Cluster structures on triangulated non-orientable surfaces

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Abstract: In 2002, Fomin and Zelevinsky introduced a cluster algebra; a dynamical system that has already proved to be ubiquitous within mathematics. In particular, it was shown by Fomin, Shapiro and Thurston [12] that some cluster algebras arise from orientable surfaces. Subsequently, Dupont and Palesi [6] extended this construction to non-orientable surfaces, giving birth to quasi-cluster algebras. The finite type cluster algebras possess the remarkable property of their exchanges graphs being polytopal [16]. We discover that the finite type quasi-cluster algebras enjoy a closely related property, namely, their exchange graphs are spherical. Revealing yet more connections we unify these two frameworks via Lam and Pylyavskyy's Laurent phenomenon algebras [26], showing that both orientable and non-orientable marked surfaces have an associated LP-algebra. The integration of these structures is attempted in two ways. Firstly we show that the quasi-cluster algebras of unpunctured surfaces have LP structures. Secondly, to obtain a connection for all marked surfaces, we consider laminations, forging the notion of the laminated quasi-cluster algebra. We show that each marked surface exhibits a lamination which supplies the laminated quasi-cluster algebra with an LP structure.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. No part of this thesis has been submitted elsewhere for any degree or qualification and it is all my own work unless referenced to the contrary in the text.

Where indicated, some of the results carried out for this degree have been quoted verbatim from my unpublished and published works [35], [36].

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Contents

Chapter 1

Introduction

Cluster algebras were introduced by Fomin and Zelevinsky with the intention of understanding a construction of canonical bases by Lusztig and Kashiwara. Subsequently it has found deep roots in diverse areas of mathematics including Poisson geometry, integrable systems, string theory, quiver representations, polytopes and the theory of surfaces.

The cluster algebra itself is a commutative ring defined by a set of generators called *cluster variables*. These cluster variables are grouped into overlapping finite subsets of the same cardinality. Given a cluster, there is an idea of *mutation*; this broadly consists of obtaining a new cluster by substituting one of the cluster variables. The *cluster structure* is the combinatorics describing how the clusters are connected via the process of mutation. In the theory of cluster algebras the main focus is usually not the underlying ring, but rather the cluster structure. In practice the set of cluster variables and clusters are not known from the outset. Instead one specifies an initial cluster together with an additional piece of combinatorial data to establish the rules of mutation – in the case of cluster algebras this data is a skew-symmetrizable matrix. The rest of the clusters are then obtained by repeated employment of mutation.

One of the most beautiful and visually comprehensible appearance of cluster algebras comes from the study of orientable surfaces [10],[12],[13]. Given an orientable marked surface we may *triangulate* it. For each triangulation T of the surface we may assign a *seed* in which the cluster variables correspond to arcs in T, and the skew symmetric matrix is obtained via the process of inscribing cycles in each triangle, with respect to the surface's orientation. These seeds form a cluster algebra structure where mutations correspond to flipping arcs in triangulations. The underlying reason for this behaviour is explained by recognising that each cluster variable actually represents the *(lambda) length* of their corresponding arc, and the matrices encode how these lengths are related.

Dupont and Palesi aimed at extending the construction to non-orientable surfaces. In this setting it is immediately apparent that the existing method of allocating a seed to a triangulation fails; for a triangulation T of an orientable surface the assignment of a seed crucially relies on the orientability of the surface. However, to avoid this hurdle, Dupont and Palesi aimed at imitating the construction for non-orientable surfaces on a purely geometrical level. Deciding upon a notion of *quasi-triangulation* that guarantees the flippability of every constituent *quasi-arc* they eliminated the requirement of combinatorial machinery by, in essence, directly defining seeds to correspond to these quasi-triangulations. To conjure a cluster structure they endowed the surface with a hyperbolic metric and discovered the various relationships between the *(lambda) lengths* of quasi-arcs. Akin to the orientable case, these relationships are invariant under change of metric, and the lengths of quasi-arcs in any given quasi-triangulation are independent. With this done, their *quasi-cluster algebras* were born. The cluster structure is initiated by fixing a quasi-triangulation T and a set of algebraically independent *cluster variables* corresponding to the quasi-arcs in T. Mutation then consists of performing flips of quasi-arcs and exchanging cluster variables under the relationship governed by how lengths of their corresponding quasi-arcs transform.

It was shown by Fomin and Zelevinsky, almost at the birth of cluster algebras, that the cluster complex of a cluster algebra has a polytopal realisation when the complex is finite. These polytopes coming from finite type cluster algebras are known as the generalised associahedra. A natural question is to ask what kind of structure the quasi-arc complex has. In an attempt to shed some light on this we prove the following theorem.

Theorem A (Corollary 4.51). Let X be a finite quasi-arc complex. Then X is spherical.

Fomin and Zelevinsky [15] proved the remarkable property that every cluster variable in a cluster algebra can be written as a Laurent polynomial in the initial cluster variables. In turn, they settled Gale and Robinson's conjecture on the integrability of generalised Somos sequences, as well as several other like-minded conjectures made by Elkies, Kleber and Propp. It is the unification of cluster algebras with the caterpillar lemma that resolve these conjectures, but cluster algebras certainly do not capture the generality which the lemma provides. In fact, Dupont and Palesi showed their quasi-cluster algebras escape the province of cluster algebras, yet still boast this surprising phenomenon. Aimed at extracting the full potential out of the lemma, Lam and Pylyavskyy concocted their own much broader cluster structure, which, by design, produces the Laurent phenomenon. As such, they befittingly named this structure the Laurent phenomenon algebra, or LP algebra for short.

Lam and Pylyavskyy discovered in [26] that these LP algebras encompass cluster algebras, and also appear naturally as coordinate rings of Lie groups. Subsequently, Gallagher and Stevens [17] demonstrated that their broken Ptolemy algebra exhibits an LP structure. Revealing yet more connections, in this thesis, we link Dupont and Palesi's quasi-cluster algebras to LP algebras. Namely, after making a minor tweak to their definition of a quasi-triangulation, see Definitions 6.6 and 6.7, we prove the following:

Theorem B (Theorem 7.12). *Let (S, M) be an unpunctured (orientable or non-orientable) marked surface. Then the LP cluster complex $\Delta_{LP}(S, M)$ is isomorphic to the quasi-arc complex $\Delta^{\otimes}(S, M)$, and the exchange graph of $\mathcal{A}_{LP}(S, M)$ is isomorphic to $E^{\otimes}(S, M)$.*

More explicitly, let T be a quasi-triangulation of (S, M) and Σ_T its associated LP seed. Then in the LP algebra $\mathcal{A}_{LP}(\Sigma_T)$ generated by this seed the following correspondence holds:

$\mathcal{A}_{\mathbf{LP}}(\mathbf{\Sigma_T})$		$(\mathbf{S}, \mathbf{M})$
Cluster variables	$\longleftrightarrow$	*Lambda lengths of quasi-arcs*
Clusters	$\longleftrightarrow$	*Quasi-triangulations*
LP mutation	$\longleftrightarrow$	*Flips*

The truth of the above theorem crucially depends on the absence of punctures. Embedded within the LP mutation process there is a requirement to *normalise* the exchange polynomials. The obstacle preventing punctured surfaces possessing an LP structure revolves around normalisation occurring when it shouldn't. In particular this instance of undesired normalisation materialises when the exchange polynomials of a quasi-triangulation are not distinct – punctured surfaces are unfortunate enough to possess this trait. To bypass this complication we consider laminated surfaces with the intention of altering the exchange polynomials. Embodying the notion of principal coefficients for orientable surfaces, we introduce *principal laminations*. Crucially, this class of laminations guarantee the uniqueness of exchange polynomials in every quasi-triangulation, allowing us to prove the following:

Theorem C (Theorem 8.50). *Let (S, M) be an orientable or non-orientable marked surface and $\mathbf{L}$ a principal lamination. Then the LP cluster complex $\Delta_{LP}(S, M, \mathbf{L})$ is* *isomorphic to the laminated quasi-arc complex* $\Delta^{\otimes}(S, M, \mathbf{L})$*, and the exchange graph of* $\mathcal{A}_{LP}(S, M, \mathbf{L})$ *is isomorphic to* $E^{\otimes}(S, M, \mathbf{L})$*.*

More explicitly, if (S, M) *is not a once-punctured closed surface, the isomorphisms may be rephrased as follows. Let* T *be a quasi-triangulation of* (S, M) *and* Σ_T *its associated LP seed. Then in the LP algebra* $\mathcal{A}_{LP}(\Sigma_T)$ *generated by this seed the following correspondence holds:*

$\mathcal{A}_{\mathbf{LP}}(\boldsymbol{\Sigma}_{\mathbf{T}})$		$(\mathbf{S}, \mathbf{M}, \mathbf{L})$
Cluster variables	$\longleftrightarrow$	*Laminated lambda lengths of quasi-arcs*
Clusters	$\longleftrightarrow$	*Quasi-triangulations*
LP mutation	$\longleftrightarrow$	*Flips*

The thesis is organised as follows. Chapter 2 presents the basics of cluster algebras including their relationship with orientable surfaces.

Chapter 3 enters the realm of non-orientable surfaces by introducing Dupont and Palesi's quasi-cluster algebras. In Chapter 4 we turn our attention to the quasi-cluster algebras of finite type. Using shellings as our tool, we show the finite type exchange graphs are spherical.

Chapter 5 consists solely of the work of Lam and Pylyavskyy, and describes the construction of their Laurent phenomenon algebras.

In Chapter 6 we embark on our main goal of linking quasi-cluster algebras to LP algebras. To initiate proceedings, in addition to generalising Dupont and Palesi's construction to include punctured surfaces, we make a small alteration to the compatibility relations – as suggested by Pylyavskyy in private communication. This change is in keeping with the flavour of cluster algebras and only affects the compatibility relations; the underlying ring is unaffected by this. By considering the orientable double cover we introduce *anti-symmetric* quivers; a key object that will, in essence, act as our book keeper. We round off the chapter by unearthing a connection between LP mutation and double mutation of anti-symmetric quivers.

In Chapter 7, for reasons explained later on, we consider only unpunctured surfaces. For this class of surface, to verify the quasi-cluster algebras exhibit an LP structure, we first restrict our attention to the quasi-triangulations that lift to triangulations, and we consider their adjacency quivers. By using the *anti-symmetric* property of these quivers we show that, when mutating amongst this type of quasi-triangulation, LP mutation agrees with quasi-cluster mutation. From here, through a case by case check, we show LP and quasi-cluster mutation agree everywhere.

In Chapter 8 we incorporate punctured surfaces into the world of LP algebras by considering laminations on the surface, with the view of defining the *laminated quasi-cluster algebra*. To achieve this we imitate the approach taken by Fomin and Thurston, and define *laminated lambda lengths*; a notion of length that takes into account the lamination as well as the underlying geometry. We call upon anti-symmetric quivers to help us compactly store the exchange relations between these lengths, and furthermore, describe how these quivers change under flips. As an intermediary step, assuming certain reasonable conditions are satisfied, we guarantee that LP mutation agrees with mutation in the laminated quasi-cluster algebra. We conclude the chapter by creating laminations resembling principal coefficients; a class of lamination which guarantees the corresponding laminated quasi-cluster algebra has an LP structure.

Chapter 2

Cluster algebras

In 2002, Fomin and Zelevinsky introduced the notion of a cluster algebra [14]. This section provides a brief description of cluster algebras of geometric type.

Let m, n be positive integers with $m \geq n$. Furthermore, let $\mathcal{F}$ be the field of rational functions in m independent variables. Fix a collection $x_{n+1}, \ldots, x_m$ of algebraically independent variables in $\mathcal{F}$. We define the *coefficient ring* to be $\mathbb{ZP} := \mathbb{Z}[x_{n+1} \ldots x_m]$.

Definition 2.1 (Cluster algebra seed)**.** A ***(cluster algebra) seed*** consists of a pair, $(\mathbf{x}, B)$, where

- $\mathbf{x} = \{x_1, \ldots x_n\}$ is a collection of variables in $\mathcal{F}$ which are algebraically independent over $\mathbb{ZP}$.
- $B = (b_{jk})$ is an $m \times n$ integer matrix whose top $n \times n$ matrix is skew-symmetrizable.

The variables in any seed are called ***cluster variables***. The variables $x_{n+1}, \ldots, x_m$ are called ***frozen variables***.

Remark 1. What we have defined here is actually a cluster algebra seed of *geometric type.* There is a more general notion of a cluster algebra seed involving an extra consideration called *coefficient variables* - however, for our purposes, it is not necessary to give the definition in full generality.

Definition 2.2 (Mutation)**.** Let $i \in \{1, \ldots, n\}$. We define a new seed $\mu_i(\mathbf{x}, B) := (\mathbf{x}', B')$, where

- $\mathbf{x}' = \{x'_1, \ldots x'_n\}$ is defined by:

$$x'_j = \begin{cases} x'_j, & \text{if } j \neq i, \\ \dfrac{\prod\limits_{b_{ki}>0} x_k^{b_{ki}} + \prod\limits_{b_{ki}<0} x_k^{-b_{ki}}}{x_i}, & \text{if } j = i, \end{cases}$$

- $B' = (b'_{jk})$ is defined by:

$$b'_{jk} = \begin{cases} -b_{jk}, & \text{if } j = i \text{ or } k = i, \\ b_{jk} + \max(0, -b_{ji})b_{ik} + \max(0, b_{ik})b_{ji}, & \text{otherwise.} \end{cases}$$

Definition 2.3 (Cluster algebra)**.** Fix an *initial seed* $(\mathbf{x}, B)$. If we label the *initial cluster variables* of $\mathbf{x}$ from $1, \ldots, n$ then we can consider the labelled n-regular tree $\mathbb{T}_n$ generated by this seed through mutations. Each vertex in $\mathbb{T}_n$ has n incident vertices labelled $1, \ldots, n$. Vertices represent seeds and the edges correspond to mutation. In particular, the label of the edge indicates which direction the seed is being mutated in.

Let $\mathcal{X}$ be the set of all cluster variables appearing in the seeds of $\mathbb{T}_n$. $\mathcal{A}_{(\mathbf{x},B)} := \mathbb{ZP}[\mathcal{X}]$ is the ***cluster algebra*** of the seed $(\mathbf{x}, B)$.

Theorem 2.4 (The Laurent phenomenon, [14])**.** Let $\mathcal{A}_{(\mathbf{x},B)}$ be a cluster algebra. Every element of $\mathcal{A}_{(\mathbf{x},B)}$ is a Laurent polynomial, over $\mathbb{ZP}$, in the initial cluster variables of $\mathbf{x}$.

2.1 Cluster algebras from surfaces

An important class of cluster algebras arises from the study of surfaces. This connection to surfaces was established by Fomin, Shapiro and Thurston in [12],[13], inspired by the earlier work of Fock and Goncharov in [10].

Definition 2.5 (Bordered surface, Definition 2.1, [12])**.** Let S be a compact orientable 2-dimensional manifold. Fix a finite set M of marked points of S such that each boundary component contains at least one marked point – marked points in the interior of S are referred to as *punctures*. The tuple (S, M) is called an ***(orientable) bordered surface***. For technical reasons, regarding both existence of triangulations, and uniqueness of flips of arcs in triangulations, we do not allow (S, M) to be an unpunctured or once-punctured monogon; digon; triangle; once, twice or thrice punctured sphere.

Definition 2.6. An ***ordinary arc*** of (S, M) is a simple curve in S connecting two (not necessarily distinct) marked points of M, which is not homotopic to a boundary arc or a marked point.

Definition 2.7 (Definition 7.1, [12])**.** A ***(tagged) arc*** γ is obtained from decorating ('tagging') an ordinary arc at each of its endpoints in one of two ways; ***plain*** or ***notched***. This tagging is required to satisfy the following conditions:

- An endpoint of γ lying on the boundary ∂S must receive a plain tagging.
- If the endpoints of γ coincide they must receive the same tagging.

Definition 2.8 (Definition 7.4, [12])**.** Let α and β be two arcs of (S, M). We say α and β are ***compatible*** *if and only if* the following conditions are satisfied:

- There exist isotopic representatives of α and β that do not intersect in the interior of S.
- Suppose the untagged versions of α and β do not coincide. If α and β share an endpoint a then the ends of α and β at a must be tagged in the same way.
- Suppose the untagged versions of α and β do coincide. Then precisely one end of α must be tagged in the same way as the corresponding end of β.

Definition 2.9 (Definition 7.7, [12])**.** A ***(tagged) triangulation*** of (S, M) is a maximal collection of pairwise compatible arcs of (S, M) containing no arcs that cut out a once-punctured monogon.

Proposition 2.10 (Theorem 7.9, [12])**.** Let T be a triangulation of (S, M). Then for any arc $\gamma \in T$ there exists a unique arc γ' of (S, M) such that $\gamma' \neq \gamma$ and $\mu_\gamma(T) := T \setminus \{\gamma\} \cup \gamma'$ is a triangulation. We call γ' the ***flip*** of γ with respect to T.

We wish to assign a seed to each triangulation of (S, M). To do this it will be helpful to introduce a different, but closely related, notion of a triangulation.

Definition 2.11 (Definition 2.6, [12])**.** An ***ideal triangulation*** of (S, M) is a maximal collection of pairwise non-intersecting ordinary arcs of (S, M).

Remark 2. An ideal triangulation cuts (S, M) into triangles. However, the sides of these triangles may not all be distinct; two sides of the same triangle may be glued together, resulting in a *self-folded triangle.*

Remark 3. To each triangulation T of (S, M) we can associate an ideal triangulation T° by performing the following operations at each puncture p:

- if there are two or more endpoints of arcs in T receiving a notch at p, then replace all these notches with plain taggings;
- if there is precisely one endpoint of an arc γ in T receiving a notch at p, then replace γ with the (unique) arc enclosing γ and p in a once-punctured monogon.

Definition 2.12 (Adjacency matrix, Definition 4.1, [12])**.** Let T be a triangulation, and consider its associated ideal triangulation T°. We may label the arcs of T from $1, \ldots, n$ – note this will also give us a canonical labelling of the arcs in T°. We define a function, $\pi_T : \{1, \ldots, n\} \to \{1, \ldots, n\}$, on the labelling of the arcs in T as follows:

$$\pi_T(i) = \begin{cases} j & \text{if } i \text{ is the glued side of a self-folded triangle in } T^\circ\text{, and } j \text{ is the remaining side;} \\ i & \text{otherwise.} \end{cases}$$

For each non-self-folded triangle Δ in T°, as an intermediary step, define the matrix $B(T)^\Delta = (b^\Delta_{jk})$ by setting

$$b^\Delta_{jk} = \begin{cases} 1 & \text{if } \Delta \text{ has sides } \pi_T(j) \text{ and } \pi_T(k), \\ & \text{and } \pi_T(k) \text{ follows } \pi_T(j) \text{ in a clockwise order;} \\ -1 & \text{if } \Delta \text{ has sides } \pi_T(j) \text{ and } \pi_T(k), \\ & \text{and } \pi_T(k) \text{ follows } \pi_T(j) \text{ in an anti-clockwise order;} \\ 0 & \text{otherwise} \end{cases}$$

The ***(signed) adjacency matrix*** $B(T) = (b_{ij})$ is then defined to be the following summation, taken over all non-self-folded triangles Δ in T°:

$$B(T) := \sum_{\Delta} B(T)^\Delta$$

Definition 2.13 (Surface cluster algebra, [12])**.** Let T be a triangulation of a bordered surface (S, M). Consider the initial seed $(\mathbf{x}, B(T))$, where $\mathbf{x}$ contains a cluster variable for each arc in T, and $B(T)$ is the adjacency matrix defined in Definition 2.12. From this seed, in view of Definition 2.3, we may define the cluster algebra $\mathcal{A}_{(\mathbf{x},B(T))}$.

Proposition 2.14 (Proposition 4.10, [12])**.** Let T_1 and T_2 be two triangulations of a bordered surface (S, M). Then $\mathcal{A}_{(\mathbf{x},B(T_1))} \cong \mathcal{A}_{(\mathbf{x},B(T_2))}$. We may therefore talk about the cluster algebra $\mathcal{A}_{(S,M)}$ of a bordered surface (S, M).

Theorem 2.15 (Theorem 6.1, [13])**.** Let (S, M) be a bordered surface. If (S, M) is not a once punctured closed surface, then in the cluster algebra $\mathcal{A}_{(S,M)}$, the following correspondence holds:

$\mathcal{A}_{(\mathbf{S},\mathbf{M})}$		$(\mathbf{S}, \mathbf{M})$
Cluster variables	$\longleftrightarrow$	*Arcs*
Clusters	$\longleftrightarrow$	*Triangulations*
Mutation	$\longleftrightarrow$	*Flips of arcs*

Remark 4. A similar statement of Theorem 2.15 can be formulated for once punctured closed surfaces. Namely, in this case, cluster variables will correspond to plain arcs (or equivalently notched arcs), and clusters will therefore correspond to triangulations consisting of only plain arcs (notched arcs).

In [13] they actually discovered a deeper connection than we have stated in Theorem 2.15. When endowing the surface with a hyperbolic metric, the cluster variables can be seen to represent the *lambda lengths* of arcs. We provide a more detailed description of this connection in Chapter 6.

Chapter 3

Dupont and Palesi's quasi-cluster algebras

This chapter recalls the work of Dupont and Palesi in [6].

Let S be a compact 2-dimensional manifold with boundary ∂S. Fix a set M of marked points in ∂S, ensuring every boundary component is allocated at least one marked point. The tuple (S, M) is called an (unpunctured) bordered surface. We wish to exclude the cases where (S, M) does not admit any triangulation. As such, we do not allow (S, M) to be a monogon, digon or triangle.

Definition 3.1. An ***arc*** is a simple curve in (S, M) connecting two (not necessarily distinct) marked points.

Definition 3.2. A closed curve in S is said to be ***two-sided*** if it admits a regular neighbourhood which is orientable. Otherwise, it is said to be ***one-sided***.

Definition 3.3. A ***quasi-arc*** is either an arc or a simple one-sided closed curve in the interior of S. Let $A^{\otimes}(S, M)$ denote the set of quasi-arcs in (S, M) considered up to isotopy.

It is well-known that a closed non-orientable surface is homeomorphic to the connected sum of k projective planes $\mathbb{R}P^2$. Such a surface is said to have (non-orientable) genus k. Recall that the projective plane is homeomorphic to a hemisphere where antipodal points on the boundary are identified. A ***cross-cap*** is a cylinder where antipodal points on one of the boundary components are identified. We represent a cross cap as shown in Figure 3.1.

Hence, a closed non-orientable surface of genus k is homeomorphic to a sphere where k open disks are removed, and have been replaced with cross-caps. More generally, a compact non-orientable surface of genus k, with boundary, is homeomorphic to a sphere where more than k open disks are removed, and k of those open disks have been replaced with cross-caps.

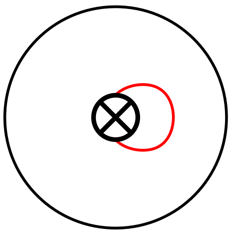


Figure 3.1: A picture of a cross-cap together with a one-sided closed curve.

Definition 3.4. Two elements in $A^{\otimes}(S, M)$ are called ***compatible*** if there exist representatives in their respective isotopy classes that do not intersect in the interior of S.

Definition 3.5. A ***quasi-triangulation*** of (S, M) is a maximal collection of mutually compatible arcs in $A^{\otimes}(S, M)$. A quasi-triangulation is called a ***triangulation*** if it consists only of arcs, i.e, there are no one-sided closed curves.

Proposition 3.6 (Proposition 3.4, [6])**.** Let T be a quasi-triangulation of (S, M). Then for any $\gamma \in T$ there exists a unique $\gamma' \in A^{\otimes}(S, M)$ such that $\gamma \neq \gamma'$ and $\mu_\gamma(T) := T \setminus \{\gamma\} \cup \{\gamma'\}$ is a quasi-triangulation of (S, M).

Definition 3.7. $\mu_\gamma(T)$ is called the ***quasi-mutation*** of T in the direction γ, and γ' is called the ***flip*** of γ with respect to T.

The flip graph of a bordered surface (S, M) is the graph with vertices corresponding to (quasi) triangulations and edges corresponding to flips. It is well known that the flip graph of triangulations of (S, M) is connected. Moreover, it can be seen that every one-sided closed curve, in a quasi-triangulation T, is bounded by an arc enclosing a Möbius strip with one marked point on the boundary. Therefore, if we perform a quasi-flip at each one-sided closed curve in T we arrive at a triangulation. As such, we get the following proposition.

Proposition 3.8 (Proposition 3.12, [6])**.** The flip graph of quasi-triangulations of (S, M) is connected.

Corollary 3.9 (Proposition 3.12, [6])**.** The number of quasi-arcs in a quasi-triangulation of (S, M) is an invariant of (S, M).

Definition 3.10. The ***quasi-arc complex*** $Arc(S, M)$ is the simplicial complex on the ground set $A^{\otimes}(S, M)$ such that k-simplices correspond to sets of k mutually compatible quasi-arcs. In particular, the vertices in $Arc(S, M)$ are the elements of $A^{\otimes}(S, M)$ and the maximum simplices are the quasi-triangulations.

Together, Corollary 3.9 and Proposition 3.6 prove the following proposition.

Proposition 3.11. $Arc(S, M)$ is a pseudo-manifold. That is, each maximal simplex in $Arc(S, M)$ has the same cardinality, and each simplex of co-dimension 1 is contained in precisely two maximal simplices.

Theorem 3.12 (Theorem 7.2, [6])**.** Given a non-orientable bordered surface (S, M) then $Arc(S, M)$ is finite *if and only if* (S, M) is M_n, the Möbius strip with n marked points on the boundary.

Moreover, $Arc(\mathrm{M}_n)$ has some seemingly nice properties. Figure 3.2 shows that for $n \in \{1, 2, 3\}$, $Arc(\mathrm{M}_n)$ is polytopal.

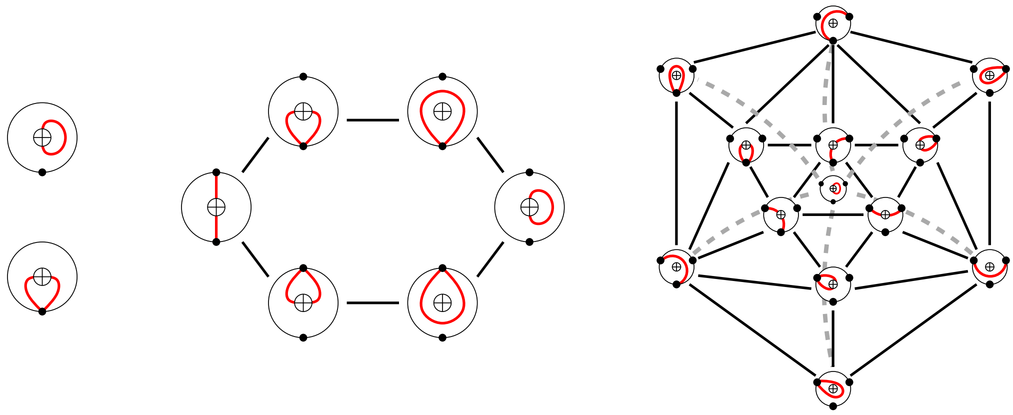


Figure 3.2: The quasi-arc complexes of M_1, M_2 and M_3.

Chapter 4

Sphericity of finite type quasi-arc complexes

This chapter contains the material of the author's article [35].

4.1 Shellability

In this section we recall some basic facts about shellability, and introduce the fundamental ideas used throughout this chapter. These well known concepts can be found in [1], [5], [31].

4.1.1 Definition of shellability and basic facts.

Definition 4.1. An n-dimensional simplicial complex is called ***pure*** if its maximal simplices all have dimension n.

Definition 4.2. Let Δ be a finite (or countably infinite) simplicial complex. An ordering $C_1, C_2 \dots$ of the maximal simplices of Δ is a ***shelling*** if the complex $B_k := \left(\bigcup_{i=1}^{k-1} C_i\right) \cap C_k$ is pure and $(\dim(C_k) - 1)$-dimensional for all $k \geq 2$.

Definition 4.3. The ***simplicial join*** $\Delta_1 * \Delta_2$ of two simplicial complexes Δ_1 and Δ_2 on disjoint ground sets has its faces being sets of the form $\sigma_1 \cup \sigma_2$ where $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$.

The following proposition is a simple and well-known result. For instance, see [31].

Proposition 4.4. The simplicial join $\Delta_1 * \Delta_2$ is shellable *if and only if* the simplicial complexes Δ_1, Δ_2 are both shellable.

In particular, Proposition 4.4 tells us that the cone over a shellable complex is itself shellable.

Proposition 4.5. If $\Delta = Arc(S, M)$ then finding a shelling for Δ is equivalent to ordering the set of triangulations T_i of (S, M) so that $\forall k$ and $\forall j < k$ $\exists i < k$ such that T_i is related to T_k by a mutation and $T_j \cap T_k \subseteq T_i \cap T_k$.

Proof. Note that triangulations T_i of S correspond to maximal simplices in $Arc(S, M)$ and that partial triangulations $T_i \cap T_j$ correspond to simplices of $Arc(S)$. Note that $T_i \cap T_k$ is a $(\dim(T_k) - 1)$-simplex *iff* T_i is a mutation away from T_k. Furthermore, since $B_k := \left(\bigcup_{i=1}^{k-1} T_i\right) \cap T_k$ must be pure and $(\dim(T_k) - 1)$-dimensional for all $k \geq 2$, it follows that B_k is the union of $(\dim(T_k) - 1)$-simplices. So we must have that $\forall j < k$ $\exists i < k$ such that T_i is a mutation away from T_k and the partial triangulation $T_j \cap T_k$ is a face of $T_i \cap T_k$ (i.e $T_j \cap T_k \subseteq T_i \cap T_k$).

□

Proposition 4.5 motivates Definition 4.6.

Definition 4.6. Given a subcollection of triangulations Γ of a surface S call Γ ***shellable*** if it admits an ordering of Γ such that $\forall k$ and $\forall j < k$ $\exists i < k$ such that T_i is related to T_k by a mutation and $T_j \cap T_k \subseteq T_i \cap T_k$.

Definition 4.7. We say two sets of triangulations A, B are ***equivalent*** if their induced simplicial complexes are isomorphic, up to taking cones. If A and B are equivalent we write $A \equiv B$.

Remark 5. Let Δ_A denote the induced simplicial complex of a set of triangulations A. Note that taking a cone over Δ_A can be thought of as disjointly adding one particular arc to every triangulation in A.

The following proposition is just a special case of Proposition 4.4.

Proposition 4.8. If $A \equiv B$ then A is shellable if and only if B is shellable.

<u>**Notation**</u>**:**

- $\operatorname*{list}_{i=1}^{n} x_i$ is the ordering $x_1, x_2, \ldots, x_n$ of the set $\{x_i | 1 \leq i \leq n\}$.

- $\operatorname*{list}_{i \in I} x_i$ is any ordering of the set $\{x_i | i \in I\}$.

- Let $C_{n,0}$ denote the cylinder with n marked points on one boundary component and no marked points on the other. Fix an orientation on the boundary component containing marked points and cyclically label them $1, \ldots, n$. Let $[i, j]$ denote the boundary segment $i \to j$.
  Note that $C_{n,0}$ arises as the partial triangulation of M_n consisting of a one-sided closed curve. We choose the canonical way of defining arcs on $C_{n,0}$.

- Let γ be an arc of $C_{n,0}$ with endpoints i, j. If γ encloses a cylinder with boundary $[j, i] \cup \gamma$ then $\gamma :=< i, j >$. If γ encloses a cylinder with boundary $[i, j] \cup \gamma$ then $\gamma :=< j, i >$, see Figure 4.1.

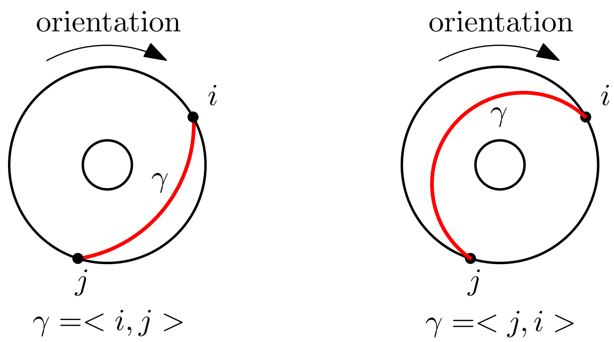


Figure 4.1: Notation for an arc γ of $C_{n,0}$

The following theorem provides a very useful application of shellability.

Theorem 4.9 (Danaraj and Klee,[5])**.** Let Δ be a simplicial complex of dimension n. If Δ is a shellable pseudo-manifold without boundary, then it is a PL n-sphere.

4.1.2 Shellability of $Arc(C_{n,0})$.

The following proposition will help to prove the shellability of $Arc(\mathrm{M}_n)$, and is introduced now to cement key ideas.

Proposition 4.10. $Arc(C_{n,0})$ is shellable for $n \geq 1$.

Proof. Consider the collection of triangulations $T(C^1_{n,0}) \subseteq T(C_{n,0})$ containing a loop at vertex 1. Note that by cutting along the loop we get the $(n+1)$-gon (and a copy of $C_{1,0}$) for $n \geq 2$. We will prove by induction on n that $T(C^1_{n,0})$ is shellable. For $n = 1$ the set $T(C^1_{1,0}) = T(C_{1,0})$ is trivially shellable. For $n = 2$ if we cut along the loop we get the triangle and $C_{1,0}$ which are both trivially shellable, so indeed $T(C^1_{2,0})$ is shellable by Proposition 4.4.

Let Block(i) be the set consisting of all triangulations in $T(C^1_{n,0})$ containing the triangle with vertices $(1, 1, i)$ for some $i \in [2, n]$, see Figure 4.2.
Note that Block(i) can be equivalently viewed as the disjoint union of triangulations of the i-gon and the $(n - i + 2)$-gon. Since $T(C^1_{k,0})$ can be viewed as triangulations of the $(k + 1)$-gon, then if we assume $T(C^1_{k,0})$ is shellable for all $k < n$, Proposition 4.4 tells us that Block(i) is shellable $\forall i \in [2, n]$. Let S(Block(i)) denote a shelling of Block(i).

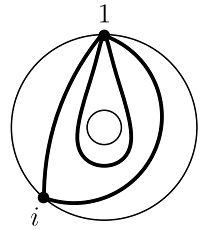


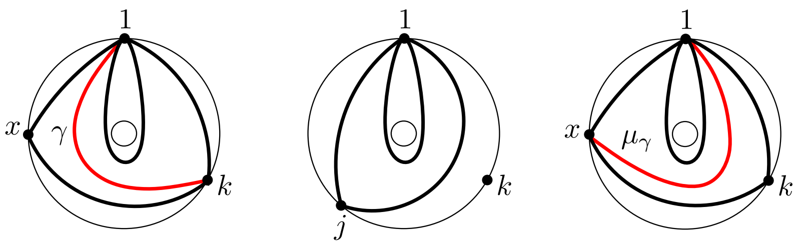
Figure 4.2: Block(i) consists of all triangulations of this partial triangulation.

Claim 1. The ordering $S(C^1_{n,0}) := \operatorname*{list}_{i=n}^{2} S(\text{Block}(i))$ is a shelling for $T(C^1_{n,0})$.

$$\boxed{S(Block(n))}\,,\ \boxed{S(Block(n-1))}\,,\ldots\ldots,\ \boxed{S(Block(2))}$$

***Proof of Claim* 1.** Let S precede T in the ordering $S(C^1_{n,0})$. Then $T \in \text{Block}(k)$ and $S \in \text{Block}(j)$ for $j \geq k$. If $j = k$ then, because $S(\text{Block}(k))$ is a shelling for Block(k), there exists $\gamma \in T$ such that $\mu_\gamma(T)$ precedes T in the ordering, and $T \cap S \subseteq T \cap \mu_\gamma(T)$. We may therefore assume $j > k$. In this case, the arc $\gamma =< k, 1 >\in T$ is not compatible with the arc $< 1, j >\in S$, so $\gamma \notin S$. Hence $T \cap S \subseteq T \cap \mu_\gamma(T)$. By Proposition 4.5 all that remains to show is that $\mu_\gamma(T)$ occurs before T in the ordering.

Note that we will have a triangle in T with vertices $(1, k, x)$ where $x \in [n, k + 1]$. And so $\mu_\gamma(T) \in \text{Block}(x)$. Since $x > k$, $\mu_\gamma(T)$ does precede T in the ordering. See Figure 4.3. Hence $T(C^1_{n,0})$ is shellable.



$T \in Block(k)$ $\qquad$ $S \in Block(j)$ $\qquad$ $\mu_\gamma(T) \in Block(x)$

Figure 4.3

End of proof of Claim 1.

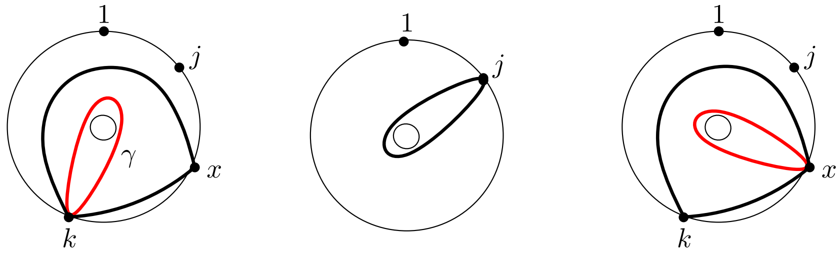
Similarly we can shell $T(C^i_{n,0})$ in the same way $\forall i \in [1, n]$. Denote a shelling by $S(C^i_{n,0})$. Combining these $S(C^i_{n,0})$, as described in Claim 2 below, we get a shelling for $Arc(C_{n,0})$, which completes the proof of the lemma.

