Diagrammatics for representation categories of quantum Lie superalgebras from skew Howe duality and categorification via foams

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Abstract: In this thesis we generalise quantum skew Howe duality to Lie superalgebras in type A, and show how this gives a categorification of certain representation categories of $\mathfrak{gl}(m|n)$.

In particular, we use skew Howe duality to describe a category of representations generated monoidally by the exterior powers of the fundamental representation. This description is in terms of MOY diagrams, with one additional local relation on $n+1$ strands. This generalises the $n=0$ case from Cautis, Kamnitzer and Morrison.

Using this, we give a categorification of this category in terms of foams, which generalises that of Queffelec, Rose and Lauda in the case $n=0$.

The Reshetikhin-Turaev procedure gives a knot polynomial associated to $\mathfrak{gl}(m|n)$, which is a specialisation of the HOMFLY polynomial $P(a,q)$ at $a=q^{m-n}$. For the case $n=0$, the polynomial can be described nicely in terms of MOY diagrams, and therefore is related strongly to skew Howe duality. This was used by Queffelec and Rose to define $\mathfrak{sl}(n)$ Khovanov-Rozansky homology by categorified skew Howe duality.

For general n, the relationship is less nice, and skew Howe duality is not sufficient to describe a homology theory associated with $\mathfrak{gl}(m|n)$ from our approach. Part of the problem is that the representation category no longer contains duals of the fundamental representations, which means that although a braid has an image in this categorified representation category, it is not possible to close this braid in the same way that Queffelec and Rose do. However, the categorified representation category does give partial progress towards the problem of defining a quantum categorification of the Alexander polynomial.

Declaration

The work in this thesis is based on research carried out in the Pure mathematics group, Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

1.1 Knot polynomials

The Alexander polynomial $\Delta_K(q) \in \mathbb{Z}[q, q^{-1}]$ is a classical knot invariant introduced by Alexander [Ale28] in 1928. This can be defined by the skein relation

$$\Delta_{\slashoverback}(q) - \Delta_{\backslashoverslash}(q) = (q - q^{-1})\Delta_{)(}(q)$$

meaning that the invariants of two oriented knots that differ only at a single crossing are related to the knot where the crossing is removed by the above formula. The value on the unknot is set to be $\Delta_{\bigcirc}(q) = 1$.

A much more recent knot invariant is the Jones polynomial [Jon85], defined as $V_K(q) \in \mathbb{Z}[q, q^{-1}]$ satisfying

$$q^2 V_{\slashoverback}(q) - q^{-2} V_{\backslashoverslash}(q) = (q - q^{-1}) V_{)(}(q)$$

and $V_{\bigcirc}(q) = q + q^{-1}$. This can be calculated by using (a variant of) the so-called Kauffman bracket [Kau87] by the rules

$$\langle \times \rangle = q \langle)(\rangle - \langle \asymp \rangle$$

$$\langle \bigcirc \rangle = q + q^{-1}$$

$$\langle D_1 \sqcup D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle$$

and setting

$$V_K(q) = (-1)^{n_{\backslashoverslash}} q^{n_{\backslashoverslash} - 2n_{\slashoverback}} \langle D \rangle$$

for any diagram D of K.

These polynomials were discovered in very different ways. The Alexander polynomial can be seen as arising from the first homology group of the cyclic covering space of the knot complement. The Jones polynomial, however, was defined in terms of a trace operator on the Hecke algebra. It may be surprising that they have such similar looking skein relations.

The Jones polynomial was quickly generalised to the HOMFLY polynomial [Fre+85] $P_K(a,q) \in \frac{a-a^{-1}}{q-q^{-1}}\mathbb{Z}[a, a^{-1}, q, q^{-1}]$

$$aP_{\times}(a,q) - a^{-1}P_{\times}(q) = (q-q^{-1})P_{)(}(a,q)$$

with $P_{\bigcirc}(a,q) = \frac{a-a^{-1}}{q-q^{-1}}$. This specialises to the Alexander polynomial at $a = 1$ and the Jones polynomial at $a = q^2$. The important special case of $a = q^n$ is called the $\mathfrak{sl}(n)$ polynomial for $n \geq 2$.

Reshetikhin and Turaev [RT90] realised that it was possible to generalise Jones's construction to produce a knot invariant for every pair $(\mathfrak{g}, V)$ where $\mathfrak{g}$ is a simple Lie algebra and V is a finite-dimensional representation of $\mathfrak{g}$. The $\mathfrak{sl}(n)$ polynomials arise as the choice $\mathfrak{g} = \mathfrak{sl}(n)$ and V the simple n-dimensional representation $\mathbb{C}^n$.

However, it is also possible to fit the Alexander polynomial into their framework by generalising to the case of Lie superalgebras $\mathfrak{gl}(m|n)$. Then the pair $(\mathfrak{gl}(1|1), \mathbb{C}^{1|1})$ gives rise to the Alexander polynomial, while in general the choice $(\mathfrak{gl}(m|n), \mathbb{C}^{m|n})$ gives rise to the HOMFLY polynomial specialised at $a = q^{m-n}$. Since $\mathfrak{gl}(m|n)$ is a Lie superalgebra of type A, we can refer to all of these quantum invariants as type A quantum knot invariants.

The Reshetikhin-Turaev invariants are calculated by first passing to the so-called quantum group $U_q(\mathfrak{g})$, which 'specialises' at $q = 1$ to $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Representations V of $\mathfrak{g}$ are in natural bijection with representations of $U_q(\mathfrak{g})$. Then the value of the Reshetikhin-Turaev invariant is the composition of morphisms of $U_q(\mathfrak{g})$-modules according to the knot diagram.

1.2 Skew Howe duality

Since the type A quantum knot invariants are defined in terms of morphisms of $U_q(\mathfrak{gl}(m|n))$ representations, it is useful to have an understanding of the morphisms of $U_q(\mathfrak{gl}(m|n))$ representations. A particularly appealing description is in terms of MOY diagrams. These were introduced to describe the morphisms that appear when one applies the Reshetikhin-Turaev process to $U_q(\mathfrak{sl}(n))$. However, Cautis, Kamnitzer and Morrison were able to use the technique of skew Howe duality to show that, in fact, all morphisms of the $U_q(\mathfrak{sl}(n))$ representations that appear can be written in terms of MOY diagrams. The method is to show that the category $\mathrm{Rep}(\mathfrak{sl}(n))$ is equivalent to a quotient of the category $\dot{U}_q(\mathfrak{gl}(\infty))$.

This was generalised by the author [Gra16] to the case of $U_q(\mathfrak{gl}(1|1))$, where a complete description of all morphisms between certain $U_q(\mathfrak{gl}(1|1))$ representations was found. This description appears in subsection 4.3.4 once the general machinery of skew Howe duality for $U_q(\mathfrak{gl}(m|n))$ is set up. The relations found were further generalised in [Gra15], and explained in subsection 4.3.5.

1.3 Categorification

Khovanov homology [Kho99] can be defined using the Kauffman bracket, by defining

$$\langle \bigcirc \rangle = V$$

where $V = \mathbb{C}[x]/(x^2)$ as a graded vector space with $\deg 1 = 1$ and $\deg x = -1$. Label the resolutions by 0 for $)($ and 1 for $\asymp$. Then a total smoothing corresponds to a vertex on the hypercube $\{0,1\}^n$ where n is the number of crossings. At each vertex, we have

$$\langle \bigcirc \sqcup \cdots \sqcup \bigcirc \rangle = V^{\otimes k}$$

where k is the number of circles in the total smoothing. Where two total smoothings differ only by one term in $\{0,1\}^n$, we add a map d to the corresponding edge in the cube, where the homogeneous map d

$$\langle \times \rangle = \Big(\; q\langle)(\rangle \xrightarrow{d} \langle \asymp \rangle \; \Big)$$

is defined by either the multiplication map $V \otimes V \to V$, or by the comultiplication

$$V \to V \otimes V : \begin{cases} 1 \mapsto 1 \otimes x + x \otimes 1 \\ x \mapsto x \otimes x \end{cases}$$

depending on whether the left-hand smoothing has one more or one fewer circle than the right-hand smoothing. Minus signs are added to the maps to ensure that $d^2 = 0$, and an overall shift in homological degree of $n_{\times}$ and shift in degree of the graded vector spaces by $n_{\times} - 2n_{\times}$ is applied. The homology of the resulting chain complex is independent of choices, giving a bigraded vector space $\mathrm{Kh}(K) = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(K)$ which is called Khovanov homology. By construction, this satisfies

$$\sum_{i,j} (-1)^i q^j \dim_{\mathbb{C}} \mathrm{Kh}^{i,j}(K) = V_K(q).$$

Khovanov homology was quickly shown to be an important tool in knot theory. It is a strictly stronger invariant than the Jones polynomial (for example, it distinguishes the knots 5_1 and 10_{132} from the Rolfsen knot table, which the Jones polynomial fails to do). Moreover, it has the property that if Σ is a knot cobordism from K_1 to K_2, ie. a smooth embedding $\Sigma : S^1 \times I \hookrightarrow S^3 \times I$ with $\Sigma(S^1 \times 0) = K_1$ and $\Sigma(S^1 \times 1) = K_2$, then it induces a map

$$\mathrm{Kh}(\Sigma) : \mathrm{Kh}(K_1) \to \mathrm{Kh}(K_2)$$

which is well-defined up to sign. This was exploited by Rasmussen [Ras04] to produce an invariant $s(K) \in \mathbb{Z}$ with

$$|s(K)| \leq 2g_*(K)$$

where $g_*(K)$ is the minimal genus surface S smoothly embedded in B^4 with $\partial S = K$. This gives a combinatorial proof of the so-called Milnor conjecture, that

$$g_*(T_{p,q}) = \frac{(p-1)(q-1)}{2}$$

where $T_{p,q}$ is the (p,q) torus knot.

Bar-Natan [BN04] gave a definition of Khovanov homology for tangles by using a category of cobordisms. Since the maps on V above give V the structure of a Frobenius algebra, it corresponds to a symmetric monoidal functor

$$\mathrm{Bord}_2 \to \mathrm{Vect}$$

where Bord_2 is the category of homeomorphism classes of oriented compact surfaces with boundary, and the objects are the boundary 1-manifolds. By linearising Bord_2 to $\mathbb{C}\,\mathrm{Bord}_2$ by taking formal linear combinations of 2-morphisms, we can take the quotient by the local relations in Figure 1.1, which all lie in the kernel of the functor, to form the category Foam_2.

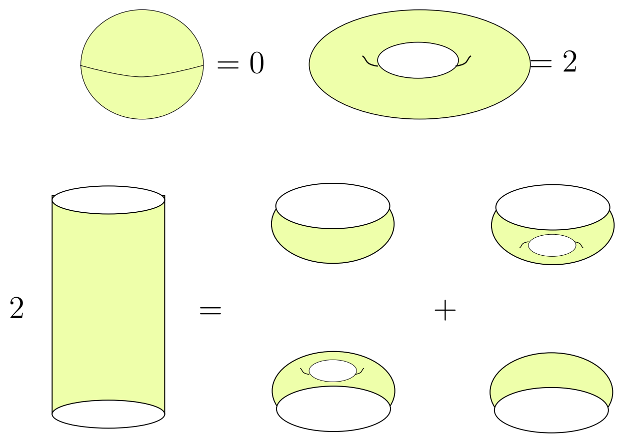


Figure 1.1: Bar-Natan's relations

We can define the Khovanov chain complex for tangles by forming chain complexes over Foam_2, where the differential is given by the saddle cobordism Figure 1.2 that

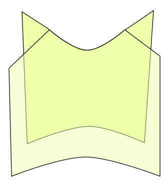


Figure 1.2: Saddle cobordism

either splits two circles or merges two circles. Of course, Foam_2 does not have kernels or cokernels in general, so we cannot take homology, but by applying the representable functor

$$\mathrm{Hom}_{\mathrm{Foam}_2}(-, \emptyset) : \mathrm{Foam}_2 \to \mathrm{Vect}$$

we recover the Khovanov chain complex in the case that we have a knot diagram.

Khovanov [Kho03] generalised the approach to define $\mathfrak{sl}(3)$ homology by allowing singular seams into the cobordisms.

Mackaay, Stošić and Vaz [MSV09] also defined foams for $\mathfrak{sl}(n)$. This involves seamed surfaces, which are also decorated with colours and dots, and further involves singular points where two singular seams can intersect. The saddle in Figure 1.2 becomes the seamed surface in Figure 1.3. The relations become difficult to find in this case, and the

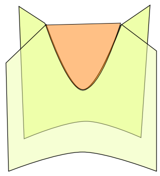


Figure 1.3: Seamed saddle cobordism

authors rely on an analytic formula to ensure they are able to evaluate all closed foams, meaning that their approach is not entirely combinatorial. However, they were able to prove their homology theory is isomorphic to one defined by Khovanov and Rozansky [KR04] by another method.

1.3.1 Categorified Skew Howe duality

In [LQR15], Lauda, Queffelec and Rose show that, in fact, the category Foam_2 arises as a 2-representation of a particular 2-category from representation theory. The 2-categories $\dot{\mathcal{U}}_Q(\mathfrak{gl}(p))$ were defined by Khovanov-Lauda and Rouquier, and can easily be generalised to $\dot{\mathcal{U}}_Q(\mathfrak{gl}(\infty))$.

The construction of the 2-representation is done by categorifying the skew Howe duality relation from section 1.2. One identifies a quotient of $\dot{\mathcal{U}}_Q(\mathfrak{gl}(\infty))$ that lifts the quotient of $\dot{U}_q(\mathfrak{gl}(\infty))$.

The foam category enters the picture as a diagram calculus for the 2-morphisms in $\dot{\mathcal{U}}_Q(\mathfrak{gl}(\infty))$. By lifting the relations that describe the quotient of $\dot{U}_q(\mathfrak{gl}(p))$ giving $\mathrm{Rep}(\mathfrak{sl}(n))$, one can give relations on $\dot{\mathcal{U}}_Q(\mathfrak{gl}(\infty))$ resulting in a categorification of $\mathrm{Rep}(\mathfrak{sl}(n))$.

With this knowledge, Queffelec and Rose [QR14] were able to define $\mathfrak{sl}(n)$ foams by defining them as the image of a 2-representation of the 2-category. This allows them to give a purely combinatorial definition of $\mathfrak{sl}(n)$ homology. This approach is explained in chapter 6.

The $\mathfrak{sl}(n)$ foams were generalised in [Gra15] to $\mathfrak{gl}(m|n)$ foams. This foam category is rather more complicated than the $\mathfrak{sl}(n)$ foam category, partly because several simplifications that were implicitly used in the definition of $\mathfrak{sl}(n)$ foam categories are not available. The general foam categories involve non-local relations, which make them harder to work with, and also makes it impossible to remove the extra rigidity in the diagrams. Furthermore, it is not clear how one can use these to categorify the $\mathfrak{gl}(m|n)$ polynomials, since the methods used to close braids are also not available in this case.

1.4 Heegaard Floer homology

Categorification of the Alexander polynomial follows a rather different route. There are several different homology theories defined from Floer theory. The first was defined by Osvath and Szabo [OS04] and Rasmussen [Ras03], called Heegaard Floer knot homology $\mathrm{HFK}(K)$. This involves taking the Heegaard Floer homology of S^3, which is defined by taking a Heegaard diagram of S^3 and defining chain groups to be generated by tuples of intersections of α and β curves, and differentials given by counting embeddings of holomorphic discs.

However, it is also possible to trivially categorify the Alexander polynomial by 'collapsing the grading' in HOMFLY homology, corresponding to simply setting $a = 1$. However, it is conjectured that there may by a spectral sequence from HOMFLY homology that converges to $\mathrm{HFK}(K)$.

One possible approach to proving such a conjecture is to construct $\mathrm{HFK}(K)$ using the same sorts of techniques as we use to construct $\mathfrak{sl}(n)$ homology. It might be hoped that the $\mathfrak{gl}(1|1)$ foam category could lead to this kind of construction. Descriptions of $\mathrm{HFK}(K)$ by cubes of resolutions involve non-local relations, so the non-locality of the relations in the foam category may be related to this.

1.5 Plan of the thesis

In chapter 2, we recall the Reshetikhin-Turaev construction of quantum knot invariants from quantum groups.

In chapter 3, we recall the type A specialisations of the Reshetikhin-Turaev construction, and how it can give both the Jones polynomial and the Alexander polynomial as a special case, and the relationship with MOY calculus.

In chapter 4, we prove the skew Howe duality theorem, and show how it is related to MOY calculus. The extra relations that describe $\mathrm{Rep}(\mathfrak{gl}(m|n))$ are also provided.

In chapter 5 we recall the Khovanov-Lauda and Rouquier definitions of categorified quantum $\mathfrak{gl}(p)$ and categorifications of its representations.

In chapter 6, we show how a categorification of the results of chapter 4 provides a foam category that lifts $\mathrm{Rep}(\mathfrak{gl}(m|n))$.

Chapter 2

Quantum knot invariants

2.1 Introduction

In this chapter we recall the basic definitions, including Lie algebras, quantum groups, and the Reshetikhin-Turaev process. This chapter is drawn mostly from [Lus93] and [CP95].

2.2 Quantum groups

2.2.1 Simple Lie algebras

Definition 2.2.1. A Cartan datum consists of

- A finite set I and a free abelian group X (called the weight lattice)
- For each $i \in I$, elements $\alpha_i \in X$ (the simple roots) and $\Lambda_i \in X$ (fundamental weights)
- A bilinear form (α, β) on X, such that $(\alpha_i, \alpha_i) \in \{2, 4, 6, \cdots\}$ and $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$.
- For each $i \in I$ elements $h_i \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (simple coroots) such that $h_i(\Lambda_j) = \delta_{ij}$ and
$$h_i(\lambda) = 2\frac{(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}.$$

The elements of the lattice X are called *weights*.

The matrix $(h_i(\alpha_j))$ is known as the Cartan matrix. Given such a Cartan datum, we can define a simple Lie algebra.

Definition 2.2.2. Let $(I, (\cdot,\cdot), X)$ be a root datum. We define the Lie algebra $\mathfrak{g}$ associated to the root datum to be generated by E_i, F_i, H_i for $i \in I$, with relations for all $i, j \in I$

- $[H_i, H_j] = 0$
- $[H_i, E_j] = a_{ij}E_j$
- $[H_i, F_j] = -a_{ij}F_j$
- $[E_i, F_j] = \delta_{ij}H_i$
- $\mathrm{ad}(E_i)^{1-a_{ij}}(E_j) = 0$
- $\mathrm{ad}(F_i)^{1-a_{ij}}(F_j) = 0$

where $a_{ij} = h_i(\alpha_j)$ are the elements of the Cartan matrix.

From the proof of the classification of simple Lie algebras, if the matrix $(h_i(\alpha_j))$ is positive-definite, then $\mathfrak{g}$ is finite-dimensional. A theorem of Serre states that every finite-dimensional simple Lie algebra has such a presentation.

From now on, $\mathfrak{g}$ will always denote a finite-dimensional simple Lie algebra.

It will later be useful to have the definition of the braid group $B_{\mathfrak{g}}$ associated to $\mathfrak{g}$ (in fact, it depends only on its Cartan datum).

Definition 2.2.3. Given a Lie algebra $\mathfrak{g}$ with Cartan datum $(I, (\cdot,\cdot))$, let $i \neq j$ in I be such that $(\alpha_i, \alpha_i)(\alpha_j, \alpha_j) - (\alpha_i, \alpha_j)^2 > 0$. Then define $h(i,j) \in \{2, 3, 4, 6\}$ by

$$\cos^2 \frac{\pi}{h(i,j)} = \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \frac{(\alpha_j, \alpha_i)}{(\alpha_j, \alpha_j)}.$$

We define the *braid group* $B_{\mathfrak{g}}$ to be the group defined by the generators T_i for $i \in I$ subject to

$$T_i T_j T_i \cdots = T_j T_i T_j$$

where both sides have $h(i,j)$ factors (that is, a relation is only imposed if $(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)-(\alpha_i,\alpha_j)^2>0$).

The *Weyl group* W is a quotient of the braid group by T_i^2 for each i.

For a finite-dimensional simple Lie algebra, the Weyl group is always finite. The length of an element $w\in W$ is defined to be the smallest number k such that w is equal to the product of k generators T_i. There is a unique element $w_0\in W$ with maximal length. In general, the expression of w_0 in terms of a minimal number of generators is not unique, but any such expression is called a *reduced expression*.

2.2.2 Universal enveloping algebra

Definition 2.2.4. We define the universal enveloping algebra $U(\mathfrak{g})$ to be the quotient of the tensor algebra

$$\mathbb{C}\oplus\mathfrak{g}\oplus\mathfrak{g}^{\otimes 2}\oplus\mathfrak{g}^{\otimes 3}\oplus\cdots$$

by the ideal generated by $a\otimes b-b\otimes a-[a,b]$ for all $a,b\in\mathfrak{g}$.

This construction is 'universal' in the sense that it is left-adjoint to the forgetful functor from associative algebras to Lie algebras. By the Poincaré-Birkhoff-Witt theorem, the inclusion $\mathfrak{g}\to U(\mathfrak{g})$ is injective and $\mathfrak{g}$ generates $U(\mathfrak{g})$ as an algebra.

Hence we can think of $U(\mathfrak{g})$ as being generated by E_i,F_i,H_i subject to the relations in definition 2.2.2. The last two relations (the so-called Serre relations) can be expressed conveniently as

$$\sum_{r=0}^{1-a_{ij}}(-1)^r\binom{1-a_{ij}}{r}E_i^{1-a_{ij}-r}E_jE_i^r$$

$$\sum_{r=0}^{1-a_{ij}}(-1)^r\binom{1-a_{ij}}{r}F_i^{1-a_{ij}-r}F_jF_i^r$$

where $a_{ij}=h_i(\alpha_j)$ is the Cartan matrix.

2.2.3 Representation Theory

Definition 2.2.5. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V equipped with a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V).$$

By the adjunction property, a Lie algebra representation is equivalent to a $U(\mathfrak{g})$-module. Let $\lambda \in X$ be a weight. Let $\mathfrak{b}$ be the Lie subalgebra generated by the elements H_i and E_i over all i.

The Verma module M_λ is defined to be

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda$ is a 1-dimensional module over $U(\mathfrak{b})$ such that $H_i \cdot v = h_i(\lambda)v$ and $E_i v = 0$.

We will only be concerned with finite-dimensional modules of $U(\mathfrak{g})$, but M_λ is clearly infinite-dimensional. However, it has a unique maximal submodule, defined as the sum of all submodules that do not contain the highest-weight vector $v = 1 \otimes 1 \in M_\lambda$. Its quotient V_λ is therefore a simple module over $U(\mathfrak{g})$. If λ satisfies $h_i(\lambda) \geq 0$ for all $i \in I$, then V_λ is finite-dimensional. Every finite-dimensional simple module can be defined this way, and every finite-dimensional module is a direct sum of simple modules. Examples for $\mathfrak{sl}(n)$ are given in chapter 3.

On any finite-dimensional simple module V_λ, the elements H_i are simultaneously diagonalisable. Given any $v \in V_\lambda$ in the eigenbasis, we define its weight $\mu \in X$ to be such that $H_i(v) = h_i(\mu)v$. The weight μ of v is denoted $\mathrm{wt}(v)$.

Definition 2.2.6. We define the category $U(\mathfrak{g})$-mod to be the category of finite dimensional $U(\mathfrak{g})$-modules.

2.2.4 Hopf algebras

An important feature of Hopf algebras is that the category of modules over a Hopf algebra has a monoidal structure and every object has duals.

Definition 2.2.7. A bialgebra (over $\mathbb{C}$) is a comonoid object in the category of algebras. That is, it is an algebra A equipped with algebra maps

$$\Delta : A \to A \otimes A$$

$$\epsilon : A \to \mathbb{C}$$

satisfying $(\Delta \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta)\Delta$ and $(\epsilon \otimes \mathbb{1})\Delta = \mathbb{1} = (\mathbb{1} \otimes \epsilon)\Delta$. Dually, it can be defined as a monoid object in the category of coalgebras.

A useful notation for coalgebras is the Sweedler notation. Since $\Delta : A \to A \otimes A$, an element v maps to a sum $\sum_i a_i \otimes b_i$. We denote this by

$$\Delta(v) = \sum v_{(1)} \otimes v_{(2)}$$

for short. Coassociativity is written

$$\sum v_{(1)} \otimes v_{(2)} \otimes v_{(3)} = \sum v_{(1)(1)} \otimes v_{(1)(2)} \otimes v_{(2)} = \sum v_{(1)} \otimes v_{(2)(1)} \otimes v_{(2)(2)}$$

and the counit relation is

$$v = \sum \epsilon(v_{(1)})v_{(2)} = \sum \epsilon(v_{(2)})v_{(1)}.$$

Definition 2.2.8. A *Hopf algebra* is a bialgebra A equipped with an *antipode*, which is a linear map $S : A \to A$ satisfying

$$S(v_{(1)})v_{(2)} = v_{(1)}S(v_{(2)}) = \epsilon(v)1$$

in Sweedler notation, for each $v \in A$.

Let M and N be two left A-modules. We can use the coproduct to define the structure of an A-module on $M \otimes_{\mathbb{C}} N$ by

$$x \cdot m \otimes n = \sum x_{(1)} \cdot m \otimes x_{(2)} \cdot n.$$

The counit implies that $\mathbb{C}$ is a module over A, with

$$x \cdot z = \epsilon(x)z.$$

Moreover, if we let $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ be the linear dual of M, we can define an

A-module structure on M^* by

$$(v \cdot f)(x) = f(S(v) \cdot x).$$

The condition on the antipode ensures that the natural maps

$$M^* \otimes M \to \mathbb{C}$$

$$\mathbb{C} \to M \otimes M^*$$

are A-module homomorphisms, implying that M^* is (left-)dual to M as A-modules.

In general, the maps

$$M \otimes M^* \to \mathbb{C}$$

$$\mathbb{C} \to M^* \otimes M$$

are not A-module morphisms under this definition. However, if Δ is cocommutative, then they are, implying that left and right duals agree.

We can equip $U(\mathfrak{g})$ with the structure of a Hopf algebra

Definition 2.2.9. We define a coproduct $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ to be

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad x \in \mathfrak{g}$$

and extended to $U(\mathfrak{g})$ multiplicatively. Define the counit $\epsilon : U(\mathfrak{g}) \to \mathbb{C}$ by $\epsilon(1) = 1$ and $\epsilon(x) = 0$ for all $x \in \mathfrak{g}$. And the antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ to be $S(x) = -x$.

With this data, $U(\mathfrak{g})$ is a Hopf algebra. In fact, it is cocommutative, although not commutative.

2.2.5 Quantised Universal Enveloping Algebras

As mentioned in subsection 2.2.4, the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Hopf algebra. It is, in fact, cocommutative because

$$\tau \circ \Delta = \Delta$$

where τ is the linear map $\tau : A \otimes A \to A \otimes A : a \otimes b \mapsto b \otimes a$. This means that there is an isomorphism

$$\phi_{M,N} : M \otimes N \to N \otimes M$$

for all modules M, N of $U(\mathfrak{g})$. Indeed this defines a symmetric structure on the category of modules, since

$$\phi_{N,M}\phi_{M,N} = \mathbb{1}_{M \otimes N}.$$

By 'quantising' this Hopf algebra, we convert this to a braided structure.

Definition 2.2.10. Let $(I, (\cdot,\cdot), X)$ be a Cartan datum. Let $a_{ij} = h_i(\alpha_j)$ and $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$. Let $q_i = q^{d_i}$. We define $U_q(\mathfrak{g})$ to be the algebra over $\mathbb{C}(q)$ generated by E_i, F_i, K_i for $i \in I$, subject to

- $K_iK_j = K_jK_i$
- $K_iE_j = q_i^{a_{ij}} E_jK_i$
- $K_iF_j = q_i^{-a_{ij}} F_jK_i$
- $E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$
- $$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r} E_j E_i^r = 0$$
- $$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r} F_j F_i^r = 0$$

where

$$[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [r]_{q_i}! = [r]_{q_i} \cdots [2]_{q_i}, \quad \text{and} \quad \begin{bmatrix} n \\ r \end{bmatrix}_{q_i} = \frac{[n]_{q_i}!}{[r]_{q_i}![n-r]_{q_i}!}.$$

The motivation for these relations is that E_i and F_i should correspond to the generators of the same name in $U(\mathfrak{g})$, and K_i should be thought of as q^{H_i} in some sense.

It is often useful to use the reduced powers

$$E_i^{(r)} = \frac{E_i^r}{[r]_{q_i}!}, \quad F_i^{(r)} = \frac{F_i^r}{[r]_{q_i}!}$$

so that the quantum Serre relations are equivalent to the slightly simpler expression

$$\sum_{r=0}^{1-a_{ij}} (-1)^r E_i^{(1-a_{ij}-r)} E_j E_i^{(r)} = 0$$

and similarly for F_i.

We can now give a comultiplication

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i$$

$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

$$\Delta(K_i) = K_i \otimes K_i$$

with counit

$$\epsilon(E_i) = \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1$$

and antipode

$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i) = K_i^{-1}.$$

This gives $U_q(\mathfrak{g})$ the structure of a Hopf algebra.

2.3 Reshetikhin-Turaev Invariants

All simple modules of $U(\mathfrak{g})$ have counterparts in $U_q(\mathfrak{g})$-mod. Since $U_q(\mathfrak{g})$ is not cocommutative, we do not have the usual symmetric structure on the category $U_q(\mathfrak{g})$-mod given by swapping two tensor factors. However, it is possible to describe a braiding.

