# *Singularities in holographic non-relativistic spacetimes*

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A Thesis presented for the degree of
Doctor of Philosophy

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*Dedicated to*

Mother and father

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## Abstract

We studied the physical meaning of tidal force singularities in non-relativistic spacetimes. Typical examples of such spacetimes include Lifshitz spacetimes, Schrödinger spacetimes and hyperscaling violation spacetimes. First I will discuss the extension of singularity-free hyperscaling violation geometry. To understand the physical meaning of singularity in the deep non-relativistic IR bulk, I will calculate string scattering amplitudes to find a field theory interpretation of bulk singularity. Since geometric quantities like singularities or horizons are not physical observables in higher spin theory, we will discuss whether it is possible to resolve such singularities in non-relativistic spacetimes from higher spin theory context. We will show singularity resolution cannot be performed in $2+1$ dimensional higher spin theory. Finally, we will give an explicit construction of Schrödinger spacetime solutions in $3+1$ dimensional higher spin theory.

# Declaration

The work in this thesis is based on research carried out at the Center for Particle Theory, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

The thesis is organised as follows. I will give an introduction to non-relativistic spacetimes in the theory of holography in Chapter 1. Chapter 2 is based on the work [1] with Simon Ross. Chapter 3 is based on the work [2] with Simon Ross and Tomas Andrade. Chapter 4 is based on another work [3] with Simon Ross. Chapter 5 is based on work [4] with Cheng Peng. I will conclude my thesis in Chapter 6.

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Research life in Durham University is rather fruitful. I really love the Msc lectures here. The contents are exactly those I am interested in. I learned how to do research and what kind of research topics are essential from Simon. Discussion with people (Mukund, Veronika, Balt, Aristomenis) in this research group broadens my knowledge and research interests. I also enjoyed the discussions with my classmates and colleagues: Wei Li, Sheng-Lan Ko, Pichet Vanichchapongjaroen, Henry Maxfield, Felix Haehl, Massimiliano Rota, Alex Peach, Reza Doobary... etc.

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# Chapter 1

# Introduction

## 1.1 Singularity

The theme that this thesis is going to explore is singularity in physics. In classical field theory or mathematics, singularity is defined as a point where physical quantities are infinite. For example, the Coulomb potential is singular at $r = 0$:

$$V(r) = \frac{q}{r}$$

It has taken people years to understand the meaning of physical singularities [5–8] in general relativity where the divergent quantity is spacetime itself. The notion of "a point" is physically meaningful if metric $g_{\mu\nu}$ is defined everywhere else around it. The singular point in spacetime is usually considered as being excluded from spacetime.

A well-known example of a metric with singularities is the Schwarzschild black hole:

$$ds^2 = -(1 - \frac{2G_N M}{r})dt^2 + (1 - \frac{2G_N M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.1.1}$$

Apparently, this metric is singular at $r_S = 2GM$ and $r = 0$. The former one is called the "Schwarzschild singularity" in pre-1960s books. From the modern perspective, we understand this singularity to be no more physical than the north/south pole singularities on earth in latitude and longitude frame. On the contrary, the $r = 0$ singularity is known as the black hole singularity, which is physical. Why do these two singularities receive unequal treatments? This question concerns the notion of physical observables, which are invariant under transformations. In general relativity, physical quantities are those which are invariant under diffeomorphism transformation, i.e. coordinate independent. A simple test to confirm that $r = 0$ is a physical singularity of the Schwarzschild metric is to calculate curvature scalars [5]:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G_N^2 M^2}{r^6} \tag{1.1.2}$$

It is obvious that the scalar curvature above is singular at $r = 0$ but regular at $r = 2G_N M$. Here $r = 0$ is not a part of spacetime.

Regularity in scalar curvature is not a sufficient condition to guarantee a surface being non-singular. There are other types of singularities which have finite scalar curvature at singular points. For example, spacetimes with component of $R_{abcd}$ blowing up in parallelly propagating orthogonal frame are also considered to be singular [9, 10]. Since we exclude singular points from spacetimes, the definition of infinite scalar curvature at this point is not obvious. Therefore, we need a more general definition of singularity. Time-like geodesics incompleteness turns out to be of physical significance to construct such definition since it implies that free moving observers probe an inextensible spacetime region in finite proper time. Inextensible means that the history of these free moving observers is lost after a moment. So we have [6] following definition:

- *Time-like and null geodesic completeness are minimal conditions for spacetimes to be singularity free.*

If a spacetime is time-like geodesics incomplete, we will say it has a singularity. We will see later the singularities in Lifshitz/Schrödinger spacetime are of this type.

## 1.2 Holography

The understanding of fundamental physics is usually improved by recognition of new principles. The holographic principle [11] may be one of the most profound discoveries in the last few decades. Just as the equivalence principle to general relativity, the correspondence principle to quantum mechanics, the holographic principle is believed to be the basic characteristic for a quantum gravity theory. The idea of holography was inspired by black hole entropy, which is proportional to area $A$ of black hole horizon [12].

$$S = \frac{A}{4G_N} \tag{1.2.3}$$

It was recognised later that black hole entropy has thermodynamical interpretation after Hawking [13] discovered thermal radiation of black hole.

Black hole entropy is rather exotic when one consider the entropy derived from a local quantum field theory, where the entropy of an ensemble is proportional to volume of the space. When we take gravitational field into consideration, the black hole entropy formula actually imposes an upper bound of entropy that a compact space with area $A$ can have. Any attempt to add more than $\frac{A}{4G_N}$ degrees of freedom in space region with area $A$ is going to create a black hole. Thus Black holes are the most entropic object in nature.

The formal statement of holographic principle is [11]:

- *A region with boundary of area $A$ is fully described by no more than $\frac{A}{4G_N}$ degrees of freedom, or about 1 bit of information per Planck area.*

The counter intuitive part of the statement above is that the degrees of freedom of bulk gravitational system are encoded non-locally at the boundary of a region. The mechanism for the non-locality is rather mysterious. As a principle for quantum gravity, holography is a very abstract concept and hard to prove. Its rapid development has been based on the appearance of AdS/CFT [14, 15], a concrete model for holography. It is recently proposed that AdS bulk information are encoded on the boundary of spacetime by the spirit of quantum error correction [16–18].

### 1.2.1 AdS/CFT

The duality between gravity theory in $d + 1$ dimensions and gauge field theory in $d$ dimensions is a remarkable realization of holography. The well studied example was proposed by Maldacena [14]: *Type II B string theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory.*

His motivation for this duality was to consider the low energy theory of D-branes in string theory and use open/closed string duality. Maldacena considered a stack of $N$ D3-branes in the limit of $\alpha' \to 0$. The dynamics in the bulk decouples from dynamics on the brane. From the open string point of view, where string coupling $g_s$ is small, $\mathcal{N} = 4$ super Yang-Mills theory lives on the worldvolume of D3-brane. The theory living in the bulk is type IIB supergravity theory in flat space. Therefore,

$$\lim_{\alpha' \to 0} S_{\text{open}} = \text{SYM on D3-brane}$$

From the closed string point of view, where $g_s N \gg 1$ with $\alpha' \to 0$, backreaction of closed strings on D-brane will support black brane solutions. In the asymptotically flat region, the theory again is again type IIB supergravity in flat space. In the near horizon limit, any string excitations are allowed since the energy measured in the infinity is redshifted to zero. Therefore, the string theory is still restricted to low energy limit. The theory near the horizon is decoupled from the theory in the asymptotic region. The near horizon geometry of black D3-brane turns out to be $AdS_5 \times S^5$. Then in the same limit

$$\lim_{\alpha' \to 0} S_{\text{closed}} = \text{string theory in } AdS_5 \times S^5$$

The open string description and closed string description are considered equivalent since they are used to described the same theory from different perspectives. By the statement of $S_{\text{open}} = S_{\text{closed}}$, we are led to the fascinating conjecture

$$\mathcal{N} = 4 \; SU(N) \text{ super Yang-Mills theory} \Leftrightarrow \text{Type IIB string on } AdS_5 \times S^5$$

This duality conjecture has now passed hundreds of tests by matching calculation results from both sides of theories. An intermediate check is the symmetry group of both theories match. On the gravity side, $AdS_5$ has isometry $SO(2,4)$, which is the conformal group of 4-dimensional CFT. The compact direction $S^5$ has symmetry group $SO(6)$ which turns out to be R-symmetry in $\mathcal{N} = 4$ super Yang-Mills theory. For the purpose of this thesis, I will review one of remarkable calculations [19] in Chapter 3, where Maldacena and Alday showed gluon scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory could be reproduced by a gravity calculation in $AdS_5$.

The duality conjecture is also checked beyond semiclassical gravity calculations. In the limit $\lambda \ll 1$ or $(\alpha' \to \infty)$, $AdS$ radius is much shorter than string length. The field theory living on the boundary is weakly coupled. A free scalar field can generate infinite number of operators with integer spin

$$\bar{\psi}\partial_\mu\psi, \qquad \bar{\psi}\partial_\mu\partial_\nu\psi, \qquad \text{other higher derivative terms}$$

On the gravity side, these operators are supposedly dual to many light higher spin states. By doing so, one has to introduce infinite tower of higher spin fields, which makes the theory very complicated. A duality between higher spin fields in AdS bulk and vector like CFT on the boundary is proposed as an extension of the semiclassical gravity version of AdS/CFT. In 4-dimensional bulk, it is conjectured Vasiliev higher spin theory in AdS is dual to $O(N)$ vector model [20–22]. Despite complexity in calculating higher spin theory, this duality is actually simpler than semiclassical gravity theory. The central charge of $O(N)$ vector model is of order $N$ at large $N$ limit whilst the central charge of CFT dual to classical gravity is of order $N^2$.

It is said gravity in 3-dimensions is dynamically trivial. However, $AdS_3/CFT_2$ duality is never a trivial duality. The first gift from this duality is to understand microscopic states corresponding to black hole entropy [23]. The integrable nature of $CFT_2$ offers the theory more fruitful structure than its higher dimensional cousins. Although bulk excitations in $AdS_3$ are pure diffeomorphisms, boundary gravitons have non-trivial dynamics. Higher spin fields in $AdS_3$ is conjectured to have a specific type of dual $CFT_2$: minimal model [24]. Locally the bulk theory can be formulated in terms of Chern-Simons theory [25], whose action is

$$S_{CS} = \frac{k}{4\pi}\int_M \mathrm{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A), \tag{1.2.4}$$

where $k$ is the Chern-Simons level. $AdS_3$ spacetime has isometry group $so(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. This allows one to decompose the gravity theory in terms of two copies of Chern-Simons gauge fields $A, \bar{A}$, resulting in $S_{\mathrm{EH}} = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}]$. Each gauge field takes values in one copy of gauge group $SL(2,\mathbb{R})$. The identification between gravitational fields and gauge fields is

$$e = \frac{1}{2}(A - \bar{A}); \qquad \omega = \frac{1}{2}(A + \bar{A})$$

Asymptotically, the symmetry group $SL(2,\mathbb{R})$ is enhanced to two copies of Virasoro algebra [26]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}m(m^2-1)\delta_{n+m,0} \tag{1.2.5}$$

If gauge fields take values in group $hs[\lambda] \times hs[\lambda]$, we will have a higher spin theory which generically has infinite tower of massless higher spin fields. The dual CFT is usually known as $\mathcal{W}_\infty[\lambda]$ CFT, where $\lambda$ is related to Chern-Simons level $k$ by

$$0 < \lambda = \frac{N}{N+k} \leq 1 \tag{1.2.6}$$

In general, $\lambda$ can take arbitrary real values. Theories with different $\lambda$ may be related [27] at quantum level. The simplification of the theory happens when $\lambda = N$ is integer: the infinite towers of higher spin fields are truncated and $hs[\lambda]$ reduce to $SL(N,\mathbb{R})$. For example, if $\lambda = N = 3$, we are dealing with spin-3 gravity theory formulated by $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons gauge fields [28].

### 1.2.2 Non-relativistic holography

The holographic duality in AdS/CFT is a strong-weak duality. In the semiclassical limit, gravity is weakly coupled while its dual theory is strongly coupled. We do not have effective tools to do calculations in the strongly coupled theory. Therefore, many endeavours have been made to use this duality to understand physics at strong coupling [29].

In reality, many strongly coupled field theories in condensed matter cannot be dual to AdS gravity. There are field theories respecting anisotropic scaling symmetry near fixed points

$$t \to \lambda^z t; \qquad x \to \lambda x \qquad (z > 1)$$

A famous example is $z = 2$ Lifshitz scalar model, whose action is

$$S = \int d^{d_s+1}x \left(-(\partial_t\phi)^2 + (\nabla^2\phi)^2\right) \tag{1.2.7}$$

These are called non-relativistic field theories due to different weight of time and space coordinates. Two theories considered as holographic duals are supposed to have the same symmetry group. To construct holographic dual to these strongly coupled field theories, we need to search for spacetime taking these non-relativistic symmetry group as its isometry. To our interest, we will discuss progress made for 3 simplest types of spacetimes:

- **Lifshitz spacetime**: The geometry is [30–32]

$$ds^2 = L^2(-r^{2z}dt^2 + r^2 d\vec{x}^2 + \frac{dr^2}{r^2}), \tag{1.2.8}$$

where there are $d_s$ spatial coordinates $\vec{x}$. The spacetime has Lifshitz isometry

$$t \to \lambda^z t, \quad x_i \to \lambda x_i, \quad r \to \frac{r}{\lambda} \tag{1.2.9}$$

where $z$ is called the dynamical critical exponent, which realises the anisotropic scaling symmetry geometrically. Lifshitz gravity is proposed to be dual to anisotropic condensed matter theory at fixed point.

- **Schrödinger spacetime** whose geometry is [33, 34]

$$ds^2 = -r^{2z}dt^2 + \frac{dr^2}{r^2} + 2r^2 dt dx^- + r^2 dx_i^2 \tag{1.2.10}$$

Note $x^-$ is an extra bulk direction, which means this holographic duality is between gravitational theory in $d_s + 3$ dimensional geometry bulk and $d_s + 1$ dimensional field theory. This is because there are two conserved quantities in non-relativistic field theory: energy $M$ and particle number $N$. The isometry group of (1.2.10) forms a symmetry group called the Schrödinger group.

For general $z \neq 2$, the isometry of the spacetime contains translation $P_\mu = \partial_\mu$, particle number $N = i\partial_\xi$, non-relativistic boosts $K_i = x_i\partial_\xi + t\partial_i$ and non-relativistic scaling $D = zt\partial_t - r\partial_r + x_i\partial_i + (2-z)\partial_\xi$. At $z = 2$, symmetry group can be enhanced. There is a special conformal generator [35]

$$C = t^2\partial_t - tr\partial_r + tx_i\partial_i + \frac{1}{2}(\frac{1}{r^2} + x_i^2)\partial_\xi$$

satisfying algebra

$$[H, C] = D, \qquad [D, C] = 2C, \qquad [D, H] = -2H$$

- **Hyperscaling violation Lifshitz spacetime** [36, 37]. The general hyperscaling violating geometry has the form [36–38]

$$ds^2 = \frac{1}{\bar{r}^{2\theta/d_s}}(-\bar{r}^{2z}dt^2 + \bar{r}^2 d\vec{x}^2 + \frac{d\bar{r}^2}{\bar{r}^2}). \tag{1.2.11}$$

The $\theta = 0$ case is the Lifshitz spacetime (1.2.8). For $\theta \neq 0$, the isometry under (1.2.9) is broken; the metric has an overall scaling under this transformation $ds^2 \to \lambda^{2\theta/d_s} ds^2$. Theories with such a hyperscaling violation have a characteristic thermodynamic behaviour which is that of a theory living in $d_s - \theta$ dimensions. As a result, it has been suggested that these metrics with $\theta = d_s - 1$ have a thermodynamic structure which may be a useful model for a field theory with a Fermi surface [37] (as the effectively one-dimensional behaviour reproduces the behaviour near a Fermi surface). After redefining radial coordinate so that it is proper size of spatial direction, hyperscaling violation spacetime above is equivalent to

$$ds^2 = -r^{2m}dt^2 + \frac{dr^2}{r^{2n}} + r^2 d\vec{x}^2 \tag{1.2.12}$$

The relation between the coordinates is $r \sim \bar{r}^{2(d_s-\theta)/d_s}$, and

$$m = \frac{d_s z - \theta}{d_s - \theta}, \quad n = \frac{d_s}{d_s - \theta}. \tag{1.2.13}$$

As one can see, if $n = 1$, (1.2.12) is exactly Lifshitz spacetime with dynamical exponent $z = m$.

Non-relativistic spacetimes are not vacuum solution of Einstein equation (with negative cosmological constant). Instead, matter fields are required to support these spacetimes. The minimal matter fields one can add to support Lifshitz and Schrödinger spacetime are massive vector fields [32]. For simplicity, let's consider gravity in 4 dimensions described by following action:

$$S = \frac{1}{2\kappa_4^2}\int d^4x(R - 2\Lambda - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{2}A^\mu A_\mu) \tag{1.2.14}$$

Lifshitz geometry (1.2.9) is supported by massive vector fields satisfying

$$A_\mu dx^\mu = \sqrt{\frac{2(z-1)}{z}}r^z dt; \qquad m^2 = 2z; \qquad \Lambda = -\frac{z^2+z+4}{2} \tag{1.2.15}$$

Time-reversal symmetry is broken by gauge field. Similarly, in general dimensions, massive vector fields can also support Schrödinger spacetimes [33,34]. For example, for $\Lambda = -3$ and $m^2 = 6$, gauge field

$$A = r^2 dt$$

together with metric (1.2.10) are a solution to action (1.2.14). In three dimensions, a gravitational Chern-Simons term can be added to Einstein-Hilbert action [39]. We then get an action:

$$S = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g} \left( R - 2\Lambda + \frac{1}{2\mu} \epsilon^{\lambda\mu\nu} \left( \Gamma^{\rho}_{\lambda\sigma} \partial_\mu \Gamma^{\sigma}_{\rho\nu} + \frac{2}{3} \Gamma^{\rho}_{\lambda\sigma} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho} \right) \right) \quad (1.2.16)$$

Variation of this action results in Einstein equation with Cotton tensor $C_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \quad (1.2.17)$$

Schrödinger spacetime can be a solution to equation of motion above considering $\Lambda = -1$ and $\mu = 2z - 1$. This theory is called topological massive gravity. In this thesis, we will show higher spin fields can also support Schrödinger spacetime [3,4] in general dimensions.

Another motivation to consider non-relativistic holography is to explore some properties of a quantum gravity theory. As it is shown in Table 1.1, different physical theories are characterised by nature constants $\hbar, G_N, c$. Theories (1)-(6) in the table are well-studied and people are trying to construct the ultimate theory (8) - a theory quantising gravity. Semiclassical approximation of quantum gravity, by combining quantum field theory and general relativity, reveals a corner of quantum gravity. However, the corner $c = \infty$ and $\hbar$, $G_N$ finite is much less-studied until recently [40]. An example of non-relativistic quantum gravity is Hořava Lifshitz gravity [41], where Lifshitz solutions are allowed without matter fields to support. In holography, the natural geometrical framework coupled to non-relativistic field theory was found out to be Torsional Newton-Cartan Gravity (TNC). Interestingly, Hořava Lifshitz gravity emerges when TNC gravity is made dynamical [42]. This is not a major topic of this thesis, but it is interesting to explore.

| | $1/c$ | $\hbar$ | $G_N$ | Theory |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | Classical Mechanics |
| 2 | Finite | 0 | 0 | Special relativity |
| 3 | 0 | Finite | 0 | Quantum mechenics |
| 4 | 0 | 0 | Finite | Newtonian Gravity |
| 5 | Finite | Finite | 0 | Quantum field theory |
| 6 | Finite | 0 | Finite | General relativity |
| 7 | 0 | Finite | Finite | Galilean quantum gravity |
| 8 | Finite | Finite | Finite | Relativistic quantum gravity |

Table 1.1: Regime of physical theory under change of constants

## 1.3 Singularities in non-relativistic spacetimes

By following AdS/CFT calculations, one can calculate correlation functions in Lifshitz/Schrödinger spacetimes to probe their causal structures [39, 43]. However, as one goes to limit $r \to 0$, one might encounter a curvature singularity even though all the curvature scalars are finite. This singularity exists in all Lifshitz spacetimes with $z \neq 1$ [44]; Schrödinger spacetimes with $1 < z < 2$ [35] and hyperscaling violation spacetimes $m < n$. We will take general hyperscaling violation spacetimes (1.2.12) as an example to illustrate its singular nature, since Lifshitz is a special case of the former.

As a solution of Einstein equations with matter fields, hyperscaling violation spacetimes are also supposed to satisfy the weak/null energy condition

$$G_{\alpha\beta}k^\alpha k^\beta = T_{\alpha\beta}k^\alpha k^\beta \geq 0$$

for all null or timelike vector $k^\alpha$ [45,46]. Violation of the weak energy condition introduces negative energy density into the theory, generically making vacuum unstable to particle pair creations. For the null vector $k^\alpha$,

$$G_{\alpha\beta}k^\alpha k^\beta = (R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R)k^\alpha k^\beta = R_{\alpha\beta}k^\alpha k^\beta - \frac{1}{2}Rk_\alpha k^\alpha = R_{\alpha\beta}k^\alpha k^\beta$$

The Ricci tensor components of (1.2.12) in $d_s = 2$ are

$$R_{tt} = m(1 + m + n)r^{2(m+n-1)}$$

$$
\begin{aligned}
R_{rr} &= -\frac{m^2+m(n-1)+2n}{r^2} \\
R_{ij} &= -(1+m+n)r^{2n}\delta_{ij}
\end{aligned}
\tag{1.3.18}
$$

For an null vector $k_\alpha$,

$$(k^t)^2 = r^{-2m-2n}(k^r)^2 + r^{2-2m}\vec{k}^2$$

Then the null energy condition is equivalent to condition

$$0 \le R_{\alpha\beta}k^\alpha k^\beta = (1+m+n)(m-1)\vec{k}^2 r^{2n} + (m-n)\frac{(k^r)^2}{r^2} \tag{1.3.19}$$

Then the null energy condition is satisfied if

$$m \ge n \tag{1.3.20}$$

$$(1+m+n)(m-1) \ge 0 \tag{1.3.21}$$

We are interested in $m \ge n \ge 1$ cases. The $r = 0$ surface looks like a singularity for generic $m, n$. If $m = n = 1$, the geometry is AdS in Poincare coordinate. Since we know AdS has a global coordinate patch to cover the whole spacetime manifold, the $r = 0$ horizon is just a coordinate singularity. A free falling observer in hyperscaling violation spacetime with generic $m, n$ can reach Poincare horizon $r = 0$ in finite proper time, (i.e.) the spacetime is geodesically incomplete. To check whether $r = 0$ is a real physical singularity for generic values of $m, n$, we need to move on to parallely-propagated-orthonormal-frame (PPON) to examine the Riemann tensor components. These components are tidal forces experienced by a freely falling observer. Consider the geodesics

$$\epsilon = -r^{2m}\dot{t}^2 + \frac{\dot{r}^2}{r^{2n}} + r^2\dot{x_i}^2$$

Since the metric is independent of the time coordinate $t$ and spatial coordinate $x_i$, the Killing energy and the Killing momentum are $E = r^{2m}\dot{t}$ and $p_i = r^2\dot{x}_i$. We can rewrite geodesics in terms of the Killing conserved energy as

$$\epsilon = -\frac{E^2}{r^{2m}} + \frac{\dot{r}^2}{r^{2n}} + \frac{p^2}{r^2} \tag{1.3.22}$$

With the geodesics above, we can construct the PPON frame as

$$(e_0)^\mu \quad = \quad -E(\partial_t)^\mu + r^{-(m+n)}\sqrt{E^2 - r^{2m}(1+\frac{p^2}{r^2})}(\partial_r)^\mu + p(\partial_x)^\mu \tag{1.3.23}$$

$$
\begin{aligned}
(e_1)^\mu &= \beta_1(\partial_t)^\mu + \beta_2(\partial_r)^\mu && (1.3.24)\\
(e_2)^\mu &= \gamma_1(\partial_t)^\mu + \gamma_2(\partial_r)^\mu + \gamma_3(\partial_x)^\mu && (1.3.25)\\
(e_i)^\mu &= r(\partial_{x_i})^\mu && (1.3.26)
\end{aligned}
$$

Constants $(\beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3)$ are chosen so that $e_\mu$ are orthogonal. We then use the constructed veilbein to transform from the static observer to the free falling observer [45]:

$$R_{abcd} = R_{\mu\nu\rho\sigma}(e_a)^\mu(e_b)^\nu(e_c)^\rho(e_d)^\sigma$$

The Riemann components include following terms:

$$R_{0i0i} = r^{2n-2m-2}[(m-n)E^2 + r^{2m-2}[(n-1)p^2 + nr^2]] \qquad (1.3.27)$$

$$R_{1i1i} = \frac{r^{2n-2m}[(m-n)E^2 - mr^{2m-2}(p^2+r^2)]}{(p^2+r^2)} \qquad (1.3.28)$$

$$R_{0i1i} = \frac{(m-n)Er^{2n-2m-1}}{p^2+r^2}\sqrt{E^2 - r^{2m}(1+\frac{p^2}{r^2})} \qquad (1.3.29)$$

As one can see, $r = 0$ would be regular if $n \geq m+1$ or $n = m$. Recall that the null energy condition requires $m \geq n$, which contradicts $n \geq m+1$ and $n \geq 2$. Therefore, the only possible non-singular hyperscaling violation spacetimes are those with $m = n \geq 2$.

One may conjecture that the singularity in $m \neq n$ spacetimes can be resolved by introducing string effects (at $\alpha'$ order). This is not true. A test string moving towards these naked singularities will be infinitely excited [44]. The physical meaning of these singularities is unclear at this stage. In a specific model, Lifshitz singularity is resolved by considering the brackreaction effect from matter fields [47]. We will discuss the geometric extension of nonsingular hyperscaling violation spacetime in chapter 2. In chapter 3, we will try to construct physical observables to probe the nature of Lifshitz singularity. In chapter 4, we will discuss the problem to resolve these tidal force singularities from higher spin theory point of view. Unfortunately there is no example in which singularity is resolved. We will be interested in knowing whether there are other resolutions.

# Chapter 2

# Extending nonsingular hyperscaling violation spacetimes

This chapter is based on paper [1], written in collaboration with Simon Ross.

## 2.1 Introduction

The application of holography to the study of field theories of relevance to condensed matter systems has been a subject of intense activity in recent years (see e.g. [29, 48] for reviews). In particular, the application to non-relativistic theories represents an interesting extension of the usual holographic dictionary. The simplest example of this type is the Lifshitz spacetime (1.2.8) [30, 32]. The case $z = 1$ gives the familiar AdS spacetime, while $z \to \infty$ gives an $\mathrm{AdS}_2 \times \mathbb{R}^{d_s}$ spacetime. These two limiting cases have a smooth extension through the apparent singularity at $r = 0$ in the geometry (1.2.8). However, this is not the case in the Lifshitz spacetime, as was already noted in [30], and was later stressed in [44, 49]. Scalar curvature invariants constructed from (1.2.8) are necessarily finite — indeed, constant — as a consequence of the Lifshitz symmetry, but there are divergent tidal forces as we approach $r = 0$ along geodesic congruences. The consequences of this singularity for observers in the spacetime were explored in [44], who argued that observers near the singularity would experience large effects.

The significance of the singularity in the Lifshitz metrics from the point of view of the dual field theory is less clear. As in the usual AdS/CFT correspondence, the natural observables to consider in the field theory are local correlation functions, which correspond to bulk correlators with their endpoints on the boundary of the spacetime at $r = \infty$. By causality, the calculation of these correlators only involves the region of spacetime $r > 0$, so they are not directly sensitive to the singularity. Indeed, the correlators can be calculated by analytic continuation from the Euclidean version of (1.2.8), which has no divergent tidal forces. In the Euclidean solution, $r = 0$ is at infinite proper distance, so the Euclidean metric in these coordinates is already geodesically complete, just as in Euclidean AdS. There is no question of extension of the solution in the Euclidean solution.

We conjecture that this singularity is reflected in the field theory in the structure of the infrared divergences appearing in scattering amplitudes. Scattering amplitudes are an intrinsically Lorentzian observable, and it is well-known that in massless theories they have infrared divergences associated with the emission of soft collinear particles. In the AdS context, non-trivial initial and final states in the Poincare patch of the geometry (corresponding to scattering amplitudes in the field theory) are associated with particles/fields crossing the Poincare horizons [50]. Further, in the work of Alday and Maldacena on gluon scattering amplitudes [19] the infrared divergence was cut off by introducing an explicit brane source in the bulk spacetime away from the Poincare horizon; the infrared divergence appears in the limit as the cutoff brane approaches the horizon. So there indeed seems to be a close connection between the $r \to 0$ limit in spacetime and infrared structure of scattering amplitudes.

To support this speculation, we need to understand what is special about cases where these tidal divergences don't arise. This is relatively easy to understand in the relativistic case; scattering amplitudes are not really a good physical observable in a relativistic conformal field theory, and one should work instead with the extension of the field theory to the Einstein static universe $\mathbb{R} \times S^{d_s}$. The extension of the spacetime beyond the Poincare horizon seems necessarily connected to this extension of the field theory. This is also connected to the existence of special conformal transformations, as it is the special conformal transformations that map the conformal boundary of Minkowski space to finite position (the inversion symmetry exchanges null infinity with the light cone of the origin).

There are two non-relativistic examples where the tidal divergences also don't arise. The first is the Schrödinger spacetimes, which we discuss in section 2.2, reviewing the extension constructed by [35]. Schrödinger with $z = 2$ follows the same pattern as the relativistic case; there is a special conformal symmetry, and the smooth extension of the spacetime is associated with an extension of the boundary geometry. Indeed, the bulk coordinate transformation was obtained in [35] by using the special conformal symmetry. However, there is a smooth extension for $z \geq 2$, and not just in the case $z = 2$ where the special conformal transformation exists. This thus seems to provide an example of a solution with an extension both in the field theory and in spacetime, but without a special conformal transformation. Our new contribution to the consideration of this case is just to note that (except for the case with three bulk dimensions) these extensions are not present once we consider asymptotically Schrödinger spacetimes with non-zero particle number. Thus, the unexpected extensions appearing in the $z > 2$ cases appear to be some special property of the field theory in an "empty box", when we consider a system with Schrödinger symmetry, but with no actual field theory particles present.

The more interesting and surprising non-relativistic example is the case found in [46], who showed that there is a particular class of hyperscaling violating spacetimes which have no tidal singularity on the horizon. This case is the main focus of our work. We will show in section 2.3 that these solutions have a smooth extension through $r = 0$, by explicitly constructing a good coordinate system there. The dual field theory has no special conformal symmetry; indeed it doesn't even have a scaling symmetry. Furthermore, we will argue that the boundary of the extended spacetime has two disconnected components, as in $AdS_2$. Thus, the extension of the spacetime is not connected to an extension of the field theory to a larger background.

Applying the usual holographic correspondence, we would expect such a spacetime to be dual to two copies of the field theory, with separate Hilbert spaces associated to the two boundaries. But the horizon separating the two asymptotic regions has zero cross-sectional area, so unlike in $AdS_2$, it seems problematic to interpret this geometry as corresponding to an entangled state in the two copies of the field theory. Thus, this example poses a challenge not just to our understanding of the significance of the singularities, but also to the picture advocated for example in [51, 52] that connectedness of the spacetime is dual to entanglement in the field theory.

In $AdS_2$, some of these issues find their solution in the fact that finite-energy excitations modify the asymptotics [53], so it is not actually possible to propagate an influence from one boundary to the other without violating the asymptotic boundary conditions.[1] In section 2.4, we will argue that there will be a similar resolution in this hyperscaling violating example. We consider the position-space Green's function for a source on one boundary, and while we are not able to fully calculate its form, we will argue that it is divergent along the horizon.

While this provides a possible resolution of the puzzle, it still seems surprising even at the level of vacuum states that we can have a connection in the spacetime between the two asymptotic boundaries without any entanglement in the field theory vacuum state. It would be interesting to understand the field theory interpretation of these cases better.

## 2.2 Extension of Schrödinger spacetimes

In this section we review known results on extension of the Schrödinger spacetimes. This will provide a useful warm-up for our later consideration of the hyperscaling violating spacetime, and this is also an interesting example worth including in the discussion in its own right.

These spacetimes were introduced in [33, 34] as duals to non-relativistic theories where the anisotropic scaling symmetry is supplemented by invariance under Galilean boosts. In the special case $z = 2$, the symmetries also include a special conformal transformation. It was shown in [35] that the Schrödinger solutions have a smooth extension through $r = 0$ for $z \geq 2$. The extension for $z = 2$ is consonant with our expectations, and indeed the smooth coordinates of [35] were constructed

[1] See e.g. [54, 55] for a discussion of the conceptual issues in $AdS_2$.

by making use of the special conformal transformation. The fact that the extension continues to exist for $z > 2$ is surprising, however. We will review the construction of [35] and comment on what happens when we consider non-vacuum states in the field theory.

Schrödinger geometry represents a holographic dual of the ground state for a field theory in $d_s$ spatial dimensions with a Schrödinger symmetry, which includes an anisotropic scaling symmetry and Galilean boosts. Realising this extended symmetry requires an extra dimension. In particular the addition of the $\xi$ coordinate enables us to realize the conserved particle number appearing in the Schrödinger algebra as momentum in the $\xi$ direction. As in the Lifshitz spacetime, this solution has an apparent singularity at $r = 0$. An extension of the spacetime beyond $r = 0$ was found in [35] for $z \geq 2$. For $z = 2$, their construction was based on the special conformal symmetry $C$ which appears for this choice of dynamical exponent. They define a new timelike coordinate $T$ such that $\partial_T = H + C = \partial_t + C$. This led them to define the new coordinates $(T, R, \vec{X}, V)$ given by

$$t = \tan T, \quad r = \frac{\cos T}{R}, \quad \vec{x} = \frac{\vec{X}}{\cos T}, \tag{2.2.1}$$

$$\xi = V + \frac{1}{2}(R^2 + \vec{X}^2)\tan T. \tag{2.2.2}$$

In these new coordinates, the metric for $z = 2$ is

$$ds^2 = -\frac{dT^2}{R^4} + \frac{1}{R^2}(-2dTdV - (R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2). \tag{2.2.3}$$

The null surfaces at $r = 0$ in the original metric correspond to surfaces $\cos T = 0$ which are evidently smooth in the new coordinates. There is still an apparent singularity at $R \to \infty$ in the new coordinates, but because of the harmonic potential in $g_{TT}$, geodesics are prevented from reaching $R \to \infty$, so this new spacetime is actually geodesically complete. From the point of view of the boundary at $r = \infty$ ($R = 0$), the extension adds regions to the future and past of the existing boundary. For $z = 2$, this extension of the boundary can be understood as a result of the special conformal transformation. Thus, the case with $z = 2$ has the same qualitative structure as for AdS.

The surprise is that this coordinate transformation also provides a smooth extension of the spacetime for $z > 2$. Applying the same coordinate transformation in the case of general $z$ gives

$$ds^2 = -(\cos T)^{2z-4}\frac{dT^2}{R^4} + \frac{1}{R^2}(-2dTdV - (R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2). \quad (2.2.4)$$

Thus, for $z > 2$, the extension is still smooth at $\cos T = 0$. The geometry no longer has a $T$-translation symmetry, which is a consequence of the absence of the special conformal transformation, but there is no obstruction to the extension, and the picture from the point of view of the causal boundary is the same as before. This is an example where the field theory unexpectedly has a smooth extension; there was no symmetry in the theory in the $t, x, \xi$ coordinates which suggested that it would be reasonable to treat $t \to \infty$ as being at finite position, but the coordinate transformation (2.2.1), which maps this to finite $T$, gives a smooth bulk geometry with an extended boundary.

A natural suspicion is that this smooth extension is a special feature of the vacuum solution. To say that an extension really exists in the field theory, we would like to see that there are excitations of the geometry which remain smooth in global coordinates, corresponding to non-trivial states of the field theory on the extended spacetime. In the next two subsections we consider two kinds of excitations; changes in the state in a sector with a given particle number, and changes in the conserved particle number of the field theory.

### 2.2.1 Excitations: mode solutions

We want to consider excitations about the Schrödinger solution, and look for excitations which remain smooth in global coordinates. In this section we consider normalizable mode solutions, corresponding to excited states of the field theory, following [35,56]. In appendix 2.A, we give some new results on position space Green's functions in this spacetime.