□

Claim 2. $S(Arc(C_{n,0})) := \operatorname*{list}_{i=1}^{n} S(C^i_{n,0})$ is a shelling for $Arc(C_{n,0})$

Proof. Let S precede T in the ordering $S(Arc(C_{n,0}))$. Then $T \in S(C^k_{n,0})$ and $S \in S(C^j_{n,0})$ for $1 \leq j \leq k$. Since $S(C^k_{n,0})$ is a shelling we may assume $j < k$. There will be a triangle in T with vertices (k, k, x) for some $x \in [1, n]\backslash\{k\}$.

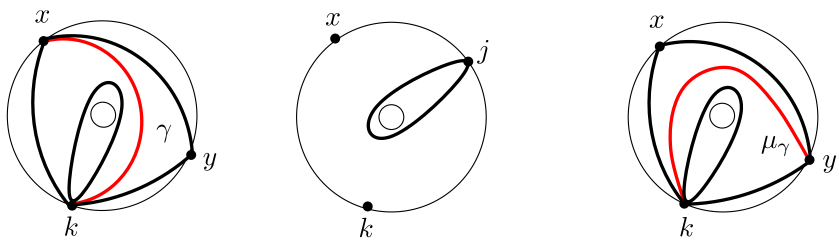
If $x \in [j, k-1]$ then mutate the loop at k to give $T' \in S(C^x_{n,0})$. T' occurs before T in the ordering because $x \in [j, k-1]$. Moreover since the loop at k cannot occur in S then $T \cap S \subseteq T \cap T'$. See Figure 4.4.



$T \in T(C^k_{n,0})$ $S \in T(C^j_{n,0})$ $\mu_\gamma(T) \in T(C^x_{n,0})$

Figure 4.4: Case when $x \in [j, k-1]$

If $x \in [k+1, j-1]$ then the arc $\gamma =< x, k >$ in T is not compatible with the loop at j in S. So $T \cap S \subseteq T \cap \mu_\gamma(T)$. Moreover the way we constructed the shelling $S(C^k_{n,0})$ in Claim 1 means that $\mu_\gamma(T)$ precedes T in the ordering. See Figure 4.5.



$T \in T(C^k_{n,0})$ $S \in T(C^j_{n,0})$ $\mu_\gamma(T) \in T(C^k_{n,0})$

Figure 4.5: Case when $x \in [k+1, j-1]$

□

Corollary 4.11. *Arc*(n-gon) is shellable for $n \geq 3$

Proof. Follows immediately from Claim 1.

□

Applying Theorem 4.9 we rediscover the classical result of Harer, [21].

Corollary 4.12. $Arc(C_{n,0})$ and $Arc(n\text{-gon})$ are PL-spheres of dimension $n-2$ and $n-4$, respectively.

4.2 Shellability of $Arc(M_n)$

In Section 4.1 we achieved shellability of a complex by grouping facets into blocks and finding a 'shelling order' in terms of these blocks. The task was then simplified to finding a shelling of the blocks themselves. Here we essentially follow the same strategy twice. However, on the second iteration of the process we require a specific shelling of the blocks since in general an arbitrary shelling would not suffice.

Definition 4.13. Let $T(\mathrm{M}_n^\circ) \subseteq T(\mathrm{M}_n)$ consist of all triangulations of M_n (i.e, no quasi-triangulations containing a one-sided curve).

Definition 4.14. Let γ be an arc in $T \in T(\mathrm{M}_n^\circ)$. Call γ a ***cross-cap arc*** (c-arc) if $\mathrm{M}_n \setminus \{\gamma\}$ is orientable. (Informally, a c-arc is an arc that necessarily passes through the cross-cap). Let (i, j) denote a c-arc with endpoints i and j.

.

Definition 4.15. Call a triangulation $T \in T(\mathrm{M}_n^\circ)$ a ***cross-cap triangulation*** (c-triangulation) if every arc in T is a c-arc. Let $T(\mathrm{M}_n^\otimes) \subseteq T(\mathrm{M}_n^\circ)$ consist of all c-triangulations.

Definition 4.16. Let γ be an arc in $T \in T(\mathrm{M}_n^\circ)$ that is not a c-arc. Call γ a ***bounding arc*** (b-arc) if it mutates to a c-arc.

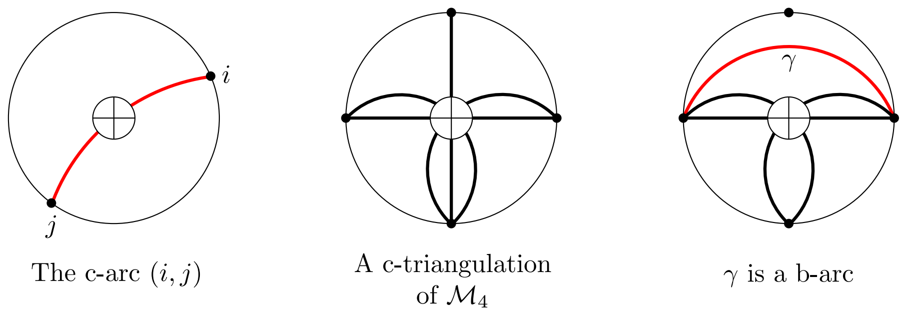


Figure 4.6

4.2.1 Reducing the problem to c-triangulations.

Lemma 4.17. If $T(\mathrm{M}_n^{\otimes})$ is shellable then so is $T(\mathrm{M}_n^{\circ})$.

Proof. Consider $I := \{i_1, \ldots, i_k\} \subseteq [1, n]$. Let $\Gamma_I^{(k)}$ consist of all triangulations $T \in T(\mathrm{M}_n^{\circ})$ such that there is a c-arc in T with endpoint j *if and only if* $j \in I$. Note that this condition implies the existence of an arc or boundary segment $< i_m, i_{m+1} >$ (where $i_{k+1} := i_1$) in every triangulation $T \in \Gamma_I^{(k)}$ $\forall m \in [1, k]$.

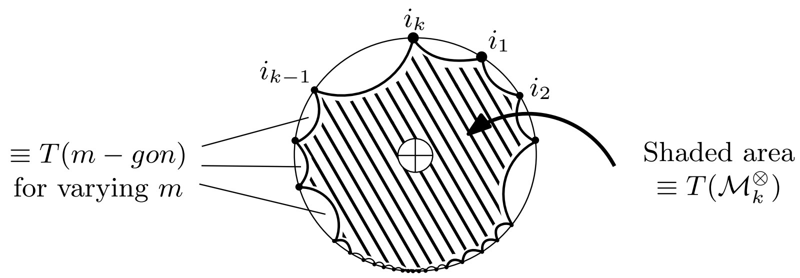


Figure 4.7: $\Gamma_I^{(k)}$

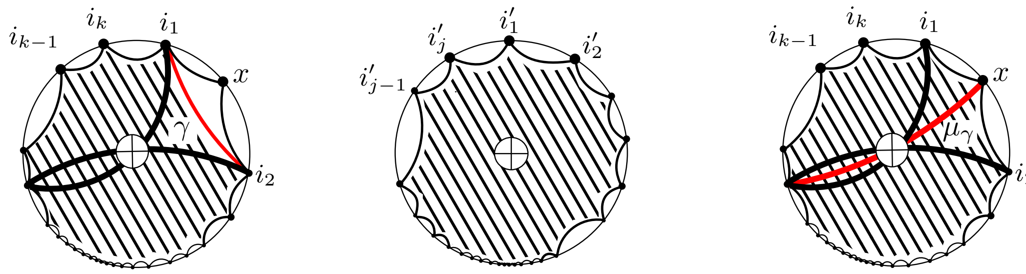
By assumption $T(\mathrm{M}_n^{\otimes})$ is shellable, and by Corollary 4.11 $T(m\text{-gon})$ is also shellable. Hence $\Gamma_I^{(k)}$ is the product of shellable collections of triangulations, and so is shellable by Proposition 4.4. Denote this shelling by $S(\Gamma_I^{(k)})$. Below, Claim 3 shows how a combination of these $S(\Gamma_I^{(k)})$ produce a shelling for $T(\mathrm{M}_n^{\circ})$. This then completes the proof of the lemma.

□

Claim 3. Let $\mathrm{Block}(k) := \underset{I \in [1,n]^{(k)}}{\mathrm{list}} S(\Gamma_I^{(k)})$. Then $\underset{k=n}{\overset{1}{\mathrm{list}}} \mathrm{Block}(k)$ is a shelling for $T(\mathrm{M}_n^{\circ})$.

Proof. Let S precede T in the ordering. Then $S \in \text{Block}(j)$ and $T \in \text{Block}(k)$ where $j \geq k$. In particular, $T \in S(\Gamma_{I_1}^{(k)})$ and $S \in S(\Gamma_{I_2}^{(j)})$ for some $I_1, I_2 \in \mathcal{P}([1,n])$ where $|I_1| \leq |I_2|$. Since $S(\Gamma_I^{(k)})$ is a shelling we may assume $I_1 \neq I_2$.

Suppose that every b-arc in T is also an arc in S. Then $I_2 \subseteq I_1$, and since $|I_1| \leq |I_2|$ this implies $I_1 = I_2$. So we may assume there is at least one b-arc $\gamma \in T$ that is not an arc in S. Since $\gamma \notin S$, $T \cap S \subsetneq T \cap \mu_\gamma(T)$. Moreover, since γ is a b-arc, $\mu_\gamma(T) \in \text{Block}(k+1)$. Hence $\mu_\gamma(T)$ precedes T in the ordering, see Figure 4.8.



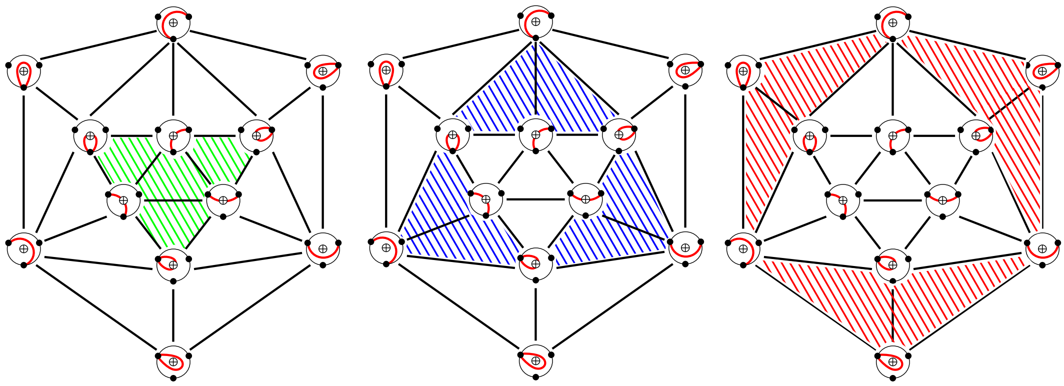
$T \in \Gamma_{I_1}^{(k)}$ $\quad S \in \Gamma_{I_2}^{(j)}$ $\quad \mu_\gamma(T) \in \Gamma_{I_1 \cup \{x\}}^{(k+1)}$

Figure 4.8

□

The idea behind Lemma 4.17 is that we are decomposing $T(\mathrm{M}_n^\circ)$ into blocks, and ordering these blocks. The ordering is chosen in such a way that if we manage to individually shell the blocks themselves, we will have a shelling of $T(\mathrm{M}_n^\circ)$. Figure 4.9 shows the block structure of $T(\mathrm{M}_3^\circ)$.

In particular, we realise that to shell a block it is sufficient to find a shelling of $T(\mathrm{M}_n^\otimes)$. We will split this into two cases: n even and n odd.

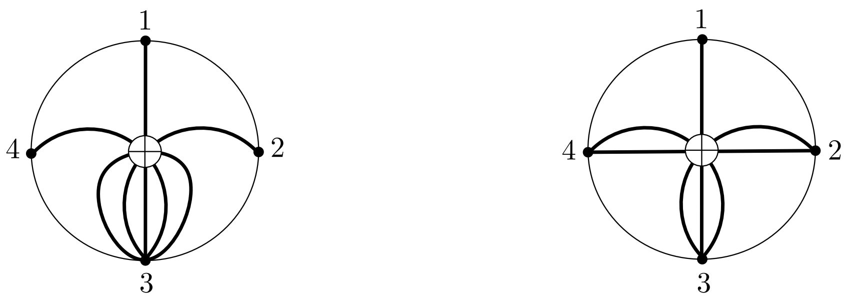


$Block(3) = \Gamma^3_{\{1,2,3\}}$ $Block(2) = \Gamma^2_{\{1,2\}} \cup \Gamma^2_{\{2,3\}} \cup \Gamma^2_{\{1,3\}}$ $Block(1) = \Gamma^1_{\{1\}} \cup \Gamma^1_{\{2\}} \cup \Gamma^1_{\{3\}}$

Figure 4.9: Block structure of $T(\mathrm{M}_3^{\circ})$

4.2.2 Shellability of $T(\mathrm{M}_n^{\otimes})$ for even n.

Let $D^n_{\{(1,\frac{n}{2}+1)\}}$ consist of all triangulations of $T(\mathrm{M}_n^{\otimes})$ containing the c-arc $(1, \frac{n}{2}+1)$ but containing no other c-arcs $(i, \frac{n}{2}+i)$ $\forall i \in [2,n]$. See Figure 4.10.



Example of a triangulation in $D^4_{\{(1,3)\}}$

Example of a triangulation not in $D^4_{\{(1,3)\}}$

Figure 4.10

Definition 4.18. Let $T \in D^n_{\{(1,\frac{n}{2}+1)\}}$ and γ a c-arc in T. $\gamma = (i,j)$ for some $i \in [1, 1+\frac{n}{2}]$ and $j \in [1+\frac{n}{2}, 1]$. Define the ***length*** of γ as follows:

- If $i = j = 1$, $l(\gamma) := n+1$.
- Otherwise, $l(\gamma) := |[i,j]|$.

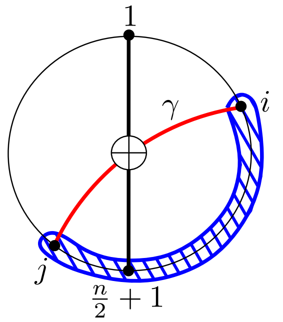


Figure 4.11: If $i \neq 1$ and $j \neq 1$ then the number of marked points in the shaded tube equals $l(\gamma)$.

Definition 4.19. Let $\mathcal{X}_1^n$ be the partial triangulation of M_n consisting of the c-arcs $(1, \frac{n}{2}+1), (2, \frac{n}{2}+1), (n, \frac{n}{2}+1)$. Additionally, let $T(\mathcal{X}_1^n)$ denote the triangulations in $D^n_{\{(1,\frac{n}{2}+1)\}}$ containing the c-arcs $(1, \frac{n}{2}+1), (2, \frac{n}{2}+1), (n, \frac{n}{2}+1)$.
Similarly, let $\mathcal{X}_2^n$ be the partial triangulation of M_n consisting of the c-arcs $(1, \frac{n}{2}+1), (1, \frac{n}{2}), (n, \frac{n}{2}+2)$. Let $T(\mathcal{X}_1^n)$ denote the triangulations in $D^n_{\{(1,\frac{n}{2}+1)\}}$ containing the c-arcs $(1, \frac{n}{2}+1), (2, \frac{n}{2}+1), (n, \frac{n}{2}+1)$. See Figure 4.12.

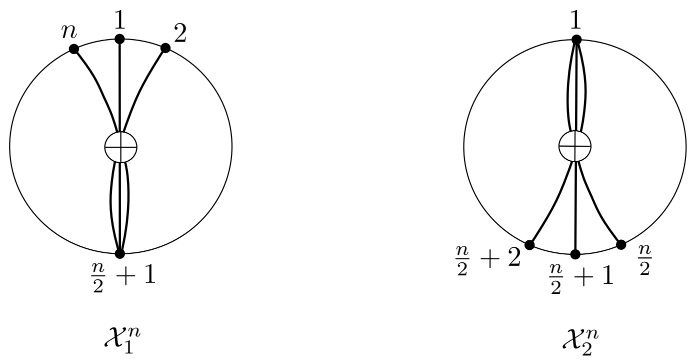


Figure 4.12

Lemma 4.20. $D^n_{\{(1,\frac{n}{2}+1)\}} = T(\mathcal{X}_1^n) \bigsqcup T(\mathcal{X}_2^n)$. Moreover, for any c-arc $\gamma \neq (1, \frac{n}{2}+1)$ in T we have the following:

- $l(\gamma) \leq \frac{n}{2}$ if $T \in T(\mathcal{X}_1^n)$.
- $l(\gamma) \geq \frac{n}{2}+2$ if $T \in T(\mathcal{X}_2^n)$.

Proof. A triangulation T in $D^n_{\{(1,\frac{n}{2}+1)\}}$ will contain either the c-arc $(2, \frac{n}{2}+1)$ or the c-arc $(1, \frac{n}{2}+2)$.

Assume the c-arc $(2, \frac{n}{2}+1)$ is in T. We will show, by induction on i, the c-arc of maximal length in T with endpoint $i \in [2, \frac{n}{2}+1]$ must be the c-arc (i, x) where $x \in [\frac{n}{2}+1, \frac{n}{2}+i-1]$.

Let γ be the c-arc in T of maximal length with endpoint 2. Let j be the other endpoint of γ and suppose for a contradiction $j \in [\frac{n}{2}+2, n]$. Since $(2, \frac{n}{2}+1) \in T$ then, as T is a c-triangulation, $(2, x) \in T$ $\forall x \in [\frac{n}{2}+1, j]$. In particular $\beta := (2, \frac{n}{2}+2) \in T$ - which contradicts $T \in D^n_{\{(1,\frac{n}{2}+1)\}}$. See Figure 4.13.

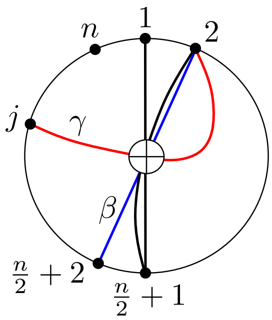


Figure 4.13

By induction, the c-arc α of maximal length in T with endpoint $i-1$ is the c-arc $(i-1, x)$ where $x \in [\frac{n}{2}+1, \frac{n}{2}+i-2]$. Let γ be the c-arc in T of maximal length with endpoint i. Let j be the other endpoint of γ and suppose $j \in [\frac{n}{2}+i, n]$. But by the maximality of α there will be a c-arc (i, y) $\forall y \in [x, j]$. In particular there will be a c-arc $\beta := (i, \frac{n}{2}+i)$. See Figure 4.14.

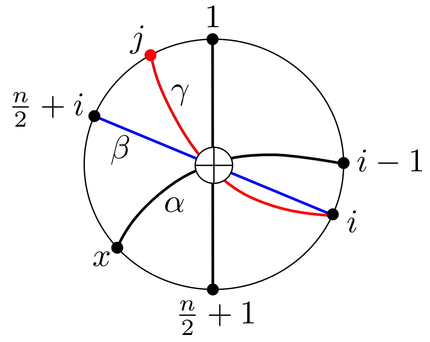


Figure 4.14

If we supposed $(1, \frac{n}{2}+2)$ was an arc in T, then an analogous argument shows that $T \in T(\mathcal{X}_2)$.

□

Corollary 4.21. Let $S \in T(\mathcal{X}_1^n)$ and $T \in T(\mathcal{X}_2^n)$ then $S \cap T = \{(1, \frac{n}{2}+1)\}$

Proof. It follows from the fact that, excluding the c-arc $(1, \frac{n}{2}+1)$, the maximal length of any c-arc in $\mathcal{X}_1^n$ is less than or equal to $\frac{n}{2}$, and the minimal length of any c-arc in $\mathcal{X}_2^n$ is greater than or equal to $\frac{n}{2}+2$.

□

Corollary 4.22. The triangulation $T_{\max}$ in Figure 4.15 is the unique triangulation in $T(\mathcal{X}_1^n)$ such that $\sum_{\gamma \in T_{\max}} l(\gamma)$ is maximal. The triangulation $T_{\min}$ is the unique triangulation in $T(\mathcal{X}_2^n)$ such that $\sum_{\gamma \in T_{\max}} l(\gamma)$ is minimal. More explicitly,

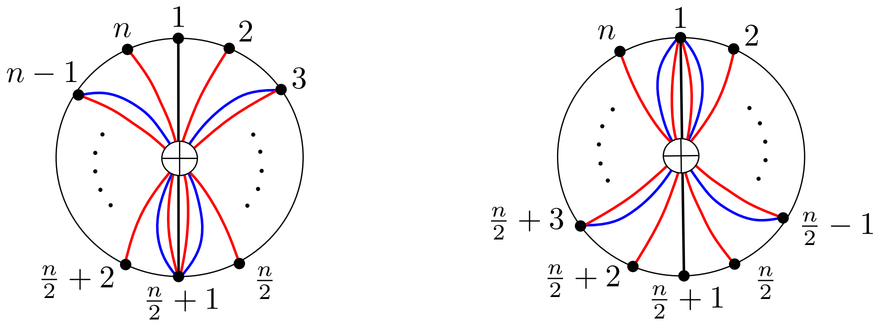
$T_{\max} := \{(1, \frac{n}{2}+1)\} \cup \{(i, \frac{n}{2}+i-1) | i \in [2, \frac{n}{2}+1]\} \cup \{(i, \frac{n}{2}+i-2) | i \in [3, \frac{n}{2}+1]\}$.

$T_{\min} := \{(1, \frac{n}{2}+1)\} \cup \{(i, \frac{n}{2}+i+1) | i \in [1, \frac{n}{2}]\} \cup \{(i, \frac{n}{2}+i+2) | i \in [1, \frac{n}{2}-1]\}$.

Proof. Consider the partial triangulation $\mathcal{P}$ of $\mathcal{X}_1^n$ consisting of all the c-arcs of maximal length. Namely the c-arcs $(i, \frac{n}{2}+i-1)$ $\forall i \in [2, \frac{n}{2}+1]$. $\mathcal{P}$ cuts M_n into (2 triangles and) quadrilaterals bounded by the two boundary segments $[i, i+1], [\frac{n}{2}+i-1, \frac{n}{2}+i]$ and the two c-arcs $(i, \frac{n}{2}+i-1), (i+1, \frac{n}{2}+i)$ $\forall i \in [3, \frac{n}{2}]$. Let T be a triangulation of $\mathcal{P}$ such that $T \in T(\mathcal{X}_1^n)$. Notice that $(i, \frac{n}{2}+i) \notin T$ by definition of $D^n_{\{(1,\frac{n}{2}+1)\}}$, hence $(i+1, \frac{n}{2}+i-1) \in T$ $\forall i \in [3, \frac{n}{2}+1]\}$ and so $T = T_{\max}$. Moreover, since

$l(i+1, \frac{n}{2}+i-1) = l(i, \frac{n}{2}+i-1) - 1$ then T is the unique triangulation in $T(\mathcal{X}_1^n)$ such that $\sum_{\gamma \in T} l(\gamma)$ is maximal.

Analogously we get the result regarding unique minimality of $T_{\min}$.



T_{max} T_{min}

Figure 4.15

□

Definition 4.23. Call a c-arc $(i, \frac{n}{2}+i)$ of M_n a ***diagonal*** arc.

Definition 4.24. Consider a c-arc γ in a triangulation of $\mathcal{X}_1^n$. If $l(\gamma) = \frac{n}{2}$ then call γ a ***max*** arc.

Definition 4.25. Consider a c-arc γ in a triangulation of $\mathcal{X}_2^n$. If $l(\gamma) = \frac{n}{2}+2$ then call γ a ***min*** arc.

Consider a partial triangulation of $\mathcal{X}_1^n$ containing two max arcs. Cutting along these max arcs we will be left with two regions. Let R be the region that does not contain the diagonal arc $(1, \frac{n}{2}+1)$. Note R will contain $2k$ marked points for some $k \in \{2, \dots, \frac{n}{2}\}$.

Lemma 4.26. The set of triangulations of R such that no max arcs occur in R is equivalent to $T(\mathcal{X}_1^{2(k-1)})$.

Proof. Collapse the quadrilateral $(1, 2, \frac{n}{2}+1, n)$ to a c-arc and relabel marked points as shown in Figure 4.16.

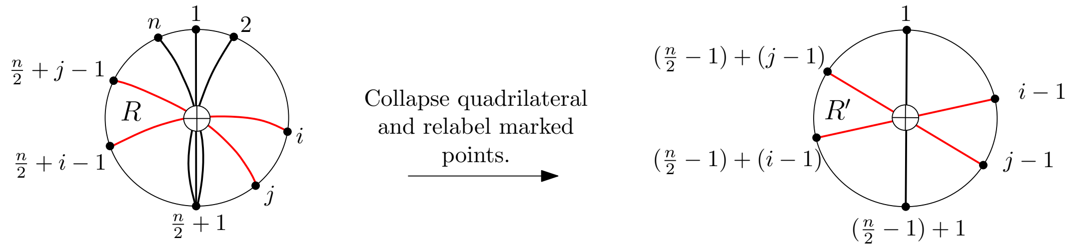


Figure 4.16

Max arcs in R correspond to diagonal arcs in R'. Furthermore, up to a relabelling of vertices, triangulating R' so that no diagonal arcs occur in the triangulation is precisely triangulating $\mathcal{X}_1^{2(k-1)}$ so that no diagonal arcs occur.

□

Remark 6. Using induction we realise that Lemma 4.26 tells us that $D^n_{\{(1,\frac{n}{2}+1)\}}$ has the same flip structure as the set of all Dyck paths of length $n-2$. In particular, triangulations in $D^n_{\{(1,\frac{n}{2}+1)\}}$ correspond to Dyck paths, and arcs appearing in those triangulations correspond to nodes in the Dyck lattice. This correspondence is indicated in Figure 4.17 and is best viewed in colour.

Definition 4.27. Let $i \in \{1,2\}$. Call an arc γ in $T \in T(\mathcal{X}_i^n)$ ***$\mathcal{X}$-mutable*** if $\mu_\gamma(T) \in T(\mathcal{X}_i^n)$.

Definition 4.28. Let γ be an $\mathcal{X}$-mutable arc in a triangulation $T \in D^n_{\{(1,\frac{n}{2}+1)\}}$, and let γ' be the arc γ mutates to. Call γ ***upper-mutable*** if $l(\gamma') > l(\gamma)$ and ***lower-mutable*** if $l(\gamma') < l(\gamma)$.

Definition 4.29. Call a shelling $\mathcal{S}$ of $T(\mathcal{X}_1^n)$ ($T(\mathcal{X}_2^n)$) an ***upper (lower)*** shelling if for any triangulation $T \in \mathcal{S}$ and any upper (lower) mutable arc γ in T, $\mu_\gamma(T)$ precedes T in the ordering.

Definition 4.30. Let $\mathcal{I}$ be the set of all max arcs of $D^n_{\{(1,\frac{n}{2}+1)\}}$, excluding the max arcs $\alpha_1 := (1, \frac{n}{2}+1), \alpha_2 := (\frac{n}{2}+1, n)$.

Lemma 4.31. If $T \in T(\mathcal{X}_1^n)$ does not contain a max arc $m \in \mathcal{I}$ then there exists an upper mutable arc γ strictly contained between the endpoints of m, see Figure 4.18.

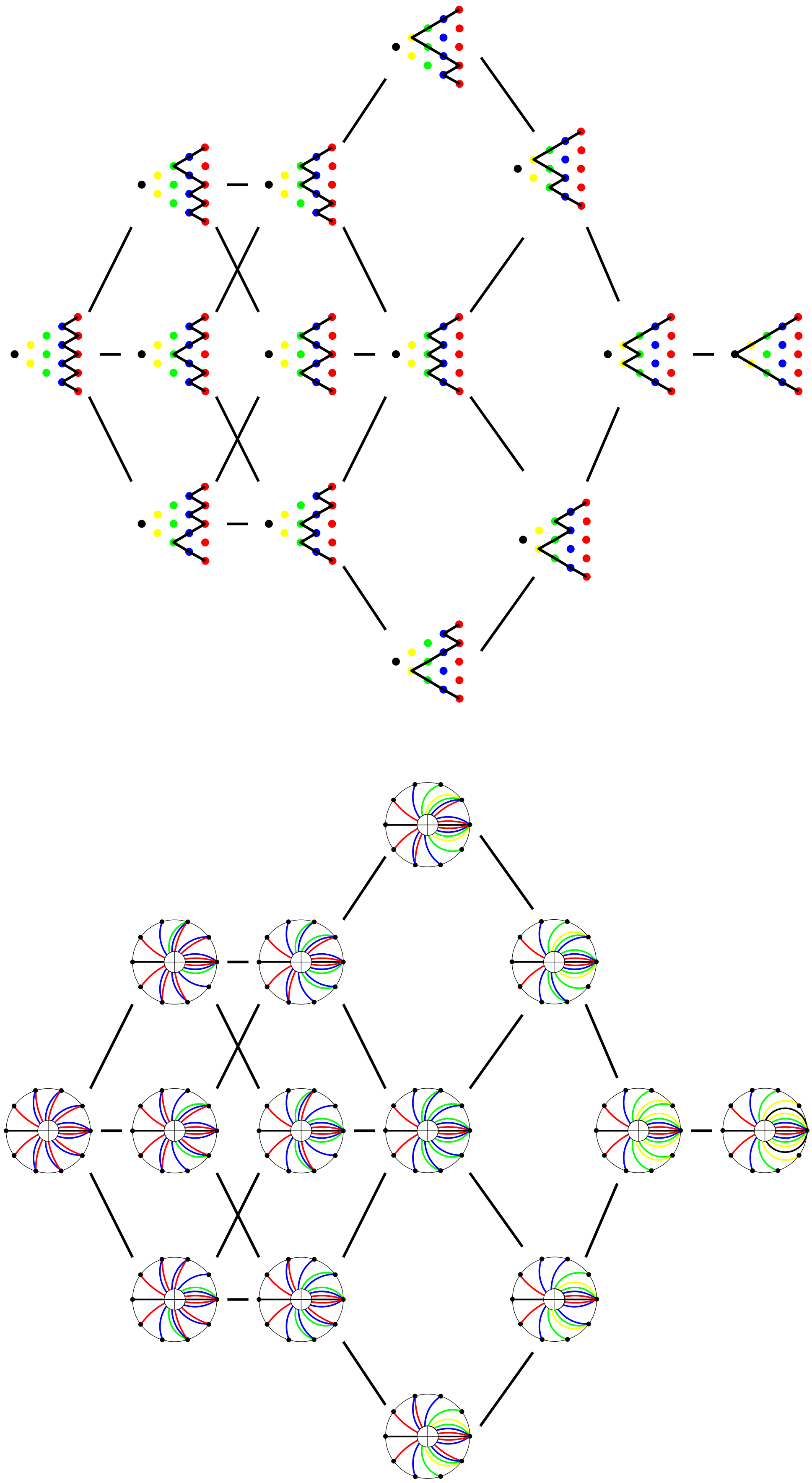


Figure 4.17: $D^n_{\{(1,\frac{n}{2}+1)\}}$ and Dyck paths. See Remark 6 for an explanation of their connection.

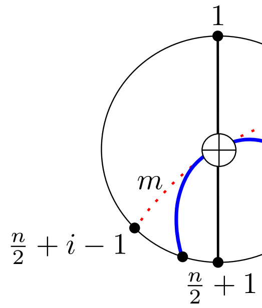


Figure 4.18

Proof. If $n \in \{2, 4\}$ then $\mathcal{I} = \emptyset$ and there is nothing to prove. So assume $n \geq 6$.

Suppose $m = (i, \frac{n}{2} + i - 1) \in \mathcal{I}$ is not in the triangulation T. We will show there exists a c-arc strictly contained between the endpoints of m.

Let (i, x) be the c-arc of maximum length in T connected to i. Since $m \neq (i, x)$ then $x \in [\frac{n}{2} + 1, \frac{n}{2} + i - 2]$. Moreover, by maximality of (i, x), $(i + 1, x) \in T$. So indeed there is a c-arc in T strictly contained between the endpoints of m, see Figure 4.19.

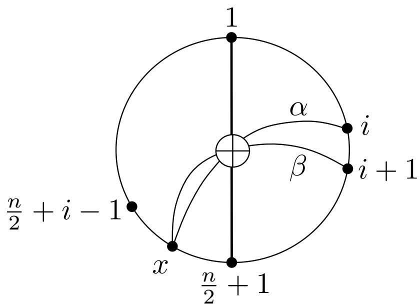


Figure 4.19

Of the c-arcs that are strictly contained between the endpoints of m, let $\gamma = (j_1, j_2)$ be an arc of minimum length. We will show that γ is upper mutable.

By minimality of γ the c-arc $(j_1, j_2 - 1)$ is not in T. Hence the c-arc $(j_1 - 1, j_2)$ must be in T. Likewise the c-arc $(j_1, j_2 + 1) \in T$. So γ is contained in the quadrilateral $(j_1, j_1 - 1, j_2, j_2 + 1)$. Hence mutating γ gives $\gamma' = (j_1 - 1, j_2 + 1)$. $l(\gamma) < l(\gamma')$ so γ is indeed upper mutable, see Figure 4.20.

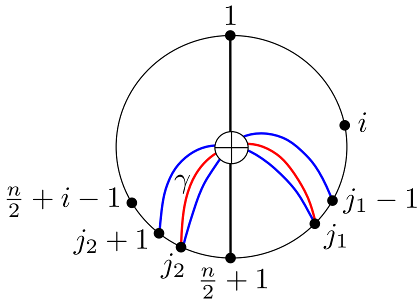


Figure 4.20

□

Lemma 4.32. There exists an upper shelling for $T(\mathcal{X}_1^n)$. Denote this by $S(\mathcal{X}_1^n)$.

Proof. Let $\Psi_{\{\gamma_1,\ldots,\gamma_k\}}$ be the collection of triangulations in $T(\mathcal{X}_1^n)$ containing the max arcs $\gamma_1,\ldots,\gamma_k,\alpha_1,\alpha_2$ and no other max arcs. By Lemma 4.26 we know that $\Psi_{\{\gamma_1,\ldots,\gamma_k\}} \equiv \prod_{i=1}^{j} \mathrm{T}(\mathcal{X}_1^{m_i})$.

Moreover, by induction on the trivial base case when $n=2$, and using Proposition 4.4, we get that there is an upper shelling for $\Psi_{\{\gamma_1,\ldots,\gamma_k\}}$. Denote this shelling by $S(\Psi_{\{\gamma_1,\ldots,\gamma_k\}})$. Merging these $S(\Psi_{\{\gamma_1,\ldots,\gamma_k\}})$ together, as shown in Claim 4 below, we get a shelling for $T(\mathcal{X}_1^n)$, and hence completes the proof of the lemma.

□

Claim 4. Let Block$(k) := \underset{J\in\mathcal{I}^{(k)}}{\text{list}} S(\Psi_J)$. Then $\underset{k=\frac{n}{2}-2}{\overset{0}{\text{list}}}$ Block(k) is an upper shelling for $T(\mathcal{X}_1^n)$.

Proof. Let $T, S \in T(\mathcal{X}_1^n)$ and suppose S precedes T in the proposed ordering. Then $T \in \Psi_{J_1}$ and $S \in \Psi_{J_2}$ where $J_1, J_2 \in \mathcal{P}([1,n])$ and $|J_1| \le |J_2|$. *W.l.o.g.* we may assume $J_1 \neq J_2$ since by induction $S(\Psi_{J_1})$ is an upper shelling.

As $|J_1| \le |J_2|$ and $J_1 \neq J_2$ there is a max arc m in S that is not in T. By Lemma 4.31 there is an upper mutable arc γ in T strictly contained between the endpoints of m. Moreover γ and m are not compatible so $S \cap T \subseteq \mu_\gamma(T) \cap T$. And $\mu_\gamma(T)$ precedes T in the ordering because of the upper shelling $S(\Psi_{J_1})$.

□

An analogous argument proves the following lemma.

Lemma 4.33. There exists a lower shelling for $T(\mathcal{X}_2^n)$. Denote this by $S(\mathcal{X}_2^n)$.

Definition 4.34. Call a c-arc γ in a triangulation $T \in D^n_{\{(1,\frac{n}{2}+1)\}}$ ***special mutable*** if any of the following is true:

- $T \in T(\mathcal{X}_1^n)$ and γ is upper mutable.
- $T \in T(\mathcal{X}_2^n)$ and γ is lower mutable.
- γ mutates to a diagonal c-arc.

Lemma 4.35. For any $T \in T(\mathcal{X}_1^n) \setminus \{T_{max}\}$, T_{max} is connected to T by a sequence of lower mutations.

Proof. By Lemma 4.31 we can keep performing mutations on upper mutable arcs until we reach a triangulation containing every max arc. By Corollary 4.22 the only triangulation in $T(\mathcal{X}_1^n)$ that contains every max arc is T_{max}. Hence T is connected to T_{max} by a sequence of upper mutations. Equivalently, T_{max} is connected to T by a sequence of lower mutations.