Definition 2.3.1. Given a monoidal category $\mathscr{C}$, a *braiding* is a natural isomorphism

$$\beta_{U,V} : U \otimes V \to V \otimes U$$

which satisfies

$$\beta_{U,V\otimes W} = (\mathbb{1}_V \otimes \beta_{U,W})(\beta_{U,V} \otimes \mathbb{1}_W)$$

$$\beta_{U\otimes V,W} = (\beta_{U,W} \otimes \mathbb{1}_V)(\mathbb{1}_U \otimes \beta_{V,W})$$

for any $U, V, W \in \mathscr{C}$.

The importance of this natural transformation is that it satisfies

$$(\mathbb{1}_W \otimes \beta_{U,V})(\beta_{U,W} \otimes \mathbb{1}_V)(\mathbb{1}_U \otimes \beta_{V,W}) = (\beta_{V,W} \otimes \mathbb{1}_U)(\mathbb{1}_V \otimes \beta_{U,W})(\beta_{U,V} \otimes \mathbb{1}_W).$$

If we draw this as a diagram involving strands labelled U, V, W and $\beta_{U,V}$ interpreted as meaning the strands labelled U and V cross one another (in a positive crossing), then the above equation can be seen to be the Reidemeister 3 relation on braids, shown in Figure 2.1. As β is an isomorphism, it has an inverse β^{-1}, which can be drawn as a negative crossing. The identity $\beta \circ \beta^{-1} = \mathbb{1}$ can be seen as the Reidemeister 2 relation.

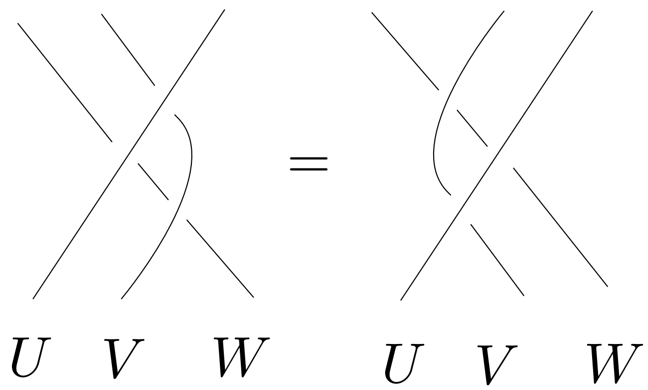


Figure 2.1: Braid relation

It is therefore possible to define maps

$$\mathrm{ev}_M^{\dagger} : M \otimes M^* \to \mathbb{C}(q) = \quad M \otimes M^* \xrightarrow{\beta_{M,M^*}} M^* \otimes M \xrightarrow{\mathrm{ev}_M} \mathbb{C}(q)$$

$$\mathrm{coev}_M^{\dagger} : \mathbb{C}(q) \to M^* \otimes M = \quad \mathbb{C}(q) \xrightarrow{\mathrm{ev}_M} M \otimes M^* \xrightarrow{\beta_{M,M^*}} M^* \otimes M$$

However, the problem is that these do not establish M as a left-dual of M^*, since we require

$$(\mathrm{ev}_M^{\dagger} \otimes \mathbb{1}_M)(\mathbb{1}_M \otimes \mathrm{coev}_M^{\dagger}) = \mathbb{1}_M, \quad \mathbb{1}_{M^*} = (\mathbb{1}_{M^*} \otimes \mathrm{ev}_M^{\dagger})(\mathrm{coev}_M^{\dagger} \otimes \mathbb{1}_{M^*})$$

which does not actually hold.

However, the category $U_q(\mathfrak{g})$ contains extra structure to let us fix this defect.

Definition 2.3.2. Let $\mathscr{C}$ be a braided monoidal category with braiding β, and duality $(*, \mathrm{ev}, \mathrm{coev})$, then a *twist* is a natural isomorphism

$$\theta_V : V \to V$$

called a twist morphism so that

$$\theta_{V\otimes W} = \beta_{W,V}\beta_{V,W}(\theta_V \otimes \theta_W)$$

and

$$(\theta_V \otimes \mathbb{1}_{V^*})\,\mathrm{coev} = (\mathbb{1}_V \otimes \theta_{V^*})\,\mathrm{coev}\,.$$

We can then modify the above maps

$$\mathrm{ev}'_M : M \otimes M^* \to \mathbb{C}(q) = \; M \otimes M^* \xrightarrow{\beta_{M,M^*}} M^* \otimes M \xrightarrow{\mathbb{1}_{M^*}\otimes\theta_M^{-1}} M^* \otimes M \xrightarrow{\mathrm{ev}_M} \mathbb{C}(q)$$

$$\mathrm{coev}'_M : \mathbb{C}(q) \to M^* \otimes M = \; \mathbb{C}(q) \xrightarrow{\mathrm{ev}_M} M \otimes M^* \xrightarrow{\beta_{M,M^*}} M^* \otimes M \xrightarrow{\mathbb{1}_{M^*}\otimes\theta_M^{-1}} M^* \otimes M$$

which do satisfy the required duality relations.

This in particular implies that M^{**} is isomorphic to M, via

$$M^{**} \xrightarrow{\mathbb{1}_{M^{**}}\otimes\mathrm{coev}'_M} M^{**} \otimes M^* \otimes M \xrightarrow{\mathrm{ev}_{M^*}\otimes\mathbb{1}_M} M$$

with inverse

$$M \xrightarrow{\mathbb{1}_M\otimes\mathrm{coev}_{M^*}} M \otimes M^* \otimes M^{**} \xrightarrow{\mathrm{ev}'_M\otimes\mathbb{1}_M} M^{**}.$$

Definition 2.3.3. A *ribbon category* is a monoidal category $(\mathscr{C}, \otimes, I)$ equipped with a braiding β, a twist θ and duals $(*, \mathrm{ev}, \mathrm{coev})$, which satisfy the compatibility axioms in definition 2.3.1 and definition 2.3.2.

One further important feature of ribbon categories is that they possess a so-called quantum trace.

Definition 2.3.4. Given a ribbon category $\mathscr{C}$, and a morphism $f : V \to V$, define the *quantum trace* to be

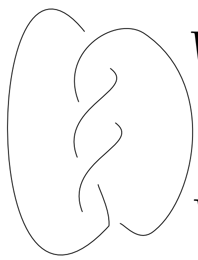
$$\mathrm{tr}_q(f) = \mathrm{ev}'_V(f \otimes \mathbb{1}_{V^*})\,\mathrm{coev}_V$$

In particular, given an object $V \in \mathscr{C}$, we define its *quantum dimension*, to be

$$\dim_q(V) = \mathrm{tr}_q(\mathbb{1}_V).$$

Ribbon categories were introduced by Reshetikhin-Turaev to describe a new family of knot invariants as follows: a framed oriented link diagram D has each component decorated with objects $V_i \in \mathscr{C}$. Perturb the diagram so that minima and maxima are isolated and there are no other inflection points. An upward-oriented strand is interpreted as V, while a downward-oriented strand is thought of as V^*. Then an oriented cup is either ev_V or ev'_V depending on orientation. Similarly, a cap is either coev_V or coev'_V. A positive crossing between strands labelled V, W is defined as $\beta_{V,W}$. An empty slice of the diagram is defined to be I, the unit of the monoidal category $\mathscr{C}$.

Applying the above procedure, we can interpret a generic framed oriented link diagram as a morphism $I \to I$ in $\mathscr{C}$ by reading from the bottom of the diagram to the top. See Figure 2.2 for this applied to the trefoil.



$$\begin{tikzcd}
I \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "\mathrm{ev}_V \otimes \mathrm{ev}'_V"'] \\
V^* \otimes V \otimes V \otimes V^* \arrow[u, "\mathbb{1}_{V^*} \otimes \beta^3_{V,V} \otimes \mathbb{1}_{V^*}"'] \\
I \arrow[u, "\mathrm{coev}'_V \otimes \mathrm{coev}_V"']
\end{tikzcd}$$

Figure 2.2: Reshetikhin-Turaev procedure applied to a diagram of the trefoil coloured by $V \in \mathscr{C}$

Theorem 2.3.5 (Reshetikhin-Turaev)**.** *The procedure described above does not depend on the choice of framed diagram D, only on the choice of decorations $\{V_i\}$. Therefore the resulting map $I \to I$ is a framed link invariant.*

The twist θ_V can be interpreted as the map associated to a framing change (or a Reidemeister 1 move). Therefore, θ_V measures the failure to be an invariant of links.

Let us apply this to the case of $U_q(\mathfrak{g})$. In general, the braiding on $U_q(\mathfrak{g})$-mod is described by an element R in a completed version of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, called the universal R-matrix. A result of Kamnitzer and Tingley [KT09](see also [ST09]) states that we can express this in the following way: Let $T_{w_0} = T_{i_1} T_{i_2} \cdots T_{i_N}$ be a reduced expression of the maximal length element in the Weyl group W. Define an action of the braid group on any representation V_λ as

$$T_i(v) = \sum_{\substack{a,b,c \geq 0 \\ -a+b+c=h_i(\mathrm{wt}(v))}} (-1)^b q_i^{-ac+b} E_i^{(a)} F_i^{(b)} E_i^{(c)} v.$$

In addition, we let $J(v) = q^{(\mathrm{wt}(v),\mathrm{wt}(v))/2+(\mathrm{wt}(v),\rho)}v$, where $\rho \in X$ is the unique weight with $(2\rho, \alpha_i) = d_i$ for all i, where $d_i = \frac{1}{2}(\alpha_i, \alpha_i)$. We set $X = JT_{w_0}$ and on any $v \otimes w \in V \otimes W$, we set

$$R(v \otimes w) = \mathrm{Flip} \circ (X^{-1} \otimes X^{-1})\Delta(X)(v \otimes w)$$

where $\mathrm{Flip}(a \otimes b) = b \otimes a$. To interpret this, we should take the representation $V \otimes W$ and split it into a direct sum of simple representations, then applying $\Delta(X)$ means applying X to each direct summand. Then (as shown in Kamnitzer and Tingley [KT09]) the operator $R : V \otimes W \to W \otimes V$ gives a braiding on $U_q(\mathfrak{g})$-mod.

The twist morphism $\theta_V : V \to V$ is just multiplication by a scalar. On V_λ, the twist θ_{V_λ} acts as $v \mapsto q^{(\lambda,\lambda+2\rho)}v$ for all $v \in V_\lambda$. Because this is only a scalar, it is possible to refine the framed link invariant from theorem 2.3.5 to a link invariant by multiplying by $q^{-(\lambda,\lambda+2\rho)w(D)}$ where $w(D)$ is the writhe of the diagram D.

Thus, for each simple Lie algebra $\mathfrak{g}$ and each representation V of $\mathfrak{g}$, we obtain a knot invariant $P_V(K) \in \mathbb{C}(q)$. In fact, it is possible to work in an integral version of $U_q(\mathfrak{g})$ to show that $P_v(K) \in \mathbb{Z}[q, q^{-1}]$. Note that the value of $P_V(K)$ when K is the unknot is the quantum dimension $\dim_q(V)$.

In chapter 3 we investigate more closely the invariants obtained by taking $\mathfrak{g} = \mathfrak{sl}_n$ and taking representations associated to fundamental weights. We also generalise slightly to those ribbon categories associated to Lie superalgebras.

Chapter 3

Type A

3.1 Introduction

In this chapter we specialise the previous chapter to the case of $\mathfrak{g} = \mathfrak{sl}(n)$, but also generalise to the Lie superalgebra case of $\mathfrak{gl}(m|n)$ that will be used throughout. This is drawn from [CKM14] and [Gra16] and [Gra15].

3.2 Quantum $\mathfrak{sl}_n$

For the Cartan datum of $\mathfrak{sl}_n$ we can take $X = \mathbb{Z}^n/(1, 1, \dots, 1)$ and $I = \{1, \dots, n-1\}$, with

$$\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0), \quad \Lambda_i = (1, \dots, 1, 1, 0, \dots, 0)$$

where in α_i the 1 occurs in place i, and there are i 1's in Λ_i. The bilinear form is given on the basis $\{\Lambda_i\}$ by $(\Lambda_i, \Lambda_j) = \min\{i, j\}$ (ie. a restriction of the dot product onto this quotient).

Thus the Hopf algebra $U_q(\mathfrak{sl}(n))$ can be presented with generators $E_i, F_i, K_i^{\pm 1}$ for $i \in \{1, \dots, n-1\}$ with

$$K_i K_i^{-1} = 1, \quad K_i K_j = K_j K_i$$

$$K_i E_i = q^2 E_i K_i, \quad K_{i\pm 1} E_i = q^{-1} E_i K_{i\pm 1}, \quad K_i E_j = E_j K_i \quad \text{if} \quad i \neq j, j \pm 1$$

$$K_i F_i = q^{-2} F_i K_i, \quad K_{i\pm 1} F_i = q F_i K_{i\pm 1}, \quad K_i F_j = F_j K_i \quad \text{if} \quad i \neq j, j \pm 1$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \quad \text{if} \quad j = i \pm 1$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \quad \text{if} \quad j = i \pm 1$$

$$E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \quad \text{if} \quad |i-j| > 1.$$

Let $\mathbb{C}(q)^n = \mathbb{C}_q^n$ be an n-dimensional vector space over $\mathbb{C}(q)$, and let $\{E_{i,j} \mid i,j \in \{1,\ldots,n\}\}$ be the elementary matrices. We define a representation of $U_q(\mathfrak{sl}(n))$ by

$$E_i \mapsto E_{i-1,i}, \quad F_i \mapsto E_{i,i-1}, \quad K_i \mapsto qE_{i,i} + q^{-1}E_{i+1,i+1} + \sum_{\substack{j\neq i \\ j\neq i+1}} E_{j,j}.$$

This can be seen to be the simple module corresponding to the fundamental weight $\Lambda_1 = (1,0,\ldots,0)$.

The R-matrix on $\mathbb{C}_q^n$ can be defined as

$$R = q\sum_i E_{i,i}\otimes E_{i,i} + \sum_{i\neq j} E_{i,i}\otimes E_{j,j} + (q-q^{-1})\sum_{i<j} E_{i,j}\otimes E_{j,i} \tag{3.2.1}$$

(see [CP95, Section 7.3]), meaning the braiding is defined as

$$\beta_{\mathbb{C}_q^n\otimes\mathbb{C}_q^n} = \mathrm{Flip}\circ R.$$

From this, we can see that

$$(\beta_{\mathbb{C}_q^n\otimes\mathbb{C}_q^n})^2 - 1 = (q-q^{-1})\beta_{\mathbb{C}_q^n\otimes\mathbb{C}_q^n}$$

giving the Skein relation

$$\beta_{\mathbb{C}_q^n\otimes\mathbb{C}_q^n} - (\beta_{\mathbb{C}_q^n\otimes\mathbb{C}_q^n})^{-1} = (q-q^{-1})\mathbb{1}_{\mathbb{C}_q^n\otimes\mathbb{C}_q^n}.$$

The twist θ_V acts as multiplication by q^{-n} since the element ρ is

$$\rho = (n-1, n-2, \ldots, 1, 0) = \sum_i \Lambda_i.$$

Thus the resulting knot polynomial satisfies the skein relation

$$q^nP_{\times}(q) - q^{-n}P_{\times}(q) = (q-q^{-1})P_{)(}(q).$$

This is the skein relation for the $\mathfrak{sl}(n)$ polynomial from section 1.1. We will see in section 3.3 that $\dim_q(\mathbb{C}_q^n) = [n]$, meaning that the polynomial agrees with the $\mathfrak{sl}(n)$ polynomial.

3.2.1 Braided exterior algebras

We can also define the module $\bigwedge_q(\mathbb{C}_q^n)$, by taking a quotient of the tensor algebra by the ideal $\mathrm{Sq}^2(\mathbb{C}_q^n) = (e_i \otimes e_i, e_i \otimes e_j + qe_j \otimes e_i)$ for all $i < j$. This ideal consists precisely of the eigenspaces of $\beta_{\mathbb{C}_q^n \otimes \mathbb{C}_q^n}$ with eigenvalue q^r for $r \in \mathbb{Z}$. The product of $x, y \in \bigwedge_q(\mathbb{C}_q^n)$ is denoted $x \wedge y$. The algebra $\bigwedge_q(\mathbb{C}_q^n)$ is graded by the grading on the tensor algebra, and the degree k subspace is denoted $\bigwedge_q^k(\mathbb{C}_q^n)$. Note that

$$\bigwedge\nolimits_q^n(\mathbb{C}_q^n) \cong \mathbb{C}(q)$$

canonically as $U_q(\mathfrak{sl}(n))$-modules. The module $\bigwedge_q^i(\mathbb{C}_q^n)$ is the simple module of highest weight Λ_i.

In general we can define simple modules $V_n(\lambda)$ for each $\lambda = \sum_i n_i \Lambda_i$ for $n_i \in \mathbb{N}$. Such λ are said to be *dominant*, and correspond to partitions of length n.

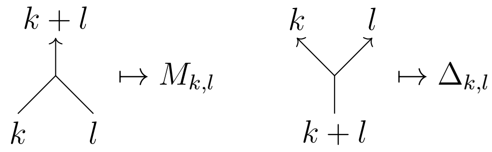
3.3 MOY diagrams for the braiding

Following Cautis-Kamnitzer-Morrison [CKM14], we define the structure of a coassociative coalgebra on $\bigwedge_q(\mathbb{C}_q^n)$ as follows: given a subset $S \subset \{1, \ldots, n\}$, let $e_S = e_{j_1} \wedge \cdots \wedge e_{j_k}$ where $S = \{j_1, \ldots, j_k\}$ with $j_1 > \ldots > j_k$. If T is a subset of S, let $l(T)$ be the smallest length of a permutation of S taking all elements of T in order before all elements of $S \setminus T$, and then we define

$$\Delta_{k,l} : \bigwedge\nolimits_q^{k+l}(\mathbb{C}_q^n) \to \bigwedge\nolimits_q^k(\mathbb{C}_q^n) \otimes \bigwedge\nolimits_q^l(\mathbb{C}_q^n) : e_S \mapsto q^{-kl} \sum_{\substack{T \subset S \\ \#T = k}} (-q)^{l(T)} e_T \otimes e_{S \setminus T}$$

so that $\Delta(x_S) = \sum_{k+l=\#S} \Delta_{k,l}(x_S)$. The counit is $\epsilon(1) = 1$ and $\epsilon(x_S) = 0$ for $S \neq \emptyset$. These two maps commute with the $U_q(\mathfrak{sl}(n))$ action.

It is useful to express the multiplication and comultiplication maps graphically:



We can also establish that $\bigwedge_q^{n-k}(\mathbb{C}_q^n)$ is dual to $\bigwedge_q^k(\mathbb{C}_q^n)$ via the maps

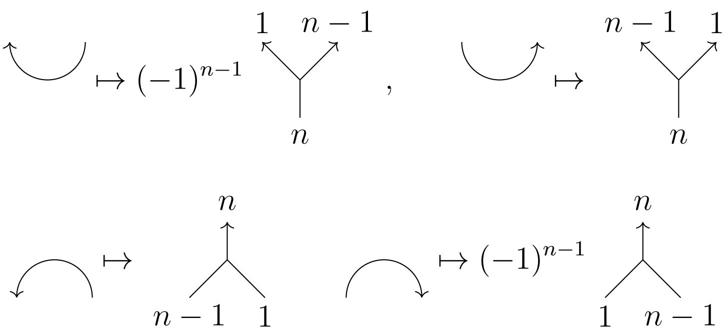
$$(-1)^{k(n-k)}\Delta_{k,n-k} : \bigwedge^n(\mathbb{C}_q^n) \to \bigwedge_q^k(\mathbb{C}_q^n) \otimes \bigwedge_q^{n-k}(\mathbb{C}_q^n)$$

$$M_{n-k,k} : \bigwedge_q^{n-k}(\mathbb{C}_q^n) \otimes \bigwedge_q^k(\mathbb{C}_q^n) \to \bigwedge_q^n(\mathbb{C}_q^n)$$

$$\Delta_{n-k,k} : \bigwedge^n(\mathbb{C}_q^n) \to \bigwedge_q^{n-k}(\mathbb{C}_q^n) \otimes \bigwedge_q^k(\mathbb{C}_q^n)$$

$$(-1)^{k(n-k)}M_{k,n-k} : \bigwedge_q^k(\mathbb{C}_q^n) \otimes \bigwedge_q^{n-k}(\mathbb{C}_q^n) \to \bigwedge_q^n(\mathbb{C}_q^n)$$

under the identification $\bigwedge_q^n(\mathbb{C}_q^n) \cong \mathbb{C}(q)$. It can be seen that these maps satisfy the duality relations establishing $\bigwedge^{n-k}(\mathbb{C}_q^n)$ as the dual representation to $\bigwedge_q^k(\mathbb{C}_q^n)$. This means we can identify the diagrams:



under the identification of $\bigwedge_q^n(\mathbb{C}_q^n) \cong \mathbb{C}(q)$ and $(\mathbb{C}_q^n)^* \cong \bigwedge_q^{n-1}(\mathbb{C}_q^n)$, and similarly for strands coloured i.

It is also possible to see that the maps $\Delta_{k,l}$ and $M_{k,l}$ satisfy the so-called MOY relations Figure 3.1, where, as before, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, and $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[2]}$.

Of course, under our identifications of the cups and caps with multiplication and comultiplication maps, we see that Move 0 is really a special case of Move 2 with $i = n$ and $j = 1$, and Move 1 is a special case of Move 5 with $k = n$, $l = i$ and $s = r = j$. Also, Move 4 can be deduced from repeated application of Move 5 and Move 3.

In particular, Move 0 implies that $\dim_q(\mathbb{C}_q^n) = [n]$. Hence, the value of the Reshetikhin-Turaev polynomial on the unknot is $[n]$, so this polynomial is precisely the $\mathfrak{sl}(n)$

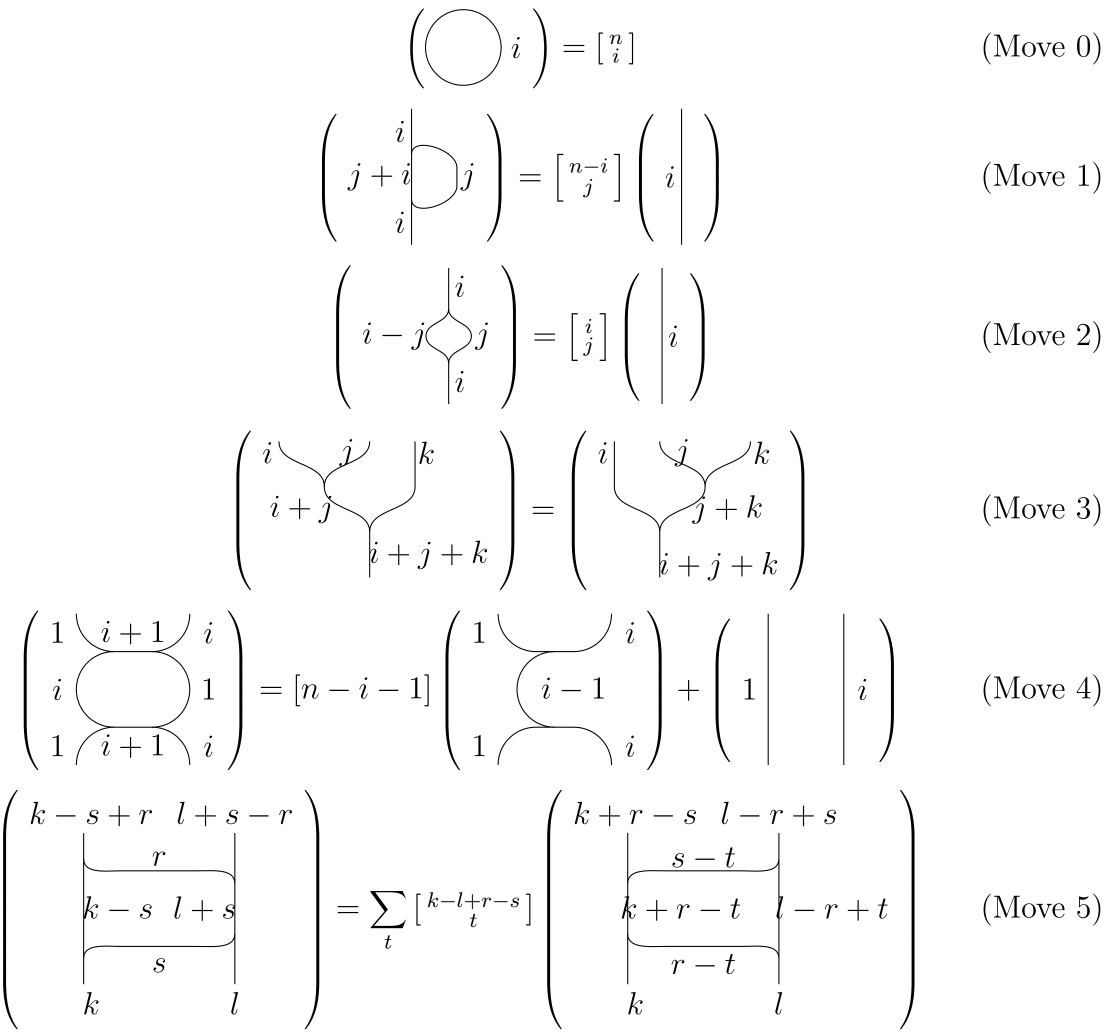


Figure 3.1: The six MOY moves for $\mathfrak{sl}(n)$ diagrams

polynomial from section 1.1.

3.3.1 Braiding

It can be seen from the equation from the R-matrix (equation 3.2.1) and the definition of the comultiplication that we can write

$$\beta_{\mathbb{C}_q^n \otimes \mathbb{C}_q^n} = q\mathbb{1}_{\mathbb{C}_q^n \otimes \mathbb{C}_q^n} - \Delta_{1,1} M_{1,1}$$

since both map

$$e_i \otimes e_j \mapsto e_j \otimes e_i$$

$$e_j \otimes e_i \mapsto e_i \otimes e_j + (q - q^{-1}) e_j \otimes e_i$$

$$e_i \otimes e_i \mapsto q e_i \otimes e_i$$

when $i < j$. This means that graphically, we have the resolutions of a crossing in

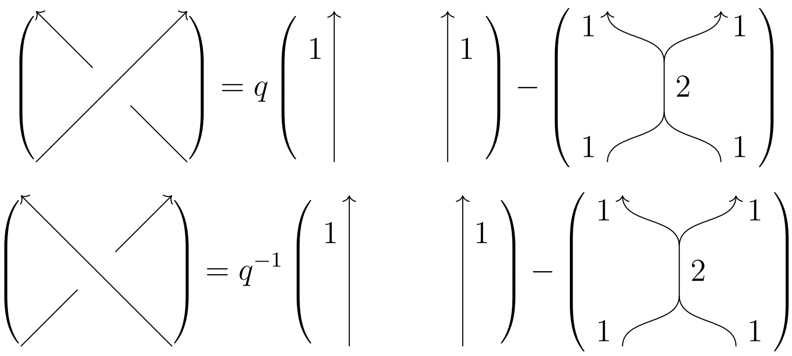
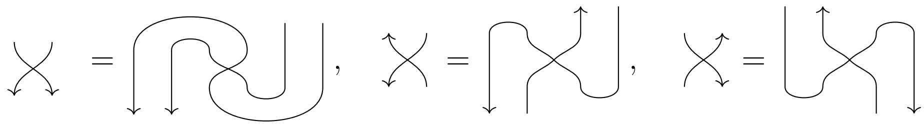


Figure 3.2: MOY resolutions of knot diagrams

Figure 3.2. Crossings that are downward oriented or have up and down orientations are defined using the duality maps



The advantage of this is that we have a completely diagrammatic way of computing the $\mathfrak{sl}(n)$ knot invariants: we simply take a diagram D for a knot, and resolve every crossing in two ways according to Figure 3.2. Then simplify the resulting diagrams according to the MOY moves in Figure 3.1 to obtain a polynomial. As a final step, multiply the resulting polynomial by $q^{-nw(D)}$ where $w(D)$ is the difference of the number of positive and negative crossings in the diagram. This method was introduced by Murakami, Ohtsuki and Yamada [MOY98], who proved that these moves are sufficient to prove the Reidemeister moves. Using this, we can prove that the MOY moves are sufficient to evaluate any diagram, as we showed in [Gra13]. We use the notation $(\Gamma)_N$ for the evaluation of the $\mathfrak{sl}_N$ MOY diagram Γ.

