On p-extensions of p-adic fields

MCCABE, KEITH,THOMAS

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Keith McCabe

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Department of Mathematical Sciences
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Abstract

Let p be an odd prime, and let K be a finite extension of $\mathbb{Q}_p$ such that K contains a primitive p-th root of unity. Let $K_{<p}$ be the maximal p-extension of K with Galois group $\Gamma_{<p}$ of period p and nilpotence class $< p$. Recent results of Abrashkin describe the ramification filtration $\left\{\Gamma_{<p}^{(v)}\right\}_{v\geqslant 0}$, and can be used to recover the structure of $\Gamma_{<p}$ [5].

The group $\Gamma_{<p}$ is described in terms of an $\mathbb{F}_p$-Lie algebra L due to the classical equivalence of categories of $\mathbb{F}_p$-Lie algebras of nilpotent class $< p$, and p-groups of period p of the same nilpotent class. In this thesis we generalise explicit calculations from [5] related to the structure of $\Gamma_{<p}$.

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Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, University of Durham. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is my own work unless referenced to the contrary in the text.

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Dedicated to

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Basic notions: Throughout this thesis we will assume that p is an odd prime. By *local field*, we mean a complete discrete valuation field with finite residue field. We will assume knowledge of the basic properties of local fields and their extensions as found in e.g. [9, 16].

If G is a topological group (resp. L is a topological Lie algebra), then for any $n \in \mathbb{N}$ we denote by $C_n(G)$ (resp. $C_n(L)$) the closure of the subgroup of commutators of length $\geqslant n$.

1 Introduction

A theme of modern number theory has been to understand the extent to which the arithmetic properties of a field are encoded in its absolute Galois group. For global fields, it is a result of Neukirch, Iwasawa, Ikeda, and Uchida that the knowledge of the absolute Galois group is equivalent to the knowledge of the field itself. That is, if the absolute Galois groups of two global fields are isomorphic as profinite groups, then the fields themselves are isomorphic [11, 15, 19]. This is not the case with local fields [21], in fact only limited invariants of the field can be recovered from the group structure alone (see introduction of [14]). However there is a local analogue of this result if one also considers the ramification filtration.

Recall that the Galois group of an extension of local fields has the additional structure of a decreasing filtration of normal subgroups $\{\Gamma^{(v)}\}_{v\geqslant 0}$ given by the ramification subgroups in the upper numbering. In the local setting then, if the absolute Galois groups of two local fields are isomorphic as profinite groups, and the isomorphism is compatible with the ramification filtrations, then the local fields themselves are isomorphic. This result was proved by Mochizuki for local fields of characteristic 0 [14], and for local fields of both positive and mixed characteristic by Abrashkin [3, 4].

For a local field then, any description of its Galois group should also describe the ramification filtration in order to be arithmetically significant. On the abelian level this is provided by local class field theory, which gives an isomorphism between $E^{\times}$ and a dense subgroup of $\Gamma_E/C_2(\Gamma_E)$, and describes the ramification filtration of $\Gamma_E/C_2(\Gamma_E)$ as the images of the subgroups of principal units of $E^\times$. At this level though, we see very little of the structure of the ramification filtration as, by the Hasse-Arf theorem, we can recover only integral breaks of the ramification filtration in the upper numbering. To get a more complete picture we must move beyond the abelian setting, and of particular interest is the maximal p-extension and its quotients.

Recall that, for a local field E, with $\Gamma_E = \mathrm{Gal}(E_{\mathrm{sep}}/E)$, the maximal p-extension $E(p)$ of E is the compositum (inside E_{sep}) of all finite Galois extensions of E whose degree is a power of p. Let $\Gamma_E(p)$ be the Galois group of $E(p)/E$, then $\Gamma_E(p)$ is a pro-p group.

The profinite group-theoretic description of $\Gamma_E(p)$ is well known. If $\mathrm{char}(E) = p$, then $\Gamma_E(p)$ is a free pro-p group with infinitely many generators [13]. If $\mathrm{char}(E) = 0$ there are two cases; if E does not contain a primitive p-th root of unity then $\Gamma_E(p)$ is a free pro-p group with $[E : \mathbb{Q}_p] + 1$ generators [17, 18], and if E does contain a primitive p-th root of unity then $\Gamma_E(p)$ is a Demushkin group with $[E : \mathbb{Q}_p] + 2$ generators and one relation (of a particular form) [8, 17]. However, as these results do not describe the ramification filtration, then from the discussion above, they do not fully reflect the appearance of $\Gamma_E(p)$ as a Galois group of a local field extension.

For fields of characteristic p, the *nilpotent Artin-Schreier theory* developed in [1, 2] allows us to work with extensions of fields of characteristic p whose Galois group has nilpotence class less than p, and period p^M for any $M \geqslant 1$. Although the general theory is applicable in a wider setting, our particular interest is in the case where E is a local field of characteristic p, and $M = 1$. In this setting the nilpotent Artin-Schreier theory can be used to give an explicit description of the Galois group of the maximal p-extension of period p modulo the subgroup of commutators of length $\geqslant p$, together with an explicit description of its ramification filtration.

The methods of the nilpotent Artin-Schreier theory cannot be applied directly to local fields of characteristic 0, however recent results [5, 6] allow the theory to be applied via the *field of norms functor* of Fontaine-Wintenberger, which for a particular class of infinite field extensions allows us to relate the Galois group of a local field of characteristic 0 to the Galois group of a local field of characteristic p [10, 20].

Outline of thesis

Let K be a finite extension of $\mathbb{Q}_p$ such that $\zeta_p \in K$, where ζ_p is a primitive p-th root of unity. Denote by $K_{<p}$ the maximal p-extension of K with Galois group of nilpotence class $< p$ and period p. Then $\Gamma_{<p} := \mathrm{Gal}(K_{<p}/K) = \Gamma_K/\Gamma_K^p C_p(\Gamma_K)$. The results in this thesis are concerned with the structure of $\Gamma_{<p}$, and are based on recent papers by Abrashkin [5, 6].

In chapter 2 we follow [1] throughout to present the relevant results of the nilpotent Artin-Schreier theory. Of particular importance to us is the case where $\mathcal{K}$ is a local field of characteristic p. Let $\mathcal{K}_{\mathrm{sep}}$ be a separable closure of $\mathcal{K}$, and $\mathcal{G} := \mathrm{Gal}(\mathcal{K}_{\mathrm{sep}}/\mathcal{K})$. Let $\mathcal{K}_{<p} \subset \mathcal{K}_{\mathrm{sep}}$ be the maximal p-extension of $\mathcal{K}$ whose Galois group has period p and nilpotence class $< p$, and let $\mathcal{G}_{<p} := \mathrm{Gal}(\mathcal{K}_{<p}/\mathcal{K})$. The nilpotent Artin-Schreier theory provides us with the following identification.

$$\eta_0 : \mathcal{G}_{<p} \longrightarrow G(\mathcal{L}), \tag{1.1}$$

where $\mathcal{L}$ is a profinite $\mathbb{F}_p$-Lie algebra with $\mathbb{F}_p$-module of generators $\mathcal{K}^*/(\mathcal{K}^*)^p$, and $G(\mathcal{L})$ is the group defined on the underlying set of $\mathcal{L}$ with operation given by the Campbell-Hausdorff group law.

We also present the techniques of the theory that can be used to lift automorphisms of $\mathcal{K}$ to automorphisms of $\mathcal{K}_{<p}$ [1, 2].

In chapter 3 we follow [5] throughout to present results related to the structure of $\Gamma_{<p}$. The main steps of [5] are the following.

(i) Fundamental exact sequence. We fix once and for all a uniformiser π_0 of K. For all $n \geqslant 1$, let $\pi_n \in \overline{K}$ be such that $\pi_n^p = \pi_{n-1}$, and let $\widetilde{K} = \bigcup_{n\geqslant 1} K(\pi_n)$, then the following sequence is exact.

$$\Gamma_{\widetilde{K}} \longrightarrow \Gamma_{<p} \longrightarrow \mathrm{Gal}(K(\pi_1)/K) = \langle \tau_0 \rangle^{\mathbb{Z}/p} \longrightarrow 1\,,$$

where $\tau_0(\pi_1) = \zeta_p \pi_1$.

(ii) Relation to characteristic p. The field extension $\widetilde{K}/K$ is *arithmetically profinite* and so the field of norms functor provides us with the construction of a local field $\mathcal{K}$ of characteristic p, and an identification $\Gamma_{\widetilde{K}} \simeq \Gamma_{\mathcal{K}}$.

(iii) Nilpotent Artin-Schreier theory. As $\mathcal{K}$ is a local field of characteristic p we can apply the identification η_0 of the nilpotent Artin-Schreier theory to obtain the following exact sequence,

$$G(\mathcal{L}) \longrightarrow \Gamma_{<p} \longrightarrow \langle \tau_0 \rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

The techniques of the nilpotent Artin-Schreier can be applied to consider lifts $\tau_{<p} \in \Gamma_{<p}$ of τ_0, which in turn can be used to recover the kernel of $G(\mathcal{L}) \to \Gamma_{<p}$ thus establishing the following exact sequence of p-groups,

$$1 \longrightarrow G(\mathcal{L}/\mathcal{L}(p)) \longrightarrow \Gamma_{<p} \longrightarrow \langle \tau_0 \rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

If we fix a lift $\tau_{<p} \in \Gamma_{<p}$ of τ_0 then the sequence splits, and the structure of $\Gamma_{<p}$ is determined by the action by conjugation of $\langle \tau_{<p} \rangle^{\mathbb{Z}/p}$ on $G(\mathcal{L}/\mathcal{L}(p))$.

(iv) Linearisation. Let L be the $\mathbb{F}_p$-Lie algebra such that $G(L) = \Gamma_{<p}$ under the Lazard correspondence (see §2.1). It is shown in [5] that the lift $\tau_{<p}$ can be recovered from the Lie structure of L. This represents a significant simplification, as it avoids the use of the Campbell-Hausdorff group law. In particular, we obtain the following exact sequence in the category of $\mathbb{F}_p$-modules,

$$0 \longrightarrow \mathcal{L}/\mathcal{L}(p) \longrightarrow L \longrightarrow \mathbb{F}_p\tau_0 \longrightarrow 0\,.$$

The sequence splits, so that if $\tau_{<p} \in L$ is a lift of τ_0, then the structure of L is determined by the derivation $\mathrm{ad}_{\tau_{<p}} \in \mathrm{Der}(\mathcal{L}/\mathcal{L}(p))/\,\mathrm{Inn}(\mathcal{L}/\mathcal{L}(p))$. It is proved in [5] that the set of lifts $\tau_{<p}$ of τ_0 are in bijection with solutions of a recurrence formula (3.6). The form of these solutions describe a lift $\tau_{<p}$

and the corresponding derivation $\mathrm{ad}_{\tau_{<p}}$, thus determining the structure of L and hence the structure of $\Gamma_{<p} = G(L)$. The main result in this thesis is the recovery of an explicit solution of the recurrence formula (3.6), which generalises explicit calculations from [5].

In chapter 4 we introduce a recurrent procedure used in [5] that will allow us to recover a solution of (3.6), and demonstrate the method by recovering an explicit solution of the recurrence formula modulo commutators of length $\geqslant 2$ (this case was stated in [5]).

The remaining chapters contain the original work of the thesis. In chapter 5 we recover a solution of the recurrence formula under a simplifying assumption, and introduce all notation and properties that will enable us to recover a general solution. In chapter 6 we demonstrate that, although the coefficients appearing in our solution are complicated in general they have an essentially combinatorial interpretation, and using this interpretation we recover some general properties of the coefficients. Finally, in chapter 7 we use our results to give a general solution of the recurrence formula (3.6).

This in principle fully determines the structure of $\Gamma_{<p}$. Unfortunately, the result is not recovered in a form from which we can also recover explicitly the generators of the ramification groups of $\Gamma_{<p}$. This is not ideal given the importance of the ramification filtration discussed above. As it is, we compare our result with explicit calculations from [5] to demonstrate that the result can be used to efficiently write down solutions of recurrence formula (3.6) modulo commutators of small degree. As we give a thorough characterisation of terms related to the structure of $\Gamma_{<p}$ it is reasonable to expect that applications will be found, and at the very least, the explicitness of the solution provides an opportunity for further study.

2 Nilpotent Artin-Schreier theory

In this chapter we present the relevant details of the nilpotent Artin-Schreier theory [1]. The theory makes use of the Lazard correspondence [12], which establishes an equivalence of categories of finitely generated Lie $\mathbb{Z}_p$-algebras of nilpotence class less than p, and finitely generated pro-p groups of the same nilpotence class.

2.1 Lazard correspondence

The following description of the Lazard correspondence can be found in [1], and is derived from [7, ex 8.4, ch 2].

Let $\mathcal{A}_{\mathbb{Q}}$ be the free associative algebra over $\mathbb{Q}$ freely generated by the (non-commuting) indeterminants $\{x, y\}$. Then $\mathcal{A}_{\mathbb{Q}}$ has the natural structure of a Lie algebra over $\mathbb{Q}$, with Lie product defined by $[a, b] = ab - ba$. Let $\mathcal{L}_{\mathbb{Q}}$ be the free Lie algebra over $\mathbb{Q}$ with free generators $\{x, y\}$, then we have a natural embedding $\mathcal{L}_{\mathbb{Q}} \subset \mathcal{A}_{\mathbb{Q}}$.

We define the degree of a monomial in $\mathcal{A}_{\mathbb{Q}}$ by setting $\deg x = \deg y = 1$, and extending in the usual way. For $n \geqslant 1$, denote by $C_n(\mathcal{A}_{\mathbb{Q}})$ the ideal generated by all monomials of degree $\geqslant n$, and let $\hat{\mathcal{A}}_{\mathbb{Q}} = \varprojlim_n \mathcal{A}_{\mathbb{Q}}/C_n(\mathcal{A}_{\mathbb{Q}})$. Similarly, for $n \geqslant 1$, let $C_n(\mathcal{L}_{\mathbb{Q}})$ be the ideal generated by all commutators of length $\geqslant n$, and let $\hat{\mathcal{L}}_{\mathbb{Q}} = \varprojlim_n \mathcal{L}_{\mathbb{Q}}/C_n(\mathcal{L}_{\mathbb{Q}})$. Note that $C_n(\mathcal{A}_{\mathbb{Q}}) \cap \mathcal{L}_{\mathbb{Q}} = C_n(\mathcal{L}_{\mathbb{Q}})$ for all $n \geqslant 1$, and thus the natural embedding $\mathcal{L}_{\mathbb{Q}} \subset \mathcal{A}_{\mathbb{Q}}$ induces an embedding $\hat{\mathcal{L}}_{\mathbb{Q}} \subset \hat{\mathcal{A}}_{\mathbb{Q}}$.

The Campbell-Hausdorff series is defined in $\hat{\mathcal{A}}_{\mathbb{Q}}$ as $H(x, y) = \log(\exp x \exp y)$. Its first few homogenous components are well known, and are given as follows,

$$x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] - \frac{1}{24}[y, [x, [x, y]]] \mod C_5(\mathcal{A}_{\mathbb{Q}}).$$

Clearly these terms are elements of $\mathcal{L}_{\mathbb{Q}}$, and in fact the result known as the Campbell-Hausdorff formula states that $H(x, y) \in \hat{\mathcal{L}}_{\mathbb{Q}}$. We can define then a group structure $G(\hat{\mathcal{L}}_{\mathbb{Q}})$ on the underlying set of $\hat{\mathcal{L}}_{\mathbb{Q}}$, with group operation given by $l_1 \circ l_2 = H(l_1, l_2)$.

Let $H_n(x, y)$ denote the homogenous terms of degree n in the Hausdorff series, then the terms of $H_n(x, y)$ are p-integral for $1 \leqslant n < p$. In particular, if $\mathcal{L}$ is a finitely generated $\mathbb{Z}_p$-Lie algebra of nilpotence class $< p$, then a group $G(\mathcal{L})$ can defined on the underlying set of $\mathcal{L}$ with group operation $l_1 \circ l_2 = H(l_1, l_2) \bmod C_p(\hat{\mathcal{L}}_{\mathbb{Q}})$. Then $G(\mathcal{L})$ is a finitely generated pro-p group of the same nilpotence class as $\mathcal{L}$, and there exist inverse Campbell-Hausdorff formulas from which we can recover the Lie structure of $\mathcal{L}$ from $G(\mathcal{L})$.

The correspondence $\mathcal{L} \to G(\mathcal{L})$ induces an equivalence of categories of finitely generated Lie $\mathbb{Z}_p$-algebras of nilpotence class less than p, and finitely generated pro-p groups of the same nilpotence class [12]. For any $\varphi \in \mathrm{End}(\mathcal{L})$ there is a unique $\varphi' \in \mathrm{End}(G(\mathcal{L}))$ such that φ and φ' are identical as set maps. Hence an automorphism of the Lie algebra $\mathcal{L}$ is automatically an automorphism of the p-group $G(\mathcal{L})$ and vice versa, acting on the underlying set in the same way. The correspondence has the following properties [12].

$$\begin{aligned} \text{Subgroups of } G(\mathcal{L}) &\longleftrightarrow \text{ Subalgebras of } \mathcal{L}\,; \\ \text{Normal subgroups of } G(\mathcal{L}) &\longleftrightarrow \text{ Ideals of } \mathcal{L}\,; \\ 1_G \in G(\mathcal{L}) &\longleftrightarrow 0 \in \mathcal{L}\,; \\ g^{-1} \in G(\mathcal{L}) &\longleftrightarrow -g \in \mathcal{L}\,. \end{aligned}$$

2.2 Main theorem of the nilpotent Artin-Schreier theory

Let K be a field of characteristic p (not necessarily local), K_{sep} a fixed separable closure of K, and $\Gamma_K = \mathrm{Gal}(K_{\mathrm{sep}}/K)$. Let $\mathcal{L}$ be an $\mathbb{F}_p$-Lie algebra of nilpotence class $< p$.

Denote by $\mathcal{L}_{K_{\mathrm{sep}}}$ the extension of scalars $\mathcal{L} \otimes_{\mathbb{F}_p} K_{\mathrm{sep}}$. Let σ be the absolute Frobenius morphism of K (i.e. $\sigma(x) = x^p$ for all $x \in K_{\mathrm{sep}}$). Then σ and Γ_K act on $\mathcal{L}_{K_{\mathrm{sep}}}$ as follows.

— $(\mathrm{id} \otimes \sigma)(l \otimes x) = l \otimes \sigma(x)$, and $\mathcal{L}_{K_{\mathrm{sep}}}\restriction_{\sigma=\mathrm{id}} = \mathcal{L}\,$;

— for $g_1, g_2 \in \Gamma_K$ we have $(\mathrm{id} \otimes g_1)((\mathrm{id} \otimes g_2)(l \otimes x)) = l \otimes (g_1(g_2(x)))$;

— The actions of σ and Γ_K commute, and $(\mathcal{L}_{K_{\mathrm{sep}}})^{\Gamma_K} = \mathcal{L} \otimes_{\mathbb{F}_p} K$.

Remarks: (1) For notational convenience, if $l \in \mathcal{L}_{K_{\mathrm{sep}}}$ and $g \in \Gamma_K$ we will generally use the shorter notation $\sigma(l)$ and $g(l)$. We will however retain the full notation in certain instances for clarity of exposition.

(2) For any topological $\mathbb{F}_p$-Lie algebra L, and any topological $\mathbb{F}_p$-module F, we will use here and below the notation $L_F = L \hat{\otimes}_{\mathbb{F}_p} F$.

The main theorem of the nilpotent Artin-Schreier theory is the following.

Theorem 2.1. [1] *Let K be a field of characteristic $p > 0$, and let $\mathcal{L}$ be an $\mathbb{F}_p$-Lie algebra of nilpotence class $< p$. Then there exists a one to one map of sets,*

$$\pi : G(\mathcal{L}_K)/R \longrightarrow \{\textit{conjugacy classes of } \mathrm{Hom}(\Gamma_K, G(\mathcal{L}))\},$$

where R is an equivalence relation on $G(\mathcal{L}_K)$ such that for $l_1, l_2 \in G(\mathcal{L}_K)$, then $l_1 \overset{R}{\sim} l_2$ if there exists $c \in G(\mathcal{L}_K)$ such that $l_2 = \sigma c \circ l_1 \circ (-c)$.

The full proof is given in [1, §1.3], here we restrict our attention to demonstrating the construction of the map π. The first step is to relate elements of $G(\mathcal{L}_K)$ to the construction of homomorphisms $\Gamma_K \to G(\mathcal{L})$.

Let $e \in G(\mathcal{L}_K)$, and let $F(e) := \{f \in \mathcal{L}_{K_{\mathrm{sep}}} : \sigma(f) = e \circ f\}$. It is proved in [1] that for any $e \in G(\mathcal{L}_K)$, the set $F(e)$ is non-empty.

Proposition 2.2. [1] *Let K be a field of characteristic $p > 0$, and let $\mathcal{L}$ be an $\mathbb{F}_p$-Lie algebra of nilpotence class $< p$. For any $e \in G(\mathcal{L}_K)$, and $f \in F(e)$, the following is a homomorphism,*

$$\begin{aligned} \pi_{e,f} : \Gamma_K &\to G(\mathcal{L}), \\ \tau &\mapsto (-f) \circ \tau(f) . \end{aligned}$$

Proof. As Γ_K acts on $G(\mathcal{L}_{K_{\mathrm{sep}}})$ by functorality then we just need to show that $\pi_{e,f}(\tau) \in G(\mathcal{L})$ for all $\tau \in \Gamma_K$, which we can do by showing that the image of τ is invariant under σ.

Note that σ and Γ_K commute as automorphisms of K_{sep}, so that for any $\tau \in \Gamma_K$ we have

$$\sigma(\tau f) = \tau(\sigma f) = \tau(e \circ f)\,. \tag{2.1}$$

Furthermore, as $\tau \in \Gamma_K$ and $e \in G(\mathcal{L}_K)$ then $\tau e = e$, and thus from (2.1) we obtain that $\sigma(\tau f) = e \circ \tau f$. Therefore we have

$$\sigma((-f)\circ\tau f) = \sigma(-f)\circ\sigma(\tau f) = -\sigma(f)\circ e\circ\tau f = (-f)\circ(-e)\circ e\circ\tau f = (-f)\circ\tau f\,.$$

As $(-f) \circ \tau f$ is σ-invariant, then $((-f) \circ \tau f) \in G(\mathcal{L})$. □

For a fixed $e \in G(\mathcal{L}_K)$, we can also show that if $f_1, f_2 \in F(e)$ then $f_1 = f_2 \circ x$ for some $x \in G(\mathcal{L})$. Recalling that elements of $G(\mathcal{L})$ are invariant under σ then we have,

$$\sigma((-f_1 \circ f_2) = -\sigma(f_1) \circ \sigma(f_2) = (-f_1) \circ -e \circ e \circ f_2 = (-f_1) \circ f_2\,.$$

Thus $f_1 = f_2 \circ x$ for some $x \in G(\mathcal{L})$. It follows that π_{e,f_1} and π_{e,f_2} are conjugated by the element x, and thus belong to the same conjugacy class of $\text{Hom}(\Gamma_K, G(\mathcal{L}))$.

Therefore, for any $e \in G(\mathcal{L}_K)$, the conjugacy class of $\pi_{e,f} \in \text{Hom}(\Gamma_K, G(\mathcal{L}))$ does not depend on the choice of $f \in F(e)$, and the map π in theorem 2.1 is then defined as follows.

$$\begin{aligned}\pi : G(\mathcal{L}_K)/R &\to \{\text{conjugacy classes of } \text{Hom}(\Gamma_K, G(\mathcal{L}))\},\\ e &\mapsto \pi(e),\end{aligned}$$

where for any $e \in G(\mathcal{L}_K)$, $\pi(e)$ *denotes the conjugacy class of* $\pi_{e,f}$.

Thereom 2.1 implies the following properties.

- For any $\eta \in \text{Hom}(\Gamma_K, G(\mathcal{L}))$ there exists $e \in G(\mathcal{L}_K)$, and $f \in F(e)$ such that $\eta = \pi_{e,f}$;
- If $\eta_1, \eta_2 \in \text{Hom}(\Gamma_K, G(\mathcal{L}))$ are such that $\eta_1 = \pi_{e_1,f_1}$, and $\eta_2 = \pi_{e_2,f_2}$, then η_1, η_2 are conjugate via an element of $G(\mathcal{L})$ if and only if $e_1 = \sigma(c) \circ e_2 \circ (-c)$ for some $c \in G(\mathcal{L}_K)$.

Remarks:

(1) The above results were presented in [1] under the assumption that the action of Γ_K on K_{sep} is given by $(g_1g_2)a = g_2(g_1a)$ where $g_1, g_2 \in \Gamma_K$ and $a \in K_{\mathrm{sep}}$. In the setting of [1] then, for any $e \in G(\mathcal{L}_K)$ the set $F^{\mathrm{op}}(e) := \{f \in G(\mathcal{L}_{K_{\mathrm{sep}}}) : \sigma(f) = f \circ e\}$ is non empty and the correspondence $g \mapsto g(f) \circ -f$ establishes a group homomorphism $\Gamma_K^{\mathrm{op}} \to G(\mathcal{L}_K)$, where Γ_K^{op} is the opposite group of Γ_K, i.e. the group defined on the underlying set of Γ_K with group composition given by $(g_1g_2)^{\mathrm{op}} = g_2g_1$. In [5] the presentation of the nilpotent Artin-Schreier theory as given in [1] is called 'contravariant', whereas the presentation used throughout this thesis (and in [5, 6]) is called 'covariant'. We note that all results established in the contravariant setting hold for the covariant setting by replacing the relevant group or Lie structures with their opposites i.e. for any group G we have $G \simeq G^{\mathrm{op}}$ via $x \mapsto x^{-1}$, and for any Lie algebra L we have $L \simeq L^{\mathrm{op}}$ via $x \mapsto -x$.

(2) Let K be a field of characteristic $p > 0$. If $\mathcal{L}$ is a one dimensional $\mathbb{F}_p$-Lie algebra, then by fixing a generator of $\mathcal{L}$ we obtain the identifications $G(\mathcal{L}) \simeq (\mathbb{Z}/p\mathbb{Z}, +)$ and $G(\mathcal{L}_K) \simeq (K, +)$. Thus, in this case, the main theorem gives the classical Artin-Schreier isomorphism, $K/(\sigma - \mathrm{id})K \simeq \mathrm{Hom}(\Gamma_K, \mathbb{Z}/p\mathbb{Z})$.

