Correlation functions, scattering amplitudes and the superconformal partial wave

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Abstract

In this thesis we explore aspects of correlation functions and scattering amplitudes in supersymmetric field theories.

Firstly, we study correlation functions and scattering amplitudes in the perturbative regime of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Here we begin by giving a new method for computing the supercorrelation functions of the chiral part of the stress-tensor supermultiplet by making use of twistor theory. We derive Feynman rules and graphical rules which involve a new set of building blocks which we can identify as a new class of $\mathcal{N} = 4$ off-shell superconformal invariants. This class of off-shell superconformal invariant is related to the known $\mathcal{N} = 4$ on-shell superconformal invariant pertinent to planar scattering amplitudes.

We then move onto the six-point tree-level NMHV scattering amplitude. Previous results are given in terms of a manifestly dual superconformal invariant basis called the R-invariant. We define and analyse a generalisation of this invariant which contains half of the dual superconformal invariance ($Q + \bar{S}$ invariant but not $\bar{Q} + S$ invariant). We apply it to the six-point tree-level NMHV scattering amplitude and find a new representation which manifestly contains half of the dual superconformal invariance and physical pole structure. This is in contrast to the R-invariant basis which manifests symmetry properties but does not manifest physical pole structures.

Finally, we find the superconformal partial wave for four-point correlation functions of scalar operators on a Grassmannian space $\mathrm{Gr}_{m|n}(2m|2n)$ for theories with space-time symmetry $\mathrm{SU}(m, m|2n)$. This contains $\mathcal{N} = 0, 2, 4$ four-dimensional superconformal field theories in analytic superspace as well as a certain class of representations for the compact SU($2n$) coset spaces. As an application we then specialise to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and use these results to perform a detailed superconformal partial wave analysis of the four-point functions of arbitrary weight $\frac{1}{2}$-BPS operators. We discuss the non-trivial separation of protected and unprotected sectors for the $\langle 2222 \rangle$, $\langle 2233 \rangle$ and $\langle 3333 \rangle$ cases in an SU(N) gauge theory at finite N (where $\langle ijkl \rangle = \big\langle \mathrm{tr}(W^i)\,\mathrm{tr}(W^j)\,\mathrm{tr}(W^k)\,\mathrm{tr}(W^l) \big\rangle$). The $\langle 2233 \rangle$ correlator predicts a non-trivial protected twist four sector for $\langle 3333 \rangle$ which we can completely determine using the knowledge that there is one such protected twist-4 operator for each spin.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, England. The results are based on the three collaborative works:

- D. Chicherin, R. Doobary, B. Eden, P. Heslop, G. P. Korchemsky, L. Mason and E. Sokatchev, *Correlation functions of the chiral stress-tensor multiplet in $\mathcal{N} = 4$ SYM*, JHEP **1506** (2015) 198, [arXiv:1412.8718]
- D. Chicherin, R. Doobary, B. Eden, P. Heslop, G. P. Korchemsky and E. Sokatchev, *Bootstrapping correlation functions in $N = 4$ SYM*, JHEP **1603** (2016) 031, [arXiv:1506.04983]
- R. Doobary and P. Heslop, *Superconformal partial waves in Grassmannian field theories*, JHEP **1512** (2015) 159 [arXiv:1508.03611]

No part of this thesis has been submitted for a degree in this or any other institution.

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Chapter 1

Introduction

Quantum field theory (QFT) is by now part of the standard toolkit for any modern theoretical physicist, which has broad applications from condensed matter physics to particle physics all the way to applications in gravity.

A major development since the discovery of QFT is superstring theory. Initially, much of the interest around superstring theory was in the hopes of gaining a way to unify all physical QFTs, and have a way to encapsulate all interactions. Whilst this remains an active area of research a related direction is to construct new QFTs from an overarching string theory set up. The most famous of these set ups actually led to an exact duality in which string theory propagating on spacetimes of the form of anti de-Sitter space times a sphere $\sim$ (AdS $\times$ S) is dual to conformally invariant gauge theories (CFT) of one dimension less than the anti de-Sitter background spacetime [1]. This is known as the AdS/CFT duality. This is in fact a strong-weak duality, whereby strong coupled physics on the CFT is related to weakly coupled physics on the AdS side. This therefore leads to a method of gaining strongly coupled data of the CFT that would otherwise be unattainable by standard QFT means.

A major concept that fell from string theory was that of planarity in SU(N) or U(N) gauge theories with coupling g. It was shown in [7] by t'Hooft that the topological structure of string theoretic diagrams are matched by the $1/N$ corrections to the $N \to \infty$ limit of the gauge theory whilst keeping g^2N fixed. The planar limit is defined to be the $N \to \infty$ limit, and leads to a very large simplification. We will often make this approximation in this thesis.

In recent times it has become fruitful to turn the microscope back on QFT by virtue of studying some of the 'simplest' possible QFTs and asking refined questions about simplicity, underlying symmetries and potentially transferable techniques and structures. These questions are often answered not by referring to the Lagrangian itself, but rather the structure of the observables.

Superconformal field theories

Famously, $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) is a QFT which possesses many remarkably simple structures. This theory contains a gluon, four fermions (plus four conjugates) and six scalars, it also has the spacetime symmetry $\mathrm{PSU}(2,2|4)$ where all fields are in the adjoint representation of gauge group $\mathrm{SU}(N)$. A characteristic example of this simplicity is that whilst the Lagrangian takes the somewhat complicated form:

$$\begin{aligned}\mathcal{L} = \mathrm{tr}\Bigg(& -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{4}D_{\alpha\dot{\alpha}}\phi^{IJ}D^{\dot{\alpha}\alpha}\phi_{IJ} + \frac{1}{8}g^2\left[\phi^{IJ},\phi^{KL}\right]\left[\phi_{IJ},\phi_{KL}\right] \\ & + i\bar{\psi}_{\dot{\alpha}I}D^{\dot{\alpha}\alpha}\psi^I_\alpha - i\left(D^{\dot{\alpha}\alpha}\bar{\psi}_{\dot{\alpha}I}\right)\psi^I_\alpha - \sqrt{2}g\psi^{\alpha I}\left[\phi_{IJ},\psi^J_\alpha\right] + \sqrt{2}g\bar{\psi}_{\dot{\alpha}I}\left[\phi^{IJ},\bar{\psi}^{\dot{\alpha}}_J\right]\Bigg),\end{aligned} \tag{1.0.1}$$

its planar one-loop four-point amplitude can be written as only one term (see [2]). In general QFT, we would expect three classes of Feynman diagrams the so-called box, bubble and triangle. However, it turns out that in $\mathcal{N}=4$ SYM the bubble and triangle contributions are absent leaving only the box contribution. A further interesting aspect is that the theory is conjectured to be integrable in the planar limit as suggested by the Yangian invariance of the amplitude and the relation of the two-point correlation function of scalars to the integrable spin chain [48,49,5] (see [3] and references therein). In fact, this conjectured integrability has gone through some precise numerical checks [4,73] by checking anomalous dimensions and three-point function structure constants.

It has therefore become incredibly fruitful to study the observables using new and interesting mathematical techniques. Twistor theory has provided a great way to study scattering amplitudes in $\mathcal{N}=4$ SYM [6,64]. The underlying integrable structure of scattering amplitudes in most easily seen via twistors through the scope of the so-called

Grassmannian formalism [69] (itself, also grounded in twistor space). Fascinatingly, the Grassmannian formalism has recently led to many famous discoveries such as the positive Grassmannian, on-shell diagrams and the Amplituhedron yielding methods related to combinatorics and projective geometry [25,65,69,71]. Some view the succession in new methods as being a revolution in QFT itself.

Finally, the notion of duality within the observables of $\mathcal{N} = 4$ SYM has played a remarkable role. It was first discovered in [39] that the scattering amplitude and the expectation value of Wilson loop were exactly dual at strong coupling. This was observed in the weak coupling regime in [45,46] and was extended to supersymmetry via the twistor formalism in [47]. It was then found in [31] that the square of the expectation value of the Wilson loop is in fact dual to the lightlike limit of a correlator in any bosonic conformal field theory. This was then applied to a correlator/amplitude duality in [50,51]. Then came a supersymmetric proposal of the correlator/amplitude duality in [36,37]. This altogether leads to a 'triality' relationship between the scattering amplitude, the Wilson loop and the correlation function. It has thus become of great importance to understand the mechanisms and structures involved in setting up these relations. This is a great motivator for computing such observables as this not only allows for a structural understanding of highly symmetric theories like $\mathcal{N} = 4$ SYM, but to potentially uncover such properties of more general QFTs.

This opens the discussion to other superconformal field theories (SCFTs). These are theories which possess superconformal symmetry. This symmetry algebra is built from Lorentz generators, a generator for dilation weight and the special conformal generator, together with supersymmetry operators and so-called special superconformal generator. For four-dimensional $\mathcal{N}$-extended SCFTs the group is $\mathrm{SU}(2,2|\mathcal{N})$. Some of the structures that have been found for the $\mathcal{N} = 4$ SYM case are similar to the so-called ABJM SCFT, which has the form of a $3d$ Chern-Simons theory [8,9].

Whilst integrability structures have not been found in general SCFTs, one can instead ask questions regarding the minimal required constraints for a consistent SCFT. The argument of [74] is that we only need the general symmetry, operator product expansion (OPE) and the four-point function. One can use the OPE to gain a basis for the four-point function called the superconformal partial wave. Then the crossing-symmetry of the four-point function of four-like operators leads to non-trivial constraints on the four-point functions. The argument claims that these constraints can be analysed to exclude inconsistent SCFTs whilst placing bounds on the allowed SCFTs. These methods are referred to as superconformal bootstrap.

The most successful application of the conformal bootstrap was in the study of the $3d$ Ising model in [20], where the theory was found on a remarkable point on the boundary of the exclusion plots. There have also been applications to SCFT with four dimensional $\mathcal{N} = 2, 4$ and six dimensional $(2, 0)$ theories [95,75,96]. In order to get these results one has to know the required superconformal partial waves, which are acquired through Lie algebraic and representation theoretic methods. The study of superconformal partial waves was studied some time ago in view of the AdS/CFT in [10,88]. However, the work in [77,79,80,87](and [94] for higher dimensions) somewhat systematised the results, whereby important pioneering work in the extraction of quantum data via the superconformal partial wave analysis was performed.

Themes

The grand goal for those who study the structure of observables in the 'simplest' QFT is to try and understand all aspects of QFT in a general sense.

We view this thesis (and the work that it is based on) as a small contribution to the very large and rich tapestry of ideas and methods. This thesis contains two major themes:

TWISTOR METHODS AND SUPERCONFORMAL PARTIAL WAVES.

Having seen how useful the twistor methods have been as applied to the study of on-shell observables (like amplitudes and Wilson loops) in planar $\mathcal{N} = 4$ SYM, it is interesting to see if one can find similarly robust structures in applying these methods to off-shell observables. In particular we will study the supercorrelation functions of the stress-tensor supermultiplet in $\mathcal{N} = 4$ SYM. The idea is simply to find some structure and see how these structures relate to the previously understood scattering amplitudes. This is a rather apt question as the scattering amplitude is known to be reproduced in the lightlike limit of correlation functions in planar $\mathcal{N} = 4$ SYM.

Within the same theme, we shall also consider the so-called NMHV tree-level scattering amplitude. This is a well known object and the form which manifests full dual superconformal symmetry is the one which does not manifest physical pole structures. We aim to use a different basis for the result which manifests pole physicality at the expense of manifesting half of the superconformal symmetry. We then find a result which manifests as much physical and symmetry related properties as possible.

We then look towards the superconformal partial waves. These objects form a basis for four-point correlation functions which are pertinent to the conformal bootstrap programme. We derive a superconformal partial wave for four-point functions of scalar operators in what we will call Grassmannian field theories which are a 2-parameter family of superspaces. These Grassmannian field theories are theories with space-time symmetry $\mathrm{SU}(m, m|2n)$ (where m and n are the two parameters) and exist on the Grassmannian space defined to be $\mathrm{Gr}_{m|n}(2m|2n)$, namely the space of $m|n$-dimensional planes in $2m|2n$-dimensions. For $m = n = 2$, this becomes a four-dimensional $\mathcal{N} = 4$ SCFT, in which we perform various superconformal partial wave analyses on various correlators.

Structure of this thesis

This thesis is built from three parts. An introduction to superspaces, a second part discussing twistor methods and a third part discussing the superconformal partial wave. Each of these parts are built from chapters which each have their own reviews and introductions which elaborate on the motivations and previous work. This means that each chapter can in principle be read independently of each other.

The first part is chapter 2 and reviews the construction of different superspaces and some of the consequences. This first part aims to provide a picture of how many different superspaces may arise from a single coset construction. This chapter is to be viewed as providing the overarching theme of superspace techniques used throughout this thesis.

The second part is chapters 3 and 4. The main theme will be twistor applications to $\mathcal{N} = 4$ SYM. In chapter 3, we use the aforementioned works in scattering amplitudes as motivation to study twistor theoretic methods to compute correlation functions of the stress-tensor supermuliplet. The main result of this chapter will be a new set of graphical rules associated to a bosonic propagator and a 'R-vertex' which is to be viewed as an off-shell generalisation to familiar superconformal invariants pertinent to scattering amplitudes.

In chapter 4, we will derive, investigate and apply a new basis for the six-point tree-level NMHV scattering amplitude which manifest properties which were previously hidden in other results. In particular, we will find a remarkably compact formula which preserves the supersymmetry and the conjugate special superconformal charge $(Q + \bar{S})$, whilst only containing manifestly physical pole structures. The main point in comparison to the previously known fully superconformal invariant result, is that we may gain manifestly physical pole structures at the expense of manifest $(\bar{Q} + S)$ symmetry.

The third part is chapter 5. The theme is the superconformal partial wave. We perform a derivation of the superconformal partial wave valid for four-point functions of scalar operators with $\mathrm{SU}(m, m|2n)$ space-time symmetry in so-called Grassmannian field theories (which in four dimensions reduces to analytic superspace). In general this will turn out to be a completely general set of results, however we apply this to $\mathcal{N} = 4$ SYM, in which we find OPE coefficients for a variety of free theory correlation functions. We will then consider the notion of operator recombination at the unitary bound for the $\langle 3333 \rangle$ correlator. The main point will be that the $\langle 2222 \rangle$ and $\langle 2233 \rangle$ can be used in conjunction with information regarding the number of operators accommodating a representation to a fix a non-trivial twist-4 protected sector.

Chapters 3, 4 and 5 are based on the papers [29], [55] and [72] respectively.

Chapter 2

Superspace

Superspaces are a natural arena in which to study supersymmetric quantum field theories. We may denote such a space as $\mathbb{F}^{m|n}$ (where $\mathbb{F}$ is some field of variables e.g. $\mathbb{C}$ or $\mathbb{R}$). As an example, $\mathbb{C}^{m|n}$ is said to be a $\mathbb{Z}_2$-graded vector space of complex variables. There exists an even and odd subspace which in this case is $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, whereby the odd subspace is built from Grassmann odd variables. Functions on such a space are generically functions of both subspaces but whilst any complex (or real subspace thereof) function may exist in the even subspace, only a finite expansion in Grassmann variables exists for the odd subspace.

This leads to the concept of a superfield where supermultiplets which are innately Lie algebraic objects may be established in a field theoretic language. Physical superfields, which correspond to physical supermultiplets are found by applying some constraint onto the superfield.

As an example, we may consider an unconstrained four dimensional $\mathcal{N} = 1$ superspace, namely $\mathbb{R}^{4|4}$. This is given by the coordinates $(x^{\alpha\dot{\alpha}}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}})$, whereby $x^{\alpha\dot{\alpha}}$ is a Grassmann even object whilst the θ^{α} and $\bar{\theta}^{\dot{\alpha}}$ are are both Grassmann odd. An unconstrained superfield takes the form:

$$S(x, \theta, \bar{\theta}) = f(x) + \theta^{\alpha}\psi_{\alpha}(x) + \cdots + F(x)\theta^2\bar{\theta}^2. \tag{2.0.1}$$

One can construct the derivative operators D_{α} and $\bar{D}_{\dot{\alpha}}$. Considering the $\mathcal{N} = 1$ superalgebra, one finds that derivative operators commute with the supercharges and among themselves but for $\left\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\right\} = 2iP_{\alpha\dot{\alpha}}$, where $P_{\alpha\dot{\alpha}}$ is the momentum operator.

The chiral supermultiplet is defined to be

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0. \tag{2.0.2}$$

This is solved by:

$$\Phi(y,\theta) = f(y) + \theta^{\alpha}\psi_{\alpha}(y) + F(y)\theta^2, \tag{2.0.3}$$

where $y^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + i\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}$. This is an example of a shortened superfield. This leads to three fields, a scalar, a fermion and another scalar which vanishes on-shell in the superspace action (since we can only have $\mathcal{N} = 1$ supersymmetry), thereby producing the physical supermultiplet.

In $\mathcal{N}$-extended supersymmetry, one introduces $\mathcal{N}$ supercharges and hence the odd subspace enlarges. This also enlarges the supersymmetry algebra to include a group of automorphisms, which we will often refer to as the internal group. This enlarges the space to $\mathbb{R}^{4|4\mathcal{N}}$, which in practice amounts to giving the Grassmann odd variables a second index, θ^I_{α}.

Upon introducing many more supercharges, it becomes possible to produce shortened supermupltiplets in a non-trivial way where the corresponding superfield is an explicit function of some mixture of θ and $\bar{\theta}$ variables, but not all Grassmann odd variables. This arises from allowing the supercharges (or some combination thereof) to directly annihilate the supermultiplet. In the context of four dimensional $\mathcal{N} = 4$ theories, these are BPS and semi-short multiplets (more in chapter 5). This leads to complicated superfield constraints which are often difficult to solve. However, one may use the so-called harmonic superspaces to project the constraint onto a subspace in which corresponding superfields are solved by chirality-like conditions like (2.0.2). Such a space will be of use throughout this entire thesis and we will discuss its construction in this chapter.

Viewing the aforementioned superspace as being useful in constraining aspects of component operators appearing in the superfield, we may also consider superspaces which constrain aspects of the kinematics appearing in a superfield. Suppose that we are dealing with observables in some theory where we are only interested in its massless sector. An example would be scattering amplitudes in some four dimensional massless theory. Scattering amplitudes are on-shell objects and do not benefit greatly from an over-complete configuration space which includes off-shell kinematic variables. Namely, one could instead pick a slice of such a configuration space which manifestly solves the corresponding masslessness constraint. In this way, all statements regarding external data in some calculation are manifestly physical statements of on-shellness and therefore masslessness.

In four dimensional massless gauge theory, twistor space is precisely such a space which has proven to yield quick and efficient results. Extending this to a superspace amounts to introducing some appropriate Grassmann odd variables. We will be using this space in chapter 3 and chapter 4 and we will review its construction in this chapter.

In this chapter we will review a coset construction for a large variety of superspaces, paying close attention to harmonic, analytic and twistor superspace, as these will appear in later chapters. We hope that this will be a guiding review into this subject, where we will prioritise practicality above formality. As a result, this should not be viewed as a complete review. An elaboration on the formal aspects of the topics discussed in this chapter can be found in [13,21].

2.1 Constructing superspaces from cosets

We begin by stating that the superconformal group in four dimensions with $\mathcal{N}$-extended supersymmetry is given by $\mathrm{SU}(2,2|\mathcal{N};\mathbb{R})$. For the discussion that follows, it is useful to consider its complexification, which makes this group $\mathrm{SL}(4|\mathcal{N};\mathbb{C})$. We will later generalise this to $\mathrm{SL}(2m|2n;\mathbb{C})$, in preparation for chapter 5 in the study of superconformal partial waves. We will also from here on out omit the $\mathbb{C}$ from any group (e.g. we will simply write $\mathrm{SL}(2m|2n)$) with the understanding that unless otherwise stated we take the complex setting to be the case.

Coset superspaces that are of interest to us are given by

$$F_{\mathfrak{p}} = \mathcal{P}\backslash\mathrm{SL}(2m|2n), \tag{2.1.1}$$

where $\mathcal{P}$ is a parabolic subgroup generated by a parabolic subalgebra. In general, we may begin by constructing an appropriately defined parabolic subalgebra (which are lower triangular block matrices) and then exponentiate the result to obtain the subgroup. The resulting supermanifold in (2.1.1) is known as a flag supermanifold.

We will present a review of this topic by starting with the purely bosonic case. We will find that the Dynkin diagram is the most succinct way to recapture not only the superspace itself, but its local coordinates and some aspects of the corresponding representation theory. We will give some immediate results for conformal and superconformal Minkowski space. Whilst leaving a more complete treatment relevant to operator spectra for chapter 5.

2.1.1 Bosonic case

A complex semi-simple Lie algebra $\mathfrak{g}$, admits a decomposition into a maximally commuting subalgebra $\mathfrak{h}$ (the Cartan subalgebra) and a set of subalgebras which are diagonalised with respect to the Cartan subalgabra with some well defined eigenvalue α in the root space Φ. This is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha . \tag{2.1.2}$$

Since for most of this thesis we will be discussing SL($2m$) and its supersymmetric extension we may keep $\mathfrak{sl}(2m)$ as our running example. We may recall the structure of the explicit Lie algebra. Taking $e_{i,j}$ to represent an explicit matrix with zeroes everywhere but for a one at position i, j, then the explicit matrix form of the Cartan subalgebra is given by

$$H_i = e_{i,i} - e_{i+1,i+1} \text{ for } i \in [1, 2m-1]. \tag{2.1.3}$$

Given the complete set of roots Φ, one can find the simple roots Δ from which one can decompose the simple roots into positive and negative roots $\Delta^+ \cup \Delta^-$. We may define such roots as

$$\begin{aligned} E^+_{\alpha_i} &= e_{i,i+1}, \\ E^-_{\alpha_i} &= e_{i+1,i}. \end{aligned} \tag{2.1.4}$$

So that the subalgebra of postive (negative) roots $\mathfrak{n}_+ (\mathfrak{n}_-)$ are the set of lower left (upper right) triangular matrices. This is opposite to the familiar convention and is because we are considering the group action from the right (see (2.1.1)) as opposed to the left.

Since the e_{ij} matrices themselves obey a $\mathfrak{gl}(2m)$ algebra:

$$[e_{i,j}, e_{k,l}] = \delta_{kj} e_{i,l} - \delta_{il} e_{k,j}, \tag{2.1.5}$$

one finds that

$$\begin{aligned}
\left[H_i, E^{\pm}_{\alpha_i}\right] &= \pm(-2) E^{\pm}_{\alpha_i}, \\
\left[E^{-}_{\alpha_i}, E^{+}_{\alpha_i}\right] &= H_i, \\
\left[E^{+}_{\alpha_i}, E^{+}_{\alpha_{i+1}}\right] &= -E^{+}_{\alpha_i+\alpha_{i+1}}, \\
\left[E^{+}_{\alpha_i+\alpha_{i+1}}, E^{+}_{\alpha_{i+1}+\alpha_{i+2}}\right] &= -E^{+}_{\alpha_i+2\alpha_{i+1}+\alpha_{i+2}},
\end{aligned} \tag{2.1.6}$$

and so on. Here we have taken $E^{+}_{\alpha_i+\alpha_{i+1}} = e_{i,i+2}, E^{+}_{\alpha_i+2\alpha_{i+1}+\alpha_{i+2}} = e_{i,i+5}$, etc. For the algebra of raising operators there is the analogous set of relations for the lower operators.

The parabolic subalgebra is defined to be the subalgebra which contains the Borel subalgebra. In the context of $\mathfrak{sl}(2m)$, the Borel subalgebra is the subalgebra spanned by the upper triangular matrices. Formally, this is given by

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha. \tag{2.1.7}$$

We can choose a specific parabolic subalgebra by selecting a set of simple roots $S_\mathfrak{p}$. In defining $\Phi(\mathfrak{l}) = \mathrm{span}(S_\mathfrak{p}) \cap \Phi$, the so-called Levi subalgebra is given by

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi(\mathfrak{l})} \mathfrak{g}_\alpha. \tag{2.1.8}$$

In the context of $\mathfrak{sl}(2m)$ this fills out a specific block diagonal part of the algebra. Schematically, if one first defines the set $S_\mathfrak{p}$, we immediately get $\mathfrak{l}$ as some block diagonal form with an overall tracelessness condition, namely for some set $\{k_1, k_2, \dots\}$ defined by $S_\mathfrak{p}$, we have

$$\mathfrak{l} = \mathfrak{s}\left(\bigoplus_{k_i \left| \sum_i k_i = 2m\right.} \oplus\mathfrak{gl}(k_i)\right) \cong \bigoplus_{k_i \left| \sum_i k_i = 2m\right.} \oplus\mathfrak{sl}(k_i) \oplus \mathbb{C}^{|\Delta/S_\mathfrak{p}|}. \tag{2.1.9}$$

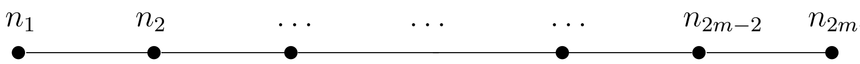
The first statement follows from the form of the $\mathfrak{l}$ as we will see in many examples to come, however the second equivalence comes from breaking the $\mathfrak{gl}(k_i)$ blocks into their traceless $\mathfrak{sl}(k_i)$ subalgebras. However, there remain elements of the overall matrices which are traceless yet not part of any $\mathfrak{sl}(k_i)$, which we associate to a charge $\mathbb{C}$. There are as many of these charges are there are simple roots which we do not take to make up $S_\mathfrak{p}$, namely there are $|\Delta/S_\mathfrak{p}|$ of them. [1]

To produce the parabolic subalgebra we simply need to fill out the lower triangular block so that the Borel subalgebra will be contained in it. Hence, we simply add this subspace to produce the parabolic subalgebra $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{l} \oplus \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\mathfrak{l})} \mathfrak{g}_\alpha \tag{2.1.10}$$

Finally, we can exponentiate this subalgebra to give the corresponding subgroup $\mathcal{P}$ and therefore produce the manifold $F_\mathfrak{p}$ as in (2.1.1).

We can label different parabolic subalgebras by the chosen set of simple roots $S_\mathfrak{p}$. This information can be represented by Dynkin diagrams. Recall that for $\mathfrak{sl}(2m)$, the Dynkin diagram is given by $2m-1$ many nodes connected by edges:



(2.1.11)

Then one can represent the parabolic subalgebra by putting a cross on those nodes whose corresponding roots do not appear in $S_\mathfrak{p}$. This is a useful way to provide information, since we require all field representations to transform under the parabolic subgroup we also learn about the index structure of those fields. This essentially follows from the fact that the field representations need to transform under the block diagonal part of the algebra defined by $S_\mathfrak{p}$, whilst transforming trivially under the off-block diagonal parts of the parabolic subalgebra. This is since the off-block diagonal parts are the raising operators which act trivially on highest weight states.

As an example at the bosonic level, we may consider complexified compactified Minkowski space in four dimensions. It is known that this space can be identified with the space of 2-planes in four dimensions, which is otherwise known as the complex Grassmannian $\mathrm{Gr}_2(4)$ [14]. The appropriate group is SL(4), whose algebra $\mathfrak{sl}(4)$ contains three simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ which correspond to the three nodes in the Dynkin

[1] for e.g. $\mathfrak{l} = \mathfrak{s}\left(\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)\right) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}$ where a basis for $\mathbb{C}$ is $\mathrm{diag}(-1,-1,1,1)$.

diagram. To produce $\mathrm{Gr}_2(4)$, we require $S_{\mathfrak{p}} = \{\alpha_1, \alpha_3\}$. This corresponds to putting a cross in the second node in the Dynkin diagram:

$$\begin{array}{ccccc} n_1 & & n_2 & & n_3 \\ \bullet & \text{———} & \times & \text{———} & \bullet \end{array} \tag{2.1.12}$$

In which the subalgabra takes the form:

$$\mathfrak{p} = \begin{pmatrix} \bullet_{2\times 2} & 0_{2\times 2} \\ \bullet_{2\times 2} & \bullet_{2\times 2} \end{pmatrix}, \tag{2.1.13}$$

where the $\bullet_2$ and 0_2 are generically non-zero and zero two by two blocks. One can then exponentiate this to produce the corresponding group. Since these are coset manifolds, local coordinates are found by fixing the degrees of freedom in the subgroup, revealing the coordinate matrix: [2]

$$X^{AB} = \begin{pmatrix} \mathbb{I}_{2\times 2} & ix_{\alpha\dot{\alpha}} \\ 0_{2\times 2} & \mathbb{I}_{2\times 2} \end{pmatrix} \text{ where } \underline{A}, \underline{B} = 1, \ldots, 4. \tag{2.1.14}$$

Here

$$x_{\alpha\dot{\alpha}} = (\sigma^{\mu})_{\alpha\dot{\alpha}} x_{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{2.1.15}$$

and we can also define $\tilde{x}^{\dot{\alpha}\alpha} = (\tilde{\sigma}_{\mu})^{\dot{\alpha}\alpha} x^{\mu}$ or put another way $\tilde{x}^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} x_{\beta\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}}$, where σ are the four two by two Pauli matrices whilst $\tilde{\sigma}$ is its conjugate. It is in fact possible to make the Grassmannian structure more obvious by not fixing all of the elements of the parabolic subgroup. This can be done by allowing the top left quadrant of the parabolic subgroup to be left unfixed. Namely,

$$\begin{aligned} &\text{If } \begin{pmatrix} A_{2\times 2} & 0_{2\times 2} \\ B_{2\times 2} & C_{2\times 2} \end{pmatrix} \in \mathcal{P} \text{ and } \begin{pmatrix} D_{2\times 2} & E_{2\times 2} \\ F_{2\times 2} & G_{2\times 2} \end{pmatrix} \in \mathrm{SL}(4), \\ &\text{we have } A_{2\times 2}\,(D_{2\times 2}, E_{2\times 2}) \in \mathcal{P}\backslash \mathrm{SL}(4) \end{aligned} \tag{2.1.16}$$

We label this local coordinate by z_a^A, where there exists a local $\mathrm{GL}(2)$ right action which acts on the a-index of z_a^A, whilst there is a global left action which acts on the A-index. Fixing the $\mathrm{GL}(2)$ freedom reveals the coordinate $(\mathbb{I}_2, ix_{\alpha\dot{\alpha}})$.

[2] This is found from exponentiating $\begin{pmatrix} 0_{2\times 2} & x_{\alpha\dot{\alpha}} \\ 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}$.

Now that we have recovered the usual space-time coordinate $x^{\alpha\dot{\alpha}}$ from this coset, we should be able to induce the non-linear transformation of $\delta x^{\alpha\dot{\alpha}}$ from the linear action of the groups that make up the coset. Note that in finding the representation (2.1.14), we had to fix the parameters of the parabolic subgroup, which we call a 'section'. If we consider the action of SL(4) (which includes the parabolic subgroup), this will perform a full transformation which will take us away from the current section. In order to restore the form in (2.1.14) we need to find the appropriate section again. Using the notation in (2.1.16), the statement is

$$X^{AC}G_C^B = P_D^A X'^{DB} \text{ for } G_B^A \in \mathrm{SL}(4) \text{ and } P_B^A \in \mathcal{P}, \tag{2.1.17}$$

where by X' we mean the SL(4) transformed point, i.e. infinitesimally $X' = X + \delta X$. We can go to the Lie algebra, by taking $G_B^A = \delta_B^A + g_B^A$ and $P_B^A = \delta_B^A + p_B^A$ for $g_B^A \in \mathfrak{sl}(4)$ and $p_B^A \in \mathfrak{p}$. By taking the infinitesimal limit of (2.1.17), we get

$$\delta X^{AB} = X^{AC}g_C^B - p_C^A X^{CB}. \tag{2.1.18}$$

In taking

$$g_B^A = \begin{pmatrix} -A_\alpha{}^\beta & iB_{\alpha\dot{\beta}} \\ iC^{\dot{\alpha}\beta} & D^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}, \tag{2.1.19}$$

we find

$$\delta x_{\alpha\dot{\alpha}} = B_{\alpha\dot{\alpha}} + A_\alpha{}^\beta x_{\beta\dot{\alpha}} + x_{\alpha\dot{\beta}} D^{\dot{\beta}}{}_{\dot{\alpha}} + x_{\alpha\dot{\beta}} C^{\dot{\alpha}\beta} x_{\beta\dot{\alpha}}, \tag{2.1.20}$$

which is the well known non-linear action of the conformal algebra upon a space-time point.

Considering representation theory concepts, we define an operator inserted at $x = 0$ to be $\mathcal{O}(0)$. We recall that a conformal primary is a highest weight state and is defined to be the specific operator $\mathcal{O}(0)$ such that $[K, \mathcal{O}(0)] = 0$. We may thus associate K with the parameter $C_{\dot{\alpha}\beta}$. We can build the infinite dimensional conformal multiplet by acting the momentum generator, namely $[P, \mathcal{O}(0)]$, thus we may associate P with the parameter $B_{\alpha\dot{\alpha}}$. This ties in with our definition of the positive(negative) subalgebra $\mathfrak{n}_+(\mathfrak{n}_-)$ in (2.1.4). We also have the dilation weight and the Lorentz generators (D, $M = \{J_1, J_2\}$) which make up the Cartan subalgbra. These generators come from the

$\mathfrak{su}(2,2)$ algebra and are explicitly given by [22]

$$\hat{\Delta} = \frac{1}{2}\begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad \hat{J}_1 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{J}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{2.1.21}$$

In previously used notation, the Cartan subalgebra of $\mathfrak{sl}(4)$ is explicitly made from

$$H_1 = e_{11} - e_{22}, \quad H_2 = e_{22} - e_{33} \text{ and } H_3 = e_{33} - e_{44}. \tag{2.1.22}$$

Each node n_i of the Dynkin diagram in (2.1.12) is associated with each element of the Cartan subalgebra H_i. Given this explicit basis, we can rewrite the n_i in (2.1.12) in terms of more familiar conformal field theory data in (2.1.21). Doing this, we find

$$n_1 = 2J_1, \quad n_2 = -\Delta - J_1 - J_2, \quad n_3 = 2J_2. \tag{2.1.23}$$

Demanding that all irreducible representations transform under the subalgebra defined by (2.1.12), implies that we are interested in irreducible representations of

$$\mathfrak{l} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}, \tag{2.1.24}$$

which we can take to be labelled by the Dynkin nodes $[n_1, n_2, n_3]$. The reason is that the off-block-diagonal part of the parabolic subalgebra are raising operators and since we define irreducible representations by highest weight states they are immediately annihilated thus transform trivially.

We can also read off the tensor structure of the corresponding field representation. A generic field representation is

$$\mathcal{O}_{\mathcal{R}(\alpha)\mathcal{R}'(\dot{\alpha})} \tag{2.1.25}$$

where α and $\dot{\alpha}$ are indices corresponding to the fundamental representation of each of the $\mathfrak{sl}(2)$s. $\mathcal{R}(\alpha)$ takes the index in some representation.

As an example, a massless scalar, fermion and vector with index structure $\mathcal{O}$, $\mathcal{O}_\alpha$ and $\mathcal{O}_{\alpha\dot{\alpha}}$ in four dimensions would have Dynkin labels $[0,-1,0]$, $[1,-1,0]$ and $[1,-3,1]$ respectively.

2.1.2 Supersymmetric case

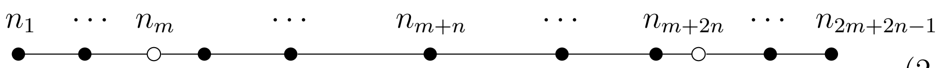
We now move onto the supersymmetric case, in particular focussing on $\mathfrak{sl}(2m|2n)$. In principle, the only thing that will change is that the commutator used to define the

Lie algebra has now generalised, namely

$$\{g_1, g_2] = -(-1)^{\deg(g_1)\deg(g_2)} \{g_2, g_1] \quad \forall g_1, g_2 \in \mathfrak{g}, \tag{2.1.26}$$

and in the current context of $\mathfrak{sl}\,(2m|2n)$, we now have supertraceless[3] matrices.

Having reviewed the machinery in the previous section, we work somewhat in reverse. Since the essential information is given by how one puts crosses in the super Dynkin diagram we can start from this point of view. However, in contrast to the bosonic case, there are now multiple distinct ways to choose simple roots. This amounts to how one distributes the odd nodes in the Dynkin diagram (see [15] for some discussion regarding $\mathcal{N} = 4$ SYM). The case of interest to us, namely $\mathfrak{sl}\,(2m|2n)$ has the form:



(2.1.27)

Where the black nodes are the even roots and the white nodes are the odd roots. Generically, one considers explicit matrices in $\mathfrak{sl}\,(2m|2n)$ to have the structural form of

$$\left(\begin{array}{c|c} 2m \times 2m & 2n \times 2m \\ \hline 2m \times 2n & 2n \times 2n \end{array}\right), \tag{2.1.28}$$

where the $2m \times 2m$ and $2n \times 2n$ blocks are Grassmann even, whilst the rest are Grassmann odd. However, it turns out to be advantageous to make a change of basis, such that the structure of the group is $\mathfrak{sl}\,(m|2n|m)$, in which case the explicit matrix structure takes the form:

$$\left(\begin{array}{c|c|c} m \times m & 2n \times m & m \times m \\ \hline m \times 2n & 2n \times 2n & m \times 2n \\ \hline m \times m & 2n \times m & m \times m \end{array}\right), \tag{2.1.29}$$

where the $m \times m$ and $2n \times 2n$ blocks are all Grassmann even, whilst the others are odd. In the current context, the advantage of this form is that the parabolic subalgebra of the cases of interest to us takes a lower block triangular form, making it closer in form to the bosonic case. This restructuring was first performed in [18].

[3] recall that the supertrace of a matrix $M \in \mathrm{Mat}_{m|n}$ is given by $\mathrm{str}(M) = \sum_{i=1}^{m} M_{ii} - \sum_{j=m+1}^{n} M_{jj}$.

Following the bosonic case, we can seek out a form of the Cartan subalgebra and its corresponding roots. We can take the Cartan subalgebra as

$$\begin{aligned} H_i &= e_{i,i} - e_{i+1,i+1} \quad \forall i \in [1, m-1]\,, \\ H_m &= -e_{m,m} - e_{m+1,m+1}\,, \\ H_i &= e_{i,i} - e_{i+1,i+1} \quad \forall i \in [m+1, m+2n-2]\,, \\ H_{m+2n} &= e_{m+2n,m+2n} + e_{m+2n+1,m+2n+1}\,, \\ H_i &= e_{i,i} - e_{i+1,i+1} \quad \forall i \in [m+2n+2, 2m+2n]\,. \end{aligned} \tag{2.1.30}$$

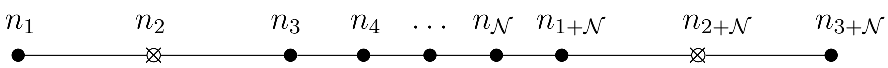
For the positive simple roots, we can choose the one used in the bosonic section, namely $E^+_{\alpha_i} = e_{i,i+1}$. However, under the change of basis as described by (2.1.29), we have two odd roots given by $E^+_{\alpha_m} = e_{m,m+1}$ and $E^+_{\alpha_{m+2n}} = e_{m+2n,m+2n+1}$.

A small cautionary remark is that the number of odd nodes in the Dynkin diagram is independent of the basis chosen. In the current case, the odd nodes directly correspond to the intersection points in the new basis in (2.1.29). However, one can indeed find similarly two odd nodes in the basis defined in (2.1.28). See [15] for two examples in $\mathcal{N} = 4$ and [16,17] for a more general discussion.

We can check some important commutation relations that distinguish this Lie superalgebra from the bosonic case. For example, whilst in similarity to the bosonic case we still have for even Lie brackets (2.1.6), we also have

$$\left\{E^+_{\alpha_m}, E^-_{\alpha_m}\right\} = -H_m \text{ and } \left\{E^+_{\alpha_{m+2n}}, E^-_{\alpha_{m+2n}}\right\} = H_{m+2n} \tag{2.1.31}$$

As an example we may consider an unconstrained four dimensional $\mathcal{N}$-extended superspace. To achieve this space we need to put crosses in the fermionic nodes of the $\mathfrak{sl}(4|\mathcal{N})$ Dynkin diagram. Namely:



(2.1.32)

Then the subalgebra in the structural form given by (2.1.29), has the form

$$\mathfrak{p} = \left(\begin{array}{c|c|c} \bullet_{2\times 2} & 0_{\mathcal{N}\times 2} & 0_{2\times 2} \\ \hline \bullet_{2\times \mathcal{N}} & \bullet_{\mathcal{N}\times \mathcal{N}} & 0_{2\times \mathcal{N}} \\ \hline \bullet_{2\times 2} & \bullet_{\mathcal{N}\times 2} & \bullet_{2\times 2} \end{array}\right), \tag{2.1.33}$$

where the zero entries $0_{\mathcal{N}\times 2}$ and $0_{2\times\mathcal{N}}$ are both Grassmann odd whilst the entry $0_{2\times 2}$ is Grassmann even. As a result, the coset can be parametrised by the local coordinates

$$X^{\mathcal{AB}} = \left(\begin{array}{c|c|c} \mathbb{I}_{2\times 2} & i\theta^I_\alpha & iX_{\alpha\dot\alpha} \\ \hline 0_{2\times\mathcal{N}} & \mathbb{I}_{\mathcal{N}\times\mathcal{N}} & i\bar\theta^{\dot\alpha}_I \\ \hline 0_{2\times 2} & 0_{\mathcal{N}\times 2} & \mathbb{I}_{2\times 2} \end{array}\right), \text{where } X_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} + \frac{i}{2}\theta^I_\alpha\bar\theta_{\dot\alpha I}, \tag{2.1.34}$$

in which we recapture the familiar coordinate system for this space.

As can be read off from (2.1.32), irreducible representations must transform under

$$\mathfrak{l} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(\mathcal{N}) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}^2. \tag{2.1.35}$$

In comparing to the real form of the group, namely $\mathrm{SU}(2,2|\mathcal{N})$ whose internal group is $\mathrm{U}(\mathcal{N})$ we can write the Dynkin nodes in terms of physical superconformal data.

To do this we need to review some aspects of superalgebras, in which a complete treatment can be found in [11]. The four dimensional $\mathcal{N}$-superconformal algebra is built from the conformal generators which have already been discussed together with the supercharges Q^α_I and $\bar Q^{\dot\alpha}_I$, and the special superconformal charges S^α_I and $\bar S^{\dot\alpha}_I$. We also have the generators of $\mathfrak{u}(\mathcal{N}) \cong \mathfrak{su}(\mathcal{N}) \oplus \mathfrak{u}(1)$, denoted $\left(R^I_J, R\right)$. The Cartan of the conformal subalgebra is already given by (2.1.21), these matrices are placed in $\mathfrak{su}(2,2|\mathcal{N})$ by putting the $0_{\mathcal{N}\times\mathcal{N}}$ block in the middle of the matrices, namely [22]

$$\hat\Delta = \frac{1}{2}\left(\begin{array}{c|c|c} -\mathbb{I}_{2\times 2} & 0_{\mathcal{N}\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{\mathcal{N}\times\mathcal{N}} & 0_{2\times\mathcal{N}} \\ \hline 0_{2\times 2} & 0_{2\times\mathcal{N}} & \mathbb{I}_{2\times 2} \end{array}\right), \quad \hat J_1 = \frac{1}{2}\left(\begin{array}{c|c|c} \sigma_3 & 0_{\mathcal{N}\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{\mathcal{N}\times\mathcal{N}} & 0_{2\times 4} \\ \hline 0_{2\times 2} & 0_{2\times\mathcal{N}} & 0_{2\times 2} \end{array}\right),$$

$$\hat J_2 = \frac{1}{2}\left(\begin{array}{c|c|c} 0_{2\times 2} & 0_{\mathcal{N}\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & 0_{\mathcal{N}\times\mathcal{N}} & 0_{2\times\mathcal{N}} \\ \hline 0_{2\times 2} & 0_{2\times\mathcal{N}} & \sigma_3 \end{array}\right). \tag{2.1.36}$$

The Cartan of $\mathfrak{su}(\mathcal{N})$ are identical to the bosonic case, namely if the $\mathfrak{su}(\mathcal{N})$ Dynkin nodes are given by $[a_1,\ldots,a_{\mathcal{N}-1}]$, then we have that $a_i = e_{i+2,i+2} - e_{i+3,i+3}$. Together with this, we also have the R-charge associated to the $\mathfrak{u}(1)$ given by

$$R = \frac{1}{2}\left(\begin{array}{c|c|c} \mathbb{I}_{2\times 2} & 0_{\mathcal{N}\times 2} & 0_{2\times 2} \\ \hline 0_{2\times 2} & \frac{4}{\mathcal{N}}\mathbb{I}_{\mathcal{N}\times\mathcal{N}} & 0_{2\times\mathcal{N}} \\ \hline 0_{2\times 2} & 0_{2\times\mathcal{N}} & \mathbb{I}_{2\times 2} \end{array}\right). \tag{2.1.37}$$

In $\mathcal{N}=4$ the R-charge is taken to vanish. This is since all of the algebra relations associated to the R-charge commute such that the generator associated to the R-charge becomes the center, and thus acts on the state space in a trivial way. This promotes the superconformal group to $\mathrm{PSU}(2,2|4)$. Finally, we can relate these physical Cartan matrices directly with the generators of (2.1.30). We get

$$n_1 = 2J_1, \quad n_{\mathcal{N}+3} = 2J_2, \quad n_{i+2} = a_i,$$
$$n_2 = \frac{1}{2}(\Delta - R) + J_1 + \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}-1} i a_i - \sum_{i=1}^{\mathcal{N}-1} a_i, \quad n_{\mathcal{N}+2} = \frac{1}{2}(\Delta + R) + J_2 - \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}-1} i a_i. \tag{2.1.38}$$

We leave the further details of unitary bounds, long and short operators to chapter 5. Similarly to (2.1.25), irreducible representations are given by

$$\mathfrak{l} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(\mathcal{N}) \oplus \mathfrak{sl}(2) \oplus \mathbb{C}^2 \tag{2.1.39}$$

and so the tensor structure is given by

$$\mathcal{O}_{\mathcal{R}(\alpha)\mathcal{R}'(\dot{\alpha})\mathcal{R}''(I)}. \tag{2.1.40}$$

This is where α and $\dot{\alpha}$ are $\mathfrak{sl}(2)$ indices in representations $\mathcal{R}$ and $\mathcal{R}'$, but now we have an $\mathfrak{sl}(\mathcal{N})$ index I in representation $\mathcal{R}''$. Note that here we also an extra $\mathbb{C}$ in comparison to previous cases, where the first one is related to the conformal dimension Δ (as in the bosonic case), and second is related to the R-charge. We may label representations as $[n_1, n_2, n_3, \ldots, n_{\mathcal{N}+3}]$.

As an example we may consider $\mathcal{N}=4$, in which $\mathcal{O}_{[IJ][KL]}$ is a scalar operator with Δ=2 and $\mathfrak{su}(4)$ representation $[0,2,0]$ which makes this a $\frac{1}{2}$-BPS. $\mathcal{O}_{(I[J)KL]}$ has the same properties but for the $\mathfrak{su}(4)$ representation, which is instead $[1,0,1]$, making this a $\frac{1}{4}$-BPS. We also give the example of $\mathcal{O}_{\alpha\beta\dot{\alpha}\dot{\beta}[IJ]}$ which is a spin-2 operator with $\Delta = 6$ and $\mathfrak{su}(4)$ representation $[0,2,0]$, making it a semi-short operator. The aforementioned examples have Dynkin labels [4]

$$\begin{aligned} \mathcal{O}_{[IJ][KL]} &= [0,0,0,2,0,0,0]\,, \\ \mathcal{O}_{(I[J)KL]} &= [0,0,1,0,1,0,0]\,, \\ \mathcal{O}_{\alpha\beta\dot\alpha\dot\beta[IJ]} &= [2,3,0,2,0,3,2]\,. \end{aligned} \tag{2.1.41}$$

2.2 Twistor superspace

Having reviewed sufficient superspace technology, we can look at four dimensional twistor superspace which we shall use extensively in chapters 3 and 4. In the previous section we gave a short discussion on the corresponding representation theory after having selected a parabolic subgroup. Since our interest in twistor superspace will be purely in the kinematic parts of correlation functions and scattering amplitudes, we shall not make any statements about the representation theory. Moreoever, the representation theory requires further mathematical concepts which we have not discussed, see [12,13].

The commonly used form of twistor superspace is a projective twistor superspace, in which the parabolic subgroup of SL(4|4) is given by the data (in the basis defined by (2.1.29)):

$$\begin{array}{ccccccccccccc} n_1 & & n_2 & & n_3 & & n_4 & & n_5 & & n_6 & & n_7 \\ \times & — & \circ & — & \bullet & — & \bullet & — & \bullet & — & \circ & — & \bullet \end{array} \tag{2.2.42}$$

$$\mathfrak{p} = \left(\begin{array}{cc|c|c} \bullet_{1\times1} & 0_{1\times1} & 0_{4\times1} & 0_{2\times1} \\ \bullet_{1\times1} & \bullet_{1\times1} & \bullet_{4\times1} & \bullet_{2\times1} \\ \hline \bullet_{1\times4} & \bullet_{1\times4} & \bullet_{4\times4} & \bullet_{2\times4} \\ \hline \bullet_{1\times2} & \bullet_{1\times2} & \bullet_{4\times2} & \bullet_{2\times2} \end{array}\right). \tag{2.2.43}$$

We can fix the coset to find local coordinates, which can be written in the form (by

[4] The index structure [••] means antisymmetrisation, whilst (••) means symmetrisation.

allowing the top left element of $\mathcal{P}$ to remain unfixed):

$$\mathcal{Z}^{\mathcal{A}} = \begin{pmatrix} \lambda^\alpha \\ \chi^I \\ \mu_{\dot\alpha} \end{pmatrix} \in \mathbb{CP}^{3|4}. \tag{2.2.44}$$

We denote $z^A = \binom{\lambda^\alpha}{\mu_{\dot\alpha}}$ to be the bosonic twistor and is comprised of two dimensional vectors, in the forthcoming context of Minkowski space-time we can take the $(\alpha, \dot\alpha)$ indices to be spinor indices. The $\mathcal{A}$-index in (2.2.44) is a fundamental index of $\mathfrak{sl}(4|4)$, whilst the index I in the bosonic subspace is in the fundamental of $\mathfrak{sl}(4)$. We also have χ^I which is a four dimensional Grassmann odd coordinate and is in the fundamental of $\mathfrak{sl}(4)$.

Twistor superspace contains a deep connection with superconformal Minkowski space. This is due to the fact that these two spaces and a third space called a 'correspondence space' fit into a structure (which we will not study here) called a double fibration [13,12,21].

We can state some of the consequences of these structures. We focus on the bosonic subspace. A first consequence is the incidence relation which is given by

$$\begin{aligned} \mu_{\dot\alpha} &= i x_{\alpha\dot\alpha}\lambda^\alpha, \\ \chi^I &= \theta^I_\alpha \lambda^\alpha, \end{aligned} \tag{2.2.45}$$

where $(x_{\alpha\dot\alpha}, \theta^I_\alpha)$ define a chiral superspace. Now, we can define two different space-time points as corresponding to the same twistor point, namely take $\mu_{\dot\alpha} = i(x_1)_{\alpha\dot\alpha}\lambda^\alpha$ and $\mu_{\dot\alpha} = i(x_2)_{\alpha\dot\alpha}\lambda^\alpha$. As a consequence we find that $(x_{12})_{\alpha\dot\alpha}\lambda^\alpha = 0$. This implies that the matrix x_{12} is of rank 1 and has determinant zero. Following (2.1.15), we have

$$\det(x) = \frac{1}{2} x^\alpha_{\dot\alpha} \tilde{x}^{\dot\alpha}_\alpha = x^2. \tag{2.2.46}$$

It follows that a twistor point corresponds to two lightlike separated space-time points. Put another way, we may solve $(x_{12})_{\alpha\dot\alpha}\lambda^\alpha = 0$ by setting $(x_{12})_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$, where $\tilde\lambda_{\dot\alpha}$ is arbitrary. Similarly for the Grassmann odd part, we end up with $(\theta_{12})^I_\alpha \lambda^\alpha = 0 \to (\theta_{12})^I_\alpha = \eta^I \lambda_\alpha$ for an arbitrary Grassmann odd parameter η^I. The constraints are equivalent to $x^2_{12} = 0$ and $(\theta)^{\alpha I}_{12}(x_{12})_{\alpha\dot\alpha} = 0$, which are manifestly space-time statements.

On the other hand, we may have two twistor points related by the same space-time point. For example, given $\mu_{1\dot{\alpha}} = ix_{\alpha\dot{\alpha}}\lambda_1^\alpha$ and $\mu_{2\dot{\alpha}} = ix_{\alpha\dot{\alpha}}\lambda_2^\alpha$, together with $\chi_1^I = \theta_\alpha^I\lambda_1^\alpha$ and $\chi_2^I = \theta_\alpha^I\lambda_2^\alpha$, we get

$$\begin{aligned} x_{\alpha\dot{\alpha}} &= -i\left(\frac{\lambda_{1\alpha}\mu_{2\dot{\alpha}} - \lambda_{2\alpha}\mu_{1\dot{\alpha}}}{\epsilon_{\alpha\beta}\lambda_1^\alpha\lambda_2^\beta}\right), \\ \theta_\alpha^I &= \frac{\lambda_{1\alpha}\chi_2^I - \lambda_{2\alpha}\chi_1^I}{\epsilon_{\alpha\beta}\lambda_1^\alpha\lambda_2^\beta}. \end{aligned} \tag{2.2.47}$$

We can manifest some of these geometric notions with some useful notation which we will use later. Focussing on the bosonic case, in order to reconstruct the notion of a point in Minkowski space-time we need two twistor points, hence we may allow a second 'local' index, namely $z^A \to z_a^A$ so that subjected to the incidence relation we have

$$z_a^A = \begin{pmatrix} \lambda_a^\alpha \\ ix_{\beta\dot{\alpha}}\lambda_a^\beta \end{pmatrix} = \begin{pmatrix} \delta_\beta^\alpha \\ ix_{\beta\dot{\alpha}} \end{pmatrix}\lambda_a^\beta. \tag{2.2.48}$$

Then we get

$$\begin{aligned} &\epsilon^{ab}z_a^Az_b^B = X^{AB}, \\ &\text{where } X^{AB} = \begin{pmatrix} \epsilon_{\alpha\beta} & -ix_\alpha^{\dot{\beta}} \\ ix_\beta^{\dot{\alpha}} & -x^2\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \end{aligned} \tag{2.2.49}$$

Note that in doing this we have taken $\lambda_a^\beta = \delta_a^\beta$. This follows from the covariance of a GL(2) action on the (a, b) indices in the left hand side of the first equation of (2.2.49). Put another way, we may think non-projectively in which we have two lines which we anti-symmetrise to produce a 2-plane. The local GL(2) action allows us to rotate two lines defining the 2-plane, however the 2-plane should not depend on the basis chosen. Indeed, we have simply recovered the Grassmannian of 2-planes in four dimensions $\mathrm{Gr}_2(4)$. In this way, we can think of $X^{AB} \in \mathbb{CP}^5$ in which we can identify these with the so-called Plücker coordinates [65].

This all implies that we can take $z_a^A = \binom{\delta_a^\alpha}{ix_{a\dot{\alpha}}}$, however we can define a conjugate object, namely $\bar{z}_{A\dot{a}} = \left(-ix_{\alpha\dot{a}}, \delta_{\dot{a}}^{\dot{\alpha}}\right)$. This gives

$$\bar{z}_{j,\dot{a}A}z_{i,a}^A = \left(-ix_{j,\alpha\dot{a}}, \delta_{\dot{a}}^{\dot{\alpha}}\right)\begin{pmatrix} \delta_a^\alpha \\ ix_{i,a\dot{\alpha}} \end{pmatrix} = ix_{ij,a\dot{a}}. \tag{2.2.50}$$

From (2.2.49), we have

$$\epsilon_{ABCD}X^{AB}X^{CD} = 0. \tag{2.2.51}$$

Now, the incidence can be taken to come from (where $\bar{X}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CD} = \epsilon_{\dot{a}\dot{b}}\bar{z}^{\dot{a}}_{A}\bar{z}^{\dot{b}}_{B}$)

$$\bar{X}_{AB}z^{B} = 0. \tag{2.2.52}$$

In this language, $\bar{X}_{AB}$ is a point in space-time. Since $\bar{X}_{AB}$ is defined by a line parametrised by z_1^A and z_2^A and $\bar{X}_{AB}$ is built out of z_1^A and z_2^A, it follows from (2.2.52):

$$\bar{X}_{AB}z^{B}(\sigma) = 0 \text{ where } z^{B}(\sigma) = z^{B}_{a}\sigma^{a} = z^{B}_{1}\sigma^{1} + z^{B}_{2}\sigma^{2}, \tag{2.2.53}$$

where $\sigma^a = (\sigma^1, \sigma^2)$ parametrises the corresponding line in twistor space. We will use this notion extensively in chapter 3.

Finally, space-time differences can be built from the X^{AB}s. Taking into account the relation in (2.2.49), we find

$$x_{ij}^2 = \frac{1}{2}\bar{X}_{iCD}X_j^{CD} = \frac{1}{4}\epsilon^{ab}\epsilon^{cd}\epsilon_{ABCD}z^{A}_{i,a}z^{B}_{i,b}z^{C}_{j,c}Zz^{D}_{j,d} = \langle z_{i,1}z_{i,2}z_{j,1}z_{j,2}\rangle\,. \tag{2.2.54}$$

We see that given two points X_i and X_j whose corresponding lines are spanned by $z^A_{i,a}$ and $z^A_{j,a}$ are non-lightlike separated if all four points are distinguishable. However, if a point is a linear combination of the other points (for example $z_{j,1} = z_{i,2}$) then by virtue of (2.2.54) we have $x_{ij}^2 = 0$. The statement is then that intersecting twistor lines correspond to lightlike separated points, whilst non-intersecting lines correspond to non-lightlike separated space-time points.

We can summarise our statements made diagrammatically from figure 2.1. An important point regarding figure 2.1 is that diagrams *ii*) and *iii*) are relevant to off-shell and on-shell physics respectively. In particular, we will see that Feynman diagrams which include diagram *ii*) as sub-diagrams are of relevance to correlation functions or partially off-shell objects. Conversely, Feynman diagrams for scattering amplitudes or partially on-shell observables will include diagram *iii*) as sub-diagrams.

Let us emphasise an important point the will feature entirely in this thesis. When considering amplitudes we will be dealing with momentum, and as we shall see in section 3.1.1 we can and will employ the relation

$$x_{i-1,\alpha\dot{\alpha}} - x_{i,\alpha\dot{\alpha}} = p_{i-1,\alpha\dot{\alpha}}, \tag{2.2.55}$$

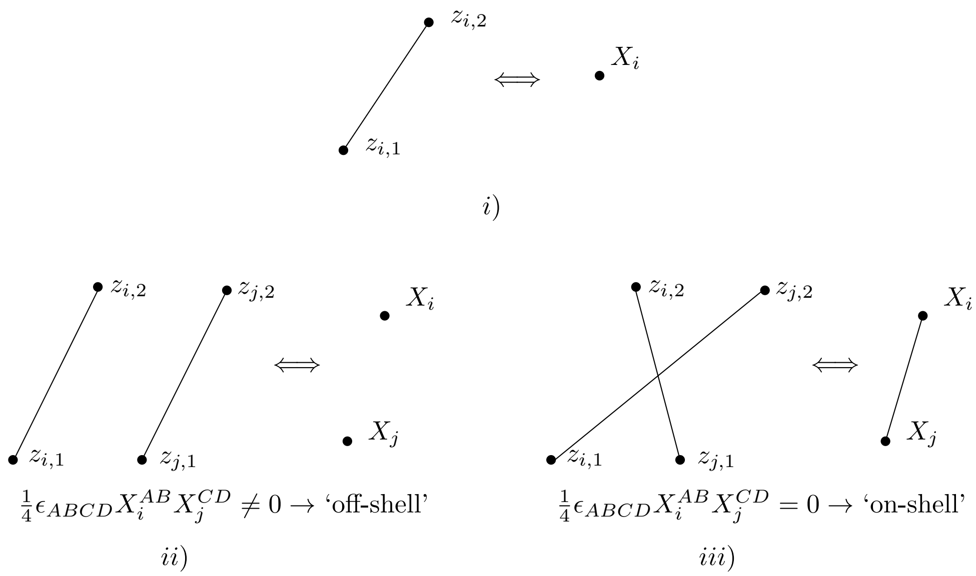


Figure 2.1: ***i)*** The basic statement that lines in twistor space correspond to points in conformal Minkowski space. ***ii)*** Non-intersecting lines leads to non-lightlike separated lines. ***iii)*** intersecting lines leads to lightlike separated lines.

where p is a momentum label. Importantly, the x variables here are to be regarded as different to the usual space-time and are to be thought of as a dual space-time. The same correspondence with twistor superspace follows and as a result the corresponding superconformal transformations are referred to as dual superconformal transformations [48]. These are the natural coordinates when dealing with Wilson loops [47].

Another coordinate system that will feature in this thesis that are related to twistors are the projective hypercone coordinates. In general the projective hypercone is useful since whilst twistor space has been constructed for a limited number of dimensions (see [23] for a six-dimensional application), the projective hypercone is valid for any dimension. For us, the projective hypercone coordinates will be useful in practical calculations. In the current context, d-dimensional Minkowski space is embedded in the projective hypercone in $d+2$-dimensions. As a result the natural coordinate is a $\mathfrak{so}(2,d)$ vectorial object X^M. In the four dimensional case we have the real isomorphism $\mathfrak{so}(2,4;\mathbb{R}) \cong \mathfrak{su}(2,2;\mathbb{R})$ from which we can recover the X^{AB} coordinates and hence the relation to twistor space.

The projective hypercone is defined in $\mathbb{R}^{d,2}$ and is given by (also known as the Klein quadric)

$$\begin{aligned}
\eta_{MN} X^M X^N &= 0 \text{ for } M, N = -1, \ldots, d+1, \\
\eta_{MN} &= \mathrm{diag}(+, +, \underbrace{-, \ldots, -}_{d}), \\
X^M &\in \mathbb{RP}^{d-1}.
\end{aligned} \tag{2.2.56}$$

The first condition above may be recast into light-cone coordinates, namely defining $X^{\pm} = X^{-1} \pm X^d$, yields the embedding

$$X^+ X^- + X^\mu X^\nu \eta_{\mu\nu} = 0, \tag{2.2.57}$$

where $\eta_{\mu\nu}$ is the d-dimensional Minkowski metric. Which allows us to declare the coordinate system as $X^M = (X^+, X^-, X^\mu) \in \mathbb{RP}^{d-1}$, where as expected X^I still enjoys scale invariance. One can fix the scale invariance by setting $X^+ = 1$:

$$X^+ = 1 \implies X^{-1} + X^{d+1} = 1, \tag{2.2.58}$$

which by virtue of the condition in (2.2.56) implies that $X^{-1} = -x^2$. We could either take the light-cone coordinates $X^M = (1, -x^2, x^\mu)$, or we could solve the system of equations for X^{-1} and X^{d+1}:

$$\begin{aligned}
X^{-1} + X^{d+1} &= 1, \\
X^{-1} - X^{d+1} &= -x^2.
\end{aligned} \tag{2.2.59}$$

In the original coordinates, solving (2.2.59) results in:

$$X^M = \left(X^{-1}, X^\mu, X^{d+1}\right) = \left(\frac{1 - x^2}{2}, x^\mu, \frac{1 + x^2}{2}\right) \tag{2.2.60}$$

In either case, one readily finds

$$\eta_{MN} X_i^M X_j^N = X_i \cdot X_j = -\frac{1}{2} x_{ij}^2. \tag{2.2.61}$$

Much like in (2.1.17), one can recover the non-linear action of the conformal algebra by first performing a SO$(2,4)$ transformation. This also includes scaling transformation, hence we require a further scale transformation (akin to P in (2.1.17)) to lie in the solution to (2.2.59).

