie manifold by using the left group action, we can identify the tangent space $T_{\tilde{\gamma}(x)} G$ with the Lie algebra – the tangent space at the identity – by now translating by $\tilde{\gamma}^{-1}$. We therefore see that the $\tilde{\gamma}^{-1} \frac{\partial \tilde{\gamma}}{\partial y_i}$ are elements of the Lie algebra of G, and that the Wess-Zumino term just pulls this back to X as stated. Note that since this gives a 3-form on a 3-manifold it is automatically closed.

The extension of ϕ to $\tilde{\phi}$, that is from S^2 to B^3 is not unique. The difference between two extensions can be seen to be equivalent to a map from $S^3 \to G$; since the two extensions $\tilde{\phi}_1$ and $\tilde{\phi}_2$ both restrict to ϕ on the boundary, the difference between them therefore gives an embedding of their union, S^3 to G. The third homotopy group $\pi_3(G)$ gives the homotopic equivalence classes of maps from $S^3 \to G$, and for a compact, connected, simple Lie group G we have $\pi_3(G) = \mathbb{Z}$. The pull-back of the 3-form to S^3 is then proportional to the winding number of the map. If the normalisation of the Wess-Zumino term is chosen such that S_{WZ} is well-defined modulo 2π, then the path integral will be equivalent for any integer multiple of S_{WZ}.

We have the Wess-Zumino term given schematically by

$$S_{WZ} = \int_{B^3} \gamma^* \omega, \tag{2.2.15}$$

where ω represents the closed 3-form and γ^* denotes the pull-back by γ. Poincaré's lemma states that the closed 3-form is locally exact – locally $G \cong \mathbb{R}^n$ and Poincaré's lemma states that all closed forms are exact on $\mathbb{R}^n$ – and hence, using this and Stoke's theorem, we have schematically

$$S_{WZ} = \int_{B^3} \gamma^* \omega = \int_{B^3} \gamma^* \,\mathrm{d}b = \int_{\partial B^3 = S^2} \gamma^* b, \tag{2.2.16}$$

where locally $\omega = \mathrm{d}b$. Using this, we can rewrite the Wess-Zumino term eq. (2.2.13) as

$$S_{WZ} = D \int_{S^2} \mathrm{d}^2 x \epsilon^{\mu\nu} b_{ab}(\phi) \partial_\mu \phi^a \partial_\nu \phi^b. \tag{2.2.17}$$

To summarise, the full action for the WZW model,

$$S_k^{WZW} = \frac{k}{16\pi\chi_R} \int_{S^2} \mathrm{d}^2 x \ \mathrm{Tr}(\partial_\mu \phi^{-1} \partial^\mu \phi) + \frac{k}{24\pi\chi_R} \int_{B^3} \mathrm{d}^3 y \, \epsilon^{ijk} \, \mathrm{Tr}(\phi^{-1}\partial_i \phi \, \phi^{-1} \partial_j \phi \, \phi^{-1} \partial_k \phi), \tag{2.2.18}$$

is conformally invariant at the quantum level [Wit84]. The Dynkin index χ_R has been introduced to allow for g to be taken in any unitary representation. The integer k is known as the *level* of the WZW model.

Furthermore, in order to make contact with the next section, we note that the action eq. (2.2.18) can equivalently be written in the form

$$S_k^{WZW} = \int_{S^2} \mathrm{d}^2 x g_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b + b_{ab}(\phi) \epsilon^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^b, \tag{2.2.19}$$

where the kinetic term has been written in terms of the metric g and the Wess-Zumino term has been written in the form of eq. (2.2.17).

The equation of motion for eq. (2.2.18) [Wit84; DMS97], written in terms of the complex variables $z, \bar{z}$,

$$\partial_z(\phi^{-1}\partial_{\bar{z}}\phi) = 0, \tag{2.2.20}$$

implies the conservation of the antiholomorphic current

$$J_{\bar{z}}(\bar{z}) := \phi^{-1}\partial_{\bar{z}}\phi, \qquad \partial_z J_{\bar{z}} = 0, \tag{2.2.21}$$

which in turn implies the conservation of the holomorphic current,

$$J_z(z) := \partial_z\phi\phi^{-1}, \qquad \partial_{\bar{z}} J_z = 0, \tag{2.2.22}$$

since

$$\begin{aligned} \partial_z(\phi^{-1}\partial_{\bar{z}}\phi) &= \partial_z\phi^{-1}\partial_{\bar{z}}\phi + \phi^{-1}\partial_z\partial_{\bar{z}}\phi, \\ &= \phi^{-1}\partial_{\bar{z}}(\partial_z\phi\phi^{-1})\phi, \end{aligned} \tag{2.2.23}$$

where we have used $\partial_\mu\phi^{-1} = -\phi^{-1}\partial_\mu\phi\phi^{-1}$, which itself follows simply from $\partial(\phi^{-1}\phi) = 0$. The conservation of the two currents results from the invariance of the action eq. (2.2.18) under local $G(z) \times G(\bar{z})$ transformations,

$$\phi(z, \bar{z}) \to A(z)\phi(z, \bar{z})\bar{A}^{-1}(\bar{z}), \tag{2.2.24}$$

for independent $A, \bar{A} \in G$.

Expanding the currents in a basis of the Lie algebra $\mathfrak{g}$ of G of dimension r,

$$J(z) = \sum_{A=1}^{r} J^A(z)T_A, \tag{2.2.25}$$

and using the methods of section 2.1.3, one can show that the currents satisfy the *current algebra*

$$J^A(z)J^B(\omega) \sim \frac{k\delta_{AB}}{(z-\omega)^2} + \sum_c f^{AB}{}_C\frac{J^C(\omega)}{z-\omega}, \tag{2.2.26}$$

where as usual $f_{AB}{}^C$ are the structure constants of the Lie algebra $\mathfrak{g}$,

$$[T_A, T_B] = f_{AB}{}^C T_C, \tag{2.2.27}$$

and we have suppressed all non-singular terms in the right-hand side of the OPE.

Expanding the currents in terms of Laurent modes,

$$J^A(z) = \sum_{n\in\mathbb{Z}} J_n^A z^{-n-1}, \tag{2.2.28}$$

and using the methods of section 2.1.4 one can then show that this OPE is equivalent to the commutation rules for the modes,

$$[J_m^A, J_n^B] = \sum_C f_{AB}{}^C J_{n+m}^C + km\delta_{AB}\delta_{m+n,0}, \tag{2.2.29}$$

which are the commutation rules for the affine Lie algebra at level k, $\hat{\mathfrak{g}}_k$.

A process known as the Sugawara construction can be used to construct the energy-momentum tensor for the WZW model as a bilinear in the currents J^A [DMS97]. The very fact that the energy-momentum tensor can be constructed out of bilinears in the currents is the reason why the resulting algebra underlying the symmetries of the WZW model is larger than the Virasoro algebra. With respect to this energy momentum tensor, the current J^a is primary with conformal dimension 1 and the central charge for the algebra is given in terms of Lie algebra $\hat{\mathfrak{g}}_k$ data as,

$$c = \frac{k \dim \mathfrak{g}}{k + h^\vee}, \qquad h^\vee := \sum_{i=1}^r {a_i}^\vee + 1, \tag{2.2.30}$$

where $h^\vee$ is known as the dual Coxeter number, r is the rank of the Lie algebra $\mathfrak{g}$ and the ${a_i}^\vee$ are known as the *comarks*, being the coefficients of the highest root of $\hat{\mathfrak{g}}_k$, θ in the basis of the coroots ${\alpha_i}^\vee$,

$$\theta = \sum_{i=1}^r {a_i}^\vee {\alpha_i}^\vee. \tag{2.2.31}$$

2.2.2 σ-models with Extended Supersymmetry

In [AF81], Alvarez-Gaume and Freedman investigated the possibility of a supersymmetric σ-model having extended supersymmetry. Beginning with a supersymmetric extension to the sigma model eq. (2.2.2) (without potential),

$$S[\Phi] = \frac{1}{4i}\int \mathrm{d}^2x\,\mathrm{d}^2\theta\; g_{ij}(\Phi)\bar{\mathrm{D}}\Phi^i\,\mathrm{D}\Phi^j, \tag{2.2.32}$$

for a superfield with component expansion,

$$\Phi^i(x,\theta) = \phi^i(x) + \bar{\theta}\psi^i(x) + \frac{1}{2}\bar{\theta}\theta F^i(x), \tag{2.2.33}$$

and spinor derivative $D_\alpha = \frac{\partial}{\partial \theta_\alpha} - i(\theta\bar{\gamma}^\mu\alpha\partial_\mu)$, the action eq. (2.2.32) can be expanded in terms of component fields as [FT81; AF81]

$$S[\phi,\psi] = \frac{1}{2}\int \mathrm{d}^2x \, \left(g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + ig_{ij}(\phi)\bar{\psi}^i\gamma^\mu D_\mu\psi^j + \frac{1}{6}R_{iklj}(\bar{\psi}^i\psi^k)(\bar{\psi}^l\psi^j)\right), \tag{2.2.34}$$

with covariant derivative $D_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{jk}\partial_\mu\phi^j\psi^k$. The bosonic fields ϕ^i still parameterise the manifold M for which g_{ij} is the metric, and the covariant derivative D_μ shows that the fermionic fields ψ^i transform as tangent vector fields on M.

The action eq. (2.2.32) is invariant under the supersymmetric transformation of the component fields defined by

$$\delta\phi^i = \bar{\epsilon}\psi^i, \qquad \delta\psi^i = -i\partial\!\!\!/\phi^i\epsilon - \Gamma^i_{jk}(\epsilon\psi^j)\psi^k, \tag{2.2.35}$$

where $\partial\!\!\!/$ uses the Feynman slash notation,

$$\partial\!\!\!/ := \gamma^\mu\partial_\mu. \tag{2.2.36}$$

The authors of [AF81] show that the requirement for the existence of additional supersymmetries is that the manifold M can be equipped with covariantly constant almost-complex structures which preserve the metric. That is, a second supersymmetry requires a degree (1,1) tensor f satisfying

$$f^2 = -1, \qquad g(f\cdot u, f\cdot v) = g(u,v), \qquad \nabla_x f = 0, \tag{2.2.37}$$

for all vectors $u,v,x \in T_pM$ and all points $p \in M$. The above requirements on f are the requirements for M to be a Kähler manifold. Furthermore, they show that the existence of a third supersymmetry, or equivalently a second almost-complex structure f' automatically implies the existence of a fourth supersymmetry given by $f'' = f\cdot f'$, and these four supersymmetries define a quaternionic structure on the tangent space of M.

As we have seen however, the general (potential-free) σ-model action described by eq. (2.2.2), which was extended to a supersymmetric action in eqs. (2.2.32) and (2.2.34) by [AF81], can be generalised to include a Wess-Zumino term as in eq. (2.2.19), defining the WZW model. One can ask therefore what the requirements for extended supersymmetry are if one starts from a supersymmetric generalisation of the WZW model action 2.2.19; this is the question discussed in [SSTV88a; STVS88; SSTV88b] which we shall now review.

Starting with the bosonic WZW action for a manifold M as described in eq. (2.2.19) and defining the spinors ψ^a as the supersymmetric partners of the bosonic fields ϕ^a,

$$\delta\phi^a = \bar{\epsilon}\psi^a, \tag{2.2.38}$$

the requirement that the commutator of two supersymmetries acts as a translation for all fields leads to the supersymmetric WZW-model with action [SSTV88a]

$$\begin{aligned} S[\phi,\psi] =& -\frac{1}{2\pi}\int \mathrm{d}^2x \ \Big(g_{ab}(\phi)\partial_\mu\phi^a\partial^\mu\phi^b + \lambda_{ab}\epsilon^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^b \\ & + g_{ab}\bar{\psi}^a \not{D}\psi^b - \frac{1}{4}\bar{\psi}^a_+\gamma_\mu\psi^b_+\bar{\psi}^c_-\gamma^\mu\psi^d_- R_{abcd}(\Gamma_+)\Big), \\ D_\mu\psi^a =& \partial_\mu\psi^a + ({\Gamma_+}^a{}_{bc}\psi^b_+ + {\Gamma_-}^a{}_{bc}\psi^b_-)\partial_\mu\phi^c. \end{aligned} \tag{2.2.39}$$

D_μ defines the covariant derivative and $\Gamma_\pm$ defines the torsionful connection,

$${\Gamma_\pm}^a{}_{bc} = \frac{1}{2}g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d} \pm 2T_{dbc}), \tag{2.2.40}$$

for a totally antisymmetric torsion tensor T. This action is invariant under the supersymmetry defined by

$$\delta\phi^a = \bar{\epsilon}\psi^a, \quad \delta\psi^a = \not{\partial}\phi^a\epsilon - \psi^b_+(\bar{\epsilon}_+{\Gamma_+}^a{}_{bc}\psi^c_- + \bar{\epsilon}_-{S_+}^a{}_{bc}\psi^c_+) + \psi^b_-(\bar{\epsilon}_-{\Gamma_-}^a{}_{bc}\psi^c_+ + \bar{\epsilon}_+{S_-}^a{}_{bc}\psi^c_-). \tag{2.2.41}$$

Similarly to the σ-model without Wess-Zumino term considered in [AF81], the authors of [SSTV88a] then investigate the restrictions on the model that the existence of further supersymmetries impose. Given a generic supersymmetric transformation of the bosonic field

$$\delta\phi^a = \bar{\epsilon}_- J_+{}^a{}_b \psi_+^b, \tag{2.2.42}$$

the invariance of the action eq. (2.2.39) under the transformation requires that $J_+{}^a{}_b$ is covariantly conserved and preserves the metric,

$$J_+{}^a{}_{b;c} = 0, \qquad J_+{}^c{}_a g_{cd} J_+{}^d{}_b = g_{ab}. \tag{2.2.43}$$

This can be used to show that the action is invariant under the supersymmetry transformation defined by replacing ψ_+^a with $J_+{}^a{}_b \psi_+^b$ in eq. (2.2.41).

Ensuring that these two supersymmetries obey the extended supersymmetry algebra imposes the requirements

$$J_+{}^b{}_a + J_+{}^a{}_b = 0, \qquad N_{ab}{}^c = 0, \tag{2.2.44}$$

where $N_{ab}{}^c$ is the Nijenhuis tensor, defined as

$$N_{ab}{}^c = J_+{}^d{}_a J_+{}^c{}_{[b,d]} - J_+{}^d{}_b J_+{}^c{}_{[a,d]}. \tag{2.2.45}$$

It is simple to show that eqs. (2.2.43) and (2.2.44) together imply that J_+ is an almost complex structure,

$$J_+{}^a{}_c J_+{}^c{}_b = -\delta^a{}_b, \tag{2.2.46}$$

and the vanishing of the Nijenhuis tensor eq. (2.2.44) is known to be the integrability condition of the almost complex structure, that is the almost complex structure is induced by a unique complex structure on M. Note that the existence of an almost complex structure implies that M must be even (real) dimensional. We now recognise the second condition of eq. (2.2.43) as the hermiticity condition for the metric and the first condition of eq. (2.2.43) as the requirement that the complex structure J is preserved by parallel transport. Equation (2.2.43) is therefore the Kähler condition, and M is therefore a Kähler manifold.

If there exist more almost complex structures J_i, then requiring the corresponding supersymmetries satisfy the extended supersymmetry algebra means imposing a

Clifford algebra structure on the complex structures,

$$J_i{}^a{}_c J_j{}^c{}_b + J_j{}^a{}_c J_i{}^c{}_b = -2\delta^a{}_b \delta_{ij}. \tag{2.2.47}$$

These almost complex structures are also required to obey the Nijenhuis conditions,

$$N_{ij}{}^a{}_{bc} := J_{(i}{}^d{}_{[b} J_{j)}{}^a{}_{c],d} + J_{(i}{}^a{}_{d,[c} J_{j)b]}{}^d = 0. \tag{2.2.48}$$

The original supersymmetry can be included in the above by identifying

$$J_0{}^a{}_b = J_{0b}{}^a = \delta^a{}_b. \tag{2.2.49}$$

If $R_{abcd} \neq 0$, eq. (2.2.43) constrains the possible total number of supersymmetries which one can define on a super WZW model. Since J is covariantly constant, it is invariant under the action of the holonomy group. The map $J_i : T_pM \to T_pM$ therefore defines an endomorphism of the holonomy module. Assuming the manifold is irreducible, that is the action of the holonomy group has no invariant subspaces, then by Schur's lemma the endomorphism ring of this module is thus a division algebra over the reals, and hence is at most four-dimensional. This shows that there can be no more than four supersymmetries defined for a super WZW model on an irreducible manifold with curvature, that is we have $\mathcal{N} \leq 4$. The authors therefore consider the case that M is a manifold without curvature and with completely antisymmetric torsion, they therefore restrict to the case of absolutely parallelisable manifolds [SSTV88a].

We now focus on the specific case of the $SU(3)$ group manifold, as this will be particularly relevant to the 'sum rules' discussed in Chapter 6. In the following example, we show how the almost-quaternionic structure (that is, the three complex structures which satisfy the Clifford algebra structure of eq. (2.2.47)) can be defined on $\mathfrak{su}(3)$.

Example 2.2.1. We denote the generators of $\mathfrak{su}(3)$ in the fundamental representation as

$$\tilde{T}_i = \frac{1}{2}\lambda^i, \tag{2.2.50}$$

where λ^i are the standard Gell-Mann matrices. Following [SSTV88a], we rename these generators as

$$\begin{aligned} T_1 &= \tilde{T}_1, \quad T_{\bar{1}} = \tilde{T}_2, \quad T_2 = \tilde{T}_4, \quad T_{\bar{2}} = \tilde{T}_5, \\ T_3 &= \tilde{T}_6, \quad T_{\bar{3}} = \tilde{T}_7, \quad T_4 = \tilde{T}_3, \quad T_{\bar{4}} = \tilde{T}_8. \end{aligned} \tag{2.2.51}$$

Since λ^3 and λ^8, are the diagonal Gell-Mann matrices, we have the Cartan subalgebra (CSA) generated by $T_4, T_{\bar{4}}$. We now define the first complex structure J on $\mathfrak{su}(3)$ as

$$JT_i = -T_{\bar{i}}, \qquad JT_{\bar{i}} = T_i, \tag{2.2.52}$$

where the second equation follows from the first since we have $J^2 = -I$. This ensures that J takes the standard general form

$$J = \begin{pmatrix} 0 & I_{d/2} \\ -I_{d/2} & 0 \end{pmatrix}, \tag{2.2.53}$$

for d the dimension of the Lie algebra, 8 in this example, and where we order the generators such that the first $d/2$ indices refer to the generators T_i and the second $d/2$ indices refer to the generators $T_{\bar{i}}$.

From the generators of the real Lie algebra defined above, we can define the generators of the complexification in the standard way,

$$\hat{T}_i = T_i + iT_{\bar{i}}, \qquad \hat{T}_{i^*} = T_i - iT_{\bar{i}} = (\hat{T}_i)^*, \tag{2.2.54}$$

where here, the $*$ denotes complex conjugation. In terms of the complex generators,

$$J\hat{T}_i = JT_i + iJT_{\bar{i}} = -T_{\bar{i}} + iT_i = i\hat{T}_i, \qquad J\hat{T}_{i^*} = -i\hat{T}_{i^*}, \tag{2.2.55}$$

hence J is diagonalised on this new basis. This satisfies the general construction of [SSTV88a], namely

$$JH = H, \qquad JE_\alpha = iE_\alpha, \qquad JE_{-\alpha} = -iE_{-\alpha}, \tag{2.2.56}$$

where $E_\alpha = \hat{T}_\alpha$ and $E_{-\alpha} = -\hat{T}_{\alpha^*}$ and hence the Nijenhuis condition eq. (2.2.44) is automatically satisfied. Note the complex basis is a Cartan-Weyl basis for $\mathfrak{su}(3)$.

The positive roots corresponding to $\hat{T}_1, \hat{T}_2$ and $\hat{T}_3$ are

$$\hat{T}_1 \to \alpha_1 = (1), \qquad \hat{T}_2 \to \alpha_2 = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \qquad \hat{T}_3 \to \alpha_3 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), \tag{2.2.57}$$

and with this (non-standard) labelling we have α_1 and α_3 as the simple roots and $\alpha_2 = \alpha_1 + \alpha_3$ as the highest root, which from here on we refer to as θ. We therefore also refer to $\hat{T}_2$ as $\hat{T}_\theta$. This set of non-zero roots for $\mathfrak{su}(3)$ is shown in fig. 2.4.

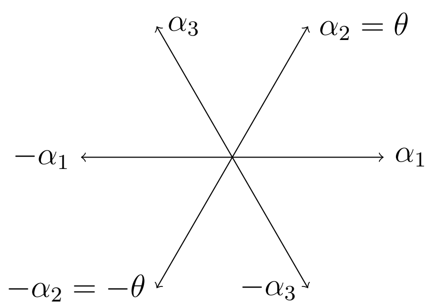


Figure 2.4: The root system of $\mathfrak{su}(3)$

The requirement to be able to define a second complex structure on the algebra is as follows. First, split the positive roots into two sets, those orthogonal to the highest root θ form a set $\Delta_{\perp\theta}$ and those positive roots not orthogonal to θ form a set Δ_θ. From $\Delta_{\perp\theta}$, take a highest root of this subset of roots perpendicular to θ, which we will call θ'. We then repeat the process, dividing $\Delta_{\perp\theta}$ into subsets depending on whether or not the roots are orthogonal to θ' and so on. The highest roots obtained at each step are known as *basic roots*. Eventually this process will terminate when we get to a basic root $\tilde{\theta}$ for which $\widetilde{\Delta_\perp}$ is empty. The hermiticity condition on the metric then gives the requirement

$$\theta^m \theta'_m = 0, \tag{2.2.58}$$

where $\theta^m = -(\theta_m)^*$. For this example of $\mathfrak{su}(3)$ we have $\theta = \alpha_2$ as the highest root, and no roots orthogonal to θ. Since θ is therefore the only basic root, there is no obstruction to defining a second complex structure.

The action of K is then given by

$$KE_\alpha = \frac{1}{4}k_\theta(1+iJ)[E_{-\theta}, E_\alpha] - \frac{1}{4}k_\theta^*(1-iJ)[E_\theta, E_\alpha], \tag{2.2.59}$$

where k_θ is a complex number whose phase is arbitrary and whose modulus is given by

$$\theta_m\theta^m|k_\theta|^2 = -1. \tag{2.2.60}$$

For our example, we have $\theta_m = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ and hence $\theta^m = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$. This gives,

$$|k_\theta| = 1, \tag{2.2.61}$$

and since the phase is irrelevant, we set the phase equal to 0. We therefore find the action of the second complex structure on the generator corresponding to the highest root (letting $\alpha = \theta$),

$$K\hat{T}_2 = \frac{1}{4}(1+iJ)(T_4 + \sqrt{3}T_{\bar{4}}) = \frac{1}{4}(1+i\sqrt{3})\hat{T}_{4^*}, \tag{2.2.62}$$

and its conjugate,

$$K\hat{T}_{2^*} = \frac{1}{4}(1-iJ)(T_4 + \sqrt{3}T_{\bar{4}}) = \frac{1}{4}(1-i\sqrt{3})\hat{T}_4, \tag{2.2.63}$$

Finally we calculate the action of K on $\hat{T}_1$ by letting $\alpha = 1$ in eq. (2.2.59),

$$\begin{aligned} K\hat{T}_1 &= \frac{1}{4}(1+iJ)[-\hat{T}_{2^*}, \hat{T}_1] - \frac{1}{4}(1-iJ)[\hat{T}_2, \hat{T}_1], \\ &= -\frac{1}{4}(1+iJ)\hat{T}_{3^*} = -\frac{1}{2}\hat{T}_{3^*}, \end{aligned} \tag{2.2.64}$$

and similarly

$$\begin{aligned} K\hat{T}_{1^*} &= \frac{1}{4}(1+iJ)[-\hat{T}_{2^*}, \hat{T}_{1^*}] - \frac{1}{4}(1-iJ)[\hat{T}_2, \hat{T}_{1^*}], \\ &= \frac{1}{4}(1-iJ)\hat{T}_3 = \frac{1}{2}\hat{T}_3. \end{aligned} \tag{2.2.65}$$

The action of K on $\hat{T}_3, \hat{T}_{3^*}, \hat{T}_4$ and $\hat{T}_{4^*}$ is now easily obtained using $K^2 = -1$. One can also easily check that J and K anticommute, for example

$$\begin{aligned} JK\hat{T}_2 &= \frac{1}{4}(1+i\sqrt{3})J\hat{T}_{4^*} = -i\frac{1}{4}(1+i\sqrt{3})\hat{T}_{4^*}, \\ KJ\hat{T}_2 &= iK\hat{T}_2 = i\frac{1}{4}(1+i\sqrt{3})\hat{T}_{4^*}, \end{aligned} \tag{2.2.66}$$

and therefore as expected we have,

$$\{J, K\}\hat{T}_2 = 0. \tag{2.2.67}$$

Finally, a third complex structure L, can be defined similarly to the second, however requiring the second and third complex structures to anticommute forces $l_\theta = ik_\theta$. Note that this is equivalent to defining

$$L := KJ, \tag{2.2.68}$$

which makes the anticommutativity of L with K and with J clear since

$$LJ = KJJ = -K, \qquad JL = JKJ = -JJK = K, \tag{2.2.69}$$

where we have used the fact that J and K are already known to anticommute. One can show the anticommutativity of L and K in the same way. We therefore have

$$\begin{aligned} L\hat{T}_1 &= -\frac{i}{2}\hat{T}_{3^*}, & L\hat{T}_2 &= \frac{i}{4}(1+i\sqrt{3})\hat{T}_{4^*}, \\ L\hat{T}_{1^*} &= -\frac{i}{2}\hat{T}_3, & L\hat{T}_{2^*} &= -\frac{i}{4}(1-i\sqrt{3})\hat{T}_4, \end{aligned} \tag{2.2.70}$$

with the action of L on the remaining generators being found using that $L^2 = -1$.

We now see that the maximum number of complex structures that can be defined on $\mathfrak{su}(3)$, or in fact on any other *non-abelian* group manifold, is at most 3. If there were a fourth complex structure Q, it would act in the same way as the second, but with $q_\theta = \pm ik_\theta$ in order for K and Q to anticommute. However, we would then have that L and Q commute rather than anticommute.

We have therefore constructed 3 anticommuting complex structures that along with $\delta^a{}_b$, satisfy the Clifford algebra structure of eq. (2.2.47). This therefore defines the almost-quaternionic structure on $\mathfrak{su}(3)$ and hence the super-WZW model on $SU(3)$

has $\mathcal{N} = 4$ supersymmetry. $\triangle$

The previous example of $\mathfrak{su}(3)$ is particularly simple, since the process of iteratively defining the second complex structure on the Lie algebra generators corresponding to elements of Δ_θ for some highest root θ then continuing from the set $\Delta_{\perp\theta}$, only has a single step, since $\Delta_{\perp\theta} = \phi$. In this case, the quaternionic structure acts between the generators E_θ, $E_{-\theta}$ and the two Cartan generators of $\mathfrak{su}(3)$, which form an $\mathfrak{su}(2) \times \mathfrak{u}(1)$ subalgebra, but also on the four non-orthogonal roots which span the tangent space at the identity coset of the homogeneous space

$$W(3) := \frac{SU(3)}{SU(2) \times U(1)}. \tag{2.2.71}$$

More generally, we always have the generators corresponding to the highest root θ and its negative transforming into 2 elements of the CSA under the second complex structure. The $\mathfrak{su}(2) \times \mathfrak{u}(1)$ subalgebra they form is hence fixed by the action of the complex structure. The quaternionic structure also acts on the other $4(h^\vee - 2)$ roots ($h^\vee$ is the dual Coxeter number) which are not orthogonal to the highest root θ and which are associated with a Wolf space. In other words, at this stage of the process, the set of generators corresponding to $\Delta_{\perp\theta}$ forms a pointwise fixed subalgebra, and the remaining Lie algebra generators corresponding to Δ_θ form a coset algebra,

$$\frac{\Delta_\theta}{\Delta_{\perp\theta} \times \mathfrak{su}(2) \times \mathfrak{u}(1)}, \tag{2.2.72}$$

where by an abuse of notation we have labelled the algebras by the sets corresponding to the appropriate roots. The associated symmetric spaces

$$\frac{G}{H \times SU(2) \times U(1)}, \tag{2.2.73}$$

correspond, up to the factor of $U(1)$ in the denominator, to the quaternionic Kähler symmetric spaces, also known as Wolf spaces [Wol65]. We recall that a symmetric space is a homogeneous space G/H where infinitesimally, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h}$. We therefore see that the quaternionic structure is defined at each stage on the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ subalgebra corresponding to the highest root, its negative, and 2 elements of the CSA and on the Wolf space spanned by the remaining generators whose roots are not orthogonal to θ. The most relevant Wolf space for our purposes is the space

$$W(N) = \frac{SU(N)}{SU(N-2) \times SU(2) \times U(1)}, \tag{2.2.74}$$

where our previous example corresponded to the case $N = 3$. We defined the quaternionic structure on the Lie algebra associated to $W(3)$ and the Lie algebra $\mathfrak{su}(2) \times \mathfrak{u}(1)$ such that the structure was then defined on $\mathfrak{su}(3)$. So the Lie group $SU(3)$ is a quaternionic group manifold of dimension 8.

Note that of the $N^2 - N$ non-zero roots of $\mathfrak{su}(N)$, $N^2 - 5N + 6$ lie in the $\mathfrak{su}(N-2)$ orthogonal to the highest root of $\mathfrak{su}(N)$ and 2 lie in the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ in which the highest root transforms, leaving $4(N-2)$ in the Wolf space $W(N)$. Indeed, in the case where $G = SU(N)$, the dual Coxeter number is $h^\vee = N$.

2.2.3 Current Algebras on Supersymmetric σ-Models

In the previous subsection we showed how one can define a supersymmetric σ-model on a group manifold which supports two supersymmetries if there is a complex structure defined on the Lie algebra and which supports four supersymmetries if it is possible to define an almost-quaternionic structure (three complex structures which satisfy the Clifford algebra structure of eq. (2.2.47)).

On group manifolds, the super-WZW action eq. (2.2.39) considered in the previous subsection can be written as

$$S = -\frac{k}{4\pi\chi_R}\left(\frac{1}{2}\int \mathrm{d}^2x \ \mathrm{Tr}[\partial_\mu g^{-1}\partial^\mu g - \bar{\psi}\partial\!\!\!/\psi] - \frac{1}{3}\int \mathrm{d}^3x \ \epsilon^{abc}\,\mathrm{Tr}[g^{-1}g_{,a}g^{-1}g_{,b}g^{-1}g_{,b}]\right), \tag{2.2.75}$$

and can be shown [STVS88] to be invariant under conformal and supersymmetry transformations, as well as under Kac-Moody and super Kac-Moody transformations. Each of these symmetries has a corresponding Noether current and just as we have seen for the Virasoro algebra in section 2.1.3, the OPEs between these currents give rise to an affine Lie algebra structure for the modes of the currents; as in section 2.1.4, the modes of the energy-momentum tensor $T(z)$ generate the Virasoro algebra, and as we saw in section 2.2.1, associated to a Kac-Moody transformation for the group G is a current $J(z)$ whose modes satisfy the $\hat{\mathfrak{g}}$ affine Lie algebra. Due to the existence of the supersymmetries present in this model, we also have dimension-$\frac{1}{2}$ currents associated with the super Kac-Moody symmetry. These super Kac-Moody currents are just given by the fermions $\psi(z)$. The Sugawara construction can be used to construct an energy-momentum tensor from these Kac-Moody currents; in the case of a quaternionic structure defined in stages as explained in the previous subsection, the energy momentum tensor is defined using the GKO coset construction [GKO86] for each stage individually. For example in the case of the simple group $SU(2M+1)$, on which the quaternionic structure may be defined in M' stages of the form $W(2M'+1)\times SU(2)\times U(1)$ for $1\leq M'\leq M$, the energy-momentum tensor at the first stage is given by

$$L = L_F + L_{SU(2M+1)} - L_{SU(2M-1)}. \tag{2.2.76}$$

Here, L_F is the contribution of the $8M$ free fermions (the dimension of $W(2M+1)\times SU(2)\times U(1)$), $4(2M-1)$ of which span the Wolf space $W(2M+1)$ and 4 corresponding to the highest root ψ_θ, its negative ψ^θ and the 2 elements of the CSA ψ^m, ψ_m which the quaternionic structure transforms ψ^θ and ψ_θ into. $L_{SU(2M+1)}$ is the energy-momentum tensor created using the standard Sugawara construction for the Kac-Moody currents associated to $SU(2M+1)$ and similarly $L_{SU(2M-1)}$ is the Sugawara energy-momentum tensor for $SU(2M-1)$.

In the case that we have an almost-quaternionic structure on the group manifold, and hence that the super-WZW model has $\mathcal{N}=4$ supersymmetry, the complex structures can be used to construct further dimension-$\frac{1}{2}$ operators. Explicitly,

$$\psi_\delta(z)\equiv\psi(z)=\delta\psi(z),\quad \psi_J(z)=J\psi(z),\quad \psi_K(z)=K\psi(z),\quad \psi_L(z)=L\psi(z). \tag{2.2.77}$$

For each operator $\psi_i(z), i \in \{\delta, J, K, L\}$ one can construct a dimension-$\frac{3}{2}$ operator associated to the superconformal transformations which is usually denoted $G_i(z)$ [STVS88]; this operator G_i should not be confused with the group G on which the super-WZW model is built. The OPEs are simplified if instead of considering the G_i, one forms the complex linear combinations [STVS88]

$$G_{\pm} = \frac{1}{2}(G_\delta + iG_J), \qquad G_{\pm K} = \frac{1}{2}(G_\delta + iG_K). \tag{2.2.78}$$

As for the energy-momentum tensor, in the quaternionic case these operators should be defined separately for the different stages of the algebra.

If all possible OPEs between these operators are considered, extra dimension-1 operators will need to be introduced in order to close the algebra. The net result is a superconformal algebra defined at each stage of the construction reviewed above, containing the dimension-2 energy momentum tensor $T(z)$, four dimension-$\frac{3}{2}$ supercharges $G_a(z)$, seven dimension-1 operators corresponding to an $\widehat{\mathfrak{su}(2)} \times \widehat{\mathfrak{su}(2)} \times \widehat{\mathfrak{u}(1)}$ Kac-Moody algebra and 4 dimension-$\frac{1}{2}$ operators $Q_a(z)$ where here $a \in \{\pm, \pm K\}$; these dimension-$\frac{1}{2}$ operators correspond to the fermions associated with the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ subalgebra. Operators defined at different stages of the algebra commute and hence one obtains a copy of this $\mathcal{N} = 4$ SCA for each stage of the algebra. This SCA is governed by 2 parameters, namely the levels of the 2 affine $\widehat{\mathfrak{su}(2)}$. The $\widehat{\mathfrak{su}(2)}$ are usually referred to as $\widehat{\mathfrak{su}(2)}^{\pm}$ and the levels are therefore denoted $k^{\pm}$. The central charge for the algebra is given in terms of these levels as

$$c = \frac{6k^+k^-}{k^+ + k^-} = \frac{6k^+k^-}{k}, \tag{2.2.79}$$

where we have defined $k = k^+ + k^-$.

This $\mathcal{N} = 4$ superconformal algebra is known in the literature as A_γ, where

$$\gamma = \frac{k^-}{k}, \tag{2.2.80}$$

and the central charge $c = 6k\gamma(1-\gamma)$ are the quantities which appear most directly in the algebra. The algebra is also referred to as the 'Large' $\mathcal{N} = 4$ SCA in order to differentiate it from the 'small' $\mathcal{N} = 4$ SCA of Ademollo et al. [Ade+76b], which contains only a single $\widehat{\mathfrak{su}(2)}$ Kac-Moody subalgebra. The OPEs for these operators can be found in the original paper [STVS88] - the commutation relations that the modes of these operators satisfy can be found in Appendix A.

In summary, all operators defined at different stages commute, and all operators defined at a given stage are primary with respect to the energy-momentum tensor at that stage. At each stage, these operators satisfy the commutation relation of the large $\mathcal{N} = 4$ superconformal algebra A_γ [STVS88]. As a consequence of this fact, one cannot assert that the super WZW model describing a superstring propagating on an arbitrary group manifold exhibits A_γ symmetry, but rather, that several A_γ SCA emerge in stages within the model. If there is more than one stage ($M' > 1$), there is no action that provides the currents representing A_γ at any stage. If there is exactly one stage however, the situation is obviously different. In fact, superstrings propagating on the 4-dimensional quaternionic group manifold $SU(2) \times U(1)$ or the 8-dimensional quaternionic group manifold $SU(3)$ are described by a super WZW model which exhibits A_γ symmetry. We will return to the $SU(3)$ quaternionic group manifold in chapter 6, where we set the scene for the potential discovery of a new moonshine phenomenon.

In the following chapter we will consider the representation theory of A_γ where the commutation relations will be of more direct use than the OPEs.

Chapter 3

The Representation Theory of A_γ

In this chapter we study the representation theory of the Large $\mathcal{N} = 4$ SCA A_γ introduced in the previous chapter. It will be advantageous to also briefly discuss the related algebra $\tilde{A}_\gamma$, which is formed from A_γ by decoupling the four dimension-$\frac{1}{2}$ operators and the dimension-1 operator corresponding to the $\widehat{\mathfrak{u}(1)}$ Kac-Moody subalgebra as described in [GS88]. The representations of interest to physics will be unitary representations whose spectrum is bounded below, that is unitary highest weight representations (UHWRs) and hence these are the representations which we will be interested in here. We will discuss the structure of UHWRs of A_γ, first studied in [GPTV89] before discussing character formulae for these representations as developed in [PT90a; PT90b]. The Ramond sector of the A_γ algebra is complicated by the fact that there is no unique highest weight state for any representation and so we first discuss the Neveu-Schwarz representations in section 3.2, before introducing an isomoprhism between the two sectors and using this to discuss the Ramond representations in section 3.3. Finally in section 3.4 we first derive the formulae for Verma modules of A_γ and sketch how one constructs the character formulae for the irreducible modules of A_γ from the characters of the associated Verma modules.

3.1 A_γ and $\tilde{A}_\gamma$

We first recall from section 2.2.3 that A_γ is a SCA containing the dimension-2 energy-momentum tensor $T(z)$, 4 dimension-$\frac{3}{2}$ supercharges $G_a(z)$, for $a \in \{\pm, \pm K\}$, 7 dimension-1 operators forming an $\widehat{\mathfrak{su}(2)}^+_{k^+} \times \widehat{\mathfrak{su}(2)}^-_{k^-} \times \widehat{\mathfrak{u}(1)}$ Kac-Moody subalgebra and 4 dimension-$\frac{1}{2}$ operators $Q_a(z)$, for $a \in \{\pm, \pm K\}$. The modes of these operators satisfy the commutation relations listed in Appendix A. The central charge of the algebra is given in terms of the levels $k^\pm$ and $k = k^+ + k^-$ as,

$$c = \frac{6k^+k^-}{k}. \tag{3.1.1}$$

In physically relevant representations of A_γ, the representations of the subalgebras $\widehat{\mathfrak{su}(2)}^\pm$ must be *integrable* representations, meaning the projection on to the $\mathfrak{su}(2)$ subalgebra associated to any real root of $\widehat{\mathfrak{su}(2)}^\pm$ must be finite dimensional. This is because correlation functions involving the primary fields associated to highest weights of non-integrable representations of $\widehat{\mathfrak{su}(2)}^\pm$ are zero, hence such primary fields decouple from the theory [DMS97]. The $\widehat{\mathfrak{su}(2)}^\pm_{k^\pm}$ affine algebras admit integrable representations if the levels satisfy,

$$k^\pm \in \mathbb{Z}_+. \tag{3.1.2}$$

This implies that the A_γ central charge c is bounded from below $(3 \leq c)$. We then see that A_γ is a one parameter family of SCAs, the one parameter being given by $\gamma = \frac{k^-}{k}$, or alternatively by $1 - \gamma = \frac{k^+}{k}$.

As shown in [GS88], one can decouple the free fermionic fields as well as the bosonic $\widehat{\mathfrak{u}(1)}$ current from the rest of the algebra, leaving a non-linear algebra known in the literature as $\tilde{A}_\gamma$ containing an energy-momentum tensor $\widetilde{T}(z)$, four fields $\widetilde{G^a}(z)$ which have weight $\frac{3}{2}$ under the new energy-momentum tensor $\widetilde{T}(z)$ and six fields $\widetilde{T}^{\pm i}(z)$ which have weight 1 under $\widetilde{T}(z)$. The central charge $\widetilde{c}$ of $\widetilde{T}(z)$ is given by

$$\widetilde{c} = c - 3, \tag{3.1.3}$$

and the weight 1 fields $\tilde{T}^{\pm i}(z)$ form an $\widehat{\mathfrak{su}(2)}^{+}_{\tilde{k}^+} \times \widehat{\mathfrak{su}(2)}^{-}_{\tilde{k}^-}$ Kac-Moody subalgebra, where the levels are given by

$$\tilde{k}^{\pm} = k^{\pm} - 1. \tag{3.1.4}$$

Explicitly, the fields in $\tilde{A}_\gamma$ are given in terms of the fields in A_γ by [GPTV89]

$$\begin{aligned} \widetilde{L} &= L + \frac{1}{k}(UU + \partial Q^a Q_a), \\ \widetilde{G}_a &= G_a + \frac{2}{k}UQ_a - \frac{2}{3k^2}\epsilon_{abcd}Q^bQ^cQ^d - \frac{4i}{k}Q^b(\alpha^{+i}_{ba}\widetilde{T}^+_i - \alpha^{-i}_{ba}\widetilde{T}^-_i), \\ \widetilde{T}^{\pm i} &= T^{\pm i} - \frac{i}{k}\alpha^{\pm i}_{ab}Q^aQ^b, \end{aligned} \tag{3.1.5}$$

where normal ordering is implicit and the non-zero values of $\alpha^{\pm i}_{ab}$ are

$$\begin{aligned} \alpha^{\pm 3}_{+\ -} &= -\frac{i}{4}, & \alpha^{+-}_{+\ +K} &= \frac{i}{2}, & \alpha^{-+}_{-\ +K} &= -\frac{i}{2}, \\ \alpha^{\pm 3}_{+K\ -K} &= \mp\frac{i}{4}, & \alpha^{++}_{-\ -K} &= -\frac{i}{2}, & \alpha^{--}_{+\ -K} &= \frac{i}{2}. \end{aligned} \tag{3.1.6}$$

3.2 Representations of A_γ in the Neveu-Schwarz Sector

In this section we describe representations of A_γ in the Neveu-Schwarz sector. Following the notation of [GPTV89; PT90a; PT90b] we derive the allowed ranges of the quantum numbers for these representations. The results are summarised in table 3.1 at the end of the section.

In a unitary highest weight representation (UHWR) of A_γ, we have a highest weight $|\Omega\rangle$ defined to satisfy

$$L_n |\Omega\rangle = T^{\pm i}_n |\Omega\rangle = U_n |\Omega\rangle = T^{\pm +}_0 |\Omega\rangle = Q^a_r |\Omega\rangle = G^a_r |\Omega\rangle = 0, \tag{3.2.1}$$

for $n \in \mathbb{Z}_+ = \{1, 2, \ldots\}$, and $r \in \mathbb{Z}_+ - \frac{1}{2}$ (Neveu-Schwarz (NS) sector) or $r \in \mathbb{Z}_+$ (Ramond (R) sector). NS representations of A_γ are classified by the eigenvalues of their highest weight state (HWS) $|\Omega\rangle$ under the zero modes of the algebra,

$$L_0 |\Omega\rangle = h |\Omega\rangle, \qquad U_0 |\Omega\rangle = -iu |\Omega\rangle, \qquad T^{\pm 3}_0 |\Omega\rangle = l^{\pm}_\Omega |\Omega\rangle, \tag{3.2.2}$$

and hence a UHWR of A_γ is labelled by the quantum numbers $h, u, l_\Omega^\pm$ as well as the central charge of the algebra c and the paramater γ. The classification of R representations is more subtle due to the non-trivial structure of the ground level, which is built by acting on a state $|\Omega\rangle$ satisfying eq. (3.2.1) with the zero modes of the A_γ generators. Besides c and γ, R representations are labelled by the maximal values of $\widehat{\mathfrak{su}(2)}^\pm$ charges appearing at ground level. Unlike in the NS sector, these charges correspond to the $T_0^{\pm 3}$ eigenvalues of different zero mode states.

Since we may also construct a representation of the sister algebra $\tilde{A}_\gamma$ on $|\Omega\rangle$, it is clear that in the Neveu-Schwarz sector, where the fermionic generators do not have zero modes, the state $|\Omega\rangle$ has the same charge under the $\widehat{\mathfrak{su}(2)}^\pm$ of A_γ as the $\widehat{\mathfrak{su}(2)}^\pm$ of $\tilde{A}_\gamma$. That is

$$\widetilde{T}_0^{\pm 3}|\Omega\rangle = \tilde{l}_\Omega^\pm |\Omega\rangle\,, \qquad T_0^{\pm 3}|\Omega\rangle = l_\Omega^\pm |\Omega\rangle\,, \qquad l_\Omega^\pm = \tilde{l}_\Omega^\pm. \tag{3.2.3}$$

We now consider the allowed values for the quantum numbers $h, u, l_\Omega^\pm$ in a UHWR. As we shall shortly see, the Ramond sector is slightly more complicated and we therefore begin by considering the Neveu-Schwarz sector. First, we consider the quantum number u; since the generator of the $\widehat{\mathfrak{u}(1)}$ current algebra was a free boson we see from Appendix A that the zero mode U_0 commutes with all the other generators of A_γ and hence all states in a UHWR have the same charge under the $\widehat{\mathfrak{u}(1)}$. We see that unitarity does not give any restrictions on u other than $u \in \mathbb{R}$,

$$|U_0|\Omega\rangle|^2 \equiv \langle\Omega| - U_0 U_0|\Omega\rangle = -(-iu)(-iu)\langle\Omega|\Omega\rangle = u^2, \tag{3.2.4}$$

where the minus sign is due to U being an antihermitian operator, so the hermitian conjugate $U_n^\dagger = -U_{-n}$ as in Appendix A. Unitarity of the representation hence requires $|U_0|\Omega\rangle|^2 \geq 0 \iff u \in \mathbb{R}$. Note that one must be careful with Bra-Ket notation when dealing with antihermitian operators such as U. Recall that the Bra-Ket notation just refers to an inner product on the Hilbert space. It is usual to think of operators between the Bra and the Ket as acting on either the left or right, as for a Hermitian operator H,

$$\langle\Omega, H\Omega\rangle = \langle H\Omega, \Omega\rangle. \tag{3.2.5}$$

For non-hermitian operators such as U this does not hold and hence we must always treat non-hermitian operators as acting on the Ket. Since this notation is common in the literature we choose to use it here regardless and hence all operators will be taken to be acting on the right.

Next we consider the allowed values for the quantum numbers $l^\pm_\Omega$, the charges under the two $\widehat{\mathfrak{su}(2)}^\pm$. Firstly, from the general theory of Kac-Moody algebras, we know that for integrable representations of $\widehat{\mathfrak{su}(2)}_k$ with highest weight $\hat{\lambda}$, the Dynkin labels satisfy

$$\lambda_i \in \mathbb{Z}_+, \tag{3.2.6}$$

and the zeroth Dynkin label is given by

$$\lambda_0 = k - (\lambda, \theta), \tag{3.2.7}$$

where λ is the finite part of the affine weight $\hat{\lambda}$ and θ is the highest root of $\mathfrak{su}(2)$. There are therefore only a finite number of integrable highest weight representations of $\widehat{\mathfrak{su}(2)}_k$ labelled by the spin l,

$$0 \leq l \leq \frac{k}{2}, \quad l \in \frac{1}{2}\mathbb{Z}, \tag{3.2.8}$$

where the factor of half is obtained in the change from the Chevalley to the spin basis of $\mathfrak{su}(2)$ where,

$$T_3 \left|\lambda\right\rangle = l \left|\lambda\right\rangle = \frac{\lambda_1}{2} \left|\lambda\right\rangle. \tag{3.2.9}$$

We therefore expect the quantum numbers $l^\pm_\Omega$ to satisfy $0 \leq l^\pm_\Omega \leq \frac{k^\pm}{2}$. As shown in [GPTV89], the $\widehat{\mathfrak{su}(2)}^\pm_{k^\pm}$ charges cannot obtain the maximum value of $\frac{k^\pm}{2}$ in the NS sector. To see this one can consider the states $T^{\pm +}_{-1}\left|\Omega\right\rangle$, assuming $l^\pm = \frac{k^\pm}{2}$. These states have norm squared

$$\left\langle\Omega\right| T^{\pm -}_{1} T^{\pm +}_{-1} \left|\Omega\right\rangle = \left\langle\Omega\right| (-2T^{\pm 3}_0 + k^\pm) \left|\Omega\right\rangle = 0, \tag{3.2.10}$$

and hence in an irreducible representation of A_γ, where we have quotiented out null-modules, these states must be identically zero and hence so must all their descendants. However, the descendant states $Q^-_{-1/2}T^{\pm+}_{-1}\left|\Omega\right\rangle \equiv -Q^{\pm K}_{-1/2}\left|\Omega\right\rangle$ have positive norm, giving a contradiction.

Given that any representation of A_γ gives rise to a representation of $\tilde{A}_\gamma$ as explained in section 3.1, we find another way to demonstrate the restricted maximum value of $l^\pm$. Writing the allowed values of $l^\pm$ as

$$0 \leq l^\pm \leq \frac{(k^\pm - 1)}{2} = \frac{\widetilde{k}^\pm}{2}, \tag{3.2.11}$$

using eq. (3.1.4), we realise that the maximum value of $l^\pm$ already obtained is the maximum one should expect using the general Kac-Moody argument for a representation of $\tilde{A}_\gamma$ with levels $\widetilde{k^\pm} = k^\pm - 1$.

Finally let us consider the conformal weight of the state $\left|\Omega\right\rangle$ in the NS sector. We obtain a unitarity bound by considering the norm of the state $Q^+_{-1/2}G^+_{-1/2}\left|\Omega\right\rangle$,

$$|Q^+_{-1/2}G^+_{-1/2}\left|\Omega\right\rangle|^2 = \frac{1}{4}\left(kh - k^+l^- - k^-l^+ - (l^+ - l^-)^2 - u^2\right), \tag{3.2.12}$$

which implies that in a unitary representation we have

$$kh \geq k^+l^- + k^-l^+ + (l^+ - l^-)^2 + u^2, \tag{3.2.13}$$

where saturation of this bound implies that the state $Q^+_{-1/2}G^+_{-1/2}\left|\Omega\right\rangle$ is identically 0 in a unitary representation, since such representations are formed by quotienting out all null modules as explained in section 2.1.5. A representation where the conformal weight saturates this bound and the aforementioned state disappears is known as a 'massless' or 'short' representation of A_γ. When the conformal weight lies above this bound the representation is said to be 'massive' or 'long'. The ground level is therefore identical for both massless and massive representations of A_γ in the NS sector and a generic such ground level is shown in fig. 3.1. The highest weight state $\left|\Omega\right\rangle$ is the top right state in this figure, which shows the ground states of a massless representation of A_γ, where the $\widehat{\mathfrak{su}(2)}^+$ charge has been plotted against the $\widehat{\mathfrak{su}(2)}^-$

charge and the states are labelled by their multiplicities. As for all representations of A_γ in the Neveu-Schwarz sector, one has a singular hws which in this example forms a triplet of $\widehat{\mathfrak{su}(2)}^+$ and a quintuplet of $\widehat{\mathfrak{su}(2)}^-$ as expected given the charges.

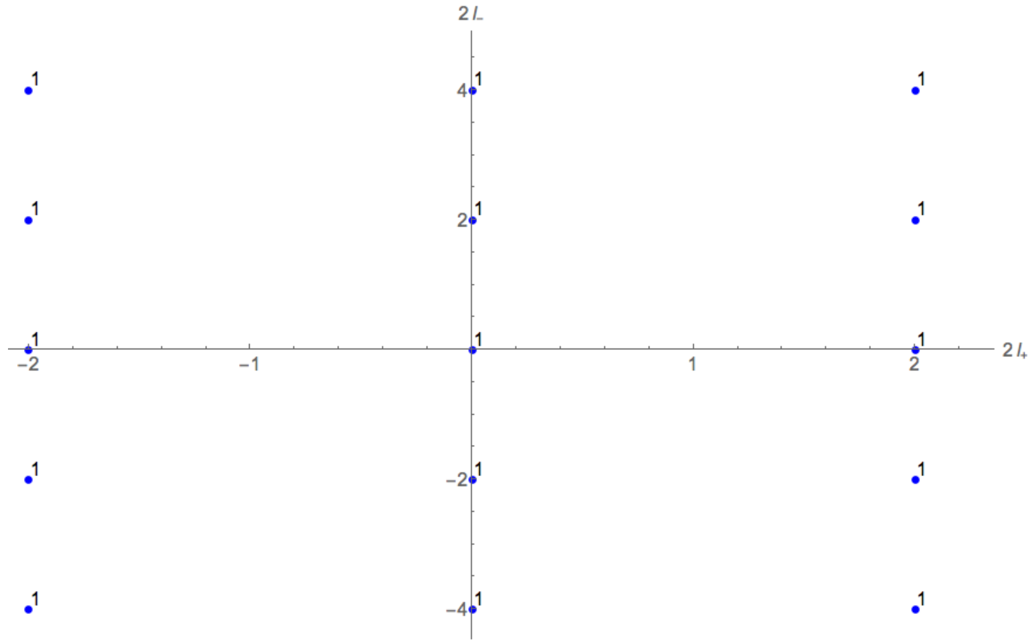


Figure 3.1: The ground states of a massless NS representation of A_γ. This representation has $k^+ = 5, k^- = 7, l^+ = 1, l^- = 2$.

At the ground level of a representation of A_γ in the NS sector, there is no difference between a massless and a massive representation. The only exception to this is when one or both of the $\widehat{\mathfrak{su}(2)}^\pm$ charges takes their maximum allowed values of $l^\pm = \frac{\widetilde{k}^\pm}{2} = \frac{k^\pm - 1}{2}$, as in this case the representation is forced to be massless; no unitary massive representations can exist where either of these charges obtains its maximum value as we will now show. We will assume that the $\widehat{\mathfrak{su}(2)}^+$ charge is the one which obtains its maximum, $l^+ = \frac{\widetilde{k}^+}{2}$ as the argument for the other charge is very similar. We consider the norm of the state $T^{++}_{-1} G^+_{-1/2} |\Omega\rangle$,

$$|T^{++}_{-1} G^+_{-1/2} |\Omega\rangle |^2 = \langle\Omega| G^-_{1/2} T^{+-}_1 T^{++}_{-1} G^+_{-1/2} |\Omega\rangle = 0, \tag{3.2.14}$$

as is easily shown using Appendix A. So in a unitary representation we must have $T^{++}_{-1} G^+_{-1/2} |\Omega\rangle$, as well as all descendants identically equal zero. In particular we therefore have

$$0 = Q^{-K}_{1/2} T^{++}_{-1} G^{+}_{-1/2} |\Omega\rangle = Q^{+}_{-1/2} G^{+}_{-1/2} |\Omega\rangle + \frac{1}{2} T^{++}_{-1} T^{-+}_{0} |\Omega\rangle \,, \tag{3.2.15}$$

but since $|\Omega\rangle$ is a hws, it is annihilated by T^{-+}_0, and so we have $Q^{+}_{-1/2} G^{+}_{-1/2} |\Omega\rangle = 0$ and hence the representation is massless. This may also be shown more simply in $\tilde{A}_\gamma$, since in this algebra we must have $\widetilde{T}^{++}_{-1} |\Omega\rangle = 0$ if $l^+ = \frac{\tilde{k}^+}{2}$ and by a similar argument to the one used above this implies we have $\widetilde{G}^{+}_{-1/2} |\Omega\rangle = 0$. This is the massless condition for $\tilde{A}_\gamma$ and implies that the corresponding representation of A_γ is also massless.

One can also see that when this bound is saturated, the level 1 states $Q^{+}_{-1/2} |\Omega\rangle$ and $G^{+}_{-1/2} |\Omega\rangle$ become linearly dependent; in this case there is therefore one less maximal $\widehat{\mathfrak{su}(2)}^+ \times \widehat{\mathfrak{su}(2)}^-$ multiplet at level 1 as for a massive representation. In fig. 3.2 we see the level 1 states for a massless and massive representation of A_γ; it is clear that in fig. 3.2a there is one less $\widehat{\mathfrak{su}(2)}^+$-quadruplet, $\widehat{\mathfrak{su}(2)}^-$-sextuplet as in fig. 3.2b.

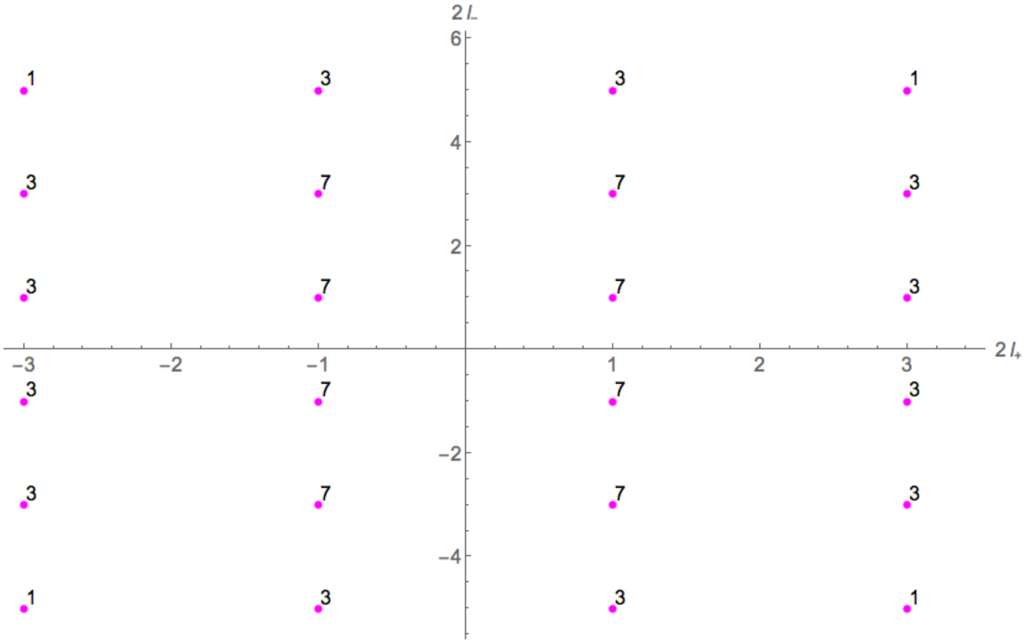
To summarise the results for the Neveu-Schwarz sector, a UHWR of A_γ for fixed γ and c is given in terms of four quantum numbers, $h, l^\pm, u$. The quantum number u is required to be a real number, $u \in \mathbb{R}$, and the other charges are as shown in table 3.1.

Type of Rep.	$\widehat{\mathfrak{su}(2)}^\pm$ Charges	Conformal Weight
A_γ Massless	$0 \le l^\pm_{NS} \le \frac{\tilde{k}^\pm}{2}$	$kh_{NS} = u^2 + (l^+_{NS} - l^-_{NS})^2 + k^- l^+_{NS} + k^+ l^-_{NS}$
A_γ Massive	$0 \le l^\pm_{NS} \le \frac{\tilde{k}^\pm - 1}{2}$	$kh_{NS} > u^2 + (l^+_{NS} - l^-_{NS})^2 + k^- l^+_{NS} + k^+ l^-_{NS}$

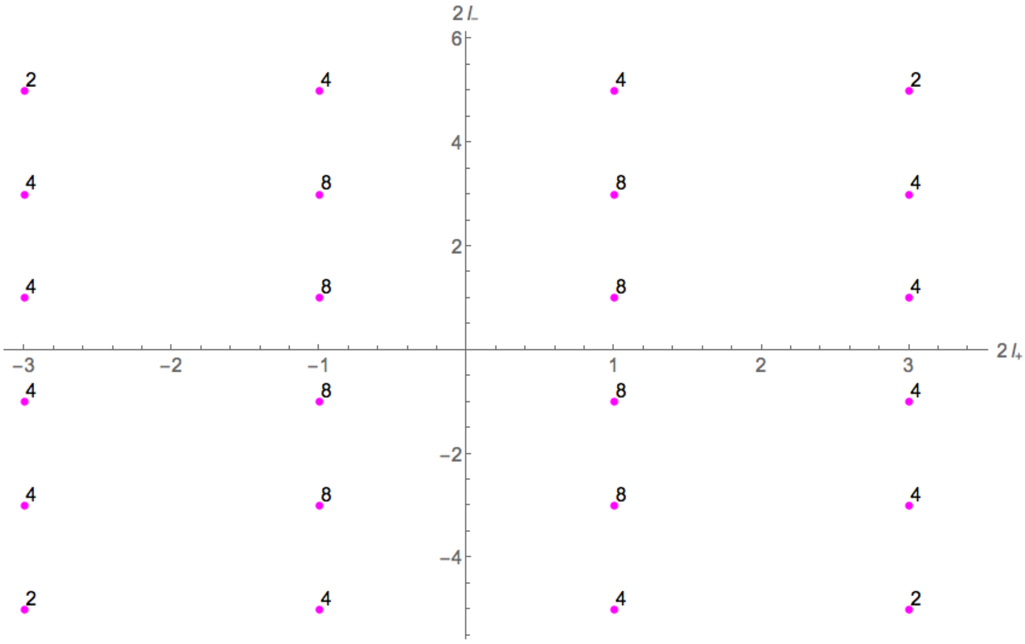
Table 3.1: A summary of the charges of A_γ representations in the NS sector

3.3 Representations of A_γ in the Ramond Sector

Having considered representations of A_γ in the Neveu-Schwarz sector in the previous subsection, we now turn to the Ramond sector. The highest weight state for such a



(a) Massless



(b) Massive

Figure 3.2: The difference between massless and massive representations of A_γ in the NS at level 1; although these figures look similar we see that the multiplicities of states differ between the two. These figures show the level 1 states for the massless and massive representations of A_γ with parameters $k^+ = 5, k^- = 7, l^+ = 1, l^- = 2$ as in the previous figure.

representation is still defined to satisfy eq. (3.2.1) and as we shall see shortly a hws $|\Omega\rangle$ for a representation in the Ramond sector also satisfies

$$Q_0^{+,+K}|\Omega\rangle = 0, \qquad G_0^{+,+K}|\Omega\rangle = 0. \tag{3.3.1}$$

To study the properties of representations in the Ramond sector we will make use of the *spectral flow* automorphism [DST88].

Proposition 3.3.1. *There exists an automorphism of A_γ known as* spectral flow. *In terms of the Laurent modes of the algebra, spectral flow is an automorphism relating different modings; explicitly we have the following automorphism,*

$$\begin{aligned}
L_m^{\rho,\eta} &= L_m - (\rho T_m^{+3} + \eta T_m^{-3}) + \left(\frac{k^+}{4k}\rho^2 + \frac{k^-}{4k}\eta^2\right)\delta_{m,0}, & U_m^{\rho,\eta} &= U_m,\\
G_{m\pm(\rho+\eta)/2}^{\rho,\eta;\pm} &= G_m^{\pm} \pm \left(\rho\frac{k^+}{k} - \eta\frac{k^-}{k}\right) Q_m^{\pm}, & Q_{m\pm(\rho+\eta)/2}^{\rho,\eta;\pm} &= Q_m^{\pm},\\
G_{m\pm(\rho-\eta)/2}^{\rho,\eta;\pm K} &= G_m^{\pm K} \pm \left(\rho\frac{k^+}{k} + \eta\frac{k^-}{k}\right) Q_m^{\pm K}, & Q_{m\pm(\rho-\eta)/2}^{\rho,\eta;\pm K} &= Q_m^{\pm K},\\
T_m^{\rho,\eta;+3} &= T_m^{+3} - \left(\rho\frac{k^+}{2}\right)\delta_{m,0}, & T_{m\pm\rho}^{\rho,\eta;+\pm} &= T_m^{+\pm},\\
T_m^{\rho,\eta;-3} &= T_m^{-3} - \left(\eta\frac{k^-}{2}\right)\delta_{m,0}, & T_{m\pm\eta}^{\rho,\eta;-\pm} &= T_m^{-\pm},
\end{aligned} \tag{3.3.2}$$

If one considers this automorphism in the case of $\rho = -1, \eta = 0$, one obtains an isomorphism between the Ramond and Neveu-Schwarz modings. In fact, this automorphism extends to an isomorphism between representations; given a hws $|\Omega; h, l^+, l^-, u\rangle$ for a representation of A_γ in the NS sector, the state $|\Omega_R\rangle :=$ $\left(T_{-1}^{++}\right)^{k^+-2l^+;NS}|\Omega\rangle$ is a hws for a representation of A_γ in the Ramond sector[1]. It is straightforward to check that the positive modes of the Ramond representation annihilate this state; it is more work to check that the zero modes in eqs. (3.2.1) and (3.3.1) annihilate this state so we shall demonstrate that this is indeed the case.

