Cooperative Models of Stochastic Growth

On a class of reinforced processes with graph-based interactions

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A Thesis presented for the degree of
Doctor of Philosophy

Department of Mathematical Sciences
Durham University
United Kingdom

May 2018

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Abstract: Consider a sequence of positive integer-valued random vectors denoted by $\mathbf{x}_n = (x_1(n), \ldots, x_N(n))$ for $n = 0, 1, 2, \ldots$. Fix $\mathbf{x}_0$, and given $\mathbf{x}_n$, choose a *random* coordinate $i_{n+1} \in \{1, \ldots, N\}$. The probability that $\{i_{n+1} = i\}$ for a particular coordinate i is proportional to a non-decreasing function f_i of $\sum_{j=1}^{N} a_{ij} x_j(n)$, where $a_{ij} \geq 0$ measures how strongly j cooperates with i. Now, on the event that $\{i_{n+1} = i\}$, update the sequence in such a way that $\mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{e}_i$, where $\mathbf{e}_i$ is the vector whose i-th coordinate is 1 and whose other coordinates are 0. Finally, given $A = (a_{ij})_{i,j=1}^{N}$ and f_i, $i = 1, \ldots, N$, what can one say about $\lim_{n\to\infty} n^{-1}\mathbf{x}_n$?

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification.

Acknowledgements

I would first like to warmly thank my supervisors Prof. Mikhail Menshikov and Dr. Andrew Wade for their advice, support and cooperation during my PhD.
Mikhail Menshikov's ingenious intuition and brilliant insights have very often times been crucial to the completion of a result, a theorem, some many lemmas and after all, to the completion of this thesis. Andrew Wade's splendid capacity for researching and teaching has greatly benefited me by providing me with mathematical rigour and sharp knowledge of relevant literature.

Secondly, I would like to thank my thesis defense examiners, Prof. Stanislav Volkov, Dr. Nicholas Georgiou and Dr. Jochen Einbeck for their invaluable feedback which has greatly improved many aspects of this thesis; and my co-authors Dr. Vadim Shcherbakov and Prof. Marina Vachovskaia for the collaboration in Chapter 4. In addition, I would also like to thank Department of Mathematical Sciences academic and research staff and students Dr. Ostap Hryniv, Dr. Sunil Chhita, Dr. Chak Hei Lo, Mr. James McRedmond, Miss Claire Wallace, Dr. Fabian Maucher, Mr. Samuel Jackson, Mr. Benjamin Lopez and Mr. Jingxi Luo for their support and friendship.

Thirdly, I would like to thank my family, Evaldo, Rachel, Felipe, Flávia, Sílvia, Luíza, Miriam and Márcia for their care for me and for being my primary examples of dedication, empathy and humility. In addition, I am very much grateful to my partner Beatriz Silva for her genuine kind-heartedness, and to my friends Pedro Merlussi, Flávio Franchello, Jeovani Brandão, Luíza Araújo, Giulia De Cesaris, Veronika Kočerová, Charlotte Neubert, Jamie Taylor and Kirsty Jones, with whom I had great pleasure to share my experience in Durham.

Last but not least, I am enormously indebted to CNPq, the Brazilian National Council for Scientific and Technological Development, for having granted me the scholarship number [248679/2013-9], without which I could not have come this far.

Yours sincerely,

Marcelo Rocha Costa

"one who seeks for methods without having a definite problem in mind seeks in the most part in vain."
D. Hilbert

This thesis is dedicated
to

Evaldo and Rachel

and also to

Felipe Rocha Costa

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Chapter 1

Introduction

The problem we want to address in this thesis is "Given an evolving population comprised of many types of individuals, how to simultaneously keep track of the size and composition of the population as time goes to infinity?". This problem has long been considered in the scientific community across several academic disciplines and there exists a huge literature on the mathematical theory of the evolution of populations we shall briefly discuss.

The mathematical formulation of the growth of a population can be traced as far back as 1760 when Leonhard Euler in his "General Researches on the Mortality and the Multiplication of the Human Race"[27] implicitly assumed that the population increases geometrically over time, leading to the equation known today as the characteristic equation of demography (or Euler-Lotka equation). The economic implications of exponential growth of the human population were the subject of the famous Thomas Malthus' writings - "Essay on the Principle of Population", 1798. A few decades after the Malthusian controversy on the exponential growth of populations, P. Verhulst introduced in 1838 the so-called logistic equation in an attempt to take into account that since the resources are limited, the death rate increases as the resources are exhausted. It was not until A. Lotka 1925 [58] (see [57]) and V. Volterra 1926 [84] that the competition between species was satisfactorily incorporated into a mathematical model.

Deterministic models are typically simpler to solve analytically or numerically whereas random models may be considerably more complicated, but it is generally acknowledged that they may also be more realistic. In this respect, deciding whether to opt for a deterministic or a stochastic model may be a delicate issue. On the one hand, one may wonder whether it is useful to approach a complex stochastic model when a deterministic description is expected to be accurate enough. On the other hand, one may consider the risk of failing to take into account the role of chance fluctuations - especially in the early stage - of the development of the process.

In this thesis we shall be concerned solely with *stochastic* growth models. Although the thesis is not focused on any particular question of demography, we sometimes apply the terminology "population", "individual", etc, for the sake of presentation purposes since the theory of stochastic processes has undergone major developments in many directions, partly in response to the challenges originally posed by modelling biological populations. Nevertheless, the reader is suggested to keep in mind that the terms "population", "individual", etc, are not necessarily restricted to any biological context and might refer to objects in the context of various fields such as computer science [1, 9, 11], economics [3, 24, 45] and game theory [18, 75, 81].

With a few exceptions, the classical stochastic population models are treated as processes in which it is assumed that different individuals reproduce independently of one another. This is a severe limitation for most applications. In contrast, in this thesis we consider some stochastic growth models in which the number of individuals of certain types may affect the growth of individuals of some other types according to a graph-based interaction.

Before starting a detailed description of the subject matter of this thesis, let us briefly mention some of the most important references in the literature and investigate how our project relates to previous research on this field.

1.1 Branching processes

One of the earliest known work on stochastic population models [1] was that of F. Galton and H. W. Watson, 1874 [38], "On the probability of the extinction of families" formulated priorly by F. Galton in [37] as follows:

"A large nation, of whom we will only concern ourselves with the adult males, who each bear separate surnames, colonise a district". Let $p_0, p_1, p_2, \ldots$ *be the respective probabilities that an adult male has* $0, 1, 2, \ldots$ *sons.*

(1) "what proportion of the surnames would become extinct after a given number of generations?"

(2) "How many instances there will be of the same surname being held by a given number of persons?"

The mathematical model of Galton and Watson (known as the branching process) was later studied by R. A. Fisher in his pioneering work [31, 32, 33] on the random variations in the frequency of genes and the survival of the progeny of a mutant gene. Fisher's investigations on the Mendelian scheme of genetic inheritance combined with the diffusion equation of S. Wright's theory [87, 88, 89] later became one the most fundamental models in population genetics known as the Fisher-Wright model.

It seems that the first authors to have used Markov processes as mathematical models for the growth of biological populations were A. McKendrick, 1914, 1927 [59, 60] on the spread of epidemics (for more on the theory of epidemics, see [10, 46]), U. Yule 1924 [90], on the evolution of new species, and W. Furry 1937 [36], on showers of electrons.

Perhaps the most well-studied branching process is the so-called birth-and-death process, where any individual, upon death, may be replaced by either 2 or no children (so that the change in the population size is ± 1). Although such a class of processes may be regarded as a particular case of a branching process, it has become a topic

[1] In 1845, I. J. Bienaymé wrote a note "On the law of multiplication and the duration of families", published in the bulletin of the Société Philomatique in Paris considering a simplified version of the problem of the extinction of families as described above.

in its own right and it has been reformulated in its own language. Since then, there has been major developments in this field and many models have been formulated so as to describe population processes with several species, competition, migration, catastrophes, etc (see [2], Chapters 8, 9 and references therein).

In Chapter 3 we get back to the theory of multi-type Markov branching processes in continuous time. These processes are the subject of the influential books by T. Harris [39] and K. Athreya and P. Ney [6] and have been taken pretty much to completion by S. Janson in his recent work [41, 42].

1.2 Pólya urn models

Another useful probabilistic model in the context of stochastic growth is the so-called Pólya urn model. Historically, the Pólya urn model first appeared in Eggenberger and Pólya, 1923 and Pólya, 1931, and was intended to model epidemics, contagion and other such spreading phenomena involving a branching mechanism. The system consists of a set of particles of N different types, and the dynamics can be described by iterating the two following steps:

- (**D1**) Drawing Mechanism: One particle is drawn uniformly at random from the system, so that the probability of drawing a particular type of particle is equal to its proportion of the whole population.
- (**R1**) Replacement Mechanism: Once a particle is drawn, its type is observed and the drawn particle is returned to the system along with a new particle of that same type.

As far as this problem is concerned, the main question is to determine how the proportion of each type of particle evolves over time. The simplest case when $N = 2$ has a remarkable property as it converges to a random limit, specifically, to a Beta distribution with parameters given by the system's initial state [30]. For arbitrary N,

it has long been known [19] that the particle type count is a Dirichlet-multinomial distribution with parameters given by the initial state and the current moment of time (For a comprehensive book on urn models, see [43]).

There are several quite natural ways to generalise this model, so before stating our results and some open problems let us briefly discuss some of the existing extensions and achievements in this field.

1.2.1 Extensions

On the one hand, one might want to start generalising this model by extending the drawing mechanism (D1). In order to make it more precise, let us denote by $X(n) = (X_1(n), ..., X_N(n)) \in \mathbb{Z}_+^N$ the numbers of particles of each type $i \in \{1, ..., N\}$ at time n. Then, instead of drawing a particle uniformly at random, let us consider the following extension.

- (**D2**) Extended Drawing Mechanism: At time $n+1$ a particle of type $i_{n+1} \in \{1, ..., N\}$ is drawn, where the random type i_{n+1} has distribution given by

$$\mathbb{P}[i_{n+1} = i \mid X(n) = (x_1, ..., x_N)] = \frac{f_i(x_i)}{\sum_{j=1}^N f_j(x_j)}, \quad i = 1 \dots, N,$$

where the $f_i : \mathbb{Z}_+ \to \mathbb{R}_+$ are non-decreasing functions called *reinforcement rule* of the dynamics [20]. When $f_i(x) = x$ for all i, we recover the previous definition (D1).

On the other hand, one might alternatively want to generalise the process by relaxing the replacement mechanism. In fact, instead of returning the drawn particle to the system along with one additional particle of its same type, let us consider that

- (**R2**) Extended Replacement Mechanism: Once a particle is drawn, it is replaced in the system by a (possibly random) set of particles of various types sampled according to a probability distribution depending on the drawn particle's type.

Extended Replacement Mechanism

In this context, the results rely on a method originally devised by K. Athreya and S. Karlin [5, 4] to analyse limit laws of such a process by means of embedding it into a continuous-time multi-type Markov branching process. In a branching process, particles branch according to offspring distributions (see Chapter 3) so that particles live, die and split independent of each other, and of the past.

If the particle lifetimes are assumed to be exponentially distributed, the resulting continuous-time process turns out to be a Markovian one. Moreover, by the memoryless property of exponential distributions and the characterization of the minimum value attained by a set of such distributions, it follows (see Theorem 3.2.12) that one can find a branching process equivalent to the Pólya urn model under (D1) and (R2).

As a consequence, one can obtain results on some generalised Pólya urn models by applying the well-known result (Theorem 3.2.10) for *irreducible* multi-type branching processes. The irreducibility assumption is a rather strong constraint and it intuitively means that for every pair of types, say i and j, given that the system starts with a single particle of type i, it is possible to find at a later time a particle of type j. This assumption greatly simplifies the analysis as it enables one to apply the Perron-Frobenius theory for irreducible non-negative matrices (see Section 3.4).

A comprehensive account on this topic was given by S. Janson, 2004 and 2005 [41, 42] where it is shown to what extent the irreducibility assumption may be weakened. As remarked by R. Pemantle in [68], "As far as I can tell, Janson's results do subsume pretty much everything previously known" on this topic. We shall get back to Janson's results in Chapter 3.

However, despite the usefulness of the aforementioned method for generalising the replacement mechanism of a Pólya urn model (subject to the irreducibility assumption, or a weakened version of it), as far as I know, it does not apply if one assumes the generalised drawing mechanism D2. In fact, by the branching process embedding,

one can only generalise the drawing mechanism by allowing the reinforcement rule to be a linear function, that is, $f_i(x) = c_i x$ for non-negative constants $c_i, i = 1, \ldots, N$. Two applications of urn models with linear reinforcement rule can be found in D. Aldous *et al*, 1988 [1].

Extended Drawing Mechanism

Many papers have been published assuming (D2) and (R1) equipped with a site-independent polynomial reinforcement rule $f(x) = x^p$ for some parameter $p > 0$ (see [64]). This class of process was apparently proposed independently by [24, 51] as a simpler variant of the so-called preferential attachment model on large networks as well as a model for competing products in an economy [23]. In the sub-linear case $p < 1$ the asymptotic proportions of particles of each type are all the same, namely, $1/N$. In the case $p = 1$ we recover the Pólya urn where the proportion of types converges to a non trivial random variable.

In the super-linear case $p > 1$ there is one type of particle that almost surely dominates the system, that is, as time goes to infinity all except finitely many particles will be of that particular type [49]. In the context of economics, the super-linear case is called a process with positive feedback (loosely speaking, the tendency that the rich get richer) and the result translated to the economic jargon reads as 'the winner takes it all'.

The tool applied to the super-linear case to prove concentration has been first published in [22] and is referred to as Rubin's embedding. Such an embedding can be proven for more general reinforcement rules (see [20, 65]), but it breaks down if one tries to relax the replacement mechanism (R1).

1.3 General framework

In this section we present our general framework that encompasses, in an unified manner, most of the features above. We also introduce an additional feature - we call interaction-based reinforcement rule - according to which different types of particles may now interact. This means that the number of particles of certain types may directly affect the growth of particles of other types. This has been inspired by V. Shcherbakov and S. Volkov's series of papers [78, 79, 80]

However, the reader should be warned beforehand that we do not treat the model in its full generality and we also do not expect that there exists any unified method capable of providing a complete description of the general model. In the next section we specialize the model to three particular (and apparently novel) cases and briefly describe the results referring the reader where the results are rigorously stated and proved.

1.3.1 Interaction-based reinforcement rules

Let us now introduce our stochastic growth model with graph-based interactions. Let $G_N = (V, E)$ be a directed graph containing N vertices. We write $v_i \to v_j$ if the oriented pair (v_i, v_j) belongs to E. For each oriented pair of vertices (v_i, v_j) we assign a non-negative real number a_{ij} such that $a_{ij} > 0$ if $v_i \to v_j$ and $a_{ij} = 0$ otherwise. Matrix $A := (a_{ij})_{i,j=1}^N$ is called the weighted adjacency matrix of G_N. Let us start by fixing an enumeration $\{v_i\}_{i=1}^N$ of vertices in V and a weighted adjacency matrix A for the underlying graph G_N.

Let us now consider a discrete time Markov chain $X(n) = (X_i(n))_{i=1}^N$ taking values in $\mathbb{Z}_+^N$ accounting for the number of particles in each vertex of G_N.

Given that at time n the current configuration is $X(n) = \mathbf{x} = (x_1, \ldots, x_N)$, a new particle is randomly placed at some vertex v_i, $i = 1, ..., N$ according to the following

transition probability

$$\bullet \ (\mathbf{D3}) \quad \mathbb{P}[X(n+1) = \mathbf{x} + \mathbf{e}_i \mid X(n) = \mathbf{x}] = \frac{\Gamma_i(\mathbf{x})}{\sum_{j=1}^{N} \Gamma_i(\mathbf{x})}, \tag{1.3.1}$$

where the $\Gamma_i : \mathbb{R}_+^N \to \mathbb{R}_+$ are mappings we call *interaction-based reinforcement rule* of the model and $\mathbf{e}_i$ is the vector whose i-th coordinate is 1 and other whose coordinates are 0. Note that the growth of a particular type of particle might take into account the whole configuration $\mathbf{x} = (x_1, \ldots, x_N)$.

More specifically, we consider that the probability of drawing a particular particle, say of type i, depends not on the whole configuration, but only on the local environment viewed from vertex v_i. Then, the interaction is defined through an weighted adjacency matrix A for a graph G_N, and a *potential* $U(\mathbf{x}) = (U_i(\mathbf{x}))_{i=1}^N$ given by

$$U_i(\mathbf{x}) := \sum_{j=1}^{N} a_{ij} x_j, \quad i = 1, \ldots, N. \tag{1.3.2}$$

Note that for specific graphs G_N and $\Gamma_i(\mathbf{x}) = U_i(\mathbf{x})$ we can recast some classical models. For example, if G_N is made only of self loops with same weight, i.e., $a_{ii} \equiv a > 0$ for all i and $a_{ij} = 0$ for all $i \neq j$, the resulting model is a Pólya urn. Also, if G_N is the complete graph with $a_{ij} = a > 0$ for all i, j, the resulting model is a uniform random deposition model. One can think of these two toy examples being on the extremes - non-cooperative and fully cooperative - of a certain class of growth processes under competition (in the sense that the more particles of a certain type, the bigger its growth rate). On the one hand, there is no cooperation in the Pólya urn model since $a_{ij} = 0$ for $i \neq j$, and on the other hand, there is full cooperation in the random deposition model since $a_{ij} = a > 0$ for all i, j.

1.4 Results

The three results in this thesis are Theorem 2.1.1 (Section 2.1); Theorem 3.1.1 (Section 3.1) and Theorem 4.2.3 (Section 4.2). In each case we compromise generality of the graph G_N and of the interaction-based reinforcement rule Γ_i, $i = 1, \ldots, N$.

Chapter 2

Following the general framework described above in Section 1.3.1; in Chapter 2, Section 2.1 we consider a system of particles of 2 different types which interact with one another according to the following weighted adjacency matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad a, b > 0.$$

Then, given a configuration of particles $\mathbf{x} = (x_1, x_1)$, we define the following interaction-based reinforcement rule

$$\Gamma_1(\mathbf{x}) := a x_1^\beta + b x_2^\beta,$$
$$\Gamma_2(\mathbf{x}) := b x_1^\beta + a x_2^\beta,$$

for some $\beta > 0$, where the constants a and b may be interpreted as the self-reinforcement and cross-reinforcement coefficients, respectively.

Let $\tilde{X}(n)$ be the process denoting the proportion of particles of each type at time n. In Theorem 2.1.1, we prove the existence of a phase transition according to which,

$$(i) \quad \text{if} \quad \left(\frac{a-b}{a+b}\right)\beta \leq 1, \quad \text{then} \quad \tilde{X}(n) \to (\tfrac{1}{2}, \tfrac{1}{2}) \quad a.s.$$
$$(ii) \quad \text{if} \quad \left(\frac{a-b}{a+b}\right)\beta > 1, \quad \text{then} \quad \tilde{X}(n) \to \Psi \quad a.s.,$$

where Ψ is a random vector supported on $\left\{\left(\frac{1}{1+c}, \frac{c}{1+c}\right), \left(\frac{c}{1+c}, \frac{1}{1+c}\right)\right\}$, and $c := c(\beta)$ is the unique root in $(0,1)$ of $\mathcal{P}(z) = z^{\beta+1} - (a/b)z^\beta + (a/b)z - 1 = 0$. In particular, in case (ii), $\mathbb{P}[\tilde{X}(n) \to (\frac{1}{2}, \frac{1}{2})] = 0$.

In addition, assuming $\beta = 1$ and denoting

$$U(n) = A \begin{pmatrix} X_1(n) \\ X_2(n) \end{pmatrix},$$

the process $(U(n))_{n\geq 0}$ is a Friedman's urn model. Also, $(\frac{1}{2}, \frac{1}{2})$ is the Perron-Frobenius eigenvector of the matrix A. It is well known that in this case, $\frac{U(n)}{U_1(n)+U_2(n)} \to (\frac{1}{2}, \frac{1}{2})$ $a.s.$, which can be obtained by part (i) of Theorem 2.1.1.

Chapter 3

Following the general framework described above in Section 1.3.1; in Chapter 3, Section 3.1 we consider a system of particles of N different types, a general strongly connected directed graph G_N, and its corresponding irreducible weighted adjacency matrix A with non-negative rational entries. Then given a configuration of particles $\mathbf{x} = (x_1, \ldots, x_N)$, we define the following interaction-based reinforcement rule

$$\Gamma_i(\mathbf{x}) = U_i(\mathbf{x}) = \sum_{j=1}^{N} a_{ij} x_j, \quad i = 1, \ldots, N.$$

In Theorem 3.1.1 we prove that the following statement holds.

$$X(n)/n \longrightarrow \mathbf{v} \quad a.s., \tag{1.4.1}$$

where $\mathbf{v}$ is a deterministic vector which can be explicitly found. In particular, $\mathbf{v}$ is a vector with strictly positive entries. That is to say, the proportion of particles of each type converges to a deterministic and strictly positive limiting constant so that the number of particles of each type genuinely grows at linear speed as time goes to infinity. It turns out that $\mathbf{v}$ is the normalised Perron-Frobenius eigenvector of matrix A, and in Theorem 3.1.1 we show how the Embedding Theorem 3.2.12 can be applied to an urn model assuming the extended drawing mechanism D3 (with $\Gamma(\mathbf{x}) = U(\mathbf{x})$) at the expense of restricting the replacement mechanism back into its simple form R1. (Recall that the Embedding is typically applied to urn models under D1 and R2)

Chapter 4

Following the general framework described above in Section 1.3.1; in Chapter 4 we fix a cycle graph G_N containing N vertices and given a configuration of particles $\mathbf{x} = (x_1, \ldots, x_N)$, we define $U_i(\mathbf{x}) = x_i + \sum_{j \sim i} x_j$ and a *site-dependent* exponential reinforcement rule

$$\Gamma_i(\mathbf{x}) = e^{\lambda_i U_i(\mathbf{x})}, \quad \lambda_i > 0, \quad i = 1, \ldots, N.$$

Our main result, Theorem 4.2.3, gives explicit criteria on the parameter space $\Lambda = \{\lambda_1, \ldots, \lambda_N\} \in \mathbb{R}_+^N$ which classifies the asymptotic behaviour of the process into two possible regimes. First, with positive probability, a single random vertex gets all but finitely many particles. Second, with positive probability, a random pair of adjacent vertices gets all but finitely many particles. Finally, one of the above events happens with probability 1.

In particular, if $\lambda_i \neq \lambda_{i+1}$ for all i, with probability one, the growth will eventually localise at a single site. Alternatively, if $\lambda_i \equiv \lambda$, with probability one, the growth will eventually localise at a pair of neighbouring sites.

1.5 Open problem

Fix a strongly connected directed graph G_N with N vertices and an associated irreducible weighted adjacency matrix A. Given a configuration of particles $\mathbf{x} = (x_1, \ldots, x_N)$, define the non-linear reinforcement rule as

$$\Gamma_i(\mathbf{x}) := \sum_{j=1}^{N} a_{ij} x_j^{\beta}, \quad \beta > 0, \quad i = 1, \ldots, N. \tag{1.5.1}$$

Open problem: Can one find the set $B = B(A)$ such that for any $\beta \in B$ there exists a deterministic vector $\mathbf{v} = \mathbf{v}_\beta(A)$ with strictly positive entries such that

$$n^{-1} X(n) \longrightarrow \mathbf{v} \quad a.s.?$$

Theorem 3.1 shows that $1 \in B$ and $\mathbf{v}_1(A)$ is the Perron-Frobenius eigenvector of A. Theorem 2.1 implies that for $N = 2$ and

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

if $a \leq b$ then $B = (0, \infty)$, otherwise $B = (0, \left(\frac{a+b}{a-b}\right)]$. In either case $\mathbf{v}_\beta(A) = (\frac{1}{2}, \frac{1}{2})$. Finally, for arbitrary N, can one find conditions on A so that $B = (0, \beta_c(A)]$ for some critical $1 \leq \beta_c(A) < \infty$?

1.6 Thesis outline

Chapter 2

Section 2.1 is the original part of the chapter containing Theorem 2.1.1, whose proof relies on the so-called stochastic approximation method.

Section 2.2 formally introduces the method and is divided into subsections (2.2.1) - (2.2.4). The first three subsections develop the concepts leading to Theorem 2.2.12, which gives sufficient conditions for applying the approximation and what is its consequences. Subsection 2.2.4 shows how to represent the sequence of increments of an urn model in order to fit it into the stochastic approximation algorithm's general form.

Section 2.3 presents a stability classification of equilibrium points that implies the very last assertion of our result.

Finally, in Section 2.4 we present the proof of Theorem 2.1

Chapter 3

Section 3.1 is the original part of the chapter containing Theorem 3.1.1, which shows how the Markov branching processes and Perron-Frobenius theory can be applied to our model.

Section 3.2 starts with an intuitive description of the multi-type Markov branching process in continuous time. The scope of the results includes only the main intermediate steps needed to prove Theorem 3.2.13 (see figure 1.1 below).

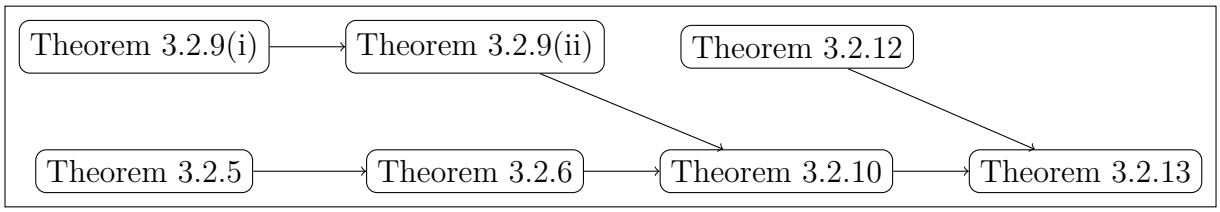

Figure 1.1: Dependence of theorems in Chapter 3

Subsection 3.2.1 presents the classical construction of the process in continuous time followed by Theorem 3.2.5 concerning the expected size of the population at time t; the Limit Theorem 3.2.6 of $Z(t)$; the Limit Theorem 3.2.9 for the splitting times, and finally the Limit Theorem 3.2.10 for $Z(t)$ embedded at the successive random splitting times τ_n.

Section 3.2.2 shows how to relate a branching process to an urn model through the so-called Embedding (Theorem 3.2.12) and it finishes with Theorem 3.2.13, which enables one to obtain results on the asymptotic behaviour of some generalised Pólya urn models.

Finally, in Section 3.3 we present the proofs of Lemma 3.1.3 and Theorem 3.1.1

The last Section 3.4 is included as a sort of an appendix on the Perron-Frobenius theory which is what all the results ultimately boil down to.

Chapter 4

Chapter 4 is the main part of this thesis and it is the result of a joint work with my supervisor Prof. Mikhail Menshikov, Prof. Vadim Shcherbakov [2] and Prof. Marina Vachkovskaia [3], which can be found in `https://arxiv.org/abs/1711.07768`.