The simplest thing to do is to consider mode solutions in the original coordinates. However, unlike on a black hole spacetime, there are no mode solutions which are regular at the horizon; that is, there is no analogue of ingoing modes on the Schrödinger background. This is trivial to see. The mode solutions in the original coordinates are

$$\phi = e^{-im\xi + i\omega t + i\vec{k}\cdot\vec{x}} f(r). \tag{2.2.5}$$

As $r \to 0$ along a generic ingoing geodesic, all of $t, \xi, \vec{x}$ diverge. For example, if we take $r \to 0$ keeping $V$ finite, $\xi$ will blow up. Whatever the dependence of $f(r)$ on $r$ is, the assumption that the mode depends separately on $t, \vec{x}$ and $r$ means that the dependence on $\xi$ cannot become a dependence on the finite coordinate $V$.[2] Thus there are no mode solutions in the original coordinates that are regular at the horizon. This is of course no obstruction to the existence of smooth solutions; it just says that the modes (2.2.5) are not a good basis for constructing them.

For the Schrödinger solution with $z = 2$, the geometry has enough symmetry to allow us to solve for mode solutions in the new coordinates. This analysis was carried out in [35], for the solutions for a probe scalar field on the Schrödinger background in the new coordinates. If we solve the massive Klein-Gordon equation with mass $\mu$ in the new coordinate system, the solutions can be decomposed in modes as

$$\phi = e^{-iET} e^{-imV} Y_L(\theta_i) \varphi_{L,n}(\rho) \phi_{L,n}(R), \tag{2.2.6}$$

where $\rho, \theta_i$ are spherical polar coordinates on the spatial $\vec{X}$ coordinates, $Y_L(\theta_i)$ are the appropriate spherical harmonics, and $\varphi_{L,n}(\rho)$ is given in terms of a generalized Laguerre polynomial. The radial function $\phi_{L,n}(R)$ satisfies

$$\phi'' - \frac{d_s + 1}{R}\phi' + (2Em - 4m(n + \frac{L}{2} + \frac{d_s}{4}) - m^2 R^2 - \frac{(m^2 + \mu^2)}{R^2})\phi = 0. \tag{2.2.7}$$

The solutions of this equation can be written in terms of confluent hypergeometric functions. The two independent solutions near $R \to \infty$ are

$$\phi \sim e^{\pm \frac{1}{2} m R^2}. \tag{2.2.8}$$

[2] In a black hole spacetime, the mode solutions are $\phi = e^{i\omega t} f(r)$, and the divergence of $t$ on the future horizon can be cancelled by choosing an ingoing solution for $f(r)$, so that $\phi \approx e^{i\omega u}$ near the horizon, where $u$ is an ingoing Eddington-Finkelstein coordinate. The point of this comment is that because more coordinates blow up on the Schrödinger horizon, no such cancellation can be engineered just by choosing $f(r)$.

Following [35], the boundary condition is taken to be that we keep only the exponentially damped falloff in the limit $R \to \infty$. The regular solution is then

$$\phi = e^{-\frac{1}{2}mR^2} R^{\Delta_+} U(a, b, mR^2), \tag{2.2.9}$$

where $U(a, b, mR^2)$ is Tricomi's confluent hypergeometric function, and

$$a = \frac{1}{2}(1+\nu) + n + \frac{L}{2} + \frac{d_s}{4} - \frac{E}{2}, \quad b = 1+\nu. \tag{2.2.10}$$

This solution is clearly regular in the interior of the spacetime. However, it only has an interpretation as a change in the state of the field theory if it only excites the normalizable (fast fall-off) part of the field near the boundary.[3] The Tricomi function is

$$U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1(a, b, z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1(a-b+1, 2-b, z). \tag{2.2.11}$$

The regular solution is purely normalizable at infinity if the first term is absent, which can happen if we encounter a pole in the $\Gamma$ function in the denominator, that is if $a - b + 1$ is a negative integer. This will select a discrete set of energies $E$,

$$E = 1 + \nu + 2n + L + \frac{d_s}{2} + r, \quad r \in \mathbb{N}. \tag{2.2.12}$$

Thus the situation in $z = 2$ Schrödinger is very similar to that in AdS. There is a discrete spectrum of smooth mode solutions with respect to the new coordinates, and we can describe smooth excitations above the vacuum state, at least at linear order in perturbations, by considering linear combinations of these modes.

It is difficult to extend this analysis to $z > 2$, as the geometry now has no time-translation symmetry in $T$, so we cannot Fourier transform in the $T$ direction. Solving the wave equation in the new coordinates would therefore requiring solving a PDE. It would be interesting if this problem could be shown not to have smooth solutions, as this would indicate a difference between the $z = 2$ and $z > 2$ cases. We will not pursue this further as we will see in the next section that both $z = 2$ and $z > 2$ encounter a problem when we consider non-zero particle number.

[3] A point which was neglected in [35].

### 2.2.2 Excitations: nonzero particle number

Since the Schrödinger algebra contains a conserved particle number, in addition to asking if the extension through $r = 0$ applies to excitations above the ground state, it is also natural to ask if it applies to the ground state in sectors of the theory with non-zero values of the particle number. Here we will consider what happens for uniform distributions of particle number, as one would expect in the ground state in a sector of fixed particle number.

For $z = 2$, one might think that exciting non-zero particle number would allow us to preserve the smoothness at $r = 0$, since the particle number operator $N$ is central in the algebra, so it commutes with both the dilatation and the special conformal transformation. However, from the geometric point of view the relevant quantity is not the total particle number but the local particle number density $\rho$; it is the dimension of this local operator that will determine the effect of particle number on the bulk spacetime. For $z = 2$, the particle number density $\rho$ has dimension $d_s$, so we would expect that giving it an expectation value will produce a deformation of the spacetime whose effect is more pronounced in the IR, modifying the structure of (1.2.10) at $r = 0$.

This is indeed what we find if we consider the geometries obtained by taking the zero-temperature limit of the black hole solutions for $d_s = 2$ found in [57–59] while holding the particle number fixed. The limiting geometry (in string frame) is

$$ds^2 = k(r)^{-1}(-r^4dt^2 + \frac{\gamma^2}{r^2}d\xi^2 - 2r^2dtd\xi) + (r^2d\vec{x}^2 + \frac{dr^2}{r^2}), \tag{2.2.13}$$

where $k(r) = 1 + \frac{\gamma^2}{r^2}$. The spacetime is asymptotically Schrödinger, with the $1/r^2$ falloff for the deviations expected for a non-zero particle number density. We can see that the introduction of the non-zero density indeed deforms the spacetime in the IR; this solution is now singular at $r = 0$. This is again a tidal divergence, with Riemann tensor components like $R_{0i0i}$ diverging in a parallelly propagated orthonormal frame along ingoing geodesics:

$$R_{0i0i} = \frac{2\gamma^2E^2}{r^6} + (1 + P_\xi^2). \tag{2.2.14}$$

This component is finite if the density $\gamma$ vanishes while becomes divergent in the finite density spacetime. Thus, there is no smooth extension through $r = 0$ for these solutions with non-zero particle number. We should note that there is also a divergent tidal force if we consider the metric in Einstein frame. Although we do not have explicit solutions for $z > 2$, we would expect a similar logic to apply there as well.

The exception to the preceding discussion is Schrödinger spacetimes with three bulk dimensions, as then $d_s = 0$, and non-zero particle number produces a marginal deformation of the geometry. Indeed, in this case the Schrödinger solution has been identified with the null warped $AdS_3$ geometry, and the solution with non-zero particle number is the spacelike warped $AdS_3$ geometry [60], with metric

$$ds^2 = -r^4 dt^2 - 2r^2 dt d\xi + \gamma^2 d\xi^2 + \frac{dr^2}{r^2}. \tag{2.2.15}$$

This metric is a fibration over $AdS_2$, as can be made manifest by defining $\rho = r^2$ and $\bar{t} = \frac{2\sqrt{\gamma^2+1}}{\gamma} t$, so

$$ds^2 = \frac{1}{4}(-\rho^2 d\bar{t}^2 + \frac{d\rho^2}{\rho^2}) + \gamma^2 (d\xi - \frac{\rho}{2(1+\gamma^2)} d\bar{t})^2. \tag{2.2.16}$$

Here, the singularity at $r = 0$ can be resolved by passing to global coordinates for the $AdS_2$ factor, both for vanishing and for non-vanishing particle number. Thus, the extension of the spacetime exists for non-zero particle number. On the other hand, the fact that the geometry involves $AdS_2$ implies that the excitations in a sector of given particle number considered in the previous section fail once we take into account back-reaction (at zero or non-zero particle number), since $AdS_2$ does not have finite excitations which are asymptotically $AdS_2$ on both asymptotic boundaries in the global coordinates [53].

Another caveat to the argument is that it applies to solutions with finite particle number density; it may be that there could be some solutions with finite total particle number (in a spatially infinite field theory) which remain smooth at $r = 0$. However, as such solutions would necessarily be time-dependent it is significantly more difficult to analyze the question, and it is the case of finite particle number density which is of real practical interest.

## 2.3 Hyperscaling violating spacetimes

In this section, we turn to our main subject, the non-singular hyperscaling violating spacetimes. We will first review the general class of spacetimes, and briefly discuss the non-singular case, before explicitly constructing a smooth extension for this case through the horizon at $r = 0$ and discussing the resulting global structure.

As mentioned in introduction, hyperscaling violation geometries generically have a curvature singularity at $r = 0$, but there is an exception; as noted in [46], the case $m = n \geq 2$ has no diverging tidal forces as we approach $r = 0$. In general, the parameters are restricted to $m \geq n$ by the null energy condition (generalizing the familiar restriction to $z \geq 1$ in the Lifshitz case). Given this, we can see that for the components of the Riemann tensor (1.3.27) to (1.3.29) to remain regular as $r \to 0$, we must have $m = n \geq 2$. It can be checked that given this condition, all the components of the Riemann tensor remain finite in the limit [45, 46].

The non-divergent case is special in the sense that it saturates the bound from the null energy condition.[4] For two spatial dimensions, the choice $z = 3/2$, $\theta = 1$, which gives $m = n = 2$, was also previously identified as special because it gives rise to a logarithmic violation of the area law for entanglement entropy [37], so it may be interesting for modelling Fermi liquids holographically.

For simplicity, we will focus mainly on the case $m = n = 2$, and comment briefly on the extension to larger values at the end. The metric is

$$ds^2 = -r^4 dt^2 + \frac{dr^2}{r^4} + r^2 dx_i^2. \tag{2.3.17}$$

The fact that the metric is non-singular precisely when $g_{rr} = 1/g_{tt}$ suggests that it will be useful to introduce a tortoise coordinate $r_*$ such that $dr_* = g_{rr} dr$, as in the Schwarzschild spacetime. Indeed, if we define

$$u = t - \frac{1}{3r^3}, \tag{2.3.18}$$

[4] Although the null energy condition is satisfied, this spacetime does require negative energy densities, and as noted in [45] it is not straightforward to construct reasonable matter Lagrangians that give rise to it as a solution. Since our interest is mainly in using this example to test our general understanding, rather than to advance it as a physically interesting model, we have not attempted to address this issue.

the metric becomes

$$ds^2 = -r^4 du^2 + 2dudr + r^2 dx_i^2. \tag{2.3.19}$$

The $u, r$ part of the metric is regular, but this cannot be the end of the story, as the metric in the $x_i$ directions is still degenerating as $r \to 0$. We can get some insight into the situation by considering the behaviour of the geodesics. The conserved energy and momentum along the geodesics are $E = r^4 \dot{t}$ and $p_i = r^2 \dot{x}_i$. Thus

$$\dot{r}^2 = E^2 - p^2 r^2 - \epsilon r^4, \tag{2.3.20}$$

where $\epsilon = 1$ for timelike geodesics and $\epsilon = 0$ for null geodesics, which implies

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = -\frac{E}{r^4\sqrt{E^2 - p^2 r^2 - \epsilon r^4}}, \tag{2.3.21}$$

$$\frac{dx_i}{dr} = \frac{\dot{x}_i}{\dot{r}} = -\frac{p_i}{r^2\sqrt{E^2 - p^2 r^2 - \epsilon r^4}}, \tag{2.3.22}$$

where we are considering ingoing geodesics. Near $r = 0$,

$$t \approx \frac{1}{3r^3} + \frac{1}{2}\frac{p^2}{E^2 r} + \dots \tag{2.3.23}$$

and

$$x_i \approx \frac{p_i}{Er} + \dots, \tag{2.3.24}$$

where the terms not written explicitly are bounded as $r \to 0$. For null geodesics, we can explicitly integrate (2.3.21) and (2.3.22) to obtain

$$t = \frac{(E^2 + 2p^2r^2)\sqrt{E^2 - p^2r^2}}{3E^3 r^3} + t_0 \tag{2.3.25}$$

and

$$x_i = p_i \frac{\sqrt{E^2 - p^2 r^2}}{E^2 r} + x_{i0}. \tag{2.3.26}$$

If we introduce $\bar{p}_i = p_i/E$, this can be rewritten as

$$t = \frac{(1 + 2\bar{p}^2 r^2)\sqrt{1 - \bar{p}^2 r^2}}{3r^3} + t_0, \quad x_i = \bar{p}_i \frac{\sqrt{1 - \bar{p}^2 r^2}}{r} + x_{i0}. \tag{2.3.27}$$

We see that the ingoing coordinate $u$ is finite (in fact constant) along the radial null geodesics with $\bar{p} = 0$, as in Eddington-Finkelstein coordinates on a black hole. However, for the general geodesics with $p \neq 0$, the coordinate transformation has removed the leading divergence in $t$ but both $u$ and the spatial coordinates $x_i$ diverge like $r^{-1}$ near $r = 0$.

To remove these divergences, we define

$$X_i = r x_i \tag{2.3.28}$$

and

$$T = u - \frac{X_i^2}{2r} = t - \frac{1}{3r^3} - \frac{1}{2} r x_i^2. \tag{2.3.29}$$

The metric can be written in a simple form by introducing polar coordinates $(R, \theta_a)$ in the transverse $X_i$ space:

$$\begin{aligned} ds^2 &= -r^4 dT^2 - \frac{1}{4} R^4 dr^2 + (1 - R^2 r^2) dR^2 + R^2 d\Omega^2_{d_s-1} \\ &\quad + (2 + R^2 r^2) dT dr + R^3 r dR dr - 2 R r^3 dR dT. \end{aligned} \tag{2.3.30}$$

We can see that the components of the metric remain finite at $r = 0$ in these coordinates; in addition, the determinant of the metric is

$$\det g_{\mu\nu} = -R^{2(d_s-1)}, \tag{2.3.31}$$

which is finite at $r = 0$, so the inverse metric is also smooth there. Thus, these coordinates provide a smooth extension of the metric through $r = 0$. The surface $r = 0$ is a null hypersurface, a smooth event horizon. We have constructed ingoing coordinates, allowing us to smoothly cross the future horizon at $t \to \infty$ as $r \to 0$; we could similarly construct outgoing coordinates by taking

$$T' = t + \frac{1}{3r^3} + \frac{1}{2} r x_i^2. \tag{2.3.32}$$

Since the metric (2.3.30) is invariant under $r \to -r$, $T \to -T$, we see that the region $r < 0$ is isometric to the region $r > 0$.

This method can also be generalized to other $n \geq 2$ cases by taking

$$T = t - \frac{1}{2n-1} r^{-(2n-1)} - \frac{r}{2} x_i^2, \quad X_i = r x_i, \tag{2.3.33}$$

which gives

$$\begin{aligned} ds^2 &= -r^{2n} dT^2 + (1 - r^{2(n-1)} R^2) dR^2 - \frac{1}{4} r^{2(n-2)} R^4 dr^2 + R^2 d\Omega^2_{d_s-1} \\ &\quad + (2 + r^{2(n-1)} R^2) dT dr - 2 r^{2n-1} R dT dR + r^{2n-3} R^3 dR dr. \end{aligned} \tag{2.3.34}$$

Note that as expected, this provides a smooth extension only for $n \geq 2$.

### 2.3.1 Global structure

To understand the meaning of this extension of the geometry from the point of view of the dual field theory, we would like to understand the relation between the regions $r > 0$ and $r < 0$; in particular, we want to understand the relation between their asymptotic boundaries at large $r$, where we conventionally think of the field theories as living. (More precisely, as the hyperscaling violating spacetime is singular as $r \to \infty$, we should introduce an explicit cutoff and work on a surface of constant $r = r_0$). In an $\mathrm{AdS}_d$ spacetime for $d > 2$, when we extend the Poincare patch to global coordinates, the boundary is connected, and there is a single Hilbert space for the dual field theory[5]; by contrast, in a black hole spacetime or in $\mathrm{AdS}_2$, there are two disconnected boundaries, which have separate field theory Hilbert spaces associated with them. The field theory dual in those cases is some entangled state in two copies of the field theory. We would like to know whether our hyperscaling violating spacetime is of the former or of the latter type.

In our smooth coordinates (2.3.30), the spacetime certainly does not look connected, but this may be just a defect of our coordinates. To consider this question in a more coordinate-independent manner, we will consider the causal structure of the spacetime. In the cases where the boundary is connected, an initial time slice in the boundary is a Cauchy surface for the full extended boundary in the field theory, and the whole of the boundary lies either to the future or to the past of this initial time slice. So if we find that there are points on the boundary which are not in the future or past of the initial data slice in one asymptotic region of the hyperscaling violating spacetime, we can conclude that the extension of the spacetime does not correspond simply to further evolution of the CFT state defined on that initial slice, but must instead involve some extension of the CFT Hilbert space.

We are therefore interested in considering the future and past of an initial time slice, which in the bulk spacetime corresponds to a constant $t$ slice of the boundary. Thus, we want to find $I^{\pm}(r = r_0, t = t_0)$. We can see from (2.3.21) that motion in the $x_i$ directions restricts the motion in $r$, so the future or past of $r = r_0$, $t = t_0$ will be

[5] The extension for the Schrödinger spacetimes reviewed above is also of this form.

bounded by the radial null geodesics. The ingoing/outgoing coordinates $u, v = t \mp \frac{1}{3r^3}$ are constant along the radial null geodesics, and the $x_i$ are constant, so $T$, although not constant, remains bounded. Thus, the ingoing radial null geodesics from $r = r_0$ will intersect the surface $r = -r_0$ beyond the future horizon at some finite value of $t$. Thus, there is indeed a part of this new asymptotic region which is spacelike separated from the initial time surface. (No part of the region beyond the future horizon is to the past of the initial surface.)

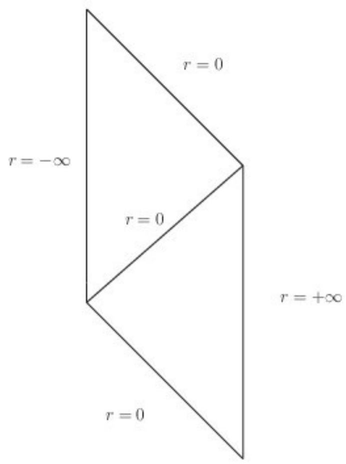



Figure 2.1: A qualitative depiction of the causal structure in the region covered by the ingoing coordinates. Note that although we draw the $T, r$ space, the geometry does not have a translational symmetry in the transverse space in the coordinates regular at the horizon, so this is not a true Penrose diagram.

This implies that the structure of the spacetime is qualitatively similar to that of $AdS_2$, as depicted in figure 2.1; there is a separate boundary at $r < 0$, disconnected from the boundary at $r > 0$.[6]

If we follow the usual holographic dictionary, we would associate these two asymptotic boundaries with two copies of the field theory Hilbert space. Now an interesting problem is that the horizon at $r = 0$ has vanishing cross-sectional area, so it is difficult to interpret the geometry as dual to an *entangled* state in two copies of the field theory. If we assumed the usual Ryu-Takayanagi prescription applied, the

[6]It is not clear if successive boundary regions at $r > 0$ are connected, as they would be in $AdS_2$; our construction has not given us a single coordinate patch covering two such regions.

entropy density in the reduced density matrix obtained by tracing over one of the boundaries should be given by the area of the horizon, as this is clearly an extremal surface [61, 62]. The field theory coordinates are $t, x^i$, so the vanishing of $g_{x^i x^i}$ at $r = 0$ in (2.3.19) appears to say that the reduced density matrix has zero entropy density.[7] Thus, the state of the field theory asociated to this spacetime would seem to have no entanglement, contrary to the general conjectures in [51, 52].

## 2.4 Excitations of the smooth spacetime

The smooth extension of the spacetime indicates that the ground state of the field theory can be thought of as naturally defined on the full asymptotic boundary of the spacetime, rather than just on the boundary in the original $r > 0$ region. As in the Schrödinger example, it is then interesting to ask if this extension has meaning also for excited states. In this section we argue that finite-energy excitations will indeed destroy the extension. We will first consider looking for mode solutions of this equation in the different coordinates, and then consider a Green's function for an operator insertion on the boundary.

### 2.4.1 Scalar fields in the static coordinate

In the original static coordinates, we can consider the plane wave modes

$$\phi(t, r, x, k, \omega) = e^{-i\omega t + i\vec{k}\cdot\vec{x}} R(r). \tag{2.4.35}$$

The Klein-Gordon equation $\nabla^2\phi - m^2\phi = 0$ then reduces to an ODE,

$$\frac{1}{r^{d_s}}\partial_r(r^{4+d_s}\partial_r R) + \frac{\omega^2}{r^4}R(r) - \frac{k^2}{r^2}R(r) - m^2 R(r) = 0, \tag{2.4.36}$$

[7] The horizon has a non-degenerate metric on the surfaces of constant $T$ in (2.3.30), $ds^2_{r=0,T=const} = dR^2 + R^2 d\Omega^2_{d_s-1}$, but since finite $R$ at $r = 0$ corresponds to infinite values of $x^i$, it seems to us that this is not naturally related to the entropy density in the field theory. However, the rules for such cases with non-compact horizons are perhaps not entirely clear. We can't easily resolve the problem by compactifying the $x_i$ coordinates as this would spoil the smoothness at the horizon, as in the Poincare patch in AdS.

The near horizon region is at $r \to 0$ and boundary is at $r \to \infty$. We can't solve this equation in closed form in general, but we are interested in the behaviour in the near horizon region. The method of solving near $r = 0$ limit of $R(r)$ is asymptotic expansion. A useful reference for introduction of this method is in [63].

To make our derivation reasonable, we need to introduce the notation of asymptotic. Two functions $f(r)$, $g(r)$ are considered asymptotically equivalent, written as $f \sim g$ at point $x = a$, if and only if

$$\lim_{r \to a} \frac{f(r)}{g(r)} = 1 \tag{2.4.37}$$

If two functions are asymptotic equivalent $f \sim g$, then $e^f \sim e^g$ if and only if $\lim_{r\to a}(f(r) - g(r)) \ll 1$. Now we can use the idea of dominant balance to derive asymptotic behavior of function $R(r)$. Assume $R(r) = e^{S(r)}$. The equation (2.4.36) reduces to

$$r^4(S''(r) + S'(r)^2) + (4 + d_s)r^3 S'(r) + \frac{\omega^2}{r^4} - \frac{k^2}{r^2} - m^2 = 0 \tag{2.4.38}$$

Near $r = 0$, for equation to be held, we should expect dominant terms are cancelled. Therefore, one can find

$$r^4 S'(r)^2 \approx \frac{\omega^2}{r^4}$$

There are two solutions to $S(r)$, corresponding to ingoing and out going coordinates we found in section 2.3. Let's take

$$S(r) = i\frac{\omega}{3r^3} + C(r)$$

where $C(r)$ is subleading terms near $r = 0$, satisfying equation

$$r^4(C'' + C'^2) - 2i\omega C' - d_s\frac{i\omega}{r} + C'(4 + d_s)r^3 - \frac{k^2}{r^2} - m^2 = 0$$

One can solve asymptotically,

$$C(r) \approx -i\frac{k^2}{2\omega r}$$

Note $C(r)$ is not negligible near $r = 0$, we should proceed this procedure. $C(r) = -i\dfrac{k^2}{2\omega r} + D(r)$, one can solve

$$D(r) = -\frac{d_s}{2}\ln r + \mathcal{O}(r)$$

Terms in $\mathcal{O}(r)$ are negligible near horizon. Therefore, the final solution of scalar equation $\phi(t,r,\vec{x})$ has asymptotics

$$\phi \sim r^{-\frac{d_s}{2}} \exp\left(-i\omega t + ikx + i\frac{\omega}{3r^3} - i\frac{k^2}{2\omega r}\right) \tag{2.4.39}$$

There are two possible ways to deal with this solution. One is to insert the geodesics expansion (2.3.23) and (2.3.24) into asymptotics and identify Killing conserved energy with $\omega, k$ here. One would find all the divergent terms in exponential function would cancel. Therefore, scalar fields propagate through horizon $r = 0$ smoothly at a point. Alternatively, since all mode solutions are independent, we can integrate over all momentum $k$ evenly. This is equivalent to an inverse Fourier transformation of our solution. We can get

$$\phi(t,x,r,\omega) = \int dk\phi(t,r,x,k,\omega) \sim r^{-\frac{d_s}{2}} \exp\left(-i\omega(t - \frac{1}{3r^3} - \frac{rx_i^2}{2})\right) = r^{-\frac{d_s}{2}} e^{-i\omega T} \tag{2.4.40}$$

There is an overall $r^{-d_s/2}$ divergence, but leaving that aside, the $e^{i\omega/3r^3}$ behaviour here is reminiscent of a black hole; it indicates that we could define "ingoing" and "outgoing" modes behaving as $e^{i\omega T}$, $e^{i\omega\tilde{T}}$, where $T, \tilde{T} = t \mp (\frac{1}{3r^3} + \frac{rx_i^2}{2})$. However, while $T$ would remain finite as we approach the horizon along geodesics, it would diverge as we approach the horizon along more generic directions. Thus, unlike in a black hole spacetime, and like in the Schrödinger example, there are no individual mode solutions which are well-behaved on the horizon. The assumption that the dependence on $t, r$ and $\vec{x}$ separates immediately implies that the modes cannot become functions of $T$ as we approach the horizon.

As in the Schrödinger case, this tells us nothing about the smoothness of the extension, but just indicates that these modes do not provide a good basis near the horizon.

## 2.4.2 The scalar fields in new coordinate

We can attempt to look for solutions of the Klein-Gordon equation in the new regular coordinate. However, this is more difficult, as there are no additional symmetries which are manifest in the new coordinates, so the wave equation does not separate in these coordinates.

The inverse metric is

$$\begin{pmatrix} \frac{1}{4}R^4 & 1-\frac{1}{2}r^2R^2 & -\frac{1}{2}rR^3 \\ 1-\frac{1}{2}r^2R^2 & r^4 & r^3R \\ -\frac{1}{2}rR^3 & r^3R & 1+r^2R^2 \end{pmatrix}, \tag{2.4.41}$$

so the equation of motion in the new coordinates is

$$\begin{aligned} & \frac{1}{4}R^4\partial_T^2\phi + (2-r^2R^2)\partial_T\partial_r\phi - rR^3\partial_T\partial_R\phi + r^4\partial_r^2\phi + 2r^3R\partial_R\partial_r\phi \quad (2.4.42) \\ + \; & (1+r^2R^2)\partial_R^2\phi - \frac{d_s+4}{2}rR^2\partial_T\phi + (d_s+4)r^3\partial_r\phi \\ + \; & (\frac{d_s-1}{R} + (d_s+4)r^2R)\partial_R\phi + \frac{\partial_\Omega^2\phi}{R^2} - m^2\phi = 0. \end{aligned}$$

The $\Omega$ stands for all the angular parts which have $d_s - 1$ dimensions. The $\Omega$ dependence is separable (as a consequence of the rotational symmetry in the $X_i$ plane), and we can take advantage of the time translation invariance in $T$ to Fourier transform in the $T$ direction, so we can write

$$\phi = e^{i\alpha T}Y_L(\Omega)H(r,R). \tag{2.4.43}$$

then we can arrange the equation into

$$\begin{aligned} & r^4\frac{\partial^2 H}{\partial r^2} + (1+r^2R^2)\frac{\partial^2 H}{\partial R^2} + 2r^3R\frac{\partial^2 H}{\partial r\partial R} + (-i\alpha r^2R^2 + (d_s+4)r^3 + 2i\alpha)\frac{\partial H}{\partial r} \\ + \; & (-i\alpha rR^3 + \frac{d_s-1}{R} + (d_s+4)r^2R)\frac{\partial H}{\partial R} \quad (2.4.44) \\ + \; & \left(-\frac{1}{4}\alpha^2R^4 - \frac{d_s+4}{2}i\alpha rR^2 - \frac{L^2}{R^2} - m^2\right)H = 0, \quad (2.4.45) \end{aligned}$$

but the $r$ and $R$ dependence in this equation does not separate, so it is not possible to make further progress analytically in general. It is possible to separate the equation for $\alpha = 0$, but this essentially reduces to the special case $\omega = 0$ of the previous analysis in the original coordinates.

It would be interesting to investigate this equation numerically. For each spherical harmonic, one should look for values of $\alpha$ such that the solution is a regular function of $r, R$ which is purely normalizable as $r \to \pm\infty$. This seems a challenging numerical problem however, so in the next section we turn to an alternative approach, studying the Green's functions for sources on the boundary.

### 2.4.3 Green's function

As an alternative to the mode solution analysis, which corresponds to considering excitations of the incoming initial state at past infinity, we can consider an excitation created by acting with some localized source on the boundary. That is, we can ask if the boundary to bulk Green's function is smooth at the horizon. We will consider first a spatially uniform source, where we can explicitly find the Green's function analytically, and we can gain some understanding of their structure. We will then argue that the Green's function for a spatially localized source is well-behaved at the horizon, although we can't do the full calculation of the Green's function explicitly in this case. It may be useful to read this section in conjunction with appendix 2.A, where the same calculation is done for $z = 2$ Schrödinger, as in that case the calculation can be carried out explicitly in full.

The hyperscaling violating spacetimes do not have a scaling symmetry; instead scaling the coordinates produces an overall rescaling of the metric. However, if we consider massless fields, this is sufficient to produce a simplification in the form of the Green's function. We will therefore restrict to the consideration of massless fields. The spacetime has a real Euclidean section defined by analytically continuing $t \to -i\tau$, so we define the Green's function in the Lorentzian spacetime by analytic continuation from this Euclidean section. In the Euclidean spacetime, the massless equation is

$$\frac{1}{r^{d_s}}\partial_r(r^{4+d_s}\partial_r\phi) + \frac{1}{r^4}\partial_\tau^2\phi + \frac{1}{r^2}\partial_i^2\phi = 0 \tag{2.4.46}$$

This equation has a symmetry under the scaling transformation

$$r \to \lambda^{-1}r; \qquad \tau \to \lambda^3\tau; \qquad x_i \to \lambda^2 x_i; \qquad ds^2 \to \lambda^2 ds^2, \tag{2.4.47}$$

as the scaling of the metric comes out as an overall factor in this massless equation.

We consider a source which is smeared over the spatial directions. By translation invariance in the original coordinates, we take the source to be at $\tau = 0$, so that the boundary condition is

$$\lim_{r\to+\infty} \phi = C\delta(\tau). \tag{2.4.48}$$

The solution with this boundary condition will be independent of the $x_i$. The delta-function in the boundary conditions breaks the symmetry under the scaling (2.4.47), but it transforms covariantly, so the solution should behave as $\phi(\lambda^3\tau, \lambda^{-1}r) = \lambda^{-3}\phi(\tau, r)$. Thus, the solution should have the form $\phi(r,t) = r^3 f(r^3\tau)$, and the problem reduces to an ODE,

$$(9x^2+1)f''(x) + (36+3d_s)xf'(x) + (18+3d_s)f(x) = 0, \tag{2.4.49}$$

where $x = r^3\tau$. The solution satisfying our boundary conditions is

$$f(x) = \frac{C}{(9x^2+1)^{1+\frac{d_s}{6}}} \tag{2.4.50}$$

that is,

$$\phi = \frac{C' r^3}{(9r^6\tau^2+1)^{1+\frac{d_s}{6}}}. \tag{2.4.51}$$

This solution satisfies the boundary conditions because it vanishes as $r \to \infty$ for $t \neq 0$, and the scaling form $\phi = r^3 f(r^3\tau)$ automatically implies that the integral $\int \phi d\tau$ over a surface of constant $r$ is independent of $r$. Explicitly, integrating against an arbitrary test function,

$$\begin{aligned}\int_{-\infty}^{\infty} \lim_{r\to\infty} \frac{r^3}{(9x^2+1)^{\frac{d_s+6}{6}}} g(\tau) d\tau &= 2\int_0^{\infty} \lim_{r\to\infty} \frac{1}{(9x^2+1)^{\frac{d_s+6}{6}}} g(\frac{x}{r^3}) dx \qquad (2.4.52)\\ &= \frac{2\sqrt{\pi}\Gamma(\frac{3+d_s}{6})}{d_s\Gamma(\frac{d_s}{6})} g(0). \qquad (2.4.53)\end{aligned}$$

We therefore get a remarkably simple result for the Lorentzian Green's function defined by analytic continuation,

$$\phi = \frac{\phi_0 r^3}{(9r^6t^2-1)^{1+\frac{d_s}{6}}}. \tag{2.4.54}$$

Note that this has a singularity along $t = \pm\frac{1}{3r^3}$, which corresponds to the radial null geodesics emanating from the point $t=0$ on the boundary; these are the light-cone singularities that we expect to see in the Lorentzian Green's function. To study the behaviour as $r \to 0$, we write

$$x = r^3 t = r^3 T + \frac{1}{3} + \frac{1}{2}R^2 r^2 \tag{2.4.55}$$

so we have

$$\lim_{r\to 0}\phi \approx \frac{r^{1-\frac{d_s}{3}}}{(6Tr+3R^2)^{1+\frac{d_s}{6}}}. \tag{2.4.56}$$

This implies the solution becomes singular at the horizon for large spatial dimension $d_s$.

Mathematically, the singularity at the horizon is related to the light-cone singularity: the function $f(x)$ must have a singularity at $x = \pm\frac{1}{3}$, as this is the bulk light cone, but by (2.4.55) we see that $x = \pm\frac{1}{3}$ on the (future/past) horizon as well, so the solution will also be singular there. There is an additional factor of $r^3$ in $\phi$ which vanishes on the horizon, but this is not sufficient to kill the singularity for large enough $d_s$. This mathematical relation makes it easy to see why we might expect the Green's function not to be regular on the horizon, but it's important to note that it's a mathematical relation, not a physical one; not all of the horizon is causally connected to the source, as discussed in the previous section.

The cases $d_s \leq 3$ seem special, as $\phi$ is then regular on the horizon for $R \neq 0$, although there is still a divergence as we approach the horizon for $R = 0$. For $d_s = 2$, which is physically the most interesting case, the Green's function on the horizon is proportional to $\delta(R)$:

$$\lim_{r\to 0} \phi \propto T^{-1/3}\delta(R^2). \tag{2.4.57}$$

However, the finiteness of $\phi$ in these cases is somewhat misleading; if we consider the stress-energy tensor, we find that we can still expect a strong back-reaction on the metric. For a massless field,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2, \tag{2.4.58}$$

and we find that in the new coordinates $T_{rr} \sim r^{-2d_s/3}$ as $r \to 0$ even for $R \neq 0$. Thus, there is a real singularity associated with this Green's function on the horizon.

However, considering a spatially uniform source can lead to divergences even in cases where generic finite-energy excitations are regular on the horizon, as we see in appendix 2.A for the Schrödinger case. We therefore need to consider a spatially localized source. Unfortunately, this problem is more difficult, and we were not able to explicitly determine the Green's function.

We consider again the massless Klein-Gordon equation, but now with a boundary condition

$$\lim_{r\to +\infty} \phi = C_1\delta(t)\delta^{d_s}(\vec{x}) \tag{2.4.59}$$

for some constant $C_1$. This boundary condition is covariant under the scaling symmetry satisfied by the equation (2.4.47), so the solution should satisfy $\phi(\lambda t, \lambda \vec{x}, \lambda^{-1} r) = \lambda^{-(2d_s+3)}\phi(t, \vec{x}, r)$. Thus, the solution should be of the form $\phi = r^{2d_s+3} H(r^3 t, r^2 \rho)$, where $\rho^2 = x_i^2$ is a radial coordinate in the plane. Thus, finding the Green's function can be reduced to a problem in two variables,

$$(9x^2 - 1)\frac{\partial^2 H}{\partial x^2} + 12xy\frac{\partial^2 H}{\partial x \partial y} + (4y^2+1)\frac{\partial^2 H}{\partial y^2} + (15d_s + 36)x\frac{\partial H}{\partial x} \tag{2.4.60}$$
$$+[(10d_s + 22)y + \frac{d_s - 1}{y}]\frac{\partial H}{\partial y} + 3(d_s+2)(2d_s+3)H = 0$$

where $\phi = r^{2d_s+3}H(x, y)$ and $x = r^3 t$, $y = r^2 \rho$, $\rho^2 = x_i^2$. The form of this equation can be slightly simplified by a change of coordinates,

$$\xi = \frac{x}{(1+4y^2)^{\frac{3}{4}}}, \qquad \eta = y, \tag{2.4.61}$$

which allows us to rewrite the equation as

$$(9\xi^2 - \frac{1}{\sqrt{1+4y^2}})\frac{\partial^2 H}{\partial \xi^2} + (4y^2+1)^2\frac{\partial^2 H}{\partial y^2} + (9d_s + 36)\xi\frac{\partial H}{\partial \xi} \tag{2.4.62}$$
$$+[(10d_s + 22)y + \frac{d_s - 1}{y}]\frac{\partial H}{\partial y}(4y^2+1) + 3(d_s+2)(2d_s+3)(4y^2+1)H = 0 \tag{2.4.63}$$

This transformation has eliminated the mixed derivative term. However, unlike in the Schrödinger case, this equation is still not separable, so we cannot solve for the Green's function exactly.