Theorem 3.3.1. *The six MOY moves uniquely determine the MOY $\mathfrak{sl}(N)$ polynomial for coloured oriented trivalent planar graphs.*

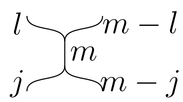
To prove this, we first specialise to $\{1,2\}$-coloured graphs:

Proposition 3.3.2. *The six MOY moves, specialised to colourings in $\{1,2\}$, determine the $\mathfrak{sl}(N)$ polynomial for oriented trivalent plane graphs coloured with $\{1,2\}$.*

Proof. Given a diagram coloured in $\{1, 2\}$, we can use Move 0 to remove closed loops coloured with 2, so suppose the remaining graph Γ has n edges coloured 2. We use the relationship with knot diagrams to construct a knot diagram D with n crossings for which the resolution with the most 2-coloured edges is Γ. With appropriate choice of crossings, we can ensure that D is a diagram for an unlink U. By the work of MOY, if the polynomial satisfies the MOY moves it also satisfies the Reidemeister moves, so the polynomial is a link invariant. Then $(U)_N = (-1)^n(\Gamma)_N + \sum_{i=1}^{2^n-1}(-1)^{k_i}(q)^{\epsilon_i}(\Gamma_i)_N$ where each Γ_i has $k_i < n$ edges coloured with 2, and ϵ_i is the number of positive crossings resolved into edges coloured 1 minus the number of negative crossings resolved into edges coloured with 1. By induction, we can use the MOY moves to calculate the values of $(\Gamma_i)_N$ for each i, and the $\mathfrak{sl}(N)$ polynomial for U is $[N]^d = \left(\frac{q^N - q^{-N}}{q - q^{-1}}\right)^d$, where d is the number of components of U. Hence we can calculate $(\Gamma)_N$ using MOY moves. $\square$

Proof of theorem 3.3.1. Suppose the largest colouring in the diagram is m, $m > 2$. The idea is that, following Wu [Wu14], we replace all the edges coloured with m with edges with colourings smaller than m.

If the diagram contains any disjoint circles, then remove them with Move 0. If there are any remaining m edges, then locally the diagram is

$\Gamma =$ 

with $j, l < m$. Then we have the equality in Figure 3.3, using Move 2 and Move 3 twice. But then we have the relation in Figure 3.4 by expanding the central portion of the first term on the right hand side of figure 3.4 using Move 5. The right hand side of figure 3.4 has no colourings larger than $m - 1$, hence we have written $(\Gamma)_N$ in terms of diagrams containing fewer m-colourings.

Thus for any coloured diagram Γ, there is a diagram Γ' that is coloured only in $\{1, 2\}$ such that $(\Gamma)_N = p_\Gamma(\Gamma')_N$ for some polynomial p_Γ determined by the MOY moves. The result then follows from Proposition 3.3.2. $\square$

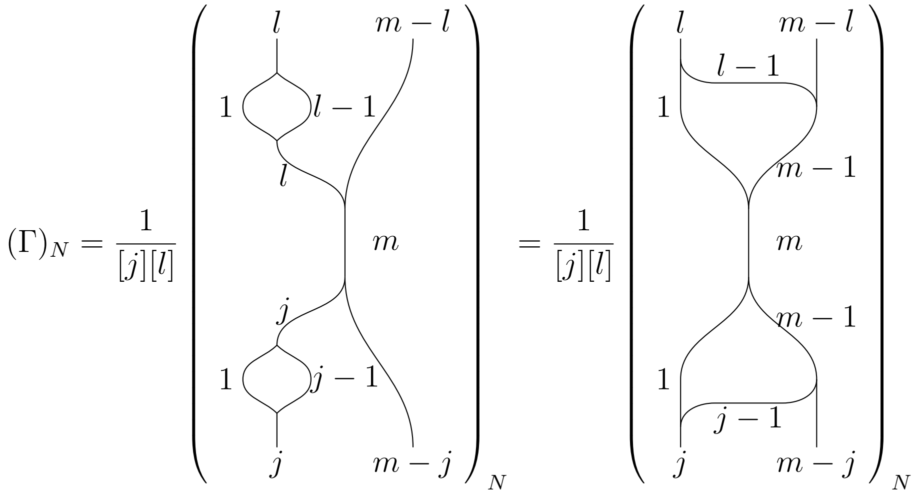


Figure 3.3: Rewriting Γ

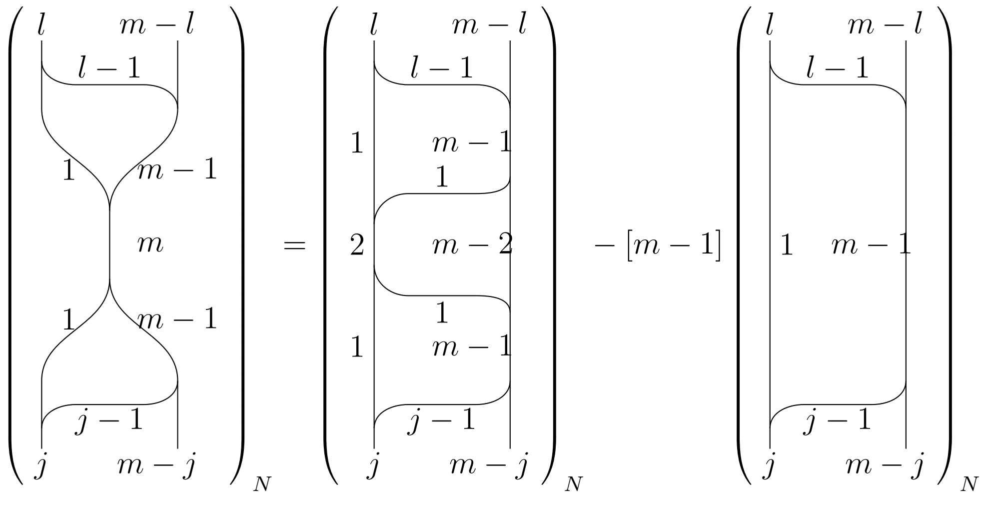
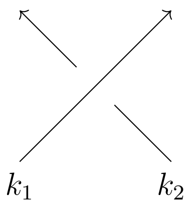


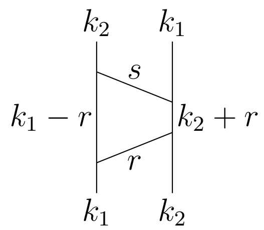
Figure 3.4: Reducing the highest colour in the diagram of Γ

3.3.2 Coloured polynomials

We can also use the above to evaluate the Reshetikhin-Turaev polynomial of links where each component is coloured by some $\bigwedge_q^i(\mathbb{C}_q^n)$. We take the braiding to be



$$= (-1)^{k_1 k_2} \sum_{\substack{r,s\geq 0 \\ r-s=k_1-k_2}} (-q)^{k_2-s} \left(\quad \right)$$



and then apply MOY moves to reduce this to a polynomial. Note the sum is finite since the diagram is taken to be 0 if any labels are negative.

3.3.3 $\mathfrak{sl}(n)$ versus $\mathfrak{gl}(n)$

In later parts of the paper we will mostly be making use of the algebra $U_q(\mathfrak{gl}(n))$ for technical reasons. This is defined as an algebra over $\mathbb{C}(q)$ with generators E_i, F_i for $1 \leq i \leq n-1$, and $L_j^{\pm 1}$ for $1 \leq j \leq n$, satisfying

$$L_i L_i^{-1} = 1, \quad L_i L_j = L_j L_i$$

$$L_i E_i = q E_i L_i, \quad L_{i+1} E_i = q^{-1} E_i L_{i+1}, \quad L_i E_j = E_j L_i \quad \text{if} \quad j \neq i, i-1$$

$$L_i F_i = q^{-1} F_i L_i, \quad L_{i+1} F_i = q F_i L_{i+1}, \quad L_i F_j = F_j L_i \quad \text{if} \quad j \neq i, i-1$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{L_i L_{i+1}^{-1} - L_i^{-1} L_{i+1}}{q - q^{-1}}$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if} \quad j = i \pm 1$$

$$F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \text{if} \quad j = i \pm 1$$

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \quad \text{if} \quad |i - j| > 1.$$

This is equivalent to adjoining to the weight lattice X the weight $\Lambda_n = (1, \ldots, 1)$. We can obtain $U_q(\mathfrak{sl}(n))$ by $K_i = L_i L_{i+1}^{-1}$ and taking the quotient by the ideal generated by $L_1 \cdots L_n$. In other words, $U_q(\mathfrak{gl}(n))$ is a central extension of $U_q(\mathfrak{sl}(n))$.

As before, we can choose a coproduct

$$\Delta(E_i) = E_i \otimes L_i L_{i+1}^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + L_i^{-1} L_{i+1} \otimes F_i, \quad \Delta(L_i) = L_i \otimes L_i$$

which along with the antipode

$$S(L_i) = L_i^{-1}, \quad S(E_i) = -E_iL_i^{-1}L_{i+1}, \quad S(F_i) = -L_iL_{i+1}^{-1}F_i$$

and counit

$$\epsilon(L_i) = 1, \quad \epsilon(E_i) = 0, \quad \epsilon(F_i) = 0$$

makes $U_q(\mathfrak{gl}(m))$ into a Hopf algebra.

The representation theory of $U_q(\mathfrak{gl}(n))$ is much the same as $U_q(\mathfrak{sl}(n))$, meaning we can define the representation $\mathbb{C}_q^n$ of $U_q(\mathfrak{gl}(n))$ by $L_i \mapsto qE_{i,i} + \sum_{j\neq i} E_{j,j}$ and E_i and F_i act as in the $U_q(\mathfrak{sl}(n))$ case. We can also define the exterior powers, and give the graphical calculus like in section 3.3, recovering the same quantum invariants of knots. However, one important difference is that the exterior power

$$\textstyle\bigwedge_q^n(\mathbb{C}_q^n) \ncong \mathbb{C}(q)$$

as $U_q(\mathfrak{gl}(n))$-modules, because the L_i act as multiplication by q on the left-hand side, but as 1 on the right-hand side. The distinction between $\bigwedge_q^n(\mathbb{C}_q^n)$ and $\mathbb{C}(q)$ is often useful, for example in chapter 4 and chapter 6, so we will mostly use $U_q(\mathfrak{gl}(n))$-modules.

3.4 General linear superalgebras

We can extend much of the above to the Lie superalgebra $\mathfrak{gl}(m|n)$.

Definition 3.4.1. Define $U_q(\mathfrak{gl}(m|n))$ to be the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}(q)$-algebra generated by $E_1, \dots, E_{n+m-1}$, $F_1, \dots, F_{n+m-1}$, $L_1^{\pm 1}, \dots, L_{m+n}^{\pm 1}$ with $\deg E_m = \deg F_m = 1$ and all other generators even. Let $K_i = L_iL_{i+1}^{-1}$ for all i. For $i \in \{1, \dots, m+n\}$ let $\{i\} = 1$ if $i \leq m$ and $\{i\} = -1$ if $i > m$. Then the relations on this algebra are

$$L_iL_j = L_jL_i, \quad \text{for all } i,j$$

$$L_iE_j = q^{\{j\}(\delta_{i,j}-\delta_{i,j+1})}E_jL_i$$

$$L_iF_j = q^{\{j\}(\delta_{i,j+1}-\delta_{i,j})}F_jL_i$$

$$E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{q^{\{i\}} - q^{-\{i\}}}, \quad i \neq m$$

$$E_mF_m + F_mE_m = \frac{K_m - K_m^{-1}}{q - q^{-1}}$$
$$E_m^2 = F_m^2 = 0$$
$$E_iE_j = E_jE_i, \quad |i-j| > 1$$
$$F_iF_j = F_jF_i, \quad |i-j| > 1$$
$$E_i^2E_{i\pm1} - [2]E_iE_{i\pm1}E_i + E_{i\pm1}E_i^2 = 0, \quad i \neq m$$
$$F_i^2F_{i\pm1} - [2]F_iF_{i\pm1}F_i + F_{i\pm1}F_i^2 = 0, \quad i \neq m$$

$$\begin{aligned}[2]E_mE_{m+1}E_{m-1}E_m &= E_{m+1}E_mE_{m-1}E_m + E_mE_{m+1}E_mE_{m-1} \\ &+ E_mE_{m-1}E_mE_{m+1} + E_{m-1}E_mE_{m+1}E_m\end{aligned}$$

$$\begin{aligned}[2]F_mF_{m+1}F_{m-1}F_m &= F_{m+1}F_mF_{m-1}F_m + F_mF_{m+1}F_mF_{m-1} \\ &+ F_mF_{m-1}F_mF_{m+1} + F_{m-1}F_mF_{m+1}F_m\end{aligned}$$

Algebras with $\mathbb{Z}/2\mathbb{Z}$-gradings where the grading introduces signs into the defining relations are often referred to as *superalgebras*. Thus the above definition is a quantisation of the universal enveloping superalgebra of the Lie superalgebra $\mathfrak{gl}(m|n)$.

The algebra $U_q(\mathfrak{gl}(m|n))$ has the structure of a Hopf superalgebra. We choose the coproduct

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,\ \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,\ \Delta(L_i) = L_i \otimes L_i$$

with counit $\epsilon(E_i) = \epsilon(F_i) = 0$ and $\epsilon(L_i) = 1$, and antipode

$$S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i, \quad S(L_i) = L_i^{-1}.$$

Using this, given two representations V, W of $U_q(\mathfrak{gl}(m|n))$, we can define a new representation $V \otimes W$ by $X \cdot (v \otimes w) = \Delta(X)v \otimes w$ with the understanding that $A \otimes B(v \otimes w) = (-1)^{\deg B \deg v} Av \otimes Bw$ on homogeneous elements.

The standard (or vector) representation of $U_q(\mathfrak{gl}(m|n))$ is defined to be the $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}(q)$-vector space $\mathbb{C}_q^{m|n} = \langle e_1, \ldots, e_m, e_{m+1}, \ldots e_{m+n}\rangle$, where $\deg e_i = 0$ for $i \leq m$ and $\deg e_i = 1$ for $i > m$, with

$$F_ie_i = e_{i+1}$$

$$E_i e_{i+1} = e_i$$

$$L_i e_i = q^{\{i\}} e_i$$

where E_i, F_i act as 0 otherwise, and L_k acts as identity otherwise. Using the elementary matrices $E_{i,j}$ as before, we can express the R-matrix

$$R = \sum_i q^{\{i\}} E_{i,i} \otimes E_{i,i} + \sum_{i \neq j} E_{i,i} \otimes E_{j,j} + (q - q^{-1}) \sum_{i<j} (-1)^{\{i\}\{j\}} E_{i,j} \otimes E_{j,i}$$

so that the braiding is

$$\beta_{\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}} = \mathrm{Flip} \circ R$$

where $\mathrm{Flip}(v \otimes w) = (-1)^{\deg(v)\deg(w)} w \otimes v$ is the symmetric structure in the category of super vector spaces. As before, we have

$$\beta_{\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}} - (\beta_{\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}})^{-1} = (q - q^{-1}) \mathbb{1}_{\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}}$$

The twist morphism on $\mathbb{C}_q^{m|n}$ is given by $\theta(v) = q^{m-n} v$, and so the Reshetikhin-Turaev knot polynomial from $U_q(\mathfrak{gl}(m|n))$ satisfies the skein relation

$$q^{m-n} P_{\diagup\!\!\!\!\diagdown}(q) - q^{n-m} P_{\diagdown\!\!\!\!\diagup}(q) = (q - q^{-1}) P_{)(}(q)$$

We can define the module $\bigwedge_q(\mathbb{C}_q^{m|n})$ as before, by taking the quotient of the tensor algebra by the ideal $\mathrm{Sq}^2(\mathbb{C}_q^{m|n})$ generated by the eigenspaces of $\beta_{\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}}$ with eigenvalue q^r for $r \in \mathbb{Z}$. We have

$$\mathrm{Sq}^2(\mathbb{C}_q^{m|n}) = (e_i \otimes e_j + (-1)^{\deg(e_i)\deg(e_j)} q e_j \otimes e_i, e_k \otimes e_k) \tag{3.4.1}$$

over all $i < j$, and $k \leq m$. Note that given an odd-degree element $w \in \mathbb{C}_q^{m|n}$, the tensor $w \otimes w \otimes \cdots \otimes w$ is non-zero in the quotient, and hence $\bigwedge_q^k(\mathbb{C}_q^{m|n})$ is non-zero for all $k \geq 0$.

The dual $(\mathbb{C}_q^{m|n})^*$ of $\mathbb{C}_q^{m|n}$ is given by the maps

$$\mathrm{ev}_{\mathbb{C}_q^{m|n}} : (\mathbb{C}_q^{m|n})^* \otimes \mathbb{C}_q^{m|n} \to \mathbb{C}(q) : e_i^* \otimes e_i \mapsto 1$$

$$\mathrm{coev}_{\mathbb{C}_q^{m|n}} : \mathbb{C}(q) \to \mathbb{C}_q^{m|n} \otimes (\mathbb{C}_q^{m|n})^* : 1 \mapsto \sum_i e_i \otimes e_i^*$$

$$\mathrm{ev}'_{\mathbb{C}_q^{m|n}} : \mathbb{C}_q^{m|n} \otimes (\mathbb{C}_q^{m|n})^* \to \mathbb{C}(q) : e_i \otimes e_i^* \mapsto \{i\} q^{m+n-\{i\}(2m-2i+1)}$$

$$\mathrm{coev}'_{\mathbb{C}_q^{m|n}} : \mathbb{C}(q) \to (\mathbb{C}_q^{m|n})^* \otimes \mathbb{C}_q^{m|n} : 1 \mapsto \sum_{i=1}^{m+n} \{i\} q^{-m-n+\{i\}(2m-2i+1)} e_i^* \otimes e_i$$

recalling that $\{i\} = 1$ if $i \leq m$ and $\{i\} = -1$ if $i > m$.

In particular, we see that the value of the Reshetikhin-Turaev invariant on the unknot is $[m-n]$, since the composition $\mathrm{ev}' \circ \mathrm{coev}$ is

$$1 \mapsto q^{m-n-1} + q^{m-n-3} + \cdots + q^{n-m+1} + \cdots + q^{-m-n+1} - q^{-m-n+1} - \cdots - q^{n-m-1}$$

In the case where $m \geq n$ the n negative terms cancel with the last n positive terms, leaving only the sum

$$q^{m-n-1} + q^{m-n-3} + \cdots + q^{n-m+1} = [m-n]$$

and similarly in the case where $m < n$ the m positive terms cancel the first m negative terms in the sum, leaving only

$$-q^{m-n+1} - \cdots - q^{n-m-1} = -[n-m] = [m-n].$$

We can no longer describe $(\mathbb{C}_q^{m|n})^*$ as some exterior power of $\mathbb{C}_q^{m|n}$. This makes the category of $U_q(\mathfrak{gl}(m|n))$-modules harder to describe diagrammatically. Moreover, in general the tensor product $(\mathbb{C}_q^{m|n})^* \otimes \mathbb{C}_q^{m|n}$ does not split into a direct sum of simple modules, meaning the category is not semi-simple.

3.4.1 The case $m = n$

Note that if $m = n$, the value of the polynomial on the unknot is 0. This implies, in fact, that the value of the polynomial on every knot or link is 0.

To resolve this, we must use the reduced polynomial. Pick some basepoint on the knot K, and cut the knot open at this basepoint and treat it as a $(1,1)$-tangle.

The Reshetikhin-Turaev procedure translates this tangle to a map

$$\mathbb{C}_q^{m|n} \to \mathbb{C}_q^{m|n}$$

which, as $\mathbb{C}_q^{m|n}$ is a simple $U_q(\mathfrak{gl}(m|n))$-module, must be

$$\Delta_K(q) \mathbb{1}_{\mathbb{C}_q^{m|n}}$$

for some $\Delta(q) \in \mathbb{C}(q)$. We take $\Delta_K(q)$ to be our Reshetikhin-Turaev polynomial, which is independent of choice of basepoint and diagram.

In fact, since $\Delta_U(q) = 1$ on the unknot U and

$$\Delta_{\times}(q) - \Delta_{\times}(q) = (q - q^{-1})\Delta_{)(}(q)$$

we see that $\Delta_K(q)$ is the Alexander polynomial of K.

3.5 MOY diagrams for superalgebras

It will follow from the results of the next section that there exists a coassociative coalgebra structure on $\bigwedge_q(\mathbb{C}_q^{m|n})$. However an explicit description does not appear in the literature except in the case $n = 0$ [CKM14] and $m = n = 1$ [Gra16]. We can give the structure explicitly as follows. To describe a basis element of $\bigwedge_q(\mathbb{C}_q^{m|n})$, let

$$Q = \{1, 2, \ldots, m\} \cup \{i_p \mid p \in \mathbb{N}, i \in \{m+1, \ldots, m+n\}\}$$

be a set ordered by $1 < 2 < \cdots < m < (m+1)_1 < (m+1)_2 < \cdots < (m+2)_1 < \cdots$. Let

$$\phi : Q \to \{1, \ldots, m+n\} : \begin{cases} t \mapsto t & \text{if } 1 \leq t \leq m \\ s_p \mapsto s & \text{if } m+1 \leq s \leq m+n \end{cases}$$

Let S be a finite subset of Q. Then let $e_S = e_{\phi(j_1)} \wedge e_{\phi(j_2)} \wedge \cdots \wedge e_{\phi(j_k)}$ where $S = \{j_1 < j_2 < \cdots < j_k\}$. We can describe a basis of $\bigwedge_q(\mathbb{C}_q^{m|n})$ as the set $\{e_S \mid S \subset Q, \#S < \infty\}$.

For $t \in Q$, we say t is even if $\phi(t) \in \{1, \ldots, m\}$ and odd if $\phi(t) \in \{m+1, \ldots, m\}$. Moreover, we say that t and s are *duplicated* if $\phi(t) = \phi(s)$.

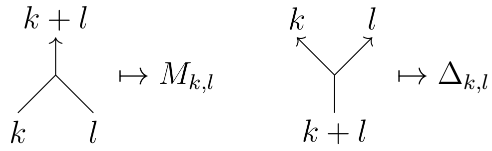
To describe the coproduct $\Delta_{k,l} : \bigwedge_q^{k+l}(\mathbb{C}_q^{m|n}) \to \bigwedge_q^k(\mathbb{C}_q^{m|n}) \otimes \bigwedge_q^l(\mathbb{C}_q^{m|n})$, we let $S \subset Q$ have cardinality $k+l$, and let $T \subset S$ have cardinality k. Let $\sigma_{T,S}$ be the smallest length permutation of S taking all elements of T in order before all elements of $S \setminus T$ in order, and write σ_T as a reduced product of transpositions $\sigma_{T,S} = \tau_1 \cdots \tau_j$. We set $l(\tau_i) = 2$ if τ_i transposes two duplicated elements, and $l(\tau_i) = 1$ otherwise. Let $l(\sigma_T) = \sum_i l(\tau_i)$.

Finally, we set

$$\Delta_{k,l}(e_S) = q^{kl} \sum_{\substack{T \subset S \\ \#T = k}} (-q)^{-l(\sigma_{T,S})} e_T \otimes e_{S \setminus T}.$$

It is not hard to see that this comultiplication is coassociative.

As before we let



Theorem 3.5.1. *These maps satisfy the version of the MOY moves in Figure 3.5.*

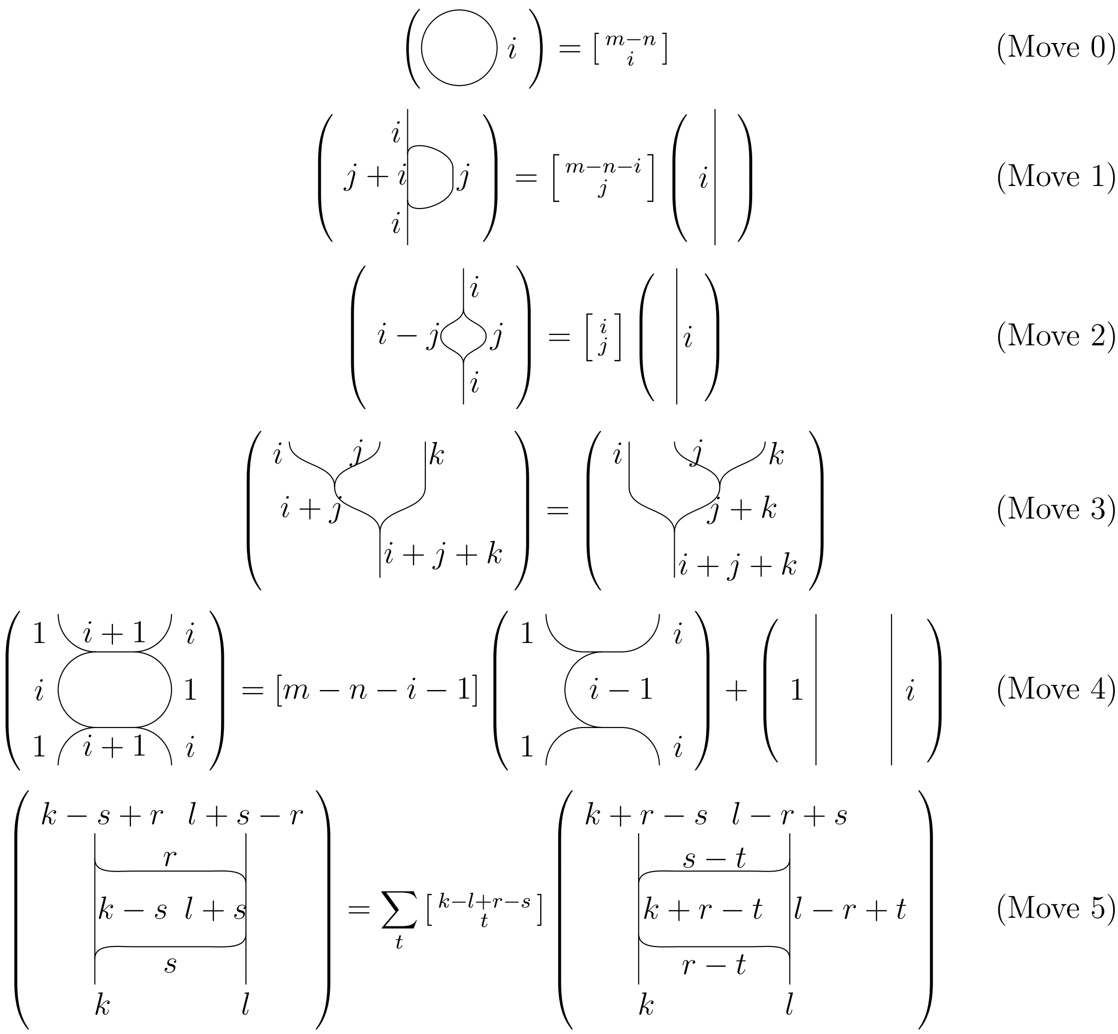


Figure 3.5: The six MOY moves for $\mathfrak{gl}(m|n)$ diagrams

We use the convention that

$$\begin{bmatrix} -i \\ j \end{bmatrix} := (-1)^j \begin{bmatrix} i+j-1 \\ j \end{bmatrix}$$

where $i, j \in \mathbb{N}$.

To prove this theorem, we note that as a corollary to Proposition 3.3.2, it suffices to prove certain special cases of the relations. We can prove Move 5 in the case $r = s = 1$, Move 2 in the case of $i = j = 1$, and Move 4 in the case $i = 1$. The remaining cases then follow from the other relations, by the argument in Proposition 3.3.2. Move 0 can also be proved from the $i = 1$ case by applying Move 1 and Move 2.

Proof. • Move 0: In the case $i = 1$, this follows from the calculation of $\mathrm{ev}' \circ \mathrm{coev}$ and $\mathrm{ev} \circ \mathrm{coev}'$.

• Move 1: We prove the case $i = 1$, $j = 1$. Let $\delta_{k,o} = 1$ if $k > m$ and 0 otherwise. The map on the left-hand side is

$$\begin{aligned}
e_k &\mapsto \sum_i e_k \otimes e_i \otimes e_i^* \\
&\mapsto -q^{-1} \sum_{i<k} e_i \wedge e_k \otimes e_i^* + \sum_{i>k} e_k \wedge e_i \otimes e_i^* \\
&\mapsto \sum_{i<k} (-e_i \otimes e_k \otimes e_i^* + q^{-1} e_k \otimes e_i \otimes e_i^*) + \sum_{i>k} (q e_k \otimes e_i \otimes e_i^* - e_i \otimes e_k \otimes e_i^*) \\
&\mapsto \Big(\sum_{i<k} \{i\} q^{m+n-1-\{i\}(2m-2i+1)} + \sum_{i>k} \{i\} q^{m+n+1-\{i\}(2m-2i+1)} \\
&\quad + \delta_{k,o} \{k\} q^{m+n-\{k\}(2m-2k+1)} [2]\Big) e_k \\
&= \left(q^{m-n-2} + \cdots + q^{2+n-m}\right) e_k = [m-n-1] e_k
\end{aligned}$$

as required.