(3) Let $\eta = \pi_{e,f}$ be such that $\eta : \Gamma_K \to G(\mathcal{L})$ is an epimorphism, If we denote by $K_e := K_{\mathrm{sep}}^{\mathrm{Ker}(\eta)}$, then η induces a group isomorphism $\Gamma_{K_e/K} \simeq G(\mathcal{L})$. Note that the field K_e is not dependent on the choice of f, since if $f, f' \in F(e)$ then the main theorem of NAS implies that $\pi_{e,f}$ and $\pi_{e,f'}$ belong to the same conjugacy class of $\mathrm{Hom}(\Gamma_K, G(\mathcal{L}))$, thus both homomorphisms are epimorphic and their kernels coincide.

2.3 Identification for local fields of positive characteristic

Let $\mathcal{K}$ be a local field of characteristic p. We fix a uniformiser t, whence $\mathcal{K} \simeq k((t))$ with $k \simeq \mathbb{F}_{p^{N_0}}$ for some $N_0 \in \mathbb{N}$. Denote by $\mathcal{K}_{<p}$ the maximal p-extension of $\mathcal{K}$ of period p and nilpotent class less than p. Then $\mathcal{G}_{<p} := \mathrm{Gal}(\mathcal{K}_{<p}/\mathcal{K}) = \Gamma_{\mathcal{K}}/C_p(\Gamma_{\mathcal{K}})\Gamma_{\mathcal{K}}^p$.

Let $\widetilde{\mathcal{L}}$ be the free profinite Lie $\mathbb{F}_p$-algebra with the $\mathbb{F}_p$-module of topological generators $\mathcal{K}^*/\mathcal{K}^{*p}$, and let $\widetilde{\mathcal{L}}_k := \widetilde{\mathcal{L}}\,\hat{\otimes}_{\mathbb{F}_p} k$ be the Lie algebra obtained by extension of scalars by k. Let $\mathbb{Z}^+(p) = \{a \in \mathbb{N} : (a,p) = 1\}$ and let $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$. It is shown in [1] that $\widetilde{\mathcal{L}}_k$ has the set of free generators $\{D_0\} \cup \{D_{an} : a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}$. In order to treat the generator D_0 in the context of all generators we fix $\alpha_0 \in k$ such that $\mathrm{Tr}_{k/\mathbb{F}_p} \alpha_0 = 1$, and for any $n \in \mathbb{Z}/N_0$ we set $D_{0n} = \sigma^n(\alpha_0)D_0$.

The generators D_{an} arise as follows. If we recall the Artin-Schreier pairing $\mathcal{K}^*/\mathcal{K}^{*p} = \mathrm{Hom}(\mathcal{K}/(\sigma - \mathrm{id})\mathcal{K}, \mathbb{F}_p)$, then fixing a uniformiser $t \in \mathcal{K}$ and an element $\alpha_0 \in k$ of trace 1 gives us an identification $\mathcal{K} = (\sigma - \mathrm{id})\mathcal{K} \oplus (\mathbb{F}_p\alpha_0) \oplus_{a\in\mathbb{Z}^+(p)} (kt^{-a})$, whence $\mathcal{K}/(\sigma - \mathrm{id})\mathcal{K} \simeq (\mathbb{F}_p\alpha_0) \oplus_{a\in\mathbb{Z}^+(p)} (kt^{-a})$. Upon extending scalars to k we obtain

$$\begin{aligned}\mathcal{K}^*/\mathcal{K}^{*p} \otimes_{\mathbb{F}_p} k &= \mathrm{Hom}_{\mathbb{F}_p}(\mathcal{K}/(\sigma - \mathrm{id})\mathcal{K}, k)\\ &= \mathrm{Hom}_{\mathbb{F}_p}((\mathbb{F}_p\alpha_0) \oplus_{a\in\mathbb{Z}^+(p)} (kt^{-a}), k)\,.\end{aligned}$$

Then D_{an} is the homomorphism such that $D_{an}(wt^{-a}) = \sigma^n(w)$ for $w \in k$, and D_{0n} is the homomorphism such that $D_{0n}(\alpha_0) = \sigma^n(\alpha_0)$. Notice that $\sigma(D_{an}) = D_{a,n+1}$, and $D_{an} = D_{a,n+N_0}$.

Proposition 2.3. [1, §5.1] *Let $\mathcal{K}$ be a local field of characteristic p, with fixed uniformiser t, and let $\mathcal{L} = \widetilde{\mathcal{L}}/C_p(\widetilde{\mathcal{L}})$. Then we have the following isomorphism*

$$\eta_0 : \Gamma_{\mathcal{K}}/(\Gamma_{\mathcal{K}})^p C_p(\Gamma_{\mathcal{K}}) \simeq G(\mathcal{L})\,,$$

where η_0 is induced by $\pi_{e,f} \in \mathrm{Hom}(\Gamma_{\mathcal{K}}, G(\widetilde{\mathcal{L}}))$ with $e = \sum_{a\in\mathbb{Z}^0(p)} t^{-a}D_{a0}$, and $f \in F(e)$.

The proof is given in [1, §5.1], and establishes for e, f chosen as above, that the homomorphism $\pi_{e,f} : \Gamma_{\mathcal{K}} \to G(\mathcal{L})$ is surjective, and $\mathrm{Ker}(\pi_{e,f}) = C_p(\Gamma_{\mathcal{K}})\Gamma_{\mathcal{K}}^p$ (see remark (3) above).

Note that the elements e, f play a very important role in the nilpotent Artin-Schreier theory. A key idea is that the relation $\sigma(f) = e \circ f$ means that the coefficients of f can be interpreted as roots of modified Artin-Schreier extensions, and thus the action of $\Gamma_{\mathcal{K}}$ on f encodes the action of $\Gamma_{\mathcal{K}}$ on p-extensions of $\mathcal{K}$ (see [6, §1.4]).

2.4 Analytic automorphisms of K

Let K be a field of characteristic p (not necessarily local), $\mathcal{L}$ an $\mathbb{F}_p$-Lie algebra of nilpotence class $< p$, and let $\eta : \Gamma_K \to G(\mathcal{L})$ be an epimorphism. Recall that $K_e \subset K_{\text{sep}}$ denotes the fixed field of $\operatorname{Ker}(\eta)$.

Suppose h is a continuous automorphism of K, and $\widehat{h}$ is a lift of h to K_{sep}, (i.e. $\widehat{h} \in \operatorname{Aut}(K_{\text{sep}})$ such that $\widehat{h}\restriction_K = h$), then we have the following result.

Proposition 2.4. [1, prop 1.5.1] *The following are equivalent.*
(1) $\widehat{h}(K_e) = K_e$;
(2) $(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) = c \circ (A \otimes \mathrm{id}_{K_{\text{sep}}})(f)$ *for some* $c \in G(\mathcal{L}_K)$ *and* $A \in \operatorname{Aut}(\mathcal{L})$ (*whereby* A *is also an automorphism of* $G(\mathcal{L})$).

Proof. Let $f_1 = (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f)$ and $e_1 = (\mathrm{id}_{\mathcal{L}} \otimes h)(e)$ (noting that $e \in G(\mathcal{L}_K)$ and $\widehat{h}\restriction_K = h$). Then since σ and $\widehat{h}$ commute as automorphisms of K_{sep} we can show that $f_1 \in F(e_1)$ as follows,

$$\begin{aligned}\sigma f_1 = \sigma((\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f)) = (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(\sigma(f)) &= (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(e \circ f) \\ &= (\mathrm{id}_{\mathcal{L}} \otimes h)(e) \circ (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) = e_1 \circ f_1 \,.\end{aligned}$$

As $f_1 \in F(e_1)$, we can introduce $\eta_1 \in \operatorname{Hom}(\Gamma_K, G(\mathcal{L}))$ such that $\eta_1 = \pi_{e_1, f_1}$. Then for any $g \in \Gamma_K$ we have

$$\begin{aligned}\eta_1(g) = -f_1 \circ g(f_1) &= (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(-f) \circ g(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) \\ &= (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})((-f) \circ (\mathrm{id}_{\mathcal{L}} \otimes \widehat{h}^{-1} g \widehat{h})(f))\,.\end{aligned} \tag{2.2}$$

Note that $(\widehat{h}^{-1} g \widehat{h}) \in \Gamma_K$, as it acts as the identify on K. Note also that $(-f) \circ (\widehat{h}^{-1} g \widehat{h})(f) = \eta(\widehat{h}^{-1} g \widehat{h})$. Hence $(-f) \circ (\widehat{h}^{-1} g \widehat{h})(f) \in G(\mathcal{L})$, and it follows that $\widehat{h}((-f) \circ (\widehat{h}^{-1} g \widehat{h})(f)) = (-f) \circ (\widehat{h}^{-1} g \widehat{h})(f)$. Thus from (2.2) we obtain for all $g \in \Gamma_K$ that

$$\eta_1(g) = \eta(\widehat{h}^{-1} g \widehat{h})\,. \tag{2.3}$$

(1) $\implies$ (2) : Recall that K_e is the fixed field of $\mathrm{Ker}(\eta)$, and since $\widehat{h}(K_e) = K_e$ equation (2.3) implies that $\mathrm{Ker}(\eta) = \mathrm{Ker}(\eta_1)$. This in turn implies the existence of $A \in \mathrm{Aut}(G(\mathcal{L}))$ (which is also an automorphism of $\mathcal{L}$) such that the following diagram commutes.

$$\begin{array}{ccc} \Gamma_K & \xrightarrow{\ \eta\ } & G(\mathcal{L}) \\ {\scriptstyle \widehat{h}^{-1} g \widehat{h}}\downarrow & \searrow^{\eta_1} & \downarrow{\scriptstyle A} \\ \Gamma_K & \xrightarrow[\ \eta\ ]{} & G(\mathcal{L}) \end{array}$$

Moreover, for all $g \in \Gamma_K$ we have,

$$\begin{aligned} \eta_1(g) = A(\eta(g)) = A((-f) \circ gf) &= (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) \circ (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(gf) \\ &= (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) \circ g((A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(f))\,. \end{aligned}$$

Since $\eta_1(g) = -f_1 \circ gf_1$ we obtain that

$$(A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) \circ g((A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(f)) = -f_1 \circ gf_1\,,$$

and upon rearrangement we have $f_1 \circ (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) = g(f_1 \circ (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f))$.

Notice that $f_1 \circ (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) \in G(\mathcal{L}_{K_{\mathrm{sep}}})^{\Gamma_K}$, hence $f_1 \circ (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) = c$ for some $c \in G(\mathcal{L}_K)$.

(2) $\implies$ (1) : Consider $\eta_1(g)$ for $g \in \Gamma_K$. Since $(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) = c \circ (A \otimes id)(f)$ we have

$$\eta_1(g) = (-f_1) \circ gf_1 = (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) \circ (-c) \circ g(c \circ (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(f))\,,$$

and since $c \in G(\mathcal{L}_K)$ we have $gc = c$, so that

$$\eta_1(g) = (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(-f) \circ g(A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(f))\,.$$

Hence $g \in \mathrm{Ker}(\eta_1) \iff g((A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(f)) = (A \otimes \mathrm{id}_{K_{\mathrm{sep}}})(f) \iff gf = f \iff g \in \mathrm{Ker}(\eta)$.

Thus the kernels of η and η_1 coincide, and it follows from (2.3) that $\widehat{h}(K_e) = K_e$. $\square$

If $h \in \mathrm{Aut}(K)$ then proposition 2.4 implies that the lifts $\widehat{h}\restriction_{K_e} \in \mathrm{Aut}(K_e)$ can be determined by the image $(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f)$ by specifying the image in the form $(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) = c \circ (A \otimes \mathrm{id})(f)$, with $A \in \mathrm{Aut}(\mathcal{L})$ and $c \in G(\mathcal{L}_K)$.

2.4.1 Lifting automorphisms of $\mathcal{K}$

In the particular case where $\mathcal{K}$ is a local field of characteristic p, then with respect to the identification η_0 from section 2.3, proposition 2.4 allows us to lift automorphisms of $\mathcal{K}$ to automorphisms of $\mathcal{K}_{<p} = \mathcal{K}_{\mathrm{sep}}^{C_p(\Gamma_{\mathcal{K}})\Gamma_{\mathcal{K}}^p}$ as follows.

Suppose $h \in \mathrm{Aut}(\mathcal{K})$ then for any lift $h_{<p} \in \mathrm{Aut}(\mathcal{K}_{<p})$ by proposition 2.4 we have that

$$(\mathrm{id}_{\mathcal{L}} \otimes h_{<p})(f) = c \circ (A \otimes \mathrm{id}_{\mathcal{K}_{<p}})(f)\,, \tag{2.4}$$

for some $c \in G(\mathcal{L}_{\mathcal{K}})$ *and* $A \in \mathrm{Aut}(\mathcal{L})$.

By applying σ to both sides of (2.4), and using that $\sigma f = e \circ f$ we obtain the following relation,

$$(\mathrm{id}_{\mathcal{L}} \otimes h)(e) \circ c = \sigma c \circ (A \otimes \mathrm{id}_{\mathcal{K}})(e). \tag{2.5}$$

Since $(\mathrm{id}_{\mathcal{L}} \otimes h)(e) = \sum_{a \in \mathbb{Z}^0(p)} h(t^{-a}) D_{a0}$, and $(A \otimes \mathrm{id}_{\mathcal{K}})(e) = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} A(D_{a0})$, then this relation gives us the opportunity to recover lifts $h_{<p}$ as follows. Modulo $C_2(\mathcal{L}_{\mathcal{K}})$ we have that

$$\sum_{a \in \mathbb{Z}^0(p)} h(t^{-a}) D_{a0} \equiv \sigma c - c + \sum_{a \in \mathbb{Z}^0(p)} t^{-a} A(D_{a0})\,. \tag{2.6}$$

As h is known, we can expand the left hand side, and recover solutions for c and A to specify $(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) \bmod C_2(\mathcal{L}_{\mathcal{K}_{<p}})$. Similarly, suppose we have recovered solutions c' and A' that specify $(\mathrm{id}_{\mathcal{L}} \otimes \widehat{h})(f) \bmod C_n(\mathcal{L}_{\mathcal{K}_{<p}})$ for $2 \leqslant n < p$, then we have that $c = c' + X_n$ for some $X_n \in C_n(\mathcal{L}_{\mathcal{K}})$ and $A = A' + A_n$ where $A_n(D_{a0}) \in C_n(\mathcal{L}_k)$ and relation (2.5) implies that

$$\sigma X_n - X_n + \sum_{a \in \mathbb{Z}^0(p)} t^{-a} A_n(D_{a0}) \equiv h(e) \circ c' - \sigma c' \circ (A')(e) \mod C_{n+1}(\mathcal{L}_{\mathcal{K}})\,. \tag{2.7}$$

All terms on the right hand side of (2.7) are known, and so solutions for X_s and A_s modulo $C_{n+1}(\mathcal{L}_\mathcal{K})$ can be recovered, thus specifying $(\mathrm{id}_\mathcal{L} \otimes \widehat{h})(f) \bmod C_{n+1}(\mathcal{L}_{\mathcal{K}_{<p}})$.

2.5 Ramification filtration of $\mathcal{G}_{<p}$

Let $\mathcal{K}$ be a local field of characteristic p, and let η_0 be the identification from section 2.3. Although not central to our main approach, we present here the description of the ramification filtration of $\mathcal{G}_{<p}$ with respect to the identification η_0.

For $v \geqslant 0$ let $\mathcal{G}_{<p}^{(v)}$ denote the ramification subgroups of $\mathcal{G}_{<p}$ in the upper numbering. Under the Lazard correspondence, for all $v \geqslant 0$ we have $\eta_0(\mathcal{G}_{<p}^{(v)}) = G(\mathcal{L}^{(v)})$ where $\mathcal{L}^{(v)} \subset \mathcal{L}$ are ideals of $\mathcal{L}$. The nilpotent Artin-Schreier theory provides an explicit description of the generators of these ideals.

Definition 2.5. For any $\gamma \geqslant 0$ and $n \in \mathbb{N}$ we define the element $\mathcal{F}^0_{\gamma,-n} \in \mathcal{L}_k$ as

$$\mathcal{F}^0_{\gamma,-n} = \sum_{\substack{1 \leqslant s < p \\ a_i, n_i}} \eta(n_1, \dots, n_s)[\dots[a_1 D_{a_1,n_1}, D_{a_2,\bar{n}_2}], \dots, D_{a_s,n_s}],$$

where the sum is taken over all $a_i \in \mathbb{Z}^0(p)$ and $0 \geqslant n_i \geqslant -n$ such that,

— $a_1 p^{n_1} + \dots + a_s p^{n_s} = \gamma$;

— If $0 = n_1 = \dots = n_{s_1} > \dots > n_{s_{m-1}+1} = \dots = n_{s_m} \geqslant -n$ (where $s_m = s$), then $\eta(n_1, \dots n_s) = (s_1! \cdots (s_m - s_{m-1}))^{-1}$, and $\eta(n_1, \dots, n_s) = 0$ otherwise.

Proposition 2.6. [1, Thm. B] *For any $v \geqslant 0$, let $G(\mathcal{L}^{(v)}) := \eta_0(\mathcal{G}_{<p}^{(v)})$. Then $\mathcal{L}^{(v)} = \mathcal{L}_k^{(v)}\lceil_{\sigma=\mathrm{id}}$, where $\mathcal{L}_k^{(v)}$ is an ideal of $\mathcal{L}_k$. For any $v \geqslant 0$ there is a natural number $\widetilde{N}(v)$ such that if $N \geqslant \widetilde{N}(v)$ is fixed, then $\mathcal{L}_k^{(v)}$ is generated by $\{\sigma^n(\mathcal{F}^0_{\gamma,-N}) : \gamma \geqslant v,\, n \in \mathbb{Z}/N_0\}$.*

Proof. The proof is given in [1] in the context of the contravariant theory. The result is adapted to the covariant theory by replacing the terms $[D_{a_s,\bar{n}_s}, \dots, [D_{a_2,\bar{n}_2}, D_{a_1,\bar{n}_1}]\dots]$ with $[\dots[D_{a_1,\bar{n}_1}, D_{a_2,\bar{n}_2}], \dots, D_{a_s,\bar{n}_s}]$ and noting that $[D_{a_s,\bar{n}_s}, \dots, [D_{a_2,\bar{n}_2}, D_{a_1,\bar{n}_1}]\dots] = (-1)^{s-1}[\dots[D_{a_1,\bar{n}_1}, D_{a_2,\bar{n}_2}], \dots, D_{a_s,\bar{n}_s}]$. □

The elements $\mathcal{F}^0_{\gamma,-n}$ will appear in chapter 7 when comparing our main result with explicit calculations in [5].

3 Main setting and fundamental exact sequences

We now turn our attention to the main setting of this thesis. Let K be a finite extension of $\mathbb{Q}_p$, with residue field $k \simeq \mathbb{F}_{p^{N_0}}$, and absolute ramification index e_K. We assume that $\zeta_p \in K$, where ζ_p is a fixed primitive p-th root of unity, and we fix once and for all a uniformiser π_0 of K. Let $\overline{K}$ be an algebraic closure of K, and $\Gamma_K := \mathrm{Gal}(\overline{K}/K)$, the absolute Galois group of K.

Let $K_{<p}$ be the maximal p-extension of K of period p and nilpotence class less than p, then $\Gamma_{<p} := \mathrm{Gal}(K_{<p}/K) = \Gamma_K/C_p(\Gamma_K)\Gamma_K^p$. The main aim of this chapter is to present recent results [5] related to the structure of $\Gamma_{<p}$, which provides a basis for the calculations and results obtained in the later chapters.

Consider the field extension $K(\pi_1)/K$, where $\pi_1 \in \overline{K}$ is such that $\pi_1^p = \pi_0$. Since $\zeta_p \in K$, then by Kummer theory we have $\mathrm{Gal}(K(\pi_1)/K) \simeq \langle\tau_0\rangle^{\mathbb{Z}/p}$ where τ_0 is uniquely defined by $\tau_0(\pi_1) = \zeta_p\pi_1$. Clearly $K(\pi_1) \subset K_{<p}$, and thus the Galois correspondence gives us the following exact sequence of p-groups,

$$1 \longrightarrow \mathrm{Gal}(K_{<p}/K(\pi_1)) \longrightarrow \Gamma_{<p} \longrightarrow \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

If we fix a lift $\tau_{<p} \in \Gamma_{<p}$ of τ_0, then the sequence splits and the structure of $\Gamma_{<p}$ is determined by $\mathrm{Gal}(K_{<p}/K(\pi_1))$ and the action of $\tau_{<p}$ on $\mathrm{Gal}(K_{<p}/K(\pi_1))$ by conjugation.

In the coming sections we will follow [5] throughout. In particular, in section 3.1 we will construct an infinite field extension $\widetilde{K}/K$ and show that the group $\mathrm{Gal}(K_{<p}/K(\pi_1))$ can be identified with a quotient group of $\Gamma_{\widetilde{K}}$. In section 3.2 we will use the *field of norms* functor of Fontaine-Wintenberger to identify the group $\Gamma_{\widetilde{K}}$ with the absolute Galois group of a local field $\mathcal{K}$ of characteristic p. Thus the group $\mathrm{Gal}(K_{<p}/K(\pi_1))$ can be understood in terms of automorphisms of a local field of characteristic p, and we can apply techniques from the NAS theory to recover the structure of $\mathrm{Gal}(K_{<p}/K(\pi_1))$, and lifts $\tau_{<p} \in \Gamma_{<p}$ of τ_0 together with its action by conjugation on $\mathrm{Gal}(K_{<p}/K(\pi_1))$, thus recovering the group structure of $\Gamma_{<p}$.

3.1 Fundamental exact sequence

For all $n \in \mathbb{N}$, let $\pi_n \in \overline{K}$ be such that $\pi_n^p = \pi_{n-1}$. Let $\widetilde{K} = \bigcup_{n\in\mathbb{N}} K(\pi_n)$, and let $\Gamma_{\widetilde{K}} = \mathrm{Gal}(\overline{K}/\widetilde{K})$.

If we fix an embedding $\Gamma_{\widetilde{K}} \subset \Gamma_K$, then the natural epimorphism $\Gamma_K \to \Gamma_{<p}$ induces a continuous group homomorphism $\tilde{i} : \Gamma_{\widetilde{K}} \to \Gamma_{<p}$. Let $j : \Gamma_{<p} \to \langle\tau_0\rangle^{\mathbb{Z}/p}$ be the natural epimorphism, then we have the following result from [5].

Proposition 3.1. [5, prop 6.1] *The following sequence is exact,*

$$\Gamma_{\widetilde{K}} \xrightarrow{\ \tilde{i}\ } \Gamma_{<p} \xrightarrow{\ j\ } \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

Proof. Following the proof in [5]. The extension $\widetilde{K}/K$ is not Galois, so we introduce its Galois closure $\widetilde{K}' = \bigcup_{n\geqslant 1} K(\pi_n, \zeta_{p^n})$, where for all $n \in \mathbb{N}$ we define $\zeta_{p^n} \in \overline{K}$ such that $(\zeta_{p^n})^p = \zeta_{p^{n-1}}$. As we have chosen coherent systems of roots $\{\zeta_{p^n}\}_{n\in\mathbb{N}}$ and $\{\pi_n\}_{n\in\mathbb{N}}$ it follows from Kummer theory that $\Gamma_{\widetilde{K}'/K}$ is topologically generated by $\langle\sigma,\tau\rangle$ where for all $n \in \mathbb{N}$ we have $\tau(\pi_n) = \zeta_{p^n}\pi_n, \tau(\zeta_{p^n}) = \zeta_{p^n}, \sigma(\pi_n) = \pi_n$, and $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{1+ps_0}$ for some $s_0 \in \mathbb{Z}$, and $\sigma^{-1}\tau\sigma = \tau^{(1+ps_0)^{-1}}$.

Note that the extension $\widetilde{K}'/\widetilde{K}$ is a Galois extension with Galois group $\langle\sigma\rangle$, hence $\Gamma_{\widetilde{K}}$ is generated by $\Gamma_{\widetilde{K}'}$ and a lift $\widehat{\sigma} \in \Gamma_{\widetilde{K}}$ of $\sigma \in \Gamma_{\widetilde{K}'/\widetilde{K}}$. To prove the proposition then, we must show that the images of $\Gamma_{\widetilde{K}'}$ and $\widehat{\sigma}$ under the natural homomorphism $\Gamma_K \to \Gamma_{<p}$ generate the kernel of $\Gamma_{<p} \to \langle\tau_0\rangle^{\mathbb{Z}/p}$.

It follows from the actions of τ and σ above that $C_2(\langle\sigma,\tau\rangle) \subset \langle\tau^p\rangle$ and $\langle\sigma,\tau\rangle^p = \langle\sigma^p,\tau^p\rangle$, hence $\Gamma^p_{\widetilde{K}'/K}C_p(\Gamma_{\widetilde{K}'/K}) = \langle\sigma^p,\tau^p\rangle$ and we have a natural exact sequence

$$\langle\sigma\rangle \longrightarrow \Gamma_{\widetilde{K}'/K}/\Gamma^p_{\widetilde{K}'/K}C_p(\Gamma_{\widetilde{K}'/K}) \longrightarrow \langle\tau\rangle \mod\langle\tau^p\rangle \longrightarrow 1\,.$$

Since $\langle\tau\rangle \bmod\langle\tau^p\rangle = \langle\tau_0\rangle^{\mathbb{Z}/p}$ it follows from the above exact sequence that the images of $\Gamma_{\widetilde{K}'}$ and $\widehat{\sigma}$ under the natural homomorphism $\Gamma_K \to \Gamma_{<p}$ generate the kernel of $\Gamma_{<p} \to \langle\tau_0\rangle^{\mathbb{Z}/p}$. □

As a result of proposition 3.1 we can identify the image of $\Gamma_{\widetilde{K}}$ in $\Gamma_{<p}$ with the group $\mathrm{Gal}(K_{<p}/K(\pi_1))$. In the next section we will see that the group $\Gamma_{\widetilde{K}}$ can be interpreted as the absolute Galois group of a local field of characteristic p.

3.2 Relation to characteristic p

The *field of norms* was developed by Fontaine-Wintenberger in [10, 20]. This remarkable construction allows us, for a certain class of field extensions, to relate the Galois group of a local field of characteristic 0 with the Galois group of a local field of characteristic p.