The content in Chapter 4 is self-contained and may be read independently of other chapters. The reader is referred to the chapter's Introduction 4.1 where an outline is provided.

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Chapter 2

A phase transition for a non-linear Friedman's urn model

2.1 The model and main result

Let us consider an urn with balls of two colours ($N = 2$) initially containing a total of n_0 balls. At each given moment of time a new ball is added to the urn. The vector $X(n) = (X_1(n), X_2(n)) \in \mathbb{Z}_+^2$ denotes the number of balls of each colour at time $n \geq 0$. The graph-based interaction is given by a potential $U(n) = (U_1(n), U_2(n))$ defined as follows

$$U_1(n) := a_{11}X_1^\beta(n) + a_{12}X_2^\beta(n), \tag{2.1.1}$$

$$U_2(n) := a_{21}X_1^\beta(n) + a_{22}X_2^\beta(n), \tag{2.1.2}$$

for some $\beta > 0$ and $a_{ij} \geq 0,\ i, j = 1, 2$. Or simply,

$$U = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} X_1^\beta \\ X_2^\beta \end{pmatrix}. \tag{2.1.3}$$

The non-negative matrix $A = (a_{ij})_{i,j=1}^2$ is the weighted adjacency matrix of an underlying graph G such that $a_{ij} > 0$ iff there is an oriented edge $i \to j$.

Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard basis of the Euclidean plane and given $X(n) = x(n)$

let us write $u_i(n)$ for $U_i(n)$. Now, we define the transition probabilities by

$$\mathbb{P}(X(n+1) = x(n) + \mathbf{e}_i \mid X(n) = x(n)) = \frac{u_i(n)}{u_1(n) + u_2(n)}, \quad i = 1, 2, \tag{2.1.4}$$

and denote the process of proportions by

$$\tilde{X}(n) = X(n)/(n_0 + n)$$

The next proposition provides checkable conditions that classify the possible limiting behaviour of $\tilde{X}_n$ as $n \to \infty$ according to the model parameters.

Theorem 2.1.1. *Let $(X(n))_{n\geq 0}$ be a two-colour urn model with interaction given by its potential $(U(n))_{n\geq 0}$ defined in* (2.1.3) *with the choice of matrix*

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad a, b > 0. \tag{2.1.5}$$

(i) *If $\left(\frac{a-b}{a+b}\right)\beta \leq 1$, then $\tilde{X}(n) \to (\frac{1}{2}, \frac{1}{2})$ a.s,*

(ii) *If $\left(\frac{a-b}{a+b}\right)\beta > 1$, then $\tilde{X}(n) \to \Psi$ a.s,*

where the random vector Ψ is supported on $\left\{\left(\frac{1}{1+c}, \frac{c}{1+c}\right), \left(\frac{c}{1+c}, \frac{1}{1+c}\right)\right\}$, and $c := c(\beta)$ is the unique root in $(0, 1)$ of

$$\mathcal{P}(z) = z^{\beta+1} - (a/b)z^{\beta} + (a/b)z - 1 = 0.$$

In particular, $\mathbb{P}[\tilde{X}(n) \to (\frac{1}{2}, \frac{1}{2})] = 0$.

Remark 2.1.2. Note that for $\beta = 1$, the process $(U(n))_{n\geq 0}$ is a Friedman's urn model. Also, $(\frac{1}{2}, \frac{1}{2})$ is the Perron-Frobenius eigenvector of the matrix (2.1.5). It is well known that for this case, $\frac{U(n)}{U_1(n)+U_2(n)} \to (\frac{1}{2}, \frac{1}{2})$ *a.s.*, which can be obtained by part (i) of Proposition 2.1.1.

The proof relies on the so-called stochastic approximation method, more specifically on Theorem 2.2.12. In the following sections, a self-contained account on the stochastic approximation method is provided. The reader who is familiar with such a method might wish to skip directly to the proof in Section 2.4.

2.2 Stochastic approximation method

The stochastic approximation method was introduced in the early 50s by Robbins and Monro (1951) [73] and Kiefer and Wolfowits (1952) [50] and has been extensively studied and applied to many fields such as urn models [16], reinforced random walks [68], neural networks [85] and game theory [35]. Generally speaking, this method concerns almost sure convergence of stochastic approximation process and how it can be related to the behaviour of an ordinary differential equation (ODE) under suitable averaging. The method applies to the context of a sequence of random vectors $\tilde{x}_n \in \mathbb{R}^N$ to be recursively updated through a sequence of random inputs ξ_n and a control sequence of "small" non-negative step sizes γ_n. The algorithm is a discrete time stochastic process with a fixed and arbitrary initial value $\tilde{x}_0$, and whose general from can be written as follows

$$\tilde{x}_{n+1} - \tilde{x}_n = \gamma_n H(\tilde{x}_n, \xi_{n+1}), \tag{2.2.1}$$

where $\gamma_n \in \mathbb{R}_+$, $H : \mathbb{R}^N \times S \to \mathbb{R}^N$ is a deterministic measurable function which characterizes the algorithm and $\xi_{n+1} \in S$ is the source of randomness that in each step is incorporated into the vector $\tilde{x}_n$. The underlying intuitive idea is that if γ_n goes to zero at a suitable rate (depending on the vector field H), it washes the random perturbation ξ_n away and the asymptotic behaviour of $\{\tilde{x}_n\}_{n\geq 0}$ can be studied in terms of the averaged ODE

$$\frac{d\tilde{x}}{dt} = \overline{H}(\tilde{x}), \tag{2.2.2}$$

where

$$\overline{H}(\tilde{x}) = \lim_{n\to\infty} \mathbb{E}[H(\tilde{x}, \xi_n)],$$

and $\mathbb{E}(\cdot)$ denotes the mathematical expectation.

It is typically assumed that equation (2.2.1) may be rewritten by decoupling H into its deterministic and random components so that $H(\tilde{x}_n, \xi_{n+1}) = F(\tilde{x}_n) + I_{n+1}$, where $F : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a continuous map, and $I_{n+1} \in \mathbb{R}^N$ a zero-mean noise.

In this perspective, the approximation algorithm can be rewritten as

$$\tilde{x}_{n+1} - \tilde{x}_n = \gamma_n(F(\tilde{x}_n) + I_{n+1}). \tag{2.2.3}$$

In this way, one then has $\overline{H}(x) = F(x)$ so the ODE (2.2.2) reduces to $dx/dt = F(x)$. Note that formula (2.2.3) resembles a perturbed version of a variable step-size Cauchy-Euler numerical approximation scheme for solving $dx/dt = F(x)$, which takes the form

$$z_{n+1} - z_n = \gamma_n F(z_n).$$

It is natural then to search for conditions under which it is possible to compare the behaviour of a sample path $\{\tilde{x}_n\}_{n\geq 0}$ with the flow induced by the vector field F.

The classical results [53, 56, 62] establishing the relationship between (2.2.1) and (2.2.2) have the following general form. Let x^* be a stable equilibrium point for the ODE (2.2.2) If $\{\gamma_n\}_{n>0}$ goes to zero at a suitable rate and if the sequence $\{\tilde{x}_n\}_{n\geq 0}$ enters infinitely often a compact subset of the domain of attraction of x^*, then $\{\tilde{x}_n\}_{n\geq 0}$ converges almost surely toward x^*. These results rely on the notion of fixed points for the ODE.

In order to describe the asymptotic behaviour of (2.2.1) in terms of the ODE (2.2.2), M. Benaïm [14, 15] proposed the following framework. First, consider the continuous-time affine interpolation $X(t)$ of the process $\tilde{x}_n$ (see (2.2.6)). Then, the question is: Without any particular assumption on the dynamics of F, is it possible to relate the random limit set

$$l(X,\omega) := \bigcap_{t\geq 0} \overline{\{X(s,\omega) \,:\, s \geq t\}} \tag{2.2.4}$$

to the equilibria of the flow Φ induced by F? Roughly speaking, this is done in three main steps (we shall define the following technical terms in due course). First, show that X is almost surely a *pseudo-trajectory* of Φ (see Definition 2.2.3). Second, show that the limit set of a pseudo-trajectory is an *internally chain-transitive set* (see definitions in Subsection 2.2.1). Third, provided a *strict Lyapunov function* for F exists (see Definition 2.2.10), then every internally chain-transitive set is contained in

the equilibria set of F. As a consequence, subject to certain assumptions, it follows that the limit set of $\tilde{x}_n$ is almost surely a connected subset of the equilibria of F, which (sometimes) can be found analytically.

2.2.1 Robbins-Monro algorithm

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 0}$ a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$. We say that the process $\{\tilde{x}_n\}$ given by (2.2.3) satisfies the *Robbins-Monro (or Martingale difference noise)* [54] condition, if

- **(A1)** $\{\gamma_n\}_{n\geq 0}$ is a deterministic sequence in $\mathbb{R}_+$, and
- **(A2)** $I_n \in \mathbb{R}^N$ is adapted to $\mathcal{F}_n$ with $\mathbb{E}(I_{n+1} \mid \mathcal{F}_n) = 0$.

The next proposition will be useful later for the characterization of pseudo-trajectory as described in Theorem 2.2.4.

Proposition 2.2.1 (Proposition 4.2, [15])**.** *Let $\{\tilde{x}_n\}$ given by* (2.2.3) *be a Robbins-Monro algorithm. Suppose that for some $q \geq 2$*

$$\sup_n \mathbb{E}(\|I_n\|^q) < \infty$$

and

$$\sum_n \gamma_n^{1+\frac{q}{2}} < \infty.$$

Then, for all $T > 0$

$$\lim_{n\to\infty} \left(\sup_{k:0\leq \tau_k - \tau_n \leq T} \left\| \sum_{i=n}^{k-1} \gamma_i I_{i+1} \right\| \right) = 0 \quad a.s., \quad \textit{where} \quad \tau_n = \sum_{i=0}^{n-1} \gamma_i.$$

Remark 2.2.2 (See [15] Remarks 4.3 and 4.5)**.** The above Proposition's conclusion remains valid if $\{\gamma_n\}_{n\geq 0}$ is a sequence of predictable random variables such that $\mathbb{E}(\sum_n \gamma_n^{1+\frac{q}{2}}) < \infty$, and provided that assumption on $\{I_n\}_{n\geq 0}$ is strengthen to $\sup_n \mathbb{E}(\|I_n\|^q \mid \mathcal{F}_{n-1}) < C$ for some deterministic constant $C > 0$.

Basic dynamical systems terminology

Let $(\mathcal{M}, d)$ be a metric space and $\Phi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ be a continuous map $(t, x) \mapsto \Phi(t, x)$ denoted by $\Phi(t, x) = \Phi_t(x)$. The family $\{\Phi_t\}_{t\in\mathbb{R}}$ is called a flow on $\mathcal{M}$ if it satisfies the group property

$$\Phi_0 = \text{Identity, and } \forall s, t \in \mathbb{R}, \;\; \Phi_s \circ \Phi_t = \Phi_{s+t}.$$

A semi-flow is defined analogously as above replacing $\mathbb{R}$ by $\mathbb{R}_+$.

Let F denote a continuous vector field defined on $\mathbb{R}^N$ with unique integral curves [1]. The flow induced by F is the family of mappings $\{\Phi_t\}$ defined on $\mathcal{M} = \mathbb{R}^N$ by

$$\frac{d}{dt}\Phi_t(x) = F(\Phi_t(x)).$$

A point x is an equilibrium if $\Phi_t(x) = x$ for all t. Note that when Φ is induced by a vector field F, equilibria of Φ coincide with zeros of F.

A point x is called periodic of period $T > 0$ if $\Phi_T(x) = x$ and $\Phi_t(x) \neq x$ for $0 < t < T$.

A set $A \subseteq \mathcal{M}$ is said to be invariant if $\Phi_t(A) \subseteq A$ for all $t \in \mathbb{R}$ (and positively invariant if for all $t \in \mathbb{R}_+$). The orbit of $x \in \mathcal{M}$ is the set $o(x) = \{\Phi_t(x) : t \in \mathbb{R}\}$ (and forward orbit $o^+(x)$ if $t \in \mathbb{R}_+$).

A point $p \in \mathcal{M}$ is an ω-limit point of x if $p = \lim_{t_k\to\infty} \Phi_{t_k}(x)$ for some subsequence $t_k \to \infty$. The ω-limit set of x denoted by $\omega(x)$ is the set of all ω-limit points of x. If $o^+(x)$ has a compact closure, $\omega(x)$ is a compact connected invariant set and $\overline{o^+(x)} = o^+(x) \cup \omega(x)$

Additionally, the α-limit set of x is denoted by $\alpha(x)$ and defined as the ω-limit set of x for the reversed flow Φ_{-t}.

The definitions below are useful if one wants to go through the proofs of the Theorems stated in the next subsections. For a classical reference on ODE's and dynamical systems see [40].

[1] An integral curve of a vector field F is a parametric curve such that $\mathbf{x}'(t) = F(\mathbf{x}(t))$

$Eq(\Phi) := \{p \in \mathcal{M} \,:\, \Phi_t(p) = p \text{ for all } t\}$ the set of equilibria,

$Per(\Phi)$ the closure of the set of periodic orbits.

$\mathcal{L}_+(\Phi) := \overline{\bigcup_{x\in M} \omega(x)}$, $\mathcal{L}_-(\Phi) := \overline{\bigcup_{x\in M} \alpha(x)}$, and

$\mathcal{L}(\Phi) := \mathcal{L}_-(\Phi) \cup \mathcal{L}_+(\Phi)$.

Asymptotic pseudo-trajectory

The notion of asymptotic pseudo-trajectory of a semi-flow has been introduced in [17] and is particularly useful for analysing the long term behaviour of stochastic approximation processes.

Definition 2.2.3. A continuous function $f : \mathbb{R}_+ \to \mathcal{M}$ is an asymptotic pseudo-trajectory for Φ if

$$\lim_{t\to\infty} \sup_{0\le h\le T} d(f(t+h), \Phi_h(f(t))) = 0 \quad \text{for any } T > 0. \tag{2.2.5}$$

The idea is that $h \mapsto f(t+h)$ "shadows" the Φ-trajectory of the point $f(t)$ over the interval $[0, T]$ with arbitrary accuracy for sufficiently large t. For further characterizations of asymptotic pseudo-trajectories see [14] (section 3.1). Now, suppose that a stochastic process $\{\tilde{x}_n\}_{n\ge0}$ given by (2.2.3) satisfies the following conditions:

(B1) $\{\gamma_n\}_{n\ge0}$ is a given deterministic sequence of non-negative numbers such that

$$\sum_n \gamma_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \gamma_n = 0;$$

(B2) $I_n \in \mathbb{R}^N$ are deterministic or random perturbations adapted to $\mathcal{F}_n$.

Let us now introduce some definitions and investigate how the sample paths of $\{\tilde{x}_n\}$ relate to the flow induced by F. Set

$$\tau_0 = 0 \quad \text{and} \quad \tau_n = \sum_{i=0}^{n-1} \gamma_i \quad \text{for} \quad n \ge 1,$$

and let X and $\overline{X} : \mathbb{R}_+ \to \mathbb{R}^N$ be the continuous-time affine and piece-wise constant

interpolated processes, such that for $s \in [0, \tau_{n+1} - \tau_n]$ and $n \geq 0$ are given by

$$X(\tau_n + s) = \tilde{x}_n + s\frac{\tilde{x}_{n+1} - \tilde{x}_n}{\tau_{n+1} - \tau_n}, \quad \text{and} \quad \overline{X}(\tau_n + s) = \tilde{x}_n \tag{2.2.6}$$

The next result ensures that under the new assumptions **B1** and **B2**, the interpolated process (2.2.6) X is an asymptotic pseudo-trajectory of the flow Φ induced by F.

Proposition 2.2.4 (Proposition 4.1, [15])**.** *Let F be a continuous vector field with unique integral curves and assume* **B1**, **B2** *and that*

- *For all $T > 0$*
$$\lim_{n\to\infty} \left(\sup_{k:0\leq\tau_k-\tau_n\leq T} \left\| \sum_{i=n}^{k-1} \gamma_i I_{i+1} \right\| \right) = 0 \quad a.s.$$
- $\sup_n \|\tilde{x}_n\| < \infty \quad a.s.$,

Then the interpolated process X (2.2.6) is almost surely an asymptotic pseudo-trajectory of the flow Φ induced by F.

Pseudo-orbits and chain-recurrence

In this subsection, a new notion of recurrence is introduced. Equilibria, periodic and omega points are certainly recurrence points. Intuitively, we may say that a point is recurrent if its flow under time evolution somehow returns to its starting point. Another notion of recurrence (introduced in [21]) is the notion of *chain-recurrence*.

Definition 2.2.5. Let δ and T be positive real numbers. A (δ, T)-pseudo-orbit from a to b in $\mathcal{M}$ is a finite sequence of partial trajectories

$$\{\Phi_t(y_i) \,:\, 0 \leq t \leq t_i\}; \quad i = 0, \ldots, k-1; \;\; t_i \geq T, \quad \text{such that}$$

$$d(y_0, a) < \delta, \quad y_k = b, \quad \text{and}$$

$$d(\Phi_{t_j}(y_j), y_{j+1}) < \delta \quad j = 0, \ldots, k-1;$$

Definition 2.2.6. A point $a \in \mathcal{M}$ is a *chain-recurrent* point if there is a (δ, T)-pseudo-orbit from a to a for every δ and T. Analogously, a pair of points a, b is *chain-transitive* if there is a (δ, T)-pseudo-orbit from a to b for every δ and T.

If $R(\Phi)$ denotes the set of chain-recurrent points for Φ, then $R(\Phi)$ is a closed, positively invariant set such that

$$Eq(\Phi) \subset Per(\Phi) \subset \mathcal{L}(\Phi) \subset R(\Phi).$$

Now we extend the definition of chain-recurrent (transitive) points to sets, and note that the following definition of a chain-recurrent (transitive) set does not coincide with the set of chain-recurrent (transitive) points.

Definition 2.2.7. Let Λ be a non-empty invariant set. Λ is called a chain-recurrent *set* for Φ if every point $a \in \Lambda$ is a chain-recurrent point for the restriction of Φ to Λ. Analogously, Λ is called a chain-transitive *set* for Φ if every pair of points $a, b \in \Lambda$ is chain-transitive for the restriction of Φ to Λ. Finally, a compact chain-recurrent (transitive) set is called an *internally chain-recurrent (transitive)* set.

Lemma 2.2.8. *Let $\Lambda \subset \mathcal{M}$. The following assertions are equivalent:*

(i) *Λ is internally chain-transitive.*

(ii) *Λ is connected and internally chain-recurrent.*

(iii) *Λ is a compact invariant set and $\Phi|\Lambda$ admits no proper attractor [2].*

2.2.2 The limit set theorem

Let $X : \mathbb{R}_+ \to \mathcal{M}$ be an asymptotic pseudo-trajectory of Φ. The limit set $l(f)$ of f is defined by

$$l(f) = \bigcap_{t \geq 0} \overline{\{f(s) \ : \ s \geq t\}}.$$

Proposition 2.2.9 (Proposition 5.7 [15])**.** .

(i) *Let f be a precompact [3] asymptotic pseudo-trajectory of Φ. Then $l(f)$ is an internally chain-transitive set.*

(ii) *Let $l \subset \mathcal{M}$ be an internally chain-transitive set and assume $\mathcal{M}$ is locally path connected. Then, there exists an asymptotic pseudo-trajectory f such that $l(f) = l$.*

[2] A non-empty subset $A \subset \mathcal{M}$ is called an attractor for Φ if it is compact, invariant and has a neighbourhood $W \subset \mathcal{M}$ such that $d(\Phi_t(x), A) \to 0$ as $t \to \infty$ uniformly in $x \in W$

[3] defined in [15] as: f is precompact if its image has compact closure in $\mathcal{M}$

2.2.3 Lyapunov functions and stochastic gradients

Let $\Lambda \subset \mathcal{M}$ be a compact invariant set for the semi flow Φ.

Definition 2.2.10. A continuous function $L : \mathcal{M} \to \mathbb{R}$ is called a Lyapunov function for Λ and Φ if the function $t \in \mathbb{R}_+ \mapsto L(\Phi_t(x))$ is constant in t for $x \in \Lambda$ and strictly monotone along any integral curve of F outside Λ. If Λ coincides with the equilibria set $Eq(\Phi)$, L is called a strict Lyapunov function and Φ a gradientlike system.

Theorem 2.2.11 (Proposition 6.4 [15])**.** *Let $\Lambda \subset \mathcal{M}$ be a compact invariant set and $L : \mathcal{M} \to \mathbb{R}$ a Lyapunov function for Λ. Assume that $L(\Lambda) \subset \mathbb{R}$ has empty interior. Then every internally chain-transitive set is contained in Λ.*

Theorem 2.2.12 (Theorem 1.2 [14])**.** *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous gradientlike vector field with unique integral curves and let Λ be its equilibria set and L a strict Lyapunov function for Λ and the flow induced by F. Let $\{\tilde{x}_n\}_{n\geq 0}$ be the solution to the recursion* (2.2.3)

$$\tilde{x}_{n+1} - \tilde{x}_n = \gamma_n(F(\tilde{x}_n) + I_{n+1}),$$

where $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n\geq 0} \gamma_n = \infty$.
In addition, assume that

- **T1)** *$\{\tilde{x}_n\}_{n\geq 0}$ is bounded.*
- **T2)** *for each $T > 0$,*

$$\lim_{n\to\infty} \left(\sup_{k:0\leq \tau_k - \tau_n \leq T} \left\| \sum_{i=n}^{k-1} \gamma_i I_{i+1} \right\| \right) = 0.$$

- **T3)** *$L(\Lambda) \subset \mathbb{R}$ has empty interior.*

Then the limit set of $\{\tilde{x}_n\}_{n\geq 0}$ is almost surely a connected subset of Λ.

2.2.4 Stochastic approximation for urn models

Let us now show how to represent the sequence of increments of an urn model in order to fit it into the stochastic approximation algorithm's general form (2.2.3).

Let $\Delta^{N-1} \subset \mathbb{R}^N_+$ be the $N-1$ simplex

$$\Delta^{N-1} = \{\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N_+ \ : \ \sum_{i=1}^N x_i = 1\}$$

and let us identify its tangent space at any point with the linear subspace

$$T\Delta^{N-1} = \{\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N \ : \ \sum_{i=1}^N x_i = 0\}$$

Consider an urn initially at time 0 containing a total of n_0 balls of colours $1, \cdots, N$. Given the configuration of balls at time n, it is added to the urn a ball of a random colour, say, $i_{n+1} \in \{1, ..., N\}$. Let $x(n) = (x_1(n), \cdots, x_N(n))$ be the configuration of balls at time n and denote

$$\tilde{x}(n) = x(n)/(n_0 + n) \in \Delta^{N-1} \quad \text{the vector of proportions.}$$

Let us first analyse how the increment of $\tilde{x}(n)$ behaves in time. Denote by $\mathbf{e}_i$ the N-dimensional vector whose entries are 1 at coordinate i and 0 otherwise. Then

$$\begin{aligned}\tilde{x}(n+1) &= \frac{x(n) + \mathbf{e}_{i_{n+1}}}{n_0 + n + 1} = \frac{(n_0 + n)\tilde{x}(n) + \mathbf{e}_{i_{n+1}}}{n_0 + n + 1} \\ &= \left(1 - \frac{1}{n_0 + n + 1}\right)\tilde{x}(n) + \frac{\mathbf{e}_{i_{n+1}}}{n_0 + n + 1}. \end{aligned} \tag{2.2.7}$$

Then

$$\tilde{x}(n+1) - \tilde{x}(n) = \frac{1}{n_0 + n + 1}(\mathbf{e}_{i_{n+1}} - \tilde{x}(n)) \tag{2.2.8}$$

Note that given $\tilde{x}(n)$, the right hand side in (2.2.8) has a deterministic component $\tilde{x}(n)$ and a random component $\mathbf{e}_{i_{n+1}}$. Little can be said about the limiting behaviour of the random component, so the idea is to rearrange equation (2.2.8) by decoupling $\mathbf{e}_{i_{n+1}}$ into its mean part and a zero mean "noise". Then, by controlling the noise, the asymptotic behaviour of $\tilde{x}(n)$ can be obtained via the deterministic part. This is the core idea of the stochastic approximation method and the algorithm's general formula (2.2.3) can be found as below. First, note that in the context of urn models, the distribution of the incoming ball's colour $\mathbb{P}(i_{n+1} = i \mid \mathcal{F}_n)$ is a function of $\tilde{x}_n$, in

which case we can write

$$F(\tilde{x}(n)) := \mathbb{E}[\mathbf{e}_{i_{n+1}} \,|\, \mathcal{F}_n] - \tilde{x}(n), \tag{2.2.9}$$

and

$$I_{n+1} := \mathbf{e}_{i_{n+1}} - \mathbb{E}[\mathbf{e}_{i_{n+1}} \,|\, \mathcal{F}_n]. \tag{2.2.10}$$

So, by setting $\gamma_n = 1/(n_0 + n + 1)$, we obtain

$$\tilde{x}(n+1) - \tilde{x}(n) = \gamma_n(F(\tilde{x}(n)) + I_{n+1}). \tag{2.2.11}$$

Observe that the distribution of the incoming ball's random colour $\mathbb{P}(i_{n+1} = i \,|\, \mathcal{F}_n)$ has not yet been specified, as long as it is a function of $\tilde{x}(n)$. Therefore, the dynamics can be arbitrarily complicated.

2.3 Stability of equilibrium points

In this subsection the points where $F(x^*) = 0$ are classified according to what kind of critical points they are for the flow induced by F. To this end, let us denote $D(F)$ the Jacobian matrix of the vector field F. If all eigenvalues of $D(F)(x^*)$ have negative real part, then x^* is an attracting point. In this case, the flow always converges to x^* as $t \to \infty$ for an initial value x_0 in some neighbourhood of x^*. If some eigenvalue of $D(F)(x^*)$ has positive real part, then x^* is called a linearly unstable critical point. Finally, if all the real parts of the eigenvalues of $D(F)(x^*)$ are positive, then x^* is a repelling point and the flow converges to x^* only if it starts there.

The results in [66, 67] assert that, under suitable conditions,

$$\mathbb{P}[\tilde{x}_n \to x^*] = 0 \quad \text{if } x^* \text{ is a linearly unstable critical point.}$$

When the equilibria set $\Lambda = \{x \ : \ F(x) = 0\}$ is discrete, this result implies that unstable points are not in the support of the random limit $\tilde{x}_n$.

Formally, let $\Delta \subseteq \mathbb{R}^N$ be an open subset of an affine subspace in $\mathbb{R}^N$. Let $F : \Delta \to$

$T\Delta$ be of class $\mathcal{C}^2$ where $T\Delta$ is a translation of Δ that contains the origin (to be identified as the tangent space of Δ).