□

Lemma 4.36. Let $T \in D^n_{\{(1,\frac{n}{2}+1)\}}$ and let P_T be the partial triangulation of M_n consisting of all the special mutable arcs in T. Then any triangulation of P_T cannot contain the diagonal c-arc $(i, \frac{n}{2}+i)$ $\forall i \in \{2, \dots, \frac{n}{2}\}$.

Proof. Assume $T \in T(\mathcal{X}_1^n)$. An analogous argument works if $T \in T(\mathcal{X}_2^n)$. We prove the lemma via induction on the upper shelling order of $T(\mathcal{X}_1^n)$.

The first triangulation in the upper shelling ordering is T_{max}. The special mutable arcs in T_{max} are $(i, \frac{n}{2}+i-2)$ $\forall i \in [3, \frac{n}{2}+1]$. However, the c-arc $(i, \frac{n}{2}+i-2)$ is not compatible with the diagonal c-arc $(i-1, \frac{n}{2}+i-1)$. And so ranging i over $3, \ldots, \frac{n}{2}+1$ proves the base inductive case.

Let γ be a lower mutable arc in a triangulation $T \in T(\mathcal{X}_1^n)$. By Lemma 4.35, to prove the lemma it suffices to show that the special mutable arcs in $\mu_\gamma(T)$ prevent the same diagonal c-arcs as the special mutable arcs in T. Let β_1, β_2 be the c-arcs containing γ in a quadrilateral. See Figure 4.21.

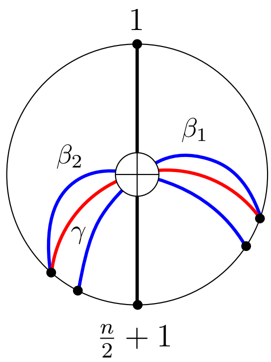


Figure 4.21

The arcs β_1 and β_2 may be special mutable in T but in $\mu_\gamma(T)$ they definitely won't be. The implication of this is that β_1 and β_2 may be c-arcs in P_T, and prevent certain diagonal arcs, but $\beta_1, \beta_2 \notin P_{\mu_\gamma(T)}$ so μ_γ needs to make up this difference. Indeed, it does make up the difference as the diagonal arcs not compatible with either β_1 or β_2 are precisely the diagonal arcs not compatible with μ_γ.

□

Lemma 4.37. In each c-triangulation T of M_n there is at least one diagonal arc.

Proof. Let us assume, for a contradiction, that there is no diagonal arc in T. Without loss of generality, we may assume that the c-arc connected to 1, of maximum length, is $\gamma = (1, j_1)$ for some $j_1 \in [1, \frac{n}{2}]$. (Otherwise just flip the picture.)

Let $\gamma_2 = (2, j_2)$ be the c-arc of maximum length in T that is connected to 2. If $j_2 > \frac{n}{2}$ then by maximality of γ_1 there is a c-arc $(2, \frac{n}{2})$. Hence, $j_2 \in [j_1, \frac{n}{2}+1]$. Inductive reasoning shows that the c-arc connected to $j_1 - 1$ in T, of maximum length, is $\gamma_{j-1} = (j-1, x)$ for some $x \in [j, \frac{n}{2} + j_1 - 2]$. However, then by the maximality of γ_{j-1} we must have $(j_1, \frac{n}{2} + j_1) \in T$. This gives a contradiction, and so the lemma is proved.

□

Lemma 4.38. $T(\mathrm{M}_n^{\otimes})$ is shellable for even n.

Proof. Let $\mathcal{K}$ be the collection of diagonal c-arcs of M_n. Consider $I = \{\gamma_1, \ldots, \gamma_k\} \subseteq \mathcal{K}$ and let D_I^n consist of all triangulations of $T(\mathrm{M}_n^{\otimes})$ containing every diagonal c-arcs in I, and no diagonal c-arcs in $\mathcal{K} \setminus I$. The set of c-triangulations $T(R)$ of a region R cut out by two diagonal c-arcs, so that no other diagonal c-arcs occur in the region, is equivalent to $D^m_{\{(1,\frac{m}{2}+1)\}}$ for some $m \in [2, n-2]$. See Figure 4.22.

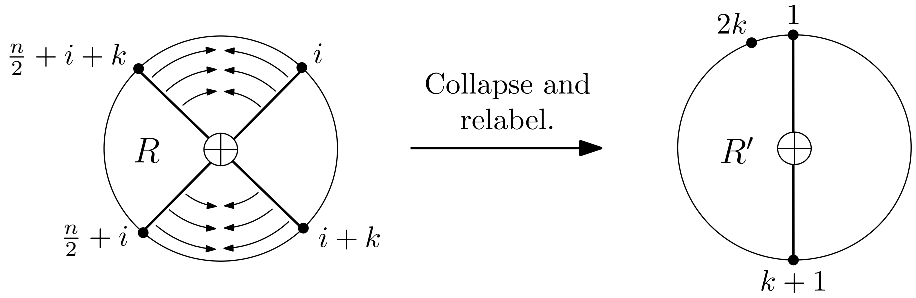


Figure 4.22: $T(R) \equiv T(R') = D^{2k}_{\{(1,k+1)\}}$

Choose $\operatorname*{list}_{i=1}^{2} S(\mathcal{X}_i^m)$ to be the ordering of $D^m_{\{(1,\frac{m}{2}+1)\}}$. Take the disjoint union of these orderings, over all the regions cut out by diagonal c-arcs in I, to get an ordering of D_I^n. Denote this ordering by $O(D_I^n)$. Below, Claim 5 shows that unifying these orderings produces a shelling of $T(\mathrm{M}_n^{\otimes})$, and this completes the proof of the lemma.

□

Claim 5. Let $\mathrm{Block}(k) := \operatorname*{list}_{I \in \mathcal{K}^{(k)}} O(D_I^n)$. Then $\operatorname*{list}_{k=\frac{n}{2}}^{1} \mathrm{Block}(k)$ is a shelling for $T(\mathrm{M}_n^{\otimes})$.

Proof. Let $T, S \in T(\mathrm{M}_n^{\otimes})$ and suppose S precedes T in the ordering. Then $T \in O(D_{I_1}^n)$ and $S \in O(D_{I_2}^n)$ for some $I_1, I_2 \in \mathcal{P}(\mathcal{K})$ where $|I_1| \leq |I_2|$.

If there is a region R in T that contains a special mutable arc γ, such that γ is not an arc in S, then $\mu_\gamma(T)$ precedes T in the ordering and $S \cap T \subseteq \mu_\gamma(T) \cap T$.

So suppose that for every region R of T all special mutable arcs in that region are also arcs in S. Then by Lemma 4.36 $I_2 \subseteq I_1$. Since $|I_1| \leq |I_2|$ we must have $I_1 = I_2$.

If $O(D_I^n)$ was a shelling for D_I^n then the proof would be finished. However, in general, it is not. To understand how we should proceed let us consider $D^n_{\{(1,\frac{n}{2}+1)\}}$.

By definition, $O(D^n_{\{(1,\frac{n}{2}+1)\}}) = \operatorname*{list}_{i=1}^{2} S(\mathcal{X}_i^n)$. Let T be the first triangulation of $S(\mathcal{X}_2)$ and let $S \in S(\mathcal{X}_1)$. Corollary 4.21 tells us that the only arc T and S share in common is the diagonal c-arc $(1, \frac{n}{2}+1)$. If $n = 2$ then $O(D^2_{\{(1,2)\}}) = S, T$ is a shelling for $D^n_{\{(1,2)\}}$. However, if $n \geq 4$ then there are at least 4 arcs in S and T. Hence, $\mu_\gamma(T) \notin S(\mathcal{X}_1^n)$ for any arc γ in T, since $\mu_\gamma(T)$ and S can share at most two arcs in common.

However, as $n \geq 4$ the first triangulation of $S(\mathcal{X}_2^n)$ contains (at least one) arc γ that mutates to a diagonal c-arc. And so $\mu_\gamma(T)$ contains more diagonal c-arcs than T. Hence $\mu_\gamma(T)$ precedes T in the overall ordering for $T(\mathrm{M}_n^{\otimes})$.

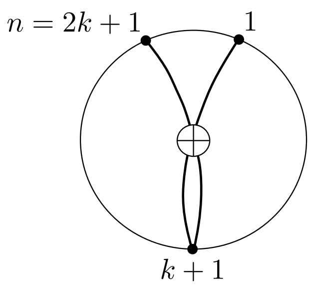
□

4.2.3 Shellability of $T(\mathrm{M}_n^{\otimes})$ for odd n.

In the even case diagonal arcs were a key ingredient in the shelling of $T(\mathrm{M}_n^{\otimes})$. We will see 'diagonal triangles' play the same role in the odd case. For the duration of this section we fix $n = 2k + 1$.

Definition 4.39. A triangle in M_n comprising of two c-arcs $(i, i+k)$, $(i, i+k+1)$ and the boundary segment $(i+k, i+k+1)$ for some $i \in [1, n]$ is called a ***diagonal triangle*** (d-triangle). Additionally, call i the ***special vertex*** of the d-triangle.

Definition 4.40. Let $\mathcal{Y}^n$ be the partial triangulation of M_n containing the d-triangle $(k+1, 2k+1, 1)$. And let $T(\mathcal{Y}^n) \subseteq T(\mathrm{M}_n^{\otimes})$ consist of all c-triangulations of M_n containing the d-triangle $(k+1, 2k+1, 1)$, and no other d-triangles. See Figure 4.23.



$\mathcal{Y}^n$

Figure 4.23

Definition 4.41. Let $T \in T(\mathcal{Y}^n)$ and γ a c-arc in T. $\gamma = (i, j)$ for some $i \in [1, k+1]$ and $j \in [k+1, n]$. Define the ***length*** of γ as $l(\gamma) := j - i + 1$, see Figure 4.24.

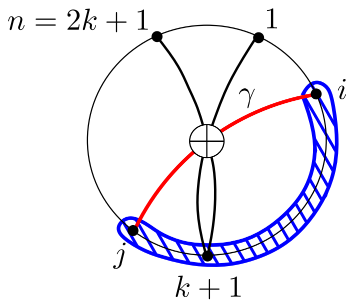


Figure 4.24: 'Number of marked points in shaded tube' $= l(\gamma)$.

Lemma 4.42. The max length of any c-arc in $T \in \mathcal{Y}^n$ is $k+1$.

Proof. Given $T \in T(\mathcal{Y}^n)$ we will prove by induction on $i \in [1, k+1]$ that there is no c-arc in T, with endpoint $k+i$, of length greater than $k+1$. For $i = 1$ this trivially holds. Now assume the statement is true for i. Then there is a c-arc $\gamma = (x, k+i)$ in T where $x \in [i, k+1]$. But the c-arc of maximum length, with endpoint $k+i+1$, that is compatible with γ is $\beta = (x, k+i+1)$. If $x \in [i+1, k+1]$ then indeed

$l(\beta) \leq k+1$. If $x = i$ then we have a d-triangle $(i, k+i, k+i+1)$ with special vertex i, which is forbidden. So indeed $l(\beta) \leq k+1$. □

Lemma 4.43. $T(\mathcal{Y}^n) \equiv T(\mathcal{X}_1^{n+1})$. As such, $T(\mathcal{X}_1^{n+1})$ induces an upper shelling of $T(\mathcal{Y}^n)$. Denote this upper shelling by $S(\mathcal{Y}^n)$.

Proof. Add a marked point to the d-triangle $(k+1, 2k+1, 1)$ in $\mathcal{Y}^n$ and relabel the marked points. Adding the c-arc $(1, k+2)$ we get $\mathcal{X}_1^{n+1}$. Lemma 4.42 tells us the maximum length of an arc in $T \in T(\mathcal{Y}^n)$ is $k+1$. And since the length of a max arc in $T(\mathcal{X}_1^{n+1})$ is also $k+1$ then $T(\mathcal{Y}^n) \equiv T(\mathcal{X}_1^{n+1})$. See Figure 4.25.

□

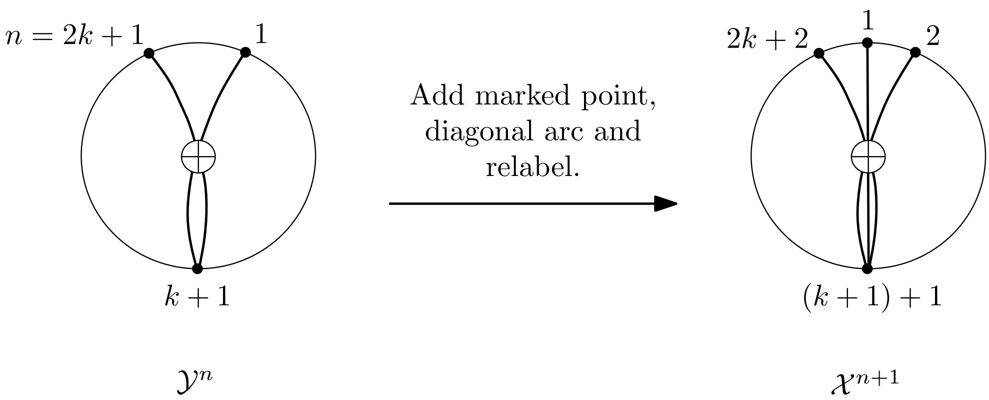


Figure 4.25

Lemma 4.44. For any $T \in T(\mathcal{Y}^n)$ there are an odd number of d-triangles in T. Moreover, the collection of triangulations of any region cut out inbetween d-triangles, such that no other d-triangles occur, is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$.

Proof. We will show that if there are two d-triangles there must in fact be a third. Additionally we'll show the collection of (legitimate) triangulations in any region cut out inbetween the three d-triarcs is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$. And applying induction on this we will have proved the lemma.

Suppose there are at least two d-triangles in a c-triangulation T. Without loss of generality we may assume the two d-triangles $(k+1, 2k+1, 1)$ and $(i, i+k, i+k+1)$ are in T, for some $i \in [1, k]$. See Figure 4.26.

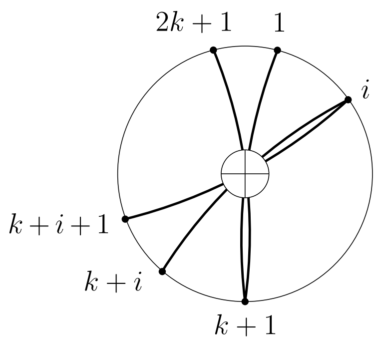


Figure 4.26

We will show there is a third d-triangle with special vertex $z \in [i+k+1, 2k+1]$. Note that if $(i+1, i+k+1) \in T$ then the d-triangle $(i+k+1, i, i+1) \in T$. Similarly, if $(k, 2k+1) \in T$ then the d-triangle $(2k+1, k+1, k) \in T$.

So suppose $(i+1, i+k+1), (k, 2k+1) \notin T$. This then implies $(i+1, x) \in T$ for some $x \in [i+k+2, 2k]$, and $(k, y) \in T$ for some $y \in [i+k+2, 2k]$. In turn, by induction, there is a d-triangle with special vertex $z \in [x, y]$. See Figure 4.27.

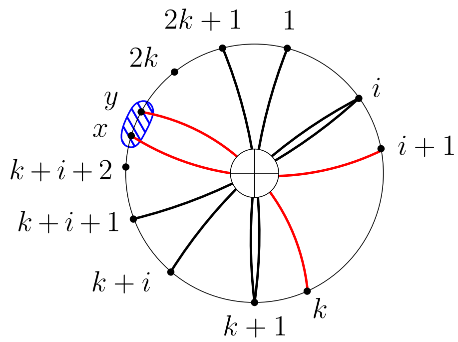


Figure 4.27: By induction there is a d-triangle with its special vertex in the shaded region.

What remains to prove is that each region cut out by these three d-triangles is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$.

Consider the d-triangles $(k+1, 2k+1, 1)$ and $(i, i+k, i+k+1)$ with special vertices $k+1$ and i, respectively. Let R be the region bounded by the c-arcs $(1, k+1)$, $(i, i+k)$ and the boundary segments $[1, i]$,$[k+1, k+i]$. Collapsing the boundary segment $[i, k+1]$ to a point and collapsing $[k+i, 1]$ to a boundary segment preserves the notion of length in R. After collapsing we see that triangulating R (so that no d-triangles occur) is equivalent to triangulating $\mathcal{Y}^{2i-1}$. See Figure 4.28.

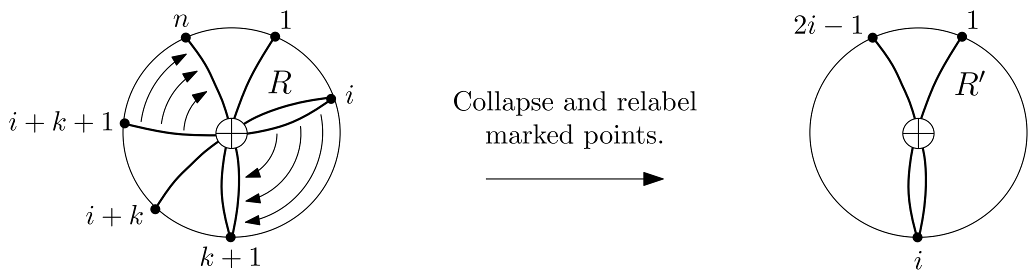


Figure 4.28

Similarly the collection of triangulations of either of the other two regions cut out by the three d-triangles is equivalent to $T(\mathcal{Y}^m)$ for some $m < n$. This completes the proof. □

Definition 4.45. Let $T \in T(\mathcal{Y}^n)$ and let γ be a c-arc in T. Call γ ***special mutable*** if it is upper mutable or $\mu_\gamma(T)$ contains more d-triangles than T.

Lemma 4.46. Let $T \in T(\mathcal{Y}^n)$ and let P_T be the partial triangulation of M_n consisting of all special mutable arcs in T. Then for any triangulation of P_T there is no d-triangle with special vertex i $\forall i \in [1, \dots, n] \setminus k+1$.

Proof. We follow the same idea used in Lemma 4.36. Namely, we will prove the lemma by induction on the shelling order of $S(\mathcal{Y}_n)$.

Let T_1 be the first triangulation in the shelling. Note $\gamma_i = (i, k+i-1)$ is a special mutable c-arc in T_1 $\forall i \in [2, k+1]$. Moreover γ_i is not compatible with the c-arc $(i-1, k+i)$. Hence there is no d-triangle with special vertex $i-1$ or $k+i$ $\forall i \in [2, k+1]$. This proves the base inductive case.

Let $T \in T(\mathcal{Y}^n)$. To prove the lemma by induction it suffices to show that for any lower mutable arc $\gamma \in T$, the d-triangles incompatible with P_T are precisely the d-triangles incompatible with $P_{\mu_\gamma(T)}$.

So let γ be a lower mutable arc in T. Let β_1, β_2 be the c-arcs of the quadrilateral containing γ. See Figure 4.29.

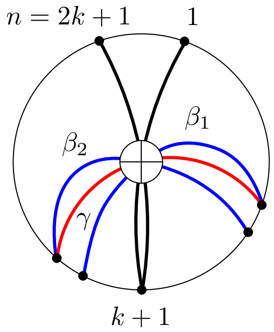


Figure 4.29

Note that β_1 and β_2 could be upper mutable in T, but they will definitely not be upper mutable in $\mu_\gamma(T)$. Analogous to the proof of Lemma 4.36, to prove the lemma it suffices to show μ_γ is incompatible with all the d-triangles incompatible with either β_1 or β_2.

This follows from the fact that a c-arc $\alpha = (x, k+y)$ of length less than k is incompatible with d-triangles with special vertex $z \in [y, x-1] \cup [k+y+1, k+x]$. See Figure 4.30.

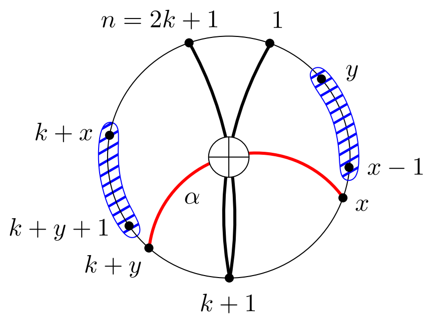


Figure 4.30: α is incompatible with d-triangles whose special vertex lies in one of the shaded regions.

□

An analogous argument to Lemma 4.37 proves the following lemma.

Lemma 4.47. In each c-triangulation T of M_n there is at least one d-triangle.

Lemma 4.48. $T(\mathrm{M}_n^{\otimes})$ is shellable for odd n.

Proof. Let $\mathcal{K}$ be the collection of d-triangles of M_n that can occur in a triangulation without containing any other d-triangles. Consider $I = \{\Delta_1, \dots, \Delta_k\} \subseteq \mathcal{K}$ and let

D_I^n consist of all triangulations of $T(\mathrm{M}_n^{\otimes})$ containing every d-triangle in I, and no d-triangles in $\mathcal{K} \setminus I$.

By Lemma 4.44, each region cut out inbetween the d-triangles in I is shellable. Taking the product of these shellings over all regions gives us a shelling for D_I^n. Denote this shelling by $S(D_I^n)$. Combining these $S(D_I^n)$ produces a shelling for $T(\mathrm{M}_n^{\otimes})$, and thus completes the proof of the lemma. See Claim 6 below.

□

Claim 6. Let $\mathrm{Block}(k) := \operatorname*{list}_{I \in \mathcal{K}^{(k)}} S(D_I^n)$. Then $\operatorname*{list}_{k=\frac{n}{2}}^{1} \mathrm{Block}(k)$ is a shelling for $T(\mathrm{M}_n^{\otimes})$.

Proof. Let $T, S \in T(\mathrm{M}_n^{\otimes})$ and suppose S precedes T in the ordering. Then $T \in S(D_{I_1}^n)$ and $S \in S(D_{I_2}^n)$ for some $I_1, I_2 \in \mathcal{P}(\mathcal{K})$ where $|I_1| \leq |I_2|$.

If there is a region R in T that contains a special arc γ, such that γ is not an arc in S, then $\mu_\gamma(T)$ precedes T in the ordering and $S \cap T \subseteq \mu_\gamma(T) \cap T$.

So suppose that for <u>every</u> region R of T <u>all</u> special arcs in that region are also arcs in S. Then by Lemma 4.46 $I_2 \subseteq I_1$. Since $|I_1| \leq |I_2|$ we must have $I_1 = I_2$. And since $S(D_{I_1}^n)$ is a shelling for D_I^n the claim is proved.

□

Lemma 4.49. $T(\mathrm{M}_n^{\circ})$ is shellable for $n \geq 1$.

Proof. Lemma 4.38 and Lemma 4.48 prove $T(\mathrm{M}_n^{\otimes})$ is shellable for all $n \geq 1$. $T(\mathrm{M}_n^{\circ})$ is therefore shellable by 4.17.

□

Returning to our example of M_3, Figure 4.31 shows a shelling of $T(\mathrm{M}_3^{\circ})$ that we can obtain through our construction.

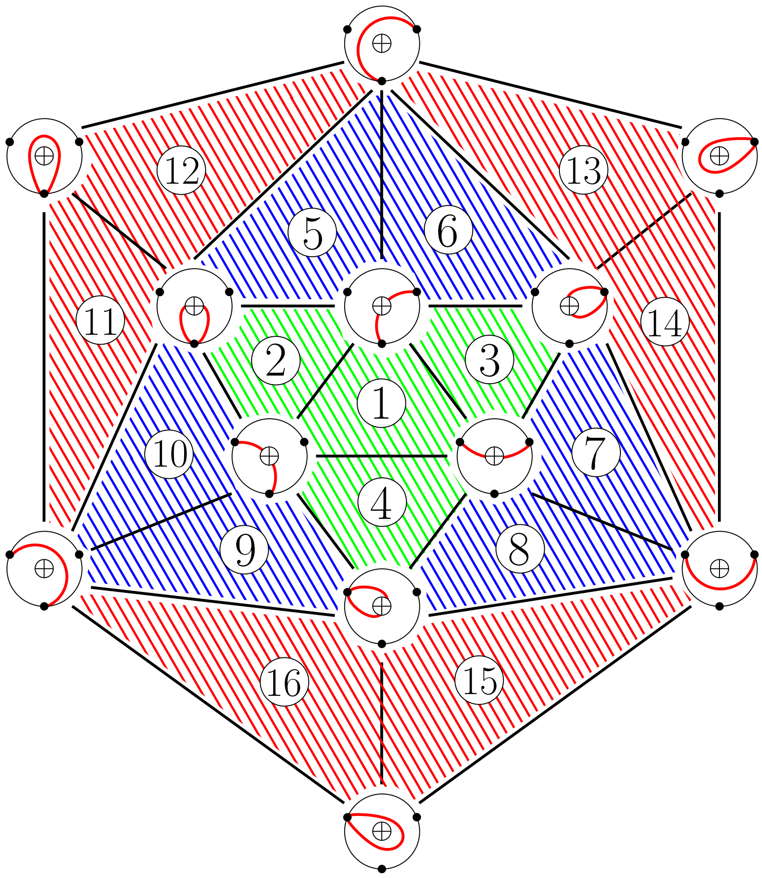


Figure 4.31: Shelling of $T(\mathrm{M}_3^\circ)$

4.2.4 Proof of Theorem A.

Theorem 4.50 (Theorem A)**.** *$Arc(\mathrm{M}_n)$ is shellable for $n \geq 1$.*

Proof. Let $\mathcal{C}$ consist of all quasi-triangulations of M_n containing the one-sided closed curve. Cutting along the one-sided curve in M_n we are left with the marked surface $C_{n,0}$. Therefore the induced simplicial complex of $\mathcal{C}$ is the cone over $Arc(C_{n,0})$. $Arc(C_{n,0})$ is shellable by Proposition 4.10 so Proposition 4.4 tells us $\mathcal{C}$ is also shellable. Let $S(\mathcal{C})$ denote a shelling for $\mathcal{C}$. Let $S(\mathrm{M}_n^\circ)$ be a shelling of $T(\mathrm{M}_n^\circ)$ guaranteed by Lemma 4.49. Coupling these two shellings, as described in Claim 7 below, provides us with a shelling of $Arc(\mathrm{M}_n)$, and this completes the proof of the theorem.

□

Claim 7. Let $S(\mathrm{M}_n) := S(\mathrm{M}_n^\circ), S(\mathcal{C})$. Then $S(\mathrm{M}_n)$ is a shelling for $Arc(\mathrm{M}_n)$.

Proof. Suppose $S, T \in S(\mathrm{M}_n)$ and S precedes T in the ordering. Without loss of generality we may assume $S \in S(\mathrm{M}_n^\circ)$ and $T \in S(\mathcal{C})$. Since T contains the one-sided closed curve γ, and $\gamma \notin S$ then $S \cap T \subseteq \mu_\gamma(T) \cap T$. Moreover, $\mu_\gamma(T) \in S(\mathrm{M}_n^\circ)$ so precedes T in the ordering.

□

Remark 7. Recall that Δ_A denotes the induced simplicial complex of a set of (quasi) triangulations A. Since $\partial\Delta_{T(\mathrm{M}_n^\circ)} \cong Arc(C_{n,0})$ and $\Delta_{\mathcal{C}} \cong \mathrm{Cone}(Arc(C_{n,0}))$ then $Arc(\mathrm{M}_n)$ is the cone over the boundary of $\Delta_{T(\mathrm{M}_n^\circ)}$.

Corollary 4.51. Let X be a finite quasi-arc complex. Then X is spherical.

Proof. Follows immediately from Theorem 3.12, Theorem 4.9 and Theorem 4.50.

□

Chapter 5

Laurent phenomenon algebras

This chapter follows the work of Lam and Pylyavskyy [26]. We will first introduce the notion of a Laurent phenomenon algebra and then conclude the section with the idea of a specialised Laurent phenomenon algebra.

Let R be a unique factorisation domain over $\mathbb{Z}$ and let $\mathcal{F}$ be the rational field of functions in $n \geq 1$ independent variables over the field of fractions $Frac(R)$.

A Laurent phenomenon (LP) ***seed*** in $\mathcal{F}$ is a pair $(\mathbf{x}, \mathbf{F})$ satisfying the following conditions:

- $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a transcendence basis for $\mathcal{F}$ over $Frac(R)$.
- $\mathbf{F} = \{F_1, \ldots, F_n\}$ is a collection of irreducible polynomials in $R[x_1, \ldots, x_n]$ such that for each $i \in \{1, \ldots, n\}$, $F_i \notin \{x_1, \ldots, x_n\}$; and F_i does not depend on x_i .

Just as in cluster algebras, $\mathbf{x}$ is called the ***cluster*** and $x_1, \ldots, x_n$ the ***cluster variables***. $F_1, \ldots, F_n$ are called the ***exchange polynomials***.

Recall that a cluster algebra seed of geometric type $(\mathbf{x}, B)$ consists of a cluster $\mathbf{x} = \{x_1, \ldots, x_n\}$ and an $m \times n$ integer matrix $B = (b_{ij})$ whose top $n \times n$ submatrix is skew-symmetrizable. We can recode this matrix into binomials defined by $F_j^B := \prod_{b_{ij}>0} x_i^{b_{ij}} + \prod_{b_{ij}<0} x_i^{-b_{ij}}$, so there is a strong similarity between the definition of cluster algebra and LP seeds. The key difference being that for LP our exchange relations can be polynomial, not just binomial. However, unlike in cluster algebras, these polynomials are required to be irreducible.

To obtain an LP algebra from our seed we imitate the construction of cluster algebras. Namely, we introduce a notion of mutation of seeds. Our LP algebra will then be defined as the ring generated by all the cluster variables we obtain throughout the mutation process. Before we present the rules of mutation we first need to introduce the idea of normalising exchange polynomials and clarify notation.

Notation:

- Let F, G be Laurent polynomials in the variables $x_1, \ldots x_n$. We denote by $F|_{x_j \leftarrow G}$ the expression obtained by substituting x_j in F by the Laurent polynomial G.
- If F is a Laurent polynomial involving a variable x then we write $x \in F$. Likewise, $x \notin F$ indicates that F does not involve x.

Definition 5.1. Given $\mathbf{F} = \{F_1, \ldots, F_n\}$ then for each $j \in \{1, \ldots, n\}$ we define $\hat{F}_j := \frac{F_j}{x_1^{a_1} \ldots x_{j-1}^{a_{j-1}} x_{j+1}^{a_{j+1}} \ldots x_n^{a_n}}$ where $a_k \in \mathbb{Z}_{\geq 0}$ is maximal such that $F_k^{a_k}$ divides $F_j|_{x_k \leftarrow \frac{F_k}{x}}$, as an element of $R[x_1, \ldots, x_{k-1}, x^{-1}, x_{k+1}, \ldots, x_n]$. The Laurent polynomials of $\hat{\mathbf{F}} := \{\hat{F}_1, \ldots, \hat{F}_n\}$ are called the ***normalised exchange polynomials***.

Example 5.2. Consider the following exchange polynomials in $\mathbb{Z}[a, b, c]$

$$F_a = b + 1, F_b = a + c, F_c = (b + 1)^2 + a^2 b.$$

Since a appears in both F_b and F_c then $\hat{F}_a = F_a$ (see Lemma 5.4). Similarly, $\hat{F}_b = F_b$. As $c \in F_b$ then $b \notin \frac{F_c}{\hat{F}_c}$. However, 2 is the maximal power of F_a that divides $F_c|_{a \leftarrow \frac{F_a}{x}}$, so $\hat{F}_c = \frac{F_c}{a^2}$.

Definition 5.3. Let $(\mathbf{x}, \mathbf{F})$ be a seed and $i \in \{1, \ldots, n\}$. We define a new seed $\mu_i(\mathbf{x}, \mathbf{F}) := (\{x'_1, \ldots, x'_n\}, \{F'_1, \ldots, F'_n\})$. Here $x'_j = x_j$ for $j \neq i$ and $x'_i = \hat{F}_i / x_i$. The exchange polynomials change as follows:

- If $x_i \notin F_j$ then $F'_j := F_j$.
- If $x_i \in F_j$ then F'_j is obtained by following the 3 step process outlined below.

  (Step 1) Define $G_j := F_j|_{x_i \leftarrow \frac{\hat{F}_i|_{x_j \leftarrow 0}}{x'_i}}$

  (Step 2) Define $H_j :=$ (G_j with all common factors with $\hat{F}_i|_{x_j \leftarrow 0}$ divided out). I.e. we have $gcd(H_j, \hat{F}_i|_{x_j \leftarrow 0}) = 1$.

  (Step 3) Let M be the unique monic Laurent monomial in $R[x_1'^{\pm 1}, \ldots, x_n'^{\pm 1}]$ such that $F'_j := H_j M \in R[x'_1, \ldots, x'_n]$ and is not divisible by any of the variables $x'_1, \ldots, x'_n$.

The new seed $\mu_i(\mathbf{x}, \mathbf{F})$ is called the ***mutation*** of $(\mathbf{x}, \mathbf{F})$ in ***direction i***. It is important to note that because of **Step 2** the new exchange polynomials are only defined up to a unit in R.

It is certainly not clear a priori that $\mu_i(\mathbf{x}, \mathbf{F})$ will be a valid LP seed due to the irreducibility requirement of the new exchange polynomials. Furthermore, due to the expression $\hat{F}_i|_{x_j \leftarrow 0}$ appearing in **Step 1** it may not be apparent that the process is even well defined. These issues are resolved by the following two lemmas.

Lemma 5.4 (Proposition 2.7, [26]). $x_i \in F_j \implies x_j \notin \frac{F_i}{\hat{F}_i}$. In particular, $x_i \in F_j$ implies that $\hat{F}_i|_{x_j \leftarrow 0}$ is well defined.

Lemma 5.5 (Proposition 2.15, [26]). F'_j is irreducible in $R[x'_1, \ldots, x'_n]$ for all $j \in \{1, \ldots, n\}$. In particular, $\mu_i(\mathbf{x}, \mathbf{F})$ is a valid LP seed.

Example 5.6. We will perform mutation μ_a at a on the LP seed

$$(\{a, b, c\}, \{F_a = b + 1, F_b = a + c, F_c = (b + 1)^2 + a^2 b\}).$$

Recall from Example 5.2 that $\hat{F}_a = F_a$. Both F_b and F_c depend on a so we are required to apply the 3 step process on each of them. We shall denote the new variable $a' := \frac{\hat{F}_a}{a}$ by d.

$$G_b = F_b\big|_{a \leftarrow \frac{\hat{F}_a|_{b\leftarrow 0}}{d}} = F_b|_{a\leftarrow \frac{1}{d}} = \frac{1}{d} + c.$$

Nothing happens at Step 2 since $\hat{F}_a|_{b\leftarrow 0} = 1$. Multiplying by the monomial d gives us our new exchange polynomial $F_b' = 1 + cd$.

$$G_c = F_c\big|_{a \leftarrow \frac{\hat{F}_a|_{c\leftarrow 0}}{d}} = F_c\big|_{a\leftarrow \frac{b+1}{d}} = (b+1)^2 + \frac{(b+1)^2 b}{d^2}.$$

Following Step 2 we divide G_c by any of its common factors with $\hat{F}_a|_{c\leftarrow 0} = b + 1$. This leaves us with $H_c = 1 + \frac{b}{d^2}$. Finally, multiplying by the monomial d^2 gives us our new exchange polynomial $F_c' = d^2 + b$.

Hence, our new LP seed is

$$(\{d, b, c\}, \{F_d = b + 1, F_b = 1 + cd, F_c = d^2 + b\}).$$

Recall that mutation in cluster algebras is an involution. In the LP setting, because mutation of exchange polynomials is only defined up to a unit in R, it is clear we can't say precisely the same thing for LP mutation. Nevertheless, we do have the following analogue.

Proposition 5.7 (Proposition 2.16, [26])**.** If $(\mathbf{x}', \mathbf{F}')$ is obtained from $(\mathbf{x}, \mathbf{F})$ by mutation at i, then $(\mathbf{x}, \mathbf{F})$ can be obtained from $(\mathbf{x}', \mathbf{F}')$ by mutation at i. It is in this sense that LP mutation is an involution.

Definition 5.8. A ***Laurent phenomenon algebra*** $(\mathcal{A}, \mathcal{S})$ consists of a collection of seeds $\mathcal{S}$, and a subring $\mathcal{A} \subset \mathcal{F}$ that is generated by all the cluster variables appearing in the seeds of $\mathcal{S}$. The collection of seeds must be connected and closed under mutation. More formally, $\mathcal{S}$ is required to satisfy the following conditions:

- Any two seeds in $\mathcal{S}$ are connected by a sequence of LP mutations.

- $\forall\ (\mathbf{x}, \mathbf{F}) \in \mathcal{S}\ \forall i \in \{1, \ldots, n\}$ there is a seed $(\mathbf{x}', \mathbf{F}') \in \mathcal{S}$ that can be obtained by mutating $(\mathbf{x}, \mathbf{F})$ at i.

Definition 5.9 (Subsection 3.6, [26])**.** The ***cluster complex*** $\Delta_{LP}(\mathcal{A})$ of an LP algebra $\mathcal{A}$ is the simplicial complex with the ground set being the cluster variables of $\mathcal{A}$, and the maximal simplices being the clusters.

Definition 5.10 (Subsection 3.6, [26])**.** The ***exchange graph*** of an LP algebra $\mathcal{A}$ is the graph whose vertices correspond to the clusters of $\mathcal{A}$. Two vertices are connected by an edge if their corresponding clusters differ by a single mutation.