• Move 2:

$$\begin{aligned}
e_S &\mapsto q^{(i-j)j} \sum_{\substack{T \subset S \\ \#T = i-j}} (-q)^{-l(\sigma_{T,S})} e_T \otimes e_{S \setminus T} \\
&\mapsto q^{(i-j)j} \sum_{\substack{T \subset S \\ \#T = j-i}} (-q)^{-l(\sigma_{T,S})} e_T \wedge e_{S \setminus T}.
\end{aligned}$$

The reduction of $e_T \wedge e_{S \setminus T}$ to e_S involves swapping elements until they are in the correct order in the wedge product. Each swap gives a factor of $-q^{-1}$. Of course, duplicated elements do not need to swap. Since $l(\tau_i) = 2$ for duplicated elements, and $l(\tau_i) = 1$ otherwise, we see that the coefficient of the T term will be simply $q^{-2l'(\sigma_T)}$ where $l'(\sigma_T)$ is just the length of the permutation. Hence the map is

$$e_S \mapsto q^{(i-j)j} \sum_{\substack{T \subset S \\ \#T = i-j}} q^{-2l'(\sigma_{T,S})} e_S = \begin{bmatrix} i \\ j \end{bmatrix} e_S.$$

- Move 3: The left-hand side is

$$\begin{aligned} e_S &\mapsto q^{(i+j)k} \sum_{\substack{T\subset S\\ \#T=i+j}} (-q)^{-l(\sigma_{T,S})} e_T \otimes e_{S\setminus T} \\ &\mapsto q^{(i+j)k+ij} \sum_{\substack{T\subset S\\ \#T=i+j}} \sum_{\substack{R\subset T\\ \#R=i}} (-q)^{-l(\sigma_{T,S})-l(\sigma_{R,T})} e_R \otimes e_{T\setminus R} \otimes e_{S\setminus T}. \end{aligned}$$

The right-hand side is

$$\begin{aligned} e_S &\mapsto q^{i(j+k)} \sum_{\substack{R\subset S\\ \#R=i}} (-q)^{-l(\sigma_{R,S})} e_R \otimes e_{S\setminus R} \\ &\mapsto q^{i(j+k)+jk} \sum_{\substack{R\subset S\\ \#R=i}} \sum_{\substack{T'\subset S\setminus R\\ \#T'=j}} (-q)^{-l(\sigma_{R,S})-l(\sigma_{T',S\setminus R})} e_R \otimes e_{T'} \otimes e_{S\setminus R\setminus T'}. \end{aligned}$$

These are equal by the association of T' in the second sum with $T \setminus R$ in the first sum, and the corresponding coefficients match, since swapping R to the front of S and then $T \setminus R$ to the front of $S \setminus R$ is the same as swapping T to the front of S and then R to the front of T.

- Move 4: It suffices to verify the case $i = 1$. The left hand side is interpreted as the composition

$$\begin{aligned} &V^* \otimes V \xrightarrow{\mathbb{1}\otimes \mathrm{coev}} V^* \otimes V \otimes V \otimes V^* \xrightarrow{m} V^* \otimes {\textstyle\bigwedge_q^2}(V) \otimes V^* \\ &\xrightarrow{\Delta} V^* \otimes V \otimes V \otimes V^* \xrightarrow{\mathrm{ev}\otimes\mathbb{1}} V \otimes V^* \xrightarrow{\mathrm{coev}'\otimes\mathbb{1}} V^* \otimes V \otimes V \otimes V^* \\ &\xrightarrow{m} V^* \otimes {\textstyle\bigwedge_q^2}(V) \otimes V^* \xrightarrow{\Delta} V^* \otimes V \otimes V \otimes V^* \xrightarrow{\mathbb{1}\otimes\mathrm{ev}'} V^* \otimes V \end{aligned}$$

We consider the action on basis elements $e_k^* \otimes e_j$ with $k \neq j$. It is easy to see the composition of the first four maps is

$$e_k^* \otimes e_j \mapsto -e_j \otimes e_k^*$$

since the coev map produces a sum over $1 \leq l \leq m+n$, but the only term that survives after the ev map is the one with $l = k$. A similar argument shows that the full composition is then the identity on $e_k^* \otimes e_j$.

The first diagram on the right-hand side acts as 0, since $\mathrm{ev}(e_k^* \otimes e_j) = 0$. Meanwhile the second diagrams acts as the identity, so the equality holds.

It remains to check the case of the basis elements $e_k^* \otimes e_k$.

We can see that the first diagram on the right-hand side acts as

$$e_k^* \otimes e_k \mapsto \sum_{j=1}^{m+n} \{j\} q^{m+n-\{j\}(2m-2j+1)}.$$

The first four maps on the left-hand side act as

$$e_k^* \otimes e_k \mapsto \sum_{l<k} q^{-1} e_l \otimes e_l^* + \sum_{l>k} e_l^* \otimes e_l + \delta_{k,o}[2] e_k^* \otimes e_k$$

where we use the notation $\delta_{k,o}$ to be 1 if $k > m$ and 0 else (in other words, whether e_k is an odd-degree element or not).

The last four maps result in a sum over all $e_j^* \otimes e_j$ with $1 \leq j \leq m+n$. The coefficient if $j < k$ is

$$\begin{aligned}\{j\}q^{m+n-\{j\}(2m-2j+1)} &\left(\sum_{l<j} \{l\} q^{-2-m-n+\{l\}(2m-2l+1)} \right. \\ &+ \sum_{j<l<k} \{l\} q^{-m-n+\{l\}(2m-2l+1)} + \sum_{l>k} \{l\} q^{2-m-n+\{l\}(2m-2l+1)} + \\ &\left. \delta_{k,o}\{k\} q^{1-m-n+\{k\}(2m-2k+1)}[2] + \delta_{j,o}\{j\} q^{-1-m-n+\{j\}(2m-2j+1)}[2] \right).\end{aligned}$$

The coefficient for $j > k$ is

$$\begin{aligned}\{j\}q^{m+n-\{j\}(2m-2j+1)} &\left(\sum_{l<k} \{l\} q^{-2-m-n+\{l\}(2m-2l+1)} \right. \\ &+ \sum_{k<l<j} \{l\} q^{-m-n+\{l\}(2m-2l+1)} + \sum_{l>k} \{l\} q^{2-m-n+\{l\}(2m-2l+1)} \\ &\left. + \delta_{k,o}\{k\} q^{-1-m-n+\{k\}(2m-2k+1)}[2] + \delta_{j,o}\{j\} q^{1-m-n+\{j\}(2m-2j+1)}[2] \right).\end{aligned}$$

and the coefficient for $e_k^* \otimes e_k$ is

$$\begin{aligned}\{k\}q^{m+n-\{k\}(2m-2k+1)} &\left(\sum_{l<k} \{l\} q^{-2-m-n+\{l\}(2m-2l+1)} \right. \\ &\left. + \sum_{l>k} \{l\} q^{2-m-n+\{l\}(2m-2l+1)} + \delta_{k,o}\{k\} q^{-1-m-n+\{k\}(2m-2k+1)}[2]^2 \right).\end{aligned}$$

In the two cases where $j \neq k$, we see that the coefficient of $e_j^* \otimes e_k$ is equal to

$\{j\}q^{m+n-\{j\}(2m-2j+1)}[m-n-2]$, since the exponent changes by two each time l increments, except at $l = j$ and $l = k$ in the case $j, k > m$, and the gaps are then filled in by the extra terms. In the case of $e_k^* \otimes e_k$ the coefficient is

$$\{k\}q^{m+n-\{k\}(2m-2k+1)}[m-n-2]+1$$

using $[2]^2 = [3]+1$.

Thus the two sides are equal, as was to be shown.

- Move 5: We prove the case $r = s = 1$. We use $\phi(\sigma_{T,S})$ for the minimum length permutation that sends every element of $\phi(T)$ in order to the left of every element of $\phi(S)$. Then the left-hand diagram acts as

$$e_S \otimes e_T \mapsto \sum_{i \in S} \sum_{j \in T \cup \{i\}} q^{x_{i,j}} e_{S \cup j \setminus \{i\}} \otimes e_{T \cup \{i\} \setminus \{j\}}$$

where the power of q is

$$x_{i,j} = k+l-1-l(\sigma_{S\setminus\{i\},S}) - l(\phi(\sigma_{\{i\},T\cup\{i\}})) - l(\sigma_{\{j\},T\cup\{i\}}) - l(\phi(\sigma_{S\setminus\{i\},S\cup\{j\}\setminus\{i\}})).$$

The other non-trivial diagram acts as

$$e_S \otimes e_T \mapsto \sum_{j \in T} \sum_{i \in S \cup \{j\}} q^{y_{i,j}} e_{S \cup \{j\} \setminus \{i\}} \otimes e_{T \cup \{i\} \setminus \{j\}}$$

where

$$y_{i,j} = k+l-1-l(\sigma_{\{j\},T}) - l(\phi(\sigma_{S,S\cup\{j\}})) - l(\sigma_{S\cup\{j\}\setminus\{i\},S\cup\{j\}}) - l(\phi(\sigma_{\{i\},T\setminus\{j\}\cup\{i\}})).$$

If $i \neq j$, then $x_{i,j} = y_{i,j}$. This is clear if $j < i$ by comparing terms $l(\sigma_{S\setminus\{i\},S})$ with $l(\sigma_{S\cup\{j\}\setminus\{i\},S\cup\{j\}}$ which will be equal. If $i > j$ then note that two of the terms will be larger by 1 in $y_{i,j}$ than the corresponding terms in $x_{i,j}$, and two will be smaller by 1, so that $x_{i,j} = y_{i,j}$.

Every pair (i, j) with $i \in S$, $j \in T$ and $i \neq j$ appears in both sums.

If $i = j$, then

$$x_{i,i} = k+l-1-2l'(\sigma_{S\setminus\{i\},S}) - 2l'(\sigma_{\{i\},T\cup\{i\}})$$

where $l'(\sigma)$ is the length of the permutation σ, and similarly

$$y_{i,i} = k + l - 1 - 2l'(\sigma_{\{i\},T}) - 2l'(\sigma_{S,S\cup\{i\}}).$$

Therefore the only surviving terms are those for which (i,i) appears only in one sum and not the other.

Thus it suffices to prove that

$$\sum_{i\in S\setminus T} q^{x_{i,i}} - \sum_{j\in T\setminus S} q^{y_{i,i}} = [k-l].$$

Now we do induction on $\min(k,l)$. When $l=0$, we get

$$\sum_{i\in S} q^{k-1-l'(\sigma_{S\setminus\{i\},S})} = [k]$$

and similarly when $k=0$ the sum gives $-[l] = [-l]$ as wanted.

For $k,l>0$, we can take a pair of elements $(s,t)\in S\times T$ that are consecutive, in the sense that there are no elements of S or T in between s and t. We have

$$x_{s,s} = k + l - 1 - 2l'(\sigma_{S\setminus\{s\},S}) - 2l'(\sigma_{\{s\},T\cup\{s\}})$$

and

$$y_{t,t} = k + l - 1 - 2l'(\sigma_{\{t\},T}) - 2l'(\sigma_{S,S\cup\{t\}}).$$

If $s<t$, then

$$l'(\sigma_{\{s\},T\cup\{s\}}) = l'(\sigma_{\{t\},T})$$

and

$$l'(\sigma_{S\setminus s,S}) = l'(\sigma_{S,S\cup\{t\}}).$$

If $s>t$, then

$$l'(\sigma_{S,S\cup\{t\}}) = l'(\sigma_{S\setminus\{s\},S}) + 1$$

and

$$l'(\sigma_{\{s\},T\cup\{s\}}) = l'(\sigma_{\{t\},T}) + 1.$$

Hence the terms corresponding to s and t cancel in the sum, and so s and t can be removed from S and T without changing the value of the sum, resulting in sets $S\setminus\{s\}$ and $T\setminus\{t\}$ of cardinality $k-1$ and $l-1$ respectively. By induction, this sum is equal to $[k-1-(l-1)] = [k-l]$, so the result follows.

$\square$

So using the resolution from Figure 3.2, we can calculate the $\mathfrak{gl}(m|n)$ polynomial by reducing diagrams using these MOY moves, then correcting for the writhe by multiplying the final result by $q^{(n-m)w(D)}$. In the case $m = n$, we again have to cut the knot or link open at a basepoint, and reduce everything to a multiple of a single upward strand. This recovers the Alexander polynomial.

Moreover, we can use the resolutions in subsection 3.3.2 to define coloured variants of these polynomials.

Chapter 4

Skew Howe Duality

4.1 Introduction

In the previous chapter we saw that the $\mathfrak{gl}(m|n)$ polynomials could be calculated in terms of MOY diagrams, which were defined as morphisms on $\bigwedge_q(\mathbb{C}_q^{m|n})$.

A surprising theorem of Cautis, Kamnitzer and Morrison [CKM14] states that, for $\mathfrak{gl}(n)$, these MOY diagrams are actually sufficient to describe every morphism

$$\bigwedge\nolimits_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge\nolimits_q^{k_m} \mathbb{C}_q^n \to \bigwedge\nolimits_q^{j_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge\nolimits_q^{j_p} \mathbb{C}_q^n$$

that commutes with the action of $U_q(\mathfrak{gl}(n))$. That is, MOY diagrams provide a generators and relations description of the category of $U_q(\mathfrak{gl}(n))$-modules monoidally generated by $\bigwedge_q^i(\mathbb{C}_q^n)$ for each i.

In this chapter, we generalise this to the case $U_q(\mathfrak{gl}(m|n))$. The main technique we use is Skew Howe duality, which we prove in theorem 4.2.3. This is a special case of a theorem that was also proved by Queffelec and Sartori [QS15, Theorem 4.2].

4.2 Skew Howe Duality

Skew Howe duality is a theorem that we use to describe $\mathrm{End}(\bigwedge_q \mathbb{C}_q^{m|n})$ in terms of the Lie algebra $U_q(\mathfrak{gl}(p))$. That is, we realise

$$\bigwedge\nolimits_q^{\lambda_1} \mathbb{C}_q^{m|n} \otimes \cdots \otimes \bigwedge\nolimits_q^{\lambda_p} \mathbb{C}_q^{m|n}$$

as the $(\lambda_1, \dots, \lambda_p)$ weight space of some $U_q(\mathfrak{gl}(p))$-module, in such a way that the $U_q(\mathfrak{gl}(p))$ acts by module homomorphisms of $U_q(\mathfrak{gl}(m|n))$ and, importantly, that every module homomorphism

$$\bigwedge\nolimits_q^{\lambda_1} \mathbb{C}_q^{m|n} \otimes \cdots \otimes \bigwedge\nolimits_q^{\lambda_p} \mathbb{C}_q^{m|n} \to \bigwedge\nolimits_q^{\mu_1} \mathbb{C}_q^{m|n} \otimes \cdots \otimes \bigwedge\nolimits_q^{\mu_p} \mathbb{C}_q^{m|n}$$

can be described by an element of $U_q(\mathfrak{gl}(p))$. This is theorem 4.2.3.

To prove this, we need the following technical lemma:

Lemma 4.2.1. *The $q = 1$ specialisation of $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ is isomorphic to $\bigwedge(\mathbb{C}^{m|n} \otimes \mathbb{C}^p)$ as modules over $U(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(p))$.*

Proof. Letting τ_{23} be the map that permutes the middle two of four tensor factors, we have that

$$R_{\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p} = \tau_{23} \circ (R_{\mathbb{C}_q^{m|n}} \otimes R_{\mathbb{C}_q^p}) \circ \tau_{23}$$

is the R-matrix on the standard module $\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p$ of $U_q(\mathfrak{gl}(p) \oplus \mathfrak{gl}(m|n))$, so it follows that

$$\mathrm{Sq}^2(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p) = \tau_{23} \left((\mathrm{Sq}^2(\mathbb{C}_q^{m|n}) \otimes \mathrm{Sq}^2(\mathbb{C}_q^p)) \oplus (\bigwedge\nolimits_q^2(\mathbb{C}_q^{m|n}) \otimes \bigwedge\nolimits_q^2(\mathbb{C}_q^p)) \right)$$

where the symmetric square is the union of the eigenspaces of R with eigenvalue q^r for some $r \in \mathbb{Z}$, as in equation 3.4.1 Letting $v_1, \dots, v_{m+n}$ be the standard basis of $\mathbb{C}_q^{m|n}$ and $x_1, \dots, x_p$ the standard basis of $\mathbb{C}_q^p$, we have a spanning set

$$(v_i \otimes v_i) \otimes (x_k \otimes x_k), \quad i \leq m$$

$$(v_i \otimes v_i) \otimes (x_k \otimes x_l + q x_l \otimes x_k), \quad i \leq m \text{ and } k < l$$

$$(v_i \otimes v_j + (-1)^{\deg(v_i)\deg(v_j)} q v_j \otimes v_i) \otimes (x_k \otimes x_k), \quad i < j$$

$$(v_i \otimes v_j + (-1)^{\deg(v_i)\deg(v_j)} q v_j \otimes v_i) \otimes (x_k \otimes x_l + q x_l \otimes x_k), \quad i < j \text{ and } k < l$$

$$(q v_i \otimes v_j - (-1)^{\deg v_i \deg v_j} v_j \otimes v_i) \otimes (q x_k \otimes x_l - x_l \otimes x_k), \quad i < j \text{ and } k < l$$

$$(v_i \otimes v_i) \otimes (q x_k \otimes x_l - x_l \otimes x_k), \quad i > m \text{ and } k < l$$

of $\tau_{23} \, \mathrm{Sq}^2(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$. Let $a = v_i \otimes x_k$, $b = v_i \otimes x_l$, $c = v_j \otimes x_k$, $d = v_j \otimes x_l$ with $\mathbb{Z}/2\mathbb{Z}$-grading determined by the degree of v_i or v_j, and we take $i < j$, $k < l$. Then in

$\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ we have

$$x^2 = \frac{1}{2}x^2 + (-1)^{1+\deg x}\frac{1}{2}x^2, \quad \text{for } x \in \{a, b, c, d\}$$

$$ab = (-1)^{\deg a} q^{1-2\deg a} ba, \quad ac = (-1)^{1+\deg a \deg c} qca$$

$$bd = (-1)^{1+\deg b \deg d} qbd, \quad ad = (-1)^{1+\deg a \deg d} da$$

$$bc + (-1)^{\deg b \deg c} cb = (q - q^{-1})(1 + (-1)^{\deg b \deg c})ad.$$

Then $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ is generated by $\{v_i \otimes x_j \mid 1 \leq i \leq m+n, 1 \leq j \leq p\}$ subject to the relations above. Let $v_{ij} = v_i \otimes x_j$. Then there is a spanning set given by elements of the form $v_{i_1 j_1} \wedge \cdots \wedge v_{i_l j_l}$, with $1 \leq i_1 < \cdots < i_l \leq m+n$ and $1 \leq j_1 < \cdots < j_l \leq p$ for all $l \neq \mathbb{N}$. This is linearly independent, since by setting $q = 1$ in the relations, we see that it is linearly independent at $q = 1$.

Hence the dimension of the $q = 1$ specialisation of $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ is equal to that of $\bigwedge(\mathbb{C}^{m|n} \otimes \mathbb{C}^p)$, and so it follows that the $q = 1$ specialisation of $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ is isomorphic to $\bigwedge(\mathbb{C}^{m|n} \otimes \mathbb{C}^p)$ as modules over $U(\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(p))$. $\square$

The above shows that quantising $\bigwedge(\mathbb{C}^{m|n} \otimes \mathbb{C}^p)$ does not involve changing too much of its structure. This means it has the same decomposition into simple modules as in the classical case, as we now show. Recall simple modules correspond to dominant weights, as in subsection 3.2.1.

Lemma 4.2.2. *There is an isomorphism*

$$\bigwedge\nolimits_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p) \cong \bigoplus_{\mu \in H} V_{m|n}(\mu^t) \otimes V_p(\mu)$$

where H is the set of dominant $\mathfrak{gl}(p)$ weights with $\mu_{n+1} \leq m$, μ^t is the reflection of the Young diagram about the diagonal, and $V_{m|n}(\mu^t)$ and $V_p(\mu)$ are highest-weight modules of $U_q(\mathfrak{gl}(m|n))$ and $U_q(\mathfrak{gl}(p))$ respectively.

Proof. By [CW01, Theorem 3.3], we have

$$\bigwedge(\mathbb{C}^{m|n} \otimes \mathbb{C}^p) \cong \bigoplus_{\mu \in H} V_{m|n}(\mu^t) \otimes V_p(\mu) \tag{4.2.1}$$

as modules over $U(\mathfrak{gl}(p) \oplus \mathfrak{gl}(m|n))$ (note this is the classical universal enveloping algebra, not the quantum one).

The module $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ decomposes into a direct sum of irreducible modules over $U_q(\mathfrak{gl}(m|n) \otimes \mathfrak{gl}(p))$, which are of the form $V \otimes W$, with V an irreducible over $U_q(\mathfrak{gl}(m|n))$ and W an irreducible over $U_q(\mathfrak{gl}(p))$, and both irreducibles are highest-weight modules. But by lemma 4.2.1, $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ specialises at $q = 1$ to $\bigwedge(\mathbb{C}^{m|n} \otimes \mathbb{C}^p)$. Since the highest-weight modules of $U_q(\mathfrak{gl}(m|n))$ specialise to highest-weight modules of $U(\mathfrak{gl}(m|n))$, and the decomposition into irreducibles is uniquely determined by the algebraic character, it follows that the highest-weight modules appearing in the decomposition of $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ correspond to the highest-weight modules appearing in the classical decomposition equation 4.2.1, so the result follows. □

Now we can prove the following major theorem, which will allow us to describe morphisms of the $U_q(\mathfrak{gl}(m|n))$-modules $\bigwedge_q^{\lambda_1} \mathbb{C}_q^{m|n} \otimes \cdots \otimes \bigwedge_q^{\lambda_p} \mathbb{C}_q^{m|n}$ in terms of elements of $U_q(\mathfrak{gl}(p))$.

Theorem 4.2.3 (Skew Howe duality)**.** *The actions of $U_q(\mathfrak{gl}(p))$ and $U_q(\mathfrak{gl}(m|n))$ on $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ generate each other's commutant. As $U_q(\mathfrak{gl}(m|n))$ representations, there is an isomorphism*

$$\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p) \cong \left(\bigwedge_q \mathbb{C}_q^{m|n}\right)^{\otimes p}$$

and the $(\lambda_1, \ldots, \lambda_p)$ weight space for the action of $U_q(\mathfrak{gl}(p))$ is identified with

$$\bigwedge_q^{\lambda_1} \mathbb{C}_q^{m|n} \otimes \cdots \otimes \bigwedge_q^{\lambda_p} \mathbb{C}_q^{m|n}.$$

Proof. By lemma 4.2.2, we have a direct sum decomposition into tensor factors, which shows that the actions of $U_q(\mathfrak{gl}(p))$ and $U_q(\mathfrak{gl}(m|n))$ on $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$ generate each others commutant.

To define the isomorphism, we let

$$\phi_j : \bigwedge_q \mathbb{C}_q^{m|n} \to \bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p) : v_i \mapsto v_i \otimes x_j$$

extended linearly, and then we can define

$$\phi : \left(\bigwedge_q \mathbb{C}_q^{m|n}\right)^{\otimes p} \to \bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p) = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_p.$$

By checking the relations in lemma 4.2.1, we can see that this is a well-defined map of $U_q(\mathfrak{gl}(m|n))$ representations, since the wedge product commutes with the $U_q(\mathfrak{gl}(m|n))$-

action, and the result is clearly a spanning set with dimension equal to the dimension of $\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)$.

The final statement follows by checking the action of each L_i on the right-hand side of the above isomorphisms. □

As a consequence, we have:

Theorem 4.2.4. *There is an isomorphism*

$$\mathrm{End}_{U_q(\mathfrak{gl}(m|n))}\left(\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)\right) \cong \bigoplus_{\mu \in H} \mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))$$

where H is the set of dominant weights $\mu \in \mathbb{Z}^p$ satisfying $\mu_{n+1} \leq m$.

Proof. By lemma 4.2.2, we have

$$\begin{aligned}\mathrm{End}_{U_q(\mathfrak{gl}(m|n))}\left(\bigwedge_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^p)\right) &\cong \mathrm{End}_{U_q(\mathfrak{gl}(m|n))}\left(\bigoplus_{\mu \in H} V_{m|n}(\mu^t) \otimes V_p(\mu)\right) \\ &\cong \bigoplus_{\mu \in H} \mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))\end{aligned}$$

since the $V_{m|n}(\mu^t)$ are simple modules of $U_q(\mathfrak{gl}(m|n))$ with no non-trivial maps between them. □

Example 4.2.5. Take $p = 2$ and $(m, n) \notin \{(1,0), (0,1), (0,0)\}$, and consider

$$\bigwedge\nolimits_q^2(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^2).$$

This module is isomorphic to

$$(V_{m|n}((1,1)) \otimes V_2((2,0))) \oplus (V_{m|n}((2,0)) \otimes V_2((1,1)))$$

by lemma 4.2.2.

Forgetting the $\mathfrak{gl}(m|n)$ action, we can write this as a $U_q(\mathfrak{gl}(2))$ module. By theorem 4.2.3, this is

$$\textstyle\bigwedge_q^0(\mathbb{C}_q^{m|n}) \otimes \bigwedge_q^2(\mathbb{C}_q^{m|n}) \underset{F}{\overset{E}{\rightleftarrows}} \bigwedge_q^1(\mathbb{C}_q^{m|n}) \otimes \bigwedge_q^1(\mathbb{C}_q^{m|n}) \underset{F}{\overset{E}{\rightleftarrows}} \bigwedge_q^2(\mathbb{C}_q^{m|n}) \otimes \bigwedge_q^0(\mathbb{C}_q^{m|n})$$

since the (λ_1, λ_2) weight space is

$$\bigwedge\nolimits_q^{\lambda_1}(\mathbb{C}_q^{m|n}) \otimes \bigwedge\nolimits^{\lambda_2}(\mathbb{C}_q^{m|n})$$

and the arrows denote the action of the $E, F \in U_q(\mathfrak{gl}(2))$.

We can write $\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n} \cong \bigwedge_q^2(\mathbb{C}^{m|n}) \oplus S_q^2(\mathbb{C}^{m|n})$ where $S_q^2(\mathbb{C}^{m|n})$ is the symmetric square. Thus the E, F above can be seen as renormalised inclusion and projections maps onto $\bigwedge_q^2(\mathbb{C}^{m|n})$, and act as 0 on $S_q^2(\mathbb{C}^{m|n})$ since there are no non-zero module maps $\bigwedge_q^2(\mathbb{C}^{m|n}) \to S_q^2(\mathbb{C}^{m|n})$ as both are irreducible and distinct.

Writing this in terms of simple $U_q(\mathfrak{gl}(2))$-modules, the representation

$$U_q(\mathfrak{gl}(2)) \to \mathrm{End}_{U_q(\mathfrak{gl}(m|n))}\left(\bigwedge\nolimits_q^2(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^2)\right)$$

is isomorphic to the direct sum $V_2(2,0) \oplus V_2(1,1)$. The module $V_2(1,1)$ is 1-dimensional, and $V_2(2,0)$ is 3-dimensional, summing to the dimension of the $U_q(\mathfrak{gl}(m|n))$ commutators in $\mathrm{End}_{\mathbb{C}(q)}\left(\bigwedge_q^2(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^2)\right)$.

4.3 Ladder Diagrams

In this section we recall the graphical calculus defined in [CKM14] for Lusztig's idempotented version $\dot{U}_q(\mathfrak{gl}(p))$ of $U_q(\mathfrak{gl}(p))$. This will translate to a graphical calculus on a representation category of modules over $U_q(\mathfrak{gl}(m|n))$, which will essentially be the MOY diagrams.

4.3.1 The algebra $\dot{U}_q(\mathfrak{gl}(p))$

We form an algebra $U_q'(\mathfrak{gl}(p))$ by adjoining to $U_q(\mathfrak{gl}(p))$ elements 1_λ for each weight $\lambda \in \mathbb{Z}^p$, with the extra relations

$$1_\lambda 1_\lambda' = \delta_{\lambda\lambda'} 1_\lambda$$

$$E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i$$

$$F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i$$

$$L_i 1_\lambda = q^{\lambda_i} 1_\lambda$$

where $\alpha_i = (0, \ldots, 1, -1, \ldots, 0)$ are the simple roots, with the 1 in position i, and λ_i is the i^{th} term of λ.

Definition 4.3.1. We define the non-unital algebra

$$\dot{U}_q(\mathfrak{gl}(p)) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^p} 1_\lambda U'_q(\mathfrak{gl}(p)) 1_\mu.$$

It is often convenient to treat $\dot{U}_q(\mathfrak{gl}(p))$ as a category, with objects 1_λ and morphisms $1_\lambda \to 1_\nu$ given by the elements of $1_\nu \dot{U}_q(\mathfrak{gl}(p)) 1_\lambda$. We shall freely switch between the two viewpoints.

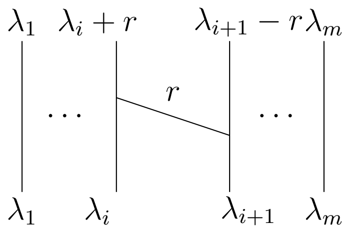
In order to relate the category $\dot{U}_q(\mathfrak{gl}(p))$ to morphisms on modules over $U_q(\mathfrak{gl}(m|n))$, we will mostly be interested in the following quotient:

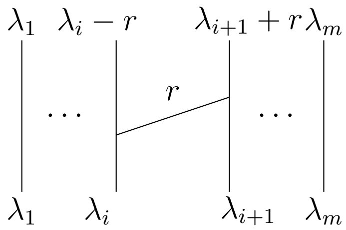
Definition 4.3.2. We define $\dot{U}_q^\infty(\mathfrak{gl}(p))$ to be the quotient of $\dot{U}_q(\mathfrak{gl}(p))$ by the two-sided ideal generated by the elements 1_λ that have $\lambda_i < 0$ for some i.

Cautis, Kamnitzer and Morrison [CKM14] defined *ladder diagrams* to describe morphisms in the category $\dot{U}_q^\infty(\mathfrak{gl}(p))$.