Theorem 2.3.1 (Theorem 1 [66]). *Let $\{\tilde{x}_n\}_{n\geq 0}$ be a discrete stochastic process satisfying Robbins-Monro conditions. Let x^* be an equilibrium point of F and $\mathcal{N}_{x^*}$ a neighbourhood of x^*. Suppose that there exist constants $p \in (1/2, 1]$ and $c_i, i = 1, \ldots, 4$ for which the following conditions hold whenever $\tilde{x}_n \in \mathcal{N}_{x^*}$ and n is sufficiently large:*

(i) x^* *is a linearly unstable critical point,*

(ii) $c_1 n^{-p} \leq \gamma_n \leq c_2 n^{-p}$,

(iii) $\mathbb{E}[(\gamma_n I_{n+1} \cdot \theta)^+ \mid \mathcal{F}_n] \geq c_3 n^{-p}$, *for every unit vector* $\theta \in T\Delta$,

(iv) $|\gamma_n I_{n+1}| \leq c_4 n^{-p}$.

Then $\mathbb{P}[\tilde{x}_n \to x^*] = 0$.

2.4 Proof of Theorem 2.1

Proof. Let us now write the process in the form of a stochastic approximation algorithm. To this end, consider the process of proportions $\tilde{X}(n) = X(n)/(n_0 + n)$ in the simplex $\Delta^1 = \{(x_1, x_2) \in \mathbb{R}^2_+ \ : \ x_1 + x_2 = 1\}$. Denote by $i_{n+1} \in \{1, \ldots, N\}$ the random location where the incoming particle is placed at time $n + 1$, given $X(n) = x(n)$. Note that the process $\tilde{X}$ is itself a Markov chain with increments

$$\tilde{X}(n+1) - \tilde{x}(n) = \frac{1}{n_0 + n + 1}(e_{i_{n+1}} - \tilde{x}(n))$$

and the same transition probabilities as the process X. Then by summing and subtracting $\mathbb{E}[\mathbf{e}_{i_{n+1}} \mid \mathcal{F}_n]$ one can write

$$\tilde{X}(n+1) - \tilde{x}(n) = \gamma_n(F(\tilde{x}(n)) + I_{n+1}),$$

where $F : \Delta^1 \to \mathbb{R}^2$, $I_{n+1} \in \mathbb{R}^2$ and $\gamma_n \in \mathbb{R}_+$ are given by

$$F(\tilde{x}(n)) := \mathbb{E}[\mathbf{e}_{i_{n+1}} \mid \mathcal{F}_n] - \tilde{x}(n), \tag{2.4.1}$$

$$I_{n+1} := \mathbf{e}_{i_{n+1}} - \mathbb{E}[\mathbf{e}_{i_{n+1}} \mid \mathcal{F}_n], \tag{2.4.2}$$

$$\gamma_n := \frac{1}{n_0 + n + 1}. \tag{2.4.3}$$

Then

$$\begin{aligned} F(\tilde{x}(n)) = \mathbb{E}[\mathbf{e}_{i_{n+1}} \mid \mathcal{F}_n] - \tilde{x}(n) &= \sum_{i=1}^{2} \mathbb{P}(i_{n+1} = i \mid \mathcal{F}_n)\mathbf{e}_i - \tilde{x}(n) \\ &= \left(\frac{u_1(n)}{u_1(n) + u_2(n)} - \tilde{x}_1(n), \frac{u_2(n)}{u_1(n) + u_2(n)} - \tilde{x}_2(n) \right). \end{aligned} \tag{2.4.4}$$

Let us now write the vector field F in (2.4.4) for the matrix $A = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$.

$$F(\tilde{x}_1, \tilde{x}_2) = \left(\frac{a\tilde{x}_1^\beta + b\tilde{x}_2^\beta}{(a+b)(\tilde{x}_1^\beta + \tilde{x}_2^\beta)} - \tilde{x}_1, \frac{b\tilde{x}_1^\beta + a\tilde{x}_2^\beta}{(a+b)(\tilde{x}_1^\beta + \tilde{x}_2^\beta)} - \tilde{x}_2 \right). \tag{2.4.5}$$

Before applying the stochastic approximation method, one needs to make sure that the hypothesis of Theorem 2.2.12 are fulfilled. Hypothesis **T1** is trivially satisfied (and this is the reason why we consider the process of proportions).

Hypothesis **T2** can be verified as follows. Define

$$M_0 = 0 \text{ and } M_n = \sum_{j=0}^{n-1} \gamma_j I_{j+1}, \quad n = 1, 2, \ldots$$

Note that

$$\mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] = \gamma_n \mathbb{E}[I_{n+1} \mid \mathcal{F}_n] = 0, \quad \text{and}$$

$$\|M_{n+1} - M_n\|^2 = \gamma_n^2 \|I_{n+1}\|^2 \leq 2\gamma_n^2$$

Therefore, for any $n \geq 0$,

$$\sum_{j=0}^{n} \mathbb{E}[||M_{j+1} - M_j||^2 \mid \mathcal{F}_j] \leq 2 \sum_{j=0}^{n} \gamma_j^2 < \infty \quad a.s.,$$

so, the sequence $\{M_n\}_{n \geq 0}$ converges almost surely to a finite random vector (see [25], Section 5.4.1). In particular, it is a Cauchy sequence in L^2. Then, Hypothesis **T2** is satisfied since T is fixed and $n \to \infty$.

Now we need to determine whether there is a strict Lyapunov function for the ODE

$$dx/dt = F(x).$$

Let $L : \Delta^1 \to \mathbb{R} \cup \{\infty\}$ be given by

$$L(x_1, x_2) = -(x_1 + x_2) + \frac{1}{a+b}\left(b\log[(a+b)x_2] + b\log x_1 - (b-a)\frac{\log(x_1^\beta + x_2^\beta)}{\beta}\right). \tag{2.4.6}$$

Note that

$$\frac{\partial L}{\partial x_1} = -1 + \frac{ax_1^\beta + bx_2^\beta}{x_1(a+b)(x_1^\beta + x_2^\beta)} = \frac{1}{x_1}F_1, \tag{2.4.7}$$

$$\frac{\partial L}{\partial x_2} = -1 + \frac{bx_1^\beta + ax_2^\beta}{x_2(a+b)(x_1^\beta + x_2^\beta)} = \frac{1}{x_2}F_2. \tag{2.4.8}$$

Denoting an integral curve of F by $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, $t \geq 0$, then

$$\frac{d(L\circ\alpha)}{dt} = \frac{\partial L}{\partial \alpha_1}\frac{d\alpha_1}{dt} + \frac{\partial L}{\partial \alpha_2}\frac{d\alpha_2}{dt} = \alpha_1\left(\frac{\partial L}{\partial \alpha_1}\right)^2 + \alpha_2\left(\frac{\partial L}{\partial \alpha_2}\right)^2 \geq 0,$$

where the equality holds in the above inequality if and only if $F(\alpha) = 0$ (see equations (2.4.7) and (2.4.8)). Thus, L is a strict Lyapunov function for F.

Now, let us investigate the equilibria set

$$\Lambda = \{(x_1, x_2) \in \Delta^1 \ : \ F(x_1, x_2) = 0\}.$$

By substituting $x_1 = cx_2$ and manipulating (2.4.5), the equilibria set Λ can be characterized by the solution of the following system:

$$\begin{cases} x_1 + x_2 = 1, \\ x_1 = cx_2, \end{cases} \tag{2.4.9}$$

where c is a positive root of the polynomial

$$\mathcal{P}(z) = z^{\beta+1} - (a/b)z^\beta + (a/b)z - 1 = 0.$$

Note that $\mathcal{P}(1) = 0$. Then, the solution $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$ is always a point of equilibrium for F. Furthermore, $\mathcal{P}(0) = -1$ and $\mathcal{P}(z) \to \infty$ as $z \to \infty$. Finally, note that when $c > 0$ is a root of $\mathcal{P}(z)$, so is $1/c$. For this reason, when looking for positive roots of $\mathcal{P}(z)$, one only needs to look for it on $(0, 1)$ or $(1, \infty)$. Now, call

$\gamma = a/b > 0$ and note that

$$\mathcal{P}'(z) = z^{\beta-1}[(\beta+1)z - \gamma\beta] + \gamma, \quad \text{and}$$

$$\mathcal{P}''(z) = \beta z^{\beta-2}[(\beta+1)z - \gamma(\beta-1)]$$

and most importantly,

$$\mathcal{P}''(1) = \beta[\beta + 1 - \gamma(\beta-1)] = \beta\mathcal{P}'(1).$$

Then, $\mathcal{P}'(1)$ and $\mathcal{P}''(1)$ change sign together.

- **(i)** $\mathcal{P}'(1)$ and $\mathcal{P}''(1) > 0$.

  In this case, note that $\mathcal{P}''(1) > 0$ implies $\mathcal{P}''(z) > 0$ for all $z > 1$. Then $\mathcal{P}'(z) > 0$ for all $z > 1$ as well. Since $\mathcal{P}(1) = 0$, then $\mathcal{P}(z) > 0$ for all $z > 1$, which implies $\mathcal{P}(z) > 0$ for all $z \in (0,1) \cup (1,\infty)$ because if $\mathcal{P}$ has no roots in $(1,\infty)$, it has no roots in $(0,1)$. It follows that the only positive root of $\mathcal{P}(z)$ is at $z = 1$.

- **(ii)** $\mathcal{P}'(1) = \mathcal{P}''(1) = 0$.

  This implies $\gamma = (\beta+1)/(\beta-1)$. In this case $\mathcal{P}''(z) = \beta(\beta+1)z^{\beta-2}(z-1)$. Note that $\mathcal{P}''(z) > 0$ for all $z > 1$. Since $\mathcal{P}'(1) = 0$, then $\mathcal{P}'(z) > 0$ for all $z > 1$. Combining this with $\mathcal{P}(1) = 0$ implies $\mathcal{P}(z) > 0$ for all $z > 1$, which implies $\mathcal{P}(z) > 0$ for all $z \in (0,1) \cup (1,\infty)$ because if $\mathcal{P}$ has no roots in $(1,\infty)$, it has no roots in $(0,1)$. Therefore, the only positive root of $\mathcal{P}(z)$ is at $z = 1$.

- **(iii)** $\mathcal{P}'(1)$ and $\mathcal{P}''(1) < 0$.

  In this case, note that $\mathcal{P}''(1) < 0$ implies $\mathcal{P}''(z) < 0$ for all $0 < z < 1$. In addition, $\mathcal{P}''(1) < 0$ implies $\gamma > 1$ and $\beta > 1$. Now, since $\mathcal{P}'(0) = \gamma := a/b > 0$ and $\mathcal{P}'(1) < 0$ and $\mathcal{P}''(z) < 0$ for all $z \in (0,1)$, then $\mathcal{P}'(z) = 0$ for only one $z \in (0,1)$. Thus, combining all the above, it follows that $\mathcal{P}(z)$ has only one root $z_0 \in (0,1)$, which implies that $z_0, 1$ and $1/z_0$ are the only positive roots of $\mathcal{P}(z)$.

Note that cases **(i)** or **(ii)** hold if and only if $\left(\frac{a-b}{a+b}\right)\beta \leq 1$, and case **(iii)**, if and only if $\left(\frac{a-b}{a+b}\right)\beta > 1$ (see Figure 2.1). In either case, the set $\Lambda = \{x \in \Delta^1 \ : \ F(x) = 0\}$ is discrete and finite, which implies that $L(\Lambda)$ has Lebesgue measure zero, and so, empty interior. This establishes condition **T3**.

Finally, let us prove that the limiting direction $(\frac{1}{2}, \frac{1}{2})$, corresponding to the root $z = 1$, is unstable in the supercritical case $\left(\frac{a-b}{a+b}\right)\beta > 1$ (see Figure 2.2). To this end, we apply Pemantle's criteria for nonconvergence to unstable points (see section 2.3).

Calculating the partial derivatives of F, the eigenvalues of its Jacobian matrix are given by

$$\begin{aligned} &\lambda_1(x_1, x_2) \equiv -1, \quad \text{and} \\ &\lambda_2(x_1, x_2) = \frac{\left(\frac{a-b}{a+b}\right)\beta x_1^{\beta-1}x_2^{\beta-1}(x_1+x_2) - (x_1^\beta + x_2^\beta)^2}{(x_1^\beta + x_2^\beta)^2}. \end{aligned} \tag{2.4.10}$$

Substituting $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$, we get

$$\lambda_2(\tfrac{1}{2}, \tfrac{1}{2}) = \left(\frac{a-b}{a+b}\right)\beta - 1,$$

Then, the point $(\frac{1}{2}, \frac{1}{2})$ is a linear unstable critical point if $\left(\frac{a-b}{a+b}\right)\beta > 1$. This is the same as Theorem's 2.3.1 hypothesis (i). Finally, the step size γ_n is of order $1/n$ and hypothesis (ii)-(iv) are evidently satisfied, which finishes the proof. □

Figure 2.1: Solutions of $\mathcal{P}(z) = 0$

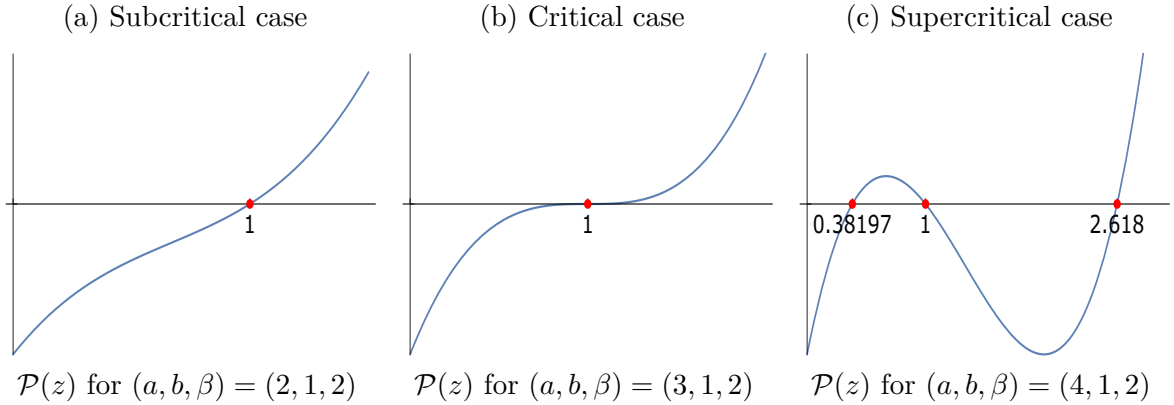
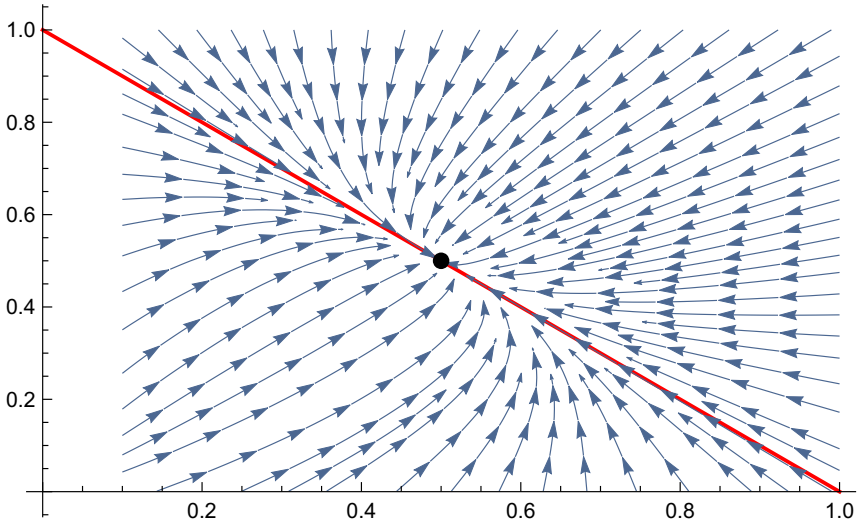
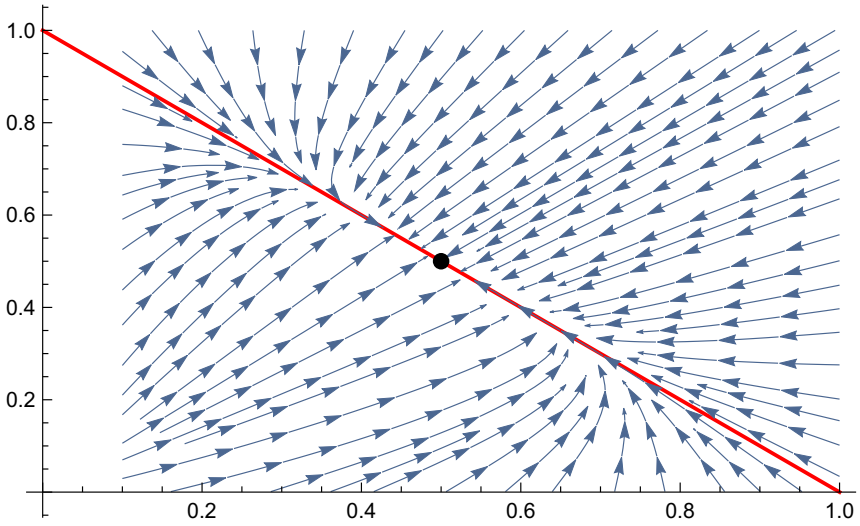


Figure 2.2: Flow lines of F on $[0,1]^2$

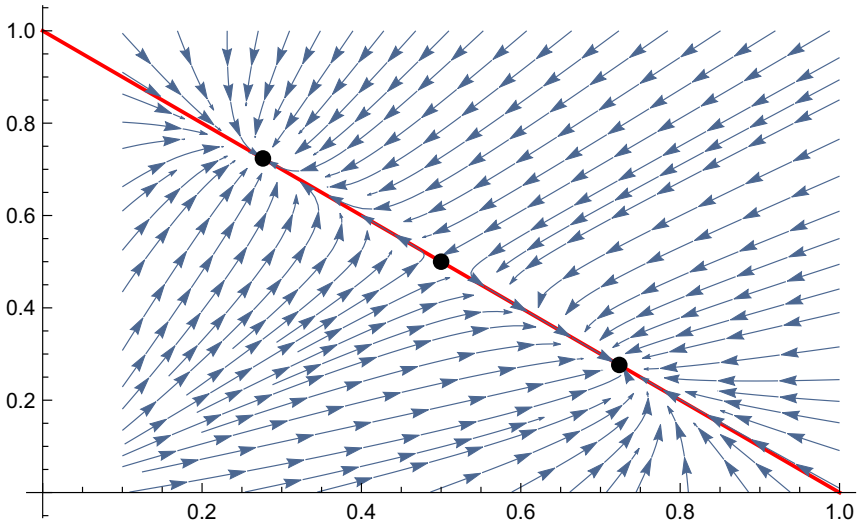
(a) Subcritical case: $(a, b, \beta) = (2, 1, 2)$



(b) Critical case: $(a, b, \beta) = (3, 1, 2)$



(c) Supercritical case: $(a, b, \beta) = (4, 1, 2)$



Chapter 3

Stabilization for a reinforced growth model with interaction

3.1 The model and main result

Let $G_N = (V, E)$ be a directed graph containing N vertices. We write $v_i \to v_j$ if the oriented pair (v_i, v_j) belongs to E. For each oriented pair of vertices (v_i, v_j) we assign a non-negative rational[1] number a_{ij} such that $a_{ij} > 0$ if $v_i \to v_j$ and $a_{ij} = 0$ otherwise. Matrix $A := (a_{ij})_{i,j=1}^N$ is called the weighted adjacency matrix of G_N. Let us start by fixing an enumeration $\{v_i\}_{i=1}^N$ of vertices in V and a weighted adjacency matrix A for the underlying graph G_N.

Let us consider a discrete time Markov chain $X(n) = (X_i(n))_{i=1}^N$ taking values in $\mathbb{Z}_+^N$ accounting for the number of particles in each vertex of G_N. The local interaction is defined through a *potential* $U(n) = (U_i(n))_{i=1}^N$ such that at each vertex v_i,

$$U_i(n) := \sum_{j=1}^{N} a_{ij} X_j(n). \tag{3.1.1}$$

[1]It is expected that the result in this section extends to non-negative real numbers a_{ij}. This is suggested by [41] Remark 4.2, 4th paragraph and also supported by A. Kyprianou, *et al* [55], Theorem 1.5 on continuous state branching processes which can be found at:
`https://arxiv.org/pdf/1707.04955.pdf`

Given that at time n the current configuration is $\mathbf{x}(n) = (x_1(n), \ldots, x_N(n))$ and corresponding potential $u(n) = (u_1(n), \ldots, u_N(n))$, a new particle is randomly placed at some vertex $v_i,\ i = 1, ..., N$ according to the following transition probability

$$\mathbb{P}[X(n+1) = \mathbf{x}(n) + \mathbf{e}_i \mid X(n) = \mathbf{x}(n)] = \frac{u_i(n)}{\sum_{j=1}^{N} u_j(n)}, \tag{3.1.2}$$

where $\mathbf{e}_i$ is the vector whose i-th coordinate is 1 and whose other coordinates are 0.

Note that the dynamics allows only one particle to be placed at a time, so that $X_k(n+1) = x_k(n) + 1$ for some k and $X_j(n+1) = x_j(n)$ for all $j \neq k$. It is worth noting that if G_N is made only of self loops with same weight, i.e., $a_{ii} \equiv a > 0$ for all i and $a_{ij} = 0$ for all $i \neq j$, the resulting model is a Pólya urn. Also, if $a_{ij} \equiv a > 0$ for all i, j the resulting model is a uniform random deposition model.

In what follows, we restrict the graph G_N to be a strongly connected directed graph so that its adjacency matrix $A := (a_{ij})_{i,j=1}^{N}$ is irreducible (see Definition 3.4.7). Note that this rules out the Pólya urn model since a diagonal matrix is not irreducible.

Before stating our result, let us recall that the Perron-Frobenius theory (see section 3.4) asserts that for an *irreducible* square matrix A with *non-negative* entries, there exists a unique largest eigenvalue which is real, positive and a simple root of the characteristic polynomial. Moreover, its associated right eigenvector can be chosen to have *strictly* positive entries. They are called the Perron-Frobenius eigenvalue and eigenvector of the matrix A and are denoted by λ_1 and $\mathbf{r}_1$, respectively. That is,

$$\lambda_1 > Re(\lambda) \quad \text{for all other eigenvalue } \lambda \text{ of } A, \tag{3.1.3}$$

$$Ar_1 = \lambda_1 \mathbf{r}_1, \text{ and } (\mathbf{r}_1)_j > 0, \quad j = 1, \ldots, N, \tag{3.1.4}$$

where we may assume that

$$\sum_{j=1}^{N} (\mathbf{r}_1)_j = 1. \tag{3.1.5}$$

The next theorem is our main result.

Theorem 3.1.1. *Let $(X(n))_{n\geq 0}$ be a Markov chain taking values in $\mathbb{Z}_+^N$ with transition probabilities as defined in* (3.1.2)*. Then, for every $X(0) \neq \mathbf{0}$*

$$n^{-1}X(n) \longrightarrow \mathbf{r}_1 \quad a.s., \tag{3.1.6}$$

where $\mathbf{r}_1$ is defined in (3.1.3)*,* (3.1.4) *and* (3.1.5) *with respect to a weighted adjacency matrix of a strongly connected directed graph G_N.*

The key observation here is that the process $\{U(n)\}_{n\geq 0}$ is a multi-type branching process with deterministic offspring distributions. We now make the following observation.

Remark 3.1.2. The adjacency matrix may be extended to assign i.i.d random weights for time-evolving oriented edges so as to mimic the split mechanism of a multi-type branching process. In this case the mean matrix $A = \mathbb{E}a_{ij}$ needs to be irreducible (or more generally, that assumptions (i)-(vi) in Section 3.2.1 hold) Nevertheless, for simplicity we restrict ourselves to the formulation above and defer the case of random a_{ij} $i, j = 1, \ldots, N$ to a future stage.

For proving this result we shall first study and derive results for the process $(U(n))_{n\geq 0}$ and translate them back to the original process $(X(n))_{n\geq 0}$. Let us denote by $i_{n+1} \in \{1, \ldots, N\}$ the random location where the next incoming particle is placed at time $n+1$. In this notation, $X(n+1) = X(n) + \mathbf{e}_{i_{n+1}}$, where the conditional distribution of i_{n+1} given $X(n) = x(n)$ is defined in (3.1.2). Therefore, the process $U(n) = (U_1(n), \ldots, U_N(n))$ is such that

$$\begin{aligned} U_i(n+1) &= \sum_{j=1}^{N} a_{ij} X_j(n+1) = \sum_{j=1}^{N} a_{ij}(x_j(n) + 1\{i_{n+1} = j\}) \\ &= u_i(n) + \sum_{j=1}^{N} a_{ij} 1\{i_{n+1} = j\}, \quad i = 1, \ldots, N, \end{aligned} \tag{3.1.7}$$

where in the above "sum" only one term survives as $1\{i_{n+1} = k\} = 1$ for some k and $1\{i_{n+1} = j\} = 0$ for all $j \neq k$. Now, since the distribution of i_{n+1} depends only on the current state $U(n) = u(n)$, the process $(U(n))_{n\geq 0}$ is itself a Markov chain with

state space $\mathbb{Q}_+^N$ whose increments are given by

$$U(n+1) = U(n) + (a_{1i_{n+1}}, a_{2i_{n+1}}, \ldots, a_{Ni_{n+1}}) \tag{3.1.8}$$

and whose transition probabilities are as follows

$$\mathbb{P}[i_{n+1} = k | U(n) = u(n)] = \frac{u_k(n)}{\sum_{j=1}^N u_j(n)}. \tag{3.1.9}$$

Lemma 3.1.3. *Under the same assumptions and notations of Theorem 3.1.1, where* $(U(n))_{n\geq 0}$ *is defined by* (3.1.1), (3.1.8) *and* (3.1.9), *and for every* $U(0) \neq \mathbf{0}$,

$$n^{-1}U(n) \longrightarrow \lambda_1 \mathbf{r}_1 \quad a.s.$$

3.2 Multi-type Markov branching processes

An N-type continuous time Markov branching process $Z(t)_{t\geq 0}$ can be interpreted as a vector in $\mathbb{Z}_+^N$ denoting the population sizes at time t of a system comprised of N types of particles evolving in the following manner:

Given a fixed initial population $Z(0) = (z_1, \ldots, z_N)$,

B1) A particle of type j has an exponentially distributed lifetime with rate parameter c_j and upon death produces offspring of particles of the N types according to the distribution $\zeta_j = (\zeta_{1j}, \ldots, \zeta_{Nj})$, where ζ_{ij} is a random number of particles of type i created upon a j-particle's death, $i, j = 1, \ldots, N$.

B2) Particles live, die and split independent of each other, and of the past.

3.2.1 Construction of the process and main results

The N-type continuous time Markov branching process is a time-homogeneous Markov process $Z(t) = (Z_1(t), \cdots, Z_N(t))$, $t \geq 0$ with respect to the σ-fields $\mathcal{F}_t = \sigma\{Z(s,\omega), s \leq t\}$, where each coordinate $Z_i(t) \in \mathbb{Z}_+$ denotes the number of particles of type i at time $t \geq 0$.