[1]Alternatively one can flow with $\rho = 0, \eta = -1$, flowing in the $\widehat{\mathfrak{su}(2)}^-$ direction instead of the $\widehat{\mathfrak{su}(2)}^+$ direction obtaining a state known in the literature as $|\Omega_-\rangle$ which could equally be treated as a highest weight state, though now annihilated by $Q_0^{\{+,-K\}}, G_0^{\{+,-K\}}$.

Firstly, showing that the zero mode of T^{-+} annihilates this state is trivial. Under the spectral flow isomorphism with $\rho = -1, \eta - 0$, we have

$$(T_m^{-+})^{NS} \leftrightarrow (T_m^{-+})^R. \tag{3.3.3}$$

We therefore need to consider

$$(T_0^{-+})^R |\Omega_R\rangle \cong (T_0^{-+})^{NS}(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle = (T_{-1}^{++})^{k^+-2l^+} T_0^{-+} |\Omega\rangle = 0, \tag{3.3.4}$$

since $[T_m^{-i}, T_n^{+i}] = 0$ and since $|\Omega\rangle$ is a highest weight state of the NS representation and hence is annihilated by T_0^{-+}.

Before we check the remaining zero modes, we prove a short lemma that will be useful for the remaining calculations.

Lemma 3.3.2. *Let $|\chi\rangle$ be a hws for a representation of A_γ. Then,*

$$\begin{aligned} |(T_{-1}^{++})^p |\chi\rangle|^2 &\equiv \langle\chi| (T_1^{+-})^p (T_{-1}^{++})^p |\chi\rangle \\ &= \begin{cases} p!\frac{(k^+-2l^+)!}{(k^+-2l^+-p)!}, & p \leq k^+ - 2l^+ \\ 0, & p > k^+ - 2l^+. \end{cases} \end{aligned} \tag{3.3.5}$$

Proof. Firstly, we have

$$\langle\chi| (T_1^{+-})^p (T_{-1}^{++})^p |\chi\rangle = p(k^+ - 2l^+ - [p-1]) \langle\chi| (T_1^{+-})^{p-1} (T_{-1}^{++})^{p-1} |\chi\rangle . \tag{3.3.6}$$

Note that if $p = k^+ - 2l^+ + 1$ this gives zero norm. Next we proceed by induction, and hence if $p > k^+ - 2l^+$ there will be a factor of 0 somewhere in this product and hence the state has zero norm. If $p \leq k^+ - 2l^+$ we have

$$\begin{aligned} \langle\chi| (T_1^{+-})^p (T_{-1}^{++})^p |\chi\rangle =& p(k^+ - 2l^+ - [p-1]) \langle\chi| (T_1^{+-})^{p-1} (T_{-1}^{++})^{p-1} |\chi\rangle , \\ =& p!\frac{(k^+ - 2l^+)!}{(k^+ - 2l^+ - p)!}. \end{aligned} \tag{3.3.7}$$

□

Next, we check that $(T_{-1}^{-+})^{k^+-2l^+;NS} |\Omega\rangle$ is annihilated by $(T_0^{++})^R$, we hence need to consider

$$(T_0^{++})^R |\Omega_R\rangle \leftrightarrow (T_{-1}^{++})^{k^+-2l^++1;NS} |\Omega\rangle = 0, \tag{3.3.8}$$

since it has zero norm by applying lemma 3.3.2.

Since under the spectral flow automorphism $G^a \to G^a + Q^a$ (schematically), it is easiest to next check that $(Q_0^{\{+,+K\}})^R |\Omega_R\rangle = 0$; the two calculations are very similar. Under spectral flow,

$$(Q_0^{\{+,+K\}})^R |\Omega_R\rangle \leftrightarrow (Q_{-1/2}^{\{+,+K\}})^{NS}(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle . \tag{3.3.9}$$

To see that this is zero, we consider the norm of these state in the NS sector. Using Appendix A and lemma 3.3.2 we obtain

$$\begin{aligned} |(Q_{-1/2}^p)^{NS}(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle|^2 &= -\langle\Omega| (T_1^{+-})^{k^+-2l^+} Q_{1/2}^m Q_{-1/2}^p (T_{-1}^{++})^{k^+-2l^+} |\Omega\rangle , \\ &= 0, \end{aligned} \tag{3.3.10}$$

where $\{m,p\} \in \{\{-,+\},\{-K,+K\}\}$. Since the states have zero norm, in a unitary representation of A_γ they must be identically zero and hence we must have

$$(Q_0^{\{+,+K\}})^R |\Omega_R\rangle = 0. \tag{3.3.11}$$

Checking that $(G_0^{\{+,+K\}})^R$ annihilates the Ramond ground state is similar to the calculation above for $Q_0^{\{+,+K\}}$. The steps for $(G_0^+)^R$ and $(G_0^{+K})^R$ are identical. We have,

$$(G_0^+)^R |\Omega_R\rangle \leftrightarrow (G_{-1/2}^+ + \frac{k^+}{k} Q_{-1/2}^+)^{NS}(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle , \tag{3.3.12}$$

and since by eq. (3.3.11) we already know that $Q_{1/2}^+$ annihilates $(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle$, we need only consider the norm of $(G_{-1/2}^+)^{NS}(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle$.

$$\begin{aligned} |(G_{-1/2}^{+,+K})^{NS}(T_{-1}^{++})^{k^+-2l^+;NS} |\Omega\rangle|^2 &= \langle\Omega| (T_1^{+-})^{k^+-2l^+} G_{1/2}^- G_{-1/2}^+ (T_{-1}^{++})^{k^+-2l^+} |\Omega\rangle , \\ &= 0, \end{aligned} \tag{3.3.13}$$

using Appendix A and lemma 3.3.2 as before. Since this is a state of zero norm, it must be identically zero in a unitary representation. Hence,

$$(G_0^{\{+,+K\}})^R \left|\Omega_R\right\rangle = 0. \tag{3.3.14}$$

A highest weight state for a Ramond representation therefore satisfies eqs. (3.2.1) and (3.3.1) as previously claimed.

Under the spectral flow isomorphism with $\rho = -1, \eta = 0$, we have

$$\begin{aligned} (L_0)^R &= (L_0)^{NS} - (T_0^{+3})^{NS} + \frac{k^+}{4}, & (U_0)^R &= (U_0)^{NS}, \\ (T_0^{+3})^R &= (T_0^{+3})^{NS} - \frac{k^+}{2}, & (T_0^{-3})^R &= (T_0^{-3})^{NS}, \end{aligned} \tag{3.3.15}$$

and hence the representation of A_γ in the R sector has quantum numbers,

$$\begin{aligned} L_0 \left|\Omega_R\right\rangle &= \left(h - l^+ + \frac{k^+}{4}\right) \left|\Omega_R\right\rangle, & U_0 \left|\Omega_R\right\rangle &= iu \left|\Omega_R\right\rangle, \\ T_0^{+3} \left|\Omega_R\right\rangle &= \left(\frac{k^+}{2} - l^+\right) \left|\Omega_R\right\rangle, & T_0^{-3} \left|\Omega_R\right\rangle &= l^- \left|\Omega_R\right\rangle, \end{aligned} \tag{3.3.16}$$

This now allows us to comment on the allowed values of the quantum numbers for unitary representations of A_γ. Since the allowed values of $l^\pm$ for a hws of an NS representation were $0 \leq l^\pm \leq \frac{\widetilde{k}^\pm}{2}$, then the charges

$$T_0^{+3} \left|\Omega_R\right\rangle = l^+_{\Omega_R} \left|\Omega_R\right\rangle, \qquad T_0^{-3} \left|\Omega_R\right\rangle = l^-_{\Omega_R} \left|\Omega_R\right\rangle, \tag{3.3.17}$$

must satisfy

$$\frac{1}{2} \leq l^+_{\Omega_R} \leq \frac{k^+}{2}, \qquad 0 \leq l^-_{\Omega_R} \leq \frac{\widetilde{k}^-}{2}. \tag{3.3.18}$$

Similarly, if one flows from a massless representation in the NS sector, whose conformal weight satisfies $hk = k^+l^- + k^-l^+ + (l^+ - l^-)^2 + u^2$, then one obtains a representation in the R sector with conformal weight

$$L_0 \left|\Omega_R\right\rangle = h_{\Omega_R} \left|\Omega_R\right\rangle, \tag{3.3.19}$$

satisfying

$$k\left(h_{\Omega_R}+\frac{k^+}{4}-l^+_{\Omega_R}\right)=k^+l^-_{\Omega_R}+k^-\left(\frac{k^+}{2}-l^+_{\Omega_R}\right)+\left(\frac{k^+}{2}-l^+_{\Omega_R}-l^-_{\Omega_R}\right)^2+u^2_{\Omega_R},$$
$$kh_{\Omega_R}=\left(l^+_{\Omega_R}+l^-_{\Omega_R}\right)+u^2_{\Omega_R}+\frac{k^+k^-}{4}. \tag{3.3.20}$$

A representation of A_γ in the R sector whose conformal weight satisfies eq. (3.3.20) is known as a massless representation. If one considers the state $Q_0^{-K}G_0^{-K}|\Omega_R\rangle$ one finds it has norm

$$|Q_0^{+K}G_0^{+K}|\Omega_R\rangle|^2=\frac{1}{4}\left(kh_{\Omega_R}-[l^+_{\Omega_R}+l^-_{\Omega_R}]^2-u^2_{\Omega_R}-\frac{k^+k^-}{4}\right), \tag{3.3.21}$$

and hence in the Ramond sector one has a unitarity bound of

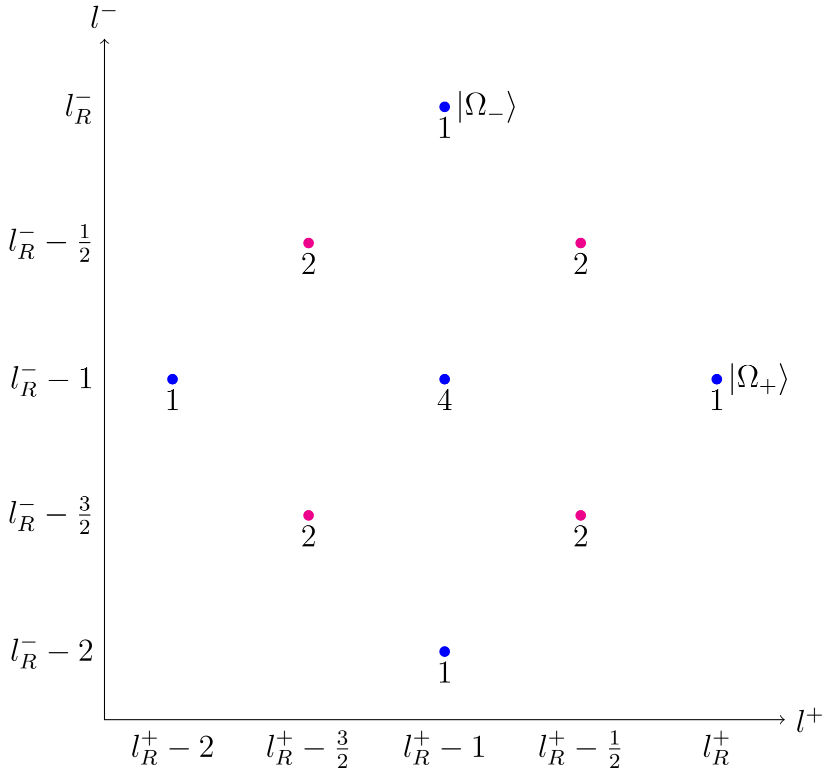
$$kh_{\Omega_R}\geq\left(l^+_{\Omega_R}+l^-_{\Omega_R}\right)+u^2_{\Omega_R}+\frac{k^+k^-}{4}. \tag{3.3.22}$$

The saturation of this bound agrees with the previously identified conformal weight for a massless representation in the Ramond sector and hence for a massless representation of A_γ in the Ramond sector one has $Q_0^{-K}G_0^{-K}|\Omega_R\rangle=0$ and the states $Q_0^{-K}|\Omega_R\rangle$ and $G_0^{-K}|\Omega_R\rangle$ become linearly dependent. In this case, the generic sixteen $\mathfrak{su}(2)\times\mathfrak{su}(2)$ hws which exist in the massive representation [PT90a] and which are shown in fig. 3.3a are reduced to eight $\mathfrak{su}(2)\times\mathfrak{su}(2)$ hws shown in fig. 3.3b.

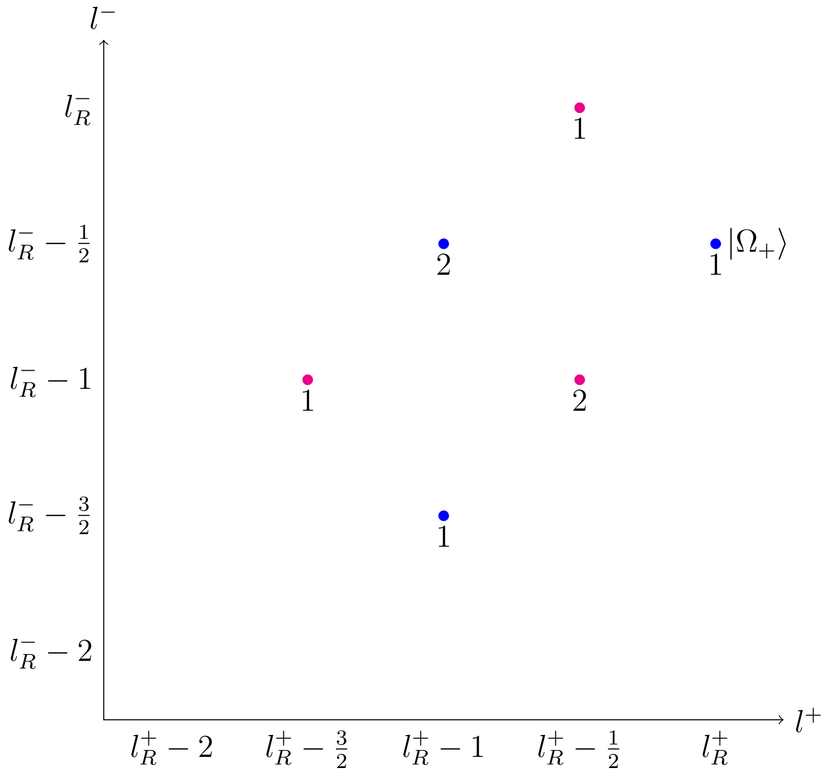
In a Ramond representation of A_γ, the state $|\Omega_R\rangle$ satisfies the definition of a highest weight state, that is it is annihilated by both T_0^{++} and T_0^{-+}. However, unlike in the NS sector, it is not the ground state with the highest $\widehat{\mathfrak{su}(2)}^-$ charge, merely the top state of the $\widehat{\mathfrak{su}(2)}^-$ multiplet from the largest $\widehat{\mathfrak{su}(2)}^+$ multiplet. In a massless R representation, the state $|\Omega_-\rangle:=G_0^{-K}|\Omega_R\rangle$ has charges

$$L_0|\Omega_-\rangle=h_{\Omega_R}|\Omega_-\rangle,\quad T_0^{+3}|\Omega_-\rangle=\left(l^+_{\Omega_R}-\frac{1}{2}\right)|\Omega_-\rangle,\quad T_0^{-3}|\Omega_-\rangle=\left(l^-_{\Omega_R}+\frac{1}{2}\right)|\Omega_-\rangle, \tag{3.3.23}$$

and is the unique ground state with highest $\widehat{\mathfrak{su}(2)}^-$ charge. In a massive R repres-



(a) The sixteen $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Ramond hws in a massive A_γ representation



(b) The eight $\mathfrak{su}(2) \times \mathfrak{su}(2)$ Ramond hws in a massless A_γ representation

Figure 3.3: The $\mathfrak{su}(2) \times \mathfrak{su}(2)$ hws of a Ramond representation of A_γ for massive compared to massless representations.

entation, the state $|\Omega_-\rangle := Q_0^{-K}G_0^{-K}|\Omega_R\rangle$ has charges

$$L_0|\Omega_-\rangle = h_{\Omega_R}|\Omega_-\rangle\,,\quad T_0^{+3}|\Omega_-\rangle = \left(l^+_{\Omega_R}-1\right)|\Omega_-\rangle\,,\quad T_0^{-3}|\Omega_-\rangle = \left(l^-_{\Omega_R}+1\right)|\Omega_-\rangle\,, \tag{3.3.24}$$

and is the unique ground state with highest $\widehat{\mathfrak{su}(2)}^-$ charge. In both cases, the state $|\Omega_R\rangle$ is known in the literature as $|\Omega_+\rangle$ and the ground state with highest $\widehat{\mathfrak{su}(2)}^-$ charge is known as $|\Omega_-\rangle$. As noted above, $|\Omega_-\rangle$ is the state we would have obtained as our hws if we had flowed in the $\widehat{\mathfrak{su}(2)}^-$ direction rather than the $\widehat{\mathfrak{su}(2)}^+$ direction. If the $\widehat{\mathfrak{su}(2)}^\pm$ charges are labelled as

$$\begin{aligned} T_0^{+3}|\Omega_+\rangle &= l^+_+|\Omega_+\rangle\,, \qquad & T_0^{-3}|\Omega_+\rangle &= l^-_+|\Omega_+\rangle\,,\\ T_0^{+3}|\Omega_-\rangle &= l^+_-|\Omega_-\rangle\,, \qquad & T_0^{-3}|\Omega_-\rangle &= l^-_-|\Omega_-\rangle\,, \end{aligned} \tag{3.3.25}$$

then the representation of A_γ is labelled by the charges

$$l^+_R := l^+_+, \qquad l^-_R := l^-_-. \tag{3.3.26}$$

In terms of the representation labels $l^\pm_R$, the allowed ranges of the $\widehat{\mathfrak{su}(2)}^\pm$ charges now take the more symmetric form

$$\frac{1}{2} \le l^\pm \le \frac{k^\pm}{2}, \tag{3.3.27}$$

and if either charge is equal to $\frac{1}{2}$ we automatically have a massless representation. The massless bound for Ramond representations should also be given in terms of the representation label $l^\pm_R$ as

$$hk = \left(l^+_R + l^-_R - \frac{1}{2}\right)^2 + u^2 + \frac{k^+k^-}{4}, \tag{3.3.28}$$

though we should perhaps note that the bound for a massive representation is

$$hk > \left(l^+_R + l^-_R - 1\right)^2 + u^2 + \frac{k^+k^-}{4}, \tag{3.3.29}$$

due to the existence of the state $Q_0^{-K}G_0^{-K}|\Omega_R\rangle$ in a massive representation.

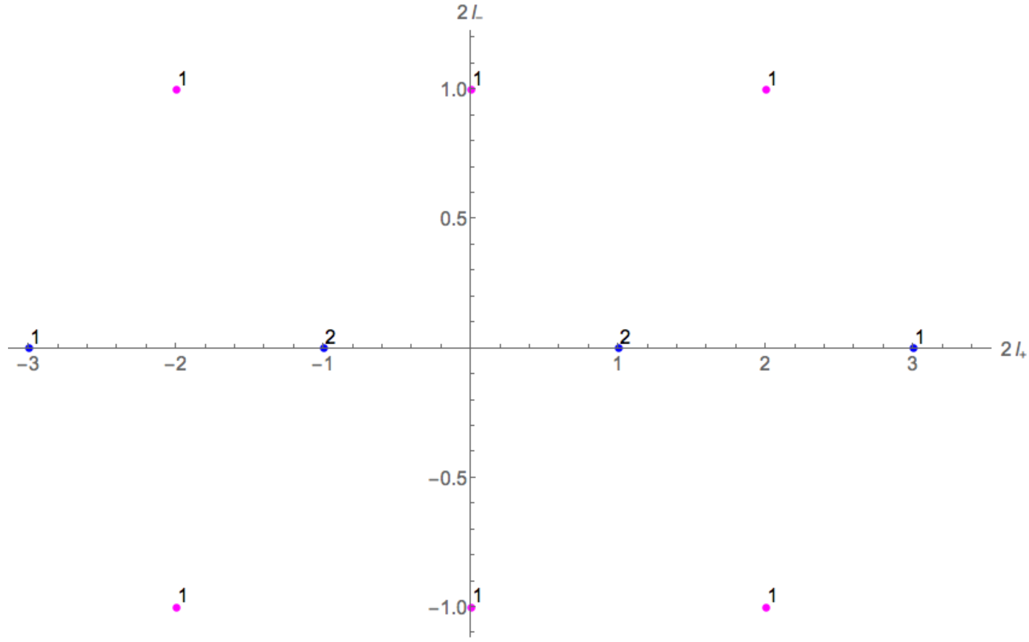
In fig. 3.4, we show the difference between the ground levels of massless and massive representations of A_γ in the Ramond sector with parameters $k^+ = 4, k^- = 3, l^+ =$

$\frac{3}{2}, l^- = 1$. The massless representation is easily distinguished due to the non-existence of a state with $(\widehat{\mathfrak{su}(2)}^+, \widehat{\mathfrak{su}(2)}^-)$ charges $(l^+_{\Omega_R} - 1, l^-_{\Omega_R} + 1)$ compared to the ground state with charges $(l^+_{\Omega_R}, l^-_{\Omega_R})$.

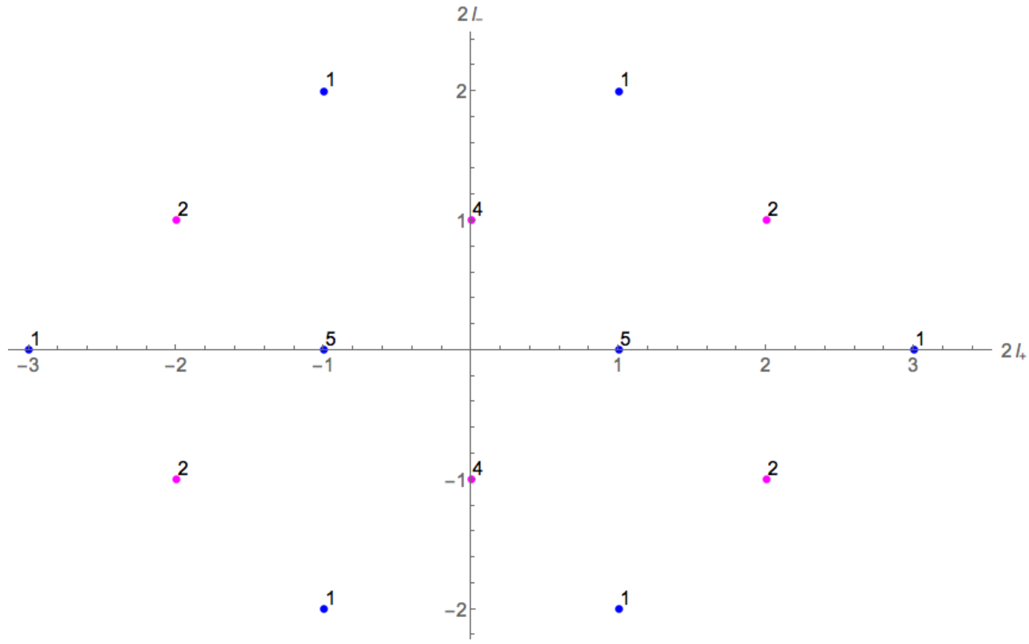
To summarise the results for the Ramond sector, a UHWR of A_γ for fixed γ and c is given in terms of four quantum numbers, $h, l^\pm, u$. The quantum number u is required to be a real number, $u \in \mathbb{R}$, and the other charges are as shown in table 3.2.

Type of Rep.	$\widehat{\mathfrak{su}(2)}^\pm$ Charges	Conformal Weight
A_γ Massless	$\frac{1}{2} \leq l^\pm_R \leq \frac{k^\pm}{2}$	$kh_R = u^2 + (l^+_R + l^-_R - \frac{1}{2})^2 + \frac{k^+k^-}{4}$
A_γ Massive	$1 \leq l^\pm_{NS} \leq \frac{k^\pm-1}{2}$	$kh_R > u^2 + (l^+_R + l^-_R - 1)^2 + \frac{k^+k^-}{4}$

Table 3.2: A summary of the charges of A_γ representations in the R sector



(a) Massless



(b) Massive

Figure 3.4: The difference between massless and massive Ramond representations of A_γ at level 0. Both representations have levels $k^+ = 4$ and $k^- = 3$. The massless representation is taken to have charges $l^+ = \frac{3}{2}$, $l^- = \frac{1}{2}$. The massive representation is taken to have charges $l^+ = \frac{3}{2}$, $l^- = 1$. Note that this means the highest weight state $|\Omega_+\rangle$ has the same charges in both representations.

3.4 Characters for A_γ

In section 2.1.5 we defined the character of a module $V(c,h)$ of the Virasoro algebra as

$$\chi_{c,h}(\tau) := \mathrm{Tr}_{V(c,h)}\, q^{L_0 - c/24}.$$

We can similarly define characters for A_γ-modules such as discussed in the preceding subsection. As discussed in sections 3.2 and 3.3, at fixed values of $k^\pm$ (or equivalently, at fixed values of c and γ), representations of A_γ are classified by quantum numbers $\{h, l^+, l^-\}$. We therefore define the characters of a module $V(c,h,l^+,l^-)$ of A_γ as [PT90a],

$$\chi^{A_\gamma,\{NS,R\}}(k^+,k^-,h,l^+,l^-;q,z_+,z_-) := \mathrm{Tr}_{V(c,h,l^+,l^-)}(q^{L_0-c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}), \quad (3.4.1)$$

where as usual we have $q = e^{2\pi i\tau}$ and now we have two further variables corresponding to the two $\widehat{\mathfrak{su}(2)}^\pm$ charges defined as $z_\pm = e^{2\pi i \omega_\pm}$ for $\omega_\pm \in \mathbb{C}$. Note that we have suppressed a possible dependence on the U_0 charge (this corresponds to setting the variable $\chi = 1$ in [PT90a] equation 2.17). In the following we will primarily be concerned with characters of Ramond modules of A_γ. Once we know the Ramond characters, we will be able to obtain the Neveu-Schwarz characters simply by utilising the spectral flow isomorphism discussed in section 3.3. Explicitly, since the modules are isomorphic, we have (for a massive representation)

$$\begin{aligned}
\chi^{A_\gamma,NS}(h,l^+,l^-,u;q,z_+,z_-) &= \mathrm{Tr}_V(q^{L_0^{NS}-c/24} z_+^{2T_0^{+3,NS}} z_-^{2T_0^{-3,NS}}), \\
&= \mathrm{Tr}_V(q^{L_0^R + T_0^{+3,R} + k^+/4 - c/24} z_+^{2T_0^{+3,R}+k^+/2} z_-^{2T_0^{-3,R}}), \\
&= q^{k^+/4} z_+^{k^+/2}\, \mathrm{Tr}_V(q^{L_0^R - c/24}[q^{1/2}z_+]^{2T_0^{+3,R}} z_-^{2T_0^{-3,R}}), \\
&= q^{k^+/4} z_+^{k^+/2} \chi^{A_\gamma,R}(h - l^+ + \frac{k^+}{4}, \frac{k^+}{2} - l^+, l^- + 1; q, q^{1/2}z_+, z_-),
\end{aligned} \quad (3.4.2)$$

where we have suppressed the levels of the subalgebras in the character as these are unaffected by the spectral flow isomorphism, and we have called the A_γ module V. Note that the character is labelled by the greatest $\widehat{\mathfrak{su}(2)}^\pm$ charges from any ground state, and hence although the spectral flow does not alter the $\widehat{\mathfrak{su}(2)}^-$ charge of our highest weight state, it does affect the representations labels as in eq. (3.3.26). This isomorphism also exists at the level of $\tilde{A}_\gamma$ [PT90b] where the characters are related by,

$$\begin{aligned}
\chi^{\tilde{A}_\gamma,NS}_{\text{Massive}}(h,\tilde{l}^+,\tilde{l}^-;q,z_+,z_-) &= \\
&q^{\tilde{k}^-/4} z_-^{\tilde{k}^-}\chi^{\tilde{A}_\gamma,R}_{\text{Massive}}\left(h-\tilde{l}^-+\frac{\tilde{k}^-}{4},\tilde{l}^++\frac{1}{2},\frac{\tilde{k}^-}{2}-\tilde{l}^-;q,z_+,q^{\frac{1}{2}}z_-\right),\\
\chi^{\tilde{A}_\gamma,NS}_{\text{Massless}}(\tilde{l}^+,\tilde{l}^-;q,z_+,z_-) &= q^{\tilde{k}^-/4} z_-^{\tilde{k}^-}\chi^{\tilde{A}_\gamma,R}_{\text{Massless}}\left(\tilde{l}^+,\frac{\tilde{k}^-}{2}-\tilde{l}^-;q,z_+,q^{\frac{1}{2}}z_-\right).
\end{aligned} \tag{3.4.3}$$

Note that these previous equations correspond to flowing in z_- rather than z_+ and equivalent expressions exist for the case one flows in z_+.

As in the case of the Virasoro algebra, the character of a Verma module of A_γ can be calculated very simply. We saw in section 2.1.5 that each of the bosonic raising modes L_n contributed a factor of $(1-q^n)^{-1}$ and hence taking all the raising modes into account gave a contribution of $\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^{-1}$. In A_γ we have four such sets of raising modes which simply increase the conformal weight, specifically L_n, U_n, T_n^{+3} and T_n^{-3}, for $n<0$. Each of these therefore contributes a factor of $\prod_{n\in\mathbb{Z}_{>0}}(1-q^n)^{-1}$ to the character of A_γ. We also have the bosonic $\widehat{\mathfrak{su}(2)}^\pm$ raising and lowering operators $T_n^{\pm,+}$ and $T_n^{\pm,-}$. Clearly we may not act with the zero modes of raising operators on the highest weight state and hence for the raising operators $T_n^{\pm,+}$ we require $n>0$. Each time we act with a raising operator, we increase the $\widehat{\mathfrak{su}(2)}^\pm$ charge by 1, and so the raising modes give a contribution to the character of $\prod_{n\in\mathbb{Z}_{>0}}(1-z_+^2q^n)^{-1}(1-z_-^2q^n)^{-1}$. For the lowering operators we are also allowed to act with the zero modes. Similarly to the raising modes, each action of a lowering operators lowers the $\widehat{\mathfrak{su}(2)}^\pm$ charge by 1 and so the lowering operators give a contribution of $\prod_{n\in\mathbb{Z}_{\geq 0}}(1-z_+^{-2}q^n)^{-1}(1-z_-^{-2}q^n)^{-1}$. Let us therefore define

$$B^{+-}(q,z_+,z_-):=\prod_{n=1}^{\infty}(1-z_+^2q^n)^{-1}(1-z_+^{-2}q^{n-1})^{-1}(1-z_-^2q^n)^{-1}(1-z_-^{-2}q^{n-1})^{-1}(1-q^n)^{-2}, \tag{3.4.4}$$

as the contribution to the character of the $\widehat{\mathfrak{su}(2)}^\pm$ operators.

For the fermionic operators we may not act with any particular mode more than once due to the anti- commutation relations these operators satisfy. If we consider one particular fermionic operator, say Q^{+K}, then the modes Q^{+K}_n for $n < 0$ raise the conformal weight by n and change the $\widehat{\mathfrak{su}(2)}^+$ and $\widehat{\mathfrak{su}(2)}^-$ charges by $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. The contribution of these modes to the character is therefore $\prod_{n=1}^{\infty}(1 + z_+ z_-^{-1} q^n)$. The contribution from all 8 fermionic operators (excluding the zero modes momentarily) is therefore $[F^R(q, z_+, z_-)]^2$, where

$$F^R(q, z_+, z_-) := \prod_{n=1}^{\infty}(1+z_+z_-q^n)(1+z_+z_-^{-1}q^n)(1+z_+^{-1}z_-q^n)(1+z_+^{-1}z_-^{-1}q^n), \tag{3.4.5}$$

since the contribution from the Q^a's is identical to the contribution from the G^a's. In the NS sector the result is similarly given by

$$\begin{aligned} F^{NS}(q, z_+, z_-) &:= \prod_{n=1/2}^{\infty}(1+z_+z_-q^n)(1+z_+z_-^{-1}q^n)(1+z_+^{-1}z_-q^n)(1+z_+^{-1}z_-^{-1}q^n), \\ &= q^{-1/12}\eta^{-2}(q)\theta_3(q, z_+z_-)\theta_3(q, z_+z_-^{-1}) \end{aligned} \tag{3.4.6}$$

where now n runs over the positive half-integers. For the second line, we use the definitions of the Jacobi theta functions as in Appendix B. Finally, in the R sector we may act on our highest weight state with fermionic zero modes. By eq. (3.3.1), we may not act with $Q_0^{\{+,+K\}}$ or $G_0^{\{+,+K\}}$, so we only have the contributions from $Q_0^{\{-,-K\}}$ and $G_0^{\{-,-K\}}$, namely $(1 + z_+^{-1}z_-)^2(1 + z_+^{-1}z_-^{-1})^2$.

Combining all these contributions, we finally get the character for the full reducible A_γ module in the Ramond sector as

$$\begin{aligned} \chi^{A_\gamma,R}_{\text{reducible}}(k^+, k^-, h, l^+, l^-; q, z_+, z_-) =& B^{+-}(q, z_+, z_-)[F^R(q, z_+, z_-)]^2(1 + z_+^{-1}z_-)^2 \\ &\times (1 + z_+^{-1}z_-^{-1})^2 \prod_{n=1}^{\infty}(1 - q^n)^{-2}. \end{aligned} \tag{3.4.7}$$

To find the characters for the irreducible modules of A_γ, we must now subtract modules built on singular vectors, the null modules mentioned at the end of section 2.1.5. As described in section 2.1.5, in order to obtain an irreducible A_γ-module, we should take the quotient of the A_γ Verma module by its maximal proper submodule. To complete the calculation for characters of the irreducible modules we therefore need to identify all the singular vectors of A_γ. It turns out to be simpler to consider the singular vectors of $\tilde{A}_\gamma$ and hence to give the irreducible characters of $\tilde{A}_\gamma$ from which the characters of A_γ can be found using

$$\mathrm{Ch}^{A_\gamma,I}(h,l_\pm^I) = \mathrm{Ch}^{A_{QU},I} \times \mathrm{Ch}^{\tilde{A}_\gamma,I}(h,\tilde{l}_\pm^I), \tag{3.4.8}$$

where $I \in \{NS, R\}$ and A_{QU} is the algebra of the four fermions and the $\widehat{\mathfrak{u}(1)}$ generator that were removed from A_γ to obtain $\tilde{A}_\gamma$ as explained in Section 3.1. We have [PT90a] $\tilde{l}_\pm^{NS} = l_\pm^{NS}$ and $\tilde{l}_\pm^{R} = l_\pm^{R} - \frac{1}{2}$ due to the fermionic zero modes in ${A_{QU}}^R$. The quantum numbers in eq. (3.4.8) are therefore equal for the NS sector, but differ by $\frac{1}{2}$ for the R sector.

Using similar reasoning to above, the character for A_{QU} is easily seen to be

$$\mathrm{Ch}^{A_{QU},NS}(u;q,z_+,z_-) = q^{u^2/k-1/8}F^{NS}(q,z_+,z_-)\prod_{n=1}^{\infty}(1-q^n)^{-1}, \tag{3.4.9}$$

since the central charge of the four fermions and one boson gives $c = 3$ and by eq. (3.2.13) the bosonic generator contributes $\frac{u^2}{k}$ to the conformal weight. In the Ramond sector we must also take into account the fermionic zero modes. The hws of A_{QU} has quantum numbers $l^+ = \frac{1}{2}$, $l^- = 0$ (see [PT90a; GPTV89; GS88] for details) and hence the contribution from the fermionic zero modes is $q^{1/4}z_+(1+z_+^{-1}z_-^{-1})(1+z_+^{-1}z_-)$. We therefore have,

$$\begin{aligned}\mathrm{Ch}^{A_{QU},R}(u;q,z_+,z_-) &= q^{u^2/k+1/8}F^{R}(q,z_+,z_-)\prod_{n=1}^{\infty}(1-q^n)^{-1}(1+z_+^{-1}z_-^{-1})(z_++z_-)\\ &= \left(q^{u^2/k}\,\frac{1}{\eta(q)}\right)\times\left(\frac{\theta_2(q,z_+z_-)\theta_2(q,z_+^{-1}z_-)}{\eta^2(q)}\right)\\ &=: \mathrm{Ch}^{A_U,R}(u;q)\times\mathrm{Ch}^{A_Q,R}(q,z_+,z_-).\end{aligned} \tag{3.4.10}$$

As stated above, we now consider singular vectors of $\tilde{A}_\gamma$. In lemma 3.3.2 we showed that the $\widehat{\mathfrak{su}(2)}^+$ operator $(T_{-1}^{++})^{k^+-2l^++1}$ annihilated the hws of the representation. Since this argument only relied on the $\widehat{\mathfrak{su}(2)}^+$ subalgebra of A_γ, which is identical in $\tilde{A}_\gamma$, it also applies to a hws of $\tilde{A}_\gamma$, with level $\tilde{k}^+$. Furthermore, the argument presented relied only on the condition that T_1^{++} annihilated the hws. It therefore holds for any singular state if one replaces the charge l^+ with the $\widehat{\mathfrak{su}(2)}^+$ charge of the singular state, since by definition singular states are also annihilated by the positive modes of operators as in section 2.1.5. Since we are considering $\tilde{A}_\gamma$ here, a state is singular (of the $+$ type) if

$$\widetilde{L}_n \left|\rho\right\rangle = \widetilde{T}_n^{\pm i} \left|\rho\right\rangle = \widetilde{G}_n^a \left|\rho\right\rangle = \widetilde{T}_0^{\pm +} \left|\rho\right\rangle = \widetilde{G}_0^{\{+,+K\}} \left|\rho\right\rangle = 0, \qquad n \in \mathbb{Z}_+,\ a \in \{\pm, \pm K\} \tag{3.4.11}$$

where the tilde operators are the operators of $\tilde{A}_\gamma$ formed from those of A_γ by removing the $\{Q^a, U\}$ system as in [GPTV89; GS88]. We therefore find that for singular $|\rho\rangle$, the state $(\widetilde{T}_{-1}^{++})^{\tilde{k}^+-2L^++1} \left|\rho\right\rangle$ is singular, where $\widetilde{T}_0^{+3} \left|\rho\right\rangle = L^+ \left|\rho\right\rangle$. Similarly the state $(\widetilde{T}_0^{--})^{2L^++1} \left|\rho\right\rangle$ is singular as one would expect for $\widehat{\mathfrak{su}(2)}^-$ and satisfies the conditions eq. (3.4.11).

As discussed in section 3.3 in the case of A_γ, given a R hws $|\Omega_+\rangle$ there is a second state which can be considered to be a R hws known as $|\Omega_-\rangle$ and which is annihilated by $G_0^{\{+,-K\}}$. Given a singular state $|\rho\rangle \equiv |\rho_+\rangle$, there also therefore exists a singular state $|\rho_-\rangle$ (of the $-$ type) which satisfies all but the final condition of eq. (3.4.11) and instead satisfies

$$\widetilde{G}_0^{\{+,-K\}} \left|\rho_-\right\rangle = 0. \tag{3.4.12}$$

From this singular state one can also construct the further singular states $(\widetilde{T}_0^{+-})^{2L^-+1} \left|\rho_-\right\rangle$ and $(\widetilde{T}_{-1}^{-+})^{\tilde{k}^--2L^-+1} \left|\rho_-\right\rangle$, where $\widetilde{T}_0^{-3} \left|\rho_-\right\rangle = L^- \left|\rho_-\right\rangle$.

As discussed in detail in [PT90b], these form chains of singular states, the modules built on which must be successively removed and added into the reducible character to obtain the irreducible character of $\tilde{A}_\gamma$. Further care must be taken as to which fermionic zero modes can be used to construct the module built on each singular state. To construct the irreducible character of the module built on a hws $|\widetilde{\Omega}\rangle$, one therefore starts with the character for the Verma module buit on $|\widetilde{\Omega}\rangle$ (given by eq. (3.4.7) divided by eq. (3.4.9) or eq. (3.4.10) depending on the sector). From this, one then subtracts characters with quantum numbers appropriate for the singular state being considered and with the appropriate fermionic zero mode factors removed, then adds characters for doubly removed states and so on (the relevant embedding diagrams are simple chains). Finally, one obtains the irreducible character formulae for $\tilde{A}_\gamma$ [PT90b],

$$\begin{aligned}\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\mathrm{Massive}}(\widetilde{k}^{+},\widetilde{k}^{+},\widetilde{l}^{+},\widetilde{l}^{-},h;q,z_+,z_-) = {} & q^{h-c/24+1/8}F^{NS}(q,z_+,z_-)B^{+-}(q,z_+,z_-) \\ & \times \prod_{n=1}^{\infty}(1-q^n)^{-1} \\ & \times \sum_{m,n=-\infty}^{\infty} q^{n^2\widetilde{k}^{+}+m^2\widetilde{k}^{-}+2n\widetilde{l}^{+}+2m\widetilde{l}^{-}}q^{m+n}z_+^{-1}z_-^{-1} \\ & \times \sum_{\epsilon_+,\epsilon_-\in\{\pm1\}} \epsilon_+\epsilon_- z_+^{\epsilon_+}z_-^{\epsilon_-}z_+^{2\epsilon_+(\widetilde{l}^{+}+n\widetilde{k}^{+})}z_-^{2\epsilon_-(\widetilde{l}^{-}+m\widetilde{k}^{-})},\end{aligned} \tag{3.4.13}$$

$$\begin{aligned}\mathrm{Ch}^{\tilde{A}_\gamma,R}_{\mathrm{Massive}}(\widetilde{k}^{+},\widetilde{k}^{+},\widetilde{l}^{+},\widetilde{l}^{-},h;q,z_+,z_-) = {} & q^{h-c/24+1/8}F^{R}(q,z_+,z_-)B^{+-}(q,z_+,z_-) \\ & \times \prod_{n=1}^{\infty}(1-q^n)^{-1}(z_+^{-1}+z_-^{-1})(1+z_+^{-1}z_-^{-1}) \\ & \times \sum_{m,n=-\infty}^{\infty} q^{n^2\widetilde{k}^{+}+m^2\widetilde{k}^{-}+2n\widetilde{l}^{+}+2m\widetilde{l}^{-}} \\ & \times \sum_{\epsilon_+,\epsilon_-\in\{\pm1\}} \epsilon_+\epsilon_- z_+^{2\epsilon_+(\widetilde{l}^{+}+n\widetilde{k}^{+})}z_-^{2\epsilon_-(\widetilde{l}^{-}+m\widetilde{k}^{-})},\end{aligned} \tag{3.4.14}$$

$$\begin{aligned}\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\mathrm{Massless}}(\widetilde{k}^{+},\widetilde{k}^{+},\widetilde{l}^{+},\widetilde{l}^{-};q,z_+,z_-) = {} & q^{h-c/24+1/8}F^{NS}(q,z_+,z_-)B^{+-}(q,z_+,z_-) \\ & \times \prod_{n=1}^{\infty}(1-q^n)^{-1} \\ & \times \sum_{m,n=-\infty}^{\infty} q^{n^2\widetilde{k}^{+}+m^2\widetilde{k}^{-}+2n\widetilde{l}^{+}+2m\widetilde{l}^{-}}q^{m+n}z_+^{-1}z_-^{-1} \\ & \times \sum_{\epsilon_+,\epsilon_-\in\{\pm1\}} \epsilon_+\epsilon_- z_+^{\epsilon_+}z_-^{\epsilon_-}z_+^{2\epsilon_+(\widetilde{l}^{+}+n\widetilde{k}^{+})}z_-^{2\epsilon_-(\widetilde{l}^{-}+m\widetilde{k}^{-})}(1+q^{n+m+1/2}z_+^{\epsilon_+}z_-^{\epsilon_-})^{-1},\end{aligned} \tag{3.4.15}$$

$$\begin{aligned}\mathrm{Ch}^{\tilde{A}_\gamma,R}_{\mathrm{Massless}}(\widetilde{k}^{+},\widetilde{k}^{+},\widetilde{l}^{+},\widetilde{l}^{-};q,z_+,z_-) = {} & q^{h-c/24+1/8}F^{R}(q,z_+,z_-)B^{+-}(q,z_+,z_-) \\ & \times \prod_{n=1}^{\infty}(1-q^n)^{-1}(z_+^{-1}+z_-^{-1})(1+z_+^{-1}z_-^{-1}) \\ & \times \sum_{m,n=-\infty}^{\infty} q^{n^2\widetilde{k}^{+}+m^2\widetilde{k}^{-}+2n\widetilde{l}^{+}+2m\widetilde{l}^{-}} \\ & \times \sum_{\epsilon_+,\epsilon_-\in\{\pm1\}} \epsilon_+\epsilon_- z_+^{2\epsilon_+(\widetilde{l}^{+}+n\widetilde{k}^{+})}z_-^{2\epsilon_-(\widetilde{l}^{-}+m\widetilde{k}^{-})}(z_+^{-\epsilon_+}q^{-n}+z_-^{-\epsilon_-}q^{-m})^{-1},\end{aligned} \tag{3.4.16}$$

where B^{+-} and F^I are as defined in eqs. (3.4.4) to (3.4.6). The characters for A_γ are then finally given by multiplying the above expressions by eq. (3.4.9) or eq. (3.4.10) depending on the relevant sector.

Finally, we note that the characters of $\tilde{A}_\gamma$ satisfy

$$\begin{aligned}\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}(\widetilde{k}^+,\widetilde{k}^+,\widetilde{l}^+,\widetilde{l}^-;q,z_+,z_-)+\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}(\widetilde{k}^+,\widetilde{k}^+,\widetilde{l}^++\frac{1}{2},\widetilde{l}^-+\frac{1}{2};q,z_+,z_-)\\ =\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massive}}(\widetilde{k}^+,\widetilde{k}^+,\widetilde{l}^+,\widetilde{l}^-,h_{\text{Massless}};q,z_+,z_-),\end{aligned} \tag{3.4.17}$$

where h_{Massless} denotes the value of h which saturates the massless bound for $\tilde{A}_\gamma$. A massive character formally evaluated at this value of h is said to be at *threshold*.

Chapter 4

The Witten Index and The Elliptic Genus

Index theory, rooted in topology, offers insights on several areas of mathematics and theoretical particle physics. The classical index theory we refer to is based on the celebrated Atiyah-Singer theorem as formulated for a smooth compact manifold M of dimension $d = 2n$. In this context, the representation theory of spin groups and of classical Lie groups plays a prominent role [Ati14]. Indices are topological invariants that can be used as tools to help decide whether two mathematical objects might be equivalent. For instance one can consider the Euler characteristic of a manifold; since this is a topological invariant, two manifolds with different Euler characteristics cannot be homeomorphic. In general however one cannot conclude that two manifolds with the same Euler characteristic are homeomorphic, as for example both the 2-sphere and Mobiüs strip have Euler characteristic 0, but the Mobiüs strip is non-orientable and so cannot be homeomorphic to the sphere. Classical index theory has been beautifully related to field theory, which describes the physics of point particles, through the work of several particle physicists building on a seminal paper by Witten, where constraints on supersymmetry breaking are explored [Wit82]. In field theories, the spectrum of massless particles is governed by index theorems – the index of the Dirac operator $\not{D}$ for example, is the difference between the number of massless fermions of positive and negative chiralities – which encode information on the topology of the manifold on which the particles evolve.

In 1984, the first 'string revolution' was triggered by the discovery that the cancellation of anomalies in string theory constrained the possible gauge groups [GS84] and physicists began to take string theory seriously as a candidate for a theory of quantum gravity. Since the Atiyah-Singer index theorem is instrumental in the discussion of anomalies in field theory, this classical theory was revisited with a view to applying it to string theory and studying conditions for anomaly-free string theories. This was the line of attack taken by Alvarez, Killingback, Mangano and Windey in 1985 in an unpublished work entitled *The index of the Ramond operator* and later revisited by them in [AKMW87b; AKMW87a]. By then, Witten had published his paper on *Elliptic genera and quantum field theory* [Wit87] and Schellekens, Warner and Pilch had published their works on anomalies [PSW87; SW86a; SW86b; SW87]. Not surprisingly, the natural framework turned out to involve loop spaces of manifolds instead of the manifolds themselves, and the representation theory of loop groups, generalising the framework of the classical index theory and leading to the theory of elliptic genera. One of the difficulties encountered when operators act on infinite dimensional spaces is that their kernel might be infinite dimensional, and the calculation of their indices is considerably easier if the kernel can be partitioned in an infinite set of finite-dimensional subspaces, which transform according to different representations of some group. This partitioning may be realised through a character-valued index, as is discussed by Witten for the equivariant Dirac-Ramond operator in [Wit85; Wit87]. There is a rich literature on this subject, which highlights the intense activity amongst mathematicians and theoretical particle physicists over almost four decades. In the following we will focus on these indices as topological invariants and their application to questions of anomalies will not be studied further here. Although the theory of elliptic genera is by now pretty much established from a mathematics point of view, we will see that the formulation of some elliptic genera in terms of mock modular forms in the context of closed superstring theory uncovers a new type of moonshine that has not yet received a satisfactory explanation within string theory.

As a build-up to the material presented in the following chapter, it feels appropriate to provide a brief review of Witten's field-theoretic approach to what has become known as the supersymmetric Witten index - and which belongs to classical index theory - and of subsequent works by several authors (including Witten) on the elliptic genus, as they illustrate the power of topological invariants both in mathematics and physics. The material in the present chapter is not new. The Witten index is introduced and discussed in [Wit82] and the relation to the Atiyah-Singer index theorem is presented in [Alv83]. The elliptic genus appears in [Wit87] and is discussed in relation to index theorems in [Wit88; AKMW87b; AKMW87a]. Since the motivation driving our thesis has been the hope to establish whether a theory with A_γ symmetry can exhibit some kind of moonshine phenomenon, we felt it was necessary to understand the elliptic genus as it applies to small $\mathcal{N} = 4$ theories before discussing indices for A_γ. We therefore review the above mentioned papers, adding material from the mathematical literature [HBJL92; Och09; Gri00] in an attempt to present a wider picture.

The first section highlights some well-known aspects of classical index theory. We consider indices which can be defined for supersymmetric field theories. In particular, we discuss the Witten index [Wit82], which for a one-dimensional σ-model turns out to be related to the Euler characteristic of the target space. We then turn to the elliptic genus [Wit87] which, for a two-dimensional σ-model, encodes the Euler characteristic of the target space as well as other classical topological invariants such as the Hirzebruch signature of the target manifold. We show how the elliptic genus may be thought of as the index of an operator on the loop space of a manifold. Special attention is given to the elliptic genus calculated from data encoded in superstring theories compactified on a K3 manifold, whose worldsheet supersymmetry is governed by the small $\mathcal{N} = (4, 4)$ superconformal algebra at central charges $(c, \bar{c}) = (6, 6)$. This allows us to summarise the emergence of Mathieu moonshine in section 4.2.3.

In Chapter 5 however we will see that applying the prescription to obtain the (field-theoretic) elliptic genus of an $\mathcal{N} = (4,4)$ SCFT to the partition function of any SCFT with large $\mathcal{N} = (4,4)$ symmetry yields a vanishing quantity. In this case one may consider instead a generalisation of the elliptic genus.

4.1 The Witten Index

In this section, we recall how the Atiyah-Singer index theorem naturally appears in the context of the one-dimensional σ-model. In particular, we highlight how the generalised Gauss-Bonnet and Hirzebruch signature theorems (theorem C.4.12) appear in this context. This section largely follows [Wit82; Alv83; FW84], though we have expanded the on the underlying topological aspects in order to make it more accessible. We provide Appendix C to introduce some of the mathematical concepts used in this chapter.

4.1.1 The Index of the Supercharge

For notational ease, we restrict to the case of $\mathcal{N} = 1$ supersymmetry (SUSY) for this section. In one dimension, the supersymmetry algebra is given by [Alv83]

$$\{Q, Q^\dagger\} = 2H, \qquad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \tag{4.1.1}$$

where H is the Hamiltonian and Q is the supercharge. From Q and $Q^\dagger$, one can form a hermitian operator,

$$\tilde{Q} = \frac{Q + Q^\dagger}{\sqrt{2}}, \tag{4.1.2}$$

which clearly satisfies $\tilde{Q}^2 = H$. We see that the energy of any state in the theory is bounded from below by 0, since $\tilde{Q}$ is hermitian and hence has real eigenvalues. In fact, in a system with unbroken supersymmetry, where the ground state is annihilated by the supercharges, this state must necessarily have zero energy. If the ground state of a theory is non-zero we therefore must have broken supersymmetry.

Given a spin operator J^3, whose eigenvalues take values in $\frac{1}{2}\mathbb{Z}$, we can define an operator $(-1)^F$ which anticommutes with the supercharges as,

$$(-1)^F = e^{2\pi i J^3}, \qquad \{(-1)^F, Q\} = \{(-1)^F, Q^\dagger\} = \{(-1)^F, \tilde{Q}\} = 0. \tag{4.1.3}$$

We now see that states of non-zero energy are paired under supersymmetry. Consider a bosonic eigenstate of the Hamiltonian $|b\rangle$, with non-zero energy E.

$$H\,|b\rangle = E\,|b\rangle\,, \qquad E > 0. \tag{4.1.4}$$

We can now act with $\tilde{Q}$ on $|b\rangle$ to obtain another state with opposite fermion number which we call $|f\rangle$. This state is also an eigenstate of the Hamiltonian with non-zero energy E,

$$H\,|f\rangle := H\tilde{Q}\,|b\rangle = \tilde{Q}H\,|b\rangle = E\,|f\rangle\,, \tag{4.1.5}$$

since clearly we have $\{\tilde{Q}, H\} = 0$. However, a bosonic state of zero energy must satisfy $\tilde{Q}\,|b\rangle = 0$, since we have $\tilde{Q}^2 = H$. We therefore see that whilst positive energy states are paired under symmetry, zero energy states are not. We denote the number of bosonic zero-energy states as $n_B^{E=0}$ and the number of fermionic zero-energy states as $n_F^{E=0}$.

If we now consider varying the parameters of the theory (the volume, the masses or the coupling constants), the states of non-zero energy will move about in energy level, remaining in Bose-Fermi pairs. If a pair of states drop to the ground energy level, then both $n_B^{E=0}$ and $n_F^{E=0}$ will increase by one. Similarly if the parameters are varied in such a way that states of zero-energy gain non-zero energy, then both $n_B^{E=0}$ and $n_F^{E=0}$ must increase by one, since as soon as a state gains a non-zero energy it must have a supersymmetric partner of the same energy level. The difference $n_B^{E=0} - n_F^{E=0}$ is therefore invariant under a change of parameters.

The invariant quantity $n_B^{E=0} - n_F^{E=0}$ may now be regarded as the trace of the operator $(-1)^F$, since states of non-zero energy do not contribute to that trace. Indeed, bosonic states have integer J_3-eigenvalues and hence each contributes $(+1)$ to the trace of $(-1)^F$, while fermionic states have J_3-eigenvalues in $\mathbb{Z}+\frac{1}{2}$ and hence each contributes

(-1) to that trace. Since the non-zero energy states always come in pairs consisting of one bosonic state and one fermionic state of the same energy, the net contribution to the trace solely comes from $n_B^{E=0} - n_F^{E=0}$. Now we note that the infinite sum over the Hilbert space is not well-defined; since the infinite series is not absolutely convergent, it depends on the ordering of the terms. We must therefore regularise the trace and this is achieved by inserting the regulator $e^{-\beta H}$ with β an arbitrary positive real number in the trace. Since states with $E \neq 0$ do not contribute to the regularised trace, the regularised trace does not depend on β and we are at leisure to evaluate it at any value of β we see fit. In the limit $\beta \to 0$ one recovers $\mathrm{Tr}(-1)^F$. We therefore have,

$$\mathrm{Tr}(-1)^F e^{-\beta H} = n_B^{E=0} - n_F^{E=0}, \tag{4.1.6}$$

a topological invariant known as the *Witten Index* of the theory. If the Witten index is non-zero, the net number of zero-energy states in the theory is non-zero and hence supersymmetry is not broken. However, a zero Witten index does not imply broken supersymmetry, since it is possible to have $n_B^{E=0} = n_F^{E=0} \neq 0$, and to therefore still have unbroken supersymmetry. The Witten index thus provides a powerful tool in the context of supersymmetry breaking: for instance, given that there is no clear evidence of supersymmetry in experiments to date, the knowledge of whether a candidate field theory exhibiting supersymmetry at tree level remains unbroken when quantum corrections are taken into account is extremely valuable.

We will now see that the quantity $\mathrm{Tr}(-1)^F$ should be thought of as the index of an operator. Let us split the Hilbert space $\mathscr{H}$ into bosonic and fermionic subspaces, $\mathscr{H} = \mathscr{H}_B \oplus \mathscr{H}_F$. Since the supercharge $\tilde{Q}$ maps bosons to fermions and vice versa it takes the form of an off-diagonal operator,

$$\tilde{Q} = \left(\begin{array}{c|c} 0 & Q^\dagger \\ \hline Q & 0 \end{array}\right), \tag{4.1.7}$$

acting on states of the form, $(B|F)^T$, where ^{T} denotes the transpose.

Now, as discussed above, since $H = \tilde{Q}^2$, the states of zero-energy are those which are annihilated by $\tilde{Q}$. Zero-energy bosonic states are therefore the states $\phi \in \mathscr{H}_B$ such that $Q\phi = 0$ and zero-energy fermionic states are therefore states $\phi \in \mathscr{H}_F$ such that $Q^\dagger \phi = 0$. Since $Q^\dagger$ is the adjoint of Q we therefore have

$$\begin{aligned}
\mathrm{Tr}(-1)^F e^{-\beta H} &= n_B^{E=0} - n_F^{E=0}, \\
&= \{\text{no. of solutions to } Q\phi = 0\} - \{\text{no. of solutions to } Q^\dagger\phi = 0\}, \\
&= \dim \mathrm{Ker}\ Q - \dim \mathrm{Ker}\ Q^\dagger = \dim \mathrm{Ker}\ Q - \dim \mathrm{Coker}\ Q, \\
&=: \mathrm{Index}(Q).
\end{aligned} \tag{4.1.8}$$

4.1.2 The Witten Index for a Simple σ-Model

We first comment on the Witten index for a σ-model with $0+1$-dimensional worldsheet. We will see that in this simple case we can already identify the Witten index with a well known topological invariant, specifically the Euler characteristic of the associated target manifold. Furthermore, we show how a path integral formulation for this index recreates the Gauss-Bonnet theorem.

We take as our starting point the supersymmetric sigma model eq. (2.2.34) with worldsheet space S^1, target space M of dimension d and fields ϕ^j, ψ^k with $j, k \in \{1, 2, \ldots, d\}$ functions of x^0 only,

$$\begin{aligned}
S[\phi,\psi] &= \frac{1}{2}\int \mathrm{d}x^0 \ \left(g_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j + i g_{ij}(\phi)\bar{\psi}^i\gamma^0 D_0\psi^j + \frac{1}{6}R_{iklj}(\bar{\psi}^i\psi^k)(\bar{\psi}^l\psi^j)\right) \\
D_0\psi^i &= \dot{\psi}^i + \Gamma^i_{jk}\dot{\phi}^j\psi^k,
\end{aligned} \tag{4.1.9}$$

where $\dot{} := \partial_0$ and the covariant derivative is with respect to the Levi-Civita connection. As noted in section 2.2.2, this shows that the fermions ψ^i transform as vector fields on M.

Following [Wit82], we change to a basis in which $\gamma^0 = \mathrm{diag}(1,-1)$ and $\psi_j = (\chi_j, \chi_j^\dagger)^T$, where $\chi_j^\dagger$ is the hermitian conjugate of χ_j and these Weyl spinors satisfy the anticommutation relations

$$\{\chi_i, \chi_j\} = \{\chi_i^\dagger, \chi_j^\dagger\} = 0, \qquad \{\chi_i, \chi_j^\dagger\} = g_{ij} \tag{4.1.10}$$

with g_{ij} functions of the fields ϕ^k. We can therefore interpret χ_j and $\chi_j^\dagger$ as annihilation and creation operators respectively. The supercharges are calculated using Noether's theorem as

$$Q = i\sum_{i=1}^{d} \chi_i^\dagger p_i, \qquad Q^\dagger = -i\sum_{i=1}^{d} \chi_i p_i, \tag{4.1.11}$$

where p_i is the momentum conjugate to ϕ^i. These supercharges satisfy the supersymmetry algebra

$$Q^2 = (Q^\dagger)^2 = 0, \qquad \{Q, Q^\dagger\} = H. \tag{4.1.12}$$

Now an important and far reaching realisation is that the Hilbert space of this model is described by the space of square-integrable differential forms on the manifold: if a state $|\Omega\rangle$ satifies $\chi_i |\Omega\rangle = 0\ \forall i$, it is bosonic and must be given by a (complex) function of only the scalar coordinates, $A(\phi^k)$; acting on such a state with $\chi_i^\dagger$ creates a fermionic state of type i, which must therefore be given by a (complex) function with one index tangent to the manifold, $A_i(\phi^k)$; applying $\chi_j^\dagger,\ j \neq i$ on the latter state yields a two-fermion state which must be given by a (complex) function $A_{ij}(\phi^k)$, antisymmetric in the indices i, j to account for the anticommutation of the creation operators $\chi_i^\dagger, \chi_j^\dagger$. In general a state containing $k < d$ fermions in this theory must be represented by an antisymmetric rank-k tensor field $A_{1,\ldots,k}(\phi^i)$, while a state with d fermions is given by a scalar function by Hodge duality. This construct is exactly the (complex valued) de Rham complex of the manifold M.

In this differential geometry context, we remark that the supercharge operator $Q = i\sum_{i=1}^{d} \chi_i^\dagger p_i$ acts as the exterior derivative of the de Rham complex decribed above. Indeed, the exterior derivative of a p-form ω is a $(p+1)$-form given by,

$$d\omega = (\partial_i \omega_{i_1,\ldots,i_p}) dx^i \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{4.1.13}$$

The action of Q on the p-form A, with components $A_{i_1,\ldots,i_p}(\phi^k)$, produces a sum of states, each with an extra fermion created by a different $\chi_i^\dagger$, and at the same time Q differentiates the original state with respect to ϕ^i since the momentum p_i conjugate to ϕ^i is contracted with $\chi_i^\dagger$ in Q. Therefore the form QA has antisymmetric components,

$$(QA)_{[i,i_1,\ldots,i_p]} = D_{[i}A_{i_1,\ldots,i_p]}(\phi^k), \tag{4.1.14}$$

where $p_i = D_i := -i\,\frac{D}{D\phi^i}$. The creation of a fermion via $\chi_i^\dagger$ has the effect of changing the form from a p to a $(p+1)$-form while differentiating the components of the p-form and hence, as announced,

$$Q = d. \tag{4.1.15}$$

In a similar manner, the operator $Q^\dagger = -i\sum_i \chi_i p_i$ can be identified with the adjoint of the exterior derivative, d^*. The Hamiltonian

$$H = QQ^\dagger + Q^\dagger Q = dd^* + d^*d, \tag{4.1.16}$$

is therefore equivalent to the Laplace-deRham operator $\Delta = (d+d^*)^2$. The states of zero energy, those satisfying $H\,|\varphi\rangle = 0$, where $|\varphi\rangle$ may be bosonic or fermionic, are therefore exactly equivalent to the harmonic forms $\Delta A_I(\phi^k) = 0$, where I is some set of fermionic indices. The Hodge theorem states that there is an isomorphism from the space of harmonic forms to the de Rham cohomology of M and hence the space of harmonic k-forms with complex coefficients and the k^{th} cohomology class $H^k(M,\mathbb{C})$ have the same complex dimension. This dimension is by definition the k^{th} Betti number b_k. See Appendix C for a brief introduction to cohomology. Since our indices $\{i_1,\ldots,i_p\}$ tell us the number of fermions in a given state, the p-forms are to be regarded as bosonic in the case that p is even and fermionic in the case that p is odd. This brings us to the result

$$\mathrm{Tr}\ (-1)^F e^{-\beta H} = n_B^{E=0} - n_F^{E=0} = \sum_{p=0}^{d}(-1)^p\ b_p = \chi(M), \tag{4.1.17}$$

where $\chi(M)$ is the Euler number of M and d is the dimension of M.