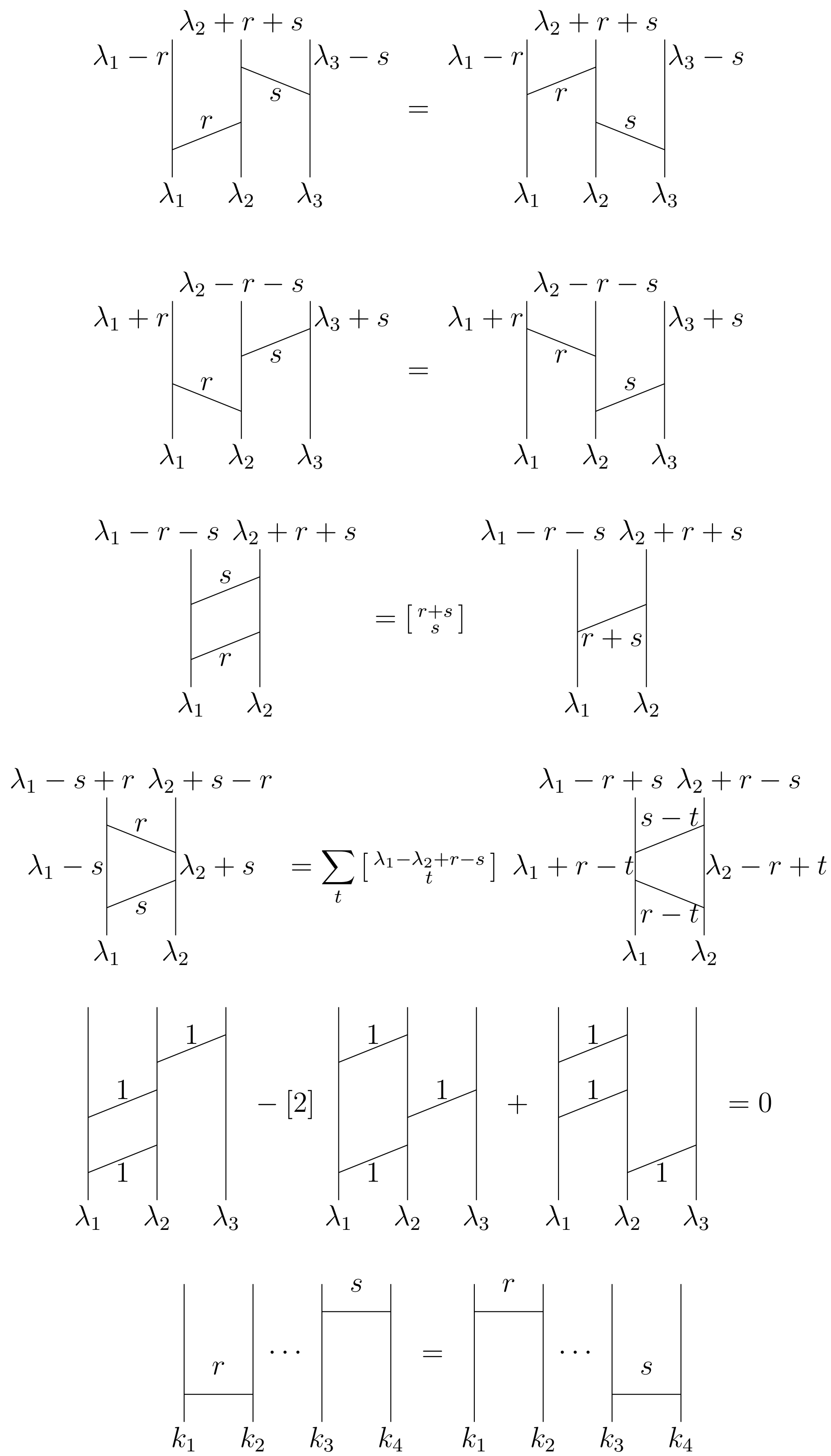
Definition 4.3.3. A *ladder* with p *uprights* is a diagram in $[0,1] \times [0,1]$ with p oriented vertical lines connecting the bottom edge to the top edge with horizontal *rungs* joining adjacent uprights. Each line segment is labelled with a natural number such that the algebraic sum of labels at a trivalent vertex is 0.

To relate this to $\dot{U}_q(\mathfrak{gl}(p))$, we associate the following ladders to morphisms in $\dot{U}_q(\mathfrak{gl}(p))$:

$E_i^{(r)} 1_\lambda \mapsto$ 

$F_i^{(r)} 1_\lambda \mapsto$ 

Then we define the following relations on ladders:



with either orientation on each of the rungs in the last relation as long as the two r-coloured rungs have the same orientation, and similarly for the two s-coloured rungs. We also take mirror images of the third and fifth relations, and include all relations with arbitrarily many uprights on each side.

Since these relations were imposed to match the relations on the morphisms of $\dot{U}_q(\mathfrak{gl}(p))$ in subsection 3.3.3, we get the following:

Lemma 4.3.4. *The category $\dot{U}_q^\infty(\mathfrak{gl}(p))$ is equivalent to the category of ladders on p uprights.*

4.3.2 Relationship with modules of $U_q(\mathfrak{gl}(m|n))$

We want a full description of the following category of $U_q(\mathfrak{gl}(m|n))$-modules:

Definition 4.3.5. We let $\mathrm{Rep}(\mathfrak{gl}(m|n))$ be the additive category monoidally generated by $\bigwedge_q^k(\mathbb{C}_q^{m|n})$ for all k. That is, objects in the category are direct sums of

$$\bigwedge\nolimits_q^{k_1}(\mathbb{C}_q^{m|n}) \otimes \cdots \otimes \bigwedge\nolimits_q^{k_p}(\mathbb{C}_q^{m|n})$$

for all $(k_1, \ldots, k_p) \in \mathbb{N}^p$, and all $p \in \mathbb{N}$. Morphisms in the category are all $U_q(\mathfrak{gl}(m|n))$-module morphisms.

As a major corollary to skew Howe duality theorem 4.2.3, we can write every morphism in $\mathrm{Rep}(\mathfrak{gl}(m|n))$ as an element of $\dot{U}_q^\infty(\mathfrak{gl}(p))$.

Theorem 4.3.6. *There is a full functor*

$$\dot{U}_q^\infty(\mathfrak{gl}(p)) \to \mathrm{Rep}(\mathfrak{gl}(m|n))$$

Proof. The functor is defined by sending 1_λ to $\bigwedge_q^{\lambda_1}(\mathbb{C}_q^{m|n}) \otimes \bigwedge_q^{\lambda_2}(\mathbb{C}_q^{m|n}) \otimes \cdots \otimes \bigwedge_q^{\lambda_p}(\mathbb{C}_q^{m|n})$ and sending

$$\begin{aligned} &1_\nu \dot{U}_q(\mathfrak{gl}(p)) 1_\lambda \to \\ &\qquad \mathrm{Hom}_{U_q(\mathfrak{gl}(m|n))}\left(\bigwedge\nolimits_q^{\lambda_1}(\mathbb{C}_q^{m|n}) \otimes \cdots \otimes \bigwedge\nolimits_q^{\lambda_p}(\mathbb{C}_q^{m|n}), \bigwedge\nolimits_q^{\nu_1}(\mathbb{C}_q^{m|n}) \otimes \cdots \otimes \bigwedge\nolimits_q^{\nu_p}(\mathbb{C}_q^{m|n})\right) \end{aligned}$$

by the action of $U_q(\mathfrak{gl}(p))$ in theorem 4.2.3. Since this action generates the commutant of the action of $U_q(\mathfrak{gl}(m|n))$, it follows that this functor is full. $\square$

However, this functor is not faithful. In the next section, we find the kernel of this functor.

4.3.3 Extra $\mathfrak{gl}(m)$ relations on ladder diagrams

The functor in theorem 4.3.6 is not faithful for all p. However, in the special case $n = 0$ (corresponding to ordinary Lie algebras $\mathfrak{gl}(m)$), the kernel is easy to describe.

By theorem 4.2.4, we have

$$\mathrm{End}_{U_q(\mathfrak{gl}(m))}\left(\bigwedge\nolimits_q(\mathbb{C}_q^m \otimes \mathbb{C}_q^p)\right) \cong \bigoplus_{\mu \in H} \mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))$$

where H is the set of partitions with $\mu_1 \leq m$. We can use the following:

Lemma 4.3.7. *Let λ be a dominant weight, and let $L(\lambda)$ be the set of all dominant weights dominated by λ. Then there is an isomorphism of algebras*

$$\dot{U}_q(\mathfrak{gl}(p))/I_\lambda \to \bigoplus_{\mu \in L(\lambda)} \mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))$$

where I_λ is the 2-sided ideal of $\dot{U}_q(\mathfrak{gl}(p))$ generated by 1_μ for all weights μ not dominated by λ.

Proof. This is Lemma 4.4.2 in [CKM14]. □

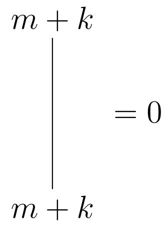
Note that we have $\dot{U}_q^\infty(\mathfrak{gl}(p)) \cong \bigoplus_{K \in \mathbb{N}} \dot{U}_q(\mathfrak{gl}(p))/I_{(K,0,\ldots,0)}$ and every dominant weight is dominated by $(K, 0, \ldots, 0)$ for some (unique) $K \in \mathbb{N}$. Thus the kernel of the map

$$\dot{U}_q^\infty(\mathfrak{gl}(p)) \to \mathrm{End}_{U_q(\mathfrak{gl}(m))}\left(\bigwedge\nolimits_q(\mathbb{C}_q^m \otimes \mathbb{C}_q^p)\right)$$

is generated by morphisms that act as 0 on every $V_p(\mu)$ for $\mu \in H$. Since H is the set of dominant weights $\mu < (m, m, \ldots, m, 0, \ldots, 0)$ for some number of m's, these are all morphisms that factor through a weight 1_λ with $\lambda_i > m$ for some i. This was proved in [CKM14].

We can completely describe the category $\mathrm{Rep}(\mathfrak{gl}(m))$ as consisting of all ladder diagrams with colours bounded above by m.

Theorem 4.3.8. *The category* $\mathrm{Rep}(\mathfrak{gl}(m))$ *is equivalent to the quotient of the category* *of ladders by the additional relations*



(4.3.1)

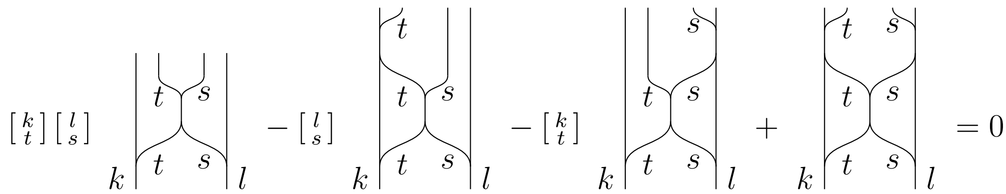
for all $k > 0$.

4.3.4 Extra $\mathfrak{gl}(1|1)$ relations on ladder diagrams

In [Gra16], we also give a description of the kernel in the case $m = n = 1$, which turns out to be a little more complicated.

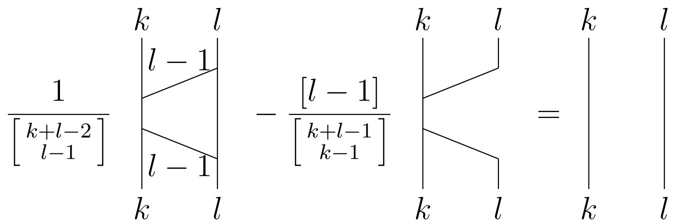
Since $\bigwedge_q^k(\mathbb{C}_q^{1|1})$ is non-zero for all $k > 0$, no relation like equation 4.3.1 can hold.

However, the MOY diagram relation



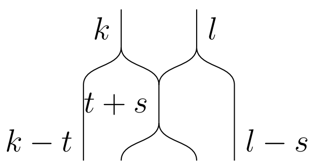
(4.3.2)

can be shown to hold for the product and coproduct on $\bigwedge_q(\mathbb{C}_q^{1|1})$ for $k, l \geq 2$ and $t, s \geq 1$. This is shown explicitly in [Gra16]. A similar relation for $k = l = 1$ also appeared in [Sar13a]. From this, we can deduce



(4.3.3)

by attaching the diagram



to the top and simplifying with the MOY moves. The left-hand side of equation 4.3.3 is idempotent.

By theorem 4.2.4, we have

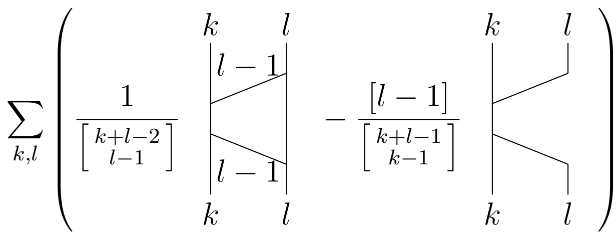
$$\mathrm{End}_{U_q(\mathfrak{gl}(1|1))}\left(\bigwedge\nolimits_q(\mathbb{C}_q^{1|1}\otimes\mathbb{C}_q^p)\right)\cong\bigoplus_{\mu\in H}\mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))$$

where H is the set of dominant weights μ with $\mu_2 \leq 1$. So, once again, the map

$$\dot{U}_q^\infty(\mathfrak{gl}(p)) \to \mathrm{End}_{U_q(\mathfrak{gl}(1|1))}\left(\bigwedge\nolimits_q(\mathbb{C}_q^{1|1}\otimes\mathbb{C}_q^p)\right)$$

is projection to a direct summand.

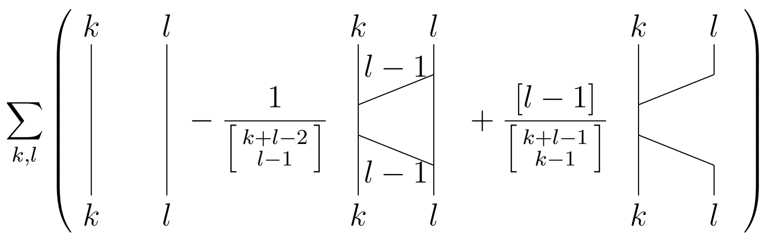
Theorem 4.3.9. *The element*



is the idempotent of $\dot{U}_q^\infty(\mathfrak{gl}(2))$ *projecting to*

$$\bigoplus_{\mu\in H}\mathrm{End}_{\mathbb{C}(q)}(V_2(\mu))=\bigoplus_{k,l}\left(\mathrm{End}_{\mathbb{C}(q)}(V_2((k+l,0)))\oplus\mathrm{End}_{\mathbb{C}(q)}(V_2((k+l-1,1)))\right)$$

Proof. Since the matrix algebras $\mathrm{End}_{\mathbb{C}(q)}(V_2(\mu))$ are simple, it is enough to show that the complementary idempotent



has non-zero action on $V_2(\mu)$ for all dominant weights μ with $\mu_2 \geq 2$. So if $\mu = (\mu_1, \mu_2)$ with $\mu_2 > 1$, the two latter terms with $k = \mu_1, l = \mu_2$ factor through the $(\mu_1 + \mu_2 - 1, 1)$ and $(\mu_1 + \mu_2, 0)$ weight spaces of $V_2(\mu)$, which are 0, so the element acts as the identity. Now we note that when $l = 1$ and $l = 0$ this element is identically 0, so the result follows. $\square$

In general, we can take equation 4.3.3 as a local relation in $\dot{U}_q^\infty(\mathfrak{gl}(p)))$ by adding uprights to either side. Then by adapting the proof of theorem 4.3.9 we find that the quotient by the relation is precisely $\bigoplus_{\mu\in H}\mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))$.

Let $\dot{U}_q^{1|1}(\mathfrak{gl}(p))$ be the quotient of $\dot{U}_q^{\infty}(\mathfrak{gl}(p))$ by the local relation equation 4.3.3. Then we have shown

Theorem 4.3.10. *The induced functor*

$$\dot{U}_q^{1|1}(\mathfrak{gl}(p)) \to \mathrm{Rep}(\mathfrak{gl}(1|1))$$

is full and faithful.

This is not an equivalence of categories because the objects in $\mathrm{Rep}(\mathfrak{gl}(1|1))$ with more than p tensor summands are not in the image. But the direct sum

$$\bigoplus_p \dot{U}_q^{1|1}(\mathfrak{gl}(p)) \to \mathrm{Rep}(\mathfrak{gl}(1|1))$$

is an equivalence of categories.

4.3.5 Extra $\mathfrak{gl}(m|n)$ relations on ladder diagrams

For general m, n it seems very difficult to give closed formulas for the extra relations (in terms of ladder diagrams), but we can describe how they arise.

As before, lemma 4.3.7 implies that there exists a system of orthogonal central idempotents $e_l \in \dot{U}_q^{\infty}(\mathfrak{gl}(p))$ corresponding to the decomposition into a direct sum of $\mathrm{End}_{\mathbb{C}(q)}(V_p(\mu))$.

By theorem 4.2.4, we have

$$\mathrm{End}_{U_q(\mathfrak{gl}(m|n))}\left(\bigwedge\nolimits_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{p})\right) \cong \bigoplus_{\mu \in H} \mathrm{End}_{\mathbb{C}(q)}(V_p(\mu)).$$

Thus there is a map

$$\dot{U}_q^{\infty}(\mathfrak{gl}(p)) \to \mathrm{End}_{U_q(\mathfrak{gl}(m|n))}\left(\bigwedge\nolimits_q(\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{p})\right)$$

which simply corresponds to projection by $\sum_{l \in H} e_l$.

So by lemma 4.3.7, we have a functor

$$\dot{U}_q^{\infty}(\mathfrak{gl}(p)) \to \mathrm{Rep}(\mathfrak{gl}(m|n))$$

which factors through

$$\dot{U}_q^\infty(\mathfrak{gl}(p))\sum_{\mu\in H} e_\mu$$

such that the induced functor

$$\dot{U}_q^\infty(\mathfrak{gl}(p))\sum_{\mu\in H} e_\mu \to \mathrm{Rep}(\mathfrak{gl}(m|n))$$

is full and faithful. This gives us our desired full description of the relations on $\mathrm{Rep}(\mathfrak{gl}(m|n))$, as any morphism in $\mathrm{Rep}(\mathfrak{gl}(m|n))$ can be described by ladder diagrams. This description is unique up to the ladder relations in subsection 4.3.1, and the relation that $\sum_{\mu\in H} e_\mu$ is the identity on $\mathrm{Rep}(\mathfrak{gl}(m|n))$.

Definition 4.3.11. We let $\dot{U}_q^{(m|n)}(\mathfrak{gl}(p))$ be the category $\dot{U}_q^\infty(\mathfrak{gl}(p))\sum_{\mu\in H} e_\mu$.

Theorem 4.3.12. *The functor*

$$\dot{U}_q^{(m|n)}(\mathfrak{gl}(p)) \to \mathrm{Rep}(\mathfrak{gl}(m|n))$$

is full and faithful, and the induced functor

$$\bigoplus_{p=2}^{\infty} \dot{U}_q^{(m|n)}(\mathfrak{gl}(p)) \to \mathrm{Rep}(\mathfrak{gl}(m|n))$$

is an equivalence of categories.

Proof. The first part is discussed above. For the second part, simply note that the functor is essentially surjective and each summand is full and faithful. □

The first non-trivial example of this is $\dot{U}_q^{(1,0)}(\mathfrak{gl}(2))$ and $\dot{U}_q^{(0,1)}(\mathfrak{gl}(2))$.

Example 4.3.13. Consider $\dot{U}_q(\mathfrak{gl}(2))/I_{(2,0)}$, which has a basis

$$1_{(2,0)},\ 1_{(1,1)},\ 1_{(0,2)},\ F1_{(2,0)},\ F^{(2)}1_{(2,0)},\ E1_{(1,1)},\ F1_{(1,1)},\ E1_{(0,2)},\ E^{(2)}1_{(0,2)},\ EF1_{(1,1)}.$$

Now $\dot{U}_q(\mathfrak{gl}(2))/I_{(2,0)} \cong \mathrm{End}_{\mathbb{C}(q)}(V_2(2,0)) \oplus \mathrm{End}_{\mathbb{C}(q)}(V_2(1,1))$, so there must exist orthogonal idempotents corresponding to this decomposition. One notes that $EF1_{(1,1)} \cdot V_2(2,0) = [2]1_{(1,1)} \cdot V_2(2,0)$, while $EF1_{(1,1)} \cdot V_2(1,1) = 0$ as this representation is 1-dimensional.

Hence, we find that $1_{(2,0)} + \frac{1}{[2]}EF1_{(1,1)} + 1_{(0,2)}$ is an idempotent projecting to the summand $\mathrm{End}_{\mathbb{C}(q)}(V_2(2,0))$ and $1_{(1,1)} - \frac{1}{[2]}EF1_{(1,1)}$ is an idempotent projecting to $\mathrm{End}_{\mathbb{C}(q)}(V_2(1,1))$.

Hence the algebra $1_{(1,1)}\dot{U}_q^{(1|0)}(\mathfrak{gl}(2))1_{(1,1)}$ is defined by

$$1_{(1,1)}\dot{U}_q^{\infty}(\mathfrak{gl}(2))(1_{(1,1)} - \frac{1}{[2]}EF1_{(1,1)})$$

and $1_{(1,1)}\dot{U}_q^{(0|1)}(\mathfrak{gl}(2))1_{(1,1)}$ is defined by

$$1_{(1,1)}\dot{U}_q^{\infty}(\mathfrak{gl}(2))(\frac{1}{[2]}EF1_{(1,1)}).$$

4.3.6 Branching rules and locality of relations

Here we wish to verify that there is a well-defined inclusion

$$\dot{U}_q^{(m|n)}(\mathfrak{gl}(p)) \hookrightarrow \dot{U}_q^{(m|n)}(\mathfrak{gl}(p+1))$$

induced by the inclusion $\dot{U}_q(\mathfrak{gl}(p)) \to \dot{U}_q(\mathfrak{gl}(p+1))$ and establish that the extra relations this imposes on $\mathrm{Rep}(\mathfrak{gl}(m|n))$ are indeed local, and do not depend on addition of strands on either side.

Definition 4.3.14. For each $j \geq 0$ define the inclusion

$$\iota_j : \dot{U}_q(\mathfrak{gl}(p)) \to \dot{U}_q(\mathfrak{gl}(p+1))$$

on objects by $1_{(\lambda_1,\ldots,\lambda_p)} \mapsto 1_{(\lambda_1,\ldots,\lambda_p,j)}$, and on morphisms by $E_i 1_{(\lambda_1,\ldots,\lambda_p)} \mapsto E_i 1_{(\lambda_1,\ldots,\lambda_p,j)}$ and $F_i 1_{(\lambda_1,\ldots,\lambda_p)} \mapsto F_i 1_{(\lambda_1,\ldots,\lambda_p,j)}$. This is well-defined since relations are mapped to relations.

It is clear that this inclusion descends to an inclusion $\dot{U}_q^{\infty}(\mathfrak{gl}(p)) \to \dot{U}_q^{\infty}(\mathfrak{gl}(p+1))$ since weights with non-negative entries are carried to the same. This inclusion gives rise to a restriction functor taking modules over $\dot{U}_q(\mathfrak{gl}(p+1))$ to modules over $\dot{U}_q(\mathfrak{gl}(p))$, by simply forgetting the action of the E_p, F_p, L_{p+1}. We denote the restriction of a $\dot{U}_q(\mathfrak{gl}(p+1))$-module M to a $\dot{U}_q(\mathfrak{gl}(p))$-module by $M|_p$.

We have the following well-known theorem:

Theorem 4.3.15. *There is an isomorphism of $\dot{U}_q(\mathfrak{gl}(p))$-modules*

$$V_{p+1}(\lambda)|_p \cong \bigoplus_{\nu} V_p(\nu)$$

where the sum is over all dominant weights ν satisfying $\lambda_{i+1} \leq \nu_i \leq \lambda_i$.

Proof. The classical case is well-known (see, for example, [IN66; Vaz13]). The quantum case then follows since irreducible highest-weight modules specialise at $q = 1$ to irreducible highest-weight modules. □

Hence we have

$$\mathrm{End}_{\mathbb{C}(q)}(V_{p+1}(\lambda)|_p) \cong \bigoplus_{\nu} \mathrm{End}_{\mathbb{C}(q)}(V_p(\nu)). \tag{4.3.4}$$

We can now state the following:

Theorem 4.3.16. *The inclusion $\iota_j : \dot{U}_q^\infty(\mathfrak{gl}(p)) \to \dot{U}_q^\infty(\mathfrak{gl}(p+1))$ descends to a well-defined inclusion*

$$\dot{U}_q^{(m|n)}(\mathfrak{gl}(p)) \hookrightarrow \dot{U}_q^{(m|n)}(\mathfrak{gl}(p+1)).$$

Thus the additional ladder relation in $\mathrm{Rep}(\mathfrak{gl}(m|n))$ *remains true if strands are added to the right of the ladder diagrams.*

Proof. If $\lambda = (\lambda_1, \dots, \lambda_{p+1})$ is such that $\lambda_{n+1} \leq m$, then $\nu_{n+1} \leq m$ for all ν in the direct sum in theorem 4.3.15. So $\lambda \in H$ implies $\nu \in H$ for such ν. Hence if $\nu \notin H$ in the decomposition in equation 4.3.4, then $\lambda \notin H$ also, so the inclusion of the quotient is well-defined. □

Note that there is also an inclusion $\dot{U}_q^\infty(\mathfrak{gl}(p)) \to \dot{U}_q^\infty(\mathfrak{gl}(p+1))$ where $1_\lambda \mapsto 1_{(j,\lambda)}$ and $E_i 1_\lambda \mapsto E_{i+1} 1_{(j,\lambda)}$, $F_i 1_\lambda \mapsto F_{i+1} 1_{(j,\lambda)}$. This inclusion also has restriction functors, and has the same branching rule as in theorem 4.3.15. Hence we have:

Corollary 4.3.17. *The additional relation in* $\mathrm{Rep}(\mathfrak{gl}(m|n))$ *remains true if strands are added to the left or to the right of the ladder diagrams.*

Hence we can deduce that the extra relation on $\mathrm{Rep}(\mathfrak{gl}(m|n))$ is a local one, and its complexity is governed by n.

Theorem 4.3.18. *The relation on* $\mathrm{Rep}(\mathfrak{gl}(m|n))$ *is generated as a local relation by the identity $\sum_{\mu \in H} e_\mu = \mathbb{1}$ in $\dot{U}_q^\infty(\mathfrak{gl}(n+1))$.*

Proof. If $\lambda \notin H$, and λ dominant, then let $\lambda' \in \mathbb{Z}^{n+1}$ be the truncation to $n+1$ terms of λ. Then $\sum_{\mu \in H} e_\mu$ acts as 0 on $V_{n+1}(\lambda')$, since $\lambda' \notin H$, so adding the uprights to the right so that the colouring along the bottom of the ladder diagram is λ, we see that this acts as 0 on $V(\lambda)$ as well, since all relations in $\dot{U}_q(\mathfrak{gl}(p))$ are local. Hence imposing the relation $\sum_{\mu\in H} e_\mu = \mathbb{1}$ with arbitrary numbers of uprights on each side kills all $\mathrm{End}_{\mathbb{C}(q)}(V(\lambda))$ with $\lambda \notin H$.

However, if $\lambda \in H$, then $e_{\lambda'}$ acts as the identity on $V_{n+1}(\lambda')$, and hence adding uprights on the right so that the colouring along the bottom is λ, we see that this element as as the identity on $V(\lambda)$.

Since these two things characterise $\mathrm{Rep}(\mathfrak{gl}(m|n))$, we see that imposing $\sum_\mu e_\mu = \mathbb{1}$ as a local relation gives $\mathrm{Rep}(\mathfrak{gl}(m|n))$.

□

4.3.7 The special case $p = 2$

As we saw in subsection 4.3.3, the additional $\mathfrak{gl}(m)$ relations are the easiest to describe, since they involve a local relation only involving one strand, namely the relation that high colours are 0.

The additional $\mathfrak{gl}(1|1)$ relations in subsection 4.3.4 are more complicated, but can still be described using only diagrams on two uprights.

By theorem 4.3.18, the additional $\mathfrak{gl}(m|1)$ relations can also be described by local relations on two uprights, since the relation is contained in $\dot{U}_q(\mathfrak{gl}(2))$. To express this, we use the idempotent element

$$e_p^{k,l} = \sum_{t=0}^{p} (-1)^t \frac{\begin{bmatrix} l-p+t \\ t \end{bmatrix}}{\begin{bmatrix} k+l-2p+t \\ l-p+t \end{bmatrix}} \frac{[k+l-2p+1]}{[k+l-2p+1+t]} F^{(l-p+t)} E^{(l-p+t)} 1_{(k,l)}$$

to produce the central idempotent

$$e_m = \sum_{k,l} \sum_{p=0}^{m} e_p^{k,l}$$

which can be written, by letting $j = p - t$, as

$$e_m = \sum_{k,l} \sum_{j=0}^{m} \sum_{p=j}^{m} (-1)^{p-j} \frac{\begin{bmatrix} l-j \\ p-j \end{bmatrix}}{\begin{bmatrix} k+l-j-p \\ l-j \end{bmatrix}} \frac{[k+l-2p+1]}{[k+l-j-p+1]} F^{(l-j)} E^{(l-j)} 1_{k,l}.$$

Of course, this is not strictly an element in $\dot{U}_q^\infty(\mathfrak{gl}(2))$ since the sum is infinite, but multiplication by e_m is a well-defined endomorphism of $\dot{U}_q^\infty(\mathfrak{gl}(2))$.