Let $\mathbf{e}_j$, $j = 1, \ldots, N$ denote the vector whose j-th coordinate is 1 and whose other coordinates are 0. Vectors in $\mathbb{Z}_+^N$ are written in bold face letters so that $\mathbf{x} = (x_1, \ldots, x_N), \mathbf{y} = (y_1, \ldots, y_N), \mathbf{1} = (1, \ldots, 1), \mathbf{0} = (0, \ldots, 0)$ and so on.

The initial state $Z(0)$ is throughout assumed to be non-random and when the process is started in state $\mathbf{e}_j$, it is denoted by $Z^{(\mathbf{e}_j)}(t)$. To define the branching mechanism of the N-type process one needs N generating functions, each in N variables $\mathbf{s} = (s_1, \ldots, s_N) \in [0,1]^N$.

$$f_j(\mathbf{s}) = \sum_{\mathbf{y} \in \mathbb{Z}_+^N} \mathsf{P}(\zeta_j = \mathbf{y}) s_1^{y_1} \ldots s_N^{y_N}, \quad j = 1, \ldots, N, \quad \text{where,} \tag{3.2.1}$$

Definition 3.2.1. ζ_j, $j = 1, \ldots, N$ are the offspring distributions taking values in $\mathbb{Z}_+^N$ so that $\mathsf{P}(\zeta_j = \mathbf{y})$ is the probability that a particle of type j, upon death, produces $y_i \in \mathbb{Z}_+$ particles of type $i = 1, \ldots, N$.

For each pair $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_+^N$, the stationary transition probabilities $P(\mathbf{x}, \mathbf{y}, t)$ are defined as

$$P(\mathbf{x}, \mathbf{y}, t) = \mathbb{P}(Z(t+s) = \mathbf{y} \mid Z(s) = \mathbf{x}),$$

which may be determined by the infinitesimal rate parameter

$$\mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{R}_+^N$$

and the offspring distributions ζ_j, $j = 1, \ldots, N$ such that for all $\mathbf{y} \in \mathbb{Z}_+^N$

$$\mathsf{P}(\zeta_j = \mathbf{y}) \geq 0, \quad \text{and} \quad \sum_{\mathbf{y} \in \mathbb{Z}_+^N} \mathsf{P}(\zeta_j = \mathbf{y}) = 1$$

as solutions of the Kolmogorov equations (3.2.2) (3.2.3) below. For notational simplicity let us first denote

$$u_j(\mathbf{s}) = c_j(f_j(\mathbf{s}) - s_j), \quad j = 1, \ldots, N,$$

$$F(\mathbf{x}, \mathbf{s}; t) = \sum_{\mathbf{y} \in \mathbb{Z}_+^N} P(\mathbf{x}, \mathbf{y}; t) s_1^{y_1} \ldots s_N^{y_N}.$$

The *Kolmogorov* equations are (see [39] p.114 or [6] p.201)

$$\frac{\partial}{\partial t} F(\mathbf{e}_j, \mathbf{s}; t) = \sum_{j=1}^{N} u_j(\mathbf{s}) \frac{\partial}{\partial s_j} F(\mathbf{e}_j, \mathbf{s}; t), \qquad \text{(forward equation)} \qquad (3.2.2)$$

$$\frac{\partial}{\partial t} F(\mathbf{e}_j, \mathbf{s}; t) = u_j(F(\mathbf{e}_1, \mathbf{s}; t), \ldots, F(\mathbf{e}_N, \mathbf{s}; t)), \qquad \text{(backward equation)} \qquad (3.2.3)$$

$j = 1, \ldots, N.$

A sufficient condition to guarantee *a.s.* non-explosion, i.e. that there cannot be infinitely many particles in finite time, is that

$$\mathbb{E}\zeta_{ij} = \frac{\partial}{\partial s_i} f_j(\mathbf{s}) \Big|_{\mathbf{s}=\mathbf{1}} < \infty \quad \text{for all } i, j. \qquad (3.2.4)$$

(For necessary and sufficient non-explosion conditions, see [74].)

Assuming (3.2.4), equations (3.2.2) and (3.2.3), subject to the boundary condition

$$P(\mathbf{x}, \mathbf{y}, 0) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y}, \\ 0 & \text{otherwise,} \end{cases}$$

have a unique solution given by the generating function of $P(\mathbf{x}, \mathbf{y}, t)$ with $\sum_{\mathbf{y} \in \mathbb{Z}_+^N} P(\mathbf{x}, \mathbf{y}, t) = 1$ and the further property that

$$\sum_{\mathbf{y} \in \mathbb{Z}_+^N} P(\mathbf{x}, \mathbf{y}, t) s_1^{y_1} \ldots s_N^{y_N} = \prod_{j=1}^{N} \left[\sum_{\mathbf{y} \in \mathbb{Z}_+^N} P(\mathbf{e}_j, \mathbf{y}, t) s_1^{y_1} \ldots s_N^{y_N} \right]^{x_j}.$$

The above identity is referred to as the *additive property*, meaning that the population at time t, started with $\mathbf{x} = (x_1, \ldots, x_N)$ particles, is distributed as the sum of x_j independent populations starting with a single particle of type j, summed over $j = 1, \ldots, N$. Thus, the additive property enables one to define the process through

the distribution of offspring of various types to be produced by a single particle of each type $j = 1, \ldots, N$ and also to write the process in its usual representation

$$Z(t+s,\omega) = \sum_{j=1}^{N} \sum_{\kappa=1}^{Z_j(s,\omega)} Z^{(\mathbf{e}_j),\kappa}(t,\omega),$$

such that for each j, given $Z(s,\omega)$, the processes $\{Z^{(\mathbf{e}_j),\kappa}(t,\omega)\}_{t\geq 0}$ are conditionally independent copies of $Z^{(\mathbf{e}_j)}(\cdot,\omega)$.

First moments

Let

$$m_{ij}(t) = \mathbb{E}(Z_i(t) \mid Z(0) = \mathbf{e}_j) \tag{3.2.5}$$

Remark 3.2.2. The indices i and j were intentionally swapped so that in what follows a matrix is regraded as an operator acting on column vectors to the right, as opposed to the standard notation for Markov chains where the transition matrix act on row vectors to the left.

It follows from (3.2.4) that $m_{ij}(t) < \infty$ for all i, j (see [39], V.6 p.103).

The *mean matrix* of the multi-type process is defined as

$$M(t) := (m_{ij}(t))_{i,j=1}^{N}. \tag{3.2.6}$$

$M(t)$ and its eigenvalues will play a key role in what follows. First, from the backward equation (3.2.3) one can show the semigroup property

$$M(t+s) = M(t)M(s), \quad s,t \geq 0, \tag{3.2.7}$$

and the continuity condition

$$\lim_{t\to 0} M(t) = I. \tag{3.2.8}$$

It is well known that (3.2.7) and (3.2.8) imply (by Hille-Yosida Theorem) the existence of a matrix $A = (a_{ij})_{i,j=1}^{N}$, called the infinitesimal generator of the semigroup,

such that

$$M(t) = \exp(At) = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \tag{3.2.9}$$

where

$$a_{ij} = c_j \left(\mathbb{E}\zeta_{ij} - \delta_{ij} \right). \tag{3.2.10}$$

Remark 3.2.3. The transposition of indices in (3.2.5) does not cause any inconsistency in the above identity since matrix transposition is a continuous map and it commutes with sums and powers.

Note that the j-th column of matrix A describes the expected change in the composition of the population for each $i = 1, \ldots, N$ when a particle of type j dies out and produces its offspring.

The eigenvalues of $M(t)$ are given by $e^{\lambda_i t}$, $i = 1, \ldots, N$ where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of A and $M(t)$ and A have the same eigenvectors. Now, it is typically assumed positive regularity of $M(t)$, i.e., that there exists a t_0, $0 < t_0 < \infty$ such that

$$m_{ij}(t_0) > 0 \quad \text{for all } i, j.$$

Positive regularity of $M(t)$ is equivalent to irreducibility of A. Then, it follows by the Perron-Frobenius theory (see section 3.4) that the eigenvalues of A can be arranged as

$$\lambda_1 > Re(\lambda_2) \geq Re(\lambda_3) \geq \ldots \geq Re(\lambda_N).$$

Moreover, the left and right eigenvectors $\mathbf{l}_1$ and $\mathbf{r}_1$ of λ_1 have strictly positive entries and might be chosen such that

$$\mathbf{l}_1 \cdot \mathbf{r}_1 = 1 \quad \text{and} \quad \mathbf{c} \cdot \mathbf{r}_1 = 1.$$

Remark 3.2.4. The process is supercritical, critical, or subcritical according as λ_1 is $>$, $=$ or $<$ than 0, respectively (see [6] V.7.4. p. 203).

The limit theorem

Recall that as in definition 3.2.1, the event $\{\zeta_j = \mathbf{y}\}$ denotes that a particle of type j, upon death, produces $y_i \in \mathbb{Z}_+$ particles of type $i = 1, \ldots, N$. Instead, for each $j = 1, \ldots, N$, let us define an N-dimensional random vector $\xi_j = (\xi_{1j}, \ldots, \xi_{Nj})$ such that $\xi_{ij} = \zeta_{ij} - \delta_{ij}$ *is the random change in the composition of particles of type* i *upon a* j*-type particle's death*, $i, j = 1, \ldots, N$. Then ξ_{ij} are integer-valued vectors such that

$$\xi_{ij} \geq 0, \quad i \neq j, \quad \text{and} \tag{3.2.11}$$

$$\xi_{jj} \geq -1 \quad a.s. \tag{3.2.12}$$

It should be emphasized now that under this new notation, the matrix A in (3.2.10) may equivalently be written as

$$A = (c_j \mathbb{E}\xi_{ij})_{i,j=1}^{N}. \tag{3.2.13}$$

We are mainly interested in the irreducible case (i.e. A is irreducible or $M(t)$ is positively regular), but since S. Janson, 2004 ([41]) extended the theory to a more general set up, let us present Janson's assumptions:

(i) (3.2.11) and (3.2.12) hold a.s., i.e. $\xi_{ij} + \delta_{ij} \geq 0$ for all i and j.

(ii) $\mathbb{E}\xi_{ij}^2 < \infty$ for all i and j.

(iii) The largest real eigenvalue λ_1 of A is strictly positive.

(iv) The largest real eigenvalue λ_1 of A is simple.

(v) There exists a type i with $Z_i(0) > 0$ such that $i \mapsto j$ for every other type j. (See definitions 3.4.3, 3.4.4)

(vi) λ_1 "belongs" to the self-communicating class of a type i satisfying item (v) above (See Remark 3.4.5)

Condition (3.2.12) means that a particle j may die out and may not produce any other particle of its own type. Condition (3.2.11) means that we cannot remove particles of other types than the one removed[2]. This implies that the matrix $A + \alpha I$ is a non-negative matrix for large enough α, which by [77] Theorem 2.6 (see section 3.4.1) is enough to apply the Perron-Forbenius theory.

Condition (ii) is essential since we need the martingale convergence theorem for L^2 bounded martingales.

Condition (iii) implies that the branching process is supercritical, which is sufficient but not necessary condition for non-extinction[3].

Finally, note that conditions (iv)-(vi) hold when A is irreducible and while (iv) is a weak restriction, (v) and (vi) are more significant ones (see [48]).

Let us now present the fundamental well-known result (cf. [6], Theorem V.8.1) which introduces the standard martingale in branching processes theory.

Theorem 3.2.5 ([41], Lemma 9.2)**.** *Assume (i)-(vi). Then $e^{-tA}Z(t)$ is a martingale for $t \geq 0$. In particular,*

$$\mathbb{E}Z(t) = e^{tA}Z(0)$$

and thus

$$\mathbb{E}Z(t) = O(e^{\lambda_1 t}).$$

The next theorem and its consequences on the limiting behaviour of the various projections of $Z(t)$ are the basis of all functional limit theorems for branching process and generalised Pólya urn models. It also leads to the classification (see [7, 8, 47]) of the asymptotic behaviour of $Z(t)$ depending on whether there is any other eigenvalue λ, besides λ_1, with a real part $Re\lambda > \lambda_1/2$. See also Artheya and Ney's

[2]If the death of one particle may imply in the removal of others (provided that such others particles exist), the resulting process is no longer a branching process since particles do not evolve independently. Nevertheless, Bagchi and Pal [9] studied the so-called *tenable* urn models where $\xi_{ij} < 0$ for some $i \neq j$ was first considered.

[3]Non-extinction is also possible in some exceptional cases with $\lambda_1 = 0$, for instance when for each j, $\xi_{jj} = -1$ and $\xi_{ij} = 1$ for exactly one $i \neq j$, the total number of particles is constant and they change type according to a Markov chain.

book [6] section V.8, pp. 209-219 and Janson's more recent account [41] Theorem 3.1, p. 185 and Lemma 9.8 p. 221.

Theorem 3.2.6 ([41], Theorem 3.1.(i)). *Assume (i)-(vi). Then*

$$e^{-\lambda_1 t} Z(t) \to W \mathbf{r}_1 \quad a.s. \quad as \quad t \to \infty,$$

where W is a non-negative random variable.

A thorough investigation on the random variable W started in Kesten and Stigum's papers [47, 48] providing necessary and sufficient condition ensuring that $\mathbb{P}(W > 0) > 0$.

Theorem 3.2.7 (Kesten-Stigum, [47]). *Let $Z(t)$ be a supercritical, positively regular and non-singular [4] process started with a single particle, that is, $Z(0) = \mathbf{e}_i$ for some i. Assume further that $\mathbb{E}\zeta_{ij} < \infty$ for all i, j. Then*

$$\mathbb{P}(W > 0) > 0 \quad \textit{if and only if} \quad \mathbb{E}[\zeta_{ij} \log \zeta_{ij}] < \infty \quad \textit{for all } i, j = 1, \ldots, N.$$

The necessary and sufficient condition $\mathbb{E}[\zeta_{ij} \log \zeta_{ij}] < \infty$ for strict positivity of W is commonly known for short as the '$x \log x$' condition. For more information on W see [47] for the irreducible case, or [48] for the reducible case. For a simpler proof of such results, see the recent paper [52].

Split times

Our primary interest is to establish functional limit theorems for $Z(t)$ at *random* times in order derive results for urn processes. To this end, Athreya and Karlin [4] originally considered the sequence of stopping times such that $\tau_0 = 0$ and $\{\tau_n\}_{n>0}$ are the nth discontinuity point of the sample path $Z(t, \omega)$. Formally,

$$\tau_0 = 0 \quad \text{and} \quad \tau_n = \inf\{t > \tau_{n-1} \,:\, Z(t) \neq Z(t^-)\}.$$

[4] A branching process is said to be singular if each particle has exactly one child.

Since τ_n is an increasing sequence of stopping times and the process $(Z(t), \sigma(Z(s,\omega), s \leq t))$ is strong Markov, it implies that $(Z(\tau_n), \mathcal{F}_{\tau_n})$ is a discrete parameter Markov chain, which, as we shall see in section 3.2.2, may be identified with a generalised Pólya urn model.

We follow Janson's extension, which more generally let the process stop when it reaches a certain number of particles. More precisely,

Definition 3.2.8 (Split times)**.** Let $\mathbf{b} \in \mathbb{R}^N$ be a fixed vector and define for $z \geq 0$

$$\tau_{\mathbf{b}}(z) := \inf\{t \geq 0 \,:\, \mathbf{b} \cdot Z(t) \geq z\},$$

with $\inf \emptyset = +\infty$.

Theorem 3.2.9 ([41], Lemma 3.14)**.** *Assume* $\mathbf{b} \cdot \mathbf{r}_1 > 0$.

(i) *Conditioned on non-extinction, it follows that* $0 \leq \tau_{\mathbf{b}}(z) < \infty$ *and*

$$\tau_{\mathbf{b}}(z) \to \infty \text{ as } z \to \infty.$$

(ii) *Moreover* ([41], Lemma 11.1),

$$\tau_{\mathbf{b}}(z) - \frac{1}{\lambda_1} \ln z \longrightarrow -\frac{1}{\lambda_1}(\ln W + \ln(\mathbf{b} \cdot \mathbf{r}_1)) \quad a.s.$$

This theorem was originally published in [4] (see cf. [6], Theorem V.7.3), and item (ii) above is an extension due to Janson [41], Lemma 11.1.

Theorem 3.2.10 ([41], Theorem 3.15)**.** *Assume (i)-(vi) and let* $\mathbf{b} \in \mathbb{R}^N$ *with* $\mathbf{b} \cdot \mathbf{r}_1 > 0$. *Then, conditioned on non-extinction it follows that*

$$z^{-1} Z(\tau_{\mathbf{b}}(z)) \to (\mathbf{b} \cdot \mathbf{r}_1)^{-1} \mathbf{r}_1 \quad a.s. \quad \text{as } z \to \infty.$$

Remark 3.2.11. For some urn processes where a fixed number of balls are added each time, the stopping times $\tau_{\mathbf{b}}(z)$ and τ_n are related in the following way. Assume the an urn starts with n_0 balls and each time a set of m new balls are added in the urn. Then, at time n there is a deterministic $mn + n_0$ number of balls and

$\tau_n = \tau_{\mathbf{b}}(mn + n_0)$ with $\mathbf{b} = (1, \ldots, 1)$. This enables one to translate the above results for $Z(\tau_n)$, and then to a Pólya urn model as described in the next section.

3.2.2 Branching processes and Pólya urn models

This section covers a method originally devised by Athreya and Karlin ([4], [5]) to analyse limit laws of generalised Pólya's urn models by means of embedding them in a multitype continuous time Markov branching process. It relates the population size of each type to the number of balls of each colour in a Polya's urn. The fundamental idea that makes it possible is that both processes are such that the relative growth rate of each type (ball) depends essentially on its proportion of the whole population (urn). The assumption on the branching dynamics that the offspring distribution is independent between any two individuals is the urn's drawing mechanism counterpart of sampling a ball uniformly at random.

Generalised Pólya urn models

Consider an urn initially containing a non-random number $X_i(0)$ of balls of colour $i = 1, \ldots, N$. Then, a single ball is drawn uniformly at random from the urn. If that ball is of colour j, then the number of balls of colour i $(i = 1, \ldots, N)$ is updated to $X_i(0) + \xi_{ij}$, where as in (3.2.11) and (3.2.12),

$$\xi_{ij} \geq 0, \quad i \neq j, \quad \text{and}$$
$$\xi_{jj} \geq -1 \quad a.s.,$$

and this procedure is recursively reiterated.

The idea of allowing $\xi_{jj} = -1$ is that we may think of not returning the ball that is drawn. Although Athreya and Karlin [4] consider the embedding only with urn process where the drawn ball is returned, i.e., when $\xi_{jj} \geq 0, \ a.s.$, the results also hold if the drawn ball is removed. Urn process not satisfying (3.2.11) and (3.2.12),

to some extent, may also be handled by modifying the embedding method but we do not treat these cases (See [41] Remark 4.2).

Let $X(n) = (X_1(n), \ldots, X_N(n))$ denote the composition of the urn at time $n \geq 0$ and let us call $X(n)$ a generalised Pólya urn model.

3.2.3 The embedding theorem

Now, let $(Z(t))_{t\geq 0}$ be a N-type continuous-time Markov branching process as discussed in this chapter. Assume that the life times of all particles of all types are exponentially distributed with parameter 1. Assume also that particles live, die and split independent of each other, and of the past, and upon death change the composition of the population according to the distribution ξ_{ij}, $i, j = 1, \ldots, N$. Finally, let τ_n denote the sequence of discontinuity points of the sample path $Z(t, \omega)$.

Theorem 3.2.12 ([5], Theorem 1)**.** *The stochastic processes* $(X(n))_{n\geq 0}$ *and* $(Z(\tau_n))_{n\geq 0}$ *are equivalent.*

Proof. First note that both processes are discrete time and discrete space Markov chains with stationary transition probabilities. Then, it remains to show that they have the same transition probabilities.

Consider the process $Z((\tau_n))_{n\geq 0}$. At time 0, there are $Z_i(0)$ particles with independent exponential lifetime distribution with mean 1 for each $i = 1, \ldots, N$. It is a well-known fact ([63], Theorem 2.3.3) that considering a family of exponential distributions $\mathcal{E}_i$ with parameter λ_i, the minimum value $\min_i \mathcal{E}_i$ is attained by a particular member $\mathcal{E}_{i_0}$ with probability given by $\lambda_{i_0}/\sum_i \lambda_i$. Furthermore, since each particle dies at rate 1, the rate at which some particle of type i dies is $Z_i(0)$. Therefore, combining the above two properties, it follows that the ensuing split in the branching process will involve a particle of type i with probability

$$\frac{Z_i(0)}{\sum_{j=1}^{N} Z_j(0)}.$$

After the first split, the process starts afresh since the memoryless property of exponentials ([63], Theorem 2.3.1) ensures that the remaining lifetime of all other particles evolves according to its original exponential distribution with mean 1. This shows that if $Z(0) = X(0)$, then, by induction, at time τ_n the ensuing split in the branching process (draw in the urn process) will involve a particle of type (a ball of colour) i with probability

$$\frac{Z_i(\tau_n)}{\sum_{j=1}^N Z_j(\tau_n)} = \frac{X(n)}{\sum_{i=1}^N X_i(n)} \quad \text{for all } n = 1, 2, \ldots. \qquad \square$$

3.2.4 The limit theorem

Finally, by Theorems 3.2.10 and 3.2.12, it follows that

Theorem 3.2.13 ([41], Theorem 3.21)**.** *Assume (i)-(vi). Conditioned on non-extinction, it follows that*

$$n^{-1}X(n) \longrightarrow \lambda_1 \mathbf{r}_1, \quad a.s. \quad as\ n \to \infty. \tag{3.2.14}$$

In particular,

$$\frac{X(n)}{\sum_{i=1}^N X_i(n)} \longrightarrow \mathbf{r}_1, \quad a.s. \quad as\ n \to \infty. \tag{3.2.15}$$

3.3 Proofs of Results

3.3.1 Proof of Lemma 3.1.3

Proof. The proof of this lemma is a consequence of the method of Arthreya and Karlin (1968) [5] to study an urn process by embedding it into a continuous time multi-type Markov branching process.

Note that the process $(U(n))_{n\geq 0}$ is a multi-type Markov branching process taking values in $\mathbb{Q}_+^N$ with deterministic offspring distributions given by the columns of matrix A.

Let us now write a_{ij} in its representation of a irreducible fraction of two integer numbers $a_{ij} = p_{ij}/q_{ij}$ and define $q := \prod_{i,j=1}^{N} q_{ij}$. We define $\hat{A} = qA$ in order to study the integer-valued process $\hat{U}(n) = \hat{A}X(n)$. First, note that

$$\mathbb{P}[\hat{U}(n+1) = \hat{U}(n) + q(a_{1k}, \ldots, a_{Nk})|\hat{U}(n) = \hat{u}(n)] = \frac{\hat{u}_k(n)}{\sum_{j=1}^{N} \hat{u}_j(n)},$$

where $\hat{u}(n) = qu(n)$.

Now, if we denote by $\hat{\lambda}_1$ and $\hat{\mathbf{r}}_1$ the Perron-Frobenius eigenvalue and corresponding eigenvector of matrix $\hat{A}$, it follows from [41], Theorem 3.21 (see 3.2.13), that

$$n^{-1}\hat{U}(n) \longrightarrow \hat{\lambda}_1 \hat{\mathbf{r}}_1 \quad a.s.$$

Finally, since $\hat{\lambda}_1 = q\lambda_1$, $\hat{\mathbf{r}}_1 = \mathbf{r}_1$ and by construction $\hat{U}(n) = qU(n)$, one can easily conclude that

$$n^{-1}U(n) \longrightarrow \lambda_1 \mathbf{r}_1 \quad a.s. \qquad \square$$

3.3.2 Proof of Theorem 3.1.1

Proof. Lemma 3.1.3 enables us to directly find the limiting behaviour of $X(n)$ when the matrix A is non-singular. In addition, it also implies that for any given matrix A the $X(n)$ mean drift vector field is asymptotically constant since

$$\begin{aligned} \mu(n) &:= \mathbb{E}[X(n+1) - X(n) \mid X(n)] \\ &= \sum_{i=1}^{N} \mathbb{P}[i_n = i \mid X(n)] e_i = \frac{U(n)}{\sum_{j=1}^{N} U_j(n)}, \end{aligned} \tag{3.3.1}$$

and,

$$\lim_{n\to\infty} \mu(n) = \lim_{n\to\infty} \frac{n^{-1}U(n)}{n^{-1}\sum_{j=1}^{N} U_j(n)} = \frac{1}{\sum_{j=1}^{N} \lambda_1 r_{1j}} \lambda_1 r_1 = r_1 \quad a.s.$$

Now, note that $X(n)$ has submartingale components, which by construction enables one to write the Doob decomposition as follows

$$X_i(n) = M_i(n) + \sum_{k=0}^{n-1} \mu_i(k), \quad i = 1, \ldots, N, \tag{3.3.2}$$

where the first term denotes the martingale factor and the second one denotes the predictable increasing sequence. So, carrying on the calculations

$$\mathbb{E}[M_i^2(n) - M_i^2(n-1)|\mathcal{F}_{n-1}] = \mathbb{E}[(M_i(n) - M_i(n-1))^2|\mathcal{F}_{n-1}]$$
$$= \mathbb{E}[(X_i(n) - X_i(n-1))^2|\mathcal{F}_{n-1}] - \mu_i^2(n-1) \leq 1. \tag{3.3.3}$$

Then, taking the expectation of both sides of equation (3.3.3) with $n = k$ and summing over $k = 1, \ldots, n-1$, it follows that $\mathbb{E}[M_i^2(n)] \leq n$. Now, (see [61], Example 2.3.17) by the Doob's maximal inequality it follows that

$$\mathbb{P}[\max_{0\leq k\leq n} |M_i(k)| \geq \lambda] \leq \lambda^{-2}\mathbb{E}[M_i^2(n)] \leq \lambda^{-2}n.$$

For $\epsilon > 0$, let $\lambda(n) = n^{1/2}(\log n)^{(1/2)+\epsilon}$ and take $n = 2^m$, $m \geq 0$, then

$$\mathbb{P}[\max_{0\leq k\leq 2^m} |M_i(k)| \geq \lambda(2^m)] \leq (m \log 2)^{-1-2\epsilon}.$$

By Borel-Cantelli lemma, it implies that, $a.s$,

$$\max_{0\leq k\leq 2^m} |M_i(k)| \leq \lambda(2^m) \quad \text{for all but finitely many } m.$$

For any natural number n there is some m_n such that $2^{m_n} \leq n \leq 2^{m_n+1}$, where $m_n \to \infty$ as $n \to \infty$. Then, $a.s.$, for all but finitely many n,

$$\max_{0\leq k\leq n} |M_i(k)| \leq \max_{0\leq k\leq 2^{m_n+1}} |M_i(k)| \leq \lambda(2n) \leq 2\lambda(n).$$

Then, for every $\epsilon > 0$, $a.s.$, for all but finitely many n

$$\max_{0\leq k\leq n} |M_i(k)| \leq 2n^{1/2}(\log n)^{(1/2)+\epsilon},$$

which in particular, implies that

$$n^{-1}M_i(n) \longrightarrow 0 \quad a.s. \quad \text{for all } i = 1, ..., N.$$

Finally, from the decomposition (3.3.2) we conclude that

$$n^{-1}(X_i(n) - \sum_{k=0}^{n-1} \mu_i(k)) \longrightarrow 0 \quad a.s.$$

Since for each $i = 1, ..., N$ we have that $\mu_i(n) \to (\mathbf{r}_1)_i$ as $n \to \infty$, its Cesàro sum $n^{-1} \sum_{k=0}^{n-1} \mu_i(k)$ also converge to the same limit as $n \to \infty$. This finally implies

$$n^{-1} X(n) \longrightarrow \mathbf{r}_1 \quad a.s. \qquad \square$$

3.4 Appendix: Perron-Frobenius theory for non-negative matrices

In this section we present a brief summary of the Perron-Frobenius theory highlighting the subtleties if one wants to extend the theory for positive to non-negative matrices. Note that a simple non-negative matrix as follows

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is such that there is no positive or largest eigenvalue, no simple root of the characteristic polynomial, and also its associated eigenvector does not have strictly positive entries. Therefore, one needs to impose further conditions in order to recast the Perron-Frobenius theorem.