Next we consider the relation that the projective hypercone coordinates have to Twistor space coordinates. It is instructive to first consider a similar situation in four dimensions, in which the pertinent isomorphism is $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$ [5]. The useful implication is that given a vectorial representation of the complexified Lorentz group, we can recover the vectorial representation of $\mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$, which corresponds to the fundamental of each algebra. In practice, this was stated in (2.1.15) as

$$x^{\mu} \to x^{\alpha\dot{\alpha}} = (\sigma_{\mu})^{\alpha\dot{\alpha}} x^{\mu}, \tag{2.2.62}$$

where σ are the four two by two Pauli matrices. In a similar sense, we have the isomorphism $\mathfrak{so}(2,4;\mathbb{R}) \cong \mathfrak{su}(2,2;\mathbb{R})$. The object of interest is then the six four by four matrices $(\Gamma_M)^{AB}$ (where M are six-dimensional whilst (A,B) are four-dimensional) which live in the Clifford algebra of $\mathfrak{su}(2,2;\mathbb{R})$ defined by the metric in $\mathfrak{so}(2,4;\mathbb{R})$. This is given by

$$\left\{(\Gamma_M)^{AB}, \left(\bar{\Gamma}_N\right)_{BC}\right\} = 2\eta_{MN}\delta^A_C, \tag{2.2.63}$$

where

$$\left(\bar{\Gamma}_M\right)_{AB} = \frac{1}{2}\epsilon_{ABCD}\left(\Gamma_M\right)^{CD}. \tag{2.2.64}$$

Then the relation between the projective hypercone and twistor coordinates is given by a similar relation to (2.2.62), namely

$$\begin{aligned} X^M &\to X^{AB} = (\Gamma_M)^{AB} X^M \\ X^M &\to \bar{X}_{AB} = \left(\bar{\Gamma}_M\right)_{AB} X^M. \end{aligned} \tag{2.2.65}$$

We see that the definition of $\left(\bar{\Gamma}_M\right)_{AB}$ is consistent with $\bar{X}_{AB} = \frac{1}{2}\epsilon_{ABCD}X^{CD}$. With this knowledge we can work out the relation between the various inner products on both sides. Let us project (2.2.63) with $X_i^M X_j^N \delta_A^C$ (where i and j is some position label, for instance particle number) in which we immediately find

$$\begin{aligned} \left\{(X_i)^{AB}, \left(\bar{X}_j\right)_{BA}\right\} &= -2\epsilon_{ABCD}X_i^{AB}X_j^{CD} = 8X_i \cdot X_j, \\ \implies -\epsilon_{ABCD}X_i^{AB}X_j^{CD} &= 4X_i \cdot X_j \end{aligned} \tag{2.2.66}$$

[5] The three real forms $\mathfrak{so}(2,2;\mathbb{R}), \mathfrak{so}(1,3;\mathbb{R})$ and $\mathfrak{so}(4;\mathbb{R})$ can be recovered from this, (see [12]).

which is consistent with (2.2.54) and (2.2.61). Note that these inner products were both found independently of this relation.

From what we have learnt so far, it also follows that

$$\left\{ (X_i)^{CB}, \left(\bar{X}_j\right)_{BA} \right\} = -x_{ij}^2 \delta_A^C. \tag{2.2.67}$$

One of the main reasons for introducing projective hypercone is more to do with practical ease in various explicit computations (see section 3.4). In particular, some observables can be naturally expressed in basis where the following object appears:

$$\epsilon_{i_1 i_2 i_3 i_4 i_5 i_6} := \epsilon_{M_1 M_2 M_3 M_4 M_5 M_6} X_{i_1}^{M_1} X_{i_2}^{M_2} X_{i_3}^{M_3} X_{i_4}^{M_4} X_{i_5}^{M_5} X_{i_6}^{M_6}. \tag{2.2.68}$$

Whilst this object can be constructed in twistor space, this is a very compact way of writing this object.

Finally, let us conclude this subsection by making some statements about the full superspace. In the full twistor superspace the coordinate is given in (2.2.44), in more common convention written as

$$\mathcal{Z}^{\mathcal{A}} = \begin{pmatrix} \lambda^\alpha \\ \mu_{\dot\alpha} \\ \chi^I \end{pmatrix} \in \mathbb{CP}^{3|4}. \tag{2.2.69}$$

Now, following [24], similar to our discussion we take two such supertwistors by introducing a new index, namely $\mathcal{Z}^{\mathcal{A}} \to \mathcal{Z}_a^{\mathcal{A}}$. We get

$$\mathcal{Z}_a^{\mathcal{A}} = \begin{pmatrix} z_a^A \\ \chi_a^I \end{pmatrix} \sim \begin{pmatrix} \mathbb{I}_2 \\ i x_{\alpha\dot\alpha} \\ \theta_\alpha^I \end{pmatrix}, \tag{2.2.70}$$

where z_a^A is the purely bosonic pair of twistors. To get the right-most hand side, we may use the incidence relation to write $\chi_a^I = \theta_\alpha^I \lambda_a^\alpha$. Then much like in (2.2.49), we can use the GL(2) covariance to take $\lambda_a^\alpha = \delta_a^\alpha$.

Recalling the paragraph after (2.2.46), in which it was found that lightlike separated lines are given by the conditions $(x_{12})_{\alpha\dot\alpha}\lambda^\alpha = 0$ and $(\theta_{12})_\alpha^I \lambda^\alpha = 0$. This implies that $x_{i,i+1}^2 = 0$ and $\left(\theta_{i,i+1}\right)^{\alpha I} \left(x_{i,i+1}\right)_{\alpha\dot\alpha} = 0$ which is a condition found by intersecting supertwistor lines. We finally note that this is a manifestly chiral superspace.

2.3 Harmonic and analytic superspace

In this subsection, we now review harmonic and analytic superspace techniques which we will make use of in chapters 3 and 5. As opposed to twistor superspace, the space-time structure remains fixed here whilst the structure of Grassmann odd variable is subject to change. The main point of constructing such superspaces is so that we may describe certain supermultiplets in an unconstrained way.

The philosophy is as follows; a full generic expansion in all possible Grassmann odd structures in a superfield will correspondingly contain all possible fields that can occur. However, we may indeed have a supermultiplet not containing all of the possible fields. Instead, we would like to consider a subspace in which we may write a completely unconstrained superfield which produces the correct supermultiplet.

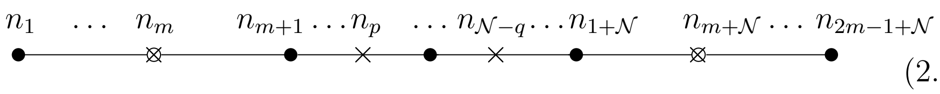
Harmonic and analytic $\mathcal{N}$-extended superspaces (with algebra $\mathfrak{sl}(2m|2n)$) do this by simply reducing the number of Grassmann odd degrees of freedom. There are two ways to do this. Firstly, in harmonic superspace one constructs the space $F_{\mathfrak{p}'} = F_{\mathfrak{p}=\{\alpha_m,\alpha_{m+2n}\}} \times \mathcal{M}$ where $F_{\mathfrak{p}=\{\alpha_m,\alpha_{m+2n}\}}$ is the generalisation of full Minkowski superspace defined by (2.1.32). $\mathcal{M}$ is a coset space of the internal group (in the current context it is $\mathcal{M} = H\backslash\mathrm{SU}(2n)$ for some subgroup H), as well shall see, this can also be constructed from an appropriate choice of simple roots $\mathfrak{p}'$. Analytic superspaces cannot be written in any such form and are written directly as a coset manifold of $\mathfrak{sl}(2m|2n)$ as in previous examples. Here, we will see that it is essentially a Grassmannian of a certain type which will be of use in chapter 5.

Finally, it is worth noting that in practice for certain choices of $\mathcal{M}$, both of these superspaces can be made to be related, namely one can achieve analytic superspace by fixing a specific local coordinate system on $\mathcal{M} = H\backslash\mathrm{SU}(2n)$.

2.3.1 From harmonic superspace...

We will be making extensive use of harmonic superspace in 3.

In general, for the supergroup $\mathrm{SL}(2m|\mathcal{N})$, a $2m$ dimensional $(\mathcal{N}, p, q)$-harmonic superspace is given by the data:



(2.3.71)

Where the crosses are on the odd nodes n_m and $n_{m+\mathcal{N}}$ but also on the even nodes corresponding to the internal group , namely n_p and $n_{\mathcal{N}-q}$. Then the subalgebra in the structural form given by (2.1.29), has the form

$$\mathfrak{p} = \left(\begin{array}{c|ccc|c} \bullet_{m\times m} & & 0_{\mathcal{N}\times m} & & 0_{m\times m} \\ \hline & \bullet_{p\times p} & 0_{\mathcal{N}-p-q\times p} & 0_{q\times p} & \\ \bullet_{m\times\mathcal{N}} & \bullet_{p\times\mathcal{N}-p-q} & \bullet_{\mathcal{N}-p-q\times\mathcal{N}-p-q} & 0_{q\times\mathcal{N}-p-q} & 0_{m\times\mathcal{N}} \\ & \bullet_{p\times q} & \bullet_{\mathcal{N}-p-q\times q} & \bullet_{q\times q} & \\ \hline \bullet_{m\times m} & & \bullet_{\mathcal{N}\times m} & & \bullet_{m\times m} \end{array}\right). \tag{2.3.72}$$

As can be read off, the irreducible representations must then be given by those of

$$\mathfrak{l} = \mathfrak{sl}(m) \oplus \mathfrak{sl}(p) \oplus \mathfrak{sl}(\mathcal{N}-p-q) \oplus \mathfrak{sl}(q) \oplus \mathfrak{sl}(m) \oplus \mathbb{C}^4. \tag{2.3.73}$$

From here, we can see that the space takes the form of $F_{\mathfrak{p}'} = F_{\mathfrak{p}=\{\alpha_m,\alpha_{m+\mathcal{N}}\}} \times \mathcal{M}$ with the subgroup in the real form $H \subset \mathrm{SU}(\mathcal{N})$ generated by the algebra of matrices given by the central block of (2.3.72). The particular choice of p and q that we will use in this thesis is by taking $\mathcal{N} = 2n$, then $p = q = n$. In chapter 3 we will specify to $m = n = 2$ but we will be general in chapter 5 in our application to superconformal partial waves where we will be able to derive very general results.

Let us also remark on the tensor structure of operators in this space, given the $(\mathcal{N}, p, q)$-harmonic superspace, the fundamental $\mathcal{N}$-dimensional vector is projected into p-dimensional ,$(\mathcal{N}-p-q)$-dimensional and q-dimensional fundamental indices, labelled $I \to (a, a', a'')$. The result is that the operator is in general described by five representations:

$$\mathcal{O}_{\mathcal{R}(\alpha)\mathcal{R}'(\dot{\alpha})\mathcal{R}''(a)\mathcal{R}'''(a')\mathcal{R}''''(a'')} \tag{2.3.74}$$

In taking $\mathcal{N} = 2n$ and $p = q = n$, we essentially get an internal structure identical to (2.1.12), namely it has the structure of Minkowski space. However, whilst in conformal Minkowski space we considered infinite dimensional representations, here we consider finite dimensional ones. One finds irreducible representation of the internal coset must transform under $\mathfrak{su}(n) \oplus \mathfrak{su}(n)' \oplus \mathfrak{u}(1)$, hence a coordinate system $u_J^I \in \mathcal{M}$ is assigned as

$$u_J^I = \left(u_J^{+a}, u_J^{-a'}\right). \tag{2.3.75}$$

Here, the lower J-index is the usual anti-fundamental index of $\mathfrak{su}(2n)$, whilst the upper index has been split in accordance with the subalgebra, and the $\pm$-charges are associated to $\mathfrak{u}(1)$. Along with the u_J^I variable we may define a conjugate $\bar{u}_J^I = \left(\bar{u}_{+a}^I, \bar{u}_{-a'}^I\right)$ defined such that

$$u_J^I \bar{u}_K^J = u_K^{+a}\bar{u}_{+a}^I + u_K^{-a'}\bar{u}_{-a'}^I = \delta_K^I. \tag{2.3.76}$$

and since $u_J^I \in H\backslash\mathrm{SU}(2m)$, by unitarity we have $\bar{u} = u^\dagger$, so that the condition above reads $uu^\dagger = \mathbb{I}$. Consequently, we get that

$$\begin{aligned} \bar{u}_{+a}^I u_I^{+b} = \delta_a^b, \quad \bar{u}_{-a'}^I u_I^{-b'} = \delta_{a'}^{b'}, \\ \bar{u}_{-a'}^I u_I^{+b} = \bar{u}_{+a}^I u_I^{-b'} = 0. \end{aligned} \tag{2.3.77}$$

Now, one can use these $(2n \times n)$ matrices to project the SU(4) indices in the Grassmann odd variables of the full Minkowski superspace into the two U(2) indices. Namely they act as,

$$\begin{aligned} \theta_\alpha^I u_I^J &= \theta_\alpha^I \left(u_I^{+a}, u_I^{-a'}\right) = \left(\theta_\alpha^{+a}, \theta_\alpha^{-a'}\right) \\ \bar{\theta}_I^{\dot\alpha} \bar{u}_J^I &= \bar{\theta}_I^{\dot\alpha}\left(\bar{u}_{+a}^I, \bar{u}_{-a'}^I\right) = \left(\bar{\theta}_{+a}^{\dot\alpha}, \bar{\theta}_{-a'}^{\dot\alpha}\right). \end{aligned} \tag{2.3.78}$$

This in turn implies that we can express θ_α^I and $\bar{\theta}_J^{\dot\alpha}$ in terms of the projected components, namely:

$$\begin{aligned} \theta_\alpha^I &= \theta_\alpha^J u_J^K \bar{u}_K^I = \theta_\alpha^{+a}\bar{u}_{+a}^I + \theta_\alpha^{-a'}\bar{u}_{-a'}^I, \\ \bar{\theta}_I^{\dot\alpha} &= \bar{\theta}_J^{\dot\alpha}\bar{u}_K^J u_I^K = \bar{\theta}_{+a}^{\dot\alpha} u_I^{+a} + \bar{\theta}_{-a'}^{\dot\alpha} u_I^{-a'} \end{aligned} \tag{2.3.79}$$

Now, in general, given that this space is $F = F_{\{\alpha_m, \alpha_{m+2n}\}} \times \mathcal{M}$ the complete coordinate system is given by the full Minkowski superspace coordinates and u_J^I. However, we can break apart the various θ and $\bar{\theta}$ variables, so that the complete coordinate system is $x_{\alpha\dot\alpha}$, $\left(\theta_\alpha^{+a}, \theta_\alpha^{-a'}\right)$, $\left(\bar{\theta}_{+a}^{\dot\alpha}, \bar{\theta}_{-a'}^{\dot\alpha}\right)$ and u_B^A. We would like to have superfields which match certain short supermultiplets, which can only happen if these fields are a function of part of the θ variables, for example, functions of θ_α^{+a} but not of $\theta_\alpha^{-a'}$. This is reminiscent of the problem of producing a chiral superfield in full Minkowksi superspace, in which the solution there was a differential constraint. The solution in the current context works in a similar way.

Constraints are built out of acting covariant derivatives on superfields, such covariants derivatives satisfy

$$\left\{D_{\alpha I}, \bar{D}^J_{\dot{\alpha}}\right\} = i\delta^J_I \partial_{\alpha\dot{\alpha}}. \tag{2.3.80}$$

We may then consider projecting these covariant derivatives along any on the harmonics, for example, we may project (2.3.80) with $\bar{u}^I_{+a}u^{-b'}_J$ or $\bar{u}^I_{-a'}u^{+b}_J$, in which we get

$$\begin{aligned}\bar{u}^I_{+a}u^{-b'}_J\left\{D_{\alpha I}, \bar{D}^J_{\dot{\alpha}}\right\} &= \left\{D_{\alpha+a}, \bar{D}^{-b'}_{\dot{\alpha}}\right\} = 0\\ \bar{u}^I_{-a'}u^{+b}_J\left\{D_{\alpha I}, \bar{D}^J_{\dot{\alpha}}\right\} &= \left\{D_{\alpha-a'}, \bar{D}^{+b}_{\dot{\alpha}}\right\} = 0\end{aligned} \tag{2.3.81}$$

So now, if we have a superfield such that it depends on the coordinates

$$\Phi\left(x, \theta^I_\alpha, \bar{\theta}^{\dot{\alpha}}_I\right) = \Phi\left(x, \theta^{+a}_\alpha, \theta^{-a'}_\alpha, \bar{\theta}^{\dot{\alpha}}_{+a}, \bar{\theta}^{\dot{\alpha}}_{-a'}\right), \tag{2.3.82}$$

then by using (2.3.80) we have that if

$$\begin{aligned}D_{\alpha-a'}\Phi\left(x, \theta^{+a}_\alpha, \theta^{-a'}_\alpha, \bar{\theta}^{\dot{\alpha}}_{+a}, \bar{\theta}^{\dot{\alpha}}_{-a'}\right) = 0 &\implies \bar{D}^{+b}_{\dot{\alpha}}\Phi\left(x, \theta^{+a}_\alpha, \theta^{-a'}_\alpha, \bar{\theta}^{\dot{\alpha}}_{+a}, \bar{\theta}^{\dot{\alpha}}_{-a'}\right) = 0\\ &\implies \Phi\left(x, \theta^{+a}_\alpha, \bar{\theta}^{\dot{\alpha}}_{-a'}\right).\end{aligned} \tag{2.3.83}$$

Under the constraint of the first equation (which is to be thought of as analogous to half the chirality condition), it follows that we have dependence on only half of the Grassmann odd variables (like the chirality condition). Fields that satisfy the first constraint in (2.3.83) are referred to as G-analytic [6] [26,19]. Depending on the $\mathfrak{su}(4)$ representation, the resulting field is one of few potential shortened representation.

When doing explicit computations, we should regard the u variables as parametrising the internal manifold. For example, in the $\mathcal{N}=2$ for $p=q=1$ one has $u \in S^2$ hence the u variables may be expressed directly in terms of spherical harmonics [7] [26]. As a result of this, these variables claim a local point in some correlation function calculation. In analogy to (2.2.49), we can construct the SU(2) invariant

$$\epsilon_{ab}u^{+a}_I u^{+b}_J = Y_{IJ}, \tag{2.3.84}$$

[6] There also exists a notion of H-analytic (referring to the subgroup of the internal coset), which is the same condition, but this time with respect to the internal u variables (see [26,19]).

[7] this is the reason for the name of this superspace.

then due to this local invariance we can take [8]

$$u^I_J = \begin{pmatrix} \delta^a_b & y^a_{b'} \\ 0 & \delta^{a'}_{b'} \end{pmatrix} \quad \text{such that} \quad u^{+a}_J = (\delta^a_b, y^a_{b'}) \ , \tag{2.3.85}$$

where we also have $\tilde{y}^{a'a} = \epsilon^{ab} y_{bb'} \epsilon^{b'a'}$. We find

$$y^2_{12} = \det(y_1 - y_2) = \frac{1}{4}\epsilon^{IJKL} Y_{1,IJ} Y_{2,KL}. \tag{2.3.86}$$

Indeed, since $\mathfrak{su}(4) \cong \mathfrak{so}(6)$, which allows us to immediately define a Euclidean $\mathfrak{so}(6)$ vector, Y^M, such that

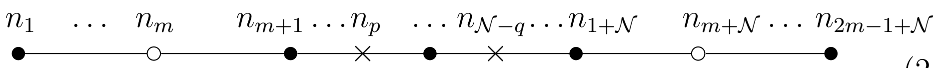
$$Y^M_i Y_{jM} = \frac{1}{2} y^2_{ij}. \tag{2.3.87}$$

whereby Y_{IJ} satisfies the same Clifford algebras as in (2.2.63), but this time on a Euclidean metric. These internal differences of points will appear in the correlation functions that we will study.

2.3.2 ...to analytic superspace

We will be making extensive use of analytic superspace in chapter 5.

Analytic superspace is similar in structure to the harmonic superspace, but instead the number of Grassmann odd variables are decreased. In particular, a $2m$ dimensional $(\mathcal{N}, p, q)$-analytic superspace is given by the data



(2.3.88)

Where as in the harmonic case, there are crosses on the nodes corresponding to the internal group, namely n_p and $n_{\mathcal{N}-q}$, but this time no crosses on the odd nodes. Then

[8]note that we can take $\bar{u}^I_J = \begin{pmatrix} \delta_b{}^a & -y_{b'}{}^a \\ 0 & \delta_{b'}{}^{a'} \end{pmatrix}$ such that $\bar{u}^I_{-a'} = \begin{pmatrix} -y_{b'}{}^a \\ \delta_{b'}{}^{a'} \end{pmatrix}$, from which it follows that $u^{+a}_{i,I} \bar{u}^I_{j,-a'} = (y_{ji})^a_{a'}$

the subalgebra in the structural form given by (2.1.29), has the form

$$\mathfrak{p} = \left(\begin{array}{c|ccc|c} \bullet_{m\times m} & \bullet_{p\times m} & 0_{\mathcal{N}-p-q\times m} & 0_{q\times m} & 0_{m\times m} \\ \hline \bullet_{m\times p} & \bullet_{p\times p} & 0_{\mathcal{N}-p-q\times p} & 0_{q\times p} & 0_{m\times p} \\ \bullet_{m\times\mathcal{N}-p-q} & \bullet_{p\times\mathcal{N}-p-q} & \bullet_{\mathcal{N}-p-q\times\mathcal{N}-p-q} & 0_{q\times\mathcal{N}-p-q} & 0_{m\times\mathcal{N}-q-p} \\ \bullet_{m\times q} & \bullet_{p\times q} & \bullet_{\mathcal{N}-p-q\times q} & \bullet_{q\times q} & \bullet_{m\times q} \\ \hline \bullet_{m\times m} & \bullet_{p\times m} & \bullet_{\mathcal{N}-p-q\times m} & \bullet_{q\times m} & \bullet_{m\times m} \end{array}\right). \tag{2.3.89}$$

As can be read off, irreducible representations must be those of

$$\mathfrak{l} = \mathfrak{sl}\,(m|p) \oplus \mathfrak{sl}\,(\mathcal{N}-p-q) \oplus \mathfrak{sl}\,(q|m) \oplus \mathbb{C}^2, \tag{2.3.90}$$

whereby we see the appearance of superalgebras as opposed to a purely bosonic algebras as in previous cases. It follows that the tensor structure is given by

$$\mathcal{O}_{\mathcal{R}(A)\mathcal{R}'(B')\mathcal{R}''(a)}, \tag{2.3.91}$$

where $\mathcal{R}, \mathcal{R}'$ and $\mathcal{R}''$ are representations of $\mathfrak{sl}\,(m|p)$, $\mathfrak{sl}\,(q|m)$ and $\mathfrak{sl}\,(\mathcal{N}-p-q)$. The A and B' indices here are really superindices and some theory has been developed and applied in [27]. We will explain some details in chapter 5, where it will be more relevant in view of protected and unprotected operators.

As in harmonic superspace, we will take $\mathcal{N} = 2n$ and $p = q = n$, in which the subalgebra in (2.3.89) takes the form

$$\mathfrak{p} = \left(\begin{array}{c|cc|c} \bullet_{m\times m} & \bullet_{n\times m} & 0_{n\times m} & 0_{m\times m} \\ \hline \bullet_{m\times n} & \bullet_{n\times n} & 0_{n\times n} & 0_{m\times n} \\ \bullet_{m\times n} & \bullet_{n\times n} & \bullet_{n\times n} & \bullet_{m\times n} \\ \hline \bullet_{m\times m} & \bullet_{n\times m} & \bullet_{n\times m} & \bullet_{m\times m} \end{array}\right). \tag{2.3.92}$$

Importantly, we see that this is identical in general structure to the conformal Minkowski case studied in (2.1.13), in which some of the information can be translated across by simply taking the 2 by 2 matrices there and enlarging them to $(m|n)$ by $(m|n)$ matrices. We find that coordinates system can be given by [9]

$$X^{\mathcal{AB}} = \begin{pmatrix} \mathbb{I}_{m|n\times m|n} & X^{AA'} \\ 0_{n|m\times m|n} & \mathbb{I}_{n|m\times n|m} \end{pmatrix}, \tag{2.3.93}$$

[9]Note that we have take the factor of i away with respect to the conformal Minkowski case, this is to fit conventions used previously, e.g. in [28]

where the matrix $X^{AA'}$ is an $n|m \times m|n$. It then follows that if an elements of $\mathfrak{sl}(2m|2n)$ is given by

$$g^{\mathcal{A}}_{\mathcal{B}} = \begin{pmatrix} -A^{A}{}_{B} & B^{AB'} \\ -C_{A'B} & D_{A'}{}^{B'} \end{pmatrix}, \tag{2.3.94}$$

We find in complete analogy to the conformal Minkowski space

$$\delta X^{AB'} = B^{AB'} + A^{A}{}_{B} X^{BB'} + X^{AC'} D_{C'}{}^{B'} + X^{AC'} C_{C'D} X^{DB'}. \tag{2.3.95}$$

A particular matrix representation that we can choose is (where we have reversed the Grassmann ordering)

$$X^{AB'} = \begin{pmatrix} x_{\alpha\dot{\alpha}} & \rho^{a}_{\alpha} \\ \bar{\rho}^{\dot{\alpha}}_{a'} & y_{a'}{}^{a} \end{pmatrix}. \tag{2.3.96}$$

This is in analogy with conformal Minkowski space where we had a Grassmannian structure, namely for group SL$(2m)$ and parabolic $\mathfrak{p} = \left(\begin{smallmatrix} \bullet_{m\times m} & 0_{m\times m} \\ \bullet_{m\times m} & \bullet_{m\times m} \end{smallmatrix}\right)$, the result is the space of m-planes in $2m$ dimensions, $\mathrm{Gr}_m(2m)$ [10]. In this case, the situation is generalised to the space of $m|n$-planes in $2m|2n$-dimension, namely the analytic superspace is equivalently described by the coordinate $\mathcal{U}^{\mathcal{A}}_{A}$, whereby the A are $m|n$ indices and the $\mathcal{A}$ is the $2m|2n$ index, then under the left action of GL$(m|n)$ on the A index we get:

$$g^{B}_{A} \mathcal{U}^{\mathcal{A}}_{B} \sim \mathcal{U}^{\mathcal{A}}_{A} = \left(\begin{array}{cc|cc} \mathbb{I}_{m\times m} & x_{\alpha\dot{\alpha}} & 0_{n\times m} & \rho^{a}_{\alpha} \\ \hline 0_{n\times n} & \bar{\rho}^{\dot{\alpha}}_{a'} & \mathbb{I}_{n\times n} & y_{a'}{}^{a} \end{array}\right) = \begin{pmatrix} \mathbb{I}_{m|n\times m|n} & X^{AA'} \end{pmatrix} \tag{2.3.97}$$

Finally, the variables of this analytic superspace can be identified with that of harmonic superspace if we identify

$$\begin{aligned} \theta^{I}_{\alpha} u^{+a}_{I} &= \rho^{a}_{\alpha}, \\ \bar{\theta}^{\dot{\alpha}}_{I} \bar{u}^{I}_{-a'} &= \rho^{\dot{\alpha}}_{a'}. \end{aligned} \tag{2.3.98}$$

We see that simply by looking at the index structure that there are no further harmonic projection onto the full Minkowski superspace that we can identify with analytic coordinates. The main point here is that by definition the analytic superspace has already

[10] Note, that we had previously taken $m = 2$, but the space generalises in the obvious way.

got a truncated Grassmann odd sector. As we will see in chapter 3, this identification allows for the chiral half of the stress tensor supermultiplet to be described by either of these superspaces.

Chapter 3

A Twistor approach to correlation functions in $\mathcal{N} = 4$ SYM

This chapter is based on the paper '*Correlation functions of the chiral stress-tensor multiplet in $\mathcal{N} = 4$ SYM*' by D.Chicherin, R.D, B. Eden, P. Heslop, G.P. Korchemsky, L. Mason and E. Sokatchev [29].

It has been known for some time that observables in planar $\mathcal{N} = 4$ SYM possess highly symmetric structures. This is most notable in the study of scattering amplitudes. The new found structures were first established in the study of scattering amplitudes by employing manifestly on-shell variables for all external data. This simplicity was first observed by Parke and Taylor [30], and the connection to twistor theory was developed by Witten in [61]. This simplicity was then used to develop rules and algorithms to compute higher point amplitudes from lower point amplitudes through new on-shell rules and recursion relations, which are known as CSW and BCFW rules [62,63].

In the meantime, a formal development from the twistor space community, in which a string of papers illuminated a top-down view of these simplified results ([64] is a review). The twistor formalism established an action $S[\mathcal{A}]$ (where $\mathcal{A}$ is a twistor superfield) together with Feynman rules. Whilst these Feynman rules were initially found in view of studying scattering amplitudes, it is indeed possible to use them to compute correlation functions.

Some applications of twistor methods to correlation functions where studied in [33,34] in view of the supercorrelator/superamplitude duality [36,37]. Particular, the construction of certain operators on twistor space was given. More recently, twistor methods have been applied to two-point functions in view of known integrable structures of certain two-point functions [35].

In this chapter, we will review this application of the Feynman rules to correlation functions of the stress-tensor multiplet studied in [29]. The main result is a new method for computing the correlation function of the chiral part of the stress-tensor supermultiplet derived from the twistor action. We gain correlation functions at Born level as a sum over these Feynman diagrams which involve propagators but no integration vertices. This in turn allows us to build new off-shell gauge dependent building blocks which in view of the supercorrelator/superamplitude duality represent an off-shell generalisation to on-shell superconformal invariants [1] used as a basis in tree-level scattering amplitudes (see [32]).

We will be focussing on the planar theory, in which we will begin by giving a short progress review in the study of the correlation functions of the chiral stress-tensor supermultiplet, which will demonstrate the motivation for this study. We then study the main result which is the derivation of the aforementioned off-shell building block. Since the stress-tensor supermultiplet is defined in harmonic superspace. The novel aspect is a projection of the appropriate twistor operator along a harmonic basis. This essentially describes a hybrid superspace.

Upon gaining our main result we can test some of its necessary consistency conditions, namely that it correctly reproduces the scattering amplitude in lightlike limit. Two other consistency checks which can be found in [29] are that a certain concatenation of yet-to-be defined graphical rules produce gauge independent results and finally that the short-distance limit correctly reproduces the required contribution from the operator product expansion.

Finally, we end the chapter with some explicit computations which exemplify not only the results (some of which were newly acquired at the time of writing [29]), but the efficiency in gaining them.

[1] since these are gauge dependent we must transform the gauge parameter accordingly to gain superconformal invariance.

3.1 Review

The stress-tensor supermultiplet $\mathcal{T}$ plays a privileged role in planar $\mathcal{N}=4$ SYM since it comprises of all local conserved currents as well as the Lagrangian of the theory. It is an example of a $\frac{1}{2}$-BPS operator (see chapter 5 for details of the general properties of protected operators). This means that it is protected by supersymmetry, namely half the number of Q_I^α and $\bar{Q}_I^{\dot\alpha}$ annihilate the operator. A further consequence is that contrary to the chiral superfield, the stress-tensor supermultiplet expands in half of the θ and $\bar\theta$ variables. In terms of the representation theory built in chapter 2 this operator is a spin-0, $\Delta=2$ (twist-2) and has $\mathfrak{su}(4)$ representation $[0,2,0]$. The Dynkin nodes corresponding to the superalgebra are given by

$$\mathcal{T}:[0,0,0,2,0,0,0]\,. \tag{3.1.1}$$

The implication of the stress-tensor supermultiplet being protected is that the two- and three-point functions are protected from quantum corrections.

In defining a superfield we follow section 2.3.1 in projecting accordingly, we take

$$\theta_\alpha^{+a}=\theta_\alpha^I u_I^{+a}, \quad \bar\theta^{\dot\alpha}_{-a'}=\bar\theta_I^{\dot\alpha}\bar u^I_{-a'}, \tag{3.1.2}$$

and defining the stress-tensor supermultiplet to be a function of θ^+ and $\bar\theta_-$ only

$$\mathcal{T}=\mathcal{T}\left(x,\theta^+,\bar\theta_-,u\right). \tag{3.1.3}$$

In what follows we will set $\bar\theta_-=0$, hence the superfield is really only a function of θ^+ and is thus chiral. The reason for this is that we would like to demonstrate and use the duality with scattering amplitudes which are chiral [2] in the superspace sense, namely it's a function of χ^I. $\mathcal{T}\left(x,\theta^+,0,u\right)$ is given by the chiral harmonic projection upon the constraint defining the full stress-tensor. In particular, the chiral half of the fermionic part of the super-curvature is given by

$$W^{IJ}(x,\theta)=\phi^{IJ}(x)+2i\sqrt{2}\theta^{\alpha[I}\psi_\alpha^{J]}(x)+i\sqrt{2}\theta_\alpha^{[I}\theta_\beta^{J]}f^{\alpha\beta}(x)+\ldots, \tag{3.1.4}$$

[2] Whilst it is true that this will allow us to exploit the duality it is worth remarking that there is no 'correlator' reason to do this simplification other than focussing on the chiral half of the $\frac{1}{2}$-BPS operator, which is still a valid study.

where the '...' imply coupling constant dependent 'g' contributions which are proportional to the non-abelian contributions.

This superfield is the chiral half of a twist-1 operator in the $\mathfrak{su}(4)$ representation $[0,1,0]$. W^{IJ} in (3.1.4) is a solution to the following constraint equation in Minkowski superspace

$$D^{\alpha}_{K} W^{IJ} = -\frac{2}{3}\delta^{[I}_{K} D^{\alpha}_{L} W^{J]L}, \tag{3.1.5}$$

where $D^{\alpha}_{I} = \frac{\partial}{\partial \theta^{I}_{\alpha}}$, and would otherwise be the usual D^{α}_{I} covariant derivative which includes $\bar{\theta}$. We may project (3.1.5) along the harmonics $\bar{u}^{K}_{-a'} u^{+a}_{I} u^{+b}_{J}$, and since $\bar{u}^{I}_{-a'} u^{+a}_{I} = 0$, we get

$$\bar{u}^{K}_{-a'} D^{\alpha}_{K} W^{IJ} u^{+a}_{I} u^{+b}_{J} = D^{\alpha}_{-a'} W^{+a+b} = 0, \tag{3.1.6}$$

where $W^{+a+b} = W^{IJ} u^{+a}_{I} u^{+b}_{J}$ and we can make a scalar by projecting (3.1.6) with ϵ_{ab}. The result is then the constraint

$$D^{\alpha}_{-a'} W^{++} = 0. \tag{3.1.7}$$

This is a G-analytic constraint as was discussed in (2.3.83). It implies that W^{++} is not an explicit function of $\theta^{-a'}_{\alpha}$, hence we write

$$W^{++} = W^{++}\left(x, \theta^{+}, u\right), \tag{3.1.8}$$

and then in performing an unconstrained expansion in θ^{+} we recapture the chiral half of the supermultiplet.

Then the chiral half of the stress-tensor supermupltiplet is given by

$$\mathcal{T}\left(x, \theta^{+}, 0, u\right) = \operatorname{tr}\left(W^{++}\left(x, \theta^{+}, u\right) W^{++}\left(x, \theta^{+}, u\right)\right). \tag{3.1.9}$$

Now, we can perform an expansion in terms of component operators

$$\begin{aligned}\mathcal{T}(x, \theta^{+}, 0, u) = {}& \mathcal{O}^{++++}(x) + \theta^{+a}_{\alpha} \mathcal{O}^{+++,\alpha}_{a}(x) + (\theta^{+})^{2}_{\alpha\beta} \mathcal{O}^{++,\alpha\beta}(x) \\ & + (\theta^{+})^{2\,ab} \mathcal{O}^{++}_{ab}(x) + (\theta^{+})^{3\,a}_{\alpha} \mathcal{O}^{+,\alpha}_{a}(x) + (\theta^{+})^{4} \mathcal{L}(x)\,,\end{aligned} \tag{3.1.10}$$

Where here we use the notation $(\theta^{+})^{2}_{\alpha\beta} = \theta^{+a}_{\alpha}\theta^{+b}_{\beta}\epsilon_{ab}$, $(\theta^{+})^{2\,ab} = \theta^{+a}_{\alpha}\theta^{+b}_{\beta}\epsilon^{\alpha\beta}$, $(\theta^{+})^{3\,a}_{\alpha} = \theta^{+b}_{\alpha}\theta^{+c}_{\beta}\theta^{+a}_{\gamma}\epsilon_{bc}\epsilon^{\beta\gamma}$ and $(\theta^{+})^{4} = \theta^{+a}_{\alpha}\theta^{+b}_{\beta}\theta^{+c}_{\gamma}\theta^{+d}_{\delta}\epsilon_{bc}\epsilon_{ad}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}$. Each component is given by [36] (we have put the coupling g back in):

$$\begin{aligned}
&\mathcal{O}^{++++} = \mathrm{tr}(\phi^{++}\phi^{++}),\\
&\mathcal{O}_a^{+++,\alpha} = 2\sqrt{2}i\,\mathrm{tr}\left(\psi_a^{+\alpha}\phi^{++}\right),\\
&\mathcal{O}^{++,\alpha\beta} = \mathrm{tr}\left(\psi^{+c(\alpha}\psi_c^{+\beta)} - i\sqrt{2}F^{\alpha\beta}\phi^{++}\right),\\
&\mathcal{O}_{ab}^{++} = -\,\mathrm{tr}\left(\psi_{(a}^{+\gamma}\psi_{b)\gamma}^{+} - g\sqrt{2}[\phi_{(a}^{+J},\bar{\phi}_{+b,J)}]\phi^{++}\right),\\
&\mathcal{O}_a^{+,\alpha} = -\frac{4}{3}\,\mathrm{tr}\left(F_\beta^\alpha\psi_a^{+\beta} + ig[\phi_a^{+J},\phi_{KL}]\psi^{L\alpha}\right),\\
&\mathcal{L} = \frac{1}{3}\,\mathrm{tr}\left\{-\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} + \sqrt{2}g\psi^{\alpha I}[\phi_{IJ},\psi_\alpha^J] - \frac{1}{8}g^2[\phi^{IJ},\phi^{KL}][\phi_{IJ},\phi_{KL}]\right\},
\end{aligned} \tag{3.1.11}$$

where the various SU(4) projections are given by [3]:

$$\begin{aligned}
&\phi_a^{+\,J} = \epsilon_{ab}u_I^{+b}\phi^{IJ}, \qquad \bar{\phi}_{+b,I} = \bar{u}_{+b}^J\phi_{IJ}, \qquad \phi^{++} = -\frac{1}{2}u_I^{+a}\epsilon_{ab}u_J^{+b}\phi^{IJ},\\
&\psi_a^{+\,\alpha} = \epsilon_{ab}u_I^{+b}\psi^{\alpha I}, \qquad \psi^{+\,a\alpha} = u_I^{+a}\psi^{\alpha I}.
\end{aligned} \tag{3.1.12}$$

Now that we have defined the stress-tensor supermultiplet we would like to consider the general form of the correlation function. The n-point supercorrelaton function is given by (where $\mathcal{T}(i) = \mathcal{T}\left(x_i,\theta_i^+,u_i\right)$):

$$G_n(1,\ldots,n) = \langle\mathcal{T}(1)\ldots\mathcal{T}(n)\rangle\,. \tag{3.1.13}$$

This supercorrelator admits an expansion in terms of polynomials in θ^+ , namely if $G_{n;p}$ is a homogeneous polynomial in θ^+ of degree $4p$, then

$$G_n = G_{n;0} + G_{n;1} + \cdots + G_{n;n-4}. \tag{3.1.14}$$

Notice that the expansion truncates to a degree $4(n-4)$ polynomial, which follows from (the chiral half) of the superconformal symmetry. Namely, we have $Q_I^\alpha G_{n;p} = \bar{S}_{\dot\alpha}^I G_{n;p} = 0$ (details in (3.1.2)).

Taking the theory to be an SU(N) gauge theory and the operator to exist in the adjoint representation means that we may take (at ℓ-loop order)

$$G_{n;p} = \sum_{\ell\geq 0} a^{p+\ell}\hat{G}_{n;p}^{(\ell)},$$

$$\text{where the t'Hooft coupling is } a = \frac{g^2N}{4\pi^2} \tag{3.1.15}$$

[3]Here $\phi^{IJ} = \frac{1}{2}\epsilon^{IJKL}\phi_{KL}$, and all symmetrisations are weighted.

where now $\hat{G}^{(\ell)}_{n;p}$ is still homogeneous in θ^+ of order $4p$. Finally, another important aspect of this operator is that it includes the on-shell Lagrangian in the supermultiplet and the main consequence of this is that we may obtain higher-loop integrands from Born level supercorrelation functions. This methodology is called the Lagrangian insertion procedure.

Let us provide some schematic details of how the Lagrangian insertion procedure works, in which a nice guideline is provided in section 7.1 of [66]. In $\mathcal{N}=4$ SYM, the correlator of the operator $\mathcal{T}$ is given by

$$\begin{aligned} G_n &= \int [\mathrm{d}\Phi]\, e^{i\int \mathcal{L}_{\mathcal{N}=4}} \mathcal{T}(1)\dots\mathcal{T}(n) \\ &= G_n^{(0)} + g^2 G_n^{(1)} + g^4 G_n^{(2)} + \dots, \end{aligned} \tag{3.1.16}$$

where Φ represents all of the fields in $\mathcal{N}=4$ SYM. It also follows that

$$g^2 \frac{\partial}{\partial g^2} G_n = g^2 G_n^{(1)} + 2g^4 G_n^{(2)} + \dots, \tag{3.1.17}$$

which implies that the first term is of order g^2 and is the one-loop correlator. We can recapture the same correlator by performing a rescaling of all of the fields. Namely, we take $\Phi \to \Phi/g$ in which the Lagrangian scales as $\mathcal{L}_{\mathcal{N}=4} \to \mathcal{L}_{\mathcal{N}=4}/g^2$ and the new Lagrangian is independent of the coupling. One can similarly rescale the stress-tensor supermultiplet as $\mathcal{T} \to \mathcal{T}/g^2$, where the new stress-tensor supermultiplet is also independent of the coupling. In which case the correlator now reads

$$G_n = \frac{1}{g^{2n}} \int [\mathrm{d}\Phi]\, e^{\frac{i}{g^2}\int \mathcal{L}_{\mathcal{N}=4}} \mathcal{T}(1)\dots\mathcal{T}(n), \tag{3.1.18}$$

however the structure of the correlator in terms of its loop expansion is identical to the second line of (3.1.16), since we also have to scale up the Feynman rules, i.e. propagators. Applying the same derivative gives

$$g^2 \frac{\partial}{\partial g^2} G_n = \frac{1}{g^{2n}} \left(\frac{-i}{g^2}\right) \int d^4x_0 \, \langle \mathcal{T}(1)\dots\mathcal{T}(n)\mathcal{L}_{\mathcal{N}=4}(0)\rangle - nG_n. \tag{3.1.19}$$

It turns out that there are contact terms from the insertion of the kinetic part of the Lagrangian which results in a cancellation with the second term on the right hand side of (3.1.19), in which we get [36]:

$$g^2 \frac{\partial}{\partial g^2} G_n = \frac{1}{g^{2n}} \left(\frac{-i}{g^2}\right) \int d^4x_0 \, \langle \mathcal{T}(1)\dots\mathcal{T}(n)\mathcal{L}(0)\rangle \,. \tag{3.1.20}$$

The cancellation has resulted in the truncation of the complete $\mathcal{N}=4$ SYM Lagrangian insertion to the on-shell action which can be found in the last line of (3.1.11) (which is independent of g due to the stress-tensor rescaling). However, since this operator is in the stress-tensor supermultiplet, we can uplift the on-shell Lagrangian insertion into a Grassmann integration over the full stress-tensor supermultiplet. From (3.1.11) we see that since

$$\mathcal{L}(0) = \int d^4\theta_0^+ \mathcal{T}(0), \tag{3.1.21}$$

it would follow that

$$g^2 \frac{\partial}{\partial g^2} G_n = \frac{-i}{(g^2)^{n+1}} \int d^4x_0 d^4\theta_0^+ \langle \mathcal{T}(1) \dots \mathcal{T}(n)\mathcal{T}(0)\rangle = -i \int d^4x_0 d^4\theta_0^+ G_{n+1}. \tag{3.1.22}$$

Hence, given (3.1.17), the $(\theta^+)^4$ component of the Born level G_{n+1} supercorrelator gives the integrand of the one-loop correction to the G_n supercorrelator. In fact, we can iterate this method to give

$$G_n^{(\ell)} = \frac{(-i)^\ell}{\ell!} \int d^4x_{n+1} \dots d^4x_{n+\ell} d^4\theta_{n+1}^+ \dots d^4\theta_{n+\ell}^+ G_{n+\ell}. \tag{3.1.23}$$

This illustrates an important aspect of this operator and thus motivates us to study its correlators further.

3.1.1 Triality

The mathematical and conceptual structure of observables in $\mathcal{N}=4$ SYM has not only received attention because of their remarkable beauty and new found simplicity but also because some of the observables are related in various limits. Whilst the simplicity in the structure of the observables certainly imply a deeper structure (or potential symmetry), and the fact that the observables are related strongly suggest this.

It was first realised that the scattering amplitude and the expectation value of the Wilson loop are exactly dual to one another in the strong coupling regime [39]. This was initially found by making use of the AdS/CFT correspondence. Meanwhile, a conjecture was made for the n-point all-loop MHV amplitude in [40], in an exponentiated form which has been dubbed the 'BDS ansatz'. By once again making use of the

AdS/CFT correspondence it was found in [41] that there is a discrepancy between the BDS ansatz and the actual answer at six-legs and onwards, this was further confirmed in [42]. Whilst the BDS ansatz is not the full answer, it does encapsulate some of the answer. There was thus a need for a correctional term which is called the 'remainder function', which together with the BDS ansatz gives the full answer. The remainder function was first introduced in [43,44]. Later, came [45,46], which showed that this duality holds at weak coupling. It remains an unproven phenomenon but has provided amazing results, in particular since the Wilson loop is a far more economical object to compute than the scattering amplitude.

We now give the main features of the duality. We take this moment to explain some basic notation of the scattering amplitude and when discussing the duality we focus on the gluonic sector (such that the bosonic Wilson loop will suffice here).

The n-particle superamplitude is given by a function of on-shell momenta p_i and a fundamental SU(4) Grassmann odd parameter χ^I:

$$\mathcal{A}_n = \mathcal{A}_{n;0} + \mathcal{A}_{n;1} + \cdots + \mathcal{A}_{n;n-4}, \tag{3.1.24}$$

where $\mathcal{A}_{n;p}$ is the N^{p}MHV amplitude which has total helicity $4 - n + 2p$ and is a homogeneous polynomial in χ^I of degree $4p$. The momentum variables are the massless (and therefore rank 1) Lorentz vectors $p_{\alpha\dot{\alpha}} = \lambda_{i,\alpha}\tilde{\lambda}_{i,\dot{\alpha}}$. Then with $P = \sum_i^n p_{i\alpha\dot{\alpha}}$, $Q = \sum_{i=1}^n \lambda_{i,\alpha}\chi^A$ and $\langle ij\rangle = \epsilon^{\alpha\beta}\lambda_{i,\alpha}\lambda_{j,\beta}$, we have

$$\mathcal{A}_{n;p} = \underbrace{\frac{\delta^{(4)}(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}}_{\mathcal{A}_{n;\text{tree}}}\hat{\mathcal{A}}_{n;p}(\lambda,\tilde{\lambda},\chi,g,N), \tag{3.1.25}$$

where

$$\hat{\mathcal{A}}_{n;p}(\lambda,\tilde{\lambda},\chi,g,N) = \sum_{\ell\geq 0} a^\ell \hat{\mathcal{A}}^{(\ell)}_{n;p}(\lambda,\tilde{\lambda},\chi), \tag{3.1.26}$$

and $\mathcal{A}^{(0)}_{n;p} = 1$. It is worth mentioning here that whilst scattering amplitudes in $\mathcal{N} = 4$ SYM do not contain UV divergences they do indeed contain IR divergences.

The expectation value of the bosonic Wilson loop (where n is parametrised by the contour) is defined to be

$$\langle W[\mathcal{C}_n]\rangle = \operatorname{tr}\mathcal{P}\exp\left[ig\oint_{\mathcal{C}_n} dx^\mu A^\mu\right] = \sum_{\ell\geq 0}\lambda^\ell W_n^{(\ell)}, \tag{3.1.27}$$

where $W_n^{(0)} = 1$. If we take the contour to be polygonal and defined by n many vertices $x_i^{\alpha\dot{\alpha}}$, whereby the edges are to be thought of as lightlike vectors and should be identified with the momentum vectors

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} = p_i^{\alpha\dot{\alpha}}, \tag{3.1.28}$$

then we may identify

$$\hat{\mathcal{A}}_{n;0}^{(\ell)} = \frac{\mathcal{A}_{n;0}^{(\ell)}}{\mathcal{A}_{n;0}^{(0)}} = W_n^{(\ell)}, \tag{3.1.29}$$

upto some minor details, e.g. the UV Wilson loop regulator must be identified with the IR amplitude regulator. The full superspace duality was considered in [47], by making use of the supersymmetric Wilson loop in twistor space. It is also important to recognise that this is a duality between two objects which are on-shell.

Famously, this duality led to the in depth study into the symmetry structures in which it was found that $\mathcal{N} = 4$ SYM was not only invariant under the superconformal symmetry but also an entirely different set of generators spanning the so-called dual superconformal symmetry algebra [48]. This eventually gave rise to the notion of Yangian structure in the amplitude [49], which has eventually led to widely believed conjecture that planar $\mathcal{N} = 4$ SYM is an integrable field theory.

Next, we review the supercorrelator/superamplitude duality. This was first studied in [31] whereby it was argued that any generic bosonic conformal field theory reproduces the square of the expectation value of the Wilson loop in an appropriate lightlike limit. Importantly, this is to be contrasted with the Wilson loop/amplitude duality, where here we are taking a limit rather than having an exact duality. [4]

Thereafter came the work in [50,51], which coupled the reproduction of the expectation value of the Wilson loop in the lightlike limit of the correlation function with the fact that the Wilson loop is exactly dual to scattering amplitudes. This is non-trivial since it is really the square of the amplitude which is dual to the correlator, and this implies that there are non-trivial relations between the basis of integrands on either side

[4] Linguistically, it would therefore be more appropriate to call this a correspondence, however we will continue to call it a duality.

which increase in complexity as one goes to higher-loops. However, there are indeed non-trivial relations due to the Lagrangian insertion procedure outlined in (3.1.23).

The work of [36,37] developed a supersymmetric extension of the bosonic proposal. Given that the Wilson loop/amplitude duality had been supersymmetrised [47], and considering (3.1.14) and (3.1.24) together leads to the proposal that at the level of the integrand (defining $a = \left(\frac{g^2 N}{4\pi^2}\right)$)

$$\lim_{x_{i,i+1}^2 \to 0} \left(\sum_{\ell \geq 0} \sum_{p=0}^{n-4} a^\ell \frac{G_{n;p}^{(\ell)}}{G_{n;0}^{(0)}} \right) = \left(\sum_{\ell \geq 0} \sum_{p=0}^{n-4} a^\ell \hat{\mathcal{A}}_{n;p}^{(\ell)} \right)^2 . \tag{3.1.30}$$

It's important to note that in (3.1.30) we have performed the scaling $G_{n;p} \to a^{-p} G_{n;p}$ so that the right hand side is not explicitly the correlation function. This is simply so that we have a formula that produces the correct result. In order to compare components, one should then compare Grassmann degree.

In [36,37], various examples of proposal was tested. In particular, there is a two step process; one can use the Lagrangian insertion procedure to express the intergrands of $G_{n;p}^{(\ell+\ell')}$ supercorrelators in terms of the integrands of $G_{n;p+\ell'}^{(\ell)}$ supercorrelators. However, one can use the supercorrelator/superamplitude duality to yield corresponding integrands for $(\ell + \ell')$-loop amplitudes. This is particularly fruitful when dealing with Born level supercorrelators (i.e. $\ell = 0$), in which Born level supercorrelators contain information about higher-loop scattering amplitudes. For example, from the six point Born level supercorrelator ($G_{6;2}^{(0)}$) we may extract the tree level six point N^{2}MHV amplitude ($\hat{\mathcal{A}}_{6;2}^{(0)}$), the integrand of the one-loop five point NMHV amplitude ($\hat{\mathcal{A}}_{5;1}^{(1)}$) and the integrand of the two-loop four point MHV amplitude ($\hat{\mathcal{A}}_{4;0}^{(2)}$).

As has become clear, a single supercorrelator $G_{n;p}^{(\ell)}$ for some fixed parameters ℓ, n and p has a profoundly large amount of information and it appears to be highly relevant to try and understand the structure of these objects.

We summarise this subsection with the diagram in figure 3.1.

3.1.2 Hidden symmetry

Over the last few years an impressively simple but powerful symmetry was discovered, namely for a particular component of the n-point supercorrelation function there exists

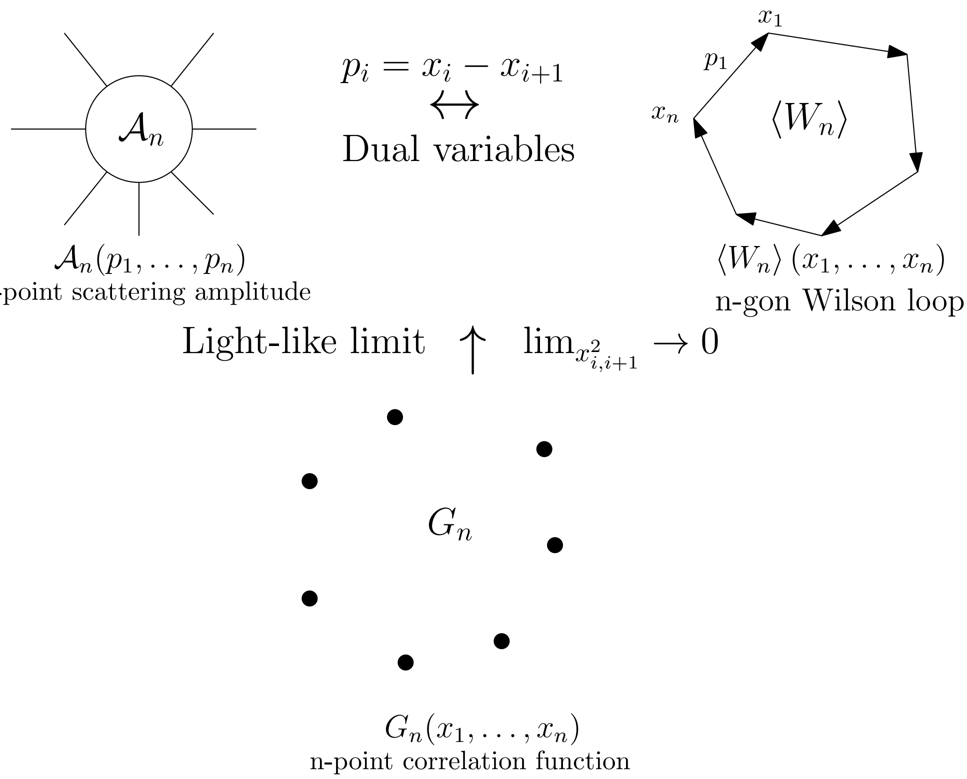


Figure 3.1: A diagrammatic summary of the triality of observables in $\mathcal{N}=4$ SYM.

a full S_n permutation invariance [52] [5]. In this thesis, we will not really need to discuss this very much, however we will take this opportunity to build up some notation regarding the structure of such supercorrelators (in particular so that we can see an amplitude analogy in section 4) whilst including the hidden symmetry for completeness.

As we explained in the previous subsection, the n-supercorrelator is a function of the spacetime and internal coordinates but also the chiral half of the harmonically projected Grassmann odd variables, namely θ^{+a}_{α}. We recall that we have $Q^{\alpha}_{A} G_{n;p} = \bar{S}^{A}_{\dot\alpha} G_{n;p} = 0$. In general, we have the $(Q+\bar{S})$ transformation

$$\delta\theta^{+a}_{\alpha} = \left(\epsilon^{A}_{\alpha} + x^{\dot\alpha}_{\alpha}\bar{\xi}^{A}_{\dot\alpha}\right) u^{+a}_{A}, \tag{3.1.31}$$

and was used in [54,55], since $\left\{Q^{\alpha}_{I}, \bar{S}^{J}_{\dot\alpha}\right\} = 0$ a superconformal invariant $\mathcal{I}_{n;p}$ can be written as

$$\mathcal{I}_{n;p}\left(x,y,\theta^{+}\right) = Q^{8}\bar{S}^{8}\mathcal{J}_{n;p+4}\left(x,y,\theta^{+}\right), \tag{3.1.32}$$

[5]This symmetry is so powerful, it has sometimes been referred to as an integrability approach (see [53])

and is a solution to the constraints $Q_I^\alpha G_{n;p} = \bar{S}^J_{\dot{\alpha}} G_{n;p} = 0$. The object $\mathcal{J}_{n;p+4}\left(x,y,\theta^+\right)$ is a homogeneous polynomial in θ^+ of order $4(p+4)$. An integral representation may be written as

$$\mathcal{I}_{n;p}\left(x,y,\theta^+\right) = \int d^8\epsilon d^8\bar{\xi} e^{\epsilon^A_\alpha Q^\alpha_A + \bar{\xi}^A_{\dot{\alpha}}\bar{S}^{\dot{\alpha}}_A} \mathcal{J}_{n;p+4}\left(x,y,\theta^+\right) = \int d^8\epsilon d^8\bar{\xi} \mathcal{J}_{n;p+4}\left(x,y,\hat{\theta}^+\right), \tag{3.1.33}$$

where $\hat{\theta}^+ = e^{\epsilon^I_\alpha Q^\alpha_I + \bar{\xi}^I_{\dot{\alpha}}\bar{S}^{\dot{\alpha}}_A}\theta^+ = \theta^+ + \delta\theta^+$. It therefore follows that in general, we have

$$G_{n;p} = \sum_i \mathcal{I}_{n;p,(i)}\left(x,y,\theta^+\right) f_{n;p,(i)}(x,y), \tag{3.1.34}$$

where the sum is over all of the possible invariants multiplied by some coefficient functions $f_{n;p,(i)}(x,y)$.

We will omit the most general discussion whilst focussing on the $p = n-4$ invariant, namely the $G_{n,n-4}$ component. From the previous discussion, we recognise that this component has the maximal number of Lagrangian insertions, in fact it is given by

$$\begin{aligned}&(\theta_5^+)^4(\theta_6^+)^4\dots(\theta_n^+)^4\\ &\times\left\langle \mathcal{O}^{++++}(1)\mathcal{O}^{++++}(2)\mathcal{O}^{++++}(3)\mathcal{O}^{++++}(4)\mathcal{L}(5)\mathcal{L}(6)\dots\mathcal{L}(n)\right\rangle + \text{perms}_{12\dots n}.\end{aligned} \tag{3.1.35}$$

Now, we can take (since $\mathcal{J}_{n;n}$ is required to be the maximally nilpotent homogeneous polynomial in θ_i^+ of degree $4n$)

$$\mathcal{J}_{n;n}\left(x,y,\theta^+\right) = g_n(x,y)\prod_{i=1}^n\left(\theta_i^+\right)^4, \tag{3.1.36}$$

for some polynomial $g_n(x,y)$. However, for the full answer to the maximally nilpotent component of the correlator we can always absorb this function into the coefficient function, such that the dependence of $G_{n;n-4}$ on the kinematics is given by

$$G_{n,n-4} \sim f_n(x,y)\int d^8\epsilon d^8\bar{\xi}\prod_{i=1}^n\left(\hat{\theta}_i^+\right)^4. \tag{3.1.37}$$

The key point is that $G_{n,n-4}$ has full S_n permutation symmetry by virtue of crossing symmetry in G_n since all the operators are identical. However, $\mathcal{J}_{n,n}$ also has S_n permutation symmetry, thus it follows that $f_n(x,y)$ must also have S_n point symmetry. Note that in the non-maximally nilpotent component correlators (i.e. $p < n-4$) we have homogeneous polynomials of less than degree $4n$, which means we that $\mathcal{J}_{n,p}$ is generally some linear combination of objects and it follows that we do not have only a single coefficient function which we can say is permutation invariant, rather we may have some non-trivial sum of unrelated terms.

Let us now unpack the integral invariant $\int d^8\epsilon d^8\bar{\xi}\prod_{i=1}^n\left(\hat{\theta}_i^+\right)^4$. A useful gauge associated to the $(Q+\bar{S})$ symmetry is given by $\theta_1^+=\theta_2^+=\theta_3^+=\theta_4^+=0$. Then one finds [52]:

$$\mathcal{I}_{n,n-4}=\prod_{i<j}^{4}x_{ij}^2R(1,2,3,4)\prod_{i=5}^{n}\left(\theta_i^+\right)^4,$$

$$\begin{aligned}\text{where } R(1,2,3,4)=&\ g_{12}g_{23}g_{34}g_{14}\left(x_{13}^2x_{24}^2-x_{12}^2x_{34}^2-x_{14}^2x_{23}^2\right)\\&+g_{12}g_{24}g_{43}g_{31}\left(x_{14}^2x_{23}^2-x_{12}^2x_{34}^2-x_{13}^2x_{24}^2\right)\\&+g_{13}g_{32}g_{24}g_{41}\left(x_{12}^2x_{34}^2-x_{14}^2x_{23}^2-x_{13}^2x_{24}^2\right)\\&+g_{12}^2g_{34}^2x_{12}^2x_{34}^2+g_{13}^2g_{24}^2x_{13}^2x_{24}^2+g_{14}^2g_{23}^2x_{14}^2x_{23}^2\end{aligned}\tag{3.1.38}$$

where $g_{ij}=\frac{y_{ij}^2}{x_{ij}^2}$. From this we find the conformal weight of $\mathcal{I}_{n,n-4}$ is (-2) at each point and the U(1) charge of $(+4)$ at each point. Since G_n must have conformal weight $(+2)$ and U(1) charge $(+4)$ we deduce from (3.1.37), that $f_n(x,y)$ does not depend on the internal structure and must have conformal weight $(+4)$ at each point. It turns out that this is enough information to deduce the general structure of this particular component, at the very least an ansatz can be given in which the basis are some permutation invariant functions. A consequence of the OPE is a further constraint that the singularity structure of $f_n(x,y)$ must only appear through factors of propagators, see [38] for details. For example, at five points we have

$$f_5(x,y)=f_5(x)=\frac{1}{\prod_{1\le i<\le 5}x_{ij}^2}.\tag{3.1.39}$$

This example is particularly appropriate at exemplifying the correspondences with amplitudes in the lightlike limit, namely it is clear the the four-point cyclic lightlike limit of $R(1,2,3,4)$ is the four point tree level correlator. Then this leaves multiplying the pre-factor of $R(1,2,3,4)$ in (3.1.38) by $f_5(x)$, the result is

$$\begin{aligned}\lim_{\left\{x_{12}^2,x_{23}^2,x_{34}^2,x_{41}^2\right\}\to 0}G_{5;1}&=\lim_{\left\{x_{12}^2,x_{23}^2,x_{34}^2,x_{41}^2\right\}\to 0}f_5(x)\prod_{i<j}^{4}x_{ij}^2R(1,2,3,4)\left(\theta_5^+\right)^4\\&=G_{4;\text{tree}}\frac{1}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}\left(\theta_5^+\right)^4,\end{aligned}\tag{3.1.40}$$

where we observe that

$$\frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{3.1.41}$$

is the famous massless box function which is the four point one-loop amplitude [37].