Definition 5.11. A ***specialised Laurent phenomenon algebra*** $(\mathcal{A}', \mathcal{S}')$ is the structure obtained from an LP algebra $(\mathcal{A}, \mathcal{S})$ when evaluating elements in the coefficient ring R at 1.

Remark 8. It is worth noting that, unlike in cluster algebras, this specialisation process does not generally produce another LP algebra. We provide a discussion on how this can happen in Section 7.2.

Chapter 6

Revised definition of a quasi-cluster algebra

This chapter continues the work of Dupont and Palesi [6]. Namely, we extend their construction to include punctured surfaces. Sections 6.1 and 6.2 contain results from the author's published work [36].

Let S be a compact 2-dimensional manifold. Fix a finite set M of marked points of S such that each boundary component contains at least one marked point – we will refer to marked points in the interior of S as *punctures*. The tuple (S, M) is called a ***bordered surface***. We wish to exclude cases where (S, M) does not admit a triangulation. As such, we do not allow (S, M) to be an unpunctured or once-punctured monogon; digon; triangle; once or twice punctured sphere; Möbius strip with one marked point on the boundary; or the once-punctured projective space. For technical reasons we also exclude the cases where (S, M) is the thrice-punctured sphere, the twice-punctured projective space, or the once-punctured Klein bottle.

To imitate the construction of cluster algebras arising from orientable surfaces we must first agree on which curves will form our notion of 'triangulation'. Our definitions will be based on the theory developed by Fomin, Shapiro and Thurston on orientable surfaces [12], [13], and Dupont and Palesi on non-orientable surfaces

[6]. As suggested by Pylyavskyy [32], we also make an adjustment to Dupont and Palesi's compatibility relations; this alteration facilitates the eventual connecting of quasi-cluster algebras to Laurent phenomenon algebras.

Definition 6.1. An ***ordinary arc*** of (S, M) is a simple curve in S connecting two (not necessarily distinct) marked points of M, which is not homotopic to a boundary arc or a marked point.

Definition 6.2. An ***arc*** γ is obtained from decorating ('tagging') an ordinary arc at each of its endpoints in one of two ways; ***plain*** or ***notched***. This tagging is required to satisfy the following conditions:

- An endpoint of γ lying on the boundary ∂S must receive a plain tagging.
- If the endpoints of γ coincide they must receive the same tagging.

Definition 6.3. A simple closed curve in S is said to be ***two-sided*** if it emits a regular neighbourhood which is orientable. Otherwise, it is said to be ***one-sided***.

Definition 6.4. A ***quasi-arc*** is either an arc or a one-sided closed curve. Throughout this chapter we shall always consider quasi-arcs up to isotopy. Let $A^{\otimes}(S, M)$ denote the set of all quasi-arcs (considered up to isotopy).

Definition 6.5 (Compatibility of arcs)**.** Let α and β be two arcs of (S, M). We say α and β are ***compatible*** *if and only if* the following conditions are satisfied:

- There exist isotopic representatives of α and β that do not intersect in the interior of S.
- Suppose the untagged versions of α and β do not coincide. If α and β share an endpoint a then the ends of α and β at a must be tagged in the same way.
- Suppose the untagged versions of α and β do coincide. Then precisely one end of α must be tagged in the same way as the corresponding end of β.

To each arc γ bounding a Möbius strip with one marked point, M_1^γ, we associate the two quasi-arcs of M_1^γ. Namely, we associate the one-sided closed curve α_γ and the arc β_γ enclosed in M_1^γ, see Figure 6.1.

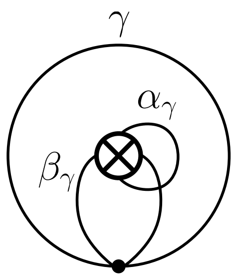


Figure 6.1: The two quasi-arcs α_γ and β_γ enclosed in the Möbius strip, M_1^γ, cut out by an arc γ.

Definition 6.6 (Compatibility of quasi-arcs)**.** We say that two quasi-arcs α and β are ***compatible*** if either:

- α and β are compatible arcs;
- α and β are not both arcs and either: α and β do not intersect, or $\{\alpha, \beta\} = \{\alpha_\gamma, \beta_\gamma\}$ for some arc γ bounding a Möbius strip M_1^γ - see Figure 6.1.

Definition 6.7. A ***quasi-triangulation*** of (S, M) is a maximal collection of pairwise compatible quasi-arcs of (S, M) containing no arcs that cut out a once-punctured monogon or a Möbius strip with one marked point on the boundary. A quasi triangulation is referred to as a ***triangulation*** if it contains no one-sided closed curves.

Definition 6.8. A ***ideal quasi-triangulation*** of (S, M) is a maximal collection of pairwise non-intersecting ordinary arcs and one-sided closed curves of (S, M). We shall refer to the curves comprising a ideal quasi-triangulation as ***ordinary quasi-arcs***.

Remark 9. After putting a hyperbolic metric on (S, M) we need only ever consider the geodesic representatives of ordinary quasi-arcs to decide which collections form ideal quasi-triangulations. This is due to the fact that ordinary quasi-arcs are non-intersecting *if and only if* their geodesic representatives do not intersect.

An analogous statement can be made when deciding which quasi-arcs form quasi-triangulations.

Let T be a quasi-triangulation of (S, M). We may associate T°, an ideal quasi-triangulation, to T as follows:

- If p is a puncture with more than one incident notch, then replace all these notches with plain taggings.
- If p is a puncture with precisely one incident notch, and this notch belongs to β, then replace β with the unique arc γ which encloses β and p in a monogon.
- If α is a one-sided closed curve in T then (by maximality of a quasi-triangulation) there exists a unique arc β in T which intersects α. Replace β with the unique arc γ enclosing α and β in a Möbius strip with one marked point.

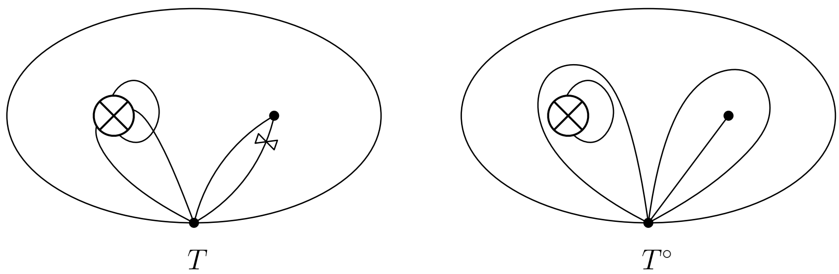


Figure 6.2: Transforming a quasi-triangulation T into an ideal triangulation T°.

Lemma 6.9. Let T be a quasi-triangulation of (S, M). Then T° cuts (S, M) into triangles and annuli.

Proof. Firstly, cut along all arcs in T° (i.e do not cut along any one-sided closed curves) to obtain a collection of connected components. Let K be one of these connected components. Note that because we have cut along arcs, K will have boundary with at least one marked point on each boundary component. Furthermore, we may assume K has only one boundary component and no punctures as otherwise this contradicts the maximality of our quasi-triangulation.

If K is non-orientable then it contains a pre-existing one-sided closed curve $\alpha \in T$. Let γ be a curve that encloses α in a Möbius strip with one marked point. By the maximality of the quasi-triangulation, γ is either isotopic to the boundary of K, forcing K to be the Möbius strip with one marked point, or γ is contractible resulting in (S, M) being the once-punctured projective space. Since we have forbidden the latter case then, if K is non-orientable, it is the Möbius strip with one marked point and a one-sided closed curve. Cutting along the one-sided closed curve yields the annulus with a marked point on one boundary component and the other empty of marked points.

What remains is to consider the case when K is orientable. K cannot be a monogon as then either (S, M) itself is a monogon, or the boundary of K is a contractible curve in (S, M) and is therefore not a valid arc. Similarly, K cannot be a digon as then one of the following situations occur: (S, M) is itself a digon; the two boundary segments of K are isotopic; or (S, M) is obtained from glueing together the boundary of K with the result being the twice punctured sphere or the once-punctured projective space. K cannot have more than four marked points as this would contradict the maximality of the quasi-triangulation. Hence, if K is orientable, it must be a triangle.

□

Proposition 6.10. Let T be a quasi-triangulation of (S, M). Then for any $\gamma \in T$ there exists a unique $\gamma' \in A^{\otimes}(S, M)$ such that $\gamma' \neq \gamma$ and $\mu_\gamma(T) := T \setminus \{\gamma\} \cup \gamma'$ is a quasi-triangulation. We call γ' the ***flip*** of γ with respect to T.

Proof. For a quasi-triangulation T of (S, M) note that performing tag changing transformations at punctures has no effect on the flippability of quasi arcs in T. Therefore, without loss of generality, we may assume that the only instance when a notched arc appears in T is when it is accompanied by its plain counterpart.

To decide the flippability of an arc in T we shall consider its local configuration. We achieve this by first considering the local configurations of quasi-arcs in the associated ideal quasi-triangulation T°, and from here we will then discover the possible local pictures in T.

By Lemma 6.9 we know that T° cuts (S, M) into triangles and annuli - for convenience we shall refer to them as *puzzle pieces*. Therefore any quasi arc of T° is the glued side of two puzzle pieces. We list these glueings in Figure 6.3 to obtain all possible neighbourhoods of a quasi-arc in T°. When the configurations in Figure 6.3 are pulled back to T the only valid local configurations, shown in Figure 6.4, are the quadrilateral, the punctured digon and the Möbius strip with two marked points - as by definition of a bordered surface we have forbidden the instance when (S, M) is the thrice punctured sphere, the twice-punctured projective space, or the once-punctured Klein bottle. Each quasi-arc in the interior of the configurations in Figure 6.4 is uniquely flippable. An important point to add is that the boundary segments of these configurations may in fact be a substituted arc bounding a punctured monogon, or a Möbius strip with one marked point. However, since this substituted arc, and the two quasi-arcs it bounds are compatible with precisely the same quasi-arcs, then this does not affect the existence or uniqueness of the flip in question.

□

Remark 10. The reason we have forbidden (S, M) to be the thrice punctured sphere, the twice-punctured projective space, or the once-punctured Klein bottle should now be clear - the glued side of their corresponding configuration in Figure 6.3 is pulled back to two arcs.

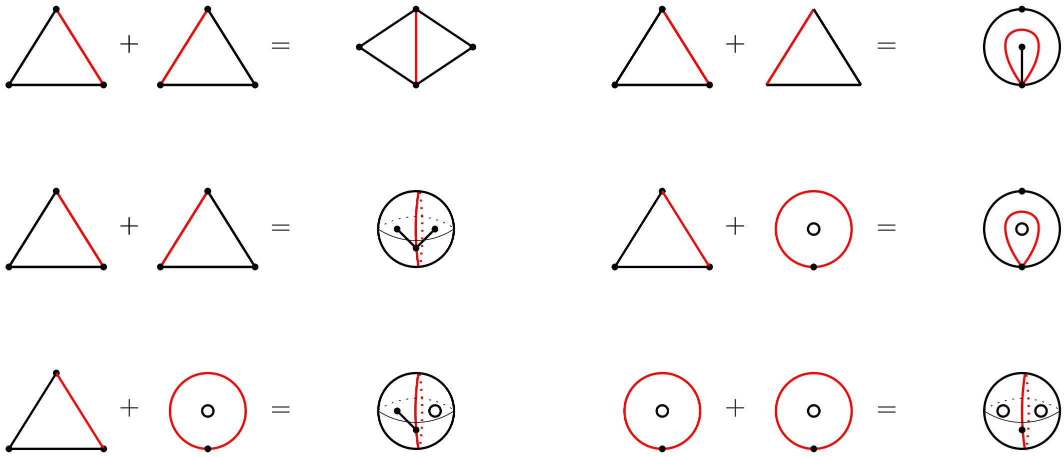


Figure 6.3: The possible glueings of puzzle pieces.

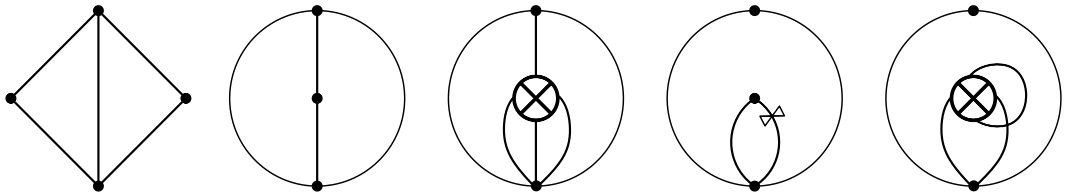


Figure 6.4: The valid pullbacks obtained from glueings of puzzle pieces.

Definition 6.11. We shall refer to a configuration in Figure 6.4 as a ***flip region***. Note that in the proof of Proposition 6.10, as an intermediary step, we saw that every quasi-arc in a quasi-triangulation belongs to the interior of a flip region, up to tag changing transformations at punctures.

Definition 6.12. The ***flip graph*** of a bordered surface (S, M) is the graph with vertices corresponding to quasi-triangulations and edges corresponding to flips.

Harer [20] proved that two ideal triangulations on an orientable surface are connected via a sequence of flips. This result applies equally well to non-orientable surfaces; for a simple proof of this see Mosher [27]. The following proposition concerning the connectivity of the flip graph follows from the result of Harer, and arguments of Fomin, Shapiro and Thurston [12] regarding the ability to flip between plain and notched arcs in triangulations.

Proposition 6.13. If (S, M) is not a closed once-punctured surface then the flip graph of (S, M) is connected. In the closed once-punctured case the flip graph has two isomorphic connected components: one containing only plain quasi-arcs, and the other containing only notched ones.

Propositions 6.10 and 6.13 tell us that the number of quasi-arcs in a quasi-triangulation is an invariant of (S, M) - this number is called the ***rank*** of (S, M).

We now introduce the notion of a seed of a bordered surface (S, M).

Quasi-seeds and mutation.

Suppose (S, M) is a bordered surface of rank n and let $b_1, \ldots, b_m$ consist of all the boundary segments of (S, M). Denote $\mathcal{F}$ as the field of rational functions in $n+m$ independent variables over $\mathbb{Q}$.

A ***quasi-seed*** of a bordered surface (S, M) in $\mathcal{F}$ is a pair $(\mathbf{x}, T)$ such that:

- T is a quasi-triangulation of (S, M).
- $\mathbf{x} := \{x_\gamma \in \mathcal{F} | \gamma \in T\}$ is an algebraically independent set in $\mathcal{F}$ over $\mathbb{ZP} := \mathbb{Z}[x_{b_1}, \ldots, x_{b_m}]$.

We call $\mathbf{x}$ the ***cluster*** of $(\mathbf{x}, T)$ and the variables themselves are called ***cluster variables***.

To define a *(quasi)-cluster structure* on (S, M) we shall consider the *decorated Teichmüller space*, $\tilde{\mathcal{T}}(S, M)$, as introduced by Penner [30]. An element of $\tilde{\mathcal{T}}(S, M)$ consists of a complete finite-area hyperbolic structure of constant curvature -1 on $S \setminus M$ together with a collection of horocycles, one around each marked point.

Fixing a decorated hyperbolic structure $\sigma \in \tilde{\mathcal{T}}(S, M)$ we may define the notion of *lambda length*, $\lambda_\sigma(\gamma)$, for each quasi-arc γ in (S, M). More explicitly,

$$\lambda_\sigma(\gamma) = \begin{cases} e^{\frac{l_\sigma(\gamma)}{2}}, & \text{if } \gamma \text{ is an arc,} \\ 2\text{sinh}(\frac{l_\sigma(\gamma)}{2}), & \text{if } \gamma \text{ is a one-sided closed curve,} \end{cases}$$

where $l_\sigma(\gamma)$ is defined as follows. If γ is a one-sided closed curve then $l_\sigma(\gamma)$ simply denotes the length of γ in σ. If γ is an arc then its endpoints are at cusps in σ, and so γ will have infinite length. However, we define $l_\sigma(\gamma)$ to be the length of γ between certain horocycles at its endpoints; the horocycle chosen at an endpoint will depend on how γ is tagged. Recall that σ comes equipped with a horocycle h_k at each marked point k. If γ has a plain tag at k then we consider precisely the horocycle h_k. If γ is notched at k then we instead consider the *conjugate horocycle* $\tilde{h}_k$, of h_k. (If h_k has length x then the conjugate horocycle $\tilde{h}_k$ is defined to be the unique horocycle at k with length $\frac{1}{x}$.

The *lambda length*, $\lambda(\gamma)$, of a quasi-arc γ is the evaluation map on $\tilde{\mathcal{T}}(S, M)$ sending decorated hyperbolic structures σ to $\lambda_\sigma(\gamma)$.

The following theorem follows from [Theorem 7.4, [13]] and [Remark 8.8, [13]].

Theorem 6.14. For any quasi-triangulation T, with quasi-arcs and boundary arcs $\gamma_1, \ldots, \gamma_{n+b}$, there exists a homeomorphism

$$\Lambda_T \colon \tilde{\mathcal{T}}(S, M) \longrightarrow \mathbb{R}^{n+b}_{>0}$$
$$\sigma \mapsto (\lambda_\sigma(\gamma_1), \ldots, \lambda_\sigma(\gamma_{n+b}))$$

As a consequence the lambda lengths of quasi-arcs and boundary arcs in a quasi-triangulation can be viewed as algebraically independent variables and we have a canonical isomorphism

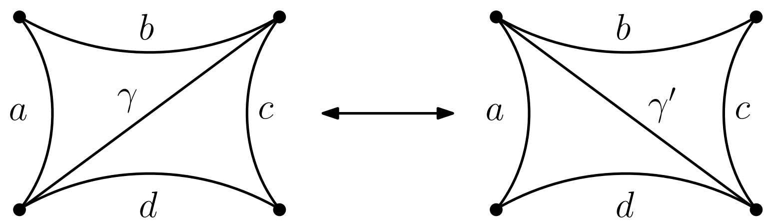
$$\mathbb{Q}(\{\lambda(\gamma) | \gamma \in T \cup B(S, M)\}) \cong \mathcal{F}.$$

We may define a (quasi)-cluster structure by calculating how these lambda lengths are related under flips. We provide these precise relations below in Definition 6.15.

Note that instead of working with lambda lengths we shall instead always consider their corresponding elements in $\mathcal{F}$.

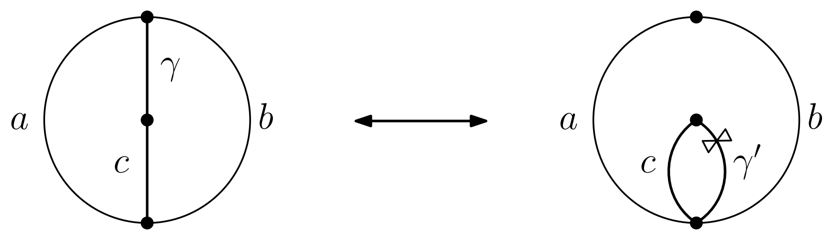
Definition 6.15. Given $\gamma \in T$ we define ***quasi-mutation*** of $(\mathbf{x}, T)$ in ***direction*** $\boldsymbol{\gamma}$ to be the pair $\mu_\gamma(\mathbf{x}, T) := (\mathbf{x}', T')$ where $T' := \mu_\gamma(T)$ and $\mathbf{x}' := \mathbf{x} \setminus \{x_\gamma\} \cup \{x_{\gamma'}\}$. The new variable $x_{\gamma'}$ depends on the combinatorial type of flip being performed. We list below the possible flips and their corresponding variable exchange relations, which may be obtained using the combined results of [6] and [13].

(1). γ is the diagonal of a quadrilateral in which no two consecutive edges are identified.



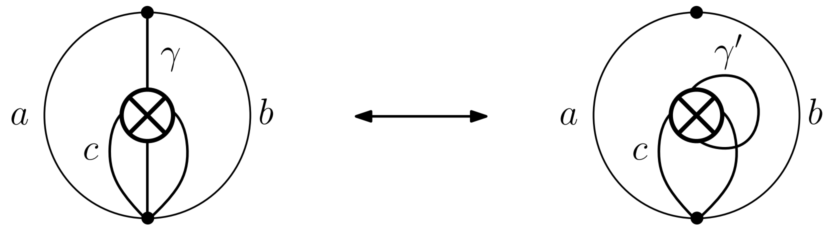
$$x_\gamma x_{\gamma'} = x_a x_c + x_b x_d$$

(2). γ is an interior arc of a punctured digon.



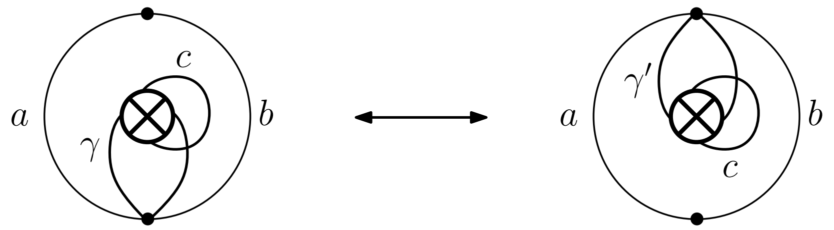
$$x_\gamma x_{\gamma'} = x_a + x_b$$

(3). γ is an arc that flips to a one-sided closed curve, or vice verca.



$$x_\gamma x_{\gamma'} = x_a + x_b$$

(4). γ is an arc intersecting a one-sided close curve c.



$$x_\gamma x_{\gamma'} = \frac{(x_a+x_b)^2+x_ax_bx_c^2}{x_c^2}$$

Figure 6.5: Combinatorial types of flips together with their corresponding exchange relations.

Let $(\mathbf{x}, T)$ be a seed of (S, M). If we label the cluster variables of $\mathbf{x}$ $1, \ldots, n$ then we can consider the labelled n-regular tree $\mathbb{T}_n$ generated by this seed through mutations. Each vertex in $\mathbb{T}_n$ has n incident vertices labelled $1, \ldots, n$. Vertices represent seeds and the edges correspond to mutation. In particular, the label of the edge indicates which direction the seed is being mutated in.

Let $\mathcal{X}$ be the set of all cluster variables appearing in the seeds of $\mathbb{T}_n$. $\mathcal{A}_{(\mathbf{x},T)}(S, M) := \mathbb{ZP}[\mathcal{X}]$ is the ***quasi-cluster algebra*** of the seed $(\mathbf{x}, T)$.

The definition of a quasi-cluster algebra depends on the choice of the initial seed. However, if we choose a different initial seed the resulting quasi-cluster algebra will be isomorphic to $\mathcal{A}_{(\mathbf{x},T)}(S, M)$. As such, it makes sense to talk about the quasi-cluster algebra of (S, M).

Definition 6.16. The ***quasi-arc complex*** $\Delta^{\otimes}(S, M)$ of the quasi-cluster algebra $\mathcal{A}(S, M)$ is the simplicial complex with the ground set being the cluster variables of $\mathcal{A}(S, M)$, and the maximal simplices being the clusters.

Definition 6.17. The ***exchange graph*** $E^{\otimes}(S, M)$ of the quasi-cluster algebra $\mathcal{A}(S, M)$ is the graph whose vertices correspond to the clusters of $\mathcal{A}(S, M)$. Two vertices are connected by an edge if their corresponding clusters differ by a single mutation.

6.1 The double cover and anti-symmetric quivers

Let (S, M) be a bordered surface. We construct an orientable double cover of (S, M) as follows. First consider the orientable surface $\tilde{S}$ obtained by replacing each cross-cap with a cylinder, see Figure 6.6.

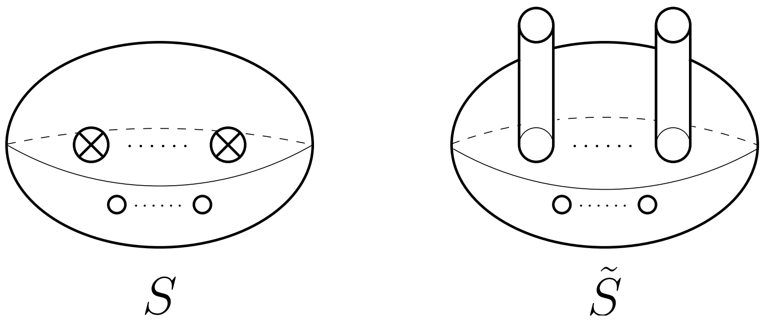


Figure 6.6: An illustration of the non-orientable surface S and the surface $\tilde{S}$ obtained by replacing each cross-cap with a cylinder. The small circles represent boundary components.

We obtain the orientable double cover $\overline{(S, M)}$ of (S, M) by taking two copies of $\tilde{S}$ and glueing each newly ajoined cylinder in the first copy, with a half twist, to the corresponding cylinder in the second copy. To clarify, we are glueing each cylinder in the first copy along their antipodal points in the second copy, see Figure 6.7. If S is orientable then the double cover is two disjoint copies of (S, M). In this case we endow the two disjoint copies with alternate orientations - this is to ensure its adjacency quiver is anti-symmetric, see Definition 6.18.

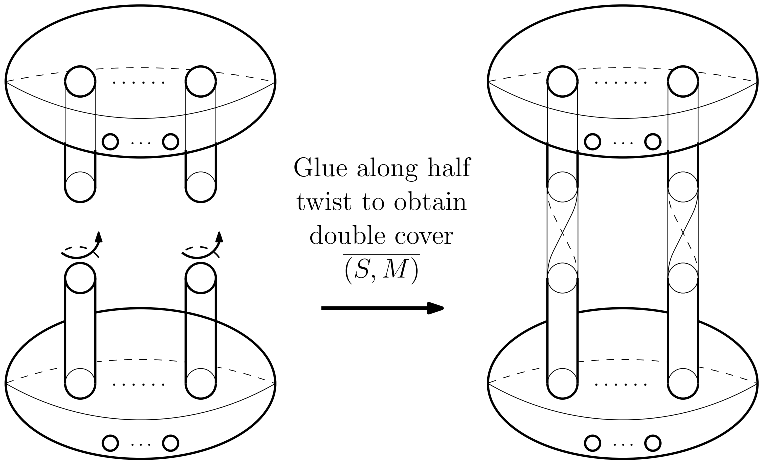


Figure 6.7: We obtain the double cover by glueing two copies of $\tilde{S}$ along the boundaries of the newly adjoined cylinders.

If T is a triangulation of (S, M) then T lifts to a triangulation $\overline{T}$ of the orientable double cover $\overline{(S, M)}$. Moreover, let i be an arc in T and, by abuse of notation, denote by i and $\tilde{i}$ the two arcs i lifts to in $\overline{T}$. Note that if i and j are arcs of a triangle Δ in $\overline{T}$, and j follows i in Δ under the agreed orientation of $\overline{(S, M)}$, then $\tilde{i}$ follows $\tilde{j}$ in the twin triangle $\tilde{\Delta}$. Hence in the quiver $Q_{\overline{T}}$ associated to $\overline{T}$ we have that $i \to j \iff \tilde{j} \to \tilde{i}$. Here we adopt the notation that $\tilde{\tilde{i}} = i$ for any $i \in \{1, \ldots, n\}$, and we shall use it throughout this thesis.

Finally, note that there is no arrow $i \to \tilde{i}$ in $Q_{\overline{T}}$ as this would imply the existence of an anti-self-folded triangle in T, which is forbidden under our definition of triangulation, see Figure 6.8.

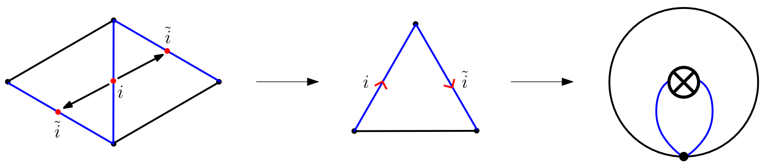


Figure 6.8: An anti-self-folded triangle; which is forbidden under our definition of triangulation.

These two observations motivate the following definition.

Definition 6.18. A quiver Q on vertices $1, \dots, n, \tilde{1}, \dots, \tilde{n}$ is called ***anti-symmetric*** if:

- For any $i, j \in \{1, \dots, n, \tilde{1}, \dots, \tilde{n}\}$ we have $i \to j \iff \tilde{j} \to \tilde{i}$.
- For any $i \in \{1, \dots, n, \tilde{1}, \dots, \tilde{n}\}$ there are no arrows $i \to \tilde{i}$.

Proposition 6.19. Let γ be an arc in a triangulation T, and by abuse of notation, denote its lifts in $\overline{T}$ by γ and $\tilde{\gamma}$. If $\mu_\gamma(T)$ is a triangulation then $\mu_\gamma \circ \mu_{\tilde{\gamma}}(\overline{T}) = \mu_{\tilde{\gamma}} \circ \mu_\gamma(\overline{T}) = \overline{\mu_\gamma(T)}$.

Proof. Consider the flip region of γ in T. The interiors of the lifted flip regions will disjoint; otherwise there would be arrows between the corresponding vertices of γ and $\tilde{\gamma}$ in $Q_{\overline{T}}$, and Figure 6.8 would then contradict the fact there are no anti-self-folded triangles in T. Finally, since $\mu_\gamma(T)$ is a triangulation, then for such triangulations, the definition of a flipping an arc will coincide on both non-orientable and orientable surfaces.

□

6.2 Mutation of anti-symmetric quivers via LP mutation

We shall now briefly leave the environment of triangulations and move to the more general setting of anti-symmetric quivers. In particular, we shall establish a connection between mutation of these quivers and LP-mutation. Recall that a quiver Q can be equivalently encoded as a skew-symmetric matrix $B = (b_{ij})$. In what follows we shall interchange between the two viewpoints.

Given an anti-symmetric quiver $Q = (b_{ij})$ we may assign an exchange polynomial to each pair of vertices $(j, \tilde{j})$ of Q.

$$F_j^Q := \prod_{b_{ij}+b_{\tilde{i}j}>0} x_i^{b_{ij}+b_{\tilde{i}j}} + \prod_{b_{ij}+b_{\tilde{i}j}<0} x_i^{-(b_{ij}+b_{\tilde{i}j})}$$

As a result we arrive at the seed $\Sigma_Q := (\{x_1, \ldots, x_n\}, \{F_1^Q, \ldots, F_n^Q\})$ associated to Q. Of course, this may not be a valid LP seed due to the requirement of irreducibility. We won't always get irreducibility, but, as the proposition below demonstrates, there are plenty of cases where Q does provide a valid LP seed.

Proposition 6.20. If $gcd(b_{1j} + b_{\tilde{1}j}, \ldots, b_{nj} + b_{\tilde{n}j}) = 1$ then F_j is irreducible in $\mathbb{Z}[x_1, \ldots, x_n]$.

Proof. The proof is identical to that of Lemma 4.1 in [26].

□

Note that if we want double mutation of our quiver to correspond to LP mutation then it is necessary for us to have $\hat{F}_i = F_i \ \forall i \in \{1, \ldots, n\}$. This is because the exchange polynomials of the arcs in the triangulations are polynomials (not strictly Laurent polynomials), so the normalisation process needs to be vacuous.

Proposition 6.21. Suppose Σ_Q is a valid LP seed and $\hat{F}_i = F_i \ \forall i \in \{1, \ldots, n\}$. Let i be a vertex in Q such that there is no path $a \to i \to \tilde{a}$ for any vertex $a \in \{1, \ldots, n, \tilde{1}, \ldots, \tilde{n}\}$. Then mutation at i and $\tilde{i}$ in Q corresponds to LP mutation of Σ_Q at i. I.e, $(\{x_1, \ldots, \frac{F_i^Q}{x_i}, \ldots, x_n\}, \{F_1^{\mu_i \circ \mu_{\tilde{i}}(Q)}, \ldots, F_n^{\mu_i \circ \mu_{\tilde{i}}(Q)}\}) = \mu_i(\{x_1, \ldots, x_n\}, \{F_1^Q, \ldots, F_n^Q\})$.

Proof. Let $j \in \{1, \ldots, n\}$. We will split the proof into two parts depending on whether $x_i \notin F_j^Q$ or $x_i \in F_j^Q$.

<u>**Case 1**</u>: $x_i \notin F_j^Q$.

If $x_i \notin F_j^Q$ then LP mutation at i does not alter the exchange polynomial F_j^Q. I.e, $(F_j^Q)' = F_j^Q$. Therefore for quiver mutation to coincide with LP mutation we require that $F_j^{\mu_i \circ \mu_{\tilde{i}}(Q)} = F_j^Q$. It suffices to show that

$$b'_{kj} + b'_{\tilde{k}j} := (\mu_i \circ \mu_{\tilde{i}}(Q))_{kj} + (\mu_i \circ \mu_{\tilde{i}}(Q))_{\tilde{k}j} = b_{kj} + b_{\tilde{k}j} \quad \forall k \in \{1, \ldots, n\}.$$

Below we check this holds when $k = i$ and $k \neq i$. Note that $x_i \notin F_j^Q \implies b_{ij} + b_{\tilde{i}j} = 0$.

- $(k = i)$ $b'_{ij} + b'_{\tilde{i}j} = -b_{ij} - b_{\tilde{i}j} = 0 = b_{ij} + b_{\tilde{i}j}$.
- $(k \neq i)$ Firstly note that because mutation at i and $\tilde{i}$ are independent of one another we have

$$b'_{kj} := (\mu_i \circ \mu_{\tilde{i}}(Q))_{kj} = (\mu_i(Q))_{kj} + (\mu_{\tilde{i}}(Q))_{kj} - b_{kj} =$$

$$b_{kj} + [-b_{ki}]_+ b_{ij} + b_{ki}[b_{ij}]_+ + [-b_{k\tilde{i}}]_+ b_{\tilde{i}j} + b_{k\tilde{i}}[b_{\tilde{i}j}]_+.$$

Now, by applying the fact that $b_{ij} = -b_{\tilde{i}j}$ we obtain the following.

$$b'_{kj} + b'_{\tilde{k}j} = b_{kj} + b_{\tilde{k}j} + b_{ij}([-b_{ki}]_+ - [-b_{k\tilde{i}}]_+ + [-b_{\tilde{k}i}]_+ - [-b_{\tilde{k}\tilde{i}}]_+)+$$

$$[-b_{ij}]_+(b_{k\tilde{i}} + b_{\tilde{k}\tilde{i}}) + [b_{ij}]_+(b_{ki} + b_{\tilde{k}i}) \overset{\text{by anti-symmetry}}{=}$$

$$b_{kj} + b_{\tilde{k}j} + b_{ij}([-b_{ki}]_+ - [b_{\tilde{k}i}]_+ + [-b_{\tilde{k}i}]_+ - [b_{ki}]_+)+$$

$$[-b_{ij}]_+(-b_{\tilde{k}i} - b_{ki}) + [b_{ij}]_+(b_{ki} + b_{\tilde{k}i}).$$

Using the fact that $[a]_+ - [-a]_+ = a$ we see that

$$b'_{kj} + b'_{\tilde{k}j} = b_{kj} + b_{\tilde{k}j}.$$

So indeed, $F_j^{\mu_i \circ \mu_{\tilde{i}}(Q)} = F_j^Q = (F_j^Q)'$ in the case $x_i \notin F_j^Q$.

<u>**Case 2**</u>: $x_i \in F_j^Q$.

If $x_i \in F_j^Q$ then *w.l.o.g* we shall assume $b_{ij} + b_{\tilde{i}j} > 0$ and $b_{ij} > 0$. By skew symmetry we have $b_{ji} < 0$. Also, $b_{\tilde{j}i} \leq 0$ follows from $b_{ij} > 0$ and the assumption that there is no path $a \to i \to \tilde{a}$. From this we get the following:

$$F_i^Q|_{x_j \leftarrow 0} = \prod_{b_{ki}+b_{\tilde{k}i}>0} x_k^{b_{ki}+b_{\tilde{k}i}}$$

From here we see **(Step 1)** of LP mutation gives us:

$$G_j^Q = \left(\prod_{\substack{b_{kj}+b_{\tilde{k}j}>0 \\ k\neq i}} x_k^{b_{kj}+b_{\tilde{k}j}} \right) \left(\frac{\prod_{b_{ki}+b_{\tilde{k}i}>0} x_k^{b_{ki}+b_{\tilde{k}i}}}{x_i'} \right)^{b_{ij}+b_{\tilde{i}j}} + \prod_{b_{kj}+b_{\tilde{k}j}<0} x_k^{-(b_{kj}+b_{\tilde{k}j})}$$

We make the observation that since $F_i^Q|_{x_j \leftarrow 0}$ is a monomial then **(Step 2)** of LP mutation can be incorporated into **(Step 3)**. Therefore to obtain $(F_j^Q)'$ we are left with the task of finding a monic Laurent monomial M such that $(F_j^Q)' := MG_j^Q \in \mathbb{Z}[x_1', \dots, x_n']$ and is not divisible by any x_k'. We shall determine the exponents of the variables x_k $(k \neq i)$ and x_i' in $(F_j^Q)'$ by splitting the task into four subcases. For each case we check the exponent agrees with the one in the exchange polynomial $F_j^{\mu_{\tilde{i}} \circ \mu_i(Q)}$ obtained via quiver mutation.

Subcase 1: $b_{ki} + b_{\tilde{k}i} \leq 0$.

This means there is no x_k term in $F_i^Q|_{x_j \leftarrow 0}$. So the x_k exponent remains unchanged from LP mutation. That being so, for LP mutation to agree with double quiver mutation we require that $b'_{kj} + b'_{\tilde{k}j} = b_{kj} + b_{\tilde{k}j}$. Since $b_{ki} + b_{\tilde{k}i} \leq 0$ and there is no path $a \to i \to \tilde{a}$ then $b_{ki}, b_{\tilde{k}i} \leq 0$. So $b'_{kj} = b_{kj}$, $b'_{\tilde{k}j} = b_{\tilde{k}j}$, and we therefore have agreement.