Let $A = \{a_{ij}\}_{i,j=1}^N$ a $N \times N$ square matrix. We write $A > 0$ (or ≥ 0) if $a_{ij} > 0$ (or ≥ 0) for all $i, j = 1, \ldots, N$.. Also, we write the matrix $A^k = (a_{ij}^{(k)})_{i,j=1}^N$.

Definition 3.4.1. A non-negative square matrix $A = \{a_{ij}\}_{i,j=1}^N$ is said to be primitive if there exists a positive integer k such that $A^k > 0$.

Theorem 3.4.2. *[[77] Theorem 1.1]*

Let A be a primitive matrix. Then, there exists an eigenvalue λ_1 such that

(i) λ_1 *is a positive real number;*

(ii) λ_1 *has strictly positive right and left eigenvectors;*

(iii) $\lambda_1 > |\lambda'|$ *for any other eigenvalue* $\lambda' \neq \lambda_1$*;*

(iv) *the eigenvectors of* λ_1 *are unique up to constant multiples;*

(v) λ_1 *is a simple root of the characteristic polynomial.*

Definition 3.4.3. For an ordered pair of indices (i, j), we write $i \mapsto j$ if there is a sequence of indices $(i, i_1, \ldots, i_{k-1}, j)$ such that

$$a_{ii_1} a_{i_1 i_2} \ldots a_{i_{k-2} i_{k-1}} a_{i_{k-1} j} > 0$$

Note that $i \mapsto j$ is equivalent to the existence of an integer $k \geq 1$ such that $a_{ij}^{(k)} > 0$.

Definition 3.4.4. An index $i \in \{1, \ldots, N\}$ is said to be essential if

(a) $i \mapsto j$ for some $j \neq i$, and

(b) If $i \mapsto j$, then $j \mapsto i$ (in which case we write $i \leftrightarrow j$ and say that i and j communicate)

An index $i \in \{1, \ldots, N\}$ is said to be inessential if it is not essential.

The set of essential and inessential indices can be further subdivided into *self-communicating* classes in such a way that if $i \leftrightarrow j$, then i and j are in the same class. Finally, there may be inessential indices which communicate with no other index, in which case these are defined as forming an inessential class by themselves.

Remark 3.4.5. Any non-negative matrix may be put into *canonical form* by simultaneous permutations of rows and columns of the matrix in such a way that
(i) Its powers are similarly transformed.
(ii) Its set of eigenvalues are unchanged.

A canonical form is always a block triangular matrix as follows

$$\left[\begin{array}{ccccc|c} C_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & C_2 & 0 & \dots & 0 & \\ 0 & 0 & C_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \\ 0 & 0 & 0 & \dots & C_k & 0 \\ \hline & & R & & & Q \end{array}\right]$$

where C_i, $i = 1, \dots, k$ are submatrices corresponding to the self-communicating essential classes and Q to all self-communicating inessential classes, with $R \neq 0$ in general.

It is not hard to check (by the multiplicative property of the determinant of block triangular matrices) that the eigenvalues of the above matrix are the combined eigenvalues of $C_1, \dots, C_k$ and Q. So, saying that an eigenvalue "belongs" to a class i means that it is an eigenvalue of C_i.

Definition 3.4.6. If $i \mapsto i$, the period of i is defined by

$$d(i) := gcd\{k \,:\, a_{ii}^{(k)} > 0\}.$$

It is not hard to verify that if $i \leftrightarrow j$ then $d(i) = d(j)$. This allows one to define the period of a self-communicating class as the period of any of its elements. An index i such that $i \mapsto i$ is said to be aperiodic if $d(i) = 1$ and periodic with period $d(i) > 1$ otherwise.

Definition 3.4.7. An $N \times N$ matrix A is said to be irreducible if it contains a single self-communicating class. Or equivalently, if for every pair of indices (i, j) there exists a positive integer $k \equiv k(i, j)$ such that $a_{ij}^{(k)} > 0$.

Theorem 3.4.8 ([77] Theorem 1.4)**.** *A matrix A is primitive if and only if it is irreducible and aperiodic. Moreover the powers of an irreducible periodic matrix may be studied in terms of powers of primitive matrices.*

Finally,

Theorem 3.4.9 ([77] Theorem 1.5)**.** *Let A be an $N \times N$ non-negative irreducible matrix. Then there exists an eigenvalue λ_1 such that all statements $(i) - (v)$ of Theorem 3.4.2 hold, with exception of (iii) which is replaced by the weaker statement* $(iii)'$ $\lambda_1 \geq |\lambda'|$ *for any eigenvalue λ' of A.*

In this case, it still holds that A has a positive eigenvalue λ_1 with algebraic and geometric multiplicity one such that

$$\lambda_1 > Re\lambda' \quad \text{for every other eigenvalue } \lambda' \text{ of } A,$$

and whose left and right eigenvectors have strictly positive entries.

3.4.1 Further extensions

The Perron-Frobenius theory may further be extended to the so-called ML-matrices.

Definition 3.4.10. A square real matrix $A = (a_{ij})_{i,j=1}^{N}$ is said to be a ML-matrix if $a_{ij} \geq 0$ for $i \neq j$.

An ML-matrix A may always be related to a non-negative matrix $T(\alpha)$ through the relation

$$T = A + \alpha I \quad \text{for large enough } \alpha.$$

For the Perron-Frobenius theorem for irreducible ML-matrices see Seneta (2006), [77], Theorem 2.6.

Chapter 4

Localization for an exponentially reinforced growth model

Abstract

This paper [1] concerns the long term behaviour of a growth model describing a random sequential allocation of particles on a finite cycle graph. The model can be regarded as a reinforced urn model with graph-based interactions. It is motivated by cooperative sequential adsorption, where adsorption rates at a site depend on the configuration of existing particles in the neighbourhood of that site. Our main result is that, with probability one, the growth process will eventually localise either at a single site, or at a pair of neighbouring sites.

4.1 Introduction

This paper concerns a probabilistic model describing a sequential allocation of particles on a finite cycle graph. The model is motivated by cooperative sequential adsorption (CSA) (see [29], [28] and references therein). CSA models are widely applied in physical chemistry for modelling adsorption processes on a material surface

[1] Joint work with M. Menshikov, V. Shcherbakov and M. Vachkovskaia

onto which particles are deposited at random. The main peculiarity of adsorption processes is that deposited particles change adsorption properties of the material. This motivates the growth rates defined in equation (4.2.1). The growth rates model a particular situation where the subsequent particles are more likely to be adsorbed around previously deposited particles.

There is typically a hard-core constraint associated with CSA. That is, the adsorption (growth) rate is zero at any location with more than a certain number of particles. The asymptotic shape of the spatial configuration of deposited particles is of primary interest in such models. Many probabilistic models of spatial growth by monolayer deposition, diffusion and aggregation dynamics present this characteristic. For instance, the Eden model [26], diffusion-limited aggregation process [86], first-passage percolation models [72] and contact interaction processes [76].

In contrast, in our model (defined in Section 4.2) we allow any number of particles to be deposited at each site. This is motivated by growing interfaces (Figure 4.1) associated with multilayer adsorption processes (see [13], [44] and [70]). Even though the random nature of these processes is usually emphasized in the physical literature, there is a limited number of rigorous formulations and published results in this field (most of them in [69] and [71]). Our model is closely related to a variant of random deposition models, but as we do not apply any of the techniques from this field, we refer the reader to the survey on surface growth [12].

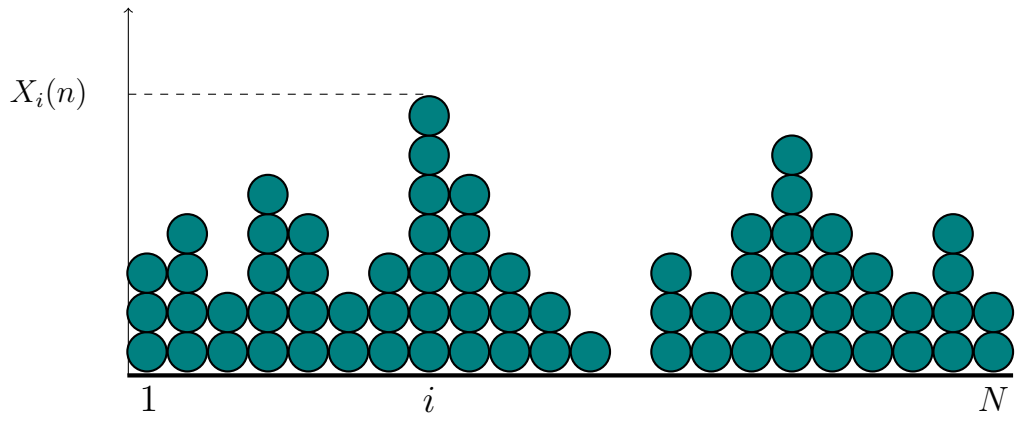


Figure 4.1: Multilayer adsorption/random deposition model

Our model can be naturally interpreted in terms of interacting urn models. In the case of no interaction, in which the growth rate at site i is given by $\Gamma(x_i)$, where x_i is the number of existing particles at site i and $\Gamma : \mathbb{Z}_+ \to (0, \infty)$ is a given function (called the reinforcement rule [20] or feedback function [65]), our model coincides with a generalised Pólya urn (GPU) model with a particular reinforcement rule Γ. Each site (with no underlying graph structure) corresponds to a different colour of ball. The growth rule corresponds to choosing an existing ball of colour i, with probability proportional to $\Gamma(x_i)$, and adding a new ball of that colour. The case $\Gamma(x) = x$ is the classical Pólya urn.

The so called Rubin's exponential embedding (first appearing in [22]) classifies the two possible limiting behaviours in the above class of GPU models. Firstly, there almost surely exists a site i that gets all but finitely many particles. Secondly, the number of particles at every site grows almost surely to infinity. For a comprehensive survey on urn models and their applications, see [68] and references therein.

In contrast, we consider growth rules with graph-based interactions (as in [79]) where the underlying graph is a cycle with N sites. In our growth model the rate of growth at site i is given by a site-dependent reinforcement rule $\Gamma_i = \exp(\lambda_i u_i)$, where $\lambda_i > 0$ and u_i is the number of existing particles in a neighbourhood of site i. This allows one to take into account the case where different sites might possibly have different reinforcement schemes (Figure 4.2). In other words, the case where each site has its own intrinsic 'capacity' parameter, which is what would be expected in many real-life situations. Although the model can easily be defined for a general graph, the results will heavily depend on its topological properties. In this paper we only address the case of a cycle graph. See [16] and [34] for results on general graphs but different growth rules.

The model with $\Gamma_i = \exp(\lambda u_i)$, i.e. $\lambda_i \equiv \lambda \in \mathbb{R}$, was first considered in [79], and an analogue of Theorem 4.2.3 (Theorem 3 in [79]) was proved for this particular case of site-independent parameter λ.

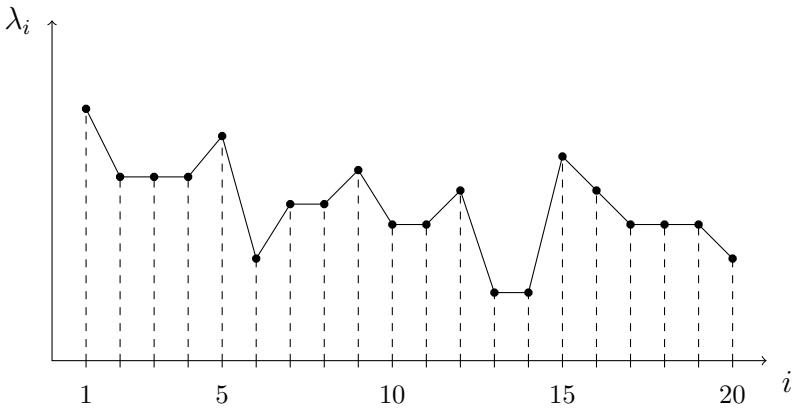


Figure 4.2: Pictorial representation of a set $\Lambda = (\lambda_i)_{i=1}^N$ for $N = 20$

The main result of the present paper classifies, in terms of the set of parameters $\Lambda = (\lambda_i)_{i=1}^N$, the two possible behaviours of the model. The first behaviour is localization of growth at a single site. This means that from a random moment of time onwards, all subsequent particles are allocated at a particular site. The second is localization of growth at a pair of neighbouring sites with equal λ parameter. Similarly as in the first case, this means that from a random moment of time onwards, all subsequent particles are allocated at a particular pair of neighbouring sites. In particular, if $\lambda_i \neq \lambda_{i+1}$ for all i, then, with probability one, the growth will eventually localise at a single site. On the other hand, if $\lambda_i \equiv \lambda$, then, with probability one, the growth will eventually localise at a pair of neighbouring sites. In the general case of a fixed and arbitrary parameter set Λ, only the above two types of limiting behaviour are possible. Theorem 4.2.3 below provides a complete characterization of the parameter set Λ and associated subsets where only one of the regimes, or both, may happen.

The paper is organised as follows. In Section 4.2, we formally define the model, fix some terminology and state Theorem 4.2.3 which is our main result. The proof of the theorem appears in Section 4.6 and relies essentially on Lemmas 4.3.1-4.3.13 stated in Section 4.3 and proved in Section 4.5. Section 4.4 contains results concerning sums of random geometric progressions, which are of interest in their own right. These results combined with stochastic domination techniques are constantly used in the proofs of Lemmas 4.3.7-4.3.13.

4.2 The model and main result

Consider a cycle graph with $N \geq 4$ vertices (sites) enumerated by the first N natural numbers such that $1 \sim 2 \sim \ldots \sim N-1 \sim N \sim 1$, where $i \sim j$ indicates that sites i and j are incident. Let $\mathbb{Z}_+$ be the set of non-negative integers and $\Lambda = \{\lambda_1, ..., \lambda_N\}$ be an arbitrary set of positive real numbers. Given $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{Z}_+^N$, define the growth rates as

$$\Gamma_i(\mathbf{x}) = e^{\lambda_i(x_i + \sum_{j\sim i} x_j)}, \quad i = 1, \ldots, N. \tag{4.2.1}$$

Consider a discrete-time Markov chain $X(n) = (X_1(n), \ldots, X_N(n)) \in \mathbb{Z}_+^N$ with the following transition probabilities

$$\mathsf{P}(X_i(n+1) = X_i(n) + 1 | X(n) = \mathbf{x}) = \frac{\Gamma_i(\mathbf{x})}{\sum_{k=1}^N \Gamma_k(\mathbf{x})},\ i = 1, \ldots, N,\ \mathbf{x} \in \mathbb{Z}_+^N.$$

The Markov chain describes the evolution of the number of particles sequentially allocated at each site of the graph. Given the configuration of particles $X(n) = \mathbf{x} \in \mathbb{Z}_+^N$ at time n, the next incoming particle is placed at site i with probability proportional to $\Gamma_i(\mathbf{x})$.

Definition 4.2.1. For $i \in \{1, \ldots, N\}$ (*modulo* N)

1. a site $\{i\}$ is a local minimum, if $\lambda_i < \min(\lambda_{i-1}, \lambda_{i+1})$;

2. a pair of sites $\{i, i{+}1\}$ is a local minimum of size 2, if $\lambda_i = \lambda_{i+1} < \min(\lambda_{i-1}, \lambda_{i+2})$;

3. a site $\{i\}$ is a local maximum, if $\lambda_i > \max(\lambda_{i-1}, \lambda_{i+1})$;

4. a pair of sites $\{i, i+1\}$ is a saddle point, if

$$\min(\lambda_{i-1}, \lambda_{i+2}) < \lambda_i = \lambda_{i+1} < \max(\lambda_{i-1}, \lambda_{i+2});$$

5. a site $\{i\}$ is a growth point, if either $\lambda_{i-1} < \lambda_i < \lambda_{i+1}$, or $\lambda_{i-1} > \lambda_i > \lambda_{i+1}$.

Definition 4.2.2. Let $\{i, i+1\}$ be a local minimum of size two. We say that it is a local minimum of size 2 and

1) type 1, if $\lambda_i = \lambda_{i+1} > \frac{\lambda_{i-1}\lambda_{i+2}}{\lambda_{i-1}+\lambda_{i+2}}$,

2) type 2, if $\lambda_i = \lambda_{i+1} \leq \frac{\lambda_{i-1}\lambda_{i+2}}{\lambda_{i-1}+\lambda_{i+2}}$.

The following theorem is the main result of the paper.

Theorem 4.2.3. *For every $X(0) = \mathbf{x} \in \mathbb{Z}_+^N$ and*

i) for every local maximum $\{k\}$, with positive probability,

$$\lim_{n\to\infty} X_i(n) = \infty \text{ if and only if } i = k;$$

ii) for every pair $\{k, k+1\}$ where $\lambda_k = \lambda_{k+1} =: \lambda$, but not a local minimum of size 2 and type 2, with positive probability,

$$\lim_{n\to\infty} X_i(n) = \infty, \text{ if and only if } i \in \{k, k+1\}, \text{ and}$$
$$\lim_{n\to\infty} \frac{X_{k+1}(n)}{X_k(n)} = e^{\lambda R},$$

where $R = \lim_{n\to\infty}[X_{k+2}(n) - X_{k-1}(n)] \in \mathbb{Z}$.

No other limiting behaviour is possible. That is, with probability 1, exactly one of the above events occurs in a random location $\{k\}$ or $\{k, k+1\}$ as described in i) and ii), respectively.

4.3 Lemmas

We start with notations that will be used throughout the proofs. Given $i = 1, ..., N$, define the following events

$$A_n^i := \{\text{at time } n \text{ a particle is placed at site } i\},\ n \in \mathbb{Z}_+,$$
$$A_n^{i,i+1} := A_n^i \cup A_n^{i+1},\ n \in \mathbb{Z}_+.$$

Define also the following events

$$A^i_{[n_1,n_2]} := \bigcap_{n=n_1}^{n_2} A^i_n,$$
$$A^{i,i+1}_{[n_1,n_2]} := \bigcap_{n=n_1}^{n_2} A^{i,i+1}_n,$$

indicating that from time n_1 to n_2 all particles are placed at site i, and at sites i or $i+1$, respectively. Further, events $A^i_{[n,\infty)}$ and $A^{i,i+1}_{[n,\infty)}$ denote the corresponding limiting cases as n_2 goes to infinity.

Let $\mathbf{e}_i \in \mathbb{Z}^N_+$ be a vector, whose i-th coordinate is 1, and all other coordinates are zero. Given $\mathbf{x} \in \mathbb{Z}^N_+$, define the following probability measure $\mathsf{P}_\mathbf{x}(\cdot) = \mathsf{P}(\,\cdot\,|\,X(0) = \mathbf{x})$.

rem In lemmas and proofs below we denote by ϵ and ε, possibly with subscripts, various positive constants whose values depend only on N and $(\lambda_i)_{i=1}^N$ and may vary from line to line. Also, the results are stated only for the essentially different cases, and whenever there are trivially symmetric situations (e.g. $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$ and $\lambda_{k-1} > \lambda_k > \lambda_{k+1}$), we state and prove only one of them in order to avoid unnecessary repetition. rem

Lemma 4.3.1. *Suppose that $\{k\}$ is a local maximum, and $\mathbf{x} \in \mathbb{Z}^N_+$ is such that $\Gamma_k(\mathbf{x}) = \max_i \Gamma_i(\mathbf{x})$. Then, with positive probability, all subsequent particles are allocated at k, i.e. $\mathsf{P}_\mathbf{x}\left(A^k_{[1,\infty)}\right) \geq \epsilon$ for some $\epsilon > 0$.*

Lemma 4.3.1 describes the only case where localisation of growth at a single site can occur, namely, at a local maximum.

Lemma 4.3.2. *Suppose that $\{k\}$ is a growth point, and $\mathbf{x} \in \mathbb{Z}^N_+$ is such that $\Gamma_k(\mathbf{x}) = \max_i \Gamma_i(\mathbf{x})$. If $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$, then there exist $n = n(\mathbf{x}, \Lambda) \in \mathbb{Z}_+$ and $\epsilon > 0$, such that $\mathsf{P}_\mathbf{x}\left(A^k_{[1,n]}\right) \geq \epsilon$ and $\Gamma_{k+1}(\mathbf{x} + n\mathbf{e}_k) = \max_i \Gamma_i(\mathbf{x} + n\mathbf{e}_k)$.*

Lemma 4.3.3. *Suppose that $\{k\}$ is a local minimum, and $\mathbf{x} \in \mathbb{Z}^N_+$ is such that $\Gamma_k(\mathbf{x}) = \max_i \Gamma_i(\mathbf{x})$. Then there exist $n = n(\mathbf{x}, \Lambda) \in \mathbb{Z}_+$ and $\epsilon > 0$, such that $\mathsf{P}_\mathbf{x}\left(A^k_{[1,n]}\right) \geq \epsilon$ and $\max(\Gamma_{k-1}(\mathbf{x} + n\mathbf{e}_k), \Gamma_{k+1}(\mathbf{x} + n\mathbf{e}_k)) = \max_i \Gamma_i(\mathbf{x} + n\mathbf{e}_k)$.*

Lemmas 4.3.2-4.3.3 describe the following effect. If the maximal rate is attained at a site which is either a growth point or a local minimum, then, with positive probability, allocating $n = n(\mathbf{x}, \Lambda)$ particles at that site results in relocation of the maximal rate to a nearest neighbour with larger parameter λ. It should be noted that the number of particles required for relocation (the relocation time) is deterministic and depends only on the starting configuration $\mathbf{x}$ and parameter set Λ.

Lemma 4.3.4. *Suppose that* $\Gamma_k(\mathbf{x}) = \max_i \Gamma_i(\mathbf{x})$.

1) $\lambda_{k-1} < \lambda_k = \lambda_{k+1} \geq \lambda_{k+2}$*; or*

2) $\lambda_{k-1} = \lambda_k = \lambda_{k+1} \geq \lambda_{k+2}$, *and* $\Gamma_{k+1}(\mathbf{x}) \geq \Gamma_{k-1}(\mathbf{x})$,

then, with positive probability, all subsequent particles are allocated at sites $\{k, k+1\}$, *i.e.* $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) \geq \epsilon$ *for some* $\epsilon > 0$.

Lemma 4.3.4 describes a case of the second possible limiting behaviour of the model, i.e. localisation of growth at a pair of neighbouring sites.

Definition 4.3.5. Define the following stopping times

$$\tau_k = \inf(n : X_k(n) = X_k(0) + 1),$$
$$w_k^+ = \min(\tau_i : \ i \neq k, k+1, k+2),$$

where the usual convention is that

$$\inf(\emptyset) = \infty \text{ and } \min(a, \infty) = a, \text{ for } a \in \mathbb{R}_+ \cup \{\infty\}.$$

The above stopping times and the quantities r, z_1 and z_2 below will appear throughout Lemmas 4.3.4-4.3.13 and their proofs.

Definition 4.3.6. Given $\mathbf{x} \in \mathbb{Z}_+^N$ define

$$r := r(\mathbf{x}) = x_{k+2} - x_{k-1}. \tag{4.3.1}$$

In addition, if a pair of sites $\{k, k+1\}$ is such that $\lambda_k = \lambda_{k+1} =: \lambda$ and

$$\lambda_{k-1} > \lambda, \text{ define } z_1 = \frac{1}{\lambda} \log \left(\frac{\lambda_{k-1} - \lambda}{\lambda} \right), \tag{4.3.2}$$

$$\lambda_{k+2} > \lambda, \text{ define } z_2 = \frac{1}{\lambda} \log \left(\frac{\lambda}{\lambda_{k+2} - \lambda} \right). \tag{4.3.3}$$

Before stating Lemma 4.3.7, let us denote by B_k the event in which a particle arrives in finite time at $k+2$ before anywhere outside $\{k, k+1, k+2\}$. That is to say,

$$B_k := \{\tau_{k+2} < w_k^+\}. \tag{4.3.4}$$

Lemma 4.3.7. *Suppose that a pair of sites $\{k, k+1\}$ is a saddle point with $\lambda_{k-1} < \lambda_k = \lambda_{k+1} =: \lambda < \lambda_{k+2}$, and $\mathbf{x} \in \mathbb{Z}_+^N$ is such that*

$$\max(\Gamma_k(\mathbf{x}), \Gamma_{k+1}(\mathbf{x})) = \max_i \Gamma_i(\mathbf{x}). \tag{4.3.5}$$

1) Then there exists $\epsilon > 0$ such that

$$\mathsf{P}_{\mathbf{x}} \left(A_{[1,\infty)}^{k,k+1} \bigcup B_k \right) = \mathsf{P}_{\mathbf{x}} \left(A_{[1,\infty)}^{k,k+1} \right) + \mathsf{P}_{\mathbf{x}} \left(B_k \right) \geq \epsilon.$$

2) If $r < z_2$, then, with positive probability, all subsequent particles are allocated at sites $\{k, k+1\}$, i.e. $\mathsf{P}_{\mathbf{x}} \left(A_{[1,\infty)}^{k,k+1} \right) \geq \varepsilon$ for some $\varepsilon > 0$.

3) If $r \geq z_2$, then $\mathsf{P}_{\mathbf{x}} \left(A_{[1,\infty)}^{k,k+1} \right) = 0$, and, hence, $\mathsf{P}_{\mathbf{x}} \left(B_k \right) \geq \epsilon$.

4) If $r > z_2$ is strict, then, with positive probability, the maximal rate relocates as follows. There exists $\varepsilon > 0$ such that

$$\mathsf{P}_{\mathbf{x}} \left(B_k, \max_{i=k+2,k+3} \Gamma_i(X(\tau_{k+2})) = \max_i \Gamma_i(X(\tau_{k+2})) \right) \geq \varepsilon, \tag{4.3.6}$$

where $\max_i \Gamma_i(X(\tau_{k+2}))$ may be attained at $k+3$ only if $\lambda_{k+3} > \lambda$.