Let us conclude this short review by bringing together the three important factors in the computational aspects of correlation functions. We have in the first instance the Lagrangian insertion procedure described in (3.1.23), followed by the correlator/amplitude duality described in (3.1.30) and finally hidden symmetry construction in this subsection. Suppose that we were interested in higher-loop scattering amplitudes, then a huge amount of information can be obtained by looking at the maximally nilpotent component ($G^{(0)}_{n,n-4}$) of the n-point Born level supercorrelator. Given some choice of ℓ (there are non-unique choices for large n), we can find the $G^{(\ell)}_{n-\ell,n-4+\ell}$ component from which we can find an amplitude via the light-limit.

3.2 Correlation functions in twistor space

In section 2.2, we spent time reviewing some technical details regarding twistor space whilst in the previous section we have provided some details about what to expect from these particular correlation functions. We can now put everything together and look towards using twistor space to provide a new approach towards computing correlation functions.

3.2.1 $\mathcal{N}=4$ SYM on twistor space

In this subsection we provide sufficient twistor technology for our purposes in the computation of correlation functions in $\mathcal{N}=4$ SYM theory. The fields of $\mathcal{N}=4$ SYM theory are described in projective twistor space by a superfield $\mathcal{A}(z,\bar{z},\chi)$. Since, the space is itself complex, we have two unrelated coordinates z^A and $\bar{z}^A$ in which $\mathcal{A}(z,\bar{z},\chi)$ is a $(0,1)$-form. As a superfield, it admits an expansion in the Grassmann odd coordinates

$$\begin{aligned}\mathcal{A}(z,\bar{z},\chi) = a(z,\bar{z}) + \chi^I\tilde{\gamma}_I(z,\bar{z}) + \frac{1}{2}\chi^I\chi^J\phi_{IJ}(z,\bar{z})\\ + \frac{1}{3!}\epsilon_{IJKL}\chi^I\chi^J\chi^K\gamma^L(z,\bar{z}) + \frac{1}{4!}\epsilon_{IJKL}\chi^I\chi^J\chi^K\chi^L g(z,\bar{z})\,,\end{aligned} \tag{3.2.42}$$

where $a(z,\bar{z})$ and $g(z,\bar{z})$ are to be identified with the self-dual and anti-self-dual states of the gluon (or positive and negative helicity states), $\phi_{IJ}(z,\bar{z})$ is the six scalars whilst $\tilde{\gamma}_I(z,\bar{z})$ and $\gamma^I(z,\bar{z})$ are the four fermions and their conjugates. It was shown in [56] that the following action reproduces the known space-time $\mathcal{N}=4$ SYM theory

$$S[\mathcal{A}] = \int_{\mathbb{CP}^{3|4}} \mathcal{D}^{3|4}\mathcal{Z} \wedge \mathrm{tr}\left(\frac{1}{2}\mathcal{A}\,\bar{\partial}\mathcal{A} + \frac{1}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right) + \int d^4x\, d^8\theta\, L_{\mathrm{int}}(x,\theta)\,, \quad (3.2.43)$$

where $\mathcal{D}^{3|4}\mathcal{Z} = \frac{1}{4!}\epsilon_{ABCD}z^A dz^B dz^C dz^D d^4\chi$ is the integration measure on the complex projective space and

$$L_{\mathrm{int}}(x,\theta) = g^2\left[\ln\det(\bar{\partial}-\mathcal{A}) - \ln\det\bar{\partial}\right]. \quad (3.2.44)$$

We note that generally $\mathcal{Z}^A$ is the full supertwistor, and we will soon double these up by putting another lower GL(2) index as in (2.2.70), namely $\mathcal{Z}^{\mathcal{A}}_{\alpha}$. Given the action we require a superpropagator and super interaction vertices in order to gain a set of Feynman rules that we may use. The theory defined in (3.2.43) is a gauge theory and as a result requires gauge fixing. A useful choice is the so-called axial gauge, in which the component of the superfield $\mathcal{A}$ in the direction of an arbitrary reference supertwistor $\mathcal{Z}_*$ vanishes.

Deriving these are quite involved tasks and so we only aim to provide the basic results here whilst referring the interested reader to [57,58,59]. Following [57,58,59], to gain the propagator one is required to solve

$$\bar{\partial}\Delta\left(\mathcal{Z}_1,\mathcal{Z}_2\right) = \bar{\delta}^{3|4}\left(\mathcal{Z}_1,\mathcal{Z}_2\right) := \int \frac{du}{u}\bar{\delta}^{4|4}\left(\mathcal{Z}_1 + u\mathcal{Z}_2\right), \quad (3.2.45)$$

where $\bar{\delta}^{p|q}$ is a 'distributional form' [57] [6]. The solution to the propagator equation is

$$\begin{aligned}\Delta^{ab}\left(\mathcal{Z}_1,\mathcal{Z}_2\right) &= \left\langle \mathcal{A}^a\left(\mathcal{Z}_1\right)\mathcal{A}^b\left(\mathcal{Z}_2\right)\right\rangle = \bar{\delta}^{2|4}\left(\mathcal{Z}_1,\mathcal{Z}_*,\mathcal{Z}_2\right)\delta^{ab} \\ &= \int \frac{ds}{s}\frac{dt}{t}\bar{\delta}^{4|4}\left(s\mathcal{Z}_1 + t\mathcal{Z}_2 + \mathcal{Z}_*\right)\delta^{ab}, \end{aligned} \quad (3.2.46)$$

where the a and b indices are fundamental gauge indices. We see the explicit emergence of $\mathcal{Z}_*$ in the propagator. All computations will involve some sum over contributing diagrams, and whilst individual diagrams may be $\mathcal{Z}_*$-dependent the corresponding sum must not be.

[6] For all intents and purposes this is essentially a delta function.

Turning to the interactions, all interaction vertices are produced by L_{int} in the axial gauge. First, one must take $\ln\det\left(\bar{\partial}-\mathcal{A}\right)-\ln\det\bar{\partial}=\ln\det\bar{\partial}\left(1-\bar{\partial}^{-1}\mathcal{A}\right)-\ln\det\bar{\partial}=\ln\det\left(1-\bar{\partial}^{-1}\mathcal{A}\right)$. Then using $\ln\det m=\operatorname{tr}\ln m$, we can perform an expansion in the object $\bar{\partial}^{-1}\mathcal{A}$.

Via [47], one finds that

$$L_{\text{int}}(x,\theta)=-g^2\sum_{n\geq 2}\frac{1}{n}\operatorname{tr}\left(\bar{\partial}^{-1}\mathcal{A}\ldots\bar{\partial}^{-1}\mathcal{A}\right) \tag{3.2.47a}$$

$$=-g^2\sum_{k\geq 2}\frac{1}{k}\int\frac{\operatorname{tr}\left(\mathcal{A}(\mathcal{Z}(\sigma_1))\wedge D\sigma_1\ldots\mathcal{A}(\mathcal{Z}(\sigma_k))\wedge D\sigma_k\right)}{\langle\sigma_1\sigma_2\rangle\ldots\langle\sigma_k\sigma_1\rangle}\,, \tag{3.2.47b}$$

where $D\sigma_i=\langle\sigma_i,d\sigma_i\rangle\equiv\epsilon_{ab}\sigma_i^a d\sigma_i^b$ is the projective measure and

$$\langle\sigma_i\sigma_j\rangle=\epsilon_{ab}\sigma_i^a\sigma_j^b\,. \tag{3.2.48}$$

In the second relation in (3.2.47a) the superfields are integrated along the line in twistor space $\mathcal{Z}(\sigma_i)=\mathcal{Z}_1\sigma_i^1+\mathcal{Z}_2\sigma_i^2$ parameterised by coordinates $\sigma_i^a\equiv(\sigma_i^1,\sigma_i^2)$ with two reference points $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of the form (2.2.69) whereby under the incidence relation we have the same $x^{\alpha\dot{\alpha}}$ and $\theta^{A\alpha}$ but different λ_α.

Now, to make contact with our correlator story, we make the following assertion

$$\mathcal{T}(x,\theta^+,u)=\int d^4\theta^-\,L_{\text{int}}(x,\theta), \tag{3.2.49}$$

where $\theta^-=\theta_\alpha^{-a'}=\theta_\alpha^I u_I^{-a'}$ is the harmonic projection of the θ variable which we do not take to be within the stress tensor supermultiplet. We now justify our assertion.

Suppose that we do indeed have such an operator $\mathcal{T}(x,\theta^+,u)$ which is implicitly defined through the superfield $\mathcal{A}$ in some way. Then we can define the correlation function, whose corresponding action is (3.2.43), namely in defining

$$G_n=\langle\mathcal{T}(1)\ldots\mathcal{T}(n)\rangle\,, \tag{3.2.50}$$

we find that

$$g^2\frac{\partial}{\partial g^2}G_n=i\int d^4x_{n+1}d^8\theta_{n+1}\,\langle\mathcal{T}(1)\ldots\mathcal{T}(n)L_{\text{int}}\left(x_{n+1},\theta_{n+1}\right)\rangle\,. \tag{3.2.51}$$

This expression bears a resemblance with (3.1.20), in fact we can identify these if we separate the full θ integration into its harmonic projections, namely we can do $\int d^4\theta^+\int d^4\theta^-=\int d^8\theta\det(u)$, where is $\det(u)$ the Jacobian when projecting the full θ

dependence into $\theta_\alpha^{+a} = \theta_\alpha^I u_I^{+a}$ and $\theta_\alpha^{-a'} = \theta_\alpha^I u_I^{-a'}$. Since, $u \in \mathrm{SU}(4)$, we get $\det(u) = 1$, hence in putting all this into (3.2.51) and imposing that we should get (3.1.22), our assertion naturally arises

$$g^2 \frac{\partial}{\partial g^2} G_n = i \int d^4x_{n+1} d^4\theta_{n+1}^+ \left\langle \mathcal{T}(1) \ldots \mathcal{T}(n) \underbrace{\int d^4\theta_{n+1}^- L_{\text{int}}\left(x_{n+1}, \theta_{n+1}\right)}_{:=\mathcal{T}(n+1)} \right\rangle ,$$

$$= \int d^4x_{n+1} d^4\theta_{n+1}^+ \left\langle \mathcal{T}(1) \ldots \mathcal{T}(n+1) \right\rangle = i \int d^4x_{n+1} d^4\theta_{n+1}^+ G_{n+1}. \quad (3.2.52)$$

Indeed we find (3.2.49), from which it follows that a twistor representation of our correlators of interest is given by

$$G_n = \int \prod_{i=1}^n d^4\theta_i^- \left\langle L_{\text{int}}(1) \ldots L_{\text{int}}(n) \right\rangle . \quad (3.2.53)$$

In practice, a correlation function is a set of twistor line X_i which correspond to space-time points where the operators lie. We plug in (3.2.47a), and at the lowest order in g^2, we may simply wick contract the fields to produce the propagators in (3.2.46).

3.2.2 Feynman rules from twistor space

In this subsection we will introduce, explain and then apply a graphical formalism for computing correlation functions in twistor superspace. In principle, this new graphical formalism is to be thought of as an off-shell generalisation of the rules used for the scattering amplitude via the Wilson loop duality used in [47,60].

The fundamental operator is $\mathcal{A}(\mathcal{Z}(\sigma))$ whose dependence of σ is through $\mathcal{Z}(\sigma) = \mathcal{Z}_\alpha^{\mathcal{A}} \sigma^\alpha$. Whilst in space-time it sits on a point, in twistor space it sits on a line parametrised by two reference points defined by $\mathcal{Z}_\alpha^{\mathcal{A}} \sigma^\alpha$, which identify the same space-time point through the incidence relation. Space-time interactions occur when multiple fields occupy the same point instantaneously. Diagrammatially, the propagation of a field occur when propagators are emitted and absorbed by a twistor line, whilst interactions are given by vertices with many propagators emitted from it. Figure 3.2 presents such diagrams whilst giving a truncated diagram which is made of bullets and lines.

In general the correlation function in (3.2.53) is therefore a collection of non-intersecting lines and bullets. An example of what such a contributing diagram is given by figure 3.3.

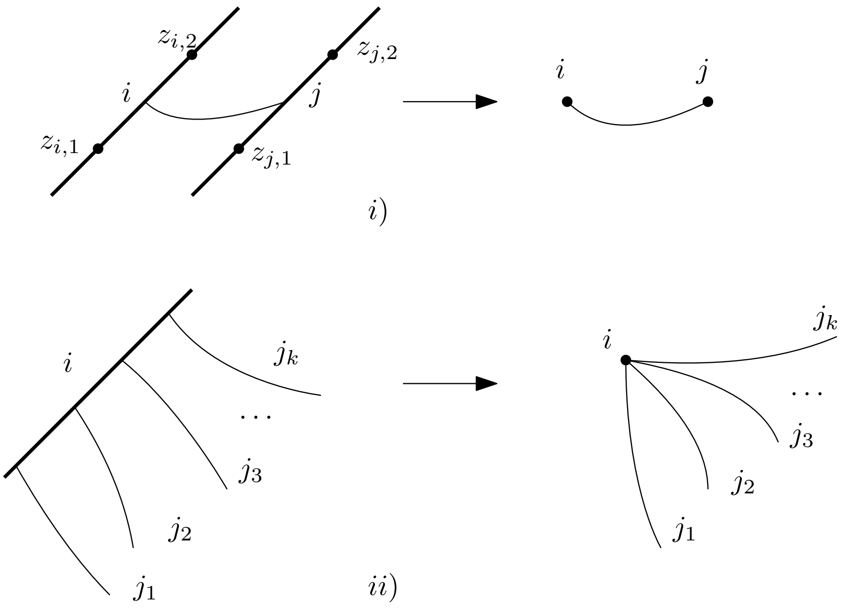


Figure 3.2: ***i)*** The diagram representing a superpropagator from twistor line i to j. ***ii)*** An interaction vertex. In both cases, bold lines are twistor lines whilst faint lines are propagators.

As a quick example we can work out how a propagator contributes to a full Feynman graph like figure 3.3. We consider an arbitrary single propagator extended between two twistor lines X_i and X_j. We denote integration variable σ_{ij} to be associated to the propagator emitted at point i and absorbed and point j [7], and we neglect fermionic integration and gauge indices here:

$$\int \langle \sigma_{ij} d\sigma_{ij} \rangle \langle \sigma_{ji} d\sigma_{ji} \rangle \bar{\delta}^{2|4} \left(\mathcal{Z}_i \left(\sigma_{ij} \right), \mathcal{Z}_* \mathcal{Z}_j \left(\sigma_{ji} \right) \right) \delta^{ab}, \tag{3.2.54}$$

with $\bar{\delta}^{2|4} \left(\mathcal{Z}_i \left(\sigma_{ij} \right), \mathcal{Z}_* \mathcal{Z}_j \left(\sigma_{ji} \right) \right) = \int \frac{ds}{s} \frac{dt}{t} \bar{\delta}^{4|4} \left(s\mathcal{Z}_1(\sigma_{ij}) + t\mathcal{Z}_2(\sigma_{ij}) + \mathcal{Z}_* \right)$. We can perform the change of variables $\sigma_{ij} \to s\sigma_{ij}$ and $\sigma_{ji} \to t\sigma_{ji}$ in which the result is

$$\int d^2\sigma_{ij} \int d^2\sigma_{ji} \bar{\delta}^{4|4} \left(\sigma_{ij}^{\alpha} \mathcal{Z}_{i,\alpha} + \sigma_{ij}^{\alpha} \mathcal{Z}_{j,\alpha} + \mathcal{Z}_* \right), \tag{3.2.55}$$

This tell us that the projective measures can always be absorbed into the integration measures of the propagator to give a full two dimensional integral.

[7] Its worth noting that this notation prescription of ‘emitting’ and ‘absorbing’ is strictly for notational ease as it helps organise integration variables, however it is physically vacuous as otherwise it would imply some notion of orientation which the lines do not have

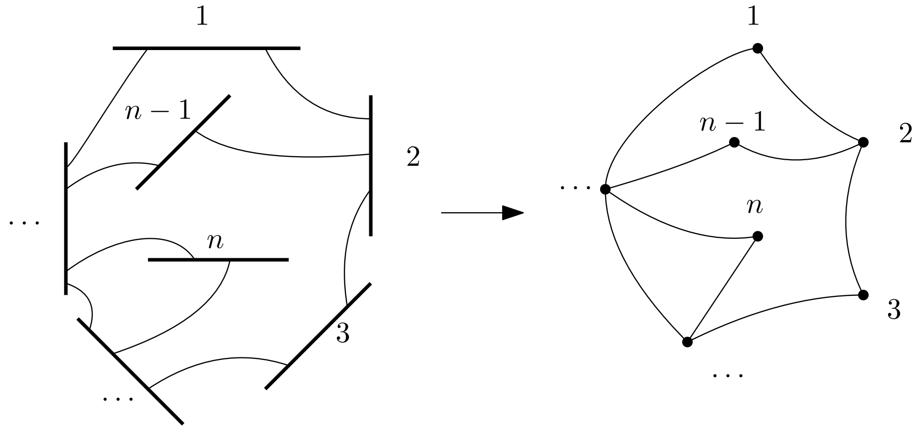


Figure 3.3: A contributing Feynman diagram to an n-point correlation function in (3.2.53). Each bold lines corresponds to a twistor line parametrised by a pair of twistors, and therefore correspond to a point in space-time (x,θ), whilst faint lines are superpropagators.

Putting together everything we have learnt so far, in particular together with (3.2.46) and (3.2.47a), we gain the following *Feynman rules*:

- To each line connecting vertices i and j we associate two pairs of spinor variables σ_{ij}^{α} and σ_{ji}^{α} (with $\alpha = 1, 2$). They define the coordinates of the end points $\sigma_{ij}^{\alpha}\mathcal{Z}_{i,\alpha}$ and $\sigma_{ji}^{\alpha}\mathcal{Z}_{j,\alpha}$ belonging to the ith and jth lines, respectively, in projective twistor space,

- A propagator connecting vertices i and j produces a graded delta function

$$\delta^{a_i a_j}\delta^{4|4}(\mathcal{Z}_* + \sigma_{ij}^{\alpha}\mathcal{Z}_{i,\alpha} + \sigma_{ji}^{\alpha}\mathcal{Z}_{j,\alpha})$$

  with a_i and a_j being $SU(N)$ colour indices,

- Each vertex comes with the following factor

$$\frac{-\operatorname{tr}(T^{a_{j_1}}T^{a_{j_2}}\cdots T^{a_{j_k}})}{\prod_{\ell=1}^{k}\langle\sigma_{ij_\ell}\sigma_{ij_{\ell+1}}\rangle}$$

  (with $j_{k+1} \equiv j_1$ and $\langle\sigma_{ij_\ell}\sigma_{ij_{\ell+1}}\rangle$ given by (3.2.48)). Since $\operatorname{tr}(T^{a_j}) = 0$, we must have at least two lines coming from each vertex,

- Finally, at each vertex $i = 1, \ldots, n$ we have to perform an integration

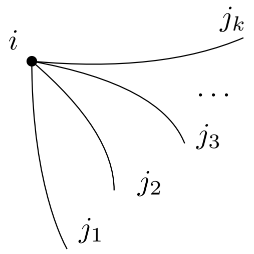
$$\int d^2\sigma_{ij_1}\ldots d^2\sigma_{ij_k}$$

over the σ-parameters of all lines attached to that vertex and, in addition, integrate out half of the Grassmann variables by $\int d^4\theta_i^-$.

Diagrammatically, the rules are summarised as in figure 3.4.

$$\delta^{a_i a_j}\bar{\delta}^{4|4}\left(\sigma^{\alpha}_{ij}\mathcal{Z}_{i,\alpha}+\sigma^{\alpha}_{ji}\mathcal{Z}_{j,\alpha}+\mathcal{Z}_*\right)$$

i)



$$-\int d^2\sigma_{ij_1}\dots d^2\sigma_{ij_k}\frac{\mathrm{tr}\left(T^{a_{j_1}}\dots T^{a_{j_k}}\right)}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\langle\sigma_{ij_2}\sigma_{ij_3}\rangle\dots\langle\sigma_{ij_k}\sigma_{ij_1}\rangle}$$

ii)

Figure 3.4: ***i)*** The Feynman rule for the superpropagator and ***ii)*** the Feynman rules for interaction vertices. In addition to this we must perform the overall integral $\prod_{i=1}^n d^4\theta^-$.

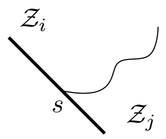
Finally, now that we know how to compute a Feynman graph, we would like to know what graphs to compute. To do this we simply need to look at the Grassmann degree of a graph. Since we require at least two propagators to each vertex, we minimally require as many propagators as there are vertices. Denoting the number of propagators as q and number of vertices as n, we take $q = n + p$ many propagators. Each propagator is Grassmann degree 4, but we perform $4n$ many fermionic integration thus the Grassmann degree of any given diagram is $4q - 4n = 4(n + p) - 4n = 4p$. So, for example if we want to compute the maximally nilpotent component of the five-point Born level supercorrelator, we require all graphs with five vertices but six propagators. Since the component $G_{n;p}$ has Grassmann degree $4p$, it follows that $G_{n;p}$ is equal to the sum of all diagrams with n vertices and $n + p$ propagators.

Before proceeding to look at the correlator in more detail let us very briefly discuss the Feynman rules for scattering amplitudes. Recalling figure 2.1, we stress the critical difference of twistor space graphs when discussing off-shell and on-shell physics. To understand the graphical difference we are required to go back to the twistor lines and propagators graphical rules (as opposed to bullets and edges) that are on left hand side of figure 3.3 and use this form in conjunction with figure 2.1. Correlation functions are built from off-shell external data and thus contain twistor lines that do not intersect.

Scattering amplitudes contain on-shell external data and thus have intersecting twistor lines. When dealing with loop level amplitudes, one has a twistor line associated to the variable in the various loops and these are off-shell and therefore cannot intersect with any other line. An ℓ-loop level scattering amplitude has ℓ off-shell points which have ℓ-many non-intersecting lines.

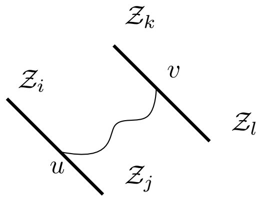
A consequence is that for any twistor line in a scattering process associated to the loop level variable, the correlator Feynman rules that we have studied in figure 3.4 are used. However, for on-shell external lines we require a further rule for the insertion of a single $\mathcal{A}$ superfield. This is simply because in scattering processes we scatter fundamental states which are in the $\mathcal{A}$ superfield as opposed to being in any composites of $\mathcal{A}$.

The new rule is simply that for a twistor line which emits a single propagator:



$$= \int \frac{ds}{s}, \tag{3.2.56}$$

for some $\mathcal{Z}(s) = \mathcal{Z}_i + s\mathcal{Z}_j$. We can consider propagators joined by two twistor lines, namely given $\mathcal{Z}(u) = \mathcal{Z}_i + u\mathcal{Z}_j$ and $\mathcal{Z}(v) = \mathcal{Z}_k + v\mathcal{Z}_l$, a contribution to the so-called NMHV amplitude is given by



$$\begin{aligned} &= \int \frac{du}{u}\frac{dv}{v}\Delta(\mathcal{Z}(u), \mathcal{Z}_*, \mathcal{Z}(v)) \\ &= \int \frac{du}{u}\frac{dv}{v}\frac{ds}{s}\frac{dt}{t}\bar{\delta}^{4|4}\left(s\mathcal{Z}(u) + t\mathcal{Z}(v) + \mathcal{Z}_*\right), \end{aligned} \tag{3.2.57}$$

and we will return to this computation in section 4.1.1, where we will compute the full n-point NMHV scattering amplitude at tree level.

These are in fact the rules used in the Wilson loop representation of the scattering amplitude used in [47,60].

3.2.3 From lower components to higher components

Having described the main method of computation, we will apply the formalism as a first test to the lowest component of the supercorrelation function, namely the n-correlator for the $\frac{1}{2}$-BPS operator $\mathcal{O}^{++++}$. Upon doing so, we will find that a direct

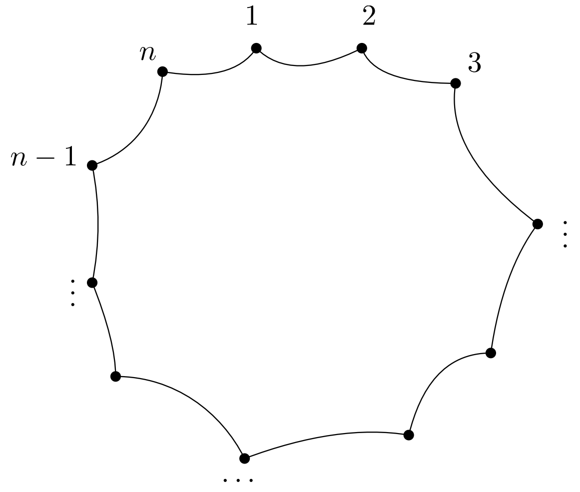


Figure 3.5: A contribution to $G_{n;0}$

applications of the aforementioned Feynman rules for higher components with non-trivial Grassmann oddness is difficult in general, hence we restructure the Feynman rules to new building blocks. This will be the main result.

Following the previous section, the Feynman graphs required to compute $G_{n;0}$ are all graphs with n vertices and n propagators. Since, every vertex requires at least two propagators, there is a structurally unique graph given by figure 3.5.

In addition to this we must sum over permutation of points (since the correlator itself has an n-point bosonic permutation symmetry) and sum over only connected graphs (otherwise we gain those which correspond to products of lower point graphs). Let us use the prescription defined in the previous section and apply it to figure 3.5, in which we get [8]

$$G_{n;0} = (N^2-1)\prod_{i=1}^{n}\int d^4\theta_i^- \int \frac{d^2\sigma_{i\ i-1} d^2\sigma_{i\ i+1}}{\langle\sigma_{i\ i-1}\sigma_{i\ i+1}\rangle^2}\bar{\delta}^{4|4}(\sigma^{\alpha}_{i\ i-1}\mathcal{Z}_{i,\alpha} + \sigma^{\alpha}_{i-1\ i}\mathcal{Z}_{i-1,\alpha} + \mathcal{Z}_*)$$
$$+ (S_n\text{−perm}), \tag{3.2.58}$$

In evaluating this result, we will need to use some of the machinery built in section 2.2. We begin by taking $\mathcal{Z}_*^{\mathcal{A}} = \left(z_*^A, \chi_*^I\right)$ and $\mathcal{Z}_{i,\alpha}^{\mathcal{A}} = \left(z_{i,\alpha}^A, \theta_{i,\alpha}^I\right)$. The required $g \in \mathrm{GL}(2)$ fixing that gives the form of (2.2.70) comes from acting $\mathcal{Z}_{i,\beta} \to g_\gamma{}^\delta \mathcal{Z}_{i,\delta}$. This variation can be compensated in (3.2.58) by the change of the integration variable $\sigma_{ik}^\beta \to (g^{-1})^\beta{}_\delta \sigma_{ik}^\delta$

[8]We will generally omit all factors associated with color from here on out.

of which the entire object remains unchanged. Let us first perform the fermionic integration, we look at a particular piece of the calculation before generalising to the rest. At points 1 and 2 the fermionic contribution to (3.2.58) is given by (setting $\chi_* = 0$):

$$\int d^4\theta_1^- \int d^4\theta_2^- \delta^{0|4}(\sigma_{12}^\alpha\theta_{1,\alpha} + \sigma_{21}^\alpha\theta_{2,\alpha})\delta^{0|4}(\sigma_{n1}^\alpha\theta_{n,\alpha} + \sigma_{1n}^\alpha\theta_{1,\alpha})\delta^{0|4}(\sigma_{23}^\alpha\theta_{2,\alpha} + \sigma_{32}^\alpha\theta_{3,\alpha}). \tag{3.2.59}$$

We can apply the decomposition with respect to harmonic projections namely $\theta_\alpha^I = \theta_\alpha^{+a}\bar{u}_{+a}^I + \theta_\alpha^{-a'}\bar{u}_{-a'}^I$. We now use the following useful identity:

$$\int d^2\theta_{1,\xi\epsilon}^- \int d^2\theta_{2,\eta\phi}^- \delta^{0|4}(\sigma_{12}^\alpha\theta_{1,\alpha} + \sigma_{21}^\alpha\theta_{2,\alpha}) \tag{3.2.60a}$$

$$= \int d^2\theta_{1,\xi\epsilon}^- \int d^2\theta_{2,\eta\phi}^- \frac{1}{4!}\epsilon_{ABCD} \times \sigma_{12}^\alpha\theta_{1\alpha}^{-a'}\bar{u}_{1,-a'}^A\sigma_{12}^\beta\theta_{1\beta}^{-b'}\bar{u}_{1,-b'}^B\sigma_{21}^\gamma\theta_{1\gamma}^{-c'}\bar{u}_{2,-c'}^C\sigma_{21}^\rho\theta_{1\rho}^{-d'}\bar{u}_{-d'}^D \tag{3.2.60b}$$

$$\sim \sigma_{12\xi}\sigma_{12\epsilon}\sigma_{21\eta}\sigma_{21\phi}\frac{1}{4}\epsilon_{IJKL}\bar{Y}_1^{IJ}\bar{Y}_2^{KL} = \sigma_{12\xi}\sigma_{12\epsilon}\sigma_{21\eta}\sigma_{21\phi}y_{12}^2. \tag{3.2.60c}$$

This requires some explanation, (3.2.60b) is the expansion into the pieces of θ^- that will be involved in the calculation. We recall (2.3.84) to get the first equation of the third line. The last equality follows from the definition of $\bar{Y}$ in terms of $\bar{u}$ used in (2.3.85), we also dropped the numerical prefactor as it is not important. Now, in making use of this identity we clearly only use half of the $d^4\theta^-$ integrals in (3.2.59), whilst using the other half to execute a similar identity with different delta functions. The result is a set of σ-variables all contracted with an $\epsilon^{\alpha\beta}$. Doing this with all the fermionic integrations gives (3.2.58) as

$$G_{n;0} = (N^2-1)\prod_{i=1}^n y_{i,i+1}^2 \int d^2\sigma_{i\,i-1}d^2\sigma_{i\,i+1}\bar{\delta}^4(\sigma_{i\,i-1}^\alpha z_{i,\alpha} + \sigma_{i-1\,i}^\alpha z_{i-1,\alpha} + z_*) + (S_n\text{−perm}). \tag{3.2.61}$$

We now need to localise the bosonic delta functions, namely we want to solve

$$\sigma_{i\,i-1}^\alpha z_{i,\alpha} + \sigma_{i-1\,i}^\alpha z_{i-1,\alpha} + z_* = 0, \tag{3.2.62}$$

in which the solutions are

$$\sigma_{i,i-1}^\alpha = \epsilon^{\alpha\beta}\frac{\langle z_{i,\beta}z_*z_{i-1,1}z_{i-1,2}\rangle}{\langle z_{i-1,1}z_{i-1,2}z_{i,1}z_{i,2}\rangle}\,, \quad \sigma_{i-1,i}^\alpha = \epsilon^{\alpha\beta}\frac{\langle z_{i-1,\beta}z_*z_{i,1}z_{i,2}\rangle}{\langle z_{i-1,1}z_{i-1,2}z_{i,1}z_{i,2}\rangle}\,,$$

in making use of (2.2.54) to write $\langle z_{j,1} z_{j,2} z_{i,1} z_{i,2} \rangle = x_{ij}^2$, we explicitly have

$$\bar{\delta}^4(\sigma_{ij}^\alpha z_{i,\alpha} + \sigma_{ji}^\alpha z_{j,\alpha} + z_*) = \frac{1}{x_{ij}^2} \bar{\delta}\left(\sigma_{ij}^1 - \epsilon^{\alpha\beta}\frac{\langle z_{i,2} z_* z_{j,1} z_{j,2}\rangle}{x_{ij}^2}\right) \bar{\delta}\left(\sigma_{ij}^2 - \epsilon^{\alpha\beta}\frac{\langle z_{i,1} z_* z_{j,1} z_{j,2}\rangle}{x_{ij}^2}\right)$$
$$\times \bar{\delta}\left(\sigma_{ji}^1 - \epsilon^{\alpha\beta}\frac{\langle z_{j,2} z_* z_{i,1} z_{i,2}\rangle}{x_{ij}^2}\right) \bar{\delta}\left(\sigma_{ji}^2 - \epsilon^{\alpha\beta}\frac{\langle z_{j,1} z_* z_{i,1} z_{i,2}\rangle}{x_{ij}^2}\right). \tag{3.2.63}$$

This means that integrating the σ-variables is (upto imposing the solution (3.2.63))

$$\int d^2\sigma_{i\ i-1} d^2\sigma_{i-1\ i} \bar{\delta}^4(\sigma_{i\ i-1}^\alpha z_{i,\alpha} + \sigma_{i-1\ i}^\alpha z_{i-1,\alpha} + z_*) = \frac{1}{x_{i-1,i}^2}, \tag{3.2.64}$$

hence we have

$$G_{n;0} = \prod_{i=1}^{n} \frac{y_{i,i+1}^2}{x_{i,i+1}^2} + (S_n\text{−perm}). \tag{3.2.65}$$

This exercise demonstrates firstly how to perform the relevant integrations but also the fact that this procedure will become increasingly inefficient if we want to compute the higher components. In general we will want to integrate a function of Grassmann degree $4(n+p)$ against a $4n$ integration. The previous example was relatively simple because we had $p = 0$, but now we wish to investigate $p > 0$.

The main result is yet another restructuring of the aforementioned Feynman rules in the previous subsection, whereby we make use of harmonic superspace.

The useful result is that we can perform the decomposition

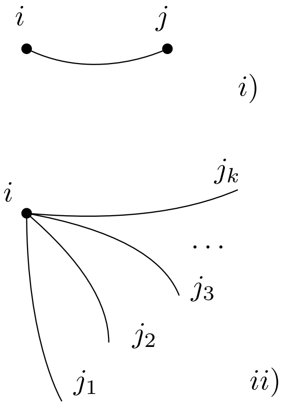
$$\delta^{0|4}\left(\chi_* + \sigma_{ij}^\alpha \theta_{i,\alpha} + \sigma_{ji}^\alpha \theta_{j,\alpha}\right) = y_{ij}^2 \delta^{0|2}\left(\sigma_{ij}^\alpha \theta_{i,\alpha}^- + A_{ij}\right) \delta^{0|2}\left(\sigma_{ji}^\alpha \theta_{j,\alpha}^- + A_{ji}\right), \tag{3.2.66}$$

where

$$A_{ij}^{a'} = \left[\chi_*^I u_{j,I}^{+b} + \sigma_{ji}^\alpha \theta_{j,\alpha}^{+b} + \sigma_{ij}^\alpha \theta_{i,\alpha}^{+b}\right] \left(y_{ij}^{-1}\right)_b^{a'} \tag{3.2.67}$$

The proof can be found in appendix A. The advantage in this modified object is that the delta function is no longer associated to both points i and j with respect to the $\int d^4\theta^-$ integrations. Instead, the object $\delta^{0|2}\left(\sigma_{ij}^\alpha \theta_{i,\alpha}^- + A_{ij}\right)$ is associated to point i only. It follows that each bivalent vertex i (connected to vertex j and k, say) brings a factor of

$$\int d^4\theta_i^- \delta^{0|2}\left(\sigma_{ij}^\alpha \theta_{i,\alpha}^- + A_{ij}\right) \delta^{0|2}\left(\sigma_{ik}^\alpha \theta_{i,\alpha}^- + A_{ik}\right) = \langle \sigma_{ij}\sigma_{ik}\rangle^2. \tag{3.2.68}$$



$$\delta^{a_i a_j} g_{ij} = \delta^{a_i a_j} \frac{y_{ij}^2}{x_{ij}^2}$$

$$\mathrm{tr}(T^{a_{j_1}} \dots T^{a_{j_k}})\, R\,(i; j_1 j_2, \dots, j_k)$$

Figure 3.6: ***i)*** The Feynman rule for the superpropagator and ***ii)*** the Feynman rules for interaction vertices

Going back to our rules in figure 3.4, we find that a k-valent vertex brings a factor of

$$R(i; j_1 j_2 \dots j_k) = -\int d^4\theta_i^- \frac{\delta^2(\sigma_{ij_1}^\alpha \theta_{i,\alpha}^- + A_{ij_1})\delta^2(\sigma_{ij_2}^\alpha \theta_{i,\alpha}^- + A_{ij_2}) \dots \delta^2(\sigma_{ij_k}^\alpha \theta_{i,\alpha}^- + A_{ij_k})}{\langle \sigma_{ij_1}\sigma_{ij_2}\rangle \langle \sigma_{ij_2}\sigma_{ij_3}\rangle \dots \langle \sigma_{ij_k}\sigma_{ij_1}\rangle}\,, \tag{3.2.69}$$

and since we already gain the factor of y_{ij}^2 from the procedure in (3.2.66) together with the $1/x_{ij}^2$ from the bosonic integration in (3.2.64), it follows that we gain the following set of new *Feynman rules*:

- A line connecting vertices i and j is associated with the propagator $g_{ij} = y_{ij}^2/x_{ij}^2$,
- Bivalent vertices are associated with $R(i; j_1 j_2)\,\mathrm{tr}\,(T^{a_{j_1}} T^{a_{j_2}}) = R(i; j_1 j_2)\delta^{a_{j_1} a_{j_2}}$ (note that $R(i; j_1 j_2) = 1$, but we write it here for generality),
- Higher valency vertices are associated with $R(i; j_1 \dots j_k)\,\mathrm{tr}\,(T^{a_{j_1}} \dots T^{a_{j_k}})$ evaluated for the σ-parameters given by

$$\sigma_{ij}^\alpha = \epsilon^{\alpha\beta} \frac{\langle z_{i,\beta} z_* z_{j,1} z_{j,2}\rangle}{x_{ij}^2}\,, \quad \sigma_{ji}^\alpha = \epsilon^{\alpha\beta} \frac{\langle z_{j,\beta} z_* z_{i,1} z_{i,2}\rangle}{x_{ij}^2}\,, \tag{3.2.70}$$

These are summarised in figure 3.6. In terms of computational organisation, we can easily organise all of the propagator factors away, whilst dealing with the $R(i; j_1 j_2 \dots j_k)$. It is this object which we can think of as superconformal invariant, in a similar way to how we think about the R-invariant in the study of scattering amplitudes [32].

3.2.4 Properties of the R-vertices

In this section we aim to provide some of the properties of these R-vertices. As we will see, the most general k-valent R-vertex can be written in terms of the product of $k-2$ trivalent R-vertices. Thus, it is computationally relevant to know the expansion of the trivalent R-vertex in its Grassmann components which we shall explore here.

The object $R\left(i;j_1j_2\ldots j_k\right)$ in (3.2.69) has a numerator which is completely permutation invariant under the exchange of points, whilst its denominator is not. Instead, it obeys the symmetries of the Parke-Taylor structure usually found in amplitudes, namely $R\left(i;j_1j_2\ldots j_k\right)$ is cyclically invariant and invariant upto a sign if we take the following shift $j_p \leftrightarrow j_{k-p+1}$ of all points:

$$\begin{aligned} R\left(i;j_1j_2\ldots j_k\right) &= R\left(i;j_2j_3\ldots j_1\right) = \cdots = R\left(i;j_kj_1\ldots j_{k-1}\right) \\ R\left(i;j_1j_2\ldots j_k\right) &= (-1)^k R\left(i;j_kj_{k-1}\ldots j_1\right). \end{aligned} \tag{3.2.71}$$

These two symmetries form the dihedral group in k-elements upto the $(-1)^k$ signature. This is particularly interesting when we take $k=3$ as the dihedral group becomes the permutation group (although still upto a sign). The result is the object $R\left(i;j_1j_2j_3\right)$ which is completely antisymmetric in the j indices. As a result when two of the indices are equivalent, say $j_1=j_2$, $R\left(i;j_2j_2j_3\right)=0$. There is also the so-called U(1)-decoupling relation:

$$\sum_{\tau\in \text{cyclc}_{k-1}} R\left(i;j_1\tau\left(j_2j_3\ldots j_k\right)\right)=0. \tag{3.2.72}$$

Moving towards the trivalent R-vertex, which is written as

$$R\left(i;j_1j_2j_3\right) = -\int d^4\theta_i^- \frac{\delta^{0|2}\left(\sigma_{ij_1}^{\alpha}\theta_{i,\alpha}^- + A_{ij_1}\right)\delta^{0|2}\left(\sigma_{ij_2}^{\alpha}\theta_{i,\alpha}^- + A_{ij_2}\right)\delta^{0|2}\left(\sigma_{ij_3}^{\alpha}\theta_{i,\alpha}^- + A_{ij_3}\right)}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\langle\sigma_{ij_2}\sigma_{ij_3}\rangle\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}, \tag{3.2.73}$$

we recognise that our σ-variables are related by the two dimensional schouten relation

$$\sigma_{ij_1}^{\alpha}\langle\sigma_{ij_2}\sigma_{ij_3}\rangle + \sigma_{ij_3}^{\alpha}\langle\sigma_{ij_1}\sigma_{ij_2}\rangle + \sigma_{ij_2}^{\alpha}\langle\sigma_{ij_3}\sigma_{ij_1}\rangle = 0, \tag{3.2.74}$$

which we may implement by rewriting the last delta function such that (3.2.73) becomes,

$$R\left(i;j_1j_2j_3\right) = -\int d^4\theta_i^- \frac{\delta^{0|2}\left(\sigma_{ij_1}^{\alpha}\theta_{i,\alpha}^- + A_{ij_1}\right)\delta^{0|2}\left(\sigma_{ij_2}^{\alpha}\theta_{i,\alpha}^- + A_{ij_2}\right)}{\left\langle\sigma_{ij_1}\sigma_{ij_2}\right\rangle\left\langle\sigma_{ij_2}\sigma_{ij_3}\right\rangle\left\langle\sigma_{ij_3}\sigma_{ij_1}\right\rangle}$$
$$\delta^{0|2}\left(\frac{-\left(\sigma_{ij_1}^{\alpha}\langle\sigma_{ij_2}\sigma_{ij_3}\rangle + \sigma_{ij_2}^{\alpha}\langle\sigma_{ij_3}\sigma_{ij_1}\rangle\right)}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle}\theta_{i,\alpha}^- + A_{ij_3}\right) \tag{3.2.75}$$

We can rewrite the third delta function on the support of the first two, namely the solution of $\sigma_{ij_1}^{\alpha}\theta_{i,\alpha}^{-a'} + A_{ij_1}^{a'} = 0$ and $\sigma_{ij_2}^{\alpha}\theta_{i,\alpha}^{-a'} + A_{ij_2}^{a'} = 0$ is given by $\theta_{i,\alpha}^{-a'} = -\frac{\sigma_{ij_1}^{\alpha}A_{ij_2}^{a'}}{\langle\sigma_{ij_2}\sigma_{ij_1}\rangle} - \frac{\sigma_{ij_2}^{\alpha}A_{ij_1}^{a'}}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle}$. We apply this solution to the third delta function, rendering it independent of $\theta_{i,\alpha}^-$. We finally apply the integration to the only remaining delta functions that depend on $\theta_{i,\alpha}^-$, from which we find

$$R(i;j_1j_2j_3) = -\frac{\delta^{0|2}\Big(\langle\sigma_{ij_1}\sigma_{ij_2}\rangle A_{ij_3} + \langle\sigma_{ij_2}\sigma_{ij_3}\rangle A_{ij_1} + \langle\sigma_{ij_3}\sigma_{ij_1}\rangle A_{ij_2}\Big)}{\left\langle\sigma_{ij_1}\sigma_{ij_2}\right\rangle\left\langle\sigma_{ij_2}\sigma_{ij_3}\right\rangle\left\langle\sigma_{ij_3}\sigma_{ij_1}\right\rangle}\,. \tag{3.2.76}$$

This process works recursively for higher valency R-vertices, however it is only for the trivalent case that we can completely integrate away $\theta_{i,\alpha}^-$ completely. Nonetheless, in the first instance we can write

$$R\left(i;j_1j_2\ldots j_k\right) = R\left(i;j_1j_2\ldots j_{k-1}\right)R\left(i;j_1j_{k-1}j_k\right), \tag{3.2.77}$$

which follows from selecting three js, in this case j_1, j_{k-1} and j_k and applying the procedure to j_k. We successfully make the delta function associated to j_k independent of $\theta_{i,\alpha}^-$ which allows for the factorisation. The remaining delta functions still have Grassmann degree of more than 4 and the integration remains unevaluated. In iterating the process, we arrive at

$$R\left(i;j_1j_2\ldots j_k\right) = \prod_{s=2}^{k-1} R\left(i;j_1j_sj_{s+1}\right). \tag{3.2.78}$$

This is the critical statement that all higher valency R-vertices can be written in terms of the trivalent R-vertex.

Having come to this realisation, we recognise the importance of expanding the trivalent R-vertex in terms of its Grassmann components. This will prove to be useful for practical computations. To proceed we find that given (3.2.70), we may define the following index-less notation

$$\begin{aligned}
y_{ijk} &:= \left(y_{ijk}\right)_a{}^b = \left(y_{ij}\right)_{ac'} \left(\tilde{y}_{jk}\right)^{c'b} , \\
y_{ijklm} &:= \left(y_{ijklm}\right)_a{}^b = \left(y_{ij}\right)^{aa'} \left(\tilde{y}_{jk}\right)^{a'c} \left(y_{kl}\right)_{cd'} \left(\tilde{y}_{lm}\right)^{d'b} , \\
(ijk) &:= \langle \sigma_{ij} \sigma_{ik} \rangle x_{ij}^2 x_{ik}^2,
\end{aligned} \tag{3.2.79}$$

where the third definition naturally follows from section 2.2 in conjunction with (3.2.70). Given the form of the trivalent R-vertex in (3.2.78), we find that

$$\begin{aligned}
R(i; j_1 j_2 j_3) =& R_1(i; j_1 j_2) + \frac{1}{2} R_2(i; j_1 j_2) + \frac{1}{2} R_3(i; j_1 j_2) \\
&+ \frac{1}{2} R_4(i; j_1 j_2 j_3) + \frac{1}{6} R_5(i; j_1 j_2 j_3) + \text{antisym}_{123} \, .
\end{aligned} \tag{3.2.80}$$

Here we have defined

$$\begin{aligned}
R_1(i; j_1 j_2) &= -\frac{\langle \sigma_{ij_1} | \theta_i^+ y_{ij_1 j_2} \theta_{j_2}^+ | \sigma_{j_2 i} \rangle}{(ij_1 j_2) g_{ij_1} g_{ij_2}} , \\
R_2(i; j_1 j_2) &= \frac{\langle \sigma_{j_1 i} | \theta_{j_1}^+ y_{j_1 i j_2} \theta_{j_2}^+ | \sigma_{j_2 i} \rangle}{(ij_1 j_2) g_{ij_1} g_{ij_2}} , \\
R_3(i; j_1 j_2) &= -\frac{\langle \sigma_{ij_1} | \left(\theta_i^+\right)^2 | \sigma_{ij_2} \rangle y_{j_1 j_2}^2}{(ij_1 j_2) g_{ij_1} g_{ij_2}} , \\
R_4(i; j_1 j_2 j_3) &= \langle \sigma_{j_1 i} | \left(\theta_{j_1}^+\right)^2 | \sigma_{j_1 i} \rangle \frac{x_{ij_1}^2 (ij_2 j_3)}{(ij_1 j_2)(ij_3 j_1) g_{ij_1}} , \\
R_5(i; j_1 j_2 j_3) &= (\theta_i^{+,\alpha} \, y_{ij_1 j_2 j_3 i} \theta_{i,\alpha}^+) \frac{1}{y_{ij_1}^2 y_{ij_2}^2 y_{ij_3}^2} ,
\end{aligned} \tag{3.2.81}$$

where we used (3.2.79) and introduced an index-less and bra-ket notation for aesthetic purposes

$$\begin{aligned}
\theta_i^{+,\alpha} \, y_{i123i} \theta_{i,\alpha}^+ &= \theta_{i,\alpha}^{+a} (y_{i123i})_a{}^b \theta_{i,\alpha+b} \\
\langle \sigma_{ij_1} | \theta_i^+ y_{i12} \theta_{j_2}^+ | \sigma_{j_2 i} \rangle &= \sigma_{ij_1}^\alpha \, \theta_{i,\alpha}^{+a} (y_{ij_1 j_2})_a{}^b \theta_{j_2,+b}^\beta \sigma_{j_2 i,\beta} \\
g_{ij} &= \frac{y_{ij}^2}{x_{ij}^2}, \ \text{etc.}
\end{aligned} \tag{3.2.82}$$

We have presented a derivation of some of the components in appendix B.

3.3 Consistency check: lightlike limit

We would like to give an example of the lightlike limit in action. The easiest example to choose is the component correlator $G_{n;1}$. At Born level the correspondence given in (3.1.30) reduces to

$$\sum_{p=0}^{n-4} \frac{G_{n;p}^{(0)}}{G_{n;0}^{(0)}} = \left(\sum_{p=0}^{n-4} \hat{\mathcal{A}}_{n;p}^{(0)} \right)^2 . \tag{3.3.83}$$

We can expand this in Grassmann degree and match the appropriate parts, namely given that

$$\lim_{x_{i,i+1}\to 0} \left[1 + \frac{1}{G_{n;0}^{(0)}} \left(G_{n;1}^{(0)} + G_{n;2}^{(0)} + \ldots G_{n;n-4}^{(0)} \right) \right] = \left(1 + \hat{\mathcal{A}}_{n;1}^{(0)} + \hat{\mathcal{A}}_{n;2}^{(0)} + \cdots + \hat{\mathcal{A}}_{n;n-4}^{(0)} \right)^2 , \tag{3.3.84}$$

we gain

$$\begin{aligned} \lim_{x_{i,i+1}^2 \to 0} \frac{G_{n;1}^{(0)}}{G_{n;0}^{(0)}} &= 2\hat{\mathcal{A}}_{n;1}^{(0)}, \\ \lim_{x_{i,i+1}^2 \to 0} \frac{G_{n;2}^{(0)}}{G_{n;0}^{(0)}} &= 2\hat{\mathcal{A}}_{n;2}^{(0)} + \left(\hat{\mathcal{A}}_{n;2}^{(0)} \right)^2 , \end{aligned} \tag{3.3.85}$$

and so on. In this subsection we will show how the correspondence works at the level of the first equivalence in (3.3.85). We recall that $\hat{\mathcal{A}}_{n;1}^{(0)}$ is a linear combination of the R-invariant:

$$\hat{\mathcal{A}}_{n;1}^{(0)} = R_n^{\text{NMHV}} = \sum_{i<j} \int \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4}{s_1 s_2 s_3 s_4} \delta^{4|4} \left(\mathcal{Z}_* + s_1 \mathcal{Z}_{i-1} + s_2 \mathcal{Z}_{j-1} + s_3 Z_{j-1} + s_4 Z_j \right) , \tag{3.3.86}$$

where $\mathcal{Z}$ are supertwistors.

$G_{n;1}$ in terms of our graphical rules are the set of all graphs that have $n+1$ edges (propagators) and n vertices (space-time points). All such graphs are given in figure 3.7. This includes both planar and non-planar contributions. Since the correspondence is a planar correspondence, we take only the planar contributions to the correlator. In section 4 of [29], a detailed analysis of the color structure is given which we will omit here. The main result of the analysis is that the kinematic parts of graphs B and C survive with an overall multiplication with $N(N^2-1)$.

We provide some of the basic intermediate points of the full analysis found in [29]. The vanishing of graph A is easiest to see, where the Kronecker deltas follow around the loop, namely we have $\delta^{a_j a_{j+1}} \delta^{a_{j+1} a_{j+2}} \ldots \delta^{\cdots} \delta^{a_{k-2} a_{k-1}} \delta^{a_{k-1} a_k} = \delta^{a_j a_k} \mathrm{tr}\left(T^{a_j} T^{a_k} T^{a_l} \right) = \mathrm{tr}\left(T^{a_j} T^{a_j} T^{a_l} \right)$. Since $T^{a_j} T^{a_j} = \left((N^2-1)/N \right) \mathbb{I}$, we then have $\mathrm{tr}\left(T^{a_{j_3}} \right) = 0$. Graph

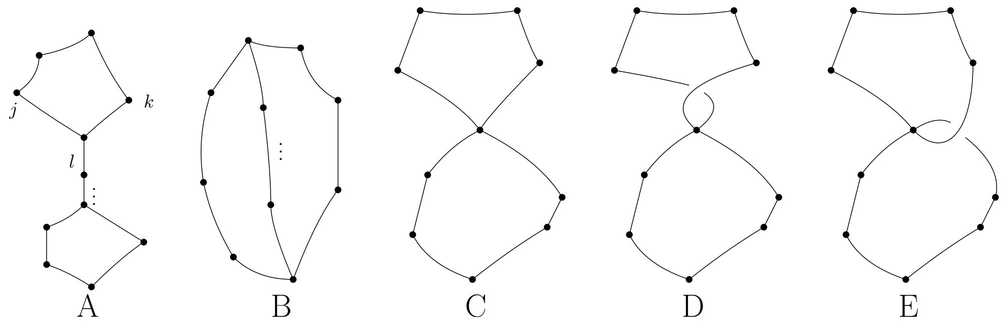


Figure 3.7: Five contributing Feynman graph topologies to $G_{n;1}$.

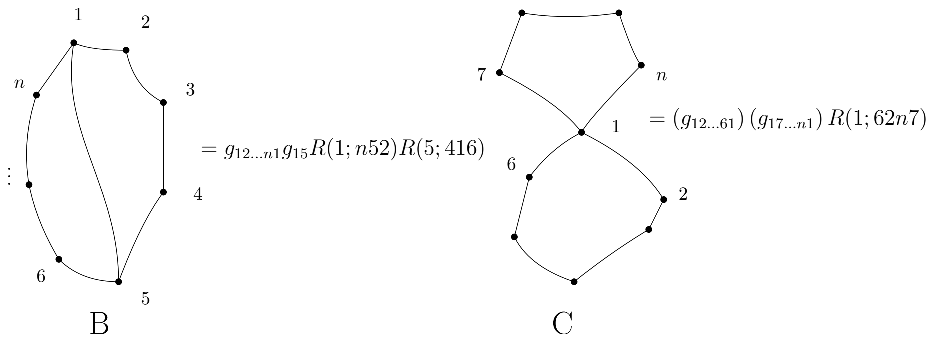


Figure 3.8: Graph B and C contributions to $G_{n;1}$.

B comes with a factor of $N(N^2-1)$. A further non-trivial fact is that in the sum of graph C, graph D and graph E, the $1/N$ dependence drops out to leave the planar result. It turns out to be no different to using the kinematics of graph B multiplied with $N(N^2-1)$. As a result we only need to work with graph B and graph C.

We would like to categorise the (sub)set of Feynman graph topologies that containing the leading singularity in the lightlike limit. Two example graphs of contribution from type B and C are given in figure 3.8 where we use the notation

$$g_{ij} := \frac{y_{ij}^2}{x_{ij}^2}, \qquad g_{ij_1j_2\ldots j_k} := g_{ij_1} g_{j_1j_2} \ldots g_{j_{k-1}j_k}, \tag{3.3.87}$$

from which we find that graph B possesses the leading singularity in the lightlike limit. The lightlike limit of the tree level $G_{n;0}^{(0)}$ is given by $\lim_{x_{i,i+1}^2 \to 0} G_{n;0}^{(0)} = g_{12\ldots n1}$, in which we therefore find that

$$\begin{aligned}\lim_{x_{i,i+1}^2 \to 0} G_{n;1}^{(0)} &= G_{n;0}^{(0)} \sum_{i \neq j} g_{ij} R\left(i; i-1 \;\; j \;\; i+1\right) R\left(j; j-1 \;\; i \;\; j+1\right) \\ &\to \lim_{x_{i,i+1}^2 \to 0} \frac{G_{n;1}^{(0)}}{G_{n;0}^{(0)}} = 2 \sum_{i<j} R_{ij},\end{aligned} \tag{3.3.88}$$

where

$$R_{ij} = g_{ij} R\,(i; i-1 \;\; j \;\; i+1)\, R\,(j; j-1 \;\; i \;\; j+1) \tag{3.3.89}$$

In order to prove that this really does equal the corresponding linear combination of R-invariants as in (3.3.86), it is easier to go to the previous incarnation of the Feynman rules, namely the corresponding result for the rules in figure 3.4. In applying those rules and performing the $\prod_{i=1}^{n} \int d\theta_i^-$ integration one finds the corresponding result to the previous found R_{ij}:

$$R_{ij} = \int \frac{d^2\sigma_{ij} d^2\sigma_{ji} \, \langle \sigma_{ii-1}\sigma_{ii+1}\rangle \langle \sigma_{jj-1}\sigma_{jj+1}\rangle}{\langle \sigma_{ii-1}\sigma_{ij}\rangle \langle \sigma_{ij}\sigma_{ii+1}\rangle \langle \sigma_{jj-1}\sigma_{ji}\rangle \langle \sigma_{ji}\sigma_{jj+1}\rangle} \delta^{4|4}(\mathcal{Z}_* + \sigma_{ij}^{\alpha}\mathcal{Z}_{i,\alpha} + \sigma_{ji}^{\alpha}\mathcal{Z}_{j,\alpha})\,. \tag{3.3.90}$$

We now need to take this object onto the super-light cone by making it on-shell. First we recall the form the σ-variables given in (3.2.70). In the lightlike limit we take $x_{i,i+1}^2 \to 0$, which immediately makes these variables singular. However, note that in (3.3.90), there is a scale invariance which allows us to scale away the singular denominator of the σ-variables, such that we have

$$\begin{aligned} \sigma_{i,i-1}^{\beta} &= \epsilon^{\beta\alpha} \langle z_* z_{i-1,1} z_{i-1,2} z_{i,\alpha}\rangle\,, \qquad & \sigma_{i,i+1}^{\beta} &= \epsilon^{\beta\alpha} \langle z_* z_{i+1,1} z_{i+1,2} z_{i,\alpha}\rangle\,, \\ \sigma_{j,j-1}^{\beta} &= \epsilon^{\beta\alpha} \langle z_* z_{j-1,1} z_{j-1,2} z_{j,\alpha}\rangle\,, \qquad & \sigma_{j,j+1}^{\beta} &= \epsilon^{\beta\alpha} \langle z_* z_{j+1,1} z_{j+1,2} z_{j,\alpha}\rangle\,. \end{aligned} \tag{3.3.91}$$

In twistor superspace the lightlike limit $x_{ii+1}^2 \to 0$ and $\theta_{i,i+1}^{A,\alpha}(x_{i,i+1})_{\alpha\dot{\alpha}} \to 0$ corresponds to the intersection of the supertwistor lines $\mathcal{Z}_{i,\alpha}$ and $\mathcal{Z}_{i+1,\alpha}$. Since there is a local GL(2) matrix action on the α-indices, we can make the choice to take

$$\mathcal{Z}_{i,2} = \mathcal{Z}_{i+1,1} \equiv \mathcal{Z}_i\,, \qquad\qquad (i = 1 \ldots n). \tag{3.3.92}$$

This identification is taking what were once free twistor lines and forcing them to intersect, recall figure 2.1 from section 2.2.

We can implement the incidence relation in full twistor space, namely

$$\mathcal{Z}_i = \left(z_i^A | \chi_i^I\right) = \left(\lambda_i^{\alpha}, i x_{i,\alpha\dot{\alpha}} \lambda^{\alpha} | \theta_i^{I\beta} \lambda_{i\beta}\right). \tag{3.3.93}$$

Applying (3.3.92) to the purely bosonic part of the supertwistors results in

$$\sigma_{i\,i+1}^{\alpha=1} = \sigma_{i\,i-1}^{\alpha=2} = 0\,, \qquad\qquad \sigma_{i\,i+1}^{\alpha=2} = -\sigma_{i+1\,i}^{\alpha=1}\,. \tag{3.3.94}$$

Using the component form $\sigma^{\alpha}_{ij} = (s_1, s_2)$ and $\sigma^{\alpha}_{ji} = (t_1, t_2)$ we find that (3.3.90) has now become:

$$R_{ij} = \int \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4}{s_1 s_2 s_3 s_4} \delta^{4|4}(\mathcal{Z}_* + s_1 \mathcal{Z}_{i-1} + s_2 \mathcal{Z}_i + s_3 \mathcal{Z}_{j-1} + s_4 \mathcal{Z}_j). \tag{3.3.95}$$

So, the conclusion of this subsection is that taking the explicit lightlike limit on $G_{n;1}$ using the twistor rules as governed by the R-vertices results in the statement given in (3.3.88). This result is equivalent to (3.3.95) which is the tree level NMHV scattering amplitude at n-points.

3.4 Computations

In this section we will apply the R-vertices towards some components of the $G_{n;1}$ piece of the supercorrelation function. We first note that we will have to deal with the reference supertwistor, which is the main price we pay for manifesting superconformal symmetry throughout our entire discussion. In certain instances it is easier to gain gauge independent results and other times it is harder. We will perform the four-, five- and six-point computation which makes use of a gauge invariant building block that we will construct for the $\left(\theta_0^+\right)^4$ component of the supercorrelation function. On the other hand, we will compute the four- and five-point correlator for $\left(\theta_0^+\right)^{2ab} \left(\theta_{0'}^+\right)^{2cd}$ (where 0 and $0'$ are two external points).

We remind the reader that we will use the previous notation used in (3.2.79), and with $g_{ij} = \frac{y_{ij}^2}{x_{ij}^2}$ and $g_{ijk\ldots ml} = g_{ij} g_{jk} \ldots g_{ml}$. This section will involve technical computations and we leave some parts for appendix C.