Subcase 2: $b_{ki} + b_{\tilde{k}i} > 0$ and $b_{kj} + b_{\tilde{k}j} \geq 0$.

This means we get an x_k term in the first monomial of G_j^Q, and it has exponent $b_{kj} + b_{\tilde{k}j} + (b_{ki} + b_{\tilde{k}i})(b_{ij} + b_{\tilde{i}j})$. To determine what happens with quiver mutation recall our assumption that $b_{ij} > 0$. Since there is no path $a \to i \to \tilde{a}$ for any vertex a of Q, then $b_{ij}, b_{i\tilde{j}} \geq 0$. Likewise, because $b_{ki} + b_{\tilde{k}i} > 0$, we get $b_{ki}, b_{\tilde{k}i} \geq 0$. Hence for quiver mutation we obtain

$$b'_{kj} = b_{kj} + b_{ki}b_{ij} - b_{j\tilde{i}}b_{\tilde{i}k}$$

$$b'_{\tilde{k}j} = b_{\tilde{k}j} + b_{\tilde{k}i}b_{ij} - b_{j\tilde{i}}b_{\tilde{i}\tilde{k}}.$$

Using anti-symmetry and skew-symmetry we see

$$b'_{kj} + b'_{\tilde{k}j} = b_{kj} + b_{\tilde{k}j} + (b_{ki} + b_{\tilde{k}i})(b_{ij} + b_{\tilde{i}j}) > 0$$

.

Consequently, LP and quiver mutation coincide for subcase 2.

<u>**Subcase 3**</u>: $b_{ki} + b_{\tilde{k}i} > 0$ and $b_{kj} + b_{\tilde{k}j} \leq 0$.

This means there will be an x_k term in both monomials of G_j^Q and after dividing out by an appropriate power of x_k, we are left with x_k having exponent $b_{kj} + b_{\tilde{k}j} + (b_{ki} + b_{\tilde{k}i})(b_{ij} + b_{\tilde{i}j})$ in $(F_j^Q)'$. The variable x_k appears in the left or right monomial of $(F_j^Q)'$ depending on whether $(b_{ki} + b_{\tilde{k}i})(b_{ij} + b_{\tilde{i}j}) \geq -(b_{kj} + b_{\tilde{k}j})$ or $(b_{ki} + b_{\tilde{k}i})(b_{ij} + b_{\tilde{i}j}) \leq -(b_{kj} + b_{\tilde{k}j})$, respectively. Just as in case 2 we observe that double mutating the quiver Q yields

$$b'_{kj} + b'_{\tilde{k}j} = b_{kj} + b_{\tilde{k}j} + (b_{ki} + b_{\tilde{k}i})(b_{ij} + b_{\tilde{i}j}).$$

Thus showing LP mutation agrees with double quiver mutation for subcase 3.

<u>**Subcase 4**</u>: The variable x'_i.

In $(F_j^Q)'$ the variable x'_i appears in the right monomial with exponent $b_{ij} + b_{\tilde{i}j}$. This agrees with quiver mutation since $b'_{ij} + b'_{\tilde{i}j} = -(b_{ij} + b_{\tilde{i}j}) < 0$.

Therefore $F_j^{\mu_i \circ \mu_{\tilde{i}}(Q)} = (F_j^Q)'$ in the case $x_i \in F_j^Q$. This concludes the proof of the proposition.

□

Chapter 7

Laurent phenomenon algebras arising from unpunctured surfaces

This chapter contains the material of the author's published work [36].

Having delved into the domain of anti-symmetric quivers, we now turn our attention back to bordered surfaces. In particular, we restrict ourselves to unpunctured surfaces with the aim of showing their corresponding quasi-cluster algebra has an LP structure.

To achieve this we first consider triangulations of (S, M), and show they slot into an LP structure. We accomplish this by proving the adjacency quiver $Q_{\overline{T}}$ satisfies the conditions demanded in Proposition 6.21, for each triangulation T of (S, M). Of course, we must also show that the exchange polynomials $F_1^{Q_{\overline{T}}}, \ldots, F_n^{Q_{\overline{T}}}$ are the exchange polynomials of their corresponding arcs in T; this is settled by Lemma 7.2. Note that, for triangulations of (S, M) to slot into an LP structure, Proposition 6.21 requires that for each triangulation T of our bordered surface we have:

- If i is a t-mutable arc in T then there is no path $k \to i \to \tilde{k}$ in $Q_{\overline{T}}$ for any vertex k.
- The exchange polynomials $F_1^{Q_{\overline{T}}}, \ldots, F_n^{Q_{\overline{T}}}$ associated to T are irreducible.

- $F_i^{Q_{\overline{T}}} = \hat{F}_i^{Q_{\overline{T}}}$ for each exchange polynomial associated to T.

The first two conditions are verified by Lemma 7.1 and Lemma 7.3, respectively. The majority of this chapter is spent proving the third condition. We achieve this by first showing the property is equivalent to the exchange polynomials of T being distinct, see Lemma 7.4. From here, via Lemmas 7.5, 7.7, 7.8, 7.9, 7.10, we discover all bordered surfaces that emit triangulations producing non-distinct exchange polynomials. In the interest of maximal generality we allow the possibility that boundary segments do not receive variables; in which case the boundary segment is instead allocated the constant value 1, and the corresponding vertex in the adjacency quiver is deleted.

Lemma 7.1. For a triangulation T of (S, M) there are vertices i, k of $Q_{\overline{T}}$ with $k \to i \to \tilde{k}$ *if and only if* T contains the Möbius strip with two marked points, M_2, with i being the non t-mutable arc of M_2. See Figure 7.2 below.

Proof. To prove this lemma we reconstruct (part of) the surface (S, M) using blocks. Namely, we use the quiver Q to determine the adjacency of triangles in T. By anti-symmetry note that $k \to i \to \tilde{k}$ implies there is the path $i \leftarrow k \to \tilde{i}$. As a consequence there must be the quadrilateral $(i, a, \tilde{i}, \tilde{b})$ with diagonal k for some a and $\tilde{b}$ not equal to $\{i, \tilde{i}, k, \tilde{k}\}$, see Figure 8.4. By antisymmetry we also have the quadrilateral $(b, i, \tilde{a}, \tilde{i})$ with diagonal $\tilde{k}$.

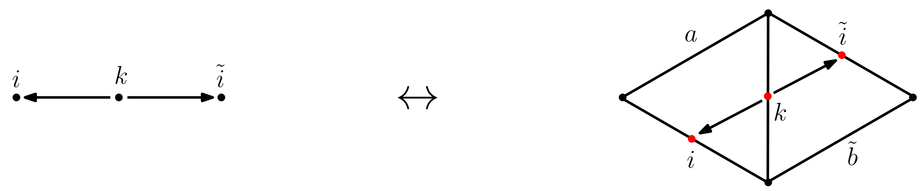


Figure 7.1: If $i \leftarrow k \to \tilde{i}$ is a sub quiver of an adjacency quiver then the surface must have the local configuration shown on the right.

Glueing these two quadrilaterals together, according to their labels, yields the cylinder shown in Figure 7.2. Taking the $\mathbb{Z}_2$-quotient of this leaves us with the Möbius strip M_2 which is also depicted in Figure 7.2.

□

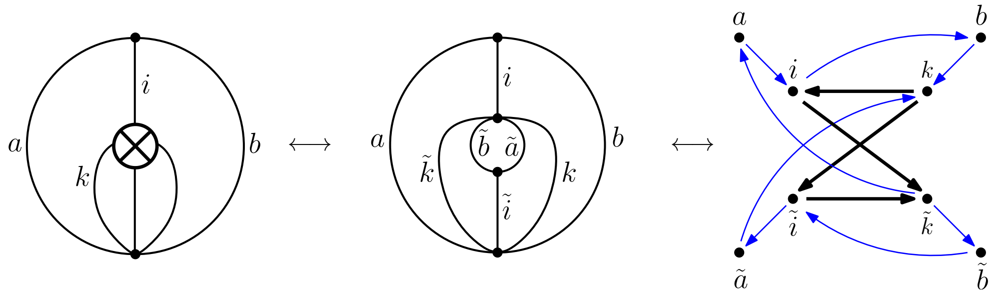


Figure 7.2: A triangulation of the Möbius strip M_2; its lifted triangulation; and the adjacency quiver of its lifted triangulation.

Lemma 7.2. Let T be a triangulation of (S, M) and $Q_{\overline{T}}$ the corresponding anti-symmetric quiver arising from the lifted triangulation $\overline{T}$. Then the exchange polynomials $\{F_1^{Q_{\overline{T}}}, \dots, F_n^{Q_{\overline{T}}}\}$ coincide with the exchange polynomials of the arcs in T they are associated with.

Proof. Let $(i, \tilde{i})$ be a twin pair of vertices in $Q_{\overline{T}}$ and consider the associated exchange polynomial $F_i^{Q_{\overline{T}}}$. If there is no path $k \to i \to \tilde{k}$ for any vertex k in $Q_{\overline{T}}$ then, by Lemma 7.1, all arcs will flip to arcs. Moreover, $F_i^{Q_{\overline{T}}} := \prod_{b_{ki}>0} x_k^{b_{ki}} + \prod_{b_{ki}<0} x_k^{-b_{ki}}$ (with the identification $x_k = x_{\tilde{k}}$) so from the standard theory of cluster algebras from surfaces we see $F_i^{Q_{\overline{T}}}$ describes how the length of the arc i changes under a flip. If there is a path $k \to i \to \tilde{k}$ then, by Lemma 7.1, locally the arc i will be contained in the triangulation of M_2 shown in Figure 7.2. In particular, it has the exchange polynomial $F_i^{Q_{\overline{T}}} = x_a + x_b$ which does indeed describe how the length of the arc i changes under a flip. □

For a seed coming from an anti-symmetric quiver Q we noted that the seed may not be a valid LP seed due to potential reducibility of the exchange polynomials. However, as shown by the following lemma, for an anti-symmetric quiver arising from a triangulation of (S, M) we always get irreducibility.

Lemma 7.3. Let T be a triangulation of (S, M). Then $F_j^{Q_{\overline{T}}}$ is irreducible in $\mathbb{ZP}[x_1, \ldots, x_n]$ for any j. In particular, $\Sigma_{Q_{\overline{T}}} := (\{x_1, \ldots, x_n\}, \{F_1^{Q_{\overline{T}}}, \ldots, F_n^{Q_{\overline{T}}}\})$ is a valid LP seed.

Proof. The quiver $Q_{\overline{T}}$ coming from the lifted triangulation $\overline{T}$ can have at most 2 ingoing and 2 outgoing arrows at any one vertex. Hence, $gcd(b_{1j}+b_{\tilde{1}j}, \ldots, b_{mj}+b_{\tilde{m}j}) \in \{1, 2, \infty\}$.

If gcd is 1 then Proposition 6.20 yields the irreducibility of $F_j^{Q_{\overline{T}}}$.

If gcd is ∞ then $b_{ij} + b_{\tilde{i}j} = 0$ for all i. So $F_j^{Q_{\overline{T}}} = 2$, which is irreducible.

If gcd is 2 then due to there being at most 2 ingoing and 2 outgoing arrows at j the only possibilities for $F_j^{Q_{\overline{T}}}$ are $x_i^2 + 1$ and $x_i^2 + x_k^2$, which are both irreducible. □

Recall that the goal of this chapter has been to show triangulations fit into an LP structure by invoking Proposition 6.21. To accomplish this we are left to prove that $\hat{F}_i = F_i$ for each F_i in $\Sigma_{Q_{\overline{T}}}$. By the following lemma we may equivalently prove that the exchange polynomials in each seed $\Sigma_{Q_{\overline{T}}}$ are distinct.

Lemma 7.4. Let T be a triangulation, $\Sigma_{Q_{\overline{T}}}$ its associated LP seed, and $i \in \{1, \ldots, n\}$. Then $\hat{F}_i = F_i$ *if and only if* $F_i \neq F_j$ for any $j \neq i$.

Proof. If $\hat{F}_i = F_i$ then, by definition of normalisation, for any $j \neq i$ we have F_j does not divide $F_i|_{x_j \leftarrow \frac{F_j}{x}}$. Hence F_j does not divide F_i and so, in particular, $F_i \neq F_j$.

Conversely, if $\hat{F}_i \neq F_i$ then there exists $j \neq i$ such that F_j divides $F_i|_{x_j \leftarrow \frac{F_j}{x}}$, which forces $x_i \notin F_j$. Suppose for a contradiction that $x_j \in F_i$. This implies the existence of a path $i \to j \to \tilde{i}$. By Proposition 7.1 and Figure 7.2 we see $F_j = x_a + x_b$ and $F_i = x_j^2 + x_a x_b$. However, this contradicts F_j dividing $F_i|_{x_j \leftarrow \frac{F_j}{x}} = \frac{F_j^2}{x^2} + x_a x_b$. Hence $x_j \notin F_i$ and F_j divides $F_i|_{x_j \leftarrow \frac{F_j}{x}} = F_i$. Moreover, since F_i is irreducible then $F_i = F_j$.

□

We now list several lemmas to help discover the heterogeneity of the exchange polynomials in $\Sigma_{Q_{\overline{T}}}$.

Lemma 7.5. If $F_i = F_j$ then there are no arrows between i and j in Q.

Proof. Since $x_i \notin F_i$ then $x_i \notin F_j$. As such, $b_{ij} + b_{\tilde{i}j} = 0$. Likewise, $b_{ji} + b_{\tilde{j}i} = 0$. Finally, since $b_{ij} = -b_{ji}$ and $b_{\tilde{i}j} = b_{\tilde{j}i}$, then, as required, $b_{ij} = 0$.

□

Definition 7.6. Let Q be a quiver and V a set of vertices of Q. We say R is the ***V-restriction*** of Q if R consists of all arrows of Q with a head or tail in V.

Lemma 7.7. Suppose R is the $\{i, j\}$-restriction of Q with $F_i^Q = F_j^Q$. Then the $\{i, j\}$-restriction of $\mu_{\tilde{i}} \circ \mu_i(R)$ is the $\{i, j\}$-restriction of $\mu_{\tilde{i}} \circ \mu_i(Q)$ where $F_i^{\mu_{\tilde{i}} \circ \mu_i(Q)} = F_j^{\mu_{\tilde{i}} \circ \mu_i(Q)}$. In particular, if R is the $\{i, j\}$-restriction of a quiver arising from (S, M) with exchange polynomials $F_i = F_j$, then so is the {i,j}-restriction of $\mu_{\tilde{i}} \circ \mu_i(R)$.

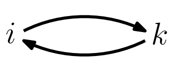
Proof. By Lemma 7.5 there are no arrows between i and j so performing mutation at i and $\tilde{i}$ in R and taking the $\{i, j\}$-restriction is the same as reversing all arrows at i and $\tilde{i}$ in R. Hence the $\{i, j\}$-restriction of $\mu_{\tilde{i}} \circ \mu_i(R)$ is the $\{i, j\}$-restriction of $\mu_{\tilde{i}} \circ \mu_i(Q)$. Moreover, the new i^{th} and j^{th} exchange polynomials remain unchanged, so are still equal.

□

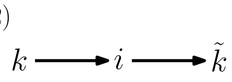
Lemma 7.8. Suppose $F_i = F_j$ for some $i \neq j$; $x_k \notin F_i$ for some k; and i and k are adjacent arcs in T. Then (S, M) is either the Möbius strip M_4 or the Klein bottle with one boundary component and two marked points, where neither surface has been allocated boundary variables.

Proof. Under the conditions of the lemma, before cancelling 2-cycles, we must have one of the following subquivers in our adjacency quiver Q:

(1)



(2)



If the subquiver (1) is in Q then we must have one of the configurations shown in Figure 7.3. In either situation, after glueing, we obtain a punctured surface.

Since we have forbidden punctures then this subquiver cannot arise from any of our triangulations.

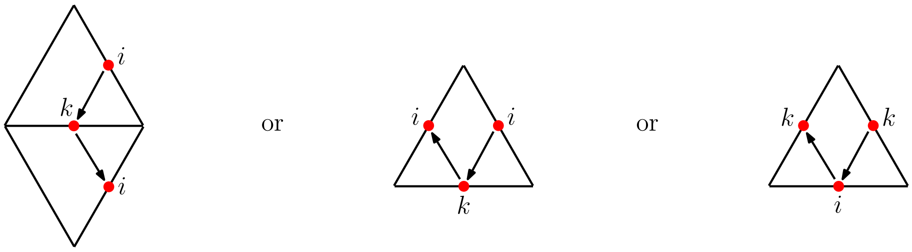


Figure 7.3: The three configurations which produce the subquiver (1).

If the subquiver (2) is in Q then by Lemma 7.1 we must have the following local picture shown on the left of Figure 7.4. Note that b cannot equal a or $\tilde{a}$ because this would give rise to a punctured surface - the twice punctured projective space $\mathbb{R}P^2$ or the once punctured Klein bottle, respectively.

Moreover, a and b cannot both be boundary segments as then there is no label j in the triangulation. Without loss of generality, suppose a is not a boundary component. As a consequence, there is an arrow $a \to i$. Since $F_i = F_j$, using Lemma 7.7, we may assume the existence of an arrow $a \to j$. Hence we arrive at the picture shown on the right of Figure 7.4.

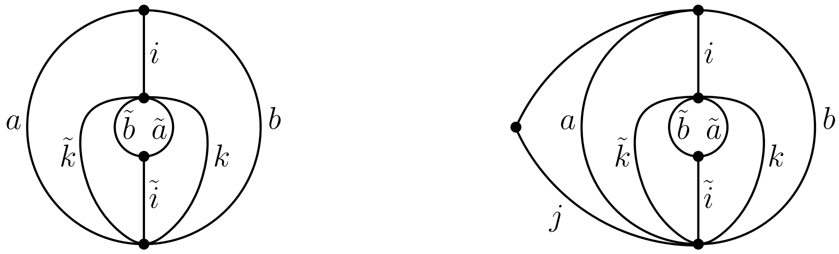


Figure 7.4: On the right we illustrate the effect on the local configuration of the surface when there is an arrow $a \to j$ in Q.

If b does not receive a variable then there is no arrow $i \to b$, and we are in one of two possible scenarios: There is a path $\tilde{m} \to j \to m$ for some m, or j is connected to only a. If there is a path $\tilde{m} \to j \to m$ then by Lemma 7.1 our surface must have the configuration shown on the left of Figure 7.5. Taking the $\mathbb{Z}_2$-quotient of this yields the Klein bottle with one boundary component and 2 marked points. Alternatively, if j is connected to only a then the arc j is the diagonal of a square with three unlabelled boundary segments and fourth side a. And we obtain the surface shown on the right of Figure 7.5. Taking the $\mathbb{Z}_2$-quotient of this yields the Möbius strip with 4 marked points.

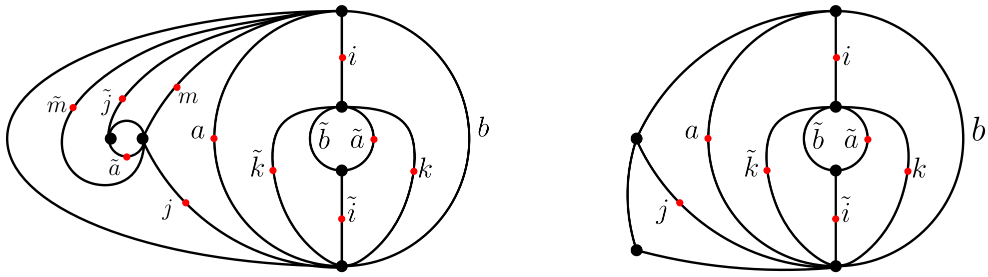


Figure 7.5: We depict the resulting surfaces when there is either: a path $\tilde{m} \to j \to m$; or j is connected only to the vertex a.

If b does receive a variable then there is an arrow $i \to b$ in Q. As such, since $F_i = F_j$, there is either an arrow $j \to b$ or an arrow $j \to \tilde{b}$. However, an arrow $j \to b$ gives rise to a punctured surface, which is forbidden. An arrow $j \to \tilde{b}$ gives rise to the configurations shown in Figure 7.6. In both cases, taking the $\mathbb{Z}_2$-quotient again yields the Klein bottle with one boundary component and two marked points.

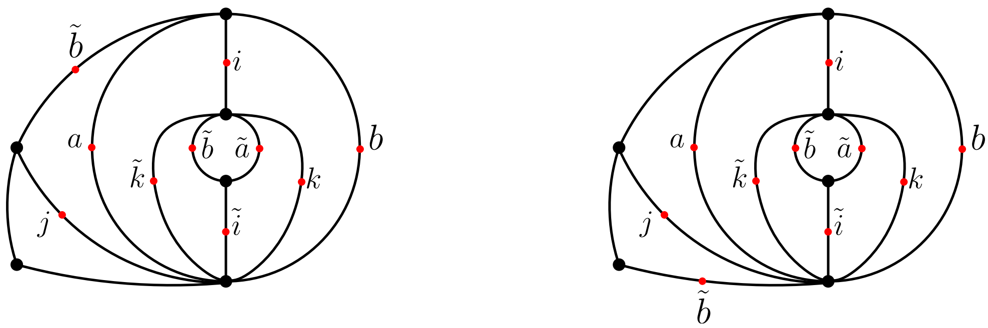


Figure 7.6: The two possibilities of the surface when there is an arrow $j \to \tilde{b}$.

□

Lemma 7.9. If $F_i = F_j$ then the quiver Q cannot contain either of the subquivers $\tilde{k} \leftarrow i \rightarrow k$ or $i \xrightarrow{2} k$, for any vertex k of Q.

Proof. If $\tilde{k} \leftarrow i \rightarrow k$ is a subquiver of Q then antisymmetry implies the existence of the path $i \rightarrow k \rightarrow \tilde{i}$. Therefore, by Lemma 7.1, we have the sub triangulation shown in Figure 7.7. Since $F_i = F_j$ then there must be an arrow $j \rightarrow k$ or $j \rightarrow \tilde{k}$. However, any triangle with side k or $\tilde{k}$ also has a side i or $\tilde{i}$. This forces an arrow between i and j or i and $\tilde{j}$, contradicting Lemma 7.5. If $i \xrightarrow{2} k$ is a subquiver of Q then since $F_i = F_j$, without loss of generality, $i \xrightarrow{2} k \xleftarrow{2} j$ is a subquiver of Q. However, this contradicts the fact that any vertex in Q can have at most 2 incoming arrows.

□

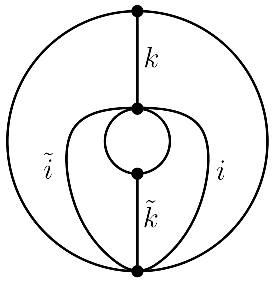


Figure 7.7: The local configuration of the surface when there is a path $\tilde{k} \leftarrow i \rightarrow k$.

Lemma 7.10. Let T be a triangulation and $\Sigma_{Q_{\overline{T}}}$ its associated LP seed. Then $\hat{F}_i = F_i$ for any $i \in \{1, \ldots, n\}$.

Proof. By Lemma 7.4 it suffices to show that $F_j \neq F_i$ for any $j \neq i$. Now, if $F_i = F_j$ for some j, by Lemma 7.5 we know there are no arrows between i and j. Due to Lemma 7.9 we also know there are no arrows of weight greater than 1 in the $\{i, j\}$-restriction of Q. Furthermore, by Lemma 7.8 and Lemma 7.9 if i (or j) is connected to both k and $\tilde{k}$ for some vertex k in Q, then the corresponding surface must be either the Möbius strip M_4 or the Klein bottle with one boundary component and two marked points, where neither surface has been allocated boundary variables. Having dealt with these cases, from here on we may therefore assume i and j are connected to at most one of k and $\tilde{k}$ for any vertex k in Q. After reversing all arrows at i if needed, i and j will locally have the same quiver up to exchanging a and $\tilde{a}$. I.e. If $i \leftarrow k$ (or $i \rightarrow k$) then $j \leftarrow k$ ($j \rightarrow k$) or $j \leftarrow \tilde{k}$ ($j \rightarrow \tilde{k}$).

To determine the remaining surfaces which emit triangulations with $F_i = F_j$ we will split our task into four cases depending on whether i and j are connected to precisely 1, 2, 3 or 4 vertices. After exchanging the roles of j and $\tilde{j}$ if necessary, we may assume there are arrows $i \leftarrow a$ and $j \leftarrow a$ for some fixed vertex a. Furthermore, note that in the quivers we draw we only include arrows between i and j. For each of these quivers R we are asking which triangulations T of (S, M) have the property that the $\{i, j\}$-restriction of $Q_{\overline{T}}$ is R.

Case 1: i and j are connected to precisely one vertex.

The only such quiver for this case is $i \leftarrow a \rightarrow j$. Since i and j are not connected to any other vertex, the arcs i and j are the diagonals of quadrilaterals with three boundary segments and fourth side a. This yields the 6-gon shown in Figure 7.8.

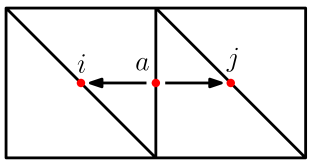


Figure 7.8: The 6-gon without boundary variables - the only surface emitting a triangulation whose adjacency quiver has the $\{i, j\}$-restriction $i \leftarrow a \rightarrow j$.

Case 2: i and j are connected to precisely two vertices.

The possible subquivers for this case are listed in Figure 7.9.

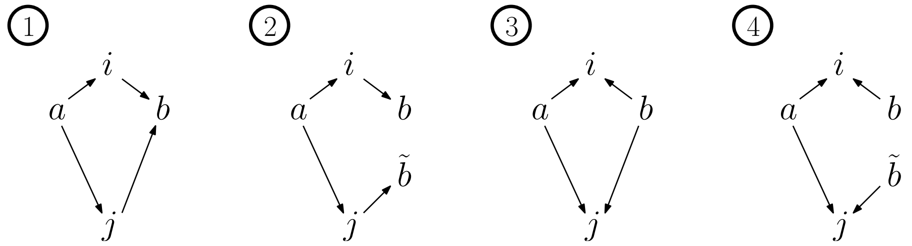
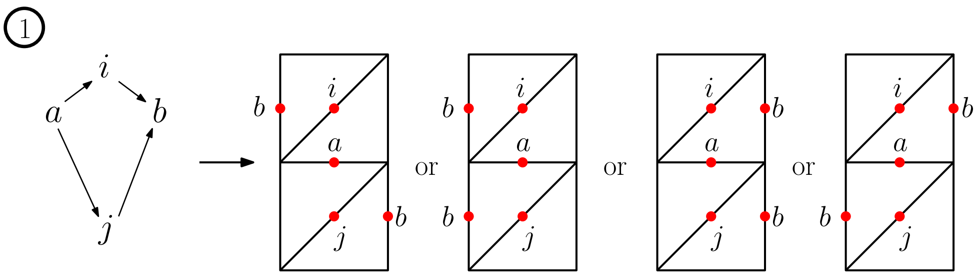
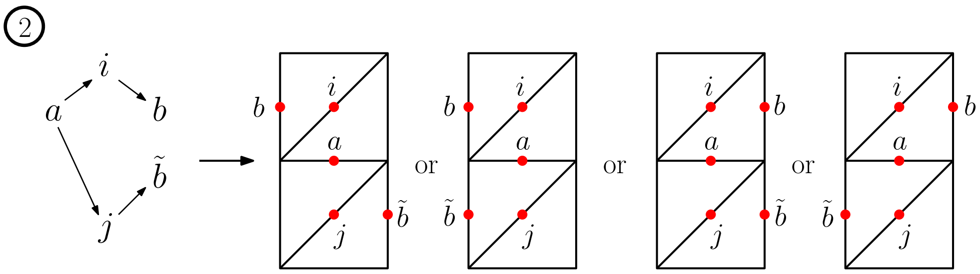
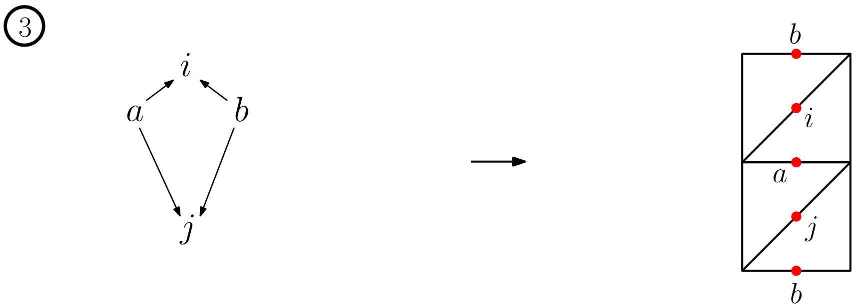


Figure 7.9: The list of the possible $\{i, j\}$-restriction quivers when i and j are connected to precisely two vertices, and $F_i = F_j$.

For each of the subquivers listed in Figure 7.9 we present below the possible triangulations/surfaces that produce them. To elaborate, we use the quiver to determine the conceivable adjacencies of triangles in the triangulation, and this is how the surface is reconstructed.







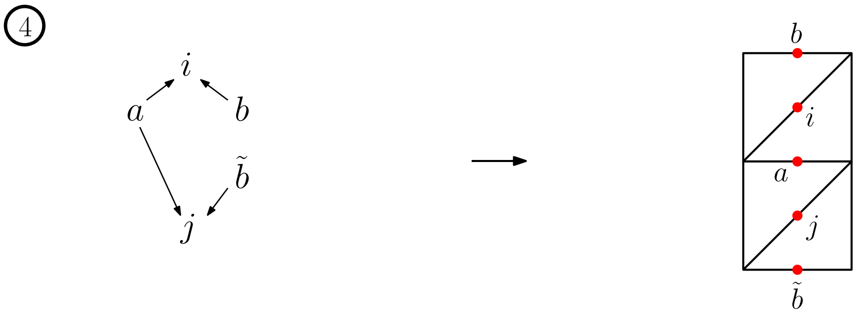


Figure 7.10: Upon glueing and taking $\mathbb{Z}_2$-quotients, the (unpunctured) surfaces we obtain in Case 2 are: The cylinder with 2 marked points on each boundary component, and the Möbius strip M_4.

For each of these Case 2 quivers we list the surfaces obtained after glueing and taking the $\mathbb{Z}_2$-quotient.

① The first and fourth give the cylinder with two marked points on each boundary component; the second and third give the once punctured square.

② All produce the Möbius strip with four marked points.

③ The cylinder with two marked points on each boundary component.

④ The Möbius strip with four marked points.

<u>**Case 3**</u>: i and j are connected to precisely three vertices.

The possible subquivers for this case are listed in Figure 7.11. Here we are using the fact that there cannot be more than two incoming/outgoing arrows at any given vertex.

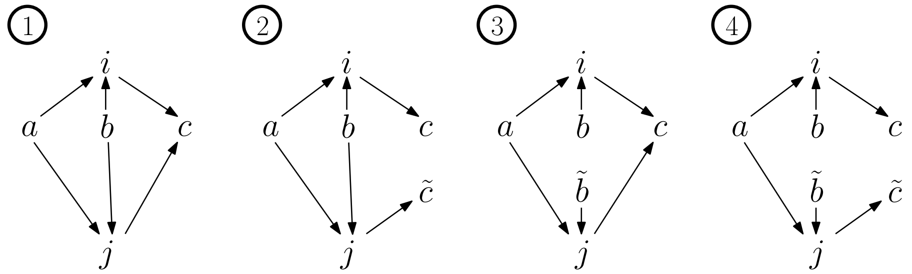
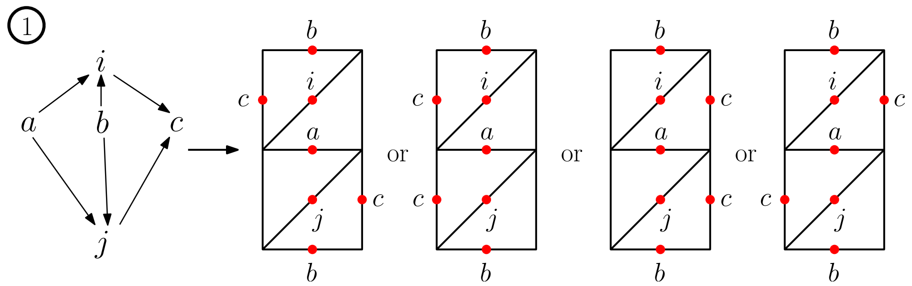
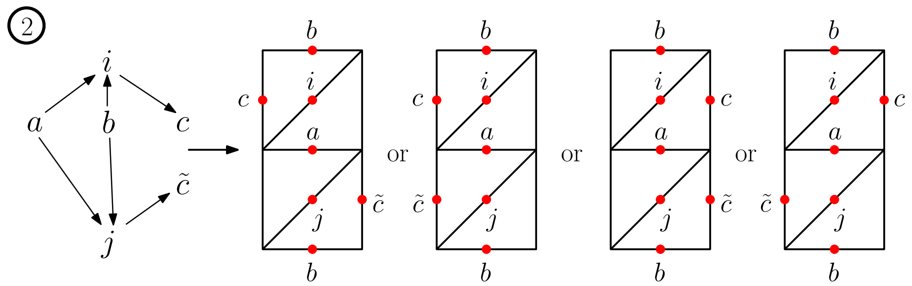


Figure 7.11: The list of the possible $\{i, j\}$-restriction quivers when i and j are connected to precisely three vertices, and $F_i = F_j$.

Note that it suffices to check only subquivers 1, 2 and 3 since 4 is equivalent to 3 after swapping the roles of a and b and using anti-symmetry. Below we present the possible surfaces producing the subquivers 1, 2 and 3.





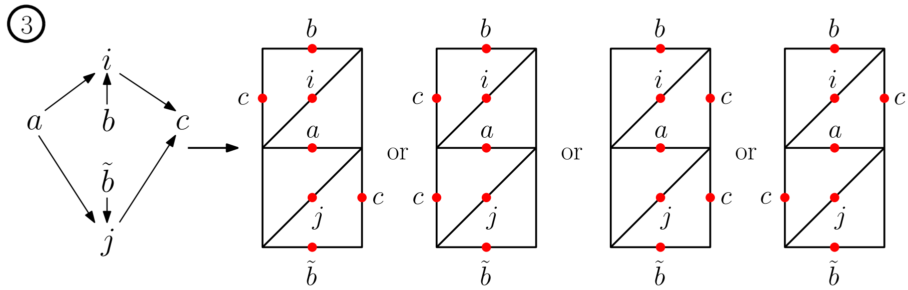


Figure 7.12: Upon glueing and taking $\mathbb{Z}_2$-quotients, the (unpunctured) surfaces we obtain in Case 3 are: The torus and Klein bottle, both with 1 boundary component and 2 marked points.

For each of these Case 3 quivers we list the surfaces obtained after glueing and taking the $\mathbb{Z}_2$-quotient.

① The first and fourth give the torus with one boundary component and two marked points; the second and third produce the twice punctured digon.

② The first and fourth give the Klein bottle with one boundary component and two marked points; the second and third produce the once punctured Möbius strip with two marked points.

③ The first and fourth both produce the Klein bottle with one boundary component and two marked points; the second and third give rise to the once punctured Möbius strip with two marked points.

<u>**Case 4**</u>: i and j are connected to precisely four vertices.

Being connected to four vertices the arc i will be the diagonal of a square with sides a, b, c and d. The arc j will therefore be the diagonal of a square with sides possessing labels from the set $\{a, \tilde{a}, b, \tilde{b}, c, \tilde{c}, d, \tilde{d}\}$. After glueing and taking the $\mathbb{Z}_2$-quotient then, if this procedure creates a surface, it will be a closed surface. However, we have forbidden punctured surfaces so none of our permitted surfaces satisfy Case 4.

In summary, the only unpunctured surfaces emitting triangulations producing non-distinct exchange polynomials are: the 6-gon; the Möbius strip with four marked points; the cylinder with two marked points on each boundary component; and the torus and the Klein bottle, both with one boundary component and two marked points. It is important to note that these surfaces only produce non-distinct exchange polynomials when their boundary segments receive no variables. In this chapter we only consider unpunctured surfaces receiving boundary variables, therefore, any triangulation of our surfaces will yield a distinct collection of exchange polynomials.

□

Proposition 7.11. Let i be a t-mutable arc in a triangulation T of (S, M). Then flipping i in T corresponds to LP mutation at i of the associated seed $\Sigma_{Q_{\overline{T}}} := (\{x_1, \ldots, x_n\}, \{F_1^{Q_{\overline{T}}}, \ldots, F_n^{Q_{\overline{T}}}\})$.

Proof. By Lemmas 7.2 and 7.10 we obtain that LP and quasi-cluster mutation agree on the level of variable change. Moreover, Lemma 7.1 tells us that if i is a t-mutable arc in T then there is no path $a \to i \to \tilde{a}$ in $Q_{\overline{T}}$ for any vertex a. Lemmas 7.3 and 7.10 confirm that $\Sigma_{Q_{\overline{T}}}$ is a valid seed and $F_j^{Q_{\overline{T}}} = \hat{F}_j^{Q_{\overline{T}}}$ for each exchange polynomial of $\Sigma_{Q_{\overline{T}}}$. Therefore we may invoke Proposition 6.19 to verify that double mutation at i and $\tilde{i}$ in $Q_{\overline{T}}$ coincides with LP mutation at i, for each t-mutable arc i in T. Finally, since Proposition 6.19 tells us that double mutation at i and $\tilde{i}$ corresponds to flipping the arc i in T, then the proof is complete.