In particular, an infinite field extension L/K is *arithmetically profinite* (APF) if the ramification groups $\Gamma_K^{(v)}\Gamma_L$ are open subgroups of Γ_K for all $v > 0$ (i.e. if the ramification groups are determined on finite field extensions). If K is a local field of characteristic 0, and L/K an infinite APF extension, then the field of norms theory gives a construction of a local field $\mathcal{X}_K(L)$ of characteristic p, and establishes an equivalence of categories between the algebraic extensions of L and the separable extensions of $\mathcal{X}_K(L)$. The following proposition shows that the field of norms theory can be applied to the extension $\widetilde{K}/K$.

Proposition 3.2. *The field extension $\widetilde{K}/K$ is arithmetically profinite.*

Proof. Let $K_0 := K$, and for all $n \geqslant 1$ let $K_n := K(\pi_n)$. For any $n \geqslant 0$ the extension K_{n+1}/K_n is a totally ramified extension of degree p. It is well known from Kummer theory that the extension K_{n+1}/K_n has the unique upper ramification break $\frac{p\,e_{K_n}}{p-1} = \frac{p^{n+1}e_{K_0}}{p-1}$ (here e_{K_n} denotes the absolute ramification index of K_n).

Using the notation of [20, 1.4.2], for any $n \geqslant 0$ let $i_n = \frac{p^{n+1}e_{K_0}}{p-1}$, and let $b_n = (i_0 + (i_1 - i_0)p^{-1} + \cdots + (i_n - i_{n-1})p^{-n})$. Clearly the sequence $\{i_n\}_{n\geqslant 0}$ is strictly increasing, and $\varinjlim_n i_n = +\infty$. Also, noting that for any $n \geqslant 1$, we have $i_n - i_{n-1} = p^n e_{K_0}$, then for all $n \geqslant 0$ we have $b_n = \frac{pe_{K_0}}{p-1} + (n-1)e_{K_0}$. Thus the sequence $\{b_n\}_{n\geqslant 0}$ is also strictly increasing, and $\varinjlim_n b_n = +\infty$.

Note that for any $n \geqslant 1$ the points $\{(i_m, b_m) : 0 \leqslant m \leqslant n\}$ are the edge points of the Herbrand function $\varphi_{K_n/K}$, and in particular b_n is the maximal upper ramification break for the extension K_n/K. Let $\{\Gamma_K^{(v)}\}_{v\geqslant 0}$ denote the ramification filtration of Γ_K in the upper numbering, then for any $v \geqslant 0$ the group $\Gamma_K^{(v)}$ acts trivially on K_n if and only if $v > b_n$. As $\{b_n\}_{n\geqslant 0}$ is strictly increasing and unbounded it follows that for any $v \geqslant 0$ we can choose some n large enough that $\Gamma_K^{(v)}$ acts non-trivially on K_m for all $m \geqslant n$. In particular, for any $v \geqslant 0$, if $K^{(v)}$ denotes the fixed field of $\Gamma_K^{(v)}$, then $K^{(v)} \cap \widetilde{K}$ is a finite extension of K, thus the group $\Gamma_K^{(v)}\Gamma_{\widetilde{K}}$ is of finite index in Γ_K, and hence is an open subgroup of Γ_K. By definition then, $\widetilde{K}/K$ is an arithmetically profinite extension. □

Let $\mathcal{K} := \mathcal{X}_K(\widetilde{K})$ denote the field of norms of the extension $\widetilde{K}/K$. Retaining temporarily the notation $K_0 := K$, and $K_n := K(\pi_n)$ for all $n \geqslant 1$, then we have the following construction of $\mathcal{K}$ [20, §2]. By definition, $\mathcal{K}^* = \varprojlim_n K_n^*$ where the limit is taken with respect to the norm maps. An element $\alpha \in \mathcal{K}$ is a norm compatible sequence of elements $(\alpha_n)_{n\geqslant 0}$ with $\alpha_n \in K_n$. For $\alpha, \beta \in \mathcal{K}$ multiplication is defined component-wise by $\alpha\beta = (\alpha_n\beta_n)_{n\geqslant 0}$, where $\alpha_n\beta_n$ denotes multiplication in the field K_n. The definition of addition is less straightforward. For $\alpha, \beta \in \mathcal{K}$, it is proved that the arithmetic profiniteness of $\widetilde{K}/K$ implies that, for any $n \geqslant 0$, the sequence $(N_{K_{n+m}/K_n}(\alpha_{n+m}+\beta_{n+m}))_{m\geqslant 0}$ converges to an an element $\gamma_n \in K_n$, and addition is then defined by setting $\alpha + \beta = (\gamma_n)_{n\geqslant 0}$.

With the operations defined above it is shown that $\mathcal{K}$ is a local field of characteristic p. Moreover, as $\widetilde{K}/K$ is totally ramified the residue field of $\mathcal{K}$ can be canonically identified with the residue field k of K. It follows that upon choosing a uniformiser t of $\mathcal{K}$ we obtain an identification $\mathcal{K} \simeq k((t))$, and we have a natural choice of uniformiser $t \in \mathcal{K}$ such that $t = (\pi_n)_{n\geqslant 0}$.

The field of norms has the following functorial properties [20, §3]. For any finite extension $F/\widetilde{K}$ in $\overline{K}$ the extension F/K is also arithmetically profinite, and its field of norms $\mathcal{X}_K(F)$ is a finite separable extension of $\mathcal{X}_K(\widetilde{K})$. If $F/\widetilde{K}$ is Galois then there is a canonical isomorphism $\mathrm{Gal}(F/\widetilde{K}) \simeq \mathrm{Gal}(\mathcal{X}_K(F)/\mathcal{X}_K(\widetilde{K}))$. Moreover, let $\mathcal{X}_K(\overline{K}) := \varinjlim \mathcal{X}_K(F)$ where the limit is taken over all finite extensions $F/\widetilde{K}$ in $\overline{K}$, then $\mathcal{X}_K(\overline{K})$ is a separable closure of $\mathcal{X}_K(\widetilde{K})$, and the field of norms theory establishes a canonical isomorphism $\Gamma_{\mathcal{X}_K(\widetilde{K})} \simeq \Gamma_{\widetilde{K}}$.

Using the notation $\mathcal{K} := \mathcal{X}_K(\widetilde{K})$, and recalling that $\mathcal{G}_{<p} := \Gamma_{\mathcal{K}}/\Gamma_{\mathcal{K}}^p C_p(\Gamma_{\mathcal{K}})$, then applying the field of norms theory we obtain the following.

Proposition 3.3. [5, prop 6.2] *The following sequence is exact,*

$$\mathcal{G}_{<p} \longrightarrow \Gamma_{<p} \longrightarrow \langle \tau_0 \rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

Proof. Recall from proposition 3.1 that we have the exact sequence

$$\Gamma_{\widetilde{K}} \xrightarrow{\tilde{i}} \Gamma_{<p} \xrightarrow{j} \langle \tau_0 \rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

By the field of norms functor we have an identification $\Gamma_{\widetilde{K}} \simeq \Gamma_{\mathcal{K}}$, and hence, as above, a natural continuous homomorphism $\tilde{i} : \Gamma_{\mathcal{K}} \to \Gamma_{<p}$. Clearly the subgroup $\Gamma_{\mathcal{K}}^p C_p(\Gamma_{\mathcal{K}})$ is in the kernel of $\tilde{i}$, and by factoring through we obtain an induced homomorphism $i : \mathcal{G}_{<p} \to \Gamma_{<p}$ such that the images $\tilde{i}(\Gamma_{\widetilde{K}})$ and $i(\mathcal{G}_{<p})$ coincide. □

Note that, if $i : \mathcal{G}_{<p} \to \Gamma_{<p}$ is the induced homomorphism from above, then as $\Gamma_{<p}$ is a finite group, the kernel of i is an open normal subgroup of $\mathcal{G}_{<p}$.

3.3 NAS identification

As $\mathcal{K}$ is a local field of characteristic p, we can apply the nilpotent Artin-Schreier theory outlined in section 2.3. In particular, we fix the uniformiser $t = (\pi_n)_{n \geqslant 0} \in \mathcal{K}$ as above, and fix $\alpha_0 \in k$ an element of trace 1, then we have the following identification (see proposition 2.3).

$$\eta_0 : \mathcal{G}_{<p} \simeq G(\mathcal{L}),$$

— where η_0 is induced by $\eta = \pi_{e,f}$ with $e = \sum_{a \in \mathbb{Z}^0(p)} t^{-a} D_{a0}$ and $f \in F(e)$;

— $\mathcal{L}_k$ has free generators $\{D_0\} \cup \{D_{an} : a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}$ over k.

We assume throughout that the elements e, f are fixed, thus fixing the identification η_0. With respect to the identification η_0, then directly from proposition 3.3 we obtain the following proposition.

Proposition 3.4. *The following sequence is exact,*

$$G(\mathcal{L}) \xrightarrow{\ i\ } \Gamma_{<p} \xrightarrow{\ j\ } \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

As a result of this, the group $\Gamma_{K_{<p}/K(\pi_1)}$ can be identified with a quotient group of $G(\mathcal{L})$, and in order to recover the identification we need to recover the kernel of the morphism $i : G(\mathcal{L}) \to \Gamma_{<p}$.

For $2 \leqslant s \leqslant p$, denote by $C_s(\Gamma_{<p})$ the commutator subgroups of $\Gamma_{<p}$, and let $i_s : G(\mathcal{L}) \to \Gamma_{<p}/C_s(\Gamma_{<p})$ be the morphisms induced by i. Clearly, for $2 \leqslant s \leqslant p$ we have that $C_s(\Gamma_{<p}) \subset \Gamma_{K_{<p}/K(\pi_1)}$, and hence, with respect to the identification η_0, we have that $\mathrm{Ker}(i_s) = G(\mathcal{L}(s))$ for some ideal $\mathcal{L}(s)$ of $\mathcal{L}$. In particular, $G(\mathcal{L}(p))$ is the kernel of the homomorphism $\mathcal{G}_{<p} \to \Gamma_{<p}$.

The issue in the recovery of the ideals $\mathcal{L}(s)$ lies in the following. Suppose $\tau \in \Gamma_K$ is a lift of τ_0, then via the field of norms functor τ appears as an automorphism of $\mathcal{K}_{\mathrm{sep}}$ and we would like to utilise the techniques of the NAS theory from section 2.4.1 to recover the lifts of τ_0 and their action on $G(\mathcal{L})$. However as our APF extension $\widetilde{K}/K$ is not a normal extension, then if $\tau \in \Gamma_K$ is a lift of τ_0, its identification under the field of norms functor to an automorphism of $\mathcal{K}_{\mathrm{sep}}$ does not induce an automorphism of $\mathcal{K}$, and thus the techniques of the NAS theory cannot be applied directly.

The solution to this problem was given in [5]. The method used was to approximate τ_0 by an automorphism $h_0 \in \mathrm{Aut}(\mathcal{K})$. The techniques of section 2.4.1 can then be applied to h_0 to recover lifts $h_{<p} \in \mathrm{Aut}(\mathcal{K}_{<p})$. Recalling that, with respect to the identification η_0, a lift $h_{<p} \in \mathrm{Aut}(\mathcal{K}_{<p})$ of h_0 is determined by its action on the element f, it is shown that under suitable conditions lifts of h_0 to $\mathcal{K}_{<p}$ correspond to lifts of τ_0 to $K_{<p}$. This allows the results obtained for lifts of h_0 to be applied to the lifts $\tau_{<p}$, and led to a recovery of the structure of $\Gamma_{<p}$.

The techniques used to establish the validity of the approximation are beyond the scope of this thesis. Therefore we state the relevant results in the next section, and give only a brief overview of the methods used in [5].

3.4 Structure of $\Gamma_{<p}$

To define the ideals $\mathcal{L}(s)$ we use the following weight function for generators of $\mathcal{L}_k$ from [5].

Definition 3.5. For any $D_{an} \in \mathcal{L}_k$, let $\mathrm{wt}(D_{an}) = s$ if $(s-1)e^* \leqslant a < se^*$.

Let $\mathcal{L}(s)_k$ be the ideal generated by all monomials $[\dots[D_{a_1,n_1}, D_{a_2,n_2}], \dots, D_{a_r,n_r}]$ such that $\sum_{1\leqslant i\leqslant r} \mathrm{wt}(D_{a_i,n_i}) \geqslant s$. Then $\mathcal{L}(1)_k = \mathcal{L}_k$, and for all $s_1, s_2 \in \mathbb{N}$ we have $\mathcal{L}(s_1)_k \subset \mathcal{L}(s_2)_k \iff s_1 \leqslant s_2$, and $[\mathcal{L}(s_1)_k, \mathcal{L}(s_2)_k] \subset \mathcal{L}(s_1+s_2)_k$. Thus the ideals $\mathcal{L}(s)_k$ give a decreasing filtration on $\mathcal{L}_k$, and induce a corresponding filtration $\mathcal{L}(s)$ on $\mathcal{L}$ where $\mathcal{L}(s) := \mathcal{L}(s)_k\restriction_{\sigma=\mathrm{id}}$.

Recall from proposition 3.4 that we have the following exact sequence,

$$G(\mathcal{L}) \xrightarrow{i} \Gamma_{<p} \xrightarrow{j} \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,. \tag{3.1}$$

For any $1 \leqslant s \leqslant p$, let $i_s : G(\mathcal{L}) \to \Gamma_{<p}/C_s(\Gamma_{<p})$ be the morphism induced by i.

Proposition 3.6. [5, §6.5] *The following sequence is exact for all $2 \leqslant s \leqslant p$,*

$$1 \longrightarrow G(\mathcal{L})/G(\mathcal{L}(s)) \xrightarrow{i_s} \Gamma_{<p}/C_s(\Gamma_{<p}) \xrightarrow{j} \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

In particular, $\mathrm{Ker}(G(\mathcal{L}) \to \Gamma_{<p}) = G(\mathcal{L}(p))$ therefore $G(\mathcal{L}/\mathcal{L}(p)) \simeq \Gamma_{K_{<p}/K(\pi_1)}$, and we obtain from (3.1) the following short exact sequence of p-groups,

$$1 \longrightarrow G(\mathcal{L}/\mathcal{L}(p)) \xrightarrow{i_p} \Gamma_{<p} \xrightarrow{j} \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,.$$

We also have, for all $2 \leqslant s < p$ that $C_s(\Gamma_{<p}) \simeq G(\mathcal{L}(s))/G(\mathcal{L}(p))$.

We give the following sketch of the methods used in [5]. Recall that $\zeta_p \in K$ is a fixed primitive p-th root of unity. As π_0 is a fixed uniformiser of K, then we have a unique expansion $\zeta_p = 1 + \sum_{i\geqslant \frac{e_K}{p-1}} \beta_i \pi_0^i$ where $\beta_i \in k$. Recall that e_K denotes the absolute ramification index of K, and note that $v_{\pi_0}(\zeta_p - 1) = \frac{e_K}{p-1}$ implies that $\beta_{e_K/p-1} \neq 0$.

Let $h_0 \in \mathrm{Aut}(\mathcal{K})$ be such that h_0 acts as the identity on the residue field of $\mathcal{K}$ and $h_0(t) = t(1 + \sum_{i \geqslant \frac{e_K}{p-1}} \beta_i^p t^{ip})$, where the coefficients $\beta_i \in k$ are given by the fixed expansion of $\zeta_p \in K$ above. For convenience in future calculations we write the automorphism h_0 in the following form $h_0(t) = t(1 + \sum_{j \geqslant 0} \widetilde{\alpha}_j t^{e^*+pj})$, where for any $j \geqslant 0$, $\widetilde{\alpha}_j = \beta_i^p$ with $i = \frac{e_K}{p-1} + j$, and $\widetilde{\alpha}_0 \neq 0$.

Let $\widetilde{\mathcal{G}}_{h_0} := \{h \in \mathrm{Aut}(\mathcal{K}_{<p}) : h\restriction_{\mathcal{K}} \in \langle h_0 \rangle\}$. Recall that $\mathcal{G}_{<p} = \mathrm{Gal}(\mathcal{K}_{<p}/\mathcal{K})$, then clearly $\mathcal{G}_{<p} \subset \widetilde{\mathcal{G}}_{h_0}$, and with respect to the identification η_0 we obtain the natural short exact sequence of profinite p-groups,

$$1 \longrightarrow G(\mathcal{L}) \longrightarrow \widetilde{\mathcal{G}}_h \longrightarrow \langle h_0 \rangle \longrightarrow 1\,. \tag{3.2}$$

The group $\widetilde{\mathcal{G}}_h$ was studied in [5, §2.4]. By the formalism of the nilpotent Artin-Schereier theory from section 2.4.1, lifts $\tilde{h}_0 \in \widetilde{\mathcal{G}}_h$ of $h_0 \in \mathrm{Aut}(\mathcal{K})$ are uniquely determined by solutions $(A_{\tilde{h}_0}, c_{\tilde{h}_0})$ of the recurrence relation $(\mathrm{id}_{\mathcal{L}} \otimes h_0)(e) \circ c = \sigma c \circ (A_{\tilde{h}_0} \otimes \mathrm{id}_{\mathcal{K}})(e)$ (see [5, Proposition 2.3]). As we saw in section 2.4.1, the automorphism $A_{\tilde{h}_0} \in \mathrm{Aut}(G(\mathcal{L}))$ corresponds to conjugation by $\tilde{h}_0$ on elements of $G(\mathcal{L})$, thus we will use the more suggestive notation $\mathrm{Ad}_{\tilde{h}_0}$.

Noting that modulo $C_2(\mathcal{L})_{\mathcal{K}}$ the Campbell-Hausdorff group law corresponds to addition in $\mathcal{L}_{\mathcal{K}}$, then the first step in the recovery of a solution of the recurrence relation is to obtain a solution of the following congruence.

$$\sigma c - c + \sum_{a \in \mathbb{Z}^0(p)} t^{-a}\,\mathrm{Ad}_{\tilde{h}_0}(D_{a0}) \equiv \sum_{\substack{a \in \mathbb{Z}^0(p) \\ i \geqslant 0}} h_0(t^{-a}) D_{a0} \mod C_2(\mathcal{L})_{\mathcal{K}}$$

Expanding the right-hand side, it was recovered that $\mathrm{Ad}_{\tilde{h}_0}(D_{00}) \equiv D_{00} \bmod C_2(\mathcal{L})_k$, and for any $a \in \mathbb{Z}^+(p)$, $\mathrm{Ad}_{\tilde{h}_0}(D_{a0}) \equiv D_{a0} - \sum_{j \geqslant 0} \widetilde{\alpha}_j a D_{e^*+pj+a,0} \bmod C_2(\mathcal{L})_k$ (cf. analogous calculations in section 4.1 below).

Using that $(\tau_{<p} \circ l \circ \tau_{<p}^{-1} \circ l^{-1}) \equiv \mathrm{Ad}_{\tau_{<p}}(l) - l \bmod C_2(G(\mathcal{L}_k))$ then it was recovered from the above that for all $s \geqslant 2$, $C_s(\widetilde{\mathcal{G}}_h) \bmod C_2(G(\mathcal{L}_k))$ is generated by all D_{an} such that $a \geqslant (s-1)e^*$. Using general properties of $\mathrm{Ad}_{\tilde{h}_0}$, this was extended to recover that for any $s \geqslant 2$ the commutator subgroups of $\widetilde{\mathcal{G}}_h$ are given by $C_s(\widetilde{\mathcal{G}}_h) = G(\mathcal{L}(s))$, where for all $s \geqslant 2$, $\mathcal{L}(s) \subset \mathcal{L}$ are the ideals given

in definition 3.5. It was shown that exact sequence (3.2) induces the following exact sequence [5, proposition 2.7].

$$1 \longrightarrow G(\mathcal{L})/G(\mathcal{L}(p)) \longrightarrow \mathcal{G}_{h_0} \longrightarrow \langle h_0\rangle \operatorname{mod}\langle h_0^p\rangle \longrightarrow 1\,, \tag{3.3}$$

where $\mathcal{G}_{h_0} := \widetilde{\mathcal{G}}_{h_0}/\widetilde{\mathcal{G}}_{h_0}^p C_p(\widetilde{\mathcal{G}}_{h_0})$.

Finally, in [5, §6] by comparing the action of $\mathcal{G}_{h_0}$ on the element f of the NAS identification η_0 with the action on f of the image of $\Gamma_{<p}$ under the field of norms functor, it was proved that if $h_0(t)$ is defined in terms of the fixed expansion of ζ_p as above, then the groups $\mathcal{G}_{h_0}$ and $\Gamma_{<p}$ are isomorphic, and thus all properties of $\mathcal{G}_{h_0}$ established in [5, §2] are valid for the group $\Gamma_{<p}$.

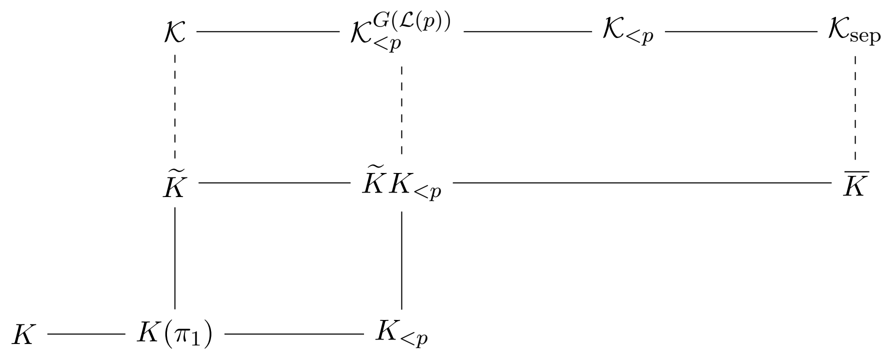

Figure 1: Diagram of relevant field extensions, where dashed lines indicate a correspondence under the field of norms functor $\mathcal{X}_{\widetilde{K}/K}$.

3.5 Linearisation

The results in the previous subsection were obtained in [5] without the need to specify a lift $\tau_{<p}$ explicitly. Although we can recover $\tau_{<p}$ from the relation $(\mathrm{id}_{\mathcal{L}} \otimes h_0)(e) \circ c = \sigma c \circ (A \otimes \mathrm{id}_{\mathcal{K}})(e)$, the complicated form of the Campbell-Hausdorff group law makes it very difficult to do so. In [5] an analogue of the recurrence relation was recovered purely in terms of Lie algebras.

Let L be the $\mathbb{F}_p$-Lie algebra such that $G(L) = \Gamma_{<p}$ under the Lazard correspondence, and let $\mathbb{F}_p\tau_0$ be the trivial $\mathbb{F}_p$-Lie algebra corresponding to $\langle\tau_0\rangle^{\mathbb{Z}/p}$,

then from proposition 3.6 we obtain the following short exact sequence of p-groups,

$$1 \longrightarrow G(\mathcal{L}/\mathcal{L}(p)) \longrightarrow G(L) \longrightarrow G(\mathbb{F}_p\tau_0) \longrightarrow 1\,. \qquad (3.4)$$

By the properties of the Lazard correspondence, exact sequence (3.4) induces the following short exact sequence of Lie algebras,

$$0 \longrightarrow \mathcal{L}/\mathcal{L}(p) \longrightarrow L \longrightarrow \mathbb{F}_p\tau_0 \longrightarrow 0\,. \qquad (3.5)$$

If $\tau_{<p} \in G(L)$ is a lift of τ_0, then the structure of $G(L) = \Gamma_{<p}$ is determined by $\mathrm{Ad}_{\tau_{<p}} \in \mathrm{Aut}(G(\mathcal{L}/\mathcal{L}(p)))$. By the properties of the Lazard correspondence, $\mathrm{Ad}_{\tau_{<p}}$ is automatically an automorphism of $\bar{\mathcal{L}} := \mathcal{L}/\mathcal{L}(p)$. Moreover, by explicit calculations in [5, Lemma 2.6] the automorphism $\mathrm{Ad}_{\tau_{<p}}$ is a unipotent automorphism of $\bar{\mathcal{L}}$ (see also sketch in previous section). Therefore, as $\bar{\mathcal{L}}$ has nilpotence class less than p, we can relate $\mathrm{Ad}_{\tau_{<p}} \in \mathrm{Aut}(\bar{\mathcal{L}})$ to a derivation $\mathrm{ad}_{\tau_{<p}} \in \mathrm{Der}(\bar{\mathcal{L}})$ via the truncated exponential i.e. $\widetilde{\exp}(\mathrm{ad}_{\tau_{<p}}) = \mathrm{Ad}_{\tau_{<p}}$.

Although there is a natural relation between $\mathrm{Ad}_{\tau_{<p}}$ and $\mathrm{ad}_{\tau_{<p}}$, this does not immediately free us from the use of the Campbell-Hausdorff group law, as we are still reliant on the recovery of solutions of the recurrence relation $(\mathrm{id}_{\mathcal{L}} \otimes h_0)(e) \circ c = \sigma c \circ (A \otimes \mathrm{id}_{\mathcal{K}})(e)$ to specify a lift $\tau_{<p}$ and the corresponding automorphism $\mathrm{Ad}_{\tau_{<p}}$. The major simplification in [5] was the recovery of an analogue of the recurrence relation given purely in terms of the Lie structure of L, which we present below.

We set $h_0(t) = t(1 + \sum_{j\geqslant 0} \widetilde{\alpha}_j t^{e^*+pj})$, where for any $j \geqslant 0$, $\widetilde{\alpha}_j = \beta_i^p$ with $i = \frac{e_K}{p-1} + j$, and $\widetilde{\alpha}_0 \neq 0$ (recall that the β_i correspond to our fixed expansion of ζ_p).

Let $\omega_\tau \in t^{e_K/(p-1)}\mathcal{O}_{\mathcal{K}}^*$ be such that $1 + \sum_{j\geqslant 0} \widetilde{\alpha}_j t^{e^*+pj} \equiv \widetilde{exp}(\omega_\tau^p) \bmod t^{e^*p}$. Then $h_0(t) \equiv \widetilde{exp}(\omega_\tau^p) \bmod t^{e^*p+1}$ (see [5, Proposition 2.1]).