Finally, we state that for the entirety of the discussion that follows we will take $\chi_* = 0$. This means that finding gauge invariant results amounts to seeking z_*-invariant results.

3.4.1 $\left(\theta_0^+\right)^4$ component

It is useful to construct a z_*-invariant quantity. Recall that the dependency is in the three-brackets (ijk) that we have defined and used in (3.2.81). It turns out that such an object is (defining some points $0, 1, \ldots, p$):

$$\begin{aligned} f(i; j_1 j_2 j_3) &= -\frac{g_{j_2 i} g_{j_1 i}}{g_{j_1 j_2}} R_3(i; j_1 j_2) + g_{i j_2} R_4(j_2; i j_3 j_1) + \text{cyc}_{123} \\ &= \frac{(x_{ij_1} \tilde{x}_{j_1 j_2} x_{j_2 j_3} \tilde{x}_{j_3 i})^{\alpha\beta}}{x^2_{j_1 i} x^2_{j_2 i} x^2_{j_3 i}} \left(\theta^+_i\right)^2_{\alpha\beta} . \end{aligned} \tag{3.4.96}$$

This is actually a basis for a much larger additive set of z_*-invariants, namely we can define

$$f(i; j_1 j_2 j_3 j_4 \ldots j_p) = \sum_{k=3}^{p} f(0; j_1 j_{k-1} j_k) = \sum_{k=3}^{p} \frac{(x_{ij_1} \tilde{x}_{j_1 j_{k-1}} x_{j_{k-1} j_k} \tilde{x}_{j_k i})^{\alpha\beta}}{x^2_{j_1 i} x^2_{i j_{k-1}} x^2_{i j_k}} \left(\theta^+_i\right)^2_{\alpha\beta} . \tag{3.4.97}$$

Some interesting results follow from this, recalling the definition of the matrix X^{AB} and $\bar{X}_{AB}$ from (2.2.49), and defining the shorthand

$$\begin{aligned} \text{tr}(j_1 \bar{j}_2 j_3 \bar{j}_4 j_5 \bar{j}_6 j_7 \bar{j}_8) &:= \text{tr}(X_{j_1} \bar{X}_{j_2} X_{j_3} \bar{X}_{j_4} X_{j_5} \bar{X}_{j_6} X_{j_7} \bar{X}_{j_8}) \\ &= X^{AB}_{j_1} \bar{X}_{j_2 BC} X^{CD}_{j_3} \ldots \bar{X}_{j_8 IA}, \end{aligned} \tag{3.4.98}$$

we get

$$f(i; j_1 j_2 j_3) f(i; j_1 j_4 j_5) = \frac{\text{tr}\,(i \bar{j}_1 j_2 \bar{j}_3 i \bar{j}_1 j_4 \bar{j}_5) - \text{tr}\,(i \bar{j}_5 j_4 \bar{j}_1 i \bar{j}_1 j_2 \bar{j}_3)}{2 x^4_{ij_1} x^2_{ij_2} x^2_{ij_3} x^2_{ij_4} x^2_{ij_5}} \left(\theta^+\right)^4 . \tag{3.4.99}$$

Recalling the Clifford algebra in (2.2.63), we find that

$$\left\{ X^A_{i,B}, X^B_{j,C} \right\} = -x^2_{ij} \delta^A_C, \tag{3.4.100}$$

which then implies that

$$\begin{aligned} &\text{tr}\,(i \bar{j}_5 j_4 \bar{j}_1 i \bar{j}_1 j_2 \bar{j}_3) = -x^2_{ij_1} \,\text{tr}\,(i \bar{j}_5 j_4 \bar{j}_1 j_2 \bar{j}_3) \\ &\text{tr}\,(i \bar{j}_1 j_2 \bar{j}_3 i \bar{j}_1 j_4 \bar{j}_5) = -x^2_{ij_1} \,\text{tr}\,(i \bar{j}_1 j_2 \bar{j}_3 j_4 \bar{j}_5) + x^2_{ij_4} \,\text{tr}\,(i \bar{j}_1 j_2 \bar{j}_1 j_3 \bar{j}_5) - x^2_{ij_5} \,\text{tr}\,(i \bar{j}_1 j_2 \bar{j}_3 j_1 \bar{j}_4) . \end{aligned} \tag{3.4.101}$$

Finally, at the point of tracing over six matrices we need to split this into a symmetric and an anti-symmetric part. Doing so and applying the Clifford algebra in (2.2.63) recursively we have the general identity

$$\begin{aligned} \text{tr}\,\left(i \bar{j} k \bar{l} m \bar{n}\right) = \frac{1}{2} \Big(&- x^2_{in} x^2_{jm} x^2_{kl} + x^2_{im} x^2_{jn} x^2_{kl} - x^2_{ij} x^2_{kl} x^2_{mn} + x^2_{in} x^2_{jl} x^2_{km} - x^2_{il} x^2_{jn} x^2_{km} \\ &- x^2_{im} x^2_{jl} x^2_{kn} + x^2_{il} x^2_{jm} x^2_{kn} - x^2_{in} x^2_{jk} x^2_{lm} + x^2_{ik} x^2_{jn} x^2_{lm} - x^2_{ij} x^2_{kn} x^2_{lm} \\ &+ x^2_{im} x^2_{jk} x^2_{ln} - x^2_{ik} x^2_{jm} x^2_{ln} + x^2_{ij} x^2_{km} x^2_{ln} - x^2_{il} x^2_{jk} x^2_{mn} + x^2_{ik} x^2_{jl} x^2_{mn} \Big) \\ &+ \text{tr}\big(\big[i \bar{j} k \bar{l} m \bar{n}\big]\big), \end{aligned} \tag{3.4.102}$$

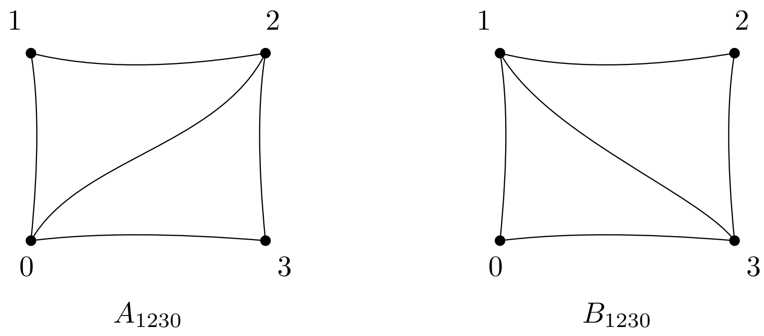


Figure 3.9: Contributions to $G_{4;1}\big|_{(\theta_0^+)^4}$.

where $\mathrm{tr}(\big[i\bar{j}k\bar{l}m\bar{n}\big])$ is the anti-symmetrised part of this six-trace, and is equal to $4i\epsilon_{ijklmn}$ as defined by the hypercone coordinates in (2.2.68).

In any case, for the computations that follow in the $(\theta^+)^4$ component of $G_{n;1}$, making use of (3.4.96) and (3.4.97) serve as an entry points to gaining a z_*-invariant results.

A computational strategy is to first recall that

$$\begin{aligned} G_{n;1} &= \big\langle \mathcal{O}^{++++}(1)\dots\mathcal{O}^{++++}(n)\mathcal{L}(0)\big\rangle \\ &= y_{12}^2 y_{23}^2 \dots y_{n1}^2 \mathcal{F}_1(x_0, x_1, \dots, x_n) + y_{12}^4 y_{34}^2 y_{45}^2 \dots y_{n3}^2 \mathcal{F}_2(x_0, x_1, \dots, x_n) \\ &\quad + \dots + \mathrm{perms}_{12\dots n}, \end{aligned} \tag{3.4.103}$$

where the '$+\dots$' are the remaining different y-structures. The y-structures are the internal basis for the result and can be made to form groups of terms related by S_n permutations, e.g. $y_{12}^2\dot{y}_{n1}^2$ and its S_n permutations are one group whilst $y_{12}^4 y_{34}^2 y_{45}^2 \dots y_{n3}^2$ and its S_n permutations are another group. Correspondingly the coefficient functions belong to the same group, thus we only need to work out one of these coefficients for each representative of a group. In (3.4.103), this corresponds to finding $\mathcal{F}_1(x_0, x_1, \dots, x_n)$ and $\mathcal{F}_2(x_0, x_1, \dots, x_n)$ first and finding the remaining terms for the other corresponding y-structure by permutations.

Four-points

Previously in figure 3.8, we saw the graph topologies that contribute to $G_{n;1}$. We can refine these for the four-point case. There are two contributing graphs given in figure 3.9.

In fact due to $\mathcal{N} = 4$ superconformal symmetry, we expect this component and indeed any Grassmann odd component of the four point Born level supercorrelator to vanish. The result is given by

$$G_{4;1}\big|_{(\theta^+)^4} \sim A_{1230} + B_{1230} + \text{perms}_{123}. \tag{3.4.104}$$

We find that

$$\begin{aligned}
A_{1230} &= g_{123}g_{10}g_{20}g_{30}R\,(2;103)\,R\,(0;321)\\
&= g_{123}g_{10}g_{20}g_{30}\,(R_4\,(2;031))\,(R_3\,(0;32) + \text{cyc}_{321})\\
&= -g_{1231}\,[g_{02}R_4(2;031)]\left[\frac{g_{01}g_{03}}{g_{13}}R_3(0;31)\right],\\
B_{1230} &= g_{1231}g_{01}g_{03}R\,(1;230)\,R\,(3;012)\\
&= g_{1231}\,[g_{01}R_4\,(1;023)]\,[g_{03}R_4\,(3;012)]\,.
\end{aligned} \tag{3.4.105}$$

In A_{1230}, going from the first equivalence to the second one, we recognise terms of the form $R_4\,(j;0kl)\,R_3\,(0;mj) \propto \sigma^{\alpha}_{0j}\sigma^{\beta}_{0j}\sigma^{\gamma}_{0m}\sigma^{\rho}_{0j}\,(\theta^+)^2_{\alpha\beta}\,(\theta^+)^2_{\gamma\rho} \propto \langle\sigma_{0j}\sigma_{0j}\rangle = 0$.

Then, in order to get all contributing terms, we need to sum over permutations. However, since all terms contribute to the y-structure in g_{1231}, we only need to cycle through points $1, 2$ and 3, this gives the result

$$\begin{aligned}
G_{4;1}\big|_{(\theta_0^+)^4} \sim -\,[g_{02}R_4(2;031)]\left[\frac{g_{01}g_{03}}{g_{13}}R_3(0;31)\right] + [g_{01}R_4(1;023)]\,[g_{03}R_4(3;012)]&\\
+\,\text{cyc}_{123}.&
\end{aligned} \tag{3.4.106}$$

It turns out that this result is proportional to a quantity which depends on various 3-brackets (ijk) (defined in (3.2.79)) which tuns out to be zero. This quantity is defined and shown to be zero in appendix C.0.4, however we take a different approach from this direct one.

An instructive approach is to make use of (3.4.96). This is useful as it directly gives us z_*-invariant results. We begin by noting that the result (3.4.106) can be rewritten as:

$$G_{4;1}\big|_{(\theta_0^+)^4} = -\,[R_4^{c_{123}}(2;031)]\,[R_3^{c_{123}}(0;31)] + \frac{1}{2}\,[R_4^{c_{123}}(1;023)]\,[R_4^{c_{123}}(3;012)]\,, \tag{3.4.107}$$

where

$$\begin{aligned}
R_4^{c_{ijk}}(k;0ij) &= g_{i0}R_4(k;0ij) + \text{cyc}_{ijk}\\
R_3^{c_{ijk}}(0;ki) &= \frac{g_{k0}g_{0i}}{g_{ki}}R_3(0;ki) + \text{cyc}_{ijk}.
\end{aligned} \tag{3.4.108}$$

Note, that $R_4(i;0jk)R_4(i;0jk) = 0$ and the factor of half is to avoid over-counting terms. Now we recall from (3.4.96) that the f-function is given by

$$\begin{aligned} f(0;123) &= -R_4^{c_{123}}(2;031) + R_3^{c_{123}}(0;31), \\ \implies \frac{1}{2}f(0;123)^2 &= \frac{1}{2}R_4^{c_{123}}(2;031)^2 + \frac{1}{2}R_3^{c_{123}}(0;31)^2 \\ &\quad - R_4^{c_{123}}(2;031)R_3^{c_{123}}(0;31), \\ \implies R_4^{c_{123}}(2;031)R_3^{c_{123}}(0;31) &= \frac{1}{2}R_4^{c_{123}}(2;031)^2 + \frac{1}{2}R_3^{c_{123}}(0;31)^2 - \frac{1}{2}f(0;123)^2. \end{aligned} \tag{3.4.109}$$

Thus, by substituting the $R_3^{c_{123}} \times R_4^{c_{123}}$- terms into (3.4.107), we retrieve

$$G_{4;1}\big|_{\left(\theta_0^+\right)^4} = \frac{1}{2}\left(f(0;123)^2 - R_3^{c_{123}}(0;31)^2\right). \tag{3.4.110}$$

Finally, we must show that $f(0;123)^2 - R_3^{c_{123}}(0;31)^2 = 0$ which is indeed true and simpler to prove than the vanishing of (3.4.106). We leave this for appendix C.0.1.

Five-points

As in the previous section, we give the contribution of graphs at five points to $G_{5;1}\big|_{\left(\theta_0^+\right)^4}$ in figure 3.10. This is the first such correlator that has a non-trivial result. The result takes the form

$$\begin{aligned} G_{5;1}\big|_{\left(\theta_0^+\right)^4} &\sim A_{12340} + B_{123450} + C_{12340} + D_{12340} + E_{12340} + \text{perms}_{1234} \\ &\sim g_{12341}\mathcal{F}_1(x_0,x_1,\dots,x_4) + g_{12}^2 g_{34}^2 \mathcal{F}_2(x_0,x_1,\dots,x_4) + \text{perms}_{1234}, \end{aligned} \tag{3.4.111}$$

for some yet to be found functions $h(x_0,x_1,\dots,x_n)$ and $m(x_0,x_1,\dots,x_n)$. We follow the basic strategy outlined around (3.4.103), namely we will find the functions $\mathcal{F}_1(x_0,x_1,\dots,x_n)$ and $\mathcal{F}_2(x_0,x_1,\dots,x_n)$ first, and use the permutation invariance to generate the remaining terms. In the following equations (3.4.112)-(3.4.116), we give the explicit result which comes from the Feynman rules, then the second line will give the contribution which applies to the $\left(\theta_0^+\right)^4$ component of the $G_{5;1}$. We will also have for the first four graphs, a third line which puts the result in a suggestive form for the

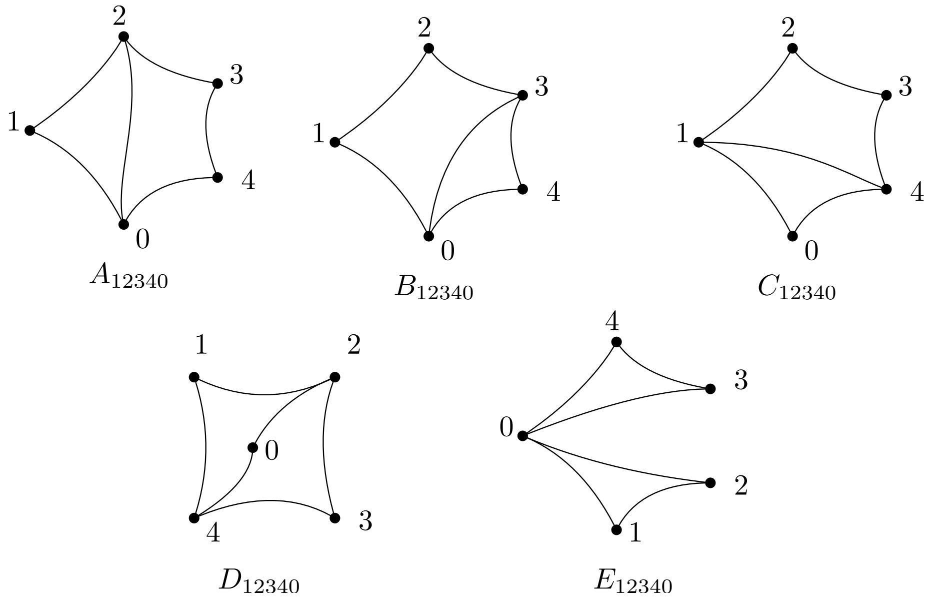


Figure 3.10: Contributions to $G_{5;1}\big|_{(\theta_0^+)^4}$.

z_*-invariant function described earlier. We find:

$$\begin{aligned} A_{12340} &= g_{1234}g_{10}g_{20}g_{20}g_{30}R(0;124)R(2;301) \\ &= g_{1234}g_{10}g_{20}g_{20}g_{30}\left[R_3(0;12)+\mathrm{cyc}_{124}\right]\left[R_4(2;013)\right] \\ &= -g_{12341}\left[\frac{g_{04}g_{01}}{g_{14}}R_3(0;41)\right]\left[g_{02}R_4(2;031)\right], \end{aligned} \tag{3.4.112}$$

$$\begin{aligned} B_{12340} &= g_{1234}g_{01}g_{03}g_{04}R(0;134)R(3;402) \\ &= g_{1234}g_{01}g_{03}g_{04}\left[R_3(0;13)+\mathrm{cyc}_{134}\right]\left[R_4(3;024)\right] \\ &= -g_{12341}\left[\frac{g_{01}g_{04}}{g_{14}}R_3(0;41)\right]\left[g_{03}R_4(3;042)\right], \end{aligned} \tag{3.4.113}$$

$$\begin{aligned} C_{12340} &= g_{12341}g_{01}g_{04}R(1;240)R(4;013) \\ &= g_{12341}g_{01}g_{04}R_4(1;024)R_4(4;013) \\ &= g_{12341}\left[g_{01}R_4(1;024)\right]\left[g_{04}R_4(4;013)\right], \end{aligned} \tag{3.4.114}$$

$$\begin{aligned} D_{12340} &= g_{12341}g_{20}g_{40}R(4;103)R(2;301) \\ &= g_{12341}g_{40}g_{20}R_4(4;031)R_4(2;013) \\ &= g_{12341}\left[g_{04}R_4(4;031)\right]\left[g_{02}R_4(2;013)\right], \end{aligned} \tag{3.4.115}$$

$$\begin{aligned}
E_{12340} &= g_{01}g_{02}g_{03}g_{04}g_{12}g_{34}R(0;4123)\\
&= g_{01}g_{02}g_{03}g_{04}g_{12}g_{34}R(0;412)R(0;234)\\
&= g_{01}g_{02}g_{03}g_{04}g_{12}\Big[\,[(R_3(0;41)+\mathrm{cyc}_{412})(R_3(0;23)+\mathrm{cyc}_{234})]\\
&+[R_5(0;412)]\,[R_5(0;234)]\Big].
\end{aligned}\tag{3.4.116}$$

In a similar way to the previous section, we now try to rewrite the result in terms of the z_*-invariant described in (3.4.96). However, graph E_{12340} involves R_5 type terms, which we can get rid of using the identity [9]

$$\begin{aligned}
&\Big[\,[(R_3(0;41)+\mathrm{cyc}_{412})(R_3(0;23)+\mathrm{cyc}_{234})]+[R_5(0;412)]\,[R_5(0;234)]\Big]\\
&=R_3(0;12)R_3(0;34)+R_3(0;41)R_3(0;23)+\frac{y_{14}^2y_{23}^2-2y_{13}^2y_{24}^2+y_{12}^2y_{34}^2}{2y_{01}^2y_{02}^2y_{03}^2y_{04}^2}
\end{aligned}\tag{3.4.117}$$

We can set the term $R_3(0;12)R_3(0;34)$ to zero as it cancels with a similar contribution which is a permutation $(12)\in S_4$ of $R_3(0;12)R_3(0;34)$. Substituting (3.4.117) back into (3.4.116) results in

$$\begin{aligned}
E'_{12340} &= g_{01}g_{02}g_{03}g_{04}g_{12}g_{34}\Big[R_3(0;41)R_3(0;23)+\frac{y_{14}^2y_{23}^2-2y_{13}^2y_{24}^2+y_{12}^2y_{34}^2}{2y_{01}^2y_{02}^2y_{03}^2y_{04}^2}\Big]\\
&= g_{12341}\Big[\Big[\frac{g_{04}g_{01}}{g_{14}}R_3(0;41)\Big]\Big[\frac{g_{02}g_{03}}{g_{23}}R_3(0;23)\Big]+\frac{1}{2}I_{14;23}\Big]\\
&+\frac{1}{2}\left[-2g_{12431}I_{13;24}+g_{12}^2g_{34}^2I_{12;34}\right],
\end{aligned}\tag{3.4.118}$$

where we have introduced a generalisation of the so-called box function:

$$I_{ij;kl}=\frac{x_{ij}^2x_{kl}^2}{x_{i0}^2x_{j0}^2x_{k0}^2x_{l0}^2}.\tag{3.4.119}$$

As can be seen in (3.4.118), the y-structures associated to the $g_{12}^2g_{34}^2$-type terms are already z_*-independent.

So now we can put together the coefficient of g_{12341} in (3.4.118), together with (3.4.112)-(3.4.115). We want to write this in a way that allows us to use the f-functions in (3.4.96). In order to do this and employ (3.4.96) we should sum over all cyclic permutations, however this will add terms which do not come from any valid Feynman

[9] This was discovered explicitly using `Mathematica`, but some details have been included in appendix C.

graph, which in turn must be explicitly taken away. We get

$$\begin{aligned}
&(3.4.112) + (3.4.113) + (3.4.114) + (3.4.115) + (3.4.118)\big|_{g_{12341}} + \text{cyc}_{1234} \\
&= -R_3^{c_{1234}}(0;12)R_4^{c_{1234}}(1;024) + \frac{1}{2}R_4^{c_{1234}}(1;024)^2 + \frac{1}{2}R_3^{c_{1234}}(0;12)^2 \\
&- \frac{1}{2}\tilde{R}_3^{c_{1234}} - \frac{1}{2}\left(I_{14;23} + I_{12;34}\right),
\end{aligned} \tag{3.4.120}$$

where $\tilde{R}_3^{c_{1234}}$ is the over counting of terms and therefore needs to be taken away, it is defined to be

$$\tilde{R}_3^{c_{1234}} := 2\left(\left(\frac{1}{2}\frac{g_{01}g_{02}}{g_{12}}\right)^2 R_3(0;12)^2 + \left(\frac{g_{01}g_{02}^2g_{03}}{g_{12}g_{23}}\right)R_3(0;12)R_3(0;23) + \text{cyc}_{1234}\right). \tag{3.4.121}$$

The factors of half in (3.4.120) are there to avoid over-counting. In making use of the f-functions from (3.4.96), we find

$$\begin{aligned}
&(3.4.112) + (3.4.113) + (3.4.114) + (3.4.115) + (3.4.118)\big|_{g_{12341}} + \text{cyc}_{1234} \\
&= \frac{1}{2}\left(f(0;1234)^2 - \tilde{R}_3^{c_{1234}} - \left(I_{14;23} + I_{12;34}\right)\right),
\end{aligned} \tag{3.4.122}$$

in which we leave the details of this computation for appendix C.0.3. It's worth noting that the object $\tilde{R}_3^{c_{1234}}$ is z_*-invariant although not manifestly so. Finally, putting everything together

$$\begin{aligned}
&(3.4.112) + (3.4.113) + (3.4.114) + (3.4.115) + (3.4.118) + \text{cyc}_{1234} \\
&\quad = g_{12341}\left[\frac{1}{2}\left(f(0;1234)^2 - \tilde{R}_3^{c_{1234}}\right) - \frac{1}{2}(I_{14;23} + I_{12;34})\right] \\
&\quad + \frac{1}{2}\left(g_{12}^2g_{34}^2I_{12;34} + g_{23}^2g_{14}^2I_{23;41}\right)
\end{aligned} \tag{3.4.123}$$

Using what we have learnt in (3.4.99), we find that

$$\frac{1}{2}\left(f(0;1234)^2 - \tilde{R}_3^{c_{1234}}\right) = -\frac{1}{2}\left[I_{12;34} - 2I_{13;24} + I_{14;23}\right] \tag{3.4.124}$$

from which it follows that (3.4.123) becomes:

$$g_{12341}\left(I_{13;24} - I_{14;23} - I_{12;34}\right) + \frac{1}{2}\left(g_{12}^2g_{34}^2I_{12;34} + g_{23}^2g_{14}^2I_{23;41}\right). \tag{3.4.125}$$

Finally, we perform two further transpositions on this result, namely $(12)+(23)$, yields the final result:

$$\begin{aligned}
G_{5;1}\big|_{(\theta_0^+)^4} &\sim g_{12341}\left(I_{13;24} - I_{14;23} - I_{12;34}\right) + g_{13241}\left(I_{12;34} - I_{14;23} - I_{13;24}\right) \\
&+ g_{21342}\left(I_{23;14} - I_{24;13} - I_{12;34}\right) + g_{12}^2g_{34}^2I_{12;34} + g_{23}^2g_{14}^2I_{23;41} + g_{13}^2g_{24}^2I_{13;24}.
\end{aligned} \tag{3.4.126}$$

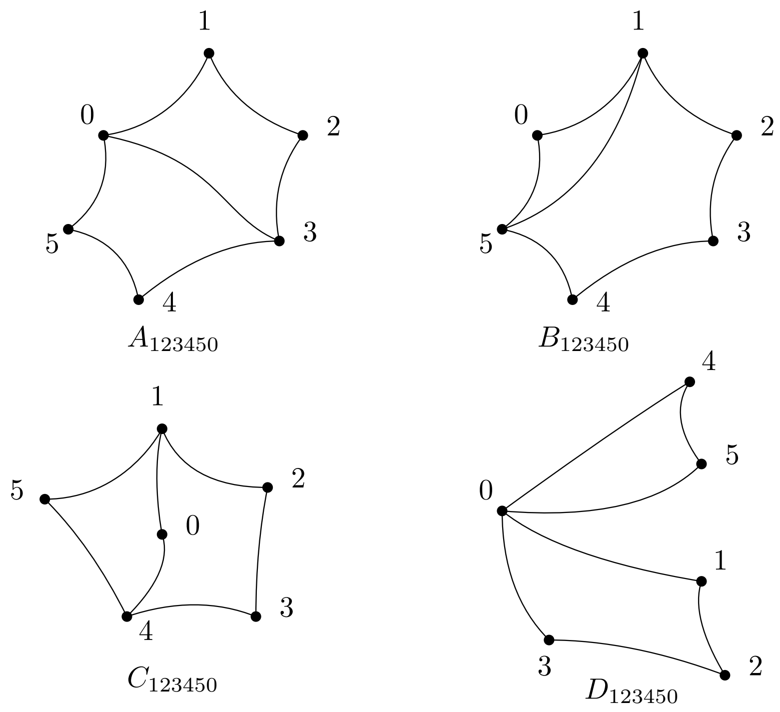


Figure 3.11: Contributions to $G_{6;1}\big|_{(\theta_0^+)^4}$.

Six-points

Having developed the basics in the previous section, we now present the result at six-points. The contributing graphs are given in figure 3.11. In a similar way to previous case, the result has the form

$$\begin{aligned} G_{6;1}\big|_{(\theta_0^+)^4} &\sim A_{123450} + B_{123450} + C_{123450} + D_{123450} + \text{perms}_{12345} \\ &\sim g_{123451}\mathcal{F}_1(x_0, x_1, \ldots, x_5) + g_{12}^3 g_{34}^2 \mathcal{F}_2(x_0, x_1, \ldots, x_5) + \text{perms}_{12345}, \end{aligned} \tag{3.4.127}$$

The result of each graph after concentrating on the relevant part for the $(\theta_0^+)^4$ component is given by

$$\begin{aligned} A_{123450} &= -g_{123451}\left[\frac{g_{05}g_{01}}{g_{15}}R_3(0;51)\right][g_{03}R_4(3;042)], \\ B_{123450} &= g_{123451}[g_{01}R_4(1;025)][g_{01}R_4(5;012)], \\ C_{123450} &= g_{123451}[g_{01}R_4(1;025)][g_{04}R_4(4;053)], \\ D_{123450} &= g_{123451}\left[\frac{g_{05}g_{01}}{g_{15}}R_3(0;51)\right]\left[\frac{g_{03}g_{04}}{g_{34}}R_3(0;34)\right] + g_{123451}\frac{I_{15;34}}{2} \\ &\quad - g_{123451}I_{14;35} + g_{1231}g_{45}^2\frac{I_{13;45}}{2}. \end{aligned} \tag{3.4.128}$$

As before, in order to group together all terms proportional to g_{123451}, we permute the last diagram by (45) to yield the result:

$$D'_{123450} = g_{123451}\left[\frac{g_{05}g_{01}}{g_{15}}R_3(0;51)\right]\left[\frac{g_{03}g_{04}}{g_{34}}R_3(0;34)\right] - g_{123451}\frac{I_{15;34}}{2} + g_{1231}g_{45}^2\frac{I_{13;45}}{2} \tag{3.4.129}$$

Further diagrams can be found by performing permutations, and we can group together terms proportional to g_{123451}. Doing this and tranferring to the f-function as we have done previously gives the result

$$\begin{aligned} G_{6;1}\big|_{(\theta_0^+)^4} \sim & \frac{1}{2}g_{12345}\left[f(0;12345)^2 - \tilde{R}_3^{c_{12345}} - (I_{15;34} + \mathrm{cyc}_{12345})\right] \\ & + \frac{1}{2}g_{1231}g_{45}^2 I_{13;45} + \mathrm{perms}_{12345} \end{aligned} \tag{3.4.130}$$

In which we can use the discussion following (3.4.96) to unpack the result, which gives:

$$\begin{aligned} G_{6;1}\big|_{(\theta_0^+)^4} \sim & g_{123451}\left[I_{13;24} + I_{14;23} - I_{12;34} + \frac{8i\epsilon_{012345}}{x_{01}^2x_{02}^2x_{03}^2x_{04}^2x_{05}^2}\right] \\ & + g_{1231}g_{45}^2 I_{13;45} + \mathrm{perms}_{12345}, \end{aligned} \tag{3.4.131}$$

where ϵ_{012345} is defined by hypercone coordinates in (2.2.68) and is given by

$$4i\epsilon_{012345} = \mathrm{tr}([0\bar{1}2\bar{3}4\bar{5}]). \tag{3.4.132}$$

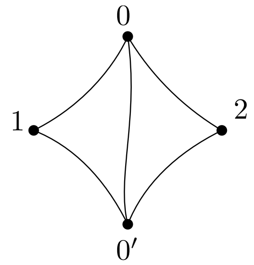
3.4.2 $\left(\theta_0^+\right)^{2(ab)}\left(\theta_{0'}^+\right)^{2(cd)}$ component

In this subsection we work out the $\left(\theta_0^+\right)^{2(ab)}\left(\theta_{0'}^+\right)^{2(cd)}$ component for $G_{n;1}$ at four- and five-points.

In contrast to the previous section, we do not need to work too hard to find a z_*-invariant object. On the other, since the component has non-trivial y-structures which are matrices (as opposed to previously being scalars), we have to work a little harder to work them out properly.

Four-points

At four points, the contributing graph is given in figure 3.12.



$A_{010'2}$

Figure 3.12: Contribution to $G_{4;1}\big|_{\left(\theta_0^+\right)^{2(ab)}\left(\theta_{0'}^+\right)^{2(cd)}}$.

The correlation function is given by the sum over the non-trivial permutations of this graph,

$$G_{4;1} \sim A_{010'2} + A_{0120'} + A_{200'4} + A_{1020'} + A_{00'12} + A_{0'021}\,. \tag{3.4.133}$$

where

$$A_{010'2} = g_{01}g_{10'}g_{0'2}g_{20}g_{00'}R(0;10'2)R(0'201), \text{ etc.} \tag{3.4.134}$$

To extract the contribution $\left(\theta_0^+\right)^{2(ab)}\left(\theta_{0'}^+\right)^{2(cd)}$, we have to replace the $R-$vertices by the corresponding R_i component, we get

$$\begin{aligned}
A_{010'2} &= \Big[\frac{g_{0'2}g_{02}(y_{010'})_{ab}(y_{010'})_{cd}}{x_{01}^2x_{10'}^2x_{00'}^2y_{00'}^2} - \frac{g_{10'}g_{02}(y_{010'})_{ab}(y_{0'20})_{cd}}{x_{01}^2x_{00'}^2x_{0'2}^2y_{00'}^2} \\
&\quad + \frac{(y_{010'20})_{ac}(y_{0'4010'})_{bd}}{2x_{01}^2x_{02}^2x_{0'2}^2x_{00'}^2x_{10'}^2y_{00'}^2} + (1\leftrightarrow 2)\Big]\left(\theta_0^+\right)^{2(ac)}\left(\theta_{0'}^+\right)^{2(bd)}, \\
A_{0120'} &= -\frac{(012)}{(010')}\frac{g_{12}(y_{010'})_{ab}(y_{0'20})_{cd}}{x_{01}^2x_{0'2}^2x_{00'}^2x_{02}^2}\left(\theta_0^+\right)^{2(ad)}\left(\theta_{0'}^+\right)^{2(bc)}, \\
A_{1020'} &= \frac{(012)(0'12)}{(200')(10'0)}\frac{g_{12}(y_{0'10})_{ab}(y_{020'})_{cd}}{x_{02}^2x_{01}^2x_{20'}^2x_{0'1}^2}\left(\theta_0^+\right)^{2(bc)}\left(\theta_{0'}^+\right)^{2(ad)},
\end{aligned} \tag{3.4.135}$$

where we remind the reader of the index-less notation

$$y_{ijk} = y_{ij}\tilde{y}_{jk}\,, \qquad y_{ijklm} = y_{ij}\tilde{y}_{jk}y_{kl}\tilde{y}_{lm}\,, \qquad (ijk) = \langle\sigma_{ij}\sigma_{ik}\rangle x_{ij}^2x_{ik}^2\,. \tag{3.4.136}$$

The expressions for the remaining terms on the right-hand side of (3.4.133) can be obtained from (3.4.135) through permutation of the indices, e.g. $A_{1020'} = A_{0120'}[0\leftrightarrow 0', 1\leftrightarrow 2]$, $A_{00'12} = A_{0120'}[1\leftrightarrow 2]$ and $A_{0'021} = A_{0120'}[0\leftrightarrow 0']$.

Note that the contribution to (3.4.133) from $A_{010'2}$ is independent of the reference twistor. Substituting (3.4.135) into (3.4.133) we use Schouten identities and perform

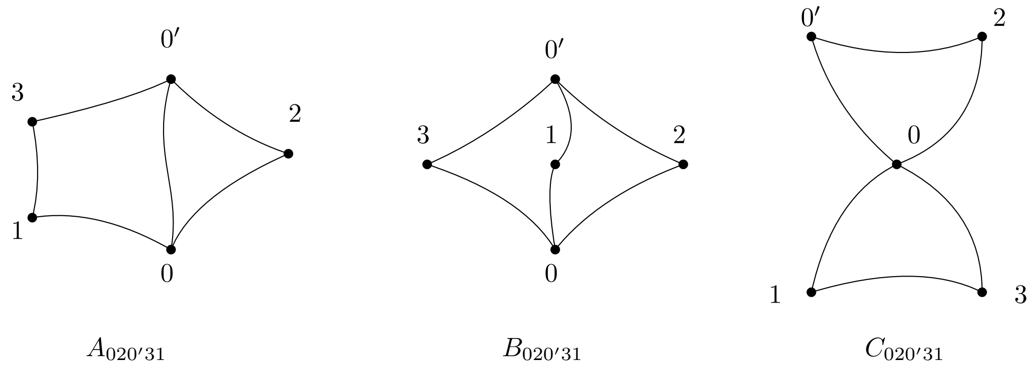


Figure 3.13: Contributions to $G_{5;1}\big|_{(\theta_0^+)^{2(ab)}(\theta_{0'}^+)^{2(cd)}}$.

some algebra to find the result:

$$\begin{aligned} G_{4;1} \sim & \frac{1}{x_{01}^2 x_{10'}^2 x_{00'}^2 x_{0'2}^2 x_{02}^2 y_{00'}^2} \Big[y_{0'2}^2 y_{02}^2 (y_{010'})_{ab} (y_{0'10})_{cd} - y_{10'}^2 y_{01}^2 (y_{020'})_{ab} (y_{00'2})_{cd} \\ & - y_{10'}^2 y_{20}^2 (y_{010'})_{ab} (y_{0'20})_{cd} - y_{20'}^2 y_{10}^2 (y_{020'})_{ab} (y_{0'20})_{cd} - y_{12}^2 y_{00'}^2 (y_{010'})_{ab} (y_{0'20})_{cd} \\ & + (y_{010'20})_{ad} (y_{0'2010'})_{bc} \Big] \left(\theta_0^+\right)^{2(ad)} \left(\theta_{0'}^+\right)^{2(bc)} . \end{aligned} \tag{3.4.137}$$

The expression inside the square brackets vanishes via a non-trivial y-identity. The easiest way to see this is to use the SU(4) covariance of (3.4.137) in order to fix the y-variables at the four points as:

$$y_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad y_1 \to \infty, \qquad y_{0'} \to 0, \qquad y_2 \to \begin{pmatrix} y & 0 \\ 0 & \bar{y} \end{pmatrix}. \tag{3.4.138}$$

Implementing this choice sets (3.4.137) to zero. Hence, the $\left(\theta_0^+\right)^{2(ad)} \left(\theta_{0'}^+\right)^{2(bc)}$ component of $G_{4;1}$ vanishes. Similarly to the $\left(\theta_0^+\right)^4$ case, we expect $G_{4;1}$ to completely vanish for all components due to $\mathcal{N} = 4$ superconformal symmetry.

Five-points

We now turn to five points, in which the set of graphs are in figure 3.13.

At five points, the correlation function $G_{5;1}\big|_{(\theta_0^+)^{2(ab)}(\theta_{0'}^+)^{2(cd)}}$ receives contributions from the graphs of three different topologies:

$$\begin{aligned} A_{020'31} &= g_{02} g_{20'} g_{00'} g_{01} g_{31} g_{0'3} R(0; 20'1) R(0'; 302) , \\ B_{020'31} &= g_{03} g_{0'3} g_{01} g_{0'1} g_{02} g_{20'} R(0; 312) R(0'; 213) , \\ C_{020'31} &= g_{02} g_{00'} g_{03} g_{01} g_{20'} g_{31} R(0; 0'31) R(0; 20'3) . \end{aligned} \tag{3.4.139}$$

$G_{5;1}\big|_{(\theta_0^+)^{2(ab)}(\theta_{0'}^+)^{2(cd)}}$ is given by their total sum symmetrised with respect to the permutations of the five points. Replacing the R-vertices in (3.4.139) by their expansion in powers of the Grassmann variables, we find that this component does not receive contributions from graphs of type C for all possible relabelings of the points. The total set of contributing graphs is

$$\begin{aligned} G_{5;1}\big|_{(\theta_0^+)^{2(ab)}(\theta_{0'}^+)^{2(cd)}} \sim & A_{020'31} + \frac{1}{2}\left(A_{100'32} + A_{10'032} + A_{300'12} + A_{30'012} + B_{10'302}\right) \\ & + \frac{1}{6}\, B_{020'31} + \text{perm}_{123}\ . \end{aligned} \tag{3.4.140}$$

Here, numerical factors are introduced to account for over-counting in the sum over permutations. The set of graphs within the parenthesis have been explicitly permuted by (13), hence a further sum over permutations will double this, thus we include the factor of a half. The graph $B_{020'31}$ sees points 1, 2 and 3 with equal footing, hence there exists a complete S_3 symmetry and thus a factor of $1/3! = 1/6$ has to be included.

Going through calculations similar to those performed in the four-point case, we obtain the following expressions for the component $\left(\theta_0^+\right)^{2(ab)}\left(\theta_{0'}^+\right)^{2(cd)}$:

$$\begin{aligned} A_{020'31} &= -\frac{y_{31}^2 (y_{01230'})_{ab}(y_{020'})_{dc}}{x_{02}^2 x_{20'}^2 x_{00'}^2 x_{0'3}^2 x_{31}^2 x_{01}^2} \left(\theta_0^+\right)^{2(ad)}\left(\theta_{0'}^+\right)^{2(bc)}, \\ A_{100'32} &= \frac{(0'31)}{(0'30)} \frac{y_{23}^2 y_{21}^2 (y_{010'})_{ab}(y_{0'30})_{cd}}{x_{01}^2 x_{0'1}^2 x_{0'3}^2 x_{00'}^2 x_{23}^2 x_{21}^2} \left(\theta_0^+\right)^{2(ad)}\left(\theta_{0'}^+\right)^{2(bc)}, \\ B_{020'31} &= \frac{(y_{02130})_{ab}(y_{0'3120'})_{cd}}{x_{02}^2 x_{20'}^2 x_{0'3}^2 x_{30}^2 x_{01}^2 x_{0'1}^2} \left(\theta_0^+\right)^{2(ab)}\left(\theta_{0'}^+\right)^{2(cd)}, \\ B_{10'302} &= \frac{(0'31)(031)}{(30'0)(100')} \frac{y_{23}^2 y_{21}^2 (y_{0'30})_{ab}(y_{010'})_{cd}}{x_{01}^2 x_{03}^2 x_{0'1}^2 x_{0'3}^2 x_{23}^2 x_{21}^2} \left(\theta_0^+\right)^{2(bc)}\left(\theta_{0'}^+\right)^{2(ad)}. \end{aligned} \tag{3.4.141}$$

The remaining graphs can be obtained by permuting the indices in these expressions.

Notice that the expressions for $A_{020'31}$ and $B_{020'31}$ do not depend on the reference twistor and have the correct conformal and SU(4) properties. Then, we examine the sum of graphs in the parentheses in (3.4.140)

$$\begin{aligned} & A_{100'32} + A_{10'032} + A_{300'12} + A_{30'012} + B_{10'302} = \\ & \frac{y_{21}^2 y_{23}^2}{\prod_{0\le i<j\le 1} x_{ij}^2} \frac{x_{02}^2 x_{20'}^2 x_{31}^2}{(30'0)(100')} (y_{0'30})_{ab}(y_{010'})_{cd} \left(\theta_0^+\right)^{2(bc)}\left(\theta_{0'}^+\right)^{2(ad)} \times \Big[(0'31)(031)x_{00'}^2 \\ & + (310)(0'10)x_{0'3}^2 + (00'3)(10'3)x_{01}^2 + (0'31)(10'0)x_{03}^2 + (310)(030')x_{0'1}^2\Big] \\ & = -\frac{y_{21}^2 y_{23}^2}{\prod_{0\le i<j\le 1} x_{ij}^2} x_{02}^2 x_{20'}^2 x_{31}^4 (y_{0'30})_{ab}(y_{0'10})_{dc} \left(\theta_0^+\right)^{2(bc)}\left(\theta_{0'}^+\right)^{2(ad)}, \end{aligned} \tag{3.4.142}$$

where in the second relation we made use of the six-term identity:

$$(10'3)(0'30)x_{01}^2 - (10'3)(013)x_{00'}^2 + (010')(10'3)x_{03}^2 + (013)(00'3)x_{10'}^2 - (010')(00'3)x_{13}^2 + (010')(013)x_{0'3}^2 = 0\,, \tag{3.4.143}$$

in which a derivation is given in C.0.4. We observe that the dependence on the reference twistor disappears in the sum of graphs.

Finally, we substitute (3.4.141) and (3.4.142) into (3.4.140) and obtain the following expression for the component $\left(\theta_0^+\right)^{2(ac)}\left(\theta_{0'}^+\right)^{2(bd)}$ of the correlation function

$$\begin{aligned} G_{5;1} = & \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}\Big[-\frac{1}{2}x_{02}^2x_{20'}^2x_{31}^4y_{21}^2y_{23}^2(y_{030'})_{ab}(y_{010'})_{cd} \\ & - x_{03}^2x_{23}^2x_{21}^2x_{0'1}^2y_{31}^2(y_{01230'})_{ab}(y_{020'})_{cd} \\ & + \frac{1}{6}x_{00'}^2x_{23}^2x_{21}^2x_{31}^2(y_{02130})_{ac}(y_{0'3120'})_{bd} + \text{perm}_{123}\Big]\left(\theta_0^+\right)^{2(ac)}\left(\theta_{0'}^+\right)^{2(bd)}. \end{aligned} \tag{3.4.144}$$

This completes the computational part of this chapter.

3.5 Conclusion

In this chapter we have developed a new approach to computing the correlation function G_n of the chiral part of the stress-tensor supermultiplet at the Born level. It relies on the reformulation of the $\mathcal{N} = 4$ SYM in twistor space and gives G_n as a sum of Feynman graph build from vertices and edges. Whilst the edges are bosonic propagators, the vertices are some concatenation of trivalent R-vertices $R(i; j_1j_2j_3)$. Throughout the calculation, superconformal invariance is always manifest however the price to pay is the existence of the reference supertwistor $\mathcal{Z}_*$, in which the complete sum of diagrams in independent of but individual diagrams are not. In the last section, we showed some computations and how some z_*-invariant blocks can be built.

An interesting connection to scattering amplitudes is through the lightlike limit. It is known that the corresponding amplitudes are not only superconformally invariant but dual superconformal invariant, which join to form Yangian invariance [49]. A consequence of this is that the lightlike limit of these correlation function are Yangian invariant, it is therefore an interesting question as to whether the twistor reformulation permits some manifest notion of enlarged symmetries.

Another interesting direction is in light of the 'Amplituhedron' in which the proposition is that scattering amplitudes can be viewed as volume forms of a positive subspace of various Grassmannians [25]. A precursor to the Amplituhedron is the Grassmannian formulation [65], which is essentially rooted in the twistor Feynman rules [32]. Now that we have a better understanding for how the twistor reformulation of the correlation function works, it would interesting to see if an analogous development can be made for the correlation function. More importantly, this development would endeavour to find some geometric notion whose lightlike limit recaptures both algebraic as well as geometric features of the amplitude.

Lastly, one could develop the method here beyond the stress-tensor supermultiplet. The next operator to check would be the twist-3 $\frac{1}{2}$-BPS operator. It would interesting to see how the graphs and the corresponding R-vertex should generalise.

Chapter 4

Scattering amplitudes: the six-point NMHV amplitude

This chapter is based on the paper '*Boostrapping correlation functions in $\mathcal{N}=4$ SYM*' by D.Chicherin, R.D, B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev [55]. In particular, we will review section 4 from that paper.

Much effort has been made in understanding and developing computational as well as conceptual progress in scattering amplitudes in planar $\mathcal{N}=4$ SYM. We developed much of the basic construction in section 3.1.1, and in this chapter we will be solely interested in the simplest possible nilpotent superamplitude, which is the NMHV amplitude at n-points in planar $\mathcal{N}=4$ SYM. We recall the result from section 3.3, as

$$\hat{\mathcal{A}}^{(0)}_{n;1} = \sum_{i<j} \int \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4}{s_1 s_2 s_3 s_4} \delta^{4|4}\left(\mathcal{Z}_* + s_1 \mathcal{Z}_{i-1} + s_2 \mathcal{Z}_{j-1} + s_3 Z_{j-1} + s_4 Z_j\right). \qquad (4.0.1)$$

The main result of the present chapter is to write a different representation of $\hat{\mathcal{A}}^{(0)}_{6;1}$ in (4.0.1) which manifests as much information about the physical pole structure and dual superconformal symmetry as possible.

The current representation in (4.0.1) is manifestly dual superconformal invariant (as we shall show). The only poles that appear in this result are bosonic and come in two types. Firstly there are the so-called physical poles which have the form

$$\langle z_{i-1} z_i z_{j-1} z_j \rangle = \epsilon_{ABCD} z^A_{i-1} z^B_i z^C_{j-1} z^D_j\,, \qquad (4.0.2)$$

using the twistor notation in (2.2.54). We also have non-physical poles which are simply poles not of the form in (4.0.2). Non-physical poles are spurious which means that over

$R_{ij} =$

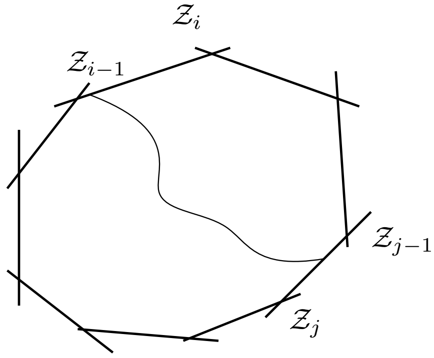


Figure 4.1: A single graphical contribution to the n-point NHMV amplitude.

the complete sum in (4.0.1), they will vanish to reveal a result which only contains physical poles. In this way, one may regard the required physical pole structure as being an emergent property in this formalism. In the current chapter we would like to find a different representation for $\hat{\mathcal{A}}^{(0)}_{6;1}$ that will manifest half of the dual superconformal symmetry whilst possessing physical poles only.

An addition, we will also find that the result is remarkably compact. It contains a single non-trivial term plus its dihedral permutations.

To get there, we will begin by familiarising ourselves with the result in (4.0.1) by deriving it from the twistor Feynman rules in (3.2.57). We shall highlight some of the basic properties of this class of superamplitude. We will develop the new but related basis for the result and we will proceed to give the result at six points.

4.1 Review

In this section we will review the construction of the n-point NMHV scattering amplitudes from the on-shell twistor Feynman rules studied in the previous chapter and given by (3.2.46) and (3.2.47b). We will also briefly explain some of the basic properties related to superconformal symmetry, identities between R-invariants in (4.0.1) and the singularity structure of the result.

4.1.1 Derivation from twistors

A single graphical contribution to the n-point NMHV superamplitude is given by figure 4.1. Using (3.2.57), we find the result in terms of dual twistors (with color structure given by $\mathrm{tr}\,(T^1T^2\dots T^n)$, but stripped away from here on out):

$$R_{ij} = \int \frac{\mathrm{d}u}{u}\frac{\mathrm{d}v}{v}\Delta\left(\mathcal{Z}(u), \mathcal{Z}_*, \mathcal{Z}(v)\right), \tag{4.1.3}$$

where $\mathcal{Z}(u) = \mathcal{Z}_{i-1} + u\mathcal{Z}_i$ and $\mathcal{Z}(v) = \mathcal{Z}_{j-1} + v\mathcal{Z}_j$, and also

$$\Delta\left(\mathcal{Z}_1, \mathcal{Z}_*, \mathcal{Z}_2\right) = \int \frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\delta^{4|4}\left(s\mathcal{Z}_1 + t\mathcal{Z}_2 + \mathcal{Z}_*\right). \tag{4.1.4}$$

To simplify the situation we perform the change of integration, namely in defining $x = su$ and $y = tv$, thus gaining $\mathrm{d}u = (\mathrm{d}x + u\mathrm{d}s)\,\frac{u}{x}$ and $\mathrm{d}v = (\mathrm{d}y + v\mathrm{d}t)\,\frac{v}{y}$, in which we find

$$R_{ij} = \int \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\delta^{4|4}\left(s\mathcal{Z}_{i-1} + x\mathcal{Z}_i + t\mathcal{Z}_{j-1} + y\mathcal{Z}_j + \mathcal{Z}_*\right). \tag{4.1.5}$$

As we have previously observed, we have four bosonic delta functions against four bosonic integrations, hence we expect to be left with a four dimensional fermionic delta function only. Using standard delta function manipulations, we find

$$\begin{aligned}
\delta^4\left(sz_{i-1} + xz_i + tz_{j-1} + yz_j + z_*\right) &= \frac{1}{\langle z_{i-1}z_iz_{j-1}z_j\rangle} \\
\times\,\delta\left(s - \frac{\langle z_iz_{j-1}z_jz_*\rangle}{\langle z_{i-1}z_iz_{j-1}z_j\rangle}\right)&\delta\left(x - \frac{\langle z_*z_{i-1}z_{j-1}z_j\rangle}{\langle z_{i-1}z_iz_{j-1}z_j\rangle}\right) \\
\delta\left(t - \frac{\langle z_{i-1}z_iz_jz_*\rangle}{\langle z_{i-1}z_iz_{j-1}z_j\rangle}\right)&\delta\left(y - \frac{\langle z_*z_{i-1}z_iz_{j-1}\rangle}{\langle z_{i-1}z_iz_{j-1}z_j\rangle}\right).
\end{aligned} \tag{4.1.6}$$

Putting this back in (4.1.5), results in

$$R_{ij} = \frac{\delta^{0|4}\left(\langle i-1\;i\;j-1\;j\rangle\,\chi_* + \mathrm{cyclic}_{i-1ij-1j*}\right)}{\langle i-1\;i\;j-1\;j\rangle\,\langle i\;j-1\;j\;*\rangle\,\langle j-1\;j\;\;*\;\;i-1\rangle\,\langle j\;\;*\;\;i-1\;i\rangle\,\langle *\;i-1\;i\;j-1\rangle}, \tag{4.1.7}$$

where we are using the shorthand $\langle ijkl\rangle = \langle z_iz_jz_kz_l\rangle$.

The $\mathcal{Z}_*$ is the reference supertwistor and is the gauge parameter as explained below (3.2.46). The sum of all contributing terms in (4.0.1) is independent of $\mathcal{Z}_*$. We may therefore set it to one of the external data points. Given R_{ij}, if we set $\mathcal{Z}_*$ to one of the other points in (4.1.7), for example setting $\mathcal{Z}_* = \mathcal{Z}_{i-1}$ makes (4.1.7) linear in zero (the numerator being fourth order, whilst the denominator is third order). This holds true when taking any of the five supertwistors making up (4.1.7) to be collinear.

The full n-point NMHV amplitude is therefore written as

$$\mathcal{A}_{n;1} = \frac{\delta^{(4)}(P)\delta^8(Q)}{\langle 12\rangle\,\langle 23\rangle\dots\langle n1\rangle}\sum_{i<j}R_{ij}, \tag{4.1.8}$$

where we recall the pre-factor from our discussion in (3.1.25), as well as the summation existing for non-zero contributions.

4.1.2 Properties

We would like to review some important properties of the n-point NMHV amplitude. We will discuss aspects of the superconformal invariance, identities and the result having the correct pole structure despite not manifestly being so.

Q and $\bar{Q}$ invariance

Following [67], in dual twistor space we can package the supercharges and superconformal charges into the objects

$$\begin{aligned} Q_I^A &:= \left(Q_I^\alpha, \bar{S}_I^{\dot\alpha}\right) = \sum_{i=1}^n z_i^A \frac{\partial}{\partial \chi_i^I} \\ \bar{Q}_A^I &:= \left(\bar{Q}_{\dot\alpha}^I, S_\alpha^I\right) = \sum_{i=1}^n \chi_i^I \frac{\partial}{\partial z_i^A}, \end{aligned} \tag{4.1.9}$$

where the definition of z^A and χ^I are given by (2.2.48),(2.2.49) and (2.2.69). One can compute the algebra

$$\left\{Q_I^A, \bar{Q}_D^J\right\} = \delta_D^A R_I^J + \delta_I^J K_D^A, \tag{4.1.10}$$

where

$$R_J^I = \sum_{i=1}^n \chi_i^I \frac{\partial}{\partial \chi_i^J} \text{ and } K_B^A = \sum_{i=1}^n z_i^A \frac{\partial}{\partial z_i^B}. \tag{4.1.11}$$

Whilst R_J^I is the twistor representation for the internal SU(4) operator, K_B^A is a generator for the conformal algebra. In this subsection we would like to apply the charges Q and $\bar{Q}$ to the object R_{ij} and see that we get zero explicitly. We will follow this with some comments on properties of general superconformal invariance.

For convenience let us take R_{35} and set $\mathcal{Z}_* = \mathcal{Z}_1$. Beginning with Q_I^A we see that it will act non-trivially on the numerator of R_{35}, namely Q_I^A invariance follows from

$$\begin{aligned} Q_I^A\left(\langle 1234\rangle \chi_5^J + \text{cyclic}_{12345}\right) &= \sum_{i=1}^5 z_i^A \frac{\partial}{\partial \chi_i^I}\left(\langle 1234\rangle \chi_5^J + \text{cyclic}_{12345}\right) \\ &= \delta_I^J\left(z_1^A\langle 2345\rangle + \text{cyclic}_{12345}\right). \end{aligned} \tag{4.1.12}$$

However, we know that

$$\left(z_1^A\langle 2345\rangle + \text{cyclic}_{12345}\right) = 0, \tag{4.1.13}$$

as this is the Schouten identity. Importantly, it follows that any function whose χ dependency is through the combination in $\left(\langle 1234\rangle\,\chi_5^J + \text{cyclic}_{12345}\right)$ is Q invariant.

Next we look at $\bar{Q}^I_A$. In this case it is advantageous to go back to the form given by (4.1.5), in which we find that

$$\bar{Q}^I_A R_{35} = \sum_{i=1}^{5} \chi_i^I \int \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\frac{\partial}{\partial z_i^A}\delta^{4|4}\left(s\mathcal{Z}_2 + x\mathcal{Z}_3 + t\mathcal{Z}_4 + y\mathcal{Z}_5 + \mathcal{Z}_1\right). \tag{4.1.14}$$

We need to split the delta function into its constituent bosonic and fermionic parts, whereby we recall the definition of the bosonic delta function[1], to get

$$\begin{aligned}\bar{Q}^I_A R_{35} &\propto \left(s\chi_2^I + x\chi_3^I + t\chi_4^I + y\chi_5^I + \chi_1^I\right)\delta^{0|4}\left(s\chi_2 + x\chi_3 + t\chi_4 + y\chi_5 + \chi_1\right)\\ &= 0,\end{aligned} \tag{4.1.15}$$

since we are multiplying the same Grassmann odd combination by its maximally nilpotent amount. In contrast to the Q case, here we require the χ dependency to be at its maximal order (of four) since if it had not been then $\bar{Q}^I_A R_{35} \neq 0$.

More generally, $\bar{Q}$ invariance of both charges required the dimensionality of the even and odd space to be the same. As noted in [70], the full set of generators associated to the superconformal group in dual twistors is packaged into the charge

$$J^{\mathcal{A}}_{\mathcal{B}} = \sum_i \left[\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{B}}} - \frac{1}{8}(-1)^{\deg(\mathcal{A})+\deg(\mathcal{C})}\delta^{\mathcal{A}}_{\mathcal{B}}\mathcal{Z}_i^{\mathcal{A}}\frac{\partial}{\partial \mathcal{Z}_i^{\mathcal{A}}}\right], \tag{4.1.16}$$

from which the diagonal parts of the algebra annihilate R_{35} and correspond to generators the maximal bosonic subalgebra, whilst the off-diagonal parts are Q and $\bar{Q}$. It can be shown that there exists a bilocal operator which is a function of $J^{\mathcal{A}}_{\mathcal{B}}$, which also annihilates the amplitude. This turns out to imply Yangian invariance which is a signature of a powerful integrable structure [70].

In fact, one can essentially build any superconformal invariant of the more general group $\mathrm{SL}(m|m)$ by demanding that the basis is built out of functions of $\delta^{m|m}\left(\sum_{i=1}^{n} s_i\mathcal{Z}_i\right)$.

[1] $\delta^4\left(sz_2 + xz_3 + tz_4 + yz_5 + z_1\right) \sim \int \mathrm{d}^4 q e^{iq_A\left(sz_2^A + xz_3^A + tz_4^A + yz_5^A + z_1^A\right)}$

Identities

Let us for the moment define

$$R_{ijklm} = \int \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\delta^{4|4}\left(s\mathcal{Z}_i + x\mathcal{Z}_j + t\mathcal{Z}_k + y\mathcal{Z}_l + \mathcal{Z}_m\right), \tag{4.1.17}$$

so that the subscripts now label the supertwistors appearing in the argument of the delta function. Then given six points labelled $i_1 \ldots i_6$, we have

$$R_{i_1i_2i_3i_4i_5} + R_{i_6i_1i_2i_3i_4} + R_{i_5i_6i_1i_2i_3} + R_{i_4i_5i_6i_1i_2} + R_{i_3i_4i_5i_6i_1} + R_{i_2i_3i_4i_5i_6} = 0, \tag{4.1.18}$$

which is true for any i_j labels.

Singularity Structure

Consider the computation of scattering amplitudes from the traditional Feynman diagram approach whereby amplitudes in a Yang-Mills theory are necessarily functions of momenta and the polarisation vectors. More specifically, we expect poles to only come from propagators which come in the form of $\sim \frac{1}{k^2}$.

Comparing this to results via the twistor formalism, it is important that the final result is in terms of 4-brackets of the form $\langle i-1\ i\ j-1\ j\rangle \sim x_{ij}^2$. Since these are dual twistors, there is a correspondence with momenta, namely $x^\mu_{i\ i+1} = k^\mu_i$ mod n and so

$$\left(x_{i\ i+1} + x_{i+1\ i+2} + \cdots + x_{j-1\ j}\right)^\mu = x^\mu_{ij} = \left(k_i + \ldots k_{j-1}\right)^\mu. \tag{4.1.19}$$

So, for any physical scattering process to make sense in the twistor formalism the pole structure of an amplitude should only come from structures like (4.1.19) and singularities should only develop in certain cases where for example an external leg is made to be soft or made to be collinear with another leg. This means that any non-physical pole (one which is not of the form $\langle i-1\ i\ j-1\ j\rangle$) in $\mathcal{A}_{n;1}$ must vanish.

Let us consider this. The six-point NMHV amplitude is given by $\hat{\mathcal{A}}^{(0)}_{n;1} = \sum_{i<j} R_{ij}$. Now, since each R_{ij} is $\mathcal{Z}_*$-dependent, but the sum given in $\hat{\mathcal{A}}^{(0)}_{n;1}$ is $\mathcal{Z}_*$-independent, we can fix it to a specific external value. However, it is also pertinent to the gauge invariance of the $\hat{\mathcal{A}}^{(0)}_{6;1}$ that apparent poles involving $\mathcal{Z}_*$ vanish. We can look at the five-point example:

$$\hat{\mathcal{A}}^{(0)}_{5;1} = R_{13} + R_{14} + R_{24} + R_{25} + R_{35}. \tag{4.1.20}$$

In setting $\mathcal{Z}_* = \mathcal{Z}_5$, R_{24} is the only surviving term and all the poles are physical. Instead, let us suppose that we didn't do this then we would expect the residue of the pole, for example $\langle 123*\rangle$ to vanish. Focussing on this example, we expect

$$\text{Residue}_{\langle 123*\rangle=0}\left(R_{13}+R_{24}\right)=0, \tag{4.1.21}$$

since these are the only R's in which this pole appears. Focussing on the χ_1^4 term, we find

$$\left(R_{13}+R_{24}\right)\Big|_{\chi_1^4}=\frac{\chi_1^4}{\langle 123*\rangle}\left(\frac{\langle 23*5\rangle^3}{\langle 5123\rangle\langle 3*51\rangle\langle *512\rangle}-\frac{\langle 234*\rangle^3}{\langle 1234\rangle\langle 34*1\rangle\langle 4*12\rangle}\right). \tag{4.1.22}$$

Then by using the Schouten identity $\langle 4512\rangle\, z_*^A + \text{cyclic}_{4512*} = 0$ repetitively on the constraint $\langle 123*\rangle = 0$, we gain the result in (4.1.21). This illustrates how the residues of non-physical singularities vanish.