Part 4) of Lemma 4.3.7 is similar to Lemmas 4.3.2-4.3.3 in that it also describes relocation of the maximal rate to a site with larger parameter λ. The main difference is that in Lemma 4.3.7 the relocation time is random. This is in contrast to Lemmas 4.3.2-4.3.3, where the relocation time is deterministic.

The proposition and definition below are intended to clarify some assumptions and simplify some notations in Lemmas 4.3.10-4.3.13 below.

Proposition 4.3.8. *Let $\{k, k+1\}$ be a local minimum of size 2 with $\lambda = \lambda_k = \lambda_{k+1}$, and let $r = r(\mathbf{x})$, z_1 and z_2 be quantities as in Definition (4.3.6). Then, $z_1 < z_2$ if and only if local minimum $\{k, k+1\}$ is of type 1, in which case there might exist $\mathbf{x}$ such that $z_1 < r < z_2$. Otherwise, if a local minimum $\{k, k+1\}$ is of type 2, then $z_2 \leq z_1$, in which case $r \geq z_2$ or $r \leq z_1$ for all $\mathbf{x}$.*

Definition 4.3.9. Recall that $\tau_k := \inf(n : X_k(n) = X_k(0) + 1)$ and let us further define the following stopping times σ_k and w_k and events D_k, D'_k and D''_k.

$$\sigma_k = \min(\tau_{k-1}, \tau_{k+2}), \quad \text{and} \quad w_k = \min(\tau_i \,:\, i \neq k \pm 1, k, k+2),$$

$$D_k = \{\sigma_k < w_k\}, \quad D'_k = \{\tau_{k-1} < \min(\tau_{k+2}, w_k)\}, \quad D''_k = \{\tau_{k+2} < \min(\tau_{k-1}, w_k)\}.$$

Note that $D'_k \cap D''_k = \emptyset$, $D_k = D'_k \cup D''_k$ and $A^{k,k+1}_{[1,\infty)} \cap D_k = \emptyset$.

Lemma 4.3.10. *Suppose that $\{k, k+1\}$ is a local minimum of size 2, and $\mathbf{x} \in \mathbb{Z}^N_+$ is such that $\max(\Gamma_k(\mathbf{x}), \Gamma_{k+1}(\mathbf{x})) = \max_i \Gamma_i(\mathbf{x})$.*

1) There exists $\epsilon > 0$ such that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)} \bigcup D_k\right) = \mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) + \mathsf{P}_{\mathbf{x}}\left(D_k\right) \geq \epsilon.$$

2) If $z_1 < r < z_2$ (only possible if $\{k, k+1\}$ is of type 1), then, with positive probability, all subsequent particles are allocated at sites $\{k, k+1\}$, i.e. $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) > \varepsilon$ for some $\varepsilon > 0$.

3) If $r \leq z_1$ or $r \geq z_2$ (always the case if $\{k, k+1\}$ is of type 2), then $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) = 0$ and, hence, $\mathsf{P}_{\mathbf{x}}\left(D_k\right) \geq \epsilon$.

Lemma 4.3.10 is analogous to Parts 1)-3) of Lemma 4.3.7 for the case of a local minimum of size 2. An analogue of Part 4) of Lemma 4.3.7 in the same situation is provided by Lemma 4.3.11 below.

Lemma 4.3.11. *Suppose that local minimum* $\{k, k+1\}$ *is of size* 2 *with* $\lambda_k = \lambda_{k+1} := \lambda$, *and* $\mathbf{x} \in \mathbb{Z}_+^N$ *is such that* $\max(\Gamma_k(\mathbf{x}), \Gamma_{k+1}(\mathbf{x})) = \max_i \Gamma_i(\mathbf{x})$.

1) If $\{k, k+1\}$ *is of type* 1 *and* $r < z_1$, *or* $\{k, k+1\}$ *is of type* 2 *and* $r < z_2$ *then*

$$\mathsf{P}_{\mathbf{x}}\left(D'_k, \max_{i=k-2,k-1} \Gamma_i(X(\tau_{k-1})) = \max_{i=1,\dots,N} \Gamma_i(X(\tau_{k-1}))\right) \geq \varepsilon > 0.$$

where $\max_i \Gamma_i(X(\tau_{k-1}))$ *may be attained at* $k-2$ *only if* $\lambda_{k-2} > \lambda$.

2) If $\{k, k+1\}$ *is of type* 1 *and* $r > z_2$, *or* $\{k, k+1\}$ *is of type* 2 *and* $r > z_1$ *then*

$$\mathsf{P}_{\mathbf{x}}\left(D''_k, \max_{i=k+2,k+3} \Gamma_i(X(\tau_{k+2})) = \max_{i=1,\dots,N} \Gamma_i(X(\tau_{k+2}))\right) \geq \varepsilon > 0,$$

where $\max_i \Gamma_i(X(\tau_{k+2}))$ *may be attained at* $k+3$ *only if* $\lambda_{k+3} > \lambda$.

3) If $\{k, k+1\}$ *is of type* 2 *and* $z_2 < r < z_1$, *then*

$$\begin{aligned}\mathsf{P}_{\mathbf{x}}&\left(D'_k, \max_{i=k-2,k-1} \Gamma_i(X(\tau_{k-1})) = \max_{i=1,\dots,N} \Gamma_i(X(\tau_{k-1}))\right) \\ &+ \mathsf{P}_{\mathbf{x}}\left(D''_k, \max_{i=k+2,k+3} \Gamma_i(X(\tau_{k+2})) = \max_{i=1,\dots,N} \Gamma_i(X(\tau_{k+2}))\right) \geq \varepsilon > 0,\end{aligned}$$

where $\max \Gamma_i$ *follows the corresponding prescriptions as above.*

Remark 4.3.12. The next lemma concerns the borderline cases in between having a local minimum $\{k, k+1\}$ of size 2 and type 1 or a saddle point. For example, in notations of Lemma 4.3.11 these cases are formally obtained by setting either $\lambda_{k-1} = \lambda$ (where $-\infty = z_1 < z_2$), or $\lambda_{k+2} = \lambda$ (where $z_1 < z_2 = \infty$). As both cases can be addressed in similar ways, the lemma below deals only with the case $\lambda_{k-1} = \lambda$.

Lemma 4.3.13. *Suppose that sites* $\{k-1, k, k+1, k+2\}$ *are such that*

$$\lambda_{k-1} = \lambda_k = \lambda_{k+1} =: \lambda < \lambda_{k+2},$$

$\mathbf{x} \in \mathbb{Z}_+^N$ *is such that* $\max(\Gamma_k(\mathbf{x}), \Gamma_{k+1}(\mathbf{x})) = \max_i \Gamma_i(\mathbf{x})$ *and, additionally,* $\Gamma_{k-1}(\mathbf{x}) \leq \Gamma_{k+1}(\mathbf{x})$.

1) There exists $\epsilon > 0$ *such that*

$$\mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty)}^{k,k+1} \bigcup D_k\right) = \mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty)}^{k,k+1}\right) + \mathsf{P}_{\mathbf{x}}\left(D_k\right) \geq \epsilon.$$

2) If $r < z_2$*, then, with positive probability all subsequent particles are allocated at sites* $\{k, k+1\}$*, i.e.* $\mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty)}^{k,k+1}\right) \geq \epsilon$ *for some* $\varepsilon > 0$.

3) If $r \geq z_2$*, then* $\mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty)}^{k,k+1}\right) = 0$ *and, hence,* $\mathsf{P}_{\mathbf{x}}\left(D_k\right) \geq \epsilon$.

4) If $r > z_2$*, then there exists* $\varepsilon > 0$ *such that*

$$\mathsf{P}_{\mathbf{x}}\left(B_k, \max_{i=k+2,k+3} \Gamma_i(X(\tau_{k+2})) = \max_i \Gamma_i(X(\tau_{k+2}))\right) \geq \varepsilon,$$

where $\max_i \Gamma_i(X(\tau_{k+2}))$ *may be attained at* $k+3$ *only if* $\lambda_{k+3} > \lambda$.

The following corollary concerns those cases covered by Parts 3) of Lemmas 4.3.7, 4.3.10 and 4.3.13, where the configuration parameter r is equal to one of the model parameters z_1 and z_2. In what follows we call them critical cases.

Corollary 4.3.14. *For the critical cases, relocation of the maximal rate to a site with larger parameter* λ *also occurs, with positive probability, in finite time.*

Remark 4.3.15. Let us remark the following.

1) It is important to emphasize that in all the above cases where the maximal rate $\max_i \Gamma_i(\mathbf{x})$ eventually relocates with positive probability, it always relocates to a site with strictly larger parameter λ.

2) Note that Lemmas 4.3.2, 4.3.3, 4.3.7 and 4.3.11 can be appropriately reformulated in order to cover the symmetric cases by simply re-labelling the graph sites in reverse order as the graph is a cycle. For example, if $\{k, k+1\}$ is a saddle point as in Lemma 4.3.7, then the corresponding symmetric case would be $\lambda_{k-1} > \lambda_k = \lambda_{k+1} > \lambda_{k+2}$, etc.

4.4 Random geometric progressions and Bernoulli measures

The statements and propositions in this section are essential building blocks for the proof of lemmas which follow. The reason is that along the proofs of Lemmas 4.3.4-4.3.13 we need to analyse the limiting behaviour of random variables of the form $\sum_{i=0}^{n} \prod_{j=1}^{i} \zeta_j$, as $n \to \infty$, where $\{\zeta_j,\ j \geq 1\}$ is an i.i.d. sequence of positive random variables. It will also be necessary to compare such variables and introduce some stochastic domination concepts to enable us to carry out uniform estimates not depending on the starting configuration $X(0) = \mathbf{x}$. We refer to [83] for standard definitions and basic properties of stochastic domination. The following notations are used throughout. Given random variables X and Y (or sequences X and Y), we write $X \geq_{st} Y$ if X stochastically dominates Y. Similarly, given two probability measures ν and μ, we write $\mu \geq_{st} \nu$ if μ stochastically dominates ν.

Random geometric progressions. In this subsection we consider random variables realised on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$. E denotes the expectation with respect to probability measure P. If X and Y are random variables or sequences such that $X \geq_{st} Y$, then we may assume that P is a coupling of probability distributions of X and Y such that $\mathsf{P}(X \geq Y) = 1$. Such a coupling exists by Strassen's theorem ([82]).

Given a random sequence $\zeta = \{\zeta_i,\, i \geq 1\}$, define

$$Y_i = \prod_{j=1}^{i} \zeta_j, i \geq 1,\ Y_0 = 1,\ \text{and}\ Z_n(\zeta) = \sum_{i=0}^{n} Y_i,\ n \geq 1, \tag{4.4.1}$$

and

$$Z(\zeta) = \sum_{i=0}^{\infty} Y_i. \tag{4.4.2}$$

Proposition 4.4.1. *1) Let $\zeta = \{\zeta_i,\, i \geq 1\}$ be an i.i.d. sequence of positive random variables such that $\mathsf{E}\left(\log(\zeta_i)\right) < 0$. Then $\mathsf{P}(Z(\zeta) < \infty) = 1$ and, consequently, $\mathsf{E}\left(e^{-Z(\zeta)}\right) > 0$.*

2) Let $\theta = \{\theta_i,\, i \geq 1\}$ be another i.i.d. sequence of positive random variables such that $\mathsf{E}\left(\log(\theta_i)\right) < 0$ and $\theta \geq_{st} \zeta$. Then $\mathsf{E}\left(e^{-Z(\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\theta)}\right)$.

Proof of Proposition 4.4.1. Denote $\mathsf{E}\left(\log(\zeta_i)\right) = a < 0$. Given $\delta > 0$ such that $a + \delta < 0$, it follows from the strong law of large numbers that $Y_n < e^{(a+\delta)n}$ for all but finitely many n almost surely. Therefore, a tail of $Z(\zeta)$ is eventually majorised by the corresponding tail of a converging geometric progression. In turn, finiteness of $Z(\zeta)$ implies positiveness of the expectation. Moreover, note that $e^{Z(\cdot)}$ is an increasing function. Therefore, $e^{-Z(\zeta)} \geq_{st} e^{-Z(\theta)}$ and hence, $\mathsf{E}\left(e^{-Z(\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\theta)}\right)$ as claimed. □

Definition 4.4.2. Let $\zeta = \{\zeta_i,\, i \geq 1\}$ and $\eta = \{\eta_j, j \geq 1\}$ be i.i.d. sequences of positive random variables. Sequence η is said to be reciprocal to ζ if η_1 has the same distribution as $1/\zeta_1$.

Proposition 4.4.3. *Let X and Y be two i.i.d. sequences of positive random variables, and let η_X and η_Y be their corresponding reciprocal sequences. If $X \geq_{st} Y$ then $\eta_X \leq_{st} \eta_Y$.*

Proposition 4.4.4. *Let $\zeta = \{\zeta_i,\, i \geq 1\}$ be an i.i.d. sequence of positive random variables such that $\mathsf{E}\left(\log(\zeta_i)\right) > 0$. Let $\{Y_i, i \geq 0\}$ and $\{Z_n(\zeta), n \geq 1\}$ be the random variables as in (4.4.1). Define the following random sequence*

$$F_n(\zeta) = Z_n(\zeta)/Y_n,\ n \geq 1.$$

Then, $F_n(\zeta)$ converges in distribution to

$$Z(\eta) = 1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \eta_j, \quad as \;\; n \to \infty, \tag{4.4.3}$$

where η is the sequence reciprocal to ζ. Moreover, $Z(\eta)$ is almost surely finite and $Z(\eta) \geq_{st} F_n(\zeta)$ for any $n \geq 1$.

Proof of Proposition 4.4.4. First, note that for every $n \geq 1$,

$$F_n(\zeta) = 1 + \sum_{i=1}^{n} \prod_{j=1}^{i} \zeta_{n-j+1}^{-1} = 1 + \sum_{i=1}^{n} \prod_{j=1}^{i} \eta_j^{(n)},$$

where $\eta_j^{(n)} = \zeta_{n-j+1}^{-1}$. This means that $F_n(\zeta)$ has the same distribution as $Z_n(\eta)$ defined for the sequence $\eta = \{\eta_i,\, i \geq 1\}$ reciprocal to ζ. Therefore, $F_n(\zeta)$ converges in distribution to $Z(\eta)$. In addition, $\mathsf{E}(\log(\eta_1)) = -\mathsf{E}(\log(\zeta_1)) < 0$. Therefore, by Proposition 4.4.1, $Z(\eta)$ is almost surely finite. Finally, it follows by construction that $Z(\eta) \geq_{st} F_n(\zeta)$, $n \geq 1$. □

Proposition 4.4.5. *Let $\zeta = \{\zeta_i,\, i \geq 1\}$ be an i.i.d. sequence of positive random variables such that $\mathsf{E}\left(\log(\zeta_i)\right) = a > 0$, and $\eta = \{\eta_i,\, i \geq 1\}$ be its reciprocal sequence. Given $0 < \gamma < 1$, define the following stopping time*

$$\widehat{m} = \min(n : \gamma Y_n \geq 1). \tag{4.4.4}$$

Then both $Z(\eta) < \infty$ and $Z_{\widehat{m}-1}(\zeta) < \infty$ almost surely, $\gamma Z_{\widehat{m}-1}(\zeta) \leq_{st} Z(\eta)$, and, hence,

$$\mathsf{E}\left(e^{-\gamma Z_{\widehat{m}-1}(\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\eta)}\right) > 0. \tag{4.4.5}$$

Proof of Proposition 4.4.5. By Proposition 4.4.4, $Z(\eta)$ is almost surely finite and $F_n(\zeta) \leq_{st} Z(\eta)$ for all $n \geq 1$. Therefore, $F_{\widehat{m}-1}(\zeta) \leq_{st} Z(\eta)$. Since $\gamma Y_{\widehat{m}-1} < 1$ we obtain that

$$\gamma Z_{\widehat{m}-1}(\zeta) < Z_{\widehat{m}-1}(\zeta)/Y_{\widehat{m}-1} = F_{\widehat{m}-1}(\zeta).$$

Consequently, $\gamma Z_{\widehat{m}-1}(\zeta) \leq_{st} Z(\eta)$, which implies (4.4.5) as claimed. □

Proposition 4.4.6. *Let $\zeta = (\zeta_i,\, i \geq 1)$ and $\theta = (\theta_i,\, i \geq 1)$ be i.i.d. sequences of positive random variables such that $\mathsf{E}\left(\log(\theta_1)\right) > 0$ and $\zeta_1 \geq_{st} \theta_1$. Let η_ζ and η_θ be*

sequences reciprocal to ζ and θ, respectively. Given $0 < \gamma < 1$, let $\widehat{m}$ be the stopping time for sequence ζ as in (4.4.4). Then

$$\mathsf{E}\left(e^{-\gamma Z_{\widehat{m}-1}(\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\eta_\theta)}\right).$$

Proof of Proposition 4.4.6. By Proposition 4.4.4 both $Z(\eta_\zeta)$ and $Z(\eta_\theta)$ are almost surely finite. Further, by Proposition 4.4.3 $\eta_\zeta \leq_{st} \eta_\theta$. Therefore

$$\mathsf{E}\left(e^{-Z(\eta_\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\eta_\theta)}\right).$$

By Proposition 4.4.5, it follows that

$$\mathsf{E}\left(e^{-\gamma Z_{\widehat{m}-1}(\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\eta_\zeta)}\right) \geq \mathsf{E}\left(e^{-Z(\eta_\theta)}\right)$$

as claimed. □

Bernoulli measures. Now, we introduce a family of Bernoulli measures and some notations that will be used throughout proofs of Lemmas 4.3.4-4.3.13.

Let $\xi = (\xi_i, i \geq 1)$ be a sequence of independent Bernoulli random variables with success probability p. Let μ_p be the distribution of ξ, that is, the product Bernoulli measure defined on the set of infinite binary sequences, and denote by $\mathbb{E}_p$ the expectation with respect to the Bernoulli measure μ_p.

Define

$$U_i = \xi_1 + \cdots + \xi_i,\ i \geq 1, \tag{4.4.6}$$

the binomial random variables corresponding to a Bernoulli sequence ξ.

Let λ_{k-1}, λ_k, λ_{k+1} and λ_{k+2} be λ-parameters corresponding to quadruples $\{k-1, k, k+1, k+2\}$ of the graph sites such that $\lambda = \lambda_k = \lambda_{k+1}$ as in Lemmas 4.3.4-4.3.13. Let us define the following i.i.d. sequences

$$\begin{aligned} \zeta_1 &= (\zeta_{1,i} = e^{\lambda_{k-1}(1-\xi_i)-\lambda},\ i \geq 1), \\ \zeta_2 &= (\zeta_{2,i} = e^{\lambda_{k+2}\xi_i-\lambda},\ i \geq 1). \end{aligned} \tag{4.4.7}$$

It is a well known fact that if $0 < p' \leq p'' < 1$, then $\mu_{p'} \leq_{st} \mu_{p''}$. This fact yields the following proposition.

Proposition 4.4.7. *Let ζ_1', ζ_2' and ζ_1'', ζ_2'' be sequences defined by (4.4.7) for Bernoulli sequences with success probabilities p' and p'', respectively. If $0 < p' \leq p'' < 1$, then $\zeta_1' \geq_{st} \zeta_1''$ and $\zeta_2' \leq_{st} \zeta_2''$.*

Note that variables Z_n (defined in (4.4.1)) corresponding to sequences ζ_1 and ζ_2 can be expressed in terms of Binomial random variables (4.4.6) as follows

$$Z_n(\zeta_1) = \sum_{i=0}^{n} e^{\lambda_{k-1}(i-U_i)-\lambda i} \quad \text{and} \quad Z_n(\zeta_2) = \sum_{i=0}^{n} e^{\lambda_{k+2}U_i-\lambda i}. \tag{4.4.8}$$

It is useful to note that if $\lambda_{k-1} = \lambda_{k+2} = \lambda$, then the above expressions are

$$Z_n(\zeta_1) = \sum_{i=0}^{n} e^{-\lambda U_i} \quad \text{and} \quad Z_n(\zeta_2) = \sum_{i=0}^{n} e^{\lambda(U_i-i)}.$$

4.5 Proofs of Lemmas

In the following proofs we show the existence of positive real constants C, c, ϵ and ε, whose exact values are immaterial and may vary from line to line, but which do not depend on the starting configuration $X(0) = \mathbf{x}$. In order to avoid notational clutter we shall denote $\Gamma_i(\mathbf{x})$ simply by Γ_i for all i. Moreover, whenever we fix index $k \in \{1, \ldots, N\}$ and consider indices in the neighbourhood of k, those indices should be interpreted as *modulo* N.

4.5.1 Proofs of Lemmas 4.3.1-4.3.3

For short, denote $B = \sum_{i \neq k, k\pm 1} \Gamma_i$ and $Z = \sum_{i=1}^{N} \Gamma_i$. By assumption, $\Gamma_k = \max_{i=1,\ldots,N} \Gamma_i$, then

$$\frac{\Gamma_{k-1}}{\Gamma_k} \leq 1, \frac{\Gamma_{k+1}}{\Gamma_k} \leq 1, \Gamma_k \geq \frac{Z}{N}, \text{ and } \frac{Z - \Gamma_k}{Z} \leq \frac{(N-1)}{N}. \tag{4.5.1}$$

It follows from the last two inequalities that

$$\frac{B}{\Gamma_k} \leq N - 1. \tag{4.5.2}$$

Proof of Lemma 4.3.1. Recall that $\lambda_k > \max(\lambda_{k-1}, \lambda_{k+1})$. We need to prove the existence of a positive number ϵ such that

$$\mathbb{P}_{\mathbf{x}}\left(A^k_{[1,\infty)}\right) = \prod_{n=0}^{\infty} \frac{\Gamma_k e^{\lambda_k n}}{\Gamma_{k-1}e^{\lambda_{k-1}n} + \Gamma_k e^{\lambda_k n} + \Gamma_{k+1}e^{\lambda_{k+1}n} + B} > \epsilon, \tag{4.5.3}$$

where $\epsilon > 0$ depends only on $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$ and N.

Indeed, rewriting the identity in (4.5.3) and applying bounds (4.5.1) and (4.5.2),

$$\begin{aligned}
&\mathbb{P}_{\mathbf{x}}\left(A^k_{[1,\infty)}\right)\\
&= \exp\left(-\sum_{n=0}^{\infty} \log\left(1 + \frac{\Gamma_{k-1}}{\Gamma_k}e^{(\lambda_{k-1}-\lambda_k)n} + \frac{\Gamma_{k+1}}{\Gamma_k}e^{(\lambda_{k+1}-\lambda_k)n} + \frac{B}{\Gamma_k}e^{-\lambda_k n}\right)\right)\\
&\geq \exp\left(-\sum_{n=0}^{\infty} \log(1 + e^{(\lambda_{k-1}-\lambda_k)n} + e^{(\lambda_{k+1}-\lambda_k)n} + (N-1)e^{-\lambda_k n})\right)\\
&\geq \exp\left(-C\sum_{n=0}^{\infty} (e^{(\lambda_{k-1}-\lambda_k)n} + e^{(\lambda_{k+1}-\lambda_k)n} + (N-1)e^{-\lambda_k n})\right) > \epsilon > 0,
\end{aligned}$$

since the series in the exponent above converges. It is not hard to see that in the last inequality, ϵ should depend only on $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$ and N. □

Proof of Lemma 4.3.2. Recall that $\lambda_{k-1} < \lambda_k < \lambda_{k+1}$. We need to prove the existence of a finite positive integer $\hat{n}$ and a positive number ϵ such that

$$\Gamma_{k+1}e^{\lambda_{k+1}\hat{n}} \geq \Gamma_k e^{\lambda_k \hat{n}} > \max\left(\Gamma_{k-1}e^{\lambda_{k-1}\hat{n}}, \max_{i\neq k,k\pm 1} \Gamma_i\right) \tag{4.5.4}$$

and

$$\mathbb{P}_{\mathbf{x}}(A^k_{[1,\hat{n}+1]}) = \prod_{n=0}^{\hat{n}} \frac{\Gamma_k e^{\lambda_k n}}{\Gamma_{k-1}e^{\lambda_{k-1}n} + \Gamma_k e^{\lambda_k n} + \Gamma_{k+1}e^{\lambda_{k+1}n} + B} > \epsilon, \tag{4.5.5}$$

where $\epsilon > 0$ depends only on $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$ and N. Note that the sequence $e^{(\lambda_{k+1}-\lambda_k)n}$, $n \geq 0$ is strictly increasing, so there exists the minimal integer $\hat{n}$ such that

$$e^{(\lambda_{k+1}-\lambda_k)\hat{n}} \geq \frac{\Gamma_k}{\Gamma_{k+1}}, \quad \text{that is,} \quad \frac{\Gamma_{k+1}(\mathbf{x} + \hat{n}\mathbf{e}_k)}{\Gamma_k(\mathbf{x} + \hat{n}\mathbf{e}_k)} \geq 1.$$

Then, there exists a positive constant such that $(\Gamma_{k+1}/\Gamma_k)e^{(\lambda_{k+1}-\lambda_k)\hat{n}} \leq C_1$, and, hence,

$$\frac{\Gamma_{k+1}}{\Gamma_k}\sum_{n=0}^{\hat{n}} e^{(\lambda_{k+1}-\lambda_k)n} \leq C_2 < \infty, \tag{4.5.6}$$

where C_2 depends only on λ_k and λ_{k+1}. Further, rewriting the identity in (4.5.5) and using bounds (4.5.1), (4.5.2) and (4.5.6), gives that

$$\begin{aligned}
&\mathbb{P}_{\mathbf{x}}(A^k_{[1,\hat{n}]}) \\
&= \exp\left(-\sum_{n=0}^{\hat{n}} \log\left(1 + \frac{\Gamma_{k-1}}{\Gamma_k}e^{(\lambda_{k-1}-\lambda_k)n} + \frac{\Gamma_{k+1}}{\Gamma_k}e^{(\lambda_{k+1}-\lambda_k)n} + \frac{B}{\Gamma_2}e^{-\lambda_k n}\right)\right) \\
&\geq \exp\left(-\sum_{n=0}^{\hat{n}} \log\left(1 + e^{(\lambda_{k-1}-\lambda_k)n} + \frac{\Gamma_{k+1}}{\Gamma_k}e^{(\lambda_{k+1}-\lambda_k)n} + (N-1)e^{-\lambda_k n}\right)\right) \\
&\geq \exp\left(-C_3\sum_{n=0}^{\hat{n}} \left(e^{(\lambda_{k-1}-\lambda_k)n} + \frac{\Gamma_{k+1}}{\Gamma_k}e^{(\lambda_{k+1}-\lambda_k)n} + (N-1)e^{-\lambda_k n}\right)\right) > \epsilon > 0.
\end{aligned}$$

□

Proof of Lemma 4.3.3. Recall that $\lambda_k < \min(\lambda_{k-1}, \lambda_{k+1})$. As in the proof of Lemma 4.3.2, we need to show existence of a finite positive integer $\hat{n}$ and a positive ϵ such that

$$\max(\Gamma_{k-1}e^{\lambda_{k-1}\hat{n}}, \Gamma_{k+1}e^{\lambda_{k+1}\hat{n}}) \geq \Gamma_k e^{\lambda_k \hat{n}} \geq \max_{i\neq k,k\pm1} \Gamma_i \tag{4.5.7}$$

and

$$\mathbb{P}_{\mathbf{x}}(A^k_{[1,\hat{n}]}) = \prod_{n=0}^{\hat{n}} \frac{\Gamma_k e^{\lambda_k n}}{\Gamma_{k-1}e^{\lambda_{k-1}n} + \Gamma_k e^{\lambda_k n} + \Gamma_{k+1}e^{\lambda_{k+1}n} + B} > \epsilon,$$

where $\epsilon > 0$ depends only on $\lambda_{k-1}, \lambda_k, \lambda_{k+1}$ and N. This can be shown similar to the proof of Lemma 4.3.2, and we skip details. □

4.5.2 Proofs of Lemmas 4.3.4-4.3.13

Notations

We start with some preliminary considerations and notations that will be used throughout the proofs of Lemmas 4.3.4-4.3.13.