□

Remark 11. Note that the LP seed $\{(a, 1+b), (b, a+c), (c, 1+b)\}$ in [Example 4.7, [26]] fails to agree with cluster algebra mutation because it arises from the 6-gon without any boundary variables. We present more of a discussion about this in Section 7.2.

7.1 Proof of Theorem B.

Theorem 7.12. Let (S, M) be an unpunctured (orientable or non-orientable) marked surface. Then the LP cluster complex $\Delta_{LP}(S, M)$ is isomorphic to the quasi-arc complex $\Delta^{\otimes}(S, M)$, and the exchange graph of $\mathcal{A}_{LP}(S, M)$ is isomorphic to $E^{\otimes}(S, M)$.

More explicitly, let T be a quasi-triangulation of (S, M) and Σ_T its associated LP seed. Then in the LP algebra $\mathcal{A}_{LP}(\Sigma_T)$ generated by this seed the following correspondence holds:

$\mathcal{A}_{\mathbf{LP}}(\mathbf{\Sigma_T})$		$(\mathbf{S}, \mathbf{M})$
Cluster variables	$\longleftrightarrow$	*Lambda lengths of quasi-arcs*
Clusters	$\longleftrightarrow$	*Quasi-triangulations*
LP mutation	$\longleftrightarrow$	*Flips*

Proof. By Proposition 7.11 all that is left to show is that LP mutation coincides with quasi-cluster mutation when:

(a) we flip an arc in a triangulation to a one-sided closed curve.

(b) we flip quasi-arcs in quasi-triangulations containing a one-sided closed curve.

Case (a).

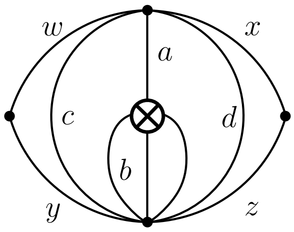
To resolve case (a) it suffices to show that flipping the arc a in Figure 7.13 agrees with LP mutation at a of the associated seed.

LP mutation at a produces the exchange polynomials:

$$F'_a = F_a \quad F'_b = (c+d)^2 + a'^2cd \quad F'_c = dy + a'bw \quad F'_d = cz + a'bx.$$

A simple computation produces the associated normalised exchange polynomials, which are recorded below. These normalised polynomials do indeed describe how lengths of quasi-arcs in the flipped quasi-triangulation exchange, so case (a) has been verified.

$$\hat{F}'_a = F_a \quad \hat{F}'_b = \frac{(c+d)^2 + a'^2 cd}{a'^2} \quad \hat{F}'_c = dy + a'bw \quad \hat{F}'_d = cz + a'bx$$



$F_a = c + d$

$F_b = a^2 + cd$

$F_c = ay + bw$

$F_d = az + bx$

Figure 7.13: A triangulation together with the associated exchange polynomials.

Case (b).

We split the task of verifying case (b) into four subcases:

1. Flipping a quasi-arc that is not enclosed in a region containing a one-sided closed curve.
2. Flipping b, c or d in the triangulation on the left of Figure 7.14.
3. Flipping a in the middle triangulation of Figure 7.14.
4. Flipping b, d or y in the triangulation on the right of Figure 7.14.

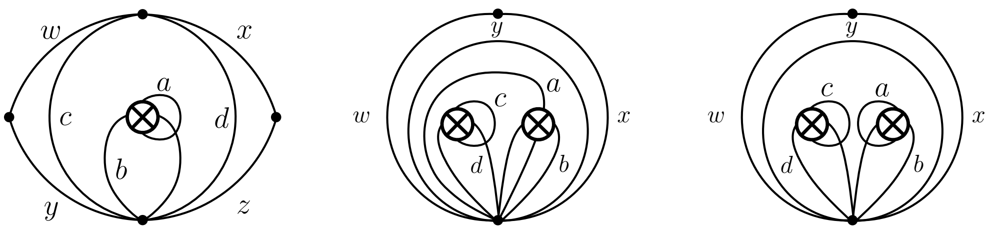


Figure 7.14: The three types of (local) configurations that contain a one-sided closed curve.

Subcase 1: Here LP mutation and surface flips coincide due to Proposition 7.11 and Case (a).

Subcase 2: To verify that LP mutation and surface flips coincide for this case, it suffices to check mutation at b and c.

The exchange polynomials corresponding to the left triangulation in Figure 7.14 are:

$$F_a = c + d \quad F_b = (c+d)^2 + a^2cd \quad F_c = dy + abw \quad F_d = cz + abx.$$

Mutating at b produces the following exchange polynomials:

$$F_a' = F_a \quad F_b' = F_b \quad F_c' = ab'y + dw \quad F_d' = ab'z + cx.$$

If instead we mutate at c we obtain the following exchange polynomials:

$$F_a' = y + c' \quad F_b' = (y+c')^2 + a^2yc' \quad F_c' = F_c \quad F_d' = wz + xc'.$$

The normalised versions of both of these sets of polynomials describe how lengths of quasi-arcs transform in their respective quasi-triangulations, so this completes subcase 2.

Subcases 3 and 4 hold analogous to case (a) and subcase $2(b)$, respectively.

□

7.2 Punctured surfaces

We confess now that we have omitted punctured surfaces throughout this chapter on account of their failure to emit an LP structure that encompasses the cluster structure already established (on orientable surfaces) in [12]. The reason why the flip/length structure of a punctured surface cannot be imitated by an LP structure is simple; if a surface is punctured then it emits a tagged triangulation containing two (distinct) arcs whose plain versions coincide. These two arcs have identical exchange polynomials, so by Lemma 7.4 the normalised exchange polynomials differ from the exchange polynomials. This ensures the LP structure and the quasi-cluster structure will not coincide.

Recall that when the boundary segments receive no variables the 6-gon and the cylinder $C_{2,2}$ have the same cluster structure as the punctured triangle and the twice punctured monogon, respectively - see Figure 7.15. From the comments made above we instantly get confirmation of the fact obtained in the proof of Lemma 7.10, that in the absence of boundary variables, there is no LP algebra producing the cluster structure of the 6-gon or the cylinder $C_{2,2}$. One might be tempted to believe the torus with one boundary component and two marked points follows suit, and shares its cluster structure with a punctured surface, however, the work of Bucher, Yakimov [3] and Gu [19] tells us that this is not the case.

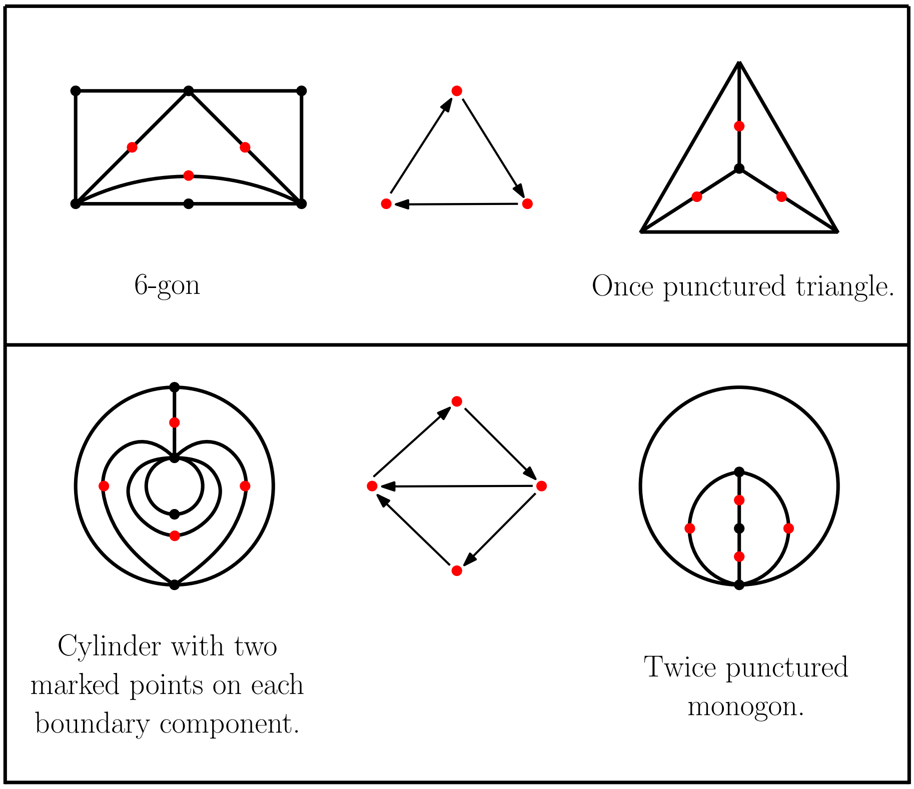


Figure 7.15: Here we list all orientable bordered surfaces which share their cluster algebra structure with a punctured surface. For each of these bordered surfaces we provide the punctured surface possessing the same cluster structure. In each case we present triangulations emitting matching adjacency quivers.

Chapter 8

Laurent phenomenon algebras arising from laminated surfaces

We just saw in Section 7.2 that the quasi-cluster algebra of a punctured surface has no LP structure. The incompatibility arises from the existence of non-distinct exchange polynomials in certain quasi-triangulations, causing undesired normalisation to take place. The goal of this chapter is to add laminations to the surface, with the intention of inserting extra variables into the exchange polynomials, ensuring normalisation only occurs when needed. We shall do this in such a way that the specialisation is precisely the underlying quasi-cluster algebra.

8.1 Laminations and shear coordinates

Definition 8.1. A ***lamination*** on a bordered surface (S, M) is a finite collection of non-self-intersecting and pairwise non-intersecting curves in (S, M), considered up to isotopy, and subject to the conditions outlined below. Each curve must be one of the following:

- A curve connecting two unmarked points in ∂S. Though we do not allow the scenario when this curve is isotopic to a piece of boundary containing one or zero marked points;

- A curve with one end being an unmarked point in ∂S, and whose other end spirals into a puncture;
- A curve with both ends spiralling into (not necessarily distinct) punctures. We forbid the case when the curve has both ends spiralling into the same puncture, and does not enclose anything else;
- A two-sided closed curve which does not bound a disk, a once-punctured disk, or a Möbius strip.

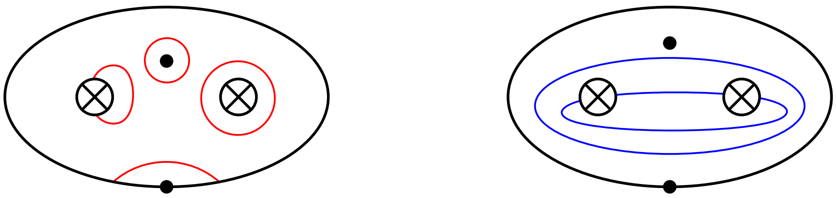


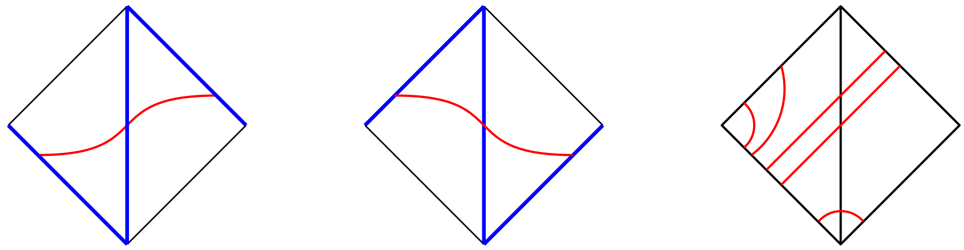
Figure 8.1: None of the curves on the left are considered laminations. All curves on the right are legitimate laminations.

We now describe W. Thurston's *shear coordinates* [33] with respect to a lamination of an ideal-triangulated orientable surface.

Definition 8.2 (S-shape and Z-shape intersections)**.** Let Q_γ be a triangulated quadrilateral with diagonal γ. Suppose C is a curve intersecting opposite sides of Q_γ (and does not intersect the boundary of Q_γ anywhere else). Denote these sides by α and β. If α, β and γ form an $'S'$ (resp. $'Z'$), then call the intersection of C with Q_γ an ***S-shape intersection*** (resp. ***Z-shape intersection***). See Figure 8.2.

Definition 8.3 (Shear coordinates for ideal triangulations)**.** Let T be an ideal triangulation of an orientable bordered surface (S, M), and L a lamination. Furthermore, let γ be an arc of T which is not the folded side of a self-folded triangle, and denote by Q_γ the quadrilateral of T whose diagonal is γ. The ***shear coordinate***, $b_T(L,\gamma)$, of L and γ, with respect to T, is defined as:

$$b_T(L,\gamma) := \#\left\{\begin{matrix} S\text{-shape intersections} \\ \text{of } L \text{ with } Q_\gamma \end{matrix}\right\} - \#\left\{\begin{matrix} Z\text{-shape intersections} \\ \text{of } L \text{ with } Q_\gamma \end{matrix}\right\}$$



S-shape intersection $B_T(L,\gamma)=1$

Z-shape intersection $B_T(L,\gamma)=-1$

$B_T(L,\gamma)=2$

Figure 8.2: S-shape and Z-shape intersections.

Remark 12. Note that even though a lamination spiralling into a puncture p will intersect any arc incident to p infinitely many times, $b_T(L,\gamma)$ will always be finite.

We explain below how Fomin and Thurston [13] extended the notion of shear coordinates to (tagged) triangulations of orientable bordered surfaces.

Definition 8.4 (Shear coordinates for triangulations)**.** Let T be a triangulation and L a lamination. If L spins into a puncture p, containing only arcs with notches at p, then reverse the direction of spinning of L at p, and replace all these notched taggings with plain ones.

Using the rule above we may convert the lamination L of T into a lamination L_1 of a triangulation T_1, with the property that any notched arc in T_1 appears with its plain counterpart. As per usual, denote by T° the ideal triangulation associated to T_1 - as hinted by the notation, this is also the ideal triangulation associated to T. Let γ be an arc of T, and denote by γ° the corresponding arc in T°. We define $b_T(L,\gamma)$ as follows:

- If γ° is not the self-folded side of a triangle in T°, then define $b_T(L,\gamma) := b_{T^\circ}(L_1,\gamma^\circ)$.
- If γ° is the self-folded side of a triangle in T°, with puncture p, then reverse the direction of spinning of L_1 at p, and denote this new lamination by L_2. Furthermore, let β denote the remaining side of the triangle in T° that is folded along γ°. We define $b_T(L,\gamma) := b_{T^\circ}(L_2,\beta)$.

Remark 13. For a lamination L of an ideal triangulation T, note that if γ is the enclosing arc of a puncture p, then $b_T(L, \gamma)$ does not depend on the direction L is spinning at any other puncture enclosed in any other monogon.

Definition 8.5. A ***multi-lamination***, $\mathbf{L}$, of a bordered surface (S, M) consists of a finite collection of laminations of (S, M).

Let T be a triangulation of an orientable bordered surface (S, M). For a multi-lamination $\mathbf{L}$, of (S, M), we extend the adjacency quiver, Q_T, to a quiver $Q_{T,\mathbf{L}}$ as follows:

- For each lamination L_i in $\mathbf{L}$ add a corresponding vertex to Q_T. Abusing notation, we shall also denote this vertex by L_i.
- Let γ denote a vertex in Q_T and its corresponding arc in T. If $b_T(L_i, \gamma)$ is positive (resp. negative) add $|b_T(L_i, \gamma)|$ arrows $L_i \to \gamma$ (resp. $L_i \leftarrow \gamma$).

Proposition 8.6 (Theorem 13.5, [13])**.** Let $\mathbf{L}$ be a multi-lamination of an orientable bordered surface (S, M). Then for any arc γ in a triangulation T, $\mu_\gamma(Q_{T,\mathbf{L}}) = Q_{\mu_\gamma(T),\mathbf{L}}$.

Proposition 8.7. For each triangulation T of a multi-laminated bordered surface $(S, M, \mathbf{L})$, $Q_{\overline{T},\overline{\mathbf{L}}}$ is an anti-symmetric quiver.

Proof. We have already verified anti-symmetry between vertices corresponding to lifted arcs. It remains to check anti-symmetry for the rest of the quiver.
For each lamination L_i of $\mathbf{L}$ we have two vertices in $Q_{\overline{T},\overline{\mathbf{L}}}$ corresponding to the lifted versions of L_i. Abusing notation, we shall denote these vertices by L_i and $\tilde{L}_i$. If the lift L_i cuts through a triangulated quadrilateral in an $'S'$ (resp. $'Z'$) shape, the other lift $\tilde{L}_i$ cuts through the twin quadrilateral in a $'Z'$ (resp. $'S'$) shape. Hence we get an arrow $L_i \to \gamma$ (resp. $L_i \leftarrow \gamma$) *if and only if* there is an arrow $\tilde{L}_i \leftarrow \tilde{\gamma}$ (resp. $\tilde{L}_i \to \tilde{\gamma}$). Furthermore, by definition of this quiver, there are no arrows

between vertices corresponding to lifted laminations. In particular, there are no arrows $L_i \to \tilde{L}_i$ for any i.

□

To utilise anti-symmetric quivers as much as possible, in certain triangulations, it will be helpful to contemplate an alternative choice of flip that our definitions had previously forbidden. This will involve considering *traditional triangulations*, which are defined below. We follow up this definition with a discussion on how this notion of triangulation arises.

Definition 8.8. A ***traditional triangulation*** consists of a maximal collection of pairwise compatible arcs, containing no arcs that cut out a once punctured monogon.

Let T be a triangulation of (S, M) and α an arc in T. Proposition 6.10 tells us there exists a unique quasi-arc α' such that $T \cup \{\alpha'\} \setminus \{\alpha\}$ is a quasi-triangulation. However, when α' is a one-sided closed curve there is an alternative flip of α we can consider (which is forbidden under our current set-up). We shall describe this alternative flip and explain how it fits in with mutation of anti-symmetric quivers. Firstly, note that by Definition 6.6, if α' is a one-sided closed curve then it intersects precisely one arc $\beta \in T$. There exists a unique arc $\alpha^* \notin T$ enclosing α' and β in M_1. If we choose to flip α to α^* (instead of α') then we will arise at a traditional triangulation.
In fact, analogous to the proof of Proposition 6.10, for any triangulation T and any arc γ of T, there exists a unique arc $\gamma' \neq \gamma$ such that $T \cup \{\gamma'\} \setminus \{\gamma\}$ is a traditional triangulation. Turning our attention back to this alternative flip, consider the lift $\overline{T}$, of T, in the double cover $\overline{(S, M)}$. If we flip both of the lifts α, $\tilde{\alpha}$ in $\overline{(S, M)}$ and take the $\mathbb{Z}_2$-quotient we will obtain precisely $T' := T \cup \{\alpha^*\} \setminus \{\alpha\}$. Therefore the existing theory of cluster algebras from surfaces, Proposition 8.6 to be precise, tells us that

$$\mu_\alpha \circ \mu_{\tilde{\alpha}}(Q_{\overline{T},\overline{\mathbf{L}}}) = Q_{\overline{T'},\overline{\mathbf{L}}}.$$

We conclude this short discussion with the following proposition.

Proposition 8.9. Fix a multi-lamination $\mathbf{L}$ of a bordered surface (S, M). Let T be a triangulation of (S, M). Then for any arc γ in T, flipping γ (with respect to traditional triangulations) corresponds to double mutation of the anti-symmetric quiver $Q_{\overline{T},\overline{\mathbf{L}}}$, at the vertices corresponding to the two lifts of γ.

Proof. By Definition 6.7, since γ is not bounded by an arc enclosing a Möbius strip, M_1, then the interiors of the flip regions containing the lifts of γ are disjoint. Therefore, flipping γ in (S, M) corresponds to simultaneously flipping both of the lifts in $\overline{(S, M)}$. Finally, by the theory of orientable surfaces, flipping an arc in the double cover corresponds to mutating the vertex in $Q_{\overline{T},\overline{\mathbf{L}}}$ representing that arc.

□

Remark 14. In general, when considering traditional triangulations, mutation does not preserve the anti-symmetric property of a quiver. In particular, after performing the flip of α to α^* discussed above, the corresponding quiver will contain (two) arrows between β and $\tilde{\beta}$, depriving it of anti-symmetry. (Flips that result in another triangulation will of course preserve the anti-symmetric property.)

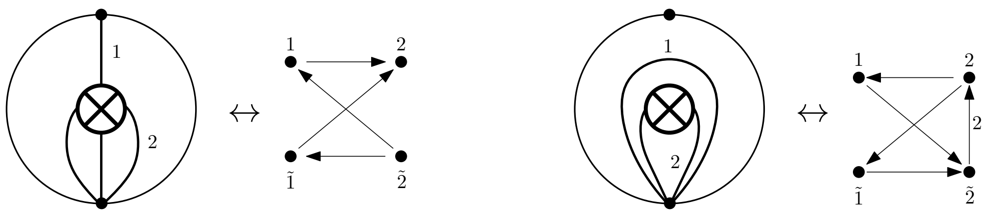


Figure 8.3: Performing a flip to an arc bounding M_1 breaks anti-symmetry.

We have already seen that the lambda length of a quasi-arc can be viewed as a formal variable. Our goal now will be to introduce new variables, called *laminated lambda lengths*, that take into account the multi-lamination $\mathbf{L}$ on the surface, not just the geometry. The procedure we shall use follows the approach taken by Fomin and Thurston [13]; it will involve rescaling the lambda length of each quasi-arc γ with respect to the intersection numbers of γ with $\mathbf{L}$. As it stands, this notion is currently ill-defined. Namely, when **L** spirals into a puncture p, it will intersect any arc incident to p infinitely many times. To bypass this problem we shall open up the punctures.

8.2 Opening the surface

Definition 8.10. Let (S,M) be bordered surface and $P \subseteq M \setminus \partial S$ be a set of punctures. The ***(partially) opened bordered surface***, (S_P, M_P) is defined as follows. S_P is obtained from S by removing a small open neighbourhood around each $p \in P$. Furthermore, to each newly created boundary component, C_p, we add a marked point m_p. We then set $M_P := (M \setminus P) \cup \{m_p\}_{p \in P}$.

It is crucial to note that our treatment of a partially opened bordered surface (S_P, M_P) throughout this chapter will differ from that of a bordered surface. I.e. we will care whether a boundary segment was the consequence of opening a puncture. In particular, the set of quasi-arcs of $A^{\otimes}(S_P, M_P)$ is defined as before, except now:

- We allow arcs to be notched at m_p for $p \in P$.
- We do not allow an arc to cut out a monogon containing C_p for $p \in P$.

With this in mind there is a canonical projection map

$$\kappa_P : A^{\otimes}(S_P, M_P) \longrightarrow A^{\otimes}(S, M)$$

that amounts to collapsing each boundary component C_p in (S_P, M_P). Any quasi-arc $\overline{\gamma} \in A^{\otimes}(S_P, M_P)$ that projects to a quasi-arc $\gamma \in A^{\otimes}(S, M)$ will be referred to as a *lift* of γ.

Definition 8.11. The ***opened bordered surface***, (S^*, M^*), is the result of opening up all the punctures. Note that $\kappa_{M \setminus \partial S} := \kappa^*$ factors through every other map κ_P.

We now describe what will be the overarching notion of a Teichmüller space with regards to opening surfaces.

Definition 8.12. A ***decorated set of punctures***, $\tilde{P}$, is a subset $P \subseteq M \setminus \partial S$ together with a choice of 'orientation' on C_p for each $p \in P$.

Remark 15. To clarify, our usage of 'orientation' means that we are choosing a direction of flow on each boundary component C_p. Being on a non-orientable surface just means that we cannot globally speak about whether this flow is clockwise or counter-clockwise.

Definition 8.13. For a decorated set of punctures $\tilde{P}$ we define the ***partially opened Teichmüller space***, $\mathcal{T}_{\tilde{P}}(S_P, M_P)$, to be the space of all finite volume, complete hyperbolic metrics on $S_P \setminus (M \setminus P)$ with geodesic boundary, up to isotopy. The ***decorated partially opened Teichmüller space***, $\tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$, consists of the same metric as in $\mathcal{T}_{\tilde{P}}(S_P, M_P)$, except now they are considered up to isotopy relative to $\{m_p\}_{p \in P}$. Additionally, there is a choice of horocycle around each point in $M \setminus P$.

Given a decorated set of punctures $\tilde{P}$ and $\sigma \in \mathcal{T}_{\tilde{P}}(S_P, M_P)$, then for any quasi-arc $\gamma \in A^{\otimes}(S_P, M_P)$ we can associate a unique non-intersecting geodesic γ_σ on S_P. If γ is a one-sided closed curve then γ_σ is just the usual geodesic representative of γ with respect to σ. If γ is an arc we define γ_σ as follows:

- For an endpoint of γ not in P, γ_σ runs out to the corresponding cusp.
- For an endpoint of γ in P that is tagged plain γ_σ should spiral (infinitely) around C_p in the chosen direction. Otherwise the endpoint is notched, and it should spiral *against* the chosen direction.

More generally, for a quasi-arc $\overline{\gamma} \in A^{\otimes}(S_P, M_P)$ we set $\overline{\gamma}_\sigma = \gamma_\sigma$, where $\gamma = \kappa_P(\overline{\gamma})$.

Definition 8.14. Let $\tilde{P}$ be a decorated set of punctures and $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$. For each $p \in P$ consider a small segment of the horocycle originating from m_p which is both perpendicular to C_p and all geodesics γ_σ that spiral into C_p in the chosen direction. Such a segment is called the ***perpendicular horocycle segment***, and is denoted h_p.

Definition 8.15 (Length of plain arcs on opened surface)**.** Let $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$. We will eventually define the lengths of all plain arcs γ in (S_P, M_P), however, for now we shall only concentrate on those whose ends twist sufficiently far around opened punctures C_p in the direction consistent with the chosen orientation of each C_p.
At the ends of γ_σ that spiral around an opened puncture C_p there will be infinitely many intersections with the horocyclic segment h_p at m_p. We describe how we pick one of these intersections:

For γ with endpoints m_p and m_q (that twist sufficiently far around the corresponding boundaries) choose the unique intersections between γ_σ and each horocyclic segment, h_p and h_q, such that the path running from

- m_p to an intersection of h_p with γ_σ, then from
- γ_σ to an intersection of γ_σ with h_q, then from
- h_q to m_q

is homotopic to the original arc γ. In the less complicated case of γ not having both endpoints in P, we leave γ_σ unmodified at the ends not in P, and, as usual, choose the unique intersection between γ_σ and the corresponding horocycle. The ***length*** of γ, $l_\sigma(\gamma)$, is defined to be the signed distance of γ_σ between the horocycles at its endpoints (with respect to the intersections described above).

This definition is extended to all plain arcs (not just those twisting sufficiently far around open punctures) by defining,

$$l_\sigma(\psi_p^{\pm 1}(\gamma)) := \pm n_p(\gamma) l_\sigma(p) + l_\sigma(\gamma) \tag{8.2.1}$$

Here $\psi_p(\gamma)$ denotes the twist of γ around C_p in the direction consistent with C_p's orientation ($\psi_p^{-1}(\gamma)$ being the twist against C_p's orientation). $n_p(\gamma)$ is the number of endpoints γ has at m_p. Finally, $l_\sigma(p)$ is the length of C_p if $p \in P$, and 0 otherwise.

Remark 16. The extended definition (8.2.1) is well defined as the distance between successive intersections of γ_σ with h_p is $l_\sigma(p)$. A proof of this can be found in [Lemma 10.7, [13]].

Definition 8.16. Let $\tilde{P}$ be a decorated set of punctures and $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$. For each $p \in P$ consider the point $\overline{m}_p$ on C_p that is a (signed) distance $v(p) := 2\ln|\lambda(p) - \lambda(p)^{-1}|$ from m_p in the direction against the orientation of C_p. The ***conjugate perpendicular horocycle segment***, $\overline{h}_p$, is the segment of the horocycle originating from $\overline{m}_p$ which is both perpendicular to C_p and all geodesics γ_σ that spiral into C_P *against* the chosen direction.

Definition 8.17 (Length of arcs on opened surface)**.** Let $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$. For an arc γ whose endpoints twist sufficiently far around opened punctures, $l_\sigma(\gamma)$ is defined as in the previous definition, except now, when there is a notched end at m_p, we consider the intersection of γ_σ with the conjugate perpendicular, $\tilde{h}_p$, instead. The definition is again extended to all arcs by using:

$$l_\sigma(\psi_p^{\pm 1}(\gamma)) := \pm n_p(\gamma) l_\sigma(p) + l_\sigma(\gamma) \tag{8.2.2}$$

Here ψ_p and $l_\sigma(p)$ are as in (8.2.1). However, we extend $n_p(\gamma)$ to all arcs by setting it as minus (resp. plus) the number of notched (resp. plain) ends of γ at m_p.

Definition 8.18 (Length of one-sided closed curves)**.** Let $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$. If γ is a one-sided closed curve then we denote by $l_\sigma(\gamma)$ the hyperbolic length of the geodesic representation of γ in σ.

Definition 8.19. Let $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(S_P, M_P)$. We define the lambda length of a quasi arc γ in (S_P, M_P) as:

$$\lambda_\sigma(\gamma) = \begin{cases} e^{\frac{l_\sigma(\gamma)}{2}}, & \text{if } \gamma \text{ is an arc,} \\ 2\sinh(\frac{l_\sigma(\gamma)}{2}), & \text{if } \gamma \text{ is a one-sided closed curve,} \end{cases}$$

In addition to this, for each puncture p of (S, M), we define $\lambda_\sigma(p) := e^{\frac{l_\sigma(p)}{2}}$.

Remark 17. Let us fix a lift $\overline{\gamma} \in (S^*, M^*)$ for each arc γ in (S, M). [Corollary 10.16, [13]] tells us that for each triangulation T of (S, M), the cluster $\mathbf{x}(T) := \{\lambda(\overline{\gamma}) | \gamma \in T\}$ may be viewed as a set of algebraically independent variables. Furthermore, [Theorem 11.1, [13]] reveals that the exchange relations between these clusters are the relations of the corresponding flips on the pre-opened surface (S, M), that have been rescaled at the situations where a flip region in (S, M) has not lifted to a flip region in (S^*, M^*). In the terminology of [13], this collection of clusters, together with the corresponding rescaled exchange relations, form a *non-normalised exchange pattern* on $E^\circ(S, M)$.

8.3 Transverse measure and tropical lambda lengths

Definition 8.20. A ***lifted lamination***, $\overline{L}$, of (S^*, M^*) consists of a choice of orientation on each opened puncture C_p, together with a finite number of non-intersecting curves with endpoints in $\partial S^* \setminus M^*$, considered up to isotopy relative to M^*. We forbid the following types of curves:

- one-sided closed curves;
- two-sided closed curves that bound a disk, a Möbius strip, or a disk containing a single opened puncture;
- curves with endpoints in ∂S^* which are isotopic to a piece of boundary containing one or zero marked points.

Remark 18. Observe that we can construct a canonical projection map taking lifted laminations, $\overline{L}$, of (S^*, M^*) to laminations, L, of (S, M). Namely, L is obtained from $\overline{L}$ by closing the opened punctures, and demanding endpoints in $\overline{L}$ that end on opened punctures, C_p, will now spiral around p in the direction *opposite* to the orientation chosen on C_p (with respect to $\overline{L}$). The reason why we demand the spiralling to oppose the orientation on C_p is to produce equation 8.3.1 - if the orientation agreed we would have to replace '$\pm$' with '$\mp$'.

Definition 8.21 (Transverse measures for plain arcs)**.** Let $\overline{L}$ be a lifted lamination of a bordered surface (S, M). Let γ be a plain arc of the opened surface (S^*, M^*), or a boundary segment. The ***transverse measure*** of γ with respect to $\overline{L}$ is the integer $l_{\overline{L}}(\gamma)$ defined as follows:

- If γ does not have ends at any m_p then $l_{\overline{L}}(\gamma)$ is the minimal number of intersection points between $\overline{L}$ and any arcs homotopic to γ.
- If γ has one or two ends at opened punctures, and γ twists sufficiently far in the direction $\overline{L}$ spirals around those opened punctures (if it even does), then $l_{\overline{L}}(\gamma)$ is again defined to be the minimal number of intersection points between $\overline{L}$ and any arcs homotopic to γ.

  We extend this definition to all plain arcs, not just to those twisting sufficiently far, by setting:

$$l_{\overline{L}}(\psi_p^{\pm 1}(\gamma)) := \pm n_p(\gamma) l_{\overline{L}}(p) + l_{\overline{L}}(\gamma) \tag{8.3.1}$$

  Here ψ_p and n_p are as in Definition 8.17, although it is key to note ψ_p is now defined with respect to the orientation on each C_p coming from L. $l_{\overline{L}}(p)$ is the number of intersections of $\overline{L}$ with C_p.

Definition 8.22 (Transverse measures for all arcs)**.** For plain arcs γ, $l_{\overline{L}}(\gamma)$ is defined as in Definition 8.21. For an arc γ which is notched at an endpoint m_p, and that twists sufficiently far *against* the direction $\overline{L}$ spirals around C_p, define $l_{\overline{L}}(\gamma)$ to be the minimal number of intersection points between $\overline{L}$ and (any arcs homotopic to) γ, plus $l_{\overline{L}}(p)$. The additional term $l_{\overline{L}}(p)$ accounts for the asymptotics of $v(p)$. We extend the definition to all arcs using equation (8.3.1), defined in Definition 8.21.

Definition 8.23 (Tropical semi-field associated with a multi lamination)**.** Let $\mathbf{L}$ be a multi lamination of a bordered surface (S, M). For each lamination L_i in $\mathbf{L}$ we introduce a variable q_i. We consider the tropical semifield $\mathbb{P}_{\mathbf{L}}$ over these variables. More specifically, $\mathbb{P}_{\mathbf{L}} = \mathrm{Trop}(q_i : i \in I)$. Note that I is just the indexing set for the laminations L_i in $\mathbf{L}$.

Definition 8.24 (Tropical lambda lengths)**.** Let $\overline{\mathbf{L}} = \{\overline{L}_i\}_{i\in I}$ be a lifted multi-lamination on an opened surface (S^*, M^*). Let γ be an arc or boundary component of (S^*, M^*). We define the ***tropical lambda length***, $c_{\overline{\mathbf{L}}}(\gamma)$, of γ as follows:

$$c_{\overline{\mathbf{L}}}(\gamma) = \prod_{i\in I} q_i^{-\frac{l_{\overline{L}_i}(\gamma)}{2}} \tag{8.3.2}$$

Note that by (8.3.1) these tropical lambda lengths satisfy

$$c_{\overline{\mathbf{L}}}(\psi_p^{\pm 1}(\gamma)) = c_{\overline{\mathbf{L}}}(p)^{\pm n_p(\gamma)} c_{\overline{\mathbf{L}}}(\gamma) \tag{8.3.3}$$

8.4 Laminated lambda lengths and the laminated quasi-cluster algebra

Recall that we began to consider the opened surface with the intention of rescaling lambda lengths of quasi-arcs using transverse measures. (Transverse measure is generally ill-defined on un-opened surfaces due to possible infinite intersections of arcs with the multi-lamination.) Our approach so far requires us to fix a lift $\overline{\gamma}$ in (S^*, M^*) for each quasi-arc γ in (S, M). As we already noted in Remark 17, the clusters $\mathbf{x}(T) := \{l(\overline{\gamma}) : \gamma \in T\}$ arising from triangulations form a non-normalised exchange pattern on $E^\circ(S, M)$. If the arcs of a flip region in (S, M) lift to another flip region in (S^*, M^*) then the exchange relations coincide. However, if they do not lift to a flip region, the exchange relations will differ. In particular, when they do not, the exchange relation on the opened surface will be a rescaled version of the original. This rescaled relation is obtained by finding a new collection of lifts such that, with respect to these new lifts, the flip region does lift to a flip region. Since the new lifts only differ from the old via spiralling at opened punctures, this rewriting is obtained using (8.2.2). The issue with the current standings is that it is quite hard to keep track of these particular rescalings. The following definition shows that by putting boundary conditions on the opened punctures, we may both achieve our goal of defining *laminated lambda lengths* that take into account the lamination on the surface, and eliminate the nasty rescaling process required when flip regions do not lift to flip regions.

Definition 8.25. The ***complete decorated Teichmüller space***, $\overline{\mathcal{T}}(S, M)$, is the disjoint union of the $\mathcal{T}_{\tilde{P}}(S_P, M_P)$ over all $3^{|\partial S \setminus M|}$ partially decorated sets $\tilde{P}$.