4.2 The six-point NMHV amplitude

We now reach the main purpose of this chapter. In the first subsection we will describe the basic ingredients whilst in the second subsection we will explain the result.

4.2.1 Constructing a new kind of basis

As given in (3.1.24), the full superamplitude can be written as

$$\mathcal{A}_n=\mathcal{A}_{n;0}\left(1+\hat{\mathcal{A}}_{n;1}+\cdots+\hat{\mathcal{A}}_{n;n-4}\right). \tag{4.2.23}$$

To construct the required invariants let us follow the construction in section 3.1.2. There we wrote the $(Q+\bar{S})$ invariant as $\mathcal{I}_{n;p}\left(x,y,\theta^+\right)=Q^8\bar{S}^8\mathcal{J}_{n;p+4}\left(x,y,\theta^+\right)$.

Similarly for the amplitude, one can invent an invariant $I_{n;p}$, such that

$$I_{n;p}(z,\chi)=Q^8\bar{S}^8 J_{n,p+4}(z,\chi), \tag{4.2.24}$$

where $J_{n,p+4}(z,\chi)$ is a homogeneous polynomial in χ of order $4(p+4)$.

Let us study the maximally nilpotent component first before going the the next-to-maximally nilpotent component. The maximally nilpotent component of the super-amplitude is given by $\hat{\mathcal{A}}_{n;n-4}$, and thus has $p=n-4$. This implies that $J_{n,n}(z,\chi)$ is a homogeneous polynomial in χ of order $4n$, from which one can only write (where we use $\chi^4 = \delta^{0|4}(\chi)$)

$$J_{n,n}(z,\chi) = \prod_{i=1}^{n} \chi_i^4 = \prod_{i=1}^{n} \delta^{0|4}(\chi_i). \tag{4.2.25}$$

Following the invariance under the operators in (4.1.9), we note that the variation of χ^I is given by

$$\chi^I \to \hat{\chi}^I = \chi^I + z^A M_A^I, \tag{4.2.26}$$

where M_A^I is a 4 by 4 matrix of parameters associated to Q_I^α and $\bar{S}_I^{\dot{\alpha}}$. It therefore follows from (4.2.24) that

$$I_{n;n-4} = \int \mathrm{d}^{16}M \prod_{i=1}^{n} \delta^{0|4}(\hat{\chi}_i). \tag{4.2.27}$$

Then for all n the result should take the form

$$\hat{\mathcal{A}}_{n;n-4} = f(z) I_{n;n-4}, \tag{4.2.28}$$

and since $I_{n;n-4}$ has homogeneity $(+4)$ at each point, $f(z)$ is required to have homogeneity (-4) at each point. [2]

Since we have seen the five-point NMHV tree-level amplitude from the basis in (4.1.7), let us re-observe the result from this point of view. At five-points the result is given by

$$\hat{\mathcal{A}}_{5;1} = R_{35} = \frac{\delta^{0|4}\left(\langle 1234\rangle \chi_5 + \text{cyclic}_{12345}\right)}{\langle 1234\rangle \langle 2345\rangle \langle 3451\rangle \langle 4512\rangle \langle 5123\rangle}, \tag{4.2.29}$$

and since this result must give $R_{35} = f(z) I_{5;1}$, we can identify

$$f(z) = \frac{1}{\langle 1234\rangle \langle 2345\rangle \langle 3451\rangle \langle 4512\rangle \langle 5123\rangle}, \tag{4.2.30}$$

and we require that

$$I_{5;1}(z,\chi) = \int \mathrm{d}^{16}M \prod_{i=1}^{5} \delta^{0|4}(\hat{\chi}_i) = \delta^{0|4}\left(\langle 1234\rangle \chi_5 + \text{cyclic}_{12345}\right). \tag{4.2.31}$$

[2] By homogeneity we mean that given a function of supertwistors $\mathcal{H}(\mathcal{Z}_1,\ldots,\mathcal{Z}_n)$, the function has homogeneity $\{a_1,\ldots,a_n\}$ at points $\{1,\ldots,n\}$ if $\mathcal{H}(\lambda_1\mathcal{Z}_1,\ldots,\lambda_n\mathcal{Z}_n) = \lambda_1^{a_1}\ldots\lambda_n^{a_n}\mathcal{H}(\mathcal{Z}_1,\ldots,\mathcal{Z}_n)$.

To see the equivalence in (4.2.31), we can either simply compute the result at each component in $\chi_i s$ on either side of (4.2.31), or more elegantly we can multiply $I_{5;1}$ by the unit insertion

$$1 = \int \mathrm{d}s_1 \ldots \mathrm{d}s_4 \delta^4 \left(z_5 + \sum_{i=1}^{4} s_i z_i \right) \langle 1234 \rangle \,, \tag{4.2.32}$$

from it which it follows that we get (4.1.5) [3].

Let us now investigate a new basis which we can use for the next-to-nilpotent component of the superamplitude, namely $\hat{\mathcal{A}}_{n,n-5}$. Here, $\hat{\mathcal{A}}_{n,n-5}$ is a homogeneous polynomial in χ of order $4(n-5)$. The invariant in (4.2.24) becomes $I_{n;n-5} = Q^8 \bar{S}^8 J_{n,n-1}(z,\chi)$ and therefore $J_{n,n-1}(z,\chi)$ is a homogeneous polynomial in χ of order $4(n-1)$. In comparison with the previous case in (4.2.25), we cannot write a single unique term as a basis given by the product of all χ_i^4. Instead, we need to strip off four units of Grassmann oddness. This can be accomplished in general by taking derivatives of the maximally nilpotent case

$$J_{ijkl;IJKL}(z,\chi) = \frac{\partial}{\partial \hat{\chi}_i^I} \frac{\partial}{\partial \hat{\chi}_j^J} \frac{\partial}{\partial \hat{\chi}_k^K} \frac{\partial}{\partial \hat{\chi}_l^L} \int \mathrm{d}^{16} M \prod_{i=1}^{n} \delta^{0|4} \left(\hat{\chi}_i \right) , \tag{4.2.33}$$

then we have

$$I_{ijkl}(z,\chi) = \epsilon^{IJKL} J_{ijkl;IJKL}(z,\chi) = \epsilon^{IJKL} \frac{\partial}{\partial \hat{\chi}_i^I} \frac{\partial}{\partial \hat{\chi}_j^J} \frac{\partial}{\partial \hat{\chi}_k^K} \frac{\partial}{\partial \hat{\chi}_l^L} \int \mathrm{d}^{16} M \prod_{i=1}^{n} \delta^{0|4} \left(\hat{\chi}_i \right) , \tag{4.2.34}$$

and so now we have a new basis relevant for the $\hat{\mathcal{A}}_{n,n-5}$ component of the superamplitude. We have the freedom to choose the set $\{i,j,k,l\}$ as we wish and therefore this is certainly a large but useful basis for the result. A key property is that I_{ijkl} is

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[3] To see this, $\int \mathrm{d}s_1 \ldots \mathrm{d}s_4 \delta^4 \left(z_5 + \sum_{i=1}^{4} s_i z_i \right) \langle 1234 \rangle I_{5;1}$ can be put on the support of the bosonic delta function which implies $\delta^{0|4}(\chi_5 + z_5 M) \to \delta^{0|4}\left(\chi_5 - \sum_{i=1}^{4} s_i z_i M \right) \to \delta^{0|4}\left(\chi_5 + \sum_{i=1}^{4} s_i \chi_i \right)$, where the second implication follows from the support of the fermionic delta functions. This renders this final delta function as independent of M, in which the remaining M-dependent delta function integrate to give $\langle 1234 \rangle^4$. The result is then

$$\int \mathrm{d}s_1 \ldots \mathrm{d}s_4 \delta^4 \left(z_5 + \sum_{i=1}^{4} s_i z_i \right) \langle 1234 \rangle I_{5;1} = \int \mathrm{d}s_1 \ldots \mathrm{d}s_4 \delta^{4|4} \left(\mathcal{Z}_5 + \sum_{i=1}^{4} s_i \mathcal{Z}_i \right) \langle 1234 \rangle^5 \,,$$

from which (4.2.31) follows.

Q invariant by construction but it is no longer $\bar{Q}$ invariant. Schematically, one can see this from the fact that the χ derivatives in I_{ijkl} will lower the Grassmann degree but the result will depend on χ through the function of form $\big(\langle 1234\rangle\, \chi_5^J + \text{cyclic}_{12345}\big)$. From the results of (4.1.12) and (4.1.15), we recall that Q invariance requires precisely the aforementioned χ dependence but $\bar{Q}$ invariance requires the maximal Grassmann order in χ which cannot be the case since we have taken derivatives in χ. We take this moment to recall that by generalising the basis at the expense of $\bar{Q}$ symmetry we will be able to find a result for the six-point NMHV scattering amplitude $\hat{\mathcal{A}}_{6;1}$ which has no spurious singularities thus containing manifests physical poles structures.

An important property of I_{ijkl} is that it satisfies the following superconformal Ward identity

$$\sum_{i=1}^{6} z_i^A I_{ijkl} = 0\,. \tag{4.2.35}$$

To see this, recall that since $\hat{\chi}_i^I = \chi_i^I + z_i^A M_A^I$ for each i, we also therefore have

$$\frac{\partial}{\partial M_A^I} = \sum_{i=1}^{n} z_i^A \frac{\partial}{\partial \hat{\chi}_i^I}. \tag{4.2.36}$$

It follows that

$$\begin{aligned}\sum_{i=1}^{n} z_i^A I_{ijkl} &= \int \mathrm{d}^{16}M \sum_{i=1}^{n} z_i^A \frac{\partial}{\partial \hat{\chi}_i^I}\left(\epsilon^{IJKL}\frac{\partial}{\partial \hat{\chi}_j^J}\frac{\partial}{\partial \hat{\chi}_k^K}\frac{\partial}{\partial \hat{\chi}_l^L}\right)\\ &= \int d^{16}M \frac{\partial}{\partial M_A^I}(\cdots) = 0\,,\end{aligned} \tag{4.2.37}$$

since M_A^I is a fermionic matrix.

So, in this subsection we have established a basis valid for the $\hat{\mathcal{A}}_{n;n-5}$ component of the superamplitude at n-points.

4.2.2 Results

We now turn to the main result of this chapter.

The tree-level six-point NMHV scattering amplitude $\hat{\mathcal{A}}_{6;1}^{(0)}$ has a well known form which is given by the basis we defined in (4.1.7). Let us use the definition in (4.1.17), where R_{ijklm} labels the external supertwistors appearing in the argument of the delta function.

Namely if we define,

$$R_{ijklm} = \int \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\delta^{4|4}\left(s\mathcal{Z}_i + x\mathcal{Z}_j + t\mathcal{Z}_k + y\mathcal{Z}_l + \mathcal{Z}_m\right), \tag{4.2.38}$$

then

$$\hat{\mathcal{A}}^{(0)}_{6;1} = R_{23456} + R_{12456} + R_{12345}, \tag{4.2.39}$$

which in terms of the new basis defined in (4.2.34) is given by

$$\begin{aligned}\hat{\mathcal{A}}^{(0)}_{6;1} = \frac{I_{1111}}{\langle 2345\rangle\langle 3456\rangle\langle 4562\rangle\langle 5623\rangle\langle 6234\rangle} &+ \frac{I_{3333}}{\langle 4561\rangle\langle 5612\rangle\langle 6124\rangle\langle 1245\rangle\langle 2456\rangle}\\ &+ \frac{I_{5555}}{\langle 6123\rangle\langle 1234\rangle\langle 2346\rangle\langle 3461\rangle\langle 4612\rangle}\,.\end{aligned} \tag{4.2.40}$$

We have shown in section 4.1.2 that these so-called R-invariants possess complete dual superconformal symmetry. However, they individually come with spurious poles and we illustrated how such poles should vanish. The new basis defined by I_{ijkl} differs from the R-invariant by containing half of the dual superconformal symmetry. In this section we want to give a result for $\hat{\mathcal{A}}^{(0)}_{6;1}$ which is given by a combination of I_{ijkl} from (4.2.34) whilst always containing manifestly physical poles.

We do this by giving an ansatz for the answer given by

$$\hat{\mathcal{A}}^{(0)}_{6;1} = \frac{\sum_{i,j,k,l=1}^{6} c_{ijkl} I_{ijkl}}{\langle 1234\rangle\langle 2345\rangle\langle 3456\rangle\langle 4561\rangle\langle 5612\rangle\langle 6123\rangle\langle 1245\rangle\langle 2356\rangle\langle 3461\rangle}, \tag{4.2.41}$$

The coefficients c_{ijkl} are polynomials of the external bosonic twistors and we can find a general form based on the homogeneity of the rest of the function. I_{ijkl} has homogeneity (+3) at point $\{i,j,k,l\}$ and (+4) otherwise, whilst the denominator of (4.2.41) has homogeneity (+6) at each point. This implies that c_{ijkl} must have homogeneity (+3) at points $\{i,j,k,l\}$ and (+2) elsewhere. So we would have

$$c_{ijkl}\left(\lambda_1 z_1, \ldots, \lambda_6 z_6\right) = \left(\lambda_1 \ldots \lambda_6\right)^2 \lambda_i\lambda_j\lambda_k\lambda_l c_{ijkl}\left(z_1,\ldots,z_6\right) \text{ for } \lambda \in \mathbb{C}. \tag{4.2.42}$$

Due to dual conformal invariance we are required to assemble the bosonic twistors in c_{ijkl} into some function of the 4-brackets . Consider the case where $\{i,j,k,l\}$ are all different, then four points are occupied with homogeneity (+3), which correspond to 12 twistors. The remaining points are necessarily different to $\{i,j,k,l\}$, and since we have six points to choose from in total, there are only two remaining points which we may take homogeneity (+2). The result is the 12 twistors corresponding to $\{i, j, k, l\}$ and $2 \times (+2) = 4$ from points not equal to this set. Thus, we have 16 twistors and arranged into 4-brackets requires the product of four 4-brackets. If we take $\{i, j, k, l\}$ to not all be different, a similar counting argument follows, but as we do this the possible ways of distributing the points increases and the corresponding ansatz enlarges.

A further consequence of this argument is that since c_{ijkl} is built from 4-brackets which are totally anti-symmetric, $c_{iiii} = c_{iiij} = 0$ for all possible labels. This is because it inevitably leads to repeating bosonic twistors in the 4-brackets.

The idea is rather simple, putting these constraints in place we would like to compare our ansatz in (4.2.41) with the known result in (4.2.40). To do this, we give every possible allowed coefficient c_{ijkl}, whilst taking away terms related by the Schouten identity in (4.1.13).

Explicitly, subject to the aforementioned constraints on c_{ijkl}, we want to equate and solve for c_{ijkl}:

$$\begin{aligned}\hat{\mathcal{A}}^{(0)}_{6;1} &= \frac{I_{1111}}{\langle 2345\rangle\langle 3456\rangle\langle 4562\rangle\langle 5623\rangle\langle 6234\rangle} + \frac{I_{3333}}{\langle 4561\rangle\langle 5612\rangle\langle 6124\rangle\langle 1245\rangle\langle 2456\rangle} \\ &+ \frac{I_{5555}}{\langle 6123\rangle\langle 1234\rangle\langle 2346\rangle\langle 3461\rangle\langle 4612\rangle} \\ &= \frac{\sum_{i,j,k,l=1}^{6} c_{ijkl} I_{ijkl}}{\langle 1234\rangle\langle 2345\rangle\langle 3456\rangle\langle 4561\rangle\langle 5612\rangle\langle 6123\rangle\langle 1245\rangle\langle 2356\rangle\langle 3461\rangle}. \end{aligned} \tag{4.2.43}$$

The problem is itself rather large and as a result we would like to make use of `Mathamatica`. The first task is to use the superconformal Ward identity to rewrite all possible I_{ijkl} in terms of a common basis, for instance in practice one could choose

$$\{I_{5555}, I_{5556}, I_{5566}, I_{5666}, I_{6666}\}\,. \tag{4.2.44}$$

This is so that in having a common I_{ijkl} basis on either (4.2.43), we only need to compare the coefficients of the choices in (4.2.44).

A simple example of this identity is found by setting $j = k = l = 5$ in (4.2.35), and projecting the result with $\epsilon_{BCDA} z_1^B z_2^C z_3^D$, so that we get

$$\begin{aligned} 0 &= \epsilon_{BCDA} z_1^B z_2^C z_3^D \sum_{i=1}^{6} z_i^A I_{i555} \\ &= \langle 1234\rangle\, I_{4555} + \langle 1235\rangle\, I_{5555} + \langle 1236\rangle\, I_{6555}, \end{aligned} \tag{4.2.45}$$

which gives I_{4555} in terms of I_{5555} and I_{6555}. An example of a more complicated result is given by

$$I_{1122} = \frac{I_{5555}\langle 1345\rangle^2\langle 2345\rangle^2}{\langle 1234\rangle^4} + \frac{2I_{5556}\left(\langle 2345\rangle\langle 2346\rangle\langle 1345\rangle^2 + \langle 1346\rangle\langle 2345\rangle^2\langle 1345\rangle\right)}{\langle 1234\rangle^4}$$
$$+ \frac{I_{5566}\left(\langle 1346\rangle^2\langle 2345\rangle^2 + 4\langle 1345\rangle\langle 1346\rangle\langle 2346\rangle\langle 2345\rangle + \langle 1345\rangle^2\langle 2346\rangle^2\right)}{\langle 1234\rangle^4}$$
$$+ \frac{2I_{5666}\left(\langle 2345\rangle\langle 2346\rangle\langle 1346\rangle^2 + \langle 1345\rangle\langle 2346\rangle^2\langle 1346\rangle\right)}{\langle 1234\rangle^4} + \frac{I_{6666}\langle 1346\rangle^2\langle 2346\rangle^2}{\langle 1234\rangle^4}. \tag{4.2.46}$$

In rewriting all of the I_{ijkl}'s into a common basis we may regard the resulting I_{ijkl}'s as a symbolic basis allowing us to work completely bosonically. We simply look at the coefficients of I_{5555}, I_{5556}, I_{5566}, I_{5666} and I_{6666} in (4.2.43) which leads to a large solvable linear system of equations.

The ansatz for the coefficients c_{ijkl} are the allowed coefficients within the criterion stated earlier with some arbitrary coefficients. We also add up all cyclic permutations and the parity transformation $\{1 \to 6, 2 \to 5, 3 \to 4, 4 \to 3, 5 \to 2, 6 \to 1\}$, which together make the dihedral group in six elements. We also impose that the all 4-brackets are physical, namely are of the form $\langle i-1\ i\ j-1\ j\rangle$. Schematically, some terms in the numerator of (4.2.41) is given by

$$A\left(I_{1244}\langle 1346\rangle\langle 2346\rangle\langle 1245\rangle^2 + I_{2355}\langle 1345\rangle\langle 2356\rangle^2\langle 1245\rangle + \dots\right)$$
$$+ B\left(I_{1244}\langle 1234\rangle\langle 1245\rangle\langle 1346\rangle\langle 2456\rangle + I_{2355}\langle 1245\rangle\langle 1356\rangle\langle 2345\rangle\langle 2356\rangle + \dots\right)$$
$$+ \dots \tag{4.2.47}$$

Then we would like to take away any over-counting due to the Schouten identity. Doing so results in 14 free coefficients.

The main computational exercise is to solve (4.2.43) for the coefficients $A, B, C \dots$. There are many solutions with varying levels of simplicity. In defining the 'simplest' solution as the one involving the least number of non-trivial terms, we find the remarkably compact form for the answer

$$\hat{\mathcal{A}}^{(0)}_{6;1} = \frac{1}{2}\frac{I_{1366}}{\langle 1234\rangle\langle 1245\rangle\langle 1256\rangle\langle 2345\rangle\langle 3456\rangle} + \text{dihedral}_{123456}\ . \tag{4.2.48}$$

where 'dihedral' denote 11 other terms with permuted indices needed to ensure the invariance of $\hat{\mathcal{A}}^{(0)}_{6;1}$ under the cyclic shift of indices and point reversal. Whilst these extra terms exists, it's pleasing that there is only one non-trivial term.

One of the main consequences of this result is further evidence of the tension between manifest full dual superconformal invariance and having manifestly physical pole structure. A similar consequence was also observed in [71].

4.3 Conclusion

In this chapter we set out to generalise the natural basis used for the six-point NMHV scattering amplitude to I_{ijkl}. The familiar basis is given by R_{ijklm} in (4.2.38) possesses full dual superconformal symmetry but the I_{ijkl} basis differs from the R_{ijklm} by not containing $\bar{Q}$ invariance.

In using I_{ijkl} as a generalisation of the R_{ijklm} basis, we sought out a different representation of the six-point NMHV amplitude which would manifest the property of only containing physical poles. The price we pay for this is that the manifest $\bar{Q}$ symmetry is lost.

From a purely aesthetic point of view it is pleasing that the new result only contains a single term whilst having an entire set of terms related by the dihedral symmetric group in six elements.

Looking forward, we can make computational progress by first extending the analysis to higher points. This may lead to the possibility of finding some structure which may lead to some predictive power. If this could be established then it would be interesting to look towards a higher level of nilpotence, again with a view towards structure and predictability.

A useful direction would note that since I_{ijkl} is Q invariant, we should collect some linear combinations of such objects to produce $\bar{Q}$ invariants. Thus some linear combination of these invariants built from I_{ijkl} should produce the amplitude. This would be a large problem but would lead to a structural understanding.

More conceptually, it would be very useful to understand why there is an apparent tension between the physical pole structure and manifesting all of the symmetries. Modern approaches like that of the Amplituhedron and the Grassmannian formalism in [69,25] usually manifest the superconformal properties whilst treating physical poles as an emergent property. However, it remains true that the Amplituhedron has the advantage that it has attached geometric meaning to this concept, namely non-physical poles are as a result of tessellating the physical space. It appears that to get a result with manifest physical pole structure we had to give up a basis with manifest $\bar{Q}$ symmetry, it would fruitful to understand this mechanism more precisely. This may lead to better understanding of the modern geometric methods.

Chapter 5

The superconformal partial wave

This chapter is based on the paper '*Superconformal partial waves in Grassmannian field theories*' by R.D and P. Heslop [72].

Operators in any given superconformal field theory (SCFT) enjoy the associative relation of the operator product expansion (OPE). The OPE in the context of a SCFT is the relation between two local operator and a finite sum of infinite dimensional local operators. The infinite dimensions come from the infinite number of momentum operators that can be applied to some highest weight state to create descendent operators. A well known version of an OPE is in the Littlewood-Richardson rule as applied to the concatenation of finite dimensional representations of $\mathfrak{su}(N)$.

Here we define a Grassmannian (or more specifically $\mathrm{Gr}_{m|n}(2m|2n)$) field theory to be an analytic superspace whose supergroup is $\mathrm{SL}(2m|2n)$, recall section 2.3.2. This space has the coordinate system

$$X^{\underline{\mathcal{AB}}} = \begin{pmatrix} \mathbb{I}_{m|n\times m|n} & X_{\underline{AA'}} \\ 0_{n|m\times m|n} & \mathbb{I}_{n|m\times n|m} \end{pmatrix}, \tag{5.0.1}$$

The superconformal partial wave is a basis for four-point supercorrelation functions which manifests data that arises from the OPE of the constituent operators. The superconformal partial wave is in general a series expansion which is an eigenfunction of the quadratic Casimir of the superconformal algabra. Schematically, one has that

$$\mathcal{D}\mathcal{O}^{\mathrm{Rep}}_{\Delta,s} = C^{\mathrm{Rep}}_{\Delta,s}\mathcal{O}^{\mathrm{Rep}}_{\Delta,s}, \tag{5.0.2}$$

where $\mathcal{D}$ is the quadratic Casimir operator, Δ is the conformal dimension, s is the spin, Rep is the internal representation and $C^{\mathrm{Rep}}_{\Delta,s}$ is the corresponding eigenvalue. One can then organise the four-point function into so-called superconformal partial waves (each associated to an operator appearing in the OPE), and these functions are eigenfunctions with the quadratic Casimir with eigenvalue $C^{\mathrm{Rep}}_{\Delta,s}$.

For particular correlators of $\mathcal{N}=4$ SCFT, Dolan and Osborn were able to perform superconformal partial wave expansions in [77,87]. This was done by making use of the superconformal Ward identities solved by Nirschl and Osborn in [83] in conjunction with unitarity and crossing symmetry constraints. Moreover, they pioneered the use of the superconformal partial wave to extract quantum data in [77,79,80].

In the very recent past, two applications which have enjoyed the use of the superconformal partial wave is the computation of three-point structure constants and the superconformal bootstrap. In the former case, Vieira and Wang used the superconformal partial wave to find corrections to the structure constants and compared against novel integrability techniques in [73]. The latter case began famously with the work of Rattazzi, Rychkov, Tonni and Vichi in [74], where they applied the crossing symmetry of the correlator of scalars in the conformal partial wave basis to deduce the minimally required conformal dimension of an operator appearing in the OPE. This leads to a basic consistency condition of any SCFT, and has been applied to theories with $\mathcal{N}=4$ superconformal symmetry by Beem, Rastelli and Van Rees in [75] and Alday and Bissi in [76].

In this chapter, we wish to solve the quadratic Casimir equation for the superconformal partial waves in analytic superspace with superconformal group $\mathrm{SL}(2m|2n)$.

Along with solving the quadratic Casimir equation the superconformal partial waves are superconformal invariant. We therefore express the superconformal partial wave as a linear combination of superconformal invariants, which turn out to be the Schur superpolynomial, $s_{\underline{\mu}}(x|y)$. Simply stated, a superconformal partial wave defined by some representation in $\mathfrak{sl}(2m|2n)$, can be expanded as

$$F^{\mathrm{rep}}(x|y) = \sum_{\underline{\mu}\geq 0} R^{\mathrm{rep}}_{\underline{\mu}} s_{\underline{\mu}}(x|y), \tag{5.0.3}$$

where by 'solving the quadratic Casimir equation' amounts to finding the numerical coefficients $R^{\mathrm{rep}}_{\underline{\mu}}$.

After providing a review of the Schur superpolynomial and various pertinent aspects of representation theory, we write the superconformal partial waves as a linear combination of such superpolynomials in which the non-trivial data are the corresponding coefficients $R_{\underline{\mu}}^{\text{rep}}$. The problem of finding $R_{\underline{\mu}}^{\text{rep}}$ turns out to be dependent on the representation yet independent of the group. This means that rather than finding $R_{\underline{\mu}}^{\text{rep}}$ for $\mathfrak{sl}(2m|2n)$ for some representation, we can instead solve the problem by considering the same representation of $\mathfrak{sl}(2m|0)$ (or $\mathfrak{sl}(0|2n)$), we thus avoid dealing with the superalgbra directly.

We then write the superconformal partial wave in a novel matrix form based on a form of the Schur superpolynomial.

We then apply the superconformal partial wave to a variety of $\mathcal{N} = 4$ supercorrelators to find the free theory OPE coefficients (the square of three-point structure constants). As an application we work through operator recombination of short operators into long ones which yields a non-trivial twist-4 sector for the the four-point function of $\text{tr}(W^3)$ in the $\text{SU}(N)$ theory.

5.1 Review

In this review we will go through some basic motivating concepts by first defining the OPE and how to gain the superconformal partial waves from the four-point function, whilst giving the basic definition of the bosonic conformal partial wave and some schematic details of the supersymmetric case as studied in [77,83]. We regard the work of Dolan and Osborn in [80] and Heslop [81] to be the leading motivating study, hence some details of that work is provided here.

5.1.1 From the OPE to the four-point function

Given two scalar operators ϕ_1 and ϕ_2 of conformal dimension Δ_i in some four dimensional CFT, the OPE is given by [79]

$$\phi_1(x_1)\phi_2(x_2) = \sum_{\mathcal{O}} C^{\mathcal{O}}_{\Delta_1\Delta_2} \left(\frac{1}{x_{12}^2}\right)^{\frac{\Delta_1+\Delta_2-\Delta}{2}} C^{\Delta;\underline{\alpha\dot{\alpha}}}(x_{12},\partial_2)\mathcal{O}^{\Delta}_{\underline{\alpha\dot{\alpha}}}(x_2), \tag{5.1.4}$$

where Δ is the conformal dimension of operators $\mathcal{O}$ in the OPE, there is also the index structure $\underline{\alpha\dot{\alpha}}$ which represents some chain in α and $\dot{\alpha}$ indices. The object $C^{\Delta;\underline{\alpha\dot{\alpha}}}(x_{12},\partial_2)$ is a formal expansion in $x_{12}^{\alpha\dot{\alpha}}$ and $\partial_{2\alpha\dot{\alpha}}$ where the derivatives act on primary operators to produce descendent states. Since the OPE is of two scalars the right hand side must also be an overall scalar and since factors of $x_{12}^{\alpha\dot{\alpha}}$ and $\partial_{2\alpha\dot{\alpha}}$ come with equal number of dotted and undotted indices it follows that the corresponding operator should also have equal numbers of dotted and undotted indices.

The two- and three-point functions are fixed by conformal symmetry, and thus we may consider the four-point function. We can take the OPE of the four-point function at points x_1 and x_2 and at points x_3 and x_4 which leads to double sum. However, since the two-point function can be used an inner product to make operators orthogonal, the four-point function may be written as

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle\\&=\frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1+\Delta_2)}(x_{34}^2)^{\frac{1}{2}(\Delta_3+\Delta_4)}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{1}{2}(\Delta_{12})}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{1}{2}(\Delta_{34})}\sum_{\mathcal{O}}\mathcal{C}_{\mathcal{O}}\mathcal{F}_{\mathcal{O}}(u,v),\end{aligned}\tag{5.1.5}$$

where

$$\begin{aligned}u=\frac{x_{12}^2x_{34}^2}{x_{24}^2x_{13}^2}=xz,\quad v=\frac{x_{14}^2x_{23}^2}{x_{24}^2x_{13}^2}=(1-x)(1-z),\\\Delta_{ij}=\Delta_i-\Delta_j,\end{aligned}\tag{5.1.6}$$

where u and v are conformal cross-ratios, and the functions $\mathcal{F}_{\mathcal{O}}(u,v)$ depend on the operator data of the contributing operator and the data of the external data. We also have $\mathcal{C}_{\mathcal{O}}$ which are the so-called OPE coefficients built from the two- and three-point structure constants. Now, the OPE that is formed by points x_1 and x_2 goes on to form the conformal partial wave built from operators which are naturally eigen-operators of the conformal quadratic Casimir. It therefore follows that each function $\mathcal{F}_{\mathcal{O}}$ is an eigenfunction of the conformal quadratic Casimir with a defined eigenvalue dependent on the data of $\mathcal{O}$, i.e. spin and conformal dimension. In fact, using [80], we can define the conformal generator as L and take $\mathcal{D}=\frac{1}{2}L^2=\frac{1}{2}(L_1+L_2)^2$ to be the quadratic Casimir. The contribution of an operator with conformal dimension Δ and spin s, satisfies the equation:

$$(\mathcal{D}-(\Delta(\Delta-4)+s(s+2)))\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle=0.\tag{5.1.7}$$

This leads to a second order differential equation on $\mathcal{F}_{\mathcal{O}}(u,v)$, which has the solution

[80]

$$\mathcal{F}_{\mathcal{O}}(u,v) = \frac{u^{\frac{\tau}{2}}}{x-z}\left(x^{1+s}\,{}_2F_1\left(\frac{1}{2}\left(s+\Delta-\Delta_{12}\right),\frac{1}{2}\left(s+\Delta+\Delta_{34}\right);s+\Delta;x\right)\right.$$
$$\left.{}_2F_1\left(\frac{1}{2}\left(-s+\Delta-\Delta_{12}-2\right),\frac{1}{2}\left(-s+\Delta+\Delta_{34}-2\right);-s+\Delta-2;z\right)-(x\leftrightarrow z)\right), \tag{5.1.8}$$

where ${}_2F_1$ is a hypergeometric function and $\tau = \Delta - s$ is known as the twist. Understanding this function is critical as this allows one to find corrections to the OPE coefficients as well as $\Delta(g)$, where g is the coupling of the theory.

A motivating feature of the work in [80,81] is that (5.1.8) is in fact a re-summed form of an infinite linear combination of Schur polynomials. In practice, the authors of [80] write an ansatz for the conformal partial wave

$$F_{\lambda_1\lambda_2} = \sum_{m\geq\lambda_1,n\geq\lambda_2} r_{mn}^{\lambda_1\lambda_2} s_{mn}, \tag{5.1.9}$$

where $\lambda_1 = \frac{1}{2}(\Delta+s)$ and $\lambda_2 = \frac{1}{2}(\Delta-s)$. Then upon acting $\mathcal{D}$ upon F_{λ_1,λ_2}, it acts upon the Schur polynomials. In imposing the eigenvalue equation in (5.1.7) for F_{λ_1,λ_2}, one finds a recursion relation on $r_{mn}^{\lambda_1\lambda_2}$, in which a solution is found. Thus one has the conformal partial wave as a series expansion in the Schur polynomial which turns out to be the series expansion of (5.1.8). We will re-derive this result in view of more general superpolynomials relevant towards SCFTs.

5.1.2 Towards superconformal partial waves

The content of this chapter is to find the superconformal partial wave in a polynomial form which we then sum up into an analytic form.

In this subsection we provide some key points regarding previous successes in this direction. There are two main result which are related, the first of which is a general superconformal partial wave analysis found in [77]. The second result involves solving the superconformal partial wave so that the result is dependent on two non-trivial functions, one of which is completely associated to the contribution of protected operators to the OPE [83].

The first of the aforementioned works considers the four-point function of the six dimensional vector of scalar operators in $\mathcal{N}=4$ SCFT

$$\left\langle \Phi^{I_1}(x_1)\Phi^{I_2}(x_2)\Phi^{I_3}(x_3)\Phi^{I_4}(x_4)\right\rangle = \frac{1}{x_{12}^2 x_{34}^2}\sum_{\text{rep}} A_{\text{rep}}(u,v)P_{\text{rep}}^{I_1I_2I_3I_4}, \tag{5.1.10}$$

where $P_{\text{rep}}^{I_1I_2I_3I_4}$ are projection operators which form the six SU(4) channels from the internal structure of $\Phi^{I_1}\times\Phi^{I_2}$, whilst $A_{\text{rep}}(u,v)$ are associated space-time functions. For the stress-tensor supermultiplet the constraints following superconformal symmetry allows in conjunction with the OPE yields a form for the functions $A_{\text{rep}}(u,v)$.

The work of [83] aimed to solve the superconformal partial wave in a very general way. It begins by essentially projecting the operators above to an index-less form, namely by taking $\Phi=\Phi^{I_1}Y_{I_1}$, then writing the result in (5.1.10), one finds

$$\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \frac{1}{x_{12}^2 x_{34}^2}\sum_{\text{rep}} A_{\text{rep}}(u,v)P_{\text{rep}}^{I_1I_2I_3I_4}Y_{1,I_1}Y_{2,I_2}Y_{3,I_3}Y_{4,I_4}, \tag{5.1.11}$$

and by defining

$$\begin{aligned} \alpha\bar{\alpha} = \frac{Y_1\cdot Y_3 Y_2\cdot Y_4}{Y_1\cdot Y_2 Y_3\cdot Y_4}, &\quad (1-\alpha)(1-\bar{\alpha}) = \frac{Y_1\cdot Y_4 Y_2\cdot Y_3}{Y_1\cdot Y_2 Y_3\cdot Y_4} \\ q = 2\alpha-1, &\quad \bar{q} = 2\bar{\alpha}-1 \\ r = \frac{2}{x}-1, &\quad \bar{r} = \frac{2}{z}-1, \end{aligned} \tag{5.1.12}$$

where $Y_i\cdot Y_j$ is the Euclidean inner product of the six dimensional vectors, the authors of [83] found that

$$\begin{aligned} \sum_{\text{rep}} A_{\text{rep}}(u,v)P_{\text{rep}}^{I_1I_2I_3I_4} =& -k + \frac{(q-r)(\bar{q}-\bar{r})\left(f(r,\bar{q})+f(\bar{r},q)\right)-(r\leftrightarrow\bar{r})}{(r-\bar{r})\left(q-\bar{q}\right)} \\ &+ (q-r)(q-\bar{r})(\bar{q}-r)(\bar{q}-\bar{r})\mathcal{H}\left(q,\bar{q},r,\bar{r}\right). \end{aligned} \tag{5.1.13}$$

Where here, the functions $f(\bullet,\bullet)$ depend on the protected operators traversing through the OPE and the function $\mathcal{H}\left(q,\bar{q},r,\bar{r}\right)$ contains contributions to from protected and unprotected operators. Importantly, $\mathcal{H}\left(q,\bar{q},r,\bar{r}\right)$ is the only relevant function when looking for quantum corrections to correlation functions, thus this is the relevant part of the answer when bootstrapping the correlation function. This is since all other parts of the result are found from the free theory. This result was also encapsulated in the partial non-renormalisation theorem studied in [78], where such a protected and unprotected sector where found.

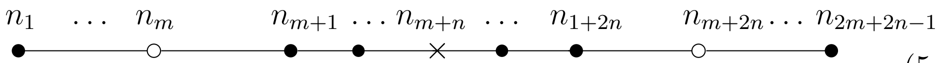
In $\mathcal{N}=4$ SCFT, $\frac{1}{2}$-BPS are built from powers of the Φ. Calling $\mathcal{O}_p = \mathrm{tr}\,(\Phi^p)$, the work of [87] completed the superconformal partial wave analysis for $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$, $\langle \mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\rangle$ and $\langle \mathcal{O}_4\mathcal{O}_4\mathcal{O}_4\mathcal{O}_4\rangle$ by finding the functions $f(\bullet,\bullet)$ and $\mathcal{H}\,(q,\bar{q},r,\bar{r})$. We aim to recapture some of these results.

5.2 Aspects of representation theory

In this section we will define operators as representations on the so-called Grassmannian superspace. We will begin by giving some basic concepts and in particular showing relevant Young tableaux for the first time. We will then move onto protected and unprotected operator classification and potential operator recombination of protected operators into unprotected operators and how the Young tableaux are used with these concepts.

5.2.1 Operator Spectra in Grassmannian superspace

Grassmannian superspace is defined to be a natural extension of analytic superspace defined in section 2.3.2, namely we define Grassmannian superspace to be a $(2n,n,n)$-analytic superspace. The Dynkin diagram is given by



(5.2.14)

Importantly, above the Dynkin nodes we have the Dynkin labels which determine irreducible representations of $\mathfrak{sl}(2m|2n)$. However, since we have a cross through the $(m+n)$'th node, we really need to consider irreducible representations of $\mathfrak{sl}(m|n)\oplus\mathfrak{sl}(m|n)\oplus\mathbb{C}$. Whilst it is true that for SCFTs, we are interested in infinite dimensional representations of $\mathfrak{sl}(2m|2n)$, we only seek finite dimensional representations of the parabolic subalgebra. The parabolic subalgebra contains $\mathfrak{sl}(m|n)\oplus\mathfrak{sl}(m|n)\oplus\mathbb{C}$ as well as the raising operators (which annihilate all highest weight states), thus highest weight state representations are given by finite dimensional representations of $\mathfrak{sl}(m|n)\oplus\mathfrak{sl}(m|n)\oplus\mathbb{C}$. We also note that the internal groups is $\mathrm{SU}(2n)$, so that $\mathbb{C}$-charge is related to the conformal dimension.

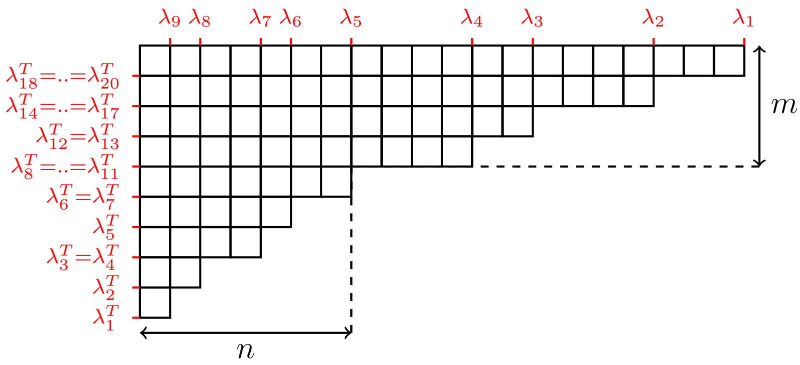


Figure 5.1: A general Young tableau associated to $\mathfrak{gl}(m|n)$

In this work we will actually consider representations of $\mathfrak{gl}(m|n) \oplus \mathfrak{gl}(m|n) \oplus \mathbb{C}$, since these will be more natural in view of Schur polynomials. The price that we have to pay for using $\mathfrak{gl}(m|n)$ is that there will exist different representations of $\mathfrak{gl}(m|n)$ that correspond to the same $\mathfrak{sl}(m|n)$.

Recalling from section 2.3.2, tensor representations in this space are given by

$$\mathcal{O}_{\mathcal{R}(\underline{A})\mathcal{R}'(\underline{B'})}, \tag{5.2.15}$$

where $\mathcal{R}$ are representations of one of the $\mathfrak{gl}(m|n)$. In the work that follows we will be dealing with OPEs of multiplets whose superconformal primaries are scalars, which here means we will only be concerned with operators which in the aforementioned tensor structure implies that we have $\mathcal{R} = \mathcal{R}'$. This means that it is sufficient to give the representation of a single $\mathfrak{gl}(m|n)$ and the charge $\mathbb{C}$.

We will be making use of Young tableaux in this chapter. We define representations of $\mathfrak{gl}(m|n)$ by the vector

$$\underline{\lambda} = [\lambda_1, \lambda_2, \dots], \tag{5.2.16}$$

where λ_i is the number of boxes in the i'th row. We define the transpose representation $\underline{\lambda}^T = [\lambda^T_1, \lambda^T_2, \dots]$ to be defined by the heights of the column of the Young tableaux, e.g. $[3, 2, 1, 1] = [4, 2, 1]^T$.

Representations of $\mathfrak{gl}(m|n)$ are given by all Young tableaux that fit into a thick hook tableau with thickness m horizontally and n vertically and an example is given in figure 5.1.

We define the charge associated to $\mathbb{C}$ to be γ, such that we may now refer to a generic representation as $\mathcal{O}^{\gamma\underline{\lambda}}$. The operator $\mathcal{O}^{\gamma\underline{\lambda}}$ defines a representation of $\mathfrak{gl}(2m|2n)$ and thus of $\mathfrak{sl}(2m|2n)$ and in turn then of the real form $\mathfrak{su}(m,m|2n)$.

Representations of $\mathfrak{su}(m,m|2n)$ are thus given via Dynkin labels associated to the $\mathfrak{su}(2n)$, $\mathfrak{sl}(m)\oplus\mathfrak{sl}(m)$ and the charge. The Dynkin nodes are given by

$$\begin{aligned} \mathfrak{su}(2n) \quad &: \quad [a_1,\ldots,a_{2n-1}] \\ \mathfrak{sl}(m)\oplus\mathfrak{sl}(m) \quad &: \quad \left[j_1^L,\ldots,j_{m-1}^L;j_1^R,\ldots,j_{m-1}^R\right], \end{aligned} \tag{5.2.17}$$

and the dilatation weight Δ (weight under $x\to\lambda x$ as usual). The translation between the labels of the operator then $\mathcal{O}^{\gamma\underline{\lambda}}$ and the corresponding representation is given by

$$\begin{aligned} a_i = a_{n-1-i} &= \lambda^T_{n-i} - \lambda^T_{n-i+1} \text{ for } 1\le i\le n{-}1, \\ a_n &= \gamma - 2\lambda_1^T, \\ j_i = j_i^L = j_i^R &= \hat\lambda_{m-i} - \hat\lambda_{m-i+1} \text{ for } 1\le i\le m{-}1, \\ \Delta &= \frac{m}{2}\gamma + \sum_{i=1}^{m-1} j_i\ , \end{aligned} \tag{5.2.18}$$

where we defined

$$\hat\lambda_i := \begin{cases} \lambda_i - n & \text{if } \lambda_i \ge n \\ 0 & \text{if } \lambda_i < n \end{cases} \tag{5.2.19}$$

In the work that follows we will be considering operators as certain powers of a basic field whose representation is called W and has $a_n = 1$, $\Delta = m/2$ with all other Dynkin labels vanishing. In this way, γ is the number of these basic field in an operator, for example $\mathcal{O}^{2\underline{\lambda}=[0]} \sim \mathrm{tr}(W^2)$.

This translation in (5.2.18) can be obtained by considering the highest weight states, and this is explained in [28]. Let us describe schematically how this comes about, we can look at the Young tableaux associated to $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$. Representations are given by single row Young tableaux in $\mathfrak{sl}(2)$, which we label as M (which is the essentially the number of boxes in that row). However, Young tableaux associated to $\mathfrak{gl}(2)$ are allowed two rows and we can relate this to the $\mathfrak{sl}(2)$ representations by taking the difference of the two rows to be M, namely if $\underline{\lambda} = [\lambda_1,\lambda_2]$ is a representation of $\mathfrak{gl}(2)$ then $M = \lambda_1 - \lambda_2$ is a representation of $\mathfrak{sl}(2)$. In this case, the degeneracy is manifested by the freedom $\lambda_i \to \lambda_i + \mu$, where μ is some constant number. In [28], this was fixed to yield so-called 'canonical representations'.

We can now consider the degeneracy in our description of operators $\mathcal{O}^{\gamma\underline{\lambda}}$ mentioned above. The vanishing supertrace of $\mathfrak{sl}(m|n)$ distinguishes it from $\mathfrak{gl}(m|n)$, thus there is a single degree of freedom more in the latter with $(m+n)$ than the former with $(m+n-1)$. This is manifested by some degeneracy in (5.2.18). Indeed we see that the relations (5.2.18) are invariant under the following shift:

$$\begin{array}{ll} (\text{if } \lambda_m \geq n+1) & (\text{if } \lambda^T_n \geq m+1) \\ \lambda_i \to \lambda_i - 1, \text{ for } 1 \leq i \leq m & \lambda^T_i \to \lambda^T_i - 1, \text{ for } 1 \leq i \leq n \\ \lambda^T_i \to \lambda^T_i + 1, \text{ for } 1 \leq i \leq n & \lambda_i \to \lambda_i + 1, \text{ for } 1 \leq i \leq m \\ \gamma \to \gamma + 2 & \gamma \to \gamma - 2\ . \end{array} \tag{5.2.20}$$

The above transformations are also valid as they stand in the two bosonic cases $m = 0$ or $n = 0$. For $n = 0$ the condition $\lambda^T_n \geq m+1$ does not make immediate sense and is interpreted as always being satisfied for any Young tableau. Then the transformation adds columns to the Young tableau in favour of reducing γ. One possibility is to use this freedom to ensure that $\gamma = 0$. This then corresponds precisely to the form chosen in [80]. Similarly in the case $m = 0$ we can ensure that $\gamma = 0$. However for general m and n, we do not perform this transformation to change γ as we would like to keep the direct connection between γ and the number of basic fields W.

Finally, let us define the notion of atypicality. This quantity is defined to be

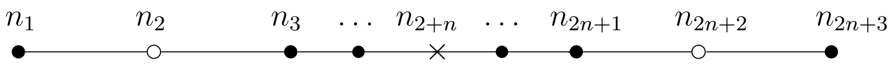
$$k = \min \{j | \lambda_j + m - n - j < 0\}\,, \tag{5.2.21}$$

and distinguishes different Young tableaux and is pertinent towards the definition of the associated Schur superpolynomials. If $k = m+1$, then the Young tableau in question is referred to as 'typical', otherwise it is referred to as 'atypical'. We can also define $k' = \min \left\{j | \lambda^T_j + n - m - j < 0\right\}$, in which the typical representations are given by $k' = n+1$.

In the physical context the measure of atypicality distinguishes short from long operators, typical representations being long and thus have a completely unconstrained expansion in superspace whilst atypical representations are short and have a shortened expansion in superspace.

5.2.2 Operator protection and Unitary bounds

Let us now return to the cases of physical interest, namely four dimensional SCFTs. Here, the Dynkin diagram takes the form



(5.2.22)

We also recall the relation to the physical SCFT data

$$n_1 = 2J_1, \quad n_{2n+3} = 2J_2, \quad n_{i+2} = a_i,$$
$$n_2 = \frac{1}{2}\Delta + J_1 + \frac{1}{2n}\sum_{i=1}^{2n-1} i a_i - \sum_{i=1}^{2n-1} a_i, \quad n_{2n+2} = \frac{1}{2}\Delta + J_2 - \frac{1}{2n}\sum_{i=1}^{2n-1} i a_i, \tag{5.2.23}$$

where $[a_1, \dots, a_{2n-1}]$ are the Dynkin labels for the representation associated to the internal algebra $\mathfrak{su}(2n)$ and (J_1, J_2) is the spin. The physically known cases are $n = 1$ and 2 which correspond to $\mathcal{N} = 2$ and 4 SCFTs.

We will be interested in operators which have the same $\mathfrak{gl}(2|2n)$ representation. In terms of (5.2.23), this implies that

$$J_1 = J_2 \quad \text{and} \quad \sum_{i=1}^{2n-1} (i - n)\, a_i = 0. \tag{5.2.24}$$

The second condition is satisfied manifestly for $n = 1$ without putting any condition on a_1, but for $n = 2$ this sets $a_1 = a_3$.

In extended supersymmetry, the supercharges enlarge in size. Shortened representation are a result of some form of operator protection. There are essentially two cases of interest for protected operators, these are the fractional BPS operators and semi-short operators. Operators which are not bound by any constraint are unprotected and are thus long. A detailed analysis relevant to four dimensions can be found in [11], we provide a qualitative description whist expressing the precise statement in terms of Dynkin labels.

If the internal group is $\mathrm{SU}(2n)$, then the there are $4n$ total supercharges. Then $\frac{p}{q}$-BPS operators are those which are annihilated by $\frac{p}{q} \times 4n$ of the supercharges. More generally we can label these by $\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$-BPS corresponding to annihilation under Q and $\bar{Q}$ separately. However, since we are interested in operators with the same $\mathfrak{gl}(2|n)$ representations in the parabolic subalgebra, we take $\frac{p}{q} = \frac{p_1}{q_1} = \frac{p_2}{q_2}$. In the $\mathcal{N} = 2$ case this corresponds to only having $\frac{1}{2}$-BPS operators, whilst for $\mathcal{N} = 4$ we can have

$\frac{1}{2}$ and as well $\frac{1}{4}$-BPS operators. In the semi-short case one has instead annihilation under a certain linear combination of the supercharges which correspond to the so-called unitary bound at which unprotected operators become protected. Importantly, unprotected operators may decompose into protected operators as they approach this unitary bound.

The so-called unitary bounds were given in [82] and in the language of Dynkin nodes in [28], they state that all operators in four dimensional SCFTs of our description with internal group SU($2n$) can be fit into the bounds:

$$\begin{aligned}
&\text{Series A}: \quad n_2 \geq n_1 + 1, \quad n_{2n+2} \geq n_{2n+3} + 1 \\
&\text{Series B}: \quad n_2 \geq n_1 + 1, \quad n_{2n+2} = 0, \quad n_{2n+3} = 0 \\
&\text{Series C}: \quad n_2 = 0, \quad n_1 = 0, \quad n_{2n+2} = 0, \quad n_{2n+3} = 0
\end{aligned} \tag{5.2.25}$$

When (5.2.24) is satisfied, which occurs when $n_1 = n_{2n+3}$, it's clear that series B operators won't ever appear in our analyses. Thus, in terms of (5.2.25), operators in series A are above the bound are generically unprotected and are long, operators at the bound of series A and series C operators are semi-short and BPS operators which are protected and are short.

The notion of protectedness is connected with the notion of atypicality in the corresponding Young tableaux mentioned in (5.2.21). To see this we apply the definition in (5.2.18) and match this against the definition of atypicality, we consider $\mathcal{N} = 4$ SCFTs for simplicity. Following (5.2.18), we have

$$\begin{aligned}
j_1^L = j_1^R = n_1 = n_7 &= \hat{\lambda}_1 - \hat{\lambda}_2 \\
&= \lambda_1 - \lambda_2 \text{ if } \lambda_1, \lambda_2 \geq 2 \text{ or} \\
&= \lambda_1 - 2 \text{ if } \lambda_1 \geq 2, \lambda_2 < 2,
\end{aligned} \tag{5.2.26}$$

and using (5.2.23), we get

$$n_2 - n_1 = \lambda_2^T, \tag{5.2.27}$$

thus the protectedness of the operator in question is dictated by the length of the second column in the Young tableau. We can now compare the definition of atypicality given in (5.2.21), with the fact that $n_2 - n_1 = \lambda_2^T$. The unprotected case is when

$n_2 > n_1 + 1 \implies \lambda_2^T \geq 2$, which requires at least $\lambda_2 \geq 2$ which leads to $k = 3$. The semi-short case is when $n_2 = n_1 + 1 \implies \lambda_2^T = 1$, which leads to $k = 2$. Finally, the BPS cases are given by $n_1 = 0,\ n_2 = 0 \implies \lambda_2^T = 0$ and this leads to $k = 1$. A similar analysis can be done for $\mathcal{N} = 2$ theories.

Indeed the measure of atypicality extends beyond the physical SCFTs and is a general representation theory concept of $\mathfrak{gl}(m|n)$, which we will touch upon in regards to Schur superpolynomials. We provide tables relating Young tableaux and operators in a four dimensional $\mathcal{N} = 0$ CFT $(n = 0)$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SCFTs in tables 5.1,5.2 and 5.3. In tables 5.2 and 5.3, in which we deal with SCFTs the Young tableaux associated to typical representations are referred to as unprotected and long whilst atypical representations protected and short.

$\underline{\lambda}$	dimension	spin
$[\lambda_1, \lambda_2]$	$\gamma + \lambda_1 + \lambda_2$	$\lambda_1 - \lambda_2$

Table 5.1: $4d$ CFT reps

$\underline{\lambda}$	dimension	spin	$\mathfrak{su}(2)$ rep	multiplet
$[0]$	γ	0	γ	$\frac{1}{2}$-BPS
$[\lambda]$ $(\lambda \geq 1)$	$\gamma{+}\lambda{-}1$	$\lambda{-}1$	$\gamma - 2$	semi-short
$[\lambda_1, \lambda_2, 1^\mu]$ $(\lambda_2 \geq 1)$	$\gamma{+}\lambda_1{+}\lambda_2{-}2$	$\lambda_1{-}\lambda_2$	$\gamma{-}2\mu - 4$	long

Table 5.2: $\mathcal{N} = 2$ SCFT reps

$\underline{\lambda}$	dimension	spin	$\mathfrak{su}(4)$ rep	multiplet
$[0]$	γ	0	$[0, \gamma, 0]$	$\frac{1}{2}$-BPS
$[\lambda, 1^\mu]$ $(\lambda \geq 2)$	$\gamma{+}\lambda{-}2$	$\lambda{-}2$	$[\mu, \gamma{-}2\mu{-}2, \mu]$	semi-short
$[1^\mu]$	γ	0	$[\mu, \gamma{-}2\mu, \mu]$	$\frac{1}{4}$-BPS
$[\lambda_1, \lambda_2, 2^{\mu_2}, 1^{\mu_1}]$ $(\lambda_2 \geq 0)$	$\gamma{+}\lambda_1{+}\lambda_2{-}4$	$\lambda_1{-}\lambda_2$	$[\mu_1{-}\mu_2, \gamma{-}2\mu_1 - 4, \mu_1{-}\mu_2]$	long

Table 5.3: $\mathcal{N} = 4$ SCFT reps

5.2.3 The Schur Superpolynomial of GL$(m|n)$

In the work to follow we will be extensively using Schur superpolynomials as a basis for the superconformal partial wave. Since these are intimately connected with the Young tableaux described previously we review these objects here.

Schur polynomials of GL(m)

Given a partition $\underline{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$, the corresponding Schur polynomial is the symmetric polynomial of m variables $x_i, i = 1 \ldots m$, given by

$$s_{\underline{\lambda}}(x) = \frac{\det\left(x_i^{\lambda_j+m-j}\right)_{1\leq i,j\leq m}}{\det\left(x_i^{m-j}\right)_{1\leq i,j\leq m}}. \tag{5.2.28}$$

The Schur polynomial is the character of the corresponding GL(m) representation described by a Young tableau with row lengths λ_i. In particular, the Schur polynomial is the trace over the representation $R_{\underline{\lambda}}$ of an element $Z \in \mathrm{GL}(m)$ written as a function of the m eigenvalues x_i of Z,

$$s_{\underline{\lambda}}(x) = \mathrm{tr}\left(R_{\underline{\lambda}}(Z)\right). \tag{5.2.29}$$

A GL(m) Schur polynomial containing a full, length m, column is equal to the Schur polynomial with that column deleted, multiplied by the product of all x's:

$$s_{[\underline{\lambda}+\delta]}(x) = \left(\prod_{i=1}^{m} x_i\right)^{\delta} \times s_{[\underline{\lambda}]}(x), \tag{5.2.30}$$

where $[\underline{\lambda}+\delta] := [\lambda_1+\delta, \lambda_2+\delta, \ldots]$.

For example for GL(2) the fundamental representation has character $\mathrm{tr}(Z) = x_1+x_2$ in agreement with the formula above for $\underline{\lambda} = [1]$. As another example, again for GL(2), consider $\underline{\lambda} = [1,1]$ corresponding to the antisymmetric representation. The trace over the representation gives

$$\mathrm{tr}\left(R_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(Z)\right) = Z_i^{[i}Z_j^{j]} = \frac{1}{2}\left(\mathrm{tr}(Z)^2 - \mathrm{tr}(Z^2)\right) = x_1x_2, \tag{5.2.31}$$

and the Schur polynomial formula (5.2.28) gives the same result $s_{[1,1]}(x) = x_1x_2$.

Schur superpolynomials of $\mathbf{GL}(m|n)$

We define the Schur superpolynomial as the characters of the supergroup $\mathrm{GL}(m|n)$ just as in (5.2.29) but this time using the supertrace

$$s_{\underline{\lambda}}(x|y) = \mathrm{str}\left(R_{\underline{\lambda}}(Z)\right) \ , \tag{5.2.32}$$

where we define the eigenvalues of $g \in \mathrm{GL}(m|n)$ to be $x_i|y_j \; i = 1 \ldots m, \; j = 1 \ldots n$. Thus for example for the fundamental representation the character is simply the supertrace of g so $s_{[1]}(x|y) = \mathrm{str}(Z) = \sum_i x_i - \sum_j y_j$ with the minus sign due to the nature of the supertrace.

In 2003 Moens and Van der Jeugt wrote down a remarkable determinantal formula for the Schur superpolynomials [84]. This formula is the analogue of the determinantal formula (5.2.28) for the standard Schur polynomials and takes the form of a $(n+k-1) \times (n+k-1)$ determinant [1]

$$s_{\underline{\lambda}}(x|y) = (-1)^{(n-1)(m+(k-1)+n/2)}\, D^{-1} \det \begin{pmatrix} X_{\underline{\lambda}} & R \\ 0 & Y_{\underline{\lambda}^T} \end{pmatrix} \ , \tag{5.2.33}$$

where

$$\begin{aligned} X_{\underline{\lambda}} &= \left(x_i^{\lambda_j + m - n - j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le k-1}} & R &= \left(\frac{1}{x_i - y_j}\right)_{1 \le i \le m, \; 1 \le j \le n} \\ Y_{\underline{\lambda}^T} &= \left((-y_j)^{\lambda_i^T + n - m - i}\right)_{\substack{1 \le i \le n-m+k-1 \\ 1 \le j \le n}} & D &= \frac{\prod_{1 \le i < j \le m}(x_i - x_j) \prod_{1 \le i < j \le n}(y_i - y_j)}{\prod_{1 \le i \le m, 1 \le j \le n}(x_i - y_j)} \ , \end{aligned} \tag{5.2.34}$$

where k is the atypicality which we restate as $k = \min\{j | \lambda_j + m - n - j < 0\}$.

There exists a form of the Schur superpolynomial in which the Y-part of (5.2.33) is instead fixed and does not depend on the representation. In [72], it was shown that given any integer p such that

$$p \ge m - n \quad \text{and} \quad p \ge \lambda_1^T \ . \tag{5.2.35}$$

[1] The minus signs here agree with those of [84] after sending $y_j \to -y_j$ (bringing a $(-1)^{n(n-1)/2}$ from D) and swapping the columns so that R appears in the top left block.

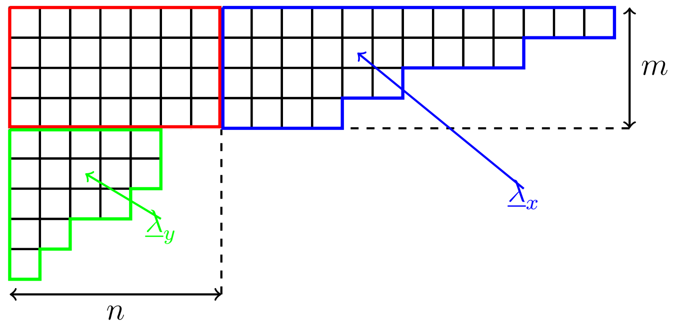


Figure 5.2: An example of a typical Young tableau associated to $\mathfrak{gl}(m|n)$

Then a different formula is given by

$$s_{\underline{\lambda}}(x|y) = (-1)^{\frac{1}{2}(2m+2p+n)(n-1)} D^{-1} \det \begin{pmatrix} \tilde{X}_{\underline{\lambda}} & R \\ K_{\underline{\lambda}} & Y \end{pmatrix} , \tag{5.2.36}$$

where D, R are as defined in (5.2.34), and

$$\tilde{X}_{\underline{\lambda}} = \left([x_i^{\lambda_j+m-n-j}] \right)_{\substack{1\le i\le m \\ 1\le j\le p}} \qquad \text{where} \qquad [x_i^a] := \begin{cases} x_i^a & a \ge 0 \\ 0 & a < 0 \, , \end{cases}$$

$$K_{\underline{\lambda}} = \left(-\delta_{i;-(\lambda_j+m-n-j)} \right)_{\substack{1\le i\le p+n-m \\ 1\le j\le p}} \qquad\qquad Y = \left(y_j^{i-1} \right)_{\substack{1\le i\le p+n-m \\ 1\le j\le n}} , \tag{5.2.37}$$

where the square brackets define the regular part, giving zero if the power is negative. We also see that whilst this form relieves Y of being representation dependent, we have introduced a matrix of -1s and 0s; K, which does depend on the representation. When we sum up the superconformal partial wave to gain an analytic form, it will be built out of this form.