As is well known, the trace appearing in the Witten index has a path integral representation. Since the Hamiltonian is the generator of time translations, the Witten index describes a path integral over the bosonic and fermionic fields of the theory. The Witten index can therefore be expressed as

$$\mathrm{Tr}\ (-1)^F e^{-\beta H} \equiv \mathrm{STr}\ e^{-\beta H} = \int_{PBC} \mathcal{D}\varphi\ \mathcal{D}\psi\ \mathcal{D}\bar{\psi} e^{-S_E}, \tag{4.1.18}$$

where the integral is taken over all field configurations with periodic boundary conditions. The $(-1)^F$ insertion comes from taking the fermions to be periodic in time. This is explained in more detail in many textbooks, but see for example [DMS97]. As shown in [Alv83], for the one-dimensional (worldsheet) σ-model considered in this subsection, if the dimension of the target space M is even, $d = 2n$ this integral can be evaluated to give

$$\begin{aligned}\chi(M) = \mathrm{Tr}\ (-1)^F e^{-\beta H} &= \frac{(-1)^n}{2^d n! \pi^n} \int_M \mathrm{d}(vol)\ \epsilon^{i_1 j_1 \dots i_n j_n} \epsilon^{k_1 l_1 \dots k_n l_n} R_{i_1 j_1 k_1 l_1} \dots R_{i_n j_n k_n l_n}, \\ &= \frac{1}{(2\pi)^n} \int_M \mathrm{Pf}(\Omega),\end{aligned} \tag{4.1.19}$$

where $\mathrm{Pf}(\Omega)$ denotes the Pfaffian of the curvature two-form,

$$\Omega_{ab} = \frac{1}{2} R_{abcd} e^c \wedge e^d. \tag{4.1.20}$$

The curvature form on a $d = 2n$ dimensional Riemannian manifold is a two-form which takes values in the Lie algebra $\mathfrak{so}(d)$ of the holonomy group $SO(d)$. We can therefore think of the curvature form in this case as a $d \times d$ skew-symmetric matrix whose elements are two-forms on M. The Pfaffian of this matrix is therefore of order n in two-forms and hence is a top form and can be integrated over the manifold. We now recognise eq. (4.1.19) as the generalised Gauss-Bonnet formula.

4.1.3 Other Indices in the 1d σ-Model

In the previous subsection we saw how the Witten index $\mathrm{Tr}(-1)^F$ for a one-dimensional (worldsheet) σ-model calculated the Euler characteristic of the target space manifold M. In this subsection we briefly show how other topological invariants of M can be calculated as indices.

The σ-model defined in eq. (4.1.9) has a discrete chiral symmetry given by $\psi \to \gamma_5\psi$ [Alv83]. Defining Q_5 to be the operator which implements this symmetry and considering two supersymmetry operators of definite chirality $Q_\pm$, we note that these operators satisfy the relation

$$Q_5 Q_\pm = \pm Q_\pm Q_5. \tag{4.1.21}$$

The condition $Q_5 Q_- = -Q_- Q_5$ can be thought of as analogous to the relation $(-1)^F Q_i = -Q_i(-1)^F$ and we can repeat similar arguments as in sections 4.1.1 and 4.1.2 with Q_5 now in place of $(-1)^F$. For states of non-zero energy we therefore have for each eigenstate $|\psi\rangle$ of Q_5 with eigenvalue $+1$, an eigenstate given by $Q_- |\psi\rangle$ which necessarily has Q_5 eigenvalue -1, since $Q_5 Q_- |\psi\rangle = -Q_- Q_5 |\psi\rangle = -Q_- |\psi\rangle$. As before, states of zero-energy are annihilated by Q_- and so form one-dimensional representations of the supersymmetry algebra and do not necessarily come in pairs. We can therefore define a quantity Tr Q_5 which is given entirely by the zero-energy states. As before, the quantity Tr Q_5 is independent of small changes to the parameters of the theory.

We can again interpret this in terms of the (complex valued) de Rham complex. The creation and annihilation operators χ and $\chi^\dagger$ were defined as eigenstates of γ_0 with eigenvalues $+1$ and -1 respectively. Now since $\gamma_5\gamma_0 = -\gamma_0\gamma_5$, the operator implementing this chiral symmetry Q_5 exchanges the operators of eigenvalue $+1$ and -1; that is, Q_5 exchanges $\chi \leftrightarrow \chi^\dagger$. Considering a 0-form in the de Rham complex, which by definition is a state annihilated by any of the annihilation operators, we realise it must be mapped under Q_5 to a state which is annihilated by all the creation operators; we therefore realise this must be an d-form if M is of dimension d. Similarly, a 1-form containing one creation operator $\chi_i^\dagger$ must be mapped to an annihilation operator χ_i acting on the n-form containing all of the creation operators, schematically

$$\chi_i^\dagger \left|\phi^k\right\rangle \xrightarrow{Q_5} \chi_i \chi_1^\dagger \chi_2^\dagger \dots \chi_d^\dagger \left|\phi^k\right\rangle. \tag{4.1.22}$$

We can now use the anticommutation relations of the χ and $\chi^\dagger$ operators to remove one of the creation operators resulting in a state containing $n-1$ creation operators, which we recognise as an $(d-1)$-form in the de Rham complex. In general Q_5 sends p-forms to $(d-p)$-forms; Q_5 should be identified as the Hodge $*$ operation.

Tr Q_5 (or Tr $Q_5 e^{-\beta H}$) is therefore given by the number of harmonic forms in the positive eigenspace of the Hodge $*$ operation minus the number of harmonic forms in the negative eigenspace of $*$. Let us consider a form in the positive eigenspace of $*$, $\alpha = \alpha_0 + \ldots + \alpha_d$, where each α_i is an i-form. Since $*$ sends p-forms to $(d-p)$-forms, it must be the case that $*\alpha_0 = \alpha_d$ and $*\alpha_d = \alpha_0$. Hence $\alpha_0 + \alpha_d$ is itself in the positive eigenspace of $*$. However we now realise that $\alpha_0 - \alpha_d$ must be in the negative eigenspace of $*$. In general for any state formed as the sum of a p-form ($p \neq \frac{d}{2}$) and an $(d-p)$-form in the positive eigenspace of $*$, there must be a state in the negative eigenspace of $*$. However for n-forms, where $d = 2n$, $*$ is a map from n-forms to n-forms. Hence we have

$$\text{Tr } Q_5 = n^{E=0}(Q_5 = +1) - n^{E=0}(Q_5 = -1) = \dim \mathcal{H}^n_+ - \dim \mathcal{H}^n_-, \tag{4.1.23}$$

where $\mathcal{H}^n_+$ is to be understood as the space of harmonic n-forms in the positive eigenspace of $*$ and similarly for the negative eigenspace.

If we consider the non-degenerate bilinear form

$$\begin{aligned} I : \mathcal{H}^n \times \mathcal{H}^n &\to \mathbb{C} \\ (\alpha, \beta) &\mapsto \int_M \alpha \wedge \bar{\beta}, \end{aligned} \tag{4.1.24}$$

where $\bar{\beta}$ denotes the complex conjugate, then we see that I is positive-definite on $\mathcal{H}^n_+ \times \mathcal{H}^n_+$ and negative-definite on $\mathcal{H}^n_- \times \mathcal{H}^n_-$, as given $\alpha \in \mathcal{H}^n_+$ and $\beta \in \mathcal{H}^n_-$

$$\begin{aligned} I(\alpha, \alpha) &= I(\alpha, *\alpha) = \int_M \alpha \wedge *\bar{\alpha} = \langle \alpha, \alpha \rangle \geq 0, \\ I(\beta, \beta) &= -\, I(\beta, *\beta) = -\int_M \beta \wedge *\bar{\beta} = -\langle \beta, \beta \rangle \leq 0, \\ I(\alpha, \beta) &= -\, I(\alpha, *\beta) = -\langle \alpha, \beta \rangle = -\langle \beta, \alpha \rangle \\ &\qquad = -I(\beta, *\alpha) = -I(\beta, \alpha) = -I(\alpha, \beta) = 0, \end{aligned} \tag{4.1.25}$$

where

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge *\bar{\beta}, \tag{4.1.26}$$

defines an inner product on p-forms.

Again, by the Hodge de Rham theorem the space of harmonic p-forms is isomorphic to the p^{th} cohomology group of M. Hence the above bilinear form I is equivalent to the intersection form

$$\begin{aligned} H : H^n(M, \mathbb{C}) \times H^n(M, \mathbb{C}) \to \mathbb{C}, \\ (\alpha, \beta) \mapsto \langle \alpha \smile \beta, [M] \rangle, \end{aligned} \tag{4.1.27}$$

where $\smile$ denotes the cup product which is defined in Appendix C. In the case that M is of dimension $d = 4k$, then the cup product on $2k$-forms is symmetric, since for α_p a p-form and β_q a q-form,

$$\alpha_p \smile \beta_q = (-1)^{pq}(\beta_q \smile \alpha_p). \tag{4.1.28}$$

In this case, the difference between the dimension of the positive eigenspace and negative eigenspace of the associated symmetric bilinear form, the signature of the form, is known as the Hirzebruch signature of the manifold (cf. definition C.4.10). Therefore for a $d = 4k$-dimensional manifold M we have

$$\mathrm{Tr}\ Q_5 e^{-\beta H} = \dim\ \mathcal{H}^{2k}_+ - \dim\ \mathcal{H}^{2k}_- = \dim\ H^{2k}_+ - \dim\ H^{2k}_- = \tau(M), \tag{4.1.29}$$

where $\tau(M)$ is the Hirzebruch signature of the manifold.

As in the previous subsection, we can compute the index density by using the functional integral form of $\mathrm{Tr}(Q_5 e^{-\beta H})$. Since the operator Q_5 splits the fermionic space into a positive and negative eigenspace, we now have to integrate over periodic boundary conditions for bosonic and fermionic configurations in the negative eigenspace, but antiperiodic boundary conditions for fermionic configurations in the positive eigenspace of Q_5. We then obtain [Alv83]

$$\tau(M) = \mathrm{Tr}(Q_5 e^{-\beta H}) = \int_M \prod_\alpha \frac{\chi_\alpha}{\tanh(\chi_\alpha)}, \tag{4.1.30}$$

where χ_α are the (skew-)eigenvalues of the skew-symmetric matrix $\frac{1}{2\pi}\Omega_{ab}$. Since these eigenvalues are two-forms for all α, as discussed in section 4.1.2, the function

$$L(M) := \prod_\alpha \frac{\chi_\alpha}{\tanh(\chi_\alpha)}, \tag{4.1.31}$$

known as the *Hirzebruch polynomial*, defined through its Taylor expansion contains a finite number of terms, terminating with a top form. We therefore obtain the signature of M by integrating this top form over M.

If we now define $Q(x) = \frac{x}{\tanh(x)}$ such that $L(M) = \prod_{i=1}^d Q(x_i)$, then as discussed in Appendix C.4, we expect to be able to write $L(M)$ as a homogeneous multiplicative sequence in the elementary symmetric polynomials $e_i(x_i^2)$; that is, $L(M)$ is a polynomial in Pontryagin classes. The top form that we should take to calculate eq. (4.1.30) is the homogeneous term of weight d from this multiplicative sequence, where as above, d is the dimension of M. This is exactly the definition of the genus associated to the characteristic series $Q(x)$, the L-genus. We therefore find

$$\tau(M) = \mathrm{Tr}(Q_5 e^{-\beta H}) = \phi_L(M), \tag{4.1.32}$$

and we have reproduced the Hirzebruch signature theorem from the σ-model.

One can similarly find the index of the Dirac operator $i\not{D}$ and the Dolbeault index, $\mathrm{ind}(\bar{\partial})$, over spin and complex manifolds respectively. The densities for these indices take a similar form to that of eq. (4.1.30),

$$\mathrm{ind}(i\not{D}) = \int_M \hat{A}(M), \qquad \mathrm{ind}(\bar{\partial}) = \int_M \mathrm{td}(M), \tag{4.1.33}$$

where

$$\hat{A}(M) = \prod_\alpha \frac{\chi_\alpha/2}{\sinh(\chi_\alpha/2)}, \qquad \mathrm{td}(M) = \prod_\alpha \frac{\omega_\alpha}{1-e^{-\omega_\alpha}}, \tag{4.1.34}$$

define *characteristic polynomials* for the indices. In each case the characteristic polynomial gives a finite polynomial in two-forms terminating in a top form and it is this form we should integrate over the manifold. The ω_α appearing in eq. (4.1.34) are the eigenvalues of the curvature form in complex coordinates,

$$\Omega_{\alpha\bar{\beta}} := \frac{i}{2\pi} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \, \mathrm{d}z^{\gamma} \, \mathrm{d}\bar{z}^{\delta}. \tag{4.1.35}$$

The previous calculations for the index densities can be combined into the following equation known as the Atiyah-Singer index theorem for compact, oriented, differentiable manifolds of dimension $d = 2n$ [HBJL92; EGH80],

$$\operatorname{ind}(D) = (-1)^n \left(\frac{\sum_{i=0}^{m} (-1)^i \operatorname{ch}(E_i)}{e(TM)} \operatorname{td}(TM \otimes \mathbb{C}) \right) [M], \tag{4.1.36}$$

where $D = (D_i : \Gamma E_i \to \Gamma E_{i+1}),\ i \in \{1, \ldots, m\}$ is an elliptic complex and $e(E) = \prod_1^n x_i$ is the Euler class of the bundle, written in terms of the Chern classes x_i of the bundle E. In the case of the Euler characteristic, where by eqs. (4.1.8) and (4.1.15) we have $\chi(M) = \operatorname{ind}(d)$, the elliptic complex is given by the complex-valued de Rham complex $\Lambda^i(T^* \otimes \mathbb{C})$ with the exterior derivative acting on the forms. In the case of the signature $\tau(M) = \operatorname{ind}(Q_-)$, the elliptic complex is given by the ± 1 eigenspaces of $* \cong Q_5$ where Q_- moves from the $+1$ eigenspace to the -1 eigenspace and vice-versa.

We do not define an elliptic complex in general, but for an elliptic complex we always have $D_i D_{i-1} = 0$, and we can therefore consider the cohomology of the elliptic complex

$$H^i = \frac{\operatorname{Ker}(D_i)}{\operatorname{Im}(D_{i-1})}. \tag{4.1.37}$$

This allows us to define the index of an elliptic complex.

Definition 4.1.1. The *index* of an elliptic complex D where $D_i : \Gamma E_i \to \Gamma E_{i+1}$ for $i \in \{1, \ldots, m\}$ is given by

$$\operatorname{ind} D = \sum_{i=0}^{m} (-1)^i \dim_{\mathbb{C}} H^i = \sum_{i=0}^{m} (-1)^i h^i, \tag{4.1.38}$$

where we define $h^i = \dim_{\mathbb{C}} H^i$.

Note that if $m = 1$ then we have a single $D_0 : \Gamma E_0 \to \Gamma E_1$ and

$$\operatorname{ind}(D) \equiv \operatorname{ind}(D_0) := \dim_{\mathbb{C}} \operatorname{Ker}\ D_0 - \dim_{\mathbb{C}} \operatorname{Coker}\ D_0. \tag{4.1.39}$$

As shown in Appendix C,

$$\operatorname{td}(TM \otimes \mathbb{C}) = \operatorname{td}(TM \oplus T^*M) = \prod_{i=1}^{n} (-1)^n \frac{x_i}{1-e^{-x_i}} \frac{x_i}{1-e^{x_i}}, \tag{4.1.40}$$

where here $n = \frac{d}{2}$, and so we can formally factor out the Euler class,

$$e(TM) = \prod_{i=1}^{n} x_i, \tag{4.1.41}$$

to give the formal equation [HBJL92],

$$\operatorname{ind}(D) = \left(\left(\sum_{i=0}^{m} (-1^i) \operatorname{ch}(E_i) \right) \prod_{j=1}^{n} \left(\frac{x_j}{1-e^{-x_j}} \frac{1}{1-e^{x_j}} \right) \right) [M]. \tag{4.1.42}$$

We now show through examples how this reduces to give the index densities for the Euler characteristic and the signature.

Example 4.1.2. By eq. (C.2.29) we have,

$$\operatorname{ch}(\sum_{i=0}^{m} \Lambda^i (T^* \otimes \mathbb{C}) y^i) = \prod_{j=1}^{n} (1 + ye^{x_j})(1 + ye^{-x_j}). \tag{4.1.43}$$

For $y = -1$, this is exactly the sum of Chern characters appearing in eq. (4.1.42) since as described above, this is the relevant elliptic complex for this case. This sum of Chern characters therefore exactly cancels the denominator from the Todd class, leaving

$$\operatorname{ind}(d) = \prod_{j=1}^{n} x_j [m] = e(M)[M] = \chi(M). \tag{4.1.44}$$

Note that we may also define the Euler class to be given by $\frac{1}{(2\pi)^n} \operatorname{Pf}(\Omega)$. This then gives

$$\chi(M) = \int_M \frac{1}{(2\pi)^n} \operatorname{Pf}(\Omega), \tag{4.1.45}$$

in agreement with eq. (4.1.19). △

Example 4.1.3. For the signature, the elliptic complex is given by the positive and negative eigenspaces of Q_5 (in the mathematical literature this is the operator $\tau = i^{p(p-1)+n} *$ which acts on the spaces $\Lambda^p(T^* \otimes \mathbb{C})$ and satisfies $\tau^2 = I$) which we call E_+ and E_- respectively. As explained in section 4.1.3, we define the signature for a manifold of dimension $d = 4k$, then we have [HBJL92]

$$\mathrm{ch}(E_+) - \mathrm{ch}(E_-) = \prod_{i=1}^{2k}(e^{x_i} - e^{-x_i}). \tag{4.1.46}$$

Substituting this into eq. (4.1.42) gives,

$$\begin{aligned}
\tau(M) &= \left(\prod_{j=1}^{2k}(e^{x_j} - e^{-x_j}) \prod_{j=1}^{2k}\left(\frac{x_j}{1-e^{-x_j}}\frac{1}{1-e^{x_j}}\right)\right)[M], \\
&= \left(\prod_{j=1}^{2k} \frac{x_j(e^{x_j/2} + e^{-x_j/2})}{e^{x_j/2} - e^{-x_j/2}}\right)[M], \\
&= \left(\prod_{j=1}^{2k} \frac{x_j}{\tanh(x_j/2)}\right)[M] = \left(\prod_{j=1}^{2k} \frac{x_j}{\tanh(x_j)}\right)[M],
\end{aligned} \tag{4.1.47}$$

where the final two products only agree in the homogeneous term of weight $2k$. This however is a cohomology class in $H^{4k}(M)$, and is therefore the class that we need to integrate over the manifold. It is therefore the only term that needs to agree.

We therefore see that

$$\tau(M) = \left(\prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i)}\right)[M] = L(M)[M] \tag{4.1.48}$$

in agreement with eqs. (4.1.30) and (4.1.31). △

4.2 The Elliptic Genus

The previous section introduced the Witten index $\mathrm{Tr}(-1)^F e^{-\beta H}$ and showed how it was given by the index of the supercharge Q. For a one-dimensional σ-model, it was shown how this was related to the Euler characteristic as well as how other topological invariants arose as indices. The σ-models which we discussed in Chapter 2 and those of relevance to string theory are two-dimensional σ-models which describe maps from the string worldsheet to a target space M. In this section, we therefore want to discuss indices for two-dimensional σ-models and in particular, we want to introduce the elliptic genus. We first give a general definition of the elliptic genus and discuss the geometric interpretation of the index. We then define the elliptic genus more specifically for the case of a two-dimensional $\mathcal{N} = (2,2)$ or $\mathcal{N} = (4,4)$

σ-models. The elliptic genus is particularly important to our story, as this is the index which exhibits moonshine in the case of $\mathcal{N}=4$ theories.

4.2.1 Elliptic Genus for a 2d Super-σ-Model

In the previous section, we regularised the Witten index with a factor of $e^{-\beta H}$. The resulting expression $Tr(-1)^F\, e^{-\beta H}$ may therefore be interpreted as the partition function of the theory with the insertion of the $(-1)^F$ operator. We then used a path integral representation of the partition function to evaluate the index of the supercharge. Let us now consider the partition function for a two-dimensional σ-model. In analogy with the one-dimensional case, we therefore consider the conformal field theory on the cylinder, compactified in the time direction. As in the one-dimensional case, the Hamiltonian H of the theory generates the time translations, but in two dimensions a state may also be translated in space and these translations are generated by the momentum P. Consider a bosonic two-dimensional conformal field theory on a torus with complex periods ω_1 and ω_2, that is a torus formed by taking the quotient of $\mathbb{C}$ by the lattice $\Lambda = \omega_1\mathbb{Z}\oplus\omega_2\mathbb{Z}$. It is common to work with the equivalent lattice $L=\frac{1}{\omega_1}\Lambda=\mathbb{Z}\oplus\tau\mathbb{Z}$. If we let $\tau=\tau_1+i\tau_2$, then we require $\tau_1\in\mathbb{R}$ and $\tau_2\in\mathbb{R}^+$ which we can always achieve by permuting ω_1 and ω_2 if necessary. We will always assume that ω_1 and ω_2 are chosen such that $\tau:=\frac{\omega_2}{\omega_1}\in\mathbb{H}$ for $\mathbb{H}$ the upper half-plane. The partition function is then given by,

$$Z(\tau):=\mathrm{Tr}_{\mathscr{H}}\, e^{-2\pi\tau_2 H+2\pi i\tau_1 P}, \tag{4.2.1}$$

where $\mathscr{H}$ is the Hilbert space of the theory and where the momentum and Hamiltonian are given by

$$H=L_0+\bar{L}_0-\frac{c}{12}, \qquad P=L_0-\bar{L}_0. \tag{4.2.2}$$

Defining $q=e^{2\pi i\tau}$ and $\bar{q}=e^{-2\pi i\bar{\tau}}$, $\bar{\tau}=\tau_1-i\tau_2$, we can therefore rewrite the partition function as,

$$Z(\tau)=\mathrm{Tr}_{\mathscr{H}}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}. \tag{4.2.3}$$

Witten first introduced the (conformal field theoretic) elliptic genus for a superstring σ-model with Ramond boundary conditions for the right-movers and Neveu-Schwarz boundary conditions for the left-movers [Wit87]. Here, the elliptic genus was defined as the index of a right-moving supercharge Q which anticommutes with a right-moving fermion operator,

$$Q(-1)^{F_R} + (-1)^{F_R} Q = 0. \tag{4.2.4}$$

As in eq. (4.1.8), by index we mean the dimension of the kernel of Q minus the dimension of the cokernel of Q. Since Q now also commutes with the momentum operator P one can consider the index of Q on individual eigenspaces of the momentum operator separately. This leads to the definition of the character-valued index.

Definition 4.2.1. Consider the subspace $\mathscr{H}_\lambda$ defined as the subspace of the Hilbert space where the momentum operator has eigenvalue λ. Denote the index of Q restricted this subspace $\mathscr{H}_\lambda$ as h_λ. We then define the *character-valued index* of Q in terms of a formal variable q to be,

$$F(q) = \sum_\lambda h_\lambda q^\lambda. \tag{4.2.5}$$

We can now write this character-valued index as the trace of an operator on the Hilbert space as

$$F(q) = \mathrm{Tr}\left((-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\right), \tag{4.2.6}$$

since $F(q)$ is only counting supersymmetric states which necessarily have $\bar{L}_0 = \frac{c}{24}$. States with $\bar{L}_0 \neq \frac{c}{24}$ drop out of the above trace due to the pairing by Q. By eq. (4.2.2), such states have momentum $L_0 - \frac{c}{24}$, and we therefore see that eq. (4.2.6) calculates the index of the eigenspaces of momentum as in definition 4.2.1. Comparing this with eq. (4.2.3), we see that this elliptic genus is an insertion of $(-1)^{F_R}$ into the partition function for a the two-dimensional conformal field theory.

Since Q acts on states representing string configurations this is the index of an operator on a space known as the loop space of the target space M, $\mathcal{L}M$.

Definition 4.2.2. The free *loop space* of M is given by

$$\mathcal{L}M = \{g : S^1 \to M | g \text{ is differentiable}\}. \tag{4.2.7}$$

The rotation of the string generated by the momentum operator P therefore defines an S^1 action on the loop space of M. Under a few assumptions, this allows us to define an S^1-equivariant index on M [HBJL92].

Definition 4.2.3. Suppose M is a complex manifold of dimension d with an elliptic complex $D = (D_i : \Gamma E_i \to \Gamma E_{i+1})$. If there is a topological group G which acts on M by holomorphic maps, and if the G action extends to the bundles E_i and commutes with D_i, then for $g \in G$ we can define an *equivariant index*,

$$\mathrm{ind}(g, D) = \sum_{i=0}^{m} (-1)^i \operatorname{Tr}_{H^i} g. \tag{4.2.8}$$

Since the trace of the identity gives the dimension of the space we clearly have

$$\mathrm{ind}(D) \equiv \mathrm{ind}(I, D), \tag{4.2.9}$$

where $\mathrm{ind}(D)$ is defined as in definition 4.1.1.

Consider a fixed point submanifold under the G-action, M_i^g. Since M is assumed to be complex, for any $p \in M$ there exists a Hermitian metric on the tangent space T_pM and hence g acts unitarily on T_pM. This tangent space therefore decomposes into a sum of eigenspaces for eigenvalues of modulus one, and one obtains an eigenbundle over M_i^g. One can show that the equivariant index can be computed as the sum over i of the fixed point submanifolds M_i^g. The contribution to the index from each fixed component can be calculated using the regular Atiyah-Singer index theorem (eq. (4.1.42)) modified to take account of the decomposition of the tangent bundle over M into eigenbundles over M_i^g [HBJL92]. When the action is a $U(1)$-action, since all representations of $U(1)$ are one-dimensional, the tangent bundle of M decomposes into a sum of bundles N_k, on each of which $g \in G$ acts as multiplication by g^k, for

$k \in Z$.

The loop space $\mathcal{L}M$ has a $U(1)$ action given by,

$$u(g(x)) = g(x-u), \tag{4.2.10}$$

for $g \in \mathcal{L}M$ and $u \in U(1)$. The fixed point set under this $U(1)$-action is M itself, embedded in $\mathcal{L}M$ as the set of constant loops $g(x) = m \in M$, $\forall x \in U(1)$. The tangent bundle of the loop space $\mathcal{L}M$ restricted to M, embedded in $\mathcal{L}M$ as above can then be shown to decompose as [HBJL92],

$$T(\mathcal{L}M)|_M = TM \bigoplus_{n>0} q^n T^{\mathbb{C}}, \tag{4.2.11}$$

with $T^{\mathbb{C}} \equiv TX^{\mathbb{C}}$ defined as in definition C.2.13. By the discussion in the preceding paragraph, we see that the $U(1)$-equivariant index of a an elliptic complex D on the loop space $\mathcal{L}M$ can be calculated using the regular Atiyah-Singer index theorem on M taking into account the contributions of the decomposition in eq. (4.2.11). After evaluating the Atiyah-Singer index theorem on M, one therefore obtains a q series where the coefficient of q^k is given by the index of D on the eigenbundle where q acts by multiplication by q^k. This is the character valued index of D.

$F(q)$, the character-valued index of Q, is now seen to be a $U(1)$-equivariant index of Q on the loop space of M. Witten [Wit87; Wit88] showed that the index of this operator (known as the Dirac-Ramond operator) on the loop space $\mathcal{L}M$ can be calculated using the Atiyah-Singer index theorem as,

$$\operatorname{ind} Q = q^{-d/16} \left(\hat{A}(M) \operatorname{ch}\Big(\bigotimes_{k \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \Lambda_{q^k} T \bigotimes_{l \in \mathbb{Z}_{>0}} S_{q^l} T \Big) \right) [M], \tag{4.2.12}$$

where $\Lambda_{q^k} T$, $S_{q^l} T$ are defined as in eq. (C.2.21) for the antisymmetric and symmetric powers of the tangent bundle to M respectively, and where d is the dimension of M. This generalises the $\hat{A}$-genus of M to a $U(1)$-equivariant index on the loop space of M. Equation (4.2.12) makes it clear that the elliptic genus is a topological invariant, since we have written it purely in terms of topological data of M.

4.2.2 Elliptic Genus for $\mathcal{N} = (2,2)$ or $\mathcal{N} = (4,4)$ Theories

In the previous subsection, we saw how the addition of a $U(1)$ action generated by the momentum, led to a graded index of the σ-model known as the elliptic genus. When calculated for a σ-model with $\mathcal{N} = (1,1)$ supersymmetry and where we take left-moving fermions with NS boundary conditions and right-moving fermions with Ramond boundary conditions we obtained a generalisation of the $\hat{A}$-genus.

If our σ-model admits a second $U(1)$ action generated by K which commutes with the supercharge, we may consider the index of the supercharge restricted to states with momentum λ and K eigenvalue k. We then obtain a character-valued index

$$\tilde{F}(q,\theta) = \sum_{\lambda,k} h_{\lambda,k} q^{\lambda} e^{i\theta k}, \tag{4.2.13}$$

where $h_{\lambda,k}$ is the index of Q restricted to $\mathscr{H}_{\lambda,k}$, the space of states of charge k and momentum λ. As in the previous subsection, we can write this new index as a trace on the Hilbert space of states,

$$\mathrm{Tr}\left((-1)^{F_R} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} e^{i\theta K}\right) \tag{4.2.14}$$

since as before, states with $L_0 = h > \frac{c}{24}, K = k$ will cancel from the trace as they will be paired by Q.

In the case of an $\mathcal{N} = (2,2)$ theory, the right moving $U(1)$-charge J_0 can be treated as the generator of the additional $U(1)$ mentioned above and so we define the elliptic genus of an $\mathcal{N} = (2,2)$ theory as the index of one of the right-moving supercharges.

Definition 4.2.4. For an $\mathcal{N} = (2,2)$ theory, we define the elliptic genus to be given by

$$\varepsilon(\tau, y) := \mathrm{Tr}_{\mathscr{H}} (-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} z^{J_0}, \tag{4.2.15}$$

where $q = e^{2\pi i \tau}$, $z = e^{2\pi i y}$, $\tau, y \in \mathbb{C}, \mathrm{Im}(\tau) > 0$. Here we take $\mathscr{H} \equiv \mathscr{H}^R \otimes \mathscr{H}^R$ to be the sector of the theory where both right and left moving fermions have Ramond boundary conditions. Note that we also now use the left-right fermion number operator $(-1)^F := e^{2\pi i (J_0 - \bar{J}_0)}$.

The elliptic genus of an $\mathcal{N}=(2,2)$ theory may also be given in terms of the partition function of the theory.

Definition 4.2.5. In an $\mathcal{N}=(2,2)$ theory, as well as having conformal weights $h, \bar{h}$, states are also charged under the $U(1)$ symmetry giving them 'isospins' $l, \bar{l}$, the charge under $J_0, \bar{J}_0$. The partition function for an $\mathcal{N}=(2,2)$ conformal field theory is then defined by

$$Z(q,\bar{q},z,\bar{z}) = \mathrm{Tr}_{\mathscr{H}}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} z^{J_0}\bar{z}^{\bar{J}_0}, \tag{4.2.16}$$

where $q=e^{2\pi i\tau}$, $z=e^{2\pi i y}$, $z,\tau\in\mathbb{C}$, $\mathrm{Im}(\tau)>0$.

The elliptic genus is then given by,

$$\varepsilon(\tau,y) = Z_{\mathscr{H}^{\tilde{R}}}(q,\bar{q},z,\bar{z}=1), \tag{4.2.17}$$

where $Z_{\mathscr{H}^{\tilde{R}}}$ denotes the partition function restricted to $\mathscr{H}^R$, with a $(-1)^F$ insertion, that is with the fermions periodic in time rather than antiperiodic.

For a theory with $\mathcal{N}=(4,4)$ SUSY, one can define the elliptic genus similarly to the $\mathcal{N}=(2,2)$ case. The $\mathcal{N}=4$ SCA contains an $\widehat{\mathfrak{su}(2)}$ subalgebra and so the zero mode T^3_0 generates a $U(1)$ algebra. We can therefore grade states by their charges under this $U(1)$, and hence define an equivariant partition function for theories with $\mathcal{N}=(4,4)$.

Definition 4.2.6. The partition function for an $\mathcal{N}=(4,4)$ conformal field theory is given by

$$Z(q,\bar{q},z,\bar{z}) = \mathrm{Tr}_{\mathscr{H}}\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} z^{2J^3_0}\bar{z}^{2\bar{J}^3_0}, \tag{4.2.18}$$

where $q=e^{2\pi i\tau}$, $z=e^{2\pi i y}$, $z,\tau\in\mathbb{C}$, $\mathrm{Im}(\tau)>0$.

The elliptic genus for an $\mathcal{N}=(4,4)$ theory is then defined in a similar way as for the $\mathcal{N}=(2,2)$ case (cf. definition 4.2.4 and eq. (4.2.17)).

Definition 4.2.7. The elliptic genus for an $\mathcal{N}=(4,4)$ theory is defined by

$$\varepsilon(q,z) := Z_{\tilde{R}}(q,\bar{q},z,\bar{z}=1) = \mathrm{Tr}_{\mathscr{H}^R}(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} z^{2J^3_0}. \tag{4.2.19}$$

Note that the right-moving contribution to the index is given by,

$$\mathrm{Tr}_{\mathscr{H}}(-1)^{F_R}\bar{q}^{\bar{L}_0-c/24}, \tag{4.2.20}$$

where now $\mathscr{H}$ is the right-moving Hilbert space and this is simply the Witten index for the right-movers.

Note that the boundary conditions in definition 4.2.7 differ to those in section 4.2.1. However, similarly to the A_γ algebra discussed in Chapter 3, the $\mathcal{N}=4$ algebra has an isomorphism known as spectral flow [SS87], which allows us to relate the NS and R sectors of the theory. Using this, we can also relate the elliptic genus to other topological invariants which we have already discussed.

Proposition 4.2.8. *The elliptic genus of an $\mathcal{N}=(4,4)$ σ-model with target space M can be evaluated at different values of y in order to obtain other topological invariants.*

$$\begin{aligned}
\varepsilon(\tau,0) &= \mathrm{Tr}_{\mathscr{H}^R}(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} = \chi(M),\\
\varepsilon(\tau,1/2) &= \mathrm{Tr}_{\mathscr{H}^R}(-1)^{F_R} q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} = F_\sigma(q),\\
\varepsilon(\tau,(\tau+1)/2) &= \mathrm{Tr}_{\mathscr{H}^{NS}\otimes\mathscr{H}^R}(-1)^{F_R} q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} = F_{\hat{A}}(q)\equiv F(q)
\end{aligned} \tag{4.2.21}$$

where $\mathscr{H}^{NS}\otimes\mathscr{H}^R$ denotes the sector with left-moving NS conditions and right-moving Ramond conditions, $F_\sigma(q)$ is the loop-space index generalising the signature, and $F_{\hat{A}}(q)\equiv F(q)$ is the index introduced in section 4.2.1 which generalises the $\hat{A}$-genus.

We now notice that the topological invariants obtained by evaluating the elliptic genus at the particular values of y discussed above, $(0,\frac{1}{2},\frac{\tau+1}{2})$ are the invariants associated with the χ_y genus at the values $(-1,1,0)$ respectively (example C.4.15). We should therefore expect the elliptic genus to be the equivariant generalisation of the χ_y genus. The χ_y genus can be given by the Atiyah-Singer formula eq. (4.1.42) as

$$\chi_y(M) = \mathrm{ch}(\Lambda_y T^*)\,\mathrm{td}(M)[M], \tag{4.2.22}$$

where $\Lambda_y T^*$ is defined as in eq. (C.2.21) and T is the holomorphic tangent bundle $T^{1,0}M$. The elliptic genus can then be defined similarly [Gri00; Wen15].

Definition 4.2.9. The (geometric) elliptic genus of a complex d-dimensional manifold M can be given in terms of characteristic classes of M as

$$\varepsilon(M) = \mathrm{ch}(\mathbb{E}_{q,y})\, \mathrm{td}(M)[M], \tag{4.2.23}$$

where $\mathbb{E}_{q,y}$ is the bundle

$$\mathbb{E}_{q,y} := y^{d/2} \bigotimes_{n\geq 1} \Lambda_{-y^{-1}q^{n-1}} T \otimes \Lambda_{-yq^n} T^* \otimes S_{q^n} T \otimes S_{q^n} T^*, \tag{4.2.24}$$

for T the holomorphic tangent bundle and where $S_n T$ is defined as in eq. (C.2.21) but for symmetric rather than antisymmetric products.

We briefly comment on the modularity of the elliptic genus for $\mathcal{N} = (2,2)$ and $\mathcal{N} = (4,4)$ theories.

Definition 4.2.10. A *weak Jacobi form* of weight k and index p is a holomorphic function ϕ on $\mathbb{H} \times \mathbb{C}$ satisfying

$$\begin{aligned} \phi(\gamma \cdot \tau, \frac{z}{c\tau + d}) &= (c\tau + d)^k e^{2\pi i p c z^2/(c\tau + d)} \phi(\tau, z), \\ \phi(\tau, z + \lambda\tau + \mu) &= (-1)^{2p(\lambda+\mu)} e^{-2\pi i t \lambda(\lambda\tau + 2z)} \phi(\tau, z), \end{aligned} \tag{4.2.25}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, $\lambda, \mu \in \mathbb{Z}$, and with Fourier expansion satisfying

$$\phi(\tau, z) = \sum_{\substack{n\geq 0 \\ l \in t+\mathbb{Z}}} f(n,l) e^{2\pi i (n\tau + lz)}. \tag{4.2.26}$$

Proposition 4.2.11. *The elliptic genus of an $\mathcal{N} = (2,2)$ or $\mathcal{N} = (4,4)$ theory on a Calabi-Yau d-fold M transforms as a weak Jacobi form of weight 0 and index $\frac{d}{2}$.*

This is proved using the field-theoretic definition of the elliptic genus in [KYY94] and using the geometric definition in [Gri00].

4.2.3 Elliptic Genus and Moonshine

The motivation for this thesis was in investigating whether a similar phenomenon to that of Mathieu moonshine exists when the conformal algebra of the theory has A_γ (i.e. large $\mathcal{N}=4$) symmetry rather than $\mathcal{N}=4$ symmetry. Having now introduced the elliptic genus for $\mathcal{N}=4$ theories, we are in a position to state what the Mathieu moonshine observation is. There is much one could discuss about moonshine and so this subsection will not try to be exhaustive, but simply mention some of the basic features of moonshine as it is known to relate to the elliptic genus.

By proposition 4.2.11, we know the elliptic genus of a σ-model with a Calabi-Yau (CY) d-fold target space is a weak Jacobi form of weight 0 and index $\frac{d}{2}$. If we consider Calabi-Yau 2-folds, then the only possibilities are tori and $K3$ surfaces.

Proposition 4.2.12. *The elliptic genus of a σ-model with target space T^4 or K^3 is given respectively by,*

$$\varepsilon_{T^4}(q,z)=0, \qquad \varepsilon_{K^3}(q,z)=8\left(\left(\frac{\theta_2(q,z)}{\theta_2(q,0)}\right)^2+\left(\frac{\theta_3(q,z)}{\theta_3(q,0)}\right)^2+\left(\frac{\theta_4(q,z)}{\theta_4(q,0)}\right)^2\right). \tag{4.2.27}$$

Proof. The elliptic genus for such a σ-model must be a weak Jacobi form of weight 0 and index 1. The space of such forms is one dimensional [Gri00], and one may take as a generator the form

$$\phi_{0,1}(q,z)=\left(\frac{\theta_2(q,z)}{\theta_2(q,0)}\right)^2+\left(\frac{\theta_3(q,z)}{\theta_3(q,0)}\right)^2+\left(\frac{\theta_4(q,z)}{\theta_4(q,0)}\right)^2, \tag{4.2.28}$$

where $\theta_i(q,z)$ are the Jacobi theta functions.

By proposition 4.2.8, the elliptic genus evaluated at $z=0$ gives the Euler characteristic. We clearly have

$$\phi_{0,1}(q,0)=3, \tag{4.2.29}$$

and hence for a σ-model with target space M a CY 2-fold,

$$\varepsilon_M(q,z)=\frac{\chi(M)}{3}\phi_{0,1}(q,z). \tag{4.2.30}$$

Since the torus T^4 has $\chi(T^4) = 0$ and $K3$ has $\chi(K3) = 24$, we therefore obtain the result. $\square$

$K3$ is a hyper-Kähler manifold, and hence by section 2.2.2 a σ-model on $K3$ must have $\mathcal{N} = (4, 4)$ SUSY. One therefore expects to be able to express the partition function for the σ-model in terms of characters of the left and right $\mathcal{N} = 4$ SCA. The elliptic genus must then also be expressible in terms of $\mathcal{N} = 4$ characters, since by definition 4.2.7 the elliptic genus is just the partition function in a particular subsector, evaluated at $\bar{z} = 1$. Although there is no known complete classification of modular invariant partition functions for $K3$ theories, some modular invariant partition functions have been calculated at particular points of the moduli space of $K3$ theories, namely for Gepner models or $T^4/\mathbb{Z}_2$ orbifold theories [EOTY89]. Since the elliptic genus is a topological invariant, it can be calculated at any point on the moduli space, and thus, for instance, at a Gepner point. It is given in terms of $\mathcal{N} = 4$ characters as [EOTY89; EH09; Oog89],

$$\varepsilon_{K3}(q, z) = 20\,\mathrm{Ch}^{\tilde{R}}_{h=1/4,\ l=0}(q, z) - 2\,\mathrm{Ch}^{\tilde{R}}_{h=1/4,\ l=1/2}(q, z) + \sum_{n\geq 1} 2A_n\,\mathrm{Ch}^{\tilde{R}}_{h=n+1/4,\ l=1/2}(q, z), \tag{4.2.31}$$

where $\mathrm{Ch}^{\tilde{R}}$ denotes a trace taken in the Ramond sector with a $(-1)^F$ insertion, and h, l indicate the conformal charge and isospin respectively. As the elliptic genus computes the Witten index for the right movers, then massive representations of $\mathcal{N} = 4$ (which necessarily have Witten index 0) do not contribute, and hence the only contribution from the right-moving sector is the Witten index of the massless representations of $\mathcal{N} = 4$. The explicit values of A_n were calculated for small n in [Oog89; EH09]. The first eight are,

n	1	2	3	4	5	6	7	8
A_n	45	231	770	2277	5796	13915	30843	65550

(4.2.32)

and in [EOT11] it was noted that the first five coefficients are dimensions of irreducible representations of the sporadic group Mathieu 24, or M_{24}. Gannon [Gan16] proved that the coefficients A_n are indeed the dimensions of representations (reducible or not) of M_{24} for all n. This is reminiscent of 'Monstrous Moonshine' in which the coefficients of the modular function $j(\tau)$ are all expressible in terms of dimensions of representations of the Monster group, which is the largest sporadic group. Furthermore, the analogue of the McKay-Thompson series which appear in Monstrous Moonshine were computed for M_{24} in [EH11; GHV10a; GHV10b; Che10] and it was confirmed that these so-called twining genera have the required properties for a 'Mathieu Moonshine' to hold. This strongly suggests the existence of an analogue to the Monster module $\mathcal{V}^\natural$, but despite sustained efforts, this module has not been constructed so far. The fact that there is no $K3$ σ-model possessing full M_{24} symmetry [GHV11] is intriguing, and the nature of the M_{24} action on the K3 σ-model model is still poorly understood.

Having discussed supersymmetric indices in general, and as applied to theories with $\mathcal{N} = 4$ SCAs, in Chapter 5 we consider the indices that may be applied to A_γ theories and the states which contribute to these indices. In the following Chapter we take a representation theoretic approach to indices for A_γ following eq. (4.2.17). In [GMMS04; Sau05] an attempt is made to give a geometric interpretation of one of these indices.

Chapter 5

Indices for A_γ Theories

Having discussed the structure of representations of A_γ in Chapter 3, and having introduced the idea of indices in Chapter 4, we can now discuss supersymmetric indices that may be applied to theories with A_γ symmetry. Although the Mathieu moonshine phenomenon seems rooted in the elliptic genus (section 4.2.3), there has been work which shows that generalisations of the elliptic genus may be instrumental in a deeper understanding on moonshine [KT17; Son17; Wen17]. As we will first show, the 'field-theoretic' elliptic genus that we have reviewed in the previous chapter is not an interesting index for such theories as it always vanishes. In 2004, Gukov, Martinec, Moore and Strominger [GMMS04] introduced an index I_1, which generalises the 'new index' of Cecotti, Fendley, Intriligator and Vafa [CFIV92] with a view to probe symmetric product theories with A_γ symmetry. In this chapter we introduce the index I_1 and study in detail which states it counts when applied to the partition function of an A_γ theory. In particular, we will show how one may calculate the contribution to the index from the representation of $\tilde{A}_\gamma$ on which a representation of A_γ is built. We will also consider which representations of the zero mode subalgebra of A_γ in the Ramond sector contribute to the index I_1. Realising that this zero mode subalgebra is described by $\mathfrak{su}(2|2)$ we will use the technique of Young supertableaux to consider the index I_1 applied to representations of the zero mode subalgebra. The work contained in this chapter is currently being prepared for publication [Fea18].

5.1 The Elliptic Genus for A_γ Theories

In section 2.2 we introduced the 'large' $\mathcal{N} = 4$ algebra A_γ and we then studied its representation theory in Chapter 3. In particular in section 3.4 we defined the character of an A_γ module $V(c, h, l^+, l^-)$ as,

$$\begin{aligned} \mathrm{Ch}^{A_\gamma,S}(q, z_+, z_-) &\equiv \chi^{A_\gamma,S}(k^+, k^-, h, l^+, l^-; q, z_+, z_-), \\ &:= \mathrm{Tr}_{V(c,h,l^+,l^-)}(q^{L_0 - c/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}), \end{aligned} \tag{5.1.1}$$

where $q = e^{2\pi i\tau}$, $z_\pm = e^{2\pi i\omega_\pm}$, $\tau, \omega_\pm \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$. Due to the two $\widehat{\mathfrak{su}(2)}$ subalgebras of A_γ under which states are charged there is some ambiguity in how one should define the elliptic genus for a theory with A_γ symmetry. For the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases we defined the elliptic genus to be given by the partition function in the $\tilde{R}$ sector, specialising the right-moving superconformal characters to $\bar{z} = 1$ (or more precisely to $\bar{\omega} = 0$ where $\bar{z} = e^{2\pi i\bar{\omega}}$). This means that the resulting quantity loses track of the right-moving $U(1)$ charge of all right-moving states. An important feature about the elliptic genus is that it counts only short (massless, BPS) right-moving multiplets. That is, the elliptic genus of a bilinear in $\mathcal{N} = 4$ characters gives the left-moving module multiplied by the Witten index of the right-moving module. This means that the elliptic genus only counts supersymmetric (or BPS) states and hence as the index of the supercharge, is constant through smooth deformations of the moduli of the theory.

Given the recent interest in Mathieu moonshine, it is tempting to explore whether theories with A_γ symmetry could hide a new moonshine phenomenon in one of their genera. In this case, we have two affine $\widehat{\mathfrak{su}(2)}$ Kac-Moody algebras and hence two $\mathfrak{u}(1)$ charges in both the left and right-moving sectors of the theory. There are therefore more choices of right-moving $\mathfrak{u}(1)$ charges to 'lose track of'. For instance, one might define an index by setting $\bar{z}_+ = 1$ whilst keeping the angular variable $\bar{z}_-$, or by setting some linear combination of $\bar{z}_+$ and $\bar{z}_-$ to 1 instead. In each case, the right-moving sector retains a dependence in one angular variable, which signals a departure from the 'small' $\mathcal{N}=4$ situation. In analogy with the latter, it is tempting to define an elliptic genus for A_γ through a natural extension of the elliptic genus of $\mathcal{N}=4$ theories.

Definition 5.1.1. The elliptic genus of a theory with A_γ symmetry for both left-movers and right-movers is given by

$$\begin{aligned}\varepsilon(q,z_+,z_-) &:= \mathrm{Tr}_{\mathscr{H}^R}(-1)^{F_L+F_R} q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}}, \\ &= Z_{\mathscr{H}^{\tilde{R}}}(q,z_+,z_-,\bar{q},\bar{z}_+=\bar{z}_-=1),\end{aligned} \tag{5.1.2}$$

where $(-1)^{F_L} := e^{2\pi i T_0^{-3}}$ and $(-1)^{F_R} := e^{2\pi i \bar{T}_0^{-3}}$. Here and in the following we take $z_\pm = e^{2\pi i\omega_\pm}$ and $\bar{z}_\pm = e^{2\pi i\bar{\omega}_\pm}$ for $\omega_\pm,\ \bar{\omega_\pm} \in \mathbb{C}$. Any state which is not annihilated by $\bar{G}_0^a$ for $a \in \{\pm, \pm K\}$ has a partnered state with the same conformal weight but opposite sign under $(-1)^F = (-1)^{F_L+F_R}$. The same is true for any state not annihilated by $\bar{Q}_0^a$ for $a \in \{\pm,\pm K\}$. This index therefore counts only states annihilated by $\bar{Q}_0^a\bar{G}_0^a$ for $a \in \{\pm,\pm K\}$.

We can also form an index which counts only states annihilated by $\bar{Q}_0^a\bar{G}_0^a$ for $a \in \{\pm K\}$ by taking the trace,

$$\begin{aligned}\phi(q,z_+,z_-) &:= \mathrm{Tr}_{\mathscr{H}^R}(-1)^F q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \bar{z}^{2(\bar{T}_0^{+3}+\bar{T}_0^{-3})}, \\ &= Z_{\mathscr{H}^{\tilde{R}}}(q,z_+,z_-,\bar{q},\bar{z}_+=\bar{z}_-=\bar{z}).\end{aligned} \tag{5.1.3}$$

Similarly, we can form an index which counts only states annihilated by $\bar{Q}_0^a\bar{G}_0^a$ for $a \in \{\pm\}$ by instead taking the partition function in the $\tilde{R}$ sector and setting $\bar{z}_+ = \bar{z}_-^{-1}$.

However, none of the elliptic genus or the other two indices mentioned above are useful invariants of A_γ theories due to the following proposition.

Proposition 5.1.2. *Both the massless and massive Ramond characters of A_γ have a zero at $z_+ = -z_-$.*

Proof. By eq. (3.4.8), the characters of A_γ factor into those of $\tilde{A}_\gamma$ and those of A_{QU}.

Recall that the characters of A_{QU} in the Ramond sector are given by eq. (3.4.10),

$$\mathrm{Ch}^{A_{QU},R}(u;q,z_+,z_-) = q^{u^2/k+1/8}F^R(q,z_+,z_-)\prod_{n=1}^{\infty}(1-q^n)^{-1}(1+z_+^{-1}z_-^{-1})(1+z_+^{-1}z_-)z_+, \tag{5.1.4}$$

where the contribution from the zero modes is

$$(1+z_+^{-1}z_-^{-1})(1+z_+^{-1}z_-)z_+. \tag{5.1.5}$$

which clearly has a zero at $z_+ = -z_-$. Therefore, both the massless and massive Ramond characters of A_γ have a zero at $z_+ = -z_-$ due to this zero mode contribution.

□

Corollary 5.1.3. *The elliptic genus of a theory with A_γ symmetry is identically 0.*

Proof. As previously stated, the partition function for an A_γ theory can be expressed in bilinears of characters of A_γ. To calculate the elliptic genus for the theory by definition 5.1.1, one should evaluate the right-moving characters of A_γ in the $\tilde{R}$ sector at $\bar{z}_+ = \bar{z}_- = \bar{z}$. Given that $(-1)^{F_R}$ was defined as $(-1)^{F_R} := e^{2\pi i \bar{T}_0^{-3}}$ in definition 5.1.1, we can flow from the Ramond sector to the $\tilde{R}$ sector by letting $\bar{\omega}_- \to \bar{\omega}_- + \frac{1}{2}$ such that $\bar{z}_-^{2\bar{T}_0^{-3}} \to (-\bar{z}_-)^{2\bar{T}_0^{-3}}$. The elliptic genus is therefore given by evaluating the right-moving Ramond characters at $\bar{z}_+ = -\bar{z}_-$. By proposition 5.1.2 all such characters are 0 and hence the elliptic genus is zero. □

Note that the above proof also shows that the other two indices mentioned above are identically zero since the proof of the above corollary relied only on $\bar{z}_+ = -\bar{z}_-$, not the explicit value of $\bar{z}_\pm$. In the case where one evaluates at $\bar{z}_+ = -\bar{z}_-^{-1}$ one uses the fact that $\bar{T}_0^{-3} \to -\bar{T}_0^{-3}$ gives an isomorphic algebra (where the roles of the various raising and lowering operators are switched).

5.2 A New Index for A_γ Theories

If one considers the characters of $\tilde{A}_\gamma$ in the Ramond sector as given in eqs. (3.4.14) and (3.4.16), then one can see that the massive $\tilde{A}_\gamma$ characters also have a zero at

$z_+ = z_-$ whereas the massless characters do not have a zero at this point. Following [GMMS04], one can therefore define an index for A_γ which we refer to as I_1.

Definition 5.2.1. Since the massless Ramond characters of A_γ have only an order one zero at $z_+ = -z_-$ one can form a non-zero index by taking a derivative. Given a theory $\mathcal{D}$, with partition function $Z^{\mathcal{D}}$, we therefore define the left-index I_1 as

$$\begin{aligned} I_1(\mathcal{D})(q, z_+, z_-, \bar{q}, \bar{z}) &:= -\bar{z}_+ \frac{\partial}{\partial \bar{z}_-} Z^{\mathcal{D}}_{\mathscr{H}^{\tilde{R}}}(q, z_+, z_-, \bar{q}, \bar{z}_+, \bar{z}_-)\Big|_{\bar{z}_+ = \bar{z}_- = \bar{z}}, \\ &= \mathrm{Tr}_{\mathscr{H}^R}\left(-F_R(-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} z_+^{2T_0^{+3}} z_-^{2T_0^{-3}} \bar{z}^{2(\bar{T}_0^{+3} + \bar{T}_0^{-3})}\right), \end{aligned} \tag{5.2.1}$$

where as before $(-1)^{F_R} := e^{2\pi i \bar{T}_0^{-3}}$, and $Z^{\mathcal{D}}_{\mathscr{H}^{\tilde{R}}}$ denotes the restriction of the partition function to the $\tilde{R}$ sector.

The index I_1 is constructed so that only massless representations of A_γ can contribute on the right, and we now consider the contribution of a massless representation of A_γ to the index. This contribution is elegantly expressed in terms of level-k theta functions which we first define.

Definition 5.2.2. The level-k theta functions $\Theta_{\mu,k}(\tau, \omega)$ for $k \in \mathbb{Z}$, $\mu \in \mathbb{Z}_{2k}$ are defined in terms of $z = e^{2\pi i \omega}, q = e^{2\pi i \tau}$ by,

$$\Theta_{\mu,k}(\tau, \omega) = \sum_{\substack{\ell \in \mathbb{Z}, \\ \ell = \mu \bmod 2k}} q^{\frac{\ell^2}{4k}} z^{\ell} = q^{\frac{\mu^2}{4k}} z^{\mu} \sum_{n \in \mathbb{Z}} q^{kn^2 + n\mu} z^{2kn}, \tag{5.2.2}$$

These theta functions satisfy

$$\Theta_{\mu,k}(\tau, -\omega) = \Theta_{-\mu,k}(\tau, \omega). \tag{5.2.3}$$

We therefore define the even and odd theta functions,

Definition 5.2.3. The even and and odd level-k theta functions are given by

$$\begin{aligned} \Theta^{\pm}_{\mu,k}(\tau, \omega) &:= \Theta_{\mu,k}(\tau, \omega) \pm \Theta_{-\mu,k}(\tau, \omega), \\ &= q^{\frac{\mu^2}{4k}} \sum_{n \in \mathbb{Z}} q^{kn^2 + n\mu}(z^{2kn+\mu} \pm z^{-2kn-\mu}), \end{aligned} \tag{5.2.4}$$

where we take the + sign for the even theta functions and the − sign for the odd theta functions.

The only non-zero contribution to I_1 from massless representations of A_γ comes from taking the derivative of the zero mode term of A_{QU}. Using the odd level-k theta functions one then obtains the following contribution to the index from a massless representation of A_γ [GMMS04].

Proposition 5.2.4. *The contribution to the index I_1 from a massless representation of A_γ is described by,*

$$-z_+ \frac{d}{dz_-} \mathrm{Ch}_0^{A_\gamma(l^+,l^-),\tilde{R}} \Big|_{z_+=z_-} = (-1)^{2l^+ +1} q^{\frac{u^2}{k}} \Theta^-_{\mu,k}(\omega,\tau), \tag{5.2.5}$$

where $k = k^+ + k^-$ is the sum of the levels of the affine $\widehat{\mathfrak{su}(2)}$'s, $\mu = 2(l^+ + l^-) - 1$ and $z = e^{2\pi i \omega}$.

Note that in a massless representation of $\tilde{A}_\gamma$, the hws is annihilated by $\tilde{G}_0^a$ for $a \in \{+, \pm K\}$. The ground level of an $\tilde{A}_\gamma$ representation therefore has a bosonic $\widehat{\mathfrak{su}(2)}^+ \times \widehat{\mathfrak{su}(2)}^-$ multiplet containing $(2[l^+ - \frac{1}{2}] + 1)(2[l^- - \frac{1}{2}] + 1)$ states, and one fermionic $\widehat{\mathfrak{su}(2)}^+ \times \widehat{\mathfrak{su}(2)}^-$ multiplet containing $(2[l^+ - 1] + 1)(2[l^- - 1] + 1)$ states. The quantity $\mu = 2(l^+ + l^-) - 1$ is therefore the Witten index of the representation of $\tilde{A}_\gamma$.

Short representations of A_γ are known to combine into long threshold ones as encoded in the character formula

$$\mathrm{Ch}_0^{A_\gamma,R}(l^+,l^-) + \mathrm{Ch}_0^{A_\gamma,R}(l^+ - \frac{1}{2}, l^- + \frac{1}{2}) = \hat{\mathrm{Ch}}_m(h, l^+, l^- + \frac{1}{2}), \tag{5.2.6}$$

where $\mathrm{Ch}_0, \hat{\mathrm{Ch}}_m$ denote massless and massive threshold characters of A_γ respectively. Using proposition 5.2.4 we can now verify that I_1 is invariant under BPS representations joining to become non-BPS representations, since

$$\begin{aligned}
&I_1(\mathrm{Ch}_0^{A_\gamma,R}(l^+,l^-) + \mathrm{Ch}_0^{A_\gamma,R}(l^+ - \frac{1}{2}, l^- + \frac{1}{2})),\\
&\qquad = I_1(\mathrm{Ch}_0^{A_\gamma,R}(l^+,l^-)) + I_1(\mathrm{Ch}_0^{A_\gamma,R}(l^+ - \frac{1}{2}, l^- + \frac{1}{2}))),\\
&\qquad = (-1)^{2l^-+1} q^{u^2/k}\Theta^-_{2(l^++l^-)-1,k}(\tau,\omega) + (-1)^{2l^-} q^{u^2/k}\Theta^-_{2(l^++l^-)-1,k}(\tau,\omega),\\
&\qquad = 0.
\end{aligned} \tag{5.2.7}$$

Furthermore, modular invariance under T-transformations requires that the bilinears appearing in the partition function satisfy $h - \bar{h} \in \mathbb{Z}$. Since the index I_1 only counts massless representations of A_γ, whose conformal weights are discrete, then under any smooth deformation of the parameters of the theory, the conformal weights of the massive representations appearing in the index must also be fixed. The index I_1 is therefore invariant under any smooth deformation of the parameters and hence truly is an index for A_γ theories, as already argued in [GMMS04].

A simple but important observation is that the contribution to the index I_1 of a right-moving massless representation of A_γ as given in proposition 5.2.4 is a $\bar{q}$-series. This index therefore counts more than just the ground states, it also receives contributions from excited states with $\bar{L}_0$ charge given by $kn^2 + n\mu$ for $n \in \mathbb{Z}$. This is qualitatively different from the elliptic genus discussed in section 4.2.2, where the right-moving massless representations contribute to the index by way of their Witten index which is simply an integer. In the next sections we therefore consider the nature of the states which contribute to I_1.

5.3 Right-Moving States Contributing to I_1

We now briefly discuss the states which contribute to the index I_1. We return to this question in section 5.4, where we consider the question from the point of view of the $\mathfrak{su}(2|2)$ representations that A_γ branches into.

Recall that the index I_1 counts only right-moving massless Ramond representations of A_γ and specifically that the index applied to such representations gives an odd level-k theta function as in proposition 5.2.4. Unlike the elliptic genus for $\mathcal{N}=2$ or $\mathcal{N}=4$ theories, which counted right moving massless representations simply by their Witten index, the index I_1 is a function of $\bar{q}$, and hence receives contributions from throughout the massless representation. We can understand the nature of these states by considering their charges. By definition 5.2.1, the power of $\bar{z}$ in eq. (5.2.5) is the charge of the state under $2(\bar{T}_0^{+3}+\bar{T}_0^{-3})$. Equation (5.2.5) then tells us that the states counted by I_1 have

$$2(\bar{T}_0^{+3}+\bar{T}_0^{-3}) = \pm\bar{\mu}\ (\mathrm{mod} 2k), \tag{5.3.1}$$

where $\bar{\mu}=2(\bar{l}^+=\bar{l}^-)-1$, the Witten index of the underlying right-moving representation of $\tilde{A}_\gamma$. Similarly, the power of $\bar{q}$ in eq. (5.2.5) tells us the charge of the states under $\bar{L}_0-\frac{\bar{c}}{24}$. We therefore have

$$\bar{L}_0-\frac{\bar{c}}{24}=\frac{u^2}{k}+\frac{1}{k}\left(\bar{T}_0^{+3}+\bar{T}_0^{-3}\right)^2. \tag{5.3.2}$$

When applied to the hws, we recognise eq. (5.3.2) as the condition for the representation to be massless. The states counted by I_1 therefore satisfy the massless eq. (3.3.20) in terms of their own charges. These 'massless' states behave like massless ground states by the following proposition.

Proposition 5.3.1. *All states contributing to the index I_1 are annihilated by $\bar{Q}_0^{-K}\bar{G}_0^{-K}$.*

Proof. This follows easily by contradiction. Assume there exists some state $|\chi\rangle\equiv|\bar{h};\bar{l}^+,\bar{l}^-\rangle$ which contributes to the index I_1 (that is, it does not cancel in the index) which is not annihilated by $\bar{Q}_0^{-K}\bar{G}_0^{-K}$. Then we have four linearly independent states $|\chi\rangle\,,\ \bar{Q}_0^{-K}|\chi\rangle\,,\ \bar{G}_0^{-K}|\chi\rangle\,,\ \bar{Q}_0^{-K}\bar{G}_0^{-K}|\chi\rangle$. Their contribution to the character $\mathrm{Ch}^{A_\gamma,R}$ is therefore given by

$$\mathrm{Ch}^{A_\gamma,R}=\bar{q}^{\bar{h}}(\bar{z}_+^{2\bar{l}^+}\bar{z}_-^{2\bar{l}^-}+2\bar{z}_+^{2(\bar{l}^+-1/2)}\bar{z}_-^{2(\bar{l}^-+1/2)}+\bar{z}_+^{2(\bar{l}^+-1)}\bar{z}_-^{2(\bar{l}^-+1)})+\ldots. \tag{5.3.3}$$

After flowing to the $\tilde{R}$ sector, and assuming without loss of generality that $\bar{l}^-\in\mathbb{Z}$, we therefore have,

$$\mathrm{Ch}^{A_\gamma,\tilde{R}} = \bar{q}^{\bar{h}}(\bar{z}_+^{2\bar{l}^+}\bar{z}_-^{2\bar{l}^-} - 2\bar{z}_+^{2(\bar{l}^+-1/2)}\bar{z}_-^{2(\bar{l}^-+1/2)} + \bar{z}_+^{2(\bar{l}^+-1)}\bar{z}_-^{2(\bar{l}^-+1)}) + \dots . \tag{5.3.4}$$

If we then take the index we have

$$-\bar{z}^+\frac{\partial}{\partial\bar{z}_-}\mathrm{Ch}^{A_\gamma,\tilde{R}}|_{\bar{z}_+=\bar{z}_-=\bar{z}} = \bar{q}^{\bar{h}}(2\bar{l}^- - 4\bar{l}^- - 2 + 2\bar{l}^- + 2)\bar{z}^{2(\bar{l}^+ + \bar{l}^-)} = 0. \tag{5.3.5}$$

Hence $|\chi\rangle$ does not contribute to the index, contradicting our initial assumption.

The index I_1 therefore only counts states annihilated by $\bar{Q}_0^{-K}\bar{G}_0^{-K}$ (for the right movers). □

As noted by [Sau05], the conditions given in eqs. (5.3.1) and (5.3.2) are invariant under symmetric spectral flow.

Proposition 5.3.2. *The conditions given in eqs.* (5.3.1) *and* (5.3.2) *are invariant under 'symmetric' spectral flow as in eq.* (3.3.2) *with* $\rho = \eta = 2n$ *for* $n \in \mathbb{Z}$.

Proof. Under this isomorphism, we have

$$L_-^{2n,2n} = L_0 - 2n(T_0^{+3}+T_0^{-3}) + kn^2, \quad T_0^{+3;2n,2n} = T_0^{+3} - nk^+, \quad T_0^{-3;2n,2n} = T_0^{-3} - nk^-. \tag{5.3.6}$$

We therefore have,

$$\begin{aligned} 2(T_0^{+3;2n,2n} + T_0^{+3;2n,2n}) &= 2(T_0^{+3} + T_0^{+3}) - 2kn, \\ &= \pm\mu + 2k(m-n), \\ L_0^{2n,2n} - \frac{c}{24} &= L_0 - 2n(T_0^{+3} + T_0^{-3}) + kn^2 - \frac{c}{24}, \\ &= \frac{u^2}{k} + \frac{1}{k}(T_0^{+3} + T_0^{-3})^2 - 2n(T_0^{+3} + T_0^{-3}) + kn^2, \\ &= \frac{u^2}{k} + \frac{1}{k}(T_0^{+3;2n,2n} + T_0^{+3;2n,2n})^2, \end{aligned} \tag{5.3.7}$$

and so we see that eqs. (5.3.1) and (5.3.2) are satisfied after the spectral flow. □

We now realise that each state counted by I_1 can be thought of as the image under spectral flow (for some n) of the states counted at the ground level, namely

$|\Omega_+\rangle$, $G^{-K}|\Omega_+\rangle \equiv |\Omega_-\rangle$, $(T_0^{+-})^{2l_+^+}(T_0^{--})^{2l_+^-}|\Omega_+\rangle$, $(T_0^{+-})^{2l_-^+}(T_0^{--})^{2l_-^-}G^{-K}|\Omega_+\rangle \equiv (T_0^{+-})^{2l_-^+}(T_0^{--})^{2l_-^-}|\Omega_-\rangle$, where l_+^+ and l_-^- are defined as in eq. (3.3.26), as per the following proposition.