Definition 3.7. Let $\mathcal{M} := \sum_{1\leqslant s<p} t^{-se^*}\mathcal{L}(s)_{\mathrm{m}} + \mathcal{L}(p)_{\mathcal{K}}$, where m denotes the maximal ideal of the valuation ring of $\mathcal{K}$. Then $\mathcal{M}$ is a Lie-subalgebra of $\mathcal{L}_{\mathcal{K}}$. For any $i \geqslant 0$, let $\mathcal{M}(i) := t^{ie^*}\mathcal{M}$, then $\{\mathcal{M}(i)\}_{i\geqslant 0}$ is a decreasing filtration of ideals of $\mathcal{M}$.

Proposition 3.8. [5, Proposition 3.7] *For any $a \in \mathbb{Z}^0(p)$ let $V_{a0} := \mathrm{ad}_{\tau_{<p}}(D_{a0})$. The set of lifts $\tau_{<p} \in \Gamma_{<p}$ of τ_0 are in bijection with the set of solutions $\left\{(\bar{c}_1, \{V_{a0}\}_{a\in\mathbb{Z}^0(p)})\right\}$ of the following recurrence formula modulo $\mathcal{M}(p-1)$.*

$$\sigma\bar{c}_1 - \bar{c}_1 + \sum_{a\in\mathbb{Z}^0(p)} t^{-a}V_a \equiv$$

$$-\sum_{\substack{1\leqslant s<p\\ j\geqslant 0}}\sum_{a_1,\dots,a_s}\frac{1}{s!}t^{-(a_1+\cdots+a_s)}\omega_\tau^p[\dots[a_1D_{a_1,0}, D_{a_2,0}],\dots,D_{a_s,0}] \tag{3.6a}$$

$$-\sum_{2\leqslant s<p}\sum_{a_1,\dots,a_s}\frac{1}{s!}t^{-(a_1+\cdots+a_s)}[\dots[V_{a_1,0}, D_{a_2,0}],\dots,D_{a_s,0}] \tag{3.6b}$$

$$-\sum_{2\leqslant s<p}\sum_{a_2,\dots,a_s}\frac{1}{(s-1)!}t^{-(a_2+\cdots+a_s)}[\dots[\sigma\bar{c}_1, D_{a_2,0}],\dots,D_{a_s,0}] \tag{3.6c}$$

where the indices $a_1,\dots,a_s$ in the above sums run over $\mathbb{Z}^0(p)$.

Proof. This formula was established in [5, §3.5] in the context of lifts of h_0, with $h_0(t) \equiv t(\widetilde{\exp}(\omega_\tau^p)) \bmod t^{pe^*+1}$. The solutions of the formula, $(\bar{c}_1, \{V_{a0}\}_{a\in\mathbb{Z}^0(p)})$, are shown to be in bijection with the set of images of lifts $h_{<p}$ in $\mathcal{G}_{h_0}$ [5, remark after proposition 3.7] . With h_0 chosen as above then by [5, proposition 6.4] the images of lifts of h_0 in $\mathcal{G}_{h_0}$ correspond precisely to the lifts $\tau_{<p} \in \Gamma_{<p}$ of τ_0, thus the result can be given in terms of $\tau_{<p}$ as above. □

Note that $\mathrm{ad}_{\tau_{<p}} \in \mathrm{End}(\mathcal{L}/\mathcal{L}(p))$, and hence for any $D_{an} \in \mathcal{L}_k$ we have $\mathrm{ad}_{\tau_{<p}}(D_{an}) = \sigma^n(\mathrm{ad}_{\tau_{<p}}(D_{a0})) = \sigma^n(V_{a0})$. Thus the elements V_{a0} fully determine the derivation $\mathrm{ad}_{\tau_{<p}}$, and hence the structure of L via the short exact sequence (3.5).

3.6 Summary

Within the context of this thesis we cannot give a full account of the results of [5] with regard to the approximation of τ_0 by h_0, and the process of linearisation. Therefore, we give the following summary to fix the ideas relevant to thesis.

1. We should like to investigate the structure of $\Gamma_{<p}$ via the exact sequence

$$1 \longrightarrow \mathrm{Gal}(K_{<p}/K(\pi_1)) \longrightarrow \Gamma_{<p} \longrightarrow \langle\tau_0\rangle^{\mathbb{Z}/p} \longrightarrow 1\,,$$

   where $\tau_0(\pi_1) = \zeta_p\pi_1$, and $\zeta_p = 1 + \sum_{i\geqslant 1}\beta_i\pi_0^i$ with $\beta_i \in k$. The group $\Gamma_{<p}$ is determined by a lift $\tau_{<p} \in \Gamma_{<p}$ of τ_0, and $\mathrm{Ad}_{\tau_{<p}}$.

2. Via the field of norms functor and the techniques of the nilpotent Artin-Schreier theory we have the following exact sequence (see (3.4)),

$$1 \longrightarrow G(\mathcal{L}/\mathcal{L}(p)) \longrightarrow G(L) \longrightarrow G(\mathbb{F}_p\tau_0) \longrightarrow 1\,.$$

   The structure of $\Gamma_{<p} = G(L)$ is determined by a solution $(\mathrm{Ad}_{\tau_{<p}}, c)$ of the recurrence relation $(\mathrm{id}_{\mathcal{L}} \otimes h_0)(e) \circ c = \sigma c \circ (A \otimes \mathrm{id}_{\mathcal{K}})(e)$, where $h_0(t) = t(1 + \sum_{i\geqslant 1}\beta_i^p t^{ip})$. The solutions of the recurrence relation are in bijection with the set of lifts $\tau_{<p} \in \Gamma_{<p}$ of τ_0 [5, Proposition 2.3].

3. In [5, §3] it was established that we can replace all involved group structures with the corresponding Lie structures (see (3.5)),

$$0 \longrightarrow \mathcal{L}/\mathcal{L}(p) \longrightarrow L \longrightarrow \mathbb{F}_p\tau_0 \longrightarrow 0\,.$$

   The structure of L is determined by a solution $(\mathrm{ad}_{\tau_{<p}}, c_1)$ of recurrence formula (3.6), with $\omega_\tau \in t^{e_K/(p-1)}\mathcal{O}_{\mathcal{K}}^*$ such that $h_0(t) \equiv t(\widetilde{\exp}(\omega_\tau^p)) \bmod t^{pe^*+1}$.

4. Moreover, any solution $(\mathrm{ad}_{\tau_{<p}}, c_1)$ of recurrence formula (3.6) corresponds to a unique solution $(\mathrm{Ad}_{\tau_{<p}}, c)$ of $(\mathrm{id}_{\mathcal{L}} \otimes h_0)(e) \circ c = \sigma c \circ (A \otimes \mathrm{id}_{\mathcal{K}})(e)$.

Thus, by the results of [5], recovering the structure of $\Gamma_{<p}$ is equivalent to recovering a solution of recurrence formula (3.6). This is the main aim of this thesis, and a solution of (3.6) is given in theorem 7.4.

Unless otherwise specified, we will assume throughout the remainder of this thesis that we are working with a fixed choice of field K, with fixed uniformiser π_0 and fixed primitive p-th root of unity $\zeta_p \in K$. Similarly, we will assume that the identification η_0 of the nilpotent Artin-Schreier theory is fixed.

4 Working with the recurrence relation

In this section we outline a recurrent procedure that will allow us to recover a solution of recurrence relation (3.6), and hence recover the structure of L. We use the continuous $\mathbb{F}_p$-linear operators $\mathcal{R}$ and $\mathcal{S}$ from [5], and we recall briefly their definitions and basic properties.

Suppose that $\mathfrak{M}$ is a profinite $\mathbb{F}_p$-module, then the continuous $\mathbb{F}_p$-linear operators $\mathcal{R}, \mathcal{S} : \mathfrak{M}_{\mathcal{K}} \to \mathfrak{M}_{\mathcal{K}}$ are defined as follows. (Recall in (ii) below, that $k \simeq \mathbb{F}_{p^{N_0}}$, and $\alpha_0 \in k$ is such that $\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1$).

Definition 4.1. [5, §2.2] For any $b \in \mathfrak{M}_k$ and $n \in \mathbb{Z}$,

(i) If $n > 0$ then $\mathcal{R}(bt^n) = 0$, and $\mathcal{S}(bt^n) = -\sum_{i\geqslant 0} \sigma^i(bt^n)\,;$

(ii) If $n = 0$ then $\mathcal{R}(b) = \alpha_0 \mathrm{Tr}_{k/\mathbb{F}_p}(b)$, and $\mathcal{S}(b) = \sum_{0\leqslant j<i<N_0} \sigma^j(\alpha_0\sigma^i(b))\,;$

(iii) If $n < 0$ and $v_p(n) = k$, then $\mathcal{R}(bt^n) = \sigma^{-k}(bt^n)$, and $\mathcal{S}(bt^n) = \sum_{0\leqslant i<k} \sigma^{-i}(bt^n)\,.$

Proposition 4.2. [5, Lemma 2.2] *For any $b \in \mathfrak{M}_{\mathcal{K}}$ we have,*
(a) $\mathcal{R}(b) + (\sigma - \mathrm{id}_{\mathfrak{M}_{\mathcal{K}}})(\mathcal{S}(b)) = b\,;$
(b) *If $b = b_1 + \sigma b_2 - b_2$, where $b_1 \in \sum_{a\in\mathbb{Z}^0(p)} t^{-a}\mathfrak{M}_k + \alpha_0\mathfrak{M}$ and $b_2 \in \mathfrak{M}_{\mathcal{K}}$, then $b_1 = \mathcal{R}(b)$ and $b_2 - \mathcal{S}(b) \in \mathfrak{M}$.*

When $n \neq 0$, then definition of the operators $\mathcal{R}$ and $\mathcal{S}$ should be familiar from the classical theory of Artin-Schreier extensions. The case $n = 0$ is less straightforward and we include the following clarification from [5].

Let $b \in \mathcal{L}_k$. Recall that $\mathrm{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1$, then $b = \sum_{0\leqslant i<N_0} \sigma^i(\alpha_0)b$. For all $0 \leqslant i < N_0$ we set $\mathcal{R}_i(b) = \alpha_0\sigma^{-i}(b)$ and $\mathcal{S}_i = \sum_{0\leqslant j<i} \sigma^j(\mathcal{R}_i(b))$. Then

$$b = \sum_{0\leqslant i<N_0} \sigma^i(\alpha_0)b = \sum_{0\leqslant i<N_0} \sigma^i(\mathcal{R}_i(b)) = \sum_{0\leqslant i<N_0} ((\sigma - \mathrm{id})\mathcal{S}_i + \mathcal{R}_i)(b)\,.$$

We set $\mathcal{R}=\sum_{0\leqslant i<N_0}\mathcal{R}_i$, and $\mathcal{S}=\sum_{0\leqslant i<N_0}\mathcal{S}_i$, whence

$$\mathcal{S}(b)=\sum_{0\leqslant j<i<N_0}\sigma^j(\alpha_0\sigma^{-i}(b))=\sum_{0<i<N_0}\Big(\sum_{0\leqslant j<N_0-i}\sigma^j(\alpha_0)\Big)\sigma^{-i}(b)\,.$$

The operators $\mathcal{R}$ and $\mathcal{S}$ are used in the following recurrent procedure. Suppose that for some $2\leqslant l<p$ we have recovered recurrence relation (3.6) as

$$\sigma\bar{c}_1-\bar{c}_1+\sum_{a\in\mathbb{Z}^0(p)}V_{a0}\equiv\mathcal{T}_l \mod \mathcal{M}(p-1)+C_l(\mathcal{L}_\mathcal{K})\,.$$

Then by proposition 4.2 we can apply the operators $\mathcal{R}$ and $\mathcal{S}$ to the right-hand side to obtain the following expressions.

$$\bar{c}_1\equiv\mathcal{S}(\mathcal{T}_l) \mod \mathcal{M}(p-1)+C_l(\mathcal{L}_\mathcal{K})\,,$$

$$\sum_{a\in\mathbb{Z}^0(p)}t^{-a}V_{a0}\equiv\mathcal{R}(\mathcal{T}_l) \mod \mathcal{M}(p-1)+C_l(\mathcal{L}_\mathcal{K})\,.$$

The first congruence provides an expression for $\bar{c}_1 \bmod \mathcal{M}(p-1)+C_l(\mathcal{L}_\mathcal{K})$ directly. In the second congruence, we note, from the definition of $\mathcal{R}$, that all terms appearing on the right-hand side are of the form $t^{-a'}b$ with $b\in\bar{\mathcal{L}}_k$ and $a'\in\mathbb{Z}^0(p)$. By equating terms with equal exponent of t in this expression we can recover $V_{a0} \bmod \mathcal{L}(p)_k+C_l(\mathcal{L}_k)$ for all $a\in\mathbb{Z}^0(p)$.

Once we have recovered a solution $(\bar{c}_1,\{V_{a0}\}_{a\in\mathbb{Z}^0(p)})$ modulo $\mathcal{M}(p-1)+C_l(\mathcal{L}_\mathcal{K})$, then we can recover the right-hand side of (3.6) explicitly modulo $\mathcal{M}(p-1)+C_{l+1}(\mathcal{L}_\mathcal{K})$, and repeating this process allows us to recover a full solution of (3.6).

Note that there is a non-unique choice of $\bar{c}_1$ at each step in the above process. The non-unique choice of $\bar{c}_1$ relates to the non-uniqueness of the choice of lift, and choosing $\bar{c}_1$ is equivalent to fixing a choice of lift.

4.1 Explicit calculations modulo $\mathcal{M}(p-1)+C_2(\mathcal{L}_\mathcal{K})$

As a brief illustration of the method we recover a solution of recurrence relation (3.6) modulo $\mathcal{M}(p-1)+C_2(\mathcal{L}_\mathcal{K})$. This case was stated in [5].

Note that $\omega_\tau^p \in t^{e^*}\mathcal{O}_{\mathcal{K}}^*$, therefore we have $\omega_\tau^p = \sum_{j\geqslant 0} A_j t^{e^*+pj}$, where $A_j \in k$ for all $j \geqslant 0$, and $A_0 \neq 0$.

Proposition 4.3. *We have the following congruences.*

$$V_0 \equiv 0 \mod \mathcal{L}(p) + C_2(\mathcal{L}),$$

$$V_{a0} \equiv -\sum_{j\geqslant 0} A_j a D_{a+e^*+pj,0} \mod \mathcal{L}(p)_k + C_2(\mathcal{L}_k), \quad \text{for } a \in \mathbb{Z}^+(p).$$

Proof. Consider recurrence relation (3.6). Sums (3.6b) and (3.6c) contain only terms of length greater than one, so that modulo $C_2(\mathcal{L}_{\mathcal{K}})$ the recurrence relation is given by

$$\sigma\bar{c}_1 - \bar{c}_1 + \sum_{a\in\mathbb{Z}^0(p)} t^{-a}V_{a0} \equiv -\sum_{j\geqslant 0}\sum_{a_1\in\mathbb{Z}^0(p)} A_j t^{e^*+pj-a_1} a_1 D_{a_1,0}\,. \tag{4.1}$$

Noting that if $a_1 = 0$ then the corresponding term in the right-hand side of (4.1) is zero, then for all non-zero terms of the sum the exponent of t is non-zero and prime to p.

Applying the operator $\mathcal{S}$ to the right-hand side of (4.1) we obtain

$$\bar{c}_1 \equiv \sum_{\substack{j\geqslant 0\\ n\geqslant 0}}\sum_{a_1<e^*+pj} \sigma^n\left(A_j t^{e^*+pj-a_1} a_1 D_{a_1,0}\right) \mod C_2(\mathcal{L}_{\mathcal{K}}).$$

Let N^* be such that $p^{N^*} > e^*(p-1)$, then modulo $\mathcal{M}(p-1)$ we can assume that $n < N^*$ in the above sum, hence we have

$$\bar{c}_1 \equiv \sum_{\substack{j\geqslant 0\\ 0\leqslant n<N^*}}\sum_{a_1<e^*+pj} \sigma^n(A_j) t^{p^n(e^*+pj-a_1)} a_1 D_{a_1,n} \mod \mathcal{M}(p-1) + C_2(\mathcal{L}_{\mathcal{K}}). \tag{4.2}$$

Applying the operator $\mathcal{R}$ to the right-hand side of (4.1) we obtain

$$V_0 \equiv 0 \mod \mathcal{L}(p) + C_2(\mathcal{L}_k),$$

$$\sum_{a\in\mathbb{Z}^+(p)} t^{-a}V_{a0} \equiv -\sum_{\substack{a\in\mathbb{Z}^+(p)\\ j\geqslant 0}} A_j t^{-a} a D_{a+e^*+pj,0} \mod \mathcal{M}(p-1) + C_2(\mathcal{L}_{\mathcal{K}}).$$

By equating terms with equal exponents of t in this expression we recover that, for all $a \in \mathbb{Z}^+(p)$,

$$V_{a0} \equiv -\sum_{j\geqslant 0} A_j a D_{a+e^*+pj,0} \mod \mathcal{L}(p)_k + C_2(\mathcal{L}_k).$$

□

4.2 Considerations modulo higher degree commutators

It is quite possible to continue with these explicit calculations modulo higher degree commutators, however the calculations naturally become more complicated. Modulo $C_3(L_k)$ the calculations are reasonably simple as many of the terms in the recurrence relation can naturally be seen to belong to $C_3(L_k)$, e.g. terms of the form $[V_{a0}, D_{a_2,0}]$. Modulo $C_4(L_k)$ this is no longer the case, and in fact all complications for the general case already appear at this level. In particular, as we progress to higher degree commutators complications arise as the same term can arise in multiple ways with different coefficients. We can see this already if we consider the recurrence formula modulo $\mathcal{M}(p-1) + C_3(\mathcal{L}_{\mathcal{K}})$.

Consider the term $-\frac{1}{2!}A_0 t^{e^*-a_1-a_2} a_1 [D_{a_1 0}, D_{a_2 0}]$ appearing in (3.6a). If $a_1 > e^*$, then by our explicit calculations the term $-A_0 a_1 D_{a_1 0}$ appears as a term of $V_{e*-a_1,0}$ modulo $C_2(\mathcal{L}_k)$, and thus we obtain the term $\frac{1}{2!}A_0 t^{e^*-a_1-a_2} a_1 [D_{a_1 0}, D_{a_2 0}]$ in (3.6b). Of course, in this case the coefficients cancel, but for general terms this is not the case. For this reason the calculations modulo higher degree commutators are best treated with a more general approach, which we present in the next chapter.

5 Solving the recurrence relation

In this chapter and the next we will work with recurrence relation (3.6) under the assumption that $\omega_{\tau_0}^p = t^c$, where $c \in p\mathbb{N}$ and $c \geqslant e^*$. It is sufficient to work with this choice as all solutions of (3.6) depend σ-linearly on $\omega_{\tau_0}^p$ (see chapter 7). As such we have the following recurrence congruence modulo $\mathcal{M}(p-1)$ for the elements $\bar{c}_1 = c_1 \operatorname{mod} \mathcal{M}(p-1)$ and $V_{a0} := \operatorname{ad}_{\tau_{<p}}(D_{a0}) \operatorname{mod} \mathcal{L}(p)_k$, for $a \in \mathbb{Z}^0(p)$.

$$\sigma\bar{c}_1 - \bar{c}_1 + \sum_{a\in\mathbb{Z}^0(p)} t^{-a}V_{a0} \equiv$$

$$-\sum_{1\leqslant s<p}\sum_{a_1,\dots,a_s} \frac{1}{s!} t^{c-(a_1+\dots+a_s)}[\dots[a_1D_{a_1,0}, D_{a_2,0}],\dots,D_{a_s,0}] \tag{5.1a}$$

$$-\sum_{2\leqslant s<p}\sum_{a_1,\dots,a_s} \frac{1}{s!} t^{-(a_1+\dots+a_s)}[\dots[V_{a_1,0}, D_{a_2,0}],\dots,D_{a_s,0}] \tag{5.1b}$$

$$-\sum_{2\leqslant s<p}\sum_{a_2,\dots,a_s} \frac{1}{(s-1)!} t^{-(a_2+\dots+a_s)}[\dots[\sigma\bar{c}_1, D_{a_2,0}],\dots,D_{a_s,0}] \tag{5.1c}$$

where the indices $a_1,\dots,a_s$ in the above sums run over $\mathbb{Z}^0(p)$.

Note that relation (5.1) makes sense only in $\bar{\mathcal{M}} := \mathcal{M}/\mathcal{M}(p-1)$, but we will retain the notation D_{an} for the images of D_{an} in $\bar{\mathcal{M}}$.

Notation: Let $(\bar{a},\bar{n}) = (a_1, n_1, \dots, a_s, n_s)$, where $1 \leqslant s < p$, all $a_i \in \mathbb{Z}^0(p)$, and $n_i \in \mathbb{Z}$. We will also use the notation $\bar{a} = (a_1,\dots,a_s)$ and $\bar{n} = (n_1,\dots,n_s)$. If we want to indicate that our vectors have length s we will use the notation $(\bar{a},\bar{n})_s$. For any $(\bar{a},\bar{n})$ we also use the following notation.

$$D_{(\bar{a},\bar{n})} = a_1[\dots[D_{a_1,n_1}, D_{a_2,n_2}],\dots,D_{a_s,n_s}]\,,$$

$$\gamma(\bar{a},\bar{n}) = a_1p^{n_1} + \dots + a_sp^{n_s}\,.$$

We are going to find the coefficients of decompositions via $D_{(\bar{a},\bar{n})}$ of the elements $\sigma\bar{c}_1$, and V_{a0} for $a \in \mathbb{Z}^0(p)$ (we recover $\sigma\bar{c}_1$ rather than $\bar{c}_1$ as this is better suited to sum (5.1c) of the recurrence formula). As noted in the previous chapter, for any $(\bar{a},\bar{n})$ the term $D_{(\bar{a},\bar{n})}$ may appear in our recurrence formula in more than one way, and from more than one sum. For this reason, we will introduce the concept of admissible partitions of $(\bar{a}, \bar{n})$. Each admissible partition of $(\bar{a}, \bar{n})$ will correspond to a different occurrence of $D_{(\bar{a},\bar{n})}$ in the recurrence formula. Then $D_{(\bar{a},\bar{n})}$ appears in the above decompositions with non-zero coefficient only if $(\bar{a}, \bar{n})$ admits at least one admissible partition. For any $(\bar{a}, \bar{n})$ we will recover the contribution of each admissible partition to the appropriate coefficient of $D_{(\bar{a},\bar{n})}$, and prove that the coefficient of $D_{(\bar{a},\bar{n})}$ is equal to the sum of those contributions.

There is a small but important subtlety with regard to the terms $D_{(\bar{a},\bar{n})}$. Recall that all D_{an} depend essentially on $n \bmod N_0$ (where $k \simeq \mathbb{F}_{p^{N_0}}$) i.e. for any $n \in \mathbb{Z}$ we have $D_{a,n+N_0} = D_{an}$, therefore it is quite possible that two distinct vectors $(\bar{a}, \bar{n})$ and $(\bar{a}', \bar{n}')$ describe the same Lie monomial, e.g. (a_1, N_0) and $(a_1, 0)$. However, we will present all decompositions in terms of vectors $(\bar{a}, \bar{n})$, with associated terms $D_{(\bar{a},\bar{n})}$, and by indexing the decompositions in this way we treat the elements $D_{(\bar{a},\bar{n})}$ as though they were distinct. Note that this is necessary in order to understand how a term arises in the recurrence formula.

5.1 Basic objects and properties

For any vector $(\bar{a}, \bar{n})$ we will be considering the associated terms $D_{(\bar{a},\bar{n})}$ in (5.1) modulo $\mathcal{M}(p-1)$. As $\mathcal{L}(p) \subset \mathcal{M}(p-1)$ we need only consider vectors $(\bar{a}, \bar{n})$ such that $\mathrm{wt}(D_{(\bar{a},\bar{n})}) < p$. Noting from definition 3.5 that for any $a \in \mathbb{Z}^0(p)$, $\mathrm{wt}(D_{an}) = [a/e^*] + 1$ we introduce the following set of vectors.

Definition 5.1. Let $\mathcal{A}(e^*)$ be the set of vectors $(\bar{a}, \bar{n})_s$ such that $a_i \in \mathbb{Z}^0(p)$, and $n_i \in \mathbb{Z}$, with $\sum_{1 \leqslant i \leqslant s} [a_i/e^*] < p - s$.

As we assume e^* to be fixed throughout this chapter, we will agree to use the simpler notation $\mathcal{A}$.

Definition 5.2. For any $(\bar{a}, \bar{n}) \in \mathcal{A}$, we say $\{(\bar{a}_i, \bar{n}_i)_{u_i} : 1 \leqslant i \leqslant l\}$ is a *partition* of $(\bar{a}, \bar{n})$ of order l where,

— for any $1 \leqslant i \leqslant l$, $\bar{a}_i = (a_{i1}, \dots, a_{iu_i}), \bar{n}_i = (n_{i1}, \dots, n_{iu_i})$;

— $\bar{a} = (a_{11}, \dots, a_{1u_1}, a_{21}, \dots, a_{2u_2}, \dots, a_{l1}, \dots, a_{lu_l})$;

— $\bar{n} = (n_{11}, \dots, n_{1u_1}, n_{21}, \dots, n_{2u_2}, \dots, n_{l1}, \dots, n_{lu_l})$.

Definition 5.3. We say a partition $\{(\bar{a}_i, \bar{n}_i)_{u_i} : 1 \leqslant i \leqslant l\}$ of $(\bar{a}, \bar{n})$ is *locally constant* if for all $1 \leqslant i \leqslant l$ we have $n_{i1} = \cdots = n_{iu_i}$. In this case we shall agree to use the notation $\bar{n}_i$ for any of the (equal) numbers $n_{i1}, \ldots, n_{iu_i}$.

Note that any $(\bar{a}, \bar{n}) \in \mathcal{A}$ admits a locally constant partition. For any $(\bar{a}, \bar{n}) \in \mathcal{A}$ we denote by $\mathcal{P}(\bar{a}, \bar{n})$ the set of all locally constant partitions of $(\bar{a}, \bar{n})$, and we denote by $\mathcal{P}(\mathcal{A})$ the set of all locally constant partitions of all $(\bar{a}, \bar{n}) \in \mathcal{A}$.

Definition 5.4. For any $(\bar{a}, \bar{n}) \in \mathcal{A}$ and any $m \in \mathbb{Z}$ we set $\sigma^m(\bar{a}, \bar{n}) = (\bar{a}, \bar{n} + m)$, where $\bar{n} + m = (n_1 + m, \ldots, n_s + m)$.