We do have some general expectations for the singularity structure. Because of the non-relativistic causal structure of the boundary, the light-cone of a point on the boundary at $t = 0$, $\vec{x} = 0$ is the same as the light cone of the surface $t = 0$; thus we would expect that the Green's function will have singularities along the light cone $t = \pm\frac{1}{3r^3}$, that is at $x = \pm\frac{1}{3}$. The future horizon corresponds to $(x, y) \to (\frac{1}{3}, 0)$, so this light cone singularity leads us to expect that $H$ diverges on the horizon as well.

Let us assume that near the horizon, we have a leading singularity $H \sim (3x-1)^\alpha$. That is, assume a double Taylor expansion around $(x, y) \to (\frac{1}{3}, 0)$ of the form

$$H = a(3x-1)^\alpha(1 + c_1 y + c_2 y^2 + c_3(3x-1) + c_4 y(3x-1) + \ldots). \tag{2.4.64}$$

Noting that near the horizon $y = rR \sim \mathcal{O}(r)$, while $3x - 1 = \frac{3}{2}r^2R^2 + 3r^3T \sim \mathcal{O}(r^2)$, the leading divergent terms in (2.4.60) near the horizon are the $\frac{\partial^2 H}{\partial x^2}$ and $\frac{\partial H}{\partial x}$ terms, which go like $(3x-1)^{\alpha-1}$. This then fixes $\alpha$:

$$18\alpha(\alpha-1)+3(5d_s+12)\alpha=0, \tag{2.4.65}$$

so $\alpha=0$ or $\alpha=-1-\frac{5d_s}{6}$. Taking the divergent solution, we would have

$$\lim_{r\to 0}\phi \sim \frac{r^{2d_s+3}}{(3x-1)^{1+\frac{5d_s}{6}}} \sim \frac{r^{\frac{d_s}{3}+1}}{(3R^2+6rT)^{1+\frac{5d_s}{6}}}. \tag{2.4.66}$$

So the solution would be regular at $r=0$ and the stress tensor would be regular at $r=0$ for $R\neq 0$ for any $d_s$, with increasing numbers of derivatives regular as we consider larger dimensions, but the solution has a singularity at $R=0$, where $\phi\sim r^{-d_s/2}$. In fact, this singularity is a mild Dirac function:

$$\begin{aligned}
\lim_{r\to 0}\phi &\sim \lim_{r\to 0}\frac{r^{\frac{d_s}{3}+1}}{(3R^2+6rT)^{1+\frac{5d_s}{6}}} && (2.4.67)\\
&\sim \lim_{r\to 0} r^{\frac{d_s}{3}+1}\int_0^{+\infty} e^{-s(R^2+2rT)}s^{\frac{5d_s}{6}}\,ds && (2.4.68)\\
&\sim \lim_{r\to 0} r^{-\frac{d_s}{2}}\int_0^{+\infty} e^{-\frac{wR^2}{r}-2wT}w^{\frac{5d_s}{6}}\,dw && (2.4.69)\\
&\sim \delta(R^2)T^{-\frac{d_s}{3}-1} && (2.4.70)
\end{aligned}$$

The singularity here is milder than the spatially uniform case, particularly for large dimensions. The big difference in the spatially localized case is that while $H$ is divergent at the horizon, this comes with a stronger suppression: the factor of $r^{2d_s+3}$ in $\phi$ weakens the singularity at the horizon. This corresponds to the physically expected effect that the energy of the disturbance can now spread out in the spatial directions. However, the solution is still singular along the horizon at $R=0$. Since the light-cone only intersects $r=0, R=0$ at $T=0$, this is not just the light cone singularity; we take it to mean that this Green function is not well-behaved, and that the smooth extension of this spacetime is a property just of the vacuum state. The geometry is concentrating some of the energy in the boundary excitation along this ray on the horizon, so it looks here more like the AdS$_2$ case.

Clearly we have not established this divergence with any real rigour, and it would be useful to explore the behaviour of excitations in more detail. However, for the present Green's function analysis it is not clear that numerical solution of (2.4.60) will be particularly useful, as the Green's function is really defined by satisfying the boundary condition in the Euclidean space and then analytically continuing to the Lorentzian section to evaluate it at the horizon. The best route to further work may be to look numerically for values of $\alpha$ such that the solution of the wave equation (2.4.44) is regular in the interior and normalizable at infinity. We conjecture that no such values exist.

## 2.A Schrödinger Green's functions

In section 2.2, we studied the smoothness of extensions above the Schrödinger spacetimes by studying mode solutions. Another approach to considering whether the extension remains smooth for finite excitations is to consider instead of a purely normalizable mode, an excitation created by acting with some localized source on the boundary. That is, we can ask if the boundary to bulk Green's function is smooth at the horizon. It is interesting to do this analysis for Schrödinger because in our analysis of the hyperscaling violating case we work with this Green's function approach, so it is useful to have the corresponding results for Schrödinger for comparison.

One can give a simple abstract argument to suggest that the Green's function will remain smooth at the horizon in the case $z = 2$; the geometry in the new coordinates (2.2.3) has a translation invariance in the $T$ direction, so the horizon at $T = \pi/2$ is not a special surface; if the Green's function insertion is at some arbitrary time, there is nothing to pick out this surface so the Green's function can't blow up there.

However, this argument misses a subtlety, so it is useful to carry out an explicit analysis. We consider for simplicity the Green's functions for a massless scalar, $\nabla^2\phi = 0$. In the Schrödinger geometry with $z = 2$, this is

$$-\frac{2}{r^2}\partial_t\partial_\xi\phi + \partial_\xi^2\phi + \frac{1}{r^2}\partial_{\vec{x}}^2\phi + \frac{1}{r^{d_s+1}}\partial_r(r^{d_s+3}\partial_r\phi) = 0. \tag{2.1.71}$$

We will always assume that the solutions are plane waves in the $\xi$ direction, $e^{-im\xi}$, corresponding to considering sources carrying particle number proportional to $m$. Consider first a source which is only localized in the time direction, and smeared uniformly with respect to $\vec{x}$. Then $\phi = e^{-im\xi}\phi(t,r)$, and

$$\frac{2im}{r^2}\partial_t\phi - m^2\phi + \frac{1}{r^{d_s+1}}\partial_r(r^{d_s+3}\partial_r\phi) = 0. \tag{2.1.72}$$

We want to impose a boundary condition

$$\lim_{r\to\infty}\phi = e^{-im\xi}\delta(t), \tag{2.1.73}$$

but we cannot impose such a delta-function boundary condition literally in the Lorentzian spacetime; the Lorentzian Green's function is divergent on the boundary not just at the point where the source is inserted but also at light like separation. The Schrödinger spacetime does not have an analytic continuation to a real Euclidean spacetime, but for the construction of the Green's function, it is sufficient to continue $t \to -i\tau$, so the wave equation becomes

$$-\frac{2m}{r^2}\partial_\tau\phi - m^2\phi + \frac{1}{r^{d_s+1}}\partial_r(r^{d_s+3}\partial_r\phi) = 0, \tag{2.1.74}$$

and require the field to satisfy

$$\lim_{r\to\infty}\phi(r,\tau) = e^{-im\xi}\delta(\tau). \tag{2.1.75}$$

The key simplification that makes it possible to solve this equation in closed form is that the scaling symmetry under $t \to \lambda^2 t$, $r \to \lambda^{-1} r$ implies that the solution is of the form

$$\phi = e^{-im\xi}r^2 f(r^2\tau). \tag{2.1.76}$$

Thus, the problem reduces to an ODE. Writing $x = r^2\tau$, the equation for $f(x)$ is

$$4x^2\partial_x^2 f + (2(d_s+8)x - 2m)\partial_x f + (2d_s+8-m^2)f = 0. \tag{2.1.77}$$

The general solution is

$$f(x) = c_1 x^{-\frac{d_s+6}{4}-\frac{\nu}{2}}{}_1F_1\left(\frac{d_s+6}{4}+\frac{\nu}{2}, 1+\nu, -\frac{m}{2x}\right) + c_2 x^{-\frac{d_s+6}{4}+\frac{\nu}{2}}{}_1F_1\left(\frac{d_s+6}{4}-\frac{\nu}{2}, 1-\nu, -\frac{m}{2x}\right), \tag{2.1.78}$$

where $\nu^2 = \frac{(d_s+2)^2}{4} + m^2$. In the asymptotic region $r \to \infty$, the first term is the normalizable solution, and the second term is the non-normalizable solution.

Since we want to impose a delta-function boundary condition, we want $\phi \to 0$ as $r \to \infty$ for $t \neq 0$, that is we want $\phi \to 0$ as $x \to \infty$, so we set $c_2 = 0$. We can formally argue that the result must be a delta-function, because the scaling form (2.1.76) implies that the integral $\int_{-\infty}^{\infty} \phi dt$ over any surface of constant $r$ is independent of $r$. However, this integral is actually badly behaved because of the divergence of (2.1.78) when $x$ approaches zero from below. Ignoring for the moment this issue, we adopt this as our definition of the 'Euclidean' Green's function.

Analytically continuing back to the Lorentzian section, the proposed Green's function is

$$\phi = ce^{-im\xi} r^2 (r^2 t)^{-\frac{d_s+6}{4}-\frac{\nu}{2}} {}_1F_1(\frac{d_s+6}{4}+\frac{\nu}{2}, 1+\nu, \frac{im}{2r^2 t}). \tag{2.1.79}$$

This solution has a singularity at $t = 0$, which can be understood as the expected light cone singularity in the Lorentzian spacetime, since surfaces of constant $t$ are null surfaces in the Schrödinger spacetime. It is easy to see that this Green's function is also singular at the horizon $r \to 0$. The argument is the same as for the mode function in section 2.2: the dependence on $\xi$ cannot be converted into dependence on the regular coordinate $V$.[8] This is surprising in light of the previous abstract argument. The resolution is that we chose to put the source at $t = 0$, which is a special point with respect to the horizon at $T = \pi/2$, and while the form of the source is invariant under the $t$-translation symmetry, it is not invariant under the $T$-translation symmetry, as this will act non-trivially on the $e^{-im\xi}$ factor.

Physically, this divergence in the response to a spatially uniform source may be interpreted as the result of the harmonic potential in the $\vec{X}$ directions in the metric (2.2.3). After half a period, this will cause particles starting at arbitrary values of $\vec{X}$ to become concentrated at a single point.

Remarkably, for Schrödinger with $z = 2$, we can go beyond this analysis for a spatially uniform source and construct the Green's function for a fully localized source. For a fully localized source, the scaling symmetry implies that the solution will be of the form $\phi = e^{-im\xi} r^{2+d_s} f(r^2 t, r\vec{x})$. As before, we make an analytic continuation to set $t = -i\tau$, and write the solution as

$$\phi = e^{-im\xi} r^{2+d_s} f(x, y), \tag{2.1.80}$$

[8] In addition, using the asymptotic form of the confluent hypergeometric function given below, we would find that $\phi \sim r^{-d_s/2}$.

where $x = r^2\tau$ as before and $y = |\vec{x}|^2/\tau$. The equation of motion then becomes

$$4x^2\partial_x^2 f + (2(3d_s+8)x - 2m)\partial_x f + (2d_s^2 + 8d_s + 8 - m^2)f \\ +\frac{1}{x}\left(4y\partial_y^2 f + (2d_s + 2my)\partial_y f\right) = 0. \tag{2.1.81}$$

This is separable; what is more, there is a separable solution which satisfies the boundary condition

$$\lim_{r\to\infty}\phi = e^{-im\xi}\delta(\tau)\delta(\vec{x}). \tag{2.1.82}$$

This solution is

$$\phi = ce^{-im\xi-\frac{my}{2}}r^{2+d_s}x^{-\frac{3}{4}(d_s+2)-\frac{\nu}{2}}{}_1F_1(\frac{d_s+6}{4}+\frac{\nu}{2}, 1+\nu, -\frac{m}{2x}), \tag{2.1.83}$$

where $\nu$ is as before. To see that this satisfies the boundary conditions, note that the $x$ dependence makes it vanish as $r\to\infty$ for $\tau\neq 0$ as before, so $\phi$ is supported only at $\tau = 0$ in the limit, and then that as $\tau\to 0$, $e^{-my/2}\to 0$ for $\vec{x}\neq\vec{0}$, so $\phi$ is supported only at $\vec{x} = 0$ in the limit. We can then argue formally as before that $\int\phi dtd^{d_s}x$ over a surface of constant $r$ is independent of $r$ as a consequence of the scaling form of the solution, and that hence it should converge to a delta function. (Note however that as before this argument is only formal due to the problem with defining the integration.)

Thus, analytically continuing back in $t$, the candidate Lorentzian Green's function is

$$\phi = ce^{-im\xi+i\frac{m\vec{x}^2}{2t}}r^{2+d_s}(r^2t)^{-\frac{3}{4}(d_s+2)-\frac{\nu}{2}}{}_1F_1(\frac{d_s+6}{4}+\frac{\nu}{2}, 1+\nu, \frac{im}{2r^2t}). \tag{2.1.84}$$

Again, this is singular at $t = 0$, which is the light-cone singularity in spacetime. This is singular at $t = 0$ for all $\vec{x}$ even for a source which is localized at $\vec{x} = 0$ because of the non-relativistic causal structure: all points at $t = 0$ are lightlike separated from this boundary point. To examine the behaviour near the horizon at $r\to 0$, use the asymptotic expansion of the confluent hypergeometric function [?]

$${}_1F_1(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b} + \frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a} \tag{2.1.85}$$

which gives

$$\phi = e^{-im\xi+i\frac{m\vec{x}^2}{2t}}\left(c'(rt)^{-d_s-2}e^{\frac{im}{2r^2t}} + c''r^{\frac{3}{2}d_s+2}(rt)^{-\frac{d_s}{2}}\right). \tag{2.1.86}$$

Making the coordinate transformation (2.2.1), we see $rt = \frac{\sin T}{R}$ is finite as $T \to \pi/2$, so the second term vanishes and the first term is finite. Using (2.2.2), the combination appearing in the exponential is

$$\xi - \frac{\vec{x}^2}{2t} - \frac{1}{2r^2 t} = V - \frac{1}{2}(R^2 + \vec{X}^2)\cot T \tag{2.1.87}$$

so as $T \to \pi/2$,

$$\phi \approx c' e^{-imV} R^{d_s+2} \tag{2.1.88}$$

is perfectly regular.

This Green's function analysis does not extend simply to $z > 2$. The particle number $m$ has a non-zero scaling dimension, so the previous argument that $\phi$ will only involve a function of $r^z t$ and $r\vec{x}$ does not apply; the function can depend separately on $r, t, \vec{x}$ with the scaling being soaked up by appropriate powers of $m$. Thus even the spatially uniform source will involve solving a PDE, and we have not explored the problem further.

## 2.B Scalar equation in HSV with curvature coupling

The analysis of the Green's function for the smooth hyperscaling violating spacetime can be extended from the massless case to consider a scalar field with a curvature coupling, as the resulting equation still satisfies the scaling symmetry (2.4.47). Consider the equation

$$\nabla^2\phi - \xi R\phi = 0. \tag{2.2.89}$$

The Ricci scalar is $R = -30r^2$, so the Ricci scalar term scales the same way as the Laplacian under (2.4.47). For the spatially uniform case, we can therefore conclude that the solution will be of the form $\phi = r^3 f(x)$ with $x = r^3\tau$, and the Euclidean problem reduces to the ODE

$$(9x^2+1)f''(x) + 42xf'(x) + (24+30\xi)f(x) = 0. \tag{2.2.90}$$

The solution satisfying the boundary condition is

$$f(x) = \frac{P(\frac{1}{6}(-3+\sqrt{5}\sqrt{5-24\xi}), \frac{4}{3}, 3ix)}{(9x^2+1)^{\frac{2}{3}}}. \tag{2.2.91}$$

where $P(a,b,x)$ is Legendre function. Analytically continuing back to the Lorentzian spacetime, we have the Green's function

$$\phi = \frac{\phi_0 r^3 P(\frac{1}{6}(-3+\sqrt{5}\sqrt{5-24\xi}), \frac{4}{3}, 3r^3 t)}{(9r^6t^2-1)^{\frac{2}{3}}}. \tag{2.2.92}$$

The singularity structure of this solution at $r \to 0$ is the same as in the previous case.

## 2.C General HSV spacetime

For general HSV spacetime with $m = n \geq 2$, the calculation would be quite similar. For a time localized boundary condition, the equation is

$$[(2n-1)^2x^2-1]f''(x)+(2n-1)(8n-4+d_s)xf'(x)+(2n-1)(d_s+4n-2)f(x)=0 \tag{2.3.93}$$

where $x = r^{2n-1}t$ determined by scaling constraint. According to regular coordinate (2.3.33), as we are approaching the horizon,

$$x = r^{2n-1}t = r^{2n-1}T + \frac{1}{2n-1} + \frac{r^{2n-2}R^2}{2} \tag{2.3.94}$$

The solution of the equation again has a simple form:

$$f(x) = [(2n-1)^2x^2-1]^{-1-\frac{d_s}{2(2n-1)}} \tag{2.3.95}$$

This means

$$\phi = \frac{\phi_0 r^{2n-1}}{[(2n-1)^2x^2-1]^{1+\frac{d_s}{2(2n-1)}}} \tag{2.3.96}$$

$$\sim \frac{r^{1-\frac{d_s(n-1)}{2n-1}}}{(R^2+2rT)^{1+\frac{d_s}{2(2n-1)}}} \tag{2.3.97}$$

which is always divergent at $R=0$ no matter $1-\frac{d_s(n-1)}{2n-1} > 0$ or not.

For a fully localized source, the solution should be $\phi = r^{2n-1+d_sn}H(x,y)$. we will have the equation:

$$[(2n-1)^2x^2-1]\frac{\partial^2 H}{\partial x^2} + 2n(2n-1)xy\frac{\partial^2 H}{\partial x \partial y} + (n^2y^2+1)\frac{\partial^2 H}{\partial y^2} \tag{2.3.98}$$

$$+ \quad (2n-1)(8n-4+2d_sn+d_s)x\frac{\partial H}{\partial x} + [(7n^2-3n+(2n^2+n)d_s)y + \frac{d_s-1}{y}]\frac{\partial H}{\partial y} \tag{2.3.99}$$

$$+ \quad (2n-1+d_s n)(4n+d_s+nd_s-2)f = 0 \tag{2.3.100}$$

Apply similar approximation shows the solution diverge like

$$H \sim \frac{r^{2n-1+d_s n}}{[(2n-1)x-1]^{1+\frac{(2n+1)d_s}{2(2n-1)}}} \tag{2.3.101}$$

$$\sim \frac{r^{1+\frac{d_s}{2n-1}}}{(R^2+2rT)^{1+\frac{(2n+1)d_s}{2(2n-1)}}} \tag{2.3.102}$$

Therefore, at the horizon $r \to 0$,

$$\lim_{r\to 0} H \sim \lim_{r\to 0} \frac{r^{1+\frac{d_s}{2n-1}}}{(R^2+2rT)^{1+\frac{(2n+1)d_s}{2(2n-1)}}} \tag{2.3.103}$$

$$\sim \lim_{r\to 0} r^{1+\frac{d_s}{2n-1}} \int_0^{+\infty} e^{-s(R^2+2rT)} s^{\frac{(2n+1)d_s}{2(2n-1)}} ds \tag{2.3.104}$$

$$\sim \lim_{r\to 0} r^{-\frac{d_s}{2}} \int_0^{+\infty} e^{-w(\frac{R^2}{r}+2T)} w^{\frac{(2n+1)d_s}{2(2n-1)}} dw \tag{2.3.105}$$

$$\sim \delta(R^2) T^{-1-\frac{d_s}{2n-1}} \tag{2.3.106}$$

# Chapter 3

# Scattering amplitudes in Lifshitz spacetimes

This chapter is based on my paper [2], written with Tomas Andrade and Simon Ross. I will first give a review of calculating scattering amplitudes in AdS gravity. Then we move on to the Lifshitz gravity case.

## 3.1 Introduction

In this chapter, I will use another form of Lifshitz metric (1.2.8), relating the original one by coordinate transformation $r \to r^{-1}$. The new form of Lifshitz geometry is

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dr^2 + d\vec{x}^2}{r^2} \tag{3.1.1}$$

where there are $d_s$ spatial dimensions $\vec{x}$, and we have set the curvature scale to one for convenience. Holographic dictionary for Lifshitz spacetime generally has a similar structure to that of AdS [31, 32, 64, 65]. It is also proposed one can learn Lifshitz holography from AdS holography by doing $z = 1 + \epsilon^2$ expansion [66]. However, while AdS is smooth at $r = 0$, Lifshitz singularity at this surface forbids any geometric extension. It is not so clear how this singularity is reflected in observables in the dual field theory. The correlation functions of local field theory operators are not sensitive to the singularity, as they can be obtained from analytic continuation of Euclidean correlators, and the Euclidean spacetime is not singular. In

Chapter 2 it was argued that the singularity could be reflected in the structure of the infrared divergences in scattering amplitudes. Scattering amplitudes are an intrinsically Lorentzian observable, and it is well-known that in massless theories they have infrared divergences associated with the emission of soft collinear particles. The singularity in the spacetime in the geometry (3.1.1) is related to the dual field theory having more soft modes, as the anisotropic scaling symmetry implies a dispersion relation $\omega \sim k^z$. The IR divergences in scattering amplitudes therefore seems a suitable place to look for observable effects of this physics.

The aim of the present chapter is to investigate this by calculating the scattering amplitudes following the pioneering work of [19] in the AdS case. In that work, the scattering amplitude was related to the calculation of a minimal surface (following [67] or more recently [68] in the flat space case). The appropriate minimal surface was obtained in [19] by working in a T-dual geometry where it is a minimal surface ending on light-like segments on the asymptotic boundary of the T-dual spacetime, whose geometry is again AdS. This gives the leading behaviour of the amplitude as

$$\mathcal{A} \sim e^{iS} \tag{3.1.2}$$

where $S$ is the action of a string wrapping the minimal surface determined by the boundary conditions; this represents a stationary point approximation to the amplitude. In the case of $\mathcal{N} = 4$ SYM, the scattering amplitude can be related to a Wilson loop [69,70], and (3.1.2) can then be understood in terms of the saddle-point calculation of the dual Wilson loop; the leading IR singularity is then related to the cusp anomalous dimension [71]. We are not claiming that such an amplitude-Wilson loop duality extends to the Lifshitz field theories; we simply want to use (3.1.2) as a convenient trick to evaluate the leading behaviour of the amplitude, working in a T-dual frame because it's easier to find the minimal surface there, in the spirit of the discussion in [19]. This can perhaps be made more rigorous in the context of the string embedding of $z = 2$ Lifshitz spacetime in [72–74], or in the construction of [75], but we will leave this as a problem for the future.

In the AdS case, the external states in the scattering amplitude have a dispersion relation $\omega = \pm k$ determined by the conformal invariance, and the amplitude is related to a closed polygonal Wilson loop made up of light-like segments whose lengths are related to the momenta of the external particle states. In [19], the expectation value of the Wilson loop related to the four-point amplitude was shown to be related by conformal invariance to a simple case with two light like segments meeting in a cusp, originally analysed in [76]. In $AdS_3$, the minimal surface corresponding to this cusp is simply

$$r^2 = 2(t^2 - x^2), \tag{3.1.3}$$

which satisfies the boundary conditions $r = 0$ at $t = \pm x$.

We will consider the analogue of this cusp in the Lifshitz case. Since we have less symmetry than in AdS, this is no longer related to the Wilson loop with four segments, and finding the appropriate minimal surface in the bulk for a full scattering amplitude is much more difficult. But considering this cusp will suffice to enable us to control the leading IR singularity in the amplitude.

In section 3.3, we will set up the calculation in Lifshitz. In the Lifshitz case, the anisotropic scaling symmetry determines the dispersion relation to be $\omega = \pm\alpha k^z$, where $\alpha$ is an undetermined parameter which would be fixed by the microscopic details of the field theory. Since we can't control these details of the field theory, We will look for minimal surfaces satisfying $r = 0$ at $t = \pm\alpha x^z$, treating $\alpha$ as a free parameter. The lines $t = \pm\alpha x^z$ are timelike in the boundary at $r \to 0$ for any $\alpha$ because of the non-relativistic causal structure in the boundary of (3.1.1). Therefore in section 3.4 we will give a brief discussion of null and timelike cusps in the AdS case, to fix expectations for the behaviour of our results as $z \to 1$.

Then in section 3.5 we find the minimal surfaces satisfying these boundary conditions, and determine the leading IR divergences in the amplitudes. We will find that these minimal surfaces have a peculiar "mushroom" shape, where the surface initially bends away to larger $x$ before turning around. The leading divergence is controlled by the near-boundary behaviour of the surface. We find this divergence is stronger than in the corresponding timelike cusps for $z = 1$, with a universal ($z$-indepenent) dependence on the cutoff with a coefficient which vanishes as we take the limit. This result is reminiscent of the behaviour of the bulk singularity, where curvature diverges as $1/\tau^2$ along the worldline of geodesics which approach the singularity (where $\tau$ is the proper time), with a coefficient which vanishes as

$z \to 1$ [49]. It should be understood in the field theory as due to the presence of the higher density of soft modes implied by the modified dispersion relation $\omega \sim k^z$.

## 3.2 Review of scattering amplitudes in AdS

Scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills should correspond to scattering processes in string theory in $AdS_5$. The leading order in strong coupling is determined by classcial string configurations. The ends of open string are quarks whilst the open string can be thought to be a gluon tube. We then will consider open string scattering process which is dual to gluon scattering in field theory. It is helpful to review how this calculation is formulated. Useful reviews of this calculation can be found in [77, 78]

### 3.2.1 Simplification of boundary conditions

Scattering amplitudes in string theory are calculated by the insertion of vertex operators at the boundary

$$A_n \sim < \prod_{i=1}^{n} V_i(\mathbf{k}_i, \mathbf{x}(\sigma_i)) > \sim \int DX e^{iS} e^{i\sum_{i=1}^{n} \mathbf{k}_i \cdot \mathbf{x}(\sigma_i)} \tag{3.2.4}$$

where S is string world sheet action in AdS spacetime and $\mathbf{k}^2 = 0$.

$$S_{\text{AdS}} = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau [\frac{1}{r^2}(\partial_\alpha x^\mu)(\partial^\alpha x_\mu) + \frac{(\partial_\alpha r)(\partial^\alpha r)}{r^2}] \tag{3.2.5}$$

where $\lambda l_s^4 = R_{\text{AdS}}^4$. The value of integral above can be approximated by the value of action at stationary point [67]. However, unless the string is propagating in flat spacetime, the equation of motions are rather difficult to solve.

In AdS spacetime, Alday and Maldacena found a beautiful method to simplify the calculation [19], which is known as T-duality transformation. Recalling that in a scattering process, momentum is considered conserved. Then we can pick a momentum in $n$th vertex operator so that

$$\mathbf{k}_n = -\sum_{j=1}^{n-1} \mathbf{k}_j \tag{3.2.6}$$

The boundary conditions can be written as

$$\begin{aligned}\sum_{j=1}^{n}\mathbf{k}_j\cdot\mathbf{x}(\sigma_j) &= \sum_{j=1}^{n}\mathbf{k}_j\cdot\int d\sigma\mathbf{x}(\sigma)\delta(\sigma-\sigma_j)\\ &= \sum_{j=1}^{n-1}\mathbf{k}_j\cdot\int d\sigma\mathbf{x}(\sigma)\left(\delta(\sigma-\sigma_j)-\delta(\sigma-\sigma_n)\right)\\ &= \sum_{j=1}^{n-1}\mathbf{k}_j\cdot\int d\sigma\mathbf{x}(\sigma)\partial_\sigma\theta(\sigma;\sigma_j,\sigma_n)\end{aligned}$$

where the distribution function is defined as

$$\theta(\sigma;\sigma_i,\sigma_j)=\begin{cases}1, & \sigma_i<\sigma<\sigma_j\\ 0, & \text{otherwise}\end{cases} \tag{3.2.7}$$

After integration by parts, we find

$$\begin{aligned}&\sum_{j=1}^{n}\mathbf{k}_j\cdot\mathbf{x}(\sigma_j)\\ =\;& \sum_{j=1}^{n-1}\mathbf{k}_j\cdot\left(-\int d\sigma\partial_\sigma\mathbf{x}(\sigma)\theta(\sigma;\sigma_j,\sigma_n)+\int d\sigma\partial_\sigma(\mathbf{x}(\sigma)\theta(\sigma;\sigma_j,\sigma_n))\right)\\ =\;& -\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}}\partial_\sigma\mathbf{x}(\sigma)\cdot(\sum_{i<j}\mathbf{k}_i+\mathbf{c})\end{aligned} \tag{3.2.8}$$

where $\mathbf{c}$ is a constant vector. Next, a T-duality transformation is performed by changing the variables in the path integral. We can do the transformation by gauging each fields $\mathbf{x}^\mu$ in action (3.2.5). The action preserving local shift symmetry with boundary condition is

$$S=\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\left(\frac{(\partial_\alpha\mathbf{x}-\mathbf{A}_\alpha)^2}{r^2}-i\mathbf{y}\cdot\mathbf{F}\right)-i\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}}(\partial_\sigma\mathbf{x}(\sigma)-\mathbf{A}_\alpha)\cdot(\sum_{i<j}\mathbf{k}_i+\mathbf{c})d\sigma \tag{3.2.9}$$

Here $\mathbf{y}$ is a Lagrange multiplier used for imposing constraints so that new action (3.2.9) is equivalent to (3.2.5). We have $\mathbf{F}=\partial_\tau\mathbf{A}_\sigma-\partial_\sigma\mathbf{A}_\tau$. By gauge transforming $\mathbf{A}_\alpha\to\mathbf{A}_\alpha+\partial_\alpha\mathbf{x}$, we can absorb $\partial_\alpha\mathbf{x}$ into gauge field $\mathbf{A}_\alpha$ . After integrating the $\mathbf{F}$ term by parts, we are left with

$$\begin{aligned}S &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau\left[\frac{\mathbf{A}_\alpha\mathbf{A}^\alpha}{r^2}+i(\mathbf{A}_\sigma\cdot\partial_\tau\mathbf{y}-\mathbf{A}_\tau\cdot\partial_\sigma\mathbf{y})\right]\\ &\quad -i\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}}\mathbf{A}_\sigma\cdot(\sum_{i<j}\mathbf{k}_i+\mathbf{c}+\frac{\sqrt{\lambda}}{4\pi}\mathbf{y})d\sigma\end{aligned} \tag{3.2.10}$$

This action will generate two equations of motion for gauge field $\mathbf{A}$. The "bulk" equation solves

$$\mathbf{A}_\alpha = ir^2\epsilon_{\alpha\beta}\partial_\beta \mathbf{y} \tag{3.2.11}$$

The boundary equation is

$$\sum_{i<j}\mathbf{k}_i + \mathbf{c} + \frac{\sqrt{\lambda}}{4\pi}\mathbf{y}(\sigma_j \leq \sigma \leq \sigma_{j+1}) = 0 \tag{3.2.12}$$

By plugging (3.2.11) into action (3.2.10), the action reduces to

$$\frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau r^2 \partial_\alpha \mathbf{y}\partial^\alpha \mathbf{y} \tag{3.2.13}$$

This means the transformed geometry is again AdS in Poincare coordinate. We can do transformation $r \to r^{-1}$ so that string world sheet action is exactly given by string propagating in AdS spacetime

$$ds^2 = \frac{dx_\mu dx^\mu + dr^2}{r^2} \tag{3.2.14}$$

The boundary equation enforces a boundary condition on $\mathbf{A}_\sigma$.

$$\mathbf{y}(\sigma_i) - \mathbf{y}(\sigma_{i+1}) = \frac{4\pi}{\sqrt{\lambda}}\mathbf{k}_i \tag{3.2.15}$$

The prefactor on the right hand side is not important. The key point is that $\mathbf{y}$ should form a closed polygon since momentum is conserved in scattering processes.

Let's summarize what's happening in above calculation. We reduce the problem of finding stationary point of action (3.2.5) to a problem of finding stationary point of action (3.2.13) under boundary conditions (3.2.15). Geometrically, this is a problem of finding minimal surface anchored on given boundary polygon segments in AdS spacetime. Massless gluons in field theory $\mathbf{k}^2 = 0$ correspond to light-like segments in the T-dual problem.

### 3.2.2 AdS scattering amplitudes

*$AdS_3$*

The simplest case is two light-like segments which meet at a cusp in $AdS_3$ geometry [19]:

$$ds^2 = \frac{-dt^2 + dx^2 + dr^2}{r^2} \tag{3.2.16}$$

The to light-like segments are given by $t = \pm x$ with $t \geq 0$. To simplify the string worldsheet action, the best strategy is to make full use of AdS isometries to parametrize the coordinate as:

$$t = e^{\tau} \cosh \sigma; \qquad x = e^{\tau} \sinh \sigma; \qquad r = e^{\tau} w$$

Boost symmetry and scaling symmetry are manifest in these coordinate: boost is a shift in $\sigma$ while scaling is a shift in $\tau$. What's more, the boundary condition is automatically satisfied by this parametrization. As one can see, as $r \to 0$,

$$t^2 - x^2 = e^{2\tau} \to 0 \qquad \text{as } \tau \to -\infty$$

The action then is a functional of $w(\tau)$.

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \frac{\sqrt{1 - (w(\tau) + w'(\tau))^2}}{w(\tau)^2}$$

The solution to the equation of motion is $w(\tau) = \sqrt{2}$. In terms of $(t, x, r)$ coordinates, the solution is

$$r^2 = 2(t^2 - x^2) \tag{3.2.17}$$

This solution was first found in [76]. Using this solution to evaluate action will tell us that the action is imaginary at stationary point. Therefore, scattering amplitudes $\mathcal{A} \sim e^{iS}$ are exponentially suppressed.

## $AdS_5$

The most fascinating part of the theory is the above result can be generalized to four light-like segments case. Consider $(r, y_0, y_1, y_2)$ with $y_3 = 0$. The string worldsheet embedded in bulk spacetime can be parametrized by any two coordinates. Let's take $y_1, y_2$ as coordinate on world sheet and $r(y_1, y_2)$, $y_0(y_1, y_2)$ as functions of $y_1, y_2$. Using scaling symmetry we can restrict the discussion to a square with segments ranging $y_1, y_2 \in (-1, 1)$. The segments are living at the boundary, so we have the boundary conditions

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0 \tag{3.2.18}$$

Besides, since the boundary segments are light-like, we also have

$$y_0(\pm 1, y_2) = \pm y_2; \qquad y_0(y_1, \pm 1) = \pm y_1 \tag{3.2.19}$$

String world sheet action is then

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma\sqrt{-\det(g_{\mu\nu}\partial_\alpha y^\mu \partial_\beta y^\nu)} \\ &= \frac{\sqrt{\lambda}}{2\pi}\int dy_1 dy_2 \frac{\sqrt{1+(\partial_i r)^2-(\partial_i y_0)^2-(\partial_1 r\partial_2 y_0-\partial_2 r\partial_1 y_0)^2}}{r^2} \end{aligned} \quad (3.2.20)$$

The equations of motion are extremely complicated non-linear partial differential equations. Miraculously, there exists a unique simple solution satisfying boundary conditions [19]

$$r^2=(1-y_1^2)(1-y_2^2); \qquad y_0=y_1y_2 \quad (3.2.21)$$

After inserting the stationary point solutions into action, one can find the scattering amplitudes calculated from gravity side exactly match BDS ansatz in $\mathcal{N}=4$ SYM. For details, see [19, 78].