Proposition 5.3.3. *The states counted by the index I_1 are the spectral flow orbits of $|\Omega_+\rangle$, $|\Omega_-\rangle$, $(T_0^{+-})^{2l_+^+}(T_0^{--})^{2l_+^-}|\Omega_+\rangle$ and $(T_0^{+-})^{2l_-^+}(T_0^{--})^{2l_-^-}|\Omega_-\rangle$ under spectral flow as in eq.* (5.3.6).

Proof. Firstly, as noted above, only states satisfying eqs. (5.3.1) and (5.3.2) can contribute to the index I_1. We now show that all such states lie in spectral flow orbits of the four states mentioned above. Let us call the states $(T_0^{+-})^{2l_+^+}(T_0^{--})^{2l_+^-}|\Omega_+\rangle$ and $(T_0^{+-})^{2l_-^+}(T_0^{--})^{2l_-^-}|\Omega_-\rangle$, $|-\Omega_+\rangle$ and $|-\Omega_-\rangle$ respectively, since the charges of the states are the negatives of $|\Omega_+\rangle$ and $|\Omega_-\rangle$ respectively.

Firstly, we note that the above four states themselves satisfy eqs. (5.3.1) and (5.3.2). In a massless Ramond representation, $|\Omega_+\rangle$ is the state with charges $|h, l^+, l^- - 1/2\rangle$. We therefore have

$$\begin{aligned} 2(T_0^{+3}+T_0^{-3})|\Omega_+\rangle &= \mu, & 2(T_0^{+3}+T_0^{-3})|\Omega_-\rangle &= \mu, \\ 2(T_0^{+3}+T_0^{-3})|-\Omega_+\rangle &= -\mu, & 2(T_0^{+3}+T_0^{-3})|-\Omega_-\rangle &= -\mu, \end{aligned} \tag{5.3.8}$$

and hence these four states all satisfy eq. (5.3.1). That $|\Omega_+\rangle$ satisfies eq. (5.3.2) is clear, since $|\Omega_+\rangle$ was our hws. Since all the states at the ground level have the same conformal charge h, and all states in the module have charge $-iu$ under the $U(1)$, then the only change in contribution to eq. (5.3.2) between the states is their charges under $\widehat{\mathfrak{su}(2)}^\pm$. However, since eq. (5.3.2) depends only on the square of the sum of these charges, then clearly all four states have the same contribution and hence all four satisfy the massless bound.

Now assume there is some state $|\chi\rangle$ which satisfies eqs. (5.3.1) and (5.3.2) and is not a ground level state already considered. Firstly, we note that $|\chi\rangle$ cannot be another ground state, since it would have to satisfy

$$(T_0^{+3}+T_0^{-3})|\chi\rangle = \pm(T_0^{+3}+T_0^{-3})|\Omega_+\rangle\,. \tag{5.3.9}$$

As discussed in section 3.3, in a massless Ramond representation of A_γ, $Q_0^{-K}\left|\Omega_+\right\rangle$ and $G_0^{-K}\left|\Omega_+\right\rangle$ are linearly dependent and $Q_0^{-K}G_0^{-K}$ annihilates $\left|\Omega_+\right\rangle$. There are therefore only two states $\left|\rho\right\rangle$ which satisfy,

$$(T_0^{+3}+T_0^{-3})\left|\rho\right\rangle=(T_0^{+3}+T_0^{-3})\left|\Omega_+\right\rangle, \tag{5.3.10}$$

namely $\left|\Omega_\pm\right\rangle$. Similarly there are only two states which satisfy

$$(T_0^{+3}+T_0^{-3})\left|\rho\right\rangle=-(T_0^{+3}+T_0^{-3})\left|\Omega_+\right\rangle, \tag{5.3.11}$$

namely $\left|-\Omega_\pm\right\rangle$. The state $\left|\chi\right\rangle$ can therefore not be a ground state unless it is one of the states already considered.

Since $\left|\chi\right\rangle$ satisfies eq. (5.3.1), then without loss of generality let us assume it satisfies

$$2(T_0^{+3}+T_0^{-3})\left|\chi\right\rangle=\mu+2mk, \tag{5.3.12}$$

for some particular $m\in\mathbb{Z}$. By eq. (5.3.7) with $n=m$, $\left|\chi\right\rangle$ is a state with $2(T_0^{+3;2m,2m}+T_0^{-3;2m,2m})\left|\chi\right\rangle=\mu$. But by proposition 5.3.2, eqs. (5.3.1) and (5.3.2) are preserved under spectral flow. Since the only states at the ground level satisfying eqs. (5.3.1) and (5.3.2) are $\left|\Omega_\pm\right\rangle$, $\left|-\Omega_\pm\right\rangle$, then $\left|\chi\right\rangle$ is in the spectral flow orbit of one of the states $\left|\Omega_\pm\right\rangle$, $\left|-\Omega_\pm\right\rangle$. □

We consider a brief example which hopefully should make this clear.

Example 5.3.4. The massless representation of A_γ with $k^+=3,\ k^-=2,\ l^+=\frac{1}{2},\ l^-=\frac{1}{2}$ is particularly simple, since for these values of $l^\pm$, the underlying representation of $\tilde{A}_\gamma$ has a singular ground state. The ground level of this representation is shown in fig. 5.1.

Clearly these states all have diagonal $\widehat{\mathfrak{su}(2)}$ charge equal to $\mu=1$, that is we have

$$\begin{aligned}2(T_0^{+3}+T_0^{-3})\left|\pm\Omega_+\right\rangle&=\pm1\left|\pm\Omega_+\right\rangle\pmod{k},\\2(T_0^{+3}+T_0^{-3})\left|\pm\Omega_-\right\rangle&=\pm1\left|\pm\Omega_-\right\rangle\pmod{k}.\end{aligned} \tag{5.3.13}$$

If we let the conformal charges of these states be given by h, then the next two contributions to the index I_1 come from states at level $h+4$ and $h+6$, as can easily

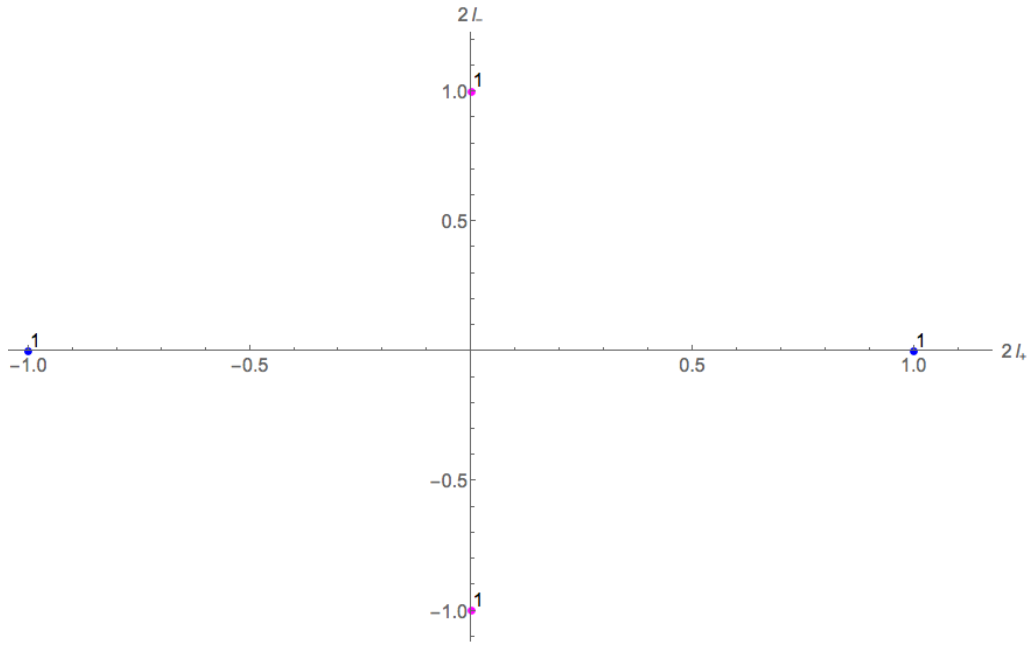


Figure 5.1: The ground level of a massless Ramond representation of A_γ with $k^+ = 3, k^- = 2, l^+ = \frac{1}{2}, l^- = \frac{1}{2}$

be calculated from eq. (5.2.5). The levels $h+4$ and $h+6$ of this representation of A_γ are shown in fig. 5.2. If we focus on the states with both $\widehat{\mathfrak{su}(2)}^\pm$ charges positive, then the states contributing to the index at level $h+4$ have

$$2(T_0^{+3} + T_0^{-3})\,|\chi_{h+4}\rangle = 9\,|\chi_{h+4}\rangle = (-\mu + 2k)\,|\chi_{h+4}\rangle\,. \tag{5.3.14}$$

Similarly those at level $h+6$ have

$$2(T_0^{+3} + T_0^{-3})\,|\chi_{h+6}\rangle = 11\,|\chi_{h+6}\rangle = (\mu + 2k)\,|\chi_{h+6}\rangle\,. \tag{5.3.15}$$

We therefore see that the states $|\chi_{h+4}\rangle$ (where we mean the pair of states of maximal diagonal charge at level $h+4$) are in the orbit of $|-\Omega_\pm\rangle$, since by eq. (5.3.6),

$$\begin{aligned} &L_0^{-2,-2}\,|-\Omega_+\rangle = (h-1+5)\,|-\Omega_+\rangle\,, \\ &T^{+3;-2,-2}\,|-\Omega_+\rangle = (-\frac{1}{2}+3)\,|-\Omega_+\rangle\,, \qquad T^{-3;-2,-2}\,|-\Omega_+\rangle = (0+2)\,|-\Omega_+\rangle\,, \end{aligned} \tag{5.3.16}$$

which are the correct charges for the bosonic state at level $h+4$. Note that since the $\widehat{\mathfrak{su}(2)}^\pm$ charges change by $k^\pm \in \mathbb{Z}$ under the symmetric spectral flow, then a bosonic state always flows to another bosonic state and a fermionic state flows to a fermionic state. We can similarly check that $|{-\Omega_-}\rangle$ flows to the maximal-diagonally charged fermionic state at level $h+4$.

Similarly, the states $|\chi_{h+6}\rangle$ are in the orbit of $|\Omega_\pm\rangle$;

$$\begin{aligned} L_0^{2,2}\,|\Omega_+\rangle &= (h+1+5)\,|\Omega_+\rangle\,, \\ T^{+3;2,2}\,|\Omega_+\rangle &= (\frac{1}{2}+3)\,|\Omega_+\rangle\,, \quad T^{-3;2,2}\,|\Omega_+\rangle = (0+2)\,|\Omega_+\rangle\,, \end{aligned} \tag{5.3.17}$$

and similarly for $|\Omega_-\rangle$. △

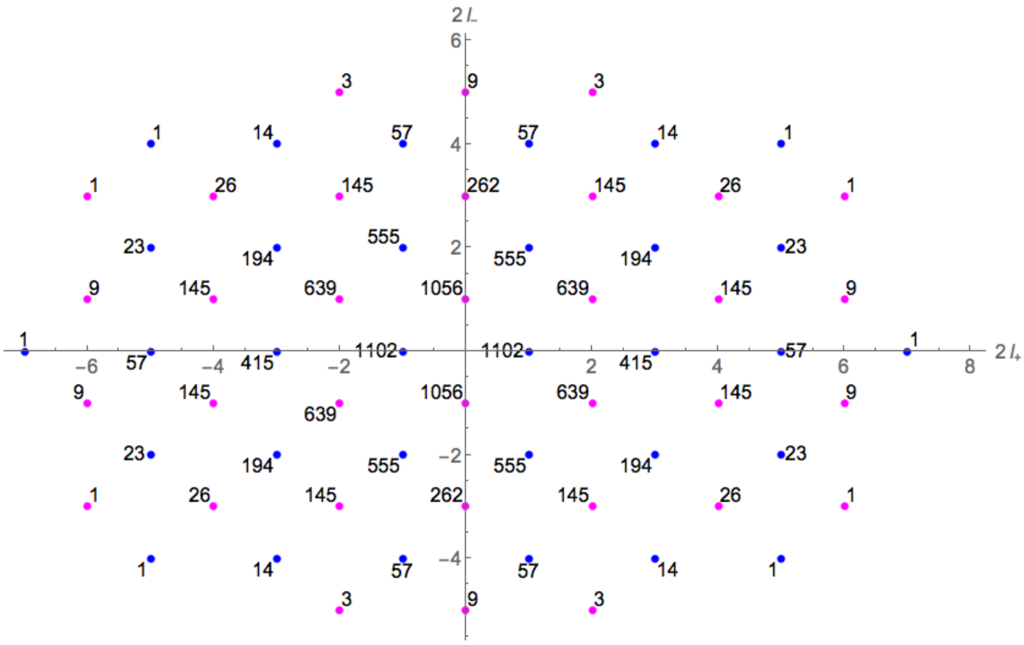
As described in Chapter 3, given a representation of A_γ one can always consider the underlying representation of $\tilde{A}_\gamma$ formed by decoupling the algebra A_{QU} of the four free fermions and the free boson from A_γ. We can therefore consider how the states which contribute to the index I_1 break into a contribution from $\tilde{A}_\gamma$ and a contribution from A_{QU} due to the factorisation of characters described in eq. (3.4.8). This leads to the following proposition.

Proposition 5.3.5. *By proposition 5.2.4 we know that the states which contribute to the index I_1 are described by the theta function $\Theta^-_{\mu,k}(q,z)$. The contributions to this index from A_{QU} and $\tilde{A}_\gamma$ are described by*

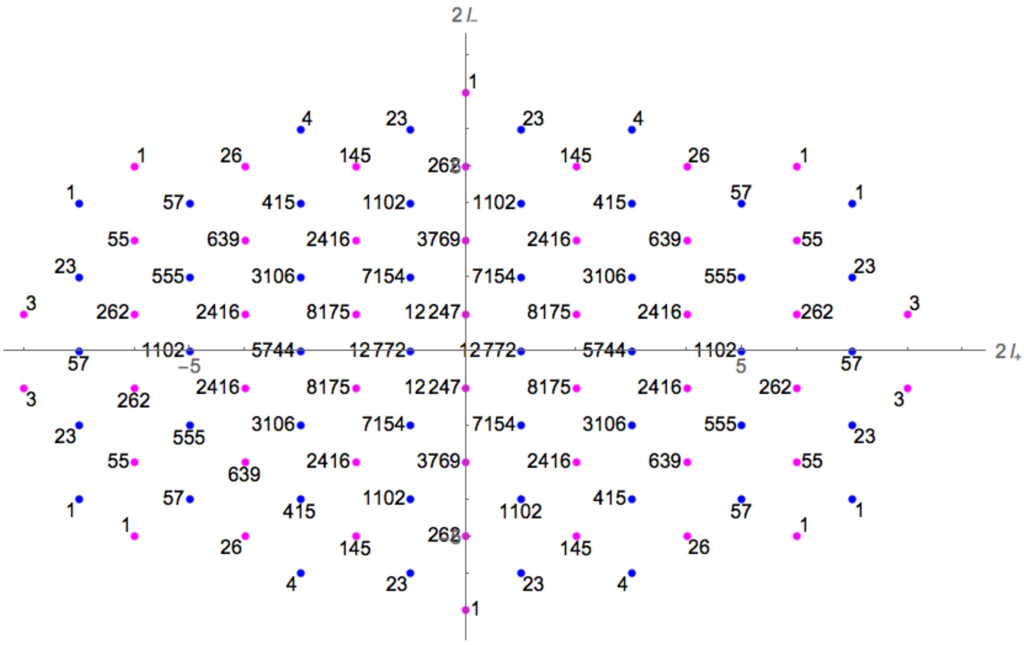
$$\begin{aligned} \Theta^-_{\mu,k}(q,z) :=& q^{\mu^2/4k}\sum_{n\in\mathbb{Z}} q^{kn^2+n\mu}(z^{2kn+\mu}-z^{-2kn-\mu}), \\ =& \underbrace{q^{1/8}}_{A_{QU}}\overbrace{q^{\tilde{h}-\tilde{c}/24}}^{\tilde{A}_\gamma}\sum_{n\in\mathbb{Z}}\underbrace{q^{n(2n+1)}}_{A_{QU}}\overbrace{q^{(\tilde{k}^++\tilde{k}^-)n^2+2n(\tilde{l}^++\tilde{l}^-)}}^{\tilde{A}_\gamma} \\ &\times\left(\underbrace{z^{4n+1}}_{A_{QU}}\overbrace{z^{2n(\tilde{k}^++\tilde{k}^-)+2(\tilde{l}^++\tilde{l}^-)}}^{\tilde{A}_\gamma}-\underbrace{z^{-4n-1}}_{A_{QU}}\overbrace{z^{-2n(\tilde{k}^++\tilde{k}^-)-2(\tilde{l}^++\tilde{l}^-)}}^{\tilde{A}_\gamma}\right), \end{aligned} \tag{5.3.18}$$

where as in Chapter 3, $\sim$'s refer to objects in $\tilde{A}_\gamma$. Since the representation of $\tilde{A}_\gamma$ is massless, $\tilde{h}$ could be written in terms of the charges $l^\pm$.

Proof. This follows since A_{QU} is itself a representation of A_γ with $k^\pm = 1$, $l^\pm = \frac{1}{2}$ [GS88; PT90a]. Since a representation A_{QU} is a representation of A_γ we can therefore consider whether the states in A_{QU} fall into 'massive' multiplets of the zero



(a) Level $h + 4$



(b) Level $h + 6$

Figure 5.2: The states contributing to the index I_1 at levels $h + 4$ and $h + 6$ from a massless Ramond representation of A_γ with $k^+ = 3, k^- = 2, l^+ = \frac{1}{2}, l^- = \frac{1}{2}$.

mode algebra as described in proposition 5.3.1 or 'massless' multiplets. If we consider a 'massive' multiplet of A_{QU} then the multiplet of A_γ formed by multiplying this multiplet against a fixed state of $\tilde{A}_\gamma$ must also cancel from the index I_1 as argued in proposition 5.3.1. We therefore realise that the only contributions to I_1 must be described by 'massless' multiplets of A_{QU}. But we already know how to count all such 'massless' multiplets which contribute to the index, by proposition 5.2.4 they are given by odd level-k theta functions. We therefore find that the index of A_{QU} is given by,

$$-z_+\frac{d}{dz_-}\mathrm{Ch}^{A_{QU},\tilde{R}}\bigg|_{z_+=z_-} = q^{\frac{u^2}{k}}\Theta^-_{1,2}(\omega,\tau) = q^{\frac{u^2}{k}+\frac{1}{8}}\sum_{n\in\mathbb{Z}} q^{n(2n+1)}\left(z^{4n+1}-z^{-4n-1}\right). \tag{5.3.19}$$

Now at any fixed power of q, we know by eq. (5.3.1) that the power of z of any state in A_γ contributing to the index must be $\pm\mu + 2kn$. Considering the positively charged states, we therefore require that the contribution to the power of z from $\tilde{A}_\gamma$ plus the power of z identified as coming from A_{QU} above satisfies

$$\begin{aligned} 4n+1+m &= \mu + 2kn, \\ m &= 2(l^+ + l^- - 1) + 2n(k-2), \\ &= 2(\tilde{l}^+ + \tilde{l}^-) + 2n(\tilde{k}^+ + \tilde{k}^-), \end{aligned} \tag{5.3.20}$$

where m is the power of z coming from $\tilde{A}_\gamma$. Similarly, by proposition 5.2.4 we know that for fixed n the power of q must be given by $\frac{\mu^2}{4k} + kn^2 + n\mu$. For fixed n, having identified the contribution to the power of q coming from A_{QU} we can therefore calculate the contribution to the power of q coming from $\tilde{A}_\gamma$. We therefore find

$$\begin{gathered} \frac{u^2}{k} + \frac{1}{8} + n(2n+1) + p = \frac{u^2}{k} + \frac{\mu^2}{4k} + kn^2 + n\mu, \\ p = (\tilde{k}^+ + \tilde{k}^-)n^2 + 2n(\tilde{l}^+ + \tilde{l}^-) + \tilde{h} - \frac{\tilde{c}}{24}, \end{gathered} \tag{5.3.21}$$

where p is the power of q coming from $\tilde{A}_\gamma$. Here we have used the expression for $\tilde{h}$

and $\tilde{c}$ in terms of the representation data $k^{\pm},\ l^{\pm}$ as [PT90b],

$$\begin{aligned} \tilde{h} &= \frac{\tilde{k}^{+}\tilde{k}^{-}}{4k} + (\tilde{l}^{+} + \tilde{l}^{-})(\tilde{l}^{+} + \tilde{l}^{-} + 1), \\ &= \frac{k^{+}k^{-} - k + \mu^{2}}{4k}, \\ \frac{\tilde{c}}{24} &= \frac{k^{+}k^{-}}{k} - \frac{1}{8}. \end{aligned} \tag{5.3.22}$$

We therefore find that the theta function which gives the contribution to the index may be written in terms of the states coming from A_{QU} and those coming from $\tilde{A}_{\gamma}$ as claimed. □

5.4 A Description of I_1 Using Supertableaux

In this section, we investigate the contributions to the index I_1 [GMMS04] considered in section 5.2, in terms of representations of the zero mode algebra of A_γ. We first show that the zero mode subalgebra of A_γ in the Ramond sector is described by the Lie superalgebra $\mathfrak{su}(2|2)$. Note that in the case of the NS sector, the finite (super) subalgebra now contains the $\frac{1}{2}$ and $-\frac{1}{2}$ modes of the odd elements and in fact is described by the sum of the finite superalgebra $D(2|1;\alpha)$ and a $\mathfrak{u}(1)$ [STV88], where $\alpha = \frac{\gamma}{1-\gamma}$. Since the index I_1 is calculated in the $\tilde{R}$ sector, we will have no use for this in the following. We then discuss the representation theory of $\mathfrak{su}(2|2)$ and describe the classification of representations of $\mathfrak{su}(2|2)$ by Young supertableaux. We then discuss the branching of $\mathfrak{su}(2|2)$ into its bosonic subalgebra, $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ and use this branching to investigate the index I_1 of A_γ.

5.4.1 From the Lie Supergroup $SU(M|N)$ to the Lie Superalgebra $\mathfrak{su}(M|N)$

We avoid going into detail about the general structure of Lie supergroups and their associated algebras, referring the interested reader to [Cor89]. In this section we show how to obtain first the real 'super' Lie algebra associated to the supergroup

$SU(M|N)$ and then the Lie superalgebra $\mathfrak{su}(M|N)$ from this real Lie algebra. This will be done more as an example, referencing the relevant theorems as we require them. For readers unfamiliar with superalgebra and supermatrices, some important definitions may be found in Appendix D.

Since we are interested specifically in $\mathfrak{su}(2|2)$ and hence $SU(2|2)$ we focus on this example. An element of the supergroup $SU(2|2)$ is an even supermatrix G satisfying

$$G^{\ddagger}G = \mathbb{I}_4, \qquad \mathrm{SDet}\, G = 1_{\mathbb{C}B_I}. \tag{5.4.1}$$

An element of the 'super' Lie algebra of $SU(2|2)$, g, is then defined by

$$g^{\ddagger} + g = 0_{M_{p|q}(\mathbb{C}B_I)}, \qquad \mathrm{STr}\, g = 0_{\mathbb{C}B_I}. \tag{5.4.2}$$

If the element g is written in terms of its submatrices as in definition D.2.1, then the first condition of eq. (5.4.2) becomes the conditions,

$$(A^{\#})^t + A = 0, \qquad (D^{\#})^t + D = 0, \qquad (B^{\#})^t + C = 0. \tag{5.4.3}$$

Similarly, the second condition of eq. (5.4.2) becomes the condition

$$\mathrm{Tr}\, A = \mathrm{Tr}\, D, \tag{5.4.4}$$

where we have used the fact that g is even – as explained in Appendix D.3 – to expand the supertrace.

The elements of A are elements of $\mathbb{C}B_{I,0}$, hence we can split A into its real and imaginary parts as

$$A = A_r + iA_i, \tag{5.4.5}$$

where now A_r and A_i are matrices whose matrix elements are elements of $\mathbb{R}B_{I,0}$. Using proposition D.1.11 we see that,

$$A^{\#} = (A_r + iA_i)^{\#} = A_r^{\#} + (iA_i)^{\#} = A_r - iA_i. \tag{5.4.6}$$

The condition $(A^{\#})^t + A = 0$ now becomes $A_r^t - iA_i^t + A_r + iA_i = 0$. We therefore have that A_r is antisymmetric and A_i is symmetric. Similarly we can split D into its real and imaginary parts D_r and D_i and find that D_r is antisymmetric and D_i is symmetric. Obviously $\mathrm{Tr}\, A_r = \mathrm{Tr}\, D_r = 0$, hence the trace condition (5.4.4) becomes

$$\mathrm{Tr}\, A_i = \mathrm{Tr}\, D_i. \tag{5.4.7}$$

Since B has its elements in $\mathbb{C}B_{I,1}$,

$$B^{\#} = B_r^{\#} + (iB_i)^{\#} = -iB_r - B_i, \tag{5.4.8}$$

so the condition $(B^{\#})^t + C = 0$ becomes $-B_i^t + C_r + i(C_i - B_r^t) = 0$, implying

$$B_r^t = C_i, \qquad B_i^t = C_r, \tag{5.4.9}$$

where we have written B and C in terms of real and imaginary parts as before.

We can now write a general element g of the 'super' Lie algebra as

$$g = \left[\begin{array}{cc|cc} iX^1 & X^2 + iX^3 & \Theta^1 + i\Theta^2 & \Theta^3 + i\Theta^4 \\ -X^2 + iX^3 & iX^4 & \Theta^5 + i\Theta^6 & \Theta^7 + i\Theta^8 \\ \hline \Theta^2 + i\Theta^1 & \Theta^6 + i\Theta^5 & iX^5 & X^6 + iX^7 \\ \Theta^4 + i\theta^3 & \Theta^8 + i\Theta^7 & -X^6 + iX^7 & iX^1 + iX^4 - iX^5 \end{array}\right]. \tag{5.4.10}$$

We therefore find the generators for the 'super' Lie algebra to be given by

$$M^1 = \left[\begin{array}{cc|cc} i\epsilon_\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\epsilon_\phi \end{array}\right], \quad M^2 = \left[\begin{array}{cc|cc} 0 & \epsilon_\phi & 0 & 0 \\ -\epsilon_\phi & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad M^3 = \left[\begin{array}{cc|cc} 0 & i\epsilon_\phi & 0 & 0 \\ i\epsilon_\phi & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \tag{5.4.11}$$

$$M^4=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & i\epsilon_\phi & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\epsilon_\phi \end{array}\right], \quad M^5=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & i\epsilon_\phi & 0 \\ 0 & 0 & 0 & -i\epsilon_\phi \end{array}\right], \quad M^6=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \epsilon_\phi \\ 0 & 0 & -\epsilon_\phi & 0 \end{array}\right],$$

$$M^7=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & i\epsilon_\phi \\ 0 & 0 & i\epsilon_\phi & 0 \end{array}\right], \quad N^1=\left[\begin{array}{cc|cc} 0 & 0 & \epsilon_\phi & 0 \\ 0 & 0 & 0 & 0 \\ \hline -i\epsilon_\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad N^2=\left[\begin{array}{cc|cc} 0 & 0 & i\epsilon_\phi & 0 \\ 0 & 0 & 0 & 0 \\ \hline -\epsilon_\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$N^3=\left[\begin{array}{cc|cc} 0 & 0 & 0 & \epsilon_\phi \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -i\epsilon_\phi & 0 & 0 & 0 \end{array}\right], \quad N^4=\left[\begin{array}{cc|cc} 0 & 0 & 0 & i\epsilon_\phi \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\epsilon_\phi & 0 & 0 & 0 \end{array}\right], \quad N^5=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon_\phi & 0 \\ \hline 0 & -i\epsilon_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$N^6=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & i\epsilon_\phi & 0 \\ \hline 0 & -\epsilon_\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad N^7=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_\phi \\ \hline 0 & 0 & 0 & 0 \\ 0 & -i\epsilon_\phi & 0 & 0 \end{array}\right], \quad N^8=\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\epsilon_\phi \\ \hline 0 & 0 & 0 & 0 \\ 0 & -\epsilon_\phi & 0 & 0 \end{array}\right], \tag{5.4.12}$$

where M and N refer to even and odd generators respectively.

In terms of these generators, the general element g in the 'super' Lie algebra can be written as

$$g = \sum_{i=1}^{7} X^i M^i + \sum_{j=1}^{8} \Theta^j N^j. \tag{5.4.13}$$

Note that the generators N^i do not satisfy the condition given in proposition D.3.2, but the combination $\Theta^i N^i$ does indeed satisfy this condition for $1 \leq i \leq 8$ as we demonstrate in the following example.

Example 5.4.1. Consider the generator

$$N^4 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & i\epsilon_\phi \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\epsilon_\phi & 0 & 0 & 0 \end{array}\right].$$

Then using definition D.2.4 we have

$$g = \Theta^4 N^4 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & i\Theta^4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \Theta^4 & 0 & 0 & 0 \end{array}\right], \qquad g^\ddagger = \left[\begin{array}{cc|cc} 0 & 0 & 0 & -i\Theta^4 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ -\Theta^4 & 0 & 0 & 0 \end{array}\right], \tag{5.4.14}$$

since Θ^4 is an odd element.

We therefore have $g^\ddagger + g = 0$ as expected $\triangle$

It should now hopefully be clear that any $g = \sum_{i=1}^{7} X^i M^i + \sum_{j=1}^{8} \Theta^j N^j$, for M^i, N^i as defined in eqs. (5.4.11) and (5.4.12), satisfies the conditions of proposition D.3.2 and hence is an element of the 'super' Lie algebra.

From the 'super' Lie algebra, we wish to construct the Lie superalgebra $\mathfrak{su}(2|2)$.

Definition 5.4.2. A complex *Lie Superalgebra* is a complex super algebra (as in definition D.1.1) $\mathcal{L}_s = \mathcal{L}_0 \oplus \mathcal{L}_1$, whose product is given by the *superbracket* $[,]_s$. The superbracket is defined to obey the following properties:

1. $[A, B]_s \in \mathcal{L}_s, \quad \forall\ A, B \in \mathcal{L}_s,$

2. $[A, \beta B + \gamma C]_s = \beta[A, B]_s + \gamma[A, C]_s, \quad \forall A, B, C \in \mathcal{L}_s,\ \beta, \gamma \in \mathbb{C},$

3. For homogeneous elements A, B, the element $[A, B]_s$ has degree $(\deg A + \deg B)$ modulo 2. This is necessary by definition, as the superbracket must respect the grading of the algebra.

4. $[B, A]_s = -(-1)^{\deg A \deg B}[A, B]_s, \quad \forall A, B \in \mathcal{L}_s,$

5. For homogeneous elements $A, B, C \in \mathcal{L}_s$ the superbracket satisfies the generalised Jacobi identity

$$(-1)^{\deg A \deg C}[A, [B, C]_s]_s + (-1)^{\deg B \deg A}[B, [C, A]_s]_s + (-1)^{\deg C \deg B}[C, [A, B]_s]_s = 0.$$

Following the common physics notation, we use $[,]$ for the superbracket if one or both of the elements is even, and $\{,\}$ for the superbracket between two odd elements of the algebra. A real Lie superalgebra is defined similarly, as a real super algebra satisfying the same properties as above, now restricted only to real linear combinations in property 2.

Given any associative superalgebra, one can define the superbracket to be the *supercommutator*

$$[A, B] = AB - (-1)^{\deg A \deg B} BA, \tag{5.4.15}$$

for all elements A, B of the superalgebra. It is easy to check that such a definition of the superbracket satisfies the properties of definition 5.4.2. When the superbracket is defined as in eq. (5.4.15) we shall refer to $[,]$ and $\{,\}$ as the commutator and anticommutator respectively.

Definition 5.4.3. We can now define the *Lie superalgebra* $\mathfrak{su}(2|2)$. If we let

$$M^i = \epsilon_\phi m^i, \qquad N^i = \epsilon_\phi n^i, \tag{5.4.16}$$

for M^i, N^i as in eqs. (5.4.11) and (5.4.12), then the complex matrices m^i, n^i are the generators of a real Lie superalgebra, $\mathfrak{su}(2|2)$.

Example 5.4.4. The following example shows how the supercommutator of the Lie superalgebra $\mathfrak{su}(2|2)$ (definition 5.4.3) appears naturally from the commutator of the 'super' Lie algebra. We shall consider the generators N^1 and N^3. First, in the 'super' Lie algebra, we calculate the commutator $[\Theta^1 N^1, \Theta^3 N^3]$, recalling that the 'bare' generators themselves are not elements of the algebra (eq. (5.4.13)). Note that by definition D.2.4 we think of Θ^1 and Θ^3 as diagonal matrices and hence we can commute Θ^i and N^j, since N^j involves only the Grassmann identity.

$$\begin{aligned}\left[\Theta^1 N^1, \Theta^3 N^3\right] &= \Theta^1 N^1 \Theta^3 N^3 - \Theta^3 N^3 \Theta^1 N^1 = \Theta^1 \Theta^3 (N^1 N^3 + N^3 N^1), \\ &= \Theta^1 \Theta^3 \epsilon_\phi (n^1 n^3 + n^3 n^1) = \Theta^1 \Theta^3 \{n^1, n^3\} = -\Theta^1 \Theta^3 m^7 = -\Theta^1 \Theta^3 M^7,\end{aligned} \tag{5.4.17}$$

△

5.4.2 $\mathfrak{su}(2|2)$ Basis Satisfying the A_γ Zero Mode Algebra

As described in section 5.4.1, $\mathfrak{su}(2|2)$ is a real Lie superalgebra, with the even basis elements given by the m^i of definition 5.4.3 and the odd basis elements given by the n^i. That is, a general element of the superalgebra can be written as

$$g = \sum_{i=1}^{7} \alpha^i m^i + \sum_{i=1}^{8} \beta^i n^i,$$

for real numbers α^i, β^i and square complex supertraceless matrices m^i, n^i. From eq. (5.4.3) and definition 5.4.3 we can see that these matrices satisfy

$$(m^i)^* + (m^i)^t = 0, \qquad i(n^j)^* + (n^j)^t = 0, \tag{5.4.18}$$

and are 'block-diagonal' and 'block-antidiagonal' respectively, in the sense that

$$m^i = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right], \qquad n^i = \left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right]. \tag{5.4.19}$$

These conditions can in fact be taken as the definition of $\mathfrak{su}(2|2)$.

The goal of this subsection will be to show that (the complexification of) this superalgebra is isomorphic to the zero mode algebra of A_γ in the Ramond sector. We will argue this in two ways, first by appealing to structure theorems of simple Lie superalgebras and the classification of such algebras [Kac77]. We also construct the isomorphism explicitly, by changing basis in $\mathfrak{su}(2|2)$ such that the new basis satisfies the commutation relations of A_γ. Since we therefore write elements of A_γ as four by four square matrices, that is we take our elements of $\mathfrak{su}(2|2)$ to be given by the fundamental representation, this clearly gives a representation of A_γ and we will see that it is the representation with $l^+ = l^- = \frac{1}{2}$. In general, one could start with a representation of $\mathfrak{su}(2|2)$ other than the fundamental in order to construct a representation of A_γ with $l^+, l^- \neq \frac{1}{2}$.

If we denote the zero mode algebra of A_γ in the Ramond sector as $A_{\gamma 0}$, then we can immediately see that $A_{\gamma 0}$ is the direct sum of the one dimensional abelian Lie (super)algebra generated by L_0 – or U_0 which is linearly dependent with L_0 – and a *simple* Lie superalgebra

$$A_{\gamma 0} = L \oplus A, \tag{5.4.20}$$

where we have denoted the abelian Lie algebra generated by L_0 as L and the simple Lie superalgebra as A. By a simple Lie superalgebra, we mean that A does not contain a $\mathbb{Z}_2$-graded ideal. This simple Lie superalgebra A is a classical Type I complex superalgebra, which means the representation of the even part of the algebra A_0 on the odd part A_1 – formed by letting A_0 act on A_1 through the adjoint action – is the direct sum of two irreducible representations of A_0. This is clear by considering the commutation relations of A_γ, specifically eqs. (A.0.11) and (A.0.14), as A_0 is the direct sum of the two $\mathfrak{su}(2)$ algebras, and the Q^a and G^a zero modes of A_1 both transform as four dimensional irreducible representations of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$. A is therefore a classical complex simple Lie superalgebra of rank 2, the Cartan subalgebra being generated by $T_0^{\pm 3}$. Considering the classification of simple superalgebras [Kac77], we see that there are four families of Type 1 superalgebras, the families known as

$$\begin{array}{llll} A(r|s), & r > s \geq 0, & A(r|r), & r \geq 1, \\ C(s), & s \geq 2, & C(r), & r \geq 2. \end{array}$$

If we consider the family members of rank 2, we find that $A(1|0)$ has a 3 dimensional even subalgebra, $C(2)$ has a four dimensional even subalgebra, $P(2)$ has an 8 dimensional even subalgebra and $A(1|1)$ has a 6 dimensional subalgebra. On dimensional grounds we therefore see that A must be isomorphic to $A(1|1)$. $A(1|1)$ has a real form given by the quotient of $\mathfrak{su}(2|2)$ by the one dimensional ideal generated by the identity I_4 and hence $A_{\gamma 0}$ is isomorphic to the complexification of $\mathfrak{su}(2|2)$ as claimed.

We now construct the isomorphism between $\mathfrak{su}(2|2)$ explicitly. The relevant commutation relations of A_γ can be found in Appendix A. Since we are trying to construct a matrix representation of A_γ, writing the generators in terms of the m^i and n^i of definition 5.4.3, we see that L and U have to be scalar multiples of the identity. By definition, L acts on the highest weight state of the representation as multiplication by the conformal dimension of the representation, h. Therefore we necessarily have

$$L = \begin{pmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & h \end{pmatrix}. \tag{5.4.21}$$

Similarly, U acts on the highest weight state as multiplication by $-iu$, so

$$U = \begin{pmatrix} -iu & 0 & 0 & 0 \\ 0 & -iu & 0 & 0 \\ 0 & 0 & -iu & 0 \\ 0 & 0 & 0 & -iu \end{pmatrix}. \tag{5.4.22}$$

In terms of the $\mathfrak{su}(2|2)$ generators m^i (using eqs. (5.4.11) and (5.4.12) and definition 5.4.3), we can write the identity as

$$I = i(m^1 + m^4 + m^5), \tag{5.4.23}$$

and hence we find

$$L = hi(m^1 + m^4 + m^5), \qquad U = u(m^1 + m^4 + m^5). \tag{5.4.24}$$

Identifying the remaining bosonic generators is also straightforward. Since we are constructing a four-dimensional representation of A_γ (using four-dimensional matrices) and the smallest representation of $\mathfrak{su}(2)$ is the fundamental two-dimensional representation, the two orthogonal $\mathfrak{su}(2)$s must both be two-dimensional representations. Recalling that the even elements are represented only in blocks A and D in the sense of eq. (5.4.19), to ensure orthogonality and without loss of generality we will assume that $\mathfrak{su}(2)^+$ is represented in submatrix A and that $\mathfrak{su}(2)^-$ is represented in submatrix D. As is well known, the two-dimensional representation of $\mathfrak{su}(2)$ can be constructed using the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.4.25}$$

as

$$T^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2), \qquad T^3 = \frac{1}{2}\sigma_3. \tag{5.4.26}$$

We can therefore represent $\mathfrak{su}(2)^{\pm}$ as

$$\begin{aligned} T^{+\pm} &= \left(\begin{array}{c|c} T^{\pm} & 0 \\ \hline 0 & 0 \end{array}\right), \quad T^{+3} = \left(\begin{array}{c|c} T^{3} & 0 \\ \hline 0 & 0 \end{array}\right), \\ T^{-\pm} &= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & T^{\pm} \end{array}\right), \quad T^{-3} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & T^{3} \end{array}\right), \end{aligned} \tag{5.4.27}$$

where $T^{\pm}, T^3$ are as in eq. (5.4.26).

In terms of the $\mathfrak{su}(2|2)$ generators, we therefore have

$$\begin{aligned} T^{++} &= \frac{1}{2}(m^2 - im^3), \quad T^{+-} = \frac{1}{2}(m^2 + im^3), \quad T^{+3} = \frac{-i}{2}(m^1 - m^4), \\ T^{-+} &= \frac{1}{2}(m^6 - im^7), \quad T^{--} = \frac{1}{2}(m^6 + im^7), \quad T^{-3} = \frac{-i}{2}(m^5), \end{aligned} \tag{5.4.28}$$

With the bosonic generators identified, knowing that the fermionic generators have entries only in submatrices B and C, we can deduce the form of the fermionic generators using the commutation relations of A_γ. We will show some of the main steps in deducing the fermionic generators; we start by considering Q_+, which as an odd element must have the general form

$$Q_+ = \begin{pmatrix} 0 & 0 & b1 & b2 \\ 0 & 0 & b3 & b4 \\ c1 & c2 & 0 & 0 \\ c3 & c4 & 0 & 0 \end{pmatrix}.$$

Now we consider the relations

$$[T^{\pm 3}, Q_+] = \frac{1}{2} Q_+,$$

and explicitly calculate the commutator on the LHS to obtain

$$\begin{pmatrix} 0 & 0 & \pm b1/2 & b2/2 \\ 0 & 0 & 0 & 0 \\ 0 & c2/2 & 0 & 0 \\ 0 & \pm c4/2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b1/2 & b2/2 \\ 0 & 0 & b3/2 & b4/2 \\ c1/2 & c2/2 & 0 & 0 \\ c3/2 & c4/2 & 0 & 0 \end{pmatrix},$$

and hence $b1 = b3 = b4 = c1 = c3 = c4 = 0$.

The relations between $T^{\pm 3}$ and each of the Q_a for $a \in \{\pm, \pm K\}$ can be used to reduce each of the Q_a to only 2 degrees of freedom (DOF). Next, the various relations between $T^{\pm +}$ and the Q_a, as well as $T^{\pm -}$ and the Q_a can be used to show that there can be only be a maximum of 2 DOF in total among all the Q_a. Finally, the relations $\{Q_+, Q_-\} = \{Q_{+K}, Q_{-K}\} = -\frac{k}{4}I$ show that there is only a single DOF for all the Q_a. Explicitly we find

$$Q_+ = \begin{pmatrix} 0 & 0 & 0 & \frac{-k}{4q} \\ 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-k}{4q} & 0 \\ 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix},$$

$$Q_{+K} = \begin{pmatrix} 0 & 0 & \frac{-k}{4q} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \end{pmatrix}, \qquad Q_{-K} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{4q} \\ q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

in terms of the one remaining DOF which we have now called q.

Similarly, the relations between the two $\mathfrak{su}(2)$s and the G_a for $a \in \{\pm, \pm K\}$ show that the G_a are of the form

$$G_+ = \begin{pmatrix} 0 & 0 & 0 & \frac{h-c/24}{g} \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad G_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{h-c/24}{g} & 0 \\ 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 \end{pmatrix},$$

$$G_{+K} = \begin{pmatrix} 0 & 0 & \frac{h-c/24}{g} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 \end{pmatrix}, \qquad G_{-K} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{h-c/24}{g} \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

in terms of one DOF g.

Finally, the relations between the Q_a and $G_{\tilde{a}}$ can be used to show that the two DOF are related as $g = \frac{2q}{k}(\frac{1}{2} + iu)$ and that the representation of A_γ must satisfy the massless requirement $k(h - \frac{c}{24}) = u^2 + \frac{1}{4}$. Note that since we are representing the two $\mathfrak{su}(2)$s as doublets, we have $l^+ = l^- = \frac{1}{2}$. Our four basis states are therefore $|\Omega_+\rangle, G_- |\Omega_+\rangle, G_{-K} |\Omega_+\rangle$ and $G_- G_{-K} |\Omega_+\rangle$, where $|\Omega_+\rangle$ is the 'highest weight state' (c.f. section 3.3).

Since $|\Omega_+\rangle$ is the highest weight state, we require

$$T^{++} |\Omega_+\rangle = T^{-+} |\Omega_+\rangle = G_+ |\Omega_+\rangle = G_{+K} |\Omega_+\rangle = 0,$$

this requires

$$|\Omega_+\rangle = (1, 0, 0, 0)^t . \tag{5.4.29}$$

Similarly,

$$T^{++} G_{-K} |\Omega_+\rangle = T^{-+} G_{-K} |\Omega_+\rangle = G_+ G_{-K} |\Omega_+\rangle = G_{-K} G_{-K} |\Omega_+\rangle = 0,$$

and therefore

$$G_{-K}\left|\Omega_+\right\rangle = (0,0,1,0)^t\,. \tag{5.4.30}$$

Solving this equation, in terms of the matrix representation of G_{-K} that we have constructed, requires us to fix $g = 1$ and so our representation is now fully determined in terms of the representation labels of A_γ.

The odd elements of A_γ (in the $l^\pm = \frac{1}{2}$ massless representation) can therefore be written in terms of $\mathfrak{su}(2|2)$ generators as

$$\begin{aligned}
Q_+ &= \frac{-q}{2}(n^6 - in^5) - \frac{k}{8q}(n^3 - in^4), & G_+ &= \frac{-1}{2}(n^6 - in^5) + \left(h - \frac{c}{24}\right)(n^3 - in^4),\\
Q_- &= \frac{-q}{2}(n^4 - in^3) - \frac{k}{8q}(n^5 - in^6), & G_- &= \frac{-1}{2}(n^4 - in^3) + \left(h - \frac{c}{24}\right)(n^5 - in^6),\\
Q_{+K} &= \frac{q}{2}(n^8 - in^7) - \frac{k}{8q}(n^1 - in^2), & G_{+K} &= \frac{1}{2}(n^8 - in^7) + \left(h - \frac{c}{24}\right)(n^1 - in^2),\\
Q_{-K} &= \frac{-q}{2}(n^2 + in^1) + \frac{k}{8q}(n^7 - in^8), & G_{-K} &= \frac{-1}{2}(n^2 - in^1) - \left(h - \frac{c}{24}\right)(n^7 - in^8),
\end{aligned} \tag{5.4.31}$$

where

$$\begin{aligned}
q &= \frac{k}{1+2iu},\\
(h - \frac{c}{24}) &= \frac{1}{k}(u^2 + \frac{1}{4}).
\end{aligned} \tag{5.4.32}$$

Hence these two algebras are isomorphic, as claimed.

5.4.3 $\mathfrak{su}(2|2)$ Representations and Supertableaux

In section 5.4.2 we saw that the zero mode algebra of A_γ is isomorphic to the Lie superalgebra $\mathfrak{su}(2|2)$. We can therefore study the branching of A_γ representations into $\mathfrak{su}(2|2)$ representations, where clearly each level of A_γ will be able to be written in terms of $\mathfrak{su}(2|2)$ representations. In this subsection we will therefore introduce the representation theory of $\mathfrak{su}(2|2)$ and show how $\mathfrak{su}(2|2)$ representations can be identified with Young supertableaux as first introduced by [BB81]. This will be seen to be very similar to the way that representations of $\mathfrak{su}(n)$ can be given by Young tableau.

We begin by considering the fundamental representation of the supergroup $SU(2|2)$. We let $SU(2|2)$ act on the complex Grassmann space $\mathbb{C}B_I^{2,2}$ using matrix multiplication as in Appendix D.2. Following the notation of [BB81] we denote the basis vectors of $\mathbb{C}B_I^{2,2}$ as,

$$\xi_A = \begin{pmatrix} \phi_a \\ \\ \psi_\alpha \end{pmatrix}, \tag{5.4.33}$$

where $a, \in \{1,2\},\ \alpha \in \{3,4\}$ run over the even and odd parts of the space. This fundamental representation is therefore a 4-dimensional representation. These basis vectors then transform under $g \in SU(2|2)$ as,

$$\xi_A \to \xi'_A = g_A^B \xi_B, \tag{5.4.34}$$

where as usual, repeated indices are to be summed over (Einstein summation). Clearly this can be expanded linearly to all of $\mathbb{C}B_I^{2,2}$, such that a vector $\Psi = \Psi^A \xi_A$ transforms under $g \in \mathbb{C}B_I^{2,2}$ as

$$\begin{aligned} \Psi \to \Psi' &= g \cdot (\Psi^B \xi_B), \\ &= (-1)^{\deg(B)\deg(A-B)} \Psi^B g_B^A \xi_A = g_B^A \Psi^B \xi_A, \end{aligned} \tag{5.4.35}$$

so we can think of the components transforming as

$$\Psi^A \to \Psi'^A = g_B^A \Psi^B, \tag{5.4.36}$$

as is common in the physics literature. Clearly, since $\mathbb{C}B_I^{2,2}$ is a complex vector space, the components Ψ^A can be taken to be complex. However, it wil be useful for us to consider $\mathbb{C}B_I^{2,2}$ as a supermodule as defined in definition D.1.14, such that the components Ψ^A can be taken to be arbitrary elements of $\mathbb{C}B_I$.

As explained in [BB81], there are actually two fundamental representations of $SU(2|2)$ which are known as Type I and Type II fundamental representations. In a Type I representation, we let $\xi_a = \phi_a$ live in the even part of the Grassmann space, $\mathbb{C}B_I^{2,0}$ and $\xi_\alpha = \psi_\alpha$ live in the odd part of the Grassmann space, $\mathbb{C}B_I^{0,2}$. In a

Type II representation, we instead let ξ_a live in the odd part of the Grassmann space and ξ_α live in the even part of the space. The representation theory of Type I representations and Type II representations can be seen to be identical [BB81], that is every Type I representation is a Type II representation with the grading reversed. If we therefore consider tensor products of Type I or Type II representations exclusively then we may choose to only consider representations of Type I. Here we will not need representations on mixed tensors and so we will assume all our fundamental representations are of Type I.

It will be convenient to associate a Young diagram to our representations as in the case for $SU(N)$, so we will associate to the (Type I) fundamental representation of $SU(2|2)$ the single box tableau in fig. 5.3.

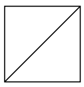


Figure 5.3: The fundamental representation of $SU(2|2)$

Similarly, one may define a conjugate fundamental representation where $g \in SU(2|2)$ is defined to act on the dual of $\mathbb{C}B_I^{2,2}$ as

$$\xi^\perp \to \xi'^\perp = g^\ddagger \xi^\perp, \tag{5.4.37}$$

for $\xi^\perp \in \mathbb{C}B_I^{2,2\ \perp}$. This is the same definition of the conjugate fundamental representation as for $SU(N)$, and following [Kin70] can be associated the single dotted Young tableau shown in fig. 5.4.

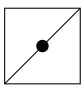


Figure 5.4: The conjugate fundamental representation of $SU(2|2)$

As in the case of $SU(N)$, more representations can be constructed from tensor products of the fundamental and conjugate fundamental representations. As before, we shall consider $\mathbb{C}B_I^{2,2}$ to be a supermodule, so we now want to define the tensor product on $\mathbb{C}B_I^{2,2}$ as a supermodule.

Definition 5.4.5. Given two rings R, S, an R-S-*bimodule* is an abelian group $(V, +)$ such that

- V is a left R-module,
- V is a right S-module,
- $(rv)s = r(vs) \forall\, r \in R, s \in S, m \in V$.

An R-R-bimodule will be simply called an R-bimodule.

Given a supercommutative (definition D.1.4) superalgebra A, then every left A-supermodule V may be regarded as an A-superbimodule by letting

$$a \cdot v \equiv (-1)^{|a||v|} v \cdot a, \tag{5.4.38}$$

for all homogeneous elements $a \in A$, $v \in V$ and extending linearly. In this manner we can think of $\mathbb{C}B_I^{2,2}$ as a superbimodule by defining the right action as above, since $\mathbb{C}B_I$ is supercommutative.

Definition 5.4.6. The tensor product of two A-superbimodules V, W can now be defined as,

$$V \otimes W := F(V \times W)/E, \tag{5.4.39}$$

where $F(V \times W)$ is the free module generated by the cartesian product of V and W, and E is the submodule generated by the equivalence relations,

$$\begin{aligned} (v_1, w_1) + (v_2, w_1) &\sim (v_1 + v_2, w_1), & (v_1 \cdot a, w_1) &\sim (v_1, a \cdot w_1), \\ (v_1, w_1) + (v_1, w_2) &\sim (v_1, w_1 + w_2), & a \cdot (v_1, v_2) &\sim (a \cdot v_1, v_2), \end{aligned} \tag{5.4.40}$$

for $v_i, w_i \in V, W$ respectively and $a \in A$.

$V \otimes W$ has a grading defined by,

$$(V \otimes W)_i = \bigoplus_{(j,k)|j+k=i \pmod 2} V_j \otimes W_k, \tag{5.4.41}$$

and is therefore a left A-supermodule.

We can now define a representation of $SU(2|2)$ on the tensor product $\mathbb{C}B_I^{2,2} \otimes \mathbb{C}B_I^{2,2}$ by letting $SU(2|2)$ act with the fundamental action on each of the factors of the tensor product. Since each fundamental representation was 4-dimensional, this tensor product representation is therefore a 16-dimensional representation. Consider $\xi \otimes \tilde{\xi} \in \mathbb{C}B_I^{2,2} \otimes \mathbb{C}B_I^{2,2}$, then $g \in SU(2|2)$ acts as,

$$(\xi \otimes \tilde{\xi}) \to (\xi \otimes \tilde{\xi})' = (g\xi \otimes g\tilde{\xi}). \tag{5.4.42}$$

This action can then be extended linearly to arbitrary elements of $\mathbb{C}B_I^{2,2} \otimes \mathbb{C}B_I^{2,2}$. As it is common in the physics literature to write this action in terms of the components of the tensor, we can use the description of $\mathbb{C}B_I^{2,2}$ as a $\mathbb{C}B_I$ module to write $\xi = \xi^A e_A$, where e_A has ϵ_ϕ in the A^{th} position and 0 in all remaining positions. Using definition 5.4.6 and eq. (5.4.35) we can therefore expand $(\xi \otimes \tilde{\xi})$ as,

$$\begin{aligned}(\xi \otimes \tilde{\xi}) &= (\xi^{A'} e_{A'} \otimes \tilde{\xi}^{B'} e_{B'}) = \xi^{A'}(e_{A'}\tilde{\xi}^{B'} \otimes e_{B'}),\\ &= \xi^{A'}(\tilde{\xi}^{B'} e_{A'} \otimes e_{B'}) = \xi^{A'}\tilde{\xi}^{B'}(e_{A'} \otimes e_{B'}),\end{aligned} \tag{5.4.43}$$

and similarly,

$$\begin{aligned}(\xi \otimes \tilde{\xi})' = (g\xi \otimes g\tilde{\xi}) &= (g_A^{A'}\xi^A e_{A'} \otimes g_B^{B'}\tilde{\xi}^B e_{B'}),\\ &= g_A^{A'}\xi^A(e_{A'} g_B^{B'}\tilde{\xi}^B \otimes e_{B'}) = g_A^{A'}\xi^A(g_B^{B'}\tilde{\xi}^B e_{A'} \otimes e_{B'}),\\ &= g_A^{A'}\xi^A g_B^{B'}\tilde{\xi}^B(e_{A'} \otimes e_{B'}),\end{aligned} \tag{5.4.44}$$

such that one may consider the components of eq. (5.4.45) transforming as,

$$\xi^A\tilde{\xi}^B \to (\xi^A\tilde{\xi}^B)' = g_{A'}^A \xi^{A'} g_{B'}^B \tilde{\xi}^{B'}. \tag{5.4.45}$$

Clearly, one can now define an action of $g \in SU(2|2)$ on $(\mathbb{C}B_I^{2,2})^{\otimes m} \otimes (\mathbb{C}B_I^{2,2})^{\perp\otimes n}$ for arbitrary $m, n \in \mathbb{Z}_+$ by applying g or $g^\ddagger$ to each factor as appropriate.

The tensor product representation is not irreducible however [BB81], as may be seen by considering a permutation operator,

$$\begin{aligned}P : V \otimes W &\to W \otimes V,\\ v \otimes w &\mapsto w \otimes v,\end{aligned} \tag{5.4.46}$$

for $v \in V, w \in W$. This can be seen to commute with the action of $SU(2|2)$ on the tensor product,

$$\begin{aligned} P(g(\xi \otimes \tilde{\xi})) &= P(g\xi \otimes g\tilde{\xi}) = (g\tilde{\xi} \otimes g\xi), \\ &= g(\tilde{\xi} \otimes \xi) = g(P(\xi \otimes \tilde{\xi})), \end{aligned} \tag{5.4.47}$$

and yet is not a multiple of the identity operator on $\mathbb{C}B_I^{2,2} \otimes \mathbb{C}B_I^{2,2}$, and so by Schur's lemma, the tensor product is not irreducible. However, as for the case of $SU(N)$, irreducible representations of $SU(2|2)$ are given by suitably symmeterised and antisymmeterised tensor products of $\mathbb{C}B_I^{2,2}$ and $(\mathbb{C}B_I^{2,2})^\perp$, each of which may be associated to a supertableau as in fig. 5.5 (the dashed diagonals are explained later in section 5.4.4). Note that since the Levi-Civita tensor is not an invariant of $SU(M|N)$, a tableau containing dotted boxes (that is a representation on tensors containing covariant indices) may not be converted to a tableau containing only undotted boxes.

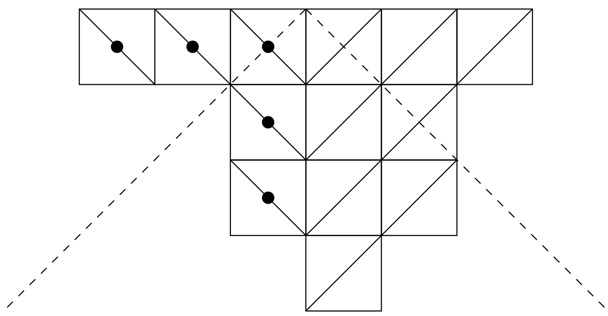


Figure 5.5: A generic representation of $SU(2|2)$

Example 5.4.7. Let us consider the example of the symmetric tensor product on two copies of $\mathbb{C}B_I^{2,2}$. This is described by the supertableau in fig. 5.6.

The symmetric tensor product of $\xi, \tilde{\xi} \in \mathbb{C}B_I^{2,2}$, which we will denote as Ξ, is given by

$$\Xi = \xi \otimes \tilde{\xi} + \tilde{\xi} \otimes \xi, \tag{5.4.48}$$

which, as in eqs. (5.4.43) and (5.4.44), we can expand in terms of components as

$$\begin{aligned}\Xi = \xi \otimes \tilde{\xi} + \tilde{\xi} \otimes \xi &= (\xi^A \tilde{\xi}^B + \tilde{\xi}^A \xi^B)(e_A \otimes e_B), \\ &= (\xi^A \tilde{\xi}^B + (-1)^{|A||B|} \xi^B \tilde{\xi}^A)(e_A \otimes e_B).\end{aligned} \tag{5.4.49}$$

We now see that the components of this tensor are symmetric unless both A and B take values in the odd part of the space (i.e $A = \alpha, B = \beta$ as in eq. (5.4.33)), in which case the components are antisymmetric. Using the usual convention of parentheses to denote symmetric indices, we therefore have

$$\Xi^{(AB)} = \xi^A \tilde{\xi}^B + (-1)^{|A||B|} \xi^B \tilde{\xi}^A. \tag{5.4.50}$$

The dimension of the symmetric space is therefore the sum of the number of independent components of Ξ^{ab}, $\Xi^{a\beta}$ and $\Xi^{\alpha\beta}$. These have three, four and one independent components respectively, since the first two are symmetric and the final one is antisymmetric, for $a, b, \in \{1, 2\}$ and $\alpha, \beta \in \{3, 4\}$, so $\Xi^{(AB)}$ has eight independent components and the symmetric space is 8-dimensional. △

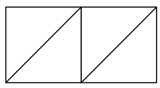


Figure 5.6: The 2-fold symmetric representation of $SU(2|2)$

It is now clear that, due to the Grassmann nature of the odd part of the space, whenever we symmeterise two indices, the components behave as antisymmetric indices when both indices lie in the odd part of the space. For this reason, [BB81] refer to the tensors as 'symmeterised' and 'supersymmeterised', to mean symmeterised on the even part of the space and antisymmeterised on the odd part of the space. Similarly, when we antisymmeterise indices, the components behave symmetrically when both indices lie in the odd part of the space; there is therefore no limit to the length of a column for a supertableau.

Definition 5.4.8. It will be useful to define the horizontal and vertical *eccentricity* of a (totally un-dotted) supertableau to be the number of boxes in the first row and first column respectively. The supertableau show in fig. 5.7 has horizontal eccentricity m and vertical eccentricity n. Such a tableau will be said to have eccentricity (m, n).

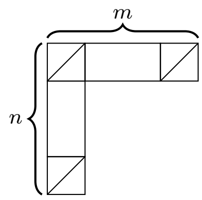


Figure 5.7: A supertableau of horizontal eccentricity m and vertical eccentricity n

Definition 5.4.9. A (totally un-dotted) supertableau of eccentricity (m, n) containing N boxes will be called *maximally eccentric* if

$$N = m + n - 1, \tag{5.4.51}$$

and *non-maximally eccentric* if

$$N \geq m + n. \tag{5.4.52}$$

Hence the tableau in fig. 5.7 is maximally eccentric, whereas the tableau shown in fig. 5.8 is non-maximally eccentric.

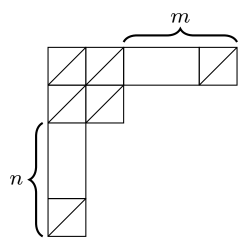


Figure 5.8: A non-maximally eccentric supertableau of eccentricity $(m + 2, n + 2)$.

Now, given the generators m^i and n^j of definition 5.4.3, we can write a generic element of the superalgebra $\mathfrak{su}(2|2)$ as

$$m = \sum_{i=1}^{7} x^i m^i + \sum_{j=1}^{8} \theta^j n^j, \tag{5.4.53}$$

for $x^i, \theta^j \in \mathbb{R}$ and similarly a generic element of the 'Super' Lie algebra as

$$M = \sum_{i=1}^{7} X^i M^i + \sum_{j=1}^{8} \Theta^j N^j = \sum_{i=1}^{7} \sum_{\mu} X^i_\mu \epsilon_\mu m^i + \sum_{j=1}^{8} \sum_{\mu} \Theta^j_\mu \epsilon_\mu n^j, \tag{5.4.54}$$

for $X^i_\mu, \Theta^j_\mu \in \mathbb{R}$. Elements of the Lie supergroup $SU(2|2)$ near the identity are then given by,

$$G = \exp(M) = \exp\left(\sum_{i=1}^{7} X^i m^i + \sum_{j=1}^{8} \Theta^j n^j\right), \tag{5.4.55}$$

where now X^i, Θ^j are elements of $\mathbb{R}B_I$ close to the identity. The Lie superalgebra elements are therefore the linear terms appearing in the expansion of the supergroup elements, and hence as for Lie groups and Lie algebras, a representation of the Lie supergroup $SU(2|2)$ gives a natural representation of the Lie superalgebra $\mathfrak{su}(2|2)$ as the linearised action of the supergroup. A more formal description of the connection between tensor representations of supergroups and the associated superalgebras is discussed in the case of $GL(M/N)$ in [Fio11].

An irreducible representation of $\mathfrak{su}(M|N)$ can therefore be described using a supertableaux which describes the suitably symmeterised tensors on which $\mathfrak{su}(M|N)$ acts as the tensor product of fundamental and conjugate fundamental representations as necessary.

In section 5.4.2 we showed how the fundamental representation of $\mathfrak{su}(2|2)$ was isomorphic to the zero mode algebra of a massless representation of A_γ with $l^\pm = \frac{1}{2}$ in the Ramond sector. Thanks to the results of this subsection, this can therefore be summarised as

$$\mathrm{Ch}^{A_\gamma,R}_{0,l^\pm=\frac{1}{2}} = \left(\boxslash\right) q^h + \dots. \tag{5.4.56}$$

5.4.4 Branching Rules for $\mathfrak{su}(2|2)$

Having shown that the $\mathfrak{su}(2|2)$ superalgebra is isomorphic to the A_γ zero mode algebra in section 5.4.2, we know that the even subalgebra of $\mathfrak{su}(2|2)$ is $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$. It is clear that given a representation (Γ, V) of an algebra $\mathfrak{g}$, with subalgebra $\mathfrak{h} \subset \mathfrak{g}$, then (Γ, V) also provides a representation of the subalgebra $\mathfrak{h}$. In general, this representation will be reducible, and so will be given by the direct sum of several irreducible representations. This decomposition,

$$(\Gamma, V) \mapsto \bigoplus_n a_n(\Gamma_n, V_n), \tag{5.4.57}$$

where (Γ_n, V_n) are irreducible representations of the subalgebra $\mathfrak{h}$ and a_n are the multiplicities at which they appear in the decomposition, is known as a *branching rule* for the algebra g. In this subsection we will show how to calculate the branching of an irreducible representation of $\mathfrak{su}(2|2)$ into irreducible representations of the bosonic subalgebra $\mathfrak{su}(2|2) \mapsto \mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ using Young (super)tableaux [BB82].