Theorem 4.3.19. *The element e_m is the projection to $\bigoplus_{\mu \in H} \operatorname{End}_{\mathbb{C}(q)}(V_2(\mu))$ in the quotient $\dot{U}_q^\infty(\mathfrak{gl}(2))$.*

Proof. If μ is a weight with $\mu_2 > m$, then the element

$$e_m^{\mu_1,\mu_2}$$

acts as 0 on $V_2(\mu)$, since every term in $e_m^{\mu_1,\mu_2}$ factors through a higher weight than μ and μ is the highest weight in $V_2(\mu)$. So since $\operatorname{End}_{\mathbb{C}(q)}(V_2(\mu))$ is a simple algebra, we see that $\operatorname{End}_{\mathbb{C}(q)}(V_2(\mu))$ is in the kernel of the projection onto the image of $e_m^{\mu_1,\mu_2}$ and hence e_m.

It remains to show that e_m acts as the identity on $V_2(\mu)$ for $\mu \in H$, ie. with $\mu_2 \leq m$. We note that on $V_2(\mu)$, $e_m^{\mu_1,\mu_2}$ acts as

$$\frac{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} \mu_1-\mu_2 \\ 0 \end{bmatrix}} \frac{[\mu_1-\mu_2+1]}{[\mu_1-\mu_2+1]} F^{(0)} E^{(0)} 1_{\mu_1,\mu_2} = 1_{\mu_1,\mu_2}$$

since only the $j = p = \mu_2$ term survives, since if j is smaller then $E^{(\mu_2 - j)}$ raises the weight above the highest weight, and if j is larger then $\begin{bmatrix} \mu_2-j \\ p-j \end{bmatrix}$ vanishes. This latter term also forces $p = j$. $\square$

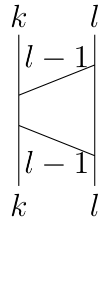
Thus one can think of the 'additional relation' on $\operatorname{Rep}(\mathfrak{gl}(m|1))$ as

$$\sum_{i=1}^{m} e_i^{k,l} = 1_{k,l}$$

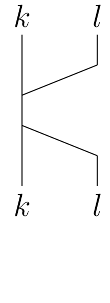
for all k, l. That is, both sides act identically on $\operatorname{Rep}(\mathfrak{gl}(m|1))$.

In the case of $U_q(\mathfrak{gl}(1|1))$, the idempotent takes on the form

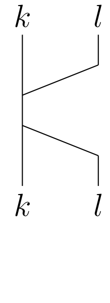
$$\sum_{k,l \in \mathbb{N}} \left(\frac{1}{\begin{bmatrix} k+l-2 \\ l-1 \end{bmatrix}} \right.$$



$$- \frac{[l][k+l-1]}{[k]\begin{bmatrix} k+l \\ k \end{bmatrix}}$$

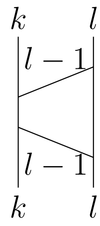


$$\left. \vphantom{\frac{1}{1}} \right) + \left(\frac{1}{\begin{bmatrix} k+l \\ k \end{bmatrix}} \right.$$

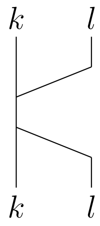


$$\left. \vphantom{\frac{1}{1}} \right)$$

$$= \sum_{k,l\in\mathbb{N}} \Bigg(\frac{1}{\left[{k+l-2 \atop l-1}\right]}$$



$$- \frac{[l-1]}{\left[{k+l-1 \atop k-1}\right]}$$



$$\Bigg)$$

This relation is the one found in subsection 4.3.4.

For other $\dot{U}_q(\mathfrak{gl}(p))$, these idempotents seem very difficult to compute explicitly, but by theorem 4.3.18 we at least know that the relation on $\mathrm{Rep}(\mathfrak{gl}(m|1))$ is generated locally by the above relation.

4.3.8 Direct Limit of $\dot{U}_q(\mathfrak{gl}(p))$

In theorem 4.3.12, we use a direct sum of $\dot{U}_q(\mathfrak{gl}(p))$ to describe an equivalence of categories. However, note that all we really needed was to ensure large enough tensor products of exterior powers were reached by the functor. There is a lot of duplication in the functor, since the object $\bigwedge^2(\mathbb{C}_q^{m|n})$ of $\mathrm{Rep}(\mathfrak{gl}(m|n))$ is reached by $1_{(2,0)}$, $1_{(2,0,0)}$, and so on. This is fine as far as an equivalence goes, but a slightly neater idea is afforded by the following:

Definition 4.3.20. Using the inclusion

$$\iota_0 : \dot{U}_q(\mathfrak{gl}(p)) \to \dot{U}_q(\mathfrak{gl}(p+1))$$

from subsection 4.3.6, we define $\dot{U}_q(\mathfrak{gl}(\infty))$ as the direct limit of the system

$$\dot{U}_q(\mathfrak{gl}(\infty)) = \varinjlim \Big(\cdots \longrightarrow \dot{U}_q(\mathfrak{gl}(p)) \longrightarrow \dot{U}_q(\mathfrak{gl}(p+1)) \longrightarrow \cdots \Big).$$

Objects in this category are elements 1_λ where λ is a sequence of integers with $\lambda_i = 0$ for all but finitely many i.

It is easy to see (cf. [Lus93, Theorem 26.3.1]) that each inclusion carries the canonical basis into the canonical basis, and therefore $\dot{U}_q(\mathfrak{gl}(\infty))$ inherits a canonical basis from the canonical basis of $\dot{U}_q(\mathfrak{gl}(p))$ for each p.

As before the inclusion takes a weight with a negative entry to a weight with a negative entry, so it also descends to a map $\dot{U}_q^\infty(\mathfrak{gl}(p)) \to \dot{U}_q^\infty(\mathfrak{gl}(p+1))$ and by theorem 4.3.16

the inclusion descends to a map

$$\dot{U}_q^{(m|n)}(\mathfrak{gl}(p)) \to \dot{U}_q^{(m|n)}(\mathfrak{gl}(p+1)).$$

Definition 4.3.21. We define $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty))$ and $\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty))$ to be the direct limits of these inclusions.

Theorem 4.3.22. *There is an equivalence of categories*

$$\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty)) \to \mathrm{Rep}(\mathfrak{gl}(m|n)).$$

We can define a highest-weight module $V_\infty(\lambda)$ over $\dot{U}_q(\mathfrak{gl}(\infty))$ in the usual way: consider a vector v_λ with weight λ, where λ is a sequence with $\lambda_1 \geq \lambda_2 \geq \cdots$ with $\lambda_i = 0$ for all but finitely many i. Now define $V_\infty(\lambda)$ to be generated by $F_i^{(k)} v_\lambda$, subject to $E_i v_\lambda = 0$ for all $i \in \mathbb{N}$, and $F_i^{(\lambda_i - \lambda_{i+1}+1)} v_\lambda = 0$. This module will be infinite dimensional in general.

These modules behave very much like their finite-dimensional counterparts, in the sense that one can define an appropriate notion of the BGG category $\mathcal{O}$ for $\mathfrak{gl}(\infty)$ and the modules $V_\infty(\lambda)$ classify all irreducible modules in $\mathcal{O}$, as proved by Du and Fu [DF09].

Then the canonical inclusion $\dot{U}_q(\mathfrak{gl}(p)) \to \dot{U}_q(\mathfrak{gl}(\infty))$ induces a restriction functor Res_p giving $V_\infty(\lambda)$ the structure of a $\dot{U}_q(\mathfrak{gl}(p))$-module. There is then a canonical inclusion of $\dot{U}_q(\mathfrak{gl}(p))$-modules $V_p(\lambda|_p) \to \mathrm{Res}_p(V_\infty(\lambda))$, where $\lambda|_p$ denotes the first p terms of λ. Hence we have the following:

Lemma 4.3.23. *The algebra* $\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty))$ *acts as* 0 *on* $V_\infty(\mu)$ *where* $\mu_{n+1} > m$.

Lemma 4.3.24. *There is an isomorphism of algebras*

$$\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty)) \cong \bigoplus_{\mu \in H} \mathrm{End}_{fr}(V_\infty(\mu))$$

where $\mathrm{End}_{fr}(V_\infty(\mu))$ *is the non-unital algebra of endomorphisms over* $\mathbb{C}(q)$ *of finite rank, the sum is the direct sum of algebras, and* H *is the set of all dominant weights with* $\mu_{n+1} \leq m$.

It seems the equivalence in theorem 4.3.22 is the most natural way to think of the action of skew Howe duality, particularly from the point of view of categorification, as we shall see.

4.3.9 Braiding

As in the previous work of Cautis, Kamnitzer and Morrison [CKM14] and the author [Gra16], the functor $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty)) \to \mathrm{Rep}(\mathfrak{gl}(m|n))$ takes a braiding on $\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty))$ to a braiding on $\mathrm{Rep}(\mathfrak{gl}(m|n))$.

A braided monoidal category is a monoidal category equipped with a natural isomorphism from the bifunctor $- \otimes -$ to the bifunctor $- \otimes^{op} -$, satisfying the two equations

$$\beta_{U\otimes V,W} = (\beta_{U,W} \otimes \mathbb{1}_V) \circ (\mathbb{1}_U \otimes \beta_{V,W})$$

$$\beta_{U,V\otimes W} = (\mathbb{1}_V \otimes \beta_{U,W}) \circ (\beta_{U,V} \otimes \mathbb{1}_W)$$

for any objects U, V, W. These equations are called the *hexagon equations*.

As mentioned in section 3.4, the category $\mathrm{Rep}(\mathfrak{gl}(m|n))$ is braided by the R-matrix.

There is also an action of the infinite braid group on $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty))$ defined by

$$1_{s_i(\lambda)} T_i 1_\lambda = (-1)^{\lambda_i \lambda_{i+1}} \sum_{s=0}^{\infty} (-q)^{\lambda_{i+1}-s} F_i^{(\lambda_i - \lambda_{i+1} - s)} E_i^{(s)} 1_\lambda, \quad \lambda_i - \lambda_{i+1} \geq 0$$

$$1_{s_i(\lambda)} T_i 1_\lambda = (-1)^{\lambda_i \lambda_{i+1}} \sum_{s=0}^{\infty} (-q)^{\lambda_{i+1}-s} E_i^{(\lambda_{i+1} - \lambda_i + s)} F_i^{(s)} 1_\lambda, \quad \lambda_i - \lambda_{i+1} \leq 0$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $s_i(\lambda)$ is λ with the ith and $(i+1)$th entries swapped. These sums are finite due to the nilpotence of the E_i and F_i in $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty))$.

Thus the image of $T_i 1_\lambda$ under $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty)) \to \mathrm{Rep}(\mathfrak{gl}(m|n))$ is precisely the braiding on $\mathrm{Rep}(\mathfrak{gl}(m|n))$.

4.4 Including dual representations into the representation category

In $\mathrm{Rep}(\mathfrak{gl}(m|n))$, representations do not have duals for $n \neq 0$, since these do not correspond to exterior powers of the fundamental representation. It is, however, possible to generalise the skew Howe duality theorem to a doubled Schur algebra, introduced by Queffelec and Sartori [QS14; QS15]. Although this allows a description of a larger category $\mathrm{Rep}'(\mathfrak{gl}(m|n))$ which now contains exterior powers of the dual representation, the description is still somewhat inflexible, and not easy to categorify.

Chapter 5

Categorified Quantum Groups

5.1 Categorified $\dot{U}_q(\mathfrak{gl}(p))$

We give a categorification of $\dot{U}_q(\mathfrak{gl}(p))$, by an easy modification of Khovanov and Lauda [KL10] and Rouquier's definition of $\dot{U}_q(\mathfrak{sl}(p))$. These two approaches were shown to be equivalent by Brundan [Bru15], up to a choice of isomorphism between left and right duals implicit in the Khovanov-Lauda definition.

5.2 KLR algebras

The definition of $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))$ makes use of relations coming from the KLR algebra. This algebra appears in other related settings, and in particular it is used to categorify highest-weight representations of $U_q(\mathfrak{gl}(p))$. This algebra was defined by Khovanov and Lauda [KL09] and independently by Rouquier [Rou08].

Let $I = \{1, \dots, p-1\}$. Given an element $\nu = (\nu_1, \dots, \nu_n) \in I^n$, we let $s_l(\nu) = (\nu_1, \dots, \nu_{l+1}, \nu_l, \dots, \nu_n)$ interchanging only the l and $(l+1)$th entries.

Let $\Bbbk$ be a commutative ring with unit, and for all $i, j \in I$ let $t_{ij} \in \Bbbk^\times$ such that $t_{ij} = t_{ji}$ when $j \neq i \pm 1$, and $t_{ii} = 1$ for all i, with $Q = \{t_{ij} \mid i, j \in I\}$.

Definition 5.2.1. The KLR algebra $R(n)$ of degree n of type A_{p-1} with the choice of scalars Q is defined to be the unital algebra over $\Bbbk$ defined by generators $e(\nu)$ ($\nu \in \{1, \dots, p-1\}^n$), x_k $(1 \leq k \leq n)$ and τ_l $(1 \leq l \leq n-1)$ with defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^n} = 1$$

$$x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu)x_k$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \text{ if } |k-l|>1$$

$$\tau_k^2 e(\nu) = \begin{cases} 0 & \text{if } \nu_k = \nu_{k+1} \\ t_{\nu_k\nu_{k+1}}e(\nu) & \text{if } |\nu_k - \nu_{k+1}| > 1 \\ (t_{\nu_k\nu_{k+1}}x_k + t_{\nu_{k+1}\nu_k}x_{k+1})e(\nu) & \text{if } \nu_k = \nu_{k+1} \pm 1 \end{cases}$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1} \\ e(\nu) & \text{if } l = k+1, \nu_k = \nu_{k+1} \\ 0 & \text{otherwise} \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} t_{\nu_k\nu_{k+1}}e(\nu) & \text{if } \nu_k = \nu_{k+2} \text{ and } \nu_{k+1} = \nu_k \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The algebra $R(n)$ is graded, with

$$\deg(e(\nu)) = 0, \quad \deg(x_k) = 2, \quad \deg(\tau_l e(\nu)) = \begin{cases} 1 & \text{if } \nu_l = \nu_{l+1} \pm 1 \\ -2 & \text{if } \nu_l = \nu_{l+1} \\ 0 & \text{else.} \end{cases}$$

5.3 Rouquier's Definition

In [Rou08], Rouquier defines the 2-category $\mathscr{U}_Q(\mathfrak{gl}(p))$ as follows:

Definition 5.3.1. The strict additive $\Bbbk$-linear 2-category $\mathscr{U}_Q(\mathfrak{gl}(p))$ is defined as follows:

- Objects: 1_λ for each $\lambda \in \mathbb{Z}^p$.
- 1-morphisms: direct sums of concatenations of $q^k E_i 1_\lambda : 1_\lambda \to 1_{\lambda+\alpha_i}$ and $q^k F_i 1_\lambda : 1_\lambda \to 1_{\lambda-\alpha_i}$ where $\alpha_i = (0, \dots, 1, -1, \dots, 0)$ and $k \in \mathbb{Z}$.
- 2-morphisms: generated by $x_{i,\lambda} : E_i 1_\lambda \to q^2 E_i 1_\lambda$, $\tau_{ij,\lambda} : E_i E_j 1_\lambda \to q^{-\alpha_i\cdot\alpha_j} E_j E_i 1_\lambda$, $\epsilon_{i,\lambda} : E_i F_i 1_\lambda \to q^{1-\lambda_i+\lambda_{i+1}} 1_\lambda$, $\eta_{i,\lambda} : 1_\lambda \to q^{1+\lambda_i-\lambda_{i+1}} F_i E_i 1_\lambda$ subject to:

  1. $\epsilon_{i,\lambda+\alpha_i}\mathbb{1}_{E_i 1_\lambda} \circ \mathbb{1}_{E_i 1_\lambda}\eta_{i,\lambda} = \mathbb{1}_{E_i 1_\lambda}$ and $\mathbb{1}_{F_i 1_\lambda}\epsilon_{i,\lambda} \circ \eta_{i,\lambda-\alpha_i}\mathbb{1}_{F_i 1_\lambda} = \mathbb{1}_{F_i 1_\lambda}$.

2. The 2-morphisms $x_{i,\lambda}$ and $\tau_{ij,\lambda}$ obey the KLR relations.

3. Let $\sigma_{ij,\lambda} = \mathbb{1}_{F_jE_i1_\lambda}\epsilon_{j,\lambda} \circ \mathbb{1}_{F_i1_{\lambda+\alpha_j}}\tau_{ij}\mathbb{1}_{F_i1_\lambda} \circ \eta_{i,\lambda+\alpha_i-\alpha_j}\mathbb{1}_{E_iF_j1_\lambda} : E_jF_i1_\lambda \to F_iE_j1_\lambda$.
   Then the 2-morphisms:

   – $\sigma_{ij,\lambda}$ for $i \neq j$.

   – If $\lambda_i - \lambda_{i+1} \geq 0$,

$$\rho_{i,\lambda} = \sigma_{ii,\lambda} \oplus \bigoplus_{k=0}^{\lambda_i-\lambda_{i+1}-1} \epsilon_{i,\lambda} \circ x^k_{i,\lambda-\alpha_i}\mathbb{1}_{F_i1_\lambda} : E_iF_i1_\lambda \to F_iE_i1_\lambda \oplus \bigoplus_{k=0}^{\lambda_i-\lambda_{i+1}-1} q^{-\lambda_i+\lambda_{i+1}+2k+1}1_\lambda$$

   – If $\lambda_i - \lambda_{i+1} \leq 0$,

$$\rho_{i,\lambda} = \sigma_{ii,\lambda} \oplus \bigoplus_{k=0}^{-\lambda_i+\lambda_{i+1}-1} \mathbb{1}_{F_i1_{\lambda+\alpha_i}}x^k_{i,\lambda} \circ \eta_{i,\lambda} : E_iF_i1_\lambda \bigoplus_{k=0}^{-\lambda_i+\lambda_{i+1}-1} q^{-\lambda_i+\lambda_{i+1}-2k-1}1_\lambda \to F_iE_i1_\lambda.$$

   are invertible.

In other words, condition 3 forces us to formally adjoin inverses to the morphisms $\sigma_{ij,\lambda}$ and $\rho_{i,\lambda}$ in addition to the generators $x_{i,\lambda}$, $\tau_{ij,\lambda}$, $\epsilon_{i,\lambda}$ and $\eta_{i,\lambda}$.

Condition 1 states that E_i1_λ is left-adjoint to $q^{1+\lambda_i-\lambda_{i+1}}F_i1_{\lambda+\alpha_i}$ by the unit $\eta_{i,\lambda}$ and counit $\epsilon_{i,\lambda+\alpha_i}$.

In order to relate this to $\dot{U}_q(\mathfrak{gl}(p))$, it is necessary first to add in 1-morphisms corresponding to idempotent 2-morphisms in the category $\mathscr{U}_Q(\mathfrak{gl}(p))$. There is a universal way to do this, called the *Karoubi envelope*

Definition 5.3.2. Given a preadditive category $\mathscr{C}$, we can define $\mathrm{Kar}(\mathscr{C})$ to have objects consisting of all pairs (A, e) for $A \in \mathscr{C}$ and $e : A \to A$ an idempotent, and morphisms $(A, e) \to (A', e')$ are triples (e', f, e) where $f : A \to A'$, and $(e', f, e) = (e', g, e)$ if $e'fe = \epsilon' g\epsilon$ as maps $A \to A'$.

Definition 5.3.3. The 2-category $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))$ is defined to be to be the Karoubi envelope of $\mathscr{U}_Q(\mathfrak{gl}(p))$.

The Grothendieck group $K_0(\mathscr{D})$ of a category $\mathscr{D}$ is defined to be the abelian group generated the isomorphism classes $[A]$ of objects $A \in \mathscr{D}$, with the relations

$$[A] = [B] + [C] \text{ if } A \cong B \oplus C.$$

If a category $\mathscr{D}$ is graded, then we can give $K_0(\mathscr{D})$ the structure of a $\mathbb{Z}[q, q^{-1}]$-module by letting q act as the grading shift functor.

We take the split Grothendieck group $K_0(\mathscr{C})$ of a 2-category $\mathscr{C}$ to be the direct sum of the split Grothendieck groups of each Hom-category.

Theorem 5.3.4 (Rouquier [Rou08]). *The 2-category $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))$ satisfies*

$$K_0(\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong \dot{U}_q(\mathfrak{gl}(p)).$$

Without tensoring with $\mathbb{C}(q)$, the algebra $K_0(\dot{\mathscr{U}}_Q(\mathfrak{gl}(p)))$ is isomorphic to the integral form of $\dot{U}_q(\mathfrak{gl}(p))$, generated over $\mathbb{Z}[q, q^{-1}]$ by $E_i^{(r)}$ and $F_i^{(r)}$ for all i.

We can categorify $\dot{U}_q^{\infty}(\mathfrak{gl}(p))$ in a straight-forward way.

Definition 5.3.5. We define $\dot{\mathscr{U}}_Q^{\infty}(\mathfrak{gl}(p))$ to be the quotient of $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p)))$ by the identity 2-morphisms of the identity 1-morphisms of the objects 1_λ with $\lambda_i < 0$ for some i.

Note that this is a direct sum of the categorified q-Schur algebras defined by Mackaay, Stošič and Vaz [MSV13].

We can also categorify $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty))$.

Lemma 5.3.6. *The inclusion*

$$\iota_0 : \dot{U}_q^{\infty}(\mathfrak{gl}(p)) \to \dot{U}_q^{\infty}(\mathfrak{gl}(p+1))$$

from subsection 4.3.6 lifts to a 2-functor

$$\dot{\mathscr{U}}_Q^{\infty}(\mathfrak{gl}(p)) \to \dot{\mathscr{U}}_Q^{\infty}(\mathfrak{gl}(p+1)).$$

Proof. The 2-functor carries 1_λ to $1_{(\lambda,0)}$ and $E_i 1_\lambda$ to $E_i 1_{(\lambda,0)}$ and similarly for $F_i 1_\lambda$. The 2-functor acts on 2-morphisms only by changing the colouring of a region from λ to $(\lambda, 0)$. Since all relations involving an i coloured strand depend only on the value of $\lambda_i - \lambda_{i+1}$, relations on 2-morphisms in $\dot{\mathscr{U}}_Q^{\infty}(\mathfrak{gl}(p))$ are mapped to relations on 2-morphisms in $\dot{\mathscr{U}}_Q^{\infty}(\mathfrak{gl}(p+1))$, so the 2-functor is well-defined. $\square$

Indecomposable objects in $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))$ are carried to indecomposable objects in $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p+1))$. Due to Webster [Web15], the indecomposable objects correspond to the elements of Lusztig's canonical basis.

Definition 5.3.7. We let $\dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(\infty))$ be the direct limit of the system of 2-categories

$$\dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(\infty)) = \varinjlim \Big(\cdots \longrightarrow \dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(p)) \longrightarrow \dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(p+1)) \longrightarrow \cdots \Big).$$

By construction, this satisfies

$$K_0(\dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(\infty))) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong \dot{U}_q^\infty(\mathfrak{gl}(\infty)).$$

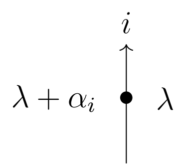
Categorification of $\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty))$ is more difficult, and relies on a notion of categorification of the modules $V_\infty(\lambda)$. This will be left to section 5.7.

5.4 Diagrammatics

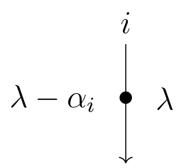
In [Lau08], Lauda defines the 2-category $\mathscr{U}_Q(\mathfrak{sl}(2))$ using diagrammatics of the nilHecke algebra. In [KL10], Khovanov and Lauda defined $\mathscr{U}_Q(\mathfrak{sl}(n))$ for all n by similar diagrammatics. We present the definition of the 2-category $\mathscr{U}_Q(\mathfrak{gl}(p))$ given as a special case of [BHLW15].

Definition 5.4.1. Let $\Bbbk$ be a commutative unital ring. Given a choice of scalars $Q' = \{c_{i,\lambda} \in \Bbbk^\times \mid i \in I, \lambda \in \mathbb{Z}^p\}$ where $c_{i,\lambda+\alpha_i}/c_{i,\lambda} =: t_{ij} \in \Bbbk^\times$ are such that $t_{ij} = t_{ji}$ when $j \neq i \pm 1$ and $t_{ii} = 1$ for all i, the 2-category $\mathscr{U}_{Q'}(\mathfrak{gl}(p))$ is defined with

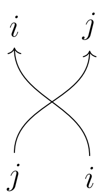
- Objects: 1_λ for each $\lambda \in \mathbb{Z}^p$.
- 1-morphisms: formal direct sums of $q^k E_i 1_\lambda$ and $q^k F_i 1_\lambda$ for $1 \leq i \leq p-1$, with $k \in \mathbb{Z}$.
- 2-morphisms: $\Bbbk$-linear combinations of compositions of the diagrams,



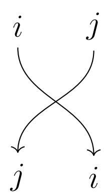
$: E_i 1_\lambda \to q^2 E_i 1_\lambda$



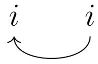
$: F_i 1_\lambda \to q^2 F_i 1_\lambda$



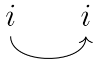
$\lambda \; : E_j E_i 1_\lambda \to q^{-\alpha_i \cdot \alpha_j} E_i E_j 1_\lambda$



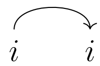
$\lambda \; : F_j F_i 1_\lambda \to q^{-\alpha_i \cdot \alpha_j} F_i F_j 1_\lambda$



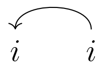
$: 1_\lambda \to q^{1+\lambda_i-\lambda_{i+1}} E_iF_i1_\lambda$



$: 1_\lambda \to q^{1-\lambda_i+\lambda_{i+1}} F_iE_i1_\lambda$



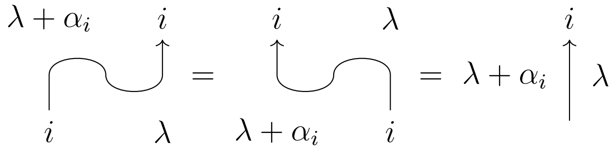
$: E_iF_i1_\lambda \to q^{1+\lambda_i-\lambda_{i+1}} 1_\lambda$

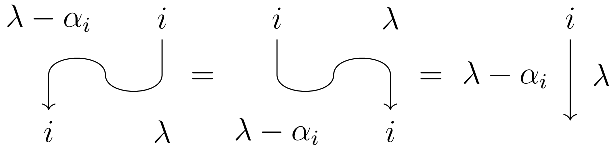


$: F_iE_i1_\lambda \to q^{1-\lambda_i+\lambda_{i+1}} 1_\lambda$

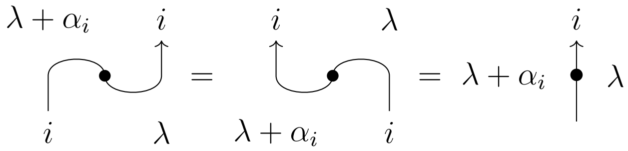
satisfying various relations depending on the $c_{i,\lambda}$ below.