## 3.3 Lifshitz amplitudes

We are interested in considering a scattering amplitude in the Lifshitz background (3.1.1). This involves insertion of on-shell particles in the boundary field theory at $t=\pm\infty$. In the bulk, it is determined by a string world sheet located near the singularity $r=\infty$ in (3.1.1); it is thus infrared divergent. This divergence can be cut off as in [19], by considering a brane at some fixed $r=r_0$ (taking $r_0\to\infty$ at the end of the calculation). Lifshitz amplitudes are calculated by string propagating in Lifshitz spacetimes, which are exactly of the form (3.2.4), with

$$\begin{aligned} S &= \int d\tau d\sigma\sqrt{-\det\begin{pmatrix} g_{\mu\nu}\partial_\tau X^\mu\partial_\tau X^\nu & g_{\mu\nu}\partial_\tau X^\mu\partial_\sigma X^\nu \\ g_{\mu\nu}\partial_\sigma X^\mu\partial_\tau X^\nu & g_{\mu\nu}\partial_\sigma X^\mu\partial_\sigma X^\nu \end{pmatrix}} \\ &= \int d\tau d\sigma\left(\frac{[(\partial_\tau t)(\partial_\sigma r)-(\partial_\tau r)(\partial_\sigma t)]^2+[(\partial_\tau t)(\partial_\sigma x)-(\partial_\tau x)(\partial_\sigma t)]^2}{r^{2z+2}}\right. \\ &\qquad \left.-\frac{[(\partial_\tau r)(\partial_\sigma x)-(\partial_\tau x)(\partial_\sigma r)]^2}{r^4}\right)^{\frac{1}{2}} \end{aligned} \quad (3.3.22)$$

being the Nambu-Goto action of the string worldsheet,[1] and the string world sheet has a boundary on the regulating brane at $r = r_0$, with Dirichlet boundary conditions on $r$ and Neumann boundary conditions on the field theory directions. We approximate the scattering amplitude by a saddle point which extremizes (3.3.22) subject to these boundary conditions.

As in [19], the minimal surface is more easily obtained by working in the T-dual coordinates, T-dualizing along the boundary directions $t, \vec{x}$. We view this as a trick to obtain the minimal surface we are interested in living in the original spacetime, so we will not carefully investigate this T-duality transformation. This has been studied extensively in the AdS case [79], and some of those results may admit extensions to the Lifshitz context, at least in the context of the supersymmetric realizations of $z = 2$ Lifshitz in [72, 73], but we will not investigate this further.

T-dualizing (3.1.1) along $t$, $\vec{x}$ to T-dual coordinates $t', \vec{x}'$ gives us back a Lifshitz spacetime in the coordinates $t', \vec{x'}, r' = 1/r$, but with a different dilaton field[2] $\phi = (z+d_s)\ln r$. The minimal surface we wanted to find thus becomes an extremum of the Nambu-Goto action (3.3.22) in terms of the T-dual coordinates, with a boundary at $r' = 1/r_0$ with Dirichlet boundary conditions in the $t', \vec{x}'$ directions. The momentum of the external states becomes separation in the $t', \vec{x}'$ directions, so the boundary of the string worldsheet is fixed to lie on a closed polygon at $r' = 1/r_0$ made up of segments with $\Delta t' = \alpha|\Delta\vec{x}'|^z$ for some $\alpha$. In the limit $r_0 \to \infty$, this is a polygon in the boundary $r' = 0$ of the T-dual spacetime.

Our main aim is to find the minimal surface satisfying these boundary conditions. Actually, this is rather difficult for a non-trivial polygon, so we will consider just the corner between two such segments; that is, we take the boundary conditions for our minimal surface to be $r' = 0$ at $t' = \pm\alpha|\vec{x}'|^z$, for $t' > 0$. Since two segments define a plane, we can orient our coordinates such that the separation is just along

[1] If we were to do a proper top-down construction this should be replaced by an appropriate superstring action, but we will neglect such details; at least in the simplest AdS context the problem reduces to finding the minimal surface which extremizes (3.3.22) as we will do here.

[2] This expression will formally have an imaginary part due to T-dualizing the time direction. However, this dilaton does not affect the evaluation of the saddle-point minimal surface.

one of the spatial directions in the Lifshitz metric (3.1.1), and the minimal surface will lie in a three-dimensional subspace of (3.1.1). One may wonder whether it is possible to find four-cusp like solutions as the one found in $AdS$ case (3.2.21). We prefer to believe such solution does not exist. $AdS$ space has global coordinate to cover the whole manifold. Single cusp solution in Poincare coordinate corresponds to four-cusp like solution in global coordinate. However, due to existence of Lifshitz singularity, global extension of Lifshitz Poincare coordinate is impossible. Then it is unlikely to have four-cusp like solution in Lifshitz spacetimes.

We will henceforth drop the primes on the dual coordinates. Our interest is then in finding a minimal surface in the three-dimensional subspace

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dr^2 + dx^2}{r^2} \tag{3.3.23}$$

satisfying the cusp boundary conditions $r = 0$ at $t = \pm\alpha x^z$ for $t > 0$.

One might be tempted to parametrize the surface by $t, x$, but we will find that it is actually not a single-valued function of $x$: the surface moves initially to larger $x$ as $r$ increases, before returning to smaller $x$. Using the fact that the action (3.3.22) is invariant under $x \to -x$, the surface satisfying our boundary conditions will be symmetric under $x \to -x$, so we can restrict attention to the surface for $x > 0$. We can then parametrize the surface for $x > 0$ by $t, r$. Using the scaling symmetry, a more convenient choice of parametrization of this surface is in terms of $\sigma, f$ where

$$t = \sigma^z; \qquad x = \sigma u(f); \qquad r = \sigma f \qquad (t \geq 0) \tag{3.3.24}$$

Our task is to determine the form of $u(f)$ which extremizes the Nambu-Goto action, subject to the boundary condition $u(0) = u_0$ for some arbitrary parameter $u_0 > 0$, where $u_0^z = \alpha$, and $u(f_0) = 0$ at some $f_0 > 0$. The Nambu-Goto action is

$$\begin{aligned} S &= \frac{1}{2\pi\alpha'} \int dX_a dX_b \sqrt{\det(G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)} \\ &= \frac{1}{2\pi\alpha'} \int \frac{d\sigma}{\sigma} \int \frac{df}{f^{z+1}} \sqrt{(z^2 - f^{2z})u'^2 + 2uf^{2z-1}u' + (z^2 - u^2 f^{2z-2})}. \end{aligned} \tag{3.3.25}$$

The stationary point equation is

$$f[f^2(f^{2z} - z^2) + f^{2z}u^2]u'' + f^2(z+1)(z^2 - f^{2z})u'^3$$

$$+f^{2z+1}(3z+1)uu'^2+[f^2(f^{2z}(z-1)+z^2(z+1))-2zf^{2z}u^2]u' \qquad (3.3.26)$$
$$-(z-1)f^{2z+1}u=0.$$

Note that the points at which

$$\Delta(f)=f[f^2(f^{2z}-z^2)+f^{2z}u^2]=0 \qquad (3.3.27)$$

are singular points of (3.3.26). The choice of parametrization (3.3.24) thus reduces the problem to an ODE. This is a complicated non-linear ODE, but it is straightforward to solve numerically when $\Delta$ does not change sign.

## 3.4 Timelike and null cusps in AdS spacetime

Before discussing the solutions of (3.3.26) in the Lifshitz case, it is useful to briefly return to AdS by setting $z=1$. This equation then simplifies, and analytic solutions can be found for $u_0=1$, corresponding to a null cusp in the boundary. There are in fact two analytic solutions satisfying the boundary conditions, $u=\sqrt{1-f^2/2}$, which corresponds to the solution (3.1.3), which is spacelike in the bulk, and $u=\sqrt{1-f^2}$, which corresponds to a null surface in the bulk spacetime. For $u=\sqrt{1-f^2}$, the action (3.3.25) vanishes identically. For $u=\sqrt{1-f^2/2}$,

$$S=\frac{1}{2\pi\alpha'}\int\frac{d\sigma}{\sigma}\int\frac{df}{f^2}\sqrt{-\frac{f^2}{2(2-f^2)}}\approx\frac{i}{4\pi\alpha'}\int\frac{d\sigma}{\sigma}\int\frac{df}{f}, \qquad (3.4.28)$$

where in the second step we have kept the part that gives the leading divergence near $f=0$. We want to introduce a cutoff $\Delta t=\Delta x=\epsilon$ to regulate this divergence. This corresponds to cutting off the $\sigma$ integral at $\sigma_{\rm min}=\epsilon$, and cutting off the range of $u$ at $u_{max}=1-\frac{\epsilon}{\sigma}$, corresponding to $f^2_{min}=4\frac{\epsilon}{\sigma}$. Thus the leading divergence is

$$S\sim\frac{i}{8\pi\alpha'}(\ln\epsilon)^2 \qquad (3.4.29)$$

In the Lifshitz case, the boundary conditions correspond to a timelike cusp in the boundary, so it will be useful to understand the minimal surfaces for timelike cusps in AdS to facilitate the comparison for the $z\to 1$ limit of our results. This corresponds to taking $u_0<1$. Here we cannot find analytic solutions; we will first consider a series expansion and then find full solutions numerically. There is a series expansion near $f = 0$ which is valid for $u_0 \neq 1$,

$$u(f) = u_0 + u_3 f^3 + \sum_{i=5} u_i f^i, \tag{3.4.30}$$

where $u_0$ and $u_3$ are free data and the first few subleading terms are

$$u_5 = -\frac{3u_3}{5(u_0^2 - 1)}, \qquad u_6 = -\frac{2u_3^2 u_0}{(u_0^2 - 1)}. \tag{3.4.31}$$

Note that the general series expansion has no $O(f^2)$ term, so it cannot match smoothly on to the $u = (1 - f^2/2)^{1/2}$ solution at $u_0 = 1$.

We construct solutions numerically by picking some value $f_0$ at which to take $u = 0$, and integrating inwards towards $f = 0$. Near $f_0$ we take an ansatz

$$u(f) = (f_0 - f)^{1/2} \sum_{i=0} b_i (f_0 - f)^i, \tag{3.4.32}$$

where $f_0$ is left undetermined by the equation of motion and the first coefficients are given by

$$b_0 = \sqrt{f_0}, \qquad b_1 = \frac{5 - 4f_0^2}{12\sqrt{f_0}(f_0^2 - 1)}. \tag{3.4.33}$$

For $f_0 = \sqrt{2}$, these coefficients agree with $u = (1-f^2/2)^{1/2}$. We find numerically that in the ranges $f_0 < 1$, $f_0 > \sqrt{2}$, $\Delta$ has a definite sign, so we can construct the surfaces integrating inwards from $u = 0$. We plot these surfaces in figure 3.1. For $f_0 < 1$ they are time-like; they approach a null surface near the turning point as $f_0 \to 1$, see fig. 3.2(a). For $f_0 > \sqrt{2}$ the minimal surfaces are space-like, see fig. 3.2(b), and as $f_0 \to \sqrt{2}$ they seem to approach the analytic solution. We were unable to find solutions for $1 < f_0 < \sqrt{2}$ using either this simple radial integration or relaxation. It was remarked in [76] that there are no minimal surfaces in AdS for $u_0$ approaching 1 from below. We note that there is a range of $u_0$ values $\sim 0.463 < u_0 < 1$ where we find no minimal surfaces, in agreement with the claim of [76]. We do not have a physical understanding of the non-existence of surfaces with $1 < f_0 < \sqrt{2}$. In principle, one could attempt to construct them by patching radial integrations from $u = 0$ to the vicinity of the singular point $\Delta = 0$ and from the singular point towards the boundary. If they exist, we believe that they would change signature in the bulk, see figure 3.2(a). Extremal surfaces with non-definite signature have been found [80], so we do not expect this feature to be an obstruction for their existence. A possible tool to attack this problem is to use Padé approximation of Taylor series [63]. We leave this investigation for future work.

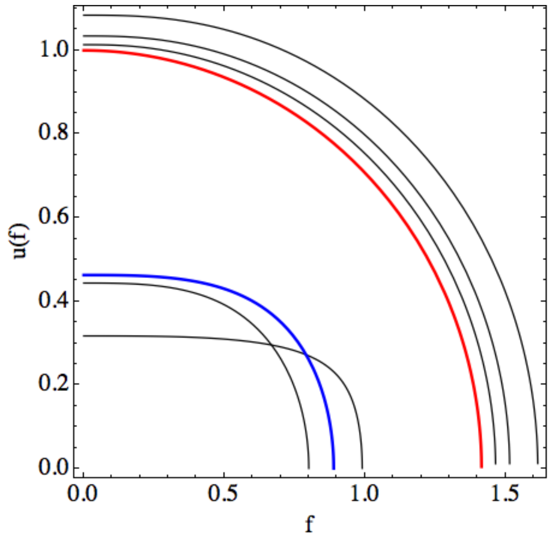



Figure 3.1: The timelike ($u_0 < 1$) and spacelike ($u_0 > 1$) minimal surfaces in AdS. The black lines are extremal surfaces for $f_0 = 0.8$, $0.99$, $\sqrt{2}+0.05$ $\sqrt{2}+0.1$, $\sqrt{2}+0.2$, the blue line corresponds to $f_0 \approx 0.889$ (which has maximum $u_0$ among our timelike surfaces) and the red line is the exact solution $u = (1 - f^2/2)^{1/2}$. Note that $u_0$ is not a monotonic function of $f_0$ in the timelike case, its maximum being $u_0 \approx 0.463$.

The leading divergence in the action comes from the behaviour near the boundary $f = 0$. For $u_0 \neq 1$ the action simplifies to

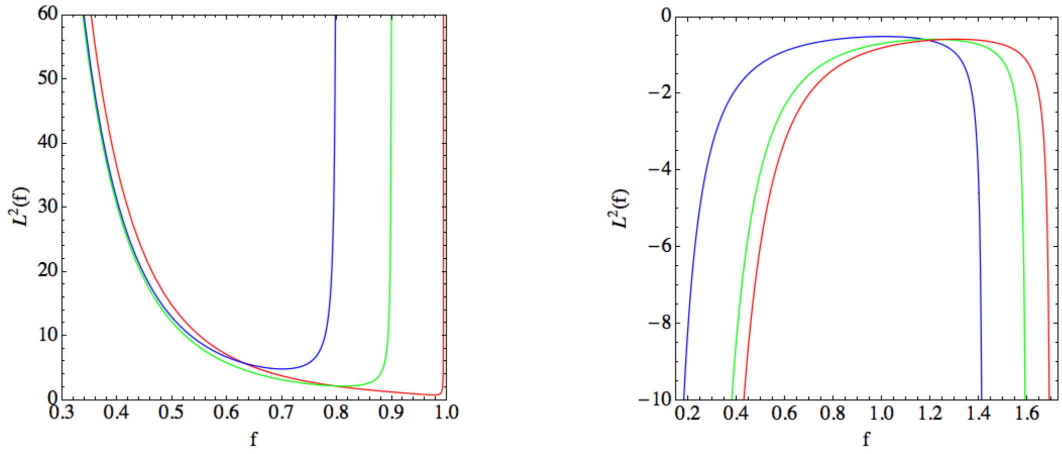
$$S \approx \frac{1}{2\pi\alpha'} \int \frac{d\sigma}{\sigma} \int \frac{df}{f^2} \sqrt{1 - u_0^2} \tag{3.4.34}$$

If we cut off $u$ at $u = u_0 - \frac{\epsilon}{\sigma}$ as before, this now corresponds by (3.4.30) to cutting off $f$ at $f_{min} = (\frac{\epsilon}{|u_3|\sigma})^{1/3}$, and[3]

$$S \sim \frac{\sqrt{1 - u_0^2}}{\epsilon^{1/3}}. \tag{3.4.35}$$

Note that the divergence for these timelike Wilson loops is stronger than in the null case. Note also that because of the $\sigma$ dependence in the cutoff for $f$, the integral

[3]Note that we assume $u_3 < 0$ to obtain a consistent form for $f_{min}$. This is consistent with our numerical solutions.




(a) (b)

Figure 3.2: (a): square of the on-shell Lagrangian for extremal surfaces in AdS with $f_0 = 0.8$ (blue), $f_0 = 0.9$ (green), $f_0 = 0.995$ (red). All surfaces are time-like ($L^2 > 0$) and they become almost null near the turning point as we approach $f_0 = 1$. (b): square of the on-shell Lagrangian for extremal surfaces in AdS with $f_0 = 1.42$ (blue), $f_0 = 1.6$ (green), $f_0 = 1.7$ (red). All surfaces are space-like ($L^2 < 0$).

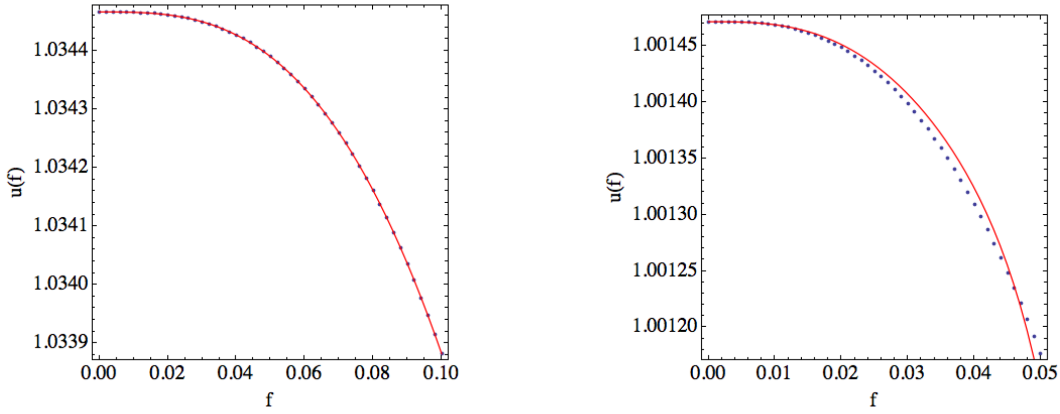
over $\sigma$ is now finite; the leading divergence in the action for the timelike case is not concentrated in the vertex of the cusp, but comes from the limit of the range of $x$ at every $t$.

Thus, the leading divergence for the AdS surfaces with $u_0 \neq 1$ is stronger than in the case $u_0 = 1$ considered previously. For the surfaces with $u_0 > 1$, the action is imaginary, corresponding to an exponential suppression of the amplitude (3.2.4), and the coefficient of this stronger divergence will vanish as we approach the null case. For $u_0 < 1$, the action is real because the surface is timelike.

Since our Lifshitz surfaces will always have timelike cusps on the boundary, we expect them to approach these timelike cusps in the limit as $z \to 1$.

## 3.5 Minimal surfaces in Lifshitz

We now turn to our main results, solving (3.3.26) to find the minimal surfaces in Lifshitz giving a saddle-point approximation to the amplitudes. We will consider generic values of $z$, focusing on the range $1 < z < 2$. As usual, there will be some




(a) (b)

Figure 3.3: Extremal surface for (a) $f_0 = \sqrt{2} + 0.1$ and (b) $f_0 = \sqrt{2} + 0.01$ in the region close to $f = 0$ and fit with the asymptotic expansion (3.4.30). Since the integration from $f = f_0$ produces surfaces connected to $u = (1 - f^2/2)^{1/2}$, the asymptotics (3.4.30) do not fit well the data for $f_0$ close to $\sqrt{2}$.

additional logarithmic terms arising for specific values such as $z = 2$; we display the asymptotic expansion in Appendix 3.B.

As in AdS, we can first consider an asymptotic expansion near the boundary $f = 0$. In this case we find

$$u(f) = u_0 + \frac{(z-1)u_0}{2z^3(2-z)} f^{2z} + b f^{z+2} + \ldots \tag{3.5.36}$$

As $z \to 1$, the coefficient of the leading non-trivial term in the series vanishes, and we recover the expansion (3.4.30) in the AdS case. Actually, the limit as $z \to 1$ of the asymptotic series expansion is somewhat subtle, as there are terms in the expansion with powers which coincide in the limit. We discuss this limit for the full series expansion in more detail in appendix 3.A.

From (3.5.36) we see a remarkable feature of the Lifshitz minimal surfaces; the value of $u$ (and hence $x$ at fixed $t$) is initially increasing for any choice of the free parameter $b$. Thus, any solution consistent with this asymptotic series solution will initially move to increasing $u$ as we move into the interior of the spacetime, even though our boundary conditions imply the surface must reach $u = 0$ at some finite $f$. Minimal surfaces satisfying these boundary conditions will thus have a "mushroom" shape. This is indeed what we find in our numerical analysis.

As in the AdS case, numerical solutions are found by starting from the point $f_0$ at which $u = 0$ and integrating in. We find that the equation can be satisfied perturbatively near this point with an expansion of the form

$$u(f) = (f_0 - f)^{1/2} \sum_{i=0} b_i (f_0 - f)^i, \tag{3.5.37}$$

which reproduces the expected behaviour that $f'(u) = 0$ at $u = 0$. At the first non-trivial order in $(f_0 - f)$, the equation of motion implies

$$b_0[2f_0 - (z+1)b_0^2](f_0^{2z} - z^2) = 0. \tag{3.5.38}$$

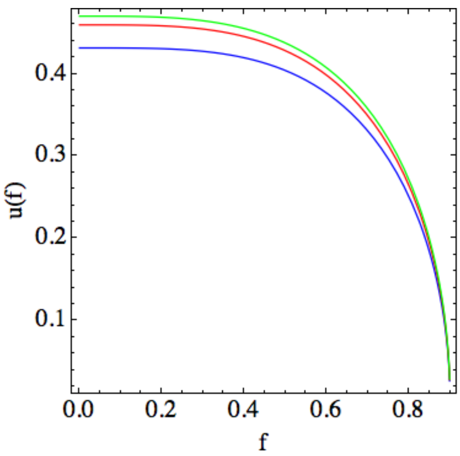
This leads to three different branches of solutions characterized by

1. $2f_0 - (z+1)b_0^2 = 0$,
2. $f_0^{2z} - z^2 = 0$,
3. or $b_0 = 0$.

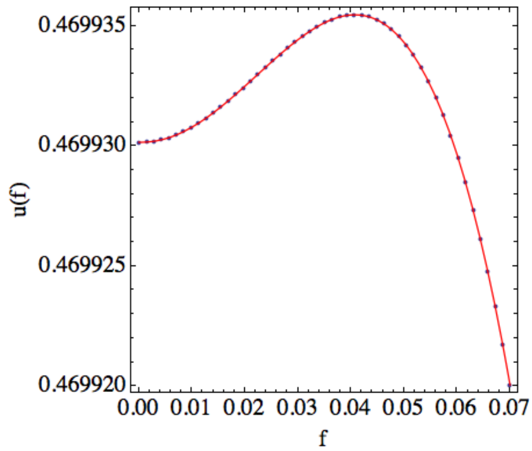
We were able to find numerical solutions satisfying our boundary conditions only for the first case, in the range $f_0 < z^{1/z}$. The equation of motion can then be easily integrated towards $f = 0$. Zooming in near the $f = 0$ region, we note that $u'(f)$ changes sign, giving rise to surfaces with a "mushroom" shape, see figures 3.4(a), 3.4(b), as expected from the asymptotics. The behaviour near the boundary is consistent with the asymptotic expansion (3.5.36). In the limit $z \to 1$, these solutions approach the timelike surfaces of section 3.4 with $f_0 < 1$.

On the other hand, for $f_0 > z^{1/z}$ the integration encounters a critical point $\Delta(f) = 0$ before reaching $u = 0$, see figure 3.5. We also attempted to find solutions in this regime by a relaxation method, but this also fails to converge. As mentioned in section 3.4, one could attempt to construct these solutions by patching two shooting procedures.

The divergence of the action is determined by the near boundary expansion (3.5.36). We are primarily considering the case $z < 2$, where the second $2z$ term dominates. If we considered instead $z > 2$, the third $z + 2$ term would dominate the near-boundary expansion. In either case, $u' \approx 0$ near $f = 0$, so the leading




(a)




(b)

Figure 3.4: (a): Extremal surfaces for $f_0 = 0.9$. The values of $z$ are (from top to bottom) $z = 9/8, 5/4, 3/2$. (b): Extremal surface for $f_0 = 0.9$ and $z = 9/8$ in the region close to $f = 0$. The points are data obtained by numerical integration from $f = 0.9$, while the solid line is the fit with the asymptotic expansion (3.5.36).

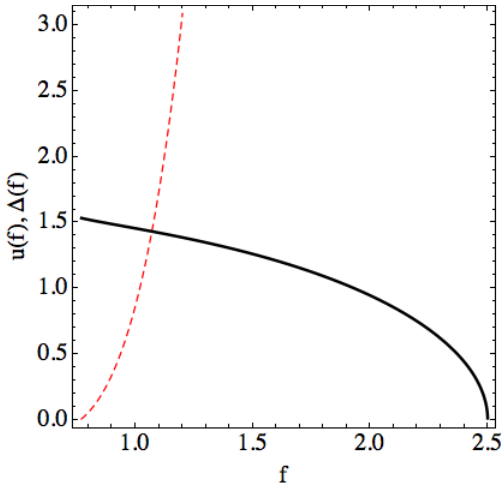



Figure 3.5: For $f_0 = 2.5$ and $z = 3/2$, we plot part of the extremal surface (solid line) and $\Delta(f)$ in (3.3.27) (dashed line). The vanishing of $\Delta$ prevents us from continuing with the shooting from $f = f_0$.

near-boundary contribution to the action is simply

$$S \approx \frac{z}{2\pi\alpha'} \int \frac{d\sigma}{\sigma} \int \frac{df}{f^{z+1}}. \tag{3.5.39}$$

We want to impose a cutoff $\epsilon$, such that $\Delta x = \epsilon$, $\Delta t = \epsilon^z$. The $\sigma$ integral will thus have a lower bound $\epsilon$, while $u - u_0$ is bounded by $\frac{\epsilon}{\sigma}$, implying

$$f_{min} = \left[\frac{2z^3(2-z)}{(z-1)u_0}\frac{\epsilon}{\sigma}\right]^{1/2z} \tag{3.5.40}$$

for $z < 2$, and

$$f_{min} = \left[\frac{1}{b}\frac{\epsilon}{\sigma}\right]^{1/(z+2)} \tag{3.5.41}$$

for $z > 2$. Thus, for $z < 2$,

$$S \sim \frac{\sqrt{z-1}}{\sqrt{\epsilon}}, \tag{3.5.42}$$

while for $z > 2$,

$$S \sim \frac{1}{\epsilon^{\frac{z}{z+2}}}. \tag{3.5.43}$$

As in the timelike AdS case, this divergence is coming from the integral over $x$ at all $t$, and there is no additional divergence from the corner contribution at small $\sigma$.

We note that the divergence here is stronger than for the timelike surfaces in AdS, but with a coefficient which goes to zero in the limit as $z \to 1$, which is consistent with these minimal surfaces reducing to the ones in AdS in this limit. The mushroom feature in the shape of the surface also goes away in this limit, as can be seen from the expansion (3.5.36). As remarked in the introduction, this stronger divergence in the Lifshitz case can be attributed to the presence of the higher density of soft modes implied by the modified dispersion relation $\omega \sim k^z$. It is interesting that this produces a divergence with a power that is independent of $z$ for $z < 2$; this is consistent with the behaviour of the curvature singularity in the bulk for geodesic probes.

## 3.A Asymptotic expansion for Lifshitz minimal surfaces

In section 3.5 we presented the leading terms in the asymptotic series around $f = 0$. Here we discuss the full series and its behaviour as $z \to 1$. The general form of solution expansion is conjectured to be

$$u(f) = u_0 + \sum_{i,j=1}^{\infty} A_{ij} f^{a_{ij}} + \sum_{m,n=1}^{\infty} B_{mn} f^{b_{mn}}, \tag{3.1.44}$$

where the possible powers appearing in the expansion are

$$a_{ij} = (2z-2)i + 2 + 2z(j-1) \tag{3.1.45}$$

and

$$b_{mn} = (2z-2)m + 4 - z + 2z(n-1), \tag{3.1.46}$$

and the first few coefficients are

$$A_{11} = \frac{(z-1)u_0}{2z^3(2-z)}; \qquad B_{11} = b \tag{3.1.47}$$

$$A_{21} = \frac{(z-1)u_0^3}{2z^4(2z-1)(3z-4)(z-2)}; \qquad B_{21} = -\frac{bu_0^2(2+z)}{6z^3} \tag{3.1.48}$$

$$A_{31} = \frac{(z-1)(2z-3)u_0^5}{2z^6(z-2)(5z-6)(3z-4)(3z-2)} \tag{3.1.49}$$

$$A_{k1} = \frac{u_0^{2k-1}(z-1)\prod_{\alpha=3}^{k}[(2\alpha-4)z-(2\alpha-3)]}{2z^{2k}[kz-(k-1)]\prod_{\beta=1}^{k}[(2\beta-1)z-2\beta]} \qquad (k \geq 3) \tag{3.1.50}$$

$$A_{12} = \frac{(z-1)(2z-1)(3z-1)u_0}{8z^6(3z-2)(2-z)} \tag{3.1.51}$$

We note that this series expansion is valid for general values of $z$; there are special rational values where some of the denominators in these expressions for the coefficients vanish. At these values, two of the powers in (3.1.44), which are in general distinct, are coinciding. There will thus be log terms in the series expansion for these values. We do not consider them further here.

Now consider the limit as $z \to 1$. This limit is clearly very special for the above series expansion, as all the terms in the summations over $i$ and $m$ will have the same power of $f$ in the limit. We are particularly interested in the leading term at $j = 1$, which would give a leading $f^2$ behaviour for the asymptotic series expansion as $z \to 1$. From (3.1.50), we see that each of these terms vanishes individually as we take the limit, but there are infinitely many of them, so it is not clear what the behaviour of the sum is in this limit.

Comparison to the asymptotic series (3.4.30) in the AdS case would lead us to expect that the coefficient of the $f^2$ term will vanish as we take the limit for our minimal surface solutions with $u_0 < 1$, and that is consistent with our numerical results, but here we want to consider if there is some other way to take the limit that could converge to the solution $u = \sqrt{1 - f^2/2}$ at $z = 1$, which does have a non-trivial $f^2$ part in its asymptotic expansion.

We therefore consider the limit of (3.1.44) assuming $u_0 \to 1$ as $z \to 1$. Let us write

$$u_0 = 1 + q\epsilon + \mathcal{O}(\epsilon^2), \tag{3.1.52}$$

and $z = 1 + \epsilon$. We want to calculate

$$A_1 = \sum_{k=1}^{\infty} A_{k1} = \frac{(z-1)u_0}{2z^3(2-z)} + \frac{(z-1)u_0^3}{2z^4(2z-1)(3z-4)(z-2)} \tag{3.1.53}$$

$$+ \sum_{k=3}^{\infty} \frac{u_0^{2k-1}(z-1)\prod_{\alpha=3}^{k}[(2\alpha-4)z-(2\alpha-3)]}{2z^{2k}[kz-(k-1)]\prod_{\beta=1}^{k}[(2\beta-1)z-2\beta]} \tag{3.1.54}$$

$$= \frac{\epsilon u_0}{2(1+\epsilon)^3(1-\epsilon)} + \frac{\epsilon u_0^3}{2(1+\epsilon)^4(1+2\epsilon)(1-3\epsilon)(1-\epsilon)} \tag{3.1.55}$$

$$+ \sum_{k=3}^{\infty} \frac{\epsilon u_0^{2k-1}\prod_{\alpha=3}^{k}[1-(2\alpha-4)\epsilon]}{2(1+\epsilon)^{2k}(1+k\epsilon)\prod_{\beta=1}^{k}[1-(2\beta-1)\epsilon]} \tag{3.1.56}$$

One can find that

$$A_1 = \sum_{k=1}^{\infty} \frac{\epsilon}{2}(1-2\epsilon)u_0^{2k-1} \quad (3.1.57)$$

$$= \frac{\epsilon}{2}(1-2\epsilon)\frac{u_0}{1-u_0^2} = -\frac{1}{4q} + \mathcal{O}(\epsilon), \quad (3.1.58)$$

where we used (3.1.52) in the last step. Thus, for $u_0 \to 1$, the sum of the series can give a non-zero answer. Note that for $u_0 \neq 1$ the sum is zero, consistent with the expansion (3.4.30) in the AdS case for $u_0 \neq 1$. To obtain the $u = \sqrt{1 - f^2/2}$ solution in the limit, we would need $q = 1$.

Thus, if there were solutions with $u_0 \to 1$ from above in the limit, they could be smoothly connected to the usual AdS minimal surface for the lightlike Wilson loop. However, numerically we have only found solutions for minimal surfaces in Lifshitz with $u_0 < 1$. In the $z \to 1$ limit these converge to the timelike AdS surfaces discussed in section 3.4.

## 3.B Asymptotic expansion at $z = 2$

If $z = 2$, two branches of solution in (3.5.36) coincide, resulting in a log term. We are not surprised by this since similar phenomenon happened in $z = 2$ holographic Lifshitz renormalization theory [81].

The series solution to equation of motion is

$$u(f) = u_0 - \frac{u_0}{16} f^4 \ln f + \frac{u_0^3}{192} f^6 \ln f + ... + bf^4 - \frac{17u_0^3 + 192u_0^2 b}{2304} f^6 + ... \quad (3.2.59)$$

# Chapter 4

# Connection/ metric description of higher spin non-AdS solutions

This chapter is based on paper [3], written with Simon Ross.

## 4.1 Introduction

There has recently been considerable interest in higher spin gravity, particularly in the context of holography [21, 24, 82]. As in Einstein gravity, the three-dimensional case is particularly simple, and provides a useful laboratory for exploring the issues. The higher spin theory in three dimensions is simply a Chern-Simons theory: in general it is based on the infinite-dimensional $hs(\lambda) \times hs(\lambda)$ gauge group, but for integer values of $\lambda$ it reduces to the finite-dimensional $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ [28, 83–86]. From the Chern-Simons perspective it is evident that this theory has no local degrees of freedom. This includes the case of pure gravity for $N = 2$. In this case it is well-known that the Chern-Simons theory corresponds to a first-order description of pure gravity with a negative cosmological constant, with the spacetime vielbein being obtained as $e_\mu = \dfrac{1}{2}(A_\mu - \bar{A}_\mu)$, where $A, \bar{A}$ are the two $SL(2,\mathbb{R})$ Chern-Simons fields [25, 87]. Similarly the theory for integer $N$ corresponds to a theory of Einstein gravity coupled to massless fields of spin up to $N$, which are all constructed from the "zuvielbein" $e_\mu = \frac{1}{2}(A_\mu - \bar{A}_\mu)$, which is now an $SL(N,\mathbb{R})$ valued one-form.

For any $N$, the solutions of the Chern-Simons theory include all the solutions of the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ theory, so pure gravity solutions are also solutions of the higher spin theories. This includes asymptotically AdS$_3$ solutions, and the higher spin theory with asymptotically AdS$_3$ boundary conditions is conjectured to be dual to a 1+1 CFT with $W_N$ symmetry [86]. But the higher-spin theory is richer, and can include solutions which are not solutions of vacuum gravity. Our discussion will focus on the realisation of spacetimes with non-relativistic symmetries, the Lifshitz spacetime [30] and the Schrödinger spacetime [33, 34].

These are of interest as potential holographic duals of field theories with non-relativistic symmetries. It would be particularly interesting to realise these as solutions of the higher-spin theories, as the large symmetry algebra may make it easier to explicitly identify the dual field theory. In addition, the IR tidal force singularities discussed in the previous two chapters (for $z \neq 1$ in the Lifshitz case [30, 44, 49] and for $1 < z < 2$ in the Schrödinger case [35]) make their interpretations doubtful in a conventional metric theory. But in a higher-spin theory, the diffeomorphism symmetry is enhanced, and these singularities could possibly be just gauge artifacts, as in [88–91].

Solutions of the higher-spin theory which give metrics of this form were obtained in [92], as we will review in section 4.2. As a simple example, a $z = 2$ Lifshitz solution can be obtained in $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory by taking the gauge connections to be

$$A = L_0 d\rho + W_2 e^{2\rho} dt + L_1 e^{\rho} dx, \quad \bar{A} = -L_0 d\rho + W_{-2} e^{2\rho} dt + L_{-1} e^{\rho} dx, \tag{4.1.1}$$

which solves the Chern-Simons equations of motion $F = \bar{F} = 0$. Defining the spacetime metric as

$$g_{\mu\nu} = \frac{1}{2}\mathrm{tr}(e_\mu e_\nu) \tag{4.1.2}$$

reproduces the metric (1.2.8), with $r = e^\rho$. In the metric language, one would expect this solution to be supported by the spin-3 field

$$\phi_{\mu\nu\lambda} = \frac{1}{6}\mathrm{tr}(e_\mu e_\nu e_\lambda). \tag{4.1.3}$$

In [93], it was found that the spin-3 field has a non-zero $\phi_{txx}$ component. It is interesting to note that this breaks time reversal symmetry, so the Lifshitz solution would have to be holographically dual to some field theory with a vacuum which is not invariant under time reversal.

But as we will discuss in section 4.2, we can choose flat connections such that the metric takes the Lifshitz form (1.2.8) but the spin-3 field identically vanishes. This is in conflict with the equations of motion in the metric formulation, as the Lifshitz metric is not a solution of the vacuum theory, and the stress tensor is constructed from terms quadratic and higher order in the spin-3 field $\phi_{\mu\nu\rho}$. It also suggests that the breaking of time-reversal symmetry is not essential to the Lifshitz solutions.