Following [BB82], we first describe the process for calculating the branching of a representation of $SU(M+N) \mapsto SU(M) \times SU(N)$. Given an irreducible representation of $SU(M)$, (Γ_1, V_1) described by a Young Tableau T_1 and an irreducible representation of $SU(N)$, (Γ_2, V_2) described by a second Young tableau T_2, then the representation $(\Gamma_1 \otimes \Gamma_2, V_1 \otimes V_2)$ of dimension $\dim(V_1)\dim(V_2)$ appears in the decomposition of an irreducible representation, (Ω, W) with multiplicity equal to the multiplicity of (Ω, W) in the decomposition of the tensor product of T_1 and T_2 now treated as representations of $SU(M+N)$. This is hopefully made clear in the following example.

Example 5.4.10. Consider the representation of $SU(3)$ described by

$$\ydiagram{2,1}$$

which has dimension 8, and the representation of $SU(4)$ described by

$$\ydiagram{2}$$

which has dimension 10. We want to check whether the 40-dimensional representation of $SU(3) \times SU(4)$ described by

$$\left(\ydiagram{2,1}\, , \ydiagram{2} \right)$$

appears in the decomposition of the 882-dimensional representation of $SU(7)$ described by

$$\ydiagram{3,2}\,.$$

We therefore want to calculate the Clebsch-Gordan decomposition of

$$\ydiagram{2,1} \times \ydiagram{2}\,,$$

where now the tableaux are understood to refer to representations of $SU(7)$. As is well known, this decomposition can easily be found using the Littlewood-Richardson rule. In this example this gives the result,

$$\begin{array}{ccccccccccccc} \ydiagram{2,1} & \times & \ydiagram{2} & = & \ydiagram{4,1} & + & \ydiagram{3,2} & + & \ydiagram{3,1,1} & + & \ydiagram{2,2,1}\,, \\ \underline{112} & \times & \underline{28} & = & \underline{1008} & + & \underline{882} & + & \underline{756} & + & \underline{490}, \end{array} \tag{5.4.58}$$

where the dimension of each representation is shown underneath the corresponding tableau.

From this calculation we conclude that the representation

$$\left(\ydiagram{2,1}\,,\ydiagram{2}\right)$$

of $SU(3) \times SU(4)$ appears with multiplicity 1 in the decomposition of the $SU(7)$ representation

$$\ydiagram{3,2}\,.$$

To fully calculate the branching from $SU(7)$ to $SU(3) \times SU(4)$, we therefore need to check which other representations of $SU(3) \times SU(4)$ contain the representation

$$\ydiagram{3,2}$$

of $SU(7)$ in their Clebsch-Gordan decomposition (when treated as tableaux of $SU(7)$). Note that since we treat the tableaux of both $SU(3)$ and $SU(4)$ a tableaux of $SU(7)$, then on the level of the diagrams the decomposition must be symmetric with respect to the factors, as the tensor product is symmetric. However after appropriately symmeterising the tableaux, one must still simplify the tableau such that no columns are of length greater than N for a tableau of $SU(N)$. The full decomposition is then

$$\begin{aligned}
\ydiagram{3,2} \quad \mapsto \quad & \left(\ydiagram{3,2}\,, 1\right) + \left(1, \ydiagram{3,2}\right) \\
+ \; & \left(\ydiagram{2,2}\,, \ydiagram{1}\right) + \left(\ydiagram{1}\,, \ydiagram{2,2}\right) \\
+ \; & \left(\ydiagram{3,1}\,, \ydiagram{1}\right) + \left(\ydiagram{1}\,, \ydiagram{3,1}\right) \\
+ \; & \left(\ydiagram{3}\,, \ydiagram{2}\right) + \left(\ydiagram{2}\,, \ydiagram{3}\right) \\
+ \; & \left(\ydiagram{2,1}\,, \ydiagram{1,1}\right) + \left(\ydiagram{1,1}\,, \ydiagram{2,1}\right) \\
+ \; & \left(\ydiagram{2,1}\,, \ydiagram{2}\right) + \left(\ydiagram{2}\,, \ydiagram{2,1}\right),
\end{aligned} \tag{5.4.59}$$

where 1 denotes the singlet representation – the empty tableau. In terms of the dimensions of the various representations this is

$$\underline{882} \mapsto \underline{15} + \underline{60} + \underline{24} + \underline{60} + \underline{60} + \underline{135} + \underline{100} + \underline{120} + \underline{48} + \underline{60} + \underline{80} + \underline{120}, \tag{5.4.60}$$

where the order of the representations has been kept the same as the tableaux in the previous equation. $\triangle$

The branching for $\mathfrak{su}(M|N) \mapsto \mathfrak{su}(M) \times \mathfrak{su}(N) \times \mathfrak{u}(1)$ works similarly, by considering the superspace $\mathbb{C}B_I^{m,n}$ to be the direct sum $\mathbb{C}B_{I,0}^m \oplus \mathbb{C}B_{I,1}^n$ [BB82]. The even part of the space therefore transforms under the $\mathfrak{su}(M)$ and is a singlet under the $\mathfrak{su}(N)$, while the odd part of the space transforms under the $\mathfrak{su}(N)$ and is a singlet under the $\mathfrak{su}(M)$. Additionally, the $\mathfrak{u}(1)$ generator is embedded in $\mathfrak{su}(M|N)$ as

$$u = \left(\begin{array}{c|c} \frac{1}{M} & 0 \\ \hline 0 & \frac{1}{N} \end{array}\right), \tag{5.4.61}$$

such that is supertraceless. Therefore a vector in the even part of the space has $\mathfrak{u}(1)$ charge $\frac{1}{M}$, while a vector in the odd part of the space has charge $\frac{1}{N}$. We can therefore branch a (totally contravariant, using only un-dotted boxes) representation of $\mathfrak{su}(M|N)$ in the same way as we branch $SU(M+N)$. However since supertableaux show supersymmeterisation of the tensor space, we should reflect the $\mathfrak{su}(N)$ tableau through its diagonal as indicated in fig. 5.5 in order to show the correct symmeterisation for the odd part of the space, as described in section 5.4.3.

We now consider an example of branching an $\mathfrak{su}(2|2)$ representation into a sum of $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ representations.

Example 5.4.11. Consider the representation

$$\ydiagram{3,2}$$

of $\mathfrak{su}(2|2)$. In example 5.4.10, we calculated the decomposition of this tableau for $SU(M+N)$ (in fact we assumed $M=3, N=4$, but on the level of the tableau the answer is valid for any M, N as long as we did not simplify the tableau, which we did not), so now to calculate the branching of $\mathfrak{su}(2|2)$, we simply have to transpose the tableau in the second part of each product on the right hand side. This gives

$$\begin{aligned}
\ydiagram{3,2} \quad \mapsto \quad & \left(\ydiagram{3,2}\,, 1\right)_{\frac{5}{2}+0} && + \left(1, \ydiagram{2,2,1}\right)_{0+\frac{5}{2}} \\
&+ \left(\ydiagram{2,2}\,, \ydiagram{1}\right)_{2+\frac{1}{2}} && + \left(\ydiagram{1}\,, \ydiagram{2,2}\right)_{\frac{1}{2}+2} \\
&+ \left(\ydiagram{3,1}\,, \ydiagram{1}\right)_{2+\frac{1}{2}} && + \left(\ydiagram{1}\,, \ydiagram{2,1,1}\right)_{\frac{1}{2}+2} \\
&+ \left(\ydiagram{3}\,, \ydiagram{1,1}\right)_{\frac{3}{2}+1} && + \left(\ydiagram{2}\,, \ydiagram{1,1,1}\right)_{1+\frac{3}{2}} \\
&+ \left(\ydiagram{2,1}\,, \ydiagram{1,1}\right)_{\frac{3}{2}+1} && + \left(\ydiagram{1,1}\,, \ydiagram{2,1}\right)_{1+\frac{3}{2}} \\
&+ \left(\ydiagram{2,1}\,, \ydiagram{2}\right)_{\frac{3}{2}+1} && + \left(\ydiagram{2}\,, \ydiagram{2,1}\right)_{1+\frac{3}{2}} ,
\end{aligned} \tag{5.4.62}$$

$$
\begin{aligned}
\begin{ytableau} \diagup & \diagup & \diagup \\ \diagup & \diagup \end{ytableau} \quad \mapsto \quad & \left(\ydiagram{1}, 1\right)_{\frac{5}{2}} && + \quad \left(1, \ydiagram{1}\right)_{\frac{5}{2}} \\
+ \quad & \left(\ydiagram{1}, 1\right)_{\frac{5}{2}} && + \quad \left(\ydiagram{2}, \ydiagram{1}\right)_{\frac{5}{2}} \\
+ \quad & \left(\ydiagram{3}, 1\right)_{\frac{5}{2}} && + \quad \left(\ydiagram{1}, 1\right)_{\frac{5}{2}} \\
+ \quad & \left(1, \ydiagram{1}\right)_{\frac{5}{2}} && + \quad \left(\ydiagram{1}, \ydiagram{2}\right)_{\frac{5}{2}} \\
+ \quad & \left(\ydiagram{2}, \ydiagram{1}\right)_{\frac{5}{2}},
\end{aligned}
\tag{5.4.63}
$$

where we have labelled each representation of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ with the $\mathfrak{u}(1)$ charges for both the even part of the space and the odd part separately in the first line, then written the total charge only in the second line while simultaneously simplifying the tableau due to any 2 box columns being equal to a singlet.

$\triangle$

Branching supertableaux also gives us a way to see that we must allow supertableaux with more than 4 rows for $\mathfrak{su}(2|2)$.

Example 5.4.12. We know that it is not possible to take the antisymmetric 5-fold representation of $\mathfrak{su}(4)$ – we cannot antisymmeterise more than four basis vectors without repetition:

$$
\dim \left(\ydiagram{1,1,1,1,1} \right) = 0. \tag{5.4.64}
$$

However, if we branch the 5-fold antisymmetric representation for $\mathfrak{su}(2|2)$, it is clear that we obtain the following branching rule (note that the $\mathfrak{u}(1)$ is clearly $\frac{5}{2}$ for each representation, so the $\mathfrak{u}(1)$ charge is not shown in the following):

$$
\begin{aligned}
\ydiagram[*(gray)]{1,1,1,1,1} \quad \mapsto \quad & \left(\ydiagram{1,1,1,1,1}\,, 1\right) \quad + \quad \left(\ydiagram{1,1,1,1}\,, \ydiagram{1}\right) \quad + \quad \left(\ydiagram{1,1,1}\,, \ydiagram{2}\right) \\
& + \left(\ydiagram{1,1}\,, \ydiagram{3}\right) + \left(\ydiagram{1}\,, \ydiagram{4}\right) + \left(1, \ydiagram{5}\right), \\
= \quad & \left(1, \ydiagram{3}\right) + \left(\ydiagram{1}\,, \ydiagram{4}\right) + \left(1, \ydiagram{5}\right),
\end{aligned}
\tag{5.4.65}
$$

and therefore

$$
\begin{aligned}
\dim\left(\ydiagram[*(gray)]{1,1,1,1,1}\right) &= (1 \times 4) + (2 \times 5) + (1 \times 6), \\
&= 20 \neq 0.
\end{aligned}
\tag{5.4.66}
$$

$\triangle$

Example 5.4.13. This branching rule also gives an easy way to show that the dimension of the n-fold symmetric tensor representation of $\mathfrak{su}(2|2)$ is $4n$.

$$
\overbrace{\ydiagram[*(gray)]{3}}^{n} \mapsto \left(\overbrace{\ydiagram{3}}^{n}, 1\right) + \left(\overbrace{\ydiagram{3}}^{n-1}, \ydiagram{1}\right) + \left(\overbrace{\ydiagram{3}}^{n-2}, 1\right), \tag{5.4.67}
$$

and hence

$$
\begin{aligned}
\dim\left(\overbrace{\ydiagram[*(gray)]{3}}^{n}\right) &= ([n+1] \times 1) + (n \times 2) + ([n-1] \times 1), \\
&= 4n.
\end{aligned}
\tag{5.4.68}
$$

$\triangle$

When we calculate the tensor product of two representations defined by tableaux, using the Littlewood-Richardson rule to decompose as a sum of irreducible representations, we do not care about the order of the two factors. Our method for checking whether an irreducible representation of $\mathfrak{su}(M) \times \mathfrak{su}(N)$ appears in the branching of a representation of $\mathfrak{su}(M+N)$ is dependent on whether the tableau describing the representation of $\mathfrak{su}(M+N)$ appears in the decomposition of the tensor product of the $\mathfrak{su}(2)$ representations. Therefore, it is clear that the representation described by (T_1, T_2) appears in the branching of a representation T of $\mathfrak{su}(4)$ if and only if the representation (T_2, T_1) also appears in the branching, where T, T_1 and T_2 are to represent suitable Young tableaux. We then branch the representation of $\mathfrak{su}(M|N)$ described by the (super)tableau T by reflecting the $\mathfrak{su}(N)$ tableau in each of the summands calculated by branching T as an $\mathfrak{su}(M+N)$ representation. This is demonstrated in eqs. (5.4.59), (5.4.62) and (5.4.63) for the case of $\mathfrak{su}(4)$ (or equivalently $SU(4)$) and $\mathfrak{su}(2|2)$ respectively. We therefore have the following propositions, the first two of which appear in [BB82].

Proposition 5.4.14. *The irreducible representation of $\mathfrak{su}(M) \times \mathfrak{su}(N)$ described by (T_1, T_2) appears in the branching of the irreducible representation of $\mathfrak{su}(N+M)$ described by T if and only if the irreducible representation described by (T_2, T_1) also appears in the branching. We summarise this rule as,*

$$(T_1, T_2) \in T \iff (T_2, T_1) \in T.$$

Proposition 5.4.15. *The irreducible representation of $\mathfrak{su}(M) \times \mathfrak{su}(N)$ described by $(T_1, (T_2)^t)$ appears in the branching of the irreducible representation of $\mathfrak{su}(M|N)$ described by T if and only if the irreducible representation described by $(T_2, (T_1)^t)$ also appears in the branching, where $(T_i)^t$ denotes the transpose of T_i on the diagonals indicated in fig. 5.5. We summarise this rule as*

$$\left(T_1, (T_2)^t\right) \in T \iff \left(T_2, (T_1)^t\right) \in T.$$

Proposition 5.4.16. *Let the branching of an irreducible representation of $\mathfrak{su}(M|N)$ described by supertableau T be given by*

$$T \mapsto \sum_i (T_{i_1}, T_{i_2}),$$

then

$$T^t \mapsto \sum_i \left((T_{i_1})^t, (T_{i_2})^t\right). \tag{5.4.69}$$

The final two propositions give the immediate corollary for $\mathfrak{su}(N/N)$:

Corollary 5.4.17. *Given an irreducible representation of* $\mathfrak{su}(N/N)$ *described by supertableau* T*, then*

$$\dim(T) = \dim(T^t).$$

Proof. By proposition 5.4.15, if the branching of T contains $(T_1, (T_2)^t)$, then it also contains $(T_2, (T_1)^t)$ of dimensions $(\dim(T_1) \times \dim((T_2)^t)$ and $(\dim(T_2) \times \dim((T_1)^t)$ respectively. Then by proposition 5.4.16, the branching of T^t contains $((T_1)^t, T_2)$ and $((T_2)^t, T_1)$ of dimensions $(\dim((T_1)^t) \times \dim(T_2))$ and $(\dim((T_2)^t) \times \dim(T_1))$. □

It will also be useful for us to note that since we are interested specifically in $\mathfrak{su}(2|2)$ and its branching into $\mathfrak{su}(2) \times \mathfrak{su}(2)$, that representations described by tableau with more that 2 rows of length strictly greater than 2, as shown in fig. 5.9, are zero representations. This is due to the supersymmeterisation of the $\mathfrak{su}(M|N)$ indices; if we branch the $\mathfrak{su}(2|2)$ representation to find the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ content, one of the two representations of $\mathfrak{su}(2)$ must be described by a tableau with at least 3 rows which is clearly a zero representation of $\mathfrak{su}(2)$.

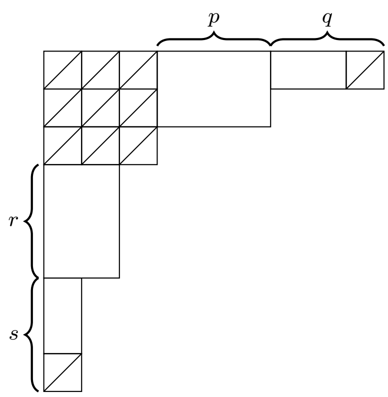


Figure 5.9: A zero representation of $\mathfrak{su}(2|2)$

5.4.5 The Index of A_γ Using Supertableaux

We have established that the zero mode algebra of A_γ in the Ramond sector is $\mathfrak{su}(2|2)$ section 5.4.2 and that we can study the $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ content of an $\mathfrak{su}(2|2)$ representation by studying the branching of the supertableau describing the $\mathfrak{su}(2|2)$ representation section 5.4.4. We can therefore now identify $\mathfrak{su}(2|2)$ representations whose $\mathfrak{su}(2) \times \mathfrak{su}(2)$ content matches representations of A_γ at a given level; The general method to do this is described in example 5.4.19.

Example 5.4.18. We have already considered the case of a massless representation of A_γ with $l^\pm = \frac{1}{2}$ in section 5.4.2, and in section 5.4.3 we identified the ground level of this A_γ representation with the fundamental representation of $\mathfrak{su}(2|2)$. Now that we have seen how to branch $\mathfrak{su}(2|2)$ supertableaux, we can branch the fundamental representation as

$$\boxslash \mapsto \Big(\square, 1\Big)_1 + \Big(1, \square\Big)_1, \tag{5.4.70}$$

and recognise the two $\mathfrak{su}(2)$ doublets (one of $\mathfrak{su}(2)^+$ and one of $\mathfrak{su}(2)^-$) which appear at ground level in A_γ as shown in fig. 5.1.

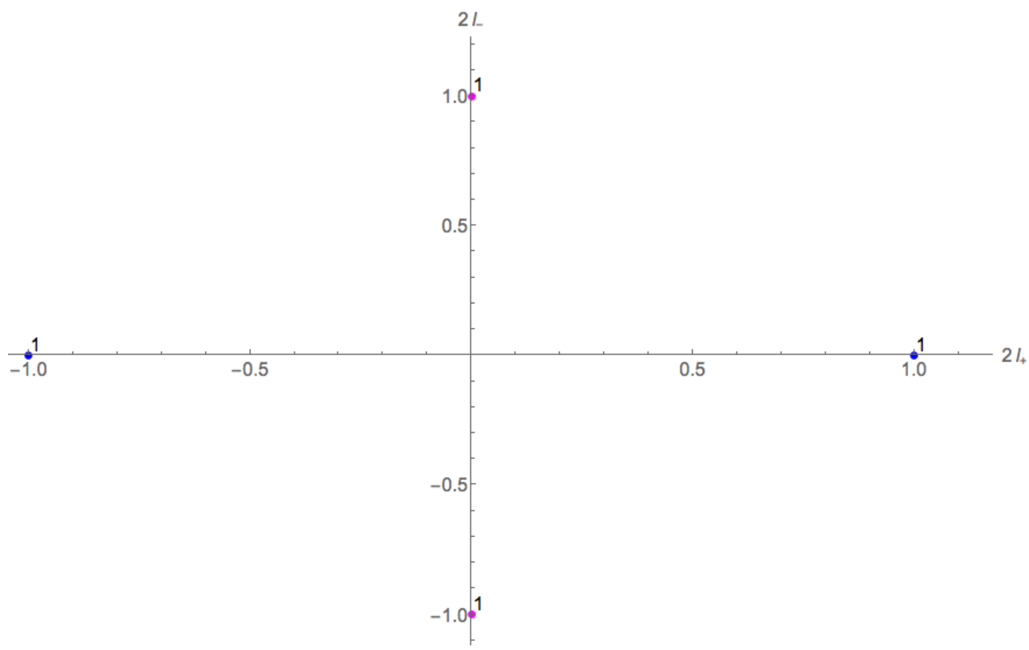


Figure 5.10: The ground level of a Ramond representation of A_γ $k^+ = 3, k^- = 2, l^+ = \frac{1}{2}, l^- = \frac{1}{2}$

As noted at the end of section 5.4.3, we therefore have

$$\mathrm{Ch}^{A_\gamma,R}_{0,k^+=3,k^-=2,l^\pm=\frac{1}{2}} = \left(\boxed{\diagup}\right) q^h + \dots . \tag{5.4.71}$$

$\triangle$

Example 5.4.19. Similarly, we can consider the level 1 states of the same representation of A_γ ($k^+ = 3, k^- = 2, l^+ = \frac{1}{2}, l^- = \frac{1}{2}$) shown in fig. 5.11.

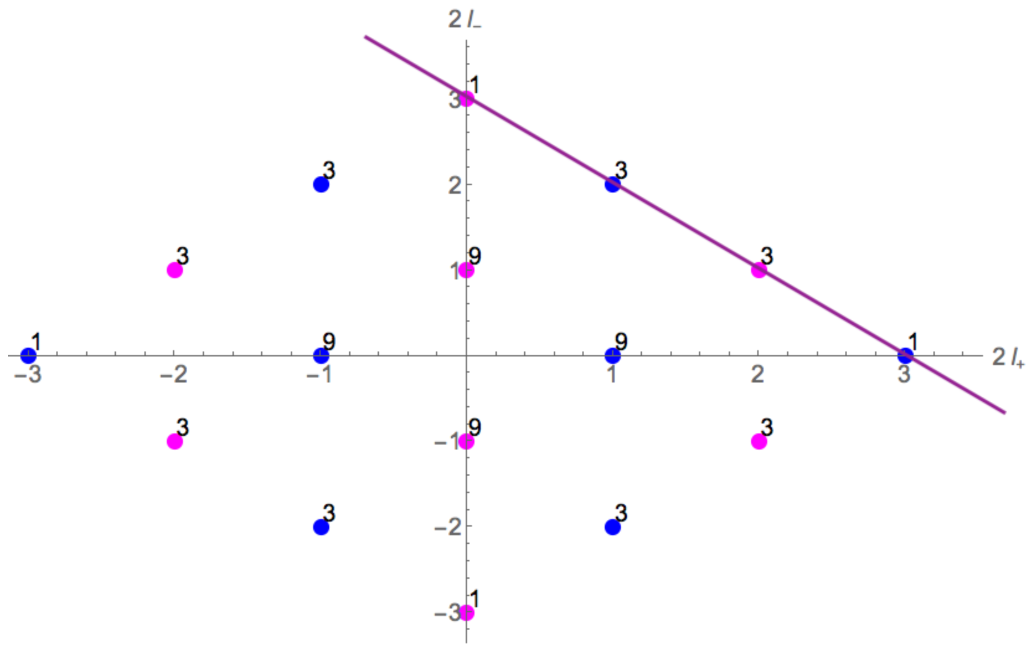


Figure 5.11: Level 1 states of a Ramond representation of A_γ with $k^+ = 3, k^- = 2, l^+ = \frac{1}{2}, l^- = \frac{1}{2}$

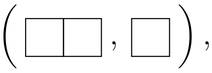
To find the $\mathfrak{su}(2|2)$ representations which contain the right $\mathfrak{su}(2) \times \mathfrak{su}(2)$ content we follow the following method: We identify the largest multiplet of $\mathfrak{su}(2)^+$, in this case the quadruplet which is a singlet of $\mathfrak{su}(2)^-$; We identify the smallest representation of $\mathfrak{su}(2|2)$ which contains this $\mathfrak{su}(2) \times \mathfrak{su}(2)$ content, in this case the representation described by

$$\boxed{\diagup}\boxed{\diagup}\boxed{\diagup}\,;$$

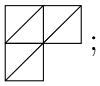
We calculate the branching of this representation of $\mathfrak{su}(2|2)$ (suppressing the $\mathfrak{u}(1)$ charge),

$$\boxed{\diagup}\boxed{\diagup}\boxed{\diagup} \mapsto \left(\Box\Box\Box\,,1\right) + \left(\Box\Box\,,\Box\right) + \left(\Box\,,1\right);$$

We now identify the next largest multiplet of $\mathfrak{su}(2)^+$, in this case the one remaining copy of

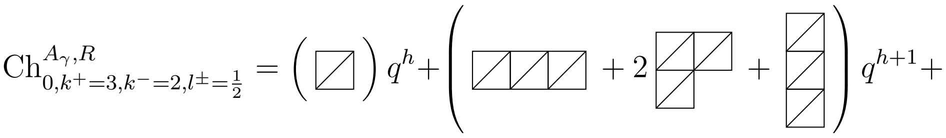


and find the smallest representation of $\mathfrak{su}(2|2)$ which contains this but does not contain any representations of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ already considered, namely



We now identify the next largest representation of $\mathfrak{su}(2)^+$ and continue this process.

Using the method described above, one finds this representation of A_γ can be branched into $\mathfrak{su}(2|2)$ representations as



(5.4.72)

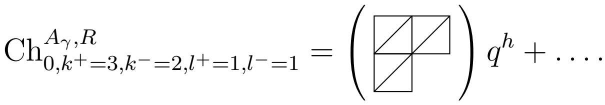
$\triangle$

This process can easily be continued to higher levels of the A_γ representation. For the massless representation with $l^\pm = \frac{1}{2}$ we have calculated branchings into $\mathfrak{su}(2|2)$ representations up to the sixth excited level.

We can also of course calculate the branching of other representations of A_γ. We first give examples showing the branching of the ground level for a few different representations of A_γ, before giving the general statement of the branching for the ground level.

Example 5.4.20. Next, we consider the ground states of the $k^+ = 3, k^- = 2, l^+ = 1, l^- = 1$ A_γ representation shown in fig. 5.12.

This can be solved in terms of $\mathfrak{su}(2|2)$ representations as



(5.4.73)

$\triangle$

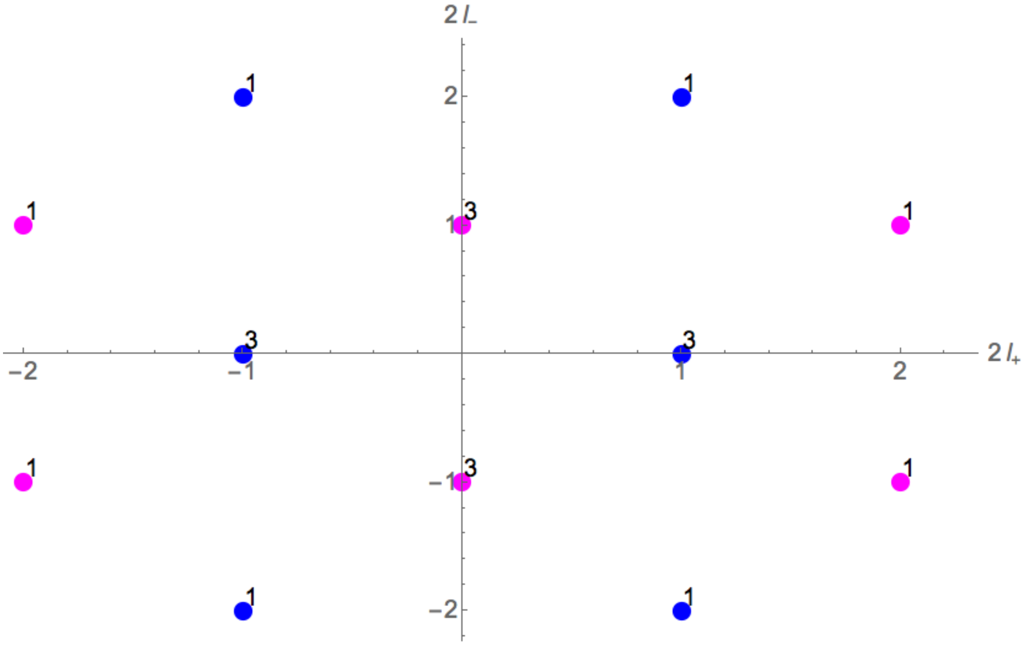
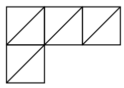


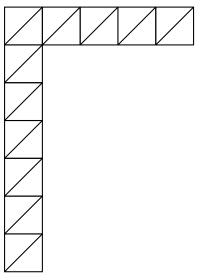
Figure 5.12: The ground level of a Ramond representation of A_γ with $k^+ = 3, k^- = 2, l^+ = 1, l^- = 1$

Example 5.4.21. The ground states of the $k^+ = 4, k^- = 3, l^+ = \frac{3}{2}, l^- = 1$ A_γ representation shown in fig. 5.13 can be solved in terms of $\mathfrak{su}(2|2)$ representations as

$$\mathrm{Ch}^{A_\gamma,R}_{0,k^+=4,k^-=3,l^+=\frac{3}{2},l^-=1} = \Bigg($$

$$\Bigg) q^h + \ldots . \tag{5.4.74}$$

△

Example 5.4.22. The ground states of the $k^+ = 5, k^- = 7, l^+ = \frac{5}{2}, l^- = \frac{7}{2}$ A_γ representation shown in fig. 5.14 can be solved in terms of $\mathfrak{su}(2|2)$ representations as

$$\mathrm{Ch}^{A_\gamma,R}_{0,k^+=5,k^-=7,l^+=\frac{5}{2},l^-=\frac{7}{2}} = \Bigg($$

$$\Bigg) q^h + \ldots . \tag{5.4.75}$$

△

It is clear that in all of these examples, the ground level is described by a single maximally eccentric supertableau. This can be made precise with the following proposition.

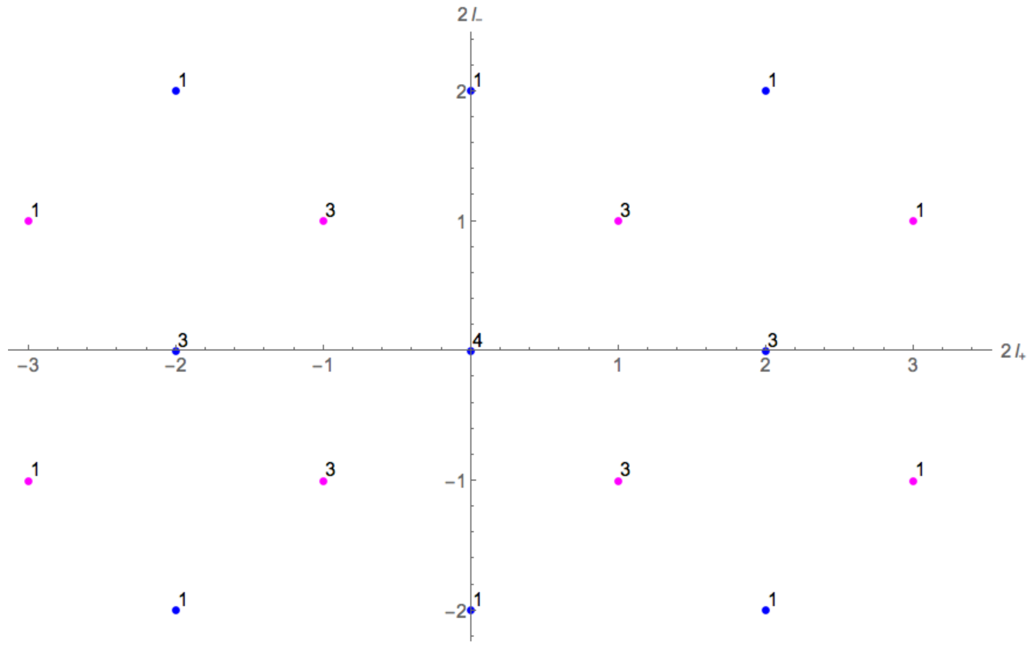


Figure 5.13: The ground level of a Ramond representation of A_γ with $k^+ = 4, k^- = 3, l^+ = \frac{3}{2}, l^- = 1$

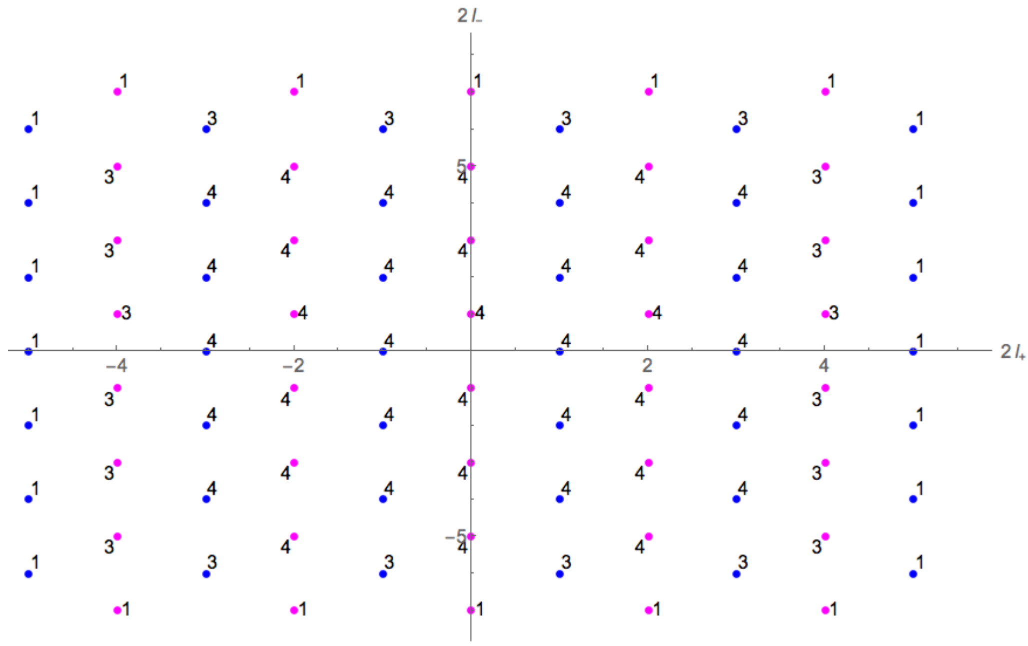


Figure 5.14: The ground level of a Ramond representation of A_γ with $k^+ = 5, k^- = 7, l^+ = \frac{5}{2}, l^- = \frac{7}{2}$

Proposition 5.4.23. *The ground level of a unitary irreducible representation of A_γ described by parameters k^+, k^- and quantum numbers l^+, l^- is described by a single representation of the superalgebra $\mathfrak{su}(2|2)$, which is in turn described by a maximally eccentric Young supertableau of eccentricity $(2l^+, 2l^-)$.*

Proof. We have already showed in section 5.4.2 that $\mathfrak{su}(2|2)$ satisfies the zero mode algebra of A_γ and so it is clear the the ground level of an irreducible representation of A_γ can be given by a representation of $\mathfrak{su}(2|2)$. This representation of $\mathfrak{su}(2|2)$ must be irreducible, since the representation of A_γ was assumed to be. We are therefore left only to show that this irreducible representation is described by a maximally eccentric supertableau of eccentricity $(2l^+, 2l^-)$.

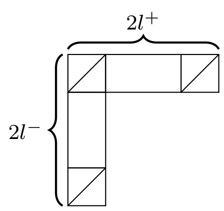
In section 3.3 we saw that the generic massless Ramond representation of A_γ had 8 highest weight states of $\mathfrak{su}(2) \times \mathfrak{su}(2)$, as shown in fig. 3.3b. We therefore have that the ground level of A_γ is given by

$$\begin{aligned}\mathrm{Ch}^{A_\gamma,R}_{0,l^+,l^-} = \Big(&\chi^+_{l^+}\chi^-_{l^--\frac{1}{2}} + \chi^+_{l^+-\frac{1}{2}}\chi^-_{l^-} + 2\chi^+_{l^+-\frac{1}{2}}\chi^-_{l^--1} + 2\chi^+_{l^+-1}\chi^-_{l^--\frac{1}{2}} \\ &+ \chi^+_{l^+-1}\chi^-_{l^--\frac{3}{2}} + \chi^+_{l^+-\frac{3}{2}}\chi^-_{l^--1}\Big)\, q^h + \dots,\end{aligned} \tag{5.4.76}$$

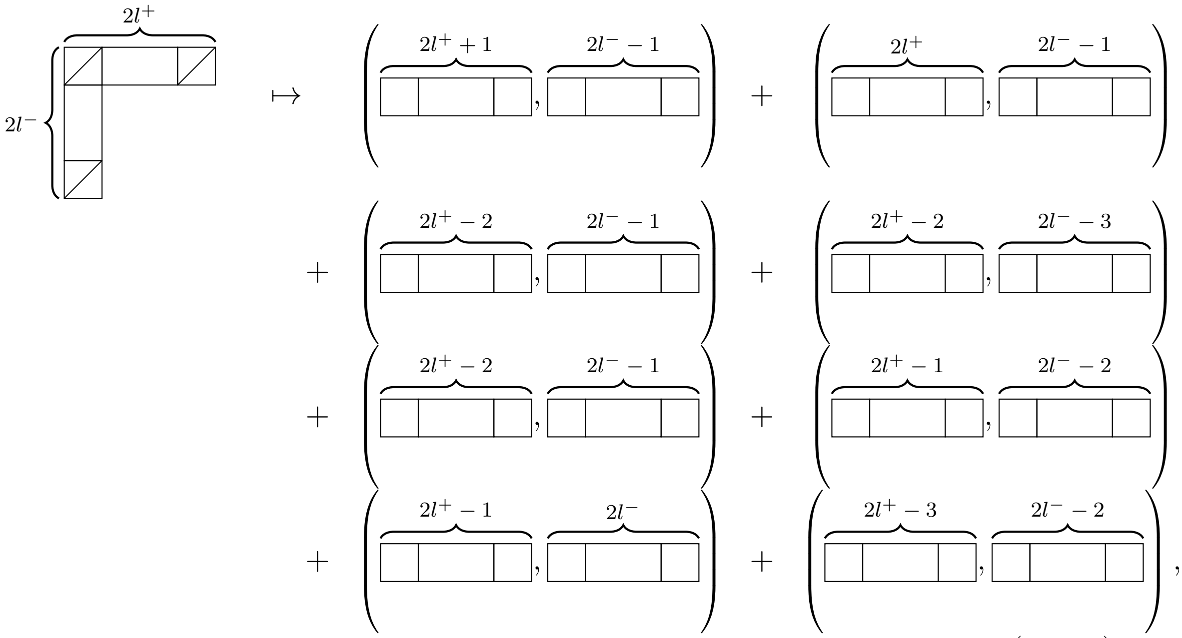
where

$$\chi^\pm_l := \chi_l(z_\pm)$$

is the $\mathfrak{su}(2)^\pm$ character for a representation of dimension $2l+1$. We now want to calculate the branching of



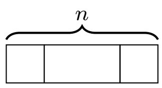
to check the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ content of this representation.



(5.4.77)

where we have suppressed the $\mathfrak{u}(1)$ charges and simplified trivial columns of length 2 on the right hand side.

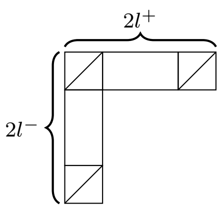
The representation of $\mathfrak{su}(2)$ described by



is the $(n+1)$-dimensional representation with character

$$\chi_{\frac{n}{2}}(z),$$

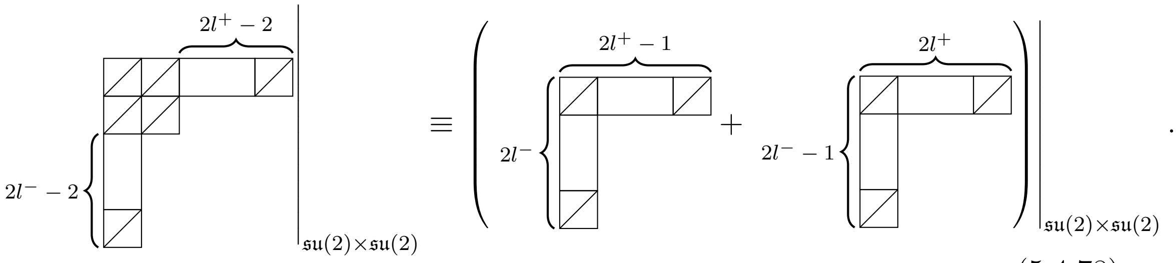
and so by comparing eq. (5.4.76) and eq. (5.4.77) we see that the ground level of a massless representation of A_γ in the Ramond sector with $\mathfrak{su}(2)$ charges l^+ and l^- is described by the supertableau



as claimed. □

Similarly we can recognise the $\mathfrak{su}(2|2)$ representation that describes the ground level of a massive representation of A_γ in the Ramond sector. We first give a lemma on the branching of non-maximally eccentric supertableau that will be useful for the massive case.

Lemma 5.4.24. *Under branching into* $\mathfrak{su}(2) \times \mathfrak{su}(2)$ *representations we have the following equivalence:*



(5.4.78)

Proof. This is proved simply by branching both sides and checking that they agree. □

Proposition 5.4.25. *The ground level of a unitary irreducible representation of* A_γ *described by parameters* k^+, k^- *and quantum numbers* l^+, l^- *is described by a single representation of the superalgebra* $\mathfrak{su}(2|2)$*, which is in turn described by a non-maximally eccentric Young supertableau of eccentricity* $(2l^+, 2l^-)$*, as shown in fig. 5.15.*

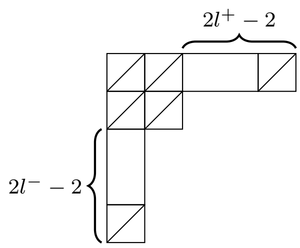
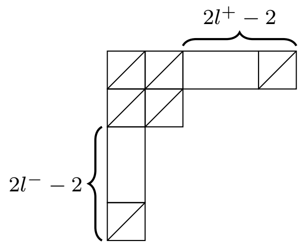


Figure 5.15: A supertableau which describes the ground level of a representation of A_γ with $\mathfrak{su}(2)$ quantum numbers l^+, l^-.

Proof. In section 3.3 we saw that the generic massive Ramond representation of A_γ had 16 highest weight states of $\mathfrak{su}(2) \times \mathfrak{su}(2)$, as shown in fig. 3.3a. We therefore have that the ground level of A_γ is given by

$$\begin{aligned}
\mathrm{Ch}^{A_\gamma,R}_{m,l^+,l^-} = \Big(&\chi^+_{l^+}\chi^-_{l^--1} + 2\chi^+_{l^+-\frac{1}{2}}\chi^-_{l^--\frac{1}{2}} + 2\chi^+_{l^+-\frac{1}{2}}\chi^-_{l^--\frac{3}{2}} \\
&+ \chi^+_{l^+-1}\chi^-_{l^-} + 4\chi^+_{l^+-1}\chi^-_{l^--1} + \chi^+_{l^+-1}\chi^-_{l^--2} \\
&+ 2\chi^+_{l^+-\frac{3}{2}}\chi^-_{l^--\frac{1}{2}} + 2\chi^+_{l^+-\frac{3}{2}}\chi^-_{l^--\frac{3}{2}} + \chi^+_{l^+-2}\chi^-_{l^--1}\Big) q^h + \dots,
\end{aligned} \tag{5.4.79}$$

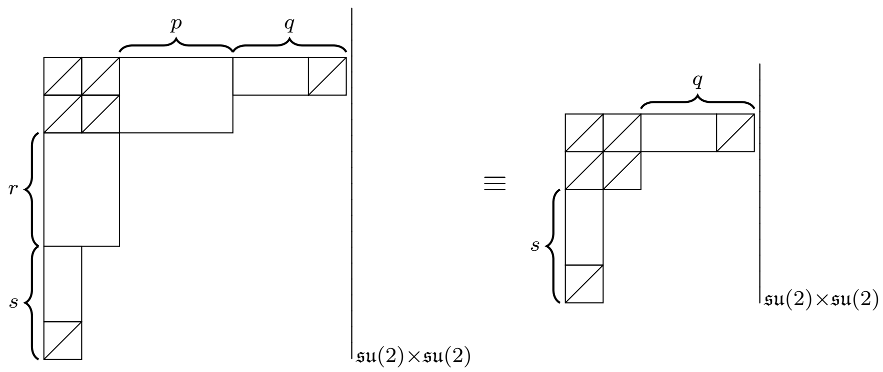
Using lemma 5.4.24 and the branching of a maximally eccentric supertableau given in the proof of proposition 5.4.23, it is simple to check that the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ multiplets which appear in the branching of the representation of $\mathfrak{su}(2|2)$ shown in fig. 5.15 agree with eq. (5.4.79). We therefore see that the ground level of a massive representation of A_γ in the Ramond sector with $\mathfrak{su}(2)$ charges l^+ and l^- is described by the supertableau



as claimed. □

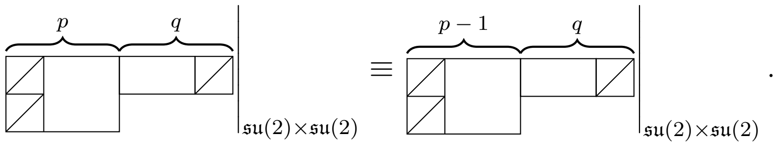
Before commenting on the index of A_γ, we note one more useful fact about the branching of $\mathfrak{su}(2|2)$ supertableaux.

Proposition 5.4.26. *The generic non-zero totally-contravariant representation of* $\mathfrak{su}(2|2)$ *described by a supertableau as shown below satisfies the following equivalence under branching into* $\mathfrak{su}(2) \times \mathfrak{su}(2)$ *representations:*



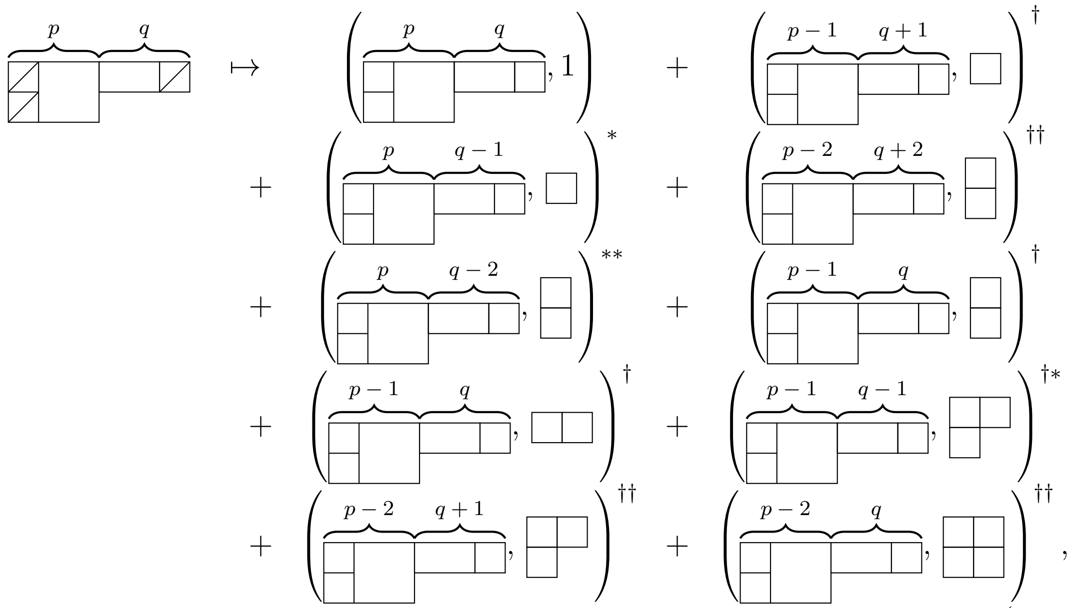
. (5.4.80)

Proof. **Step 1:** We argue that we have the branching equivalence for $p > 2$



(5.4.81)

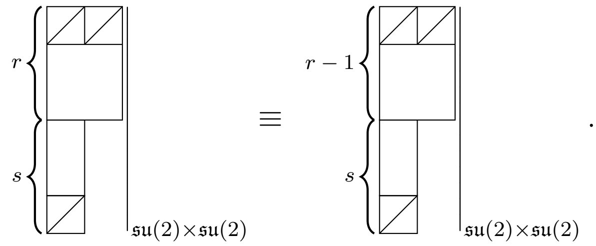
To show this, we first calculate the branching of a supertableau of the type shown in eq. (5.4.81):



(5.4.82)

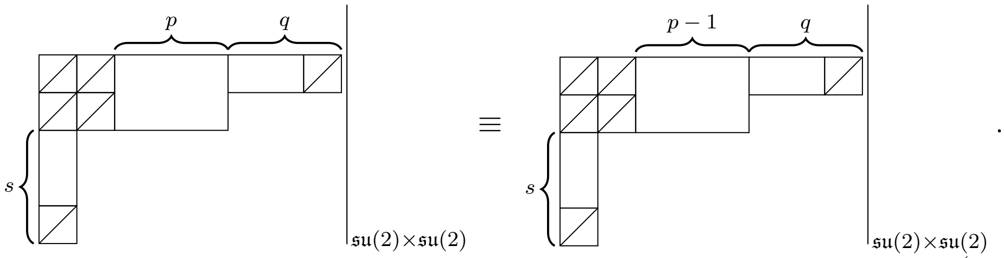
where we have not simplified trivial columns of two boxes on the right hand side. The representations indicated by * will appear only for $q \geq 1$ and the representation indicated by ** will appear only for $q \geq 2$. Similarly, the representations indicated by † will only appear for $p \geq 1$ and the representations indicated by †† will appear only for $p \geq 2$. Therefore all representations appear when $p \geq 2, q \geq 2$. Since the block of columns of length two may be trivially cancelled for $\mathfrak{su}(2)$, we will get an equivalent set of representations of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ on both sides of eq. (5.4.81) if $p > 2$.

Now as noted in the proof of corollary 5.4.17, if a supertableau contains the branching component $(T_1, (T_2)^t)$ then the transposed supertableau T^t contains the component $((T_2)^t, T_1)$. We therefore immediately get the following equivalence for $r > 2$ as a corollary to the previous equivalence:



(5.4.83)

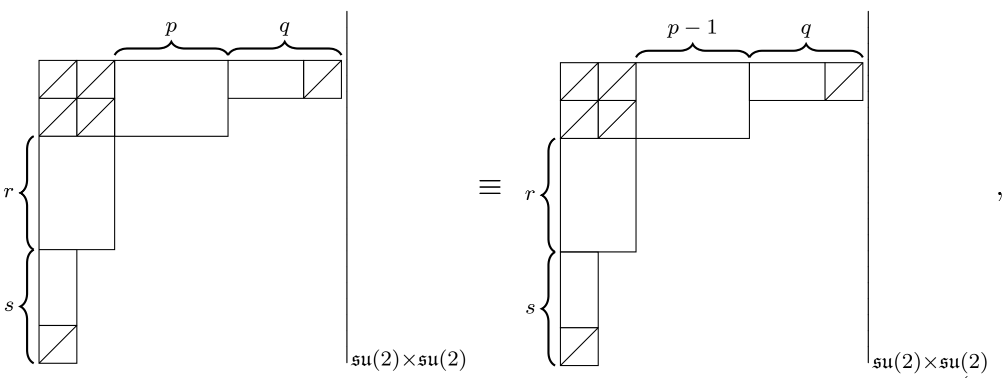
Step 2: We argue that the equivalence



(5.4.84)

for $p > 1$ follows easily from eq. (5.4.81). This is done by noting that the column of length s must be moved to the right hand factor of $\mathfrak{su}(2)$ and transposed, otherwise the left hand factor of $\mathfrak{su}(2)$ will have a column of length > 3 and hence will be a zero representation. Clearly the result of moving this column over to the right hand factor, taking the appropriate tensor products where necessary, does not affect the p dependence of the branching. It is therefore clear that after cancelling trivial columns of length two, the branching of the two sides of eq. (5.4.84) agree as long as we have $p > 1$.

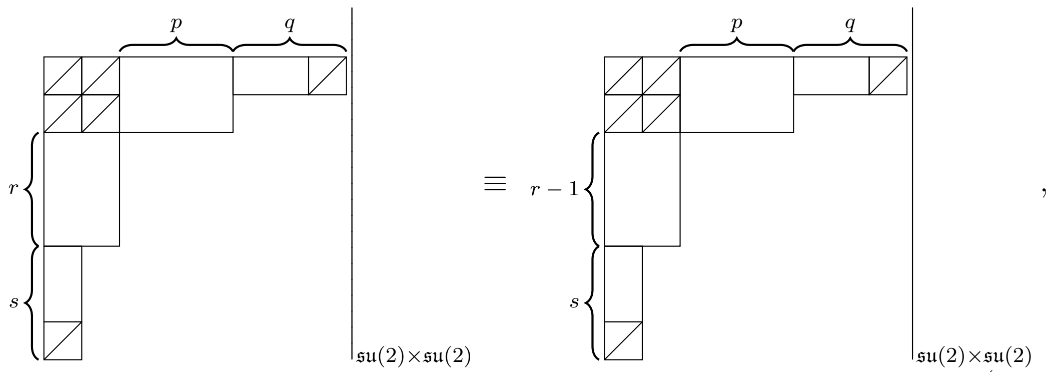
Step 3: This previous step can trivially be extended to give the branching equivalence



(5.4.85)

for $p > 1$ using the same argument as for the previous step, except for now we must clearly take the two columns of lengths $r + s$ and s to the right hand side, again taking tensor products where necessary. Again, this will not affect the p dependence of the branching and so after cancelling trivial columns of length two, the branching of the two sides of eq. (5.4.85) agree as long as we have $p > 1$.

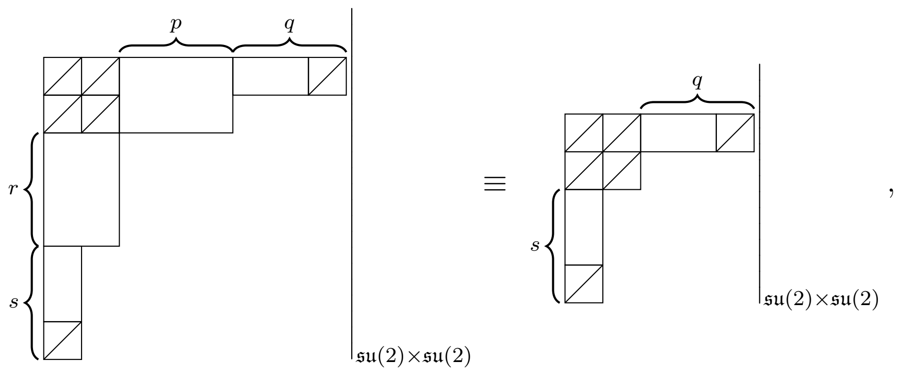
Step 4: Finally we again use the proof of corollary 5.4.17 to obtain the equivalence



(5.4.86)

for $r > 1$.

Using Step 3 p times and Step 4 r times we now obtain



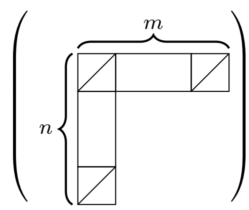
(5.4.87)

as required.

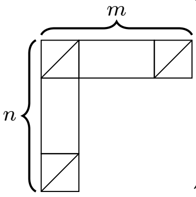
□

We can now finally calculate the index I_1 for all totally covariant supertableaux as appear in our decompositions of A_γ representations.

Proposition 5.4.27.

$\mathcal{I}_1$  $= (-1)^n \left(z^{-m-n+1} - z^{m+n-1}\right).$ (5.4.88)

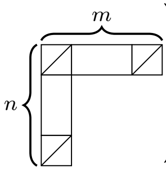
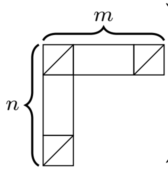
Proof. In the proof of proposition 5.4.23 we calculated the branching of a maximally extremal supertableau into $\mathfrak{su}(2) \times \mathfrak{su}(2)$ representations and checked that the $\mathfrak{su}(2)$ characters contained in this branching agree with the $\mathfrak{su}(2)$ characters that appear in a massless representation of A_γ at the ground level as given in eq. (5.4.76). We therefore have

$$\mathrm{Ch}\left(\right.$$

$$\left.\right) = \chi^+_{\frac{m}{2}}\chi^-_{\frac{n-1}{2}} + \chi^+_{\frac{m-1}{2}}\chi^-_{\frac{n}{2}} + 2\chi^+_{\frac{m-1}{2}}\chi^-_{\frac{n}{2}-1} + 2\chi^+_{\frac{m}{2}-1}\chi^-_{\frac{n-1}{2}} + \chi^+_{\frac{m}{2}-1}\chi^-_{\frac{n-3}{2}} + \chi^+_{\frac{m-3}{2}}\chi^-_{\frac{n}{2}-1}. \tag{5.4.89}$$

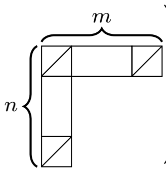
In this sense, we think of the supertableaux as describing the representation content of A_γ in the Ramond sector. Recall that the contribution to the index of a representation of A_γ is given by

$$\mathcal{I}_1\left(\mathrm{Ch}^{A_\gamma,R}\right) := -z_+\frac{\partial}{\partial z_-}\,\mathrm{Ch}^{A_\gamma,\tilde{R}}\Big|_{z_-=z_+\equiv z}, \tag{5.4.90}$$

therefore to calculate the index we need to flow to the $\tilde{R}$ sector, that is to consider the supercharacter rather than the character of the representation of $\mathfrak{su}(2|2)$,

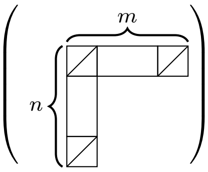
$$\mathrm{SCh}\left(\right.$$

$$\left.\right)(z_+, z_-) := \mathrm{Ch}\left(\right.$$

$$\left.\right)(z_+, -z_-). \tag{5.4.91}$$

By some straightforward algebra one then obtains,

$$\mathrm{SCh}\left(\right.$$

$$\left.\right)(z_+, z_-) = \left(\chi_{\frac{1}{2}}(z_+) - \chi_{\frac{1}{2}}(z_-)\right)\Big((-1)^{n-1}\chi_{\frac{m-1}{2}}(z_+)\chi_{\frac{n-1}{2}}(z_-) + (-1)^n\chi_{\frac{m}{2}-1}(z_+)\chi_{\frac{n}{2}-1}(z_-)\Big). \tag{5.4.92}$$

The index I_1 as defined in eq. (5.4.90) is evaluated at $z_+ = z_-$ and clearly we have $\left(\chi_{\frac{1}{2}}(z_+) - \chi_{\frac{1}{2}}(z_-)\right)\Big|_{z_+=z_-} = 0$. Therefore we need only consider the term where the differential $\frac{\partial}{\partial z_-}$ is applied to this zero. We therefore have

$\mathcal{I}_1$



$$= (-1)^{n+1}\left(z^{-1}-z\right)\Big(\chi_{\frac{m}{2}-1}(z)\chi_{\frac{n}{2}-1}(z) - \chi_{\frac{m-1}{2}}(z)\chi_{\frac{n-1}{2}}(z)\Big). \tag{5.4.93}$$

We now use the identity

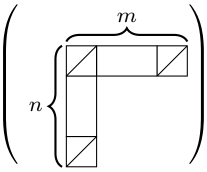
$$\chi_l(z) = \frac{z^{-2l} - z^{2(l+1)}}{1-z^2}, \tag{5.4.94}$$

to show that

$$\Big(\chi_{\frac{m}{2}-1}(z)\chi_{\frac{n}{2}-1}(z) - \chi_{\frac{m-1}{2}}(z)\chi_{\frac{n-1}{2}}(z)\Big) = -\chi_{\frac{m+n}{2}-1}(z). \tag{5.4.95}$$

Substituting this into eq. (5.4.93) we finally obtain

$\mathcal{I}_1$



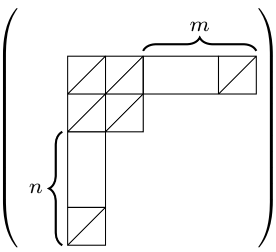
$$\begin{aligned} &= (-1)^{n}\left(z^{-1}-z\right)\chi_{\frac{m+n}{2}-1}(z), \\ &= (-1)^{n}\left(z^{-m-n+1}-z^{m+n-1}\right). \end{aligned} \tag{5.4.96}$$

□

We now have the immediate corollary due to lemma 5.4.24.

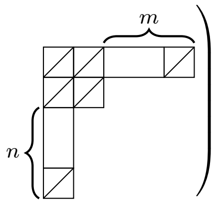
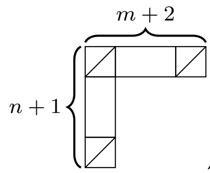
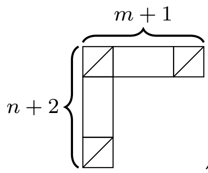
Corollary 5.4.28.

$\mathcal{I}_1$



$$= 0. \tag{5.4.97}$$

Proof. Since the index at a given level of A_γ is dependent only on the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ information, we simply use the branching of the supertableau in lemma 5.4.24 to obtain

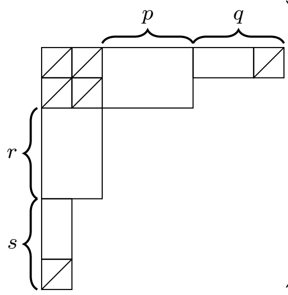
$\mathcal{I}_1 \Bigg($  $\Bigg) = \mathcal{I}_1 \Bigg($  $\Bigg) + \mathcal{I}_1 \Bigg($  $\Bigg),$

$$=(-1)^{n+1}\left(z^{-m-n-2} - z^{m+n+2}\right) + (-1)^{n}\left(z^{-m-n-2} - z^{m+n+2}\right),$$
$$=0. \tag{5.4.98}$$

□

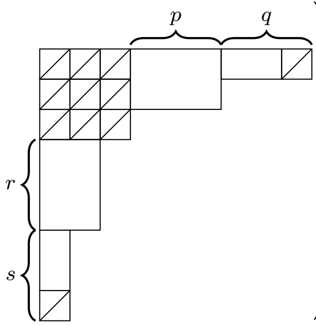
We also have the following corollary due to proposition 5.4.26.

Corollary 5.4.29.

$\mathcal{I}_1 \Bigg($  $\Bigg) = 0,$ (5.4.99)

which follows immediately from corollary 5.4.28 and proposition 5.4.26.

Finally, since they give zero representations, we clearly have

$\mathcal{I}_1 \Bigg($  $\Bigg) = 0,$ (5.4.100)

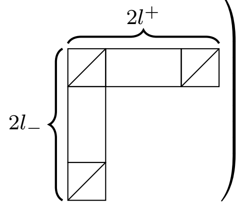
as well as

$$\mathcal{I}_1(T) = 0, \tag{5.4.101}$$

for any tableau larger than those already considered.

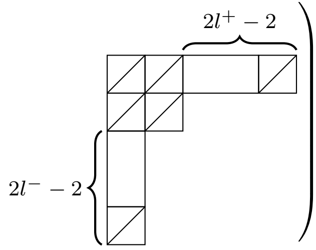
Since they are the only supertableaux with non-zero index, we now see that the only contributions to I_1 from representations of A_γ come from these maximally eccentric representations of the zero mode subalgebra $\mathfrak{su}(2|2)$. Using eq. (5.2.5) we can therefore summarise the $\mathfrak{su}(2|2)$ representation content of A_γ relevant to I_1 as

$$\mathrm{Ch}^{A_\gamma,R}_{0,l^+,l^-} =$$



$$q^h + (\text{non-maximally eccentric } \mathfrak{su}(2|2) \text{ representations})\, q^{h+1} + \ldots + q^{\frac{\mu^2+4u^2}{4k}+k-\mu}(\text{maximally eccentric representation with } 2k-\mu \text{ boxes}) + \ldots$$

$$\mathrm{Ch}^{A_\gamma,R}_{m,l^+,l^-} =$$



$$q^h + (\text{non-maximally eccentric } \mathfrak{su}(2|2) \text{ representations})\, q^{h+1} + \ldots . \tag{5.4.102}$$

Now that we have a more comprehensive understanding of the states contributing to I_1, we calculate the index for a set of A_γ theories in the following chapter.

Chapter 6

A_γ Character Sum Rules and the Index I_1

Having developed a deeper understanding of the type of states counted by the new index I_1 in the context of 'abstract' unitary highest weight representations of A_γ and $\tilde{A}_\gamma$ in the previous chapter, we now pave the way for the potential discovery of a new moonshine phenomenon. In order to do so, we first return in Section 6.1 to the super WZW model describing superstrings propagating on the 8-dimensional quaternionic group manifold $SU(3)$ that was reviewed in sections 2.2.2 and 2.2.3 and exploit character sum rules introduced in [OPT92; PT93] in this context. One of the challenges with the sum rules is to identify their explicit dependence in A_γ massive characters, with a view to constructing explicit modular invariant partition functions in terms of A_γ massless and massive characters. Once such a partition function has been constructed, the new index I_1 may be applied to the right moving massless and massive A_γ characters, in analogy with the procedure followed with the Witten index in $K3$ theories which led to Mathieu moonshine. In Section 6.2, we identify an infinite set of A_γ massive characters appearing in every sum rule, encoded as threshold characters multiplied by q-series $F_i(q)$ ($q = \exp 2\pi i\tau$) for which we do not have a closed form in most cases at present. One can however proceed and apply the index I_1 on diagonal partition functions of interest and analyse the results obtained. This is sketched in Section 6.3, and is work in progress [FTT18].