Similarly, if $\pi = \{(\bar{a}_i, \bar{n}_i)_{u_i} : 1 \leqslant i \leqslant l\} \in \mathcal{P}(\bar{a}, \bar{n})$, then for any $m \in \mathbb{Z}$ we set $\sigma^m(\pi) = \{(\bar{a}_i, \bar{n}_i + m)_{u_i} : 1 \leqslant i \leqslant l\}$.

Note that if $\pi \in \mathcal{P}(\bar{a}, \bar{n})$, then $\sigma^m(\pi) \in \mathcal{P}(\sigma^m(\bar{a}, \bar{n}))$, for any $m \in \mathbb{Z}$.

Suppose $\pi = \{(\bar{a}_i, \bar{n}_i)_{u_i} : 1 \leqslant i \leqslant l\} \in \mathcal{P}(\mathcal{A})$, we will use below the notation $\pi[i] = (\bar{a}_i, \bar{n}_i)$, $\pi = \{\pi[i] : 1 \leqslant i \leqslant l\}$, and $\pi_{\leqslant t} = \{\pi[i] : 1 \leqslant i \leqslant t\}$ for any $1 \leqslant t \leqslant l$. We will also use the notation $\gamma(\pi) = \gamma(\bar{a}, \bar{n})$, $\gamma(\pi[i]) = \gamma(\bar{a}_i, \bar{n}_i)$, and $\gamma(\pi_{\leqslant t}) = \gamma(\bar{a}_1, \bar{n}_1) + \cdots + \gamma(\bar{a}_t, \bar{n}_t)$. By convention we will set $\gamma(\pi_{\leqslant 0}) = 0$.

We will often use the property that if $\pi \in \mathcal{P}(\mathcal{A})$ and $\pi' = \sigma^m(\pi)$ for some $m \in \mathbb{Z}$ then $\gamma(\pi'_{\leqslant t}) = p^m \gamma(\pi_{\leqslant t})$ for all $0 \leqslant t \leqslant l$.

5.2 Admissible partitions

With all notation and properties of the previous section we now introduce the concept of admissible partitions by induction on the order l (recall in (b)(ii) below that $k \simeq \mathbb{F}_{p^{N_0}}$).

Definition 5.5. Let $\pi \in \mathcal{P}(\mathcal{A})$, then π is *admissible* if,

(a) $\gamma(\pi) - cp^{\bar{n}_1} \in \mathbb{Z}$;

(b) one of the following holds,

  (i) $\gamma(\pi) < cp^{\bar{n}_1}$ and $\bar{n}_l \geqslant 1$;

  (ii) $\gamma(\pi) = cp^{\bar{n}_1}$ and $-N_0 < \bar{n}_l \leqslant 0$;

  (iii) $\gamma(\pi) > cp^{\bar{n}_1}$ and $\bar{n}_l \leqslant 0$;

(c) and if $l > 1$ then $\sigma^{-\bar{n}_l}(\pi_{\leqslant l-1})$ is admissible.

Definition 5.6. For any $\pi = \{\pi[i] : 1 \leqslant i \leqslant l\} \in \mathcal{P}(\mathcal{A})$,

(a) Let $l_{(+)}(\pi) = \max\{0 \leqslant i \leqslant l : \gamma(\pi_{\leqslant i}) < cp^{\bar{n}_1}\}$.

(b) Let $l_{(0)}(\pi) = \max\{0 \leqslant i \leqslant l : \gamma(\pi_{\leqslant i}) \leqslant cp^{\bar{n}_1}\}$.

The following properties follow easily from basic definitions

Proposition 5.7. *For any* $\pi \in \mathcal{P}(\mathcal{A})$*,*
(a) $l_{(+)}(\pi_{\leqslant t}) = \min\{t, l_{(+)}(\pi)\}$*, and* $l_{(0)}(\pi_{\leqslant t}) = \min\{t, l_{(0)}(\pi)\}$ *for any* $1 \leqslant t \leqslant l$*.*
(b) $l_{(+)}(\pi) = l_{(+)}(\sigma^m(\pi))$ *and* $l_{(0)}(\pi) = l_{(0)}(\sigma^m(\pi))$ *for any* $m \in \mathbb{Z}$*.*

Remark: We will often use the simpler notation $l_{(+)}$ and $l_{(0)}$ when the partition π is clear from context. Similarly, by part (b) of the above proposition we can unambiguously use the notation $l_{(+)}$ and $l_{(0)}$ for both π and $\sigma^m(\pi)$.

For any $(\bar{a}, \bar{n}) \in \mathcal{A}$ we denote by $\mathcal{P}_{\text{adm}}(\bar{a}, \bar{n})$ the subset of all admissible partitions in $\mathcal{P}(\bar{a}, \bar{n})$, and denote by $\mathcal{P}_{\text{adm}}$ the set of all admissible partitions of all $(\bar{a}, \bar{n}) \in \mathcal{A}$. With the above notation we can give the following explicit characterisation of admissible partitions.

Proposition 5.8. *Let* $\pi \in \mathcal{P}(\mathcal{A})$ *be such that* $\gamma(\pi) - cp^{\bar{n}_1} \in \mathbb{Z}$*, then* $\pi \in \mathcal{P}_{\text{adm}}$ *if and only if*
(a) $\bar{n}_1 > \bar{n}_2 > \cdots > \bar{n}_{l_{(+)}} > \bar{n}_{l_{(+)}+1} \leqslant \cdots \leqslant \bar{n}_l \leqslant 0$;
(b) *if* $l_{(+)} < i \leqslant l_{(0)}$ *then* $\bar{n}_i - \bar{n}_{i+1} > -N_0$*.*

Remark: In (a), if $l_{(+)} = l$ we set $\bar{n}_{l_{(+)}+1} = 0$. Similarly, in (b), if $l_{(0)} = l$ we set $\bar{n}_{l_{(0)}+1} = 0$.

Proof. If $\pi \in \mathcal{P}(\mathcal{A})$ is a partition of order 1 such that $\gamma(\pi) - cp^{\bar{n}_1} \in \mathbb{Z}$, then the proposition follows directly from definition 5.5. Let $\pi = \{\pi[i] : 1 \leqslant i \leqslant l\} \in \mathcal{P}(\mathcal{A})$ be a partition of order $l > 1$ such that $\gamma(\pi) - cp^{\bar{n}_1} \in \mathbb{Z}$, and assume for induction that the proposition is true for all partitions of order $l - 1$.

As the admissibility of π depends partly on the admissibility of $\sigma^{-\bar{n}_l}(\pi_{\leqslant l-1})$ we begin by setting $\pi'_{\leqslant l-1} = \sigma^{-\bar{n}_l}(\pi_{\leqslant l-1})$, and use our induction hypothesis to recover the conditions under which $\pi'_{\leqslant l-1}$ is admissible.

Let $l'_{(+)} := l_{(+)}(\pi'_{\leqslant l-1})$, and $l'_{(0)} := l_{(0)}(\pi'_{\leqslant l-1})$. Then by our induction assumption, $\pi'_{\leqslant l-1} \in \mathcal{P}_{\mathrm{adm}}$ if and only if the following conditions hold:

$$\bar{n}'_1 > \cdots > \bar{n}'_{l'_{(+)}+1} \leqslant \cdots \leqslant \bar{n}'_{l-1} \leqslant 0; \text{ and } \bar{n}'_i - \bar{n}'_{i+1} < N_0 \text{ for all } l'_{(+)} < i \leqslant l'_{(0)}. \tag{5.2}$$

As $\bar{n}'_i = \bar{n}_i - \bar{n}_l$ for all $1 \leqslant i \leqslant l-1$ it follows from (5.2) that $\pi'_{\leqslant l-1} \in \mathcal{P}_{\mathrm{adm}}$ if and only if the following conditions hold:

$$\bar{n}_1 > \cdots > \bar{n}_{l'_{(+)}+1} \leqslant \cdots \leqslant \bar{n}_{l-1} \leqslant \bar{n}_l; \text{ and } \bar{n}_i - \bar{n}_{i+1} < N_0 \text{ for all } l'_{(+)} < i \leqslant l'_{(0)}. \tag{5.3}$$

Using definition 5.5 and (5.3) then we have the following conditions for π to be admissible.

(i) If $\gamma(\pi) < cp^{\bar{n}_1}$, then $l'_{(+)} = l-1$. Hence $\pi \in \mathcal{P}_{\mathrm{adm}}$ if and only if $\bar{n}_1 > \bar{n}_2 > \cdots > \bar{n}_l > 0$.

(ii) If $\gamma(\pi) \geqslant cp^{\bar{n}_1}$, then $l'_{(+)} = l_{(+)}(\pi)$. Hence $\pi \in \mathcal{P}_{\mathrm{adm}}$ if and only if $\bar{n}_1 > \bar{n}_2 > \cdots > \bar{n}_{l_{(+)}} > \bar{n}_{l_{(+)}+1} \leqslant \cdots \leqslant \bar{n}_l \leqslant 0$; and $\bar{n}_i - \bar{n}_{i+1} > -N_0$ for all $l_{(+)} < i \leqslant l_{(0)}$.

This completes the inductive step, and the proposition follows by induction on l. □

The following propositions will be of use when applying the operators $\mathcal{R}$ and $\sigma\mathcal{S}$ to terms of the recurrence relation, and both follow easily from basic definitions and proposition 5.8.

Proposition 5.9. *Let $\pi \in \mathcal{P}(\mathcal{A})$ be such that $\gamma(\pi) < cp^{\bar{n}_1}$, and let $i \in \mathbb{Z}$. Then $\sigma^i(\pi) \in \mathcal{P}_{\mathrm{adm}}$ if and only if $\bar{n}_1 > \cdots > \bar{n}_l$ and $\bar{n}_l + i \geqslant 1$.*

Proposition 5.10. *Let $\pi \in \mathcal{P}_{\mathrm{adm}}$ be such that $\gamma(\pi) \geqslant cp^{\bar{n}_1}$, and let $i \in \mathbb{Z}$.*

(a) *if $l_{(0)} = l$ then $\sigma^i(\pi) \in \mathcal{P}_{\mathrm{adm}}$ if and only if $-N_0 < i + \bar{n}_l \leqslant 0$.*

(b) *if $l_{(0)} < l$ then $\sigma^i(\pi) \in \mathcal{P}_{\mathrm{adm}}$ if and only if $-v_p(\gamma(\pi) - cp^{\bar{n}_1}) \leqslant i \leqslant -\bar{n}_l$.*

In theorem 5.13 below we show that the elements $\sigma\bar{c}_1$ and V_{a0} for $a \in \mathbb{Z}^0(p)$ can be expressed as k-linear combinations of terms associated to admissible partitions. We introduce the following sets to distinguish further between terms of $\sigma\bar{c}_1$ and V_{a0}.

Definition 5.11. We define the following subsets of $\mathcal{P}_{\mathrm{adm}}$.
$\mathcal{P}_\sigma = \{\pi \in \mathcal{P}_{\mathrm{adm}} : \gamma(\pi) - cp^{\bar{n}_1} \in p\mathbb{Z}\}$,
$\mathcal{P}_V = \{\pi \in \mathcal{P}_{\mathrm{adm}} : \gamma(\pi) - cp^{\bar{n}_1} \in \mathbb{Z}^0(p)\}$.

Note that $\mathcal{P}_{\mathrm{adm}} = \mathcal{P}_\sigma \cup \mathcal{P}_V$, and $\mathcal{P}_\sigma \cap \mathcal{P}_V = \{\pi \in \mathcal{P}_{\mathrm{adm}} : \gamma(\pi) = cp^{\bar{n}_1}\}$.

Proposition 5.12. *Let $\pi \in \mathcal{P}_{\mathrm{adm}}$ such that $\gamma(\pi) > cp^{\bar{n}_1}$.*

(a) *$\sigma^i(\pi) \in \mathcal{P}_V$ if and only if $i = -v_p(\gamma(\pi) - cp^{\bar{n}_1})$.*

(b) *$\sigma^i(\pi) \in \mathcal{P}_\sigma$ if and only if $-v_p(\gamma(\pi) - cp^{\bar{n}_1}) < i \leqslant -\bar{n}_l$.*

Proof. As $\pi \in \mathcal{P}_{\mathrm{adm}}$ then by proposition 5.10 (b) we have that $\sigma^i(\pi) \in \mathcal{P}_{\mathrm{adm}}$ if and only if $-v_p(\gamma(\pi) - cp^{\bar{n}_1}) \leqslant i \leqslant -\bar{n}_l$. Moreover as $\gamma(\pi) > cp^{\bar{n}_1}$ then $\sigma^i(\pi) \in \mathcal{P}_{\mathrm{adm}}$ belongs to precisely one of $\mathcal{P}_V$ or $\mathcal{P}_\sigma$. Therefore it is enough to prove statement (a), which follows easily from the fact that $v_p(\gamma(\pi') - cp^{\bar{n}'_1}) = v_p(\gamma(\pi) - cp^{\bar{n}_1}) + i$. □

If $\pi \in \mathcal{P}(\bar{a}, \bar{n})$ we will use below the notation $D_\pi = D_{(\bar{a},\bar{n})}$.

Theorem 5.13. (a) *For all $\pi \in \mathcal{P}_\sigma$ there are $\kappa_\sigma(\pi) \in k$ such that*

$$\sigma\bar{c}_1 = \sum_{\pi \in \mathcal{P}_\sigma} \kappa_\sigma(\pi) t^{cp^{\bar{n}_1} - \gamma(\pi)} D_\pi \, .$$

(b) *For all $\pi \in \mathcal{P}_V$ there are $\kappa_V(\pi) \in k$ such that for any $a \in \mathbb{Z}^0(p)$*

$$V_{a0} = \sum_{\pi \in \mathcal{P}_{V_a}} \kappa_V(\pi) D_\pi \, ,$$

where $\mathcal{P}_{V_a} := \{\pi \in \mathcal{P}_V : \gamma(\pi) = cp^{\bar{n}_1} + a\}$.

Proof. Consider first sum (5.1a) of the recurrence formula. All terms of the sum are of the form $-\frac{1}{s!} t^{c-\gamma(\bar{a},\bar{n})} D_{(\bar{a},\bar{0})_s}$, and hence we can associate each term with a unique locally constant partition of order one. Therefore sum (5.1a) can be written in terms of partitions as follows.

$$-\sum_{1 \leqslant s < p} \frac{1}{s!} \sum_{\pi_s} t^{c-\gamma(\pi_s)} D_{\pi_s}$$

where for $1 \leqslant s < p$ the sum runs over all partitions $\pi_s = \{(\bar{a}_1, \bar{0})_s\} \in \mathcal{P}(\mathcal{A})$ of length s and order one.

Clearly all coefficients are in k, and hence we need only consider the form of the involved terms when applying the operators $\sigma\mathcal{S}$ and $\mathcal{R}$ to recover the contributions to the elements $\sigma\bar{c}_1$, and V_{a0} for $a \in \mathbb{Z}^0(p)$ respectively.

If $\pi_s = \{(\bar{a}_1, \bar{0})_s\}$ is such that $c - \gamma(\pi_s) > 0$ then the partition is not admissible (as $\bar{n}_l = 0$). However for such terms $\mathcal{R}(t^{c-\gamma(\pi_s)} D_{\pi_s}) = 0$ and so there are no contributions to V_{a0} for any $a \in \mathbb{Z}^0(p)$. Applying the operator $\sigma\mathcal{S}$ we obtain terms of the form $t^{cp^{\bar{n}'_1} - \gamma(\pi'_s)} D_{\pi'_s}$ where $\pi'_s = \sigma^i(\pi_s)$ with $i \geqslant 1$, and π'_s is admissible by definition, as $\bar{n}'_1 = i \geqslant 1$. In particular all contributions to $\sigma\bar{c}_1$ from terms of this form correspond to σ-admissible partitions, and occur with coefficient in k.

If $\pi_s = \{(\bar{a}_1, \bar{0})_s\}$ is such that $\gamma(\pi_s) = c$ then the partition π_s is admissible. Applying the operator $\mathcal{R}$ we obtain contributions to V_0 of the form $D_{\pi'}$ where $\pi'_s = \sigma^{-i}(\pi_s)$ with $0 \leqslant i < N_0$, and π'_s is admissible by proposition 5.10. In particular $\pi'_s \in \mathcal{P}_{V_0}$. Applying $\sigma\mathcal{S}$ we obtain contributions to $\sigma\bar{c}_1$ of the form $D_{\pi'}$ where $\pi'_s = \sigma^{-i}(\pi)$ with $0 \leqslant i < N_0 - 1$, and again such terms are admissible by proposition 5.10, and occur with coefficient in k.

Finally, if $\pi_s = \{(\bar{a}_s, \bar{0}_s)\}$ is such that $\gamma(\pi_s) > c$ then the partition π_s is admissible. Let $v_p(c - \gamma(\pi_s)) = M$, then applying the operator $\mathcal{R}$ we obtain terms of the form $t^{cp^{\bar{n}'_1} - \gamma(\pi')} D_{\pi'_s}$, where $\pi'_s = \sigma^{-M}(\pi_s)$, and hence $\pi'_s \in \mathcal{P}_V$ by proposition 5.12. More specifically, $\pi'_s \in \mathcal{P}_{V_a}$ where $a = \gamma(\pi'_s) - cp^{\bar{n}'_1} \in \mathbb{Z}^+(p)$, and the term $D_{\pi'_s}$ contributes to V_{a0}. Applying the operator $\sigma\mathcal{S}$ we obtain contributions to $\sigma\bar{c}_1$ of the form $t^{cp^{\bar{n}'_1} - \gamma(\pi')} D_{\pi'_s}$, where $\pi'_s = \sigma^i(\pi_s)$ for some $-M < i \leqslant 0$ and thus $\pi'_s \in \mathcal{P}_\sigma$ by proposition 5.12.

Therefore, all contributions to V_{a0} and $\sigma\bar{c}_1$ from terms of (5.1a) are of the appropriate form. In particular, as all terms of sums (5.1b) and (5.1c) are elements of $C_2(\mathcal{L}_\mathcal{K})$, then the proposition is true modulo $\mathcal{M}(p-1) + C_2(\mathcal{L}_\mathcal{K})$, and we can proceed by induction.

Let $2 \leqslant r < p$ and assume that the proposition is true modulo $\mathcal{M}(p-1) + C_r(\mathcal{L}_\mathcal{K})$. As we have already established the relevant properties for terms of (5.1a), then we need only consider terms in (5.1b) and (5.1c).

Consider sum (5.1b). For any $a \in \mathbb{Z}^0(p)$ by our induction assumption all terms of $V_{a0} \bmod \mathcal{M}(p-1)+C_{r+1}(\mathcal{L}_\mathcal{K})$ correspond to V_a-admissible partitions. Therefore any term of (5.1b) is of the following form.

$$-\frac{\kappa_V(\pi')}{(s+1)!}t^{cp^{\bar{n}'_1}-\gamma(\pi')-\gamma(\bar{a},\bar{0})_s}[\dots[D_{\pi'},D_{a_2,0}],\dots,D_{a_s,0}]\,,$$

where $\pi' \in \mathcal{P}_{V_a}$ *with* $a=\gamma(\pi')-cp^{\bar{n}'_1}$.

To any such term we can associate a unique partition $\pi \in \mathcal{P}(\mathcal{A})$ such that $2 \leqslant l \leqslant r$ with $\pi[l]=(a_2,0,\dots,a_s,0)$ and $\pi_{\leqslant l-1}=\pi' \in \mathcal{P}_V$, therefore sum (5.1b) can be written in terms of partitions as follows.

$$-\sum_{\pi}\frac{1}{(u_l+1)!}\kappa_V(\pi_{\leqslant l-1})t^{cp^{\bar{n}_1}-\gamma(\pi)}D_\pi \tag{5.4}$$

where π *runs over all* $\pi \in \mathcal{P}(\mathcal{A})$ *such that* $2 \leqslant l \leqslant r$ *with* $\bar{n}_l=0$ *and* $\pi_{\leqslant l-1} \in \mathcal{P}_V$.

Note that $cp^{\bar{n}_1}-\gamma(\pi) \in \mathbb{Z}_{\leqslant 0}$, as $\pi_{\leqslant l-1}$ is V-admissible, and $-\gamma(\pi[l]) \in \mathbb{Z}_{\leqslant 0}$. It follows that all partitions of this form are admissible by definition 5.5, and by inductive assumption all terms occur with coefficient in k.

Similarly, if we consider sum (5.1c), then by our induction assumption the sum can be written as

$$-\sum_{\pi}\frac{1}{(u_l)!}\kappa_\sigma(\pi_{\leqslant l-1})t^{cp^{\bar{n}_1}-\gamma(\pi)}D_\pi \tag{5.5}$$

where π *runs over all* $\pi \in \mathcal{P}(\mathcal{A})$ *such that* $2 \leqslant l \leqslant r$ *with* $\bar{n}_l=0$ *and* $\pi_{\leqslant l-1} \in \mathcal{P}_\sigma$.

If π in (5.5) is such that $\gamma(\pi)<cp^{n_1}$ then the partition π is not admissible, however, as $\gamma(\pi_{\leqslant l-1})<cp^{\bar{n}_1}$ and $\pi_{\leqslant l-1} \in \mathcal{P}_\sigma$ by our induction assumption, then for such π we have $\bar{n}_1>\dots>\bar{n}_l=0$. Upon applying the operator $\sigma\mathcal{S}$, all contributions to $\sigma\bar{c}_1$ are of the form $t^{cp^{\bar{n}'_1}-\gamma(\pi')}D_{\pi'}$, where $\pi'=\sigma^i(\pi)$ for some $i \geqslant 1$ and thus $\pi' \in \mathcal{P}_\sigma$ by proposition 5.9. As in the case for terms of (5.1a), we can see that upon applying the operator $\mathcal{R}$ such terms do not contribute to V_{a0} for any $a \in \mathbb{Z}^0(p)$.

All remaining terms π in (5.5) are such that $\gamma(\pi) \geqslant cp^{\bar{n}_1}$ and π is admissible for the same reasons as terms of (5.4). For these terms, and for all terms of (5.4) we can follow precisely the same reasoning as we did for terms of (5.1a) to conclude from proposition 5.10 that upon applying the operators $\mathcal{R}$ and $\sigma\mathcal{S}$ all contributions to the elements V_{a0} and $\sigma\bar{c}_1$ occur with coefficient in k and correspond to the appropriate admissible partitions modulo $\mathcal{M}(p-1)+C_{r+1}(\mathcal{L}_{\mathcal{K}})$. This completes the inductive step, and the theorem follows by induction on r. □

5.3 Coefficients

In this section we give an explicit description of the coefficients $\kappa_\sigma(\pi)$ and $\kappa_V(\pi)$ from theorem 5.13. For any admissible partition π such that $\gamma(\pi)=cp^{\bar{n}_1}$ the partition is both V-admissible and σ-admissible, and the coefficient is more complicated in this case. As such we introduce the notion of non-degenerate partitions, and recover the coefficient for such partitions first.

Definition 5.14. We say a partition $\pi=\{\pi[i] : 1\leqslant i\leqslant l\}\in\mathcal{P}(\mathcal{A})$ is *non-degenerate* if $\gamma(\pi_{\leqslant t})\neq cp^{\bar{n}_1}$ for all $1\leqslant t\leqslant l$.

We will denote by $\mathcal{P}^{\mathrm{nd}}(\mathcal{A})$ the subset of all non-degenerate partitions in $\mathcal{P}(\mathcal{A})$, and similarly we will use the notation $\mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}$, $\mathcal{P}^{\mathrm{nd}}_{\sigma}$, $\mathcal{P}^{\mathrm{nd}}_{V}$ to denote the subsets of non-degenerate partitions of the appropriate sets.

If $\pi\in\mathcal{P}^{\mathrm{nd}}(\mathcal{A})$, then $\pi_{\leqslant t}\in\mathcal{P}^{\mathrm{nd}}(\mathcal{A})$ for all $1\leqslant t<l$, and $\sigma^m(\pi)\in\mathcal{P}^{\mathrm{nd}}(\mathcal{A})$ for all $m\in\mathbb{Z}$. We also note that $\mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}=\mathcal{P}^{\mathrm{nd}}_{\sigma}\cup\mathcal{P}^{\mathrm{nd}}_{V}$, and $\mathcal{P}^{\mathrm{nd}}_{\sigma}\cap\mathcal{P}^{\mathrm{nd}}_{V}=\emptyset$. In particular, if $\pi\in\mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}$, then $\sigma^{-\bar{n}_{t+1}}(\pi_{\leqslant t})\in\mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}$ for all $1\leqslant t<l$, and hence $\sigma^{-\bar{n}_{t+1}}(\pi_{\leqslant t})$ belongs to precisely one of $\mathcal{P}_V$ or $\mathcal{P}_\sigma$.

Definition 5.15. For any $\pi\in\mathcal{P}_{\mathrm{adm}}$ set $\delta_\pi[0]=0$, and for any $1\leqslant t\leqslant l$ we define

$$\delta_\pi[t]=\begin{cases}1, & \text{if } l_{(0)}<t\leqslant l, \text{ and } \sigma^{-\bar{n}_{t+1}}(\pi_{\leqslant t})\in\mathcal{P}_V.\\ 0, & \text{otherwise.}\end{cases}$$

Remark: If π is non-degenerate then $\delta_\pi[t]=0$ if and only if $\sigma^{-\bar{n}_{t+1}}(\pi_{\leqslant t})\in\mathcal{P}_\sigma$.

Definition 5.16. For any $\pi\in\mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}$ we define

$$\kappa(\pi)=(-1)^{l-l_{(0)}}\prod_{1\leqslant t\leqslant l}\left((u_t+\delta[t-1])!\right)^{-1}.$$

Proposition 5.17. *Let $\pi, \pi' \in \mathcal{P}_{\text{adm}}$ be such that $\pi' = \sigma^m(\pi)$ for some $m \in \mathbb{Z}$. Then $\delta_\pi[t] = \delta_{\pi'}[t]$ for all $0 \leqslant t < l$.*

Proof. As $l_{(0)}(\pi) = l_{(0)}(\pi')$, then $\delta_\pi[t] = \delta_{\pi'}[t] = 0$ for all $0 \leqslant t \leqslant l_{(0)}$. For any $l_{(0)} < t < l$, since $\bar{n}'_{t+1} = \bar{n}_{t+1} + m$ and $\pi'_{\leqslant t} = \sigma^m(\pi_{\leqslant t})$ then we have $\sigma^{-\bar{n}'_{t+1}}(\pi'_{\leqslant t}) = \sigma^{-\bar{n}_{t+1}}(\pi_{\leqslant t})$, and hence $\delta_\pi[t] = 1$ if and only if $\delta_{\pi'}[t] = 1$. □

Corollary 5.18. *Let $\pi, \pi' \in \mathcal{P}^{\text{nd}}_{\text{adm}}$ be such that $\pi' = \sigma^m(\pi)$ for some $m \in \mathbb{Z}$. Then $\kappa(\pi) = \kappa(\pi')$.*

With respect to the decompositions by admissible partitions in theorem 5.13 we have the following result.