Definition 8.26. Let $\mathbf{L} = \{L_i\}_{i \in I}$ be a multi-lamination of (S, M). Fix a lift $\overline{\mathbf{L}}$, of $\mathbf{L}$, on the opened surface (S^*, M^*). A point (σ, q) of the ***laminated Teichmüller space***, $\overline{\mathcal{T}}(S, M, \overline{\mathbf{L}})$, consists of a decorated hyperbolic structure $\sigma \in \overline{\mathcal{T}}(S, M)$ and a collection of positive real numbers $q := (q_1, \dots q_{|I|})$ subject to the following condition on all punctures $p \in \partial S \setminus M$:

$$\lambda(p) = c_{\overline{\mathbf{L}}}(p).$$

Definition 8.27. Let (S, M) be a bordered surface, and $\mathbf{L}$ a multi-lamination. Fix a lift $\overline{\mathbf{L}}$ of $\mathbf{L}$. For each quasi-arc γ of (S, M) choose a lift $\overline{\gamma}$. We define the ***laminated lambda length***, $x_{\overline{\mathbf{L}}}(\gamma)$, of γ to be:

$$x_{\overline{\mathbf{L}}}(\gamma) := \frac{\lambda(\overline{\gamma})}{c_{\overline{\mathbf{L}}}(\overline{\gamma})} \tag{8.4.1}$$

Due to the enforced 'boundary' condition $\lambda(p) = c_{\overline{\mathbf{L}}}(p)$ for each puncture p, from equations (8.2.2) and (8.3.3) we realise that $x_{\overline{\mathbf{L}}}(\gamma)$, as the notation suggests, is independent of the choice of lift $\overline{\gamma}$. It is worth noting that this definition *does* depend on the choice of lift $\overline{\mathbf{L}}$.

The following theorem follows from [Corollary 15.5, [13]].

Theorem 8.28. Let $\mathbf{L} = \{L_i\}_{i \in I}$ be a multi-lamination of (S, M) and $\overline{\mathbf{L}}$ a lift. For any quasi-triangulation T with quasi-arcs and boundary arcs $\gamma_1, \dots, \gamma_{n+b}$ there exists a homeomorphism

$$\Lambda_T \colon \overline{\mathcal{T}}(S,M,\overline{\mathbf{L}}) \longrightarrow \mathbb{R}^{n+b+|I|}_{>0}$$
$$(\sigma, q) \mapsto (x_{\overline{\mathbf{L}}}(\gamma_1), \dots, x_{\overline{\mathbf{L}}}(\gamma_1), q_1, \dots q_{|I|})$$

Theorem 8.28 allows us to simultaneously view the laminations in a multi-lamination, and the laminated lambda lengths of any quasi-triangulation, as algebraically independent variables. With this in mind, given a laminated bordered surface $(S, M, \mathbf{L})$, we can consider a seed, $(\mathbf{x}, T)$, consisting of a quasi triangulation T and a collection of algebraically independent *cluster variables* $\mathbf{x} := \{x_\gamma | \gamma \in T\}$. Furthermore, consider the coefficient ring $\mathbb{ZP}$ generated (over $\mathbb{Z}$) by the algebraically independent *frozen variables* x_b and x_{L_i} corresponding to each boundary segment b of (S, M) and each lamination L_i of $\mathbf{L}$.

Performing flips of quasi-arcs, and using the exchange relations in Definition 6.15 coupled with the equation (8.4.1) of a laminated lambda length, we can generate all other seeds with respect to our initial seed $(\mathbf{x}, T)$.

Let $\mathcal{X}$ be the set of all cluster variables appearing in all of these seeds. $\mathcal{A}_{(\mathbf{x},T)}(S, M, \overline{\mathbf{L}}) := \mathbb{ZP}[\mathcal{X}]$ is the ***laminated quasi-cluster algebra*** of the seed $(\mathbf{x}, T)$.

The definition of a quasi-cluster algebra depends on the choice of the initial seed and of the lift $\overline{\mathbf{L}}$. However, [Definition 15.3, [13]] reassures us that if we choose a different initial seed, or a different lift, the resulting laminated quasi-cluster algebra will be isomorphic to $\mathcal{A}_{(\mathbf{x},T)}(S, M, \overline{\mathbf{L}})$. As such, it makes sense to talk about the laminated quasi-cluster algebra, $\mathcal{A}(S, M, \mathbf{L})$, of $(S, M, \mathbf{L})$.

Definition 8.29. The ***laminated quasi-arc complex*** $\Delta^{\otimes}(S, M, \mathbf{L})$ of the laminated quasi-cluster algebra $\mathcal{A}(S, M, \mathbf{L})$ is the simplicial complex with the ground set being the cluster variables of $\mathcal{A}(S, M, \mathbf{L})$, and the maximal simplices being the clusters.

Definition 8.30. The ***exchange graph*** $E^{\otimes}(S, M)$ of the laminated quasi-cluster algebra $\mathcal{A}(S, M, \mathbf{L})$ is the graph whose vertices correspond to the clusters of $\mathcal{A}(S, M, \mathbf{L})$. Two vertices are connected by an edge if their corresponding clusters differ by a single mutation.

8.5 Connecting laminated quasi-cluster algebras to LP algebras

8.5.1 Finding exchange relations of quasi-arcs via quivers

The following proposition, which is a subcase of [Theorem 15.6, [13]], tells us that when we are looking at flips between traditional triangulations, then the exchange polynomials of arcs can be obtained by looking at the ingoing and outgoing arrows of the associated quiver.

Proposition 8.31. Let T be a triangulation of (S, M), $\mathbf{L}$ a multi-lamination and $\overline{\mathbf{L}}$ a lift. Label the arcs, boundary segments and laminations $1, \ldots, m$ and consider the associated quiver $Q_{\overline{T}, \overline{\mathbf{L}}}$.

Let γ be an arc in T and consider the unique arc $\gamma' \neq \gamma$ such that $T \cup \{\gamma'\} \setminus \{\gamma\}$ is a traditional triangulation. Suppose the lifts of γ receives the labels j and $\tilde{j}$. Then the exchange polynomial of γ with respect to this flip is:

$$F_j = \prod_{\substack{b_{ij}>0 \\ i \in \{1,\ldots,m,\tilde{1}\ldots,\tilde{m}\}}} x_i^{b_{ij}} + \prod_{\substack{b_{ij}<0 \\ i \in \{1,\ldots,m,\tilde{1}\ldots,\tilde{m}\}}} x_i^{-b_{ij}}$$

Proof. Follows from [Theorem 15.6, [13]].

□

We are now at the stage where we know that for a laminated surface $(S, M, \mathbf{L})$, flipping arcs in a triangulation T corresponds to double-mutation of the associated anti-symmetric quiver $Q_{\overline{T},\overline{\mathbf{L}}}$. Moreover, we know that the associated exchange relations of each vertex of $Q_{\overline{T},\overline{\mathbf{L}}}$ describe how the laminated lambda lengths change under flip. It is crucial to note that to get the correspondence above we have been allowing flips to arcs bounding M_1 instead of to one-sided closed curves. It turns out that if we make an adjustment to how we 'read off' polynomials from $Q_{\overline{T},\overline{\mathbf{L}}}$ then we can obtain the exchange relations regarding the flip to a one-sided closed curve instead of the arc bounding M_1.

Definition 8.32. Let Q be an anti-symmetric quiver with $2m$ vertices, of which $m - n$ pairs are frozen. The ***shortened exchange matrix*** of Q is the matrix $\overline{B} = (\bar{b}_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$, where $\bar{b}_{ij} := b_{ij} + b_{\tilde{i}j}$. Each column $1 \le j \le n$ of $\overline{B}$ is naturally associated to the polynomial

$$\overline{F}_j^Q := \prod_{\substack{\bar{b}_{ij} > 0 \\ i \in \{1,\ldots,m\}}} x_i^{\bar{b}_{ij}} + \prod_{\substack{\bar{b}_{ij} < 0 \\ i \in \{1,\ldots,m\}}} x_i^{-\bar{b}_{ij}}.$$

We wish to show that these exchange relations from $\overline{B}$ describe how laminated lambda lengths change when flipping arcs. To achieve this we require the following two lemmas.

Lemma 8.33. Let T be a triangulation of $(S, M, \mathbf{L})$ and $Q_{\overline{T},\overline{\mathbf{L}}}$ its associated quiver. Furthermore, let i be a vertex of $Q_{\overline{T},\overline{\mathbf{L}}}$ corresponding to an arc. Then there is a path $k \to i \to \tilde{k}$ in $Q_{\overline{T},\overline{\mathbf{L}}}$ for some vertex k *if and only if* i flips to a one-sided closed curve and k is an arc.

Proof. If there is a path $k \to i \to \tilde{k}$ in $Q_{\overline{T},\overline{\mathbf{L}}}$, then, by sign coherence, k must be an arc and not a lamination Furthermore, by anti-symmetry there is also the path $i \leftarrow k \to \tilde{i}$. This implies the existence of the quadrilateral $(a, \tilde{i}, \tilde{b}, i)$ shown in Figure 8.4, where a and $\tilde{b}$ may not be arcs in T, but the associated arc bounding an arc and its notched counterpart. We see that $a, \tilde{b} \notin \{i, \tilde{i}\}$ as this would then imply T

contains either a punctured monogon or M_1, both of which are forbidden. Applying anti-symmetry again we find the existence of the quadrilateral $(\tilde{a}, i, b, \tilde{i})$. Glueing these two quadrilaterals together and taking the $\mathbb{Z}_2$-quotient yields the picture in Figure 8.5, confirming that i flips to a one-sided closed curve.

The proof of the other direction is trivial.

□

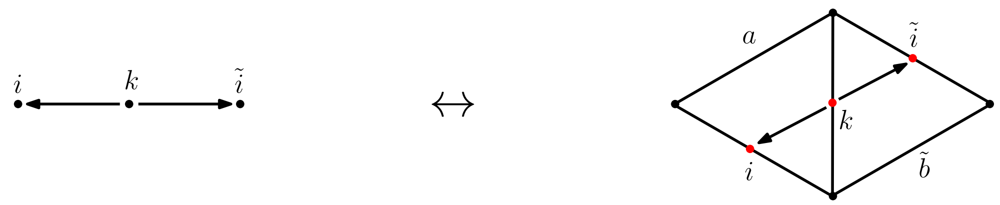


Figure 8.4: The local configuration of the surface if $i \leftarrow k \rightarrow \tilde{i}$ is a path in $Q_{\overline{T},\overline{\mathbf{L}}}$.

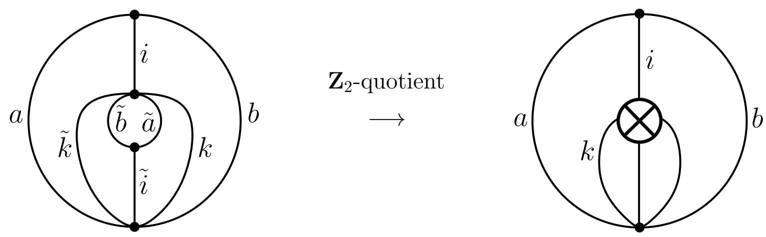


Figure 8.5: The quasi-triangulation induced by the path $k \rightarrow i \rightarrow \tilde{k}$ in $Q_{\overline{T},\overline{\mathbf{L}}}$.

Lemma 8.34. Let α^* be an arc bounding a Möbius strip with one marked point, M_1, and β the unique arc in M_1. Consider the flip of β to the one-sided closed curve α. Then $x_{\overline{L}}(\alpha)x_{\overline{L}}(\beta) = x_{\overline{L}}(\alpha^*)$.

Proof. There are two elementary laminations of M_1 - these are shown in Figure 8.6.

For the lamination on the left in Figure 8.6: $c_{\overline{L}}(\beta) = c_{\overline{L}}(\alpha) = X^{\frac{1}{2}}$, $c_{\overline{L}}(\alpha^*) = X$.

For the lamination on the right in Figure 8.6: $c_{\overline{L}}(\beta) = c_{\overline{L}}(\alpha^*) = X$, $c_{\overline{L}}(\alpha) = 1$.

Recall that by Definition 6.15 the lambda lengths of α, β, α^* are related by $\lambda(\alpha)\lambda(\beta) = \lambda(\alpha^*)$. Therefore, employing equation (8.4.1), for any lamination $\overline{L}$, we obtain

$$x_{\overline{L}}(\alpha)x_{\overline{L}}(\beta) = \frac{x_{\overline{L}}(\alpha^*)c_{\overline{L}}(\alpha^*)}{c_{\overline{L}}(\alpha)c_{\overline{L}}(\beta)} = x_{\overline{L}}(\alpha^*).$$

□

Remark 19. Note that the truth of Lemma 8.34 crucially depends on our exclusion, in Definition 8.1, of closed curves that are one-sided, or bound a Möbius strip. If L is one of these forbidden curves contained in M_1, then $c_{\overline{L}}(\alpha)c_{\overline{L}}(\beta) = X \neq 1 = c_{\overline{L}}(\alpha*)$. Consequently, $x_{\overline{L}}(\alpha)x_{\overline{L}}(\beta) \neq x_{\overline{L}}(\alpha^*)$.

Figure 8.6: The two *elementary* laminations of M_1. (Meaning every lamination of M_1 will be some union of these laminations.)

__Notation:__ From here onwards, by abuse of notation, for each quasi-arc γ of a laminated bordered surface $(S, M, \overline{\mathbf{L}})$, we shall denote the laminated lambda length $x_{\overline{\mathbf{L}}}(\gamma)$ by γ itself.

Proposition 8.35. Let $Q_{\overline{T},\overline{\mathbf{L}}}$ be an anti-symmetric quiver of a triangulation T of $(S, M, \overline{\mathbf{L}})$. Then the polynomials $\overline{\mathbf{F}}$ from Definition 8.32 are the exchange relations describing how laminated lambda lengths change under flips of arcs in T.

Proof. Currently by Proposition 8.31 we know that the polynomial

$$F_j^{Q_{\overline{T},\overline{\mathbf{L}}}} = \prod_{\substack{b_{ij}>0 \\ i\in\{1,\dots,m,\tilde{1}\dots,\tilde{m}\}}} x_i^{b_{ij}} + \prod_{\substack{b_{ij}<0 \\ i\in\{1,\dots,m,\tilde{1}\dots,\tilde{m}\}}} x_i^{-b_{ij}}$$

describes how the laminated lambda length of an arc γ_j in T changes under flip when we allow flips to arcs bounding M_1 instead of one-sided closed curves. By Lemma 8.33, if γ_j does not flip to a one-sided closed curve then $F_j^{Q_{\overline{T},\overline{\mathbf{L}}}} = \overline{F}_j^{Q_{\overline{T},\overline{\mathbf{L}}}}$.

If γ_j does flip to a one-sided closed curve α, then by Lemma 8.33 we know $\bar{b}_{kj} := b_{kj} + b_{\tilde{k}j} = 0$ for some k (where $b_{kj} = -b_{\tilde{k}j} = \pm 1$), and $b_{ij}, b_{\tilde{i}j}$ are both simultaneously non-positive or non-negative for all $i \in [m] \setminus \{k\}$. Hence $\overline{F}_j^{Q_{\overline{T},\overline{\mathbf{L}}}} = \frac{F_j^{Q_{\overline{T},\overline{\mathbf{L}}}}}{\beta}$, where β is the arc corresponding to k. Proposition 8.31 tells us that when γ_j flips instead to the arc α^* enclosing M_1, then $\gamma_j \alpha^* = F_j^{Q_{\overline{T},\overline{\mathbf{L}}}}$. Moreover, by Lemma 8.34 we know that $\alpha\beta = \alpha^*$. This gives us the desired relation $\gamma_j \alpha = \frac{F_j^{Q_{\overline{T},\overline{\mathbf{L}}}}}{\beta} = \overline{F}_j^{Q_{\overline{T},\overline{\mathbf{L}}}}$.

□

Proposition 8.35 tells us that for a triangulation T of a laminated bordered surface $(S, M, \overline{\mathbf{L}})$, for any arc $\gamma \in T$, the exchange polynomial $\overline{F}_\gamma$ is obtained from considering (sums of) the ingoing and outgoing arrows of the vertex γ (or equivalently $\tilde{\gamma}$) in $Q_{\overline{T},\overline{\mathbf{L}}}$. However, currently we have no such combinatorial method that provides us with the exchange polynomials of quasi-arcs in quasi-triangulations containing one-sided closed curves. The following lemma addresses this.

Before we state the lemma let us fix some notation. Recall that a one-sided closed curve α in a quasi-triangulation T will intersect precisely one arc $\beta \in T$. As it always will throughout this chapter, α^* denotes the unique arc enclosing α and β in M_1. For each quasi-triangulation T we can therefore uniquely associate a traditional triangulation T^* by replacing each one-sided closed curve $\alpha \in T$ with α^*.

For the rest of this chapter, by an abusive of notation, for each quasi-arc γ we shall also denote its laminated lambda length by γ – previously written as $x_{\overline{L}}(\gamma)$. Similarly, for a lamination L_i of a multi-lamination $\mathbf{L}$, we also denote its corresponding variable by L_i – previously written as q_i.

Lemma 8.36. Let T be a quasi-triangulation containing a one-sided closed curve α, and let β denote the unique arc in T intersecting α. Consider the associated traditional triangulation T^* and its corresponding quiver $Q_{\overline{T}^*}$, shown in Figure

8.7. Furthermore, denote by b_{ij} and b'_{ij} the coefficients of $Q_{\overline{T}^*}$ and $\mu_{\alpha^*} \circ \mu_{\tilde{\alpha}^*}(Q_{\overline{T}^*})$, respectively. Then the exchange polynomials of the quasi arcs α and β in T are given by:

$$\alpha\alpha' = \Big(\prod_{\bar{b}_{L_i\alpha^*}>0} L_i^{\bar{b}_{L_i\alpha^*}}\Big)y + \Big(\prod_{\bar{b}_{L_i\alpha^*}<0} L_i^{-\bar{b}_{L_i\alpha^*}}\Big)x$$

$$\beta\beta' = \frac{\Big(\prod_{\bar{b}'_{L_i\beta}>0} L_i^{\bar{b}'_{L_i\beta}}\Big)\bigg(\Big(\prod_{\bar{b}_{L_i\alpha^*}>0} L_i^{\bar{b}_{L_i\alpha^*}}\Big)y + \Big(\prod_{\bar{b}_{L_i\alpha^*}<0} L_i^{-\bar{b}_{L_i\alpha^*}}\Big)x\bigg)^2 + \Big(\prod_{\bar{b}'_{L_i\beta}<0} L_i^{-\bar{b}'_{L_i\beta}}\Big)xy\alpha^2}{\alpha^2}.$$

Proof. The exchange relation of α follows from Proposition 8.9 and Proposition 8.35. To obtain the exchange polynomial of β we (first) need to consider $\mu_{\alpha^*} \circ \mu_{\tilde{\alpha}^*}(Q_{\overline{T}^*})$ instead of $Q_{\overline{T}^*}$ – this is because β flips to β' in $\mu_{\alpha^*}(T^*)$, but not in T^*. By Proposition 8.35 we get that:

$$\beta\beta' = \Big(\prod_{\bar{b}'_{L_i\beta}>0} L_i^{\bar{b}'_{L_i\beta}}\Big)\alpha'^2 + \Big(\prod_{\bar{b}'_{L_i\beta}<0} L_i^{-\bar{b}'_{L_i\beta}}\Big)xy.$$

Rewriting α' using the exchange relation already obtained for α yields

$$\beta\beta' = \frac{\Big(\prod_{\bar{b}'_{L_i\beta}>0} L_i^{\bar{b}'_{L_i\beta}}\Big)\bigg(\Big(\prod_{\bar{b}_{L_i\alpha^*}>0} L_i^{\bar{b}_{L_i\alpha^*}}\Big)y + \Big(\prod_{\bar{b}_{L_i\alpha^*}<0} L_i^{-\bar{b}_{L_i\alpha^*}}\Big)x\bigg)^2 + \Big(\prod_{\bar{b}'_{L_i\beta}<0} L_i^{-\bar{b}'_{L_i\beta}}\Big)xy\alpha^2}{\alpha^2}.$$

□

Remark 20. More generally, for any quasi-arc γ in a quasi-triangulation T, the exchange polynomial for γ is still obtained by the formulae of Propositions 8.35 and 8.36 – we just have to remember that if a variable α^* appears in the exchange relation we must replace it with $\alpha\beta$.

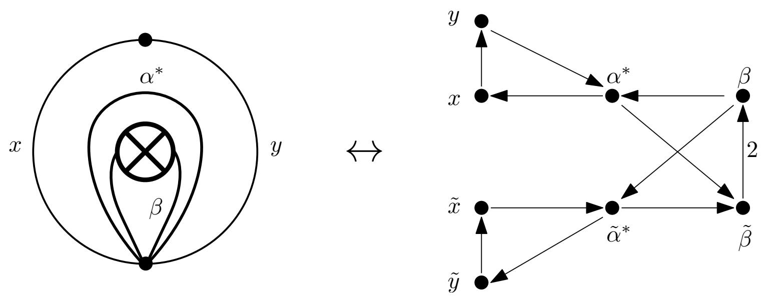


Figure 8.7: The traditional triangulation T^* together with its associated quiver $Q_{\overline{T}^*}$.

8.5.2 Laminated quasi-cluster algebras via LP mutation

Let γ be a quasi-arc in a quasi-triangulation T. We have seen in Propositions 8.35 and 8.36 that the exchange relation of γ is a Laurent polynomial; we denote by F_γ the numerator of this polynomial. To each quasi-triangulation T we assign the 'LP' seed $(\mathbf{x}, \mathbf{F}_T)$ where $\mathbf{x} := \{x_{\overline{L}}(\gamma) | \gamma \in T\}$ and $\mathbf{F}_T := \{F_\gamma | \gamma \in T\}$. Of course, due to the irreducibility conditions, $(\mathbf{x}, \mathbf{F}_T)$ may not be a valid LP seed - this will be addressed later.

The following lemma assures us that, if the polynomials in $\mathbf{F}_T$ are distinct, then the normalisations of these polynomials are the exchange relations of their corresponding quasi-arcs.

Lemma 8.37. Let T be a quasi-triangulation and suppose $F_{\gamma_i} \neq F_{\gamma_j}$ for any quasi-arcs γ_i and γ_j in T $(i \neq j)$. If $\gamma \in T$ intersects a one-sided closed curve $\alpha \in T$ then $\hat{F}_\gamma = \frac{F_\gamma}{\alpha^2}$, otherwise $\hat{F}_\gamma = F_\gamma$.

Proof. Let $\gamma_1, \ldots, \gamma_n$ be the quasi-arcs in T. Recall that $\hat{F}_{\gamma_j} := \frac{F_{\gamma_j}}{\gamma_1^{a_1} \ldots \gamma_{j-1}^{a_{j-1}} \gamma_{j+1}^{a_{j+1}} \ldots \gamma_n^{a_n}}$ where $a_k \in \mathbb{Z}_{\geq 0}$ is maximal such that $F_{\gamma_k}^{a_k}$ divides $F_{\gamma_j}|_{\gamma_k \leftarrow \frac{F_{\gamma_k}}{x}}$.

Hence, $a_k > 0$ *if and only if* F_{γ_k} divides the constant term of F_{γ_j} when viewed as a polynomial in γ_k.

If γ_j does not intersect a one-sided closed curve in T then F_{γ_j} is a binomial. As a consequence, when viewed as a polynomial in γ_k, the constant term is either a monomial ($\gamma_k \in F_{\gamma_j}$), or the whole binomial F_{γ_j} ($\gamma_k \notin F_{\gamma_j}$). If it is a monomial then it is not divisible by F_{γ_k}. From our assumptions in the lemma we know that F_{γ_j} is irreducible and $F_{\gamma_j} \neq F_{\gamma_k}$. So if the constant term is F_{γ_j}, this also cannot be divisible by F_{γ_k}. Hence $\hat{F}_{\gamma_j} = F_{\gamma_j}$.

If γ_j intersects a one-sided closed curve $\alpha \in T$, then γ_j has the flip region shown in Figure 8.8. Moreover, by Lemma 8.36, it has the exchange polynomial

$$F_{\gamma_j} = \Big(\prod_{\bar{b}_{i\gamma_j}>0} L_i^{\bar{b}_{i\gamma_j}}\Big)F_\alpha^2 + \Big(\prod_{\bar{b}_{i\gamma_j}<0} L_i^{-\bar{b}_{i\gamma_j}}\Big)xy\alpha^2$$

where

$$F_\alpha = \Big(\prod_{\bar{b}_{i\alpha}>0} L_i^{\bar{b}_{i\alpha}}\Big)y + \Big(\prod_{\bar{b}_{i\alpha}<0} L_i^{-\bar{b}_{i\alpha}}\Big)x$$

.

Accordingly, for any quasi-arc $\gamma_k \in T\setminus\{\alpha\}$, the constant term of F_{γ_j}, when viewed as a polynomial in γ_k, is a monomial or F_{γ_j}. Just as before, this implies $\gamma_k \notin \frac{\hat{F}_{\gamma_j}}{F_{\gamma_j}}$. However, when F_{γ_j} is viewed as a polynomial in α the constant term is $\Big(\prod_{\bar{b}_{i\gamma_j}>0} L_i^{\bar{b}_{i\gamma_j}}\Big)F_\alpha^2$, and the degree 1 term is 0. Thus, $\hat{F}_{\gamma_j} = \frac{F_{\gamma_j}}{\alpha^2}$.

□

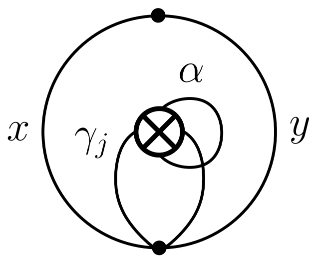


Figure 8.8: The flip region of γ_j if it intersects a one-sided closed curve.

Lemma 8.38. Let T be the traditional triangulation of M_2 obtained from glueing together a triangle and an anti self-folded triangle. If we label the lifted arcs as in

Figure 8.9, then for any lamination L of M_2, in the quiver $Q_{\overline{T},\overline{L}}$ we have $\overline{b}_{L\beta} \geq 0$ and either:

$$b_{L\alpha^*} \geq b_{\tilde{L}\beta} \text{ and } b_{\tilde{L}\alpha^*} \geq b_{L\beta} \quad \textit{or} \quad b_{L\alpha^*} \leq b_{\tilde{L}\beta} \text{ and } b_{\tilde{L}\alpha^*} \leq b_{L\beta}.$$

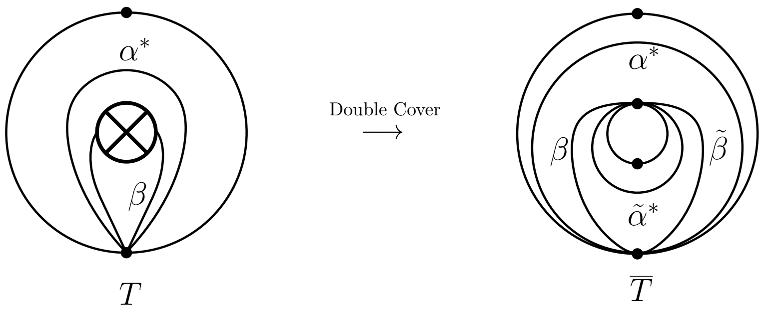


Figure 8.9: The traditional triangulation T obtained from glueing a triangle with an anti-self triangle, and its lift $\overline{T}$.

Proof. Let us consider the lifted triangulation $\overline{T}$ of T, and suppose we have labelled the arcs as shown in Figure 8.9. We shall first determine when L adds weight to β or α^*.

Recall that for a lamination L to add positive (resp. negative) weight to β it needs to cut the quadrilateral in $\overline{T}$ containing β in an $'S'$ (resp. $'Z'$) shape. In Figure 8.10, for each shape type, we show the local configuration of the lamination within the quadrilateral containing β, and we denote its accompanying twin lamination with a dotted line.

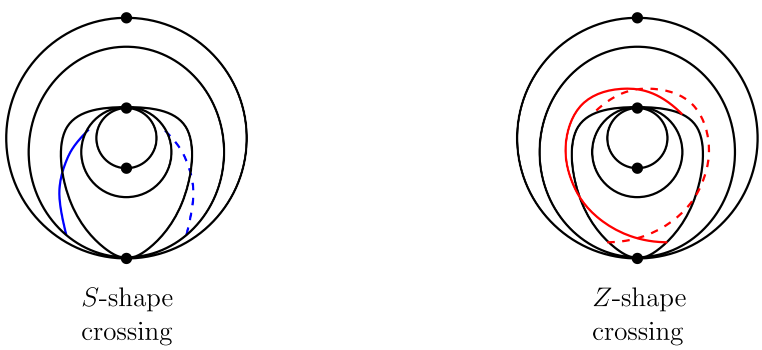


Figure 8.10: The instances where the lamination cuts β in an 'S' or 'Z' shape.

For the case when L cuts β in an $'S'$ shape the partial lamination shown in Figure 8.10 can be extended (without self-intersections) in three ways. The $\mathbb{Z}_2$-quotients of these extensions are shown in Figure 8.11. However, note that when L cuts β in a $'Z'$ shape the partial 'lamination' shown in Figure 8.10 is self intersecting, and therefore will not form a legitimate lamination. As a consequence, for any lamination L, when we label the arcs as in Figure 8.9, then we can only ever have $\overline{b}_{L\beta} \geq 0$. (If we labelled β and $\tilde{\beta}$ the other way around we would only ever have $\overline{b}_{L\beta} \leq 0$.)

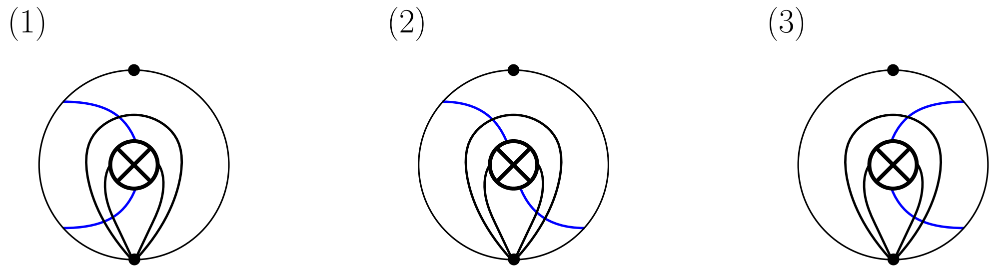


Figure 8.11: The possible (elementary) laminations adding weight to β.

Now suppose L adds weight to α^*. Locally within the quadrilateral containing α^*, depending on which shape L cuts α^*, L will have one of the configurations shown in Figure 8.12.

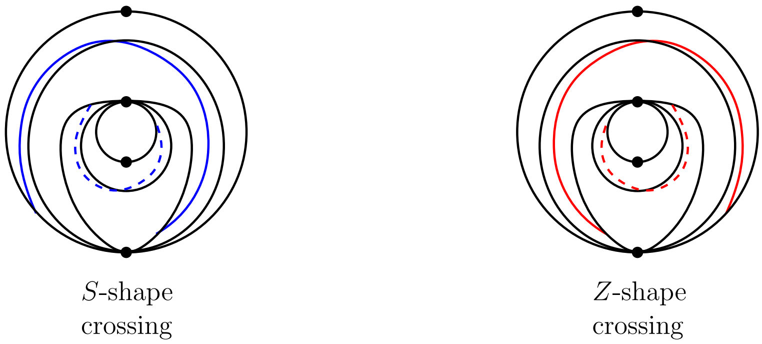


Figure 8.12: The instances where the lamination cuts α^* in an 'S' or 'Z' shape.

We can see that each configuration can be extended to a (non-intersecting) lamination in precisely two ways. Taking the $\mathbb{Z}_2$-quotient leaves us with the laminations shown in Figure 8.13.

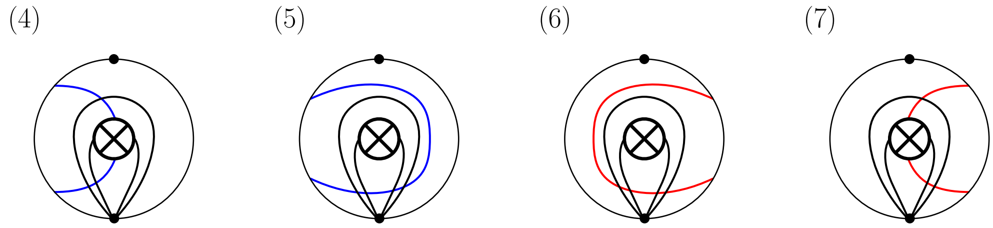


Figure 8.13: The possible (elementary) laminations adding weight to α^*.

If $b_{L\alpha^*} > b_{\tilde{L}\beta} = -b_{L\tilde{\beta}}$ then L must contain the curve (5).

If $-b_{L\tilde{\alpha}^*} = b_{\tilde{L}\alpha^*} < b_{L\beta}$ then L must contain (at least) one of the curves (2), (3) = (7) or (6).

Since (5) intersects each of (2), (3) = (7) and (6), these inequalities cannot be simultaneously satisfied. Consequently, $b_{L\alpha^*} > b_{\tilde{L}\beta}$ implies $b_{\tilde{L}\alpha^*} \geq b_{L\beta}$. An analogous argument shows that $b_{L\alpha^*} < b_{\tilde{L}\beta}$ implies $b_{\tilde{L}\alpha^*} \leq b_{L\beta}$.

□

Remark 21. Note that if $\overline{b}_{\gamma L} = 0$ for some arc $\gamma \in T$, then since the lamination L is not self-intersecting, this implies $b_{\gamma L} = b_{\tilde{\gamma} L} = 0$.

Having realised how to obtain lamination coefficients of exchange polynomials of any quasi-triangulation T we will now describe how these change under flips. If T is a triangulation then these coefficients will change in accordance to usual quiver mutation formulae. We are therefore left with the task of describing how coefficients change when we perform flips in regions containing a one-sided closed curve. For a quiver $Q_{\overline{T}}$ arising from a traditional triangulation T containing anti self-folded triangles, Lemma 8.38 puts a restriction on the possible extended quivers, $Q_{\overline{T},\overline{\mathbf{L}}}$, that can arise from a multi lamination $\mathbf{L}$ on the surface. We shall use this lemma to sift out these 'obvious' impossible extended quivers. After this sifting, on the surviving extended quivers, we will describe in the following lemma how the quiver changes with respect to flips of arcs in T.

Lemma 8.39. Let $(S, M, \mathbf{L})$ be a laminated bordered surface. Consider the quasi-triangulation T and its associated lift $\overline{T}^*$, both shown in Figure 8.14. Let us denote the coefficients of $Q_{\overline{T}^*,\overline{\mathbf{L}}}$ by $\bar{b}_{ij}$. Then:

(a) the lamination coefficients, $\bar{b}'_{ij}$, corresponding to α^*, β and x in $Q_{\overline{\mu_{\alpha^*}(T)}^*,\overline{\mathbf{L}}}$ can be written as:

$$\bar{b}'_{L_i x} = \bar{b}_{L_i x} + \max(0, \bar{b}_{L_i \alpha^*}) \quad , \quad \bar{b}'_{L_i \alpha^*} = -\bar{b}_{L_i \alpha^*} \quad , \quad \bar{b}'_{L_i \beta} = \bar{b}_{L_i \beta} - |\bar{b}_{L_i \alpha^*}|$$

(b) The lamination coefficients, $\bar{b}'''_{ij}$, of α^*, β and x in $Q_{\overline{\mu_{\beta}(T)}^*,\overline{\mathbf{L}}}$ can be written as:

$$\bar{b}'''_{L_i \alpha^*} = \bar{b}_{L_i \alpha^*} \quad , \quad \bar{b}'''_{L_i \beta} = -\bar{b}_{L_i \beta},$$

$$\bar{b}'''_{L_i x} = \bar{b}_{L_i x} + \max(0, \bar{b}_{L_i \alpha^*} + \bar{b}_{L_i \beta}) + \max(0, \bar{b}_{L_i \alpha^*} - \bar{b}_{L_i \beta})$$

Proof. To validate these formulae we must:

- perform mutation, μ_{α^*}, at α^* and $\tilde{\alpha}^*$ in $Q_{\overline{T}^*,\overline{\mathbf{L}}}$ to obtain $Q_{\overline{\mu_{\alpha}(T)}^*,\overline{\mathbf{L}}}$
- perform the sequence of mutations $\mu_{\alpha^*} \circ \mu_{\beta} \circ \mu_{\alpha^*}$ on $Q_{\overline{T}^*,\overline{\mathbf{L}}}$ to obtain $Q_{\overline{\mu_{\beta}(T)}^*,\overline{\mathbf{L}}}$.

By Lemma 8.38, we can divide our task into four natural cases:

1. $b_{L\alpha}, b_{L\tilde{\alpha}} \geq 0, \quad b_{L\alpha^*} \geq b_{\tilde{L}\beta}, \quad b_{\tilde{L}\alpha^*} \geq b_{L\beta}$
2. $b_{L\alpha}, b_{L\tilde{\alpha}} \geq 0, \quad b_{L\alpha^*} \leq b_{\tilde{L}\beta}, \quad b_{\tilde{L}\alpha^*} \leq b_{L\beta}$
3. $b_{L\alpha}, b_{L\tilde{\alpha}} \leq 0, \quad b_{L\alpha^*} \geq b_{\tilde{L}\beta}, \quad b_{\tilde{L}\alpha^*} \geq b_{L\beta}$
4. $b_{L\alpha}, b_{L\tilde{\alpha}} \leq 0, \quad b_{L\alpha^*} \leq b_{\tilde{L}\beta}, \quad b_{\tilde{L}\alpha^*} \leq b_{L\beta}$

Using the matrix mutation exchange relation, $b'_{kj} = \text{sgn}(b_{ij})[b_{ki}b_{ij}]_+$, it is easily verified that, in each of the four cases, the resulting coefficients agree with the claimed formulae.