6.1 Character Sum Rules

We saw in sections 2.2.2 and 2.2.3 that the super WZW model with the $SU(3)$ group manifold as target space possesses A_γ symmetry. The underlying SCA can be characterised by two integers $k^\pm$ which are the levels of the two $\widehat{\mathfrak{su}(2)}^\pm$ Kac-Moody subalgebras of A_γ. We also know from Section 3.1 that four free fermions and one boson corresponding to the $\widehat{\mathfrak{u}(1)}$ subalgebra of A_γ can be decoupled, leaving the non-linear SCA $\tilde{A}_\gamma$ characterised by the levels $\tilde{k}^\pm = k^\pm - 1$ of its two $\widehat{\mathfrak{su}(2)}^\pm$ Kac-Moody algebras. We are interested in realisations of $\tilde{A}_\gamma$ on $SU(3)$, built by defining the action of $\tilde{A}_\gamma$ on $W(3)$ and on $SU(2) \times U(1)$ where $W(3)$ is a 4-dimensional Wolf space to which 4 free fermionic fields are associated as in section 2.2.3. We naturally call these fermions the 'Wolf space fermions'. These are used to construct a realisation of the $\widehat{\mathfrak{su}(2)}^-$ currents, while the $\widehat{\mathfrak{su}(2)}^+$ currents may be constructed from the currents associated to the highest root $\widehat{\mathfrak{su}(2)}$ subalgebra of $\widehat{\mathfrak{su}(N)}_{\tilde{k}^+}$, where $N := k^- + 1$. Hereafter we choose $k^- = 2$ for simplicity.

In [PT93], the authors consider the space,

$$\mathcal{H}(\Lambda, \tilde{k}^+) := \mathcal{H}^{WS} \otimes \mathcal{H}_\Lambda^{\widehat{su(3)}_{\tilde{k}^+}}, \tag{6.1.1}$$

where $\mathcal{H}^{WS}$ is the Fock space of the 4 Wolf space fermions and $\mathcal{H}_\Lambda^{\widehat{su(3)}_{\tilde{k}^+}}$ is an $\widehat{\mathfrak{su}(3)}_{\tilde{k}^+}$ module with highest weight Λ. This space is shown to provide not only representations for $\tilde{A}_\gamma$, but also the rational torus algebra $\mathcal{A}_{3k}$, an extension of the $\hat{\mathfrak{u}}(1)$ subalgebra of A_γ.

The aforementioned equivalence between representation spaces,

$$\mathcal{H}(\Lambda, \tilde{k}^+) = \bigoplus_i \left({\mathcal{H}^{\tilde{A}_\gamma}}_{0,l_i^+,l_i^-} \otimes \mathcal{H}^{\mathcal{A}_{3k}}_{m_i} \right) \bigoplus_j \left(\oplus_n {\mathcal{H}^{\tilde{A}_\gamma}}_{h_n,l_j^+} \otimes \mathcal{H}^{\mathcal{A}_{3k}}_{m_j} \right) \tag{6.1.2}$$

leads to the character sum rules, which can be expressed in the NS sector as

$$\chi^{WS,NS}\chi_{\Lambda}^{\mathfrak{su}(3)} = \{\chi^{WS,NS}\chi_{\Lambda}^{\mathfrak{su}(3)}\}_0 + \{\chi^{WS,NS}\chi_{\Lambda}^{\mathfrak{su}(3)}\}_m, \tag{6.1.3}$$

where $\{\chi^{WS,NS}\chi_{\Lambda}^{\mathfrak{su}(3)}\}_0$ contains the contributions from the massless representations of $\tilde{A}_\gamma$ and $\{\chi^{WS,NS}\chi_{\Lambda}^{\mathfrak{su}(3)}\}_m$ contains the massive contributions. We will now explain this sum rule in more detail, showing how the massless and massive parts are written in terms of $\tilde{A}_\gamma$ and $\mathcal{A}_{3k}$ characters.

Firstly, we will need expressions for the various characters involved in the sum rules. The character for the Wolf space fermions in the NS sector is given by

$$\begin{aligned}\chi^{WS,NS}(q,z_-,z_y) &= q^{-1/12}\prod_{n\geq 1}(1+z_-z_yq^{n-1/2})(1+z_-^{-1}z_y^{-1}q^{n-1/2}) \\ &\quad\times(1+z_-z_y^{-1}q^{n-1/2})(1+z_-^{-1}z_yq^{n-1/2}), \\ &= \frac{\theta_3(q,z_-z_y)}{\eta(q)}\frac{\theta_3(q,z_-z_y^{-1})}{\eta(q)},\end{aligned} \tag{6.1.4}$$

where $\theta_3(q,z)$ is defined as in Appendix B.

The other character we need to construct the left-hand side of the character sum rule is that of an $\widehat{\mathfrak{su}(3)}_{\tilde{k}+}$ representation with highest (affine) weight Λ. We therefore make a few definitions for characters of affine Lie algebras.

Definition 6.1.1. The character of an integrable highest weight representation of an affine Lie algebra $\hat{\mathfrak{g}}$ with highest weight Λ is given by,

$$\mathrm{ch}(\Lambda) = \sum_{\lambda\in\Omega_\Lambda}\mathrm{mult}_\Lambda(\lambda)e^\lambda, \tag{6.1.5}$$

where Ω_Λ is the weight space of the representation and $\mathrm{mult}_\Lambda(\lambda)$ denotes the multiplicity of the weight λ in the representation.

Such a character may be evaluated at an affine weight $\xi = -2\pi i(\zeta,\tau,t)$ where – up to the factor of $-2\pi i$ – ζ is the finite weight, τ the level and t the grade (of ξ), by defining

$$e^\lambda(\xi) := e^{(\lambda,\xi)}. \tag{6.1.6}$$

In order to calculate the $\widehat{\mathfrak{su}(3)}_{\tilde{k}+}$ characters of interest, we will make use of the *affine*

Weyl-Kac formula, written in terms of generalised theta functions.

Definition 6.1.2. Given two affine weights Λ and $\xi = -2\pi i(\zeta, \tau, t)$ of an affine Lie algebra $\widehat{\mathfrak{g}}$, we define the *generalised theta function* evaluated at ξ as,

$$\Theta_\Lambda(\xi) = e^{-2\pi i k t} \sum_{\alpha^\vee \in Q^\vee} e^{-\pi i[2k(\alpha^\vee,\zeta)+2(\lambda,\zeta)-\tau k|\alpha^\vee+\lambda/k|^2]}, \tag{6.1.7}$$

where $\alpha^\vee$ is a coroot in the coroot lattice $Q^\vee$ and λ is the finite part of Λ.

The affine Weyl-Kac formula now gives us an expression for *normalised* characters $\chi_\Lambda(\xi)$ of the module Λ of $\widehat{\mathfrak{g}}$ evaluated at the point ξ as [DMS97],

$$\chi_\Lambda(\xi) = \frac{\sum_{\omega \in W} \epsilon(\omega)\Theta_{\omega(\Lambda+\hat{\rho})}(\xi)}{\sum_{\omega \in W} \epsilon(\omega)\Theta_{\omega\hat{\rho}}(\xi)}, \tag{6.1.8}$$

where W is the (finite) Weyl group of $\widehat{\mathfrak{g}}$, $\epsilon(\omega) \in \{\pm 1\}$ is the *signature* of ω and $\hat{\rho}$ is the affine Weyl vector. We evaluate the character at the point $\xi = -2\pi i(\sum_{i=1}^r z_i \alpha_i^\vee, \tau, 0)$ such that this character then calculates [DMS97]

$$\chi_\Lambda(\tau, z) := \chi_\Lambda(\xi) = \mathrm{Tr}_\Lambda\, e^{2\pi i \tau L_0} e^{-2\pi i \sum_{j=1}^r z_j h^j}, \tag{6.1.9}$$

where h^j is a generator of the affine Lie algebra in the Chevalley basis and we used the shorthand $z = (z_1, \ldots, z_r)$.

The derivation of the characters for $\tilde{A}_\gamma$ was outlined in section 3.4, and the formulae for the massless and massive characters of $\tilde{A}_\gamma$ in the NS sector may be found in eqs. (3.4.13) and (3.4.15). In the character formulas in [PT93; OPT92] however, the massless $\tilde{A}_\gamma$ characters appear in the linear combinations,

$$\begin{aligned}
\mathrm{Ch}_0^{\tilde{A}_\gamma,NS}(L=0;q,z_\pm) :=& -\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}\left(\tilde{l}^+ = 0, \tilde{l}^- = \frac{1}{2}; q, z_\pm\right),\\
\mathrm{Ch}_0^{\tilde{A}_\gamma,NS}(L=1,\ldots,k-3;q,z_\pm) :=& \frac{1}{2}\left[\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}\left(\tilde{l}^+ = \frac{1}{2}(L-1), \tilde{l}^- = 0; q, z_\pm\right)\right.\\
&\left. - \mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}\left(\tilde{l}^+ = \frac{L}{2}, \tilde{l}^- = \frac{1}{2}; q, z_\pm\right)\right],\\
\mathrm{Ch}_0^{\tilde{A}_\gamma,NS}(L=k-2;q,z_\pm) :=& -\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}\left(\tilde{l}^+ = \frac{1}{2}(k-3), \tilde{l}^- = 0; q, z_\pm\right),
\end{aligned} \tag{6.1.10}$$

where the $\tilde{k}^\pm$ labels and $(q, z_\pm)$ dependency has been suppressed for legibility.

We consider only *dominant highest weight* representations of $\widehat{\mathfrak{su}(3)}$, which are necessarily integrable representations and this guarantees the unitarity of the representation spaces.

Definition 6.1.3. A highest weight representation of an affine Lie algebra $\widehat{\mathfrak{g}}$ with highest weight Λ is said to be *dominant* if all the Dynkin labels of the highest weight are non-negative integers. That is if in the basis of fundamental weights

$$\Lambda = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i + l\delta, \tag{6.1.11}$$

where $\hat{\omega}_i$ are the fundamental weights and δ is the imaginary root, the Dynkin labels λ_i are all non-negative integers.

Since the zeroth Dynkin index satisfies

$$\lambda_0 = k - (\lambda, \theta), \tag{6.1.12}$$

where λ is the finite part of Λ and θ is the longest root, we see that for a dominant highest weight representation we have

$$k \geq (\lambda, \theta) = \sum_{i=1}^{r} a_i^\vee \lambda_i. \tag{6.1.13}$$

There are therefore a finite number of dominant highest weight representations of $\widehat{\mathfrak{g}}$. For $\widehat{\mathfrak{su}(N)}$, the comarks $a_i^\vee$ are all 1, and hence we have

$$\lambda_0 = k - \sum_{i=1}^{r} \lambda_i, \tag{6.1.14}$$

with the requirement that all $\lambda_i \geq 0$. We label the set of dominant weights at level k P_+^k. If we therefore consider $\widehat{\mathfrak{su}(3)}$ at level 2, the possible dominant highest weights of interest are then given in terms of Dynkin labels as

$$P_+^2 = \{[0,0,2], [0,1,1], [0,2,0], [1,0,1], [1,1,0], [2,0,0]\}. \tag{6.1.15}$$

For consistency with [PT93] we label the finite part of Λ as $\lambda = (a_1, a_2)$ with

$a_{1,2} \in \mathbb{Z}_{\geq 0}$ and by the above, satisfying

$$a_1 + a_2 \leq \tilde{k}^+, \tag{6.1.16}$$

for a representation of $\widehat{\mathfrak{su}(3)}_{\tilde{k}^+}$.

Massive $\tilde{A}_\gamma$ characters appear in the sum rules at threshold.

Definition 6.1.4. A massive character of A_γ ($\tilde{A}_\gamma$) is said to be at *threshold* if the conformal charge h ($\tilde{h}$) is taken at the saturating value of the massless bound eq. (3.2.13). These characters (written here in the case of A_γ but equivalently for $\tilde{A}_\gamma$) are then denoted,

$$\hat{\mathrm{Ch}}_m^{A_\gamma,NS}(k^\pm, l^\pm; q, z+\pm) = q^{-\Delta h}\, \mathrm{Ch}^{A_\gamma,NS}(k^\pm, l^\pm, h; q, z_\pm), \tag{6.1.17}$$

where

$$\Delta h = h - \frac{1}{k}(k^+ l^- + k^- l^+ + (l^+ - l^-)^2 + u^2). \tag{6.1.18}$$

Finally, the characters of $\mathcal{A}_{3k}$ are given by [DVVV89],

$$\chi_m^{3k}(q, z_y) := \frac{1}{\eta(q)} \sum_{n\in\mathbb{Z}} q^{3k\left(n+\frac{m}{6k}\right)^2} z_y^{2k\left(n+\frac{m}{6k}\right)} = \frac{\theta_{m,3k}(q, z_y^{2/3})}{\eta(q)}. \tag{6.1.19}$$

We can now give the expressions for the massive and massless parts of the sum rule as [PT93],

$$\{\chi^{WS,NS}(q, z_-, z_y)\chi_\Lambda^{\mathfrak{su}(3)}\}_m = \sum_{2\tilde{l}^+=0}^{\tilde{k}^+-1} \sum_{n\in\mathbb{Z}_k} \hat{M}_{2l_+,n}^\Lambda(q, z_+, z_-, z_y) F_{2\tilde{l}^+,n}^\Lambda(q), \tag{6.1.20}$$

where the massive matrix $\hat{M}_{2l_+,n}^\Lambda$ is given by,

$$\hat{M}_{2l_+,n}^\Lambda(q, z_+, z_-, z_y) = \hat{\mathrm{Ch}}_m^{\tilde{A}_\gamma,NS}(l^\pm; q, z_\pm)\chi_{-2a_1+2a_2+6\tilde{l}^++6n}^{3k}(q, z_y). \tag{6.1.21}$$

and

$$\{\chi^{WS,NS}(q, z_-, z_y)\chi_\Lambda^{\mathfrak{su}(3)}(q, z_+, z_y\}_0 = \sum_{L=0}^{k-2} M_\Lambda^L(q, z_y)\, \mathrm{Ch}_0^{\tilde{A}_\gamma,NS}(L; q, z_\pm), \tag{6.1.22}$$

where the massless matrix M^L_Λ is given by,

$$\begin{aligned}M^L_\Lambda(q,z_y) =& -\delta_{L,0}\Big[\delta_{a_1,0}\chi^{3k}_{2a_2+3}(q,z_y)+\delta_{a_2,0}\chi^{3k}_{-2a_1-3}(q,z_y)\\&+\delta_{a_1+a_2,k-3}\chi^{3k}_{a_1-a_2+3k}(q,z_y)\Big]\\&+\delta_{L,k-2}\Big[\delta_{a_1,0}\chi^{3k}_{2a_2-3(k-1)}(q,z_y)+\delta_{a_2,0}\chi^{3k}_{-2a_1+3(k-1)}(q,z_y)\\&+\delta_{a_1+a_2,k-3}\chi^{3k}_{a_1-a_2}(q,z_y)\Big]\\&+(1-\delta_{L,0})(1-\delta_{L,k-2})\Big[\delta_{L,k-2-a_1}\chi^{3k}_{a_1+2a_2-3(k-1)}(q,z_y)\\&+\delta_{L,k-2-a_2}\chi^{3k}_{-2a_1-a_2+3(k-1)}(q,z_y)+\delta_{L,a_1+a_2+1}\chi^{3k}_{a_1-a_2}(q,z_y)\\&-\delta_{L,a_1}\chi^{3k}_{a_1+2a_2+3}(q,z_y)-\delta_{L,a_2}\chi^{3k}_{-2a_1-a_2-3}(q,z_y)\\&-\delta_{L,k-3-a_1-a_2}\chi^{3k}_{a_1-a_2+3k}(q,z_y)\Big]\,,\end{aligned} \tag{6.1.23}$$

6.2 The Functions $F_i(q)$

The only n dependence in the massive matrix given by eq. (6.1.21) is in the m index of the rational torus model character χ^{3k}_m, which gives three times the $\mathfrak{u}(1)$ charge of the highest weight state of the torus module modulo $6k$. These characters are therefore identical under $m \to m+6kn$ for $n \in \mathbb{Z}$, as can clearly be seen from their character formula in eq. (6.1.19). This implies that the q-series $F^\Lambda_{2\tilde{l}^+,n}$ – which when multiplied by the finite number of threshold characters in the massive matrix give the contribution from an infinite number of massive characters not at threshold – are also only well defined for n modulo $\mathbb{Z}_k$. That is, these functions satisfy

$$F^\Lambda_{2\tilde{l}^+,n}(q) = F^\Lambda_{2\tilde{l}^+,n+k}(q). \tag{6.2.1}$$

These functions can also be shown to satisfy the following equivalences,

$$\begin{aligned}&F^\Lambda_{2\tilde{l}^+,n}(q) = F^{\Lambda_C}_{2\tilde{l}^+,-n-2\tilde{l}^+}(q), \qquad F^\Lambda_{2\tilde{l}^+,n}(q) = F^\Lambda_{\tilde{k}^+-1-2\tilde{l}^+,n+2\tilde{l}^++2}(q),\\&F^\Lambda_{2\tilde{l}^+,n}(q) = F^{\phi^\epsilon(\Lambda)}_{2\tilde{l}^+,n+\epsilon+\frac{1}{2}(\epsilon-1)a_1+\frac{1}{2}(\epsilon+1)a_2}(q),\end{aligned} \tag{6.2.2}$$

where for $\Lambda = \Big((a_1,a_2),\tilde{k}^+,0\Big)$, Λ_C is the conjugate representation given by

$$\Lambda_C = \Big((a_2,a_1),\tilde{k}^+,0\Big)\,, \tag{6.2.3}$$

and with $\epsilon = \pm 1$, the order 3 function ϕ is defined by

$$\phi(a_1, a_2) := (a_2, \tilde{k}^+ - (a_1 + a_2)), \qquad \phi(\Lambda) := (\phi(a_1, a_2), \tilde{(k^+)}, 0). \tag{6.2.4}$$

As an example we now give the sum rule for the representation $\Lambda = ((0,0),2,0)$ of $\widehat{\mathfrak{su}(3)}_2$. Although the sum rules in [PT93] are written in terms of the linear combinations of massless characters introduced in eq. (6.1.10), since we will shortly discuss partition functions we find it clearer to have all modules appearing with positive integer multiplicities. We therefore use eq. (3.4.17) to re-write massive threshold characters as sums of massless characters in order to achieve this.

Example 6.2.1. As stated at the start of this subsection, we take $\tilde{k}^- = 1$ and we now also fix $\tilde{k}^+ = 2$. We then consider the singlet representation of $\widehat{\mathfrak{su}(3)_2}$ with $\Lambda = ((0,0),2,0)$. The sum rule in this case is then,

$$\begin{aligned}
\chi^{WS,NS}(q, z_-, z_y)\chi_\Lambda^{\mathfrak{su}(3)}(q, z_+, z_y) = {} & \mathrm{Ch}_0^{\tilde{A}_\gamma, NS}\left(0, \frac{1}{2}; q, z_\pm\right)\left[\chi_3^{15}(q, z_y) + \chi_{-3}^{15}(q, z_y)\right] \\
& + \mathrm{Ch}_0^{\tilde{A}_\gamma, NS}(0, 0; q, z_\pm)\,\chi_0^{15}(q, z_y) + \mathrm{Ch}_0^{\tilde{A}_\gamma, NS}\left(1, \frac{1}{2}; q, z_\pm\right)\chi_{15}^{15}(q, z_y) \\
& + \mathrm{Ch}_0^{\tilde{A}_\gamma, NS}(1, 0; q, z_\pm)\left[\chi_{12}^{15}(q, z_y) + \chi_{-12}^{15}(q, z_y)\right] \\
& + \hat{\mathrm{Ch}}_m(0, 0; q, z_\pm)\Big[F_1(q)\chi_0^{15}(q, z_y) + F_2\left(\chi_6^{15}(q, z_y) + \chi_{-6}^{15}(q, z_y)\right) \\
& \quad + F_3(q)\left(\chi_{12}^{15}(q, z_y) + \chi_{-12}^{15}(q, z_y)\right)\Big] \\
& + \hat{\mathrm{Ch}}_m\left(\frac{1}{2}, 0; q, z_\pm\right)\Big[F_1(q)\chi_{15}^{15}(q, z_y) + F_2\left(\chi_9^{15}(q, z_y) + \chi_{-9}^{15}(q, z_y)\right) \\
& \quad + F_3(q)\left(\chi_3^{15}(q, z_y) + \chi_{-3}^{15}(q, z_y)\right)\Big],
\end{aligned} \tag{6.2.5}$$

where we used the slightly condensed notation,

$$\mathrm{Ch}_0^{\tilde{A}_\gamma, NS}\left(\tilde{l}^+, \tilde{l}^-; q, z_\pm\right) \equiv \mathrm{Ch}_{\mathrm{Massless}}^{\tilde{A}_\gamma, NS}\left(\tilde{k}^+, \tilde{k}^-, \tilde{l}^+, \tilde{l}^-; q, z_\pm\right).$$

We also rename the equivalence classes of the $F_{2\tilde{l}^+, n}^\Lambda$ functions where,

$$\begin{aligned}
&F_1(q) \sim F_{0,0}^{((0,0),2,0)}(q) - \frac{1}{2}, \quad F_2(q) \sim F_{0,1}^{((0,0),2,0)}(q), \quad F_3(q) \sim F_{0,2}^{((0,0),2,0)}(q), \\
&F_4(q) \sim F_{0,0}^{((1,0),2,0)}(q), \quad F_5(q) \sim F_{0,1}^{((1,0),2,0)}(q) - \frac{1}{2}, \quad F_6(q) \sim F_{0,2}^{((1,0),2,0)}(q),
\end{aligned} \tag{6.2.6}$$

up to equivalence. The factors of $\frac{1}{2}$ appearing in F_1 and F_5 come from rewriting the massless combinations of eq. (6.1.10) which appear in the sum rules into a sum of massless representations with positive inter coefficients using eq. (3.4.17). △

We now want to make some comments on the functions $F^{\Lambda}_{2\tilde{l}^+,n}$. Firstly, we can easily count how many independent functions there are.

Lemma 6.2.2. *Let the number of functions $F^{\Lambda}_{2\tilde{l}^+,n}$, ignoring the relations in eq.* (6.2.2), *be given by $N_D(\tilde{k}^+)$, then*

$$N_D(\tilde{k}^+) = \frac{1}{2}\tilde{k}^+(\tilde{k}^+ + 1)(\tilde{k}^+ + 2)(\tilde{k}^+ + 3). \tag{6.2.7}$$

Proof. First note that the number of pairs $(a_1, a_2) \in Z^2_{\geq 0}$ such that $a_1 + a_2 \leq \tilde{k}^+$ is the triangular number given by

$$\frac{(\tilde{k}^+ + 1)(\tilde{k}^+ + 2)}{2}.$$

Then $\tilde{l}^+$ lies in the range $0 \leq 2\tilde{l}^+ \leq \tilde{k}^+ - 1$ containing $\tilde{k}^+$ values and $n \in \mathbb{Z}_k$ has $k = \tilde{k}^+ + 3$ values and hence the number of (dependent) functions $F^{\Lambda}_{2\tilde{l}^+,n}$ is given by lemma 6.2.2. □

We now calculate the effect of the relations given in eq. (6.2.2). In the following, we let $\Lambda = ((a_1, a_2), \tilde{k}^+, 0)$ and recall that $0 \leq 2\tilde{l}^+ \leq \tilde{k}^+ - 1$.

Proposition 6.2.3. *Let the number of independent functions $F^{\Lambda}_{2\tilde{l}^+,n}$ under the equivalences of eq.* (6.2.2) *be given by $N_I(\tilde{k}^+)$ then,*

$$N_I(\tilde{k}^+) = \frac{1}{24}\tilde{k}^+(\tilde{k}^+ + 1)(\tilde{k}^+ + 2)(\tilde{k}^+ + 3) + \frac{1}{2}\left\lceil \frac{\tilde{k}^+}{2} \right\rceil \left\lceil \frac{\tilde{k}^+ + 2}{2} \right\rceil. \tag{6.2.8}$$

Proof. The first observation we make is that a general $F^{\Lambda}_{2\tilde{l}^+,n}$ lies in an equivalence class of 12 elements under the relations given in eq. (6.2.2). This can be seen by noting that the first and second of these transformations commute, for fixed ϵ the second and third commute, and that the effect of applying the first transformation followed by the third with $\epsilon = 1$ is equivalent to first applying the third with $\epsilon = -1$

and then applying the first. Since the first 2 transformations are of order 2 and the third is of order 3, then the general equivalence class is of order 12.

However, there are two cases in which an equivalence class can be smaller than 12 elements. The first of the equivalence relations in eq. (6.2.2) has fixed points when,

$$a_1 = a_2, \qquad 2n + 2\tilde{l}^+ \equiv 0 \ (k). \tag{6.2.9}$$

When $\tilde{k}^+$ is even, such that k is odd, then 2 is an invertible element in $\mathbb{Z}_k$ and hence for each value of $\tilde{l}^+$, there is a unique value of n such that $2n + 2\tilde{l}^+ \equiv 0(k)$. When $\tilde{k}^+$ is odd, then $2n + 2\tilde{l}^+ \equiv 0(k)$ has 2 solutions when $\tilde{l}^+ \in \mathbb{Z}$ and 0 solutions when $\tilde{l}^+ \in \mathbb{Z} + \frac{1}{2}$. These fixed points are paired under the action of the second equation in eq. (6.2.2), which itself never has fixed points, so one obtains shorter equivalence classes of 6 elements, each of which contains 2 elements fixed under the first equation in eq. (6.2.2).

The other case leading to smaller equivalence classes is if the image of $F^{\Lambda}_{2\tilde{l}^+,n}$ under the first relation in eq. (6.2.2) is the same as the image under the second equation. This occurs when

$$a_1 = a_2, \qquad 4\tilde{l}^+ = k - 4, \qquad 2n \equiv 2 \ (k). \tag{6.2.10}$$

This can only occur when $\tilde{k}^+$ is odd as $2\tilde{l}^+ \in \mathbb{Z}$. When $\tilde{k}^+$ is odd there are then two solutions to eq. (6.2.10) which necessarily lie in the same equivalence class.

There are $\left\lfloor \frac{\tilde{k}^+}{2} \right\rfloor + 1$ representations Λ which are invariant under conjugation, that is with $a_1 = a_2$. Putting this all together, when $\tilde{k}^+$ is even, there are $\frac{\tilde{k}^+}{2} + 1$ pairs $a_1 = a_2$, no solutions to eq. (6.2.10) and $\frac{\tilde{k}^+}{2}$ pairs of solutions to eq. (6.2.9) for each pair $a_1 = a_2$, giving $\frac{1}{4}\tilde{k}^+ \left(\tilde{k}^+ + 2\right)$ short equivalence classes of 6 elements. When $\tilde{k}^+$ is odd, there are $\left\lfloor \frac{\tilde{k}^+}{2} \right\rfloor + 1$ pairs of solutions to eq. (6.2.9) and one pair of solutions to eq. (6.2.10) for each pair $a_1 = a_2$, giving a total of $\frac{1}{4}\left(\tilde{k}^+ + 1\right)\left(\tilde{k}^+ + 3\right)$ short equivalence classes of 6 elements. The total number of short classes, $N_S(\tilde{k}^+)$ can then be given by

$$N_S(\tilde{k}^+) = \left\lceil \frac{\tilde{k}^+}{2} \right\rceil \left\lceil \frac{\tilde{k}^+ + 2}{2} \right\rceil. \tag{6.2.11}$$

Finally we can count the number of independent functions $F^{\Lambda}_{2\tilde{l}^+,n}$, $N_I(\tilde{k}^+)$ as,

$$\begin{aligned} N_I(\tilde{k}^+) &= \frac{N_D(\tilde{k}^+)}{12} + \frac{N_S(\tilde{k}^+)}{2}, \\ &= \frac{1}{24}\tilde{k}^+(\tilde{k}^+ + 1)(\tilde{k}^+ + 2)(\tilde{k}^+ + 3) + \frac{1}{2}\left\lceil \frac{\tilde{k}^+}{2} \right\rceil \left\lceil \frac{\tilde{k}^+ + 2}{2} \right\rceil. \end{aligned} \tag{6.2.12}$$

□

We can now use the sum rules eqs. (6.1.3), (6.1.20) and (6.1.22) to calculate the early terms in the q-expansion for the independent $F^{\Lambda}_{2\tilde{l}^+,n}$ which we will henceforth refer to simply as F_i for i in an indexing set of size $N_I(\tilde{k}^+)$. We have carried out these calculations using Mathematica and we give our results for the case $\tilde{k}^+ = 2$ below.

Example 6.2.4. As explained above in proposition 6.2.3, when $\tilde{k}^+ = 2$ there are $N_I(2) = 6$ independent functions F_i. We have calculated the first 29 terms for each of these 6 functions which we give the first 16 of below, along with a representative of each equivalence class.

$$\begin{aligned} F_1(q) \sim F^{((0,0),2,0)}_{0,0}(q) - \frac{1}{2} &= q(1 + q + q^3 + q^4 + 2q^5 + q^6 + q^7 + q^8 + 3q^9 \\ &\quad + 2q^{10} + 3q^{11} + 3q^{12} + 3q^{13} + 3q^{14} + 5q^{15} + \ldots), \\ F_2(q) \sim F^{((0,0),2,0)}_{0,1}(q) &= q^{2/5}(1 + q^2 + q^3 + q^4 + 2q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} \\ &\quad + 2q^{11} + 3q^{12} + 2q^{13} + 4q^{14} + 4q^{15} + \ldots), \\ F_3(q) \sim F^{((0,0),2,0)}_{0,2}(q) &= q^{8/5}(1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + q^7 + 2q^8 + 2q^9 \\ &\quad + 3q^{10} + 2q^{11} + 4q^{12} + 3q^{13} + 4q^{14} + 4q^{15} + \ldots), \\ F_4(q) \sim F^{((1,0),2,0)}_{0,0}(q) &= q^{1/5}(1 + q + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 \\ &\quad + 3q^{10} + 3q^{11} + 5q^{12} + 5q^{13} + 5q^{14} + 6q^{15} + \ldots), \end{aligned} \tag{6.2.13}$$

$$
\begin{aligned}
F_5(q) \sim F_{0,1}^{((1,0),2,0)}(q) - \frac{1}{2} &= q(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9) \\
&\quad + 4q^{10} + 4q^{11} + 5q^{12} + 5q^{13} + 6q^{14} + 6q^{15} + \ldots), \\
F_6(q) \sim F_{0,2}^{((1,0),2,0)}(q) &= q^{3/5}(1 + 2q^2 + q^3 + q^4 + 2q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 \\
&\quad + 4q^{10} + 3q^{11} + 5q^{12} + 4q^{13} + 6q^{14} + 7q^{15} + \ldots).
\end{aligned} \tag{6.2.14}
$$

Note that the factors of $\frac{1}{2}$ appearing here in F_1 and F_5 are as in 6.2.1. △

Remark 6.2.5. Up to the offset, the power of q factored out at the front in eqs. (6.2.13) and (6.2.14), the coefficients of these functions agree – to as many coefficients as we have been able to calculate, ∼30 – with some known functions. In particular, the q-expansions we have obtained agree with,

$$
\begin{aligned}
F_2(q) &\sim q^{2/5}\frac{f(-q^5)^2}{f(-q^2,-q^3)}, \qquad & F_3(q) &\sim q^{-2/5}\Psi_1(q), \\
F_4(q) &\sim q^{1/5}\frac{f(-q^5)^2}{f(-q,-q^4)}, \qquad & F_5(q) &\sim \Psi_0(q),
\end{aligned} \tag{6.2.15}
$$

where

$$
f(a,b) = \sum_{n\in\mathbb{Z}} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad f(-a) = f(-a,-a^2), \tag{6.2.16}
$$

is the Ramanujan general theta function [Ber12], and $\Psi_0(q)$ and $\Psi_1(q)$ are 5^{th} order mock theta functions. The expansions of these four functions may be found at [Somb; Hicb; Soma; Hica] respectively. The 5^{th} order mock-theta functions used here seem to have first appeared in [Hic88]. Although we have not been able so far to identify a compact form for the series $F_1(q)$ and $F_6(q)$, it is clear from the sum rules eq. (6.2.5) that we expect forms of weight 1/2 (modular or mock-modular). For the complete theory of mock-modular forms, see [Zwe02; B+04; Zag09].This could be inferred from the identifications eq. (6.2.15) as mock-theta functions are mock-modular forms of weight 1/2. We will comment further on this in the next section, after reviewing the concept of modular invariance and covariance.

As noted in [PT93], for $\tilde{k}^- = 1$, $\tilde{k}^+ = 5$, the central charge from A_γ is

$$c = 6\frac{(\tilde{k}^+ + 1)(\tilde{k}^- + 1)}{\tilde{k}^+ + \tilde{k}^- + 2} = 9, \tag{6.2.17}$$

and could therefore be relevant for string compactification. We have therefore calculated the first 11 terms in the q-expansion for each independent $F_i(q)$ for $\tilde{k^+} \in \{2,3,4,5\}$. The results for $\tilde{k}^+ = 2$ are presented above in 6.2.4 and the results for $\tilde{k^+} \in \{3,4,5\}$ are tabulated in Appendix E.

One would like to obtain exact analytic expressions for these functions rather than simply the first terms in a q-expansion. Hoping to derive the modular transformations of these functions we have begun work to derive the modular transformation properties of the $\tilde{A}_\gamma$ characters directly rather than in linear combinations as used in [PT93]. Since the authors of [PT93] make an assumption as to the independence of their massless and massive sectors under modular transformations, a direct calculation would provide a useful alternative method. Since the modular transformation properties of the other characters in the sum rules are known, one could then determine the modular properties of the F_i functions and then hope to use known properties of the vector spaces they inhabit to deduce the exact form of these functions from their early coefficients. Calculating the S-transformations of the massless $\tilde{A}_\gamma$ characters has proven to be technically challenging and so this work is still ongoing. We hope to publish these S-transformations alongside the implications for the functions F_i soon [FTT18].

6.3 Diagonal Theories and the Index I_1

The quantisation of the bosonic string can be performed through the path integral formalism, and leads, among others, to the so-called vacuum-to-vacuum amplitude for closed bosonic strings, interacting or not. It is given by,

$$Z(\tau,\bar{\tau}) = \sum_{\gamma=0}^{\infty} (g_s)^{2(\gamma-1)} \int \mathcal{D}^d\Sigma^a\, \mathcal{D}h_{\mu\nu}\, e^{-S^{\text{Polyakov}}(\Sigma,h)}, \tag{6.3.1}$$

where $S^{\text{Polyakov}}(\Sigma, h)$ was introduced in eq. (2.2.1). The discrete sum is over compact Riemann surfaces, classified by their genus γ and weighted by a power of the string coupling constant g_s, with an offset of g_s^{-2} to ensure Z has the right dimensions. The combination $2(\gamma - 1)$ which therefore appears as the power of the string coupling is the topological quantity known as the Euler characteristic discussed in Chapter 4. This discrete sum therefore counts all possible worldsheets swept by a closed string evolving in spacetime from the infinite past to the infinite future – the string state at infinity is given by the insertion of the appropriate vertex operator (the identity operator in the case of the vacuum to vacuum amplitude) and a conformal transformation allows us to bring the leg at infinity to a finite distance, we therefore have a worldsheet given by a Riemann surface as expected. For genus $\gamma = 0$, the Riemann surface is a sphere and models free closed strings. The torus ($\gamma = 1$) is the worldsheet of a closed string splitting into two closed strings that later rejoin – the relative power of g_s^2 compared to the vacuum weighs the splitting and the rejoining. The integral $\int \mathcal{D}h_{\mu\nu}$ is over all intrinsic shapes of a particular Riemann surface (that is, over all metrics for the Riemann surface), while the integral $\int \mathcal{D}^d\Sigma^a$ is over all different ways to embed the two-dimensional surface in spacetime.

In the context of string theory, the *one-loop partition function* is the name given to the vacuum-to-vacuum amplitude on the torus and this is often simply referred to as *the partition function*. The torus is parameterised by a complex variable τ (its modulus), and one expects the partition function not to depend on how the torus is parameterised. The group of τ-transformations that do not change the shape of the torus (i.e. that do not change the underlying lattice) is $SL(2, \mathbb{Z})$. This is readily seen when expressing the modulus in terms of the periods ω_1 and ω_2 of the underlying lattice, say $\tau = \frac{\omega_2}{\omega_1}$ [BBS06]. These transformations are global conformal transformations called *modular transformations*. Since $\{\pm I\} \subset SL(2, \mathbb{Z})$ yield the same transformed modulus, it is common to consider the modular transformations in $PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z})/\{\pm I\}$ instead, which are generated by the two transformations $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ with $S^2 = (ST)^3 = 1$.

For the bosonic string, the partition function takes the general form,

$$Z_B(\tau,\bar{\tau}) = \sum_{a,\bar{a}} M_{a\bar{a}}\, \chi_a^{\text{Vir}}(\tau)\, \chi_{\bar{a}}^{\text{Vir}}(\bar{\tau}), \tag{6.3.2}$$

where $\chi_a^{\text{Vir}}(\tau)$ is the character of the unitary representation of the Virasoro algebra with highest weight state $|a\rangle$ corresponding to the (left-moving, holomorphic) primary field labelled by ϕ^a, that is,

$$\chi_a^{\text{Vir}}(\tau) = \text{Tr}_{\mathscr{H}_a}\, e^{2\pi i\tau(L_0 - \frac{c}{24})} \tag{6.3.3}$$

with the trace taken over the Hilbert space of all positive norm states descending from the highest weight state $|a\rangle$. Similar conventions are taken for the right-moving, anti-holomorphic sector. The partition function sums over all the physical states of the theory, and therefore the multiplicities $M_{a\bar{a}} \in \mathbb{Z}_{\geq 0}$. If $(a,\bar{a}) = (0,0)$ labels the vacuum state, then one requires $M_{0,0} = 1$ to ensure unicity of the vacuum.

Characters of the Virasoro algebra transform *covariantly* under the modular group, by which we mean that each transforms into a (discrete) sum of all Virasoro characters of that theory under $PSL(2,\mathbb{Z})$. In particular,

$$\chi_a^{\text{Vir}}(\tau+1) = \sum_b T_{ab}\, \chi_b^{\text{Vir}}(\tau) \quad \text{and} \quad \chi_a^{\text{Vir}}(-1/\tau) = \sum_b S_{ab}\, \chi_b^{\text{Vir}}(\tau), \tag{6.3.4}$$

with T a diagonal matrix of phases and S is a unitary, symmetric matrix. If the states of a particular theory are then organised into such representations of the Virasoro algebra, then it is reasonably easy to construct a modular invariant partition function, i.e. to find multiplicities $M_{a\bar{a}}$ that ensure eq. (6.3.2) is modular *invariant.* In matrix notations, modular invariance of the partition function requires $[M,T] = [M,S] = 0$, where we have already argued that M is a matrix of non-negative integers with $M_{00} = 1$. The classification of modular invariant partition functions has been successfully performed in certain classes of theories, for which the first was in [CIZ87b; CIZ87a].

In the case where the symmetries of a theory are governed by an extended Virasoro algebra – in particular, a SCA – one argues along the same lines as before, with the Virasoro characters replaced by the characters of the SCA in question. If one is interested in constructing modular invariant partition functions that could correspond to theories whose symmetries are governed by superconformal algebras, the torus partition function must sum over all states in the theory, including summing over all spin structures, which correspond to the various periodicity conditions for the left-moving and right-moving fermions when they wrap the two cycles of the torus. We note in passing that being able to exhibit a modular invariant partition function does not guarantee the existence of an underlying physical theory: alongside the modular invariant partition function, the correlation functions must also be modular invariant.

The task we want to perform here is to construct modular invariant partition functions built on the $\tilde{A}_\gamma$ characters presented in Chapter 3. We are in the process of calculating analytically the behaviour of $\tilde{A}_\gamma$ characters under S and T [FTT18], but we already know that the S-transformation of the *massless* characters exhibits features that are reminiscent of those encountered in dealing with the S-transformation of massless $\mathcal{N} = 4$ characters, and which are actually at the origin of the Mathieu moonshine observation. In fact, in the sector of the theory relevant to a potential new moonshine, the massless $\tilde{A}_\gamma$ characters can be written as finite sums where each term is the product of a ratio of Jacobi theta functions and Dedekind functions (transforming with weight $k' = -1$) and of an Appell function at level $\ell = 2k^+k^-$ [STT05], which is a mock modular form of weight $k = 1$. In practical terms, this means that the S-transformation of our massless $\tilde{A}_\gamma$ characters at levels $k^\pm - 1$ involve an integral over the massive $\tilde{A}_\gamma$ characters alongside a finite sum of massless $\tilde{A}_\gamma$ characters. This integral complicates matters greatly if we were to use brute force techniques to construct a modular invariant partition function, and this is out of scope at present, as it was in the case of $K3$ theories. In order to circumvent this difficulty, we resort to looking for special types of theories where the building blocks of the partition function are discrete – albeit infinite – sums, where each term is the product of an $\tilde{A}_\gamma$ character and some rational torus character, and which can be expressed in terms of a finite set of characters from other theories that transform among themselves under $PSL(2,\mathbb{Z})$. What we are looking for is precisely the structure encountered in the character sum rules of Section 6.1.

The sum rules were shown to relate modules of $\tilde{A}_\gamma$ to those of $\widehat{su(3)}_{\tilde{k}^+}$, and we can use invariants of $\widehat{su(3)}$ to construct invariants for 'extended' $\tilde{A}_\gamma$ theories. Having constructed modular invariant partition functions we can then calculate the index I_1 for these models using definition 5.2.1 and proposition 5.2.4.

By definition 5.2.1, the index I_1 depends only on the $\tilde{R}$ sector of the partition function. We therefore need to write the sum rules of eq. (6.1.3) in the $\tilde{R}$ sector, rather than the NS sector. As discussed in sections 3.3 and 3.4, there exists an isomorphism of the algebra $\tilde{A}_\gamma$ which relates the characters for representations of $\tilde{A}_\gamma$ through spectral flow eq. (3.4.3). For instance, we may flow from the NS sector to the $\tilde{R}$ sector by evaluating the NS sum rules at $-z_-q^{-1/2}$ instead of z_- with the characters of $\tilde{A}_\gamma$ flowing as,

$$\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\mathrm{Massless}}(\tilde{l^+},\tilde{l^-};q,z_+,-z_-q^{-1/2}) = q^{-\tilde{k}^-/4}(-z_-)^{\tilde{k}^-}\,\mathrm{Ch}^{\tilde{A}_\gamma,\tilde{R}}_{\mathrm{Massless}}(\tilde{l^+},\frac{\tilde{k}^-}{2}-\tilde{l^-};q,z_+,z_-). \tag{6.3.5}$$

Under spectral flow, the Wolf space fermion character eq. (6.1.4) also 'flows' as,

$$\begin{aligned}\chi^{\mathrm{WS},NS}(q,-z_-q^{-1/2},z_y) &:= \eta^{-2}(q)\theta_3(q,-q^{-1/2}z_-z_y)\theta_3(q,-q^{-1/2}z_-z_y^{-1}),\\ &= -\,q^{-1/4}z_-\eta^{-2}(q)\theta_1(q,z_-z_y)\theta_1(q,z_-z_y^{-1}),\\ &= -\,q^{-1/4}z_-\chi^{\mathrm{WS},\tilde{R}}(q,z_+,z_-),\end{aligned} \tag{6.3.6}$$

as is easily shown using the product formulae for the θ-functions as in Appendix B.

We are ultimately interested in partition functions involving A_γ characters rather than $\tilde{A}_\gamma$ characters in the present framework of sum rules. A close look at eq. (6.1.2) reveals that the sum rules we have considered involve a rational torus theory with characters $\chi^{3k}_m(q,z_y)$ eq. (6.1.19), which is the extension of a $\widehat{\mathfrak{u}(1)}$ algebra by a dimension-3k operator and its hermitian conjugate [DVVV89]. We may view this $\widehat{\mathfrak{u}(1)}$ affine algebra as the subalgebra of A_γ whose current U decouples alongside the four fermionic currents $Q_\pm, Q_{\pm K}$ when considering $\tilde{A}_\gamma$ [OPT92]. So to obtain sum rules with A_γ characters, we multiply both sides of the sum rules by the character for the four free fermions. In the $\tilde{R}$ sector, these are given by,

$$\mathrm{Ch}^{A_Q,\tilde{R}}(q,z_+,z_-) = \frac{\theta_1(q,z_+z_-)\theta_1(q,z_+^{-1}z_-)}{\eta^2(q)}, \tag{6.3.7}$$

as can be checked by sending $z_- \to -z_-$ in $\mathrm{Ch}^{A_Q,R}(q,z_+,z_-)$ from eq. (3.4.10).

Defining $\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;q,z_\pm)$ as the spectral flow of $\mathrm{Ch}_0^{\tilde{A}_\gamma,NS}(L;q,z_\pm)$ multiplied by $\mathrm{Ch}^{A_Q,\tilde{R}}(q,z_+,z_-)$, we write the sum rules for A_γ in the $\tilde{R}$ sector as,

$$\begin{aligned}
&\frac{\theta_1(q,z_+z_-)\theta_1(q,z_+^{-1}z_-)}{\eta^2(q)}\cdot\frac{\theta_1(q,z_-z_y)\theta_1(q,z_-z_y^{-1})}{\eta^2(q)}\chi_\Lambda^{\mathfrak{su}(3)}(q,z_+,z_y)\\
&= \sum_{L=0}^{k-2}\eta(q)M_\Lambda^L(q,z_y)\ \mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;q,z_\pm)\\
&\quad+\sum_{2\tilde{l}^+=0}^{\tilde{k}^+-1}\sum_{n\in\mathbb{Z}_k}\hat{\mathrm{Ch}}_m^{A_\gamma,\tilde{R}}(l^\pm;q,z_\pm)\,\eta(q)\chi^{3k}_{-2a_1+2a_2+6\tilde{l}^++6n}(q,z_y)F^\Lambda_{2\tilde{l}^+,n}(q),
\end{aligned}\tag{6.3.8}$$

where we have set $u=0$. By $\hat{\mathrm{Ch}}_m^{A_\gamma,\tilde{R}}$, we mean the massive character of A_γ in the $\tilde{R}$ sector taken at the threshold value of the conformal weight h, as for the NS sector. As explained in section 6.2, although the massless characters appear in linear combinations with negative coefficients in eq. (6.3.8), in practice we can always use eq. (3.4.17) to rewrite the sum rule in such a way that all characters appear with positive integer coefficients.

Before proceeding, we wish to come back on the nature of the $F^\Lambda_{2\tilde{l}^+,n}(q)$ series that appear in the sum rules eq. (6.1.20), as promised at the end of the previous section. Examining the sum rules in the $\tilde{R}$ sector, we note that the LHS is a product of the character of four Wolf space fermions (which is invariant under S) and of characters which transform covariantly under S with weight 0. The RHS has terms which are either products of rational torus characters and massless $\tilde{A}_\gamma$ characters, or products of rational torus characters, massive $\tilde{A}_\gamma$ characters and $F_i(q)$ functions. The rational torus characters transform covariantly under S with weight 0 and the massless $\tilde{A}_\gamma$ characters are mock modular forms of weight 0. On the other hand, the massive $\tilde{A}_\gamma$

characters, which are not mock, transform with weight $-1/2$. We therefore expect at least some of the functions $F^{\Lambda}_{2\tilde{l}^+,n}(q)$ to be mock modular forms of weight 1/2, while the rest are modular of weight 1/2.

Using Appendix B as well as the well known diagonal $\widehat{\mathfrak{su}(3)}$ invariant [Gan94], we see that we can write the $\tilde{R}-\tilde{R}$ sector of a modular invariant partition function for a theory with 'extended' $\tilde{A}_\gamma$ symmetry as,

$$Z_{\tilde{R},\tilde{R}} = \sum_{\Lambda \in P_+^{\tilde{k}^+}} \left| \frac{\theta_1(q, z_+ z_-)\theta_1(q, z_+^{-1} z_-)}{\eta^2(q)} \frac{\theta_1(q, z_- z_y)\theta_1(q, z_- z_y^{-1})}{\eta^2(q)} \chi_\Lambda^{\mathfrak{su}(3)}(q, z_+, z_y) \right|^2 , \tag{6.3.9}$$

where $P_+^{\tilde{k}^+}$ is as in eq. (6.1.15). Having finally constructed the $\tilde{R}-\tilde{R}$ sector of a modular invariant partition function, we can now calculate the index I_1 for such a theory.

As discussed in section 5.2, the index I_1 annihilates right-moving massive representations of A_γ and counts right-moving massless representations as per eq. (5.2.5). Recall that as well as the quantum numbers flowing under spectral flow, the representation labels can also change when multiplying by a representation of A_{QU} as discussed in section 3.4. One can easily check that starting from an $\tilde{A}_\gamma$ NS representation, flowing to the R sector then tensoring against A_{QU} gives the same representation as first tensoring against A_{QU} and then flowing to R. Either way the net result is,

$$\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massless}}(\tilde{l}^+,\tilde{l}^-) \to \mathrm{Ch}^{A_\gamma,R}_{\text{Massless}}\left(\tilde{l}^+ + \frac{1}{2}, \frac{k^-}{2} - \tilde{l}^-\right), \tag{6.3.10}$$

$$\mathrm{Ch}^{\tilde{A}_\gamma,NS}_{\text{Massive,threshold}}(\tilde{l}^+,\tilde{l}^-) \to \mathrm{Ch}^{A_\gamma,R}_{\text{Massive,threshold}}\left(\tilde{l}^+ + 1, \frac{k^-}{2} - \tilde{l}^-\right). \tag{6.3.11}$$

Labelling the $\tilde{R}-\tilde{R}$ sector of the partition function of the diagonal theory as $Z^{D_{\tilde{k}^+}}_{\tilde{R},\tilde{R}}$

and the theory itself as $\mathcal{D}_{\tilde{k}^+}$, we can now calculate the index I_1 of this theory as,

$$\begin{aligned} I_1(\mathcal{D}_{\tilde{k}^+})(q,z_+,z_-,z_y;\bar{q},\bar{z},\bar{z}_y) :=& -\bar{z}_+ \frac{\partial}{\partial \bar{z}_-} Z^{D_{\tilde{k}^+}}_{\tilde{R},\tilde{R}}\Big|_{\bar{z}_+=\bar{z}_-}, \\ = |\eta(q)|^2 \sum_{\Lambda\in P_+^{\tilde{k}^+}} & \left(\sum_{L=0}^{k-2} M^L_\Lambda(q,z_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;q,z_\pm) \right. \\ & \left. + \sum_{2\tilde{l}^+=0}^{\tilde{k}^+-1} \sum_{n\in\mathbb{Z}_k} \hat{M}^\Lambda_{2l_+,n}(q,z_+,z_-,z_y) F^\Lambda_{2\tilde{l}^+,n}(q) \right) \cdot I_1\left(\sum_{L=0}^{k-2} M^L_\Lambda(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right). \end{aligned} \tag{6.3.12}$$

We abuse notation in the above by defining,

$$I_1\left(\sum_{L=0}^{k-2} M^L_\Lambda\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L)\right) := -\bar{z}_+\frac{\partial}{\partial\bar{z}_-}\left(\sum_{L=0}^{k-2} M^L_\Lambda\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L)\right)\Bigg|_{\bar{z}_+=\bar{z}_-}, \tag{6.3.13}$$

to mean the contribution to the index from the massive part of the sum rule.

Example 6.3.1. Let us consider the case $\tilde{k}^+ = 1$. By proposition 6.2.3, we see that there are only 2 independent functions in this case and they have been calculated as [OPT92],

$$F_1(q) \sim F^{((0,0),1,0)}_{0,0}(q) - \frac{1}{2} = \sum_{n=1}^{\infty} q^{n^2}, \qquad F_2(q) \sim F^{((0,0),1,0)}_{0,1}(q) = \sum_{n=1}^{\infty} q^{(2n-1)^2/4}. \tag{6.3.14}$$

The contributions $I_1\left(\sum_{L=0}^{k-2} M^L_\Lambda\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L)\right)$ can then be calculated by eq. (5.2.5), recalling that we have set $u=0$ (or equivalently cancelled the $q^{u^2/k}$ term from both sides of eq. (6.3.8)) as,

$$\begin{aligned} I_1\left(\sum_{L=0}^{2} M^L_{((0,0),1,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) =& \ \theta^-_{2,4}(\bar{q},\bar{z})\chi^{12}_0(\bar{q},\bar{z}_y) - \theta^-_{2,4}(\bar{q},\bar{z})\chi^{12}_{12}(\bar{q},\bar{z}_y) \\ - \theta^-_{3,4}(\bar{q},\bar{z})\left(\chi^{12}_9(\bar{q},\bar{z}_y) + \chi^{12}_{-9}(\bar{q},\bar{z}_y)\right) &+ \theta^-_{1,4}(\bar{q},\bar{z})\left(\chi^{12}_3(\bar{q},\bar{z}_y) + \chi^{12}_{-3}(\bar{q},\bar{z}_y)\right), \end{aligned} \tag{6.3.15}$$

$$\begin{aligned} I_1\left(\sum_{L=0}^{2} M^L_{((0,1),1,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) =& \ \theta^-_{2,4}(\bar{q},\bar{z})\chi^{12}_{-4}(\bar{q},\bar{z}_y) - \theta^-_{2,4}(\bar{q},\bar{z})\chi^{12}_8(\bar{q},\bar{z}_y) \\ - \theta^-_{3,4}(\bar{q},\bar{z})\left(\chi^{12}_{-7}(\bar{q},\bar{z}_y) + \chi^{12}_{-1}(\bar{q},\bar{z}_y)\right) &+ \theta^-_{1,4}(\bar{q},\bar{z})\left(\chi^{12}_5(\bar{q},\bar{z}_y) + \chi^{12}_{11}(\bar{q},\bar{z}_y)\right), \\ I_1\left(\sum_{L=0}^{2} M^L_{((1,0),1,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) =& \ \theta^-_{2,4}(\bar{q},\bar{z})\chi^{12}_{-8}(\bar{q},\bar{z}_y) - \theta^-_{2,4}(\bar{q},\bar{z})\chi^{12}_4(\bar{q},\bar{z}_y) \\ - \theta^-_{3,4}(\bar{q},\bar{z})\left(\chi^{12}_1(\bar{q},\bar{z}_y) + \chi^{12}_7(\bar{q},\bar{z}_y)\right) &+ \theta^-_{1,4}(\bar{q},\bar{z})\left(\chi^{12}_{-11}(\bar{q},\bar{z}_y) + \chi^{12}_{-5}(\bar{q},\bar{z}_y)\right). \end{aligned} \tag{6.3.16}$$

$\triangle$

Example 6.3.2. When $\tilde{k}^+ = 2$ we only have the first terms of the q-expansions for the functions $F_i,\ i \in \{1, \ldots, 6\}$ as given in example 6.2.4.

The massive contributions to the sum rules can be calculated as in example 6.3.1 to give,

$$\begin{aligned}
&I_1\left(\sum_{L=0}^{3} M^L_{((0,0),2,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) = \theta^-_{1,5}(\bar{q},\bar{z})\left(\chi^{15}_3(\bar{q},\bar{z}_y)+\chi^{15}_{-3}(\bar{q},\bar{z}_y)\right)\\
&+\theta^-_{2,5}(\bar{q},\bar{z})\chi^{15}_0(\bar{q},\bar{z}_y)+\theta^-_{3,5}(\bar{q},\bar{z})\chi^{12}_{15}(\bar{q},\bar{z}_y)+\theta^-_{4,5}(\bar{q},\bar{z})\left(\chi^{15}_{12}(\bar{q},\bar{z}_y)+\chi^{12}_{-12}(\bar{q},\bar{z}_y)\right),\\
&I_1\left(\sum_{L=0}^{3} M^L_{((0,1),2,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) = \theta^-_{1,5}(\bar{q},\bar{z})\chi^{15}_5(\bar{q},\bar{z}_y)\\
&-\theta^-_{2,5}(\bar{q},\bar{z})\left(\chi^{15}_{-4}(\bar{q},\bar{z}_y)+\chi^{15}_{14}(\bar{q},\bar{z}_y)\right)-\theta^-_{3,5}(\bar{q},\bar{z})\left(\chi^{15}_{11}(\bar{q},\bar{z}_y)+\chi^{15}_{-1}(\bar{q},\bar{z}_y)\right)\\
&+\theta^-_{4,5}(\bar{q},\bar{z})\chi^{15}_{-10}(\bar{q},\bar{z}_y),\\
&I_1\left(\sum_{L=0}^{3} M^L_{((1,0),2,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) = \theta^-_{1,5}(\bar{q},\bar{z})\chi^{15}_{-5}(\bar{q},\bar{z}_y)\\
&-\theta^-_{2,5}(\bar{q},\bar{z})\left(\chi^{15}_{4}(\bar{q},\bar{z}_y)+\chi^{15}_{-14}(\bar{q},\bar{z}_y)\right)-\theta^-_{3,5}(\bar{q},\bar{z})\left(\chi^{15}_{-11}(\bar{q},\bar{z}_y)+\chi^{15}_{1}(\bar{q},\bar{z}_y)\right)\\
&+\theta^-_{4,5}(\bar{q},\bar{z})\chi^{15}_{10}(\bar{q},\bar{z}_y),\\
&I_1\left(\sum_{L=0}^{3} M^L_{((1,1),2,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) = \theta^-_{1,5}(\bar{q},\bar{z})\chi^{15}_{15}(\bar{q},\bar{z}_y)\\
&-\theta^-_{2,5}(\bar{q},\bar{z})\left(\chi^{15}_{6}(\bar{q},\bar{z}_y)+\chi^{15}_{-6}(\bar{q},\bar{z}_y)\right)-\theta^-_{3,5}(\bar{q},\bar{z})\left(\chi^{15}_{9}(\bar{q},\bar{z}_y)+\chi^{15}_{-9}(\bar{q},\bar{z}_y)\right)\\
&+\theta^-_{4,5}(\bar{q},\bar{z})\chi^{15}_{0}(\bar{q},\bar{z}_y),\\
&I_1\left(\sum_{L=0}^{3} M^L_{((0,2),2,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) = \theta^-_{1,5}(\bar{q},\bar{z})\left(\chi^{15}_7(\bar{q},\bar{z}_y)+\chi^{15}_{-17}(\bar{q},\bar{z}_y)\right)\\
&+\theta^-_{2,5}(\bar{q},\bar{z})\chi^{15}_{10}(\bar{q},\bar{z}_y)+\theta^-_{3,5}(\bar{q},\bar{z})\chi^{12}_{-5}(\bar{q},\bar{z}_y)+\theta^-_{4,5}(\bar{q},\bar{z})\left(\chi^{15}_{-8}(\bar{q},\bar{z}_y)+\chi^{12}_{-2}(\bar{q},\bar{z}_y)\right),\\
&I_1\left(\sum_{L=0}^{3} M^L_{((2,0),2,0)}(\bar{q},\bar{z}_y)\,\mathrm{Ch}_0^{A_\gamma,\tilde{R}}(L;\bar{q},\bar{z}_\pm)\right) = \theta^-_{1,5}(\bar{q},\bar{z})\left(\chi^{15}_{-7}(\bar{q},\bar{z}_y)+\chi^{15}_{17}(\bar{q},\bar{z}_y)\right)\\
&+\theta^-_{2,5}(\bar{q},\bar{z})\chi^{15}_{-10}(\bar{q},\bar{z}_y)+\theta^-_{3,5}(\bar{q},\bar{z})\chi^{12}_{5}(\bar{q},\bar{z}_y)+\theta^-_{4,5}(\bar{q},\bar{z})\left(\chi^{15}_{8}(\bar{q},\bar{z}_y)+\chi^{12}_{2}(\bar{q},\bar{z}_y)\right).
\end{aligned} \tag{6.3.17}$$

$\triangle$

From these examples, we see that the index I_1 counts an infinity of states in the right-moving sector, and although we gained an understanding of the A_γ states counted in Chapter 5, much remains to be studied, even within these simple examples. If there is a moonshine phenomenon here, one would expect it to appear in the term of the form

$$\left(\sum_{2\tilde{l}^+=0}^{\tilde{k}^+-1} \sum_{n\in\mathbb{Z}_k} \hat{M}^{\Lambda}_{2l_+,n}(q, z_+, z_-, z_y) F^{\Lambda}_{2\tilde{l}^+,n}(q) \right) \cdot I_1 \left(\sum_{L=0}^{k-2} M^L_{\Lambda}(\bar{q}, \bar{z}_y) \, \mathrm{Ch}_0^{A_\gamma, \tilde{R}}(L; \bar{q}, \bar{z}_\pm) \right), \tag{6.3.18}$$

from eq. (6.3.12). This is work in progress.

Chapter 7

Conclusion

This thesis was motivated by Mathieu moonshine, where the decomposition of the elliptic genus for a $K3$ theory into characters of the 'small' $\mathcal{N}=4$ superconformal algebra at central charge $c=6$ has revealed a surprising connection with the sporadic group M_{24}. Since the underlying $\mathcal{N}=4$ symmetry is important to this observation in the context of strings propagating on a $K3$ manifold, it is natural to ask whether a theory with the 'large' $\mathcal{N}=4$ symmetry governed by the A_γ SCA could exhibit a similar phenomenon. The algebra A_γ was discovered in the late 1980's [IKL88b; IKL88a; Sch88] and studied in [SSTV88a; STVS88; STV88] in the context of super Wess-Zumino-Witten models describing superstring propagating on group manifolds allowing a quaternionic structure.

The A_γ SCA has 8 bosonic operators which, apart from the energy-momentum tensor of conformal weight 2, includes the 6 bosonic generators of two $\widehat{\mathfrak{su}(2)^\pm}_{k^\pm}$ subalgebras and the bosonic generator $U(z)$ of a $\mathfrak{u}(1)$ subalgebra, as well as 4 conformal weight $3/2$ operators corresponding to the supersymmetry generators and 4 generators $Q^a(z)$ of conformal weight $1/2$ corresponding to free fermions. Unitary representations require $\gamma = \frac{k^-}{k^-+k^+}$ with $k^\pm \in \mathbb{N}$.

Amongst the elementary quaternionic group manifolds, only two yield a theory with a single energy-momentum tensor: $SU(2)\times U(1)$ and $SU(3)$. In the first case, the bosonic currents of the σ-model satisfy the OPEs of an $\widehat{\mathfrak{su}(2)}$ algebra at integer level

n and of a $\widehat{\mathfrak{u}(1)}$ algebra at level 1. These bosonic currents together with their four fermionic superpartners – whose OPEs are standard free fermion OPEs – generate a representation of A_γ for $k^- = 1$ and $k^+ = n+1$, which yields a central charge $c = 6(n+1)/(n+2)$. Although the limit $n \to \infty$ is known to reduce the A_γ algebra to the 'small' $\mathcal{N} = 4$ algebra at $c = 6$ [STV88], this corresponds to a limit where the $SU(2) \times U(1)$ group manifold becomes $U(1)^4$, and such a theory describing superstrings propagating on a 4-torus has elliptic genus zero [NW01; Wen00]. The elliptic genus of a theory with A_γ symmetry is also zero for all finite values of $k^\pm$, as shown in Chapter 5 and discussed already in [GMMS04]. The emergence of a moonshine phenomenon through the elliptic genus in this framework is therefore doomed. A similar conclusion can be reached starting with the group manifold $SU(3)$, where the 8 σ-model bosonic currents and their fermionic superpartners generate a representation of A_γ at $k^- = 2$ and $k^+ = n+1$, with the $n \to \infty$ limit being 'small' $\mathcal{N} = 4$ with central charge $c = 12$. In this case however, the theory is richer than in the previous situation, as the root diagram of $\mathfrak{su}(3)$ has a non-empty set of non-zero roots, non-orthogonal to the highest root, and which span the 4-dimensional Wolf space $SU(3)/(SU(2) \times U(1))$ as in section 2.2.2. We have thus chosen the super WZW model for $SU(3)$ as a test bed for ideas that might lead to a new moonshine.