Theorem 5.19. *For any $\pi \in \mathcal{P}^{\text{nd}}_{\text{adm}}$,*
(a) *if $\pi \in \mathcal{P}^{\text{nd}}_\sigma$ then $\kappa_\sigma(\pi) = \kappa(\pi)$;*
(b) *if $\pi \in \mathcal{P}^{\text{nd}}_{V_a}$ then $\kappa_V(\pi) = \kappa(\pi)$.*

Proof. To prove the theorem we must prove for any non-degenerate $\pi \in \mathcal{P}_{V_a}$, that the associated term D_π occurs in V_{a0} with coefficient $\kappa(\pi)$, and for any non-degenerate $\pi \in \mathcal{P}_\sigma$, that the associated term $t^{cp^{\bar{n}_1} - \gamma(\pi)} D_\pi$ occurs in $\sigma \bar{c}_1$ with coefficient $\kappa(\pi)$. We prove this by induction on the order l of the partition.

In the proof of theorem 5.13 we saw that any term associated with a partition of order one must arise from sum (5.1a), which is given as follows.

$$-\sum_{1 \leqslant s < p} \frac{1}{s!} \sum_{\pi_s} t^{c - \gamma(\pi_s)} D_{\pi_s}$$

where for any $1 \leqslant s < p$ the index π_s runs over all $\pi_s = \{(\bar{a}_1, \bar{0})_s\} \in \mathcal{P}(\mathcal{A})$.

Note that a partition of order one is non-degenerate if and only if $\gamma(\pi) \neq cp^{\bar{n}_1}$.

If $\gamma(\pi_s) < c$, then the contributions to $\sigma \bar{c}_1$ are given by $\sum_{i \geqslant 1} \frac{1}{s!} t^{p^i (c - \gamma(\pi_s))} \sigma^i (D_{\pi_s})$. Suppose $\pi'_s = \sigma^i(\pi_s)$ for some $i \geqslant 1$ then $\pi'_s \in \mathcal{P}^{\text{nd}}_\sigma$ and since $\delta_{\pi'_s}[0] = 0$ and $l_{(0)}(\pi'_s) = l = 1$ then $\kappa(\pi'_s) = \frac{1}{u_1!}$.

If $\gamma(\pi_s) > c$, then π_s is admissible and as $\delta_{\pi_s}[0] = 0$ and $l_{(0)}(\pi_s) = 0$ then $\kappa(\pi_s) = -\frac{1}{u_1!}$. If $v_p(cp^{\bar{n}_1} - \gamma(\pi)) = M$, then by corollary 5.18 the contributions to $\sigma \bar{c}_1$ are given by $\kappa(\pi'_s) t^{cp^{\bar{n}'_1} - \gamma(\pi')} D_{\pi'_s}$ where $\pi'_s = \sigma^{-i}(\pi_s)$ for some $0 \leqslant i < M$, and the contribution to V_{a0} is given by $\kappa(\pi'_s) D_{\pi'_s}$ where $\pi'_s = \sigma^{-M}(\pi_s)$, and $a = \gamma(\pi'_s) - cp^{\bar{n}_1}$.

Therefore the theorem is true for all non-degenerate admissible partitions of order one.

Let $2 \leqslant l < p$ and assume for induction that the theorem is true for all partitions of order $l-1$. Consider the explicit description of sums (5.1b) and (5.1c) given in equations (5.4) and (5.5) respectively. For any non-degenerate term appearing in these sums, as $\pi_{\leqslant l-1}$ is non-degenerate then $\pi_{\leqslant l-1}$ belongs to precisely one of $\mathcal{P}_V$ or $\mathcal{P}_\sigma$, therefore the non-degenerate terms of order l in (5.1) are given by the following sum.

$$-\sum_{\pi} \frac{\kappa(\pi_{\leqslant l-1})}{(u_l + \delta_{\pi_{\leqslant l-1}}[l-1])!} t^{cp^{\bar{n}_1} - \gamma(\pi)} D_\pi \tag{5.6}$$

where π runs over all $\pi \in \mathcal{P}^{\mathrm{nd}}(\mathcal{A})$ of order l such that $\pi_{\leqslant l-1} \in \mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}$ and $\bar{n}_l = 0$.

If $\gamma(\pi) < cp^{\bar{n}_1}$ in the above sum, then π is not admissible and therefore the coefficient $\kappa(\pi)$ is not defined, however, as $\delta_\pi = 0$ for all $0 \leqslant t \leqslant l$ then π occurs in (5.6) with coefficient $\prod_{1 \leqslant t \leqslant l} (u_t!)^{-1}$, and by proposition 5.17 we see that the (admissible) contributions to $\sigma \bar{c}_1$ are given by $\kappa(\pi') t^{cp^{\bar{n}'_1} - \gamma(\pi')} D_{\pi'}$ where $\pi' = \sigma^i(\pi)$ for $i \geqslant 1$.

If $\gamma(\pi) > cp^{\bar{n}_1}$ then the term is admissible and appears in sum (5.6) as $\kappa(\pi) t^{cp^{\bar{n}_1} - \gamma(\pi)} D_\pi$. We can then follow the same reasoning as we did for partitions of order one to conclude that all terms contributing to $\sigma \bar{c}_1$ and V_{a0} occur with the appropriate coefficient.

Therefore the proposition is true for all non-degenerate admissible partitions of order l, and the theorem follows by induction on l. □

To recover the coefficient in full we must consider all possible cases to remove the assumption about the non-degeneracy of π. As noted above, the coefficient for degenerate partitions is more complicated as admissible components such that $\gamma(\pi_{\leqslant t}) = cp^{\bar{n}_1}$ can appear from both V_0 and $\sigma \bar{c}_1$. Our recurrent procedure implies the following inductive definition.

Definition 5.20. For any $\pi \in \mathcal{P}_{\mathrm{adm}}$, we define inductively $\kappa_V(\pi) = \kappa_V(\pi_{\leqslant l})$ and $\kappa_\sigma(\pi) = \kappa_\sigma(\pi_{\leqslant l})$, where for any $l_{(+)} \leqslant i \leqslant l$,

$$\kappa_V(\pi_{\leqslant i}) = \begin{cases} 0 & \text{if } i = l_{(+)} \\ \sigma^{\bar{n}_{i+1}}(\alpha_0)\cdot\widetilde{\kappa}(\pi_{\leqslant i}) & \text{if } l_{(+)} < i \leqslant l_{(0)} \\ (\delta_\pi[i])\cdot\widetilde{\kappa}(\pi_{\leqslant i}) & \text{if } l_{(0)} < i \leqslant l \end{cases}$$

$$\kappa_\sigma(\pi_{\leqslant i}) = \begin{cases} \prod\limits_{1\leqslant i\leqslant l_{(+)}} (u_i!)^{-1} & \text{if } i = l_{(+)} \\ \left(\sum\limits_{\bar{n}_{i+1}<j<N_0+\bar{n}_i} \sigma^j(\alpha_0)\right)\cdot\widetilde{\kappa}(\pi_{\leqslant i}) & \text{if } l_{(+)} < i \leqslant l_{(0)} \\ (1-\delta_\pi[i])\cdot\widetilde{\kappa}(\pi_{\leqslant i}) & \text{if } l_{(0)} < i \leqslant l \end{cases}$$

$$\textit{where for any}\quad l_{(+)} < i \leqslant l, \quad \widetilde{\kappa}(\pi_{\leqslant i}) = -\left(\frac{\kappa_V(\pi_{\leqslant i-1})}{(u_i+1)!} + \frac{\kappa_\sigma(\pi_{\leqslant i-1})}{(u_i)!}\right).$$

Remark: By our usual convention, if $l_{(0)} = l$ then we set $\bar{n}_{l_{(0)}+1} = 0$.

We prove in theorem 5.23 below, that these coefficients are precisely those implied by theorem 5.13. In preparation for the proof, we establish a property of the coefficients with respect to σ, and prove that for a non-degenerate partition the coefficient defined in 5.20 agrees with the coefficient as given in definition 5.16.

Proposition 5.21. *Let $\pi \in \mathcal{P}_{\text{adm}}$ with $\gamma(\pi) \geqslant cp^{\bar{n}_1}$ and $\bar{n}_l = 0$,*

(a) *if $\gamma(\pi) = cp^{\bar{n}_1}$, then $\sigma^{-i}(\widetilde{\kappa}(\pi)) = \widetilde{\kappa}(\sigma^{-i}(\pi))$ for any $0 \leqslant i < N_0$.*

(b) *if $\gamma(\pi) - cp^{\bar{n}_1} \in \mathbb{N}$, then $\sigma^{-i}(\widetilde{\kappa}(\pi)) = \widetilde{\kappa}(\sigma^{-i}(\pi))$ for any $0 \leqslant i \leqslant v_p(\gamma(\pi) - cp^{\bar{n}_1})$.*

Proof. Let $\pi \in \mathcal{P}_{\text{adm}}$ with $\gamma(\pi) \geqslant cp^{\bar{n}_1}$ and $\bar{n}_l = 0$, and let $\pi' = \sigma^{-i}(\pi) \in \mathcal{P}_{\text{adm}}$ for some $i \in \mathbb{Z}$, then by proposition 5.10 any such i necessarily satisfies the conditions in the statements. With the above notation we prove that $\sigma^{-i}(\widetilde{\kappa}(\pi_{\leqslant t})) = \widetilde{\kappa}(\pi'_{\leqslant t})$ for all $l_{(+)} < t \leqslant l$, (recall that $l_{(+)}(\pi) = l_{(+)}(\pi')$).

Let $t = l_{(+)} + 1$, then as $\kappa_V(\pi_{\leqslant l_{(+)}}) = \kappa_V(\pi'_{\leqslant l_{(+)}}) = 0$ we have

$$\widetilde{\kappa}(\pi_{\leqslant t}) = -\frac{\kappa_\sigma(\pi_{\leqslant l_{(+)}})}{u_t!} \quad \text{and} \quad \widetilde{\kappa}(\pi'_{\leqslant t}) = -\frac{\kappa_\sigma(\pi'_{\leqslant l_{(+)}})}{u'_t!}.$$

Since $\sigma^{-i}(\kappa_\sigma(\pi_{\leqslant l_{(+)}})) = \kappa_\sigma(\pi'_{\leqslant l_{(+)}})$ the statement is clear in this case.

If $l = l_{(+)} + 1$ then we are done. Otherwise, let $l_{(+)} + 1 < t \leqslant l$ and assume for induction that $\sigma^{-i}(\widetilde{\kappa}(\pi_{\leqslant t-1})) = \widetilde{\kappa}(\pi'_{\leqslant t-1})$. To recover $\widetilde{\kappa}(\pi_{\leqslant t})$ we must first recover $\kappa_V(\pi_{\leqslant t-1})$ and $\kappa_\sigma(\pi_{\leqslant t-1})$, and similarly for $\widetilde{\kappa}(\pi'_{\leqslant t})$.

Case 1: $\gamma(\pi_{\leqslant t-1}) = cp^{\bar{n}_1}$. In this case the coefficients $\kappa_V(\pi_{\leqslant t})$ and $\kappa_\sigma(\pi_{\leqslant t})$ are given as follows.

$$\kappa_V(\pi_{\leqslant t}) = \sigma^{\bar{n}_{t+1}}(\alpha_0)\widetilde{\kappa}(\pi_{\leqslant t}), \quad \text{and} \quad \kappa_\sigma(\pi_{\leqslant t}) = \left(\sum_{\bar{n}_{t+1}<j<N_0+\bar{n}_t} \sigma^j(\alpha_0)\right)\widetilde{\kappa}(\pi_{\leqslant t})\,.$$

Similarly, as $\gamma(\pi'_{\leqslant t}) = cp^{\bar{n}'_1}$ then by definition we have

$$\kappa_V(\pi'_{\leqslant t}) = \sigma^{\bar{n}'_{t+1}}(\alpha_0)\widetilde{\kappa}(\pi'_{\leqslant t}), \quad \text{and} \quad \kappa_\sigma(\pi'_{\leqslant t}) = \left(\sum_{\bar{n}'_{t+1}<j<N_0+\bar{n}'_t} \sigma^j(\alpha_0)\right)\widetilde{\kappa}(\pi'_{\leqslant t})\,.$$

Since $\bar{n}'_t = \bar{n}_t - i$, and $\bar{n}'_{t+1} = \bar{n}_{t+1} - i$ then it follows from our inductive assumption that $\sigma^{-i}(\kappa_V(\pi_{\leqslant t-1})) = \kappa_V(\pi'_{\leqslant t-1})$, and $\sigma^{-i}(\kappa_\sigma(\pi_{\leqslant t-1})) = \kappa_\sigma(\pi'_{\leqslant t-1})$. Therefore,

$$\sigma^{-i}(\widetilde{\kappa}(\pi_{\leqslant t+1})) = -\left(\frac{\sigma^{-i}(\kappa_V(\pi_{\leqslant t}))}{(u_{t+1}+1)!} + \frac{\sigma^{-i}(\kappa_\sigma(\pi_{\leqslant t}))}{(u_{t+1})!}\right) = \widetilde{\kappa}(\pi'_{\leqslant t+1})\,.$$

Case 2: $\gamma(\pi_{\leqslant t-1}) > cp^{\bar{n}_1}$. By proposition 5.17 we have $\delta_\pi[t-1] = \delta_{\pi'}[t-1]$. If $\delta_\pi[t-1] = 1$ we have $\widetilde{\kappa}(\pi_{\leqslant t}) = -\frac{\widetilde{\kappa}(\pi_{\leqslant t-1})}{(u_t+1)!}$ and $\widetilde{\kappa}(\pi'_{\leqslant t}) = -\frac{\widetilde{\kappa}(\pi'_{\leqslant t-1})}{(u'_t+1)!}$.
Similarly, if $\delta_\pi[t-1] = 0$ we have $\widetilde{\kappa}(\pi_{\leqslant t}) = -\frac{\widetilde{\kappa}(\pi_{\leqslant t-1})}{u_t!}$ and $\widetilde{\kappa}(\pi'_{\leqslant t}) = -\frac{\widetilde{\kappa}(\pi'_{\leqslant t-1})}{u'_t!}$.

For both values of $\delta_\pi[t-1]$ our inductive assumption implies that $\sigma^{-i}(\widetilde{\kappa}(\pi_{\leqslant t})) = \widetilde{\kappa}(\pi'_{\leqslant t})$, which completes the inductive step. It follows by induction on t, that $\sigma^{-i}(\widetilde{\kappa}(\pi_{\leqslant t})) = \widetilde{\kappa}(\pi'_{\leqslant t})$ for all $l_{(+)} < t \leqslant l$, and in particular $\sigma^{-i}(\widetilde{\kappa}(\pi)) = \widetilde{\kappa}(\pi')$. □

Proposition 5.22. *Let $\pi \in \mathcal{P}^{\mathrm{nd}}_{\mathrm{adm}}$ be such that $\gamma(\pi) > cp^{\bar{n}_1}$, then $\widetilde{\kappa}(\pi) = \kappa(\pi)$.*

Proof. Consider $\widetilde{\kappa}(\pi_{\leqslant t})$ for $t = l_{(+)} + 1$. As $\delta_\pi[i] = 0$ for all $0 \leqslant i \leqslant l_{(+)}$, and $\widetilde{\kappa}_V(\pi_{\leqslant l_{(+)}}) = 0$ then it follows that $\widetilde{\kappa}(\pi_{\leqslant t}) = \prod_{1\leqslant i\leqslant t}((u_i + \delta_\pi[t-1])!)^{-1}$.

If $l = l_{(+)} + 1$ then clearly $\widetilde{\kappa}(\pi_{\leqslant l}) = \kappa(\pi)$ and we are done. Otherwise we note that for any $l_{(+)} + 1 < t \leqslant l$, then as π is non-degenerate we have $\gamma(\pi_{\leqslant t-1}) > cp^{\bar{n}_1}$. As we saw in case 2 in the proof of proposition 5.21, if $\delta_\pi[t-1] = 1$ we have $\widetilde{\kappa}(\pi_{\leqslant t}) = -\frac{\widetilde{\kappa}(\pi_{\leqslant t-1})}{(u_t+1)!}$, and if $\delta_\pi[t-1] = 0$ we have $\widetilde{\kappa}(\pi_{\leqslant t}) = -\frac{\widetilde{\kappa}(\pi_{\leqslant t-1})}{u_t!}$.

It follows easily that $\widetilde{\kappa}(\pi_{\leqslant l}) = \prod_{1\leqslant i\leqslant l}((u_i + \delta_\pi[i-1])!)^{-1}$, which is precisely the expression for $\kappa(\pi)$ in definition 5.16. □

Theorem 5.23. *For any $\pi \in \mathcal{P}_{\mathrm{adm}}$ let $\kappa_\sigma(\pi)$ and $\kappa_V(\pi)$ be as given in definition 5.20. Then*

$$\sigma\bar{c}_1 = \sum_{\pi\in\mathcal{P}_\sigma} \kappa_\sigma(\pi) t^{cp^{\bar{n}_1}-\gamma(\pi)} D_\pi \,,$$

and for all $a \in \mathbb{Z}^0(p)$,

$$V_{a0} = \sum_{\pi\in\mathcal{P}_{V_a}} \kappa_V(\pi) D_\pi \,.$$

Proof. To prove the theorem we must prove for any $\pi \in \mathcal{P}_{V_a}$, that the associated term D_π occurs in V_{a0} with coefficient $\kappa_V(\pi)$, and for any $\pi \in \mathcal{P}_\sigma$, that the associated term $t^{cp^{\bar{n}_1}-\gamma(\pi)}D_\pi$ occurs in $\sigma\bar{c}_1$ with coefficient $\kappa_\sigma(\pi)$. If π is a non-degenerate partition, then these statements follow from proposition 5.22 and theorem 5.19. As such we need only establish the theorem for degenerate partitions, which we do by induction on the order l of the partition.

We consider first terms of (5.1a). Note that $\pi = \{(\bar{a}_1, \bar{0})_s\}$ is a degenerate partition if and only if $\gamma(\pi) = c$. Therefore the degenerate terms of (5.1a) are given as follows.

$$-\sum_{\pi:\gamma(\pi)=c} \frac{1}{u_1!} D_\pi$$

Note that all such π are admissible with $l_{(+)} = 0$ and $l_{(0)} = 1$. Therefore, $\kappa_V(\pi_{\leqslant l_{(+)}}) = 0$ and $\kappa_\sigma(\pi_{\leqslant l_{(+)}}) = 1$, and it follows that $\widetilde{\kappa}(\pi_{\leqslant 1}) = -\frac{1}{u_1!}$. Using proposition 5.21, it follows that for any π in the above sum, the contributions to V_0 are given by $\alpha_0\widetilde{\kappa}(\pi')D_{\pi'}$ where $\pi' = \sigma^{-i}(\pi)$ for some $0 \leqslant i < N_0$, and the contributions to $\sigma\bar{c}_1$ are given by $(\sum_{0<j<N_0-i}\sigma^j(\alpha_0))\widetilde{\kappa}(\pi')D_{\pi'}$ where $\pi' = \sigma^{-i}(\pi)$ for some $0 \leqslant i < N_0 - 1$.

Note here, that the operator $\sigma\mathcal{S}$ only produces terms $\sigma^{-i}(\pi)$ for $0 \leqslant i < N_0 - 1$, and therefore not all σ-admissible terms appear in the image of $\sigma\mathcal{S}$. However, for the σ-admissible partition $\pi' = \sigma^{-N_0+1}(\pi)$ the coefficient $\kappa_\sigma(\pi') = 0$, as the sum $\sum_{0<j<1} \sigma^j(\alpha_0)$ is empty. Therefore the non-occurrence of the term $\sigma^{-N_0+1}(\pi)$ is reflected in the definition of the coefficient. Thus the theorem is true for partitions of order one.

Let $2 \leqslant l < p$ and assume for induction that the theorem is true for all partitions of order $< l$. As all terms obtained from (5.1a) are associated with partitions of order one, then we need only consider sums (5.1b) and (5.1c). Noting that $\kappa_V(\pi_{\leqslant l-1})$ (resp. $\kappa_\sigma(\pi_{\leqslant l-1})$) is non-zero only if $\pi_{\leqslant l-1}$ is V-(resp. σ-)admissible, then by our inductive assumption, the terms associated with partitions of order l in (5.1) are given by

$$-\sum_{\pi} \frac{\kappa_V(\pi_{\leqslant l-1})}{(u_l+1)!} t^{cp^{\bar{n}_1}-\gamma(\pi)} D_\pi - \sum_{\pi} \frac{\kappa_\sigma(\pi_{\leqslant l-1})}{u_l!} t^{cp^{\bar{n}_1}-\gamma(\pi)} D_\pi \tag{5.7}$$

where the sums run over all $\pi \in \mathcal{P}_{\text{adm}}$ *of order* l *such that* $n_l = 0$.

Clearly then, any π in the above sum occurs with associated term $\widetilde{\kappa}(\pi)\, t^{cp^{\bar{n}_1}-\gamma(\pi)} D_\pi$.

For any term such that $\gamma(\pi) = cp^{\bar{n}_1}$, then using proposition 5.21 we can follow the same reasoning as we did for partitions of order one to establish that the contributions to V_0 and $\sigma\bar{c}_1$ occur with the relevant coefficient. Similarly, for terms $\gamma(\pi) > cp^{\bar{n}_1}$ it follows easily from proposition 5.21 that all contributions to V_{a0} and $\sigma\bar{c}_1$ occur with the appropriate coefficient.

This completes the inductive step, and the theorem follows by induction on l. □

5.4 Main theorem

For any vector $(\bar{a}, \bar{n}) \in \mathcal{A}$ we denote by $\mathcal{P}_\sigma(\bar{a}, \bar{n})$ the set of all σ-admissible partitions of $(\bar{a}, \bar{n})$, and for any $a \in \mathbb{Z}^0(p)$ we denote by $\mathcal{P}_{V_a}(\bar{a}, \bar{n})$ the set of all V_a-admissible partitions of $(\bar{a}, \bar{n})$. We will say a vector $(\bar{a}, \bar{n})$ is admissible if it admits an admissible partition, and similarly we will say a vector $(\bar{a}, \bar{n})$ is V_a(resp. σ)-admissible if it admits a V_a(resp. σ)-admissible partition. Combining the results of the previous sections we can present the elements $\sigma\bar{c}_1$, and

V_{a0} for $a \in \mathbb{Z}^0(p)$, as decompositions of terms $D_{(\bar{a},\bar{n})}$ associated to admissible vectors $(\bar{a}, \bar{n})$.

Definition 5.24. For any $(\bar{a}, \bar{n}) \in \mathcal{A}$ we define

$$\kappa_\sigma(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_\sigma(\bar{a},\bar{n})} \kappa_\sigma(\pi), \quad \text{and} \quad \kappa_{V_a}(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_{V_a}(\bar{a},\bar{n})} \kappa_V(\pi)\,.$$

Theorem 5.25. *A solution of recurrence formula* (5.1) *is given by the following elements.*

(a) $$\sigma\bar{c}_1 = \sum_{(\bar{a},\bar{n})} \kappa_\sigma(\bar{a}, \bar{n}) t^{cp^{n_1} - \gamma(\bar{a},\bar{n})} D_{(\bar{a},\bar{n})},$$

(b) *for any* $a \in \mathbb{Z}^0(p)$, $$V_{a0} = \sum_{(\bar{a},\bar{n})} \kappa_{V_a}(\bar{a}, \bar{n}) D_{(\bar{a},\bar{n})}.$$

Proof. Consider first statement (a). Note that for any $(\bar{a}, \bar{n})$ the coefficient $\kappa_\sigma(\bar{a}, \bar{n})$ is non-zero only if $(\bar{a}, \bar{n})$ is σ-admissible. Moreover, for all σ-admissible $(\bar{a}, \bar{n})$, any occurrence of the term $t^{cp^{n_1} - \gamma(\bar{a},\bar{n})} D_{(\bar{a},\bar{n})}$ corresponds to a σ-admissible partition of $(\bar{a}, \bar{n})$. By theorem 5.23, all such partitions of $(\bar{a}, \bar{n})$ occurs with coefficient $\kappa_\sigma(\pi)$ and hence statement (a) follows by the definition of $\kappa_\sigma(\bar{a}, \bar{n})$.

Similarly, for statement (b) we note that for any $a \in \mathbb{Z}^0(p)$ and $(\bar{a}, \bar{n})$ the coefficient $\kappa_{V_a}(\bar{a}, \bar{n})$ is non-zero only if $(\bar{a}, \bar{n})$ is V-admissible and $\gamma(\bar{a}, \bar{n}) = cp^{n_1} + a$. For any $a \in \mathbb{Z}^0(p)$, any occurrence of the term $D_{(\bar{a},\bar{n})}$ in V_{a0} corresponds to a V-admissible partition of $(\bar{a}, \bar{n})$ such that $\gamma(\pi) = cp^{\bar{n}_1} + a$. By theorem 5.23, all such partitions of $(\bar{a}, \bar{n})$ occurs with coefficient $\kappa_V(\pi)$ and hence statement (b) follows by the definition of $\kappa_{V_a}(\bar{a}, \bar{n})$. $\square$

6 Properties of coefficients

In this chapter we investigate further the coefficients $\kappa_{V_a}(\bar{a},\bar{n})$ and $\kappa_\sigma(\bar{a},\bar{n})$ from theorem 5.25. The properties of a vector $(\bar{a},\bar{n})$ are intimately linked with the properties of its partitions, therefore we rephrase in terms of vectors some of the results and properties that were established in the previous chapter in terms of partitions. As the coefficients $\kappa_{V_a}(\bar{a},\bar{n})$ and $\kappa_\sigma(\bar{a},\bar{n})$ are defined as the sum of the coefficients of all appropriate admissible partitions of $(\bar{a},\bar{n})$, then the problem of recovering the coefficients $\kappa_{V_a}(\bar{a},\bar{n})$ and $\kappa_\sigma(\bar{a},\bar{n})$ is essentially the combinatorial problem of recovering all admissible partitions of a given vector. We will show that if $(\bar{a},\bar{n})$ is an admissible vector such that $\gamma(\bar{a},\bar{n}) < cp^{n_1}$ then it admits a unique admissible partition, and thus the coefficient $\kappa_\sigma(\bar{a},\bar{n})$ has a very simple expression in this case. If $(\bar{a},\bar{n})$ is an admissible vector with $\gamma(\bar{a},\bar{n}) \geqslant cp^{n_1}$ we will show that, in general, the vector admits a large number of admissible partitions, and thus in this case the coefficients $\kappa_{V_a}(\bar{a},\bar{n})$ and $\kappa_\sigma(\bar{a},\bar{n})$ are more complicated.