5.2.4 Schur superpolynomial relations and decompositions of typical representations

In supergroups, representations occur as typical or atypical representations. Typical representations of $\mathfrak{gl}(m|n)$ are ones for which the atypicality measure $k = m + 1$ and so the first m rows and first n columns are fully occupied and $\lambda_m \ge n, (\lambda^T)_n \ge m$. An example of such a representation is given by figure 5.2. In this example the $m \times n$ block is bounded in red. If one deleted this block we would be left with two Young tableaux, one which we call $\underline{\lambda}_x$ and the other $\underline{\lambda}_y$. So the full Young tableau is given in terms of $\underline{\lambda}_x$ and $\underline{\lambda}_y$ as

$$\underline{\lambda} = [\underline{\lambda}_x + n, \underline{\lambda}_y], \tag{5.2.38}$$

where by $\underline{\lambda}_x + n$ we simply mean add n to each row. Using the form of the Schur superpolynomial in (5.2.33), we see that the matrix becomes block diagonal with $X_{\underline{\lambda}}$ being $m \times m$ and $Y_{\underline{\lambda}^T}$ being $n \times n$ from which we see that for $\underline{\lambda}$ in figure 5.2 we have

$$s_{\underline{\lambda}}(x|y) = s_{\underline{\lambda}_x}(x) s_{\underline{\lambda}_y^T}(-y) \times \prod_{1 \leq i \leq m,\, 1 \leq j \leq n} (x_i - y_j), \tag{5.2.39}$$

where $s_{\underline{\lambda}_x}(x)$ and $s_{\underline{\lambda}_y}(-y)$ are bosonic Schur polynomials. Since for these forms we have $s_{\underline{\lambda}_x}(x) = s_{[\underline{\lambda}_x+\delta]}(x) \left(\prod_{i=1}^m x_i\right)^{-\delta}$ and $s_{\underline{\lambda}_y}(-y) = s_{[\underline{\lambda}_y+\delta]}(-y) \left(\prod_{i=1}^n -y_i\right)^{-\delta}$. It follows that

$$s_{[\underline{\lambda}_x+n, \underline{\lambda}_y]} = \left(\frac{\prod_{i=1}^m x_i}{\prod_{i=1}^n -y_i} \right)^{\delta} s_{[\underline{\lambda}_x - \delta + n, n^\delta, \underline{\lambda}_y]}, \tag{5.2.40}$$

Another important identity among the Schur superpolynomials is one which manifest the decomposition of typical representations into two atypical representations. More specifically, the sum of atypical representations with $k = m$ can sum to a factorised form. Let $\underline{\lambda}_x$ be an $m-1$ row Young tableau and similarly let $\underline{\lambda}_y^T$ be an $n-1$ row Young tableau. Then consider the three $\mathfrak{gl}(m|n)$ Young tableaux $\underline{\lambda}, \underline{\lambda}_1, \underline{\lambda}_2$ with $\underline{\lambda}$ the typical representation defined in (5.2.38) and $\underline{\lambda}_1, \underline{\lambda}_2$ the two atypical Young tableaux

$$\begin{aligned}
\underline{\lambda}_1 &= [\underline{\lambda}_x + (n{-}1), n{-}1, \underline{\lambda}_y], \\
\underline{\lambda}_2 &= [\underline{\lambda}_x + n, \underline{\lambda}_y], \\
\underline{\lambda} &= [\underline{\lambda}_x + n, n, \underline{\lambda}_y] \ .
\end{aligned} \tag{5.2.41}$$

Then we have

$$\left(\prod_{i=1}^m x_i \right) \times s_{\underline{\lambda}_1}(x|y) + \left(\prod_{j=1}^n -y_j \right) \times s_{\underline{\lambda}_2}(x|y) = s_{\underline{\lambda}}(x|y). \tag{5.2.42}$$

We will use this relation extensively as an application to $\mathcal{N} = 4$ so let us elaborate on how it works by virtue of an example. The minimal example is taking $(m, n) = (2, 2)$ and $\underline{\lambda}_x = \underline{\lambda}_y = 0$ so that we have

$$\begin{aligned}
s_{[2,2]}(x|y) &= (x_1 x_2) s_{[1,1]}(x|y) + (y_1 y_2) s_{[2]}(x|y) \\
\ydiagram{2,2} &= (x_1 x_2) \ydiagram{1,1} + (y_1 y_2) \ydiagram{2},
\end{aligned} \tag{5.2.43}$$

which can be checked and if we divide through by (x_1x_2), we therefore have

$$\frac{1}{(x_1x_2)}s_{[2,2]}(x|y) = s_{[1,1]}(x|y) + \left(\frac{y_1y_2}{x_1x_2}\right) s_{[2]}(x|y),$$

but since $\frac{1}{(x_1x_2)}s_{[2,2]}(x|y) = \lim_{\rho\to 1} s_{[\rho,\rho]}(x|y)$, we can write

$$\lim_{\rho\to 1} s_{[\rho,\rho]}(x|y) = s_{[1,1]}(x|y) + \left(\frac{y_1y_2}{x_1x_2}\right) s_{[2]}(x|y), \tag{5.2.44}$$

where $s_{[\rho,\rho]}(x|y)$ is taken to be a typical representation i.e. $k = 3$. The limit is understood for arbitrary real ρ via an analytic continuation of the results for the long representations $\rho = 2, 3, 4, \ldots$,etc.

From the point of view of protected and unprotected operators, we can view the limit as taking an unprotected operator to the unitary bound. Results of the form (5.2.44) generalise to

$$\lim_{\rho\to 1} s_{[\lambda+\rho,\rho,1^\nu]}(x|y) = s_{[\lambda+1,1^{\nu+1}]}(x|y) + \left(\frac{y_1y_2}{x_1x_2}\right) s_{[\lambda+2,1^\nu]}(x|y), \tag{5.2.45}$$

which are a 2-parameter family of identities parametrised by λ and ν.

5.3 The superconformal partial wave in Grassmannian field theories

In this section we consider four-point functions of scalar operators of arbitrary weight on the Grassmannian and in particular obtain the superconformal partial wave associated with any operator occurring in the OPE of two of them. We will obtain explicit formulae for the partial waves, both as an expansion in Schur superpolynomials with given coefficients, and in a summed up form.

We remind the reader that coordinates on Grassmannian field theories are given by

$$X^{AB'} = \begin{pmatrix} x_{\alpha\dot\alpha} & \rho_\alpha^a \\ \bar\rho_{a'}^{\dot\alpha} & y_{a'}{}^a \end{pmatrix}, \tag{5.3.46}$$

where x is an $m \times m$ matrix, y is an $n \times n$ matrix, ρ is an $m \times n$ matrix and $\bar\rho$ is an $n \times m$ matrix.

5.3.1 The OPE and its relation to an expansion in Schur polynomials

Here, we examine the connection between the OPE and superconformal partial waves of four-point functions in a general $\mathrm{Gr}_{m|n}(2m|2n)$ field theory. We take the OPE of two scalar operators, $\mathcal{O}^{p_1}, \mathcal{O}^{p_2}$ with arbitrary integer weight p_1, p_2. In the $\mathcal{N}=4$ context this corresponds to taking two $\frac{1}{2}$-BPS operators with dimension p_i and lying in the SU(4) representation with Dynkin labels $[0, p_i, 0]$.

The OPE takes the general form [89]

$$\mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2) = \sum_{\mathcal{O}} C^{\mathcal{O}}_{p_1p_2}\, g_{12}^{\frac{p_1+p_2-\gamma}{2}}\, C^{\gamma,\underline{\lambda};\underline{AA'}}(X_{12},\partial_2)\mathcal{O}^{\gamma\underline{\lambda}}_{\underline{AA'}}(X_2),$$
$$\gamma = |p_{21}|, |p_{21}|+2, \ldots, p_1+p_2\,, \tag{5.3.47}$$

where we define $p_{ij} = p_i - p_j$ and where

$$g_{ij} = \mathrm{sdet}(X_i - X_j)^{-1} \tag{5.3.48}$$

which becomes the superpropagator in the physical cases where $m=2$. Here the sum is over all superconformal primary operators in the theory. The index structure $\underline{A}$ and $\underline{A'}$ is a string of superindices dictated by the representation of $\mathcal{O}$.

The object $C^{\gamma,\underline{\lambda};\underline{AA'}}(X_{12},\partial_2)$ is a formal expansion in powers of $X_{12}^{AA'}$ and derivatives $(\partial/\partial X_2)_{AA'}$ which act on the primary operator (thus producing descendant operators). It takes the form

$$C^{\gamma,\underline{\lambda};\underline{AA'}}(X_{12},\partial_2)\mathcal{O}^{\gamma\underline{\lambda}}_{\underline{AA'}}(X_2) = \sum_{\underline{\mu}\geq\underline{\lambda}} C^{\gamma\underline{\lambda}}_{\underline{\mu}}\left(X_{12}^{|\underline{\mu}|}\right)^{\underline{BB'}}\left[\partial_2^{|\underline{\mu}|-|\underline{\lambda}|}\mathcal{O}^{\gamma\underline{\lambda}}\right]_{\underline{BB'}}, \tag{5.3.49}$$

where the sum is over all Young tableaux $\underline{\mu}$ which contain $\underline{\lambda}$, where $|\underline{\mu}| = \sum_i \mu_i$ the number of boxes in the Young tableau $\underline{\mu}$. There are $|\underline{\mu}|$ powers of X_{12} and both primed and unprimed indices are symmetrised into the representation $\underline{\mu}$ according to the Young tableau. $\underline{B}$ and $\underline{B'}$ denote the correspondingly symmetrised string of indices. Descendent operators are built from $\mathcal{O}$ and its derivatives which yield a total of $|\underline{\mu}|$ primed and unprimed downstairs indices with $\underline{B}$, $\underline{B'}$ index structure. Since the left hand side is a scalar, the right hand side must have all indices contracted.

The first term in this expansion is always normalised to one

$$C^{\gamma\underline{\lambda}}_{\underline{\lambda}} = 1, \tag{5.3.50}$$

whilst the remaining coefficients will be fixed by symmetry.

To obtain the contribution of operators to the four-point function, we insert the OPE into the four-point function twice (once at points X_1, X_2 and once at points X_3, X_4) and use the two-point functions which is also fixed by symmetry to be [2]

$$\langle \mathcal{O}^{\gamma\underline{\lambda}}_{\underline{AA'}}(X_1)\tilde{\mathcal{O}}^{\gamma\underline{\lambda}}_{\underline{BB'}}(X_2)\rangle = C_{\mathcal{O}\tilde{\mathcal{O}}}\, g_{12}^{\gamma}(X_{12}^{-|\underline{\lambda}|})_{\underline{A'}\,\underline{B}}(X_{12}^{-|\underline{\lambda}|})_{\underline{B'}\,\underline{A}}, \tag{5.3.51}$$

from which we obtain

$$\begin{aligned}\langle \mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)\rangle &= \sum_{\mathcal{O},\tilde{\mathcal{O}}} C^{\mathcal{O}}_{p_1p_2}C^{\tilde{\mathcal{O}}}_{p_3p_4}C_{\mathcal{O}\tilde{\mathcal{O}}}\, g_{12}^{\frac{p_1+p_2-\gamma}{2}} g_{34}^{\frac{p_3+p_4-\gamma}{2}}\\ \times\, C^{\gamma,\underline{AA'}}(X_{12},\partial_2)C^{\gamma,\underline{BB'}}(X_{34},\partial_4)g_{24}^{\gamma}&(X_{24}^{-|\underline{\lambda}|})_{\underline{A'}\,\underline{B}}(X_{24}^{-|\underline{\lambda}|})_{\underline{B'}\,\underline{A}}\ .\end{aligned} \tag{5.3.52}$$

Here for $C_{\mathcal{O}\tilde{\mathcal{O}}}$ to be non-zero, the representations of $\mathcal{O}$ and $\tilde{\mathcal{O}}$ must be the same. In particular γ takes on values appearing both in the range for the OPE $\mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)$, ($|p_{12}| \le \gamma \le p_1+p_2$) as well as for the OPE $\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)$, ($|p_{34}| \le \gamma \le p_1+p_2$). If we assume (without loss of generality) that $p_1+p_2 \le p_3+p_4$ then there are two inequivalent cases to consider

$$\begin{array}{llcl} \text{Case 1:} & |p_{12}| \ge |p_{34}| & \Rightarrow & |p_{12}| \le \gamma \le p_1+p_2 \\ \text{Case 2:} & |p_{12}| \le |p_{34}| & \Rightarrow & |p_{34}| \le \gamma \le p_1+p_2 \end{array} \tag{5.3.53}$$

The superconformal partial wave expansion given in (5.3.52) is not manifestly superconformal invariant. It is however possible to re-expand the superconformal partial wave in a way that makes the superconformal symmetry manifest in terms of Schur superpolynomials.

$$\begin{aligned}&\langle \mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)\rangle\\ &= \sum_{\gamma,\underline{\lambda}} \mathcal{C}^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}}\, g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac12 p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac12 p_{43}} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac12\gamma} F^{\alpha\beta\gamma\underline{\lambda}}(Z),\\ &\qquad \alpha = \tfrac12(\gamma-p_{12}) \quad \beta = \tfrac12(\gamma+p_{34})\ ,\end{aligned} \tag{5.3.54}$$

[2]Two examples of this formula are if in $\mathcal{N}=4$ and $\underline{\lambda}=[0]$, $\gamma=2$ then $\langle \mathcal{O}^{2[0]}(X_1)\tilde{\mathcal{O}}^{2[0]}(X_2)\rangle = C_{\mathcal{O}\tilde{\mathcal{O}}}g_{12}^2$, if $\underline{\lambda}=[1]$ and $\gamma=2$ then $\langle \mathcal{O}^{2[1]}_{AA'}(X_1)\tilde{\mathcal{O}}^{2[1]}_{BB'}(X_2)\rangle = C_{\mathcal{O}\tilde{\mathcal{O}}}g_{12}^2\left(X_{12}^{-1}\right)_{A'B}\left(X_{12}^{-1}\right)_{B'A}$, in this second case it may be more familiar to consider $\mathrm{Gr}_2(4) \implies$ 4d CFT, in which we get $\langle \mathcal{O}^{2[1]}_{\alpha\dot\alpha}(x_1)\tilde{\mathcal{O}}^{2[1]}_{\beta\dot\beta}(x_2)\rangle = C_{\mathcal{O}\tilde{\mathcal{O}}}\frac{1}{(x_{12}^2)^2}\left(x_{12}\right)_{\dot\alpha\beta}\left(x_{12}\right)_{\dot\beta\alpha}$

where

$$\mathcal{C}^{p_1 p_2 p_3 p_4}_{\gamma \underline{\lambda}} = \sum_{\mathcal{O}^{\gamma \underline{\lambda}}, \tilde{\mathcal{O}}^{\gamma \underline{\lambda}}} C^{\mathcal{O}}_{p_1 p_2} C^{\tilde{\mathcal{O}}}_{p_3 p_4} C_{\mathcal{O}\tilde{\mathcal{O}}} \tag{5.3.55}$$

is the OPE coefficient. The superconformal partial wave is given as a sum over Schur superpolynomials $s_\mu(Z) = Z^{\mu(A)}{}_{\mu(A)}$ (traces over irreps as described in the next section)

$$F^{\alpha\beta\gamma\underline{\lambda}}(Z) = \sum_{\underline{\mu} \geq 0} R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} Z^{\mu(A)}{}_{\mu(A)} \ , \tag{5.3.56}$$

of the GL$(m|n)$ cross-ratio matrix Z

$$Z = X_{12} X_{24}^{-1} X_{43} X_{31}^{-1} \ , \tag{5.3.57}$$

of which it is shown in appendix F that taking traces over this Z matrix manifestly solves the superconformal ward identities. In addition, there are numerical coefficients $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ with

$$R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\lambda}} = 1 \ . \tag{5.3.58}$$

Here we have restricted ourselves to two cases without loss of generality

$$\begin{aligned}
\text{Case 1:} \quad & \big(p_1 + p_2 \leq p_3 + p_4,\ p_1 \geq p_2,\ p_3 \geq p_4,\ p_{12} \geq p_{34}\big) \\
& \alpha = \big(0, 1, \ldots p_2\big) \\
& \beta = \big(\tfrac{1}{2}(p_{12} + p_{34}), \tfrac{1}{2}(p_{12} + p_{34}) + 1, \ldots, \tfrac{1}{2}(p_1 + p_2 + p_{34})\big) \\
& \gamma = \big(p_{12}, p_{12} + 2, \ldots, p_1 + p_2\big) \\
\text{Case 2:} \quad & \big(p_1 + p_2 \leq p_3 + p_4,\ p_2 \geq p_1,\ p_4 \geq p_3,\ p_{21} \leq p_{43}\big) \\
& \alpha = \big(\tfrac{1}{2}(p_{21} + p_{43}), \tfrac{1}{2}(p_{21} + p_{43}) + 1, \ldots p_2\big) \\
& \beta = \big(0, 1, \ldots, \tfrac{1}{2}(p_1 + p_2 + p_{34})\big) \\
& \gamma = \big(p_{43}, p_{43} + 2, \ldots, p_1 + p_2\big)
\end{aligned} \tag{5.3.59}$$

It is one of the main purposes of this chapter to derive a formula for the numerical coefficients in (5.3.56), $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$. Furthermore we would like to sum up the superconformal partial wave expansion.

Crucially the coefficients $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ only depend on α, β, γ and the Young tableaux $\underline{\mu}, \underline{\lambda}$ but are independent of the group. This fact can be seen by considering the limit of the

GL$(2m|2n)$ Grassmannian field theory to either GL$(2(m-1)|2n)$ or GL$(2m|2(n-1))$. In this limit $F^{\alpha\beta\gamma\underline{\lambda}}(Z)$ simply becomes the equivalent partial wave for the reduced group (or vanishes if the corresponding representation $\underline{\lambda}$ does not exist for the reduced isotropy group GL$(m-1|n)$ or GL$(m|n-1)$ respectively). Similarly the Schur superpolynomials $Z^{\mu(A)}{}_{\mu(A)}$ become the equivalent Schur superpolynomial for the reduced Z (or otherwise vanishes). We thus conclude that the coefficients of the Schur superpolynomials in the partial wave must reduce directly, and hence be independent of m, n.

It is instructive to see how the first term of the expansion in trace structures over the Z-matrix arises. We begin with the form of the four-point function in (5.3.52) and input (5.3.49) whilst taking $\underline{\mu} = \underline{\lambda}$, we obtain (recall that $C^{\gamma\underline{\lambda}}_{\underline{\lambda}} = 1$)

$$\begin{aligned}
&\langle \mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)\rangle \\
&= \sum_{\mathcal{O},\tilde{\mathcal{O}}} C^{\mathcal{O}}_{p_1p_2} C^{\tilde{\mathcal{O}}}_{p_3p_4} C_{\mathcal{O}\tilde{\mathcal{O}}}\, g_{12}^{\frac{p_1+p_2-\gamma}{2}} g_{34}^{\frac{p_3+p_4-\gamma}{2}} g_{24}^{\gamma} (X_{12}^{|\underline{\lambda}|})^{\underline{AA'}} (X_{34}^{|\underline{\lambda}|})^{\underline{BB'}} (X_{24}^{-|\underline{\lambda}|})_{\underline{A'}\underline{B}} (X_{24}^{-|\underline{\lambda}|})_{\underline{B'}\underline{A}} \\
&\qquad + O(X_{12}, X_{34}). \qquad (5.3.60)
\end{aligned}$$

We can write $(X_{12}^{|\underline{\lambda}|})^{\underline{AA'}} (X_{34}^{|\underline{\lambda}|})^{\underline{BB'}} (X_{24}^{-|\underline{\lambda}|})_{\underline{A'}\underline{B}} (X_{24}^{-|\underline{\lambda}|})_{\underline{B'}\underline{A}} \sim (X_{12}X_{24}^{-1}X_{34}X_{24}^{-1})^{\underline{\lambda}(A)}{}_{\underline{\lambda}(A)}$, which we can view as the double OPE limit taken on a single Z matrix given in (5.3.56). From this we expect some correction terms dependent on X_{12} and X_{34}, we thus arrive at

$$\sum_{\mathcal{O},\tilde{\mathcal{O}}} C^{\mathcal{O}}_{p_1p_2} C^{\tilde{\mathcal{O}}}_{p_3p_4} C_{\mathcal{O}\tilde{\mathcal{O}}}\, g_{12}^{\frac{p_1+p_2-\gamma}{2}} g_{34}^{\frac{p_3+p_4-\gamma}{2}} g_{24}^{\gamma} Z^{\underline{\lambda}(A)}{}_{\underline{\lambda}(A)} + O(X_{12}, X_{34})\,. \qquad (5.3.61)$$

The object $Z^{\underline{\lambda}(A)}{}_{\underline{\lambda}(A)}$ is the trace over the representation $\underline{\lambda}$ of Z and is hence equal to the Schur superpolynomial $s_{\underline{\lambda}}(x|y)$.

5.3.2 Free field theory OPE and Wick's theorem

The discussion of the OPE in section 5.3.1 is completely general and essentially only uses symmetry. In the free theory, we can be explicit about the operators appearing in the OPE.

As described in [88] the easiest way to derive the OPE in a free field theory context is to simply use Wick's theorem. The time ordered product of two operators $\mathcal{O}_{p_1}(X_1)\mathcal{O}_{p_2}(X_2)$ is equal to the normal ordered product, together with the sum over contractions multiplied by appropriate powers of propagators. In this context, we get that (for $p_1 \leq p_2$)

$$\mathcal{O}_{p_1}(X_1)\mathcal{O}_{p_2}(X_2) =: \mathcal{O}_{p_1}(X_1)\mathcal{O}_{p_2}(X_2) : \; + \sum_{p=0}^{p_1-1} g_{12}^{p_1-p}\mathcal{O}_{p_2-p_1+2p}(X_1,X_2)\,, \tag{5.3.62}$$

where for example $\mathcal{O}_{p_1+p_2-2}$ is the result of a single contraction

$$\mathcal{O}_{p_2-p_1+2p}(X_1,X_2) = \mathrm{tr}(W^{p_1-1}W)(X_1)\,\mathrm{tr}(WW^{p_2-1})(X_2) : \,, \tag{5.3.63}$$

whereas $\mathcal{O}_{p_1-p_2-4}$ will involve two contractions etc. Here the contractions simply give a Kronecker delta in the corresponding adjoint gauge index.

Now we Taylor expand the right hand side and rearrange the result into primaries and descendants to obtain (5.3.47) but with explicit expressions for the operators which appear.

So if $\gamma = p_1 + p_2$, the operators are double trace operators from the product (in general with derivatives) of $\mathcal{O}_{p_1}$ and $\mathcal{O}_{p_2}$. If however $\gamma = p_1 + p_2 - 2$, then in the U(N) theory the single Wick contraction will glue together the two traces to form a single trace. Similarly for the SU(N) theory in the large N limit. For finite N in the SU(N) theory however there will be a $\frac{1}{N}$ correction (from writing the Kronecker delta's in adjoint indices back in terms of fundamental gauge indices via $T^a_{ij}T^a_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}$) giving back a double trace operator.

5.3.3 Deriving the superconformal partial wave for Grassmannian field theories

In this subsection we finally derive and give a summary of the results for the superconformal partial wave in Grassmannian field theories. In the later part of section 5.3.1, it was explained that the coefficients of the Schur superpolynomials, namely $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ are independent of the m and n. From this argument, we conclude that the coefficients relevant for the full superconformal partial wave are no different from the coefficient for the conformal partial wave, thus we aim to find $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ for GL(m). We find the conformal partial wave for the GL(m) case in a determinant form inherited from the determinant form of the GL(m) Schur polynomial. We then use the same numerical coefficients for the superconformal partial wave in which we find a novel determinant form which is in this case inherited from (5.2.33).

The GL(m) conformal partial wave

We now find the coefficient $R_{\underline{\mu}}^{\alpha\beta\gamma\underline{\lambda}}$ associated to GL(m) conformal partial waves, we give the key formulae whilst leaving the details for appendix G. The proof follows a similar procedure to that of [80] for the conformal $4d$ case $(m = 2, n = 0)$.

We take the space-time coordinate $x^{\alpha\dot{\alpha}}$ to be an m-dimensional matrix, where

$$x_{ij}^2 := \det(x_{ij}) = \frac{1}{m!}\left(x_{ij\dot{\alpha}_1}^{\alpha_1} \dots x_{ij\dot{\alpha}_m}^{\alpha_m}\right)\epsilon^{\dot{\alpha}_1\dots\dot{\alpha}_m}\epsilon_{\alpha_1\dots\alpha_m}. \tag{5.3.64}$$

We may then consider some scalar operators $\Phi_\Delta(x)$ which take a representation of $\mathfrak{sl}(m)$. The four-point function of these operators is given by

$$\langle\Phi_{\Delta_1}(x_1)\Phi_{\Delta_2}(x_2)\Phi_{\Delta_3}(x_3)\Phi_{\Delta_4}(x_4)\rangle = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{1}{2}\Delta_{21}}\left(\frac{x_{13}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{43}} F(x). \tag{5.3.65}$$

Where $F(x)$ is a function of the m many eigenvalues of $z = x_{12}x_{24}^{-1}x_{43}x_{31}^{-1}$ labeled x_i. We are considering the Grassmannian $\mathrm{Gr}_m(2m)$ which can be viewed as the space of $2m \times m$ matrices given by u_α^A. This is where the small Greek indices refer to the isotropy group whilst the big Latin indices refer to the global group. Explicitly, one can put coordinates on this by using the section

$$u_\alpha^A = \left(\delta_\alpha^\beta, x_\alpha^{\dot{\beta}}\right) \ , \ \bar{u}_A^{\dot{\alpha}} = \begin{pmatrix} -x_\alpha^{\dot{\alpha}} \\ \\ \delta_{\dot{\beta}}^{\dot{\alpha}} \end{pmatrix}, \tag{5.3.66}$$

So that we have $u_{i\alpha}^A\bar{u}_{jA}^{\dot{\alpha}} = x_{ij\alpha}^{\dot{\alpha}}$. In the $m = 2$ case, we may view u_α^A as being a pair of twistors, as was used in a similar context in [24]. The benefit of this is that the generators of GL(m) are given by

$$D_B^A = u_A^\alpha\frac{\partial}{\partial u_B^\alpha}, \tag{5.3.67}$$

which satisfies the algebra:

$$\left[D_B^A, D_D^C\right] = \delta_B^C D_D^A - \delta_D^A D_B^C. \tag{5.3.68}$$

The conformal partial waves are eigenfunctions of the quadratic Casimir operator which will act on the four-point function (5.3.65) at points x_1 and x_2. This is given by

$$\frac{1}{2}D_{12}^2 = \frac{1}{2}(D_{1B}^A + D_{1B}^A)(D_{1A}^B + D_{1A}^B). \tag{5.3.69}$$

In order to find the coefficients $R^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1,\dots,\mu_m}$, in an expansion in Schur polynomials we will proceed by doing two things. Firstly we will re-express (5.3.69) in terms of the eigenvalues of z; namely x_i, by considering its action on GL(m) Schur polynomials of z. We can then apply it to the correlation function (5.3.65). This will lead to an action upon the conformal partial wave $F^{\underline{\lambda}}(x) = \sum_{\underline{\mu}\geq\underline{\lambda}} t^{\underline{\lambda}}_{\mu_1,\dots,\mu_m} s_{\underline{\mu}}(x)$ (where $F(x) = \sum_{\lambda_{i+1}\geq\lambda_i} F^{\underline{\lambda}}(x)$) , which in turn leads to a recursion relation on $t^{\underline{\lambda}}_{\mu_1,\dots,\mu_m}$, which we solve and apply to the superconformal case.

Leaving the details for section in appendix G, Defining $D^{(m)} := \frac{1}{2}D^2_{12}$, we find that

$$\begin{aligned}D^{(m)} = \frac{1}{\mathrm{vdet}^{(m)}(x)}\Bigg[\sum_{i=1}^{m}\Bigg[x_i\left(-x_i\left(\frac{1}{2}\left(\Delta_{34}-\Delta_{12}\right)-2m+3\right)-2m+2\right)\frac{\partial}{\partial x_i}\\ +(1-x_i)x_i^2\frac{\partial^2}{\partial x_i^2} - \left(\frac{1}{2}\Delta_{21}-m+1\right)\left(\frac{1}{2}\Delta_{34}-m+1\right)x_i\Bigg]\\ +\frac{m}{3}(m-1)(2m-1)\Bigg]\mathrm{vdet}^{(m)}(x),\end{aligned} \tag{5.3.70}$$

where we define the Vandermonde determinant:

$$\mathrm{vdet}^{(m)}(x) = (-1)^{\binom{m}{2}}\det_{ij}(x_i^{j-1}) = \det_{ij}(x_i^{m-j}) = \prod_{1\leq i<j\leq m}(x_i-x_j). \tag{5.3.71}$$

The action of the Casimir operator corresponding to the contribution of an operator in the OPE yields the eigenvalue equation on the four-point function

$$D^{(m)}\left\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\right\rangle = \sum_{i=1}^{m}\lambda_i(\lambda_i-(2i-1))\left\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\right\rangle. \tag{5.3.72}$$

This eigenvalue is simply the value of the Casimir for the corresponding representation of SL($2m$).

We define the GL(m) conformal partial wave in (5.3.65) to have the form $F(x) = \sum_{\lambda_{i+1}\geq\lambda_i} F^{\underline{\lambda}}(x)$, where

$$F^{\underline{\lambda}} = \sum_{\underline{\mu}\geq\underline{\lambda}} t^{\underline{\lambda}}_{\mu_1,\dots,\mu_m} s_{\underline{\mu}}(x)\ . \tag{5.3.73}$$

By noting the action of the Casimir upon the Schur polynomial

$$\begin{aligned}D^{(m)}s_{\underline{\mu}}(x) = \left(\sum_{i=1}^{m}\mu_i(\mu_i-(2i-1))\right)s_{\underline{\mu}}(x)\\ -\left(\sum_{i=1}^{m}(\mu_i-(i-1)-\frac{1}{2}\Delta_{12})(\mu_i-(i-1)+\frac{1}{2}\Delta_{34})s_{(\dots,\mu_i+1,\dots)}(x)\right),\end{aligned} \tag{5.3.74}$$

and following (5.3.72), it follows that the action of the quadratic Casimir operator upon the four-point function yields the recursion relation on $t^{\underline{\lambda}}_{\mu_1,\ldots,\mu_m}$

$$\sum_{i=1}^{p}\Bigg((\mu_i-\lambda_i)\left(\lambda_i+\mu_i-(2i-1)\right)t^{\underline{\lambda}}_{\mu_1,\ldots,\mu_m} \\ -\left(\mu_i-i-\frac{1}{2}\Delta_{12}\right)\left(\mu_i-i+\frac{1}{2}\Delta_{34}\right)t^{\underline{\lambda}}_{\mu_1,\ldots,\mu_i-1,\ldots,\mu_m}\Bigg)=0 \tag{5.3.75}$$

which is solved by:

$$t^{\underline{\lambda}}_{\mu_1\ldots\mu_m}=\prod_{i=1}^{m}\frac{\left(\lambda_i+1-i+\frac{1}{2}\Delta_{21}\right)^{\mu_i-\lambda_i}\left(\lambda_i+1-i+\frac{1}{2}\Delta_{34}\right)^{\mu_i-\lambda_i}}{(\mu_i-\lambda_i)!\,(2\lambda_i-2i+2)^{\mu_i-\lambda_i}} \tag{5.3.76}$$

where $(x)^y$ is the ascending Pochhammer symbol. In taking $m=2$, we find agreement with [80]. However, in the supersymmetric case the conformal partial wave is accompanied with the super-cross ratio

$$\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}\gamma}F^{\alpha\beta\gamma\underline{\lambda}}(Z)=\operatorname{sdet}(Z)^{\frac{1}{2}\gamma}F^{\alpha\beta\gamma\underline{\lambda}}(Z). \tag{5.3.77}$$

In view of this we instead consider a shifted conformal partial wave

$$F^{\underline{\lambda}+m}=\sum_{\underline{\mu}\geq 0}t^{\underline{\lambda}+m}_{\mu_1,\ldots,\mu_m}s_{\underline{\mu}+m}(x), \tag{5.3.78}$$

where $\underline{\lambda}+m=[\lambda_1+m,\lambda_2+m,\ldots,\lambda_m+m]$. Noting that $s_{\underline{\lambda}+m}=\left(\prod_{i=1}^m x_i\right)^m s_{\underline{\lambda}}=\det(z)^m s_{\underline{\lambda}}$, we find that

$$F^{\underline{\lambda}+m}=\left(\prod_{i=1}^{m}x_i\right)^m\sum_{\underline{\mu}\geq\underline{\lambda}}t^{\underline{\lambda}+m}_{\mu_1,\ldots,\mu_m}s_{\underline{\mu}}(x) \tag{5.3.79}$$

where we may now define the resulting coefficients by $r^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1,\ldots,\mu_m}$

$$r^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1,\ldots,\mu_m}:=t^{\underline{\lambda}+m}_{\mu_1,\ldots,\mu_m}=\prod_{i=1}^{m}\frac{(\lambda_i+1-i+\alpha)^{\mu_i-\lambda_i}(\lambda_i+1-i+\beta)^{\mu_i-\lambda_i}}{(\mu_i-\lambda_i)!\,(2\lambda_i+2-2i+\gamma)^{\mu_i-\lambda_i}} \tag{5.3.80}$$

Where here, $\alpha=\frac{1}{2}(2m-\Delta_{12})$, $\beta=\frac{1}{2}(2m+\Delta_{34})$ and $\gamma=2m$.

The sum in (5.3.79) sums over all Young tableaux $\underline{\mu}$ that fit in $\underline{\lambda}$ in an unconstrained way, i.e. μ_{i+1} is not always greater than μ_i which leads to the appearance of nonsensical representations. The appearance of such representations is circumvented by the so-called affine Weyl reflection symmetry of the Schur polynomials (upto a sign) [13].

This symmetry is generated by the discrete action w_σ where $\sigma \in S_m$ and acts on the Young tableaux labels as

$$w_\sigma(\mu_1, \dots, \mu_m) = (\mu_{\sigma_1} + 1 - \sigma_1, \mu_{\sigma_2} + 2 - \sigma_2, \dots, \mu_{\sigma_m} + m - \sigma_m). \tag{5.3.81}$$

This essentially re-orders the Young tableaux labels to make sense whenever they do not (i.e. $\mu_{i+1} \geq \mu_i$). We therefore find:

$$F^{\alpha\beta\gamma\underline{\lambda}} = \sum_{\underline{\mu}\geq 0} r^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1,\dots,\mu_m} s_{\underline{\mu}} = \sum_{\substack{\underline{\mu}\geq 0\\ \mu_{i+1}\geq\mu_i}} R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} s_{\underline{\mu}}, \tag{5.3.82}$$

where $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} := \sum_{\sigma\in S_m} (-1)^{|\sigma|} r^{\alpha\beta\gamma\underline{\lambda}}_{w_\sigma(\mu_1,\dots,\mu_m)}$. Notice that because of the form of $r^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1,\dots,\mu_m}$, we do not need to impose the sum to be to $\underline{\mu} \geq \underline{\lambda}$ since if $\underline{\lambda} \geq \underline{\mu}$ then $r^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1,\dots,\mu_m} = 0$

The coefficients $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ are our sought after coefficients.

The summation of the GL(m) conformal partial waves

Given the result in the GL(m) case, we can re-sum the whole result in (5.3.82) to retrieve an analytic form. In particular we use the fact that since

$$\begin{aligned} &x_i^{\lambda_j+m-j}\,{}_2F_1(\lambda_j{+}1{-}j{+}\alpha, \lambda_j{+}1{-}j{+}\beta; 2\lambda_j{+}2{-}2j{+}\gamma; x_i) \\ &= \sum_{\mu_j=0}^{\infty} \frac{(\lambda_j{+}1{-}j{+}\alpha)^{(\mu_j-\lambda_j)}(\lambda_j{+}1{-}j{+}\beta)^{(\mu_j-\lambda_j)}}{(\mu_j{-}\lambda_j)!(2\lambda_j{+}2{-}2j{+}\gamma)^{(\mu_j-\lambda_j)}} x_i^{\mu_j+m-j}, \end{aligned} \tag{5.3.83}$$

where $a^{(n)} = a(a+1)\dots(a+n-1)$ is the ascending Pochhammer symbol. Now using the determinate form of the Schur polynomial from (5.2.28) we get that

$$\begin{aligned} F^{\alpha\beta\gamma\underline{\lambda}} &= \sum_{\substack{\underline{\mu}\geq 0\\ \mu_{i+1}\geq\mu_i}} R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} s_{\underline{\mu}} = \sum_{\substack{\underline{\mu}\geq 0\\ \mu_{i+1}\geq\mu_i}} R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} \frac{\det\left(x_i^{\mu_j+m-j}\right)_{1\leq i,j\leq m}}{\det\left(x_i^{m-j}\right)_{1\leq i,j\leq m}} \\ &= \frac{\det\left(x_i^{\lambda_j+m-j}\,{}_2F_1(\lambda_j{+}1{-}j{+}\alpha, \lambda_j{+}1{-}j{+}\beta; 2\lambda_j{+}2{-}2j{+}\gamma; x_i)\right)_{1\leq i,j\leq m}}{\det\left(x_i^{m-j}\right)_{1\leq i,j\leq m}}. \end{aligned} \tag{5.3.84}$$

Indeed, in the case $m = 2$ we recapture the well known form in (5.1.8).

It is worth remarking there exists the relation

$$F^{\alpha\beta\gamma\underline{\lambda}}(x) = \left(\prod_{i=1}^{m} x_i\right)^{-\delta} F^{(\alpha-\delta)(\beta-\delta)(\gamma-2\delta)[\underline{\lambda}+\delta]}(x). \tag{5.3.85}$$

To see this, one recalls that $s_{\underline{\mu}+\delta}(x) = \left(\prod_{i=1}^m x_i\right)^\delta s_{\underline{\mu}}(x)$. We invert this and plug it into (5.3.84), and noting the form of the coefficients, namely (5.3.80) we find the necessary transformations in α, β, γ and $\underline{\lambda}$.

5.3.4 The GL$(m|n)$ superconformal partial wave

The coefficients of the Schur superpolynomials in any GL$(m|n)$ partial wave expansion are universal, which implies that they do not depend on the group but only on the representations defined by Young tableau. This means that having obtained the GL(m) partial waves for any m, we can immediately write down the GL$(m|n)$ partial waves as an explicit expansion over Schur superpolynomials.

We can now state the superconformal partial wave associated to GL$(m|n)$. This subsection encapsulates the main results of this entire chapter so let us restate the correlator and what have found before summing the result. We have found that the contribution of an operator $\mathcal{O}^{\gamma\underline{\lambda}}$ to a four-point function $\langle p_1p_2p_3p_4\rangle$ is given by

$$\begin{aligned}
&\langle p_1p_2p_3p_4\rangle := \\
&\langle \mathcal{O}^{p_1}(X_1)\mathcal{O}^{p_2}(X_2)\mathcal{O}^{p_3}(X_3)\mathcal{O}^{p_4}(X_4)\rangle \\
&= \sum_{\gamma,\underline{\lambda}} \mathcal{C}^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}}\, g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac{1}{2}p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{1}{2}p_{43}} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}\gamma} F^{\alpha\beta\gamma\underline{\lambda}}(Z), \\
&\qquad \alpha = \tfrac{1}{2}(\gamma - p_{12}) \quad \beta = \tfrac{1}{2}(\gamma + p_{34})\ , \qquad (5.3.86)
\end{aligned}$$

where, in terms of OPE coefficients,

$$\mathcal{C}^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}} = \sum_{\mathcal{O}^{\gamma\underline{\lambda}},\tilde{\mathcal{O}}^{\gamma\underline{\lambda}}} C^{\mathcal{O}}_{p_1p_2} C^{\tilde{\mathcal{O}}}_{p_3p_4} C_{\mathcal{O}\tilde{\mathcal{O}}}. \qquad (5.3.87)$$

Here we have that

$$\begin{aligned}
F^{\alpha\beta\gamma\underline{\lambda}}(x|y) &= \sum_{\underline{\mu}\geq 0} R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}\, s_{\underline{\mu}}(x|y), \\
R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} &= \sum_{\sigma\in S_m} (-1)^{|\sigma|}\, r^{\alpha\beta\gamma\underline{\lambda}}_{w_\sigma(\mu_1,\dots,\mu_m)}, \\
r^{\alpha\beta\gamma\underline{\lambda}}_{\mu_1\dots\mu_m} &= \prod_{j=1}^m \frac{(\lambda_j+1-j+\alpha)^{(\mu_j-\lambda_j)}(\lambda_j+1-j+\beta)^{(\mu_j-\lambda_j)}}{(\mu_j-\lambda_j)!(2\lambda_j+2-2j+\gamma)^{(\mu_j-\lambda_j)}}\ , \qquad (5.3.88)
\end{aligned}$$

where $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ are exactly the same numerical coefficients as defined and used in the GL(m) case and $s_{\underline{\mu}}(x|y)$ are the GL$(m|n)$ Schur superpolynomials defined in (5.2.33).

Indeed in the practical computation of OPE coefficients – as we will do for $\mathcal{N}$=4 SYM in section 5.4 – this form of the partial wave is the most useful one.

Following the form of the summed up GL(m) conformal partial wave and its close relation with the corresponding Schur polynomial, we give the summed up GL($m|n$) result based on the Schur superpolynomial in (5.2.36). We find

$$F^{\alpha\beta\gamma\underline{\lambda}}(x|y) \;=\; (-1)^{\frac{1}{2}(2m+2p+n)(n-1)}D^{-1}\det\begin{pmatrix} F^X_{\underline{\lambda}} & R \\ K_{\underline{\lambda}} & F^Y \end{pmatrix}\,, \tag{5.3.89}$$

where here we define

$$p = \min\{\alpha,\beta\} \tag{5.3.90}$$

and D, R are just as defined previously for the Schur superpolynomial, in (5.2.34), $K_{\underline{\lambda}}$ is defined in (5.2.37) and $F^X_{\underline{\lambda}}$ and F^Y are matrices of hypergeometric functions

$$\begin{aligned} F^X_{\underline{\lambda}} &= \Big([x_i^{\lambda_j+m-n-j}{}_2F_1(\lambda_j+1-j+\alpha,\lambda_j+1-j+\beta;2\lambda_j+2-2j+\gamma;x_i)]\Big)_{\substack{1\le i\le m\\1\le j\le p}} \\ F^Y &= \Big((y_j)^{i-1}{}_2F_1(i+m-n-\alpha,i+m-n-\beta;2i+2(m-n)-\gamma;y_j)\Big)_{\substack{1\le i\le p+n-m\\1\le j\le n}}\,. \end{aligned} \tag{5.3.91}$$

Here we again define the square brackets to mean "the regular part at $x = 0$" i.e.

$$\begin{aligned} [x^{-\ell}{}_2F_1(a,b;c;x)] &:= x^{-\ell}{}_2F_1(a,b;c;x) - \sum_{k=0}^{\ell-1}\frac{a^{(k)}b^{(k)}}{k!\,c^{(k)}}x^{k-\ell} \\ &= \sum_{k=0}^{\infty}\frac{a^{(k+\ell)}b^{(k+\ell)}}{(k+\ell)!\,c^{(k+\ell)}}x^k\,. \end{aligned} \tag{5.3.92}$$

5.3.5 Superconformal partial wave relations and decompositions of typical representations

In section 5.2.4, we reviewed various relations that exist between Schur superpolynomials and here we see similar relations on the full superconformal partial wave. One first observes the factorisation into external and internal groups, namely for a typical representation in which $\underline{\lambda} = [\underline{\lambda}_x + n, \underline{\lambda}_y]$, one has

$$F^{\alpha\beta\gamma\underline{\lambda}}(x|y) = F^{(\alpha+n)(\beta+n)(\gamma+2n)\underline{\lambda}_x}(x|0) \times F^{(\alpha-m)(\beta-m)(\gamma-2m)\underline{\lambda}_y}(0|-y)\prod_{\substack{1\le i\le m,\\1\le j\le n}}(x_i-y_j). \tag{5.3.93}$$

We also have the relation

$$F^{\alpha\beta\gamma\underline{\lambda}}(x|y) = \frac{\prod_{i=1}^{m} x_i}{\prod_{j=1}^{n}(-y_j)} F^{(\alpha+1)(\beta+1)(\gamma+2)\underline{\lambda}'}(x|y), \tag{5.3.94}$$

for when

$$\underline{\lambda} = [\underline{\lambda}_x+n, \underline{\lambda}_y],\ \ \underline{\lambda}' = [\underline{\lambda}_x-1+n, n, \underline{\lambda}_y]\ . \tag{5.3.95}$$

Another relation is that if given two atypical representations $\underline{\lambda}_1$ and $\underline{\lambda}_2$ and a typical one $\underline{\lambda}$, where $\underline{\lambda}_x$ and $\underline{\lambda}_y$ are $m-1$ and $n-1$ row Young tabluex respectivly, namely

$$\begin{aligned}\underline{\lambda}_1 &= [\underline{\lambda}_x + (n-1), n-1, \underline{\lambda}_y],\\ \underline{\lambda}_2 &= [\underline{\lambda}_x + n, \underline{\lambda}_y],\\ \underline{\lambda} &= [\underline{\lambda}_x + n, n, \underline{\lambda}_y]\ ,\end{aligned} \tag{5.3.96}$$

we have

$$\begin{aligned}&\left(\prod_{i=1}^{m} x_i\right) \times F^{\alpha\beta\gamma\underline{\lambda}_1}(x|y) + \left(\prod_{j=1}^{n} -y_j\right) \times F^{(\alpha-1)(\beta-1)(\gamma-2)\underline{\lambda}_2}(x|y)\\ =&F^{(\alpha+n-1)(\beta+n-1)(\gamma+2n-2)\underline{\lambda}_x}(x) \times F^{(\alpha-m)(\beta-m)(\gamma-2m)\underline{\lambda}_y}(0|-y) \times \prod_{\substack{1\le i\le m,\\ 1\le j\le n}} (x_i - y_j)\ .\end{aligned} \tag{5.3.97}$$

Let us refine this final relation for $\mathcal{N}=4$ SCFT, where here we take $(m,n)=(2,2)$ and as a representative example let us consider $\underline{\lambda}_x = [0]$ and $\underline{\lambda}_y = [0]$. As an application to (5.3.97), we get

$$\begin{aligned}&(x_1x_2) \times F^{\alpha\beta\gamma[1,1]}(x|y) + (y_1y_2) \times F^{(\alpha-1)(\beta-1)(\gamma-2)[2]}(x|y)\\ =&F^{(\alpha+1)(\beta+1)(\gamma+2)[0]}(x) \times F^{(\alpha-2)(\beta-2)(\gamma-4)[0]}(0|-y) \times \prod_{\substack{1\le i\le 2,\\ 1\le j\le 2}} (x_i - y_j)\ .\end{aligned} \tag{5.3.98}$$

We divide through by (x_1x_2), and we use the fact that from (5.3.85) we have

$$\frac{1}{(x_1x_2)} F^{(\alpha+1)(\beta+1)(\gamma+2)[0]}(x) = F^{(\alpha+2)(\beta+2)(\gamma+4)[-1,-1]}(x), \tag{5.3.99}$$

then by putting this together and using (5.3.93) we find that

$$F^{\alpha\beta\gamma[1,1]}(x|y) + \left(\frac{y_1y_2}{x_1x_2}\right) F^{(\alpha-1)(\beta-1)(\gamma-2)[2]}(x|y) = \lim_{\rho\to 1} F^{\alpha\beta\gamma[\rho,\rho]}(x|y), \tag{5.3.100}$$

where $F^{\alpha\beta\gamma[\rho,\rho]}(x|y)$ is the superconformal partial wave for a typical representation. This relation generalises to

$$\begin{aligned}
F_{\text{long}}^{\alpha\beta\gamma[\lambda+1,1^{\nu+1}]} &:= \lim_{\rho\to 1} F^{\alpha\beta\gamma[\lambda+\rho,\rho,1^{\nu}]}(x|y) \\
&= \left(\frac{y_1 y_2}{x_1 x_2}\right) F^{(\alpha-1)(\beta-1)(\gamma-2)[\lambda+2,1^{\nu}]}(x|y) + F^{\alpha\beta\gamma[\lambda+1,1^{\nu+1}]}(x|y), \qquad (5.3.101)
\end{aligned}$$

where the limit is understood for arbitrary real ρ via an analytic continuation of the results for the long representations $\rho = 2, 3, 4, \ldots$, etc. We define a separate superconformal partial wave $F_{\text{long}}^{\alpha\beta\gamma[\lambda+1,1^{\nu+1}]}$ since it has Young tableaux labels that would have previously been described by an atypical representation, but here by writing the subscript 'long', we are instructed to take $k = 3$. We will make extensive use of the relation (5.3.101) in our application to $\mathcal{N} = 4$ SCFT.

5.4 Application to $\mathcal{N} = 4$ SYM

For this section we specialise to $\mathcal{N} = 4$ SYM. We thus take the partial waves of the previous section and set $(m, n) = (2, 2)$. We begin by giving the summed up form for the superconformal partial wave pertinent to the study of $\mathcal{N} = 4$. We then consider the free theory OPE coefficients of various correlators.

5.4.1 The $\mathcal{N} = 4$ superconformal partial wave

In section 5.3.4, we provided the superconformal partial waves corresponding to a theory with $\mathrm{SU}(m, m|2n)$ symmetry. We now write the result specifically for $\mathcal{N} = 4$ SCFT.

The results can be rewritten in terms of two functions, a one variable (in each of x and y) function, $f(x, y)$, and a two-variable function $f(x_1, x_2, y_1, y_2)$. Recalling that $p = \min\{\alpha, \beta\}$, the full superconformal partial wave is written in terms of these simply as

$$\begin{aligned}
&F^{\alpha\beta\gamma\underline{\lambda}}(x|y) \\
&= (-1)^{\sum_i \lambda_i}\Bigg[\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{-p} \left(\delta_{\underline{\lambda};0} + D^{-1}\left[\left(\frac{f(x_2,y_2)}{x_1-y_1} \ - \ y_1 \leftrightarrow y_2\right) \ - \ x_1 \leftrightarrow x_2\right]\right) \\
&\qquad\qquad + \ D^{-1} f(x_1, x_2, y_1, y_2).\Bigg] \qquad (5.4.102)
\end{aligned}$$

where here

$$D^{-1} = \frac{(x_1-y_1)(x_1-y_2)(x_2-y_1)(x_2-y_2)}{(x_1-x_2)(y_1-y_2)} \,. \tag{5.4.103}$$

We define the following functions:

$$\begin{aligned} F^{\alpha\beta\gamma}_{\lambda}(x) &:= x^{\lambda-1}{}_2F_1(\lambda+\alpha,\lambda+\beta;2\lambda+\gamma;x), \\ G^{\alpha\beta\gamma}_{\lambda'}(y) &:= y^{\lambda'-1}{}_2F_1(\lambda'-\alpha,\lambda'-\beta;2\lambda'-\gamma;y). \end{aligned} \tag{5.4.104}$$

Then the functions given in the superconformal partial wave defined in (5.4.102), are given by

$\boldsymbol{\lambda_2 > 1}$ **(long)** :

$$\begin{aligned} &f(x,y) = 0 \\ &f(x_1,x_2,y_1,y_2) = (-1)^{\lambda_1'+\lambda_2'}\Big(F^{\alpha\beta\gamma}_{\lambda_1}(x_1)\,F^{\alpha\beta\gamma}_{\lambda_2-1}(x_2) - x_1 \leftrightarrow x_2\Big) \\ &\qquad\qquad\qquad \times \Big(G^{\alpha\beta\gamma}_{\lambda_1'}(y_1)\,G^{\alpha\beta\gamma}_{\lambda_2'-1}(y_2) - y_1 \leftrightarrow y_2\Big)\,, \end{aligned} \tag{5.4.105}$$

$\boldsymbol{\lambda_2 = 0, 1}$ $\left(\textbf{semi-short and } \frac{1}{4}\textbf{-BPS}\right)$:

$$\begin{aligned} f(x,y) &= \left(\frac{x}{y}\right)^p (-1)^{\lambda_1'} F^{\alpha\beta\gamma}_{\lambda_1}(x) G^{\alpha\beta\gamma}_{\lambda_1'}(y) \\ f(x_1,x_2,y_1,y_2) &= \sum_{j=\lambda_1'+1}^{p} (-1)^{\lambda_1'}\Big(F^{\alpha\beta\gamma}_{1-j}(x_2)\,F^{\alpha\beta\gamma}_{\lambda_1}(x_1) - (x_1\leftrightarrow x_2)\Big) \\ &\qquad\qquad \times \Big(G^{\alpha\beta\gamma}_{j}(y_2)\,G^{\alpha\beta\gamma}_{\lambda_1'}(y_1) - (y_1\leftrightarrow y_2)\Big) \\ &\quad + \sum_{j=2}^{\lambda_1'} (-1)^{\lambda_1'}\Big(F^{\alpha\beta\gamma}_{2-j}(x_2)\,F^{\alpha\beta\gamma}_{\lambda_1}(x_1) - (x_1\leftrightarrow x_2)\Big) \\ &\qquad\qquad \times \Big(G^{\alpha\beta\gamma}_{j-1}(y_2)\,G^{\alpha\beta\gamma}_{\lambda_1'}(y_1) - (y_1\leftrightarrow y_2)\Big)\,, \end{aligned} \tag{5.4.106}$$

$\underline{\lambda} = \mathbf{0}$ $\left(\frac{1}{2}\textbf{-BPS}\right)$:

$$\begin{aligned} f(x,y) &= -\left(\frac{x}{y}\right)^p \sum_{i=1}^{p} F^{\alpha\beta\gamma}_{1-i}(x) G^{\alpha\beta\gamma}_{i}(y) \\ f(x_1,x_2,y_1,y_2) &= \sum_{1\le i<j\le p} \Big(F^{\alpha\beta\gamma}_{1-i}(x_2)\,F^{\alpha\beta\gamma}_{1-j}(x_1) - (x_1\leftrightarrow x_2)\Big) \\ &\qquad\qquad \times \Big(G^{\alpha\beta\gamma}_{i}(y_1)\,G^{\alpha\beta\gamma}_{j}(y_2) - (y_1\leftrightarrow y_2)\Big)\,. \end{aligned} \tag{5.4.107}$$

5.4.2 OPE coefficients in $\mathcal{N}=4$ SYM

We wish to perform a superconformal partial wave expansion on free theory correlation functions in order to illustrate and confirm the partial waves of the previous section, and obtain new results in this theory.

A general free theory correlation function of four arbitrary charge $\frac{1}{2}$-BPS operators is given by a sum of products of propagators

$$g_{ij} = \det\left(X_j - X_j\right)^{-1} = \frac{y_{ij}^2}{x_{ij}^2} + O(\rho\bar{\rho}) \ . \tag{5.4.108}$$

Any free theory correlation function can be written, by observing that

$$\operatorname{sdet}\left(1-Z\right) = \left(\frac{g_{14}g_{23}}{g_{13}g_{24}}\right)^{-1} , \tag{5.4.109}$$

in the general form:

$$\begin{aligned}&\langle p_1p_2p_3p_4\rangle \\ &= g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac{1}{2}p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{1}{2}p_{43}} \sum_{\gamma} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}\gamma} \sum_{i=0}^{\lfloor\frac{1}{2}\gamma\rfloor} a_{\gamma i} \operatorname{sdet}\left(1-Z\right)^{-i}\end{aligned} \tag{5.4.110}$$

where $p_{ij} = p_i - p_j$ and where $a_{\gamma\, i}$ are colour factors which can be computed using Wick contractions. The restrictions on γ are the same as in (5.3.59).

On the other hand we wish to compare this with the conformal partial wave expansion (5.3.54)

$$\begin{aligned}&\langle p_1p_2p_3p_4\rangle \\ &= \sum_{\mathcal{O},\tilde{\mathcal{O}}} C^{\mathcal{O}}_{p_1p_2} C^{\tilde{\mathcal{O}}}_{p_3p_4} C_{\mathcal{O}\tilde{\mathcal{O}}}\, g_{12}^{\frac{p_1+p_2}{2}} g_{34}^{\frac{p_3+p_4}{2}} \left(\frac{g_{24}}{g_{14}}\right)^{\frac{1}{2}p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{\frac{1}{2}p_{43}} \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{\frac{1}{2}\gamma} F^{\alpha\beta\gamma\underline{\lambda}}(Z).\end{aligned} \tag{5.4.111}$$

The exercise is then to equate

$$\sum_{i=0}^{\lfloor\frac{1}{2}\gamma\rfloor} a_{\gamma i} \operatorname{sdet}\left(1-Z\right)^{-i} = \sum_{\underline{\lambda}\geq 0} \mathcal{C}_{\gamma\underline{\lambda}} F^{\alpha\beta\gamma\underline{\lambda}}(Z) \tag{5.4.112}$$

in order to find the OPE coefficients $\mathcal{C}_{\gamma\underline{\lambda}} = C^{\mathcal{O}}_{p_1p_2} C^{\tilde{\mathcal{O}}}_{p_3p_4} C_{\mathcal{O}\tilde{\mathcal{O}}}$.

The simplest way to proceed is to use the so-called Cauchy identity to rewrite the right hand side of (5.4.112) as an infinite sum over the Schur superpolynomials. This then allows for a direct comparison with the superconformal partial wave expansion (which we also view as a sum over Schur superpolynomials) and thus allows us to solve for the OPE coefficients. Remarkably, this means we never in fact need to know the form of the Schur superpolynomials themselves, both sides are given as expansions in Schur superpolynomials and since we know these are independent this allows us to equate the coefficients of each Schur superpolynomial.

5.4.3 The Cauchy Identity

The Cauchy identity provides a way to write functions of $\mathrm{sdet}(1-Z)^{-q}$ for some q as an expansion in super Schur polynomials. Cauchy's identity states that (see appendix A of [90]):

$$\frac{1}{\prod_{i,j}(1-x_i z_j)} = \sum_{\underline{\lambda}} s_{\underline{\lambda}}(x) s_{\underline{\lambda}}(z), \tag{5.4.113}$$

where $\underline{\lambda}$ is some Young tableau. If we set the z_j's to 1 we gain the following formula relevant to the bosonic case:

$$\det(1-Z)^{-p} = \frac{1}{\prod_i (1-x_i)^p} = \sum_{\lambda} s_{\underline{\lambda}}(x) d_{\underline{\lambda}}^{\mathrm{GL}(p)}, \tag{5.4.114}$$

where $d_{\underline{\lambda}}^{\mathrm{GL}(p)}$ is the dimension of some Young tableau $\underline{\lambda}$ in $\mathrm{GL}(p)$. In particular this means we can never see Young tableaux with more than p rows.

In the supersymmetric case, this formula generalises naturally to

$$\prod_i \left(\frac{1-y_i}{1-x_i}\right)^p = \sum_{\underline{\lambda}} s_{\underline{\lambda}}(x|y) d_{\underline{\lambda}}^{\mathrm{GL}(p)}. \tag{5.4.115}$$

The standard Hook dimension formula gives

$$d_{\underline{\lambda}}^{\mathrm{GL}(p)} = \frac{\prod_{i=1}^{p}(p-i+1)^{(\lambda_i)}}{\prod_{i=j}^{p}\prod_{j=1}^{p}(\lambda_j - \lambda_i + (i-j+1))^{(\lambda_i - \lambda_{i+1})}}, \tag{5.4.116}$$

where $x^{(n)}$ is the ascending Pochhammer symbol. Implicitly, this formula has a label for $p+1$ which we must switch off, namely $\lambda_{p+1} = 0$.

For example for $p = 1$, in $\mathcal{N} = 4$ SYM, one finds that

$$\mathrm{sdet}(1-Z)^{-1} = \frac{(1-y_1)(1-y_2)}{(1-x_1)(1-x_2)} = \sum_{\lambda=0}^{\infty} s_{[\lambda,0,\dots]}(x|y). \tag{5.4.117}$$

whereas for $p=2$, we get

$$\mathrm{sdet}(1-Z)^{-2} = \frac{(1-y_1)^2(1-y_2)^2}{(1-x_1)^2(1-x_2)^2} = \sum_{\lambda_1\geq\lambda_2\geq 0}^{\infty}(\lambda_1-\lambda_2+1)s_{[\lambda_1,\lambda_2,0,..]}(x|y)\ .$$

Using the above results it is now straightforward to obtain the OPE coefficients in the free theory. In the next section we give a number of low weight examples of this. Note that at this stage we are not considering the fact that in the interacting theory certain short multiplets can combine together to become long. We will consider this in the following subsection.

Let us outline a basic example for precisely how this works. In the example of $\langle 1111\rangle$ which we study in the next subsection, we will encounter the function

$$A(1+\mathrm{sdet}(1-Z)^{-1}), \tag{5.4.118}$$

which we want to compare with a linear combination of superconformal partial wave expansions of the form $F^{112[\lambda]}$ (corresponding to twist-2 operators). So using the Cauchy identity we equate

$$A(1+\mathrm{sdet}(1-Z)^{-1}) = 2As_{[0]}(x|y) + A\sum_{i\geq 1}s_{[\lambda]}(x|y) = \sum_{\lambda\geq 0}\mathcal{C}_{2[\lambda]}F^{112[\lambda]} \tag{5.4.119}$$

We can expand the rightmost-side explicitly using (5.3.88) giving

$$\begin{aligned}2As_{[0]}(x|y) + A\sum_{i\geq 1}s_{[\lambda]}(x|y) &= \mathcal{C}_{2[0]}\underbrace{\left(s_{[0]}(x|y)+\frac{1}{2}s_{[1]}(x|y)+\frac{1}{3}s_{[2]}(x|y)+\dots\right)}_{F^{112[0]}}\\ &+\mathcal{C}_{2[1]}\underbrace{\left(s_{[1]}(x|y)+\frac{1}{2}s_{[2]}(x|y)+\frac{9}{10}s_{[3]}(x|y)+\dots\right)}_{F^{112[1]}}\\ &+\mathcal{C}_{2[2]}\underbrace{\left(s_{[2]}(x|y)+\frac{3}{2}s_{[3]}(x|y)+\frac{12}{7}s_{[4]}(x|y)+\dots\right)}_{F^{112[2]}}+\dots\end{aligned} \tag{5.4.120}$$

Comparing the coefficients of $s_{[0]}(x|y)$ requires that $\mathcal{C}_{2[0]} = 2A$. A consequence of this is that this automatically sets coefficient of $s_{[1]}(x|y)$ to A on the right hand side, which yields an overall equality if we set $\mathcal{C}_{2[1]} = 0$. We may continue to the next order to find $\mathcal{C}_{2[2]}$ and there onwards to find the rest of the coefficients. With enough terms, one can spot a pattern and write a general formula. As we will see in the next subsection, it turns out that the only non-zero OPE coefficients in this case are

$\lambda \in \mathbb{Z}_{\text{even}}$, corresponding to even spin operators. All results are found in this way. Note that as mentioned previously, one never even needs to know the explicit form of the Schur superpolynomials for this.

5.4.4 Results: Free theory OPE coefficients (before recombination)

The purpose of this section is to display the OPE coefficients before taking into account any recombination in the interacting theory. We do this for the list of the correlation functions $\langle 1111\rangle$, $\langle 1122\rangle$, $\langle 2222\rangle$, $\langle 2233\rangle$ and $\langle 3333\rangle$ whilst leaving $\langle 2433\rangle$ and $\langle 3544\rangle$ for appendix D. Clearly the first two correlators can only exist in the U(N) gauge theory (since $\mathrm{tr}(W^1) = 0$ for SU(N)) whilst the others may exist in either U(N) or SU(N).