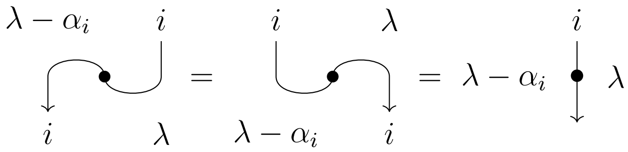
Biadjointness of E_i1_λ and F_i1_λ

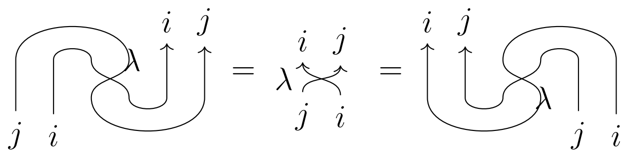


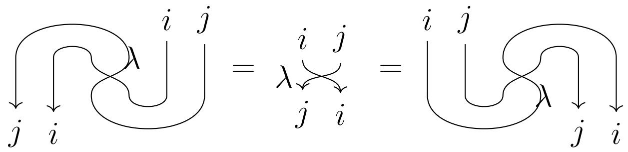


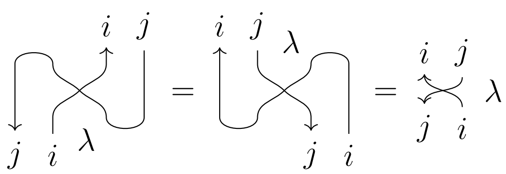
Cyclicity of 2-morphisms with respect to the biadjoint structure

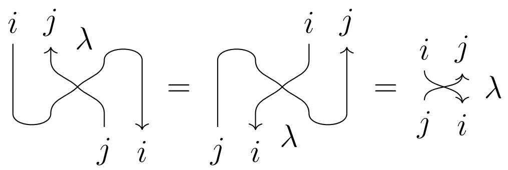




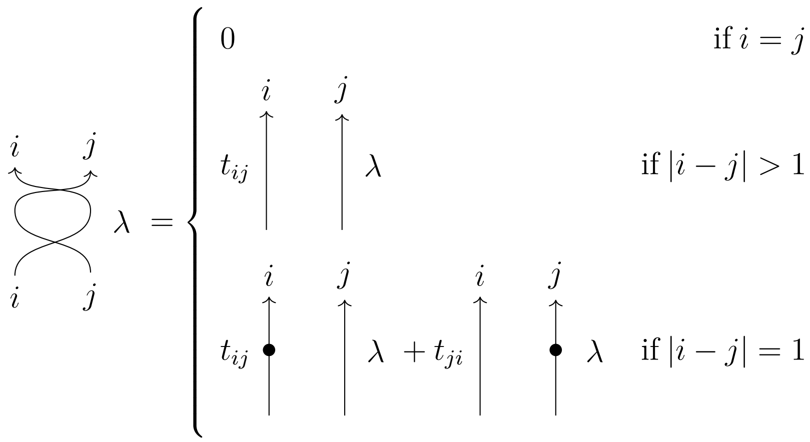


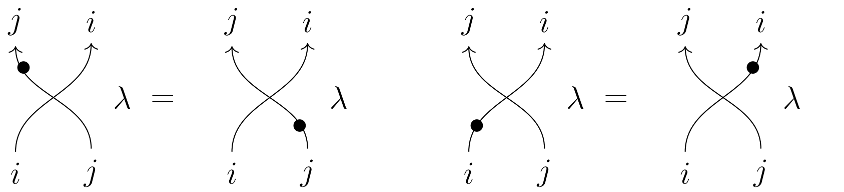




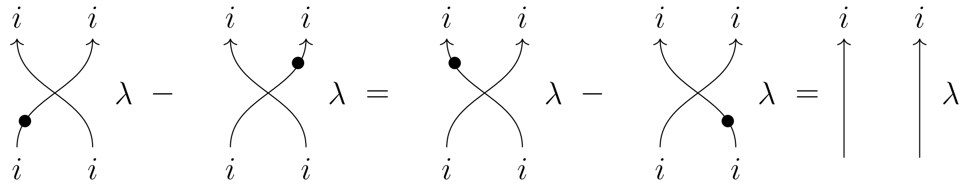


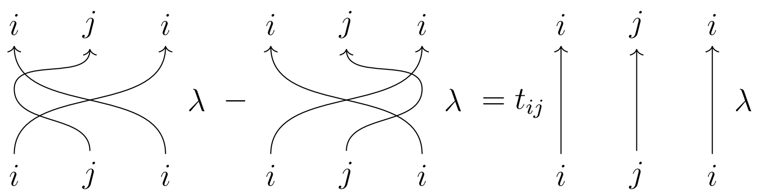
Action of KLR algebra on $E_i 1_\lambda$



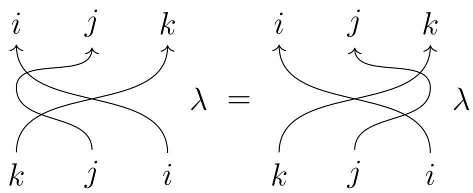


if $i \neq j$



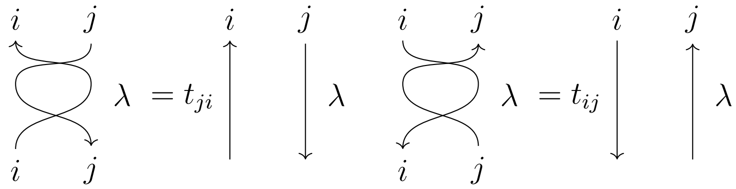


if $|i - j| = 1$

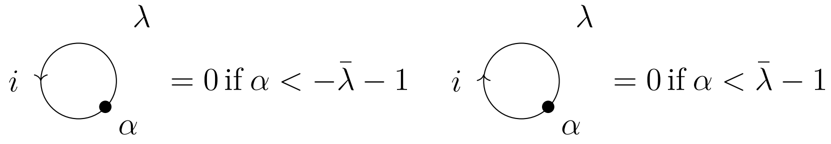


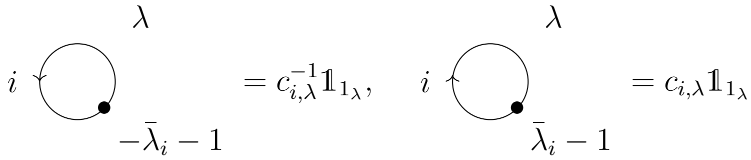
if $i \neq k$ or $|i - j| \neq 1$

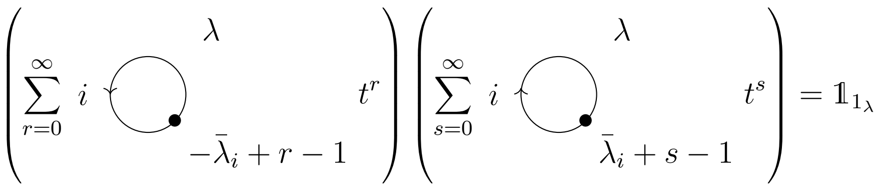
Mixed relations between $E_iF_j1_\lambda$ and $F_jE_i1_\lambda$ for $i \neq j$



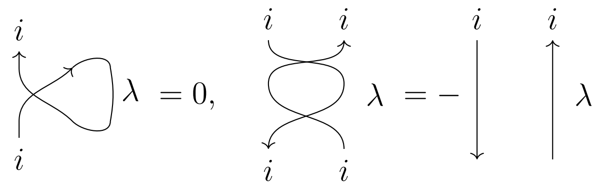
Bubble relations

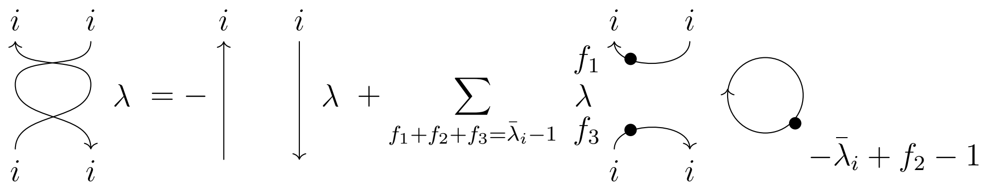






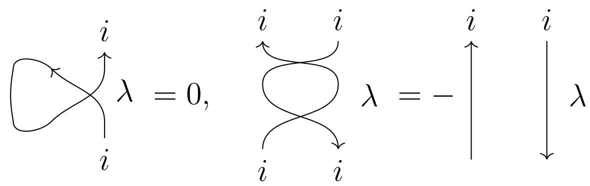
Extended $\mathfrak{sl}_2$ relations with $\lambda_i - \lambda_{i+1} > 0$

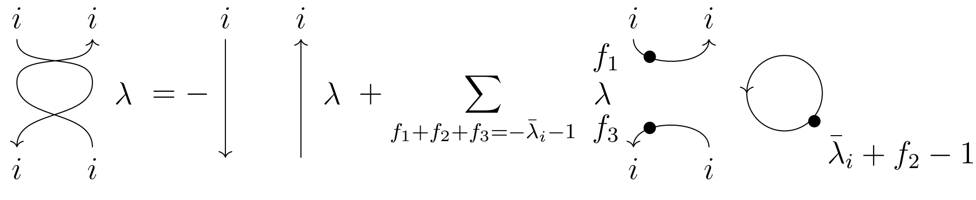




where $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$.

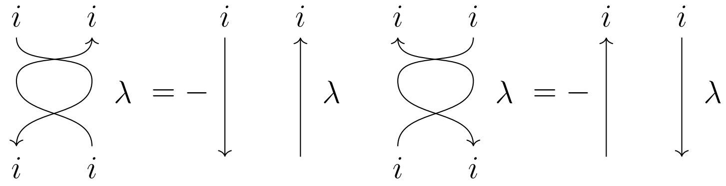
Extended $\mathfrak{sl}_2$ relations with $\lambda_i - \lambda_{i+1} < 0$

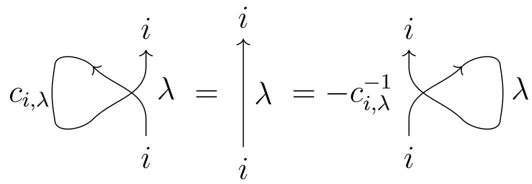




where $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$.

Extended $\mathfrak{sl}_2$ relations with $\lambda_i - \lambda_{i+1} = 0$





As before, we can take the Karoubi envelope of this 2-category to form $\dot{\mathcal{U}}_{Q'}(\mathfrak{gl}(p))$.

Theorem 5.4.2 (Khovanov-Lauda [KL10])**.** *The 2-category $\dot{\mathcal{U}}_{Q'}(\mathfrak{gl}(p))$ satisfies*

$$K_0(\dot{\mathcal{U}}_Q(\mathfrak{gl}(p))) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong \dot{U}_q(\mathfrak{gl}(p)).$$

In terms of the planar diagrams above, we can interpret the quotient $\dot{\mathcal{U}}_Q^{\infty}(\mathfrak{gl}(p))$ as meaning a 2-morphism is 0 if it contains a region coloured with a weight λ with $\lambda_i < 0$ for some i.

5.5 Cyclicity and the equivalence of the two definitions

Rouquier section 5.3 defines the 2-category $\mathcal{U}_Q(\mathfrak{gl}(p))$ by declaring that 2-morphisms between the E_i's are given by the KLR algebra and that $q^{1+\lambda_i-\lambda_{i+1}}F_i 1_{\lambda+\alpha_i}$ is right-adjoint to $E_i 1_\lambda$, and formally inverting certain maps between the compositions of $E_i 1_\lambda$ and $F_i 1_\lambda$. Remarkably, this forces that $E_i 1_\lambda$ is also right-adjoint to $q^{1-\lambda_i+\lambda_{i+1}}F_i 1_{\lambda-\alpha_i}$, but not canonically. The relationship between the left and the right adjunction is determined by the value of the bubbles, and is called the *pivotal structure*.

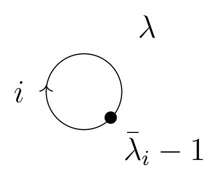
The definition in section 5.4 imposes the two adjunctions from the beginning. However, the relationship between the two adjunctions is not preserved by equivalences of 2-categories, so we still have the following:

Theorem 5.5.1 (Brundan [Bru15]). *For each choice of scalars Q' lifting scalars Q, there is a 2-functor from the Khovanov-Lauda* 2*-category* $\mathscr{U}_{Q'}(\mathfrak{gl}(p))$ *to the Rouquier* 2*-category* $\mathscr{U}_Q(\mathfrak{gl}(p))$*, which is an equivalence of 2-categories.*

Thus the two definitions are equivalent as categories with duality once one fixes the adjunction $q^{1-\lambda_i+\lambda_{i+1}}F_i1_{\lambda-\alpha_i} \dashv E_i1_\lambda$ in Rouquier's definition. This is entirely determined by the relations on the bubbles, since the cups and caps are the units and counits of the adjunctions. Hence this amounts to lifting the scalars Q to scalars Q' and imposing these as the bubble relations.

The advantage of the Khovanov-Lauda definition in section 5.4 is that the choice of adjunction is completely explicit, meaning the duality is part of the structure. Duals of 2-morphisms are found diagrammatically by rotating a diagram 180 degrees.

The original Khovanov-Lauda definition [KL10] had the choice of coefficients $c_{i,\lambda} = 1$ for all i, λ. However, it was realised that the 2-representation they construct to prove the non-degeneracy of their 2-category does not obey this relation. To fix this, unappealing coefficients were introduced into the cyclicity relations to preserve the relation



$$= \mathbb{1}_{1_\lambda}.$$

Therefore, in the version of the categorified quantum group appearing in [CL15], the symmetry determined by pushing a 2-morphism around cups and caps does not square to the identity.

However, it was realised in [BHLW15] that this is due to the choice of adjunction

$$q^{1-\lambda_i+\lambda_{i+1}}F_i1_{\lambda-\alpha_i} \dashv E_i1_\lambda$$

and by modifying it so that the compositions of the adjunction maps, which are bubbles, are given the values as in section 5.4, we can recover the cyclicity of all 2-morphisms.

Although Rouquier's 2-category is equivalent to the 2-category defined in this way, the Khovanov-Lauda definition comes with a choice of pivotal structure, which is not preserved under equivalence since adjunctions are only unique up to isomorphism.

5.6 2-representations

In this section we define an appropriate 'higher' analogue of a representation of $\mathfrak{gl}(p)$. Rather than studying the action on a vector space, here we study the action of $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))$ on a category. Hence we define a mapping of $\dot{\mathscr{U}}_Q(\mathfrak{gl}(p))$ into a 2-category, that can be thought of as a category of functors and natural transformations acting on another category.

Definition 5.6.1 (Rouquier [Rou08])**.** A 2-representation of $U_q(\mathfrak{gl}(p))$ is a graded additive $\Bbbk$-linear 2-category $\mathscr{C}$, and a strict 2-functor $\mathscr{U}_Q(\mathfrak{gl}(p)) \to \mathscr{C}$. This is equivalent to the following:

- There exists a family of objects $(V_\lambda)_{\lambda\in\mathbb{Z}^p}$ of $\mathscr{C}$.
- There exist 1-morphisms $E_iV_\lambda : V_\lambda \to V_{\lambda+\alpha_i}$ and $F_iV_\lambda : V_\lambda \to V_{\lambda-\alpha_i}$ in $\mathscr{C}$.
- For all λ, there are 2-morphisms $x_{i,\lambda} : E_i1_\lambda \to E_i1_\lambda$ and $\tau_{ij,\lambda} : E_jE_iV_\lambda \to E_iE_jV_\lambda$ satisfying the KLR relations.
- E_iV_λ is left-adjoint to $q^{1+\lambda_i-\lambda_{i+1}}F_iV_{\lambda+\alpha_i}$.
- The maps $\rho_{i,\lambda}$ and $\sigma_{ij,\lambda}$ for $i \neq j$ map to isomorphisms.

Definition 5.6.2. A 2-representation of $U_q(\mathfrak{gl}(p))$ is said to be integrable if for each $\lambda \in \mathbb{Z}^p$ and each object $M \in V_\lambda$ there exists n such that for all i we have

$$E_i^n(M) = F_i^n(M) = 0.$$

The following criterion of an integrable 2-representation were given by Cautis and Lauda [CL15].

Theorem 5.6.3. *Let $\Bbbk$ be a field, and $\mathscr{C}$ be a graded additive $\Bbbk$-linear idempotent-complete 2-category. The following data gives rise to an integrable 2-representation $\mathscr{C}$ of $U_q(\mathfrak{gl}(p))$:*

- *There exists a family of objects $(V_\lambda)_{\lambda \in \mathbb{Z}^p}$ of $\mathscr{C}$.*
- *There exist 1-morphisms $E_i V_\lambda : V_\lambda \to V_{\lambda+\alpha_i}$ and $F_i V_\lambda : V_\lambda \to V_{\lambda-\alpha_i}$ in $\mathscr{C}$.*
- *$V_{\lambda+r\alpha_i}$ is isomorphic to 0 for all but finitely many $r \in \mathbb{Z}$.*
- *$E_i V_\lambda$ has a left adjoint and a right adjoint, and its right adjoint is $q^{1+\lambda_i-\lambda_{i+1}} F_i V_{\lambda+\alpha_i}$.*
- *$\mathrm{Hom}_{\mathscr{C}}(\mathbb{1}_{V_\lambda}, q^l \mathbb{1}_{V_\lambda})$ is 0 if $l < 0$ and generated by the identity 2-morphism if $l = 0$. All hom-spaces between 1-morphisms are finite dimensional.*
- *In $\mathscr{C}$,*

$$F_i E_i V_\lambda \cong E_i F_i V_\lambda \bigoplus_{k=0}^{-\lambda_i+\lambda_{i+1}-1} q^{-\lambda_i+\lambda_{i+1}+2k+1} V_\lambda, \text{ if } \lambda_i - \lambda_{i+1} \leq 0$$

$$E_i F_i V_\lambda \cong F_i E_i V_\lambda \bigoplus_{k=0}^{\lambda_i-\lambda_{i+1}-1} q^{\lambda_i-\lambda_{i+1}-2k-1} V_\lambda, \text{ if } \lambda_i - \lambda_{i+1} \geq 0$$

  where multiplication by q denotes the grading shift functor on $\mathscr{C}$.

- *There are 2-morphisms $x_{i,\lambda} : E_i V_\lambda \to E_i V_\lambda$ and $\tau_{ij,\lambda} : E_j E_i V_\lambda \to E_i E_j V_\lambda$ satisfying the KLR relations.*
- *If $i \neq j$, then $F_j E_i V_\lambda \cong E_i F_j V_\lambda$.*

Note that it follows that $F_i V_{\lambda+\alpha_i}$ is also left-adjoint to $E_i V_\lambda$.

Since $\mathscr{U}_Q(\mathfrak{gl}(\infty))$ looks 'locally' like $\mathscr{U}_Q(\mathfrak{gl}(p))$, we can extend the notion of 2-representation to $\mathscr{U}_Q(\mathfrak{gl}(\infty))$.

Definition 5.6.4. A 2-representation of $U_q(\mathfrak{gl}(\infty))$ is a graded additive $\Bbbk$-linear 2-category $\mathscr{C}$, and a strict 2-functor $\mathscr{U}_Q(\mathfrak{gl}(\infty)) \to \mathscr{C}$. This is equivalent to the following:

- There exists a family of objects $(V_\lambda)_{\lambda \in \mathbb{Z}^\infty}$ of $\mathscr{C}$.
- There exist 1-morphisms $E_i V_\lambda : V_\lambda \to V_{\lambda+\alpha_i}$ and $F_i V_\lambda : V_\lambda \to V_{\lambda-\alpha_i}$ in $\mathscr{C}$.

- For all λ, there are 2-morphisms $x_i : E_i 1_\lambda \to E_i 1_\lambda$ and $\tau_{ij} : E_j E_i 1_\lambda \to E_i E_j 1_\lambda$ satisfying the KLR relations.

- $E_i V_\lambda$ is left-adjoint to $F_i V_\lambda$.

- The maps ρ_i and σ_{ij} for $i \neq j$ map to isomorphisms.

Definition 5.6.5. A 2-representation of $U_q(\mathfrak{gl}(\infty))$ is said to be integrable if for each $\lambda \in \mathbb{Z}^\infty$ and each object $M \in V_\lambda$ there exists n such that for all i we have

$$E_i^n(M) = F_i^n(M) = 0.$$

Since theorem 5.6.3 holds for all $p \geq 2$, it is easy to see that it also holds for $p = \infty$. The notion of an integrable 2-representation of $U_q(\mathfrak{gl}(\infty))$ is then a categorification of the notion of an integrable weight-module over $U_q(\mathfrak{gl}(\infty))$, in the sense of Du and Fu [DF09].

5.7 Categorification of Irreducible Highest Weight Modules

In this section, we review the categorification of highest-weight $U_q(\mathfrak{gl}(p))$-modules using cyclotomic KLR algebras.

From now on, we choose the base ring $\Bbbk$ to be a field.

Definition 5.7.1. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a dominant weight. We define the cyclotomic KLR algebra $R^\lambda(n)$ of degree n of type A_{p-1} to be the quotient of the KLR algebra $R(n)$ by the 2-sided ideal generated by $\sum_{\nu \in I^n} x_1^{\lambda_{\nu_1} - \lambda_{\nu_1 + 1}} e(\nu)$. We let $R^\lambda = \bigoplus_n R^\lambda(n)$.

The importance of the cyclotomic KLR algebras is that they categorify highest-weight modules of $U_q(\mathfrak{sl}_p)$ (or equivalently, $U_q(\mathfrak{gl}_p)$), as conjectured in [KL09] and proven (for general quantum groups $U_q(\mathfrak{g})$) by Kang and Kashiwara [KK12]. An alternative proof has been given by Webster [Web13], and in the case of type A_p a proof was also given by Vaz [Vaz13].

Theorem 5.7.2. *The category of projective modules* p-mod R^λ *is a 2-representation of* $U_q(\mathfrak{gl}(p+1))$ *and satisfies*

$$K_0(\text{p-mod}\, R^\lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong V_{p+1}(\lambda)$$

where $V_{p+1}(\lambda)$ *is the irreducible module of* $U_q(\mathfrak{gl}(p+1))$ *of highest-weight* λ*, and* $K_0(\mathcal{C})$ *denotes the split Grothendieck group of the category* $\mathcal{C}$*.*

In fact, p-mod $R^\lambda \cong \bigoplus_{\nu\in\mathbb{N}[I]}$ p-mod $R^\lambda(\nu)$ where $R^\lambda(\nu)$ is the subalgebra generated by $|\nu|$ strands, with ν_i strands labelled i. This gives a decomposition of the categorified highest-weight module into categorified weight spaces. Note that p-mod $R^\lambda(0)$ corresponds to the highest weight space.

Remark 5.7.3. Without the tensor product with $\mathbb{C}(q)$, $K_0(\text{p-mod}\, R^\lambda)$ is isomorphic to the integral form of the representation $V_{p+1}(\lambda)$, which is a module over $\mathbb{Z}[q,q^{-1}]$.

Kang and Kashiwara [KK12] also give an action of the categorified quantum group $\mathcal{U}_q(\mathfrak{gl}(p+1))$ as follows:

The functors

$$E_i 1_{\lambda-\sum_k \nu_k\alpha_k-\alpha_i} : \text{p-mod}\, R^\lambda(\nu+i) \to \text{p-mod}\, R^\lambda(\nu)$$

are defined by a graded shift of the restriction of modules by the inclusion $R^\lambda(\nu) \hookrightarrow R^\lambda(\nu+i)e(\nu,i)$ where $e(\nu,i)$ is the sum over all colourings of identity strands except with the last strand coloured i. In other words, $N \mapsto e(\nu,i)N$. Also

$$F_i 1_{\lambda-\sum_k \nu_k\alpha_k} : \text{p-mod}\, R^\lambda(\nu) \to \text{p-mod}\, R^\lambda(\nu+i)$$

is defined by induction

$$F_i 1_{\lambda-\sum_k \nu_k\alpha_k}(M) = R^\lambda(\nu+i)e(\nu,i) \otimes_{R^\lambda(\nu)} M.$$

Kang and Kashiwara show that these are exact, projective, and are left and right adjoint to each other.

Note also that these functors carry an action of the KLR algebra as follows: consider a sequence $\mathbf{i} = (i_1,\dots,i_n) \in \{1,\dots,p\}^n$, and consider the functor

$$E_{\mathbf{i}} 1_{\lambda-\sum_k \nu_k\alpha_k-\sum_{i\in\mathbf{i}}\alpha_i} = E_{i_n}\cdots E_{i_1} 1_{\lambda-\sum_k \nu_k\alpha_k-\sum_{i\in\mathbf{i}}\alpha_i} : \text{p-mod}\, R^\lambda(\nu+\mathbf{i}) \to \text{p-mod}\, R^\lambda(\nu)$$

which has the effect of projecting to strands with the left-most coloured by colourings according to ν, and the right-most coloured in the fixed sequence $\mathbf{i}$. Then the action of the KLR algebra on n strands $R(i)$ intertwines the structure as a $R^\lambda(\nu)$-module, so defines a natural transformation $E_{\mathbf{i}}^n 1_{\lambda-\sum_k \nu_k\alpha_k-\sum_{i\in\mathbf{i}}\alpha_i} \to E_{\mathbf{i}}^n 1_{\lambda-\sum_k \nu_k\alpha_k-\sum_{i\in\mathbf{i}}\alpha_i}$.

There is a similar action for the F's, with 'added strands' giving the KLR action.

5.7.1 Categorification of highest-weight modules of $U_q(\mathfrak{gl}(\infty))$

Here we extend the categorification of highest-weight modules of $U_q(\mathfrak{gl}(p))$ to a categorification of the highest-weight modules $V_\infty(\lambda)$ of $U_q(\mathfrak{gl}(\infty))$ defined in subsection 4.3.8.

Given a dominant weight $\lambda \in \mathbb{Z}^p$, there is a well-defined inclusion of non-unital $\Bbbk$-algebras

$$R^\lambda \hookrightarrow R^{(\lambda,0)}$$

given by taking the projectors $e(\nu) \in R^\lambda$ to $e(\nu) \in R^{(\lambda,0)}$.

Definition 5.7.4. Let $\lambda \in \mathbb{Z}^\infty$ be a dominant weight with all but finitely many entries equal to zero, and suppose that $\lambda_i = 0$ for $i > p$. Then we define R^λ to be the direct limit

$$R^\lambda = \varinjlim \left(R^{(\lambda_1,\ldots,\lambda_p)} \longrightarrow R^{(\lambda_1,\ldots,\lambda_p,0)} \longrightarrow R^{(\lambda_1,\ldots,\lambda_p,0,0)} \longrightarrow \cdots \right).$$

For a more concrete description the algebra $R^\lambda(n)$ of degree n is generated by $e(\nu)$ with $\nu \in \mathbb{N}^n$, x_k with $1 \le k \le n$ and τ_l with $1 \le l \le n-1$, subject to the relations in section 5.2, and the cyclotomic relation

$$x_1^{\lambda_{\nu_1}-\lambda_{\nu_1+1}} e(\nu) = 0$$

for all $\nu \in \mathbb{N}^n$. The algebra R^λ is non-unital, since the sum over all orthogonal idempotents is infinite.

We define $\mathbb{N}[\mathbb{N}]$ to be the commutative semi-group consisting of elements

$$\beta = \sum_{i\in\mathbb{N}} \beta_i \cdot i$$

where $\beta_i \in \mathbb{N} \cup \{0\}$ is 0 for all but finitely many i. We let $|\beta| = \sum_i \beta_i$.

Given $\beta \in \mathbb{N}[\mathbb{N}]$ we define $R^\lambda(\beta) = R^\lambda \sum_\nu e(\nu)$ where the sum is over all $\nu \in \mathbb{N}^{|\beta|}$ such that the entry i appears β_i times in the entries of ν. We also let

$$e(\beta, i) = \sum_{\nu \in \mathbb{N}^{|\beta|+1}, \nu_{|\beta|+1} = i} e(\nu).$$

Denote by p-mod R^λ the category of finite-dimensional projective left R^λ-modules. As before, p-mod R^λ breaks into a direct sum

$$\text{p-mod}\, R^\lambda \cong \bigoplus_\beta \text{p-mod}\, R^\lambda(\beta)$$

over all $\beta \in \mathbb{N}[\mathbb{N}]$, corresponding to the direct sum $R^\lambda = \bigoplus_\beta R^\lambda(\beta)$.

Theorem 5.7.5. *The category* p-mod R^λ *is a 2-representation of* $U_q(\mathfrak{gl}(\infty))$ *which categorifies the representation* $V_\infty(\lambda)$.

Proof. For each $\beta \in \mathbb{N}[\mathbb{N}]$ and $i \in \mathbb{N}$ there exist functors

$$\begin{aligned} E_i 1_{\lambda - \sum_k \beta_k \alpha_k - \alpha_i} &: \text{p-mod}\, R^\lambda(\beta + i) \to \text{p-mod}\, R^\lambda(\beta) \\ &: N \mapsto q^{1-\lambda_i+\lambda_{i+1}+\beta_i-\beta_{i+1}} e(\beta, i) R^\lambda(\beta + i) \otimes_{R^\lambda(\beta+i)} N \end{aligned}$$

$$F_i 1_{\lambda - \sum_k \beta_k \alpha_k} : \text{p-mod}\, R^\lambda(\beta) \to \text{p-mod}\, R^\lambda(\beta + i) : M \mapsto R^\lambda(\beta + i) e(\beta, i) \otimes_{R^\lambda(\beta)} M$$

which are well-defined since $R^\lambda(\beta + i)e(\beta, i)$ is a projective right $R^\lambda(\beta)$-module and $e(\beta, i)R^\lambda(\beta + i)$ is a projective left $R^\lambda(\beta)$-module by the main theorem of Kang and Kashiwara [KK12].

The 2-morphism $x_{i,\lambda-\sum_k \beta_k\alpha_k-\alpha_i} : E_i 1_{\lambda-\sum_k \beta_k\alpha_k} \to E_i 1_{\lambda-\sum_k \beta_k\alpha_k}$ is given by left-multiplication by $x_{|\beta|+1}$ on the module $e(\beta, i)R^\lambda(\beta + i) \otimes_{R^\lambda(\beta+i)} N$. For the action of τ_{ij} we note that

$$E_i E_j 1_{\lambda - \sum_k \beta_k \alpha_k - \alpha_i - \alpha_j}(N) \cong e(\beta, j, i) R^\lambda(\beta + i + j) \otimes_{R^\lambda(\beta+i+j)} N$$

where $e(\beta, j, i) = \sum_{\nu \in \mathbb{N}^{|\beta|+2}, \nu_{|\beta|+2}=i, \nu_{|\beta|+1}=j} e(\nu)$, and so $\tau_{ij,\lambda-\sum_k \beta_k\alpha_k-\alpha_i-\alpha_j}$ is given by left-multiplication by τ_{n+1}.

Each algebra $R^\lambda(\beta)$ is exactly the same as $R^{(\lambda_1,\ldots,\lambda_p)}(\beta)$ where p is such that $\beta_i = 0$ for $i > p$, and therefore relations on the natural transformations between these functors are exactly the same as in the case of $U_q(\mathfrak{gl}(p))$. Hence by theorem 5.7.2, we must have

$E_i 1_\mu$ left-adjoint to $F_i 1_{\mu+\alpha_i}$ and the morphisms $\rho_{i,\mu}$ and $\sigma_{ij,\mu}$ map to isomorphisms for $i \neq j$.

Hence p-mod R^λ is a 2-representation of $U_q(\mathfrak{gl}(\infty))$. Since this is generated by the category $R^\lambda(0) \cong$ p-mod$(\Bbbk) = \mathrm{Vect}_\Bbbk$, we conclude that this is a categorification of $V_\infty(\lambda)$. $\square$

Remark 5.7.6. As noted in [CL15] for the finite case, this could be proven more directly if we knew that the centre of $R^\lambda(\beta)$ contained no negative degree elements, and a 1-dimensional space of degree 0 elements.