Although some results are used in chapter 7 to compare our approach to explicit calculations from [5], this chapter is not central to our main result. Rather, its purpose is to show that, although the coefficients are complicated, they are accessible, and this is most easily demonstrated under the simplifying assumption of chapter 5.

6.1 Definitions

Definition 6.1. For any $(\bar{a},\bar{n})_s \in \mathcal{A}$ we define the following:
(a) For any $1 \leqslant i \leqslant s$, $\gamma_i(\bar{a},\bar{n}) = a_1p^{n_1} + \cdots + a_ip^{n_i}$. By convention we set $\gamma_0(\bar{a},\bar{n}) = 0$.
(b) $s_{(+)}(\bar{a},\bar{n}) = \max\{0 \leqslant i \leqslant s : cp^{n_1} - \gamma_i(\bar{a},\bar{n}) > 0\}$.
(c) $s_{(0)}(\bar{a},\bar{n}) = \max\{0 \leqslant i \leqslant s : cp^{n_1} - \gamma_i(\bar{a},\bar{n}) \geqslant 0\}$.

Remark: Note that if $(\bar{a},\bar{n}') = \sigma^m(\bar{a},\bar{n})$ for some $m \in \mathbb{Z}$ then $\gamma_i(\bar{a},\bar{n}') = p^m(\gamma_i(\bar{a},\bar{n}))$ for all $0 \leqslant i \leqslant s$, and it follows easily that $s_{(+)}(\bar{a},\bar{n}') = s_{(+)}(\bar{a},\bar{n})$ and $s_{(0)}(\bar{a},\bar{n}') = s_{(0)}(\bar{a},\bar{n})$.

Recall that we defined a vector $(\bar{a}, \bar{n})$ to be admissible if it admits at least one admissible partition. The following proposition follows directly from our explicit characterisation of admissible partitions in proposition 5.8.

Proposition 6.2. *$(\bar{a}, \bar{n}) \in \mathcal{A}$ is admissible if and only if $\gamma(\bar{a}, \bar{n}) - cp^{n_1} \in \mathbb{Z}$, and one of the following holds:*

(a) *$\gamma(\bar{a}, \bar{n}) < cp^{n_1}$ and $n_1 \geqslant \cdots \geqslant n_s > 0$.*

(b) *$\gamma(\bar{a}, \bar{n}) \geqslant cp^{n_1}$ with $n_1 \geqslant \cdots \geqslant n_{s_{(+)}+1} \leqslant \ldots \leqslant n_s \leqslant 0$, and $n_i - n_{i+1} < N_0$ for any $s_{(+)} < i \leqslant s_{(0)}$.*

Definition 6.3. For any $(\bar{a}, \bar{n}) \in \mathcal{A}$ there is some (unique) $1 \leqslant m \leqslant s$ and $1 \leqslant s_1 < \cdots < s_m = s$ such that

$$n_1 = \cdots = n_{s_1} \neq n_{s_1+1} = \cdots = n_{s_2} \neq \cdots \neq n_{s_{m-1}+1} = \cdots = n_s.$$

We say the elements $s_1, \ldots, s_m$ are *structural points*, and for any vector we denote by $S(\bar{a}, \bar{n}) = \{s_1, \ldots, s_m\}$ the set of structural points of $(\bar{a}, \bar{n})$.

Of course, any locally constant partition of $(\bar{a}, \bar{n})$ must respect these structural points, and in fact these structural points determine a minimal (in the sense of order) locally constant partition of $(\bar{a}, \bar{n})$, i.e. $\pi \in \mathcal{P}(\bar{a}, \bar{n})$ such that $\pi[i] = (a_{s_{i-1}+1}, n_{s_{i-1}+1}, \ldots, a_{s_i}, n_{s_i})$ for all $1 \leqslant i \leqslant l(= m)$. For admissible vectors such that $\gamma(\bar{a}, \bar{n}) < cp^{n_1}$ we can show that such a partition is in fact the only admissible partition of $(\bar{a}, \bar{n})$.

Proposition 6.4. *Let $(\bar{a}, \bar{n})$ be admissible such that $\gamma(\bar{a}, \bar{n}) < cp^{n_1}$, and let $S(\bar{a}, \bar{n}) = \{s_1, \ldots, s_m\}$. Then $(\bar{a}, \bar{n})$ admits the unique admissible partition $\pi = \{\pi[i] : 1 \leqslant i \leqslant l\}$, where $l = m$, $u_1 = s_1$ and $u_i = s_i - s_{i-1}$ for all $2 \leqslant i \leqslant l$.*

Proof. Consider the partition $\pi = \{\pi[i] : 1 \leqslant i \leqslant l\}$, where $l = m$, $u_1 = s_1$ and $u_i = s_i - s_{i-1}$ for all $2 \leqslant i \leqslant l$. As $(\bar{a}, \bar{n})$ is admissible and $\gamma(\bar{a}, \bar{n}) < cp^{n_1}$, then $n_1 \geqslant \cdots \geqslant n_s > 0$ by proposition 6.2, and by definition of the structural points it follows that for the partition π we have $\bar{n}_1 > \cdots > \bar{n}_l > 0$, and thus π is admissible by proposition 5.8.

Now let $\pi' \in \mathcal{P}(\bar{a}, \bar{n})$ be a partition distinct from π, then there is some $1 \leqslant t \leqslant l'$ such that $u'_1 + \cdots + u'_t \neq s_i$ for any $s_i \in S(\bar{a}, \bar{n})$. But for such t we have $\bar{n}'_t = \bar{n}'_{t+1}$, and since $cp^{\bar{n}'_1} > \gamma(\pi')$ then π' is not admissible by proposition 5.8. □

Using the expression for $\kappa_\sigma(\bar{a}, \bar{n})$ from definition 5.20 we have the following corollary.

Corollary 6.5. *Let $(\bar{a}, \bar{n})$ be admissible such that $\gamma(\bar{a}, \bar{n}) < cp^{n_1}$, and let $S(\bar{a}, \bar{n}) = \{s_1, \ldots, s_m\}$. Then $\kappa_\sigma(\bar{a}, \bar{n}) = (s_1!(s_2 - s_1)! \cdots (s_m - s_{m-1})!)^{-1}$.*

As we saw in definition 2.5 the coefficient in the above corollary appears in the nilpotent Artin-Schreier as $\eta(\bar{n})$ in connection with the generators of the ramification ideals, and will be of use in chapter 7 when we compare results with [5]. As such we give the following definition of $\eta(\bar{n})$ in terms of our notation.

Definition 6.6. Let $(\bar{a}, \bar{n}) \in \mathcal{A}$, and let $S(\bar{a}, \bar{n})$ be its set of structural points. If $n_{s_1} > \cdots > n_{s_m}$ then $\eta(\bar{n}) = (s_1!(s_2 - s_1)! \cdots (s_m - s_{m-1})!)^{-1}$, and $\eta(\bar{n}) = 0$ otherwise. By convention we set $\eta(\emptyset) = 1$.

As noted at the beginning of the chapter, if $(\bar{a}, \bar{n})$ is an admissible vector such that $\gamma(\bar{a}, \bar{n}) \geqslant cp^{n_1}$ then the situation is not as simple.

Proposition 6.7. *Let $(\bar{a}, \bar{n})$ be an admissible vector such that $n_1 = \cdots = n_s$, and $\gamma(\bar{a}, \bar{n}) \geqslant cp^{n_1}$. Then for any $\pi \in \mathcal{P}(\bar{a}, \bar{n})$, π is admissible if and only if $u_1 > s_{(+)}$.*

Proof. By assumption we have $\gamma(\bar{a}, \bar{n}) - cp^{n_1} \in \mathbb{Z}_{\geqslant 0}$, and hence $s_{(+)} < s$. Let $\pi \in \mathcal{P}(\bar{a}, \bar{n})$ and assume that $u_1 > s_{(+)}$, Then for all $1 \leqslant t \leqslant l$ we have $\gamma(\pi_{\leqslant t}) - cp^{\bar{n}_1} \geqslant 0$, and $\bar{n}_t = \bar{n}_{t+1}$, thus $\pi \in \mathcal{P}_{\mathrm{adm}}(\bar{a}, \bar{n})$ by proposition 5.8.

On the other hand, if $u_1 \leqslant s_{(+)}$, then $\gamma(\pi_{\leqslant 1}) - cp^{\bar{n}_1} < 0$, and since $\bar{n}_1 = \bar{n}_2$ then $\pi \notin \mathcal{P}_{\mathrm{adm}}(\bar{a}, \bar{n})$ by proposition 5.8. □

The above implies that if $(\bar{a}, \bar{n})_s$ is as stated in the proposition then any choice of partition in the 'non-positive part' is admissible. In fact, in this case one can show that $(\bar{a}, \bar{n})_s$ admits 2^r admissible partitions, where $r = s - s_{(+)} - 1$.

A similar result holds for a general admissible vector, and because of the large number of admissible partitions for such vectors it is unlikely that we will find a simple expression for the coefficients $\kappa_{V_a}(\bar{a},\bar{n})$ and $\kappa_\sigma(\bar{a},\bar{n})$ in general. However, we can recover a reasonably simple recurrence formula for 'non-degenerate' vectors, from which we can deduce some properties of the coefficients $\kappa_{V_a}(\bar{a},\bar{n})$ and $\kappa_\sigma(\bar{a},\bar{n})$, and give some special cases.

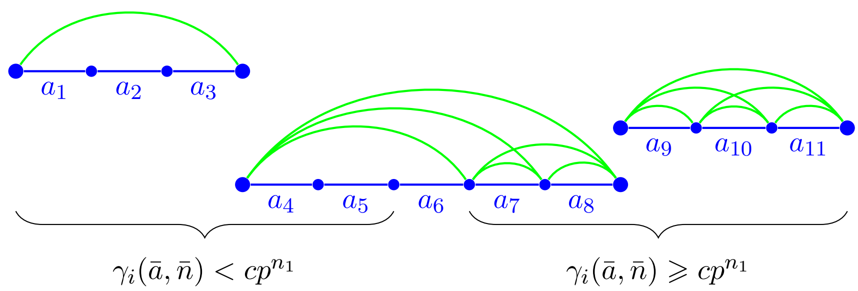

Figure 2: Heuristic interpretation of all admissible partitions of a vector $(\bar{a},\bar{n})$*, with* $S(\bar{a},\bar{n}) = \{3,8,11\}$ *and* $s_{(+)}(\bar{a},\bar{n}) = 5$.

6.2 Coefficient for non-degenerate vectors

Definition 6.8. We say a vector $(\bar{a},\bar{n})$ is non-degenerate if $\gamma_i(\bar{a},\bar{n}) \neq cp^{n_1}$ for all $1 \leqslant i \leqslant s$.

Note that any partition of a non-degenerate vector $(\bar{a},\bar{n})$ is necessarily non-degenerate (see definition 5.14).

Definition 6.9. For any $1 \leqslant i \leqslant s$ let $\lambda_i(\bar{a},\bar{n}) = 0$ if $n_1 = \cdots = n_i$, and $\lambda_i(\bar{a},\bar{n}) = \max\{s_j \in S(\bar{a},\bar{n}) : s_j < i\}$ otherwise.

Definition 6.10. Let $(\bar{a},\bar{n})_s$ be admissible. For any $0 \leqslant i < s$, let $\delta_i(\bar{a},\bar{n}) = 1$ if $v_p(cp^{n_1} - \gamma(\bar{a},\bar{n})_i) = n_{i+1}$, and $\delta_i(\bar{a},\bar{n}) = 0$ otherwise.

The following properties follow easily from basic definitions.

Proposition 6.11. *Let* $(\bar{a},\bar{n})$ *and* $(\bar{a},\bar{n}')$ *be admissible vectors such that* $(\bar{a},\bar{n}') = \sigma^m(\bar{a},\bar{n})$ *for some* $m \in \mathbb{Z}$*, then:*
(a) $S(\bar{a},\bar{n}') = S(\bar{a},\bar{n})$.
(b) $\lambda_i(\bar{a},\bar{n}') = \lambda_i(\bar{a},\bar{n})$ *for all* $1 \leqslant i \leqslant s$.
(c) $\delta_i(\bar{a},\bar{n}') = \delta_i(\bar{a},\bar{n})$ *for all* $0 \leqslant i < s$.

Definition 6.12. For any non-degenerate admissible vector $(\bar{a}, \bar{n})$ such that $\gamma(\bar{a}, \bar{n}) > cp^{n_1}$ we define recursively the coefficient $\kappa(\bar{a}, \bar{n}) = \kappa_s(\bar{a}, \bar{n})$, where for any $s_{(+)} < i \leqslant s$:

$$\kappa_i(\bar{a}, \bar{n}) = \begin{cases} -\eta(n_1, \ldots, n_i) - \displaystyle\sum_{s_{(+)} < j < i} \frac{\kappa_j(\bar{a}, \bar{n})}{(i - j + \delta_j)!} & \text{if } \lambda_i(\bar{a}, \bar{n}) \leqslant s_{(+)} \\ - \displaystyle\sum_{\lambda_i \leqslant j < i} \frac{\kappa_j(\bar{a}, \bar{n})}{(i - j + \delta_j)!} & \text{if } \lambda_i(\bar{a}, \bar{n}) > s_{(0)} \end{cases}$$

Remark: For any non-degenerate vector $s_{(+)} = s_{(0)}$, but the use of $s_{(0)}$ in the above definition is deliberate, as that case is also valid for degenerate vectors.

From the properties stated in proposition 6.11 then we have the following property of $\kappa(\bar{a}, \bar{n})$.

Proposition 6.13. *Let $(\bar{a}, \bar{n})$ and $(\bar{a}, \bar{n}')$ be non-degenerate admissible vectors such that $(\bar{a}, \bar{n}') = \sigma^m(\bar{a}, \bar{n})$ for some $m \in \mathbb{Z}$, then $\kappa(\bar{a}, \bar{n}) = \kappa(\bar{a}, \bar{n}')$.*

Theorem 6.14. *Let $(\bar{a}, \bar{n})$ be a non-degenerate admissible vector such that $\gamma(\bar{a}, \bar{n}) > cp^{n_1}$.*
(a) *If $(\bar{a}, \bar{n})$ is V_a-admissible then $\kappa_{V_a}(\bar{a}, \bar{n}) = \kappa(\bar{a}, \bar{n})$.*
(a) *If $(\bar{a}, \bar{n})$ is σ-admissible then $\kappa_\sigma(\bar{a}, \bar{n}) = \kappa(\bar{a}, \bar{n})$.*

Proof. Recall from definition 5.24 that for any $(\bar{a}, \bar{n}) \in \mathcal{A}$,

$$\kappa_\sigma(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_\sigma(\bar{a}, \bar{n})} \kappa_\sigma(\pi), \quad \text{and} \quad \kappa_{V_a}(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_{V_a}(\bar{a}, \bar{n})} \kappa_V(\pi).$$

As $(\bar{a}, \bar{n})$ is non-degenerate, then all partitions of $(\bar{a}, \bar{n})$ are non-degenerate and we can use the simpler formula for $\kappa_\sigma(\pi)$ and $\kappa_V(\pi)$ from definition 5.16. Moreover, if $(\bar{a}, \bar{n})$ is V_a-admissible then all admissible partitions of $(\bar{a}, \bar{n})$ are V_a-admissible, and similarly if $(\bar{a}, \bar{n})$ is σ-admissible. Therefore to prove the proposition it is sufficient to prove for any non-degenerate admissible $(\bar{a}, \bar{n})$, that $\kappa(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_{\mathrm{adm}}(\bar{a}, \bar{n})} \kappa(\pi)$.

Let $(\bar{a}, \bar{n}) \in \mathcal{A}$ be admissible such that $\gamma(\bar{a}, \bar{n}) > cp^{n_1}$. By proposition 6.13 we can also assume that $n_s = 0$. Under these assumptions it follows easily from basic definitions that for any $\pi \in P(\bar{a}, \bar{n})$ of order $\geqslant 2$, π is admissible if and only if $\pi_{\leqslant l-1}$ is admissible.

If $s = s_{(+)} + 1$, then $\gamma_i(\bar{a}, \bar{n}) > cp^{n_1}$ for all $1 \leqslant i < s$, and it follows in the same way as proposition 6.4 that $(\bar{a}, \bar{n})$ admits a unique admissible partition π corresponding to the structural points $S(\bar{a}, \bar{n})$. Moreover, as $s = s_{(+)} + 1$ we have $l - l_{(0)} = 1$ and $\delta_\pi[t] = 0$ for all $0 \leqslant t < l$. Thus by definition 5.16 we have $\sum_{\pi \in \mathcal{P}_{\text{adm}}(\bar{a},\bar{n})} \kappa(\pi) = -\eta(\bar{n})$, which is precisely the coefficient $\kappa(\bar{a}, \bar{n})$ as given in definition 6.12.

Suppose then that $s > s_{(+)} + 1$, and that the proposition is true for all vectors $(\bar{a}', \bar{n}')_j$ such that $\lambda_s \leqslant j < s$ and $(\bar{a}', \bar{n}')_j = (a_1, n_1, \ldots, a_j, n_j)$, i.e. the elements of the vector $(\bar{a}', \bar{n}')$ agree with the elements a_i and n_i of the vector $(\bar{a}, \bar{n})$ for $1 \leqslant i \leqslant j$. For any $\lambda_s \leqslant j < s$, if we fix $\pi[l] = (a_{j+1}, n_{j+1}, \ldots, a_s, n_s)_{s-j}$, then the set of all partitions in $P(\bar{a}, \bar{n})$ such that $\pi[l] = (a_{j+1}, n_{j+1}, \ldots, a_s, n_s)_{s-j}$ is in bijection with $P(\bar{a}', \bar{n}')_j$ via the map $\pi \mapsto \pi_{\leqslant l-1}$. In particular, if $(\bar{a}', \bar{n}')_j$ is admissible, then by our induction assumption the contribution to the coefficient $\kappa(\bar{a}, \bar{n})$ is given by $-\frac{1}{(s-j+\delta_j)!}\kappa(\bar{a}', \bar{n}')_j$, and if $(\bar{a}', \bar{n}')_j$ is not admissible then there is no contribution to the coefficient $\kappa(\bar{a}, \bar{n})$.

The proposition follows from the above by noting that if $\lambda_s \leqslant s_{(+)}$, then $\delta_{\lambda_s} = 0$, and $(\bar{a}', \bar{n}')_j = (a_1, n_1, \ldots, a_j, n_j)$ is admissible if and only if $j = \lambda_s$ or $s_{(+)} < j < s$, and if $\lambda_s > s_{(+)}$ then $(\bar{a}', \bar{n}')_j = (a_1, n_1, \ldots, a_j, n_j)$ is admissible for all $\lambda_s \leqslant j < s$. ∎

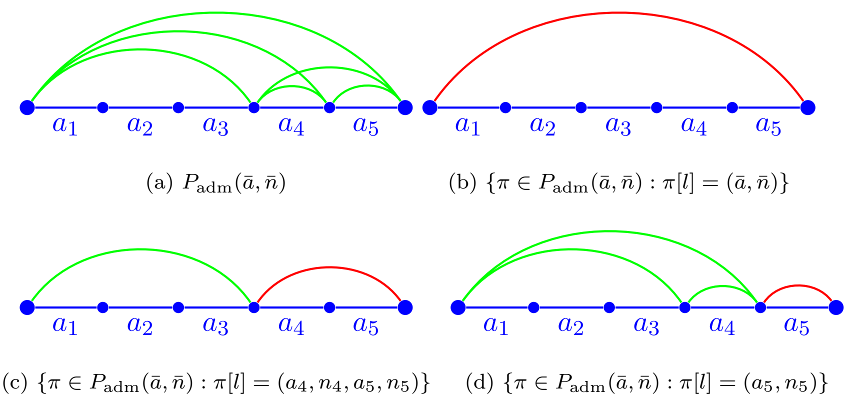

Figure 3: Heuristic interpretation of the recovery of $\kappa(\bar{a}, \bar{n})$, where $S(\bar{a}, \bar{n}) = \{5\}$, and $s_{(+)}(\bar{a}, \bar{n}) = 2$.

By the above proposition, we can study properties of the coefficients $\kappa_\sigma(\bar{a}, \bar{n})$ and $\kappa_{V_a}(\bar{a}, \bar{n})$ for non-degenerate vectors by studying the coefficient $\kappa(\bar{a}, \bar{n})$.

6.3 Properties of the coefficient $\kappa(\bar{a}, \bar{n})$.

Example 1: In the case that $(\bar{a}, \bar{n})_s$ is admissible and non-degenerate with $s = s_{(+)} + 1$ then $\kappa(\bar{a}, \bar{n})$ has a particularly simple form,

$$\kappa(\bar{a}, \bar{n}) = -\eta(\bar{n}).$$

Although $\gamma(\bar{a}, \bar{n}) > cp^{n_1}$, this is related to the situation in proposition 6.4; for the same reasons the vector still admits a unique admissible partition in this case.

Example 2: Let $(\bar{a}, \bar{n})_s$ be admissible. In general, if $\kappa_i(\bar{a}, \bar{n}) = 0$ for some $1 \leqslant i < s$ this does not imply that $\kappa(\bar{a}, \bar{n}) = 0$. However, if $i \in S(\bar{a}, \bar{n})$ and $i > s_{(0)}$ then the implication does hold.

Proposition 6.15. *Let $(\bar{a}, \bar{n})_s$ be a non-degenerate admissible vector. If $\kappa_{s_i}(\bar{a}, \bar{n}) = 0$ for some $s_i \in S(\bar{a}, \bar{n})$ with $s_i > s_{(0)}$, then $\kappa(\bar{a}, \bar{n}) = 0$.*

Proof. If $s_i = s$ then there is nothing to prove, so assume that $s_{(0)} < s_i < s$. It is sufficient to prove that $\kappa_{s_i}(\bar{a}, \bar{n}) = 0$ implies that $\kappa_{s_{i+1}}(\bar{a}, \bar{n}) = 0$.

Note that for any $s_i < i \leqslant s_{i+1}$ we have $\lambda_i = s_i$, and as $s_i > s_{(0)}$ then by definition 6.12 for any $s_i < i \leqslant s_{i+1}$ we have

$$\kappa_i(\bar{a}, \bar{n}) = \sum_{s_i \leqslant j < i} \frac{\kappa_j(\bar{a}, \bar{n})}{(i - j + \delta_j)!} \, .$$

If $i = s_i + 1$ then $\kappa_i(\bar{a}, \bar{n}) = \frac{\kappa_{s_i}(\bar{a}, \bar{n})}{(1+\delta_{s_i})!} = 0$. Similarly, for any $s_i < i \leqslant s_{i+1}$ if $\kappa_j(\bar{a}, \bar{n}) = 0$ for all $s_i \leqslant j < i$ then $\kappa_i(\bar{a}, \bar{n}) = 0$. By induction then $\kappa_{s_i}(\bar{a}, \bar{n}) = 0 \implies \kappa_{s_{i+1}}(\bar{a}, \bar{n}) = 0$.

The proposition follows by noting that if $\kappa_{s_i}(\bar{a}, \bar{n}) = 0$ then $\kappa_{s_j}(\bar{a}, \bar{n}) = 0$ for all $i \leqslant j \leqslant m$, and in particular $\kappa(\bar{a}, \bar{n}) = 0$. □

This proposition can be useful when performing explicit calculations, as in some cases we can use it to rule out the occurrence of vectors of a particular form by showing that the coefficient is 0 without needing to recover the coefficient in full.

Example 3

Proposition 6.16. *Let $(\bar{a}, \bar{n})_s$ be a non-degenerate admissible vector. If there is some $2 \leqslant i \leqslant s$ such that $i - \lambda_i \geqslant 2$ with $\delta_{\lambda_i} = 0$ and $\delta_{\lambda_i+1} = 1$, then $\kappa(\bar{a}, \bar{n}) = 0$.*

Proof. Note that $\lambda_i \in S(\bar{a}, \bar{n})$ by definition, therefore we set $\lambda_i = s_i$ and prove that the conditions in the proposition imply that $\kappa_{s_{i+1}}(\bar{a}, \bar{n}) = 0$, in which case $\kappa(\bar{a}, \bar{n}) = 0$ by proposition 6.15.

Case 1: $s_i \leqslant s_{(+)}$.
By assumption $\delta_{s_i+1} = 1$, and hence $\kappa_{s_1+1}(\bar{a}, \bar{n}) = -\eta(n_1, \ldots, n_{s_i+1})$, and $\kappa_{s_i+2}(\bar{a}, \bar{n}) = -\eta(n_1, \ldots, n_{s_i+2}) + \frac{1}{2!}\eta(n_1, \ldots, n_{s+1})$.

As $s_i > s_i + 1$ then $\eta(n_1, \ldots, n_{s_i+1}) = \eta(n_1, \ldots, n_{s_i}) \cdot 1$ and $\eta(n_1, \ldots, n_{s_i+2}) = \eta(n_1, \ldots, n_{s_i}) \cdot \frac{1}{2!}$, (if $s_i = 0$ we recall that $\eta(\emptyset) = 0$). Hence $\kappa_{s_i+2}(\bar{a}, \bar{n}) = 0$.