Although the elliptic genus for such a theory is trivial, another supersymmetric index I_1 (defined in section 5.2) was introduced in the context of theories with A_γ symmetries in [GMMS04] as a generalisation of the new index of [CFIV92]. The motivation there was to identify a holographic dual to a string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ [GMMS05; BPS99; EFGT99]. In this thesis, we have not followed this line, but instead revisited the approach initiated in [OPT92; PT93] of constructing modular invariant partition functions for theories with A_γ symmetry. The building blocks of these partition functions are provided by the character sum rules of Chapter 6, which exploit realisations of $\tilde{A}_\gamma$ on cosets [SSTV88a; STVS88; GPTV89; Van89; ST90], in particular on $SU(N)/SU(N-2)$ via a realisation on $W(N)$ and on $SU(2) \times U(1)$ where $W(N)$ is a Wolf space – we have focussed on the case $N = 3$, as in this case there is a single energy-momentum tensor. In section 3.1, we recall how the non-linear algebra $\tilde{A}_\gamma$ is obtained by decoupling the four free fermion generators $Q^a(z)$ and the $\widehat{\mathfrak{u}(1)}$ bosonic generator $U(z)$ from A_γ. In the case of interest ($N = 3$), the four $Q^a(z)$ generators correspond to the superpartners of the bosonic generators associated with the $SU(2) \times U(1)$ factor, which means that after they decouple and A_γ reduces to $\tilde{A}_\gamma$, the only remaining fermionic generators are those corresponding to the Wolf space $W(3)$, i.e. to the four non-zero roots of $\mathfrak{su}(3)$ other than the highest root and it negative. The character sum rules of interest to us encode realisations of $\tilde{A}_\gamma$ extended by a rational torus algebra – itself obtained through extension of the $\widehat{\mathfrak{u}(1)}$ subalgebra of A_γ by a dimension $3(k^+ + k^-)$ operator and its hermitian conjugate – where one uses the four free Wolf space fermions and the currents of $\widehat{\mathfrak{su}(3)}_{k^+-1}$ with $k^+ - 1 = n$.

Manufacturing modular invariant partition functions is reasonably straightforward since the sum rules relate the $\tilde{A}_\gamma$ characters to those of free Wolf space fermions multiplied by $\widehat{\mathfrak{su}(3)}_{k^+-1}$ characters, for which the modular transformations are known [Kac94; Gan94]. Here we have considered diagonal invariants for the two simplest values of $k^+ = 2$ and $k^+ = 3$, building on the work of [OPT92; PT93]. The case $k^+ = 2$ is somehow special as it corresponds to $\gamma = \frac{1}{2}$ (since $k^- = 2$ here) and yields the $\mathcal{N} = 4$ SCA with $\mathfrak{so}(4)$ subalgebra at $c = 6$ of Ademollo et al. [Ade+76b; Ade+76a]. We have in particular concentrated on the massive sector of the sum rules, i.e. on the contribution from massive $\tilde{A}_\gamma$ characters to the sum rules. In the $\tilde{R}$ sector, these appear as infinite sums of the form $\sum_{n=1}^{\infty} c_n \operatorname{Ch}_m^{A_\gamma,\tilde{R}}(\tilde{h} + n, l^\pm)$, for $\tilde{h}$ the threshold conformal weight and with all coefficients c_n positive, and correspond to unitary representations of A_γ built on primary fields with conformal weights $\tilde{h} + n\ \forall n \in \mathbb{N}$. These sums of characters can be written in terms of threshold massive characters using $\sum_{n=1}^{\infty} c_n \operatorname{Ch}_m^{A_\gamma,\tilde{R}}(\tilde{h} + n, l^\pm) = \hat{\operatorname{Ch}}_m^{A_\gamma,\tilde{R}}(\tilde{h}, l^\pm) \sum_{n=1}^{\infty} c_n q^n$. These series $\sum_{n=1}^{\infty} c_n q^n$, generically labelled $F(q)$ for $q = e^{2\pi i\tau}$, are expected to be either modular or mock modular forms of weight 1/2 as discussed in section 6.3. We have compelling evidence that in the case $k^+ = 3$, some are 5^{th}-order mock theta functions, and some are ratios of Ramanujan theta functions, but we are still studying the nature of these series for a range of k^+-values. We provide the first 11 (16 in the case $k^+ = 3$) coefficients of the series $F_i(q)$ for $k^+ \in \{3, 4, 5, 6\}$ in Appendix E.

As mentioned in section 6.2, we had hoped to find formulae for the S-transformations of the massless and massive characters of $\tilde{A}_\gamma$. Although the calculations for massive characters are reasonably straightforward, those for the massless characters of $\tilde{A}_\gamma$ have proven technically challenging and so this work is still ongoing. Once we have calculated these S-transformations, we hope to be able to provide exact expressions for the functions $F_i(q)$, which at present are only accessible via their Fourier expansions within the character sum rules (see Appendix E). We are encouraged to pursue this avenue, as some of the forms appearing in umbral moonshine can be written similarly in terms of mock theta functions [CDH14b]. Ultimately, we want to understand the dimension of the vector space spanned by the functions $F_i(q)$ pertaining to a particular theory characterised by $k^- = 2$ and arbitrary integer $k^+ \geq 2$.

Also in Chapter 6, and in analogy with the pathway to the Mathieu moonshine discovery, which involves the calculation of the elliptic genus, we have calculated the supersymmetric index I_1 on the partition functions constructed from the character sum rules of section 6.3. In the case of Mathieu moonshine, the elliptic genus was found to be a weak Jacobi form for $\mathbb{H} \times \mathbb{C}$ (where $\mathbb{H}$ is the upper half-plane). The contribution to the elliptic genus from right-moving massless representations of $\mathcal{N} = 4$ is the Witten index of these representations (integers). The partition function is there reduced from a function from $\mathbb{H}^2 \times \mathbb{C}^2$ to a function from $\mathbb{H} \times \mathbb{C}$ under the elliptic genus. The factor multiplying the massive threshold characters in the elliptic genus, the mock modular form identified in section 4.2.3, is just a function of q. The index I_1 for A_γ is more complicated. As we saw in section 5.2, massless representations of A_γ contribute to the index I_1 in the form of theta functions $\theta^-_{\mu,k}(\bar{q}, \bar{z})$, where μ is the Witten index of the underlying $\tilde{A}_\gamma$ representation. The contribution of massive threshold characters of A_γ to I_1 is therefore a function of the form $A(q, z_\pm, z_y; \bar{q}, \bar{z}_y, \bar{z})$ as in eq. (6.3.12). We expect any moonshine phenomenon, if any, to emerge from this massive threshold contribution corresponding to the left-moving massive A_γ characters multiplied by I_1 applied to the right-moving A_γ characters. This is quite a complex analysis, which is still ongoing.

In our journey to Chapter 6, we have had to learn a substantial amount of background material, which is not new but which we present from our personal viewpoint in Chapters 1 to 4 and some appendices, focusing on aspects which are essential for our work in Chapter 6. This abundance of background material reflects how moonshine phenomena lie at the interface of group and representation theory, algebraic geometry and number theory, as well as string theory. We go from the emergence of A_γ symmetry in super WZW models on group manifolds with quaternionic structures introduced in Chapter 2 to the representation theory of A_γ and $\tilde{A}_\gamma$ in Chapter 3, and to the notion of supersymmetric indices as topological invariants in one- and two-dimensional σ-models, embedded in a synthesis of the mathematical and physical literature surrounding these supersymmetric indices in Chapter 4.

Chapter 5, which builds on the broad ideas of index theory reviewed in Chapter 4, contains original work. Contributions to the supersymmetric index I_1 were shown to come from spectral flow orbits of four particular ground states, a fact observed by Saulina earlier [Sau05]. We also demonstrate how the contributions to the index I_1 split into contributions from $\tilde{A}_\gamma$ and from A_{QU}, the algebra of 4 fermions $Q^a(z)$ and one boson $U(z)$. After introducing the notion of a Lie supergroup and its associated Lie superalgebra, it was proved that the superalgebra $\mathfrak{su}(2|2)$ describes the zero mode subalgebra of A_γ in the Ramond sector by constructing a basis of $\mathfrak{su}(2|2)$ which satisfies the commutation relations of A_γ. Young supertableaux were used to classify representations of $\mathfrak{su}(2|2)$ and, by considering the branching of $\mathfrak{su}(2|2)$ into $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$, we have shown how one may organise the finite-dimensional representations of the zero mode subalgebra of A_γ occurring at any given level, into $\mathfrak{su}(2|2)$ representations. We also indicated how to consider the index I_1 acting on a supertableau and showed that the only supertableaux which contribute to I_1 are the maximally eccentric ones which contain the 'massless' states of A_γ – those in the spectral flow orbits of massless A_γ highest weight states (and their $\mathfrak{su}(2) \times \mathfrak{su}(2)$ lowest weight counterparts) [Fea18].

Appendix A

The Commutation Relations of the Large $\mathcal{N}=4$ Algebra A_γ

We present here the commutation relations for the 'large' $\mathcal{N}=4$ algebra, A_γ first discovered in [STVS88] with the commutation relations first given in [STV88]. This algebra contains the energy-momentum operator $T(z)$ of conformal dimension 2, four supercurrents $G^a(z)$ of dimension $\frac{3}{2}$, as well as operators $T^{\pm i}(z)$ and $U(z)$ of dimension 1 and $Q^a(Z)$ dimension $\frac{1}{2}$.

The Virasoro modes satisfy the usual Virasoro algebra with central charge c,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{A.0.1}$$

with commutation relations between the Virasoro modes and the modes of the other operators fixed by the conformal dimensions of the operators,

$$[L_m, \phi_n] = [(d_\phi - 1)m - n]\phi_{m+n}, \tag{A.0.2}$$

where d_ϕ is the conformal dimension of ϕ, i.e $d_G = \frac{3}{2}, d_T^{\pm i} = d_U = 1, d_Q = \frac{1}{2}$ for $i \in \{+,-,3\}$.

The modes of the supercharges satisfy the commutation relations,

$$\begin{aligned}
\{G_{+,m}, G_{-,n}\} &= L_{m+n} + \frac{c}{6}(m^2 - \frac{1}{4})\delta_{m+n,0} - (n-m)\left[\gamma^+ T^{+3}_{m+n} + \gamma^- T^{-3}_{m+n}\right], \\
\{G_{+K,m}, G_{-K,n}\} &= L_{m+n} + \frac{c}{6}(m^2 - \frac{1}{4})\delta_{m+n,0} - (n-m)\left[\gamma^+ T^{+3}_{m+n} - \gamma^- T^{-3}_{m+n}\right], \\
\{G_{\pm,m}, G_{+K,n}\} &= \pm\gamma^{\pm}(n-m)T^{\pm\pm}_{m+n}, \quad \{G_{\pm,m}, G_{-K,n}\} = \pm\gamma^{\mp}(n-m)T^{\mp\pm}_{m+n},
\end{aligned} \tag{A.0.3}$$

where we use the notation,

$$\gamma^+ = \gamma = \frac{k^-}{k}, \qquad \gamma^- = 1 - \gamma = \frac{k^+}{k}, \qquad k = k^+ + k^-, \tag{A.0.4}$$

where k^+, k^- are the levels of the two affine $\widehat{\mathfrak{su}(2)}$ algebras known as $\widehat{\mathfrak{su}(2)}^{\pm}$ respectively. The central charge of the Virasoro algebra is defined in terms of the levels of the affine algebras as,

$$c = 6\frac{k^+ k^-}{k}. \tag{A.0.5}$$

The modes of the affine $\widehat{\mathfrak{su}(2)}^{\pm}$ satisfy the usual affine commutation relations,

$$\begin{aligned}
[T^{\pm+}_m, T^{\pm-}_n] &= 2T^{\pm3}_{m+n} + mk^{\pm}\delta_{m+n,0}, \qquad & [T^{\pm3}_m, T^{\pm+}_n] &= T^{\pm+}_{m+n}, \\
[T^{\pm3}_m, T^{\pm3}_n] &= \frac{1}{2}mk^{\pm}\delta_{m+n,0}, & [T^{\pm3}_m, T^{\pm-}_n] &= -T^{\pm-}_{m+n}.
\end{aligned} \tag{A.0.6}$$

The modes of the four dimension $\frac{1}{2}$ operators $Q_{\pm,\pm K}$ satisfy the following relations with the modes of the $\widehat{\mathfrak{su}(2)}^{+}$,

$$\begin{aligned}
[T^{++}_m, Q_{+,n}] &= 0, & [T^{++}_m, Q_{-K,n}] &= -Q_{+,m+n}, \\
[T^{++}_m, Q_{+K,n}] &= 0, & [T^{++}_m, Q_{-,n}] &= Q_{+K,m+n}, \\
[T^{+-}_m, Q_{+,n}] &= -Q_{-K,m+n}, & [T^{+-}_m, Q_{-K,n}] &= 0, \\
[T^{+-}_m, Q_{+K,n}] &= Q_{-,m+n}, & [T^{+-}_m, Q_{-,n}] &= 0, \\
[T^{+3}_m, Q_{+,n}] &= \frac{1}{2}Q_{+,m+n}, & [T^{+3}_m, Q_{-K,n}] &= -\frac{1}{2}Q_{-K,m+n}, \\
[T^{+3}_m, Q_{+K,n}] &= \frac{1}{2}Q_{+K,m+n}, & [T^{+3}_m, Q_{-,n}] &= -\frac{1}{2}Q_{-,m+n},
\end{aligned} \tag{A.0.7}$$

and similarly with the modes of the $\widehat{\mathfrak{su}(2)^-}$,

$$\begin{aligned}
[T_m^{-+}, Q_{+,n}] &= 0, & [T_m^{-+}, Q_{+K,n}] &= -Q_{+,m+n}, \\
[T_m^{-+}, Q_{-K,n}] &= 0, & [T_m^{-+}, Q_{-,n}] &= Q_{-K,m+n}, \\
[T_m^{--}, Q_{+,n}] &= -Q_{+K,m+n}, & [T_m^{--}, Q_{+K,n}] &= 0, \\
[T_m^{--}, Q_{-K,n}] &= Q_{-,m+n}, & [T_m^{--}, Q_{-,n}] &= 0, \\
[T_m^{-3}, Q_{+,n}] &= \frac{1}{2}Q_{+,m+n}, & [T_m^{-3}, Q_{+K,n}] &= -\frac{1}{2}Q_{+K,m+n}, \\
[T_m^{-3}, Q_{-K,n}] &= \frac{1}{2}Q_{-K,m+n}, & [T_m^{-3}, Q_{-,n}] &= -\frac{1}{2}Q_{-,m+n}.
\end{aligned} \tag{A.0.8}$$

These equations can be condensed considerably by forming the doublets,

$$\begin{aligned}
(Q_1, Q_2) &= (Q_+, -Q_{-K}) \quad \text{or} \quad (Q_{+K}, Q_-) \quad \text{for } \widehat{\mathfrak{su}(2)^+}, \\
(Q_1, Q_2) &= (Q_+, -Q_{+K}) \quad \text{or} \quad (Q_{-K}, Q_-) \quad \text{for } \widehat{\mathfrak{su}(2)^-},
\end{aligned} \tag{A.0.9}$$

and letting τ^i for $i \in \{+, -, 3\}$ be the standard doublet representation of $SU(2)$,

$$\tau^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{A.0.10}$$

such that these doublets transform under the $\widehat{\mathfrak{su}(2)^\pm}$ as,

$$[T_m^{\pm i}, (Q_1, Q_2)_n] = (Q_1, Q_2)_{m+n} \tau^i. \tag{A.0.11}$$

The supercharge modes transform under the $SU(2)^+$ modes as

$$\begin{aligned}
[T^{++}_m, G_{+,n}] &= 0, & [T^{+3}_m, G_{+,n}] &= \frac{1}{2}G_{+,m+n} - \gamma^- m Q_{+,m+n},\\
[T^{++}_m, G_{+K,n}] &= 0, & [T^{+3}_m, G_{+K,n}] &= \frac{1}{2}G_{+K,m+n} - \gamma^- m Q_{+K,m+n},\\
[T^{+-}_m, G_{-K,n}] &= 0, & [T^{+3}_m, G_{-K,n}] &= -\frac{1}{2}G_{-K,m+n} + \gamma^- m Q_{-K,m+n},\\
[T^{+-}_m, G_{-,n}] &= 0, & [T^{+3}_m, G_{-,n}] &= -\frac{1}{2}G_{-,m+n} + \gamma^- m Q_{-,m+n},\\
[T^{++}_m, G_{-K,n}] &= -G_{+,m+n} + 2\gamma^- m Q_{+,m+n}, \\
[T^{++}_m, G_{-,n}] &= G_{+K,m+n} - 2\gamma^- m Q_{+K,m+n}, \\
[T^{+-}_m, G_{+,n}] &= -G_{-K,m+n} + 2\gamma^- m Q_{-K,m+n}, \\
[T^{+-}_m, G_{+K,n}] &= G_{-,m+n} - 2\gamma^- m Q_{-,m+n},
\end{aligned} \tag{A.0.12}$$

and under the $SU(2)^-$ modes as

$$\begin{aligned}
[T^{-+}_m, G_{+,n}] &= 0, & [T^{-3}_m, G_{+,n}] &= \frac{1}{2}G_{+,m+n} + \gamma^+ m Q_{+,m+n},\\
[T^{-+}_m, G_{-K,n}] &= 0, & [T^{-3}_m, G_{-K,n}] &= \frac{1}{2}G_{-K,m+n} + \gamma^+ m Q_{-K,m+n},\\
[T^{--}_m, G_{+K,n}] &= 0, & [T^{-3}_m, G_{+K,n}] &= -(\frac{1}{2}G_{+K,m+n} + \gamma^+ m Q_{+K,m+n}),\\
[T^{--}_m, G_{-,n}] &= 0, & [T^{-3}_m, G_{-,n}] &= -(\frac{1}{2}G_{-,m+n} + \gamma^+ m Q_{-,m+n}),\\
[T^{-+}_m, G_{+K,n}] &= -(G_{+,m+n} + 2\gamma^+ m Q_{+,m+n}), \\
[T^{-+}_m, G_{-,n}] &= G_{-K,m+n} + 2\gamma^+ m Q_{-K,m+n}, \\
[T^{--}_m, G_{+,n}] &= -(G_{+K,m+n} + 2\gamma^+ m Q_{+K,m+n}), \\
[T^{--}_m, G_{-K,n}] &= G_{-,m+n} + 2\gamma^+ m Q_{-,m+n}.
\end{aligned} \tag{A.0.13}$$

As for the modes of the Q_a, these equations can be condensed considerably as,

$$[T^{\pm i}_m, (G_1, G_2)_n] = \Big((G_1, G_2)_{m+n} \mp 2\gamma^{\mp} m (Q_1, Q_2)_{m+n}\Big)\,\tau^i, \tag{A.0.14}$$

where $i \in \{+, -, 3\}$ and (G_1, G_2) given similarly to equation (A.0.9)

$$\begin{aligned}
(G_1, G_2) &= (G_+, -G_{-K}) \quad \text{or} \quad (G_{+K}, G_-) \quad \text{for } \widehat{\mathfrak{su}(2)}^+,\\
(G_1, G_2) &= (G_+, -G_{+K}) \quad \text{or} \quad (G_{-K}, G_-) \quad \text{for } \widehat{\mathfrak{su}(2)}^-.
\end{aligned} \tag{A.0.15}$$

Finally, the remaining non-trivial commutation relations are given by

$$\begin{aligned}
\{Q_{\pm,m}, G_{\mp,n}\} &= \mp\frac{1}{2}(T^{+3}_{m+n} - T^{-3}_{m+n}) + \frac{1}{2}U_{m+n}, & \{Q_{\pm,m}, G_{+K,n}\} &= \frac{1}{2}T^{\pm\pm}_{m+n}, \\
\{Q_{\pm K,m}, G_{\mp K,n}\} &= \mp\frac{1}{2}(T^{+3}_{m+n} + T^{-3}_{m+n}) + \frac{1}{2}U_{m+n}, & \{Q_{\pm,m}, G_{-K,n}\} &= -\frac{1}{2}T^{\mp\pm}_{m+n}, \\
\{Q_{+K,m}, G_{\pm,n}\} &= -\frac{1}{2}T^{\pm\pm}_{m+n}, & \{Q_{-K,m}, G_{\pm,n}\} &= \frac{1}{2}T^{\mp\pm}_{m+n}, \\
[U_m, G_{a,n}] &= mQ_{a,m+n}, & [U_m, U_n] &= -\frac{1}{2}mk\delta_{m+n,0}, \\
\{Q_{+,m}, Q_{-,n}\} &= -\frac{1}{4}k\delta_{m+n,0}, & \{Q_{+K,m}, Q_{-K,n}\} &= -\frac{1}{4}k\delta_{m+n,0},
\end{aligned} \tag{A.0.16}$$

where $a \in \{\pm, \pm K\}$.

The hermiticity properties of the modes are given by

$$\begin{aligned}
L^\dagger_n &= L_{-n}, & U^\dagger_n &= -U_{-n}, & G^\dagger_{+,n} &= G_{-,-n}, & Q^\dagger_{+,n} &= -Q_{-,-n}, \\
G^\dagger_{+K,n} &= G_{-K,-n}, & Q^\dagger_{+K,n} &= -Q_{-K,-n}, & (T^{\pm 3}_n)^\dagger &= T^{\pm 3}_{-n}, & (T^{\pm +}_n)^\dagger &= T^{\pm -}_{-n}.
\end{aligned} \tag{A.0.17}$$

Appendix B

Theta Functions and the Dedekind Eta Function

This short appendix is included as a reference for the definitions of the Jacobi theta functions and the Dedekind eta function, as well as their transformation properties under the modular group. The modular transformations either follow immediately from the definitions (in the case of the T-transformations), or are proved using Poisson summation (in the case of the S-transformations). These are classical results and as such are available in many places including for example [DMS97].

In this appendix we repeatedly abuse notation by writing functions of τ in the upper half-plane $\mathbb{H}$ and $\omega \in \mathbb{C}$ in terms of the *nome* $q = e^{2\pi i\tau}$ and $z = e^{2\pi i\omega}$. In all the following definitions we always take $\tau \in \mathbb{H}$ and $\omega \in \mathbb{C}$.

Definition B.0.1. The Dedekind η-function is defined as,

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{B.0.1}$$

The Dedekind η-function is also commonly written in terms of the Euler ϕ-function as,

$$\eta(\tau) = q^{1/24}\phi(\tau), \qquad \phi(\tau) := \prod_{n=1}^{\infty} (1 - q^n). \tag{B.0.2}$$

We also define the Jacobi theta functions as functions from $\mathbb{H} \times \mathbb{C} \to \mathbb{C}$.

Definition B.0.2. The four Jacobi theta functions are defined as,

$$
\begin{aligned}
\theta_1(\tau,\omega) &:= i\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} z^{n-\frac{1}{2}},\\
\theta_2(\tau,\omega) &:= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} z^{n-\frac{1}{2}},\\
\theta_3(\tau,\omega) &:= \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} z^{n},\\
\theta_4(\tau,\omega) &:= \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} z^{n}.
\end{aligned}
\tag{B.0.3}
$$

Using the Jacobi triple product, one can write product formulae for these theta functions as,

$$
\begin{aligned}
\theta_1(\tau,\omega) &= iq^{1/8}z^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1}z)(1-q^n z^{-1}),\\
\theta_2(\tau,\omega) &= q^{1/8}z^{-1/2}\prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1}z)(1+q^n z^{-1}),\\
\theta_3(\tau,\omega) &= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2}z)(1+q^{n-1/2} z^{-1}),\\
\theta_4(\tau,\omega) &= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2}z)(1-q^{n-1/2} z^{-1}).
\end{aligned}
\tag{B.0.4}
$$

Proposition B.0.3. *The modular transformations of $\eta(\tau)$ as well as the Jacobi theta functions are given by,*

$$
\begin{aligned}
\eta(\tau)^S &:= \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau), & \eta(\tau)^T &:= \eta(\tau+1) = e^{\frac{\pi i}{12}}\eta(\tau),\\
\theta_i(\tau,z)^S &:= \theta_i(-\frac{1}{\tau},\frac{\omega}{\tau}), & \theta_i(\tau,z)^T &:= \theta_i(\tau+1,\omega),\\
\theta_1(\tau,z)^S &= -i(-i\tau)^{\frac{1}{2}}e^{\pi i\frac{\omega^2}{\tau}}\theta_1(\tau,\omega), & \theta_1(\tau,\omega)^T &= e^{\frac{\pi i}{4}}\theta_1(\tau,\omega),\\
\theta_2(\tau,\omega)^S &= (-i\tau)^{\frac{1}{2}}e^{\pi i\frac{\omega^2}{\tau}}\theta_4(\tau,\omega), & \theta_2(\tau,\omega)^T &= e^{\frac{\pi i}{4}}\theta_2(\tau,\omega),\\
\theta_3(\tau,\omega)^S &= (-i\tau)^{\frac{1}{2}}e^{\pi i\frac{\omega^2}{\tau}}\theta_3(\tau,\omega), & \theta_3(\tau,\omega)^T &= \theta_4(\tau,\omega),\\
\theta_4(\tau,\omega)^S &= (-i\tau)^{\frac{1}{2}}e^{\pi i\frac{\omega^2}{\tau}}\theta_2(\tau,\omega), & \theta_4(\tau,\omega)^T &= \theta_3(\tau,\omega).
\end{aligned}
\tag{B.0.5}
$$

Appendix C

Characteristic Classes and Genera

The aim of this appendix is to introduce some mathematical notions, which while standard in the mathematical literature are less well known in the physics literature. In particular, we wish to define a genus as a homomorphism from the cobordism ring to some unital $\mathbb{Q}$-algebra and this section introduces the basic definitions that we need for this. We start by briefly introducing the dual notions of homology and cohomology, as cohomology classes will be key for much of this chapter. We then introduce the notion of characteristic classes, which associate cohomology classes to bundles over a base space. Next we define the equivalence notion of bordism as a courser way to classify manifolds than through diffeomorphically equivalent manifolds. In particular we show that the equivalences classes of manifolds up to bordisms has a ring structure. Finally we use characteristic classes to define a genus as a homomorphism from the (oriented) bordism ring to some $\mathbb{Q}$-algebra Λ. We finish this appendix by explaining what it means for a genus to be elliptic and give some examples of elliptic genera.

C.1 Homology and Cohomology

C.1.1 Simplicial Homology

In general, homology theory is a procedure for associating a sequence of abelian groups to an object such as a topological space. There are many different types of homology theory that one can consider, but here we will consider *simplicial homology* as this enables us to consider the homology in a visual way. For spaces which admit a triangulation, simplicial homology is isomorphic to singular homology which we consider in Appendix C.1.2.

Definition C.1.1. The *convex hull* of a set T in Euclidian space is the smallest convex set that contains T.

Definition C.1.2. A *k-simplex* is a k-dimensional polytope in Euclidian space which is the convex hull of its $k+1$ vertices in general position as taken with the subspace topology. This simply gives us the point, the closed line, the triangle, the tetrahedron etc. Further, we refer to the convex hull of any subset of $(m-1)$ points of a k-simplex as an *m-face* of the simplex.

Note that this definition does not require the simplex to be regular, if the simplex is regular then it is called a regular simplex.

Definition C.1.3. A *simplicial complex* Σ, is a topological space formed as the union of k-simplices of possibly different dimensions which satisfies the following two conditions:

1. The face of any simplex $X \in \Sigma$ is also a simplex in Σ.

2. The intersection of any two simplices σ_1, $\sigma_2 \in \Sigma$ is a face of both σ_1 and σ_2.

A simplicial complex can easily be constructed from k-simplices of increasing dimension. If we begin with a point set (0-simplices), then we may attach lines (1-simplices) to the set freely, as long as we ensure that each line ends on a simplex in order to satisfy property 1 of definition C.1.3 and as long as the lines do not intersect away from the vertex set in order to satisfy condition 2 of definition C.1.3 . We may now attach triangles (2-simplices) to our complex in a similar manner; the triangles must have their edges as lines in the complex and must not intersect each other at any points not already in our complex. We can continue attaching simplices of higher dimensions in this manner until we stop after attaching a set of n-simplices. We then refer to our complex as having *dimension* n.

If we label the vertices of our simplex as $0, \dots, n$, then our simplices naturally inherit an orientation. The edges are oriented as (i, j) if $i < j$ and as (j, i) otherwise, where i, j are the vertices. Similarly, a k-simplex is oriented in increasing order of its vertices.

Realising a topological space in terms of a simplicial complex is known as a *triangulation* of the space. Any real, differentiable manifold can be seen to admit a triangulation by the Whitney embedding theorem.

Definition C.1.4. The *chain group* $C_k(X)$ of a simplicial complex X, is defined as the $\mathbb{Q}$-vector space over X_k, where X_k is the set of all k-simplices in X.

An element $x \in C_k(X)$ is called a *k-chain* on X.

Example C.1.5. It is clear that for fig. C.1 we have

$$e_1 = (v_1, v_2), \qquad e_5 = (v_3, v_4), \qquad \sigma_1 = (v_1, v_2, v_3). \tag{C.1.1}$$

The chain groups are

$$\begin{aligned} &C_0(X) = \mathrm{Span}\{v_1, v_2, v_3, v_4\}, \qquad C_1(X) = \mathrm{Span}\{e_1, e_2, e_3, e_4, e_5\}, \\ &C_2(X) = \mathrm{Span}\{\sigma_1, \sigma_2\}. \end{aligned} \tag{C.1.2}$$

△

We introduced homology in order to help us describe the topology of a space. The preceding definitions have been introduced to allow us to define the notion of a boundary in a manner that agrees with our intuitive understanding of the notion of

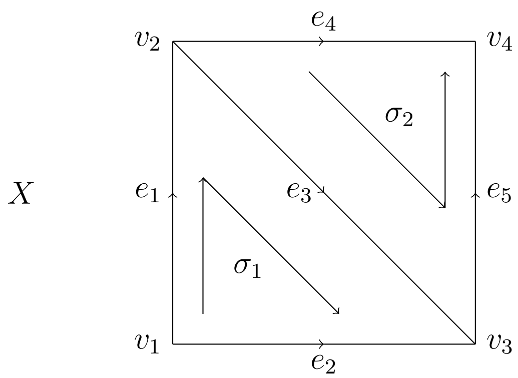


Figure C.1: A simple chain complex

a boundary. We define a boundary operator as a linear map $\partial : C_k(X) \to C_{k-1}(X)$ for some complex X.

Definition C.1.6. Let $(v_0, v_1, \ldots, v_k) \equiv [0, 1, \ldots, k]$ and let $[0, \ldots, \hat{n}, \ldots, k]$ be the simplex with all the vertices $0, \ldots, k$ aside from n, then we can define,

$$\partial[0, \ldots, k] = \sum_{n=0}^{k} (-1)^n [0, \ldots, \hat{n}, \ldots, k]. \tag{C.1.3}$$

One can easily check that this agrees with our standard notion of boundary.

Example C.1.7. Consider the closed path $e_3 + e_5 - e_4$.

$$\begin{aligned} \partial(e_3 + e_5 - e_4) &:= \partial[2,3] + \partial[3,4] - \partial[2,4], \\ &= [2] - [3] + [3] - [4] - [2] + [4] = 0, \end{aligned} \tag{C.1.4}$$

which shows a closed path has no boundary as expected. △

Lemma C.1.8.

$$\partial^2 = 0. \tag{C.1.5}$$

Proof. $\partial^2[0, \ldots, k] = \sum_{m,n=0}^{k} [(-1)^{n+m} + (-1)^{n+m-1}][0, \ldots, \hat{m}, \ldots, \hat{n}, \ldots, k] = 0$ □

Definition C.1.9. *k-cycles* on X are chains in Ker ∂, denoted $Z_k(X)$

$$Z_k(X) = \mathrm{Ker}(\partial_k).$$

k-boundaries are chains in Im ∂, denoted $B_k(X)$

$$B_k(X) = \mathrm{Im}(\partial_{k+1}).$$

Now since we have proved that ∂ necessarily satisfies $\partial^2 = 0$, by lemma C.1.8, we have that k-cycles are boundariless, that is $B_k(x) \subset Z_k(X)$, so $B_k(X)$ is a linear subspace of $Z_k(X)$. Since our chain groups are abelian groups and we have $Z_k(x) \subset C_k(X)$, then $Z_k(X)$ is abelian and hence $B_k(X)$ is normal in $Z_k(X)$.

It therefore makes sense to define the quotient group and this is what we define as the k^{th} homology group of a simplicial complex X.

Definition C.1.10. We define the k^{th} *homology group* of a simplicial complex X as

$$H_k(X) = \frac{Z_k(X)}{B_k(X)} \tag{C.1.6}$$

and we can construct a chain complex for our example C.1.5

$$\dots \xrightarrow{0} 0 \xrightarrow{0} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{0} 0 \xrightarrow{0} \dots$$

that satisfies $\partial^2 = 0$, or more precisely $\partial_1 \circ \partial_2 = 0$.

Geometrically we can think of 'holes' in the space X as cycles in $Z_k(X)$ that are not the boundaries of higher dimensional simplices. This is exactly the information captured by the homology groups of X.

Definition C.1.11. The n^{th} Betti number b_n of a topological space X is the rank of the n^{th} homology group $H_n(X)$

$$b_n(X) = \text{rank } H_n(X).$$

The Betti numbers can be thought of as the number of k-dimensional holes in X. The Betti number b_0 tells us the number of connected components of X.

C.1.2 Singular Homology

In singular homology we begin by considering *singular chains* of *standard n-simplices*.

Definition C.1.12. The *standard n-simplex* is the set

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1,\ t_i \geq 0,\ i = 0, \ldots, n\}$$

This definition agrees with our understanding of the simplex as defined in definition C.1.2. Note that we could also choose to define the standard n-simplex in $\mathbb{R}^n$ instead, but this embedding into $\mathbb{R}^{n+1}$ automatically defines us a *regular* n-simplex.

Definition C.1.13. Given a space X, a *singular n-simplex* is a continuous map $\sigma : \Delta^n \to X$. This map need not be injective. For instance, any constant map from a n-simplex to X may be viewed as a singular n-simplex.

As in the case of simplicial homology (Appendix C.1.1), we can now use the singular n-simplices to form singular n-chains.

Definition C.1.14.

- A *singular n-chain* is a formal linear combination of singular n-simplices, $\sum_i n_i \sigma_i$.
- The *singular chain group*, $S_n(X)$, is the free abelian group with a basis of the set of all singular n-simplices on X. A singular n-chain is therefore any element of the chain group $S_n(X)$.

The advantage of considering singular homology rather than simplicial homology is that in simplicial homology we needed to state that the homology groups obtained are independent of the triangulation of the space. In singular homology, since we are considering all maps from the standard simplices into the space X, then this ambiguity is removed, but in return we are left with much larger (usually uncountable) chain groups.

We can now define the boundary of a singular n-simplex as a formal sum of the singular $(n-1)$-simplices obtained by restricting the singular simplex σ to the faces of the standard n-simplex and taking an alternating sign to take account of orientation as in the simplicial case. That is, for the vertices, we have

$$\partial_n \sigma_n(\Delta^n) = \sum_{i=0}^{n} (-1)^i \sigma \circ \iota_{(0,\ldots,\hat{i},\ldots,n)}, \tag{C.1.7}$$

where ι_S is the natural embedding of the standard simplex spanned by S into the standard n-simplex, and as before, $S = (0, \ldots, \hat{i}, \ldots, n)$ is taken to mean the set of points of S not including i.

C.1.3 Cohomology

Cohomology arises by considering the algebraic dualisation of homology. Rather than considering the chain groups C_k and the linear boundary maps ∂_k between them, we consider the dual spaces C_k^*, the *cochain groups*, and the transpose of the boundary operators, $\delta^n : C_{n-1}^* \to C_n^*$, the *coboundary operators* or *differentials*. The coboundary maps clearly satisfy $\delta^{n+1}\delta^n = 0$ since $\delta^{n+1}\delta^n = (\partial_{n+1})^t(\partial_n)^t = (\partial_n \partial_{n+1})^t = (0)^t = 0$ and hence we can form a cochain complex

$$\ldots \xleftarrow{\delta^{n+1}} C_n^* \xleftarrow{\delta^n} C_{n-1}^* \xleftarrow{\delta^{n-1}} \ldots$$

We can then define the n^{th} cohomology group $H^n(X) = \frac{\mathrm{Ker}(\delta^{n+1})}{\mathrm{Im}(\delta^n)}$.

In cohomology it is possible to define a product operation on the elements of the cohomology groups in order to obtain a cohomology ring. This operation, known as the *cup product*, defines a way of combining a p-cocycle with a q-cocycle to obtain a $p+q$-cocycle. It can then be shown that this cup product on the cocycles induces a product on the cohomology classes. Having introduced singular homology in Appendix C.1.2 we will now use this to define the cup product in singular cohomology. In singular cohomology the cup product of a p cocycle, c^p, and a q cocycle, d^q is given by

$$(c^p \smile d^q)(\sigma_{p+q}) = c^p(\sigma_{p+q} \circ \iota_{0,\ldots,p}) \cdot d^q(\sigma_{p+q} \circ \iota_{p,\ldots,p+q}), \tag{C.1.8}$$

where ι_S is the natural embedding of the standard S-simplex into the standard $(p+q)$-simplex. $\sigma_{p+q} \circ \iota_{0,\ldots,p}$ and $\sigma_{p+q} \circ \iota_{p+1,\ldots,p+q}$ are often referred to as the p^{th} *front* *face* and q^{th} *back face* of σ_{p+q} respectively.

Since the coboundary map δ is the dual of the boundary map ∂, we have

$$\delta c(\sigma) = \ c(\partial\sigma) = \sum_i (-1)^i c(\sigma \circ \iota_{0,\ldots,\hat{i},\ldots,n+1}). \tag{C.1.9}$$

This leads to the following standard lemma,

Lemma C.1.15.

$$\delta(c^p \smile d^q) = (\delta c^p \smile d^q) + (-1)^p (c^p \smile \delta d^q). \tag{C.1.10}$$

From this we have two simple corollaries.

Corollary C.1.16. *The cup product of two cocylces, c^p, d^q is a cocycle, $\delta(c^p \smile d^q) = 0$.*

Corollary C.1.17. *The cup product of a cocycle and a coboundary is a coboundary.*

The cup product therefore induces a product for the cohomology classes

$$H^p(X) \times H^q(X) \to H^{p+q}(X). \tag{C.1.11}$$

One important example of a cohomology theory is de Rham cohomology. In de Rham cohomology, the cochains are the elements of the spaces $\Omega^k(M)$ of k-forms on the smooth manifold M and the differential is the exterior derivative $d : \Omega^k \to \Omega^{k+1}$. We call a form α *exact* if it is the image of a form under the exterior derivative, $\alpha = d\beta$. We call a form α *closed* if its exterior derivative is equal to 0, $d\alpha = 0$. Since $d^2 = 0$, all exact forms are closed and we can consider the quotient space of closed forms modulo exact forms. We call these spaces the *de Rham cohomology groups*, $H^k_{dR}(M)$. In de Rham cohomology, one can realise the cup product as the wedge product of differential forms.

C.2 Characteristic Classes

Characteristic classes are a way to associate cohomology classes of a space X to vector and principal bundles on the space. Specifically we wish to consider the Chern classes and the Pontryagin classes.

Chern classes are characteristic classes which are associated to complex vector bundles – bundles whose fibres are complex vector spaces with $\mathbb{C}$-linear transition functions. The Chern classes of a complex vector bundle E over a manifold M are elements of the cohomology ring $H^*(M, \mathbb{Z})$. Specifically, the i^{th} Chern class $c_i(E) \in H^{2i}(M, \mathbb{Z})$.

We take the Chern classes to be defined in the following way.

Definition C.2.1. To a complex vector bundle E of complex rank n over a smooth manifold M, we may associate distinguished elements of the cohomology ring $H^*(M, \mathbb{Z})$ known as *Chern classes.* The Chern classes may be given by the coefficients of the characteristic equation of the curvature form Ω on E,

$$\det\left(\frac{it\Omega}{2\pi} + I\right) = \sum_i c_i(E)t^i. \tag{C.2.1}$$

This follows from the Chern-Weil homomorphism and hence is independent of the choice of connection on E.

Definition C.2.2. The *total Chern class* for a rank n complex bundle E is given by

$$c(E) = \sum_{i=1}^{\infty} c_i(E) \in H^*(X, \mathbb{Z}). \tag{C.2.2}$$

Note that we always have $c_0(E) = 1$ for all complex bundles E.

An alternate axiomatic definition is given in [HBJL92].

Definition C.2.3. Given a complex vector bundle E over a manifold X, we may define the *Chern classes* as characteristic classes uniquely satisfying the following properties:

1. $c_i(E) \in H^{2i}(X, \mathbb{Z}),\ c_0(E) = 1$.

2. $c_i(f^*E) = f^*c_i(E)$, where $f : Y \to X$ is a continuous map.

3. $c(E \oplus F) = c(E) \cdot c(F)$, where F is another complex vector bundle over X and $E \oplus F$ is the Whitney sum of the two bundles, that is the fibrewise direct sum of the bundles.

4. $c(H) = 1 - g$, where H is the *Hopf bundle* over $\mathbb{C}P^n$ and $g \in H^2(\mathbb{C}P^n, \mathbb{Z})$ is the generating element of the cohomology ring of $\mathbb{C}P^n$. $(\mathbb{C}^n)^\times \equiv \mathbb{C}^n \setminus \{0\}$ naturally fibres over $\mathbb{C}P^n$; If $\mathbb{C}P^n$ is seen as the image of the map $\rho : (\mathbb{C}^n)^\times \to \mathbb{C}P^n$ defined by $\rho(\lambda z_1, \lambda z_2) = (z_1, z_2)$ for non-zero $\lambda \in \mathbb{C}$, then the restriction of ρ to the unit norm elements of $\mathbb{C}^n$ defines the fibration known as the *Hopf fibration*.

For notational ease we also make the following standard definition.

Definition C.2.4. We can define the Chern classes of a smooth manifold M as the Chern classes of the tangent bundle to the manifold,

$$c_i(M) := c_i(TM). \tag{C.2.3}$$

We use the following proposition, taken from [Hat03] without proof.

Proposition C.2.5. *Given a manifold M, the group of isomorphism classes of complex line bundles on M under tensor products is isomorphic to the second cohomology group of M, $H^2(M, \mathbb{Z})$. Specifically the isomorphism is given by c_1, so for complex line bundles L_1, L_2 we have*

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2). \tag{C.2.4}$$

This has the following simple corollary,

Corollary C.2.6. *Given a manifold M, consider a complex line bundle L with first Chern class $c_1(L)$. Then the first Chern class of the dual bundle L^* is given by $c_1(L^*) = -c_1(L)$.*

Proof. We note that the tensor product of L with L^*, $L \otimes L^* = \mathrm{Hom}(L, L)$ is a trivial bundle, since it is a rank 1 bundle with a nowhere vanishing section given by the identity map. We then have

$$0 = c_1(\mathrm{Hom(L,L)}) = c_1(L \otimes L^*) = c_1(L) + c_1(L^*), \tag{C.2.5}$$

and hence $c_1(L^*) = -c_1(L)$. □

We note the following theorem about the tangent bundle to $\mathbb{C}P^n$, a proof of which may be found in [Coh98].

Theorem C.2.7.

$$T\mathbb{C}P^n \oplus \epsilon_1 \cong \oplus_{n+1} H^*, \tag{C.2.6}$$

where ϵ_1 is a trivial line bundle and H^* is the dual to the Hopf bundle defined in definition C.2.3.

Using this theorem we can prove the following lemma.

Lemma C.2.8. *The total Chern class of $\mathbb{C}P^n$ is given by*

$$c(\mathbb{C}P^n) = (1 + g)^{n+1}, \tag{C.2.7}$$

where g is the generating element of the cohomology ring of $\mathbb{C}P^n$ as in definition C.2.3, and where by the Chern class of a manifold we implicitly mean the Chern class of the tangent bundle.

Proof. We have

$$\begin{aligned} c(\mathbb{C}P^n) &:= c(T\mathbb{C}P^n) = c(T\mathbb{C}P^n \oplus \epsilon_1), \\ &= c(\oplus_{n+1} H^*) = c(H^*)^{n+1}, \\ &= (1 + g)^{n+1}, \end{aligned} \tag{C.2.8}$$

where in the first line we use proposition C.2.5 to show that $c(\epsilon_1) = 1 + c_1(\epsilon_1) = 1$. Since the trivial line bundle is the identity of the group structure on the isomorphism classes of line bundles it gets mapped to zero under the isomorphism to $H^2(M, \mathbb{Z})$

given by c_1. We then use the Whitney sum rule as defined in definition C.2.3 to split the Chern classes of the direct sum bundles into the products of their Chern classes. In the final line we also used corollary C.2.6. $\square$

A theorem known as the 'splitting principle' [HBJL92], means one can consider any statement about Chern classes as a statement on sums of line bundles.

Theorem C.2.9. *Consider a vector bundle $\pi : E \to X$ of rank n. Then there exists a space $Fl(E)$, known as the flag bundle of E and a map $\rho : Fl(E) \to X$ such that*

- *the induced homomorphism of cohomology, $\rho^* : H^*(X) \to H^*(Fl(E))$ is injective.*
- *the pullback bundle $\rho^*\pi : \rho^*(E) \to Fl(E)$ breaks up as a direct sum of line bundles, $\rho^*(E) = E_1 \oplus E_2 \oplus \ldots \oplus E_n$.*

The point is that we may always consider any statement about the Chern classes on the flag bundle $Fl(E)$ and then push forward to the space X with ρ. The fact that ρ^* is injective on the cohomology rings means that any equation which the Chern classes satisfy in $H^*(Fl(E))$ holds in $H^*(X)$.

Using this principle we can calculate the total Chern class of some specific vector bundles.

Example C.2.10. Consider a rank n complex vector bundle E. By the splitting principle, in any calculation about Chern classes, we may consider E to be given by the direct sum of n complex line bundles, $E = E_1 \oplus E_2 \oplus \ldots \oplus E_n$. These line bundles have total Chern class

$$c(E_i) = 1 + c_1(E_i), \tag{C.2.9}$$

and so let us define $x_i := c_1(E_i)$. The total Chern class of E is then given by the

Whitney sum rule as,

$$\begin{aligned} c(E) &= c(E_1 \oplus E_2 \oplus \ldots \oplus E_n), \\ &= c(E_1) \cdot \ldots \cdot c(E_n), \\ &= \prod_{i=1}^{n} (1 + x_i). \end{aligned} \tag{C.2.10}$$

Now let us calculate the Chern class of the dual bundle $c(E^*) = c(E_1^* \oplus \ldots \oplus E_n^*)$,

$$\begin{aligned} c(E^*) &= c(E_1^* \oplus \ldots \oplus E_n^*), \\ &= c(E_1^*) \cdot \ldots \cdot c(E_n^*), \\ &= \prod_{i=1}^{n} (1 - x_i). \end{aligned} \tag{C.2.11}$$

and so we have

$$c(E^*) = \sum_{i=1}^{n} (-1)^n c_i(E). \tag{C.2.12}$$

Let us also calculate the Chern class of $\Lambda^k E$,

$$\begin{aligned} c(\Lambda^k E) &= c(\bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} L_{i_1} \otimes \ldots \otimes L_{i_k}), \\ &= \prod_{1 \leq i_1 < \ldots < i_k \leq n} (1 + x_{i_1} + \ldots + x_{i_k}). \end{aligned} \tag{C.2.13}$$

Combining these two results gives

$$c(\Lambda^k E^*) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} (1 - (x_{i_1} + \ldots + x_{i_k})). \tag{C.2.14}$$

△

In order to make contact with the Atiyah-Singer index theorem, we will also need to define the Chern character.

Definition C.2.11. Given a complex vector bundle E of rank n, the splitting principle (theorem C.2.9) shows that it has total Chern class

$$c(E) = \prod_{i=1}^{n} (1 + x_i), \tag{C.2.15}$$

where $x_i := c_1(E_i)$. We then define the *Chern character* as

$$\mathrm{ch}(E) := \sum_{i=1}^{n} e^{x_i}. \tag{C.2.16}$$

By definition we have

$$\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2), \tag{C.2.17}$$

for the Chern character.

Example C.2.12. We can now calculate the Chern character of the bundles considered in the previous example.

For E^*, we showed that $c(E^*) = \prod_{i=1}^{n}(1 - x_i)$, and hence we have

$$\mathrm{ch}(E^*) = \sum_{i=1}^{n} e^{-x_i}. \tag{C.2.18}$$

For $\Lambda^k E$, we showed that $c(\Lambda^k E) = \prod_{1\leq i_1<\ldots<i_k\leq n}(1 + x_{i_1} + \ldots + x_{i_k})$, and hence

$$\mathrm{ch}(\Lambda^k E) = \sum_{1\leq i_1<\ldots<i_k\leq n} e^{x_{i_1}+\ldots+x_{i_k}}, \tag{C.2.19}$$

and similarly

$$\mathrm{ch}(\Lambda^k E^*) = \sum_{1\leq i_1<\ldots<i_k\leq n} e^{-(x_{i_1}+\ldots+x_{i_k})}. \tag{C.2.20}$$

If we now define

$$\Lambda_t E := \sum_{i=0}^{\infty}(\Lambda^i E)\cdot t^i, \tag{C.2.21}$$

then we have the following formula for the Chern character of the dual bundle to E,

$$\begin{aligned} \mathrm{ch}(\Lambda_t E^*) &:= \mathrm{ch}(\sum_{i=0}^{n}(\Lambda^i E^*)t^i), \\ &:= \sum_{i=0}^{n} \mathrm{ch}(\Lambda^i E^*)t^i, \\ &= \prod_{i=1}^{n}(1 + te^{-x_i}), \end{aligned} \tag{C.2.22}$$

where in the second line we used the additivity for the Chern character noted in definition C.2.11. △

Given a real vector bundle, one can form a complex vector bundle.

Definition C.2.13. Given a real vector bundle E, one can form a complex vector bundle known as the *complexification* of E,

$$E^{\mathbb{C}} := E \otimes \epsilon_1, \tag{C.2.23}$$

where as in theorem C.2.7, ϵ_1 is the 1-dimensional trivial complex line bundle. We will also denote the complexification by $E^{\mathbb{C}} = E \otimes \mathbb{C}$, where here $\mathbb{C}$ stands for the trivial complex line bundle.

Similarly, given a complex vector bundle, we can easily create a real vector bundle.

Definition C.2.14. Given a complex vector bundle E of rank n, we can form a real vector bundle of rank $2n$ which we denote $E_{\mathbb{R}}$ by forgetting the complex structure on the fibres.

Lemma C.2.15. *Given a complex vector bundle E of rank n, we have*

$$(E_{\mathbb{R}})^{\mathbb{C}} := E_{\mathbb{R}} \otimes \mathbb{C} = E \oplus \bar{E}, \tag{C.2.24}$$

where $\bar{E}$ is a complex vector bundle known as the conjugate bundle to E *which is isomorphic to E as a real vector bundle, but where complex numbers act on the fibres through their conjugates.*

Proof. E has structure group $GL_n(\mathbb{C})$. On the underlying real bundle $E_{\mathbb{R}}$, we let $m = m_r + im_i \in GL_n(\mathbb{C})$ act through the map $r(m)$, where

$$r(m) = \begin{pmatrix} m_r & -m_i \\ m_i & m_r \end{pmatrix}. \tag{C.2.25}$$

On the complexification of this bundle $(E_{\mathbb{R}})^{\mathbb{C}}$, we should be able to act with arbitrary complex numbers, and hence the structure group is $GL_{2n}(\mathbb{C})$. Denote the inclusion map from $GL_n(\mathbb{R})$ to $GL_n(\mathbb{C})$ by C. Consider the matrix $\eta \in GL_{2n}(\mathbb{C})$,

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}, \tag{C.2.26}$$

then we have

$$\eta C(r(m))\eta^{-1} = \begin{pmatrix} m & 0 \\ 0 & \bar{m} \end{pmatrix}. \tag{C.2.27}$$

We therefore see that acting on the vector space $(E_{\mathbb{R}})^{\mathbb{C}}$ with a $m \in GL_n\mathbb{C}$ is equivalent to acting on the vector space $E \oplus E$ with $m \oplus \bar{m}$, and therefore we have the bundle isomorphism $(E_{\mathbb{R}})^{\mathbb{C}} \cong E \oplus \overline{E}$ as claimed. □

We can now calculate the total Chern class and Chern character for real vector bundles.

Example C.2.16. For a real rank $2n$ bundle E, we can use the splitting principle to treat it as the direct sum of n rank 2 bundles $E = \oplus_{i=1}^{n} E_i$. The 2-dimensional fibres all have a natural complex structure, and so we can define $x_i = c_1(E_i)$ as before. The Chern character for $E^{\mathbb{C}}$ is therefore given by

$$\begin{aligned} c(E^{\mathbb{C}}) &= c((E_1 \otimes \mathbb{C}) \oplus \ldots \oplus (E_n \otimes \mathbb{C})), \\ &= c(E_1 \otimes \mathbb{C}) \ldots c(E_n \otimes \mathbb{C}), \\ &= c(E_1 \oplus \overline{E_1}) \ldots c(E_n \oplus \overline{E_n}), \\ &= c(E_1 \oplus E_1^*) \ldots c(E_n \oplus E_n^*), \\ &= \prod_{i=1}^{n} (1 + x_i)(1 - x_i), \end{aligned} \tag{C.2.28}$$

where we used lemma C.2.15 since the E_i are the underlying real bundles of complex line bundles. Note also that a Hermitian metric on the bundle gives an isomorphism between $\overline{E}$ and E^*.

The Chern character of $E^{\mathbb{C}}$ is therefore zero as one would expect by the addition property given in definition C.2.11.

Finally, we can calculate the Chern character for $\Lambda_t(E^{*\mathbb{C}}) := \sum_{i=0}^{n}(\Lambda^i(E^*\otimes\mathbb{C}))t^i$ as,

$$\begin{aligned}\mathrm{ch}(\Lambda_t(E^{*\mathbb{C}})) &= \mathrm{ch}(\sum_{i=0}^{n}(\Lambda^i(E^*\otimes\mathbb{C}))t^i),\\ &= \prod_{i=1}^{n}((1+te^{x_i})(1+te^{-x_i})),\end{aligned} \tag{C.2.29}$$

using eq. (C.2.22) △

The *Pontryagin classes*, p_i, are characteristic classes associated to real vector bundles over a manifold. We define the Pontryagin classes as follows:

Definition C.2.17. Given a real vector bundle E over a manifold M, we define the *Pontryagin classes*, p_i, by

$$p_i(E) = (-1)^i c_{2i}(E\otimes\mathbb{C}) \in H^{4i}(X,\mathbb{Z}). \tag{C.2.30}$$

Similarly to the case for Chern classes we then define the *total Pontryagin class* as

$$p(E) = \sum_{i=0}^{\infty} p_i(E) \in H^*(X,\mathbb{Z}). \tag{C.2.31}$$

As for the Chern classes, we also define the Pontryagin classes of a manifold M to be the Pontryagin classes of the tangent bundle to the manifold,

$$p_i(M) := p_i(TM). \tag{C.2.32}$$

Lemma C.2.18. *For a complex vector bundle E,*

$$\sum_{i=0}^{\infty}(-1)^i p_i(E_{\mathbb{R}}) = c(E)\cdot\sum_{i=0}^{\infty}(-1)^i c_i(E) \tag{C.2.33}$$

Proof. By lemma C.2.15, given a complex vector bundle E we have $E_{\mathbb{R}}\otimes\mathbb{C}\cong E\oplus\overline{E}$. We therefore have

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^i p_i(E_{\mathbb{R}}) =& \sum_{i=0}^{\infty} c_{2i}(E_{\mathbb{R}}\otimes\mathbb{C}) = \sum_{i=0}^{\infty} c_{2i}(E\oplus\overline{E})\\ =& \sum_{i=0}^{\infty}(c_{2i}(E) + c_{2i-1}(E)\cdot c_1(\overline{E}) + \dots\\ & + c_1(E)\cdot c_{2i-1}(\overline{E}) + c_{2i}(\overline{E})\\ =& c(E)\cdot\sum_{i=0}^{\infty}(-1)^i c_i(E).\end{aligned} \tag{C.2.34}$$

□

Lemma C.2.19.

$$p(\mathbb{C}P^n) = (1+g^2)^{n+1} \tag{C.2.35}$$

Proof. By lemma C.2.18, we have

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^i p_i(\mathbb{C}P^n) = c(\mathbb{C}P^n)\sum_{i=0}^{\infty}(-1)^i c_i(\mathbb{C}P^n) &= (1+g)^{n+1}(1-g)^n+1, \\ &= (1-g^2)^{n+1},\end{aligned} \tag{C.2.36}$$

using lemma C.2.8. Then by considering the dimensions of the Pontryagin classes as elements of the cohomology ring we obtain,

$$p(\mathbb{C}P^n) = (1+g^2)^{n+1} \tag{C.2.37}$$

as claimed. □

We can use the Pontryagin classes to obtain *Pontryagin numbers* by pairing them with the fundamental class of the manifold, which we will first define.

Proposition C.2.20. *The top homology group of a closed, connected, orientable manifold $H_n(X)$ is isomorphic to $\mathbb{Z}$.*

Proof. First consider a triangulation of M. If the manifold is of dimension n, then by definition there are no simplices of dimension $(n+1)$ in the triangulation. Hence the top homology group, $H^n(X)$, is generated by $\mathrm{Ker}(\partial_n)$. The set of n-cycles of the manifold is generated by the sum of all the n-simplices. Hence the top homology group is isomorphic to $\mathbb{Z}$. □

Definition C.2.21. The *fundamental class* of an oriented manifold M of dimension n is an element of the top homology group of M. By Proposition C.2.20 we know that this homology group is isomorphic to $\mathbb{Z}$. Therefore this homology group has a single generator, which we call the fundamental class, $[M] \in H_n(M, \mathbb{Z})$.

Definition C.2.22. Consider a smooth manifold, M, of dimension $4n$ and a partition of n, $(i_1, i_2, \dots, i_k)$, i.e. $\sum_{j=1}^{k} i_j = n$. Recall that the cup product endows the cohomology groups with the structure of a graded ring,

$$\smile : H^p(M) \times H^q(M) \to H^{p+q}(M). \tag{C.2.38}$$

If we take the cup product of the Pontryagin classes then we have $\prod_{j=1}^{k} p_{i_j}(M) \in H^{4n}(M)$. Now we recall that the cohomology classes are dual spaces to the homology classes, hence we can evaluate the above product of Pontryagin classes against an element of $H_{4n}(M)$; we have a natural such element - the fundamental class $[M]$. This product of Pontryagin classes is known as the *Pontryagin number* of M^{4n} for a partition $(i_1, \dots, i_k)$ of n

$$P_{i_1,i_2,\dots,i_k} = p_{i_1}(M) \smile p_{i_2}(M) \smile \dots \smile p_{i_k}(M)[M]. \tag{C.2.39}$$

We will now consider the Chern classes on a direct sum of line bundles and see that we may think of the Chern classes as symmetric functions. Consider a complex vector bundle E of rank n, which is the direct sum of complex line bundles E_i, $E = E_1 \oplus E_2 \oplus \dots \oplus E_n$. The total Chern class of the bundle E is by definition

$$c(E) = 1 + c_1(E) + \dots c_n(E). \tag{C.2.40}$$

Now since the bundles E_i are line bundles, the only non-zero Chern classes are $c_0(E_i) = 1$ and $c_1(E_i) := x_i$. We therefore have $c(E_i) = (1 + x_i)$. Now by property 3 of Definition C.2.3, we have

$$\begin{aligned} c(E) &= c(E_1) \cdot c(E_2) \cdot \dots \cdot c(E_n) \\ &= (1 + x_1)(1 + x_2) \dots (1 + x_n) \\ &= 1 + \sum_{m=1}^{n} e_m(x_1, \dots, x_n), \end{aligned} \tag{C.2.41}$$

where $e_m(x_1, \dots, x_n)$ is the m^{th} elementary symmetric polynomial in the x_i's. Now by considering the dimensions of the cohomology classes we see that we have

$$c_i(E) = e_i(x_1, \ldots, x_n). \tag{C.2.42}$$

We may do something similar for the Pontryagin classes. We consider the real vector bundle $E_{\mathbb{R}}$ of rank $2n$. It can be shown that the Pontryagin classes satisfy $p(E \oplus F) = p(E) \cdot p(F) \bmod 2$. By the splitting principle we therefore have

$$p(E_{\mathbb{R}}) = p(E_{\mathbb{R},1}) \ldots p(E_{\mathbb{R},n}) \tag{C.2.43}$$

for 2-dimensional bundles $E_{i,\mathbb{R}}$. Now for the 2-dimensional bundles $E_{i,\mathbb{R}}$ we have

$$\begin{aligned} p(E_{i,\mathbb{R}}) &= 1 + p_1(E_{\mathbb{R},i}) = 1 - c_2(E_{\mathbb{R},i} \otimes \mathbb{C}) \\ &= 1 - c_2(E_i \oplus \bar{E}_i) = 1 - c_2(E_i) - c_1(E_i) \cdot c_1(\bar{E}_i) - c_2(\bar{E}_i). \end{aligned} \tag{C.2.44}$$

Now since $E_i, \bar{E}_i$ are of complex dimension 1, then $c_2(E_i) = c_2(\bar{E}_i) = 0$. By eq. (C.2.12) we have $c_i(\bar{E}) = (-1)^i c_i(E)$ and hence

$$p(E_{i,\mathbb{R}}) = (1 + c_1(E_i)^2) = (1 + x_i^2), \tag{C.2.45}$$

where we let $x_i := c_1(E_i)$ as before. As before, by considering the dimensions of the cohomology classes, we therefore see that we have

$$p_i(E) = e_i(x_1^2, \ldots, x_n^2). \tag{C.2.46}$$

C.3 Oriented Bordism

In order to make a precise definition for a *genus*, we need to briefly introduce the notion of oriented bordism. For more details, see for instance [Wal60] or the lecture notes by Alexander Kupers [Kup12].

It is known that it is impossible to algorithmically decide if two manifolds are diffeomorphic. One way to proceed is to introduce invariants, which can tell whether two manifolds are not diffeomorphic. Another way to proceed is to introduce a new equivalence relation, which can then be used to classify manifolds; this relation is oriented bordism. As in the diffeomorphic case, invariants can then be found which can be used to decide whether two manifolds are equivalent up to bordism. For the following, by manifold we mean a smooth compact manifold, possibly with boundary.

Definition C.3.1. Given two oriented d-dimensional manifolds M^d, N^d, we say M and N are *bordant* if $\exists\, W^{d+1}$ such that ∂W is diffeomorphic to $M \sqcup \overline{N}$ as an oriented d-dimensional manifold, where $\overline{N}$ is defined as N with the opposite orientation, and $\sqcup$ denotes the disjoint union. We refer to W as a *cobordism* (or simply *bordism*) between M and N.

Example C.3.2. In fig. C.2, we have a cobordism from the disjoint union of two S^1's to a single S^1. △

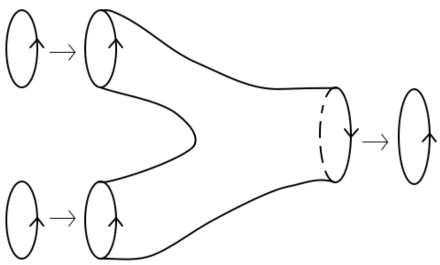


Figure C.2: A cobordism from the union of two copies of S^1 to a single S^1.

Note that since $\partial^2 M = 0$, then our cobordism must be a manifold with boundary by definition, but the manifolds M and N which are bordant must be manifolds without boundary.

Lemma C.3.3. *Bordism is an equivalence relation between manifolds, that is it satisfies the properties of identity, symmetry and reflexitivity.*

Proof. The manifold $M \times I$ gives a cobordism between M and itself for any oriented M, so the identity property is clear. If W provides a cobordism between M and N, then $\overline{W}$ defines a cobordism between N and M, so cobordism satisfies the symmetry property. Finally, if W defines a cobordism between M and N, and W' defines a cobordism between N and O, then the manifold W'', formed by gluing together W and W' on the boundary given by N for each cobordism, provides a cobordism between M and O. □

Since bordism is an equivalence relation between manifolds, it defines equivalence classes. We denote the class of all manifolds equivalent under bordism to M as $[M]$.

The set of bordism classes forms an abelian group under the disjoint union of manifolds. We think of the empty set ϕ as an n-dimensional manifold for any n and the disjoint union of any manifold M with ϕ clearly has the same cobordism class as M. ϕ is therefore the identity in each bordism class. The inverse of a manifold M is then $\overline{M}$, since $M \times I$ with boundary $M \sqcup \overline{M}$ gives a bordism between M and M, and hence also gives a cobordism between $M \sqcup \overline{M}$ and ϕ. We require that the group operation respects the equivalency condition, but it clearly does since if $M_1 \sim M_2$ and $N_1 \sim N_2$ then $M_1 \sqcup N_1 \sim M_2 \sqcup N_2$ by taking the disjoint union of the cobordisms.

Definition C.3.4. Ω_d^{SO} is the set of all d-dimensional oriented manifolds modulo bordism. As we will only discuss oriented bordism here we will henceforth drop the superscript SO.

We can enrich Ω_d with a ring structure by considering the cartesian product of manifolds. Given M^m and N^n with orientation, $M \times N$ is an $(m+n)$-dimensional manifold with a canonical orientation inherited from its factors. As long as this product respects the equivalence relations then we have a graded ring with identity $\Omega_* = \bigoplus_{i \in \mathbb{N}} \Omega_i$. We now check that this is well-defined.