Let $\{k, k+1\}$ be a pair of sites such that $\lambda_k = \lambda_{k+1} =: \lambda$. If, as defined in Definition 4.2.2, $r = r(\mathbf{x}) = x_{k+2} - x_{k-1}$, then $\frac{\Gamma_{k+1}(\mathbf{x})}{\Gamma_k(\mathbf{x})} = e^{\lambda r}$. Therefore, given that the next particle is allocated at either k or $k+1$, the conditional $\mathsf{P}_{\mathbf{x}}$-probability to choose $k+1$ is equal to

$$p := p(r) = \frac{\Gamma_{k+1}(\mathbf{x})}{\Gamma_k(\mathbf{x}) + \Gamma_{k+1}(\mathbf{x})} = \frac{e^{\lambda r}}{1 + e^{\lambda r}}. \tag{4.5.8}$$

We henceforth denote $q = 1 - p$. Furthermore, probability p does not change by adding particles at sites k and $k+1$ since configuration parameter r remains constant.

Note that $p(z)$, considered as a function of $z \in \mathbb{R}$, is monotonically increasing. A direct computation gives that unique solutions of equations $\lambda_{k-1} - \lambda = p(z)\lambda_{k-1}$ and $\lambda_{k+2}p(z) = \lambda$ are quantities z_1 and z_2 (defined in (4.3.2)) respectively.

Let S_n be the number of additional particles at site $k+1$ at time $n \geq 1$. Let $S_0 = 0$ and $s(n) = (s_0, s_1, \ldots, s_n)$ be a fixed trajectory of a finite random sequence $S(n) = (S_0, S_1, \ldots, S_n)$. Note that, by construction, any trajectory $s(n)$ is a sequence of non-negative integers such that $s_0 = 0$ and $s_i - s_{i-1} \in \{0, 1\}$, $i = 1, \ldots, n$.

For short, denote

$$\Gamma_i = \Gamma_i(\mathbf{x}),\ \widetilde{\Gamma}_k = \sum_{i \neq k, k\pm 1, k+2} \Gamma_i,$$

$$\gamma_{k,1} = \frac{\Gamma_{k-1}}{\Gamma_k + \Gamma_{k+1}},\ \gamma_{k,2} = \frac{\Gamma_{k+2}}{\Gamma_k + \Gamma_{k+1}},\ \widetilde{\gamma}_k = \frac{\widetilde{\Gamma}_k}{\Gamma_k + \Gamma_{k+1}}.$$

In the rest of this section we are going to derive expressions for probabilities $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right)$, $n \geq 1$, in terms of expectations with respect to a Bernoulli product measure on $\{0,1\}^\infty$ with parameter p defined in (4.5.8). These expressions allow one to obtain lower and upper bounds for the above probabilities. We start with the case of fixed n and then extend it to the case where n is a stopping time.

In the above notations

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{i+1}, S_{i+1} = s_{i+1} \middle| A^{k,k+1}_{[1,i]}, S_i = s_i\right) = \frac{p^{s_{i+1}-s_i} q^{1-(s_{i+1}-s_i)} (\Gamma_k + \Gamma_{k+1}) e^{\lambda i}}{(\Gamma_k + \Gamma_{k+1}) e^{\lambda i} + \Gamma_{k-1} e^{\lambda_{k-1}(i-s_i)} + \Gamma_{k+2} e^{\lambda_{k+2} s_i} + \widetilde{\Gamma}_k}$$

$$= \frac{p^{s_{i+1}-s_i} q^{1-(s_{i+1}-s_i)}}{1+\gamma_{k,1} e^{\lambda_{k-1}(i-s_i)-\lambda i} + \gamma_{k,2} e^{\lambda_{k+2} s_i - \lambda i} + \widetilde{\gamma}_k e^{-\lambda i}}.$$

Then, given n we obtain by repeated conditioning that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}, S_{n+1} = s_{n+1}, \ldots, S_1 = s_1\right) = p^{s_{n+1}} q^{n+1-s_{n+1}} W_n(s_1, \ldots, s_n),$$

where

$$W_n(s_1, \ldots, s_n) = \prod_{i=0}^{n} \frac{1}{1+\gamma_{k,1} e^{\lambda_{k-1}(i-s_i)-\lambda i} + \gamma_{k,2} e^{\lambda_{k+2} s_i - \lambda i} + \widetilde{\gamma}_k e^{-\lambda i}}. \tag{4.5.9}$$

Consequently, we get that

$$\begin{aligned} \mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) &= \sum_{s(n+1)} p^{s_{n+1}} q^{n+1-s_{n+1}} W_n(s_1, \ldots, s_n), \\ &= \sum_{s(n)} (p+q) p^{s_n} q^{n-s_n} W_n(s_1, \ldots, s_n), \\ &= \sum_{s(n)} p^{s_n} q^{n-s_n} W_n(s_1, \ldots, s_n), \end{aligned} \tag{4.5.10}$$

where the sum in the first line is over all possible trajectories $s(n+1) = (s_1, \ldots, s_{n+1})$ of $S(n+1) = (S_1, \ldots, S_{n+1})$ and the other two are over all possible trajectories $s(n) = (s_1, \ldots, s_n)$ of $S(n) = (S_1, \ldots, S_n)$. Therefore, we arrive to the following equation

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) = \mathbb{E}_p(W_n(U_1, \ldots, U_n)), \tag{4.5.11}$$

where $\mathbb{E}_p$ is the expectation with respect to the Bernoulli measure μ_p defined in Section 4.4 and U_i, $i \geq 1$, are Binomial random variables defined in (4.4.6).

Further, assumptions of Lemmas 4.3.4-4.3.13 imply that $\frac{\Gamma_i}{\Gamma_k + \Gamma_{k+1}} \leq 1$, $i = 1, \ldots, N$. Therefore,

$$\widetilde{\gamma}_k e^{-\lambda i} \leq (N-4) e^{-\lambda i} \leq c_1 e^{-c_2 i}, \tag{4.5.12}$$

for some $c_1, c_2 > 0$. Using bound (4.5.12) and inequality $\log(1+z) \leq z$ for all $z \geq 0$ we obtain that

$$\begin{aligned} W_n(s_1, \ldots, s_n) &\geq \prod_{i=0}^{n} \frac{1}{1+\gamma_{k,1} e^{\lambda_{k-1}(i-s_i)-\lambda i} + \gamma_{k,2} e^{\lambda_{k+2} s_i - \lambda i} + c_1 e^{-c_2 i}} \\ &= e^{-\sum_{i=0}^{n} \log\left(1+\gamma_{k,1} e^{\lambda_{k-1}(i-s_i)-\lambda i} + \gamma_{k,2} e^{\lambda_{k+2} s_i - \lambda i} + c_1 e^{-c_2 i}\right)} \end{aligned}$$

$$\geq e^{-\left(\sum_{i=0}^{n}\gamma_{k,1}e^{\lambda_{k-1}(i-s_i)-\lambda i}+\gamma_{k,2}e^{\lambda_{k+2}s_i-\lambda i}+c_1e^{-c_2 i}\right)}$$

$$\geq \delta e^{-\gamma_{k,1}\sum_{i=0}^{n}e^{\lambda_{k-1}(i-s_i)-\lambda i}}e^{-\gamma_{k,2}\sum_{i=0}^{n}e^{\lambda_{k+2}s_i-\lambda i}}, \tag{4.5.13}$$

for some $\delta > 0$ not depending on the configuration $\mathbf{x}$. On the other hand, note that

$$W_n(s_1,\ldots,s_n) \leq \prod_{i=0}^{n}\frac{1}{1+\gamma_{k,1}e^{\lambda_{k-1}(i-s_i)-\lambda i}+\gamma_{k,2}e^{\lambda_{k+2}s_i-\lambda i}}. \tag{4.5.14}$$

The above inequalities yield the following lower and upper bounds

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) \geq \delta\mathbb{E}_p\left(e^{-\gamma_{k,1}\sum_{i=0}^{n}e^{\lambda_{k-1}(i-U_i)-\lambda i}}e^{-\gamma_{k,2}\sum_{i=0}^{n}e^{\lambda_{k+2}U_i-\lambda i}}\right), \tag{4.5.15}$$

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) \leq \mathbb{E}_p\left(\prod_{i=0}^{n}\frac{1}{1+\gamma_{k,1}e^{\lambda_{k-1}(i-U_i)-\lambda i}+\gamma_{k,2}e^{\lambda_{k+2}U_i-\lambda i}}\right). \tag{4.5.16}$$

We will also need a generalisation of lower bound (4.5.15) for probabilities $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\tau]}\right)$, where τ is one of the following stopping times, $\min(n : S_n - c_1 n \geq c_2)$, $\min(n : n - S_n \geq c_3)$, and the minimum of two such stopping times. At the moment, we shall not further specify such stopping times as it will be clear later which one it refers to. Arguing similarly as in equation (4.5.10), one can obtain that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\tau]}\right) = \sum_{n=0}^{\infty}\sum_{s(n)}p^{s_n}q^{n-s_n}W_n(s_1,\ldots,s_n)1_{\{\mathsf{M}_n\}}, \tag{4.5.17}$$

where M_n is a set of paths $s(n) = (s_1,\ldots,s_n)$ for which $\tau = n+1$. Furthermore, similar to equation (4.5.11), we can rewrite equation above as

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\tau]}\right) = \mathbb{E}_p\left(W_{\tilde{\tau}}(U_1,\ldots,U_{\tilde{\tau}})\right), \tag{4.5.18}$$

where $\tilde{\tau}$ is a stopping time defined by replacing S_n by U_n in the same way as τ but in terms of random variables U_n. Proceeding similarly to how we got lower bound (4.5.15) we obtain the following lower bound

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\tau]}\right) \geq \delta\mathbb{E}_p\left(e^{-\gamma_{k,1}\sum_{i=0}^{\tilde{\tau}-1}e^{\lambda_{k-1}(i-U_i)-\lambda i}}e^{-\gamma_{k,2}\sum_{i=0}^{\tilde{\tau}-1}e^{\lambda_{k+2}U_i-\lambda i}}\right). \tag{4.5.19}$$

Let us rewrite the lower bounds in terms of random sequences ζ_1, ζ_2 and Z_n as defined in (4.4.7) and (4.4.8). In these notations, lower bounds (4.5.15) and (4.5.19)

take the following form

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) \geq \delta \mathbb{E}_p\left(e^{-\gamma_{k,1}Z_n(\zeta_1)}e^{-\gamma_{k,2}Z_n(\zeta_2)}\right) \tag{4.5.20}$$

and

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\tau]}\right) \geq \delta \mathbb{E}_p\left(e^{-\gamma_{k,1}Z_{\tilde{\tau}-1}(\zeta_1)}e^{-\gamma_{k,2}Z_{\tilde{\tau}-1}(\zeta_2)}\right) \tag{4.5.21}$$

respectively.

Finally, letting $n \to \infty$ in (4.5.15) and (4.5.20) we obtain the following bound

$$\begin{aligned}\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) &\geq \delta \mathbb{E}_p\left(e^{-\gamma_{k,1}\sum_{i=0}^{\infty}e^{\lambda_{k-1}(i-U_i)-\lambda i}}e^{-\gamma_{k,2}\sum_{i=0}^{\infty}e^{\lambda_{k+2}U_i-\lambda i}}\right)\\ &= \delta \mathbb{E}_p\left(e^{-\gamma_{k,1}Z(\zeta_1)}e^{-\gamma_{k,2}Z(\zeta_2)}\right).\end{aligned} \tag{4.5.22}$$

Proof of Lemma 4.3.4

We start with the following proposition.

Proposition 4.5.1. *Let μ_p be the Bernoulli measure defined in Section 4.4, and let U_n, $n \geq 1$, be the corresponding Binomial random variables (defined in (4.4.6)). Then*

1) given $\varepsilon \in (0,1)$ and $\kappa > 0$, there exist positive constants c_1 and c_2 such that

$$\inf_{p\in(0,1)} \mu_p\left(\bigcap_{n=M}^{\infty}\left\{\frac{n}{2}p(1-\kappa) - c_1 \leq U_n \leq np(1+\kappa) + c_2\right\}\right) \geq \varepsilon, \tag{4.5.23}$$

where $M = [p^{-1}]$ is the integer part of p^{-1};

2) given $\lambda > 0$, there exists $\varepsilon_1 > 0$ such that

$$\inf_{p\in(0,1)} \mathbb{E}_p\left(e^{-p\sum_{i=0}^{\infty}e^{-\lambda U_i}}\right) \geq \varepsilon_1.$$

Proof of Proposition 4.5.1. Set $U_0 = 0$ and define the following random variables

$$V_j = U_{jM} - U_{(j-1)M} = \sum_{i=(j-1)M+1}^{jM} \xi_i,\ j \geq 1,$$

$$Y_j = V_1 + \cdots + V_j,\ j \geq 1, \quad \text{and } Y_0 = 0.$$

First, denote $a(p) := \mathbb{E}_p(V_i) = pM = p[p^{-1}]$ and note that $a(p) \in [1/2, 1]$ for all $p \in (0,1)$. Moreover, $Var(V_i) = \mathbb{E}_p(V_i^2) - (\mathbb{E}_p(V_i))^2 = p(1-p)[p^{-1}] \leq 1$. Now, consider the auxiliary process $\chi_n := Y_n - n(1-\kappa)/2 + c'$, with $\chi_0 = c'$. Note that $\mathbb{E}_p(\chi_{n+1} - \chi_n \,|\, \chi_n = \chi) = a(p) - (1-\kappa)/2 > 0$. Moreover, if we define the stopping time $t_x = \min_{n\geq 0}\{\chi_n < x\}$, it follows from Theorem 2.5.18 in [61] that there exist x_1 and $\alpha > 0$ such that

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty}\{Y_n \geq \frac{n}{2}(1-\kappa) - (c' - x_1)\}\right) = \mathbb{P}(t_{x_1} = \infty) \geq 1 - \left(\frac{1+x_1}{1+\chi_0}\right)^{\alpha}.$$

So, for every $\varepsilon \in (0,1)$ and $\kappa > 0$, we can appropriately choose α and $\chi_0 = c' > x_1$ such that the probability in the above display is greater than $\varepsilon/2$. Analogously, if we define $\chi_n = -Y_n + n(1+\kappa)$, the upper bound can be found exactly as above, yielding

$$\mu_p\left(\bigcap_{n=1}^{\infty}\{\frac{n}{2}(1-\kappa) - c \leq Y_n \leq n(1+\kappa) + c\}\right) \geq \varepsilon. \tag{4.5.24}$$

Further, fix $n \geq M$. Let m_n and l_n be integers such that $n = m_n M + l_n$, where $l_n < M$. Then on event $\bigcap_{n=1}^{\infty}\{\frac{n}{2}(1-\kappa) - c \leq Y_n \leq n(1+\kappa) + c\}$ the following bounds hold

$$U_n \geq Y_{m_n} \geq \frac{1}{2}\left(\frac{n}{M} - \frac{l_n}{M}\right)(1-\kappa) - c \geq \frac{1}{2}np(1-\kappa) - c_1, \tag{4.5.25}$$

and

$$U_n \leq Y_{m_n+1} \leq \left(\frac{n}{M} + \frac{M - l_n}{M}\right)(1+\kappa) + c \leq np(1+\kappa) + c_2. \tag{4.5.26}$$

Inequalities (4.5.24), (4.5.25) and (4.5.26) yield bound (4.5.23).

Recall that $M = [p^{-1}]$, and so,

$$p\sum_{i=0}^{M-1} e^{-\lambda U_i} \leq pM \leq 1.$$

By combining this bound with bound (4.5.23), it follows that given $\varepsilon \in (0,1)$ and

$\kappa > 0$ we can find $c_1 > 0$ such that with μ_p-probability at least ε

$$p\sum_{i=0}^{\infty} e^{-\lambda U_i} \leq 1 + p\sum_{i=M}^{\infty} e^{-\lambda(\frac{1}{2}pi(1-\kappa)-c_1)} \leq C \tag{4.5.27}$$

for some deterministic constant $C = C(\varepsilon, \lambda)$ and all $p \in (0,1)$. Therefore

$$\inf_{p\in(0,1)} \mathbb{E}_p\left(e^{-p\sum_{i=0}^{\infty} e^{-\lambda U_i}}\right) \geq \varepsilon e^{-C} = \varepsilon_1 > 0,$$

as required. □

We are now ready to proceed with the proof of the lemma. Recall that $\lambda_k = \lambda_{k+1} =: \lambda$.

Proof of Part 1) of Lemma 4.3.4. Recall that in this case $\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \lambda$, $\lambda \geq \lambda_{k+2}$ and $\Gamma_k = \max_i \Gamma_i$. Then,

$$\gamma_{k,1} Z(\zeta_1) = \gamma_{k,1}\sum_{i=0}^{\infty} e^{\lambda_{k-1}(i-U_i)-\lambda i} \leq \sum_{i=0}^{\infty} e^{-(\lambda-\lambda_{k-1})i} \leq C_1 < \infty, \tag{4.5.28}$$

where $C_1 > 0$ is a deterministic constant and we used that $\gamma_{k,1} \leq 1$.

Further, if $\lambda > \lambda_{k+2}$, then

$$\gamma_{k,2} Z(\zeta_2) = \gamma_{k,2}\sum_{i=0}^{\infty} e^{\lambda_{k+2}U_i-\lambda i} \leq \sum_{i=0}^{\infty} e^{-(\lambda-\lambda_{k+2})i} \leq C_2 < \infty, \tag{4.5.29}$$

where $C_2 > 0$ is a deterministic constant and we used that $\gamma_{k,2} \leq 1$. Then, using bounds (4.5.28) and (4.5.29) in lower bound (4.5.22) gives that $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) \geq \varepsilon$ for some $\varepsilon > 0$, as claimed.

If $\lambda = \lambda_{k+2}$, then bound (4.5.29) cannot be used, and we proceed as follows. Note that in this case

$$\gamma_{k,2} Z(\zeta_2) = \gamma_{k,2}\sum_{i=0}^{\infty} e^{\lambda(U_i-i)} \leq q\sum_{i=0}^{\infty} e^{\lambda(U_i-i)}, \tag{4.5.30}$$

as

$$\gamma_{k,2} = \frac{\Gamma_{k+2}}{\Gamma_k + \Gamma_{k+1}} \leq \frac{\Gamma_k}{\Gamma_k + \Gamma_{k+1}} = q = 1 - p, \tag{4.5.31}$$

where p is defined in (4.5.8). Further, combining bounds (4.5.28) and (4.5.30) in

(4.5.22) we get that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) \geq \varepsilon_1 \mathbb{E}_p\left(e^{-q\sum_{i=0}^{\infty} e^{\lambda(U_i-i)}}\right) = \varepsilon_1 \mathbb{E}_p\left(e^{-p\sum_{i=0}^{\infty} e^{-\lambda U_i}}\right), \tag{4.5.32}$$

where the equality holds by symmetry. It is left to note that the expectation in the right side of the last equation is bounded below uniformly over $p \in (0,1)$ by Part 2) of Proposition 4.5.1. □

Proof of Part 2) of Lemma 4.3.4. Recall that in this case $\lambda_{k-1} = \lambda_k = \lambda_{k+1} = \lambda \geq \lambda_{k+2}$, $\Gamma_k = \max_i \Gamma_i$ and $\Gamma_{k-1} \leq \Gamma_{k+1}$. These conditions give that $e^{\lambda_{k-1}(i-U_i)-\lambda i} = e^{-\lambda U_i}$, $e^{\lambda_{k+2}U_i - \lambda i} \leq e^{\lambda(U_i-i)}$, and

$$\gamma_{k,1} = \frac{\Gamma_{k-1}}{\Gamma_k + \Gamma_{k+1}} \leq \frac{\Gamma_{k+1}}{\Gamma_k + \Gamma_{k+1}} = p. \tag{4.5.33}$$

Recall also that $\gamma_{k,2} \leq q = 1-p$ (see (4.5.31)). Using all these inequalities in lower bound (4.5.22) gives the following lower bound

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) \geq \delta \mathbb{E}_p\left(e^{-p\sum_{i=0}^{\infty} e^{-\lambda U_i}} e^{-q\sum_{i=0}^{\infty} e^{\lambda(U_i-i)}}\right). \tag{4.5.34}$$

We have already shown in (4.5.27) that for any $\varepsilon \in (0,1)$ there exists constant $C = C(\varepsilon) > 0$ such that

$$\mu_p\left(p\sum_{i=0}^{\infty} e^{-\lambda U_i} \leq C\right) \geq \varepsilon \quad \text{and} \quad \mu_p\left(q\sum_{i=0}^{\infty} e^{\lambda(U_i-i)} \leq C\right) \geq \varepsilon \tag{4.5.35}$$

for all p, where the second bound holds by symmetry. Choosing $\varepsilon > 0.5$ we get that

$$\mu_p\left(p\sum_{i=0}^{\infty} e^{-\lambda U_i} \leq C,\, q\sum_{i=0}^{\infty} e^{\lambda(U_i-i)} \leq C\right) \geq 2\varepsilon - 1 > 0,$$

for all p. Combining this bound with equation (4.5.34) we finally obtain that $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty)}\right) \geq \varepsilon_2$ for some $\varepsilon_2 > 0$, as claimed. □

Proof of Lemma 4.3.7

Proof of Part 1) of Lemma 4.3.7. Note that at every time a particle is added to site k or $k+1$, the allocation rates at these sites are multiplied by e^{λ}. In particular,

if a particle is added to site k, then the allocation rate at $k-1$ is multiplied by $e^{\lambda_{k-1}}$. Otherwise, if a particle is added to site $k+1$, then the allocation rate at $k+2$ is multiplied by $e^{\lambda_{k+2}}$. Other rates remain unchanged. Thus, by allocating a particle at k or $k+1$, the sum of rates at $k, k+1$ and $k+2$ over the sum of rates at all other sites is increased by a multiple constant. This yields the following exponential bound

$$\mathsf{P}_{\mathbf{x}}\left(\bigcup_{i\neq k,k+1,k+2} A^{i}_{n+1}\middle| A^{k,k+1}_{[1,n]}\right) \leq C_1 e^{-C_2 n}, \tag{4.5.36}$$

for some $C_1, C_2 > 0$. In turn, bound (4.5.36) implies that with a positive probability (not depending on $\mathbf{x}$) event $A^{k,k+1}_{[1,\infty)} \cup \{\tau_{k+2} < w^{+}_{k}\}$ occurs as claimed. Note also that events $A^{k,k+1}_{[1,\infty)}$ and $\{\tau_{k+2} < w^{+}_{k}\}$ are mutually exclusive. Thus, with a positive probability either all particles will be allocated at k and $k+1$, or a particle is eventually placed at $k+2$. Placing a particle at $k+2$ can violate condition (4.3.5) because the maximal allocating probability can be now attained at sites $k+2$ and $k+3$ as well. Part 1) of Lemma 4.3.7 is proved. □

Proof of Part 2) Lemma 4.3.7. Note that $e^{\lambda_{k-1}(i-U_i)-\lambda i} < e^{-(\lambda-\lambda_{k-1})i}$ and $\lambda_{k-1} < \lambda$. Consequently, for any n

$$Z_n(\zeta_1) = \sum_{i=0}^{n} e^{\lambda_{k-1}(i-U_i)-\lambda i} \leq \sum_{i=0}^{\infty} e^{-(\lambda-\lambda_{k-1})i} < C < \infty. \tag{4.5.37}$$

Note also that $\gamma_{k,1} \leq 1$ and $\gamma_{k,2} \leq 1$. Combining these inequalities with equation (4.5.37) and letting $n \to \infty$ in (4.5.20) gives that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty]}\right) \geq \varepsilon_1 \mathbb{E}_p\left(e^{-Z(\zeta_2)}\right),$$

for some $\varepsilon_1 > 0$. Further, assumption $r < z_2$ implies that $\lambda_{k+2}p - \lambda < 0$. Recall that parameter $r = x_{k+2} - x_{k-1}$ takes integer values, and $p = p(r)$ is a monotonically increasing function of r. Let r_0 be the maximal integer such that $r < z_2$ and $p_0 = p(r_0)$, so that $\lambda_{k+2}p_0 - \lambda < 0$. It follows from Proposition 4.4.1 and Proposition

4.4.7 that for all $0 < p < p_0$

$$\mathbb{E}_p\left(e^{-Z(\zeta_2)}\right) \geq \mathbb{E}_{p_0}\left(e^{-Z(\zeta_2)}\right) > 0, \tag{4.5.38}$$

and, hence, $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty]}\right) \geq \varepsilon$ for some uniform $\varepsilon > 0$ over configurations $\mathbf{x}$ satisfying $r < z_2$. Part 2) of Lemma 4.3.7 is proved. □

Proof of Part 3) of Lemma 4.3.7. We are going to use the following relaxation of upper bound (4.5.16)

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) \leq \mathbb{E}_p\left(\prod_{i=0}^{n} \frac{1}{1+\gamma_{k,2}e^{\lambda_{k+2}U_i-\lambda i}}\right). \tag{4.5.39}$$

Next, assumption $r \geq z_2$ implies that $\lambda_{k+2}p - \lambda \geq 0$. Therefore, by the strong law of large numbers, we get that μ_p-*a.s.* $\lambda_{k+2}U_i - \lambda i \geq 0$ for infinitely many i and, hence, $\prod_{i=0}^{n} \frac{1}{1+\gamma_{k,2}e^{\lambda_{k+2}U_i-\lambda i}} \to 0$. The product is bounded by 1, therefore, by the Lebesgue's dominated convergence theorem, the expectation in the right side of (4.5.39) tends to 0 as $n \to \infty$, which implies that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty]}\right) = \lim_{n\to\infty} \mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,n+1]}\right) = 0, \tag{4.5.40}$$

as claimed. Note that equation (4.5.40) combined with Part 1) of the lemma further yields that $\mathsf{P}_{\mathbf{x}}\left(\tau_{k+2} < w^+_k\right) > \epsilon$ for some ϵ. □

Proof of Part 4) of Lemma 4.3.7. Define

$$\widehat{n} = \min\left(n : \gamma_{k,2}e^{\lambda_{k+2}S_n-\lambda n} \geq 1\right). \tag{4.5.41}$$

In other words, $\hat{n}$ is the first time when the allocation rate at site $k+2$ exceeds the sum of allocation rates at sites k and $k+1$, becoming therefore, the maximal rate.