□

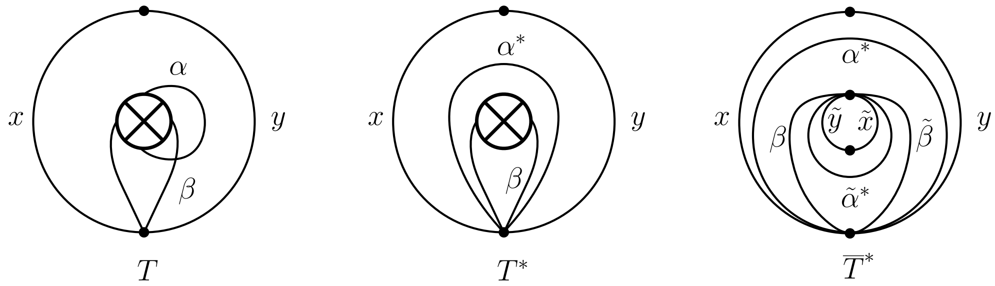


Figure 8.14: A quasi-triangulation T together with its associated traditional triangulation T^* and lift $\overline{T}^*$.

Having discovered how exchange polynomials of any quasi-triangulation change via the consideration of quivers, we are now ready to show that (under certain conditions) LP mutation is also describing how these polynomials change under flips.

Theorem 8.40. Suppose $(S, M, \overline{\mathbf{L}})$ is a laminated bordered surface such that for each quasi-triangulation T, $(\mathbf{x}, \mathbf{F}_T)$ is a valid seed, and $F_{\gamma_i} \neq F_{\gamma_j}$ for any quasi-arcs γ_i and γ_j in T $(i \neq j)$. Then LP mutation amongst seeds corresponds to flipping quasi-arcs. Specifically, for any seed, $(\mathbf{x}, \mathbf{F}_T)$, and quasi-arc $\gamma \in T$ we have that $\mu_\gamma(\mathbf{x}, \mathbf{F}_T) = (\mu_\gamma(\mathbf{x}), \mathbf{F}_{\mu_\gamma(T)})$. Here $\mu_\gamma(\mathbf{x}) := \mathbf{x} \setminus \{x_{\overline{\mathbf{L}}}(\gamma)\} \cup \{x_{\overline{\mathbf{L}}}(\gamma')\}$ where γ' is the flip of γ with respect to T.

Proof. In Proposition 6.10 we classified the type of flip regions of quasi-triangulations T - these are shown in Figure 6.4. It is crucial to note that the sides of these flip regions may not be arcs (or boundary segments) in T, but rather an arc bounding M_1 or a punctured digon. In which case this arc is representing the two quasi-arcs it bounds. Propositions 6.21, 8.35 and Lemma 8.37 tells us that LP mutation describes how the exchange polynomials of arcs change when flipping amongst triangulations. It remains to check this is the case when mutating to, and amongst, quasi-triangulations containing one-sided closed curves. For now we shall only consider flip regions whose sides are arcs in T. Note that when we perform a flip only the exchange polynomials of the interior and boundary quasi-arcs of the flip region can change. Therefore, for each flip, we just need to show that LP mutation describes how these polynomials change (as LP mutation will also leave all other exchange polynomials unchanged).

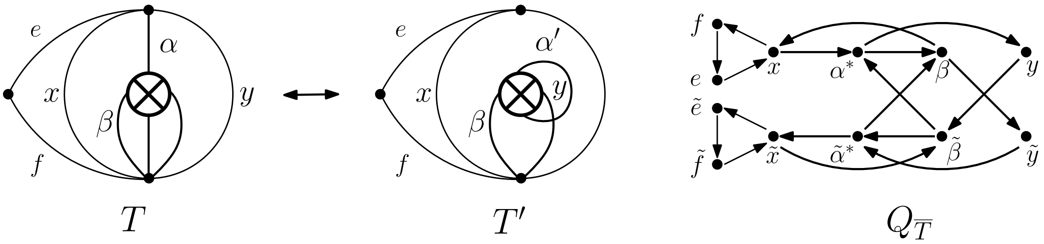


Figure 8.15: A triangulation T, in which α flips to a one-sided closed curve α', and the corresponding quiver $Q_{\overline{T}}$.

Case 1: The flip of an arc, α, to a one-sided closed curve.

Suppose an arc α in a triangulation of (S, M) flips to a one-sided closed curve α'. Let x and y be the boundaries of this region, and β the other interior arc. Without loss of generality it suffices to show that LP mutation describes how F_x and F_β change under this flip. Furthermore we may assume that x is not a boundary segment of (S, M), as otherwise it has no exchange polynomial, and there would be nothing to check. We will therefore have the local picture shown in Figure 8.15, moreover, for our chosen labelling of $Q_{\overline{T}}$, by Proposition 8.35, we will get:

$$F_\alpha = \Big(\prod_{\bar{b}_{L_i\alpha}>0} L_i^{\bar{b}_{L_i\alpha}}\Big)x + \Big(\prod_{\bar{b}_{L_i\alpha}<0} L_i^{-\bar{b}_{L_i\alpha}}\Big)y$$

$$F_\beta = \Big(\prod_{\bar{b}_{L_i\beta}>0} L_i^{\bar{b}_{L_i\beta}}\Big)\alpha^2 + \Big(\prod_{\bar{b}_{L_i\beta}<0} L_i^{-\bar{b}_{L_i\beta}}\Big)xy$$

$$F_x = \Big(\prod_{\bar{b}_{L_ix}>0} L_i^{\bar{b}_{L_ix}}\Big)\beta e + \Big(\prod_{\bar{b}_{L_ix}<0} L_i^{-\bar{b}_{L_ix}}\Big)\alpha f.$$

Let us consider the quasi-triangulation $T' := \mu_\alpha(T)$ and denote the coefficients in $Q_{\overline{T'}}$ by $\overline{b}'_{ij}$. By Propositions 8.35 and 8.36, we are required to show that LP mutation changes F_β and F_x to the following polynomials:

$$F'_\beta = \Big(\prod_{\bar{b}_{L_i\beta}>0} L_i^{\bar{b}_{L_i\beta}}\Big)F_\alpha^2 + \Big(\prod_{\bar{b}_{L_i\beta}<0} L_i^{-\bar{b}_{L_i\beta}}\Big)xy\alpha'^2$$

$$F'_x = \Big(\prod_{\bar{b}'_{L_ix}>0} L_i^{\bar{b}'_{L_ix}}\Big)\beta e\alpha' + \Big(\prod_{\bar{b}'_{L_ix}<0} L_i^{-\bar{b}'_{L_ix}}\Big)yf.$$

Since $F_\alpha|_{\beta\leftarrow 0} = F_\alpha$ then,

$$G_\beta = F_\beta|_{\alpha\leftarrow\frac{F_\alpha}{\alpha'}} = \frac{\Big(\prod_{\bar{b}_{L_i\beta}>0} L_i^{\bar{b}_{L_i\beta}}\Big)F_\alpha^2 + \Big(\prod_{\bar{b}_{L_i\beta}<0} L_i^{-\bar{b}_{L_i\beta}}\Big)xy\alpha'^2}{\alpha'^2}$$

Hence $MG_\beta = F'_\beta$, as required.
Since $\hat{F}_\alpha|_{x\leftarrow 0} = \Big(\prod_{\bar{b}_{L_i\alpha}<0} L_i^{-\bar{b}_{L_i\alpha}}\Big)y$ we obtain:

$$G_x = F_x\big|_{\alpha\leftarrow\frac{\hat{F}_\alpha|_{x\leftarrow 0}}{\alpha'}} = \frac{\Big(\prod_{\bar{b}_{L_ix}>0} L_i^{\bar{b}_{L_ix}}\Big)\beta e\alpha' + \Big(\prod_{\bar{b}_{L_ix}<0} L_i^{-\bar{b}_{L_ix}}\Big)\Big(\prod_{\bar{b}_{L_i\alpha}<0} L_i^{-\bar{b}_{L_i\alpha}}\Big)yf}{\alpha'}$$

From here we see that the exponent of L_i in MG_x will be $|\bar{b}_{L_ix} - \max(0, -\bar{b}_{L_i\alpha})|$. Moreover, L_i will appear in the left or right monomial of MG_x respective of whether $\bar{b}_{L_ix} - \max(0, -\bar{b}_{L_i\alpha})$ is positive or negative. Lemma 8.39 tells us that $\bar{b}_{L_ix} = \bar{b}'_{L_ix} + \max(0, \bar{b}_{L_i\alpha})$ and $\bar{b}_{L_i\alpha} = -\bar{b}'_{L_i\alpha}$. So $\bar{b}'_{L_i\beta} = \bar{b}_{L_ix} - \max(0, -\bar{b}_{L_i\alpha})$ and LP mutation indeed describes how the exchange polynomials change for this flip.

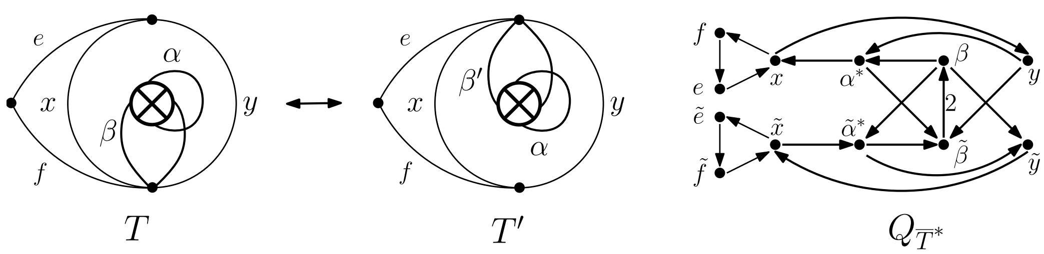


Figure 8.16: A triangulation T, in which β intersects a one-sided closed curve α, and the corresponding quiver $Q_{\overline{T}^*}$.

Case 2: The flip of an arc, β, intersecting a one-sided closed curve.

Suppose an arc β in a quasi-triangulation of (S, M) intersects a one-sided closed curve α. Let x and y be the boundary segments of the flip region, and denote by β' the arc β flips to. As before, it is enough to show that LP mutation describes how F_α and F_x change under this flip, and we may assume x is not a boundary segment of (S, M). We therefore arrive at the sub quasi-triangulation, T, and sub quiver, $Q_{\overline{T}^*}$, shown in Figure 8.16. For our chosen labelling of $Q_{\overline{T}^*}$, by Propositions 8.35 and 8.36, we obtain:

$$F_\alpha = \Big(\prod_{\overline{b}_{L_i\alpha^*}>0} L_i^{\overline{b}_{L_i\alpha^*}} \Big) y + \Big(\prod_{\overline{b}_{L_i\alpha^*}<0} L_i^{-\overline{b}_{L_i\alpha^*}} \Big) x$$

$$F_\beta = \Big(\prod_{\overline{b}'_{L_i\beta}>0} L_i^{\overline{b}'_{L_i\beta}} \Big) F_\alpha^2 + \Big(\prod_{\overline{b}'_{L_i\beta}<0} L_i^{-\overline{b}'_{L_i\beta}} \Big) xy\alpha^2.$$

$$F_x = \Big(\prod_{\overline{b}_{L_ix}>0} L_i^{\overline{b}_{L_ix}} \Big) \alpha\beta e + \Big(\prod_{\overline{b}_{L_ix}<0} L_i^{-\overline{b}_{L_ix}} \Big) fy$$

Note that here we represent the coefficients of $Q_{\overline{\mu_\alpha(T)}} = \mu_{\alpha^*} \circ \mu_{\tilde{\alpha}^*}(Q_{\overline{T}^*})$ by $\overline{b}'_{L_i\beta}$. Furthermore, for $T' := \mu_\beta(T)$, if we denote the coefficients of $Q_{\overline{T'}^*}$ by $\overline{b}'''_{ij}$ then, by Propositions 8.35 and 8.36, we are required to show that LP mutation changes F_α and F_x to the following polynomials:

$$F'_\alpha = \Big(\prod_{\overline{b}'''_{L_i\alpha^*}>0} L_i^{\overline{b}'''_{L_i\alpha^*}} \Big) y + \Big(\prod_{\overline{b}'''_{L_i\alpha^*}<0} L_i^{-\overline{b}'''_{L_i\alpha^*}} \Big) x$$

$$F'_x = \Big(\prod_{\overline{b}'''_{L_ix}>0} L_i^{\overline{b}'''_{L_ix}} \Big) ey + \Big(\prod_{\overline{b}'''_{L_ix}<0} L_i^{-\overline{b}'''_{L_ix}} \Big) f\alpha\beta'$$

Since $\beta \notin F_\alpha$ then we need $F_\alpha = F'_\alpha$. This is the case since Lemma 8.39 tells us that $\overline{b}'''_{L_i\alpha^*} = \overline{b}_{L_i\alpha^*}$. It remains to check how F_β changes under LP mutation.

$$\hat{F}_\beta|_{x \leftarrow 0} = \frac{\Big(\prod_{\overline{b}'_{L_i\beta}>0} L_i^{\overline{b}'_{L_i\beta}} \Big) \Big(\prod_{\overline{b}_{L_i\alpha^*}>0} L_i^{2\overline{b}_{L_i\alpha^*}} \Big) y^2}{\alpha^2}$$

$$= \frac{\Big(\prod_{\max(0,\bar{b}_{L_i\alpha^*}+\bar{b}_{L_i\beta})+\max(0,\bar{b}_{L_i\alpha^*}-\bar{b}_{L_i\beta})>0} L_i^{\max(0,\bar{b}_{L_i\alpha^*}+\bar{b}_{L_i\beta})+\max(0,\bar{b}_{L_i\alpha^*}-\bar{b}_{L_i\beta})}\Big)y^2}{\alpha^2}$$

The last equality follows from Lemma 8.39 which tells us that $\bar{b}'_{L_i\beta} = \bar{b}_{L_i\beta} - |\bar{b}_{L_i\alpha^*}|$, and the inequalities of Lemma 8.38. For convenience, let us define $K_i := \max(0, \bar{b}_{L_i\alpha^*} + \bar{b}_{L_i\beta}) + \max(0, \bar{b}_{L_i\alpha^*} - \bar{b}_{L_i\beta})$. As a consequence we obtain:

$$G_x = F_x|_{\beta \leftarrow \frac{\hat{F}_\beta|_{x\leftarrow 0}}{\beta'}} = \frac{\Big(\prod_{\bar{b}_{L_ix}>0} L_i^{\bar{b}_{L_ix}}\Big)\Big(\prod_{K_i>0} L_i^{K_i}\Big)ey + \Big(\prod_{\bar{b}_{L_ix}<0} L_i^{-\bar{b}_{L_ix}}\Big)f\alpha\beta'}{\frac{\alpha\beta'}{y}}$$

From here we see L_i will have exponent $|\bar{b}_{L_ix} + K_i|$ in MG_x. Moreover, L_i will appear in the left or right monomial of MG_x respective of whether $\bar{b}_{L_ix} + K_i$ is positive or negative. From Lemma 8.39 we saw $\bar{b}'''_{L_i\beta} = \bar{b}_{L_ix} + K_i$, so LP mutation does indeed describe how the exchange polynomials change for this flip.

For the cases when the boundaries of flip regions are not all arcs, analogous calculations show that LP mutation still describes how the exchange polynomials change. □

8.5.3 Principal laminations

Theorem 8.40 asserts that for a laminated quasi-cluster algebra $\mathcal{A}(S, M, \mathbf{L})$, if the exchange polynomials in each seed are irreducible and distinct then flips coincide with LP mutations. Therefore, to establish an LP structure on a bordered surface (S, M) we must concoct a multi-lamination which guarantees irreducibility and uniqueness of the exchange polynomials in any quasi-triangulation. This multi-lamination will follow the flavour of principal coefficients, but to introduce it we will first need some preliminaries.

Definition 8.41. An arc γ of (S, M) is called ***orientable*** if it has an orientable neighbourhood. Otherwise γ is said to be ***non-orientable***.

Remark 22. Note that an arc γ will be non-orientable *if and only if* it has a unique endpoint and crosses through an odd number of cross-caps.

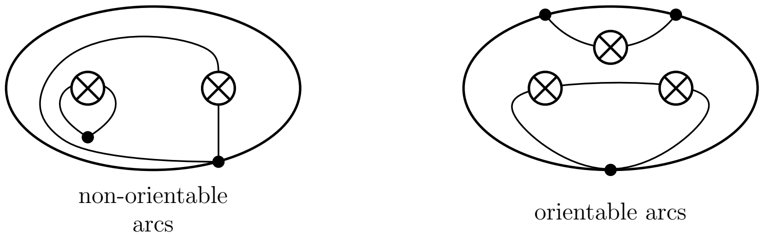


Figure 8.17: Examples of orientable and non-orientable arcs.

Definition 8.42. Define the parity, $p(\gamma)$, of an arc γ of (S, M) to be $+1$ or -1 respective of whether γ passes through an even or odd number of cross-caps.

Lemma 8.43. Let $\Delta = (\alpha, \beta, \gamma)$ be a triangle in (S, M). Then $p(\alpha)p(\beta)p(\gamma) = 1$.

Proof. Consider the slightly smaller triangle $\Delta' = (\alpha', \beta', \gamma')$ lying in the interior of Δ. The parity of the arcs in Δ' remain the same since they are only slight perturbations of their original versions. Moreover, although two sides of Δ may be glued together in (S, M), all arcs in Δ' will be distinct. As a consequence, the neighbourhood of Δ' is orientable, implying that $p(\alpha)p(\beta)p(\gamma) = p(\alpha')p(\beta')p(\gamma') = 1$

□

Lemma 8.44. Let T be a triangulation of (S, M) and γ be a non-orientable arc in T with unique endpoint $m \in M$. Then there exists an orientable arc $\beta \in T$ with (at least one) endpoint m.

Proof. The non-orientable arc γ belongs to a triangle Δ in T° (see Figure 6.2 regarding definition of T°). Lemma 8.43 ensures there exists an orientable arc $\beta \in \Delta$. If $\beta \in T$ then we are done, so consider the other possibility of β enclosing two arcs $\beta_1, \beta_2 \in T$ which only differ by a tagging at one puncture. β_1 and β_2 have distinct endpoints which ensures they are orientable, and this concludes the proof.

□

Definition 8.45 (Principal lamination)**.** Let T be a triangulation of (S, M). We define a ***principal (multi) lamination***, $\mathbf{L}_T := \{L_\gamma | \gamma \in T\}$, as follows:

- If γ is an orientable plain arc then L_γ is taken to be the lamination that runs along γ in a small neighbourhood thereof, which consistently spirals around the endpoints of γ both clockwise (or anti-clockwise). For endpoints of γ which are not punctures, the spiralling of L_γ ends when it reaches the boundary.

- If γ is an orientable arc with some notched endpoints, L_γ is defined as above, except now, at notched endpoints the direction of spinning is reversed.

- If γ is a non-orientable arc with (unique) endpoint m situated on the boundary, then consider two points on the boundary, m_1 and m_2, that lie either side of m in a small neighbourhood thereof. L_γ is the lamination with endpoints m_1 and m_2, which runs along a small neighbourhood of γ - note that L_γ will intersect γ once.

- If γ is a non-orientable arc situated at a puncture, p, then, by Lemma 8.44, γ has an incident orientable arc $\beta \in T$. Let L_γ be a lamination which:

  - spirals out of the puncture p, then
  - runs parallel to γ after intersecting γ and then β, (After L_γ intersects γ it is allowed to intersect both endpoints of β before running parallel to γ, it is just not allowed to intersect one endpoint of γ and then run parallel to the other endpoint of γ, without intersecting β inbetween.) then,
  - intersects γ and continues to run parallel to it, then
  - when it arrives back to a neighbourhood of p, it should run against the orientation of spiralling at p until it reaches an endpoint of β, then
  - runs along a small neighbourhood of β, and at the endpoint spirals depending on the type of tagging of β: if the endpoints of β receive the same tagging then the direction of spiralling should be consistent with

the spiralling of γ at p; if the endpoints of β receive different taggings then the direction of spiralling should oppose the spiralling of γ at p.

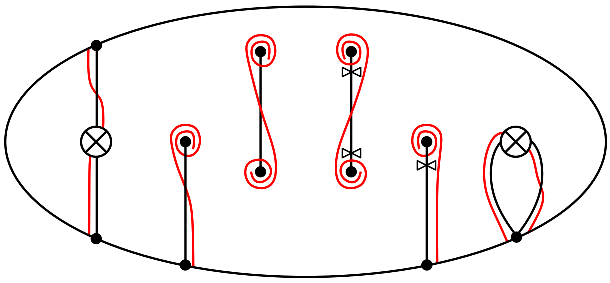


Figure 8.18: Examples of various types of laminations occurring in a principal lamination.

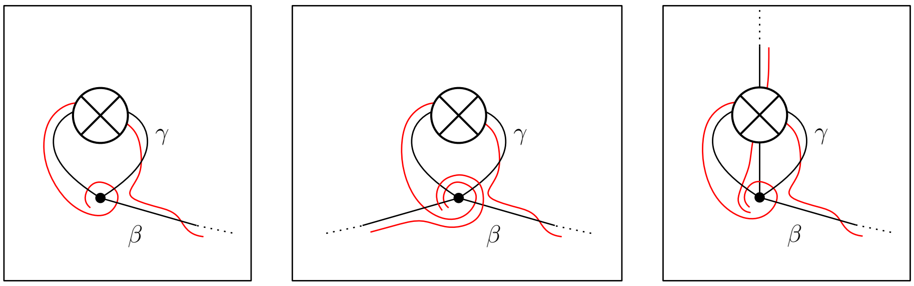


Figure 8.19: The types of laminations, L_γ, when γ is a non-orientable arc situated at a puncture, and β is the chosen incident orientable arc.

Remark 23. In Definition 8.45, β is required to be orientable so that if it has unique endpoint p, when L_γ runs parallel to it (and intersects it) and arrives back at p, it is able to spiral back around p. If β was non-orientable it wouldn't be able to spiral back around p without self-intersections. Likewise, the conditions that L_γ must

(a) intersect γ and then β before running parallel to L_γ;

(b) (when moving against the orientation of spiralling at p) run parallel to β as soon as it meets an endpoint β;

are required, otherwise self-intersections would occur if β also has unique endpoint p.

Remark 24. In general, for a given arc $\gamma \in T$ the choice of L_γ is not unique. We are just concerned about choosing some lamination L_γ that satisfies the rules demanded in Definition 8.45. The motivation behind the definition is to ensure the shortened exchange matrix associated to T and L_γ is of full rank.

Proposition 8.46. Let (S, M) be a bordered surface and T a triangulation. If $\mathbf{L}_T$ is a principal lamination of T then the shortened exchange matrix $\overline{B}$ is of full rank, and the gcd of each column is 1.

Proof. By the bigon criterion [7], since L_i does not form a bigon with any arc in T, then it is in minimal position (regarding intersections). Therefore, L_i will add weight ± 1 to γ_i, and to β_i if γ_i is non-orientable and situated at a puncture. Moreover, L_i will not add weight to any other arcs. Consequently, after rearranging columns of $\overline{B}$, the bottom $n \times n$ submatrix will be upper triangular with ± 1 entries on its diagonals. This confirms $\overline{B}$ has full rank, and that the gcd of each column is 1.

□

In Proposition 8.48 we will show that the rank of the shortened exchange matrix is preserved under mutation. For this we need the following technical Lemma 8.47.

Lemma 8.47. Let $i \in \{1, \dots, n\}$ be a vertex in an anti-symmetric quiver Q, and suppose there is no path $k \to i \to \tilde{k}$ for any vertex k in Q.

If $\bar{b}_{ij} \geq 0 \geq \bar{b}_{ji}$ or $\bar{b}_{ij} \leq 0 \leq \bar{b}_{ji}$ for every $j \in \{1, \dots, n\}$, then, for any $j, k \in \{1, \dots, n\} \setminus \{i\}$, mutation at i and $\tilde{i}$ in Q gives:

$$\bar{b}'_{jk} = \bar{b}_{jk} + \max(0, -\bar{b}_{ji})\bar{b}_{ik} + \max(0, \bar{b}_{ik})\bar{b}_{ji}. \tag{8.5.1}$$

Proof. Without loss of generality we may assume $\bar{b}_{ji} := b_{ji} + b_{\tilde{j}i} \geq 0$ (otherwise we could just reverse all the arrows, as this has no effect on the truth of the proposed equation 8.5.1). Since there are no paths $k \to i \to \tilde{k}$ for any k, then $b_{ji}, b_{\tilde{j}i} \geq 0$. Moreover, by using this path condition again, and anti-symmetry, we realise either

(a) $b_{ik} \leq 0$ and $b_{\tilde{i}k} \geq 0$, or

(b) $b_{ik} \geq 0$ and $b_{\tilde{i}k} \leq 0$.

The respective local configurations of the quiver for cases (a) and (b) are shown in Figure 8.20. In case (a) we see that mutating the quiver (at i and $\tilde{i}$) adds no new arrows between $j, k, \tilde{j}, \tilde{k}$, so

$$\overline{b}'_{jk} = \overline{b}_{jk} \tag{8.5.2}$$

.

In case (b) we see mutation produces

$$\overline{b}'_{jk} := b'_{jk} + b'_{\tilde{j}k} = (b_{jk} + b_{ji}b_{ik} - b_{j\tilde{i}}b_{\tilde{i}k}) + (b_{\tilde{j}k} + b_{\tilde{j}i}b_{ik} - b_{\tilde{j}\tilde{i}}b_{\tilde{i}k}) = \overline{b}_{jk} + \overline{b}_{ji}\overline{b}_{ik}. \tag{8.5.3}$$

With this knowledge at hand, we will now check agreement of the proposed equation 8.5.1 and quiver mutation. We shall achieve this by splitting the task into two parts, depending on whether

1. $\text{sgn}(\overline{b}_{ji}) = \text{sgn}(\overline{b}_{ik}) = \pm 1$, or

2. $\text{sgn}(-\overline{b}_{ji}) = \text{sgn}(\overline{b}_{ik})$, or at least one of $\overline{b}_{ji}, \overline{b}_{ik}$ is zero.

For case 1(a), $\overline{b}_{ki} := b_{ki} + b_{\tilde{k}i} > 0$. Since $b_{ik} = 1 > 0$, this contradicts the conditions of the lemma, meaning case 1(a) is redundant.

For cases 1(b) and 2(a) our proposed equation 8.5.1 produces exactly what is written in (8.5.3) and (8.5.2), respectively.

For case 2(b) we have $\overline{b}_{ki} := b_{ki} + b_{\tilde{k}i} \leq 0$. However, since $0 \leq \text{sgn}(-\overline{b}_{ji}) = \text{sgn}(\overline{b}_{ik})$, then $\overline{b}_{ik} \leq 0$, so by the conditions of the lemma we deduce that $\overline{b}_{ik} = 0$. In turn, this implies $b_{ki} = b_{\tilde{i}k}$, and equation (8.5.3) reduces to $\overline{b}'_{jk} = \overline{b}_{jk}$. This is exactly what our proposed equation 8.5.1 produces.

□

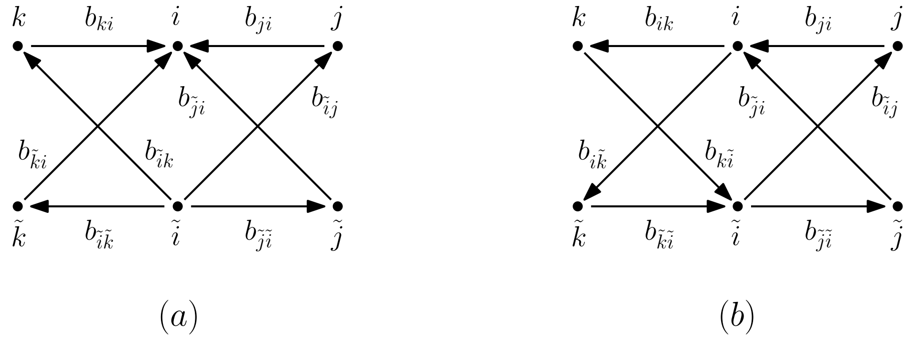


Figure 8.20: The two possible (local) configurations of Q with respect to i, j, k. (Here all coefficients present are ≥ 0.)

Proposition 8.48. Let $i \in \{1, \ldots, n\}$ be a vertex in an anti-symmetric quiver Q, and suppose there is no path $k \to i \to \tilde{k}$ for any vertex k in Q. Then mutation at i and $\tilde{i}$ in Q preserves the rank of the shortened exchange matrix $\overline{B}$.

Proof. We would like to apply Lemma 8.47 to understand how the coefficients in $\overline{B}$ change under mutation. However, it may be that $\overline{B}$ does not satisfy the conditions demanded in the lemma. Explicitly, there may exist $j \in \{1, \ldots, n\}$ such that $\overline{b}_{ij}, \overline{b}_{ji} > 0$ or $\overline{b}_{ij}, \overline{b}_{ji} < 0$. However, swapping the labels $j \leftrightarrow \tilde{j}$ in Q gives us a different shortened exchange matrix $\overline{B}^*$; for any $k \in \{1, \ldots, n\}$ we get:

$$\overline{b}^*_{jk} = b^*_{jk} + b^*_{\tilde{j}k} = b_{\tilde{j}k} + b_{jk} = \overline{b}_{jk}$$

$$\overline{b}^*_{kj} = b^*_{kj} + b^*_{\tilde{k}j} = b_{k\tilde{j}} + b_{\tilde{k}\tilde{j}} = -(b_{\tilde{k}j} + b_{kj}) = -\overline{b}_{kj}$$

.

In particular, we obtain $\overline{b}^*_{ji} = \overline{b}_{ji} > 0 > -\overline{b}_{ij} = \overline{b}^*_{ij}$. This means that we can perform a relabelling, $j \leftrightarrow \tilde{j}$, of the quiver for any j which fails the condition demanded in Lemma 8.47. The new corresponding shortened exchange matrix $\overline{B}^*$ will then satisfy the desired conditions. Note that this relabelling process only multiplies the j^{th} column by -1, so it preserves the rank of the matrix, and the corresponding exchange polynomials remain unchanged.

Therefore, without loss of generality, we may assume $\overline{B}$ satisfies the conditions of Lemma 8.47. As a consequence of Lemma 8.47 the following equations holds.

$$\left(\begin{array}{c|c} -1 & 0\cdots 0 \\ \hline \max(0,-\bar{b}_{21}) & \\ \vdots & I_{m-1} \\ \max(0,-\bar{b}_{m1}) & \end{array}\right) \overline{B} \left(\begin{array}{c|c} -1 & \max(0,\bar{b}_{12})\cdots\max(0,\bar{b}_{1n}) \\ \hline 0 & \\ \vdots & I_{n-1} \\ 0 & \end{array}\right) = \left(\begin{array}{c|c} 0 & -\bar{b}_{12} \ \cdots \ -\bar{b}_{1n} \\ \hline -\bar{b}_{21} & \\ \vdots & \left(\bar{b}'_{jk}\right)_{j,k\geq 2} \\ -\bar{b}_{m1} & \end{array}\right)$$

The matrix on the right is, $\overline{B}'$, the shortened exchange matrix of the mutated quiver $Q' = \mu_1 \circ \mu_{\tilde{1}}(Q)$. Moreover, since the matrices we are multiplying $\overline{B}$ by are invertible, $\overline{B}$ and $\overline{B}'$ have the same rank.

□

Lemma 8.49. Let $\mathbf{L}$ be a principal lamination of (S, M) and T a quasi-triangulation. Then the exchange polynomials of the quasi-arcs in T are distinct.

Proof. Let $T_{\mathbf{L}}$ be the triangulation that L is constructed from. By construction we know that the shortened exchange matrix of $T_{\mathbf{L}}$ will have full rank. As a direct consequence, the exchange polynomials of $T_{\mathbf{L}}$ will be distinct. Moreover, by Lemma 8.33 and Proposition 8.48 we know that the shortened exchange matrix of any triangulation will have full rank, in turn implying the desired uniqueness of the exchange polynomials. It remains to show the exchange polynomials of quasi-triangulations containing one-sided closed curves are distinct. Since any quasi-triangulation can be mutated into a triangulation by successive mutations at one-sided closed curves, it suffices to show that mutating to a one-sided closed curve in a quasi-triangulation preserves the uniqueness of the exchange polynomials.

Let α' be an arc in a quasi-triangulation T that flips to a one-sided closed curve α. Denote by β the unique arc intersecting α, and let x and y denote the boundary segments of the flip region. Assuming uniqueness of the exchange polynomials of T, we will argue why all exchange polynomials in the quasi-triangulation $T' := \mu_{\alpha'}(T)$ are also distinct. Suppose for now that x and y are not arcs enclosing M_1 or a punctured monogon.

Since $F_{\alpha'} = F'_{\alpha}$ and $\alpha' \in F_x, F_y$, then as all other exchange polynomials remain unchanged, we have $F'_\alpha \neq F_\gamma$ for any quasi-arc $\gamma \in T' \setminus \{\alpha\}$. The only exchange polynomials of T' depending on α are F'_x, F'_y and F'_β. Furthermore, when viewed as a polynomial in α, F_β is the only degree 2 polynomial in T', so our task is reduced to showing that $F'_x \neq F'_y$.

Consider the sub quiver Q of $Q_{\overline{T'}^*}$ obtained from looking solely at the flip region in question. We see that there is an arrow between x (or $\tilde{x}$) and y (or $\tilde{y}$) in Q, however, for F'_x and F'_y to be equal there cannot be any arrows between them in the global quiver $Q_{\overline{T'}^*}$. It must therefore be the case that the arrow in Q gets cancelled, and our quasi-triangulation must contain the configuration shown in Figure 8.21.

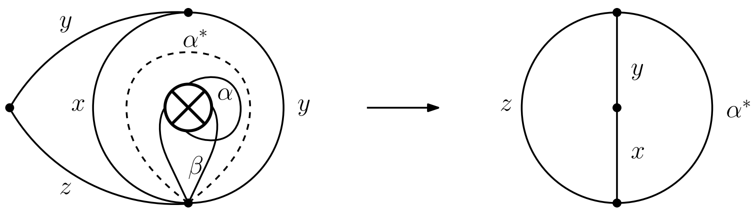


Figure 8.21: On the left we illustrate the (local) configuration of T' required to ensure $\bar{b}_{xy} = \bar{b}_{yx} = 0$ in $Q_{\overline{T}^*}$. The digon embodying this configuration is shown on the right.

However, from here we realise that x and y are the interior arcs of a punctured digon (with boundary segments z and α^*). By construction of our principal lamination **L** there is a lamination spiralling into every puncture of (S, M), so there will be a lamination spiralling into the puncture of this digon. In turn this implies $F'_x \neq F'_y$.

Finally, we need to turn our attention back to the possibility that x or y is an arc enclosing M_1 or a punctured monogon. Without loss of generality, suppose that x is such an arc, and let x_1 and x_2 be the quasi-arcs it bounds. Analogous to the reasoning employed in our proof thus far, we may deduce that the only possibility for non-uniqueness of the polynomials in T' is if $F_{x_1} = F_{x_2}$. However, if x bounds M_1 then F_{x_1} and F_{x_2} have different degrees. If x bounds a punctured monogon then x_1 and x_2 are the interior arcs of a punctured digon, and since there is a lamination spinning into this puncture we obtain $F_{x_1} \neq F_{x_2}$.

□

8.5.4 Proof of Theorem C

Theorem 8.50. Let (S, M) be an orientable or non-orientable marked surface and $\mathbf{L}$ a principal lamination. Then the LP cluster complex $\Delta_{LP}(S, M, \mathbf{L})$ is isomorphic to the laminated quasi-arc complex $\Delta^{\otimes}(S, M, \mathbf{L})$, and the exchange graph of $\mathcal{A}_{LP}(S, M, \mathbf{L})$ is isomorphic to $E^{\otimes}(S, M, \mathbf{L})$.

More explicitly, if (S, M) is not a once-punctured closed surface, the isomorphisms may be rephrased as follows. Let T be a quasi-triangulation of (S, M) and Σ_T its associated LP seed. Then in the LP algebra $\mathcal{A}_{LP}(\Sigma_T)$ generated by this seed the following correspondence holds:

$\mathcal{A}_{\mathbf{LP}}(\mathbf{\Sigma_T})$		$(\mathbf{S}, \mathbf{M}, \mathbf{L})$
Cluster variables	$\longleftrightarrow$	*Laminated lambda lengths of quasi-arcs*
Clusters	$\longleftrightarrow$	*Quasi-triangulations*
LP mutation	$\longleftrightarrow$	*Flips*

Proof. This is a consequence of Proposition 6.20, Theorem 8.40 and Lemma 8.49. □

Remark 25. If (S, M) is a closed once-punctured bordered surface then Proposition 6.13 tells us that $E^{\otimes}(S, M, \mathbf{L})$ has two connected components. In this case, Theorem 8.50 reveals the cluster variables correspond to the laminated lambda lengths of one-sided closed curves and plain arcs (or equivalently notched arcs), and the clusters will therefore correspond to quasi-triangulations consisting of one-sided closed curves and plain arcs (notched arcs).

Corollary 8.51. Let (S, M) be a bordered surface. Then the quasi-cluster algebra $\mathcal{A}(S, M)$ is a specialised LP algebra.

Proof. Let $\mathbf{L}$ be a principal lamination of (S, M). Theorem 8.50 yields that $\mathcal{A}(S, M, \overline{\mathbf{L}})$ is an LP algebra. Specialising the lamination coefficients yields the desired result.

□

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