Lemma C.3.5.

$$M \sim M',\ N \sim N' \implies M \times N \sim M' \times N' \tag{C.3.1}$$

Proof. Let W be the cobordism from M to M' and let V be the cobordism from N to N'. Then $W \times N$ is a cobordism from $M \times N$ to $M' \times N$. Similarly $M' \times V$ is a cobordism from $M' \times N$ to $M' \times N'$. Taking the union of these manifolds at their common boundary $M' \times N$ gives a cobordism from $M \times N$ to $M' \times N'$ and hence this product is well-defined on the bordism groups. □

One can consider the tensor product of an abelian group with the rationals, $\mathbb{Q}$, to obtain the so called *free* or *torsion-free* part.

Example C.3.6. Consider $\mathbb{Z}_2 \otimes \mathbb{Q}$. Elements in this group are of the form $a \otimes b$ where $a \in \mathbb{Z}_2, b \in \mathbb{Q}$. Then we have $a \otimes b = a \otimes \frac{2b}{2} = 2a \otimes \frac{b}{2} = 0 \otimes \frac{b}{2} = 0$, by bilinearity. △

Then, due to the fundamental theorem of finitely generated abelian groups, any finitely generated abelian group G is isomorphic to the direct sum of primary cyclic groups and infinite cyclic groups. We can write $G \cong \mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_r}$, where q_i is a prime power and n is the rank of the group.

In [Tho54] Thom studied the structure of $\Omega_* \otimes \mathbb{Q}$. From this we have the following theorem which we will not prove here.

Theorem C.3.7. *The structure of* $\Omega_* \otimes \mathbb{Q}$*:*

- $\Omega_n \otimes \mathbb{Q} = 0$ *for* $4 \nmid n$
- Ω_{4k} *is a finitely generated abelian group of rank equal to the number of partitions of* k*.*
- *The spaces* $\mathbb{C}P^{2n}$*, of dimension* $4n$*, are a* basis sequence *of* $\Omega_* \otimes \mathbb{Q}$*. That is,* $\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$.

We can now define a genus, following [HBJL92; Och09].

Definition C.3.8. Let Λ be a unital $\mathbb{Q}$-algebra. A *genus* is a ring homomorphism

$$\phi : \Omega_* \to \Lambda. \tag{C.3.2}$$

C.4 Genera and Multiplicative Sequences

We have already defined a genus in Appendix C.3 as a homomorphism from the oriented bordism ring to some other ring Λ. In this section we will develop the idea of a genus by considering multiplicative genera and define the notion of an elliptic genus as introduced by [Och09].

From definition C.2.22, we have that the Pontryagin numbers are a dual space for the oriented bordism ring $\Omega_* \otimes \mathbb{Q}$, which in turn (by theorem C.3.7) has the spaces $\mathbb{C}P^{2n}$ as a basis sequence, i.e. $\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$. In definition C.3.8 we defined a genus, ϕ, as a homomorphism from the cobordism ring to a ring Λ, $\phi : \Omega_* \to \Lambda$. Since we have a good understanding of the structure of $\Omega_* \otimes \mathbb{Q}$, we will now consider our genus only on this free part, which is mapped through ϕ to an integral domain R. Our goal will be to be able to express this genus in terms of a power series. We want to find a polynomial expression related to the power series that, when we substitute the Pontryagin numbers of a manifold M into the expression, gives us the value of the genus on M.

Consider an even power series $Q(x)$ with constant term 1 and coefficients $a_i \in R$,

$$Q(x) = 1 + a_2x^2 + a_4x^4 + \ldots \tag{C.4.1}$$

A product $Q(x_1)Q(x_2)\ldots Q(x_n)$ is symmetric in the x_i.

Example C.4.1.

$$\begin{aligned} Q(x_1)Q(x_2)Q(x_3) = 1 &+ a_2(x_1^2 + x_2^2 + x_3^2) + a_2^2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\ &+ a_4(x_1^4 + x_2^4 + x_3^4) + a_2^3(x_1^2x_2^2x_3^2) \\ &+ a_2a_4(x_1^4x_2^2 + x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_2^4 + x_1^2x_3^4 + x_2^2x_3^4) \\ &+ a_6(x_1^6 + x_2^6 + x_3^6) + \ldots \end{aligned} \tag{C.4.2}$$

We now use

$$\begin{aligned} (x_1^4 + x_2^4 + x_3^4) &= (x_1^2 + x_2^2 + x_3^2)^2 - 2(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\ &= e_1^2 - 2e_2, \end{aligned} \tag{C.4.3}$$

where, as before, e_i are the elementary symmetric polynomials though now in terms of the x_i^2, as well as

$$(x_1^4x_2^2 + x_1^4x_3^2 + x_2^4x_3^2 + x_1^2x_2^4 + x_1^2x_3^4 + x_2^2x_3^4) = e_1e_2 - 3e_3, \tag{C.4.4}$$

and

$$(x_1^6 + x_2^6 + x_3^6) = e_1^3 - 3e_1e_2 + 3e_3, \tag{C.4.5}$$

to obtain

$$\begin{aligned} Q(x_1)Q(x_2)Q(x_3) = 1 + a_2e_1 + a_4e_1^2 + (a_2^2 - 2a_4)e_2 + (a_2^3 - 3a_4a_2 + 3a_6)e_3 \\ + (a_4a_2 - 3a_6)e_1e_2 + a_6e_1^3 + \dots \end{aligned} \tag{C.4.6}$$

Hence we have

$$Q(x_1)Q(x_2)Q(x_3) = 1 + K_1(e_1) + K_2(e_1, e_2) + K_3(e_1, e_2, e_3) + \dots \tag{C.4.7}$$

where the terms $K_i(e_1, \dots, e_i)$ are homogeneous polynomials in x_i^2, written in terms of the elementary symmetric polynomials e_i. △

In general [HBJL92], we have

$$\begin{aligned} Q(x_1)\dots Q(x_n) = 1 + K_1(e_1) + K_2(e_1, e_2) + \dots \\ + K_n(e_1, \dots, e_n) + K_{n+1}(e_1, \dots, e_n, 0) + \dots \end{aligned} \tag{C.4.8}$$

where the polynomials K_i for $1 \le i \le n$ do not depend on n. We call the sequence $\{K_i\}$ of polynomials K_i, the *multiplicative sequence* of polynomials associated to the power series $Q(x)$.

Lemma C.4.2. *The polynomials K_i are* multiplicative *in the sense that the identity*

$$\sum_i e_i z^i = \sum_j e'_j z^j \cdot \sum_k e''_k z^k \tag{C.4.9}$$

implies that

$$\sum_i K_i(e_1, \dots, e_i) z^i = \sum_j K_j(e'_1, \dots, e'_j) z^j \cdot \sum_k K_i(e''_1, \dots, e''_k) z^k. \tag{C.4.10}$$

We do not give a proof of this lemma here, but it may be found in [HBS66].

Example C.4.3. Consider the K_i we obtained in the previous example

$$
\begin{aligned}
&K_1 = a_2 e_1, \qquad K_2 = a_4 e_1^2 + (a_2^2 - 2a_4)e_2, \\
&K_3 = (a_2^3 - 3a_4 a_2 + 3a_6)e_3 + (a_4 a_2 - 3a_6)e_1 e_2 + a_6 e_1^3.
\end{aligned}
\tag{C.4.11}
$$

Considering the powers of z, eq. (C.4.9) implies we have $e_1 = e_1' + e_1''$. Then $K_1(e_1) = a_2(e_1' + e_1'') = K_1(e_1') + K_1(e_1'')$. Similarly, eq. (C.4.9) implies we have $e_2 = e_2' + e_2'' + e_1' e_1''$. If we substitute this into K_2 we get

$$
\begin{aligned}
K_2(e_1, e_2) &= a_4 e_1^2 + (a_2^2 - 2a_4)e_2 \\
&= a_4(e_1' + e_1'')^2 + (a_2^2 - 2a_4)(e_2' + e_2'' + e_1' e_1'') \\
&= K_2(e_1', e_2') + K_2(e_1'', e_2'') + a_2^2 e_1' e_1'' \\
&= K_2(e_1', e_2') + K_2(e_1'', e_2'') + K_1(e_1') \cdot K_1(e_1''),
\end{aligned}
\tag{C.4.12}
$$

which satisfies equation (C.4.10). Similarly equation (C.4.9) implies that we have $e_3 = e_3' + e_3'' + e_1' e_2'' + e_2' e_1''$. Substituting this into K_3 shows that K_3 also satisfies eq. (C.4.10). △

Now we can define how to obtain a genus $\phi : \Omega_* \otimes \mathbb{Q} \to R$ from a power series $Q(x)$.

Definition C.4.4. The *genus corresponding to a power series* $Q(x), \phi_Q$, is defined for every compact oriented differentiable manifold M of dimension $4n$ by

$$
\phi_Q(M) := K_n(p_1, \ldots, p_n)[M] \in R, \tag{C.4.13}
$$

where $p_i = p_i(M) \in H^{4i}(M, \mathbb{Z})$, and we set $\phi_Q = 0$ if the dimension of the manifold is not divisible by four.

The genus belonging to a power series Q is therefore a linear combination of Pontryagin numbers.

Lemma C.4.5. *A genus ϕ_Q defined from a power series Q is a well-defined homomorphism, $\phi_Q : \Omega_* \otimes \mathbb{Q} \to R$, for R an integral domain.*

We do not prove this here, but the proof is found in [HBJL92].

We now want to define the logarithm of a genus ϕ_Q.

Definition C.4.6. Given an even power series, $Q(x)$, with constant term 1 and coefficients in R, we define $f(x) := \frac{x}{Q(x)}$. This is an odd power series with first term x and coefficients in R. Now let $y = f(x)$ and put $g = f^{-1}$ as the inverse of $f, g(f(x)) = g(y) = x$. The power series g is known as the *logarithm of the genus* ϕ_Q.

The logarithm gives us important information about the genus by the following lemma.

Lemma C.4.7.

$$g'(y) = \sum_{n=0}^{\infty} \phi_Q(\mathbb{C}P^n) \cdot y^n. \tag{C.4.14}$$

Proof. By lemma C.2.19, the total Pontryagin class of $\mathbb{C}P^n$ is given by,

$$p(\mathbb{C}P^n) = (1 + x^2)^{n+1}, \tag{C.4.15}$$

where we have renamed the generator of $H^*(\mathbb{C}P^n, \mathbb{Z})$ from g to x as we want to interpret the Pontryagin classes of $T\mathbb{C}P^n$ as symmetric polynomials in x as in eq. (C.2.46).

Since the sequence of polynomials associated with the power series $Q(x)$ is multiplicative, we therefore have

$$K(p_1, \ldots, p_n) = K(p_1)^{n+1} = Q(x)^{n+1}. \tag{C.4.16}$$

Therefore

$$\begin{aligned} \phi_Q(\mathbb{C}P^n) &= \left(\frac{x}{f(x)}\right)^{n+1} [\mathbb{C}P^n] = \text{the coefficient of } x^n \text{ in } \left(\frac{x}{f(x)}\right)^{n+1} \\ &= \frac{1}{2\pi i} \int_\kappa \left(\frac{1}{f(x)}\right)^{n+1} dx = \frac{1}{2\pi i} \int_{f(\kappa)} \frac{1}{y^{n+1}} g'(y) dy \\ &= \text{the coefficient of } y^n \text{ in } g'(y) \end{aligned} \tag{C.4.17}$$

by the residue theorem. □

Example C.4.8. Consider the power series

$$Q(x) = \frac{x}{\tanh(x)} = 1 + \frac{x^2}{3} - \frac{1}{45}x^4 + \frac{2}{945}x^6 + \dots \tag{C.4.18}$$

Let $f(x) := \frac{x}{Q(x)} = \tanh(x)$. Then we have

$$f'(x) = 1 - f(x)^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 - y^2, \qquad g'(y) = \frac{1}{1-y^2} = 1 + y^2 + y^4 + \dots \tag{C.4.19}$$

Hence we have a genus that takes that value 1 on all the spaces $\mathbb{C}P^n$. This genus is known as the *L-genus*, ϕ_L.

Using the general multiplicative sequence calculations from example C.4.3, we can write the first three elements of the multiplicative sequence associated to the L-genus in terms of the Pontryagin classes as

$$\begin{aligned} K_1(p_1) &= \frac{1}{3}p_1, \qquad K_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2), \\ K_3(p_1, p_2, p_3) &= \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_1^3) \end{aligned} \tag{C.4.20}$$

△

Example C.4.9. Consider the power series

$$Q(x) = \frac{x/2}{\sinh(x/2)} = 1 - \frac{x^2}{24} + \frac{7}{5760}x^4 - \frac{31}{967680}x^6 + \dots. \tag{C.4.21}$$

The genus associated to this power series is known as the $\hat{A}$-genus. The first three elements of the multiplicative sequence associated to the $\hat{A}$-genus in terms of the Pontryagin classes are

$$\begin{aligned} K_1(p_1) &= -\frac{1}{24}p_1, \qquad K_2(p_1, p_2) = \frac{1}{5760}(-4p_2 + 7p_1^2), \\ K_3(p_1, p_2, p_3) &= \frac{1}{967680}(-16p_3 + 44p_1p_2 - 31p_1^3). \end{aligned} \tag{C.4.22}$$

△

Definition C.4.10. Let M be a connected oriented manifold of dimension $4n$. Then the cup product induces a non-degenerate, symmetric bilinear form

$$H^{2n}(M, \mathbb{R}) \otimes H^{2n}(M, \mathbb{R}) \to H^{4n}(M, \mathbb{R}) \cong \mathbb{R}. \tag{C.4.23}$$

The *signature* $\sigma(M)$ of M is the signature of this bilinear form. We can trivially extend this definition to non-connected manifolds of dimension $4n$ by adding the signatures of the components.

The signature satisfies the following important lemma.

Lemma C.4.11. *If M and N are bordant manifolds of dimension $4n$, $[M] = [N]$, then $\sigma(M) = \sigma(N)$.*

The signature is therefore an invariant of bordism of the type mentioned at the start of Appendix C.3. The signature can therefore be used to check if two manifolds are not bordant; if the signature for the two manifolds does not agree then the two manifolds can not be bordant. Having the same signature is not enough to ensure that two manifolds are bordant however.

It is known that the signature of the spaces $\mathbb{C}P^n$ is equal to 1 $\forall n$, and hence we have:

Theorem C.4.12. ***Hirzebruch Signature Theorem***

For all $4n$-dimensional manifolds M

$$\sigma(M) = \phi_L(M). \tag{C.4.24}$$

Since the spaces $\mathbb{C}P^n$ form a basis sequence for the cobordism ring, it is sufficient to check the result on these spaces. We know that the signature of these spaces is always 1, and by our example C.4.8 the L-genus is always 1 on these spaces too, so we are done.

Since the signature is given by a genus, specifically the L-genus, we now have a proof of lemma C.4.11 since by lemma C.4.5, such a genus is a well defined homomorphism from the (free) bordism classes to the ring R.

Having shown that the signature is a genus, it is natural to wonder whether the Euler characteristic is also a genus.

Lemma C.4.13. *The Euler characteristic is not a genus.*

Proof. This follows quickly by contradiction. Suppose the Euler characteristic is a genus, then by lemma C.4.5 it must be equivalent for different representatives of a given bordism class. All handle-bodies are bordant to the empty manifold, since they are the boundary of the solid handle-body, and so form a single bordism class. However $2 = \chi(S^2) \neq \chi(T^2) = 0$, and we have a contradiction. The Euler characteristic is therefore not a genus. □

We should perhaps be careful here to clarify which bordism ring we are discussing the Euler characteristic as a possible genus from. The context in this section has been that of oriented bordism, though as a sum of Betti numbers the Euler characterstic may be defined for unoriented manifolds such as real projective space in even dimensions. One can define unoriented bordism and hence the unoriented bordism ring Ω_*^O and one might wonder if the Euler characteristic is a genus for the unoriented bordism ring. The same argument as used in the proof of lemma C.4.13 shows that it cannot be a genus for this bordism ring either. However $\chi(\text{mod} 2)$ is a genus for unoriented manifolds.

Another cobordism ring one can form is the complex cobordism ring. We can therefore define complex genera as homomorphisms from the complex cobordism ring to some other ring R. Given a complex manifold M of real dimension $2n$ and a power series $Q(x)$, one can now define the genus corresponding to $Q(x)$ as

$$\phi(M) = K_n(c_1, \ldots, c_n)[M], \tag{C.4.25}$$

where as usual the c_i are the Chern classes of M and K_n is the homogenous polynomial of degree n formed from $Q(x)$ as in eq. (C.4.8). Note that this complex genus is now defined for $Q(x)$ not necessarily even.

Example C.4.14. Consider the power series

$$Q(x) = \frac{x}{1 - e^{-x}}. \tag{C.4.26}$$

The genus associated to this power series is known as the *Todd genus*, $\mathrm{td}(M)$. △

Example C.4.15. Given $y \in \mathbb{C}$, consider the power series

$$Q(x) = \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}}. \tag{C.4.27}$$

This is the characteristic power series associated with a genus known as Hirzebruch χ_y genus. This complex genus is particularly interesting, since at particular values of y it reduces to other characteristic power series that we have already seen.

When $y = 0$

$$Q(x, 0) = \frac{x}{1 - e^{-x}}, \tag{C.4.28}$$

and we have

$$\chi_0(M) \equiv \mathrm{td}(M). \tag{C.4.29}$$

When $y = 1$

$$Q(x, 1) = \frac{x(1 - e^{-2x})}{1 - e^{-2x}} = \frac{x}{\tanh(x)}, \tag{C.4.30}$$

and we have

$$\chi_1(M) \equiv \tau(M). \tag{C.4.31}$$

There is one other value of y for which we want to consider $Q(x)$, namely $y = -1$. At this value of y, both the numerator and denominator go to 0 and the value of $Q(x)$ is undetermined. However we can consider $Q(x)$ in the limit $y \to -1$ and use l'Hôpital's rule,

$$\lim_{y \to -1} Q(x) = \lim_{y \to -1} \frac{\partial_y x(1 + ye^{-x(1+y)})}{\partial_y 1 - e^{-x(1+y)}} = 1 + x. \tag{C.4.32}$$

This power series can be associated with the Euler characteristic $\chi(M)$ [HBJL92].

We also note that

$$Q(x, 0) = \frac{x}{1 - e^{-x}} = e^{x/2} \frac{x/2}{\sinh(x/2)}, \tag{C.4.33}$$

and so

$$\chi_0(M) = \mathrm{td}(M) = e^{c_1/2} \hat{A}(M). \tag{C.4.34}$$

△

We can now give the definition of an elliptic genus.

Definition C.4.16. A genus ϕ is called an *elliptic genus* if its odd power series $f(x) = \frac{x}{Q(x)}$ satisfies one of the three following equivalent conditions:

- $f'^2 = 1 - 2\delta \cdot f^2 + \epsilon \cdot f^4$
- $f(u+v) = \frac{f(u)f'(v)+f'(u)f(v)}{1-\epsilon \cdot f(u)^2 f(v)^2}$
- $f(2u) = \frac{2f(u)f'(u)}{1-\epsilon \cdot f(u)^4}$

Example C.4.17. Consider the L-genus defined by $f(x) = \tanh(x)$, then we have $f'(x) = 1 - f(x)^2$ i.e. $f'(x)^2 = 1 - 2f^2(x) + f^4(x)$ which is therefore elliptic with $\delta = \epsilon = 1$. △

Example C.4.18. Consider the $\hat{A}$-genus defined by $f(x) = 2\sinh(x/2)$, then

$$\begin{aligned} f'(x) &= \cosh(x/2), \\ (f'(x))^2 &= \cosh^2(x/2) = 1 + \sinh^2(x/2), \\ &= 1 + \frac{1}{4}f^2(x), \end{aligned} \tag{C.4.35}$$

hence the $\hat{A}$-genus is elliptic with $\delta = -\frac{1}{8},\ \epsilon = 0$. △

Given a lattice $L = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, the Weierstrauss $\wp$-function,

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L\backslash\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \tag{C.4.36}$$

is an elliptic function which satisfies the differential equation

$$\wp'(z) = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3), \tag{C.4.37}$$

where

$$e_1 := \wp(\frac{\omega_1}{2}), \qquad e_2 := \wp(\frac{\omega_2}{2}), \qquad e_3 := \wp(\frac{\omega_1 + \omega_2}{2}). \tag{C.4.38}$$

These points are known as the 2-division points of L, since they are the unique points p satisfying $-p \equiv p \pmod{L}$ or equivalently $2p \equiv 0 \pmod{L}$.

The function $f(z) = 1/\sqrt{(\wp(z) - e_1)}$ then satisfies [HBJL92]

$$f'(z)^2 = 1 - 2\delta f(z)^2 + \epsilon f(z)^4, \tag{C.4.39}$$

where $\delta = -\frac{3}{2}e_1$ and $\epsilon = (e_1 - e_2)(e_1 - e_3)$. The function $Q(z) = \frac{1}{f(z)} = \sqrt{(\wp(z) - e_1)}$ is therefore an elliptic genus. We therefore have an elliptic genus associated to every lattice L, or equivalently if we have an elliptic genus, we may find δ and ϵ and hence determine an associated lattice L. Note that the elliptic genus discussed explicitly so far, the L-genus with $\delta = \epsilon^2 = 1$ and the $\hat{A}$-genus with $\delta = -\frac{1}{8}, \epsilon = 0$ arise from degenerate lattices.

If we consider an equivalent lattice $\Omega' = \alpha\Omega, \alpha \in \mathbb{C} \setminus \{0\}$, then the 2-division point e_i' for this new lattice is

$$e_i'(\alpha\Omega) = \alpha^{-2} e_i(\Omega), \tag{C.4.40}$$

since clearly

$$\wp(\alpha z; \alpha L) = \alpha^{-2}\wp(z; L). \tag{C.4.41}$$

We therefore find that $\delta = -\frac{3}{2}e_1$ is a lattice invariant of weight 2 and $\epsilon = (e_1 - e_2)(e_1 - e_3)$ is a lattice invariant of weight 4. Due to the differential equation given in definition C.4.16 that $f(z)$ satisfies, $f(z)$ is therefore a homogenous polynomial in δ and ϵ. $Q(z) = \frac{z}{f(z)}$ is therefore also a homogeneous polynomial in δ and ϵ, where the coefficient of z^{2n} in $Q(z)$ is a polynomial of weight $2n$. Since the genus associated to $Q(z)$ is obtained by substituting homogenous polynomials of weight k in Pontryagin classes in place of powers of z^{2k}, and then evaluating against the fundamental class, the genus picks out the z^{2n} term from $Q(z)$. The elliptic genus of a manifold M of dimension $4n$ is therefore a homogeneous polynomial in δ and ϵ of weight $2n$.

Given a homogeneous lattice function of weight k, we can associate to it a modular form.

Lemma C.4.19. *Consider a function*

$$F : \{(L, \omega)|\ L \textit{ a lattice},\ \omega \in L,\ N\omega \equiv 0 \ (\mathrm{mod} L)\} \to \mathbb{C}, \tag{C.4.42}$$

which is homogeneous of degree k,

$$F(\alpha L, \alpha\omega) = \alpha^{-k} F(L, \omega). \tag{C.4.43}$$

Then the function $f(\tau) := F(\tau\mathbb{Z} \oplus \mathbb{Z}, 1/N)$ satisfies the modular transformation

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \tag{C.4.44}$$

for all $\gamma \in \Gamma_1(N)$, where

$$\Gamma_1(N) = \{\gamma \in SL(2, \mathbb{Z}) | \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\mathrm{mod} N)\}. \tag{C.4.45}$$

Proof.

$$\begin{aligned} f(\gamma\tau)(c\tau + d)^{-k} &= F\left(\frac{a\tau + b}{c\tau + d}\mathbb{Z} \oplus \mathbb{Z}, 1/N\right)(c\tau + d)^{-k}, \\ &= F\left((a\tau + b)\mathbb{Z} \oplus (c\tau + d)\mathbb{Z}, \frac{c\tau + d}{N}\right), \\ &= F\left(\tau\mathbb{Z} \oplus \mathbb{Z}, \frac{c}{N}\tau + \frac{d}{N}\right), \\ &= F(\tau\mathbb{Z} \oplus \mathbb{Z}, 1/N), \\ &= f(\tau). \end{aligned} \tag{C.4.46}$$

□

Note that $\Gamma_1(2) \equiv \Gamma_0(2)$, where

$$\Gamma_0(2) := \{\gamma \in SL(2, \mathbb{Z}) | \; c \equiv 0 \; (\mathrm{mod} 2)\}, \tag{C.4.47}$$

and hence $e_1 := \wp(\frac{\omega_1}{2}; L)$ defines a modular form of weight 2 for $\Gamma_0(2)$. Since $\delta = -\frac{3}{2}e_1$, δ similarly defines a modular form of weight 2 for $\Gamma_0(2)$. In the same way, ϵ defines a modular form of weight 4 for $\Gamma_0(2)$. In fact, using the valence formula, one can show that the ring of modular forms for $\Gamma_0(2)$ is generated by δ and ϵ,

$$M_*(\Gamma_0(2)) \cong \mathbb{C}[\delta, \epsilon]. \tag{C.4.48}$$

Since the elliptic genus of a manifold of dimension $4n$ is a homogeneous polynomial of weight $2n$ in δ and ϵ, it is therefore a modular form of weight $2n$ for $\Gamma_0(2)$.

Appendix D

Superalgebras and Supermatrices

D.1 Superalgebras

The algebra A_γ is an example of an affine Lie superalgebra. To make clear the connection between A_γ and $\mathfrak{su}(2|2)$, we shall first introduce the group $SU(M|N)$ which will be the focus of this section. All the material in this section is standard and may be found for example in [Cor89]. We include it here for reference. Although we introduce $\mathfrak{su}(2|2)$ starting from the supergroup $SU(2|2)$, note that one may also use *super Harish-Chandra pairs*, which consist of a real (resp. complex) Lie group and a Lie superalgebra (satisfying certain natural conditions) without requiring the notion of a Lie supergroup [Kos82].

Definition D.1.1. A *superalgebra* A is an associative $\mathbb{Z}_2$-graded algebra.

Such a superalgebra A, admits a decomposition into its even and odd parts,

$$A = A_0 \oplus A_1, \tag{D.1.1}$$

where A_0 is the subspace of A generated by the even elements of the basis and similarly A_1 is the odd subspace.

Definition D.1.2. An element a of a superalgebra A is *homogeneous* if $a \in A_0$ or $a \in A_1$.

Definition D.1.3. The *degree* of a homogeneous element a of a superalgebra A is given by

$$\deg a := \begin{cases} 0 & a \in A_0, \\ 1 & a \in A_1. \end{cases}$$

We also use the notation $\deg a \equiv |a|$.

Definition D.1.4. A superalgebra A is *supercommutative* if

$$y \cdot x = (-1)^{|x| \cdot |y|} x \cdot y,$$

for $x, y \in A$.

The particular Lie superalgebras we are interested in may be defined in terms of supermatrices. These are matrices whose elements are elements of a Grassmann algebra.

Definition D.1.5. A *Grassmann algebra* is a supercommutative associative algebra generated by a finite set of I elements ω_i for $i \in \{1, \ldots, I\}$, such that

$$\omega_i \cdot \omega_j = -\omega_j \cdot \omega_i,$$

for $i, j \in \{1, \ldots, I\}$. Clearly therefore $\omega_i \cdot \omega_i = -\omega_i \cdot \omega_i = 0, \ \forall\ \omega_i$.

Note that by *generated* by the elements ω_i, we mean that the elements of the algebra are polynomials in the elements ω_i and the identity 1, with coefficients in $\mathbb{C}$ (or similarly $\mathbb{R}$). The identity element is defined by

$$1 \cdot \omega_i = \omega_i \cdot 1 = \omega_i, \qquad \forall\ \omega_i. \tag{D.1.2}$$

Note that every element of the Grassmann algebra can be uniquely put into the form $\omega_i \cdot \omega_j \cdot \ldots \cdot \omega_k$, where $1 \leq i < j < k \leq I$. We shall call an element ordered in this way in terms of the generators *in standard ordering.*

Definition D.1.6. A non-zero product of the generators of the Grassmann algebra, in standard ordering, has *level* equal to the number of generators. We use the notation $N(\epsilon_i)$ for the level of the element ϵ_i.

Definition D.1.7. To each set of integers $\mu = \{\mu_1, \ldots, \mu_{N(\mu)}\}$, ordered such that $1 \leq \mu_1 < \mu_2 < \ldots < \mu_{N(\mu)} \leq I$ we can uniquely associate an element of the Grassmann algebra in standard ordering

$$\epsilon_\mu = \omega_{\mu_1} \cdot \omega_{\mu_2} \cdot \omega_{\mu_{N(\mu)}}. \tag{D.1.3}$$

Note that this element has level $N(\epsilon_\mu) = N(\mu)$. In this notation, the identity element $1_{\mathbb{C}B_I}$ is given by ϵ_ϕ.

Using this notation, we see that

$$\epsilon_\mu \cdot \epsilon_\nu = \begin{cases} 0 & \mu \cap \nu \neq \phi, \\ \pm\epsilon_\rho := \pm\epsilon_{\mu\cup\nu} & \mu \cap \nu = \phi, \end{cases} \tag{D.1.4}$$

where the sign of ϵ_ρ depends on the number of exchanges of generators required to put ϵ_ρ into standard ordering.

Definition D.1.8. The *Complex Grassmann Superalgebra*, denoted by $\mathbb{C}B_I$, is the Grassmann algebra defined in definition D.1.5, over $\mathbb{C}$, with grading given by the *degree*

$$\deg \epsilon_i := N(\epsilon_i) \pmod 2. \tag{D.1.5}$$

The algebra $\mathbb{C}B_I$ is of complex dimension 2^I and both the even and odd subspaces are of dimension 2^{I-1}.

Concretely, using the notation of definition D.1.7 elements of $\mathbb{C}B_I$ are of the form

$$E = \sum_\mu E_\mu \epsilon_\mu, \tag{D.1.6}$$

for E_μ in $\mathbb{C}$.

We can similarly define the real Grassmann superalgebra by taking the base field to be $\mathbb{R}$. This is denoted $\mathbb{R}B_I$ and is of real dimension 2^I.

To define the supergroup $SU(M|N)$, we clearly need some notion of an adjoint. We define this first for $\mathbb{C}B_I$ in two steps as follows:

Definition D.1.9. The *complex conjugate* of an element $E = \sum_\mu E_\mu \epsilon_\mu$ of $\mathbb{C}B_I$ is given by

$$E^* = \sum_\mu E^*_\mu \epsilon_\mu, \tag{D.1.7}$$

where E^*_μ is the complex conjugate of E_μ.

Definition D.1.10. The *adjoint* of an element $E = \sum_\mu E_\mu \epsilon_\mu$ of $\mathbb{C}B_I$ is given by

$$E^\# = \sum_\mu E^*_\mu \epsilon^\#_\mu, \tag{D.1.8}$$

where we define

$$\epsilon^\#_\mu = \begin{cases} \epsilon_\mu & \deg \epsilon_\mu = 1, \\ -i\epsilon_\mu & \deg \epsilon_\mu = -1. \end{cases} \tag{D.1.9}$$

Proposition D.1.11. *Two important properties that the Grassmann adjoint satisfies are, given* $E, E' \in \mathbb{C}B_I$*,*

1. $(E + E')^\# = E^\# + E'^\#$.

2. $(E \cdot E')^\# = E'^\# \cdot E^\#$.

The proof may be found in [Cor89] for example.

Let us now define the real Grassmann space $\mathbb{R}B_I^{m,n}$.

Definition D.1.12. The *real Grassmann space* $\mathbb{R}B_I^{m,n}$ is defined to be m copies of the even subspace $\mathbb{R}B_{I,0}$ and n copies of the odd subspace $\mathbb{R}B_{I,1}$. Following [Cor89], an element of the space $\mathbb{R}B_I^{m,n}$ will be written as $(X^1, X^2, \ldots, X^m, \Theta^1, \Theta^2, \ldots, \Theta^n)$, or more concisely as $(\mathbf{X}; \boldsymbol{\Theta})$, where

$$(\mathbf{X}; \boldsymbol{\Theta}) = (X^1, X^2, \ldots, X^m, \Theta^1, \Theta^2, \ldots, \Theta^n), \tag{D.1.10}$$

in the obvious way. Each of the components X^i and Θ^i are real Grassmann elements, with the X^i even and the Θ^i odd, and therefore have expansions of the as described in definition D.1.7.

$$X^i = \sum_\mu X^i_\mu \epsilon_\mu, \qquad \Theta^i = \sum_\mu \Theta^i_\mu \epsilon_\mu. \tag{D.1.11}$$

We similarly define the *complex Grassmann space* $\mathbb{C}B_I^{m,n}$, where now the X^i and Θ^i are complex Grassmann elements.

$\mathbb{R}B_I^{m,n}$ and $\mathbb{C}B_I^{m,n}$ form real and complex vector spaces of dimension $m+n$ respectively, but we would like to be able to multiply elements of $\mathbb{R}B_I^{m,n}$ and $\mathbb{C}B_I^{m,n}$ by elements of $\mathbb{R}B_I$ and $\mathbb{C}B_I$ respectively. Since these Grassmann algebras are not fields, we cannot define $\mathbb{R}B_I^{m,n}$ and $\mathbb{C}B_I^{m,n}$ to be 'Grassmann vector spaces', but instead we can define them to be Grassmann supermodules.

Definition D.1.13. Given a ring R, a *left R-module* V, is an abelian group $(V,+)$ with an R-action $\cdot_V : R \times V \to V$ which, $\forall\, r, s \in R$ and $v, w \in V$ satisfies

- $r \cdot_V (v+w) = r \cdot_V v + r \cdot_V w$,
- $(r+s) \cdot_V v = r \cdot_V v + s \cdot_V v$,
- $(r \cdot_R s) \cdot_V v = r \cdot_V (s \cdot_V v)$,
- $1_R \cdot_V v = v$,

where 1_R is the multiplicative identity of R.

Similarly a *right R-module* is an abelian group $(V,+)$ with an R-action $\cdot_V : V \times R \to V$ satisfying the same requirements, except with the R action now from the right rather than the left.

Definition D.1.14. Given a superalgebra G, a *left G-supermodule* V is a left G-module with a direct sum decomposition

$$V = V_0 \oplus V_1, \tag{D.1.12}$$

such that under the left action of G,

$$G_i \cdot_G V_j \subseteq V_{i+j}, \tag{D.1.13}$$

for $i, j \in \mathbb{Z}_2$.

$\mathbb{C}B_I^{m,n}$ is then a left $\mathbb{C}B_I$-supermodule.

D.2 Supermatrices

The supergroup $SU(M|N)$ is a matrix group like its non-super counterpart. We therefore need to define the set of even supermatrices.

Definition D.2.1. A block matrix of dimensions $(p+q) \times (r+s)$

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{D.2.1}$$

where

- A is a $p \times r$ matrix and D is a $q \times s$ matrix both with entries in $\mathbb{C}B_{I,0}$,
- B is a $p \times s$ matrix and C is a $q \times r$ matrix both with entries in $\mathbb{C}B_{I,1}$,

is an *even supermatrix* of dimension $(p|q) \times (r|s)$.

Similarly. one can define an *odd supermatrix* by letting A and D take values in the odd part of the Grassmann algebra and B and C take values in the even part of the Grassmann algebra.

We can put a grading on the space of supermatrices by introducing a degree

Definition D.2.2. The *degree* of a supermatrix M is given by

$$\deg M = \begin{cases} 0 & M \text{ even}, \\ 1 & M \text{ odd}. \end{cases} \tag{D.2.2}$$

We then define the set $M_{p|q}(\mathbb{C}B_I)$ to be the set of all supermatrices with $p = r,\ q = s$. These are sometimes known as *square supermatrices*, though we should note that whilst A and D (in the sense of definition D.2.1) are square matrices, B and C are not required to be.

We can define a group structure for invertible matrices, using the standard rules for multiplying two matrices. Note that the multiplication of supermatrices respects the $\mathbb{Z}_2$-grading on the space and hence the set of even supermatrices is closed under multiplication.

Definition D.2.3. We call the element $\mathbb{I}_{p+q} = \mathrm{Diag}(1_{\mathbb{C}B_I}, \ldots, 1_{\mathbb{C}B_I})$ the *identity supermatrix.* Note that this is an even supermatrix. For any other supermatrix $M_{p|q}$

$$\mathbb{I}_{p+q} \cdot M_{p|q} = M_{p|q} \cdot \mathbb{I}_{p+q} = M_{p|q} \tag{D.2.3}$$

We shall need to define the scalar multiplication between a supermatrix and a homogeneous element of the Grassmann algebra $B \in \mathbb{C}B_{I,0}$ or $\mathbb{C}B_{I,1}$.

Definition D.2.4. Let $M \in M_{p|q}(\mathbb{C}B_I$ be partitioned as in definition D.2.1, then given a $B \in \mathbb{C}B_{I,0}$ or $\mathbb{C}B_{I,1}$ we define

$$B \cdot M = \left[\begin{array}{c|c} B\mathbb{I}_p & 0 \\ \hline 0 & (-1)^{\deg B} B\mathbb{I}_q \end{array}\right] \cdot \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \tag{D.2.4}$$

and

$$M \cdot B = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c|c} B\mathbb{I}_p & 0 \\ \hline 0 & (-1)^{\deg B} B\mathbb{I}_q \end{array}\right]. \tag{D.2.5}$$

Definition D.2.5. An *invertible supermatrix* is an $M \in M_{p|q}(\mathbb{C}B_I)$ such that there exists an *inverse* matrix $M^{-1} \in M_{p|q}(\mathbb{C}B_I)$ satisfying

$$M \cdot M^{-1} = M^{-1} \cdot M = \mathbb{I}_{p+q}. \tag{D.2.6}$$

It should now be clear that the set of square even invertible supermatrices, which we denote $GL_{p|q}(\mathbb{C}B_I)$ (for supermatrices of dimension $p|q$) forms a group under multiplication. It is not enough to simply check the determinant to see if an even supermatrix is invertible, though a theorem which may be found with proof in [Cor89] states,

Theorem D.2.6.

- *An element $M \in M_{p/0}(\mathbb{C}B_I)$ may be decomposed as*

$$M = \sum_{\mu} M_{\mu} \epsilon_{\mu}, \tag{D.2.7}$$

where M_{μ} is a complex matrix of dimension $p \times p$. M is invertible if the Grassmann identity component M_{ϕ} is invertible as a complex matrix.

- *Let $M \in M_{p|q}(\mathbb{C}B_I)$. M is invertible if its even submatrices A, D, as in definition D.2.1, are invertible as elements of $M_{p|0}(\mathbb{C}B_I)$ and $M_{q|0}(\mathbb{C}B_I)$ respectively.*

To define $SU(M|N)$, and subsequently its Lie Algebra $\mathfrak{su}(M|N)$, we need to define the trace, determinant and adjoint as used for supermatrices.

Definition D.2.7. The *supertrace* for a supermatrix M, in terms of its submatrices as in definition D.2.1, is given by

$$\mathrm{STr}\, M = \mathrm{Tr}\, A - (-1)^{\deg M}\, \mathrm{Tr}\, D, \tag{D.2.8}$$

where Tr is the usual matrix trace. Since we are mainly interested in even supermatrices, we note that for an even supermarix M the supertrace is given by

$$\mathrm{STr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D. \tag{D.2.9}$$

We now see why we defined scalar multiplication as in definition D.2.4, since if the scalar $B \in \mathbb{C}B_I$ is odd and the supermatrix M is even, then BM is odd. The scalar multiplication we have defined ensures we still have the expected properties

$$\mathrm{STr}(BM) = B\, \mathrm{STr}(M), \qquad \mathrm{STr}(MB) = \mathrm{STr}(M)B. \tag{D.2.10}$$

Definition D.2.8. The *superdeterminant* of an $M \in GL_{p|q}(\mathbb{C}B_I)$, in terms of its submatrices as in definition D.2.1, is given by

$$\mathrm{SDet}\, M = \frac{\mathrm{Det}(A - BD^{-1}C)}{\mathrm{Det}\, D}. \tag{D.2.11}$$

Two properties of the superdeterminant, found in [Cor89], which we state here without proof, are

Theorem D.2.9.

1. $\mathrm{SDet}(MN) = (\mathrm{SDet}\, M)(\mathrm{SDet}\, N)$,

2. $\mathrm{SDet}(\exp[M]) = \exp[\mathrm{STr}\, M]$.

The final definition we need is the notion of an adjoint for the supermatrices

Definition D.2.10. The *superadjoint* of a supermatrix $M \in M_{p|q}(\mathbb{C}B_I)$ is

$$M^{\ddagger} = \left[\begin{array}{c|c} (A^{\#})^t & (C^{\#})^t \\ \hline (B^{\#})^t & (D^{\#})^t \end{array}\right], \tag{D.2.12}$$

where $(A^{\#})^t$ denotes the transpose of the matrix $(A^{\#})$, which in turn is the matrix whose components are the Grassmann adjoints D.1.10 of A. That is,

$$[(A^{\#})^t]_{ab} = (A_{ba})^{\#}. \tag{D.2.13}$$

Proposition D.2.11. *Given $M, N \in M_{p|q}(\mathbb{C}B_I)$, the superadjoint satisfies*

$$(M \cdot N)^{\ddagger} = N^{\ddagger} \cdot M^{\ddagger}. \tag{D.2.14}$$

D.3 The Supergroup $SU(M|N)$ and the Associated 'Super' Lie Algebra

We can finally define the supergroup $SU(M|N)$.

Definition D.3.1.

$$SU(M|N) = \{G \in GL_{M|N}(\mathbb{C}B_I) \mid G^{\ddagger}G = \mathbb{I}_{p+q},\ \mathrm{SDet}\, G = 1_{\mathbb{C}B_I} \equiv \epsilon_{\phi}\} \tag{D.3.1}$$

We note that this is indeed a subgroup of $GL_{M|N}(\mathbb{C}B_I)$, since by D.2.9 and D.2.11, given $M, N \in SU(M|N)$, $\mathrm{SDet}(M \cdot N) = \mathrm{SDet}(M) \cdot \mathrm{SDet}(N) = 1_{\mathbb{C}B_I} \equiv \epsilon_{\phi}$ and $(M \cdot N)^{\ddagger} = N^{\ddagger} \cdot M^{\ddagger} = \mathbb{I}_{p+q}$.

Since all Lie supergroups can be seen to also be real Lie groups (essentially since $\mathbb{R}B_I^{m,n}$ can be thought of as a real vector space of dimension $(m+n)2^{I-1}$), then $SU(M|N)$ has an associated real Lie algebra, which we refer to as the real 'super' Lie algebra. This is not a real Lie superalgebra. Since elements of the real Lie group near the identity can be found by exponentiating elements of the real Lie algebra, we can write elements $G(\mathbf{X}, \mathbf{\Theta})$ of $SU(M|N)$ as

$$G(\mathbf{X}, \mathbf{\Theta}) = \exp[g(\mathbf{X}, \mathbf{\Theta})], \tag{D.3.2}$$

for $g(\mathbf{X}, \mathbf{\Theta})$ a generic element of the real 'super' Lie algebra.

Proposition D.3.2. *The defining relations of the 'super' Lie algebra of $SU(N/M)$ are*

$$g^\ddagger + g = 0_{M_{p|q}(\mathbb{C}B_I)}, \qquad \mathrm{STr}\, g = 0_{\mathbb{C}B_I}. \tag{D.3.3}$$

The proof of this is standard and follows using theorem D.2.9 as well as standard properties of exp.

Note that since the even element $M \in SU(M|N)$ is given by $M = \exp[g]$, then g must also be an even supermatrix in order to guarantee the evenness of M after exponentiating.

Appendix E

Coefficient Data for the Functions $F_i(q)$ with $\tilde{k}^+ \in \{3, 4, 5\}$

In this Appendix, we present the coefficient data for the functions $F_i(q)$ of section 6.2 in the cases $\tilde{k}^+ \in \{2, 3, 4, 5\}$. In each table below, one for each value of $\tilde{k}^+$, we label the function F_i alongside a representative of its equivalence class and give the coefficients up to q^{10}. In each case we have factored out an overall power of q which we call the *offset* such that the first term is q^0.

E.1 $\tilde{k}^+ = 2$

Name	Representative		Coefficient of q^i										
F_i	$F^{\Lambda}_{2\tilde{l}^+,n}$	offset	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}
F_1	$F^{((0,0),2,0)}_{0,0} - \frac{1}{2}$	q	1	1	0	1	1	2	1	1	1	3	2
F_2	$F^{((0,0),2,0)}_{0,1}$	$q^{2/5}$	1	0	1	1	1	0	2	1	2	2	2
F_3	$F^{((0,0),2,0)}_{0,2}$	$q^{8/5}$	1	0	1	1	1	1	2	1	2	2	3
F_4	$F^{((1,0),2,0)}_{0,0}$	$q^{1/5}$	1	1	1	1	2	1	2	2	3	3	3
F_5	$F^{((1,0),2,0)}_{0,1} - \frac{1}{2}$	q	1	1	1	1	2	2	2	2	3	3	4
F_6	$F^{((1,0),2,0)}_{0,2}$	$q^{3/5}$	1	0	2	1	1	2	2	2	3	3	4

Table E.1: The coefficients of the 6 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 2$

E.2 $\tilde{k}^+ = 3$

Name	Representative		Coefficient of q^i										
F_i	$F^{\Lambda}_{2\tilde{l}^+,n}$	offset	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}
F_1	$F^{((0,0),3,0)}_{0,0} - \frac{1}{2}$	q	1	1	1	1	1	3	3	4	4	6	6
F_2	$F^{((0,0),3,0)}_{0,1}$	$q^{1/2}$	1	0	1	1	2	1	3	2	5	5	7
F_3	$F^{((0,0),3,0)}_{0,2}$	q^2	1	0	1	2	2	2	4	4	6	7	9
F_4	$F^{((0,0),3,0)}_{0,3}$	$q^{5/2}$	1	0	2	1	2	3	5	4	7	7	11
F_5	$F^{((0,0),3,0)}_{1,0}$	$q^{5/3}$	1	1	1	2	3	3	5	6	7	10	12
F_6	$F^{((0,0),3,0)}_{1,1}$	$q^{2/3}$	1	0	1	2	2	2	4	4	6	8	9
F_7	$F^{((1,0),3,0)}_{0,0}$	$q^{1/6}$	1	1	2	1	3	3	5	5	8	9	13
F_8	$F^{((1,0),3,0)}_{0,1} - \frac{1}{2}$	q	1	1	2	2	3	4	6	7	9	11	15
F_9	$F^{((1,0),3,0)}_{0,2}$	$q^{5/6}$	1	0	2	2	3	3	6	6	9	10	15
F_{10}	$F^{((1,0),3,0)}_{0,3}$	$q^{5/3}$	1	1	2	2	4	5	6	8	11	13	17
F_{11}	$F^{((1,0),3,0)}_{0,4}$	$q^{5/2}$	1	1	2	2	4	4	7	7	11	12	17
F_{12}	$F^{((1,0),3,0)}_{0,5}$	$q^{1/3}$	1	1	1	2	3	3	5	5	8	10	13
F_{13}	$F^{((1,0),3,0)}_{1,0} - \frac{1}{2}$	q	1	2	2	3	4	6	8	10	13	16	20
F_{14}	$F^{((1,0),3,0)}_{1,1}$	$q^{1/3}$	1	1	2	2	4	5	6	8	11	14	18
F_{15}	$F^{((1,0),3,0)}_{1,2}$	$q^{2/3}$	1	1	2	3	4	5	7	9	12	15	19
F_{16}	$F^{((1,1),3,0)}_{0,0}$	$q^{1/2}$	1	1	2	3	4	5	7	9	12	16	19
F_{17}	$F^{((1,1),3,0)}_{0,1} - \frac{1}{2}$	q	1	2	2	3	5	6	8	11	13	17	22
F_{18}	$F^{((1,1),3,0)}_{1,0}$	$q^{1/6}$	1	2	2	4	5	6	10	12	15	20	26

Table E.2: The coefficients of the 18 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 3$

E.3 $\tilde{k}^+ = 4$

Name F_i	Representative $F^{\Lambda}_{2\tilde{l}^+,n}$	offset	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}
			Coefficient of q^i										
F_1	$F^{((0,0),4,0)}_{0,0} - \frac{1}{2}$	q	1	1	1	2	1	3	4	6	7	11	11
F_2	$F^{((0,0),4,0)}_{0,1}$	$q^{4/7}$	1	0	1	1	2	2	4	3	7	8	12
F_3	$F^{((0,0),4,0)}_{0,2}$	$q^{16/7}$	1	0	1	2	3	3	6	6	11	13	18
F_4	$F^{((0,0),4,0)}_{0,3}$	$q^{22/7}$	1	0	2	2	3	4	8	8	13	16	23
F_5	$F^{((0,0),4,0)}_{1,0}$	$q^{12/7}$	1	1	2	2	4	5	8	10	14	19	26
F_6	$F^{((0,0),4,0)}_{1,1}$	$q^{6/7}$	1	0	1	2	3	3	6	7	11	14	20
F_7	$F^{((0,0),4,0)}_{1,2}$	$q^{15/7}$	1	1	1	3	4	6	8	11	15	22	27
F_8	$F^{((0,0),4,0)}_{1,3}$	$q^{18/7}$	1	1	2	3	5	5	10	13	18	23	32
F_9	$F^{((1,0),4,0)}_{0,0}$	$q^{1/7}$	1	1	2	2	3	4	7	8	13	16	23
F_{10}	$F^{((1,0),4,0)}_{0,1} - \frac{1}{2}$	q	1	1	2	3	4	5	9	11	16	21	29
F_{11}	$F^{((1,0),4,0)}_{0,2}$	q	1	0	2	2	4	5	8	10	16	20	28
F_{12}	$F^{((1,0),4,0)}_{0,3}$	$q^{15/7}$	1	1	2	3	6	7	11	15	21	28	39
F_{13}	$F^{((1,0),4,0)}_{0,4}$	$q^{17/7}$	1	1	3	3	6	8	12	16	24	30	42
F_{14}	$F^{((1,0),4,0)}_{0,5}$	$q^{13/7}$	1	1	2	3	5	7	11	13	20	26	36
F_{15}	$F^{((1,0),4,0)}_{0,6}$	$q^{3/7}$	1	1	1	2	4	4	7	9	13	17	25
F_{16}	$F^{((1,0),4,0)}_{1,0} - \frac{1}{2}$	q	1	2	3	4	6	9	13	18	26	34	46
F_{17}	$F^{((1,0),4,0)}_{1,1}$	$q^{3/7}$	1	1	2	3	5	7	11	14	21	28	39
F_{18}	$F^{((1,0),4,0)}_{1,2}$	q	1	1	2	4	6	8	13	17	25	33	45
F_{19}	$F^{((1,0),4,0)}_{1,3}$	$q^{12/7}$	1	2	3	5	7	11	16	22	30	42	55

Table E.3: The coefficients of the 38 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 4$, Part 1

Name	Representative		Coefficient of q^i										
F_i	$F^{\Lambda}_{2\tilde{l}^+,n}$	offset	0	1	2	3	4	5	6	7	8	9	10
F_{20}	$F^{((1,0),4,0)}_{1,4}$	$q^{11/7}$	1	2	3	4	8	10	15	21	30	39	54
F_{21}	$F^{((1,0),4,0)}_{1,5}$	$q^{4/7}$	1	1	2	3	5	8	11	15	21	30	40
F_{22}	$F^{((1,0),4,0)}_{1,6}$	$q^{5/7}$	1	1	3	3	6	8	12	16	23	31	43
F_{23}	$F^{((2,0),4,0)}_{0,0}$	$q^{2/7}$	1	1	2	2	4	5	9	10	16	20	30
F_{24}	$F^{((2,0),4,0)}_{0,1}$	$q^{10/7}$	1	1	3	3	6	7	12	15	23	28	41
F_{25}	$F^{((2,0),4,0)}_{0,2}$	$q^{12/7}$	1	1	3	3	6	8	13	16	24	30	44
F_{26}	$F^{((2,0),4,0)}_{0,3}$	$q^{8/7}$	1	0	2	3	5	5	11	13	20	25	37
F_{27}	$F^{((2,0),4,0)}_{1,0}$	$q^{2/7}$	1	1	3	3	6	8	13	17	25	33	47
F_{28}	$F^{((2,0),4,0)}_{1,1} - \frac{1}{2}$	q	1	2	3	5	7	11	15	23	30	43	56
F_{29}	$F^{((2,0),4,0)}_{1,2}$	$q^{6/7}$	1	1	3	4	7	10	15	20	30	39	55
F_{30}	$F^{((2,0),4,0)}_{1,6}$	$q^{5/7}$	1	2	2	5	7	10	14	21	28	39	52
F_{31}	$F^{((1,1),4,0)}_{0,0}$	$q^{3/7}$	1	1	3	3	6	8	12	16	23	32	43
F_{32}	$F^{((1,1),4,0)}_{0,1} - \frac{1}{2}$	q	1	2	3	4	7	10	14	20	28	37	51
F_{33}	$F^{((1,1),4,0)}_{0,2}$	$q^{5/7}$	1	1	2	4	6	8	13	17	25	34	46
F_{34}	$F^{((1,1),4,0)}_{0,3}$	$q^{11/7}$	1	2	3	5	8	11	17	23	32	44	59
F_{35}	$F^{((1,1),4,0)}_{1,0}$	$q^{1/7}$	1	2	3	5	8	11	17	24	34	47	64
F_{36}	$F^{((1,1),4,0)}_{1,1}$	$q^{2/7}$	1	2	3	5	8	12	18	25	35	49	66
F_{37}	$F^{((1,1),4,0)}_{1,2}$	$q^{4/7}$	1	2	3	6	9	13	19	28	38	53	72
F_{38}	$F^{((1,1),4,0)}_{1,3} - \frac{1}{2}$	q	1	3	4	7	10	15	23	32	44	60	82

Table E.4: The coefficients of the 38 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 4$, Part 2

E.4 $\tilde{k}^+ = 5$

Name	Representative		Coefficient of q^i										
F_i	$F^{\Lambda}_{2\tilde{l}^+,n}$	offset	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}
F_1	$F^{((0,0),5,0)}_{0,0} - \frac{1}{2}$	q	1	1	1	2	2	3	4	7	9	14	16
F_2	$F^{((0,0),5,0)}_{0,1}$	$q^{5/8}$	1	0	1	1	2	2	5	4	8	10	16
F_3	$F^{((0,0),5,0)}_{0,2}$	$q^{5/2}$	1	0	1	2	3	4	7	8	14	18	26
F_4	$F^{((0,0),5,0)}_{0,3}$	$q^{29/8}$	1	0	2	2	4	5	10	11	19	24	36
F_5	$F^{((0,0),5,0)}_{0,4}$	q^4	1	0	2	3	4	5	11	13	20	27	39
F_6	$F^{((0,0),5,0)}_{1,0}$	$q^{7/4}$	1	1	2	3	4	6	10	13	19	27	37
F_7	$F^{((0,0),5,0)}_{1,1}$	q	1	0	1	2	3	4	7	9	14	20	28
F_8	$F^{((0,0),5,0)}_{1,2}$	$q^{5/2}$	1	1	1	3	5	7	11	15	22	32	44
F_9	$F^{((0,0),5,0)}_{1,3}$	$q^{13/4}$	1	1	2	4	6	8	14	19	28	40	54
F_{10}	$F^{((0,0),5,0)}_{2,0}$	$q^{9/4}$	1	1	2	3	5	8	12	17	24	34	48
F_{11}	$F^{((0,0),5,0)}_{2,1}$	$q^{9/8}$	1	0	1	2	4	4	8	10	18	22	34
F_{12}	$F^{((0,0),5,0)}_{2,3}$	$q^{21/8}$	1	1	3	3	7	8	14	18	29	38	56
F_{13}	$F^{((1,0),5,0)}_{0,0}$	$q^{1/8}$	1	1	2	2	4	4	8	10	16	21	32
F_{14}	$F^{((1,0),5,0)}_{0,1} - \frac{1}{2}$	q	1	1	2	3	5	6	10	14	21	28	41
F_{15}	$F^{((1,0),5,0)}_{0,2}$	$q^{9/8}$	1	0	2	2	4	6	10	12	21	27	41
F_{16}	$F^{((1,0),5,0)}_{0,3}$	$q^{5/2}$	1	1	2	3	7	9	14	20	31	42	60
F_{17}	$F^{((1,0),5,0)}_{0,4}$	$q^{25/8}$	1	1	3	4	8	10	18	24	37	50	73
F_{18}	$F^{((1,0),5,0)}_{0,5}$	q^3	1	1	3	4	7	11	17	23	36	49	70
F_{19}	$F^{((1,0),5,0)}_{0,6}$	$q^{17/8}$	1	1	2	3	6	8	14	18	28	37	56
F_{20}	$F^{((1,0),5,0)}_{0,7}$	$q^{1/2}$	1	1	1	2	4	5	8	11	17	23	34
F_{21}	$F^{((1,0),5,0)}_{1,0} - \frac{1}{2}$	q	1	2	3	5	7	11	16	24	35	49	69
F_{22}	$F^{((1,0),5,0)}_{1,1}$	$q^{1/2}$	1	1	2	3	6	8	13	19	28	40	57
F_{23}	$F^{((1,0),5,0)}_{1,2}$	$q^{5/4}$	1	1	2	4	7	10	16	24	35	50	71
F_{24}	$F^{((1,0),5,0)}_{1,3}$	$q^{9/4}$	1	2	3	6	9	15	23	33	48	69	96
F_{25}	$F^{((1,0),5,0)}_{1,4}$	$q^{5/2}$	1	2	4	6	11	16	24	36	53	74	105

Table E.5: The coefficients of the 76 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 5$, Part 1

Name F_i	Representative $F^{\Lambda}_{2\tilde{l}^+,n}$	offset	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}
F_{26}	$F^{((1,0),5,0)}_{1,5}$	q^2	1	2	3	5	9	14	21	31	45	64	90
F_{27}	$F^{((1,0),5,0)}_{1,6}$	$q^{3/4}$	1	1	2	3	6	9	14	20	30	43	61
F_{28}	$F^{((1,0),5,0)}_{1,7}$	$q^{3/4}$	1	1	3	4	6	10	15	21	32	44	63
F_{29}	$F^{((1,0),5,0)}_{2,0}$	$q^{13/8}$	1	2	4	5	10	14	23	31	48	66	96
F_{30}	$F^{((1,0),5,0)}_{2,1}$	$q^{3/4}$	1	1	2	4	6	10	16	23	34	49	70
F_{31}	$F^{((1,0),5,0)}_{2,2}$	$q^{9/8}$	1	1	3	4	8	11	19	26	40	55	81
F_{32}	$F^{((1,0),5,0)}_{2,3}$	$q^{7/4}$	1	2	4	6	10	15	23	34	49	70	98
F_{33}	$F^{((2,0),5,0)}_{0,0}$	$q^{1/4}$	1	1	2	3	4	6	11	14	22	30	44
F_{34}	$F^{((2,0),5,0)}_{0,1}$	$q^{11/8}$	1	1	3	4	7	9	16	20	33	43	64
F_{35}	$F^{((2,0),5,0)}_{0,2}$	$q^{7/4}$	1	1	3	4	7	10	17	23	35	48	69
F_{36}	$F^{((2,0),5,0)}_{0,3}$	$q^{11/8}$	1	0	2	3	6	7	14	18	30	40	60
F_{37}	$F^{((2,0),5,0)}_{0,4}$	$q^{9/4}$	1	1	3	4	8	11	19	26	39	54	79
F_{38}	$F^{((2,0),5,0)}_{0,5}$	$q^{19/8}$	1	1	4	4	9	12	20	27	43	56	84
F_{39}	$F^{((2,0),5,0)}_{0,6}$	$q^{7/4}$	1	1	3	4	7	10	17	23	35	48	69
F_{40}	$F^{((2,0),5,0)}_{0,7}$	$q^{3/8}$	1	1	2	2	5	6	11	14	23	30	46
F_{41}	$F^{((2,0),5,0)}_{1,0}$	$q^{1/4}$	1	1	3	4	7	10	17	24	36	52	74
F_{42}	$F^{((2,0),5,0)}_{1,1} - \frac{1}{2}$	q	1	2	3	6	9	14	21	32	46	67	94
F_{43}	$F^{((2,0),5,0)}_{1,2}$	q	1	1	3	5	8	13	21	30	45	65	92
F_{44}	$F^{((2,0),5,0)}_{1,3}$	$q^{5/4}$	1	1	3	5	9	14	22	32	49	69	99
F_{45}	$F^{((2,0),5,0)}_{1,4}$	$q^{7/4}$	1	2	4	7	11	17	27	40	58	83	117
F_{46}	$F^{((2,0),5,0)}_{1,5}$	$q^{3/2}$	1	2	4	6	11	16	25	37	54	77	110
F_{47}	$F^{((2,0),5,0)}_{1,6}$	$q^{1/2}$	1	1	3	4	7	11	18	26	39	55	79
F_{48}	$F^{((2,0),5,0)}_{1,7}$	$q^{3/4}$	1	2	3	5	9	13	20	30	43	62	88
F_{49}	$F^{((2,0),5,0)}_{2,0} - \frac{1}{2}$	q	1	2	4	6	10	16	24	36	53	76	108
F_{50}	$F^{((2,0),5,0)}_{2,1}$	$q^{3/8}$	1	1	3	4	8	11	20	27	43	59	89

Table E.6: The coefficients of the 76 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 5$, Part 2

Name	Representative		Coefficient of q^i										
F_i	$F^{\Lambda}_{2\tilde{l}^+,n}$	offset	q^0	q^1	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}
F_{51}	$F^{((2,0),5,0)}_{2,2}$	q	1	2	3	6	10	16	24	36	53	76	108
F_{52}	$F^{((2,0),5,0)}_{2,3}$	$q^{7/8}$	1	1	4	5	10	14	24	33	52	71	105
F_{53}	$F^{((1,1),5,0)}_{0,0}$	$q^{3/8}$	1	1	3	4	6	10	15	21	32	45	63
F_{54}	$F^{((1,1),5,0)}_{0,1} - \frac{1}{2}$	q	1	2	3	5	8	12	18	27	39	55	78
F_{55}	$F^{((1,1),5,0)}_{0,2}$	$q^{7/8}$	1	1	2	4	7	10	16	24	35	51	72
F_{56}	$F^{((1,1),5,0)}_{0,3}$	q^2	1	2	3	6	10	15	24	35	50	73	102
F_{57}	$F^{((1,1),5,0)}_{0,4}$	$q^{19/8}$	1	2	4	7	11	17	27	39	57	82	114
F_{58}	$F^{((1,1),5,0)}_{1,0}$	$q^{1/8}$	1	2	3	6	9	14	22	33	48	70	101
F_{59}	$F^{((1,1),5,0)}_{1,1}$	$q^{3/8}$	1	2	3	6	9	15	24	35	52	75	107
F_{60}	$F^{((1,1),5,0)}_{1,2}$	$q^{7/8}$	1	2	3	7	11	17	28	41	60	88	125
F_{61}	$F^{((1,1),5,0)}_{1,3}$	$q^{13/8}$	1	3	5	9	15	23	36	54	78	112	159
F_{62}	$F^{((1,1),5,0)}_{2,0}$	$q^{5/8}$	1	2	4	7	12	19	29	44	65	94	134
F_{63}	$F^{((1,1),5,0)}_{2,1}$	$q^{1/2}$	1	2	4	6	11	18	28	42	62	90	128
F_{64}	$F^{((1,1),5,0)}_{2,3} - \frac{1}{2}$	q	1	3	5	9	14	22	34	51	75	108	152
F_{65}	$F^{((2,1),5,0)}_{0,0}$	$q^{5/8}$	1	1	3	4	8	11	19	26	40	56	82
F_{66}	$F^{((2,1),5,0)}_{0,1}$	$q^{3/2}$	1	2	4	6	11	16	25	36	54	76	108
F_{67}	$F^{((2,1),5,0)}_{0,2}$	$q^{13/8}$	1	2	4	6	11	16	26	37	56	77	112
F_{68}	$F^{((2,1),5,0)}_{0,3}$	q	1	1	2	5	8	12	20	28	44	62	89
F_{69}	$F^{((2,1),5,0)}_{0,7} - \frac{1}{2}$	q	1	2	4	5	9	14	22	31	46	65	94
F_{70}	$F^{((2,1),5,0)}_{1,0}$	$q^{1/2}$	1	2	4	7	12	19	29	44	65	94	134
F_{71}	$F^{((2,1),5,0)}_{1,1} - \frac{1}{2}$	q	1	3	5	9	14	23	35	53	77	112	158
F_{72}	$F^{((2,1),5,0)}_{1,2}$	$q^{3/4}$	1	2	4	7	13	20	31	47	70	101	144
F_{73}	$F^{((2,1),5,0)}_{1,6}$	$q^{1/4}$	1	2	4	6	11	17	27	40	60	87	124
F_{74}	$F^{((2,1),5,0)}_{2,0}$	$q^{1/8}$	1	2	4	6	12	18	29	42	66	94	137
F_{75}	$F^{((2,1),5,0)}_{2,1}$	$q^{1/4}$	1	2	4	7	12	19	30	46	68	99	142
F_{76}	$F^{((2,1),5,0)}_{2,2}$	$q^{5/8}$	1	3	4	9	14	23	35	54	77	115	160

Table E.7: The coefficients of the 76 independent functions F_i appearing in the sum rules when $\tilde{k}^+ = 5$, Part 3

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