Applying lower bound (4.5.21) gives that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \geq \delta\mathbb{E}_p\left(e^{-\gamma_{k,1}Z_{\widehat{m}-1}(\zeta_1)}e^{-\gamma_{k,2}Z_{\widehat{m}-1}(\zeta_2)}\right), \tag{4.5.42}$$

where

$$\widehat{m} = \min\left(m : \gamma_{k,2} e^{\lambda_{k+2} U_m - \lambda m} \geq 1\right). \tag{4.5.43}$$

Equation (4.5.37) yields that $\gamma_{k,1} Z_{\widehat{m}-1}(\zeta_1) < Z(\zeta_1) < C < \infty$. This allows us to rewrite bound (4.5.42) as follows $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \geq \varepsilon_2 \mathbb{E}_p\left(e^{-\gamma_{k,2} Z_{\widehat{m}-1}(\zeta_2)}\right)$, for some ε_2. By assumption $r > z_2$. Let now r_0 be the minimal integer such that $r_0 > z_2$ and $p_0 = p(r_0)$. Then $\lambda_{k+2} p - \lambda > \lambda_{k+2} p_0 - \lambda > 0$ for any $p > p_0$. It follows from Proposition 4.4.6 and Proposition 4.4.7 that for all $p > p_0$

$$\mathbb{E}_p\left(e^{-\gamma_{k,2} Z_{\widehat{m}-1}(\zeta_2)}\right) \geq \mathbb{E}_{p_0}\left(e^{-Z(\eta_2)}\right) > 0, \tag{4.5.44}$$

where η_2 is the sequence reciprocal to ζ_2. Hence $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \geq \varepsilon_2 \epsilon_2 > 0$.

Next, recall event B_k defined in (4.3.4). Note that $A^{k,k+1}_{[1,\widehat{n}]} \cap A^{k+2}_{\widehat{n}+1} \subseteq B_k$, so that $\mathsf{P}_{\mathbf{x}}(B_k) \geq \varepsilon_2 \epsilon_2 / N > 0$ as well.

It is left to show that the maximal rate $\max_i \Gamma_i$ relocates as described in (4.3.6). Clearly, this is always the case if $\lambda < \min(\lambda_{k+2}, \lambda_{k+3})$. This might not be the case in the following particular situation. Namely, suppose that $\lambda_{k+3} \leq \lambda$ and initial configuration $\mathbf{x}$ is such that $\Gamma_k = \max_i \Gamma_i$ and $\Gamma_{k+3} e^{\lambda_{k+3}} \geq \Gamma_k$. In this case, if $\tau_{k+2} = 1$, then the maximal rate might move to $k+3$. However, note that $\tau_{k+2} \geq 2$ on event $A^{k,k+1}_{[1,\widehat{n}]}$. Indeed, by definition (4.5.41) $\widehat{n} \geq 1$, and, hence, on this event $\tau_{k+2} \geq 2$ as $\tau_{k+2} > \widehat{n}$, so that at least one particle is deposited at $\{k, k+1\}$ by time τ_{k+2}. It is not hard to check that placing one particle at $\{k, k+1\}$ makes impossible that relocation of $\max_i \Gamma_i$ to $k+3$ when $\lambda_{k+3} \leq \lambda$. □

Proof of Lemma 4.3.10

First, note that the proof of Part 1) of Lemma 4.3.10 is analogous to the proof of Part 1) of Lemma 4.3.7 and we omit technical details. For simplicity of notation we denote $\lambda = \lambda_k = \lambda_{k+1}$ in the rest of the proof.

Proof of Part 2) of Lemma 4.3.10. Recall lower bound (4.5.22)

$$\mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty]}^{k,k+1}\right) \geq \delta \mathbb{E}_p\left(e^{-\gamma_{k,1}Z(\zeta_1)}e^{-\gamma_{k,2}Z(\zeta_2)}\right).$$

Note that $z_1 < r < z_2$ if and only if both $\lambda_{k-1}(1-p) - \lambda < 0$ and $\lambda_{k+2}p - \lambda < 0$. Therefore, it follows from Proposition 4.4.1 that μ_p-*a.s.* both $Z(\zeta_1) < \infty$ and $Z(\zeta_2) < \infty$. Consequently,

$$\mathbb{E}_p\left(e^{-\gamma_{k,1}Z(\zeta_1)}e^{-\gamma_{k,2}Z(\zeta_2)}\right) \geq \mathbb{E}_p\left(e^{-Z(\zeta_1)}e^{-Z(\zeta_2)}\right) \geq \varepsilon(p) > 0,$$

as $\gamma_{k,i} \leq 1$, $i = 1, 2$, so that $\mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty]}^{k,k+1}\right) \geq \delta\varepsilon(p)$. It is left to note that there is a finite number (depending only on λ's) of possible values of integer-valued parameter r satisfying $z_1 < r < z_2$, and, hence, the same number of possible values of probability p. Therefore, constant $\varepsilon(p)$ can be chosen as the minimal one for those values of p. This concludes the proof of the second part of the lemma. □

Proof of Part 3) of Lemma 4.3.10. Let us start by noting the following. Assumption $r \leq z_1$ implies that $\lambda_{k-1}(1-p) - \lambda \geq 0$, and assumption $r \geq z_2$ implies that $\lambda_{k+2}p - \lambda \geq 0$. Therefore, the law of large numbers yields that μ_p-*a.s.* at least one of the following events $\{\lambda_{k-1}(i - U_i) - \lambda i \geq 0\}$ and $\{\lambda_{k+2}U_i - \lambda i \geq 0\}$ occurs for infinitely many i. Consequently, μ_p-*a.s.* $\prod_{i=0}^{n} \frac{1}{1+\gamma_{k,1}e^{\lambda_{k-1}(i-U_i)-\lambda i}+\gamma_{k,2}e^{\lambda_{k+2}U_i-\lambda i}} \to 0$, as $n \to \infty$. Using bound (4.5.16) and the Lebesgue dominated convergence theorem, we obtain that

$$\mathsf{P}_{\mathbf{x}}\left(A_{[1,n+1]}^{k,k+1}\right) \leq \mathbb{E}_p\left(\prod_{i=0}^{n} \frac{1}{1+\gamma_{k,1}e^{\lambda_{k-1}(i-U_i)-\lambda i}+\gamma_{k,2}e^{\lambda_{k+2}U_i-\lambda i}}\right) \to 0,$$

as $n \to \infty$. Hence, $\mathsf{P}_{\mathbf{x}}\left(A_{[1,\infty]}^{k,k+1}\right) = 0$, and, hence, $\mathsf{P}_{\mathbf{x}}\left(D_k\right) \geq \varepsilon$, as claimed. □

Proof of Lemma 4.3.11

The proof here is similar to the proof of Part 4) of Lemma 4.3.7. The common starting point is the lower bound (4.5.21) where τ and $\tilde{\tau}$ are appropriately chosen stopping times.

Proof of Part 1) and 2) of Lemma 4.3.11. First, note that the random variables $Z(\zeta_1)$ and $Z(\zeta_2)$ are finite if $\lambda_{k-1}(1-p)-\lambda<0$ and $\lambda_{k+2}p-\lambda<0$, respectively. In fact, by our assumptions, precisely one of these conditions is necessarily satisfied so that one of $Z(\zeta_1)$ and $Z(\zeta_2)$ is almost surely finite. Then we apply bound (4.5.21) with the corresponding pair of stopping times $(\tau,\tilde{\tau})=(\widehat{n}_2,\widehat{m}_2)$ or $(\tau,\tilde{\tau})=(\widehat{n}_1,\widehat{m}_1)$ respectively, where

$$\begin{aligned}
\widehat{n}_1 &= \min\left(n : \gamma_{k,1}e^{\lambda_{k-1}(n-S_n)-\lambda n}\geq 1\right),\\
\widehat{n}_2 &= \min\left(n : \gamma_{k,2}e^{\lambda_{k+2}S_n-\lambda n}\geq 1\right),\\
\widehat{m}_1 &= \min\left(m : \gamma_{k,1}e^{\lambda_{k-1}(m-U_m)-\lambda m}\geq 1\right),\\
\widehat{m}_2 &= \min\left(m : \gamma_{k,2}e^{\lambda_{k+2}U_m-\lambda m}\geq 1\right).
\end{aligned}$$

For concreteness, consider the case where $\{k,k+1\}$ is of type 2 and $r>z_1\geq z_2$, in which case $\lambda_{k-1}(1-p)-\lambda<0$ and $\lambda_{k+2}p-\lambda>0$. Applying bound (4.5.21) with $(\tau,\tilde{\tau})=(\widehat{n}_2,\widehat{m}_2)$ yields that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}_2]}\right)\geq\delta\mathbb{E}_p\left(e^{-\gamma_{k,1}Z_{\widehat{m}_2-1}(\zeta_1)}e^{-\gamma_{k,2}Z_{\widehat{m}_2-1}(\zeta_2)}\right).$$

Condition $\lambda_{k-1}(1-p)-\lambda<0$ and Proposition 4.4.1 imply that $Z(\zeta_1)<\infty$ μ_p-a.s. Therefore, we can bound $\gamma_{k,1}Z_{\widehat{m}_2-1}(\zeta_1)\leq Z(\zeta_1)$, as $\gamma_{k,1}\leq 1$. Also, condition $\lambda_{k+2}p-\lambda>0$ and Proposition 4.4.5 imply that $\gamma_{k,2}Z_{\widehat{m}_2-1}(\zeta_2)<\infty\,\mu_p$-a.s. Combining the above, we get to the following lower bound

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}_2]}\right)\geq\delta\mathbb{E}_p\left(e^{-Z(\zeta_1)}e^{-\gamma_{k,2}Z_{\widehat{m}_2-1}(\zeta_2)}\right).$$

Moreover, let η_2 be the sequence reciprocal to ζ_2. Then, applying Proposition 4.4.5 again, we get that $Z(\eta_2)<\infty$ μ_p-a.s., $Z(\eta_2)\geq_{st}\gamma Z_{\widehat{m}-1}(\zeta_2)$ and

$$\mathbb{E}_p\left(e^{-Z(\zeta_1)}e^{-\gamma Z_{\widehat{m}-1}(\zeta_2)}\right)\geq\mathbb{E}_p\left(e^{-Z(\zeta_1)}e^{-Z(\eta_2)}\right)>0.$$

Let us show that, when $r>z_1$, the expectation in the right side of the preceding display is uniformly bounded below over $p=p(r)$. To this end, take the minimal integer r_0 such that $r_0>z_1$ so that condition $r>z_1$ implies $p>p_0=p(r_0)$, and,

hence, $\lambda_{k-1}(1-p)-\lambda < \lambda_{k-1}(1-p_0)-\lambda < 0$ and $\lambda_{k+2}p-\lambda > \lambda_{k+2}p_0-\lambda > 0$. This implies the following. First, consider the random variable $Z(\zeta_1)$ with distribution determined by parameter p_0. By Propositions 4.4.1 and 4.4.7, it follows that $Z(\zeta_1)$ is almost surely finite, and, moreover, it stochastically dominates any other random variable $Z(\zeta_1)$ with distribution determined by $p > p_0$. Second, consider the random variable $Z(\eta_2)$, where η_2 is a sequence reciprocal to sequence ζ_2 whose distribution is determined by parameter p_0. By Propositions 4.4.1, 4.4.3 and 4.4.7, it follows that $Z(\eta_2)$ is almost surely finite and, moreover, it stochastically dominates any other random variable $Z(\eta_2)$, where η_2 is reciprocal to ζ_2 whose distribution is determined by $p > p_0$.

Therefore, $\mathbb{E}_p\left(e^{-Z(\zeta_1)}e^{-Z(\eta_2)}\right) \geq \mathbb{E}_{p_0}\left(e^{-Z(\zeta_1)}e^{-Z(\eta_2)}\right)$. Summarizing the above, we finally obtain that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}_2]}\right) \geq \delta\mathbb{E}_{p_0}\left(e^{-Z(\zeta_1)}e^{-Z(\eta_2)}\right) > 0. \tag{4.5.45}$$

We have considered here only the case where $\{k, k+1\}$ is of type 2 and $r > z_1$, but by rearranging the stopping times above, one should note that for all the remaining cases stated in Parts 1) and 2) of Lemma 4.3.11, the reasoning is exactly the same as above. □

Proof of Part 3) of Lemma 4.3.11. Let us obtain the lower bound in Part 3) of Lemma 4.3.11. In this case $\{k, k+1\}$ is a local minimum of type 2 and $z_2 < r < z_1$. The double inequality implies that both $\lambda_{k-1}(1-p)-\lambda > 0$ and $\lambda_{k+2}p-\lambda > 0$. As a result, both $Z(\zeta_1)$ and $Z(\zeta_2)$ are infinite. In this case we modify bound (4.5.21) with stopping times $\tau = \widehat{n} = \min(\widehat{n}_1, \widehat{n}_2)$ and $\tilde{\tau} = \widehat{m} = \min(\widehat{m}_1, \widehat{m}_2)$, as follows

$$\begin{aligned}\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) &\geq \delta\mathbb{E}_p\left(e^{-\gamma_{k,1}Z_{\widehat{m}-1}(\zeta_1)}e^{-\gamma_{k,2}Z_{\widehat{m}-1}(\zeta_2)}\right)\\ &\geq \delta\mathbb{E}_p\left(e^{-\gamma_{k,1}Z_{\widehat{m}_1-1}(\zeta_1)}e^{-\gamma_{k,2}Z_{\widehat{m}_2-1}(\zeta_2)}\right),\end{aligned}$$

where in the last inequality we bounded $\widehat{m} = \min(\widehat{m}_1, \widehat{m}_2)$ by $\widehat{m}_1$ and $\widehat{m}_2$ respectively. By Proposition 4.4.5 μ_p-*a.s.* both $\gamma_{k,1}Z_{\widehat{m}_1-1}(\zeta_1) < \infty$ and $\gamma_{k,2}Z_{\widehat{m}_2-1}(\zeta_2) < \infty$.

Therefore, $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \geq \varepsilon(p) > 0$. Further, there are finitely many integers r such that $z_2 < r < z_1$. Consequently, there are finitely many corresponding values of probability p, and $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \geq \varepsilon$ for some $\varepsilon > 0$ uniformly over all values of p in this finite set.

Finally, relocation of the maximal rate in all cases covered by Lemmas 4.3.11 can be shown by modifying the argument used in the proof of Part 4) of Lemma 4.3.7. □

Proof of Lemma 4.3.13

We skip proofs of Parts 1) and 3) as they are analogous to the proofs of Parts 1) and 3) of Lemma 4.3.7. Proofs of Parts 2) and 4) can be obtained by appropriately modifying proofs of Parts 2) and 4) of Lemma 4.3.7 and combining them with the ideas in the proof of Lemma 4.3.4. Modifications are due to condition $\lambda_{k-1} = \lambda$ implying that $z_1 = -\infty < z_2$ (see Remark 4.3.12).

Proof of Part 2) of Lemma 4.3.13. Recall that in this case $r < z_2$, so that $\lambda_{k+2}p - \lambda < 0$ and $p < p_0$, where p_0 is defined in Part 2) of Lemma 4.3.7. Repeating the proof of Part 2) of Lemma 4.3.7 and using that $\gamma_{k,1} \leq p$ and $\gamma_{k,2} \leq 1$ (see (4.5.33) and (4.5.31)) we obtain the following lower bound

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty]}\right) \geq \mathbb{E}_p\left(e^{-pZ(\zeta_1)}e^{-Z(\zeta_2)}\right), \tag{4.5.46}$$

Our assumptions imply that both $Z(\zeta_1)$ and $Z(\zeta_2)$ are almost surely finite by Proposition 4.4.1. Fix $\varepsilon > 0.5$, let $C_1 = C_1(\varepsilon) > 0$ be such that

$$\mu_p\left(pZ(\zeta_1) \leq C_1\right) = \mu_p\left(p\sum_{i=0}^{\infty} e^{-\lambda U_i} \leq C_1\right) \geq \varepsilon \tag{4.5.47}$$

for all $p \in (0,1)$ (see (4.5.35)), and let $C_2 = C_2(\varepsilon)$ be such that $\mu_{p_0}\left(Z(\zeta_2) \leq C_2\right) \geq \varepsilon$. The last inequality yields that $\mu_p\left(Z(\zeta_2) \leq C_2\right) \geq \mu_{p_0}\left(Z(\zeta_2) \leq C_2\right) \geq \varepsilon$, as $Z(\zeta_2)$, with distribution determined by parameter p_0, dominates any random variable $Z(\zeta_2)$ with distribution determined by parameter $p < p_0$. Finally, by using the same elementary argument as in the proof of Lemma 4.3.4, we get that

$\mu_p\left(pZ(\zeta_1) \le C_1, Z(\zeta_2) \le C_2\right) \ge 2\varepsilon - 1$, which implies that the expectation in the right side of (4.5.46) is bounded below away from zero, so that $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\infty]}\right) \ge \varepsilon_1$ for some uniform $\varepsilon_1 > 0$ over configurations $\mathbf{x}$ satisfying $r < z_2$. □

Proof of Part 4) of Lemma 4.3.13. Recall that in this case $r > z_2$, so that $\lambda_{k+2}p - \lambda > 0$ and $p > p_0$, where p_0 is now defined in Part 4) of Lemma 4.3.7. Repeating the proof of Part 4) of Lemma 4.3.7 and using again that $\gamma_{k,1} \le p$ we obtain the following lower bound

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \ge \delta \mathbb{E}_p\left(e^{-pZ(\zeta_1)} e^{-\gamma_{k,2} Z_{\widehat{m}-1}(\zeta_2)}\right),$$

where $\widehat{n}$ and $\widehat{m}$ are defined in (4.5.41) and (4.5.43) respectively. Our assumptions imply that both $Z(\zeta_1)$ and $Z_{\widehat{m}-1}(\zeta_2)$ are almost surely finite by Propositions 4.4.1 and 4.4.5. Further, Proposition 4.4.5 yields that

$$\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \ge \delta \mathbb{E}_p\left(e^{-pZ(\zeta_1)} e^{-Z(\eta_2)}\right), \tag{4.5.48}$$

where η_2 is the random sequence reciprocal to ζ_2.

Let $\varepsilon > 0.5$ and $C_1 = C_1(\varepsilon) > 0$ be such that (4.5.47) holds, and let $C_2 = C_2(\varepsilon)$ be such that $\mu_{p_0}\left(Z(\eta_2) \le C_2\right) \ge \varepsilon$. The last inequality yields that

$$\mu_p\left(Z(\eta_2) \le C_2\right) \ge \mu_{p_0}\left(Z(\eta_2) \le C_2\right) \ge \varepsilon,$$

as $Z(\eta_2)$, with distribution determined by parameter p_0, dominates any random variable $Z(\eta_2)$ with distribution determined by parameter $p > p_0$.

As at the same stage of the proof in Part 2) we can now conclude that the expectation in the right side of (4.5.48) is bounded below away from zero, which implies that $\mathsf{P}_{\mathbf{x}}\left(A^{k,k+1}_{[1,\widehat{n}]}\right) \ge \varepsilon_2$ for some uniform $\varepsilon_2 > 0$ over configurations $\mathbf{x}$ satisfying $r > z_2$. □

Proof of Corollary 4.3.14

The critical cases where $r = z_1$ or $r = z_2$ need to be treated separately since these cases can not be proven directly by the above arguments. However, by a slight

modification one can amend the proof of each lemma in order to encompass such critical cases.

The modification is the same for all lemmas, but for the sake of concreteness let us consider the critical case described in Part 3) of Lemma 4.3.7 assuming that $r = z_2$. We start by commenting on the same effect that we already discussed in the proof of Part 4) of Lemma 4.3.7. Namely, recall that if $\lambda_{k+3} < \lambda_k = \lambda_{k+1}$, $\Gamma_{k+3}e^{\lambda_{k+3}} \geq \Gamma_k$, and $\Gamma_k = \max_i \Gamma_i$, then $\tau_{k+2} = 1$ makes the maximal rate move to $k+3$. One can check that the above situation is the only one that can possibly relocate the maximal rate to a site with smaller λ. In order to avoid such case, it is simply a matter of placing a particle at k at the first step, which can be done with probability at least $1/N$. Therefore, without loss of generality we can exclude this case.

Next, if at time τ_{k+2} the maximal rate relocates either to $k+2$, or to $k+3$ (provided $\lambda_{k+3} > \lambda_k = \lambda_{k+1}$) then we are done. Suppose the opposite, namely, that at time τ_{k+2} the maximal allocation rate remains where it was, that is, at k or at $k+1$. It is left to note that given event $A^{k,k+1}_{[1,\tau_{k+2}-1]}$, placing a particle at site $k+2$ at moment τ_{k+2} increases the configuration parameter $r = x_{k+2} - x_{k-1}$ by 1, so that the resulting configuration is such that $r > z_2$. By Part 4) of Lemma 4.3.7, the next allocated particles at $\{k, k+1\}$ will end up by relocating the maximal rate as prescribed.

Other critical cases can be handled similarly, and we skip straightforward technical details.

4.6 Proof of Theorem 4.2.3

The idea of the proof goes briefly as follows. Given any initial state $X(0) = \mathbf{x}$, the site k where $\Gamma_k(\mathbf{x}) = \max_{i=1,\dots,N}(\Gamma_i(\mathbf{x}))$ is identified. Then, a particle allocation strategy is drawn so that it always results in localization of growth as described in Theorem 4.2.3. Lemmas 4.3.1-4.3.13 enable us to identify the corresponding strategy for each particular case and bound its probability from below *uniformly over initial*

configurations. Should a particular strategy fail to happen, which means that at a certain step n a particle is not allocated according to that strategy, but somewhere else, a new one is drawn and this procedure reiterates from $X(n)$. Since there is a finite number of possible strategies it follows from the renewal argument below that almost surely one of them eventually succeeds.

In what follows, when referring to Lemma 4.3.2 or one of Lemmas 4.3.4-4.3.13, this automatically includes the symmetric cases by re-labelling the graph in reverse order (as explained in Remark 4.3.15). Also, local minima of size 2 and type 1 automatically include the limiting case described in Remark 4.3.12.

Let $X(n) = \mathbf{x}$ be a fixed and arbitrary configuration, and:

1) Assume that $\Gamma_k(\mathbf{x}) = \max_{i=1,\dots,N}(\Gamma_i(\mathbf{x}))$ and $\lambda_{k-1} \neq \lambda_k \neq \lambda_{k+1}$.

1.1) Let k be a local maximum. By Lemma 4.3.1, with positive probability, all subsequent particles are allocated at k.

1.2) Let k be either a growth point, or a local minimum. By Lemmas 4.3.2 and 4.3.3, with positive probability, the maximal rate relocates in finite time to one of its nearest neighbours having parameter $\lambda > \lambda_k$.

2) Assume that $\Gamma_k(\mathbf{x}) = \max_{i=1,\dots,N}(\Gamma_i(\mathbf{x}))$ and that additional assumptions of Lemma 4.3.4 are satisfied. Lemma 4.3.4 yields that, with positive probability, all subsequent particles are allocated at sites $\{k, k+1\}$

3) Assume that $\max(\Gamma_k(\mathbf{x}), \Gamma_{k+1}(\mathbf{x})) = \max_i \Gamma_i(\mathbf{x})$, where $\{k, k+1\}$ is either a saddle point, or a local minimum of size 2 and type 1. Additional assumptions on $\mathbf{x}$, as described in Part 2) of Lemmas 4.3.7, 4.3.10 and 4.3.13, guarantee that, with positive probability, all subsequent particles are allocated at sites $\{k, k+1\}$.

4) Assume that $\max(\Gamma_k(\mathbf{x}), \Gamma_{k+1}(\mathbf{x})) = \max_i \Gamma_i(\mathbf{x})$, where $\{k, k+1\}$ is either a saddle point of size 2, or a local minimum of size 2 of *either* type. Assume also that configuration $\mathbf{x}$ is such that assumptions as in the preceding item do *not* hold. Such

cases are covered by Lemmas: 4.3.7, Part 3) and 4); 4.3.10 Part 3); 4.3.11 and 4.3.13; and finally, 4.3.13 Part 3) and 4) complemented by Corollary 4.3.14. In all those cases, with positive probability, the maximal rate eventually relocates in a random but *finite* time to a site with larger parameter λ.

5) Finally, for the remaining cases of local minima, maxima or saddle points of size greater than 2, it is not hard to check that such cases can be reduced to one, or a combination, of the above items.

Thus, for every configuration $\mathbf{x}$ and every set of positive real parameters $\Lambda = (\lambda_k)_{k=1}^N$, we have identified two types of events. First, there are events resulting in localisation of growth at either a single site or a pair of neighbouring sites (as described in Theorem 4.2.3 Part 1) and 2) respectively). Call such events L-events. Second, there are events resulting in relocation of the maximal rate. Call such events R-events.

The next step of the proof is to define a sequence of random moments of time $(T_j)_{j\geq 0}$ called renewal moments. First, set $T_0 = 0$. Now, given T_j, let us define T_{j+1}. Suppose that at time T_j the process is at state $\mathbf{x}$. We identify an event $R_1 \dots R_m L$ (strategy) formed by a sequence of m R-events (possibly none) ending at an L-event. At the fist moment of time $t > T_j$ a particle is not allocated according to $R_1 \dots R_m L$, we set $T_{j+1} = t$.

Note that R-events are defined in a way so that the maximal rate always relocates to a site with strictly larger parameter λ. It follows that the number of R-events preceding any L-event is bounded by the number of different values of λ_i, $i = 1 \dots N$. Then, by Lemmas 4.3.1-4.3.13, probabilities of events $R_1 \dots R_m L$ are bounded below uniformly over configurations, where $m \leq N$.

Further, let $j_{max} := \max\{j \geq 0 : T_j < \infty\}$. Lemmas 4.3.1-4.3.13 imply the existence of an uniform bound $\epsilon > 0$ such that $\mathsf{P}(T_j = \infty) \geq \epsilon$ on $\{T_{j-1} < \infty\}$. Therefore, $\mathsf{P}(T_j < \infty) \leq 1-\epsilon$ on $\{T_{j-1} < \infty\}$, or equivalently, $\mathsf{P}(j_{max} \geq j \,|\, j_{max} \geq j-1) < 1-\epsilon$. Thus, $\mathsf{P}(j_{max} < \infty) = 1$. This implies that $T_j = \infty$ for some j, so that, with

probability one, a certain allocation strategy $R_1 \dots R_m L$ eventually succeeds, that is the growth process localises as claimed.

Finally, the long term behaviour of ratio $X_{k+1}(n)/X_k(n)$ described in item ii) of the theorem is implied by the law of large numbers for the Binomial distribution. This follows straightforwardly from the proofs of Lemma 4.3.4 and Parts 2) of Lemmas 4.3.7, 4.3.10 and 4.3.13. The theorem is proved. □

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