If $s_{i+1} = s_i + 2$ then we are done, otherwise assume that $s_i + 2 < i \leqslant s_{i+1}$, and that $\kappa_j(\bar{a}, \bar{n}) = 0$ for all $s_i + 2 \leqslant j < i$. We have the following expression for $\kappa_i(\bar{a}, \bar{n})$.

$$\begin{aligned} \kappa_i(\bar{a}, \bar{n}) &= -\eta(n_1, \ldots, n_i) - \sum_{s_i < j < i} \frac{\kappa_j(\bar{a}, \bar{n})}{(i - j + \delta_j)!} \\ &= -\eta(n_1, \ldots, n_i) - \frac{\kappa_{s_i+1}(\bar{a}, \bar{n})}{(i - s_i)!} \end{aligned}$$

But $\kappa_{s_i+1}(\bar{a}, \bar{n}) = -\eta(n_1, \ldots, n_{s_i}) \cdot 1$, and $\eta(n_1, \ldots, n_i) = \eta(n_1, \ldots, n_{s_i}) \cdot \frac{1}{(i-s_i)!}$, and thus $\kappa_i(\bar{a}, \bar{n}) = 0$.

It follows (for case 1) by induction that $\kappa_{s_{i+1}}(\bar{a}, \bar{n}) = 0$ and thus $\kappa(\bar{a}, \bar{n}) = 0$ by proposition 6.15.

Case 2: $s_i > s_{(+)}$.
This follows in a similar way. By assumption, $\delta_{s_i} = 0$ and $\delta_{s_i+1} = 1$, therefore $\kappa_{s_i+1}(\bar{a}, \bar{n}) = -\kappa_{s_i}(\bar{a}, \bar{n})$, and $\kappa_{s_i+2}(\bar{a}, \bar{n}) = -\frac{1}{2!}\kappa_{s_i}(\bar{a}, \bar{n}) + \frac{1}{2!}\kappa_{s_i}(\bar{a}, \bar{n}) = 0$.

If $s_{i+1} = s_i + 2$ then we are done, otherwise assume that $s_i + 2 < i \leqslant s_{i+1}$, and that $\kappa_j(\bar{a}, \bar{n}) = 0$ for all $s_i + 2 \leqslant j < i$. We have the following expression for $\kappa_i(\bar{a},\bar{n})$.

$$\begin{aligned}\kappa_i(\bar{a},\bar{n}) &= -\sum_{s_i \leqslant j < i} \frac{\kappa_j(\bar{a},\bar{n})}{(i-j+\delta_j)!} \\ &= -\frac{\kappa_{s_i}(\bar{a},\bar{n})}{(i-s_i)!} - \frac{\kappa_{s_i+1}(\bar{a},\bar{n})}{(i-s_i)!}\end{aligned}$$

But $\kappa_{s_i+1}(\bar{a},\bar{n}) = -\kappa_{s_i}(\bar{a},\bar{n})$, thus $\kappa_i(\bar{a},\bar{n}) = 0$. It follows (for case 2) by induction that $\kappa_{s_{i+1}}(\bar{a},\bar{n}) = 0$, hence $\kappa(\bar{a},\bar{n}) = 0$ by proposition 6.15. □

We can apply the above proposition to recover the following special case.

Proposition 6.17. *Let $(\bar{a},\bar{n})_s$ be an admissible vector such that $s \geqslant 2$ and $a_1 > c$. Then $\kappa(\bar{a},\bar{n}) = 0$.*

Proof. Note that, as $a_1 > c$, then $s_{(+)} = s_{(0)} = 0$, therefore $(\bar{a},\bar{n})$ is non-degenerate, and as $(\bar{a},\bar{n})$ is admissible then it follows from proposition 6.2 that $n_1 \leqslant \cdots \leqslant n_s \leqslant 0$.

By definition, $\delta_0(\bar{a},\bar{n}) = 0$. We will also establish that $\delta_1(\bar{a},\bar{n}) = 1$ and $n_1 = n_2$, in which case the result will follow from proposition 6.16.

As $a_1 > c$, then $a_1 \in \mathbb{Z}^+(p)$, and as $c \in p\mathbb{N}$ it follows that $v_p(cp^{n_1} - a_1p^{n_1}) = n_1$. As stated above, $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_s \leqslant 0$. Suppose for contradiction that $n_1 < n_2$, then $v_p(cp^{n_1} - \gamma(\bar{a},\bar{n})) = n_1 < 0$ (use that $v_p(a_2p^{n_2} + \cdots a_sp^{n_s}) \geqslant n_2$). But $(\bar{a},\bar{n})$ is admissible with $\gamma(\bar{a},\bar{n}) - cp^{n_1} \in \mathbb{N}$, and we have a contradiction.

It follows that $n_1 = n_2$, and $\delta_1(\bar{a},\bar{n}) = 1$ (as $v_p(cp^{n_1} - a_1p^{n_1}) = n_1$). Moreover, $\lambda_2(\bar{a},\bar{n}) = 0$ and thus $\kappa(\bar{a},\bar{n}) = 0$ by proposition 6.16. □

Example 4 As a final example we have the following special case.

Proposition 6.18. *Let $(\bar{a},\bar{n})$ be a non degenerate admissible vector. If $1 \leqslant i \leqslant s$ is such that $\delta_{\lambda_i} = \cdots = \delta_{i-1} = 1$, then*

$$\kappa_i(\bar{a},\bar{n}) = \kappa_{\lambda_i}(\bar{a},\bar{n}) \cdot \frac{B_{(i-\lambda_i)}}{(i-\lambda_i)!}$$

where B_n denotes the n-th Bernoulli number.

Proof. As $\delta_i = 0$ for all $i \leqslant s_{(+)}$ then $\delta_{\lambda_i} = 1$ implies that $\lambda_i > s_{(+)}$, and as $\delta_{\lambda_i} = \cdots = \delta_{i-1} = 1$ then $\kappa_i(\bar{a}, \bar{n}) = \sum_{\lambda_i \leqslant j < i} \frac{\kappa_j(\bar{a},\bar{n})}{(i-j+1)!}$. We prove the proposition by induction on the length $i - \lambda_i$.

Induction base: Let $i = \lambda_i + 1$. As $\delta_{\lambda_i} = 1$, then $\kappa_{\lambda_i+1}(\bar{a}, \bar{n}) = -\frac{1}{2!}\kappa_{\lambda_i}(\bar{a}, \bar{n})$. To establish the inductive base it is enough to note that $\frac{B_1}{1!} = -\frac{1}{2!}$.

Inductive step: Let $i > \lambda_i + 1$ and assume the proposition is true for all $\lambda_i \leqslant j < i$. The key to proving the inductive step is the following well known recurrence relation for the Bernoulli numbers, which holds for all $n \geqslant 1$.

$$\sum_{0 \leqslant j \leqslant n} \binom{n+1}{j} B_j = 0$$

Dividing through by $(n+1)!$ and rearranging we obtain the following identity, which holds for any $n \geqslant 1$.

$$\frac{B_n}{n!} = -\sum_{0 \leqslant j < n} \frac{B_j}{(n-j+1)!j!} \tag{6.1}$$

As $\delta_j = 1$ for all $\lambda_i \leqslant j < i$ then by our induction assumption we have

$$\begin{aligned}\kappa_i(\bar{a}, \bar{n}) &= -\kappa_{\lambda_i}(\bar{a}, \bar{n}) \cdot \sum_{\lambda_i \leqslant j < i} \frac{B_{j-\lambda_i}}{(i-j+1)!(j-\lambda_i)!} \\ &= -\kappa_{\lambda_i}(\bar{a}, \bar{n}) \cdot \sum_{0 \leqslant j' < i-\lambda_i} \frac{B_{j'}}{(i-\lambda_i-j'+1)!(j')!}\end{aligned}$$

Using (6.1) with $n = i - \lambda_i$ we see that $\kappa_i(\bar{a}, \bar{n}) = \kappa_{\lambda_i}(\bar{a}, \bar{n}) \cdot \dfrac{B_{i-\lambda_i}}{(i-\lambda_i)!}$ as required. This completes the inductive step, and the proposition follows for any length $i - \lambda_i$ by induction. □

In particular, as $B_n = 0$ for all odd $n > 1$, then the above proposition can be combined effectively with proposition 6.15 when carrying out explicit calculations.

The appearance of the Bernoulli numbers is not surprising, as the recovery of the coefficient is essentially a combinatorial problem, in which context the Bernoulli numbers frequently arise.

6.4 General remarks

The results of this chapter demonstrate that, for non-degenerate vectors, although the coefficient is complicated in general it is still possible to work explicitly with the associated terms, and extract general properties. The recovery of the coefficient is very accessible to combinatorial methods, and for vectors of short length the recurrence formula is easily and quickly handled by programs such as Mathematica or Maple.

It is possible to define a recurrence formula for the coefficients $\kappa_{V_a}(\bar{a}, \bar{n})$ and $\kappa_\sigma(\bar{a}, \bar{n})$ for degenerate vectors $(\bar{a}, \bar{n})$ in a very similar way, however the expression is not concise due to the involved coefficients α_0, and the fact that terms arise from both V_0 and $\sigma\bar{c}_1$. For this reason we do not give an explicit formula, but note that if $(\bar{a}, \bar{n})$ is a degenerate vector such that $\gamma(\bar{a}, \bar{n}) > cp^{n_1}$ then for any $s_{(0)} < i \leqslant s$ such that $\lambda_i > s_{(0)}$ that portion of the coefficient is recovered as per definition 6.12, thus many of the properties of this section can also be applied to degenerate terms.

7 General solution of the recurrence formula

All results in chapters 5 and 6 were obtained under the simplifying assumption that $\omega_\tau^p = t^c$ with $c \in p\mathbb{N}$. In the general setting we need to consider $\omega_\tau^p = \sum_{j\geqslant 0} A_j t^{e^*+pj}$, where $A_j \in k$ and $A_0 \neq 0$, therefore in order to provide a general solution of recurrence formula (3.6) we need to make some formal adjustments to our notation.

Firstly, for any given vector $(\bar{a}, \bar{n}) \in \mathcal{A}$, all properties established in the previous chapters correspond to $c = e^* + pj$ for a fixed choice of $j \geqslant 0$. For example, suppose $(\bar{a}, \bar{n}) \in \mathcal{A}$ is such that $\bar{n} = \bar{0}$ and $\gamma(\bar{a}, \bar{n}) = e^* + 1$. Then $(\bar{a}, \bar{n})$ is an admissible vector with respect to $c = e^*$, but not with respect to $c = e^* + p$ (as $\gamma(\bar{a}, \bar{n}) < e^* + p$ and $\bar{n} = \bar{0}$). As such, we adapt the notation of chapter 5 to reflect a specific choice of $j \geqslant 0$.

Definition 7.1. Let $\omega_\tau^p = \sum_{j\geqslant 0} A_j t^{e^*+pj}$, then for any $j \geqslant 0$ we denote by $\mathcal{P}_{\mathrm{adm},j}$ the set of all admissible partitions with respect to $c = e^* + pj$. Similarly we set $\mathcal{P}_{\sigma,j} = \{\pi \in \mathcal{P}_{\mathrm{adm},j} : \gamma(\pi) - (e^* + pj)p^{\bar{n}_1} \in p\mathbb{Z}\}$, and for any $a \in \mathbb{Z}^0(p)$ we denote by $\mathcal{P}_{V_a,j} = \{\pi \in \mathcal{P}_{\mathrm{adm},j} : \gamma(\pi) = (e^* + pj)p^{\bar{n}_1} + a\}$.

Secondly, for any $j \geqslant 0$ in the expression for ω_τ^p we have an associated coefficient $A_j \in k$, which appears in (5.1a). For any partition $\pi \in \mathcal{P}_{\mathrm{adm},j}$ if one formally follows the coefficient A_j through the recurrent procedure one sees that the cumulative effect of σ on A_j is precisely the cumulative effect of σ on $\pi[1]$. Therefore we introduce the following definitions.

Definition 7.2. Let $\omega_\tau^p = \sum_{j\geqslant 0} A_j t^{e^*+pj}$, then for any $j \geqslant 0$ and $\pi \in \mathcal{P}_{\mathrm{adm},j}$

$$\kappa_{V,j}(\pi) = \sigma^{\bar{n}_1}(A_j) \cdot \kappa_V(\pi), \quad \text{and} \quad \kappa_{\sigma,j}(\pi) = \sigma^{\bar{n}_1}(A_j) \cdot \kappa_\sigma(\pi)\,,$$

where the coefficients $\kappa_V(\pi)$ and $\kappa_\sigma(\pi)$ are given in definition 5.20, with respect to $c = e^* + pj$.

Definition 7.3. Let $\omega_\tau^p = \sum_{j\geqslant 0} A_j t^{e^*+pj}$, then for any $j \geqslant 0$, and $(\bar{a}, \bar{n}) \in \mathcal{A}$

$$\kappa_{\sigma,j}(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_{\sigma,j}(\bar{a},\bar{n})} \kappa_{\sigma,j}(\pi), \quad \text{and} \quad \kappa_{V_a,j}(\bar{a}, \bar{n}) = \sum_{\pi \in \mathcal{P}_{V_a,j}(\bar{a},\bar{n})} \kappa_{V,j}(\pi)\,.$$

With these adjustments, for any fixed $j \geqslant 0$, the corresponding terms contributing to V_{a0} or $\sigma\bar{c}_1$ are given by theorem 5.23, and thus the following general solution of the recurrence formula follows directly from the results of chapter 5 by linearity.

Theorem 7.4. *Let $\omega_\tau^p = \sum_{j\geqslant 0} A_j t^{e^*+pj}$, then a solution of* (3.6) *is given by the following elements.*

(a) $$\sigma\bar{c}_1 = \sum_{j\geqslant 0}\sum_{(\bar{a},\bar{n})} \kappa_{\sigma,j}(\bar{a},\bar{n}) t^{(e^*+pj)p^{n_1}-\gamma(\bar{a},\bar{n})} D_{(\bar{a},\bar{n})}\,.$$

(b) *For any $a \in \mathbb{Z}^0(p)$,* $$V_{a0} = \sum_{j\geqslant 0}\sum_{(\bar{a},\bar{n})} \kappa_{V_a,j}(\bar{a},\bar{n}) D_{(\bar{a},\bar{n})}\,.$$

From our discussion in section 3.6, the elements V_{a0} in theorem 7.4 define a class of derivations $\mathrm{ad}_{\tau_{<p}} \in \mathrm{Der}(\mathcal{L}/\mathcal{L}(p))/\,\mathrm{Inn}(\mathcal{L}/\mathcal{L}(p))$, thus fully describing the structure of L, and hence the structure of $\Gamma_{<p}$.

It is regrettable that we are unable to present an immediate application of the results in this thesis. However, we have obtained a very precise characterisation of the elements V_{a0}; they appear as k-linear combinations of terms $D_{(\bar{a},\bar{n})}$ associated to admissible vectors, which are of a very predictable form. We have also demonstrated in chapter 6 that, for any $(\bar{a},\bar{n}) \in \mathcal{A}$, the recovery of the coefficients $\kappa_{\sigma,j}(\bar{a},\bar{n})$ and $\kappa_{V_a,j}(\bar{a},\bar{n})$ can be reduced to essentially combinatorial methods, from which general properties of the elements $D_{(\bar{a},\bar{n})}$ can be recovered. Due to the explicitness of the nilpotent Artin-Schreier theory, and our explicit recovery of the terms V_{a0}, we should certainly expect that the results of this thesis can be applied to study further the group $\Gamma_{<p}$.

As it is, in the next section we compare our description of the elements V_{a0} from theorem 7.4 with explicit calculations from [5], and in the final section we discuss briefly some opportunities for further study.

7.1 Comparison with explicit calculations in [5]

The following expressions for the elements $V_{a0} \bmod \mathcal{L}(3)_k$ for $a \in \mathbb{Z}^0(p)$ were given in [5, proposition 3.9].

$$V_{00} \equiv -\alpha_0 \sum_{\substack{j\geqslant 0\\ 0\leqslant n<N_0}} \sigma^{-n}(A_j\mathcal{F}^0_{e^*+pj,0}) \mod \mathcal{L}(3)_k\,,$$

$$V_{a0} \equiv -\sum_{\substack{n\geqslant 1\\ j\geqslant 0}} \sigma^n(A_j\mathcal{F}^0_{e^*+pj+ap^{-n},-n}) - \sum_{\substack{m\geqslant 0\\ j\geqslant 0}} \sigma^{-m}(A_j\mathcal{F}^0_{e^*+pj+ap^m,0}) \mod \mathcal{L}(3)_k\,.$$

Recall from section 2.5 that the terms $\mathcal{F}^0_{\gamma,-n}$ appear in connection with generators of the ramification groups under the identification η_0 of the nilpotent Artin-Schreier theory. Modulo $\mathcal{L}(3)_k$ the terms of V_{a0} can quite naturally be grouped in this way, as at this level the terms involved admit a unique admissible partition. Because the terms $\mathcal{F}^0_{\gamma,-n}$ appear naturally in the NAS-theory, as a first approach to the recovery of a solution of (3.6) we sought to recover a general solution involving these elements. Although this was achieved, the grouping of terms led to complicated expressions, and a complicated exposition (cf. [5, §5.2] for associated difficulties in recovering V_{00} in this form). Moreover, there was no meaningful control on the appearance of $\mathcal{F}^0_{\gamma,-n}$ in the various terms. The exposition given in this thesis is more natural with respect to the recurrent procedure, and hopefully provides a clearer picture of how the terms of V_{a0} and $\sigma\bar{c}_1$ and their coefficients arise.

Whilst the above expressions for $V_{a0} \bmod \mathcal{L}(3)_k$ from [5] can be recovered by continuing the explicit calculations from chapter 4, it is informative to recover the elements from theorem 7.4 above. We show that our solution matches the expression for V_{00} given above, and note that similar calculations can be carried out to verify that our solution matches the expression V_{a0} for any $a \in \mathbb{Z}^+(p)$.

Proposition 7.5.

$$\sum_{j\geqslant 0}\sum_{(\bar{a},\bar{n})} \kappa_{V_0,j}(\bar{a},\bar{n})D_{(\bar{a},\bar{n})} \equiv -\alpha_0 \sum_{\substack{j\geqslant 0\\ 0\leqslant n<N_0}} \sigma^{-n}(A_j\mathcal{F}^0_{e^*+pj,0}) \mod \mathcal{L}(3)_k\,.$$

Proof. It is sufficient to prove that the congruence in the proposition holds for any fixed $j \geqslant 0$. Suppose then that $j \geqslant 0$ is fixed, and consider first the sum on the left-hand side of the congruence. Note that for any $(\bar{a}, \bar{n}) \in \mathcal{A}$ the coefficient $\kappa_{V_0,j}(\bar{a}, \bar{n})$ is non-zero only if $(\bar{a}, \bar{n})$ is (V_0, j)-admissible. If $(\bar{a}, \bar{n})_1 \in \mathcal{A}$ we have $\gamma(\bar{a}, \bar{n}) \notin p\mathbb{N}$, and therefore there are no (V_0, j)-admissible vectors of length one. For any $(\bar{a}, \bar{n})_2 \in \mathcal{A}$, we have $\gamma(\bar{a}, \bar{n}) \in p\mathbb{N}$ only if $n_1 = n_2$. Therefore, modulo $\mathcal{L}(3)_k$ any (V_0, j)-admissible $(\bar{a}, \bar{n})$ is of the form $(a_1, -r, a_2, -r)$ with $a_1 + a_2 = e^* + pj$ and $0 \leqslant r < N_0$. Note that any such term admits the unique admissible partition $\pi = \pi[1] = \{(a_1, -r, a_2, -r)\}$ (use that $\gamma_1(\bar{a}, \bar{n}) < (e^* + pj)$ and $n_1 \leqslant 0$). Therefore, by definition 7.3 we have $\kappa_{V_0,j}(\bar{a}, \text{-}\bar{r}) = -\alpha_0 \sigma^{-r}(A_j)\frac{1}{2!}$.

Now consider the sum on the right-hand side of the congruence. By definition 2.5 the element $\mathcal{F}^0_{e^*+pj,0}$ is given as follows.

$$\mathcal{F}^0_{e^*+pj,0} \equiv \sum_{a_1 = e^*+pj} a_1 D_{a_1,0} + \sum_{a_1+a_2=e^*+pj} \frac{1}{2!} a_1 [D_{a_1,0}, D_{a_2,0}] \mod \mathcal{L}(3)_k$$

where $a_1, a_2 \in \mathbb{Z}^0(p)$.

Again, no term of length one can occur, so the first sum is empty. All terms of length two correspond to (V_0, j)-admissible vectors $(\bar{a}, \bar{0})_2$. It follows that for any $0 \leqslant r < N_0$ we have

$$\sum_{(\bar{a},\text{-}\bar{r})} \kappa_{V_0,j}(\bar{a}, \text{-}\bar{r}) D_{(\bar{a},\text{-}\bar{r})} \equiv -\alpha_0\, \sigma^{-r}(A_j \mathcal{F}^0_{e^*+pj,0}) \mod \mathcal{L}(3)_k\,.$$

As $(\bar{a}, \text{-}\bar{r})$ is not (V_0, j)-admissible for any $r \geqslant N_0$, then the congruence in the proposition holds for any choice of $j \geqslant 0$. □

7.2 Opportunities for further study

As noted in the introduction, our solution can not be considered completely satisfactory in terms of ramification properties, as the lift $\tau_{<p} \in$ of τ_0 described by the elements in theorem 7.4 is not of a form from which we recover explicitly the generators of the ramification groups of $\Gamma_{<p}$. In order to fully describe the ramification filtration for the group $\Gamma_{<p}$ additional care must be taken when specifying the choice of lift $\tau_{<p}$ via a solution of (3.6). Such lifts are described in [5] as *arithmetic lifts*, and although the full technical considerations are more

involved, the situation is completely analogous to the classical case for finite Galois extensions, where the incompatibility of the upper numbering with subgroups means that different lifts of τ_0 may belong to different ramification groups of $\Gamma_{<p}$ (see e.g [16, IV §3]).

In [5, §6.5] a description of the ramification filtration of $\Gamma_{<p}$ was given as follows. The nilpotent Artin-Schreier theory describes the ramification groups of $\mathcal{G}_{<p}$ as groups defined on ideals $\mathcal{L}^{(v)}$ of $\mathcal{L}$ (see section 2.5), and due to the compatibility of the field of norms functor with ramification, the images of $\mathcal{L}^{(v)}$ in $\bar{\mathcal{L}} = \mathcal{L}/\mathcal{L}(p)$ describe the ramification groups of the extension $K_{<p}/K(\pi_1)$ via the Herbrand function $\varphi_{\widetilde{K}/K}$ of the APF extension $\widetilde{K}/K$. Furthermore, the ramification filtration of the extension $K(\pi_1)/K$ has a simple description, with unique break in the upper (and lower) numbering corresponding to $e^* = pe_K/p-1$.

An arithmetic lift of $\tau_{<p}$ corresponds to a lift such that $\tau_{<p} \in L^{(e^*)}$, and $\tau_{<p} \notin L^{(v)}$ for any $v > e^*$. Therefore in [5, §6] it was recovered that for all $v > e^*$ the ramification groups of $\Gamma_{<p}$ are given by $G(\mathcal{L}^{(v')}/\mathcal{L}(p))$ where $v' = e^* + p(v - e^*)$ (using the Herbrand function for the extension $K(\pi_1)/K$). If $\tau_{<p}$ is an arithmetic lift of τ then for $v \leqslant e^*$ the ramification group $\Gamma_{<p}^{(v)}$ is generated by $\tau_{<p}$ and $G(\mathcal{L}^{(v)}/\mathcal{L}(p))$. As an application in [5] it was shown for $K[s] := K_{<p}^{C_{s+1}(L)}$ that the maximal upper ramification number of $K[s]/K$ is e^* if $s = 1$, and $e^* + (e^*(s-1)-1)/p$ if $2 \leqslant s < p$.

The following criterion was established in [5] to determine whether a lift $\tau_{<p}$, given by a solution $(\bar{c}_1, \{V_{a0}\}_{a\in\mathbb{Z}^0(p)})$, of (3.6) is arithmetic.

Theorem 7.6. [5, Theorem 4.8] *The following properties are equivalent.*

(a) $\tau_{<p}$ *is arithmetic.*

(b) $(\mathrm{Ad}_{\tau_{<p}} - \mathrm{id}_{\mathcal{L}})\mathcal{L} \subset \mathcal{L}^{(e^*)}$ *and for a sufficiently large* N*,*

$$\bar{c}_1 \equiv \sum_{\gamma,j\geqslant 0} \sum_{0\leqslant i<N} \sigma^i(A_j \mathcal{F}^0_{\gamma,-i} t^{-\gamma+e^*+pj}) \mod \mathcal{L}_{\mathcal{K}}^{(e^*)} + \mathcal{M}(p-1)\,.$$

(c) $(\mathrm{Ad}_{\tau_{<p}} - \mathrm{id}_{\mathcal{L}})\mathcal{L} \subset \mathcal{L}^{(e^*)}$ *and for a sufficiently large* N,

$$\bar{c}_1(0) \equiv \sum_{j \geqslant 0} \sum_{0 \leqslant i < N} \sigma^i(A_j \mathcal{F}^0_{e^*+pj,-i}) \mod \mathcal{L}_k^{(e^*)} + \mathcal{M}(p-1)\,.$$

Remark: For statement (c) of the theorem, we note that $\bar{c}_1(0)$ is used in [5] to denote the terms of $\bar{c}_1$ corresponding to t^0.

It would be interesting to establish whether our solution in theorem 7.4 corresponds to an arithmetic lift. With respect to the above criterion, any work in either showing the the solution given in theorem 7.4 is arithmetic, or making a different choice of lift to recover an arithmetic lift (i.e. by choosing a different $\bar{c}_1$), should come essentially within the 'degenerate portion' of a term, by which we mean the section of $(\bar{a}, \bar{n})$ such that $\gamma_i(\bar{a}, \bar{n}) = cp^{n_1}$. This reduces essentially to recovering the element V_{00} and the terms of $\bar{c}_1(0)$ in the appropriate form. Again, this is related to difficulties discussed in [5, §5.2].

As mentioned above, the maximal upper break of the ramification filtration for the extensions $K[s]/K$ was given in [5]. Another interesting avenue for further study would be to recover all ramification breaks for these extensions. As remarked in [5, Introduction], the ramification filtration of $\Gamma_K^p C_2(\Gamma_K)/\Gamma_K^p C_3(\Gamma_K)$ was studied in [22], however the explicit nature of the nilpotent Artin-Schreier theory, together with a complete description of the generators of the ramification ideals of L should allow for a deeper study of the ramification breaks of $\Gamma_{<p}$.

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