In section 4.3, we will argue that the solution of this puzzle is that the relation between the Chern-Simons and metric formulations fails for the solution (4.1.1). In the pure gravity case $N = 2$, it is well-known that there are solutions of the Chern-Simons theory which do not correspond to regular solutions in the metric description: the vielbein $e = \frac{1}{2}(A - \bar{A})$ may fail to be invertible, implying that the metric is degenerate. These are singular configurations in Chern-Simons theory [94]. The relation between the Chern-Simons and metric formulations for the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ Chern-Simons theory was studied in [95–97]. In particular, [96, 97] give a generalization of the non-degeneracy condition for the vielbein. We will see that this condition is not satisfied for the Chern-Simons fields (4.1.1). Thus, we do not have access to a metric-like formulation for this case. The cases which give a Schrödinger metric involve $N > 3$, so we need to analyse the equivalence between Chern-Simons and metric formulations from first principles; we will find that the $z = 2$ Schrödinger solutions are non-degenerate but the $1 < z < 2$ solutions are degenerate. We will also comment in passing that the realisations of AdS via non-principal embeddings [98] also have a degenerate frame.

One might hope that this is basically a technical issue and that one could still use these solutions to explore non-relativistic holography in a Chern-Simons language: the connections (4.1.1) are solutions of the flatness conditions, and they manifestly exhibit a non-relativistic scaling. However, as we will discuss in section 4.4, the set of gauge transformations that leaves (4.1.1) invariant is a global $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ subgroup of the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ gauge group, just as in the AdS case. This is because the solutions have no holonomies, so they can be related to $A = \bar{A} = 0$

globally by a single-valued gauge transformation. As a result, the symmetry group is the same as that of $A = \bar{A} = 0$. This provides a general understanding of a fact which was uncovered as something of a surprise in the analysis of asymptotically Lifshitz solutions in [93].

If we could legitimately pass to a metric formulation, this could be separated into the Lifshitz isometries of the metric (1.2.8) and some higher-spin gauge transformations, but in the Chern-Simons language there is nothing to pick out the Lifshitz subgroup of $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ as special. Thus, purely in the Chern-Simons formulation, it is not clear how we identify these backgrounds as non-relativistic, in the sense that their field theory duals would have a non-relativistic symmetry. This is consistent with the results of [93], which concluded that the dual of the Lifshitz cases is a field theory with $W_N$ symmetry, just as in the AdS case.

For the Lifshitz case, asymptotically Lifshitz boundary conditions based on the solution (4.1.1) have been described in [93, 99–102]. In section 4.5, we comment on the extension of our analysis to asymptotically Lifshitz solutions, and argue that the boundary conditions of [93] could be re-interpreted as a novel kind of asymptotically AdS boundary conditions. Finally, we conclude in section 6 with a discussion of the significance of the degeneracy we find and prospects for further work.

## 4.2 Non-relativistic solutions in the higher spin theory

The $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ Chern-Simons theory has action

$$S = S_{CS}[A] - S_{CS}[\bar{A}], \tag{4.2.4}$$

where the Chern-Simons action is (1.2.4). The equations of motion are the flatness conditions

$$F = dA + A \wedge A = 0; \qquad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = 0. \tag{4.2.5}$$

The theory is invariant under $SL(N,\mathbb{R})$ gauge transformations

$$A \to A' = g^{-1}Ag + g^{-1}dg, \tag{4.2.6}$$

and similarly for the barred sector. Since the connection is flat on-shell, it is locally gauge-equivalent to $A = 0$, that is in open regions we can write $A = g^{-1}dg$ for some $g$. If the gauge field has holonomies they form an obstruction to writing $A$ as pure gauge globally.

We will write solutions in the "radial gauge" , where we choose a radial coordinate $\rho$ and write

$$A = b^{-1}ab + b^{-1}db, \qquad \bar{A} = b\bar{a}b^{-1} + bdb^{-1} \tag{4.2.7}$$

where $b = e^{\rho L_0}$, and $a$ is a one-form with no $d\rho$ component, which is furthermore independent of $\rho$, and a similar form is taken for the barred sector.

This theory can be related to a higher spin gravitational theory by introducing the "zuvielbein" and spin connection

$$e_\mu = \frac{l}{2}(A_\mu - \bar{A}_\mu), \quad \omega_\mu = \frac{1}{2}(A_\mu + \bar{A}_\mu), \tag{4.2.8}$$

where we introduce an arbitrary length scale $l$ in defining the zuvielbein. The equations of motion then become in terms of these variables

$$de + e \wedge \omega + \omega \wedge e = 0, \tag{4.2.9}$$

$$d\omega + \omega \wedge \omega + \frac{1}{l^2} e \wedge e = 0. \tag{4.2.10}$$

In the $N = 2$ case, writing $e_\mu = e^a_\mu t_a$, $e^a_\mu$ is a $3 \times 3$ matrix which we can interpret as the gravitational vielbein, and these are the equations of motion of pure gravity in a frame field formalism [25, 87], with Newton constant $G_N = l/16k$. For $N > 2$, $e_\mu$ is an $SL(N, \mathbb{R})$ valued one-form, with $3(N^2 - 1)$ independent components, and it can be traded for a metric and higher-spin fields up to spin $N$. For example, for $N = 3$ [28], we have a metric defined by

$$g_{\mu\nu} = \frac{1}{2}\mathrm{tr}(e_\mu e_\nu) \tag{4.2.11}$$

and the spin-3 field

$$\phi_{\mu\nu\lambda} = \frac{1}{6}\mathrm{tr}(e_\mu e_\nu e_\lambda). \tag{4.2.12}$$

Henceforth we will take units with $l = 1$. Above map between frame fields and metric-like fields is supposed to be invertible. In $N = 3$ case, there are $3 \times 8 = 24$

independent components in veilbein. $g_{\mu\nu}$ has 6 independent components while spin-3 field $\phi_{\mu\nu\rho}$ has 10 independent components. The 8 extra independent components in frames are fixed by requiring metric-like fields invariant under local Lorentz transformation.

A simple class of solutions of this theory is constructed by taking the principal embedding $SL(2,\mathbb{R}) \subset SL(N,\mathbb{R})$ and considering flat $SL(2,\mathbb{R})$ connections, corresponding to vacuum gravity solutions. The global AdS$_3$ solution in Poincare coordinates is obtained by taking

$$a = L_1 dx^+, \quad \bar{a} = L_{-1} dx^-, \tag{4.2.13}$$

where $L_0, L_{\pm 1}$ are the usual $SL(2,\mathbb{R})$ generators. Our conventions are set out in appendix 4.A. In the metric description $x^\pm$ become null coordinates on the surfaces of constant $\rho$.

We are interested in the non-AdS solutions constructed in [92], in particular the Lifshitz and Schrödinger solutions. There it was found that one can construct a Lifshitz solution with integer $z$ by taking

$$a = a_1 W_+ dt + L_1 dx, \quad \bar{a} = W_- dt + a_2 L_{-1} dx \tag{4.2.14}$$

where $W_\pm$ are required to satisfy

$$[W_\pm, L_0] = \pm z W_\pm, \quad [W_\pm, L_{\pm 1}] = 0, \quad \mathrm{tr}(W_+ W_-) \neq 0, \tag{4.2.15}$$

and $a_1, a_2$ are normalization factors. For example, by taking $W_\pm = W_{\pm 2}$ in $SL(3,\mathbb{R})$ we can realise Lifshitz with $z = 2$; this produces the solution in (4.1.1).

A Schrödinger solution with integer $z$ is obtained by taking

$$a = (a_1 L_1 + a_2 W_+) dt, \quad \bar{a} = W_- dt + L_{-1} dx^-. \tag{4.2.16}$$

With the same condition on $W_\pm$, and appropriate choices of $a_1, a_2$, this gives the metric (1.2.10), with $r = e^\rho$. We will focus on the realisation of $z = 2$ Schrödinger in $SL(3,\mathbb{R})$ as an example of this class of solutions. Schrödinger solutions with fractional weights are obtained by taking

$$a = (a_1 W_+^{[1]} + a_2 W_+^{[2]}) dt, \quad \bar{a} = W_-^{[2]} dt + W_-^{[1]} dx^-, \tag{4.2.17}$$

where

$$[W_\pm^{[i]}, L_0] = \pm h^{[i]} W_\pm^{[i]}, \quad [W_-^{[1]}, W_-^{[2]}] = 0, \quad tr(W_+^{[i]} W_-^{[j]}) = t_i \delta_{ij}, t_i \neq 0. \tag{4.2.18}$$

We will take the case with $z = 3/2$ in $SL(4,\mathbb{R})$ as an example of this class of solutions, where

$$a = (U_3 + W_2)dt; \qquad \bar{a} = -\frac{5}{72} U_{-3} dt + \frac{5}{24} W_{-2} dx^- \tag{4.2.19}$$

The corresponding metric is

$$ds^2 = \frac{5}{8}\left(-r^3 dt^2 - 2r^2 dt dx^- + \frac{dr^2}{r^2}\right) \tag{4.2.20}$$

after replacing $r = e^{2\rho}$.

In addition to these non-relativistic cases, we will also comment on the non-principal embeddings of AdS: for example, in $SL(3,\mathbb{R})$ we can realize AdS by taking [98]

$$a = W_2 dx^+, \quad \bar{a} = W_{-2} dx^-. \tag{4.2.21}$$

### 4.2.1 A puzzle

In the above solutions, we introduced some normalization constants to cancel trace factors to make the metric take the usual form with no additional numerical factors. These can be thought of as a suitable scaling of the boundary coordinates ($t, x$ or $t, \xi$ respectively). But we could go further: for example, in the $z = 2$ Lifshitz case we could take

$$a = a_1 W_2 dt + b_2 L_1 dx, \quad \bar{a} = b_1 W_{-2} dt + a_2 L_{-1} dx. \tag{4.2.22}$$

This is still a flat connection for any values of the constants. The metric is

$$ds^2 = -a_1 b_1 e^{4\rho} dt^2 + d\rho^2 + a_2 b_2 e^{2\rho} dx^2. \tag{4.2.23}$$

We can re-absorb the constants here in redefinitions of the coordinates. But the change in the spin-3 field is more significant: the only non-vanishing component is

$$\phi_{txx} = -\frac{1}{4}(b_1 b_2^2 - a_1 a_2^2) e^{4\rho}. \tag{4.2.24}$$

(Note that our conventions for the generators are different from [93], as set out in appendix 4.A.) In [93], this term was interpreted as supporting the Lifshitz spacetime. It was also noted that it breaks time reversal symmetry. However, if we choose $b_1 b_2^2 = a_1 a_2^2$, we set the three-form field to zero. How can we have a Lifshitz metric with no matter field to support it? Note that we can keep the metric fixed and change the value of the three-form field by varying the constants appropriately, so we expect that the metric equations of motion fail to be satisfied for generic values of the parameters; there might at best be some special choice of $a_1, a_2, b_1, b_2$ such that the resulting $\phi$ correctly sources the metric.

## 4.3 Degeneracy of the non-relativistic solutions

The puzzle noted above suggests that there is a problem in the relation between the Chern-Simons and metric descriptions in the Lifshitz solution. In this section we will see that there is indeed a problem for Lifshitz, some of the Schrödinger solutions, and AdS with non-principal embeddings.

The issue is one that was already noted in the pure gravity case in [25]: the Chern-Simons description includes solutions, such as for example $A = \bar{A}$, for which the vielbein $e^a_\mu$ is degenerate, and hence not invertible. For pure gravity, such solutions are not acceptable solutions in the metric formulation. In addition, it is not possible to determine the spin connection in terms of the vielbein, because the vielbein is not invertible. It is this latter issue which will generalize to our case. Clearly the problem for the Lifshitz solutions is not that the metric is not invertible. But in the higher spin context, even when the metric is invertible the zuvielbein $e^a_\mu$ can fail to determine the connection $\omega^a_\mu$.

In general, the issue is that to convert from a frame formulation of the equations to a second-order metric formulation, we want to solve the torsion-free condition (4.2.9) to determine the spin connection $\omega$ in terms of the zuvielbein $e$. The spin connection is an $SL(N, \mathbb{R})$ valued one-form, so it has $3(N^2 - 1)$ independent components. The equation is an $SL(N, \mathbb{R})$ valued two-form, so it also has $3(N^2 - 1)$ independent components. This is a linear algebraic system for the components of

$\omega$, so generically it has a unique solution, and knowing $e$ is sufficient to determine $\omega$. In passing to the metric formulation, we exchange the information in $e$ for the metric and higher-spin fields, as in (4.2.11, 4.2.12), and this data is then equivalent to the connections $A$, $\bar{A}$.

But there can be special values of $e$ such that the solution of (4.2.9) is not unique. (If we obtain $e = A - \bar{A}$ as the difference of two flat connections, then $\omega = A + \bar{A}$ is always a solution of (4.2.9), so it can't happen that there's no solution.) The metric formulation, where we retain only the data in $e$, is then not equivalent to the Chern-Simons formulation. The two pictures are equivalent only when we can solve (4.2.9) for $\omega$ uniquely.

In the $N = 2$ case, we can solve (4.2.9) explicitly by multiplying it by the inverse frame field, so the uniqueness of solutions is equivalent to the invertibility of $e^a_\mu$. For $N > 2$, $e^a_\mu$ is not a square matrix, so we cannot express the problem in terms of its invertibility. In [96, 97], this was addressed for $N = 3$ by introducing additional auxiliary quantities $e^a_{\mu\nu}$ constructed out of $e^a_\mu$ such that the collection $e^a_\mu, e^a_{\mu\nu}$ forms a square matrix, and (4.2.9) was again explicitly solved using the matrix inverse.

These additional quantities are constructed by first defining the symmetric tensor

$$\hat{e}_{\mu\nu} = \frac{1}{2}\{e_\mu, e_\nu\} - \frac{2}{3} g_{\mu\nu} I_3 \tag{4.3.25}$$

where $I_3$ is the identity matrix, which is added to ensure traceless of $\hat{e}$ as a group element. Then we define the traceless tensor

$$e_{(\mu\nu)} = \hat{e}_{\mu\nu} - \frac{1}{3} g_{\mu\nu}\hat{\rho}; \qquad \hat{\rho} = g^{\lambda\beta}\hat{e}_{\lambda\beta} \tag{4.3.26}$$

There are five independent components of $e_{(\mu\nu)}$. Thus the combination $(e^a_\mu, e^a_{(\mu\nu)})$ can be treated as a square matrix. In [96], it is shown that invertibility of this matrix is necessary and sufficient for $\omega$ to be uniquely determined by $e$. For the AdS realisation in (4.2.13), [96] show that this matrix is indeed invertible.

Thus, for the $SL(3,\mathbb{R})$ cases, checking degeneracy reduces to checking the invertibility of this matrix. For the Lifshitz $z = 2$ case, the matrix is not invertible, as

$$e_{t\rho} = \hat{e}_{t\rho} = \frac{1}{2}\{e_t, e_\rho\} = \frac{1}{2} e^{2\rho}\{a_1 W_2 - b_1 W_{-2}, L_0\} = 0, \tag{4.3.27}$$

so the matrix has a row of zeros. This explains why the metric-like fields we obtained in (4.2.23,4.2.24) don't solve the equations of motion in the metric formulation: from the Chern-Simons point of view there's a higher-spin component in $\omega$ which is not determined by $g, \phi$ which plays a role in satisfying the flatness conditions. The general solution of the torsion-free condition (4.2.9) in this case is

$$\omega = \frac{1}{2}(A + \bar{A}) + \lambda_1[-e^{\rho} L_0 dt + (W_1 + W_{-1})dx + \frac{1}{2}e^{-\rho}(-W_2 + W_{-2})d\rho] + \lambda_2 W_0 dx \tag{4.3.28}$$

where the $\lambda_i$ are arbitrary constants parametrising the non-uniqueness of the solution.

For the $z = 2$ Schrödinger solution (4.2.16), by contrast, the matrix is invertible, so the Chern-Simons and metric formulations are equivalent. The explicit calculation is given in appendix 4.B.1; the determinant is

$$\det(e^a_\mu, e^a_{(\mu\nu)}) = -\frac{1}{32}e^{10\rho} \tag{4.3.29}$$

which is non-zero for finite $\rho$. We can also check that the equations of motion in the metric formulation are satisfied by the $z = 2$ Schrödinger fields $g, \phi$; this is discussed in appendix 4.B.2.

For the AdS solution in the non-principal embedding (4.2.21), the matrix is again not invertible. It is not hard to show $e_{++} = e_{--} = 0$. Therefore, we again have zero rows leading to vanishing determinant. The general solution for the connection $\omega$ in this case is

$$\omega = \frac{1}{2}(A + \bar{A}) + W_0\Theta \tag{4.3.30}$$

where $\Theta$ is an undetermined one-form.

Finally, we would like to consider the non-integer Schrödinger solutions. To do so we need to go to $N > 3$, so we cannot use the description from [96]. But for a given $e$, it is a simple linear algebra problem to check if (4.2.9) has a unique solution for $\omega$ or not. In the case of the $z = \frac{3}{2}$ Schrödinger solution in (4.2.19), we find that it does not have a unique solution. The general solution for the connection $\omega$ in this case is

$$\omega = \frac{1}{2}(A + \bar{A}) + \hat{\omega}, \tag{4.3.31}$$

where the extra term $\hat{\omega}$ written in components is

$$\begin{aligned}
\hat{\omega}_t &= -\frac{25}{768}\lambda_1 e^{4\rho} W_{-2} + \lambda_2 L_0 - \frac{5}{8}\lambda_1 U_3 + \frac{10}{3}\lambda_2 U_0 - \frac{25}{144}\lambda_3 U_{-2} - \frac{25}{432}\lambda_1 U_{-3} \\
\hat{\omega}_x &= \frac{5}{32}\lambda_1 e^{3\rho} - \frac{25}{64}\lambda_1 e^{3\rho} U_{-1} \\
\hat{\omega}_\rho &= -\frac{5}{8}\lambda_1 e^{\rho} W_1 + \lambda_3 L_1 + \lambda_1 L_0 + \frac{5}{3}\lambda_3 U_1 + \frac{10}{3}\lambda_1 U_0
\end{aligned} \tag{4.3.32}$$

The constants $\lambda_i$ again parametrise the non-uniqueness of the solution.

## 4.4 Symmetries of the Chern-Simons solutions

In the previous section, we found that the Lifshitz solution (4.2.14) and the fractional $z$ Schrodinger solution (4.2.19) do not have a metric formulation, as the connection $\omega$ is not determined uniquely by $e$. Can we formulate a duality relating them to non-relativistic theories directly in the Chern-Simons formulation? In this section we will argue that this is challenging because the Chern-Simons formulation does not associate a distinguished set of non-relativistic symmetries with these backgrounds.

Originally, the Lifshitz and Schrödinger metrics (1.2.8) and (1.2.10) were constructed to have the corresponding symmetries as isometry groups. In the higher-spin context, these diffeomorphism isometries are supplemented by some higher-spin gauge transformations that also leave the background invariant, but one could argue that in the metric formulation we can draw a distinction between diffeomorphisms and the higher-spin gauge transformations and still regard the backgrounds as having a non-relativistic symmetry. But in the Chern-Simons formulation, it is not clear how to make such a distinction. All of the symmetries are simply gauge transformations that leave the given flat connection unchanged.

In the discussion of asymptotically Lifshitz solutions in [93], it was found that the higher-spin gauge transformations extend the Lifshitz symmetry of (1.2.8) to a global $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ symmetry group. In fact, there is a simple argument to see that the same happens in all cases. The symmetries are the gauge transformations $\epsilon$ such that

$$\delta_\epsilon A = d\epsilon + [A, \epsilon] = 0, \tag{4.4.33}$$

and similarly in the barred sector. The Lifshitz and Schrödinger metrics (1.2.8) and (1.2.10) are analogous to AdS in Poincare coordinates, so the boundary coordinates are non-compact, and cannot be compactified without eliminating the anisotropic scaling symmetry (with the exception of the Schrödinger $z = 2$ case, where we can compactify $\xi$). Thus, in the Chern-Simons formulation there can be no non-trivial holonomies, as there are no non-trivial topological cycles in the spacetime to measure holonomies around. As a result, the connection is globally gauge-equivalent to zero, that is each of our solutions is of the form $A = g^{-1}dg$, $\bar{A} = \bar{g}^{-1}d\bar{g}$ for some globally defined group elements $g$, $\bar{g}$. Now if we use $A = g^{-1}dg$, and set $\epsilon = g^{-1}\epsilon' g$, (4.4.33) reduces to

$$d\epsilon' = 0 \tag{4.4.34}$$

which is satisfied by arbitrary constant $\epsilon'$, forming a global $SL(N,\mathbb{R})$ subgroup of the gauge group. Thus the $\epsilon$ that leave $A$ invariant will always form a global $SL(N,\mathbb{R})$ group (although for a given $A$, the gauge transformations $\epsilon = g^{-1}\epsilon' g$ are not themselves constants). Thus, the symmetry of any Chern-Simons solution with no holonomies is always $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$.

Explicitly, for the $z = 2$ Lifshitz solution, $d\epsilon' = 0$ can be solved by writing

$$\epsilon' = \sum_{i=-1}^{1} \epsilon^{L_i} L_i + \sum_{i=-2}^{2} \epsilon^{W_i} W_i \tag{4.4.35}$$

where $\epsilon^{L_i}$ and $\epsilon^{W_i}$ are constants. The relevant group element $g$ such that $A = g^{-1}dg$ gives the Chern-Simons field in (4.1.1) is $g = e^{W_2 t + L_1 x} e^{\rho L_0}$. Thus the symmetries $\epsilon = g^{-1}\epsilon' g$ are

$$\begin{aligned}
\epsilon \;=\;& e^{\rho}(-x\epsilon^{L_0} + \epsilon^{L_1} + x^2\epsilon^{L_{-1}} + t\epsilon^{W_{-1}} - 4tx\epsilon^{W_{-2}})L_1 \\
+\;& (\epsilon^{L_0} - 2x\epsilon^{L_{-1}} + 4t\epsilon^{W_{-2}})L_0 + e^{-\rho}\epsilon^{L_{-1}}L_{-1} \\
-\;& e^{2\rho}(2t\epsilon^{L_0} - 4tx\epsilon^{L_{-1}} - x^2\epsilon^{W_0} + x\epsilon^{W_1} - \epsilon^{W_2} + x^3\epsilon^{W_{-1}} + 4t^2\epsilon^{W_{-2}} - x^4\epsilon^{W_{-2}})W_2 \\
+\;& e^{\rho}(-4t\epsilon^{L_{-1}} - 2x\epsilon^{W_0} + \epsilon^{W_1} + 3x^2\epsilon^{W_{-1}} - 4x^3\epsilon^{W_{-2}})W_1 \\
+\;& (\epsilon^{W_0} - 3x\epsilon^{W_{-1}} + 6x^2\epsilon^{W_{-2}})W_0 + e^{-\rho}(\epsilon^{W_{-1}} - 4x\epsilon^{W_{-2}})W_{-1} + e^{-2\rho}\epsilon^{W_{-2}}W_{-2}
\end{aligned} \tag{4.4.36}$$

reproducing the result of [93]. If we interpreted these symmetries in terms of diffeomorphisms using $\epsilon = -\xi^{\mu}A_{\mu}$, as suggested in [93], $\epsilon^{W_2}$, $\epsilon^{L_1}$, $\epsilon^{L_0}$ parametrize time-translation, spatial translation and Lifshitz scaling respectively, although it is not clear if this is valid given that the frame is degenerate [25].

For the AdS solutions (4.2.13), (4.2.21), the appearance of an $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ symmetry is expected. But for the Lifshitz and Schrödinger solutions it implies that we cannot identify a non-relativistic isometry group from the Chern-Simons perspective. For $z=2$ Schrödinger, we can pass to a metric formulation, and identify the Schrödinger algebra as the subgroup of this $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ which is realised as diffeomorphisms. But for the other cases with no metric formulation there is no clear sense in which they are non-relativistic, despite the manifest scaling properties of (4.1.1); this scaling is only one of a set of $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ symmetries.

A possible subtlety in this argument is that when we take a background and define asymptotic boundary conditions where the fields approach the background asymptotically, the isometries of the background may not form a subgroup of the asymptotic symmetry algebra (see [26] for an example of this). So the non-relativistic symmetry could potentially be picked out by a notion of asymptotically Lifshitz/Schrödinger boundary conditions. But a choice of boundary conditions such that the asymptotic symmetry algebra does not include the symmetries of the background is usually considered undesirable. In particular, this does not happen for the asymptotically Lifshitz solutions of [93], where the full $SL(3,\mathbb{R})\times SL(3,\mathbb{R})$ symmetry is included in the asymptotic symmetry algebra.

### 4.4.1 Map to AdS

One way of thinking about this result is that since all the topologically trivial solutions are gauge-equivalent to $A=\bar{A}=0$, the Lifshitz and Schrödinger solutions can be related to the usual AdS solution by a suitable gauge transformation; so the fact that they have the same symmetries can be seen as a reflection of their just being AdS in a different gauge. Let us give this transformation explicitly in the Lifshitz case. For the AdS solution (4.2.13), $A_{AdS}=g^{-1}dg$ with $g=e^{L_1 x^+}e^{L_0\rho}$, while for the Lifshitz solution (4.2.14), $A_{Lif}=h^{-1}dh$ with $h=e^{W_2 t+L_1 x}e^{\rho L_0}$. Identifying the AdS coordinate $x^+$ with $t+x$ in the Lifshitz solution, the transformation is then

$$A_{Lif}=f^{-1}df+f^{-1}A_{AdS}f, \tag{4.4.37}$$

with

$$f = g^{-1}h = \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{2}e^{\rho}t & 1 & 0 \\ t(t+2)e^{2\rho} & -\sqrt{2}e^{\rho}t & 1 \end{pmatrix}. \tag{4.4.38}$$

A similar argument in the barred sector produces

$$\bar{f} = \begin{pmatrix} 1 & -\sqrt{2}e^{\rho}t & t(t+2)e^{2\rho} \\ 0 & 1 & -\sqrt{2}e^{\rho}t \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.4.39}$$

We have assumed that we work with non-compact $x$, as compactifying it breaks the scaling symmetry, but it is interesting to note that compactifying $x$ does not obstruct this relation.

## 4.5 Asymptotically Lifshitz solutions

So far, we have focused on the non-relativistic backgrounds, and seen that some interesting examples fail to have a corresponding metric description. Holographically, such solutions are dual to the vacuum state in the dual field theory, and it is essential to consider solutions which asymptotically approach these backgrounds to define the holographic dictionary. Since the failure of the metric description is non-generic, one would expect that considering these more generic solutions could also offer a resolution of it. In addition, imposing a given asymptotic boundary conditions partially fixes the gauge in the asymptotic region, eliminating those gauge transformations that take us out of this choice of boundary conditions. Since the bulk theory has no local degrees of freedom, it is these gauge transformations that are broken by the choice of boundary conditions that provide the physical content of the bulk theory - the higher spin analogue of the boundary gravitons.

In this section, we will consider spacetimes which asymptotically approach the Lifshitz background (4.2.14). We will first consider the asymptotically Lifshitz boundary conditions of [93], which are the most well developed, and then consider alternatives. In [93], asymptotically Lifshitz solutions were defined in the radial gauge as Chern-Simons solutions with

$$A = b^{-1}db + b^{-1}(\hat{a}^{(0)} + a^{(0)} + a^{(1)})b, \quad \bar{A} = b^{-1}db + b^{-1}(\hat{\bar{a}}^{(0)} + \bar{a}^{(0)} + \bar{a}^{(1)})b, \tag{4.5.40}$$

where $b = e^{\rho L_0}$, and $\hat{a}^{(0)}$, $\hat{\bar{a}}^{(0)}$ is the background solution (4.2.14). The first fluctuations $a^{(0)}, \bar{a}^{(0)}$ have only an $x$ component. Let's take unbarred sector as example, and expand it in terms of generators

$$a_x^{(0)} = a^{L_0}L_0 + a^{L_1}L_1 + a^{L_{-1}}L_{-1} + a^{W_2}W_2 + a^{W_1}W_1 + a^{W_0}W_0 + a^{W_{-1}}W_{-1} + a^{W_{-2}}W_{-2}$$

Imposing equation of motion, $a_x^{(0)}$ satisfies equations

$$\begin{aligned} \dot{a}^{L_0} - 4a^{W_{-2}} &= 0 \\ a^{W_{-1}} = \dot{a}^{L_{-1}} = \dot{a}^{W_0} = \dot{a}^{W_{-2}} &= 0 \\ \dot{a}^{W_2} + 2a^{L_0} &= 0 \\ \dot{a}^{W_1} + 4a^{L_{-1}} &= 0 \end{aligned} \tag{4.5.41}$$

Similar equations can be written for barred sector. Solution is determined in terms of four functions $\mathcal{L}(x), \bar{\mathcal{L}}(x), \mathcal{W}(x), \bar{\mathcal{W}}(x)$,

$$a_x^{(0)} = 4t\mathcal{W}L_0 - \mathcal{L}L_{-1} - 4t^2\mathcal{W}W_2 + 4t\mathcal{L}W_1 + \mathcal{W}W_{-2}, \tag{4.5.42}$$

$$\bar{a}_x^{(0)} = -4t\bar{\mathcal{W}}L_0 - \bar{\mathcal{L}}L_1 - 4t^2\bar{\mathcal{W}}W_{-2} - 4t\bar{\mathcal{L}}W_{-1} + \bar{\mathcal{W}}W_2 \tag{4.5.43}$$

(the constant coefficients here are different from in [93] because we use a different convention for the $SL(3,\mathbb{R})$ generators, as set out in appendix 4.A). The second subleading terms $a^{(1)}, \bar{a}^{(1)}$ are general, having arbitrary $t$ and $x$ components, but are required to fall off at large $\rho$, $a^{(1)}, \bar{a}^{(1)} \sim o(1)$.

In [93], this definition of the asymptotic boundary condition was shown to lead to finite, conserved canonical charges (constructed from the boundary functions $\mathcal{L}, \bar{\mathcal{L}}, \mathcal{W}, \bar{\mathcal{W}}$ and the gauge transformations preserving the boundary conditions) which generate a $\mathcal{W}_3 \oplus \mathcal{W}_3$ asymptotic symmetry algebra, containing the $SL(3,\mathbb{R}) \times SL(3,\mathbb{R})$ symmetries of the background (4.2.14).

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}'\epsilon_L + 2\mathcal{L}\epsilon_L' + 3\mathcal{W}\epsilon_W' + 2\mathcal{W}'\epsilon_W - \frac{1}{2}\epsilon_L''' \qquad (4.5.44) \\ \delta\mathcal{W} &= 3\mathcal{W}\epsilon_L' + \mathcal{W}'\epsilon_L - \frac{1}{6}\mathcal{L}'''\epsilon_W - \frac{3}{4}\mathcal{L}''\epsilon_W' - \frac{5}{4}\mathcal{L}'\epsilon_W'' - \frac{5}{6}\mathcal{L}\epsilon_W''' \end{aligned}$$

$$+\quad \frac{8}{3}\mathcal{L}^2\epsilon'_W + \frac{8}{3}\mathcal{L}\mathcal{L}'\epsilon_W + \frac{1}{24}\epsilon_W^{(5)} \tag{4.5.45}$$

with similar expressions for the barred sector. We put the detailed derivation in Appendix 4.C.

Because the first subleading terms do not affect the $a_t$ component, the extended vielbein at this order is still degenerate:

$$e_{t\rho} = \frac{1}{2}\{e_t, e_\rho\} \approx 0, \tag{4.5.46}$$

up to terms coming from $a^{(1)}, \bar{a}^{(1)}$. Thus, it would seem that there are solutions with non-zero values of the charges here where the metric formulation is still not possible. For solutions with sufficiently general $a^{(1)}, \bar{a}^{(1)}$, the extended vielbein may be non-degenerate in the bulk, but as these terms vanish as we approach the boundary, we would expect that the inverse vielbeins of [96] will blow up there. Thus, the degeneracy is a real obstacle to the construction of a good metric description for this class of asymptotically Lifshitz boundary conditions.

It was argued in [93] that these boundary conditions are distinct from the usual asymptotically AdS boundary conditions [28]. Two main arguments were given: one relied on the breaking of time-reversal invariance in the Lifshitz solution, but as we have seen it is possible to take the generalised backgrounds in (4.2.22) such that the spin-three field vanishes, eliminating the breaking of time-reversal symmetry. The other was that the asymptotically Lifshitz boundary conditions involve functions of $x$, while asymptotically AdS boundary conditions involve functions of $x^{\pm}$. This indeed shows that asymptotically Lifshitz solutions are distinct from the asymptotically AdS solutions, if we relate the two backgrounds using the gauge transformation (4.4.38).

However, given the failure of the metric description in the gauge (4.2.14), we think it may be more straightforward to understand the physical significance of these boundary conditions if we apply this gauge transformation to re-express them in terms of the AdS solution (4.2.13). That is, let us take the solutions (4.5.42,4.5.43) and apply the gauge transformation (4.4.38). We then obtain a family of solutions of the form (4.5.40), but where now $\hat{a}^{(0)}$, $\hat{\bar{a}}^{(0)}$ are the AdS background (4.2.13), and

$$a_x^{(0)} \quad = \quad -\mathcal{L}t^2L_1 - 2\mathcal{L}tL_0 - \mathcal{L}L_{-1} + \mathcal{W}t^4W_2 + 4\mathcal{W}t^3W_1 \tag{4.5.47}$$

$$
\begin{aligned}
& +6\mathcal{W}t^2W_0+4\mathcal{W}tW_{-1}+\mathcal{W}W_{-2}, \\
\bar{a}_x^{(0)} \;=\; & -\bar{\mathcal{L}}t^2L_{-1}-2\bar{\mathcal{L}}tL_0-\bar{\mathcal{L}}L_1+\bar{\mathcal{W}}t^4W_{-2}+4\bar{\mathcal{W}}t^3W_{-1} \\
& +6\bar{\mathcal{W}}t^2W_0+4\bar{\mathcal{W}}tW_1+\bar{\mathcal{W}}W_2. \qquad (4.5.48)
\end{aligned}
$$

Thus, the asymptotic boundary conditions of [93] can be rewritten in a different gauge as a new kind of asymptotically AdS boundary conditions. Since in this gauge the relation to the metric formulation is possible, the physics of the boundary conditions may be clearer in this gauge. Note the asymptotic symmetry algebra (4.5.44), (4.5.45) is unaffected when we shift from Lifshitz gauge solution to AdS gauge solution.

An alternative asymptotically Lifshitz boundary condition was given in [100]. The connection is taken to have the form

$$a_t = W_2-2\mathcal{L}W_0+\frac{2}{3}\mathcal{L}'W_{-1}-2\mathcal{W}L_{-1}+(\mathcal{L}^2-\frac{1}{6}\mathcal{L}'')W_{-2}, \qquad (4.5.49)$$

$$a_x = L_1-\mathcal{L}L_{-1}+\mathcal{W}W_{-2}, \qquad (4.5.50)$$

where $\mathcal{L}$ and $\mathcal{W}$ are now functions of both $t$ and $x$, subject to the consistency conditions

$$\dot{\mathcal{L}} = 2\mathcal{W}', \qquad (4.5.51)$$

$$\dot{\mathcal{W}} = \frac{4}{3}(\mathcal{L}^2)'-\frac{1}{6}\mathcal{L}'''. \qquad (4.5.52)$$

Similarly, for the barred fields

$$\bar{a}_t = W_{-2}-2\bar{\mathcal{L}}W_0-\frac{2}{3}\bar{\mathcal{L}}'W_1+2\bar{\mathcal{W}}L_1+(\bar{\mathcal{L}}^2-\frac{1}{6}\bar{\mathcal{L}}'')W_2, \qquad (4.5.53)$$

$$\bar{a}_x = L_{-1}-\bar{\mathcal{L}}L_1-\bar{\mathcal{W}}W_2, \qquad (4.5.54)$$

with consistency constraints

$$\dot{\bar{\mathcal{L}}} = -2\bar{\mathcal{W}}', \qquad (4.5.55)$$

$$\dot{\bar{\mathcal{W}}} = -\frac{4}{3}(\bar{\mathcal{L}}^2)'+\frac{1}{6}\bar{\mathcal{L}}'''. \qquad (4.5.56)$$

In these asymptotic boundary conditions, the degeneracy of the generalised frame is resolved for generic $\mathcal{L}$, $\mathcal{W}$. The determinant is

$$-\frac{\mathcal{W}^3}{8r^{14}}(r^2+\mathcal{L})^4[(r^2+\mathcal{L})^3-2\mathcal{W}^2][(r^2+\mathcal{L})(r^2-\mathcal{L})^2-2\mathcal{W}^2]. \qquad (4.5.57)$$

There are some specific points $r$ where the determinant vanishes. These singularities would not spoil the non-degeneracy property and can be avoided by method of fibre bundle [96]. Since the determinant is not vanishing even at large $r$, one would expect a metric formulation is possible even in the asymptotic region. It may be interesting to explore these boundary conditions further; it was noted in [93] that the canonical charges in this case are finite but not conserved.