The categories p-mod $R^\lambda(\beta)$ categorify the weight-space of $V_\infty(\lambda)$ of weight $\lambda - \sum_i \beta_i \alpha_i$.

5.8 Categorification of $\mathrm{Rep}(\mathfrak{gl}(m|n))$

Now we return to the programme of categorifying $\mathrm{Rep}(\mathfrak{gl}(m|n))$. The idea in this section is that we can use the decomposition of $\dot{U}_q^\infty(\mathfrak{gl}(\infty))$ into a direct sum indexed by dominant weights satisfying $\mu_{n+1} \leq m$, and categorify each direct summand separately.

Given a dominant weight μ with $\mu_{n+1} \leq m$, we have, by the results of the previous section, a 2-functor

$$\Phi_\mu : \dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(\infty)) \to \text{p-mod}\, R^\mu.$$

Let $\mathscr{E}(V_\infty(\mu))$ be the quotient of $\dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(\infty))$ by the kernel of Φ_μ (that is, quotient by the 2-morphisms acting as 0 on p-mod R^μ). By construction, there is an integrable 2-representation

$$\mathscr{E}(V_\infty(\mu)) \to \text{p-mod}\, R^\mu.$$

By theorem 5.7.5 we have the following:

Theorem 5.8.1. *There is a canonical isomorphism*

$$K_0(\mathscr{E}(V_\infty(\mu))) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong \mathrm{End}_{fr}(V_\infty(\mu)).$$

Proof. The algebra $\mathrm{End}_{fr}(V_\infty(\mu))$ inherits a basis from the canonical basis of $\dot{U}_q(\mathfrak{gl}(\infty))$. Each non-zero basis element corresponds to an indecomposable object of $\dot{\mathscr{U}}_Q(\mathfrak{gl}(\infty))$

that acts non-trivially on p-mod R^μ by theorem 5.7.5, so in particular its identity 2-morphism is not killed. This gives rise to an injective map

$$\mathrm{End}_{fr}(V_\infty(\mu)) \to K_0(\mathscr{E}(V_\infty(\mu))) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q)$$

sending the basis element to the indecomposable object. Any indecomposable object not reached by the map acts trivially on p-mod R^μ by theorem 5.7.5, so the map is also surjective. □

Hence we define the categorification of $\mathrm{Rep}(\mathfrak{gl}(m|n))$ as follows:

Definition 5.8.2. Let

$$\mathscr{R}(\mathfrak{gl}(m|n)) = \bigoplus_{\mu \in H} \mathscr{E}(V_\infty(\mu))$$

where the sum indicates the direct sum of 2-categories.

Theorem 5.8.3. *The 2-category* $\mathscr{R}(\mathfrak{gl}(m|n))$ *satisfies*

$$K_0(\mathscr{R}(\mathfrak{gl}(m|n))) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong \mathrm{Rep}(\mathfrak{gl}(m|n)).$$

Proof. This follows from theorem 5.8.1 and the decomposition of $\mathrm{Rep}(\mathfrak{gl}(m|n))$ into $\bigoplus_{\mu \in H} \mathrm{End}(V_\infty(\mu))$ from lemma 4.3.24 and theorem 4.3.22. □

Chapter 6

Foams

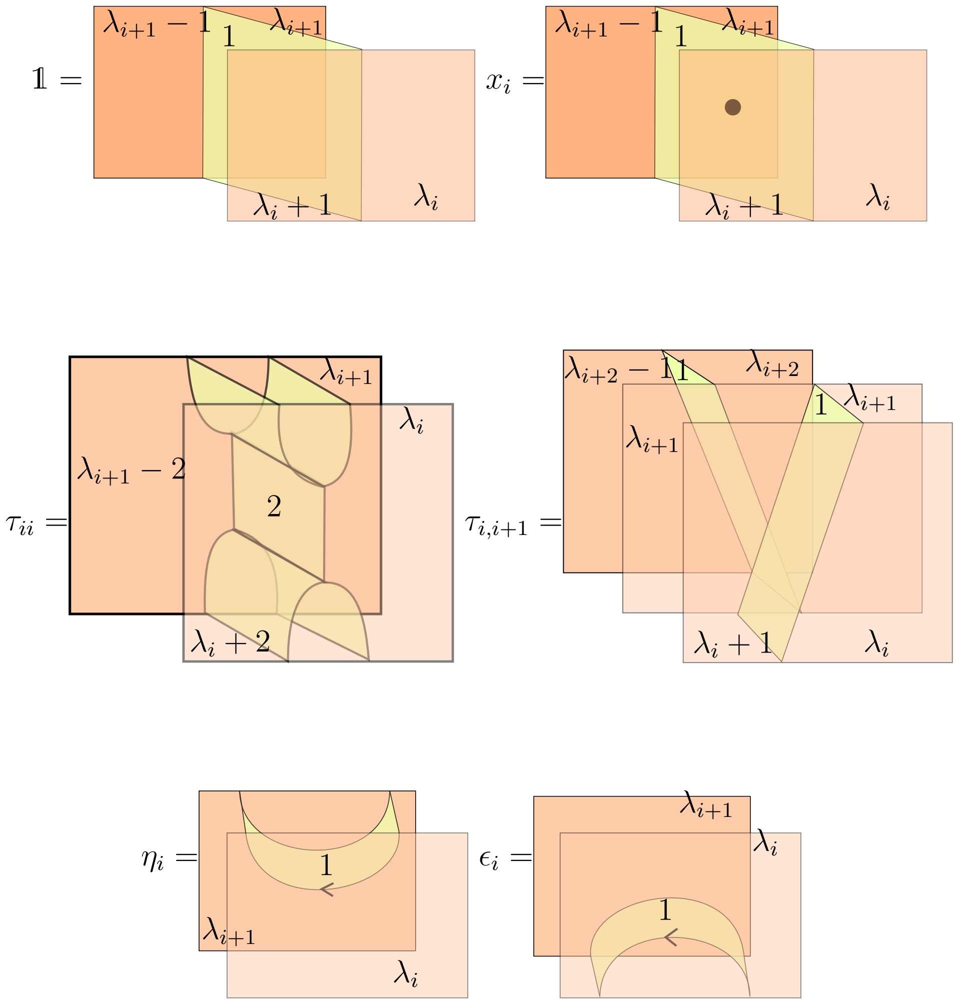
6.1 Rigid foams

We have seen that lemma 4.3.4 tells us that $\dot{U}_q^{\infty}(\mathfrak{gl}(p))$ is equivalent to the category of ladders on p uprights, and therefore in subsection 4.3.8 we showed that $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty))$ is equivalent to the category of ladders with any finite number of uprights, which we first pursued in [Gra15].

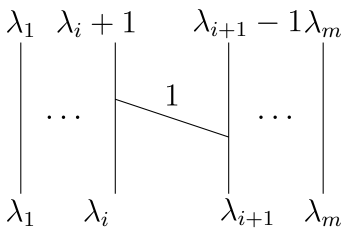
In this section we relate the categorification $\dot{\mathscr{U}}_Q^{\infty}(\mathfrak{gl}(\infty))$ of $\dot{U}_q^{\infty}(\mathfrak{gl}(\infty))$ to a categorification of the category of ladders. This was first proposed by Lauda, Queffelec and Rose [LQR15] and developed by Queffelec and Rose [QR14]. As before we let $\Bbbk$ be a commutative unital ring.

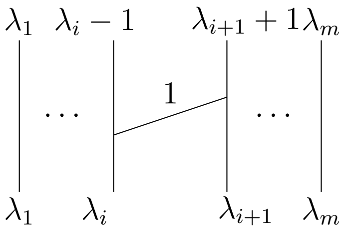
Definition 6.1.1. The 2-category Foam is defined as follows:

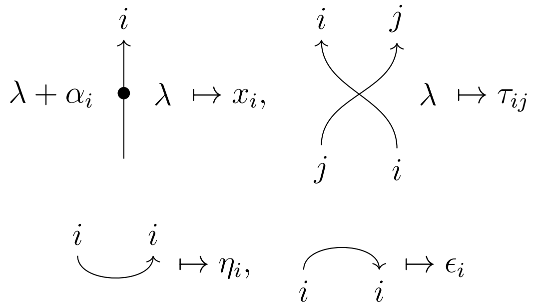
- Objects are sequences $(\lambda_1, \lambda_2, \ldots)$ with $\lambda_i = 0$ for all but finitely many λ_i.
- 1-morphisms $\lambda \to \mu$ are ladder diagrams with uprights at the bottom coloured by λ and uprights at the top coloured by μ.
- 2-morphisms are $\Bbbk$-linear combinations of labelled, decorated singular surfaces with oriented seams whose generic slices are ladder diagrams, generated by the following:



along with foams $\tau_{i,j}$ for $|i-j|>1$ that involve switching distant rungs, and duals for the η_i and ϵ_i foams involving switching orientations, such that the 2-functor $\mathscr{U}_Q^\infty(\mathfrak{gl}(\infty)) \to \text{Foam}$ defined by

$E_i 1_\lambda \mapsto$ 

$F_i 1_\lambda \mapsto$ 



is an equivalence of 2-categories.

Remark 6.1.2. The definition of the foam 2-category in [QR14] and [LQR15] is less tautological, and in particular avoids the more rigid setting of ladder diagrams, using instead web diagrams. This is why we call our foams *rigid*. However, the resulting 2-categories are equivalent as noted in [QR14, Proposition 3.22]. The use of ladder-based foams will be unavoidable in what follows, hence we use this definition.

Remark 6.1.3. Note that dots can only be placed on the yellow facets, which are traces of rungs of the ladder diagrams obtained by slicing horizontally through each foam diagram. In [QR14] and [LQR15] (and indeed [MSV09]) there are algebraic relations allowing dots to migrate to adjacent facets. However, this uses the fact that they are working specifically with $\mathfrak{sl}(n)$ foams, and in particular uses the rule that n dots on a 1-labelled facet gives the 0 foam. In general, dot migration is more complicated, as we shall see in subsection 6.1.3.

6.1.1 The Foam Description of $\mathcal{E}(V_\infty(\lambda))$

One advantage of this categorification is that each 2-category $\mathcal{E}(V_\infty(\lambda))$ admits a description by foams, since we have the equivalence of 2-categories Foam $\cong \dot{\mathcal{U}}_Q^\infty(\mathfrak{gl}(\infty))$ and $\mathcal{E}(V_\infty(\lambda))$ occurs as a quotient of $\dot{\mathcal{U}}_Q^\infty(\mathfrak{gl}(\infty))$.

The relation

$$x_1^{\lambda_{\nu_1}-\lambda_{\nu_1+1}} e(\nu) = 0$$

translates to a relation on foams, given by Figure 6.1 where $\lambda = (\lambda_1, \lambda_2, \ldots)$, and $\nu_1 = i$. So sufficiently large numbers of dots on horizontal facets give the 0 foam.

This effectively means that foams provide a diagram calculus for natural transformations

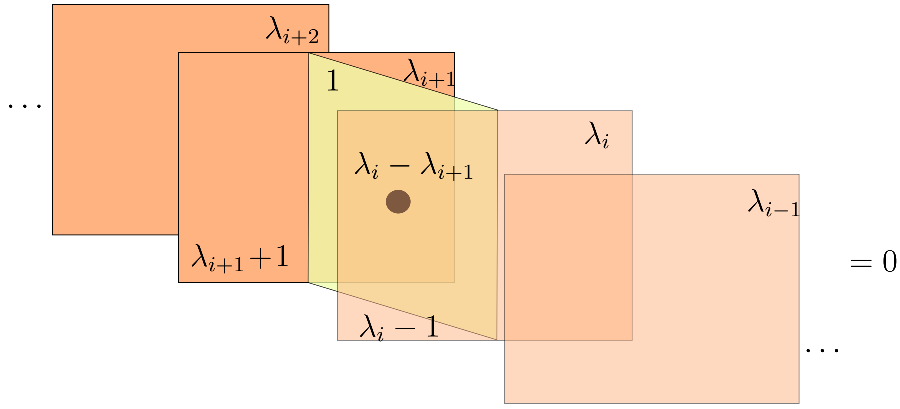


Figure 6.1: Cyclotomic relation on foams

of compositions of the endofunctors $F_i 1_\lambda$ and $E_i 1_\lambda$ in the category of finite-dimensional projective left R^λ-modules.

6.1.2 Braiding

In Section 4.3.9, we defined a braiding on $\mathrm{Rep}(\mathfrak{gl}(m|n))$ by the braiding on $\dot{U}_q^\infty(\mathfrak{gl}(\infty))$. There is a categorification of this braiding $\mathcal{T}_i 1_\lambda$ as a chain complex in $\dot{\mathscr{U}}_Q^\infty(\mathfrak{gl}(\infty))$ by

$$F_i^{(\lambda_i-\lambda_{i+1})}1_\lambda \xrightarrow{d_1} qF_i^{(\lambda_i-\lambda_{i+1}+1)}E_i 1_\lambda \xrightarrow{d_2} \cdots \xrightarrow{d_t} q^s F_i^{(\lambda_i-\lambda_{i+1}+s)}E_i^{(s)}1_\lambda \xrightarrow{d_{t+1}} \cdots$$

if $\lambda_i - \lambda_{i+1} \geq 0$, with grading shifted by $q^{(m-n)\lambda_i\lambda_{i+1}-\lambda_i}$, and by

$$E_i^{(\lambda_{i+1}-\lambda_i)}1_\lambda \xrightarrow{d_1} qE_i^{(\lambda_{i+1}-\lambda_i+1)}F_i 1_\lambda \xrightarrow{d_2} \cdots \xrightarrow{d_t} q^s E_i^{(\lambda_{i+1}-\lambda_i+s)}F_i^{(s)}1_\lambda \xrightarrow{d_{t+1}} \cdots$$

if $\lambda_{i+1} - \lambda_i \geq 0$, with grading shifted by $q^{(m-n)\lambda_i\lambda_{i+1}-\lambda_i}$. The differentials d_t in the second complex are explicitly defined using 'thick calculus' by Lauda, Queffelec and Rose [LQR15, Section 2.2]. Each complex is finite for the same reason the sum defining the braiding in Section 4.3.9 is finite: $E_i^{(s)}1_\lambda$ and $F_i^{(s)}1_\lambda$ are 0 for sufficiently large s in the category $\dot{U}_q^\infty(\mathfrak{gl}(\infty))$. The complex $\mathcal{T}_i 1_\lambda$ is invertible up to chain homotopy, so there also exists a chain complex $1_\lambda \mathcal{T}_i^{-1}$ with $T_i^{-1}T_i 1_\lambda \sim 1_\lambda$.

The complexes $\mathcal{T}_i 1_\lambda$ and $\mathcal{T}_i^{-1}1_\lambda$ descend to a complex over $\mathcal{E}(V_\infty(\mu))$ for each μ. This gives a braid group action on $\mathscr{R}(\mathfrak{gl}(m|n))$ given by a direct sum of the actions on each of the $\mathcal{E}(V_\infty(\mu))$ factors, which categorifies the braid-group action on $\mathrm{Rep}(\mathfrak{gl}(m|n))$ given by the R-matrix.

6.1.3 Examples And Non-local Behaviour

To understand this categorification, let us give some examples.

Consider the identity I on the standard representation $\mathbb{C}_q^{m|n} \to \mathbb{C}_q^{m|n}$. This is represented as a ladder by a single upright labelled 1, that is, by the element $1_{(1,0,\ldots)}$ in $\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty))$. So in $\mathscr{R}(\mathfrak{gl}(m|n))$, the space of 2-morphisms that map $I \to I$ are given by the 2-morphisms in $\mathscr{E}(V_\infty(1,0,\ldots))$ as $V_\infty(1,0,\ldots)$ is the only representation with a $(1,0,\ldots)$-weight space. But then $(1,0,\ldots)$ is also the highest-weight, so there are no non-trivial endomorphisms of $1_{(1,0,\ldots)}$ in $\mathscr{E}(V_\infty(1,0,\ldots))$. Therefore there is only a 1-dimensional space of 2-morphisms $I \to I$.

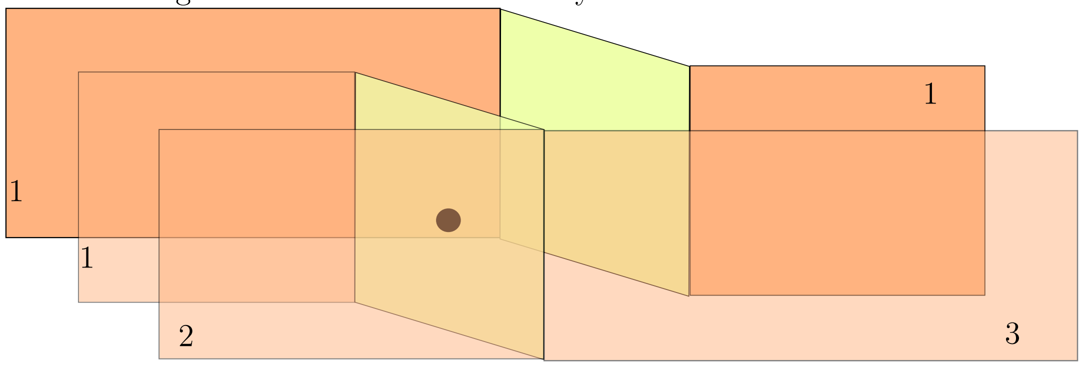
In general, the identity $\bigwedge_q^k \mathbb{C}_q^{m|n} \to \bigwedge_q^k \mathbb{C}_q^{m|n}$ has a 1-dimensional space of 2-morphisms, since it is the space of 2-morphisms $1_{(k,0,\ldots)} \to 1_{(k,0,\ldots)}$ in $\mathscr{E}(V_\infty(k,0,\ldots))$.

However, note that not all ladders with label 1 have 1-dimensional endomorphism spaces. Consider instead the identity $I : \mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n} \to \mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}$. This is represented by $1_{(1,1,0,\ldots)}$ in $\dot{U}_q^{(m|n)}(\mathfrak{gl}(\infty))$. In this case there is a $(1,1,0,\ldots)$ weight-space in two representations, namely $V_\infty(2,0,\ldots)$ and $V_\infty(1,1,0,\ldots)$. Then the space of 2-morphisms $1_{(1,1,0,\ldots)} \to 1_{(1,1,0,\ldots)}$ in $\mathscr{E}(V_\infty(2,0,\ldots))$ is 4-dimensional, since each strand can have at most one dot on it. However, the space of 2-morphisms $1_{(1,1,0,\ldots)} \to 1_{(1,1,0,\ldots)}$ in $\mathscr{E}(V_\infty(1,1,0,\ldots))$ is again 1-dimensional. So this time $\mathrm{End}(I)$ is 1-dimensional if $(m,n) = (1,0)$, 4-dimensional if $(m,n) = (0,1)$, and 5-dimensional otherwise.

This demonstrates an important property of this categorification: the relations on foams are non-local. In the special case of $\mathcal{E}(V_\infty(n,n,\ldots,n,0,\ldots))$, it turns out every 1-labelled facet can be treated the same way: an n-dotted 1-labelled facet is 0. However, in a category like $\mathcal{E}(V_\infty(3,1,0,\ldots))$, it is clear that there are no dots allowed on the 1-facet appearing in the identity foam of $1_{(3,1,0,\ldots)}$ since this is the highest weight space, but applying F_1F_2 to get to the $(2,1,1,\ldots)$ weight space, there is an endomorphism which has a dotted facet, since the cyclotomic relation gives $x_2^2 e(12) = 0$, see Figure 6.2. There is also an endomorphism corresponding to a τ foam, which maps $F_1F_2 1_{(3,1,0,\ldots)}$ to $F_2F_1 1_{(3,1,0,\ldots)}$. Therefore, these two 1-labelled facets play different roles.

Another aspect of the non-locality is that there is a dependence on what has come above or below the ladder in question. For instance, we said there is a 5-dimensional

Figure 6.2: A dot on the other yellow facet results in the 0 foam.



space of 2-morphisms $I : \mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n} \to \mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}$, but if this map is preceded or followed by a map $\bigwedge_q^2 \mathbb{C}_q^{m|n} \to \mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n}$ or $\mathbb{C}_q^{m|n} \otimes \mathbb{C}_q^{m|n} \to \bigwedge_q^2 \mathbb{C}_q^{m|n}$, then it is only 4-dimensional, since we are then working solely in $\mathcal{E}(V_\infty(2, 0, \ldots))$.

6.2 Important special cases

6.2.1 Relationship to Foams for $\mathfrak{sl}(n)$ homology

The Queffelec-Rose foam 2-category $N\,\mathrm{Foam}_n$ is equivalent to our category defined previously as $\mathcal{E}(V_\infty(n, \ldots, n, 0, \ldots, 0)$ with N terms labelled n, since the additional relation becomes any 1-facet containing n dots is equal to 0.

However, we have

$$\mathrm{Rep}(\mathfrak{gl}(n)) \cong \bigoplus_{\mu \in H} \mathrm{End}_{fr}(V_\infty(\mu)) \tag{6.2.1}$$

where H is the set of dominant weights with $\mu_i \leq n$ for all i.

Therefore

$$\mathscr{R}(\mathfrak{gl}(n)) \cong \bigoplus_{\mu \in H} \mathcal{E}(V_\infty(n, \ldots, n, 0, \ldots, 0))$$

Using this, Lauda, Queffelec and Rose [LQR15], defined a category of foams $N\,\mathrm{Foam}_n$ by the relation that a foam is 0 if it has a facet labelled 1 with n dots on it.

The reason this 2-category is suitable for studying link homology is that the resolutions of any link written as ladder diagrams will have some number of uprights coloured n at the bottom, and at the top. Hence the ladder represents an element of

$$1_{(n,n,\ldots,n,0,\ldots)} \dot{U}_q^\infty(\mathfrak{gl}(\infty)) 1_{(n,n,\ldots,n,0,\ldots)}$$

$$\cong 1_{(n,n,\ldots,n,0,\ldots)} \operatorname{End}_{fr}(V_\infty(n,n,\ldots,n,0,\ldots))1_{(n,n,\ldots,n,0,\ldots)}$$

since $V_\infty(n,n,\ldots,n,0,\ldots)$ is the only highest weight space with highest weight in H containing a non-zero $(n,n,\ldots,n,0,\ldots)$ weight space. Thus if we want to think of diagrams that are local pictures of a link diagram, then it makes sense to restrict only to $\mathcal{E}(V_\infty(n,n,\ldots,n,0,\ldots))$, rather than considering the whole of $\mathscr{R}(\mathfrak{gl}(n))$.

The highest-weight space of $V_\infty(n,n,\ldots,n,0,\ldots)$ is 1-dimensional, so the space of its endomorphisms is also 1-dimensional, since an endomorphism of the highest-weight space is determined by where the element 1 gets sent.

Similarly, the category p-mod $R^{(n,n,\ldots,n,0,\ldots)}(0)$ is equivalent to the category $\mathrm{Vect}_{\Bbbk}$ of vector spaces over the ground field $\Bbbk$. Therefore a functor $G \in \mathcal{E}(V_\infty(n,n,\ldots,n,0,\ldots))$ on p-mod $R^{(n,n,\ldots,n,0,\ldots)}(0)$ is determined by where $\Bbbk$ is sent. In fact, it is possible to determine the image of G using foams by the Yoneda lemma, since

$$G\Bbbk \cong \operatorname{Nat}(\operatorname{Hom}(\Bbbk,-),G) \cong \operatorname{Nat}(\mathbb{1}_{\mathrm{Vect}_{\Bbbk}},G) \cong \operatorname{Foam}(1_{(n,n,\ldots,n,0,\ldots)},G)$$

where Nat is the vector space of natural transformations, all isomorphisms are isomorphisms of vector spaces, and $\operatorname{Foam}(1_{(n,n,\ldots,n,0,\ldots)},G)$ is the space of foams from the ladder $1_{(n,n,\ldots,n,0,\ldots)}$ to the ladder representing the 1-morphism G. This shows that the tautological functor introduced by Bar-Natan [BN04] is a natural categorification of the evaluation of a MOY diagram. In fact, the same isomorphism works if the ground ring $\Bbbk$ is $\mathbb{Z}$, since then p-mod $R^{(n,n,\ldots,n,0,\ldots)}(0)$ is the category of free abelian groups.

Note that restricting to $\mathcal{E}(V_\infty(n,n,\ldots,n,0,\ldots))$ has a number of advantages. All facets labelled 1 can be thought of as having essentially the same 'role', and we have the uniform rule that n dots on a 1-labelled facet gives the 0 foam. A similar rule applies to all higher coloured facets. This remarkable feature makes $\mathfrak{sl}(n)$ foams much nicer to work with, and means we can forget the ladder structure and work with a more flexible 2-category. Dot migration relations can be defined which allow all facets to be treated equally, rather than separated strictly into those that sit below uprights (orange) and those that sit below rungs (yellow). This is described by Queffelec and Rose [QR14, Section 3.1].

6.2.2 Symmetric powers of the standard representation of $\mathfrak{sl}(n)$

The special case of $\mathfrak{gl}(n|0)$ is shown above to be particularly interesting. Another important case is $\mathfrak{gl}(0|n)$. In fact, we have $U_q(\mathfrak{gl}(n|0)) \cong U_q(\mathfrak{gl}(0|n))$ by an isomorphism that takes $q \mapsto q^{-1}$.

The standard representation $\mathbb{C}_q^{0|n}$ consists of only odd-degree elements, so in fact we have $\bigwedge_q^i \mathbb{C}_q^{0|n} \cong S_q^i \mathbb{C}_q^n$ where S_q^i is the ith symmetric power.

We have

$$\mathrm{Rep}(\mathfrak{gl}(0|n)) \cong \bigoplus_{\mu \in H} \mathrm{End}_{fr}(V_\infty(\mu))$$

where H consists of dominant weights with at most n parts by Lemma 4.3.24. This gives a diagram calculus for the category $\mathrm{Sym}(\mathfrak{sl}(n))$, consisting of ladder diagrams satisfying an additional relation corresponding to the condition on H.

In the case of $n = 2$, diagrams for symmetric powers of $\mathfrak{sl}(2)$ have been studied by Rose and Tubbenhauer [RT15].

Also, one can mix exterior and symmetric powers of $\mathfrak{sl}(n)$ in a single graphical calculus. The category which mixes exterior powers and symmetric powers seems a lot easier to describe completely, involving only a single 'dumbbell' relation that relates the two types of representation. This was proved by Tubbenhauer, Vaz and Wedrich [TVW15].

The connection between knot homology for exterior powers and for symmetric powers is particularly interesting because of a conjectural relationship at the level of HOMFLY homology due to Gukov and Stošić [GS12]:

Conjecture 6.2.1 (Gukov, Stošić)**.** For a knot K, there is an isomorphism between anti-symmetric coloured HOMFLY homology and symmetric coloured HOMFLY homology

$$H^{\Lambda^r}_{i,j,*}(K) \cong H^{S^r}_{i,-j,*}(K).$$

This conjecture is based on computational evidence, although the homology theory $H^{S^r}_{i,j,k}(K)$ is not yet formally defined. Reduced coloured HOMFLY homology has been defined by Wedrich [Wed16].

6.2.3 Non-local relations in Heegaard Floer homology

Part of our interest in categorifying Rep($\mathfrak{gl}(m|n)$) was to understand the case $m = n = 1$. The knot polynomial associated to the standard representation of $U_q(\mathfrak{gl}(1|1))$ is the Alexander polynomial (see [Sar13b], or [Gra16]). There are several known link homologies with graded Euler characteristic equal to the Alexander polynomial, such as Heegaard Floer knot homology and Instanton Floer knot homology, but one arising purely from categorified representation theory has not yet been found. Heegaard Floer knot homology instead is usually defined in terms of analytic geometry.

Gilmore [Gil10] gave a description of Heegaard Floer knot homology that more closely resembles some constructions of Khovanov-Rozansky $\mathfrak{sl}(n)$ homology, in the sense that it was based on Ozsváth and Szabó's cube of resolutions description of Heegaard Floer homology [OS09], and uses MOY diagrams to prove invariance of the homology under Reidemeister moves. It is interesting that in Gilmore's algebraic setting, it is also necessary to make use of non-local relations.

We hope that is is possible to use our categorification $\mathscr{R}(\mathfrak{gl}(1|1))$ to define a link homology theory categorifying the Alexander polynomial. It would be especially interesting if this theory was related to Gilmore's construction of Heegaard Floer knot homology.

Some progress towards understanding the quantum nature of Heegaard Floer homology has been made by Ellis, Petkova and Vértesi [EPV15]. This was based on [PV14], where the authors use an algebra $A(P)$ for $P \in \{1, -1\}^n$ to define bimodules that give a description of Heegaard Floer homology. It was shown that

$$K_0(A(P)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q) \cong V_P \otimes L(\lambda_{n+1})$$

where V_P is the tensor product of standard and dual standard representations of $\mathfrak{gl}(1|1)$ corresponding to the sign sequence P, and $L(\lambda_{n+1})$ is the $\mathfrak{gl}(1|1)$ module of highest weight $1 - \sum_i P_i$. They also define bimodules E and F such that the derived tensor product with E and F induces the action of E and F on $K_0(A(P)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q)$. Of course, they cannot define a categorical action, since there is not currently a categorification of $\mathfrak{gl}(1|1)$.

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