For notational convenience we have defined

$$f_\gamma\left(a_{\gamma 0}, a_{\gamma 1}, \ldots, a_{\gamma\lfloor\frac{1}{2}\gamma\rfloor}\right) := \sum_{i=0}^{\lfloor\frac{1}{2}\gamma\rfloor} a_{\gamma i}\,\mathrm{sdet}\,(1-Z)^{-i} \tag{5.4.121}$$

where $a_{\gamma i}$ are the associated colour factors.

We consider all $\frac{1}{2}$-BPS operators, both single- and multi-trace at finite N. We denote $A_\gamma = \mathrm{tr}(W^\gamma)$ so the multi-trace operator $\mathrm{tr}(W^2)^2$ is denoted $(A_2)^2$ etc. Finally, we tabulate all (or almost all) colour factors for both U(N) as well as SU(N) gauge theories, however for some cases these tables are very large and so we relegate these tables to appendix E.

$\langle 1111\rangle$

This correlator can only exist in the U(N) gauge theory and is given by

$$\langle 1111\rangle = A\left(g_{14}g_{23} + g_{13}g_{24} + g_{12}g_{34}\right) = g_{12}g_{34}\left(f_0(A) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) f_2(A,A)\right)$$

The colour factor is given by

$$A = N^2 \tag{5.4.122}$$

Correlator type	A	B
$\langle A_1 A_1 A_2 A_2 \rangle$	$2N^3$	$4N$
$\langle A_1 A_1 (A_1)^2 A_2 \rangle$	$2N^2$	$4N^2$
$\langle A_1 A_1 (A_1)^2 (A_1)^2 \rangle$	$2N^3$	$4N^3$

Table 5.4: U(N) colour factors associated to the $\langle 1122 \rangle$ correlator

In comparing with the superconformal partial wave expansion, one finds that

$$\langle 1111 \rangle = g_{12} g_{34} \left(A + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right) \sum_{\lambda \geq 0} \mathcal{C}_{2[\lambda]} F^{112[\lambda]} \right)$$
$$\text{with } \mathcal{C}_{2[\lambda]} = \frac{2A(\lambda!)^2}{(2\lambda)!} \text{ for } \lambda \in \mathbb{Z}_{\text{even}} \text{ and zero otherwise.} \tag{5.4.123}$$

$\langle 1122 \rangle$

$$\langle 1122 \rangle = A g_{12} g_{34}^2 + B \left(g_{14} g_{23} g_{34} + g_{13} g_{24} g_{34} \right) = g_{12} g_{34}^2 \left(f_0(A) + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right) f_2(B, B) \right) \tag{5.4.124}$$

The colour factors for U(N) for the various types of correlators are tabulated in table 5.4.

Since $p_{12} = p_{34} = 0$ (which means we use the same set of superconformal partial waves), we see that this result is structurally identical to the (5.4.123), but for the change

$$\mathcal{C}_{2[\lambda]} = \frac{2B(\lambda!)^2}{(2\lambda)!}, \tag{5.4.125}$$

which is simply a change in the colour factor.

$\langle 1133 \rangle$

$$\langle 1133 \rangle = A g_{12} g_{34}^3 + B \left(g_{14} g_{23} g_{34}^2 + g_{13} g_{24} g_{34}^2 \right) = g_{12} g_{34}^3 \left(f_0(A) + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right) f_2(B, B) \right) \tag{5.4.126}$$

The U(N) colour factors for the various types of correlators is given in table 5.5.

The result of the superconformal partial wave expansion is identical to the $\langle 1122 \rangle$ previously shown but for the precise colour factors.

Correlator type	A	B
$\langle A_1A_1A_3A_3\rangle$	$3N^2(1+N^2)$	$18N^2$
$\langle A_1A_1(A_1A_2)A_3\rangle$	$6N^3$	$6N(2+N^2)$
$\langle A_1A_1(A_1A_2)(A_1A_2)\rangle$	$2N^2(2+N^2)$	$2N^2(8+N^2)$
$\langle A_1A_1(A_1A_2)(A_1)^3\rangle$	$6N^3$	$18N^3$
$\langle A_1A_1(A_1)^3(A_3)\rangle$	$6N^2$	$18N^2$
$\langle A_1A_1(A_1)^3(A_1)^3\rangle$	$6N^4$	$18N^4$

Table 5.5: U(N) colour factors associated to the $\langle 1133\rangle$ correlator

$\langle 2222\rangle$

This is the first case where we have a correlator which may exist in either the U(N) or SU(N) gauge theory. The correlator is given by

$$\begin{aligned}\langle 2222\rangle &= A(g_{12}^2g_{34}^2+g_{13}^2g_{24}^2+g_{14}^2g_{23}^2)+B(g_{12}g_{23}g_{34}g_{41}+g_{13}g_{32}g_{21}g_{14}+g_{13}g_{34}g_{42}g_{21})\\ &= g_{12}^2g_{34}^2\left(f_0(A)+\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)f_2(B,B)+\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 f_4(A,B,A)\right)\end{aligned} \tag{5.4.127}$$

For the SU(N) theory, there is only one possible colour structure where the operator is A_2 which is given in table 5.6.

Correlator type $SU(N)$	A	B
$\langle A_2A_2A_2A_2\rangle$	$4(N^2-1)^2$	$16(N^2-1)$

Table 5.6: SU(N) colour factors associated to the $\langle 2222\rangle$ correlator

On the other hand there are a few variations in the U(N) theory, which are given in table 5.7.

Comparing to a superconformal partial wave expansion yields

$$\langle 2222\rangle = g_{12}^2g_{34}^2\left(A+\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\sum_{\lambda\geq 0}\mathcal{C}_{2[\lambda]}F^{112[\lambda]}+\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2\sum_{\lambda_1\geq\lambda_2\geq 0}\mathcal{C}_{4[\lambda_1,\lambda_2]}F^{224[\lambda_1,\lambda_2]}\right), \tag{5.4.128}$$

Correlator type $U(N)$	A	B
$\langle A_2 A_2 A_2 A_2 \rangle$	$4N^4$	$16N^2$
$\langle (A_1)^2 A_2 A_2 A_2 \rangle$	$4N^3$	$16N$
$\langle (A_1)^2 (A_1)^2 A_2 A_2 \rangle$	$4N^4$	$16N^2$
$\langle (A_1)^2 (A_1)^2 (A_1)^2 A_2 \rangle$	$4N^3$	$16N^3$
$\langle (A_1)^2 (A_1)^2 (A_1)^2 (A_1)^2 \rangle$	$4N^4$	$16N^4$
$\langle (A_1)^2 A_2 (A_1)^2 A_2 \rangle$	$4N^2$	$16N^2$

Table 5.7: U(N) colour factors associated to the $\langle 2222 \rangle$ correlator

where the coefficients are given by

$$\begin{aligned}
\mathcal{C}_{2[\lambda]} &= \frac{2B(\lambda!)^2}{(2\lambda)!} \text{ for } \lambda \in \mathbb{Z}_{\text{even}} \text{ zero otherwise,} \\
\mathcal{C}_{4[\lambda_1,\lambda_2]} &= \frac{\lambda_1!\,(\lambda_1+1)!\,(\lambda_2!)^2\left(A\,(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+2)+B(-1)^{\lambda_2}\right)}{(2\lambda_2)!\,(2\lambda_1+1)!} \\
&\text{for } \lambda_1-\lambda_2 \in \mathbb{Z}_{\text{even}} \geq 0,\ \lambda_2 \in \mathbb{Z} \geq 0 \text{ and zero otherwise.}
\end{aligned} \tag{5.4.129}$$

$\langle 2233 \rangle$

One may write the free theory correlator as

$$\begin{aligned}
\langle 2233 \rangle &= A g_{12}^2 g_{34}^3 + B\left(g_{14}^2 g_{34} g_{23}^2 + g_{13}^2 g_{24}^2 g_{34}\right) + C\left(g_{12} g_{14} g_{23} g_{34}^2 + g_{12} g_{13} g_{24} g_{34}^2\right) \\
&+ D g_{13} g_{14} g_{23} g_{24} g_{34}, \\
&= g_{12}^2 g_{34}^3 \left(f_0(A) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) f_2(C,C) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 f_4(B,D,B) \right)
\end{aligned} \tag{5.4.130}$$

The colour factors for SU(N) can only come from one correlator and is given in table 5.8.

Correlator type $SU(N)$	A	B	C	D
$\langle A_2 A_2 A_3 A_3 \rangle$	$\frac{6(N^2-1)^2(N^2-4)}{N}$	0	$\frac{36(N^2-1)(N^2-4)}{N}$	$\frac{72(N^2-1)(N^2-4)}{N}$

Table 5.8: SU(N) colour factors associated to the $\langle 2233 \rangle$ correlator

For the U(N) theory we have 18 possible ways of partitioning the p_is into local operators, and we leave this for table in table E.3 in appendix E

We see that this result here is structurally identical to the $\langle 2222 \rangle$ case, the only difference is as in previous cases the precise difference in the colour factors. Namely, the result is identical to (5.4.128), but instead we have

$$\mathcal{C}_{2[\lambda]} = \frac{2C(\lambda!)^2}{(2\lambda)!} \text{ for } \lambda \in \mathbb{Z}_{\text{even}} \text{ zero otherwise,}$$
$$\mathcal{C}_{4[\lambda_1,\lambda_2]} = \frac{\lambda_1!\,(\lambda_1+1)!\,(\lambda_2!)^2\left(B\,(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+2)+D(-1)^{\lambda_2}\right)}{(2\lambda_2)!\,(2\lambda_1+1)!}$$
$$\text{for } \lambda_1-\lambda_2 \in \mathbb{Z}_{\text{even}} \geq 0,\ \lambda_2 \in \mathbb{Z} \geq 0 \text{ and zero otherwise.} \tag{5.4.131}$$

$\langle 3333 \rangle$

The free theory correlator is given by

$$\begin{aligned}\langle 3333\rangle &= A\left(g_{14}^3 g_{23}^3 + g_{13}^3 g_{24}^3 + g_{12}^3 g_{34}^3\right) + B(g_{13}g_{14}^2 g_{24}g_{23}^2 + g_{12}g_{14}^2 g_{34}g_{23}^2\\ &+ g_{13}^2 g_{14}g_{24}^2 g_{23} + g_{12}^2 g_{14}g_{34}^2 g_{23} + g_{12}^2 g_{13}g_{24}g_{34}^2 + g_{12}g_{13}^2 g_{24}^2 g_{34}) + C g_{12}g_{13}g_{14}g_{23}g_{24}g_{34},\\ &= g_{12}^3 g_{34}^3\Bigg(f_0(A) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) f_2(B,B) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 f_4(B,C,B)\\ &\qquad + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3 f_6(A,B,B,A)\Bigg)\end{aligned} \tag{5.4.132}$$

There is only one SU(N) correlator which has colour factors given in table 5.9.

Correlator type	A	B	C
$\langle A_3A_3A_3A_3\rangle$	$\frac{9(N^2-4)^2(N^2-1)^2}{N^2}$	$\frac{81(N^2-4)^2(N^2-1)}{N^2}$	$\frac{162(N^2-4)(N^2-1)\left(N^2-12\right)}{N^2}$

Table 5.9: SU(N) colour factors associated to the $\langle 3333\rangle$ correlator

For the U(N) theory we have 17 possible ways of partitioning the p_is into local operators, and we leave this for table in table E.4 in appendix E.

Upon comparing to a superconformal partial wave expansion we get

$$\begin{aligned}\langle 3333\rangle = g_{12}^3 g_{34}^3\Bigg(A &+ \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\sum_{\lambda\geq 0}\mathcal{C}_{2[\lambda]}F^{112[\lambda]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2\sum_{\lambda_1\geq\lambda_2\geq 0}\mathcal{C}_{4[\lambda_1,\lambda_2]}F^{224[\lambda_1,\lambda_2]}\\ &+ \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3\sum_{\lambda_1\geq\lambda_2\geq\lambda_3\geq 0}\mathcal{C}_{6[\lambda_1,\lambda_2,\lambda_3]}F^{336[\lambda_1,\lambda_2,\lambda_3]}\Bigg),\end{aligned} \tag{5.4.133}$$

Similarly to previous examples we see structures repeating again. Namely, the $\gamma=2$ is identical to (5.4.129) and $\gamma=4$ sector is structurally identical to (5.4.129) but for the change of colour factor $A\to B$ and $B\to C$. We also get a $\gamma=6$ sector where the

OPE coefficients are

$$\mathcal{C}_{6[\lambda_1,\lambda_2]} = m_{\lambda_1,\lambda_2}\frac{1}{2}\Big(A\,(\lambda_1+2)\,(\lambda_1+3)\,(\lambda_1-\lambda_2+1)\,(\lambda_2+1)\,(\lambda_2+2)\,(\lambda_1+\lambda_2+4)$$
$$+\,4B\,\Big(\left((-1)^{\lambda_2}+1\right)\lambda_1\,(\lambda_1+5)+8(-1)^{\lambda_2}+\left((-1)^{\lambda_2}-1\right)\lambda_2\,(\lambda_2+3)+4\Big)\Big)$$
$$\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{even}}\geq 0, \lambda_2\geq 0 \text{ and zero otherwise,} \tag{5.4.134}$$

$$\mathcal{C}_{6[\lambda_1,\lambda_2,1]} = m_{\lambda_1,\lambda_2}\frac{1}{4}\Big(A\,(\lambda_1+1)\,(\lambda_1+4)\,(\lambda_1-\lambda_2+1)\,\lambda_2\,(\lambda_2+3)\,(\lambda_1+\lambda_2+4)$$
$$+\,4B\,\left((-1)^{\lambda_2}-1\right)(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+4)\Big)$$
$$\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{odd}}\geq 1, \lambda_2\geq 1 \text{ and zero otherwise,} \tag{5.4.135}$$

$$\mathcal{C}_{6[\lambda_1,\lambda_2,2]} = m_{\lambda_1,\lambda_2}\frac{1}{12}\Big(A\lambda_1\,(\lambda_1+5)\,(\lambda_1-\lambda_2+1)\,(\lambda_2-1)\,(\lambda_2+4)\,(\lambda_1+\lambda_2+4)$$
$$+\,4B\,\Big(\left((-1)^{\lambda_2}+1\right)\lambda_1\,(\lambda_1+5)+\left((-1)^{\lambda_2}-1\right)(\lambda_2-1)\,(\lambda_2+4)\Big)\Big)$$
$$\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{even}}\geq 0, \lambda_2\geq 2 \text{ and zero otherwise,} \tag{5.4.136}$$

where

$$m_{\lambda_1,\lambda_2} = \frac{(\lambda_1+2)!^2\,(\lambda_2+1)!^2}{(2\lambda_2+2)!\,(2\lambda_1+4)!}. \tag{5.4.137}$$

We give two further cases in appendix D, namely $\langle 4233\rangle$ and $\langle 5344\rangle$.

5.4.5 Consistency checks for the above OPE coefficients

It is possible to perform non-trivial consistency checks for the above results if we have some information concerning the number of operators in each representation.

To see where these consistency checks come from, consider writing the OPE coefficients as follows,

$$\mathcal{C}^{p_1p_2p_3p_4}_{\gamma\underline{\lambda}} = \langle C_{p_1p_2}, C_{p_3p_4}\rangle := \sum_{\mathcal{O}^{\gamma\underline{\lambda}},\tilde{\mathcal{O}}^{\gamma\underline{\lambda}}} C^{\mathcal{O}}_{p_1p_2}C^{\tilde{\mathcal{O}}}_{p_3p_4}C_{\mathcal{O}\tilde{\mathcal{O}}}\ , \tag{5.4.138}$$

such that the rightmost-side defines an inner product of the structure constants of the three-point function with a metric defined by the two point function. Here we sum over all operators in the same representation defined by γ and $\underline{\lambda}$ and we take $C_{p_ip_j}$ to be a vector in which the dimension is equal to the number of operators in this representation. If we choose a basis for the operators where we have diagonalised the two-point functions, then we have simply $C_{\mathcal{O}\tilde{\mathcal{O}}} \sim \delta_{\mathcal{O}\tilde{\mathcal{O}}}$ and this becomes the standard scalar product.

The first observation that follows from this is that

$$\cos^2(\theta) = \frac{\langle C_{p_1p_2}, C_{p_3p_4}\rangle^2}{\langle C_{p_1p_2}, C_{p_1p_2}\rangle \langle C_{p_3p_4}, C_{p_3p_4}\rangle}, \tag{5.4.139}$$

where θ is the angle between the two vectors $C^{\mathcal{O}}_{p_1p_2}$ and $C^{\mathcal{O}}_{p_3p_4}$, and so it follows that

$$0 \le \frac{(\mathcal{C}^{p_1p_2p_3p_4})^2}{\mathcal{C}^{p_1p_2p_1p_2}\mathcal{C}^{p_3p_4p_3p_4}} \le 1 \tag{5.4.140}$$

for all OPE coefficients. [3]

Furthermore, if there is only one operator $\mathcal{O}$ in the representation in question, then the vector space has dimension 1 and it follows that

$$\frac{(\mathcal{C}^{p_1p_2p_3p_4})^2}{\mathcal{C}^{p_1p_2p_1p_2}\mathcal{C}^{p_3p_4p_3p_4}} = 1. \tag{5.4.141}$$

Indeed if we know how many operators there are in a particular representation, M, (so we know the dimension of the relevant inner product space) then we know that any Gram determinant of dimension $M+1$ must vanish [4]. So

$$\det\left(\mathcal{C}^{p_ip_jp_kp_l}\right)_{\substack{(p_i,p_j)\in S\\(p_k,p_l)\in S}}, \tag{5.4.142}$$

where S is any set of pairs (p_i,p_j) such that $|S| = M+1$.

So for the previously mentioned case where the number of operators is one we let $S = \{(p_1,p_2),(p_3,p_4)\}$ and then

$$\text{Gram} = \det\begin{pmatrix} \mathcal{C}^{p_1p_2p_1p_2} & \mathcal{C}^{p_1p_2p_3p_4} \\ \mathcal{C}^{p_1p_2p_3p_4} & \mathcal{C}^{p_3p_4p_3p_4}\end{pmatrix} = \mathcal{C}^{p_1p_2p_1p_2}\mathcal{C}^{p_3p_4p_3p_4} - (\mathcal{C}^{p_1p_2p_3p_4})^2 = 0, \tag{5.4.143}$$

which is equivalent to equation (5.4.140) being equal to one. For the case where we have two operators the Gram determinate becomes

$$\text{Gram} = \det\begin{pmatrix} \mathcal{C}^{p_1p_2p_1p_2} & \mathcal{C}^{p_1p_2p_3p_4} & \mathcal{C}^{p_1p_2p_5p_6} \\ \mathcal{C}^{p_1p_2p_3p_4} & \mathcal{C}^{p_3p_4p_3p_4} & \mathcal{C}^{p_3p_4p_5p_6} \\ \mathcal{C}^{p_1p_2p_5p_6} & \mathcal{C}^{p_3p_4p_5p_6} & \mathcal{C}^{p_5p_6p_5p_6}\end{pmatrix} = 0\ . \tag{5.4.144}$$

[3] For long operators, this need only be true after taking (5.2.20) into account.

[4] Recall the the Gram determinant is the determinant of some inner product, i.e. given some inner product $\langle \nu_i, \nu_j\rangle$, we define Gram $:= \det(\langle \nu_i,\nu_j\rangle)$

Let us check these conditions in a few cases. Firstly, consider the case with only one operator. This is the case for all twist two operators $\mathcal{O}^{2[\lambda]}$ in the SU(N) theory. Looking back at the results above one can straightforwardly check that indeed

$$\begin{aligned}
&\mathcal{C}^{2222}_{2[\lambda]}\mathcal{C}^{3333}_{2[\lambda]} - (\mathcal{C}^{2233}_{2[\lambda]})^2 \\
&= \left(\frac{2(\lambda!)^2}{(2\lambda)!}\right)^2 \left[16(N^2-1) \times \frac{81(N^2-4)^2(N^2-1)}{N^2} - \left(\frac{36(N^2-1)(N^2-4)}{N}\right)^2\right] = 0\ .
\end{aligned} \tag{5.4.145}$$

Similarly in the U(N) case there are two twist 2 operators $\mathcal{O}^{2[\lambda]}$ for each spin λ (a single-trace and a double-trace one). Thus the following 3×3 Gram determinant should vanish

$$\det\begin{pmatrix} \mathcal{C}^{1111}_{2[\lambda]} & \mathcal{C}^{1122}_{2[\lambda]} & \mathcal{C}^{1133}_{2[\lambda]} \\ \mathcal{C}^{1122}_{2[\lambda]} & \mathcal{C}^{2222}_{2[\lambda]} & \mathcal{C}^{2233}_{2[\lambda]} \\ \mathcal{C}^{1133}_{2[\lambda]} & \mathcal{C}^{2222}_{2[\lambda]} & \mathcal{C}^{3333}_{2[\lambda]} \end{pmatrix} = 0 \tag{5.4.146}$$

which can be seen to be true via previously found results.

In the next section we will show how similar considerations give information about the disentangling of protected and unprotected operators. We will make use of such relations in order to completely disentangle the protected and unprotected sectors in the $\langle 3333\rangle$ correlator.

5.4.6 Physical OPE coefficients: recombination of operators in SU(N)

It is known that free theory supermultiplets in $\mathcal{N}=4$ SYM combine together to form long supermultiplets, which are then free to develop an anomalous dimension. In order to separate out the OPE coefficients into free and interacting pieces, it is useful to be able to disentangle the genuine short multiplets from those which become part of long multiplets. This is also required in the conformal bootstrap programme where one requires the contribution of all protected operators to the free correlator [76].

It is impossible to uniquely disentangle this information from the free theory alone, one requires some information from the interacting theory. At least in some situations however, knowledge of mixed charge correlators, together with simply the knowledge of the number of long/short operators (the precise form of them is however not required) allows us to uniquely disentangle the protected and unprotected sectors. The number of short and long operators can be obtained by an examination of the classical interacting theory [27,91]. We will give an example of this in the current section, and we will obtain the precise separation of the free SU(N) correlator $\langle 3333\rangle$ into protected and unprotected sectors by making use of the $\langle 2233\rangle$ and $\langle 2222\rangle$ correlators.

In order to gain the correct answer, we make repetitive use of the reducibility equation at the unitary bound (5.3.101) which we restate for convenience

$$\begin{aligned} F_{\text{long}}^{\alpha\beta\gamma[\lambda+1,1^{\nu+1}]}(x|y) &:= \lim_{\rho\to 1} F^{\alpha\beta\gamma[\lambda+\rho,\rho,1^{\nu}]}(x|y) \\ &= \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^{-1} F^{(\alpha-1)(\beta-1)(\gamma-2)[\lambda+2,1^{\nu}]}(x|y) + F^{\alpha\beta\gamma[\lambda+1,1^{\nu+1}]}(x|y). \end{aligned} \tag{5.4.147}$$

There then remains the question as to how to decide which operators become long without doing explicit computations.

In this subsection we present the physical OPE coefficients of gauge group SU(N), in particular for $\langle 2222\rangle$, $\langle 2233\rangle$ and $\langle 3333\rangle$. Let us begin with the $\langle 2222\rangle$ case.

$\langle 2222\rangle$

Stating the result again, we had

$$\langle 2222\rangle = g_{12}^2 g_{34}^2 \left(A + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) \sum_{\lambda\geq 0} \mathcal{C}_{2[\lambda]} F^{112[\lambda]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 \sum_{\lambda_1\geq\lambda_2\geq 0} \mathcal{C}_{4[\lambda_1,\lambda_2]} F^{224[\lambda_1,\lambda_2]} \right), \tag{5.4.148}$$

where the coefficients are given by (5.4.129), but for convenience we repeat them

$$\begin{aligned} \mathcal{C}_{2[\lambda]} &= \frac{2B(\lambda!)^2}{(2\lambda)!} \text{ for } \lambda \in \mathbb{Z}_{\text{even}} \text{ zero otherwise,} \\ \mathcal{C}_{4[\lambda_1,\lambda_2]} &= \frac{\lambda_1!\,(\lambda_1+1)!\,(\lambda_2!)^2\left(A\,(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+2) + B(-1)^{\lambda_2}\right)}{(2\lambda_2)!\,(2\lambda_1+1)!} \\ &\text{for } \lambda_1-\lambda_2 \in \mathbb{Z}_{\text{even}} \geq 0,\ \lambda_2 \in \mathbb{Z}\geq 0 \text{ and zero otherwise.} \end{aligned} \tag{5.4.149}$$

with

Correlator type $SU(N)$	A	B
$\langle A_2A_2A_2A_2\rangle$	$4(N^2-1)^2$	$16(N^2-1)$

(5.4.150)

We recognise the term $F^{112[2]}$ as being the Konishi operator. Famously, the Konishi operator gains an anomalous dimension in the interacting theory, hence it should be long whilst as it stands it is short. By looking at the structure of the Wick contractions, one also observes that the semi-short operators that follow, namely $F^{112[\lambda\geq 4]}$ are all long in the interacting theory and have the form $\mathrm{tr}(W_{AB}(\partial)^\lambda \bar{W}^{AB})$ [27]. The operator corresponding to $F^{112[0]}$, on the other hand, corresponds to the stress-tensor multiplet, and is the only $\gamma = 2$ protected operator. It will remain short in the interacting theory.

In order to manifest these points one may make use of the reducibility equation

$$F^{112[\lambda]} = \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\left(F^{224[\lambda-1,1]}_{\text{long}} - F^{224[\lambda-1,1]}\right). \tag{5.4.151}$$

In which we get

$$\begin{aligned}\langle 2222\rangle = g_{12}^2 g_{34}^2 \Bigg(A + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) 2BF^{112[0]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 \Bigg(\sum_{\lambda\geq 0}^{\infty} \mathcal{C}_{4[\lambda]} F^{224[\lambda]} \\ + \sum_{\lambda\geq 1}^{\infty} \mathcal{C}'_{4[\lambda,1]} F^{224[\lambda,1]} + \sum_{\lambda_1\geq\lambda_2\geq 2}^{\infty} \mathcal{C}_{4[\lambda_1,\lambda_2]} F^{224[\lambda_1,\lambda_2]} + \sum_{\lambda\geq 1}^{\infty} \mathcal{C}_{2[\lambda+1]} F^{224[\lambda,1]}_{\text{long}} \Bigg)\Bigg),\end{aligned} \tag{5.4.152}$$

where

$$\mathcal{C}'_{4[\lambda,1]} = \mathcal{C}_{4[\lambda,1]} - \mathcal{C}_{2[\lambda+1]}. \tag{5.4.153}$$

Here the second line consists of unprotected operators, whereas the first line corresponds to genuine short operators.

So we have used qualitative knowledge (essentially that all twist two operators become long) to disentangle the protected and unprotected sectors. This result is consistent with [77].

$\langle 2233\rangle$

As we discussed above, the structural form of $\langle 2233\rangle$ is the same as that of $\langle 2222\rangle$. The reason for this is that we are computing the overlap of the A_2A_2 OPE with the A_3A_3 OPE, which in fact contains all the sectors of the A_2A_2 OPE. With coefficients given by

$$\begin{aligned}
\mathcal{C}_{2[\lambda]} &= \frac{2C(\lambda!)^2}{(2\lambda)!} \text{ for } \lambda\in\mathbb{Z}_{\text{even}} \text{ zero otherwise,}\\
\mathcal{C}_{4[\lambda_1,\lambda_2]} &= \frac{\lambda_1!\,(\lambda_1+1)!\,(\lambda_2!)^2\left(B\,(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+2)+D(-1)^{\lambda_2}\right)}{(2\lambda_2)!\,(2\lambda_1+1)!}\\
&\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{even}}\geq 0,\ \lambda_2\in\mathbb{Z}\geq 0 \text{ and zero otherwise.}
\end{aligned} \tag{5.4.154}$$

with table 5.8.

Correlator type $SU(N)$	A	B	C	D
$\langle A_2A_2A_3A_3\rangle$	$\frac{6(N^2-1)^2(N^2-4)}{N}$	0	$\frac{36(N^2-1)(N^2-4)}{N}$	$\frac{72(N^2-1)(N^2-4)}{N}$

(5.4.155)

The multiplet recombination is then identical to the $\langle 2222\rangle$ case: essentially remove all $F^{112[\lambda]}$ (except for the $\frac{1}{2}$-BPS case $F^{112[0]}$) in favour of long operators.

The result of performing this is:

$$\begin{aligned}
\langle 2233\rangle = g_{12}^2 g_{34}^3 \Bigg(A + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) 2CF^{112[0]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 \Bigg(\sum_{\lambda\geq 0}^{\infty} \mathcal{C}_{4[\lambda]} F^{224[\lambda]}\\
+ \sum_{\lambda\geq 1}^{\infty} \mathcal{C}'_{4[\lambda,1]} F^{224[\lambda,1]} + \sum_{\lambda_1\geq\lambda_2\geq 2}^{\infty} \mathcal{C}_{4[\lambda_1,\lambda_2]} F^{224[\lambda_1,\lambda_2]} + \sum_{\lambda\geq 1}^{\infty} \mathcal{C}_{2[\lambda+1]} F^{224[\lambda,1]}_{\text{long}} \Bigg)\Bigg),
\end{aligned} \tag{5.4.156}$$

where

$$\mathcal{C}'_{4[\lambda,1]} = \mathcal{C}_{4[\lambda,1]} - \mathcal{C}_{2[\lambda+1]}, \tag{5.4.157}$$

and again the first line consists of protected operators and the second line unprotected operators.

Interestingly, the coefficient $\mathcal{C}'_{4[1,1]}$ of $F^{224[1,1]}$, namely $\frac{1}{6}(4B-2C-D)$ is subleading in the planar limit, whereas for the $\langle 2222\rangle$ case it is not. This can be understood as follows. The coefficient $\mathcal{C}'_{4[1,1]}$ is related to the OPE coefficient of the genuine twist four quarter BPS operator. In the large N limit this is a double trace operator (see [27,92]). As described in section 5.3.2 the twist-4 operators arising from the A_2A_2 OPE are double trace operators whereas the twist four operators arising from the A_3A_3 OPE on the other hand involve a Wick contraction, which in the large N limit reduces to a single trace operator.

Also note that the presence of non-zero coefficients $\mathcal{C}_{4[\lambda]}$ and $\mathcal{C}'_{4[\lambda,1]}$ imply that the OPE coefficient $C_{33}^{\mathcal{O}^{\text{twist }4}}$ where $\mathcal{O}^{\text{twist }4}$ are the protected twist-4 operators, can not be zero. This in turn has some unexpected implications for the twist four part of the protected sector of the $\langle 3333\rangle$ correlator as we shall see.

$\langle 3333\rangle$

Now we come to a more non-trivial case, the $\langle 3333\rangle$ correlator which contains operators up to twist-6.

Firstly we restate the result before recombination from the previous section. The OPE coefficients here are as in (5.4.129) and (5.4.134) where for the $A_{4\underline{\lambda}}$ coefficient of the former, we must do the change $A \to B$ and $B \to C$.

$$\langle 3333\rangle = g_{12}^3 g_{34}^3 \Bigg(A + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) \sum_{\lambda\geq 0} \mathcal{C}_{2[\lambda]} F^{112[\lambda]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 \sum_{\lambda_1\geq\lambda_2\geq 0} \mathcal{C}_{4[\lambda_1,\lambda_2]} F^{224[\lambda_1,\lambda_2]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3 \sum_{\lambda_1\geq\lambda_2\geq\lambda_3\geq 0} \mathcal{C}_{6[\lambda_1,\lambda_2,\lambda_3]} F^{336[\lambda_1,\lambda_2,\lambda_3]} \Bigg), \tag{5.4.158}$$

with coefficients

$$\begin{aligned}
\mathcal{C}_{2[\lambda]} &= \frac{2B(\lambda!)^2}{(2\lambda)!} \text{ for } \lambda \in \mathbb{Z}_{\text{even}} \text{ zero otherwise,} \\
\mathcal{C}_{4[\lambda_1,\lambda_2]} &= \frac{\lambda_1!\,(\lambda_1+1)!\,(\lambda_2!)^2\left(B\,(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+2)+C(-1)^{\lambda_2}\right)}{(2\lambda_2)!\,(2\lambda_1+1)!} \\
&\text{for } \lambda_1-\lambda_2 \in \mathbb{Z}_{\text{even}} \geq 0,\ \lambda_2 \in \mathbb{Z} \geq 0 \text{ and zero otherwise.}
\end{aligned} \tag{5.4.159}$$

and exactly as is given in (5.4.134), with colour factors

Correlator type	A	B	C
$\langle A_3A_3A_3A_3\rangle$	$\frac{9(N^2-4)^2(N^2-1)^2}{N^2}$	$\frac{81(N^2-4)^2(N^2-1)}{N^2}$	$\frac{162(N^2-4)(N^2-1)\left(N^2-12\right)}{N^2}$

(5.4.160)

Here, the first manoeuver is to use the reducibility equation (5.4.151) to replace the short Konishi and the succession of $\gamma = 2$ semi-short operators by long operators as in the previous two cases.

However, now we need some additional information to help us with the twist four ($\gamma = 4$) sector. In particular we need to know how many genuine short twist four operators there are in the theory (we already know from the $\langle 2233\rangle$ correlator that it can not be zero). This can be answered by appealing to the classical interacting theory [27]. In analytic superspace the short twist four operators $\mathcal{O}^{4[\lambda]}$ and $\mathcal{O}^{4[\lambda-1,1]}$ are double trace operators of the form $A_2\partial^\lambda A_2$ whereas those which combine to become long operators are single trace operators. Just as for the twist two operators, there is precisely one such operator for all even λ. The first few cases can also be checked with table 6 in the appendix of [91].

Armed with this knowledge that there is only one protected twist four operator for each case, we can then use the considerations of section 5.4.5 to predict the OPE coefficients, $\tilde{\mathcal{C}}^{3333}_{4\underline{\lambda}}$, after multiplet recombination, using the corresponding coefficients from $\langle 2222\rangle$ and $\langle 2233\rangle$ via (5.4.143).

Namely we predict that

$$\tilde{\mathcal{C}}_{4[\lambda]} = \frac{\left(\mathcal{C}^{2233}_{4[\lambda]}\right)^2}{\mathcal{C}^{2222}_{4[\lambda]}} = \frac{1296\,(N^2-4)^2\,(N^2-1)\,\lambda!(\lambda+1)!}{N^2(2\lambda+1)!\,(-\lambda(\lambda+3)+(\lambda+1)(\lambda+2)N^2+2)}, \tag{5.4.161}$$

$$\tilde{\mathcal{C}}_{4[\lambda,1]} = \frac{\left(\mathcal{C}'^{2233}_{4[\lambda,1]}\right)^2}{\mathcal{C}'^{2222}_{4[\lambda,1]}} = \frac{5184\,(N^2-4)^2\,(N^2-1)\,((\lambda+1)!)^2}{N^2(2\lambda+2)!\,(\lambda(\lambda+3)\,(N^2-1)-12)}, \tag{5.4.162}$$

where we have explicitly put in the colour factors.

We therefore deduce that we must use the reducibility equations to send part of the $\gamma=4$ superconformal partial waves to the $\gamma=6$ sectors, leaving the above coefficients. Moreover we find another consistency check in the fact that $\tilde{\mathcal{C}}_{4[1,1]} = \mathcal{C}'_{4[1,1]}$ corresponding to a protected $\frac{1}{4}$-BPS operator which can not be combined with any higher weight operators to become long.

Altogether, this requires the use of the three reducibility equations, and the final equation comes from the redundancy of the Dynkin labels

$$\begin{aligned}
F^{112[\lambda]} &= \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\left(F^{224[\lambda-1,1]}_{\text{long}} - F^{224[\lambda-1,1]}\right),\\
F^{224[\lambda]} &= \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\left(F^{336[\lambda-1,1]}_{\text{long}} - F^{336[\lambda-1,1]}\right),\\
F^{224[\lambda,1]} &= \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\left(F^{336[\lambda-1,1,1]}_{\text{long}} - F^{336[\lambda-1,1,1]}\right),\\
F^{224[\lambda_1,\lambda_2]} &= \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\left(F^{336[\lambda_1-1,\lambda_2-1,2]}\right).
\end{aligned} \tag{5.4.163}$$

We thus obtain

$$\frac{\langle 3333\rangle}{g_{12}^3 g_{34}^3} = \text{protected} + \text{unprotected}, \tag{5.4.164}$$

where

$$\begin{aligned}
\text{protected} = A &+ \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right) 2BF^{112[0]} \\
&+ \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 \left[(2B+C)F^{224[0]} + \sum_{\lambda\geq 2} \tilde{\mathcal{C}}_{4[\lambda]}F^{224[\lambda]} + \sum_{\lambda\geq 1} \tilde{\mathcal{C}}_{4[\lambda,1]}F^{224[\lambda,1]}\right] \\
&+ \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3 \left[\sum_{\lambda\geq 0} \mathcal{C}_{6[\lambda]}F^{336[\lambda]} + \frac{1}{10}(18A-14B-C)F^{336[1,1]} \right. \\
&\qquad \left. + \sum_{\lambda\geq 3} \mathcal{C}'_{6[\lambda,1]}F^{336[\lambda,1]} + \sum_{\lambda\geq 2} \mathcal{C}'_{6[\lambda,1,1]}F^{336[\lambda,1,1]}\right]
\end{aligned} \tag{5.4.165}$$

and

$$\begin{aligned}
\text{unprotected} = &\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 \left[\sum_{\lambda\geq 2} \mathcal{C}_{2[\lambda,2]}F^{224[\lambda,2]} + \sum_{\lambda\geq 1} \mathcal{C}_{2[\lambda+1]}F^{224[\lambda,1]}_{\text{long}}\right] \\
&+ \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3 \left[\sum_{\lambda_1\geq\lambda_2\geq 2} \mathcal{C}_{6[\lambda_1,\lambda_2]}F^{336[\lambda_1,\lambda_2]} + \sum_{\lambda_1\geq\lambda_2\geq 2} \mathcal{C}_{6[\lambda_1,\lambda_2,1]}F^{336[\lambda_1,\lambda_2,1]} \right. \\
&\qquad + \sum_{\lambda_1\geq\lambda_2\geq 2} \mathcal{C}'_{6[\lambda_1,\lambda_2,2]}F^{336[\lambda_1,\lambda_2,2]} + \sum_{\lambda\geq 2} \mathcal{C}''_{6[\lambda,1,1]}F^{336[\lambda,1,1]}_{\text{long}} + \\
&\qquad \left. + \sum_{\lambda\geq 1} \mathcal{C}'''_{6[\lambda+1]}F^{336[\lambda,1]}_{\text{long}}\right]
\end{aligned} \tag{5.4.166}$$

where

$$\begin{aligned}
\mathcal{C}'_{6[\lambda,1]} &= \mathcal{C}_{6[\lambda,1]} - \mathcal{C}_{4[\lambda+1]} + \tilde{\mathcal{C}}_{4[\lambda+1]}, \\
\mathcal{C}'_{6[\lambda,1,1]} &= \mathcal{C}_{6[\lambda,1,1]} - \mathcal{C}_{4[\lambda+1,1]} + \mathcal{C}_{2[\lambda+2]} + \tilde{\mathcal{C}}_{4[\lambda+1,1]}, \\
\mathcal{C}'_{6[\lambda_1,\lambda_2,2]} &= \mathcal{C}_{6[\lambda_1,\lambda_2,2]} + \mathcal{C}_{4[\lambda_1+1,\lambda_2+1]}, \\
\mathcal{C}''_{6[\lambda,1,1]} &= \mathcal{C}_{4[\lambda+1,1]} - \mathcal{C}_{2[\lambda+2]} - \tilde{\mathcal{C}}_{4[\lambda+1,1]}, \\
\mathcal{C}'''_{6[\lambda,1,1]} &= \mathcal{C}_{4[\lambda+1]} - \tilde{\mathcal{C}}_{4[\lambda+1]}
\end{aligned} \tag{5.4.167}$$

The existence of a non-trivial protected twist-4 sector, $\tilde{\mathcal{C}}$, differs from the assumption made in [87] that these should be absent and absorbed further into long operators using the third line of (5.4.163).

Note that both the results here and the results of [87] are consistent with positivity of the OPE coefficients (we have checked and indeed all these coefficients remain non-negative). Furthermore these results agree with [87] in the large N limit, since the coefficients $\tilde{\mathcal{C}}$ are subleading.

5.5 Conclusion

In this chapter we have provided the superconformal partial waves relevant for four-point functions of scalar operators in so-called Grassmannian of the form $\mathrm{Gr}_{m|n}(2m|2n)$. These are interesting mathematical objects in their own right, however they gain physical relevance for some selected values of the (m, n) parameters, which yields $\mathcal{N} = 4$, $\mathcal{N} = 2$ and bosonic (super)conformal partial waves in four dimensions together with the purely internal conformal partial wave (which we omitted here but can be found in [72]). This all comes from the very same coefficient function $R_{\underline{\mu}}^{\alpha\beta\gamma\underline{\lambda}}$ which does not depend on any particular group, but rather the Young tableaux only. The precise group only comes in via the Schur superpolynomials. We performed a summation on the expansion in Schur superpolynomials to give a novel determinant form for general (m, n).

We then considered $\mathcal{N} = 4$ analytic superspace and initiated a detailed analysis of mixed charge $\frac{1}{2}$-BPS four-point functions in the free theory. We analysed the free theory OPE coefficients in both the SU(N) and the U(N) gauge theory for a number of correlators. We finally considered the multiplet rearrangement due to the recombination of short operators into long operators for the SU(N) theory. In particular, using the $\langle 2233\rangle$ correlator in the SU(N) gauge theory implied that there must be a non-trivial twist-4 sector appearing in the $\langle 3333\rangle$ correlator which remains protected.

We also remark upon a result obtained in the paper in which this chapter is based but omitted here. It was also found section 3.4.4 of [72] that one can set $m = 0$ in the summed up form of the superconformal partial wave (5.3.89) and expand in internal Schur polynomials. The result is a completely different set of coefficients that works such that we can compare the external with the internal coefficients. We found that the numerical coefficient of the external representation $\underline{\lambda}$ is equal to the coefficient of the internal representation $\underline{\lambda}^T$.

Looking forward, there are a number of directions to take. Computationally, in the $\mathcal{N} = 4$ SYM case there is much data – anomalous dimensions and structure constants – to be extracted, which can then be compared to those computed via integrability. Moreover, by understanding what the dimensionality of the vectors $C^{\mathcal{O}}_{p_1 p_2}$ are and using its inner product we could go ahead and work out the precise OPE coefficients for further correlators, in particular those which we have not studied all the way here.

On the bootstrap side it would be interesting to revisit and continue the work of [75,76] analysing the superconformal bootstrap in $\mathcal{N} = 4$ SYM for higher charge correlators.

Other superconformal theories not covered by the Grassmannian theories here is the six-dimensional $(2, 0)$ theory. A superconformal partial wave analysis of the energy-momentum correlator in the $(2, 0)$ theory was performed in [93] and superconformal partial waves were also considered in [94]. On the bootstrap side there has been recent work analysing the restrictions on anomalous dimensions for this theory in [96].

It would therefore be interesting to see if the method presented here can be modified to $(2, 0)$ SCFTs and related theories. In the work presented here, we made use of Schur superpolynomials. These polynomials belong to a one-parameter family of polynomials called the Jack polynomial $J_\alpha(x)$, parametrised by α [85], where for the Schur polynomials one takes $\alpha = 1$. For a completely analogous study for $(2, 0)$ SCFTs in six dimensions one would instead use $J_{\frac{1}{2}}(x)$ instead of the Schur polynomials. Some aspects of this have been studied in [93,80]

Finally, our work has been solely considering the OPE of scalar operators. An important question would be to try and find out how much of what we have previously learnt follows into the OPE of non-scalar OPEs. A starting point for $\mathcal{N} = 2$ SCFTs has begun in [97].

Chapter 6

Outlook

In this final chapter we would like to bring this thesis to a close and remark on the most important points found in this thesis.

This thesis covered two major themes. The first was in the application of twistor theoretic methods in the perturbative regime of $\mathcal{N} = 4$ SYM. The second was the derivation and application of superconformal partial waves on what we have called Grassmannian field theories.

Beginning with the twistor methods we studied a novel approach to the supercorrelation functions of the stress-tensor supermultiplet in chapter 3. Here, the main result was a complete re-writing of the Feynman rules with some new graphical methods. The new set of rules contained essentially two rules, for every propagator we have a factor of $g_{ij} = \frac{y_{ij}^2}{x_{ij}^2}$ and for every p-vertex we have a factor of $R(i; j_1 \dots j_p)$. Schematically, we found that the lightlike limit of the n-point Grassmann degree four correlator, namely $G_{n;1}^{(0)}/G_{n;0}^{(0)}$ is the on-shell superconformal invariant relevant to scattering amplitude (namely, the R-invariant), thus we may think of $R(i; j_1 \dots j_p)$ as an off-shell generalisation.

The main lesson that we gained from this work was that much like the twistor approach to scattering amplitudes, the application here has led to a clear simplification that has given computational ease and efficiency. Most importantly, it was provided some structure in its formation. Critically, the main obstruction in comparison to amplitudes is that it is not as easy to gain gauge invariant ($\mathcal{Z}_*$-independent) results.

Looking forward, we recall that the scattering amplitudes in $\mathcal{N} = 4$ SYM are conjectured to be integrable as suggested by the Yangian invariance. In view of this, the supercorrelation functions are objects whose lightlike limit reproduces this Yangian invariance. An outstanding yet pertinent question is therefore, what is the precise mechanism in which lightlike limit reproduces the Yangian invariance? It would even be interesting to reverse the question and quantify the aspects of the full Yangian invariance that is preserved in the correlator, if at all. Understanding this may lead to a mirroring of progress for the correlator as has been done for the amplitude. The twistor approach demonstrated in chapter 3 provides a first step towards gaining the correlator analogy to the Grassmannian formulation studied in [69].

In chapter 4, we considered the six-points tree-level NMHV scattering amplitude. Previous methods which have come directly from the twistor approach gave results which manifest full dual superconformal symmetry but needs to be summed up to see the emergent physical pole structure. The main result was by making use of a new basis. In expressing the result in terms of this new basis we found a result which contained manifestly physical poles and manifestly preserved half of the dual superconformal symmetry. The new result contained only one non-trivial term whilst the others are related by the six-point dihedral symmetry.

The main lesson is that we found further evidence that in the twistor language one cannot write a result which contains manifestly full dual superconformal symmetry as well as physical pole structure. A remark is that most modern methods prioritise manifest dual superconformal symmetry whilst allowing the physical pole structure to be an emergent property. The Amplituhedron studied in [25] is the only method that attributes the non-physical pole structure to a geometric property, namely the tessellations of the Amplituhedron.

The second major theme was the derivation and application of the superconformal partial wave.

In chapter 5 we found the superconformal partial wave associated to four-point functions of scalar operators for Grassmannian field theories with $\mathrm{SU}(m,m|2n)$ symmetry. In particular for $m=2$ and $n=0,1,2$ we get $\mathcal{N}=0,2,4$ four dimensional (S)CFTs in which the four-point function in question is that of $\frac{1}{2}$-BPS operators. By considering the structure of the the OPE as applied to the four-point function, we found that the superconformal partial wave can be expressed as an infinite linear combination of GL$(m|n)$ Schur superpolynomials, i.e. $F^{\alpha\beta\gamma\underline{\lambda}}(x|y) = \sum_{\underline{\mu}\geq 0} R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}} s_{\underline{\mu}}(x|y)$. The problem then reduces to finding the coefficients $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$. By a series of arguments, we found that $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ can be obtained from the purely bosonic GL(m) problem. By constructing a conformal quadratic Casimir operator, we found a recursion relation on $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ which we solved and thus found the superconformal partial wave.

We then went on to apply the superconformal partial wave to the $\mathcal{N} = 4$ SCFT case in which we performed superconformal partial wave analyses for various correlators in the free theory. We then considered the possibility of the recombination of two short operators into long operators. By making use of the $\langle 2222\rangle$ and $\langle 2233\rangle$ correlator, we were then able to deduce a non-trivial twist-4 sector for the $\langle 3333\rangle$ which were previously undiscovered [87].

There were two major lessons in this work which are rather remarkable. The first is that at the core of this problem are the coefficients $R^{\alpha\beta\gamma\underline{\lambda}}_{\underline{\mu}}$ which are attained in the bosonic sector and can be applied to the supersymmetric case. The second major lesson was in the re-derivation of the OPE coefficients of free theory correlator $\langle 3333\rangle$ after any operator recombination had taken place. This showed that the result are required to satisfy consistency conditions which rely on knowledge of other correlators. This input was not included in previous works.

The problem as solved in chapter 5 was done completely as an exercise in group theory and representation theory, particularly that of SU$(m,m|2n)$. It is only for particular values of m and n that we gain physically relevant theories. More generally, we have found the infinite dimensional representations that follow from the concatenation of certain representations, all in SU$(m,m|2n)$. In view of this more general problem, there are two directions we can take. Firstly in the current context of SU$(m,m|2n)$, we can consider non-scalar operators, such as $\mathcal{O}_{AA'}$. Secondly, and more interestingly we can attempt to go beyond this group, namely another relevant supergroup is Osp$(8|2\mathcal{N})$ which corresponds to six-dimensional $(\mathcal{N},0)$ SCFTs. In our work, we could very easily switch m or $n = 0$, to reveal superconformal partial waves for the corresponding maximal bosonic subgroups of SU$(m,m|2n)$, namely SU$(2n)$ and SU(m,m) respectively. In the Osp$(8|2\mathcal{N})$ case, the two maximal bosonic subgroups are rather different, in this case being SO(8) and Sp$(2\mathcal{N})$. It would very fruitful in view of more general groups to see if a similar methodology can be employed to find the superconformal partial waves.

Appendix A

Proof of (3.2.66) in section 3.2.3

In section 3.2.3, we claimed that

$$\delta^{0|4}\left(\chi_* + \sigma^\alpha_{ij}\theta_{i,\alpha} + \sigma^\alpha_{ji}\theta_{j,\alpha}\right) = y^2_{ij}\delta^{0|2}\left(\sigma^\alpha_{ij}\theta^-_{i,\alpha} + A_{ij}\right)\delta^{0|2}\left(\sigma^\alpha_{ji}\theta^-_{j,\alpha} + A_{ji}\right), \tag{A.0.1}$$

where

$$A^{a'}_{ij} = \left[\chi^I_* u^{+b}_{j,I} + \sigma^\alpha_{ji}\theta^{+b}_{j,\alpha} + \sigma^\alpha_{ij}\theta^{+b}_{i,\alpha}\right]\left(y^{-1}_{ij}\right)^{a'}_{b}. \tag{A.0.2}$$

In this appendix we wish to prove it. We begin by remarking that

$$\delta^4\left(\chi_* + \sigma^\alpha_{ij}\theta_{i,\alpha} + \sigma^\alpha_{ji}\theta_{j,\alpha}\right) = \int d^4\omega e^{\omega_I\left(\chi^I_* + \sigma^\alpha_{ij}\theta^I_{i,\alpha} + \sigma^\alpha_{ji}\theta^I_{j,\alpha}\right)}, \tag{A.0.3}$$

then we perform the change of variables

$$\omega_I = u^{+a}_{i,I}v^{(i)}_a + u^{+a}_{j,I}v^{(j)}_a,$$

in which we get

$$d^4\omega = \underbrace{\det\left(u^{+a}_{i,I}, u^{+a}_{j,I}\right)^{-1}}_{y^{-2}_{ij}} d^2v^{(i)}d^2v^{(j)}. \tag{A.0.4}$$

This result follows from the basic fact that if we have some Grassmann odd n dimensional vector p^I, then we may consider the integral $\int d^n p f(p)$. Then the contributing term from $f(p)$ will be maximally nilpotent and under some change of variables $p^I \to m^I_J p^J$ we have $f(mp) \to \det(m)f(p)$. For this to be a valid change of variables, we require $d^n(mp) = \det(m)^{-1}d^n p$. One can check from (2.3.85) that $\det\left(u^{+a}_{i,I}, u^{+a}_{j,I}\right) = y^2_{ij}$.

Applying all of this gives back

$$
\begin{aligned}
&\int d^4\omega e^{\omega_I\left(\chi_*^I+\sigma_{ij}^\alpha\theta_{i,\alpha}^I+\sigma_{ji}^\alpha\theta_{j,\alpha}^I\right)} \\
&= y_{ij}^{-2}\int d^2v^{(i)}d^2v^{(j)}e^{v_a^{(i)}\left(u_{i,I}^{+a}\chi_*^I+\sigma_{ij}^\alpha\theta_{i,\alpha}^{+a}+\sigma_{ji}^\alpha\theta_{j,\alpha}^I u_{i,I}^{+a}\right)}e^{v_a^{(j)}\left(u_{j,I}^{+a}\chi_*^I+\sigma_{ji}^\alpha\theta_{j,\alpha}^{+a}+\sigma_{ij}^\alpha\theta_{i,\alpha}^I u_{j,I}^{+a}\right)} \\
&= y_{ij}^2\delta^{0|2}\left(u_{i,I}\chi_*^I+\sigma_{ij}^\alpha\theta_{i,\alpha}+\sigma_{ji}^\alpha\theta_{j,\alpha}^I u_{i,I}\right)\delta^{0|2}\left(u_{j,I}\chi_*^I+\sigma_{ji}^\alpha\theta_{j,\alpha}+\sigma_{ij}^\alpha\theta_{i,\alpha}^I u_{j,I}\right)
\end{aligned}
\tag{A.0.5}
$$

In each delta function there exists the term $\sigma_{ij}^\alpha\theta_{i,\alpha}^A u_{j,A}^{+a}$ which we can write as

$$
\begin{aligned}
\sigma_{ij}^\alpha\theta_{i,\alpha}^I u_{j,I}^{+a} &= \sigma_{ij}^\alpha\left(\theta_{i,\alpha}^{+b}\bar{u}_{i,+b}^I+\theta_{i,\alpha}^{-b'}\bar{u}_{i,-b'}^I\right)u_{i,I}^{+a} \\
&= \sigma_{ij}^\alpha\theta_{i,\alpha}^{+a}+\sigma_{ij}^\alpha\theta_{i,\alpha}^{-a'}y_{ij,a'}^{\ a},
\end{aligned}
\tag{A.0.6}
$$

where once again we make use of (2.3.85), namely we used $\bar{u}_{i,+a}^I u_{j,I}^{+b}=\delta_a^b$ and $\bar{u}_{i,-a'}^I u_{j,I}^{+b}=y_{ji,a'}^{\ b}$. Performing the analogous manoeuvres to $\sigma_{ji}^\alpha\theta_{j,\alpha}^I u_{i,I}^{+a}$, and putting everything back in leads to the last line of (A.0.5) being equivalent to

$$
\begin{aligned}
&y_{ij}^{-2}\delta^{0|2}\left(y_{ij,a'}^{\ a}\left[\left(u_{i,I}^{+b}\chi_*^I+\sigma_{ji}^\alpha\theta_{i\alpha}^{+b}+\sigma_{ji}^\alpha\theta_{j,\alpha}^{+b}\right)\left(y_{ij}^{-1}\right)_a^{\ a'}+\sigma_{ji}^\alpha\theta_{j,\alpha}^{-a'}\right]\right) \\
&\times\delta^{0|2}\left(y_{ji,a'}^{\ a}\left[\left(u_{j,I}^{+b}\chi_*^I+\sigma_{ij}^\alpha\theta_{j\alpha}^{+b}+\sigma_{ij}^\alpha\theta_{i,\alpha}^{+b}\right)\left(y_{ji}^{-1}\right)_a^{\ a'}+\sigma_{ij}^\alpha\theta_{i,\alpha}^{-a'}\right]\right).
\end{aligned}
\tag{A.0.7}
$$

Using the property of fermionic delta functions that $\delta^{0|n}(Mp)=\det(M)\delta(p)$, we apply this to the matrix y_{ij} in the argument of the delta functions to find that (A.0.7) is equal to

$$
y_{ij}^2\delta^{0|2}\left(\sigma_{ij}^\alpha\theta_{i,\alpha}^-+A_{ij}\right)\delta^{0|2}\left(\sigma_{ji}^\alpha\theta_{j,\alpha}^-+A_{ji}\right), \tag{A.0.8}
$$

where

$$
A_{ij}^{a'}=\left[\chi_*^I u_{j,I}^{+b}+\sigma_{ji}^\alpha\theta_{j,\alpha}^{+b}+\sigma_{ij}\theta_i^{+b}\right]\left(y_{ij}^{-1}\right)_b^{\ a'} \tag{A.0.9}
$$

Appendix B

Component expansion of R-vertex

In this appendix, we would like to show some of the derivation regarding the expansion (3.4.116) of the R-vertex used in section 3.2. For convenience we repeat the findings:

$$
\begin{aligned}
R(i;j_1j_2j_3) &= -\frac{\delta^{0|2}\Big(\langle\sigma_{ij_1}\sigma_{ij_2}\rangle A_{ij_3} + \langle\sigma_{ij_2}\sigma_{ij_3}\rangle A_{ij_1} + \langle\sigma_{ij_3}\sigma_{ij_1}\rangle A_{ij_2}\Big)}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\,\langle\sigma_{ij_2}\sigma_{ij_3}\rangle\,\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}\ . \\
&= R_1(i;j_1j_2) + \frac{1}{2}R_2(i;j_1j_2) + \frac{1}{2}R_3(i;j_1j_2) + \frac{1}{2}R_4(i;j_1j_2j_3) \\
&+ \frac{1}{6}R_5(i;j_1j_2j_3) + \mathrm{antisym}_{123}\ ,
\end{aligned} \tag{B.0.1}
$$

where

$$
A_{ij}^{a'} = \left[\chi_*^I u_{j,I}^{+b} + \sigma_{ji}^{\alpha}\theta_{j,\alpha}^{+b} + \sigma_{ij}^{\alpha}\theta_{i,\alpha}^{+b}\right]\left(y_{ij}^{-1}\right)_b^{\ a'}\,, \tag{B.0.2}
$$

gives

$$
\begin{aligned}
R_1(i;j_1j_2) &= -\frac{\langle\sigma_{ij_1}|\theta_i^+ y_{ij_1j_2}\theta_{j_2}^+|\sigma_{j_2i}\rangle}{(ij_1j_2)g_{ij_1}g_{ij_2}}\,, \\
R_2(i;j_1j_2) &= \frac{\langle\sigma_{j_1i}|\theta_{j_1}^+ y_{j_1ij_2}\theta_{j_2}^+|\sigma_{j_2i}\rangle}{(ij_1j_2)g_{ij_1}g_{ij_2}}\,, \\
R_3(i;j_1j_2) &= -\frac{\langle\sigma_{ij_1}|\left(\theta_i^+\right)^2|\sigma_{ij_2}\rangle y_{j_1j_2}^2}{(ij_1j_2)g_{ij_1}g_{ij_2}}\,, \\
R_4(i;j_1j_2j_3) &= \langle\sigma_{j_1i}|\left(\theta_{j_1}^+\right)^2|\sigma_{j_1i}\rangle\frac{x_{ij_1}^2(ij_2j_3)}{(ij_1j_2)(ij_3j_1)g_{ij_1}}\,, \\
R_5(i;j_1j_2j_3) &= (\theta_i^{+,\alpha}\, y_{ij_1j_2j_3i}\theta_{i,\alpha}^+)\frac{1}{y_{ij_1}^2y_{ij_2}^2y_{ij_3}^2}\,,
\end{aligned} \tag{B.0.3}
$$

where we have also used the index-less notation is used:

$$
\begin{aligned}
y_{ijk} &:= (y_{ijk})_{ab} = (y_{ij})_{ac'} \, (\tilde{y}_{jk})^{c'}_{\ b} \, , \\
y_{ijklm} &:= (y_{ijklm})_{ab} = (y_{ij})_{aa'} \, (\tilde{y}_{jk})^{a'c} \, (y_{kl})_{cd'} \, (\tilde{y}_{lm})^{d'}_{\ b} \, , \\
(ijk) &:= \langle \sigma_{ij} \sigma_{ik} \rangle x_{ij}^2 x_{ik}^2 .
\end{aligned} \tag{B.0.4}
$$

Since χ_*^A is a gauge parameter we can set it zero, thus we take

$$
A_{ij}^{a'} = \left[\sigma_{ji}^{\alpha} \theta_{j,\alpha}^{+b} + \sigma_{ij}^{\alpha} \theta_{i,\alpha}^{+b} \right] \left(y_{ij}^{-1} \right)^{a'}_{\ b} . \tag{B.0.5}
$$

We would like to derive $R_5(i; j_1 j_2 j_3)$ and $R_3(i; j_1 j_2)$ explicitly. First we note that $R(i; j_1 j_2 j_3)$ has Grassmann degree two which means one has

$$
\begin{aligned}
R(i; j_1 j_2 j_3) = {} & C_i^{\alpha\beta} \left(\theta_i^+ \right)^2_{\alpha\beta} + C_{j_1}^{\alpha\beta} \left(\theta_{j_1}^+ \right)^2_{\alpha\beta} + C_{iab} \left(\theta_i^+ \right)^{2ab} + C_{ij_1 ab}^{\alpha\beta} \theta_{i\alpha}^{+a} \theta_{j_1 \beta}^{+b} \\
& + C_{j_1 j_2 ab}^{\alpha\beta} \theta_{j_1,\alpha}^{+a} \theta_{j_2,\beta}^{+b} + \ldots
\end{aligned} \tag{B.0.6}
$$

Throughout the forthcoming exercise one must be very careful with numerical factors as these can arise subtly and lead to precise simplifications, we recall that

$$
\begin{aligned}
\frac{1}{2} y_{aa'} y_{bb'} \epsilon^{ab} \epsilon^{b'a'} &= y^2 \\
y_{aa'} y_{bb'} \epsilon^{ab} &= y^2 \epsilon_{b'a'} \\
\left(y^{-1} \right)^{a'a} &= y^{-2} \tilde{y}^{a'a} ,
\end{aligned} \tag{B.0.7}
$$

where $\tilde{y}^{a'a} = \epsilon^{ab} y_{bb'} \epsilon^{b'a'}$.