In [100], there was also a further generalization to turn on some source terms, taking

$$\begin{aligned} a_t &= \mu_2 W_2 + \mu_1 L_1 - 2\mathcal{L}\mu_2 W_0 - (2\mathcal{W}\mu_2 + \mathcal{L}\mu_1)L_{-1} + (\mathcal{L}^2\mu_2 + \mathcal{W}\mu_1)W_{-2}, \\ a_x &= L_1 - \mathcal{L}L_{-1} + \mathcal{W}W_{-2}, \end{aligned} \tag{4.5.58}$$

and barred sector

$$\begin{aligned} \bar{a}_t &= \mu_2 W_{-2} - \mu_1 L_{-1} - 2\bar{\mathcal{L}}\mu_2 W_0 + (2\bar{\mathcal{W}}\mu_2 + \bar{\mathcal{L}}\mu_1)L_1 + (\bar{\mathcal{L}}^2\mu_2 + \bar{\mathcal{W}}\mu_1)W_2, \\ \bar{a}_x &= L_{-1} - \bar{\mathcal{L}}L_1 - \bar{\mathcal{W}}W_2. \end{aligned} \tag{4.5.59}$$

The presence of the sources $\mu_1$, $\mu_2$ makes the determinant of the generalized vielbein non-zero even for vanishing $\mathcal{L}, \mathcal{W}$, so this deformation away from Lifshitz resolves the degeneracy of the generalized vielbein even in the vacuum. The metric formulation is well-defined in this case since metric-like fields solve Einstein equations by the method in appendix 4.B.2. We leave further study of these deformations to future work.

# 4.A Conventions

## 4.A.1 $sl(3,R)$ Algebra

The conventions in two cases are different. The $sl(3,R)$ generators satisfy algebra

$$[L_n, L_m] = (n-m)L_{n+m} \tag{4.1.60}$$

$$[L_n, W_m] = (2n-m)W_{n+m} \tag{4.1.61}$$

$$[W_n, W_m] = \sigma(n-m)(2n^2+2m^2-mn-8)L_{m+n} \tag{4.1.62}$$

In our calculation $\sigma = -\frac{1}{12}$. Our generators are

$$L_{-1} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$W_{-2} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad W_0 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$W_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

### 4.A.2 $sl(4,R)$ algebra

Our representation of $sl(4,R)$ algebra is slightly different from [92].

$$L_{-1} = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

Quintet:

$$W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 2\sqrt{3} & 0 & 0 \end{pmatrix}, \quad W_{-2} = \begin{pmatrix} 0 & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_{-1} = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Septet:

$$U_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & 0 \end{pmatrix}, \quad U_1 = \frac{2}{5}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

$$U_0 = \frac{1}{10}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \quad U_{-1} = \frac{2}{5}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_{-2} = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_{-3} = \begin{pmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

# 4.B Schrödinger higher spin calculations

## 4.B.1 Determinant

We consider the most general form of Schrödinger solution after normalization:

$$a_t = kW_2 + cL_1; \qquad a_{x^-} = 0 \tag{4.2.63}$$

$$\bar{a}_t = \frac{1}{k}W_{-2}; \qquad \bar{a}_{x^-} = \frac{2}{c}L_{-1} \tag{4.2.64}$$

Dreibein $e$ can be found to be

$$e = L_0 d\rho + \frac{1}{2}(ke^{2\rho}W_2 + ce^{\rho}L_1 - \frac{1}{k}e^{2\rho}W_{-2})dt - \frac{1}{c}e^{\rho}L_{-1}dx^- \tag{4.2.65}$$

The extra introduced 5 tetrads are

$$\begin{aligned} e_{(x^-x^-)} &= \frac{1}{c^2}e^{2\rho}W_{-2} && (4.2.66)\\ e_{(\rho\rho)} &= W_0 - \frac{1}{3c^2}e^{2\rho}W_{-2} - \frac{k}{3c}e^{\rho}L_1 && (4.2.67)\\ e_{(tx^-)} &= -\frac{1}{2}e^{2\rho}W_0 + \frac{k}{6c}e^{3\rho}L_1 - \frac{1}{3c^2}e^{4\rho}W_{-2} && (4.2.68) \end{aligned}$$

$$e_{(tt)} = \frac{c^2}{4}e^{2\rho}W_2 - e^{4\rho}W_0 + \frac{1}{3c^2}e^{6\rho}W_{-2} + \frac{k}{3c}e^{5\rho}L_1 + \frac{c}{2k}e^{3\rho}L_{-1} \quad (4.2.69)$$

$$e_{(t\rho)} = \frac{1}{2}ce^{\rho}W_1 \quad (4.2.70)$$

$$e_{(\rho x^-)} = -\frac{1}{c}e^{\rho}W_{-1} \quad (4.2.71)$$

We only need 5 of these tetrad since they are linearly dependent due to the traceless condition $g^{\mu\nu}e_{(\mu\nu)} = 0$. In this specific case,

$$e_{(x^-x^-)} + e_{(\rho\rho)} + 2e^{-2\rho}e_{(tx^-)} = 0$$

Therefore, we calculate the determinant of $8 \times 8$ matrix with spacetime indices excluding $(\rho\rho)$.

$$\det(e^a_\mu, e^a_{(\mu\nu)}) = -\frac{1}{32}e^{10\rho} \quad (4.2.72)$$

We find this nonvanishing value is independent of the choice of $k$ and $c$. Then we should be able to map frame-like Schrödinger solution (4.2.63) (4.2.64) to metric-like fields.

### 4.B.2 Einstein equation in D=3 higher spin theory

We showed that the zuvielbein of $z = 2$ Schrödinger solution in $SL(3, R)$ has non-vanishing determinant. One would then expect the fields constructed from it to solve the equations of motion in the metric formulation. In terms of metric-like fields $g, \phi$, Lagrangian of (4.2.4) can be written as [95]

$$\mathcal{L} = \mathcal{L}_{\text{E-H}} + \mathcal{L}_F, \quad (4.2.73)$$

where $\mathcal{L}_{\text{E-H}} = R + \frac{2}{l^2}$ and $\mathcal{L}_F$ contains terms depending on $\phi$ (note that we set $l = 1$). $\mathcal{L}_F$ was worked out to quadratic order in $\phi$ terms in [95], with general expression:

$$\begin{aligned}\mathcal{L}_F(\phi^2) &= \phi^{\mu\nu\rho}(\mathcal{F}_{\mu\nu\rho} - \frac{3}{2}g_{(\mu\nu}\mathcal{F}_{\rho)}) + m_1\phi_{\mu\nu\rho}\phi^{\mu\nu\rho} + m_2\phi_\mu\phi^\mu \quad (4.2.74)\\ &+ 3R_{\rho\sigma}(k_1\phi^\rho_{\mu\nu}\phi^{\sigma\mu\nu} + k_2\phi^{\rho\sigma}_\mu\phi^\mu + k_3\phi^\rho\phi^\sigma) + 3R(k_4\phi_{\mu\nu\rho}\phi^{\mu\nu\rho} + k_5\phi_\mu\phi^\mu)\end{aligned}$$

where $\phi_\rho = \phi_{\rho\mu}{}^\mu$, $\mathcal{F}_\rho = \mathcal{F}_{\rho\mu}{}^\mu$ and $\mathcal{F}_{\mu\nu\rho}$ is the Fronsdal tensor defined by

$$\mathcal{F}_{\mu\nu\rho} = \nabla^\sigma\nabla_\sigma\phi_{\mu\nu\rho} - \frac{3}{2}(\nabla^\sigma\nabla_{(\mu}\phi_{\nu\rho)\sigma} + \nabla_{(\mu}\nabla^\lambda\phi_{\nu\rho)\lambda}) + 3\nabla_{(\mu}\nabla_\nu\phi_{\rho)} \quad (4.2.75)$$

The mass coefficients $m_i$ are determined by requiring invariance under gauge transformations, which gives

$$m_1 = 6(k_1 + 3k_4 - 1); \qquad m_2 = 6(k_2 + k_3 + 3k_5 + \frac{9}{4}) \tag{4.2.76}$$

Different $k_i$s may parametrize the same theory if one performs a redefinition of metric and spin-3 fields. For convenience, let's take those values of $k_i$ in [95],

$$k_1 = \frac{3}{2}; \quad k_2 = 0; \quad k_3 = -\frac{3}{4}; \quad k_4 = -\frac{1}{2}; \quad k_5 = 0 \tag{4.2.77}$$

The unique choice of $k_i$ were determined by requiring asymptotically AdS solution solving Einstein equation.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_F)}{\delta g^{\mu\nu}} \tag{4.2.78}$$

Exact expression of $T_{\mu\nu}$ is accessible in [103]. We perform the calculation by the help of *xAct* package [104, 105], and the result is

$$\begin{aligned}
T_{\mu\nu} = {} & 3m_1\phi_a{}^{cd}\phi_{bcd} + 9k_4R\phi_a{}^{cd}\phi_{bcd} + \tfrac{3}{2}k_2R^{cd}\phi_a\phi_{bcd} + 6k_1R^{cd}\phi_{ac}{}^e\phi_{bde} + m_2\phi_a\phi_b \\
& + \tfrac{3}{2}k_2R^{cd}\phi_{acd}\phi_b + 3k_2R^{cd}\phi_{ab}{}^e\phi_{cde} + \tfrac{3}{2}R_b{}^c\phi_a{}^{de}\phi_{cde} + 3k_1R_b{}^c\phi_a{}^{de}\phi_{cde} + 6R_{bcdf}\phi_a{}^{cd}\phi^f \\
& + \tfrac{3}{2}R_a{}^c\phi_b{}^{de}\phi_{cde} + 3k_1R_a{}^c\phi_b{}^{de}\phi_{cde} + 2m_2\phi_{ab}{}^c\phi_c + 6k_5R\phi_{ab}{}^c\phi_c + \tfrac{3}{4}R_b{}^c\phi_a\phi_c + 3k_3R_b{}^c\phi_a\phi_c \\
& + \tfrac{3}{4}R_a{}^c\phi_b\phi_c + 3k_3R_a{}^c\phi_b\phi_c - 3R_{bedf}\phi_a{}^{cd}\phi_c{}^{ef} - 3R_{aedf}\phi_b{}^{cd}\phi_c{}^{ef} - \tfrac{1}{2}m_1g_{ab}\phi_{cde}\phi^{cde} \\
& - \tfrac{3}{2}k_4g_{ab}R\phi_{cde}\phi^{cde} + 3g_{ab}R_{dfeg}\phi_c{}^{fg}\phi^{cde} + 3k_1g_{ab}R_{dfeg}\phi_c{}^{fg}\phi^{cde} - \tfrac{1}{2}m_2g_{ab}\phi^d\phi_d \\
& - \tfrac{3}{2}g_{ab}R^{cd}\phi_c{}^{ef}\phi_{def} + \tfrac{9}{2}R^{cd}\phi_{abc}\phi_d + 6k_3R^{cd}\phi_{abc}\phi_d + 3k_2R_b{}^c\phi_{ac}{}^d\phi_d + 3k_2R_a{}^c\phi_{bc}{}^d\phi_d \\
& + 3k_5R_{ab}\phi^d\phi_d - \tfrac{3}{2}k_5g_{ab}R\phi^d\phi_d - \tfrac{3}{4}g_{ab}R^{cd}\phi_c\phi_d - 3k_3g_{ab}R^{cd}\phi_c\phi_d - \tfrac{3}{2}k_2g_{ab}R^{cd}\phi_{cd}{}^e\phi_e \\
& + 6R_{acdf}\phi_b{}^{cd}\phi^f - \nabla_a\phi^{cde}\nabla_b\phi_{cde} - 6k_4\nabla_a\phi^{cde}\nabla_b\phi_{cde} + 3\nabla_a\phi^d\nabla_b\phi_d - 6k_5\nabla_a\phi^d\nabla_b\phi_d \\
& - 6k_4\phi^{cde}\nabla_b\nabla_a\phi_{cde} - 6k_5\phi^d\nabla_b\nabla_a\phi_d - 3\nabla_b\phi_d\nabla_c\phi_a{}^{cd} - \tfrac{3}{2}k_2\nabla_b\phi_d\nabla_c\phi_a{}^{cd} - 3\nabla_a\phi_d\nabla_c\phi_b{}^{cd} \\
& - \tfrac{3}{2}k_2\nabla_a\phi_d\nabla_c\phi_b{}^{cd} - \tfrac{3}{2}k_2\nabla_a\phi_b{}^{cd}\nabla_d\phi_c - \tfrac{3}{2}k_2\nabla_b\phi_a{}^{cd}\nabla_d\phi_c + \tfrac{15}{8}\phi^d\nabla_d\nabla_a\phi_b - \tfrac{3}{2}k_3\phi^d\nabla_d\nabla_a\phi_b \\
& + 3\phi_b{}^{cd}\nabla_d\nabla_a\phi_c - 3k_1g_{ab}R^{cd}\phi_c{}^{ef}\phi_{def} - \tfrac{3}{2}k_2\phi_b{}^{cd}\nabla_d\nabla_a\phi_c + \tfrac{15}{8}\phi^d\nabla_d\nabla_b\phi_a - \tfrac{3}{2}k_3\phi^d\nabla_d\nabla_b\phi_a \\
& + 3\phi_a{}^{cd}\nabla_d\nabla_b\phi_c - \tfrac{3}{2}k_2\phi_a{}^{cd}\nabla_d\nabla_b\phi_c - \tfrac{9}{8}\nabla_b\phi_d\nabla^d\phi_a + 3\nabla_c\phi_d\nabla^d\phi_{ab}{}^c + 3\nabla_d\phi_c\nabla^d\phi_{ab}{}^c \\
& + 3k_2\nabla_d\phi_c\nabla^d\phi_{ab}{}^c + 3k_3\nabla_d\phi_b\nabla^d\phi_a + 3\nabla_c\phi_a{}^{cd}\nabla_e\phi_{bd}{}^e - 3\nabla_c\phi_{ab}{}^c\nabla_e\phi^e - \tfrac{3}{2}k_3\nabla_b\phi_d\nabla^d\phi_a \\
& - \tfrac{3}{4}\nabla_d\phi_b\nabla^d\phi_a - \tfrac{9}{8}\nabla_a\phi_d\nabla^d\phi_b - \tfrac{3}{2}k_3\nabla_a\phi_d\nabla^d\phi_b - \tfrac{3}{4}\nabla_a\phi_b{}^{cd}\nabla_e\phi_{cd}{}^e - \tfrac{3}{2}k_1\nabla_a\phi_b{}^{cd}\nabla_e\phi_{cd}{}^e \\
& - \tfrac{3}{4}\nabla_b\phi_a{}^{cd}\nabla_e\phi_{cd}{}^e - \tfrac{3}{2}k_1\nabla_b\phi_a{}^{cd}\nabla_e\phi_{cd}{}^e + \tfrac{15}{8}\nabla_a\phi_b\nabla_e\phi^e - \tfrac{3}{2}k_3\nabla_a\phi_b\nabla_e\phi^e + \tfrac{15}{8}\nabla_b\phi_a\nabla_e\phi^e
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{3}{2}k_3\nabla_b\phi_a\nabla_e\phi^e-\tfrac{3}{4}\phi^{cde}\nabla_e\nabla_a\phi_{bcd}-\tfrac{3}{2}k_1\phi^{cde}\nabla_e\nabla_a\phi_{bcd}-\tfrac{3}{2}k_2\phi^d\nabla_e\nabla_a\phi_{bd}{}^e-\tfrac{3}{4}\phi_b{}^{cd}\nabla_e\nabla_a\phi_{cd}{}^e\\
&-\tfrac{3}{2}k_1\phi_b{}^{cd}\nabla_e\nabla_a\phi_{cd}{}^e-\tfrac{21}{8}\phi_b\nabla_e\nabla_a\phi^e-\tfrac{3}{2}k_3\phi_b\nabla_e\nabla_a\phi^e+3k_4R_{ab}\phi_{cde}\phi^{cde}-\tfrac{3}{4}\phi^{cde}\nabla_e\nabla_b\phi_{acd}\\
&-\tfrac{3}{2}k_1\phi^{cde}\nabla_e\nabla_b\phi_{acd}-\tfrac{3}{2}k_1\phi_a{}^{cd}\nabla_e\nabla_b\phi_{cd}{}^e-\tfrac{21}{8}\phi_a\nabla_e\nabla_b\phi^e-\tfrac{3}{2}k_3\phi_a\nabla_e\nabla_b\phi^e+3\phi^{cde}\nabla_e\nabla_d\phi_{abc}\\
&-3\phi^d\nabla_e\nabla_d\phi_{ab}{}^e-3\phi_b{}^{cd}\nabla_e\nabla_d\phi_{ac}{}^e+3\phi_b\nabla_e\nabla_d\phi_a{}^{de}-3\phi_a{}^{cd}\nabla_e\nabla_d\phi_{bc}{}^e+3\phi_a\nabla_e\nabla_d\phi_b{}^{de}\\
&+3\phi_{ab}{}^c\nabla_e\nabla_d\phi_c{}^{de}-3g_{ab}\phi^{cde}\nabla_e\nabla_d\phi_c+\tfrac{3}{2}k_2g_{ab}\phi^{cde}\nabla_e\nabla_d\phi_c+\tfrac{3}{2}k_2\phi^d\nabla_e\nabla^e\phi_{abd}\\
&+\tfrac{3}{4}\phi_b{}^{cd}\nabla_e\nabla^e\phi_{acd}+\tfrac{3}{2}k_1\phi_b{}^{cd}\nabla_e\nabla^e\phi_{acd}-\tfrac{15}{8}\phi_b\nabla_e\nabla^e\phi_a+\tfrac{3}{2}k_3\phi_b\nabla_e\nabla^e\phi_a+\tfrac{3}{4}\phi_a{}^{cd}\nabla_e\nabla^e\phi_{bcd}\\
&+\tfrac{3}{2}k_1\phi_a{}^{cd}\nabla_e\nabla^e\phi_{bcd}-\tfrac{15}{8}\phi_a\nabla_e\nabla^e\phi_b+\tfrac{3}{2}k_3\phi_a\nabla_e\nabla^e\phi_b-3\phi_{ab}{}^c\nabla_e\nabla^e\phi_c+\tfrac{3}{2}k_2\phi_{ab}{}^c\nabla_e\nabla^e\phi_c\\
&+\tfrac{9}{4}\nabla_b\phi_{cde}\nabla^e\phi_a{}^{cd}-\tfrac{3}{2}k_1\nabla_b\phi_{cde}\nabla^e\phi_a{}^{cd}-3\nabla_d\phi_{bce}\nabla^e\phi_a{}^{cd}-\tfrac{3}{2}\nabla_e\phi_{bcd}\nabla^e\phi_a{}^{cd}+3k_1\nabla_e\phi_{bcd}\nabla^e\phi_a{}^{cd}\\
&+\tfrac{9}{4}\nabla_a\phi_{cde}\nabla^e\phi_b{}^{cd}-\tfrac{3}{2}k_1\nabla_a\phi_{cde}\nabla^e\phi_b{}^{cd}-\tfrac{3}{8}g_{ab}\nabla_d\phi_e\nabla^e\phi^d+\tfrac{3}{2}k_3g_{ab}\nabla_d\phi_e\nabla^e\phi^d-\tfrac{3}{2}g_{ab}\nabla_e\phi_d\nabla^e\phi^d\\
&+6k_5g_{ab}\nabla_e\phi_d\nabla^e\phi^d+\tfrac{3}{4}g_{ab}\nabla_c\phi^{cde}\nabla_f\phi_{de}{}^f+\tfrac{3}{2}k_1g_{ab}\nabla_c\phi^{cde}\nabla_f\phi_{de}{}^f+3k_2g_{ab}\nabla^e\phi^d\nabla_f\phi_{de}{}^f\\
&+\tfrac{3}{2}k_3g_{ab}\nabla_d\phi^d\nabla_f\phi^f+\tfrac{3}{4}g_{ab}\phi^d\nabla_f\nabla_d\phi^f+3k_3g_{ab}\phi^d\nabla_f\nabla_d\phi^f+\tfrac{3}{2}g_{ab}\phi^{cde}\nabla_f\nabla_e\phi_{cd}{}^f\\
&+\tfrac{3}{2}k_2g_{ab}\phi^d\nabla_f\nabla_e\phi_d{}^{ef}+6k_4g_{ab}\phi^{cde}\nabla_f\nabla^f\phi_{cde}+6k_5g_{ab}\phi^d\nabla_f\nabla^f\phi_d-\tfrac{3}{4}g_{ab}\nabla_e\phi_{cdf}\nabla^f\phi^{cde}\\
&+\tfrac{3}{2}k_1g_{ab}\nabla_e\phi_{cdf}\nabla^f\phi^{cde}+6k_4g_{ab}\nabla_f\phi_{cde}\nabla^f\phi^{cde}+3k_5R\phi_a\phi_b+\tfrac{1}{2}g_{ab}\nabla_f\phi_{cde}\nabla^f\phi^{cde}\\
&-\tfrac{3}{4}\phi_a{}^{cd}\nabla_e\nabla_b\phi_{cd}{}^e-\tfrac{3}{2}k_2\phi^d\nabla_e\nabla_b\phi_{ad}{}^e+\tfrac{3}{8}g_{ab}\nabla_d\phi^d\nabla_f\phi^f+3k_1g_{ab}\phi^{cde}\nabla_f\nabla_e\phi_{cd}{}^f
\end{aligned}
\tag{4.2.79}
$$

Schrödinger spacetime is not asymptotically AdS. However, one can consider it as perturbative deformation of AdS [39]. The zuveilbein to our interest would be

$$a_t = L_1 + \sigma W_2; \qquad a_{x^-} = 0 \tag{4.2.80}$$

$$\bar{a}_t = \sigma W_{-2}; \qquad \bar{a}_{x^-} = 2L_{-1} \tag{4.2.81}$$

which corresponds to metric

$$ds^2 = -\sigma^2 r^4 dt^2 + \frac{dr^2}{r^2} + 2r^2 dt dx^- \tag{4.2.82}$$

and spin-3 field

$$\phi_{t--} = \frac{\sigma}{3} r^4; \qquad \phi_{ttt} = -\frac{\sigma}{4} r^4 \tag{4.2.83}$$

$\sigma$ measures deformation from pure AdS in lightcone frame. Apparently metric fields would solve Einstein equation if $\sigma = 0$.

After substituting (4.2.82) and (4.2.83) into (4.2.78), one can find the equation holds at the lowest order of $\sigma$. Similarly, one can also check the equation of motion about $\phi_{\mu\nu\rho}$ [95] can be solved at the same order of $\sigma$.

## 4.C $\mathcal{W}_3$ algebra from asymptotically Lifshitz boundary condition

Let's consider gauge transformations $\epsilon$ preserving boundary conditions (4.5.42) and take the case of unbarred sector. The barred sector can be derived in exactly the same procedure. Denote $\epsilon = b^{-1}\epsilon_0 b$. Gauge transformation on field $a$ is described by equation

$$\delta a^{(0)} = d\epsilon_0 + [a, \epsilon_0] \tag{4.3.84}$$

Expand gauge parameter $\epsilon_0$ in terms of

$$\epsilon_0 = \epsilon^{L_0}L_0 + \epsilon^{L_1}L_1 + \epsilon^{L_{-1}}L_{-1} + \epsilon^{W_2}W_2 + \epsilon^{W_1}W_1 + \epsilon^{W_0}W_0 + \epsilon^{W_{-1}}W_{-1} + \epsilon^{W_{-2}}W_{-2}$$

and note $\delta a^{(0)} = (4t\delta\mathcal{W}L_0 - \delta\mathcal{L}L_{-1} - 4t^2\delta\mathcal{W}W_2 + 4t\delta\mathcal{L}W_1 + \delta\mathcal{W}W_{-2})dx$, we will be left with following equations:

- $t$ component equations

$$\begin{aligned}
&\dot{\epsilon}^{L_1} - \epsilon^{W_{-1}} = 0; \qquad \dot{\epsilon}^{L_0} - 4\epsilon^{W_{-2}} = 0\\
&\dot{\epsilon}^{W_2} + 2\epsilon^{L_0} = 0; \qquad \dot{\epsilon}^{W_1} + 4\epsilon^{L_{-1}} = 0\\
&\dot{\epsilon}^{L_{-1}} = \dot{\epsilon}^{W_0} = \dot{\epsilon}^{W_{-1}} = \dot{\epsilon}^{W_{-2}} = 0
\end{aligned} \tag{4.3.85}$$

- $x$ component equations

$$\begin{aligned}
L_1: &\quad \partial_x\epsilon^{L_1} + \epsilon^{L_0} - 4\mathcal{W}t\epsilon^{L_1} + 2\mathcal{L}t\epsilon^{W_0} + 4\mathcal{W}t^2\epsilon^{W_{-1}} = 0\\
L_0: &\quad \partial_x\epsilon^{L_0} + 2\mathcal{L}\epsilon^{L_1} + 2\epsilon^{L_{-1}} + 4\mathcal{W}\epsilon^{W_2} + 2\mathcal{L}t\epsilon^{W_{-1}} + 16\mathcal{W}t^2\epsilon^{W_{-2}} = 4t\delta\mathcal{W}\\
L_{-1}: &\quad \partial_x\epsilon^{L_{-1}} + \mathcal{L}\epsilon^{L_0} + 4\mathcal{W}t\epsilon^{L_{-1}} + \mathcal{W}\epsilon^{W_1} - 4\mathcal{L}t\epsilon^{W_{-2}} = -\delta\mathcal{L}\\
W_2: &\quad \partial_x\epsilon^{W_2} - 8\mathcal{W}t^2\epsilon^{L_0} - 4\mathcal{L}t\epsilon^{L_1} + \epsilon^{W_1} - 8\mathcal{W}t\epsilon^{W_2} = -4t^2\delta\mathcal{W}\\
W_1: &\quad \partial_x\epsilon^{W_1} + 4\mathcal{L}t\epsilon^{L_0} - 16\mathcal{W}t^2\epsilon^{L_{-1}} + 2\epsilon^{W_0} - 4\mathcal{W}t\epsilon^{W_1} + 4\mathcal{L}\epsilon^{W_2} = 4t\delta\mathcal{L}\\
W_0: &\quad \partial_x\epsilon^{W_0} + 3(4\mathcal{L}t\epsilon^{L_{-1}} + \mathcal{L}\epsilon^{W_1} + \epsilon^{W_{-1}}) = 0\\
W_{-1}: &\quad \partial_x\epsilon^{W_{-1}} - 4\mathcal{W}\epsilon^{L_1} + 2\mathcal{L}\epsilon^{W_0} + 4\mathcal{W}t\epsilon^{W_{-1}} + 4\epsilon^{W_{-2}} = 0\\
W_{-2}: &\quad \partial_x\epsilon^{W_{-2}} - 2\mathcal{W}\epsilon^{L_0} + \mathcal{L}\epsilon^{W_{-1}} + 8\mathcal{W}t\epsilon^{W_{-2}} = \delta\mathcal{W}
\end{aligned} \tag{4.3.86}$$

Solutions are parametrized by two free $x$ dependent functions $\epsilon_L(x), \epsilon_W(x)$

$$\epsilon^{L_{-1}} = \frac{1}{2}\epsilon_L'' - 2\mathcal{W}\epsilon_W - \mathcal{L}\epsilon_L \tag{4.3.87}$$

$$\epsilon^{W_0} = \frac{1}{2}\epsilon''_W - 2\mathcal{L}\epsilon_W \tag{4.3.88}$$

$$\epsilon^{W_{-1}} = \frac{5}{3}\mathcal{L}\epsilon'_W - \frac{1}{6}\epsilon'''_W + \frac{2}{3}\mathcal{L}'\epsilon_W \tag{4.3.89}$$

$$\epsilon^{W_{-2}} = \mathcal{W}\epsilon_L + \mathcal{L}^2\epsilon_W - \frac{1}{6}\mathcal{L}''\epsilon_W - \frac{7}{12}\mathcal{L}'\epsilon'_W - \frac{2}{3}\mathcal{L}\epsilon''_W + \frac{1}{24}\epsilon''''_W \tag{4.3.90}$$

$$\epsilon^{L_0} = 4\epsilon^{W_{-2}}t - \epsilon'_L \tag{4.3.91}$$

$$\epsilon^{L_1} = \epsilon^{W_{-1}}t + \epsilon_L \tag{4.3.92}$$

$$\epsilon^{W_1} = -4t(\frac{1}{2}\epsilon''_L - 2\mathcal{W}\epsilon_W - \mathcal{L}\epsilon_L) - \epsilon'_W \tag{4.3.93}$$

$$\epsilon^{W_2} = -4t^2\epsilon^{W_{-2}} + 2\epsilon'_L t + \epsilon_W \tag{4.3.94}$$

Insert the solution into equation (4.3.86) we will get transformations laws (4.5.47) for conserved charges $\mathcal{L}$ and $\mathcal{W}$.

# Chapter 5

# 4D Schrödinger higher spin solution

This chapter is based on paper [4]. To understand whether degeneracy problem in 3D non-relativistic solution is a result of special property of 3D gravity, it is necessary to construct non-relativistic higher spin solutions in higher dimensional spacetimes. Field theory with non-relativistic higher spin symmetry was studied in [106–108] to model unitary Fermi gas. The lesson we learnt from 3D construction is Schrödinger spacetime with dynamical exponent $z$ is supported by higher spin fields with spin $s = z + 1$ [92]. Every spin-$s$ field will back react on lightcone AdS geometry and deform it by a factor $r^{-2(s-1)}dt^2$ (Here we use $r \to r^{-1}$ to define radial coordinate in Schrödinger metric (1.2.10)). To have a Schrödinger spacetime with dynamical exponent $z$, we need to truncate the infinite tower of higher spin fields.

In 3D higher spin theory, truncation of higher spin tower can be realized by tuning $\lambda$ to be integer [24]. This trick is not allowed in $D \geq 4$ manifold. On the other hand, that Schrödinger gauge fields (4.2.80) and (4.2.81) can solve flatness equation is independent of $SL(3, R)$ representation of $W, L$ generators. Instead, it depends on commutativity of $W_{2n}$ and $L_n$. This immediately leads to a conclusion: for generic value of $\lambda$, even higher spin fields are not truncated, (4.2.80) and (4.2.81) still solve flatness equation. This sounds more like an analogy to higher dimensional case.

There should exist another scheme to truncate infinite tower of higher spin fields even in the theory with all $s > 2$ spin fields. We will review this truncation scheme in section 5.1 by analysing 3D Schrödinger solution in $hs[\lambda]$ theory. Based on this idea, we will give an explicit construction of 4D higher spin Schrödinger solutions in 5.2.

## 5.1 3D Schrödinger higher spin solution

### 5.1.1 Vasiliev formulation

We would like to reformulate our 3D higher spin Schrödinger solution (4.2.80)-(4.2.83) in Vasiliev theory.

Our normalization in this section is slightly different from [109] but is self-consistent. Let us introduce oscillators $\hat{y}_\alpha$ $(\alpha = 1, 2)$ fulfilling

$$[\hat{y}_\alpha, \hat{y}_\beta] = \frac{1}{2}\epsilon_{\alpha\beta}(1 + \nu k)\,, \qquad k\hat{y}_\alpha = -\hat{y}_\alpha k\,, \qquad k^2 = 1\,, \tag{5.1.1}$$

where $\nu$ is a free parameter and $k$ is the Klein operator. Define bilinear oscillators $T_{\alpha\beta}$

$$T_{\alpha\beta} = \{\hat{y}_\alpha, \hat{y}_\beta\}\,, \tag{5.1.2}$$

that generate a $sl(2)$ algebra

$$[T_{\alpha\beta}, T_{\gamma\sigma}] = \epsilon_{\beta\gamma}T_{\alpha\sigma} + T_{\beta\sigma}\epsilon_{\alpha\gamma} + T_{\alpha\gamma}\epsilon_{\beta\sigma} + \epsilon_{\alpha\sigma}T_{\beta\gamma}\,. \tag{5.1.3}$$

Higher (symmetric) powers of these oscillators give the higher spin generators. The connection with the Chern-Simons formulation is explained in section 5.A.

In the current case, the gravitational connection

$$W = \omega + \frac{1}{l}\psi e\,, \quad \psi^2 = 1\,, \quad [\psi, \hat{y}_\alpha] = 0\,, \tag{5.1.4}$$

where $\psi$ is the central involutive element and $l$ is the AdS radius, satisfies the equation of motion [109]

$$dW + W \wedge W = 0\,.$$

The $z = 2$ Schrödinger gauge fields (4.2.16) translate to the oscillator form

$$e \;=\; l(\frac{1}{4}rT_{11} + \frac{\sigma}{8}r^2T_{11}T_{11} - \frac{\sigma}{8}r^2T_{22}T_{22})dt - \frac{l}{2}rT_{22}d\xi + \frac{l}{2r}T_{12}dr\,, \tag{5.1.5}$$

$$\omega \;=\; (\frac{1}{4}rT_{11} + \frac{\sigma}{8}r^2T_{11}T_{11} + \frac{\sigma}{8}r^2T_{22}T_{22})dt + \frac{1}{2}rT_{22}d\xi\,. \tag{5.1.6}$$

via (5.1.60). It is then trivial to check that they solve the above equation of motion (setting $l = 1$), which in component form reads

Torsion free equations

$$\begin{aligned}
\psi\,T_{\alpha\beta}: &\quad de^{\alpha\beta} + e^{\alpha\kappa}\wedge\omega^{\gamma\beta}\epsilon_{\kappa\gamma} + e^{\kappa\beta}\wedge\omega^{\gamma\alpha}\epsilon_{\kappa\gamma} = 0\,, & (5.1.7)\\
\psi\,\sigma T_{\alpha\beta}T_{\gamma\kappa}: &\quad de^{\alpha\beta\gamma\kappa} + 2\omega^{\alpha n\gamma\kappa}\wedge e^{m\beta}\epsilon_{nm} + 2\omega^{\alpha\beta\gamma n}\wedge e^{m\kappa}\epsilon_{nm} & \\
&\quad +2e^{\alpha n\gamma\kappa}\wedge\omega^{m\beta}\epsilon_{nm} + 2e^{\alpha\beta\gamma n}\wedge\omega^{m\kappa}\epsilon_{nm} = 0\,, & (5.1.8)\\
\psi\,T_{\alpha\beta}T_{\gamma\kappa}T_{mn}: &\quad \omega^{\alpha\beta\gamma\kappa}\wedge e^{mncd} = 0\,, & (5.1.9)
\end{aligned}$$

Curvature equations:

$$\begin{aligned}
T_{\alpha\beta}: &\quad d\omega^{\alpha\beta} + \omega^{\alpha\kappa}\wedge\omega^{\gamma\beta}\epsilon_{\kappa\gamma} + \frac{1}{l^2}e^{\alpha\kappa}\wedge\omega^{\gamma\beta}\epsilon_{\kappa\gamma} = 0\,, & (5.1.10)\\
\sigma T_{\alpha\beta}T_{\gamma\kappa}: &\quad d\omega^{\alpha\beta\gamma\kappa} + 2\omega^{\alpha n\gamma\kappa}\wedge\omega^{m\beta}\epsilon_{nm} + 2\omega^{\alpha\beta\gamma n}\wedge\omega^{m\kappa}\epsilon_{nm} & \\
&\quad +\frac{1}{l^2}(2e^{\alpha n\gamma\kappa}\wedge e^{m\beta}\epsilon_{nm} + 2e^{\alpha\beta\gamma n}\wedge e^{m\kappa}\epsilon_{nm}) = 0\,, & (5.1.11)\\
T_{\alpha\beta}T_{\gamma\kappa}T_{mn}: &\quad \omega^{(4)}\wedge\omega^{(4)} + e^{(4)}\wedge e^{(4)} = 0\,. & (5.1.12)
\end{aligned}$$

This solution has no non-trivial holonomy, so one can do a large gauge transformation to relate this solution to empty AdS [3].

### 5.1.2 Scalar equations

In this section, we consider the motion of a scalar in the above $3D$ Schrödinger background, characterized by

$$dC + A * C - C * \bar{A} = 0\,. \tag{5.1.13}$$

We briefly review the analysis of [110] in terms of the lone-star product in this subsection. The notation and its relation with the previously mentioned oscillator formalism is explained in Appendix 5.A.

All the fields take value in the higher spin algebra

$$C = \sum_{s=1}^{\infty}\sum_{|m|<s} C^s_m V^s_m\,, \qquad A = \sum_{s=2}^{\infty}\sum_{|m|<s} A^s_m V^s_m\,, \qquad \bar{A} = \sum_{s=2}^{\infty}\sum_{|m|<s} \bar{A}^s_m V^s_m\,, \tag{5.1.14}$$

with $C_0^1$ being the physical scalar. We now extract the equation of motion of $C_0^1$. If $A$ and $\bar{A}$ span pure $AdS_3$ gravity, equation (5.1.13) reduces to Klein-Gordon equation. Now consider $z = 2$ Schrödinger spacetime [92, 99]

$$A = (\sigma e^{2\rho} V_2^3 + e^{\rho} V_1^2) dt + V_0^2 d\rho \,, \quad \bar{A} = \sigma e^{2\rho} V_{-2}^3 dt + 2 e^{\rho} V_{-1}^2 d\xi - V_0^2 d\rho \,, \tag{5.1.15}$$

where the constant source $\sigma$ parametrizes the higher spin deformation. Plugging these expansions into the scalar equation (5.1.13) we get an infinite set of equations, one from each term proportional to $V_m^s dx^\mu \equiv V_{m,\mu}^s$. Remarkably, as shown in [110], we can choose a set of equations, being the coefficients of $\{V_{0,\rho}^1, V_{0,\bar{t}}^1, V_{1,x}^2, V_{0,\rho}^2, V_{0,x}^1, V_{-2,x}^3, V_{-1,x}^2, V_{-1,\rho}^2, V_{-1,\rho}^3\}$,[1] that reduce to the explicit equation of motion for $C_0^1$

$$\begin{aligned}&(\sigma e^{4\rho} \partial_\rho^4 + 8\sigma e^{4\rho} \partial_\rho^3 + 2\sigma(11 - \lambda^2) e^{4\rho} \partial_\rho^2 - 8\sigma e^{4\rho} (\lambda^2 - 3) \partial_\rho + \sigma e^{4\rho} (\lambda^2 - 1)(\lambda^2 - 9) \\ &+ 2e^{2\rho}(1 - \lambda^2)\partial_x + 4e^{2\rho}\partial_\rho\partial_x + 2e^{2\rho}\partial_x\partial_\rho^2 - \sigma\partial_x^4 + 4\partial_t\partial_x^2) C_0^1 = 0 \,.\end{aligned} \tag{5.1.16}$$

Furthermore, as $\sigma \to 0$, one gets the $x$-derivative of the Klein-Gordon equation in AdS background [110]; thus, we can solve the full equation perturbatively with respect to $\sigma$.

## 5.2 $4D$ Schrödinger solution

### 5.2.1 Star product in 4D

Most of the notation in this section will follow [21], where $x^\mu$ ($\mu = 0, 1, 2, 3$) denote spacetime Poincaré coordinates with $x_2 = r$. In this coordinate, the AdS spacetime metric is

$$ds^2 = \frac{-dx_0^2 + dx_1^2 + dr^2 + dx_3^2}{r^2} \,. \tag{5.2.17}$$

The internal twistor space is parametrized by spinors $(Y, Z) = (y^\alpha, \bar{y}^{\dot{\alpha}}, z^\alpha, \bar{z}^{\dot{\alpha}})$, $\alpha, \dot{\alpha} = 1, 2$. Here $z^\alpha, \bar{z}^{\dot{\alpha}}$ are auxiliary coordinates; physical fields are those with constraints $z^\alpha = \bar{z}^{\dot{\alpha}} = 0$.

The star product of two spinor-valued functions can be defined as [21]

$$f(Y,Z) * g(Y,Z) \;\; = \;\; f(Y,Z) \exp \Big[ \epsilon^{\alpha\beta} (\overleftarrow{\partial}_{y^\alpha} + \overleftarrow{\partial}_{z^\alpha})(\overrightarrow{\partial}_{y^\beta} - \overrightarrow{\partial}_{z^\beta})$$

[1] Our choice is slightly different from that in [110].

$$+\epsilon^{\dot\alpha\dot\beta}(\overleftarrow{\partial}_{\bar y^{\dot\alpha}} + \overleftarrow{\partial}_{\bar z^{\dot\alpha}})(\overrightarrow{\partial}_{\bar y^{\dot\beta}} - \overrightarrow{\partial}_{\bar z^{\dot\beta}})\Big] g(Y,Z) \,. \qquad (5.2.18)$$

There are in addition Klein operators $K(t) = e^{tz^\alpha y_\alpha}$ and $\bar K(t) = e^{t\bar z^{\dot\alpha}\bar y_{\dot\alpha}}$.

Vasiliev master fields include a gravitational connection $W = W_\mu(x|y,\bar y,z,\bar z)dx^\mu$, an auxiliary fields $S = dz^\alpha S_\alpha(x|y,\bar y,z,\bar z) + d\bar z^{\dot\alpha} S_{\dot\alpha}(x|y,\bar y,z,\bar z)$ [2] and a scalar field $B(x|y,\bar y,z,\bar z)$. The equations of motion that determine the dynamics of the system are

$$d_x W + W * \wedge W = 0 \,, \qquad (5.2.19a)$$

$$d_Z W + d_x S + \{W,S\}_* = 0 \,, \qquad (5.2.19b)$$

$$d_Z S + S * S = B * K dz^2 + B * \bar K d\bar z^2 \,, \qquad (5.2.19c)$$

$$d_x B + W * B - B * \pi(W) = 0 \,, \qquad (5.2.19d)$$

$$d_Z B + S * B - B * \pi(S) = 0 \,, \qquad (5.2.19e)$$

where $\pi(H)$ flips the signs of unbarred spinors $(y,z,dz)$ in $H$ while it preserves the signs of barred coordinates $(\bar y,\bar z,d\bar z)$. These master fields also satisfy

$$[R,W]_* = \{R,S\}_* = [R,B]_* = 0 \,, \qquad (5.2.20)$$

where $R = K\bar K$. This implies $W,B$ are even functions of $(Y,Z)$ while $S$ is an odd function of $(Y,Z)$.

In this section, we will discuss the vacuum solutions of master equation (5.2.19), i.e. $B = 0\,,\ S = dz^\alpha\, z_\alpha + d\bar z^{\dot\alpha}\,\bar z_{\dot\alpha}$ and $W(Y,Z) = W(Y)$ from (5.2.19b).

### 5.2.2 AdS solution in lightcone coordinate

Vacuum $AdS_4$ spacetime

$$B = 0\,, \quad S = dz^\alpha\, z_\alpha + d\bar z^{\dot\alpha}\,\bar z_{\dot\alpha}\,, \quad W = e_{\alpha\dot\beta} y^\alpha \bar y^{\dot\beta} + \omega_{\alpha\beta} y^\alpha y^\beta + \omega_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta}\,, \qquad (5.2.21)$$

[2]The spinor indices are raised and lowered by the antisymmetric tensor $\epsilon_{\alpha\beta}$,

$$A^\alpha = \epsilon^{\alpha\beta} A_\beta; \quad A_\alpha = A^\beta \epsilon_{\beta\alpha}, \quad \epsilon_{12} = \epsilon^{12} = 1\,.$$

is a solution to the Vasiliev equations (5.2.19), which reduces to the component form

$$y^\alpha \bar{y}^{\dot\alpha}: \qquad de_{\alpha\dot\alpha} + 4e_{\gamma\dot\alpha} \wedge \omega_{\alpha\beta}\epsilon^{\gamma\beta} - 4e_{\alpha\dot\beta} \wedge \omega_{\dot\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta} = 0\,, \tag{5.2.22}$$

$$y^\alpha y^\beta: \qquad d\omega_{\alpha\beta} + e_{\alpha\dot\gamma} \wedge e_{\beta\dot\kappa}\epsilon^{\dot\gamma\dot\kappa} + 4\omega_{\beta\kappa} \wedge \omega_{\alpha\gamma}\epsilon^{\kappa\gamma} = 0\,, \tag{5.2.23}$$

$$\bar{y}^{\dot\alpha}\bar{y}^{\dot\beta}: \qquad d\omega_{\dot\alpha\dot\beta} - e_{\kappa\dot\alpha} \wedge e_{\gamma\dot\beta}\epsilon^{\gamma\kappa} + 4\omega_{\dot\beta\dot\kappa} \wedge \omega_{\dot\alpha\dot\gamma}\epsilon^{\dot\kappa\dot\gamma} = 0\,. \tag{5.2.24}$$

Explicitly, we have

$$e_{\alpha\dot\beta} = \tfrac{1}{4}e^a(\sigma_a)_{\alpha\dot\beta}\,, \quad \omega_{\alpha\beta} = -\omega^a(\sigma_{a2}\epsilon)_{\alpha\beta}\,, \quad \bar\omega_{\dot\alpha\dot\beta} = -\omega^a(\epsilon\bar\sigma_{a2})_{\dot\alpha\dot\beta}\,, \tag{5.2.25}$$

where $e^a = \frac{\delta^a_\mu}{r}dx^\mu$, $\omega^a = \frac{\delta^a_\mu}{8r}dx^\mu$ are the veilbein and the spin connection of AdS spacetime (5.2.17) in the lightcone Poincaré coordinate

$$ds^2 = \frac{2dtd\xi + dr^2 + dx^2}{r^2}\,, \qquad \xi = \frac{x_1 - x_0}{\sqrt{2}}\,,\ t = \frac{x_1 + x_0}{\sqrt{2}}\,,\ x = x_3\,. \tag{5.2.26}$$

We have further employed Pauli matrices in the lightcone coordinate in (5.2.25)

$$\sigma_t = \frac{\sigma_0+\sigma_1}{\sqrt{2}}, \qquad \sigma_\xi = \frac{-\sigma_0+\sigma_1}{\sqrt{2}}, \qquad \sigma_r = \sigma_2, \qquad \sigma_x = \sigma_3\,,$$
$$\sigma_{t\mu} = \frac{\sigma_{0\mu}+\sigma_{1\mu}}{\sqrt{2}}, \quad \sigma_{\xi\mu} = \frac{-\sigma_{0\mu}+\sigma_{1\mu}}{\sqrt{2}}, \quad \bar\sigma_{t\mu} = \frac{\bar\sigma_{0\mu}+\bar\sigma_{1\mu}}{\sqrt{2}}, \quad \bar\sigma_{\xi\mu} = \frac{-\bar\sigma_{0\mu}+\bar\sigma_{1\mu}}{\sqrt{2}}\,.$$

Further notice that we work in the Minkowski signature, so the Pauli matrices are the familiar ones that are hermitian. As a consequence, the parity action is our convention is then $y_\alpha \leftrightarrow \bar{y}_{\dot\alpha}\,, z_\alpha \leftrightarrow \bar{z}_{\dot\alpha}\,,$ and further accompanied with hermitian conjugation of the coefficients of the oscillators.

### 5.2.3 Schrödinger solution with $z = 2$

We are now ready to construct 4D Schrödinger geometry (1.2.10) in Vasiliev higher spin theory. The simplest non-trivial example is the $z = 2$ Schrödinger geometry which turns out to be supported by extra $s = 3$ higher spin fields. We consider a variant form of the Schrödinger metric

$$ds^2 = -\frac{\sigma^2 dt^2}{r^{2z}} + \frac{2dtd\xi + dr^2 + dx^2}{r^2}\,, \qquad z = 2\,, \quad \sigma \in \mathbb{R}\,, \sigma \neq 0\,, \tag{5.2.27}$$

which can be converted from (1.2.10) by field redefinition $t \to \sigma t,\ \xi \to \xi\sigma^{-1}$.

### General solution

We try to find a ground state solution to (5.2.19) of the form

$$B = 0\,, \qquad S = dz^\alpha\, z_\alpha + d\bar{z}^{\dot\alpha}\, \bar{z}_{\dot\alpha}\,, \qquad W(Y,Z|x) = W(Y|x)\,, \tag{5.2.28}$$

with some spin-3 fields turned on in $W$. We simply take $W = W_2 + W_3$, where $W_2$ is the spin-2 piece (5.2.21), (5.2.25), and $W_3$ encodes spin-3 fields that are quartic in the $y, \bar{y}$ oscillators

$$W_3 = \omega_{\alpha\beta\gamma\kappa} y^\alpha y^\beta y^\gamma y^\kappa + \omega_{\alpha\beta\gamma\dot\kappa} y^\alpha y^\beta y^\gamma \bar{y}^{\dot\kappa} + \omega_{\alpha\beta\dot\gamma\dot\kappa} y^\alpha y^\beta \bar{y}^{\dot\gamma} \bar{y}^{\dot\kappa} + \omega_{\alpha\dot\beta\dot\gamma\dot\kappa} y^\alpha \bar{y}^{\dot\beta} \bar{y}^{\dot\gamma} \bar{y}^{\dot\kappa} + \omega_{\dot\alpha\dot\beta\dot\gamma\dot\kappa} \bar{y}^{\dot\alpha} \bar{y}^{\dot\beta} \bar{y}^{\dot\gamma} \bar{y}^{\dot\kappa}\,. \tag{5.2.29}$$

The only nontrivial equation (5.2.19a) decomposes schematically to

$$y^2: \quad d_x W_2 + W_2 * \wedge W_2 = 0\,, \tag{5.2.30a}$$

$$y^4: \quad d_x W_3 + W_2 * \wedge W_3 + W_3 * \wedge W_2 = 0\,, \tag{5.2.30b}$$

$$y^6: \quad W_3 * \wedge W_3 = 0\,. \tag{5.2.30c}$$

The equation (5.2.30a) simply means we can take $W_2$ as the AdS connection (5.2.21) and (5.2.25). The equation (5.2.30c) is very restrictive and can only be solved due to the wedge product: we take $W_3$ to be proportional to $dt$ in the light of our aimed solution (1.2.10). The only remaining equation to be solved, namely (5.2.30b), decomposes to

$$\begin{aligned}
y^4: \quad & d\omega_{\alpha\beta\gamma\kappa} + 2e_{\alpha\dot\xi} \wedge \omega_{\beta\gamma\kappa\dot\delta}\epsilon^{\dot\xi\dot\delta} + 16\omega_{\alpha\xi} \wedge \omega_{\beta\gamma\kappa\delta}\epsilon^{\xi\delta} = 0\,,\\
y^3\bar{y}: \quad & d\omega_{\alpha\beta\gamma\dot\kappa} + 8e_{\xi\dot\kappa} \wedge \omega_{\alpha\beta\gamma\delta}\epsilon^{\xi\delta} + 4e_{\alpha\dot\xi} \wedge \omega_{\beta\gamma\dot\delta\dot\kappa}\epsilon^{\dot\xi\dot\delta} + 12\omega_{\alpha\xi} \wedge \omega_{\beta\gamma\delta\dot\kappa}\epsilon^{\xi\delta}\\
& +4\omega_{\dot\kappa\dot\xi} \wedge \omega_{\alpha\beta\gamma\dot\delta}\epsilon^{\dot\xi\dot\delta} = 0\,,\\
y^2\bar{y}^2: \quad & d\omega_{\alpha\beta\dot\gamma\dot\kappa} + 6e_{\xi\dot\gamma} \wedge \omega_{\alpha\beta\delta\dot\kappa}\epsilon^{\xi\delta} + 6e_{\alpha\dot\xi} \wedge \omega_{\beta\dot\gamma\dot\kappa\dot\delta}\epsilon^{\dot\xi\dot\delta} + 8\omega_{\alpha\xi} \wedge \omega_{\beta\delta\dot\gamma\dot\kappa}\epsilon^{\xi\delta}\\
& +8\omega_{\dot\gamma\dot\xi} \wedge \omega_{\alpha\beta\dot\kappa\dot\delta}\epsilon^{\dot\xi\dot\delta} = 0\,,\\
y\bar{y}^3: \quad & d\omega_{\alpha\dot\beta\dot\gamma\dot\kappa} + 4e_{\xi\dot\beta} \wedge \omega_{\alpha\delta\dot\gamma\dot\kappa}\epsilon^{\xi\delta} + 8e_{\alpha\dot\xi} \wedge \omega_{\dot\beta\dot\gamma\dot\kappa\dot\delta}\epsilon^{\dot\xi\dot\delta} + 4\omega_{\alpha\xi} \wedge \omega_{\delta\dot\beta\dot\gamma\dot\kappa}\epsilon^{\xi\delta}\\
& +12\omega_{\dot\beta\dot\xi} \wedge \omega_{\alpha\dot\gamma\dot\kappa\dot\delta}\epsilon^{\dot\xi\dot\delta} = 0\,,\\
\bar{y}^4: \quad & d\omega_{\dot\alpha\dot\beta\dot\gamma\dot\kappa} + 2e_{\xi\dot\alpha} \wedge \omega_{\delta\dot\beta\dot\gamma\dot\kappa}\epsilon^{\xi\delta} + 16\omega_{\dot\alpha\dot\xi} \wedge \omega_{\dot\beta\dot\gamma\dot\kappa\dot\delta}\epsilon^{\dot\xi\dot\delta} = 0\,.
\end{aligned} \tag{5.2.31}$$

Considering only time independent, spherical symmetric solution, this set of equations is solved to get

$$\omega_{2222} = \frac{C_1}{4r^2}\,, \qquad \omega_{222\dot2} = \frac{-iC_1}{r^2}\,, \qquad \omega_{22\dot2\dot2} = \frac{-3C_1}{2r^2}\,, \qquad \omega_{2\dot2\dot2\dot2} = \frac{iC_1}{r^2}\,, \quad \omega_{\dot2\dot2\dot2\dot2} = \frac{C_1}{4r^2}\,,$$

$$
\begin{array}{lllll}
\omega_{2221} = \dfrac{-C_2}{6r^2}\,, & \omega_{222\dot{1}} = \dfrac{2iC_2}{3r^2}\,, & \omega_{22\dot{2}1} = \dfrac{2iC_2}{3r^2}\,, & \omega_{22\dot{2}\dot{1}} = \dfrac{C_2}{r^2}\,, & \\
\omega_{2\dot{2}\dot{2}1} = \dfrac{C_2}{r^2}\,, & \omega_{2\dot{2}\dot{2}\dot{1}} = \dfrac{-2iC_2}{3r^2}\,, & \omega_{\dot{2}\dot{2}\dot{2}1} = \dfrac{-2iC_2}{3r^2}\,, & \omega_{\dot{2}\dot{2}\dot{2}\dot{1}} = \dfrac{-C_2}{6r^2}\,, & \\
\omega_{1111} = \dfrac{C_3}{4r^2}\,, & \omega_{111\dot{1}} = \dfrac{-iC_3}{r^2}\,, & \omega_{11\dot{1}\dot{1}} = \dfrac{-3C_3}{2r^2}\,, & \omega_{1\dot{1}\dot{1}\dot{1}} = \dfrac{iC_3}{r^2}\,, & \omega_{\dot{1}\dot{1}\dot{1}\dot{1}} = \dfrac{C_3}{4r^2}\,, \\
\omega_{1122} = \dfrac{-C_4}{6r^2}\,, & \omega_{112\dot{2}} = \dfrac{2iC_4}{3r^2}\,, & \omega_{11\dot{2}\dot{2}} = \dfrac{C_4}{r^2}\,, & \omega_{1\dot{1}22} = \dfrac{2iC_4}{3r^2}\,, & \omega_{1\dot{1}2\dot{2}} = \dfrac{C_4}{r^2}\,, \\
\omega_{\dot{1}\dot{1}22} = \dfrac{C_4}{r^2}\,, & \omega_{1\dot{1}\dot{2}\dot{2}} = \dfrac{-2iC_4}{3r^2}\,, & \omega_{\dot{1}\dot{1}\dot{2}2} = \dfrac{-2iC_4}{3r^2}\,, & \omega_{\dot{1}\dot{1}\dot{2}\dot{2}} = \dfrac{-C_4}{6r^2}\,, & \\
\omega_{1112} = \dfrac{-C_5}{6r^2}\,, & \omega_{111\dot{2}} = \dfrac{2iC_5}{3r^2}\,, & \omega_{11\dot{1}2} = \dfrac{2iC_5}{3r^2}\,, & \omega_{11\dot{1}\dot{2}} = \dfrac{C_5}{r^2}\,, & \\
\omega_{1\dot{1}\dot{1}2} = \dfrac{C_5}{r^2}\,, & \omega_{1\dot{1}\dot{1}\dot{2}} = \dfrac{-2iC_5}{3r^2}\,, & \omega_{\dot{1}\dot{1}\dot{1}2} = \dfrac{-2iC_5}{3r^2}\,, & \omega_{\dot{1}\dot{1}\dot{1}\dot{2}} = \dfrac{-C_5}{6r^2}\,, &
\end{array}
\qquad (5.2.32)
$$

where $C_i$ $(i = 1, ..., 5)$ are arbitrary real constants. Furthermore, this solution is manifestly parity invariant.

We would like to remark that, in general, once spin-3 generators in $D > 3$ dimensional higher spin theory are included, one is forced to include the infinite tower of higher spin fields to solve the equation. This problem is avoided in our construction since the spin-3 fields are only turned on in the $t$ direction and $dt \wedge dt = 0$. For this reason we are able to isolate a single spin-s field, which back-reacts and supports the $z = s - 1$ Schrödinger spacetime. The spinorial index structure of $\omega_{(4)}$ fields implies that the above solution can be expanded in a basis consisting of tensors of two Pauli matrices. Making use of the identity [111]

$$\sigma^{\mu}_{\alpha\dot{\gamma}}\sigma^{\nu}_{\beta\dot{\kappa}} + \sigma^{\nu}_{\alpha\dot{\gamma}}\sigma^{\mu}_{\beta\dot{\kappa}} = \eta^{\mu\nu}\sigma^{r}_{\alpha\beta}\sigma^{r}_{\dot{\gamma}\dot{\kappa}} + 4(\sigma^{l\mu}\epsilon)_{\alpha\beta}(\epsilon\bar{\sigma}^{l\nu})_{\dot{\alpha}\dot{\beta}}, \qquad (5.2.33)$$

the $W_3$ field can be recast into

$$W_3 = \big(e^{ab}\sigma_a\sigma_b + H^{ab}_{\mathrm{ew}}\,\sigma_a(\sigma_{b2}\epsilon) + H^{ab}_{\mathrm{e\bar{w}}}\,\sigma_a(\epsilon\bar{\sigma}_{b2}) + H^{ab}_{\mathrm{ww}}\,(\sigma_{a2}\epsilon)(\sigma_{b2}\epsilon) + H^{ab}_{\mathrm{\bar{w}\bar{w}}}\,(\epsilon\bar{\sigma}_{a2})(\epsilon\bar{\sigma}_{b2})\big)dt\,. \qquad (5.2.34)$$

We have checked that the $e^{ab}, H^{ab}$ fields can be determined for the Schrödinger spacetime (5.2.32). However, the result is not much simpler than (5.2.32) and is not very illuminating so we do not show them explicitly.

Another comment is that given a generalised vielbein

$$E = e_{\alpha\dot{\beta}}y^{\alpha}\bar{y}^{\dot{\beta}} + \omega_{\alpha\beta\dot{\gamma}\dot{\kappa}}y^{\alpha}y^{\beta}\bar{y}^{\dot{\gamma}}\bar{y}^{\dot{\kappa}}\,, \qquad (5.2.35)$$

which means fixing the $C_i$, $i = 1, \ldots, 5$ parameters, the $W$ field is fully determined. This is equivalent to the statement that (generalised) spin-connection can be fully determined by the (generalised) veilbein from "torsion free" equations. Therefore, our $z = 2$ Schrödinger solution is free from degeneracy problem [3].

**The metric**

As we have briefly explained in the previous section, we do not treat the spin-3 fields as probe but take their backreaction on the geometry into account. We thus propose the following formula to compute the metric from the (generalised) vielbein

$$g = \mathrm{Tr}(E * E)\,, \tag{5.2.36}$$

where the trace is defined in (5.1.59). Notice that this definition reduces to the more familiar definition $g = \mathrm{Tr}(e * e)$ in general relativity when the higher spin fields are turned off.

This formula is determined by requiring the invariance of the metric under generalised local Lorentz transformations that rotate the local Lorentz indices and thus the local basis. This idea was first proposed in 3-dimensional [28] and we simply generalise it to higher dimension. To justify our proposal, we start with the general gauge transformation of any solution of the set of Vasiliev equations (5.2.19)

$$\delta W = d\epsilon + [W\,, \epsilon]_*\,, \quad \delta B = B * \pi(\epsilon) - \epsilon * B\,, \quad \delta S = [S, \epsilon]_*\,. \tag{5.2.37}$$

Since we have $B = 0$ and $\epsilon = \epsilon(Y|x)$, we only consider the first transformation. From which we can read off the general transformation $\delta E$ of our definition $E$ (5.2.36). Then we want to decompose the gauge transformation as

$$\epsilon = \xi + \Lambda + \Lambda_{\text{extra}}\,, \tag{5.2.38}$$

where $\xi$ parametrizes the generalised diffeomorphisms, $\Lambda$ parametrizing the generalized local Lorentz transformations and $\Lambda_{\text{extra}}$ parametrizes the extra gauge transformation associated to the extra auxiliary fields and other terms from higher spin generators [3]. The difference between the latter two is that the $\Lambda$ only rotates the

[3]Although the equation of motion is truncated by wedge product, the symmetry group is not

index in the first row in the two-row Young tableaux notation while $\Lambda_{\text{extra}}$ rotates indices in both the two rows. We thus require the metric to be invariant under all transformations parametrized by $\Lambda$.[4] It can be explicitly checked that our proposal (5.2.36) fulfils this requirement: the extra variation of the vielbeins under the local higher spin transformation is cancelled by the variation of the generalised vielbeins $\omega_{\alpha\beta\dot{\gamma}\dot{\kappa}}$. In fact, there is a much easier way to demonstrate this invariance. The variation takes a nice form $\delta E = [E, \Lambda]_*$, then it is trivial to verify the invariance of the metric by cyclicity of the trace [5]

$$\delta_\Lambda g = \mathrm{Tr}([E, \Lambda]_* * E + E * [E, \Lambda]_*) = 0\,. \tag{5.2.39}$$

With this definition, the solution we have found gives the following metric

$$ds^2 = -(72C_4^2 - 64C_2C_5 + 144C_1C_3)\frac{dt^2}{r^4} + \frac{2dtd\xi + dr^2 + dx^2}{r^2}\,. \tag{5.2.40}$$

**Higher spin fields**

The spin-3 metric like field can be determined similarly

$$\Phi = \mathrm{Tr}(E * E * E)\,, \tag{5.2.41}$$

which is again invariant under the higher spin generalisation of the local Lorentz transformation. Linearising the above spin-3 field leads to traceless symmetric tensor

$$\Phi_{\mu\nu_1\nu_2} \sim \mathrm{Tr}(e_{\alpha_1\dot{\beta}_1} y^{\alpha_1}\bar{y}^{\dot{\beta}_1} * e_{\gamma_1\dot{\kappa}_1} y^{\gamma_1}\bar{y}^{\dot{\kappa}_1} * \omega_{\alpha_2\beta_2\dot{\gamma}_2\dot{\kappa}_2} y^{\alpha_2} y^{\beta_2}\bar{y}^{\dot{\gamma}_2}\bar{y}^{\dot{\kappa}"}) \sim \sigma_{\nu_1}^{\alpha_1\dot{\gamma}_2}\sigma_{\nu_2}^{\beta_2\dot{\kappa}_2}\omega_{\mu|\alpha_2\beta_2\dot{\gamma}_2\dot{\kappa}_2}\,, \tag{5.2.42}$$

which agrees with the expression given in [21] up to normalization. The authors are acknowledged there are some nontriviality with this definition. However, (5.2.41) is shown to be invariant under local Lorentz transformation. Considering it matches the known result at linearised level, the definition is a potential candidate for spin-3 field at least in this Schrödinger vacuum case.

---

truncated. Commutator between spin-3 generator in master field $W$ and gauge transformation $\Lambda$ can result in terms with spin $s > 3$

[4] The metric does transform under $\Lambda_{\text{extra}}$, which is the higher dimensional analogue of phenomena discussed in, e.g. [88, 98].

[5] We thank Stefen Theisen to point this out to us.

We can further evaluate the fully nonlinear spin-3 fields (5.2.41) explicitly

$$\begin{aligned}
\phi_{ttt} &= \tfrac{3\big((3C_1+8C_2+3C_3-12C_4-8C_5)r^2+512\big(4C_4^3-9C_1C_3C_4-8C_2C_5C_4-6C_1C_5^2-6C_2^2C_3\big)\big)}{2r^6}\,, \\
\phi_{tt\xi} &= \tfrac{-4C_4-3(C_1+C_3)}{r^4}\,, \quad \phi_{t\xi\xi} = -\tfrac{-3C_1+8C_2-3C_3+12C_4+8C_5}{2r^4}\,, \quad \phi_{txx} = \tfrac{(3C_1+3C_3+4C_4)}{r^4}\,, \\
\phi_{ttx} &= \tfrac{\sqrt{2}(3C_1+4C_2-3C_3-4C_5)}{r^4}\,, \qquad \phi_{tx\xi} = -\tfrac{3C_1-4C_2-3C_3+4C_5}{\sqrt{2}r^4}\,,
\end{aligned} \tag{5.2.43}$$

with all other components vanish. Notice that in most of the terms the power at the boundary is exactly the dimension $\Delta = 4$ of a conserved spin-3 currents in the dual field theory. The only exception is the $r^{-6}$ term in $\Phi_{ttt}$ which has cubic coefficients $C_iC_jC_k$; both its scaling behaviour and its coefficient structure indicate the non-linear nature of this term.

As we have shown explicitly, the metric and the spin-3 metric like fields can be uniquely determined. To determine metric like higher spin fields with $s > 3$, more information is needed, which is similar to what happens in 3D [112], in addition to the requirement of local Lorentz invariance and the correct linearisation limit. This is because there are more than one combinations of veilbeins satisfying the above constraints. For example, for $s = 4$, $(\mathrm{tr}(E * E))^2$ and $\mathrm{tr}(E * E * E * E)$ are both local Lorentz invariant. Only a linear combination of these two terms gives the right Fronsdal field

$$\Phi^{(4)} = \mathrm{tr}\big((E * E * E * E)_s\big) + c\ \mathrm{tr}(E * E)\,\mathrm{tr}(E * E)\,,$$

where $(a * b)_s = a * b + b * a$ is the totally symmetric star product. The coefficient $c$ can be fixed by imposing the double-traceless condition or by imposing a Fefferman-Graham-like gauge condition $\Phi_{rrrr} = 0$ [113]. Remarkably, the two conditions lead to the same value $c = -\frac{1}{2}$.[6] This result agrees with our expectation and also agrees with what happens in 3D.

We comment here that even though we only turn on spin-2 and spin-3 components of the frame like field $W$ (5.2.29), there can be a nonzero spin-4 metric like field as constructed above. This property can only be seen at the fully nonlinear

[6]The exact value of $c$ depends on our definition of the trace, but the conclusion that the two conditions lead to the same value is independent of our definition of the trace; the latter can be checked explicitly.

level; the linearised spin-4 field, defined similarly as (5.2.42), vanishes. Moreover, we believe the whole tower of the metric like fields of arbitrary spin are nonzero unless protected by some hidden symmetries.

### Symmetries of the solution

**Relation with the AdS spacetime** One immediate question is if the solution we have got is gauge equivalent to the AdS vacuum. This is a reasonable question since both of them are solutions of equation (5.2.19a). However, we can show that the two solutions are physically distinct.

- Indeed, the following transformation

$$\delta W = d\epsilon + [W, \epsilon]_* \,, \tag{5.2.44}$$

maps a solution $W$ of (5.2.19a) into another solution $W + \delta W$ of (5.2.19a). For the case we are interested in, the AdS solution can be mapped to our Schrödinger solution with the parameter $\epsilon = \epsilon_{abcd} y^a y^b y^c y^d$ ,

$$\begin{aligned}
&\epsilon_{2222} = \frac{i}{4}\epsilon_{222\dot{2}} = \frac{-1}{6}\epsilon_{22\dot{2}\dot{2}} = \frac{-i}{4}\epsilon_{2\dot{2}\dot{2}\dot{2}} = \epsilon_{\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{tC_1 + d_1}{4r^2}\\
&\epsilon_{1222} = \frac{i}{4}\epsilon_{222\dot{1}} = \frac{i}{4}\epsilon_{122\dot{2}} = \frac{-1}{6}\epsilon_{12\dot{2}\dot{2}} = \frac{-1}{6}\epsilon_{22\dot{1}\dot{2}} = \frac{-i}{4}\epsilon_{1\dot{2}\dot{2}\dot{2}} = \frac{-i}{4}\epsilon_{2\dot{1}\dot{2}\dot{2}} = \epsilon_{\dot{1}\dot{2}\dot{2}\dot{2}} = -\frac{tC_2 + d_2}{6r^2}\\
&\epsilon_{1111} = \frac{i}{4}\epsilon_{111\dot{1}} = \frac{-1}{6}\epsilon_{11\dot{1}\dot{1}} = \frac{-i}{4}\epsilon_{1\dot{1}\dot{1}\dot{1}} = \epsilon_{\dot{1}\dot{1}\dot{1}\dot{1}} = \frac{tC_3 + d_3}{4r^2}\\
&\epsilon_{1122} = \frac{i}{4}\epsilon_{122\dot{1}} = \frac{i}{4}\epsilon_{112\dot{2}} = \frac{-1}{6}\epsilon_{11\dot{2}\dot{2}} = \frac{-1}{6}\epsilon_{12\dot{1}\dot{2}} = \frac{-1}{6}\epsilon_{22\dot{1}\dot{1}} = \frac{-i}{4}\epsilon_{1\dot{1}\dot{2}\dot{2}} = \frac{-i}{4}\epsilon_{2\dot{1}\dot{1}\dot{2}}\\
&= \epsilon_{\dot{1}\dot{1}\dot{2}\dot{2}} = -\frac{tC_4 + d_4}{6r^2}\\
&\epsilon_{1112} = \frac{i}{4}\epsilon_{111\dot{2}} = \frac{i}{4}\epsilon_{112\dot{1}} = \frac{-1}{6}\epsilon_{11\dot{1}\dot{2}} = \frac{-1}{6}\epsilon_{12\dot{1}\dot{1}} = \frac{-i}{4}\epsilon_{2\dot{1}\dot{1}\dot{1}} = \frac{-i}{4}\epsilon_{1\dot{1}\dot{1}\dot{2}}\\
&= \epsilon_{\dot{1}\dot{1}\dot{1}\dot{2}} = -\frac{tC_5 + d_5}{6r^2}\,.
\end{aligned} \tag{5.2.45}$$

However, as discussed in [21,114], any transformation relating two solutions with different boundary falloff behavior is not a true gauge transformation. The Schrödinger solution we found has $t$ component being

$$W = W_2 + W_3 \to \frac{1}{r^2} \sim W_3 \qquad \text{as} \quad r \to 0$$

which is different from AdS boundary condition. This fact can also be seen from the parameters characterizing the transformation (5.2.46); the parameters diverge at the boundary $r = 0$, which means they are non-trivial on the boundary. Such transformation relates two different physical solutions, which means our Schrödinger solution is not equivalent to the AdS solution.

• It is intuitive to have an interpretation of the fields in terms of Einstein classical gravity theory. It is confirmed [3, 95] by perturbation calculations that 3D Einstein equation can be solved by $z = 2$ Schrödinger metric and its spin-3 matter fields. In the current 4D example, we again expect the spin-3 fields to be responsible for supporting the non-AdS metric solutions

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \,. \tag{5.2.46}$$

The solution in Vasiliev frame equation is a strong evidence indicating that the higher spin fields give the correct stress-energy tensor $T_{\mu\nu}$, although it is not simple to compute it explicitly due to the lack of an action. This nonvanishing $T_{\mu\nu}$ tensor also indicates that this solution is physically different from AdS vacuum solution. We expect this solution can be a simple model to study the interaction between spin-2 metric and higher spin fields. [7]

**Spacetime symmetry** We can find the spacetime symmetry of the full solution by finding all the Killing vectors of both the metric and the higher spin metric like fields. By definition, the Lie derivative of the fields along the direction of any killing vector $\chi^\mu$ vanishes

$$\mathcal{L}_\chi g_{\mu\nu} = 0 \,, \qquad \mathcal{L}_\chi \phi_{\mu\nu\rho} = 0 \,, \qquad \mathcal{L}_\chi \phi_{\mu\nu\rho\sigma} = 0 \;\; \ldots \,. \tag{5.2.47}$$

Solving the first equations, we find the follow killing vectors generating the Schrödinger isometry of the spacetime in our $z = 2$ example

$$\chi_H = \partial_t \,, \qquad \chi_M = \partial_\xi \,, \qquad \chi_P = \partial_x \,, \qquad \chi_K = x\partial_\xi - t\partial_x \,, \tag{5.2.48}$$

[7] In another known example, Schrödinger spacetime in $D \geq 4$ can be obtained by coupling a gauge field $A_\mu$ to the Einstein gravity and then turning on finite background $A_\mu$ field [34]. (Notice this gauge field also only has non-vanishing component in $t$ direction.)