Corresponding the R_3 and R_5 are $C_i^{\alpha\beta}$ and C_{iab} respectively. Let us begin with $C_i^{\alpha\beta}$, namely we can take $\theta_{k,\alpha}^{+a} = 0$ for $k \in \{j_1, j_2, j_3\}$, and in applying

$$
A_{ij}^{a'} = \left[\sigma_{ji}^{\alpha} \theta_{j,\alpha}^{+b} + \sigma_{ij}^{\alpha} \theta_{i,\alpha}^{+b} \right] \left(y_{ij}^{-1} \right)^{a'}_{\ b} \tag{B.0.8}
$$

to (B.0.3), we get

$$
\begin{aligned}
& R(i; j_1 j_2 j_3) \big|_{\theta_i^+} \\
& = - \frac{\delta^{0|2} \Big(\Big[\langle \sigma_{ij_1} \sigma_{ij_2} \rangle \sigma_{ij_3}^{\alpha} \left(y_{ij_3}^{-1} \right)^{a'}_{\ b} + \langle \sigma_{ij_2} \sigma_{ij_3} \rangle \sigma_{ij_1}^{\alpha} \left(y_{ij_1}^{-1} \right)^{a'}_{\ b} + \langle \sigma_{ij_3} \sigma_{ij_1} \rangle \sigma_{ij_2}^{\alpha} \left(y_{ij_2}^{-1} \right)^{a'}_{\ b} \Big] \theta_{i,\alpha}^{+b} \Big)}{\langle \sigma_{ij_1} \sigma_{ij_2} \rangle \, \langle \sigma_{ij_2} \sigma_{ij_3} \rangle \, \langle \sigma_{ij_3} \sigma_{ij_1} \rangle}
\end{aligned} \tag{B.0.9}
$$

then from the Schouten identity

$$
\langle \sigma_{ij_1} \sigma_{ij_2} \rangle \sigma_{ij_3}^{\alpha} + \langle \sigma_{ij_3} \sigma_{ij_1} \rangle \sigma_{ij_2}^{\alpha} + \langle \sigma_{ij_2} \sigma_{ij_3} \rangle \sigma_{ij_1}^{\alpha} = 0, \tag{B.0.10}
$$

we can solve for $\langle\sigma_{ij_3}\sigma_{ij_1}\rangle\sigma^{\alpha}_{ij_2}$ and substitute this in, and noting that

$$y_{ik}^{-1} - y_{il}^{-1} = y_{il}^{-1}(y_{li} + y_{ik})y_{ki}^{-1} = y_{il}^{-1}y_{lk}y_{ki}^{-1}, \tag{B.0.11}$$

where we have omitted the indices, we arrive at

$$\begin{aligned} &R(i;j_1j_2j_3)\big|_{\theta_i^+} \\ &= -\frac{\delta^{0|2}\Big(\Big[\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\sigma^{\alpha}_{ij_3}\left(y_{ij_2}^{-1}y_{j_2j_3}y_{j_3i}^{-1}\right)_b^{a'} + \langle\sigma_{ij_2}\sigma_{ij_3}\rangle\sigma^{\alpha}_{ij_1}\left(y_{ij_2}^{-1}y_{j_2j_1}y_{j_1i}^{-1}\right)_b^{a'}\Big]\theta^{+b}_{i,\alpha}\Big)}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\langle\sigma_{ij_2}\sigma_{ij_3}\rangle\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}. \end{aligned} \tag{B.0.12}$$

We can write the delta function in the previous object as

$$\delta^{0|2}\left(\mathcal{M}_a^{\alpha}\theta^{+a}_{i,\alpha}\right) = \epsilon_{a'b'}\mathcal{M}_a^{\alpha,a'}\mathcal{M}_b^{\beta,b'}\theta^{+a}_{i,\alpha}\theta^{+b}_{i,\beta}. \tag{B.0.13}$$

We may then write the Grassmann odd objects in terms of irreducible Grassmann objects of degree two,

$$\theta^{+a}_{\alpha}\theta^{+b}_{\beta} = \frac{1}{2}\epsilon_{\alpha\beta}(\theta^+)^{2ab} + \frac{1}{2}\epsilon^{ab}(\theta^+)^2_{\alpha\beta}. \tag{B.0.14}$$

In this way, we get both of the components that we want. Let us focus $\epsilon_{\alpha\beta}(\theta^+)^{2ab}$ first in which we directly get :

$$C_{iab} = -\frac{1}{2}\frac{\left(y_{ij_1j_2j_3i} - (j_1 \leftrightarrow j_3)\right)_{ab}}{y_{ij_1}^2y_{ij_2}^2y_{ij_3}^2}. \tag{B.0.15}$$

However, we note that $\left(y_{ij_3j_2j_1i}\right)_{ab} = -\left(y_{ij_1j_2j_3i}\right)_{ba}$ which is seen easiest by using what we have learnt in section 2.3.1 to write $\left(y_{ij_3j_2j_1i}\right)_{ab} = \epsilon_{ac}\epsilon_{bd}u^{+c}_{iI}\bar{Y}^{IJ}_{j_3}Y_{j_2,JK}\bar{Y}^{KL}_{j_1}u^{+d}_{i,L}$. It follows that

$$\begin{aligned} u^{+c}_{i,I}\bar{Y}^{IJ}_{j_3}Y_{j_2,JK}\bar{Y}^{KL}_{j_1}u^{+d}_{i,L} &= u^{+d}_{i,L}\bar{Y}^{IJ}_{j_3}Y_{j_2,JK}\bar{Y}^{KL}_{j_1}u^{+c}_{i,I} \\ &= u^{+d}_{i,L}\bar{Y}^{KL}_{j_1}Y_{j_2,JK}\bar{Y}^{IJ}_{j_3}u^{+c}_{i,I} \\ &= -u^{+d}_{i,L}\bar{Y}^{LK}_{j_1}Y_{j_2,KJ}\bar{Y}^{JI}_{j_3}u^{+c}_{i,I}, \end{aligned} \tag{B.0.16}$$

where we used that $\bar{Y}^{IJ}$ and Y_{IJ} are anti-symmetric in their indices. Using

$$\left(y_{ij_3j_2j_1i}\right)_{ab} = -\left(y_{ij_1j_2j_3i}\right)_{ba} \text{ and } \left(\theta^+\right)^{2ab} = \left(\theta^+\right)^{2ba}, \tag{B.0.17}$$

we have

$$C_{iab} = -\frac{1}{6}\frac{\left(y_{ij_1j_2j_3i}\right)_{ab}}{y_{ij_1}^2y_{ij_2}^2y_{ij_3}^2}, \tag{B.0.18}$$

where we have put in a $\frac{1}{6}$ since all the external legs are anti-symmetric, this corresponds to $R_5(i;j_1j_2j_3)$. [1]

To get $C_i^{\alpha\beta}$, we simply look at the corresponding component in (B.0.14). A direct computation yields

$$C_i^{\alpha\beta} = \frac{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle}{\langle\sigma_{ij_2}\sigma_{ij_3}\rangle\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}\sigma_{ij_3}^{\alpha}\sigma_{ij_3}^{\beta}\left(\frac{y_{j_2j_3}^2}{y_{ij_2}^2y_{ij_3}^2}\right) + \frac{\langle\sigma_{ij_2}\sigma_{ij_3}\rangle}{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}\sigma_{ij_1}^{\alpha}\sigma_{ij_1}^{\beta}\left(\frac{y_{j_1j_2}^2}{y_{j_1i}^2y_{j_2i}^2}\right)$$
$$+ \frac{\sigma_{ij_1}^{\alpha}\sigma_{ij_3}^{\beta}}{\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}\frac{\mathrm{tr}(y_{ij_1}\tilde{y_{j_1j_2}}y_{j_2j_3}\tilde{y_{j_3i}})}{y_{ij_1}^2y_{ij_2}^2y_{j_3i}^2}. \tag{B.0.19}$$

Recalling the structures investigated in section 2.3.1, we have (defining $Y_{ij} := \bar{Y}_i^{IJ}Y_{j,IJ} = \frac{1}{2}\epsilon^{IJKL}Y_{i,IJ}Y_{j,KL}$):

$$\mathrm{tr}\left(y_{ij_1}\tilde{y}_{j_1j_2}y_{j_2j_3}\tilde{y}_{j_3i}\right) = \mathrm{tr}\left(Y_i\bar{Y}_{j_1}Y_{j_2}\bar{Y}_{j_3}\right) = \frac{1}{4}\left(Y_{ij_3}Y_{j_1j_2} - Y_{ij_2}Y_{j_1j_3} + Y_{ij_1}Y_{j_2j_3}\right), \tag{B.0.20}$$

which follows from the fact that Y_{AB} obeys the Clifford algebra in six dimensions, namely

$$\left\{Y_{i,J}^{I}, Y_{j,I}^{K}\right\} = \frac{1}{2}Y_{ij}\delta_J^K. \tag{B.0.21}$$

Since, $Y_{ij} = 2y_{ij}^2$ it follows that

$$\mathrm{tr}\left(y_{ij_1}\tilde{y}_{j_1j_2}y_{j_2j_3}\tilde{y}_{j_3i}\right) = y_{ij_3}^2y_{j_1j_2}^2 - y_{ij_2}^2y_{j_1j_3}^2 + y_{ij_1}^2y_{j_2j_3}^2. \tag{B.0.22}$$

Now that all y-structures are in the same basis, we need to use the Schouten identity to simplify. For example, consider the coefficient of $y_{j_2j_3}^2$ in (B.0.19). It is given by

$$\frac{\langle\sigma_{ij_1}\sigma_{ij_2}\rangle}{\langle\sigma_{ij_2}\sigma_{ij_3}\rangle\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}\sigma_{ij_3}^{\alpha}\sigma_{ij_3}^{\beta}\left(\frac{y_{j_2j_3}^2}{y_{ij_2}^2y_{ij_3}^2}\right) + \frac{\sigma_{ij_1}^{\alpha}\sigma_{ij_3}^{\beta}}{\langle\sigma_{ij_3}\sigma_{ij_1}\rangle}\frac{y_{j_2j_3}^2}{y_{ij_2}^2y_{j_3i}^2} = -\frac{\sigma_{ij_2}^{\alpha}\sigma_{ij_3}^{\beta}}{\langle\sigma_{ij_2}\sigma_{ij_3}\rangle}\frac{y_{j_2j_3}^2}{y_{ij_2}^2y_{j_3i}^2} \tag{B.0.23}$$

Where we have used $\langle\sigma_{ij_1}\sigma_{ij_2}\rangle\sigma_{ij_3}^{\alpha} + \langle\sigma_{ij_2}\sigma_{ij_3}\rangle\sigma_{ij_1}^{\alpha} = -\langle\sigma_{ij_3}\sigma_{ij_1}\rangle\sigma_{ij_2}^{\alpha}$. By implementing (B.0.4), we gain $R_3(i;j_2,j_3)$ where the symmetry factor of $\frac{1}{2}$ comes from the anti-symmetry $R_3(i;j_2,j_3) = -R_3(i;j_3,j_2)$.

[1] Note that aside from the permutation we have discussed already we also have (omitting indices) $u_i\bar{Y}_{j_1}Y_{j_3}\bar{Y}_{j_2}u_i = -u_i\bar{Y}_{j_1}Y_{j_2}\bar{Y}_{j_3}u_i - \frac{1}{2}y_{j_2j_3}^2u_i\bar{Y}_{j_1}u_i$ which follows from the fact that Y satisfy the Clifford algebra in six dimensions, however $u_i\bar{Y}_{j_1}u_i = 0$ since $\bar{Y}_{j_1}$ is antisymmetric in its indices. This is an example of permutation invariance upto a sign under neighbouring points, in this case points j_2 and j_3.

The remaining components can be found in a similar manner. However, there exists a package called `grassmann.m` which is to be used on `Mathematica` [98] and can easily be used to find the expansion of R-vertex.

Appendix C

Some computational details from section 3.4

In this appendix we gather up important results relevant to section 3.4. We put the relevant equation as subheadings in this appendix.

C.0.1 (3.4.110)

We would like to show that

$$f(0;123)^2 - R_3^{c_{123}}(0;31)^2 = 0. \tag{C.0.1}$$

Recalling (3.4.99), we note that

$$f(0;123)^2 = \frac{1}{2}\left(2\frac{x_{12}^2 x_{23}^2}{x_{10}^2 x_{20}^4 x_{30}^2} - \frac{x_{12}^4}{x_{10}^4 x_{20}^4} + \mathrm{cyc}_{123}\right)\left(\theta_0^+\right)^4. \tag{C.0.2}$$

Given that

$$R_3^{c_{123}}(0;31) = \frac{g_{30}g_{01}}{g_{12}} R_3(0;31) + \mathrm{cyc}_{123} \tag{C.0.3}$$

a direct calculation using $\left(\theta^+\right)^2_{(\alpha\beta)}\left(\theta^+\right)^2_{(\gamma\rho)} = \frac{1}{2}\left(\epsilon_{\gamma\alpha}\epsilon_{\beta\rho} + \epsilon_{\rho\alpha}\epsilon_{\beta\gamma}\right)\left(\theta^+\right)^4$ gives the z_*-independent result

$$R_3^{c_{123}}(0;31)^2 = \left(\frac{x_{12}^2 x_{23}^2}{x_{01}^2 x_{02}^4 x_{03}^2} - \frac{x_{12}^4}{2x_{10}^4 x_{20}^4} + \mathrm{cyc}_{123}\right)\left(\theta_0^+\right)^4. \tag{C.0.4}$$

The fact that $R_3^{c_{123}}(0;31)^2$ is z_*-independent works term by term, there are no non-trivial intermediate identities required, for example one such result is

$$\left(\frac{g_{30}g_{01}}{g_{13}}\right)^2 R_3(0;31)^2 = -\frac{x_{13}^2}{2x_{30}^4 x_{10}^4}\left(\theta^+\right)^4. \tag{C.0.5}$$

Since we have (C.0.4) and (C.0.2), it follows that $f(0;123)^2 - R_3^{c_{123}}(0;31)^2 = 0$.

C.0.2 (3.4.117)

We turn to the result in (3.4.117), which we display again as

$$\Big[[(R_3(0;41) + \text{cyc}_{412})(R_3(0;23) + \text{cyc}_{234})] + [R_5(0;412)]\,[R_5(0;234)] \Big]$$
$$= R_3(0;12)R_3(0;34) + R_3(0;41)R_3(0;23) + \frac{y_{14}^2 y_{23}^2 - 2y_{13}^2 y_{24}^2 + y_{12}^2 y_{34}^2}{2y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2}, \tag{C.0.6}$$

We would like to show some details of how this result comes about. More generally, this came from a five-point correlator graph which had a four-point R-vertex. One has:

$$\begin{aligned}
&R(0;j_1j_2j_3j_4) = R(0;j_1j_2j_3)R(0;j_3j_4j_1) = \\
&\Big[R_3(0;j_1j_2) + \text{cyc}_{j_1j_2j_3} + R_5(0;j_1j_2j_3)\Big]\Big[R_3(0;j_3j_4) + \text{cyc}_{j_3j_4j_1} + R_5(0;j_3j_4j_1)\Big] \\
&= (R_3(0;j_1j_2) + \text{cyc}_{j_1j_2j_3})(R_3(0;j_3j_4) + \text{cyc}_{j_3j_4j_1}) + R_5(0;j_1j_2j_3)R_5(0;j_3j_4j_1)
\end{aligned} \tag{C.0.7}$$

Now, there exists a nice simplification if we focus our attention on the term which is quadratic in R_5's, in which we can write

$$R_5(0;j_1j_2j_3)R_5(0;j_3j_4j_1) = R_5(0;j_1j_2j_3)R_5(0;j_1j_3j_4) \tag{C.0.8}$$

It turns out that $R_5 \times R_5$ terms can be expanded into a sum of $R_3 \times R_3$: [1]

$$\begin{aligned}
&R_5(0;j_1j_2j_3)R_5(0;j_3j_4j_1) = -R_3(0;j_2j_3)R_3(0;j_3j_4) + R_3(0;j_2j_3)R_3(0;j_3j_1) \\
&+ R_3(0;j_2j_1)R_3(0;j_1j_3) + R_3(0;j_4j_3)R_3(0;j_3j_1) - R_3(0;j_2j_1)R_3(0;j_1j_4) \\
&+ R_3(0;j_3j_1)R_3(0;j_1j_4) - R_3(0;j_3j_1)R_3(0;j_1j_3) + \frac{y_{j_2j_1}^2 y_{j_3j_4}^2 - 2y_{j_2j_4}^2 y_{j_3j_1}^2 + y_{j_2j_3}^2 y_{j_4j_1}^2}{y_{0j_2}^2 y_{0j_3}^2 y_{0j_4}^2 y_{0j_1}^2}.
\end{aligned} \tag{C.0.9}$$

When we put all this back into (C.0.7), we arrive at the result:

$$\begin{aligned}
R(0;j_1j_2j_3j_4) &= R_3(0;j_2j_3)R_3(0;j_4j_1) + R_3(0;j_1j_2)R_3(0;j_3j_4) \\
&+ \frac{y_{j_2j_1}^2 y_{j_3j_4}^2 - 2y_{j_2j_4}^2 y_{j_3j_1}^2 + y_{j_2j_3}^2 y_{j_4j_1}^2}{2y_{0j_2}^2 y_{0j_3}^2 y_{0j_4}^2 y_{0j_1}^2},
\end{aligned} \tag{C.0.10}$$

which is what we used in (3.4.117).

[1] This was discovered explicitly using the `grassmann.m` package on `Mathematica` [98]

C.0.3 (3.4.122)

Here we would like to show that (3.4.122) follows from (3.4.120). Essentially, we wish to show that

$$-R_3^{c1234}(0;12)R_4^{c1234}(1;024)+\frac{1}{2}R_4^{c1234}(1;024)^2+\frac{1}{2}R_3^{c1234}(0;12)^2=\frac{1}{2}f(0;1234)^2 \tag{C.0.11}$$

Now recall the f-function at 4-points.

$$f(0;1234)=-R_3^{c1234}(0;12)+R_4^{c1234}(1;024),$$

from which we find that

$$\begin{aligned}\implies \frac{1}{2}f(0;1234)^2=&\frac{1}{2}R_3^{c1234}(0;12)^2+\frac{1}{2}R_4^{c1234}(1;024)^2\\&-R_3^{c1234}(0;12)R_4^{c1234}(1;024).\end{aligned} \tag{C.0.12}$$

C.0.4 (3.4.143)

The non-trivial six-term identity

$$\begin{aligned}&(234)(341)x_{12}^2-(234)(124)x_{13}^2+(123)(234)x_{14}^2\\&\quad+(124)(134)x_{23}^2-(123)(134)x_{24}^2+(123)(124)x_{34}^2=0,\end{aligned} \tag{C.0.13}$$

was of use in (3.4.143) as well as (3.4.106). We wish to provide a proof here. It is convenient to introduce an auxiliary dual reference twistor t_* normalised as $t_{*A}z_*^A=1$. It then allows us to define two sets of dual variables

$$t_{iA}=X_{i,AB}z_*^B\,,\qquad\qquad h_i^A=X_i^{AB}t_{*B}\,, \tag{C.0.14}$$

with $X_i^{BC}=z_{i,1}^B z_{i,2}^C-z_{i,1}^C z_{i,2}^B$ and $X_{i,AB}=\frac{1}{2}\epsilon_{ABCD}X_i^{CD}$. They satisfy the relations

$$t_{jA}z_*^A=h_i^At_{*A}=0\,. \tag{C.0.15}$$

We also notice that since the X_{AB} takes values in the Clifford algebra of SU(4), the following holds true:

$$h_i^At_{jA}+h_j^At_{iA}=-t_{*A}z_*^C(X_i^{AB}X_{jBC}+X_j^{AB}X_{iBC})=-(X_i\cdot X_j)\,, \tag{C.0.16}$$

where the left-most hand side is the dot product defined on hypercone coordinates in (2.2.61), such that $X_i \cdot X_i = -\frac{1}{2}x_{ij}^2$, but as we shall see later we can omit the prefactor. Using the dual variables (C.0.14) we can obtain two equivalent representations for (ijk)

$$(ijk) = \frac{1}{2}\epsilon^{ABCD}t_{iA}t_{jB}t_{kC}t_{*D} = \frac{1}{2}\epsilon_{ABCD}h_i^A h_j^B h_k^C z_*^D \equiv \langle\, ijk* \,\rangle \,. \qquad \text{(C.0.17)}$$

According to (C.0.15), the twistors t_{jA} with $j = 1, \ldots, 4$ are all orthogonal to z_*^A, therefore, they are linear dependent. The same is true for h_j^A with $j = 1, \ldots, 4$. This yields two identities

$$\begin{aligned} &t_{1A}\,\langle 234*\rangle + t_{2A}\,\langle 34{*}1\rangle + t_{3A}\,\langle 4{*}12\rangle + t_{4A}\,\langle *123\rangle = 0\,, \\ &h_1^A\,\langle 234*\rangle + h_2^A\,\langle 34{*}1\rangle + h_3^A\,\langle 4{*}12\rangle + h_4^A\,\langle *123\rangle = 0 \end{aligned} \qquad \text{(C.0.18)}$$

Finally we multiply the expressions on the left-hand side and contract the SU(4) indices to get

$$\begin{aligned} &(234)(341)(X_1 \cdot X_2) - (123)(134)(X_2 \cdot X_4) - (234)(124)(X_1 \cdot X_3) \\ &+ (123)(234)(X_1 \cdot X_4) + (124)(134)(X_2 \cdot X_3) + (123)(124)(X_3 \cdot X_4) = 0\,. \end{aligned} \qquad \text{(C.0.19)}$$

where we made use of (C.0.16) and took into account that $(X_i \cdot X_i) = 0$. Since the last relation is homogenous in X's we can simply replace $(X_i \cdot X_j) \to x_{ij}^2$.

Appendix D

Further free theory coefficients

In this section, we give the free theory OPE coefficients of correlation functions $\langle 4233 \rangle$ and $\langle 5344 \rangle$ relevant to section 5.4.2. These cases distinguish themselves from the cases studied in the main text. Firstly, we now have $p_{12} = 2 \neq 0$. Secondly, for the first time there can be more than one type of $\frac{1}{2}$-BPS operator in the SU(N) gauge theory (e.g. at charge four $\mathrm{tr}(W^4)$ as well as $\mathrm{tr}(W^2)^2$.)

$\langle 4233 \rangle$

The correlator is written as

$$\begin{aligned}
\langle 4233 \rangle &= A \left(g_{14} g_{24}^2 g_{13}^3 + g_{14}^3 g_{23}^2 g_{13} \right) + B g_{13}^2 g_{23} g_{24} g_{14}^2 + C g_{12}^2 g_{13} g_{34}^2 g_{14} \\
&\quad + D \left(g_{12} g_{14} g_{24} g_{34} g_{13}^2 + g_{12} g_{14}^2 g_{23} g_{34} g_{13} \right) \\
&= g_{12}^3 g_{34}^3 \frac{g_{14}}{g_{24}} \left(\left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right) f_2(C,0) + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right)^2 f_4(D,D,0) \right. \\
&\qquad \left. + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right)^3 f_6(A,B,A,0) \right)
\end{aligned} \tag{D.0.1}$$

We tabulate the SU(N) colour factors in table D.1, whilst leaving some of the U(N) factors for table E.2 in appendix E since these tables are a lot larger.

In comparing with the appropriate SCPW expansion one finds the result

$$\begin{aligned}
\langle 4233 \rangle = g_{12}^3 g_{34}^3 \frac{g_{14}}{g_{24}} &\left(\left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right) \sum_{\lambda \geq 0} \mathcal{C}_{2[\lambda]} F^{012[\lambda]} + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right)^2 \sum_{\lambda_1 \geq \lambda_2 \geq 0} \mathcal{C}_{4[\lambda_1,\lambda_2]} F^{124[\lambda_1,\lambda_2]} \right. \\
&\left. + \left(\frac{g_{13} g_{24}}{g_{12} g_{34}} \right)^3 \sum_{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0} \mathcal{C}_{6[\lambda_1,\lambda_2,\lambda_3]} F^{236[\lambda_1,\lambda_2,\lambda_3]} \right),
\end{aligned} \tag{D.0.2}$$

Correlator type	B	C	D
$\langle A_4 A_2 A_3 A_3\rangle$	$\frac{72(N^2-1)(N^2-4)(N^2-6)}{N^2}$	$\frac{72(N^2-1)(N^2-4)(2N^2-3)}{N^2}$	$\frac{144(N^2-1)(N^2-4)(N^2-6)}{N^2}$
$\langle (A_2)^2 A_2 A_3 A_3\rangle$	$\frac{144(N^2-1)(N^2-4)}{N}$	$\frac{72(N^2-1)(N^2-4)(1+N^2)}{N}$	$\frac{288(N^2-1)(N^2-4)}{N}$

Table D.1: SU(N) colour factors associated to the $\langle 4233\rangle$ correlator, note that A is always zero.

with the following coefficients

$$\begin{aligned}
\mathcal{C}_{2[0]} &= C \text{ all else } 0, \\
\mathcal{C}_{4[\lambda_1]} &= \frac{D\lambda_1!(\lambda_1+2)!}{(2\lambda_1+1)!} \text{ for } \lambda_1 \in \mathbb{Z}_{\text{even}} \text{ and all else } 0, \\
\mathcal{C}_{6[\lambda_1,\lambda_2]} &= \frac{4(-1)^{\lambda_2}(\lambda_1+2)(\lambda_1+3)(\lambda_2+2)\left((\lambda_1+2)!\right)^2\left((\lambda_2+1)!\right)^2}{\left(2(-1)^{\lambda_2}\lambda_1+5(-1)^{\lambda_1}-(-1)^{\lambda_2}\right)(2\lambda_1+4)!\,(2\lambda_2+2)!} \\
&\quad\times\left(\frac{1}{24}A\left(12\left(\lambda_1-3\right)\lambda_1+\left(96\lambda_1-12\lambda_2\left(\lambda_2+3\right)+25\right)+23\right)+B(-1)^{\lambda_2}\right) \\
&\quad \text{for } \lambda_1-\lambda_2 \in \mathbb{Z}_{\text{even}} \geq 0, \lambda_2 \geq 0\ .
\end{aligned} \tag{D.0.3}$$

All other coefficients are vanishing.

As a non-trivial check we can compute the OPE coefficients for the correlator $\langle 3342\rangle$. We find the the explicit ingredient of the SCPW expansion change, namely one uses $F^{122[\underline{\lambda}]}$, $F^{234[\underline{\lambda}]}$ and $F^{346[\underline{\lambda}]}$ instead of the SCPW's used in (D.0.2). However, critically the result for the OPE coefficients give identically the same result as in (D.0.2). Furthermore we also note that the results for $\mathcal{C}_{6[\lambda_1,\lambda_2]}$ agree perfectly in the large N limit with those obtained from free three-point functions in [73] (see the first row of table 5).

$\langle 5344 \rangle$

The correlator is given by

$$
\begin{aligned}
\langle 5344\rangle &= A(g_{14}g_{24}^3g_{13}^4 + g_{14}^4g_{23}^3g_{13}) + B(g_{14}^2g_{23}g_{24}^2g_{13}^3 + g_{14}^3g_{23}^2g_{24}g_{13}^2) \\
&\quad + C(g_{12}g_{14}g_{24}^2g_{34}g_{13}^3 + g_{12}g_{14}^3g_{23}^2g_{34}g_{13}) \\
&\quad + D(g_{12}g_{13}^2g_{14}^2g_{23}g_{24}g_{34}) + E(g_{12}^2g_{13}g_{14}^2g_{23}g_{34}^2 + g_{12}^2g_{13}^2g_{14}g_{24}g_{34}^2) + F(g_{12}^3g_{13}g_{14}g_{34}^3) \\
&= g_{12}^4g_{34}^4\frac{g_{14}}{g_{24}}\left(\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)f_2(F,0) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2 f_4(E,E,0)\right. \\
&\qquad\qquad \left. + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3 f_6(C,D,C,0) + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^4 f_8(A,B,B,A,0)\right)
\end{aligned}
\tag{D.0.4}
$$

We have given some of the colour factors in table E.1 and table E.5 in appendix E. The SCPW expansion is given by

$$
\begin{aligned}
\langle 5344\rangle = g_{12}^3g_{34}^3\frac{g_{14}}{g_{24}}&\left(\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)\sum_{\lambda\geq 0}\mathcal{C}_{2[\lambda]}F^{012[\lambda]} + \left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^2\sum_{\lambda_1\geq\lambda_2\geq 0}\mathcal{C}_{4[\lambda_1,\lambda_2]}F^{124[\lambda_1,\lambda_2]}\right. \\
&+\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^3\sum_{\lambda_1\geq\lambda_2\geq\lambda_3\geq 0}\mathcal{C}_{6[\lambda_1,\lambda_2,\lambda_3]}F^{236[\lambda_1,\lambda_2,\lambda_3]} \\
&\left.+\left(\frac{g_{13}g_{24}}{g_{12}g_{34}}\right)^4\sum_{\lambda_1\geq\lambda_2\geq\lambda_3\geq\lambda_4\geq 0}\mathcal{C}_{8[\lambda_1,\lambda_2,\lambda_3,\lambda_4]}F^{348[\lambda_1,\lambda_2,\lambda_3,\lambda_4]}\right),
\end{aligned}
\tag{D.0.5}
$$

whereby the result is structurally identical to (D.0.3) for the $\gamma = 2, 4$ and 6 but for changes in the precise colour factors:

$$
\begin{aligned}
\mathcal{C}_{2[0]} &= F \text{ all else } 0, \\
\mathcal{C}_{4[\lambda_1]} &= \frac{E\lambda_1!(\lambda_1+2)!}{(2\lambda_1+1)!} \text{ for } \lambda_1\in\mathbb{Z}_{\text{even}} \text{ and all else } 0, \\
\mathcal{C}_{6[\lambda_1,\lambda_2]} &= \frac{4(-1)^{\lambda_2}(\lambda_1+2)(\lambda_1+3)(\lambda_2+2)((\lambda_1+2)!)^2((\lambda_2+1)!)^2}{(2(-1)^{\lambda_2}\lambda_1+5(-1)^{\lambda_1}-(-1)^{\lambda_2})(2\lambda_1+4)!(2\lambda_2+2)!} \\
&\quad\times\left(\frac{1}{24}C\left(12(\lambda_1-3)\lambda_1+(96\lambda_1-12\lambda_2(\lambda_2+3)+25)+23\right)+D(-1)^{\lambda_2}\right) \\
&\quad\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{even}}\geq 0, \lambda_2\geq 0 \text{ and all else zero.}
\end{aligned}
\tag{D.0.6}
$$

For the $\gamma = 8$ sector we get:

$$
\begin{aligned}
&\mathcal{C}_{8[\lambda_1,\lambda_2]} = n_{\lambda_1,\lambda_2}\frac{1}{6}\left(\lambda_1+4\right)\left(2\lambda_2+5\right)\\
&\times\Big(A\left(\lambda_1+2\right)\left(\lambda_1+5\right)\left(\lambda_1-\lambda_2+1\right)\left(\lambda_2+1\right)\left(\lambda_2+4\right)\left(\lambda_1+\lambda_2+6\right)\\
&+12B\left(\left((-1)^{\lambda_2}+1\right)\left(\lambda_1+2\right)\left(\lambda_1+5\right)+\left((-1)^{\lambda_2}-1\right)\left(\lambda_2+1\right)\left(\lambda_2+4\right)\right)\Big)\\
&\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{even}}\geq 0, \lambda_2\geq 0 \text{ and zero otherwise,}\\
&\mathcal{C}_{8[\lambda_1,\lambda_2,1]} = n_{\lambda_1,\lambda_2}\frac{1}{12}\left(\lambda_1+4\right)\left(\lambda_1-\lambda_2+1\right)\left(\lambda_1+\lambda_2+6\right)\left(2\lambda_2+5\right)\\
&\times\Big(A\left(\lambda_1+1\right)\left(\lambda_1+6\right)\lambda_2\left(\lambda_2+5\right)+12B\left((-1)^{\lambda_2}-1\right)\Big)\\
&\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{odd}}\geq 1, \lambda_2\geq 1 \text{ and zero otherwise,}\\
&\mathcal{C}_{8[\lambda_1,\lambda_2,2]} = n_{\lambda_1,\lambda_2}\frac{1}{30}\left(\lambda_1+4\right)\left(2\lambda_2+5\right)\\
&\times\Big(A\lambda_1\left(\lambda_1+7\right)\left(\lambda_1-\lambda_2+1\right)\left(\lambda_2-1\right)\left(\lambda_2+6\right)\left(\lambda_1+\lambda_2+6\right)\\
&+12B\left(\left((-1)^{\lambda_2}+1\right)\lambda_1^2+7\left((-1)^{\lambda_2}+1\right)\lambda_1+\left((-1)^{\lambda_2}-1\right)\left(\lambda_2-1\right)\left(\lambda_2+6\right)\right)\Big)\\
&\text{for } \lambda_1-\lambda_2\in\mathbb{Z}_{\text{even}}\geq 0, \lambda_2\geq 2 \text{ and zero otherwise,}
\end{aligned} \tag{D.0.7}
$$

where

$$
n_{\lambda_1,\lambda_2} = \frac{\left((\lambda_1+3)!\right)^2\left((\lambda_2+3)!\right)^2}{(2\lambda_1+6)!\,(2\lambda_2+6)!}. \tag{D.0.8}
$$

Appendix E

Colour factors used in free theory correlators

In this appendix we collect the tables that are too large to fit in the main text of section 5.4 and appendix D.

Note that in the following $M = (N^2 - 4)(N^2 - 1)$

Correlator type	B	C	D
$\langle A_5 A_3 A_4 A_4 \rangle$	$\frac{240M\left(N^2-6\right)\left(N^4-6N^2+36\right)}{N^4}$	$\frac{480M\left(N^2-6\right)\left(N^4-6N^2+36\right)}{N^4}$	$\frac{480M\left(N^6+3N^4+72N^2-864\right)}{N^4}$
$\langle (A_2 A_3) A_3 A_4 A_4 \rangle$	$\frac{480M\left(N^2-6\right)\left(2N^2-9\right)}{N^3}$	$\frac{960M\left(N^2-6\right)\left(2N^2-9\right)}{N^3}$	$\frac{2880M\left(2N^4-21N^2+72\right)}{N^3}$
$\langle A_5 A_3 (A_2 A_2) A_4 \rangle$	$\frac{1440M\left(N^2-6\right)\left(N^2-2\right)}{N^3}$	$\frac{2880M\left(N^2-6\right)\left(N^2-2\right)}{N^3}$	$\frac{5760M\left(N^4-7N^2+24\right)}{N^3}$
$\langle (A_2 A_3) A_3 (A_2 A_2) A_4 \rangle$	$\frac{160M\left(N^2-6\right)\left(N^2+9\right)}{N^2}$	$\frac{320M\left(N^2-6\right)\left(N^2+9\right)}{N^2}$	$\frac{960M\left(N^3+16N^2-6N-78\right)}{N^2}$

Correlator type	E	F
$\langle A_5 A_3 A_4 A_4 \rangle$	$\frac{480M\left(N^6-6N^4+99N^2-378\right)}{N^4}$	$\frac{480M\left(N^2-2\right)\left(N^4-6N^2+18\right)}{N^4}$
$\langle (A_2 A_3) A_3 A_4 A_4 \rangle$	$\frac{2160M\left(N^4-10N^2+42\right)}{N^3}$	$\frac{480M\left(N^4-6N^2+18\right)}{N^3}$
$\langle A_5 A_3 (A_2 A_2) A_4 \rangle$	$\frac{1440M\left(3N^4-13N^2+42\right)}{N^3}$	$\frac{960M\left(N^2-2\right)\left(2N^2-3\right)}{N^3}$
$\langle (A_2 A_3) A_3 (A_2 A_2) A_4 \rangle$	$\frac{4320M\left(2N^2-7\right)}{N^2}$	$\frac{960M\left(2N^2-3\right)}{N^2}$

Table E.1: Colour factors for $\langle 5344 \rangle$ in $\mathrm{SU}(N)$, note that A is always zero.

Correlator type	A	B	C	D
$\langle A_4A_2A_3A_3\rangle$	$216N^2(1+N^2)$	$72N^2(5+N^2)$	$144N^2(2+N^2)$	$144N^2(5+N^2)$
$\langle(A_2A_2)A_2A_3A_3\rangle$	$432N^3$	$144N(1+2N^2)$	$72N(2+N)(1+N^2)$	$288N(1+2N^2)$
$\langle(A_1A_3)A_2A_3A_3\rangle$	$54N^3(7+N^2)$	$216N(1+N^2)$	$108N(1+3N^2)$	$432N(1+N^2)$
$\langle(A_1^2A_2)A_2A_3A_3\rangle$	$216N^2(1+N^2)$	$432N^2$	$36N^2(9+2N+N^2)$	$864N^2$
$\langle A_4(A_1)^2(A_1A_2)(A_1A_2)\rangle$	$432N^3$	$16N(12+13N^2+2N^4)$	$48N(6+N+2N^2)$	$96N(4+5N^2)$
$\langle(A_1)^4(A_1)^2(A_1)^3(A_1)^3\rangle$	$432N^6$	$432N^6$	$432N^6$	$864N^6$
$\langle(A_1^2A_2)(A_1)^2(A_1A_2)A_3\rangle$	$72N^2(5+N)$	$24N^2(14+N+3N^2)$	$48N^2(5+4N^2)$	$48N^2(15+N+2N^2)$
$\langle(A_2A_2)(A_1)^2(A_1A_2)A_3\rangle$	$144N(2+N)$	$48N(4+4N^2+N^3)$	$48N(4+5N^2)$	$96N(6+N+2N^2)$

Table E.2: U(N) colour factors associated to the $\langle 4233\rangle$ correlator

Correlator type $U(N)$	A	B	C	D
$\langle A_2A_2A_3A_3\rangle$	$6N^3(1+N^2)$	$36N^3$	$36N(1+N^2)$	$72N(1+N^2)$
$\langle(A_1)^2A_2A_3A_3\rangle$	$6N^2(1+N^2)$	$36N^2$	$72N^2$	$72N(1+N^2)$
$\langle(A_1)^2(A_1)^2A_3A_3\rangle$	$6N^3(1+N^2)$	$36N$	$72N^3$	$144N$
$\langle A_2A_2(A_1A_2)A_3\rangle$	$12N^4$	$12N^2(2+N^2)$	$72N^2$	$144N^2$
$\langle A_2A_2(A_1A_2)(A_1A_2)\rangle$	$4N^3(2+N)$	$4N(2+N^2)^2$	$24N(2+N^2)$	$48N(2+N^2)$
$\langle A_2A_2(A_1)^3A_3\rangle$	$12N^3$	$36N^3$	$72N$	$144N$
$\langle A_2A_2(A_1)^3(A_1)^3\rangle$	$12N^5$	$36N^3$	$72N^3$	$144N^3$
$\langle A_2A_2(A_1)^3(A_1A_2)\rangle$	$12N^4$	$12N^2(2+N^2)$	$72N^2$	$144N^2$
$\langle(A_1)^2A_2(A_1)^3A_3\rangle$	$12N^2$	$36N^2$	$72N^2$	$144N^2$
$\langle(A_1)^2A_2(A_1A_2)A_3\rangle$	$12N^3$	$12N(2+N^2)$	$24N(2+N^2)$	$48N(2+N^2)$
$\langle(A_1)^2A_2(A_1)^3(A_1A_2)\rangle$	$12N^3$	$36N^3$	$72N^3$	$144N^3$
$\langle(A_1)^2A_2(A_1)^3(A_1)^3\rangle$	$12N^4$	$36N^4$	$72N^4$	$144N^4$
$\langle(A_1)^2A_2(A_1A_2)(A_1A_2)\rangle$	$4N^2(2+N)$	$12N^2(2+N^2)$	$8N^2(8+N^2)$	$16N^2(8+N^2)$
$\langle(A_1)^2(A_1)^2(A_1)^3A_3\rangle$	$12N^4$	$36N^4$	$72N^4$	$144N^4$
$\langle(A_1)^2(A_1)^2(A_1A_2)A_3\rangle$	$12N^4$	$36N^2$	$24N^2(2+N^2)$	$144N^2$
$\langle(A_1)^2(A_1)^2(A_1)^3(A_1A_2)\rangle$	$12N^4$	$36N^4$	$72N^4$	$144N^4$
$\langle(A_1)^2(A_1)^2(A_1A_2)(A_1A_2)\rangle$	$4N^3(2+N^2)$	$36N^3$	$8N^3(8+N^2)$	$144N^3$
$\langle(A_1)^2(A_1)^2(A_1)^3(A_1)^3\rangle$	$12N^5$	$36N^5$	$72N^5$	$144N^5$

Table E.3: U(N) colour factors associated to the $\langle 2233\rangle$ correlator

Correlator type	A	B	C
$\langle A_3A_3A_3A_3\rangle$	$9N^2(1+N^2)^2$	$81N^2(3+N^2)$	$162N^2(7+N^2)$
$\langle (A_1)^3A_3A_3A_3\rangle$	$18N^2(1+N^2)$	$108N^2(2+N)$	$1296N^2$
$\langle (A_1)^3(A_1)^3A_3A_3\rangle$	$18N^4(1+N^2)$	$324N^2$	$1296N^2$
$\langle (A_1)^3(A_1)^3(A_1)^3A_3\rangle$	$36N^4$	$324N^4$	$1296N^4$
$\langle (A_1)^3A_3(A_1)^3A_3\rangle$	$36N^2$	$324N^2$	$1296N^2$
$\langle (A_1A_2)A_3A_3A_3\rangle$	$18N^3(1+N^2)$	$108N(2+N^3)$	$1296N$
$\langle (A_1A_2)(A_1A_2)A_3A_3\rangle$	$6N^2(1+N^2)(2+N^2)$	$36N^2(8+N^2)$	$72N^2(17+N^2)$
$\langle (A_1A_2)(A_1A_2)(A_1A_2)A_3\rangle$	$12N^3(2+N^2)$	$12N(12+14N^2+N^4)$	$48N(14_13N^2)$
$\langle (A_1A_2)A_3(A_1A_2)A_3\rangle$	$36N^4$	$36N^2(8+N^2)$	$72N^2(17+N^2)$
$\langle (A_1)^3(A_1A_2)(A_1A_2)(A_1A_2)\rangle$	$12N^3(2+N^2)$	$36N^3(8+N^2)$	$48N^3(26+N^2)$
$\langle (A_1)^3(A_1)^3(A_1A_2)(A_1A_2)\rangle$	$12N^4(2+N^2)$	$324N^4$	$1296N^4$
$\langle (A_1)^3(A_1)^3(A_1)^3(A_1A_2)\rangle$	$36N^5$	$324N^5$	$1296N^5$
$\langle (A_1)^3(A_1)^3(A_1)^3(A_1)^3\rangle$	$36N^6$	$324N^6$	$1296N^6$
$\langle (A_1)^3(A_1A_2)(A_1)^3(A_1A_2)\rangle$	$36N^4$	$324N^4$	$1296N^4$
$\langle (A_1)^3(A_1A_2)A_3(A_1)^3\rangle$	$36N^3$	$108N^3(2+N^2)$	$1296N^3$
$\langle (A_1)^3(A_1A_2)A_3(A_1A_2)\rangle$	$36N^4$	$108N^2(2+N^2)$	$144N^2(8+N^2)$
$\langle (A_1A_2)(A_1A_2)(A_1A_2)(A_1A_2)\rangle$	$4N^2(2+N^2)^2$	$4N^2(60+20N^2+N^4)$	$48N^2(22+5N^2)$

Table E.4: U(N) colour factors associated to the $\langle 3333\rangle$ correlator

Correlator type	A	B	C
$\langle A_5A_3A_4A_4\rangle$	$34560N^2(1+N^2)(5+N^2)$	$240N^2(2+N^2)(23+N^2)$	$480N^2(47+24N^2+N^4)$
$\langle (A_2A_3)A_3A_4A_4\rangle$	$69120N(1+N^2)(1+2N^2)$	$480N(9+23N^2+4N^4)$	$480N(19+46N^2+7N^4)$
$\langle A_5A_3(A_2A_2)A_4\rangle$	$69120N^3(5+N^2)$	$480N(8+N^2)(1+3N^2)$	$960N(8+N^2)(1+3N^2)$
$\langle (A_2A_3)A_3(A_2A_2)A_4\rangle$	$138240N^2(1+2N^2)$	$160N^2(63+38N^2+N^4)$	$320N^2(64+37N^2+N^4)$
$\langle A_5(A_1A_2)A_4A_4\rangle$	$69120N^3(5+N^2)$	$480N(8+N^2)(1+3N^2)$	$960N(8+N^2)(1+3N^2)$

Correlator type	D	E	F
$\langle A_5A_3A_4A_4\rangle$	$480N^2(158+57N^2+N^4)$	$480N^2(74+33N^2+N^4)$	$480N^2(13+10N^2+N^4)$
$\langle (A_2A_3)A_3A_4A_4\rangle$	$480N(65+139N^2+12N^4)$	$240N(55+146N^2+15N^4)$	$1440N(1+6N^2+N^4)$
$\langle A_5A_3(A_2A_2)A_4\rangle$	$1920N(16+35N^2+3N^4)$	$1440N(10+23N^2+3N^4)$	$1920N(1+4N^2+N^4)$
$\langle (A_2A_3)A_3(A_2A_2)A_4\rangle$	$960N^2(77+5N+25N^2+N^3)$	$17280N^2(2+N^2)$	$5760N^2(1+N^2)$
$\langle A_5(A_1A_2)A_4A_4\rangle$	$1920N(16+35N^2+3N^4)$	$1440N(10+23N^2+3N^4)$	$640N(3+13N^2+2N^4)$

Table E.5: Colour factors for $\langle 5344\rangle$ in U(N) gauge theory

Appendix F

Superconformal invariant on Grassmannian space

In section 5.3, we took the Schur superpolynomial over the matrix

$$Z = X_{12}X_{24}^{-1}X_{43}X_{31}^{-1} \ , \tag{F.0.1}$$

from which we claimed traces over such matrices yields a manifestly superconformal invariant basis of variables. Here we would like to show some of the steps towards this. This is essentially a mini-review of [28].

The superconformal transformations of a point in $\mathrm{Gr}_{m|n}(2m|2n)$ is given by

$$\delta X^{AB'} = B^{AB'} + A^{A}{}_{B}X^{BB'} + X^{AC'}D_{C'}{}^{B'} + X^{AC'}C_{C'D}X^{DB'}, \tag{F.0.2}$$

where the Lie superalgebra of $\mathfrak{sl}(m|n)$ is given by

$$\begin{pmatrix} -A^{A}{}_{B} & B^{AB'} \\ -C_{A'B} & D_{A'}{}^{B'} \end{pmatrix} \in \begin{pmatrix} \mathfrak{gl}(m|n) & \mathfrak{gl}(n|m) \\ \mathfrak{gl}(m|n) & \mathfrak{gl}(n|m) \end{pmatrix} \ \text{with} \ -\mathrm{str}(A) + \mathrm{str}(D) = 0. \tag{F.0.3}$$

We would like to provide some steps towards finding a function at four points, namely

$$\mathcal{F}(X_1, X_2, X_3, X_4), \tag{F.0.4}$$

that is invariant under the superconformal transformations given in (F.0.2). We will use an index-less notation for simplicity where we will use $(A \cdot X) = A_{BB'}X^{BB'}$ to imply a scalar quantity whilst $(AX) = A^{B}_{B}X^{BC'}$ implies a matrix quantity.

In general we have $\delta(TT^{-1}) = 0$ for some matrix T, and from this we have

$$\delta T^{-1} = -(T^{-1}\,[\delta T]\,T^{-1}) \tag{F.0.5}$$

Beginning with the B transformation, we may take the basis of superpoints to $\mathcal{F}(X_1, X_{12}, X_{13}, X_{14})$. In doing so we find that since $\delta_B X_{ij} = 0$, we must have

$$\delta_B \mathcal{F}(X_1, X_{12}, X_{13}, X_{14}) = B \cdot \frac{\partial}{\partial X_1}\mathcal{F}(X_1, X_{12}, X_{13}, X_{14}) = 0, \tag{F.0.6}$$

from which we conclude that $\mathcal{F}(X_1, X_{12}, X_{13}, X_{14})$ does not depend on X_1 since B is arbitrary.

Now since $\delta_C X = (XCX)$ it follows that $\delta_C X_{1i} = (X_1CX_1) - (X_iCX_i) = (X_1CX_{1i}) + (X_{1i}CX_1) - (X_{1i}CX_{1i})$, from (F.0.5) we have

$$\delta_C X_{1i}^{-1} = C - (X_{1i}^{-1}X_1C) - (CX_1X_{1i}^{-1}). \tag{F.0.7}$$

Returning back to the function in question we can take it to be

$$\mathcal{F}(X_1, X_{12}^{-1}, X_{13}^{-1}, X_{14}^{-1}) \to \mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4), \tag{F.0.8}$$

where $Q_i^{-1} = X_{1i}^{-1} - X_{12}^{-1}$. This leads to $Q_i = (X_{12}X_{2i}^{-1}X_{i1})$ which can be checked directly and by using $(X_{12}X_{2i}^{-1}X_{i1}) = (X_{1i}X_{2i}^{-1}X_{21})$. By direct computations we get

$$\delta_C Q_i = (Q_iCX_1) + (X_1CQ_i). \tag{F.0.9}$$

From the point of view of $\delta_C \mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4)$ the infinitesimal transformation associated to X_{12}^{-1} and Q_i both have pieces dependent on X_1 in (F.0.7) and (F.0.9), however we concluded in (F.0.6) that $\mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4)$ is independent of X_1, and so we may set $X_1 = 0$. We conclude that

$$\delta_C \mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4) = C \cdot \frac{\partial}{\partial X_{12}^{-1}}\mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4) = 0, \tag{F.0.10}$$

we therefore find that $\mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4)$ is independent of X_{12}^{-1} since C is arbitrary, leading to

$$\mathcal{F}(X_1, X_{12}^{-1}, Q_3, Q_4) \to \mathcal{F}(Q_3, Q_4). \tag{F.0.11}$$

Finally, we have $\delta_{A+D}X = (AX) + (XD)$ which leads to $\delta_{A+D}Q_i = (AQ_i) + (Q_iD)$ and by once again using (F.0.5) we have $\delta_{A+D}Q_i^{-1} = -(Q_i^{-1}A) - (DQ_i^{-1})$. Now, we may consider $R = Q_4 Q_3^{-1}$, in which $\delta_{A+D} R = (AR) - (RA)$. It can be shown that $R = \left(X_{12} \left[\mathbb{I} - Z\right] X_{12}^{-1}\right) = \mathbb{I} - X_{12} Z X_{12}^{-1}$ [1]. Since $\mathbb{I}$ is trivially invariant under δ, we define $T = X_{12} Z X_{12}^{-1}$ and write

$$\mathcal{F}(Q_3, Q_4) \to \mathcal{F}(Q_3, T). \tag{F.0.12}$$

where $\mathcal{F}(Q_3, T)$ is an invariant if

$$\left[(AQ_3 + Q_3 D) \cdot \frac{\partial}{\partial Q_3} + (TA - AT) \cdot \frac{\partial}{\partial T}\right] \mathcal{F}(Q_3, T) = 0. \tag{F.0.13}$$

Now the details of the group numbers m and n become important. Consider the case where $m \neq n$, where in order to guarantee $\mathrm{str}(A) = \mathrm{str}(D)$, we may take $A = \mathrm{str}(D) \frac{\mathbb{I}}{m-n}$. It follows that $(TA - AT) = 0$ and $(AQ_3 + Q_3 D) = \left(Q_3 \left[\mathrm{str}(D) \frac{\mathbb{I}}{m-n} + D\right]\right)$ which is arbitrary, and therefore forces $\mathcal{F}(Q_3, T)$ to be independent of Q_3, thus we have $\mathcal{F}(T)$.

If m=n the situation is different, as taking $A \propto \mathbb{I}$ renders it supertraceless which necessarily makes D supertraceless. We gain the action of the schematic form

$$M \cdot \frac{\partial}{\partial Q_3} \mathcal{F}(Q_3, T), \tag{F.0.14}$$

which is only valid for when M is supertraceless, hence we cannot completely get rid of the Q_3 dependence. However, the best we can do is get rid of all parts of Q_3 but for its supertrace, $K = \mathrm{str}\,(Q_3)$. We now get

$$\left[\mathrm{str}\,(A + D)\, K \frac{\partial}{\partial K} + (TA - AT) \cdot \frac{\partial}{\partial T}\right] \mathcal{F}(Q_3, T) = 0. \tag{F.0.15}$$

In [28] it was shown that after a set of redefinitions one ends up with what would be found in the $m \neq n$ case, namely

$$\delta \mathcal{F}(T) = [A, T] \frac{\partial}{\partial T} \mathcal{F}(T) = 0. \tag{F.0.16}$$

The main consequence of this is that T transforms under the adjoint representation of $\mathfrak{gl}(m|n)$. It therefore follows that any and all trace structures of the matrix T yield $\delta F = 0$, of which a basis of such polynomials is given by the Schur superpolynomial. It

[1] This is since $R = (X_{12} X_{24}^{-1} X_{41} X_{31}^{-1} X_{23} X_{12}^{-1})$, one can take $X_{41} = X_{43} + X_{31}$ and $X_{23} = X_{21} + X_{13}$ to get $\mathbb{I} - X_{12} Z X_{12}^{-1}$

also follows that since we are using traces and $T = X_{12} Z X_{12}^{-1}$, we can take $T \to Z$ and so $\mathcal{F}(T) \to \mathcal{F}(Z)$. For similar reasons the superdeterminant is also an invariant, namely $\delta \mathrm{sdet}(Z) = 0$. We therefore have that $\mathcal{F}(Z)$ is a combination of Schur superpolynomials and superdeterminants of Z.

Appendix G

The eigenvalue basis of the quadratic Casimir for the GL(m) conformal partial wave from section 5.3.3

In section 5.3.3, we found the form of the conformal partial wave for theories with SL$(2m)$ 'conformal' symmetry. In this appendix, we elaborate on the details of the derivation.

Following the discussion there we begin with the statement of the generator

$$D^A_B = u^\alpha_A \frac{\partial}{\partial u^\alpha_B}, \tag{G.0.1}$$

from which we derive the quadratic Casimir in terms of the eigenvalues of the matrix cross-ration z.

In the first instance it is useful to consider the inverse cross-ration $\omega = z^{-1}$ whose eigenvalues are the inverse of z. If the eigenvalues of z are $x_1, \ldots, x_m$, then the eiganvalues of ω are $w_1 = 1/x_1, \ldots, w_m = 1/x_m$.

Now let us consider the entire correlator function in (5.3.65), in which we take the function $F(w)$ to a be linear combination of Schur polynomials, a direct application of the Casimir gives

$$\frac{1}{2}D_{12}^2 \langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{x_{14}^2}{x_{24}^2}\right)^{\frac{1}{2}\Delta_{21}} \left(\frac{x_{13}^2}{x_{14}^2}\right)^{\frac{1}{2}\Delta_{43}}$$
$$\times \left[\left(\frac{1}{2}(\Delta_{34}-\Delta_{12})\frac{\partial}{\partial\,\mathrm{tr}(\omega)} - \frac{1}{4}\Delta_{34}\Delta_{12}\sum_{i=1}^{m}\frac{1}{w_i}\right)F(w) + \frac{1}{2}D_{12}^2 F(w)\right]. \qquad \text{(G.0.2)}$$

Since $F(\omega)$ is a linear combination of Schur polynomials it is useful to consider the action of the Casimir upon these first. We note that since

$$D_{12B}^{A}u_{iC}^{\alpha} = u_{iB}^{\alpha}\delta_C^A \text{ and } D_{12A}^{B}\bar{u}_{i\dot{\delta}}^{C} = -\delta_A^C\bar{u}_{i\dot{\delta}}^{B} \qquad \text{(G.0.3)}$$

for $i=1$ and 2, it follows that

$$\begin{aligned} D_{12}^2\omega_\beta^\alpha &= 2(2m\omega_\beta^\alpha - m\delta_\beta^\alpha),\\ D_{12J}^{I}\omega_\beta^\alpha D_{12I}^{J}\omega_\rho^\gamma &= 2\omega_\rho^\alpha\omega_\beta^\gamma - \omega_\rho^\alpha\delta_\beta^\gamma - \delta_\beta^\gamma\omega_\rho^\alpha. \end{aligned} \qquad \text{(G.0.4)}$$

We find

$$\frac{1}{2}D_{12}^2 s_{\underline{\lambda}}(w) = \left(2m\omega_\beta^\alpha - m\delta_\beta^\alpha\right)\frac{\partial s_{\underline{\lambda}}(w)}{\partial\omega_\beta^\alpha} + \omega_\rho^\alpha\left(\omega_\beta^\gamma - \delta_\beta^\gamma\right)\frac{\partial^2 s_{\underline{\lambda}}(w)}{\partial\omega_\rho^\gamma\partial\omega_\beta^\alpha}. \qquad \text{(G.0.5)}$$

In order to retrieve the usual form in terms of m variables w_i, one simply diagonalises the ω matrices.

The first two terms of (G.0.5) are linear in differential operators and are therefore trivial to diagonalise. The corresponding eigenvalue result will also be in terms of linear differential operators. The results are

$$\begin{aligned} 2m\omega_\beta^\alpha\frac{\partial s_{\underline{\lambda}}(w)}{\partial\omega_\beta^\alpha} &= 2m\left[\sum_{i=1}^{n} w_i\frac{\partial}{\partial w_i}\right]s_{\underline{\lambda}}(w) = 2m\sum_{i=1}^{m}\lambda_i s_{\underline{\lambda}}(w),\\ m\delta_\beta^\alpha\frac{\partial s_{\underline{\lambda}}(w)}{\partial\omega_\beta^\alpha} &= m\frac{\partial s_{\underline{\lambda}}(w)}{\partial\,\mathrm{tr}(\omega)} = m\left[\sum_{i=1}^{m}\frac{\partial}{\partial w_i}\right]s_{\underline{\lambda}}(w)\\ &= m\sum_{i=1}^{m}(\lambda_i - i + m)s_{(\lambda_1,\lambda_2,\ldots,\lambda_i-1,\ldots,\lambda_m)}(w). \end{aligned} \qquad \text{(G.0.6)}$$

A proof of the of the second expression can be found in appendix A of [86].

The last two terms of (G.0.5) are slightly more non-trivial than the previous cases, since these are quadratic in differentials, however in the eigenvalue basis it may include quadratic as well as linear differentials. Instead, we can apply the matrix action of quadratic differential terms upon $\prod_{i=1}^{m}\mathrm{tr}(\omega^i)^{a_i}$, and consider as many different values of m in which in it takes to find a consistent differential operator in terms of w_i. It is good enough to consider $\prod_{i=1}^{m} \mathrm{tr}(\omega^i)^{a_i}$ since this produces symmetric polynomials upon diagonalisation.

We begin by defining the Vandermonde determinant:

$$\mathrm{vdet}^{(m)}(w) = (-1)^{\binom{m}{2}} \det_{ij}(w_i^{j-1}) = \det_{ij}(w_i^{m-j}) = \prod_{1\leq i<j\leq m} (w_i - w_j), \qquad \text{(G.0.7)}$$

one then finds that

$$\begin{aligned}
&\omega^\alpha_\rho \omega^\gamma_\beta \frac{\partial^2}{\partial\omega^\gamma_\rho \partial\omega^\alpha_\beta} \prod_{i=1}^{m} \mathrm{tr}(\omega^i)^{a_i} \\
&= \Bigg[-\sum_{j=1}^{n} \frac{j^2 a_j \,\mathrm{tr}\left(\omega^{2j}\right)}{\mathrm{tr}\left(\omega^j\right)^2} + \sum_{j=1}^{m}\sum_{k=0}^{j-2} \frac{j a_j \,\mathrm{tr}\left(\omega^{k+1}\right) \mathrm{tr}\left(\omega^{j-k-1}\right)}{\mathrm{tr}\left(\omega^j\right)} \\
&\qquad + \sum_{k=1}^{m}\sum_{j=1}^{m} \frac{jk a_j a_k \,\mathrm{tr}\left(\omega^{j+k}\right)}{\mathrm{tr}\left(\omega^j\right)\mathrm{tr}\left(\omega^k\right)} \Bigg] \prod_{i=1}^{m} \mathrm{tr}(\omega^i)^{a_i}, \qquad \text{(G.0.8)}
\end{aligned}$$

by putting in various examples for m, we find that the following operator always gives the correct result

$$\begin{aligned}
\omega^\alpha_\rho \omega^\gamma_\beta \frac{\partial^2}{\partial\omega^\gamma_\rho \partial\omega^\alpha_\beta} &= \frac{1}{\mathrm{vdet}^{(m)}(w_i)} \sum_{i=1}^{n} w_i^2 \frac{\partial}{\partial w_i^2} \mathrm{vdet}^{(m)}(w_i) - 2(m-1)\sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i} \\
&\quad - \frac{m}{3}(m-1)(m-2). \qquad \text{(G.0.9)}
\end{aligned}$$

Similarly we find

$$\begin{aligned}
&\omega^\alpha_\rho \delta^\gamma_\beta \frac{\partial^2}{\partial\omega^\gamma_\rho \partial\omega^\alpha_\beta} \prod_{i=1}^{m} \mathrm{tr}(\omega^i)^{a_i} \\
&\Bigg[-\sum_{j=1}^{m} \frac{j^2 a_j \,\mathrm{tr}\left(\omega^{2j-1}\right)}{\mathrm{tr}\left(\omega^j\right)^2} + \sum_{j=1}^{m}\sum_{k=0}^{j-2} \frac{j a_j \,\mathrm{tr}\left(\omega^{k}\right) \mathrm{tr}\left(\omega^{j-k-1}\right)}{\mathrm{tr}\left(\omega^j\right)} \\
&\qquad + \sum_{k=1}^{m}\sum_{j=1}^{m} \frac{jk a_j a_k \,\mathrm{tr}\left(\omega^{j+k-1}\right)}{\mathrm{tr}\left(\omega^j\right)\mathrm{tr}\left(\omega^k\right)} \Bigg] \prod_{i=1}^{m} \mathrm{tr}(\omega^i)^{a_i}, \qquad \text{(G.0.10)}
\end{aligned}$$

in which with various different values of m, always agrees with the operator:

$$\omega^\alpha_\rho \delta^\gamma_\beta \frac{\partial^2}{\partial\omega^\gamma_\rho \partial\omega^\alpha_\beta} = \frac{1}{\mathrm{vdet}^{(m)}(w)} \sum_{i=1}^{m} \frac{\partial}{\partial w_i} w_i \frac{\partial}{\partial w_i} \mathrm{vdet}^{(m)}(w) - m \sum_{i=1}^{m} w_i \frac{\partial}{\partial w_i}. \qquad \text{(G.0.11)}$$

Putting this together with (G.0.2), inverting the coordinates so that the Casimir is in terms of x_i where $x_i = \frac{1}{w_i}$, namely with $D^{(m)} := \frac{1}{2} D^2_{12}|_{w_i \to \frac{1}{x_i}}$, we find that

$$\begin{aligned} D^{(m)} = \frac{1}{\mathrm{vdet}^{(m)}(x)} \Bigg[\sum_{i=1}^{m} \Bigg[& x_i \left(-x_i \left(\frac{1}{2} \left(\Delta_{34} - \Delta_{12} \right) - 2m + 3 \right) - 2m + 2 \right) \frac{\partial}{\partial x_i} \\ & + (1 - x_i) x_i^2 \frac{\partial^2}{\partial x_i^2} - \left(\frac{1}{2} \Delta_{21} - m + 1 \right) \left(\frac{1}{2} \Delta_{34} - m + 1 \right) x_i \Bigg] \\ & + \frac{m}{3} (m-1)(2m-1) \Bigg] \mathrm{vdet}^{(m)}(x). \end{aligned} \tag{G.0.12}$$

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