Table 5.1: Symmetry enhancement and metric like fields

| | Killing vectors | $-g_{tt}r^4$ | spin-3 fields |
|---|---|---|---|
| (a) | $\chi_K$ | $162C_3^2$ | $\phi_{ttt}=\frac{72C_3}{r^4}$ |
| (b) | $\chi_D$ | $162C_3^2$ | $\phi_{ttt}=-\frac{3072C_3^3}{r^6}\,,\ \phi_{tt\xi}=\frac{-8C_3}{r^4}\,,\ \phi_{txx}=\frac{8C_3}{r^4}$ |
| (c) | $\chi_D-\sqrt{2}\chi_K$ | $\frac{9}{2}C_3^2$ | $\phi_{ttt}=\frac{6\left(3C_3r^2-8C_3^3\right)}{r^6}\,,\ \phi_{tt\xi}=\frac{-2C_3}{r^4}$ <br> $\phi_{ttx}=\frac{6\sqrt{2}C_3}{r^4}\,,\ \phi_{txx}=\frac{2C_3}{r^4}$ |
| (d) | $\chi_D+\frac{2\left(C_1\mp\sqrt{C_1}\sqrt{C_3}\right)\chi_K}{\sqrt{2}C_1+\sqrt{2}\sqrt{C_3}\sqrt{C_1}}$ | $\frac{9}{2}(C_1^2+34C_3C_1+C_3^2)$ | $\phi_{ttt}=\frac{6\left(3\left(\pm\sqrt{C_1}-\sqrt{C_3}\right)^2r^2-8\left(\pm\sqrt{C_1}+\sqrt{C_3}\right)^6\right)}{r^6}$ <br> $\phi_{tt\xi}=\frac{-2\left(\pm\sqrt{C_1}+\sqrt{C_3}\right)^2}{r^4},\ \phi_{ttx}=\frac{6\sqrt{2}(C_1-C_3)}{r^4}$ <br> $\phi_{txx}=\frac{2\left(\pm\sqrt{C_1}+\sqrt{C_3}\right)^2}{r^4}\,.$ |

$$\chi_D=2t\partial_t+x\partial_x+r\partial_r\,,\qquad \chi_C=t^2\partial_t-\tfrac{1}{2}(x^2+r^2)\partial_\xi+tx\partial_x+tr\partial_r\,. \tag{5.2.49}$$

Applying the Lie derivatives associated with these vectors to the spin-3 fields, we find in general only $H, M, P$ remain symmetry of the spin-3 fields. However, for special choice of the parameters $C_i$, $i=1,\ldots,5$, the symmetry of the system could get enhanced. These extra enhanced symmetries can be summarized in Table 5.1 where the coefficients take the following values in different cases:

$$\begin{aligned}
(a):&\quad C_2\to\tfrac{3}{2}C_3,\quad C_1\to C_3,\quad C_4\to-\tfrac{3}{2}C_3,\quad C_5\to\tfrac{3}{2}C_3\,, \\
(b):&\quad C_2\to 0,\quad C_1\to C_3,\quad C_4\to\tfrac{1}{2}C_3,\quad C_5\to 0\,,\\
(c):&\quad C_1\to 0,\quad C_2\to 0,\quad C_4\to-\tfrac{1}{4}C_3,\quad C_5\to\tfrac{3}{4}C_3\,,\\
(d):&\quad C_2\to\tfrac{3}{4}\left(C_1\mp\sqrt{C_1C_3}\right),\,C_4\to\tfrac{1}{4}\left(-C_1\pm4\sqrt{C_1C_3}-C_3\right),\,C_5\to\tfrac{3}{4}\left(\mp\sqrt{C_1C_3}+C_3\right)\,.
\end{aligned} \tag{5.2.50}$$

Thus we see that in case (a) the boost $K$ generator restores and the symmetry is enhanced to a Galilean group.[8] For another choice of the parameters (b), the

[8] In our convention, the Galilean group is generated by translations, rotations and boosts. One could also add in a dilatation generator, but the particle number will not be conserved under this scaling transformation for $z\neq 2$. Therefore in this paper we do not include this dilatation generator to be part of the Galilean group and consider it as part of the extension to the Schrödinger group at $z=2$.

scaling symmetry is respected. Furthermore, it is possible for some other choices of parameters (c), (d) that a linear combination of boost and scaling becomes a symmetry. But it is impossible that both of them become symmetry simultaneously; there are at most 4 generators in the symmetry of the solution.

The solutions (a), (b) and (c) have different boundary behaviour and hence are different physical solutions. While in case (d) the parameter $C_1$ is a gauge parameter that relates the solutions (d) to (c).

We then consider the symmetries of the spin-4 metric like fields. Astoundingly, the previously determined symmetries of the metric and spin-3 metric like fields are all symmetries of the spin-4 metric like field as well. This is very likely to be a consequence of the fact that in the frame like field $W$, only spin-3 components of the higher spin fields are turned on; even though the spin $s > 3$ metric like fields are non-vanishing, they do not carry new physical information.[9] Therefore we believe the symmetries we have found previously are symmetries of the full solution that we have constructed.

**Global internal symmetry** Global symmetry of a vacuum solution to the Vasiliev equation can be extracted from the equation

$$d\epsilon(y|x) + [W, \epsilon(y|x)]_* = 0\,, \tag{5.2.51}$$

which determines how does a given symmetry parameter $\epsilon_0(y)$ at any fixed spacetime point extend to a small neighborhood around this point. Since $W$ is a solution to the flatness equation, it is always possible to rewrite the vacuum solution in the form of a pure gauge in this neighborhood [115–117].

$$W = g^{-1}(y|x) * dg(y|x)\,. \tag{5.2.52}$$

The solution to the equation (5.2.51) in this gauge can be trivially solved as

$$\epsilon(y|x) = g^{-1}(y|x) * \epsilon_0(y) * g(y|x)\,, \tag{5.2.53}$$

where $\epsilon_0(y)$ does not depend on spacetime coordinates and fully determines the global (internal) symmetry. It is concluded in [3] that the symmetry of Schrödinger

[9] We thank Wei Li for a discussion on similar situations in 3D.

higher spin solution in 3D Chern-Simons theory is just $SL(N, R) \times SL(N, R)$ by applying the gauge function method above. In the current higher dimensional case, one could also conclude that $\epsilon_0(y)$ exhausts the whole Vasiliev higher spin symmetry group.

### 5.2.4 Solutions with other scaling factors

As we have mentioned in the introduction, $z = 2$ Schrödinger spacetime has a larger isometry group than Schrödinger spacetime with $z \neq 2$. To demonstrate that our construction is universal for all integer $z$ rather than merely a result of the larger symmetry group at $z = 2$, we have also constructed the $z = 3$ Schrödinger spacetime in a similar way. The $z = 3$ Schrödinger spacetime turns out to be supported by spin-4 fields in the $t$ direction. I put the solution in appendix.

From the construction, we find explicitly that the back-reaction of spin-$s$ fields "deforms" $AdS_4$ to Schrödinger spacetime in 4D with $z = s - 1$.

A general spin-$s$ field $W_{(2s-2)} = \left\{\omega_{\alpha_1 \ldots \alpha_{2s-2}},\ \ldots,\ \omega_{\dot{\alpha}_1 \ldots \dot{\alpha}_{2s-2}}\right\}$ has $N_s = \dfrac{s}{3}(4s^2 - 1)$ independent components, which is the same as the number of independent equations in (5.2.32). In other words, if one specifies a group of parameters as "boundary conditions" of the differential equations, all the components of master field $W$ can be uniquely determined. Furthermore, if this group of parameters can be fixed from a given set of generalised vielbein, as in our spin-3 example, there is no degeneracy problem. This property can only be checked case by case.

We have considered solution to the Vasiliev equation that corresponds to spacetime with Schrödinger isometry. These solutions are derived by turning on higher spin fields with one given spin. One immediate question is what if we turn on fields with different spins in a similar manner. [10]

From the above construction, we notice that the higher spin fields only enter equation (5.2.30b) and hence fields with different spins are in general independent to each other. Therefore, the general solution with different higher spin fields turned on is simply a linear combination of the previous solutions where only one single

[10] We thank Matthias Gaberdiel for pointing this direction to us.

higher spin field is turned on. Thus the general solution gives the following metric

$$ds^2 = (\sum_{i=i_{\min}}^{i_{\max}} \frac{f_i}{r^{2i-2}})dt^2 + \frac{2dtd\xi + dr^2 + dx^2}{r^2}\,, \tag{5.2.54}$$

where the index $i_{\min}$ and $i_{\max}$ are the minimal and maximal spins we have turned on in the $t$-direction. The number of independent parameter $f_i$ agrees with the number of different higher spin fields. Higher spin Fronsdal fields can be similarly determined.

Geometrically, these solutions interpolate between Schrödinger-like geometries with different dynamic exponents. This can be easily verified not only for the metric but also the higher spin Fronsdal fields. The existence of this type solution is due to the presence of higher spin fields, as well studied in the pure AdS case [98].

## 5.A Higher spin algebra in $D = 3$

We will follow the notation in [24, 118]. The higher spin algebra $hs[\lambda]$ generator $V^s_m$ are defined to be

$$V^s_m = (-1)^{s-1-m}\frac{(s+m-1)!}{(2s-2)!}\left[\underbrace{V^2_{-1}, ...[V^2_{-1}, [V^2_{-1}}_{s-m-1 \text{ terms}}, (V^2_1)^{s-1}]]\right]\,, \tag{5.1.55}$$

where

$$V^2_1 = L_1\,, \quad V^2_0 = L_0\,, \quad V^2_{-1} = L_{-1}\,.$$

If $\lambda = N$, the algebra is truncated to $sl(N)$ and all the $s > N$ generators can be removed. The lone star product between generators has a closed form

$$V^s_m * V^t_n = \frac{1}{2}\sum_{u=1}^{s+t-|s-t|-1} g^{st}_u(m, n; \lambda)V^{s+t-u}_{m+n}\,, \tag{5.1.56}$$

with

$$g^{st}_u(m, n; \lambda) = (\frac{1}{4})^{u-2}\frac{1}{2(u-1)!}\phi^{st}_u(\lambda)N^{st}_u(m, n)\,,$$

where

$$N^{st}_u(m, n) = \sum_{k=0}^{u-1}(-1)^k\binom{u-1}{k}[s-1+m]_{u-1-k}[s-1-m]_k[t-1+n]_k[t-1-n]_{u-1-k}\,,$$

$$\phi_u^{st}(\lambda) = {}_4F_3 \left[ \begin{matrix} \frac{1}{2}+\lambda, \frac{1}{2}-\lambda, \frac{2-u}{2}, \frac{1-u}{2} \\ \frac{3}{2}-s, \frac{3}{2}-t, \frac{1}{2}+s+t-u \end{matrix} \mid 1 \right] .$$

Here $[a]_n = a(a-1)...(a-n+1)$ are the descending Pochhammer symbol. The commutator of two generators are defined as

$$[X, Y] = X * Y - Y * X . \tag{5.1.57}$$

$V_m^s$ transforms in the $(2s-1)$ dimensional representation of $sl(2)$ Lie algebra

$$[V_m^2, V_m^s] = (-n + m(s-1))V_{m+n}^s , \tag{5.1.58}$$

which is also one of the useful formulas used in verifying Schrödinger solution. The trace of lone star product is defined to be

$$\mathrm{tr}(X * Y) = X * Y \mid_{V_m^s=0, s>0} . \tag{5.1.59}$$

The relation with the oscillator realization is via the identification

$$V_1^2 = \frac{1}{2}T_{11} , \qquad V_0^2 = \frac{1}{2}T_{12} , \qquad V_{-1}^2 = \frac{1}{2}T_{22} . \tag{5.1.60}$$

Other higher spin generators $V_m^s$ are related to $T_{\alpha\beta}$ via equation (5.1.55).

## 5.B Prove local Lorentz invariance of metric-like fields in 4D

We are going to show in the section that metric-like fields defined in section 3 are invariant under generalized local Lorentz transformation. Take the following ansatz:

$$E = e_2 + e_3 = e_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} + \omega_{\alpha\beta\dot{\gamma}\dot{\kappa}} y^\alpha y^\beta \bar{y}^{\dot{\gamma}} \bar{y}^{\dot{\kappa}}$$

We will take spin-2 metric-like fields $g_{\mu\nu}$ as example. Invariance of higher spin fields can be proved in similar way, but requires more texts to explain.

It is very straightforward to check $g = \mathrm{Tr}(e_2 * e_2)$ is invariant under local Lorentz transformation $\Lambda_2$ if only spin-2 fields are involved. In this case, we can confirm

$$e_2 = e_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}}; \qquad \omega = \omega_{\alpha\beta} y^\alpha y^\beta + \omega_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} \tag{5.2.61}$$

$$\xi = \bar{\epsilon}_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}}; \qquad \Lambda_2 = \bar{\epsilon}_{\alpha\beta} y^\alpha y^\beta + \bar{\epsilon}_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} \tag{5.2.62}$$

Then

$$\delta e_2 = d\xi + [e_2, \Lambda_2] + [\omega, \xi]$$

$$\delta g = \mathrm{Tr}([e_2, \Lambda_2]_* * e_2 + e_2 * [e_2, \Lambda_2]_*) = 0$$

Spin-3 case is more complicated. For clarity, we will try to prove its invariance under the basis of oscillator $y^\alpha$. Denote the generalized local Lorentz transformation as $\Lambda = \Lambda_2 + \Lambda_3$. $\Lambda_3$ are those terms whose commutator with $E$ would vary it by $\delta E$.

$$\begin{aligned} & \delta_\Lambda E * E = [E, \Lambda]_* * E \\ = \; & [e_2, \Lambda_2]_* * e_3 + [e_3, \Lambda_3]_* * e_3 + [e_2, \Lambda_3]_* * e_2 + [e_3, \Lambda_2]_* * e_2 \qquad (5.2.63) \\ & + [e_2, \Lambda_2]_* * e_2 + [e_3, \Lambda_3]_* * e_2 + [e_2, \Lambda_3]_* * e_3 + [e_3, \Lambda_2]_* * e_3 \qquad (5.2.64) \end{aligned}$$

Note finally, we need to prove $\delta_\Lambda g = 0$. The trace structure helps us simplify the calculation. Note all the 4 terms in (5.2.63) would not have contribution to $\delta_\Lambda g$. Take first term as an example. The commutator results in terms with odd numbers of $\epsilon_{ab}$ tensor, so $[e_2, \Lambda_2]_*$ only has terms with two $y$s. The trace contraction of $y^2$ and $y^4$ by star product is always zero.

We are interested in those spin-3 gauge transformation terms whose commutator with $E$ change the value of $E$. These terms are

$$\Lambda_3 \sim \bar{\epsilon}_{\alpha\beta\gamma\dot{\kappa}} y^\alpha y^\beta y^\gamma \bar{y}^{\dot{\kappa}} + \bar{\epsilon}_{\alpha\dot{\beta}\dot{\gamma}\dot{\kappa}} y^\alpha \bar{y}^{\dot{\beta}} \bar{y}^{\dot{\gamma}} \bar{y}^{\dot{\kappa}}$$

We take the calculation of first term as example. By calculation, $[e_2, \Lambda_3]_* * e_3 + e_3 * [e_2, \Lambda_3]_*$ has a term

$$[e_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}}, \bar{\epsilon}_{\gamma\kappa\sigma\dot{\tau}} y^\gamma y^\kappa y^\sigma \bar{y}^{\dot{\tau}}]_* * e_3 = 2 e_{\alpha\dot{\beta}} \bar{\epsilon}_{\gamma\kappa\sigma\dot{\tau}} (3\epsilon^{\alpha\gamma} y^\kappa y^\sigma \bar{y}^{\dot{\beta}} \bar{y}^{\dot{\tau}} + \epsilon^{\dot{\beta}\dot{\tau}} y^\alpha y^\gamma y^\kappa y^\sigma) * e_3 \tag{5.2.65}$$

The second term above is not important since it vanishes after taking trace. The first term results in

$$\begin{aligned} & \mathrm{Tr}(6 e_{\alpha\dot{\beta}} \bar{\epsilon}_{\gamma\kappa\sigma\dot{\tau}} \epsilon^{\alpha\gamma} y^\kappa y^\sigma \bar{y}^{\dot{\beta}} \bar{y}^{\dot{\tau}} * e_{ab\dot{c}\dot{d}} y^a y^b \bar{y}^{\dot{c}} \bar{y}^{\dot{d}}) \\ = \; & 96 e_{\alpha\dot{\beta}} \bar{\epsilon}_{\gamma\kappa\sigma\dot{\tau}} \epsilon^{\alpha\gamma} \epsilon^{\kappa a} \epsilon^{\sigma b} \epsilon^{\dot{\beta}\dot{c}} \epsilon^{\dot{\tau}\dot{d}} \qquad (5.2.66) \end{aligned}$$

This term is exactly cancelled by its counter partners in $[e_3, \Lambda_3]_* * e_2 + e_2 * [e_3, \Lambda_3]_*$. Since

$$[e_3, \bar{\epsilon}_{\alpha\beta\gamma\dot{\kappa}} y^\alpha y^\beta y^\gamma \bar{y}^{\dot{\kappa}}]_* = \mathcal{O}(y^6) + 48 e_{ab\dot{c}\dot{d}} \bar{\epsilon}_{\gamma\kappa\sigma\dot{\tau}} \epsilon^{a\kappa} \epsilon^{b\sigma} \epsilon^{\dot{d}\dot{\tau}} y^\gamma \bar{y}^{\dot{c}}$$

The first term has no influence on the result. The second term contracts with $e_2$ and gives $96 e_{\alpha\dot{\beta}} \bar{\epsilon}_{\gamma\kappa\sigma\dot{\tau}} e_{ab\dot{c}\dot{d}} \epsilon^{a\kappa} \epsilon^{b\sigma} \epsilon^{\dot{d}\dot{\tau}} \epsilon^{\gamma\alpha} \epsilon^{\dot{c}\dot{\beta}}$, which exactly cancels the term (5.2.66). The cancellation of other term related to $\bar{\epsilon}_{\alpha\dot{\beta}\dot{\gamma}\dot{\kappa}}$ can be shown in similar way. The other two terms in (5.2.64) can be trivially cancelled by counter partners in $E * [E, \Lambda]_*$. Putting all these results together, we prove the metric defined by star-product trace has local Lorentz transformation invariance.

## 5.C $z = 3$ Schrödinger solution

We will show explicitly a solution to spin-4 $t$ component Vasiliev equation. Consider $W_4$ $t$ component perturbation near AdS vacuum (5.2.30), then equation (5.2.30a) is modified as

$$d_x W_4 + W_2 * \wedge W_4 + W_4 * \wedge W_2 = 0 \tag{5.3.67}$$

Expand in terms of components, $W_4$ is

$$\begin{aligned} W_{4,t} \;\; = \;\; & \omega_{\alpha\beta\gamma\kappa\delta\tau} y^\alpha y^\beta y^\gamma y^\kappa y^\delta y^\tau + \omega_{\alpha\beta\gamma\kappa\delta\dot{\tau}} y^\alpha y^\beta y^\gamma y^\kappa y^\delta \bar{y}^{\dot{\tau}} + \omega_{\alpha\beta\gamma\kappa\dot{\delta}\dot{\tau}} y^\alpha y^\beta y^\gamma y^\kappa \bar{y}^{\dot{\delta}} \bar{y}^{\dot{\tau}} \\ & + \omega_{\alpha\beta\gamma\dot{\kappa}\dot{\delta}\dot{\tau}} y^\alpha y^\beta y^\gamma \bar{y}^{\dot{\kappa}} \bar{y}^{\dot{\delta}} \bar{y}^{\dot{\tau}} + \omega_{\alpha\beta\dot{\gamma}\dot{\kappa}\dot{\delta}\dot{\tau}} y^\alpha y^\beta \bar{y}^{\dot{\gamma}} \bar{y}^{\dot{\kappa}} \bar{y}^{\dot{\delta}} \bar{y}^{\dot{\tau}} + \omega_{\alpha\dot{\beta}\dot{\gamma}\dot{\kappa}\dot{\delta}\dot{\tau}} y^\alpha \bar{y}^{\dot{\beta}} \bar{y}^{\dot{\gamma}} \bar{y}^{\dot{\kappa}} \bar{y}^{\dot{\delta}} \bar{y}^{\dot{\tau}} \\ & + \omega_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\kappa}\dot{\delta}\dot{\tau}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}} \bar{y}^{\dot{\gamma}} \bar{y}^{\dot{\kappa}} \bar{y}^{\dot{\delta}} \bar{y}^{\dot{\tau}} \end{aligned} \tag{5.3.68}$$

The solution to (5.3.67) turns out to be

$$\begin{aligned}
&\omega_{\dot{2}\dot{2}\dot{2}\dot{2}\dot{2}\dot{2}} = -\frac{iC_1}{6r^3}\,, \quad \omega_{\dot{1}\dot{2}\dot{2}\dot{2}\dot{2}\dot{2}} = -\frac{iC_2}{6r^3}\,, \quad \omega_{\dot{1}\dot{1}\dot{2}\dot{2}\dot{2}\dot{2}} = -\frac{C_5}{15r^3}\,, \quad \omega_{\dot{1}\dot{1}\dot{1}\dot{2}\dot{2}\dot{2}} = \frac{iC_7}{20r^3}\,, \\
&\omega_{\dot{1}\dot{1}\dot{1}\dot{1}\dot{2}\dot{2}} = -\frac{C_6}{15r^3}\,, \quad \omega_{\dot{1}\dot{1}\dot{1}\dot{1}\dot{1}\dot{2}} = -\frac{iC_3}{6r^3}\,, \quad \omega_{\dot{1}\dot{1}\dot{1}\dot{1}\dot{1}\dot{1}} = -\frac{iC_4}{6r^3}\,, \quad \omega_{2\dot{2}\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{C_1}{r^3}\,, \\
&\omega_{2\dot{1}\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{C_2}{r^3}\,, \quad \omega_{2\dot{1}\dot{1}\dot{2}\dot{2}\dot{2}} = -\frac{2iC_5}{5r^3}\,, \quad \omega_{2\dot{1}\dot{1}\dot{1}\dot{2}\dot{2}} = -\frac{3C_7}{10r^3}\,, \quad \omega_{2\dot{1}\dot{1}\dot{1}\dot{1}\dot{2}} = -\frac{2iC_6}{5r^3}\,, \\
&\omega_{2\dot{1}\dot{1}\dot{1}\dot{1}\dot{1}} = \frac{C_3}{r^3}\,, \quad \omega_{22\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{5iC_1}{2r^3}\,, \quad \omega_{22\dot{1}\dot{2}\dot{2}\dot{2}} = \frac{5iC_2}{2r^3}\,, \quad \omega_{22\dot{1}\dot{1}\dot{2}\dot{2}} = \frac{C_5}{r^3}\,, \\
&\omega_{22\dot{1}\dot{1}\dot{1}\dot{2}} = -\frac{3iC_7}{4r^3}\,, \quad \omega_{22\dot{1}\dot{1}\dot{1}\dot{1}} = \frac{C_6}{r^3}\,, \quad \omega_{222\dot{2}\dot{2}\dot{2}} = -\frac{10C_1}{3r^3}\,, \quad \omega_{222\dot{1}\dot{2}\dot{2}} = -\frac{10C_2}{3r^3}\,, \\
&\omega_{222\dot{1}\dot{1}\dot{2}} = \frac{4iC_5}{3r^3}\,, \quad \omega_{222\dot{1}\dot{1}\dot{1}} = \frac{C_7}{r^3}\,, \quad \omega_{2222\dot{2}\dot{2}} = -\frac{5iC_1}{2r^3}\,, \quad \omega_{2222\dot{1}\dot{2}} = -\frac{5iC_2}{2r^3}\,,
\end{aligned}$$

$$
\begin{aligned}
&\omega_{2222\dot{1}\dot{1}} = -\frac{C_5}{r^3}\,, \qquad \omega_{22222\dot{2}} = \frac{C_1}{r^3}\,, \qquad \omega_{22222\dot{1}} = \frac{C_2}{r^3}\,, \qquad \omega_{222222} = \frac{iC_1}{6r^3}\,, \\
&\omega_{1\dot{2}\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{C_2}{r^3}\,, \qquad \omega_{1\dot{1}\dot{2}\dot{2}\dot{2}\dot{2}} = -\frac{2iC_5}{5r^3}\,, \qquad \omega_{1\dot{1}\dot{1}\dot{2}\dot{2}\dot{2}} = -\frac{3C_7}{10r^3}\,, \qquad \omega_{1\dot{1}\dot{1}\dot{1}\dot{2}\dot{2}} = -\frac{2iC_6}{5r^3}\,, \\
&\omega_{1\dot{1}\dot{1}\dot{1}\dot{1}\dot{2}} = \frac{C_3}{r^3}\,, \qquad \omega_{1\dot{1}\dot{1}\dot{1}\dot{1}\dot{1}} = \frac{C_4}{r^3}\,, \qquad \omega_{12\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{5iC_2}{2r^3}\,, \qquad \omega_{12\dot{1}\dot{2}\dot{2}\dot{2}} = \frac{C_5}{r^3}\,, \\
&\omega_{12\dot{1}\dot{1}\dot{2}\dot{2}} = -\frac{3iC_7}{4r^3}\,, \qquad \omega_{12\dot{1}\dot{1}\dot{1}\dot{2}} = \frac{C_6}{r^3}\,, \qquad \omega_{12\dot{1}\dot{1}\dot{1}\dot{1}} = \frac{5iC_3}{2r^3}\,, \qquad \omega_{122\dot{2}\dot{2}\dot{2}} = -\frac{10C_2}{3r^3}\,, \\
&\omega_{122\dot{1}\dot{2}\dot{2}} = \frac{4iC_5}{3r^3}\,, \qquad \omega_{122\dot{1}\dot{1}\dot{2}} = \frac{C_7}{r^3}\,, \qquad \omega_{122\dot{1}\dot{1}\dot{1}} = \frac{4iC_6}{3r^3}\,, \qquad \omega_{1222\dot{2}\dot{2}} = -\frac{5iC_2}{2r^3}\,, \\
&\omega_{1222\dot{1}\dot{2}} = -\frac{C_5}{r^3}\,, \qquad \omega_{1222\dot{1}\dot{1}} = \frac{3iC_7}{4r^3}\,, \qquad \omega_{12222\dot{2}} = \frac{C_2}{r^3}\,, \qquad \omega_{12222\dot{1}} = -\frac{2iC_5}{5r^3}\,, \\
&\omega_{122222} = \frac{iC_2}{6r^3}\,, \qquad \omega_{11\dot{2}\dot{2}\dot{2}\dot{2}} = \frac{C_5}{r^3}\,, \qquad \omega_{11\dot{1}\dot{2}\dot{2}\dot{2}} = -\frac{3iC_7}{4r^3}\,, \qquad \omega_{11\dot{1}\dot{1}\dot{2}\dot{2}} = \frac{C_6}{r^3}\,, \\
&\omega_{11\dot{1}\dot{1}\dot{1}\dot{2}} = \frac{5iC_3}{2r^3}\,, \qquad \omega_{11\dot{1}\dot{1}\dot{1}\dot{1}} = \frac{5iC_4}{2r^3}\,, \qquad \omega_{112\dot{2}\dot{2}\dot{2}} = \frac{4iC_5}{3r^3}\,, \qquad \omega_{112\dot{1}\dot{2}\dot{2}} = \frac{C_7}{r^3}\,, \\
&\omega_{112\dot{1}\dot{1}\dot{2}} = \frac{4iC_6}{3r^3}\,, \qquad \omega_{112\dot{1}\dot{1}\dot{1}} = -\frac{10C_3}{3r^3}\,, \qquad \omega_{1122\dot{2}\dot{2}} = -\frac{C_5}{r^3}\,, \qquad \omega_{1122\dot{1}\dot{2}} = \frac{3iC_7}{4r^3}\,, \\
&\omega_{1122\dot{1}\dot{1}} = -\frac{C_6}{r^3}\,, \qquad \omega_{11222\dot{2}} = -\frac{2iC_5}{5r^3}\,, \qquad \omega_{11222\dot{1}} = -\frac{3C_7}{10r^3}\,, \qquad \omega_{112222} = \frac{C_5}{15r^3}\,, \\
&\omega_{111\dot{2}\dot{2}\dot{2}} = \frac{C_7}{r^3}\,, \qquad \omega_{111\dot{1}\dot{2}\dot{2}} = \frac{4iC_6}{3r^3}\,, \qquad \omega_{111\dot{1}\dot{1}\dot{2}} = -\frac{10C_3}{3r^3}\,, \qquad \omega_{111\dot{1}\dot{1}\dot{1}} = -\frac{10C_4}{3r^3}\,, \\
&\omega_{1112\dot{2}\dot{2}} = \frac{3iC_7}{4r^3}\,, \qquad \omega_{1112\dot{1}\dot{2}} = -\frac{C_6}{r^3}\,, \qquad \omega_{1112\dot{1}\dot{1}} = -\frac{5iC_3}{2r^3}\,, \qquad \omega_{11122\dot{2}} = -\frac{3C_7}{10r^3}\,, \\
&\omega_{11122\dot{1}} = -\frac{2iC_6}{5r^3}\,, \qquad \omega_{111222} = -\frac{iC_7}{20r^3}\,, \qquad \omega_{1111\dot{2}\dot{2}} = -\frac{C_6}{r^3}\,, \qquad \omega_{1111\dot{1}\dot{2}} = -\frac{5iC_3}{2r^3}\,, \\
&\omega_{1111\dot{1}\dot{1}} = -\frac{5iC_4}{2r^3}\,, \qquad \omega_{11112\dot{2}} = -\frac{2iC_6}{5r^3}\,, \qquad \omega_{11112\dot{1}} = \frac{C_3}{r^3}\,, \qquad \omega_{111122} = \frac{C_6}{15r^3}\,, \\
&\omega_{11111\dot{2}} = \frac{C_3}{r^3}\,, \qquad \omega_{11111\dot{1}} = \frac{C_4}{r^3}\,, \qquad \omega_{111112} = \frac{iC_3}{6r^3}\,, \qquad \omega_{111111} = \frac{iC_4}{6r^3}
\end{aligned}
\tag{5.3.69}
$$

This field will backreact on AdS geometry to give a $z = 3$ Schrödinger spacetime metric.

# Chapter 6

# Discussions and outlook

We explored nonsingular hyperscaling violation spacetimes in chapter 2 and found that these geometries have smooth extensions beyond Poincare horizon in vacuum case. We conjectured singularity in singular non-relativistic spacetimes is reflected as IR divergences of field theory scattering amplitudes. By applying Maldacena and Alday's trick [19] in chapter 3, we found scattering amplitudes in Lifshitz spacetimes have universal stronger IR divergences than those in AdS spacetime, which is considered as a result of higher density of soft modes $\omega \sim k^z$.

In chapter 4, we want to discuss whether it's possible to resolve tidal force singularity in higher spin theory. We have seen that the Lifshitz and non-integer Schrödinger solutions of [92] have degenerate generalized vielbeins, so they are not equivalent to some solution in the metric formulation of the higher spin theory. We also found that in all cases the symmetries of the backgrounds in the Chern-Simons formulation are $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$, generalizing and simplifying an observation of [93]. These seem significant obstacles to interpreting these backgrounds as non-relativistic solutions. The Schrödinger solutions with integer $z$ have non-degenerate generalized vielbeins, so they remain as non-trivial examples of non-relativistic backgrounds in the higher spin context. But our results prevent us from studying several interesting questions about these backgrounds, such as identifying examples of Lifshitz field theories or addressing the physical meaning of the IR singularities in the metrics (1.2.8, 1.2.10).

These problems could be moderated by considering classes of solutions which asymptotically approach these backgrounds, although one would be concerned that the problem with the vacuum solution would reappear in the asymptotic region. For the most well-developed example of asymptotically Lifshitz boundary conditions in the higher spin context [93], we find that the generalized vielbein is still degenerate at first subleading order. We have proposed that these boundary conditions may be more usefully viewed instead as a novel asymptotically AdS boundary condition. In that gauge a metric formulation is available, and it would be interesting to understand the differences from the usual asymptotically AdS boundary condition. For the boundary conditions of [100], the degeneracy of the generalized vielbeins was lifted, and it appeared that an inverse could exist even in the asymptotic region. It would be interesting to understand this case further.

The problems we have found are likely to be special to the case of three bulk dimensions, as the Chern-Simons formulation is particular to this case, and the absence of bulk degrees of freedom also obstructs obtaining richer families of solutions. Therefore, we showed explicitly how to construct Schödinger solution in 4D. These solutions of the Vasiliev higher spin theory have Galilean symmetry in $D = 4$ dimensions. Generalization to other dimensional spacetimes is straightforward by using vectorial construction. We show that the spacetime symmetry group can be the Galilean group or a non-relativistic scaling symmetry group. The field theory interpretation of this solution can be considered as an analogue of massive vector case [39]. Turning on spin-3 fields in the bulk corresponds to spin-3 current. Since bulk AdS higher spin theory corresponds to free 3D boundary CFT, the Schrödinger solution is expected to dual to a deformed CFT with spin-3 current. Therefore the immediate next step is to consider correlation functions of the bulk higher spin system on the Schrödinger background and in the dual field theories. This would provide another piece of strong evidence of whether our proposal is sensible or not. This is currently under investigation.

It is possible to construct $z = 2$ $3D$ Lifshitz spacetime by dimensional reduction. One can show that if one adds a constant one-form $\eta = \eta_t dt$ to the AdS gravitational connection

$$e = e_{\alpha\dot{\beta}} y^\alpha \bar{y}^{\dot{\beta}} + \eta; \qquad \omega = \omega_{\alpha\beta} y^\alpha y^\beta + \omega_{\dot{\alpha}\dot{\beta}} \bar{y}^{\dot{\alpha}} \bar{y}^{\dot{\beta}}\,, \tag{6.0.1}$$

the master field $W$ still solves Vasiliev equation. It turns out that the corresponding metric represents the $z = 0$ Schrödinger spacetime

$$ds^2 = -\eta_t^2 dt^2 + \frac{2dtd\xi + dr^2 + dx^2}{r^2}\,. \tag{6.0.2}$$

To proceed, we use the fact that $D-1$ dimensional $z = 2$ Lifshitz spacetime emerges from $z = 0$ Schrödinger spacetime in $D \geq 4$ dimension by dimensional reduction in the $t$ direction [73, 74, 81, 119]. Those $3D$ Lifshitz spacetimes are solutions of Einstein equation with supporting matter fields and therefore safe from degeneracy problem in higher spin theory [3]. One may be able to study how higher spin transformation operates on the Lifshitz geometry, and understand the physical meaning of IR singularity.

Even though we offer many calculations and discussions about the nature of Lifshitz/Schrödinger singularity in this thesis, we are still unable to draw any conclusion to tell how to resolve it. This singularity does not affect many holographic computations, thus many constructions of non-relativistic holography are then valid without exploring this issue. Apart for this topic, there are many unanswered questions in non-relativistic holography. A few interesting topics are mentioned in [32]. Let me list several important ones.

It would be interesting to know whether Schrödinger black hole solution exists in $4D$ Vasiliev theory. The known higher spin solution in 3 dimension [120, 121], the charged black hole solution with asymptotic Schrödinger geometry [57–59] together with the reformulation of $AdS_4$ Kerr black hole solution into the unfolding formalism [122] hint on possibility of finding black hole solutions with asymptotic Schrödinger geometry in higher spin theory. We will leave this for future work.

One important question is whether Lifshitz solution exists in higher spin theory? Although we prefer to give a negative answer to current construction [3], Newton-Carton gravity is potentially able to contain such a theory. The advantages of Newton-Carton gravity contain two perspectives. One statement is Horava Lifshitz gravity emerge from Newton-Carton gravity if the latter is made dynamical [42]. Horava Lifshitz gravity allows Lifshitz solution without matter to support. Therefore, (4.2.22) may be able to solve the modified Einstein equation. On the other hand, Newton-Carton gravity allows degenerate metrics $g_{\mu\nu} = -\tau_\mu \tau_\nu + h_{\mu\nu}$. Then Lifshitz higher spin frame solution may have well-defined metric-like interpretation [123].

Newton-Cartan gravity as a non-relativistic limit of AdS/CFT, may be more suitable to model non-relativistic holography. In this formulation, the symmetry of holographic Lifshitz theories can be enhanced [124–126]. One may wonder what kind of holographic Lifshitz may allow such constructions or further the enhanced symmetry. This is still an open question.

Calculations of non-relativistic holography is technically more challenging. Perturbation methods are introduced to study non-relativistic holographic theories which are small deformation of $AdS$ holography: Lifshitz geometry near $z = 1$ and boosted Schrödinger geometry (4.2.82) [39, 66]. However, $z \to \infty$ limit Lifshitz is less explored. This is due to back reaction [53,127] and other subtleties of the limit geometry $AdS_2$. Backreaction effect can deform the $AdS_2$ geometry to be a Lifshitz like spacetime [128] in some models. An interesting question is to what extent do Lifshitz spacetimes inherit backreaction effect from $AdS_2$ geometry? Besides this, one can see that the asymptotic boundary geometry of Schrödinger spacetime also contains an $AdS_2$ factor [129]. Understanding $AdS_2$ gravity can promote the study of non-relativistic holography.

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