A Singular Theta Lift and the Shimura Correspondence

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Abstract

Modular forms play a central and critical role in the study of modern *number theory*. These remarkable and beautiful functions have led to many spectacular results including, most famously, the proof of Fermat's Last Theorem. In this thesis we find connections between these enigmatic objects. In particular, we describe the construction and properties of a *singular theta lift*, closely related to the well known *Shimura correspondence*.

We first define a (twisted) lift of *harmonic weak Maass forms* of weight $3/2-k$, by integrating against a well chosen kernel Siegel theta function. Using this, we obtain a new class of automorphic objects in the upper-half plane of weight $2-2k$ for the group $\Gamma_0(N)$. We reveal these objects have intriguing *singularities* along a collection of geodesics. These singularities divide the upper-half plane into Weyl chambers with associated *wall crossing formulas*. We show our lift is harmonic away from the singularities and so is an example of a *locally harmonic Maass form*. We also find an explicit Fourier expansion.

The Shimura/Shintani lifts provided very important correspondences between half-integral and even weight modular forms. Using a natural differential operator we link our lift to these. This connection then allows us to derive the properties of the Shimura lift. The nature of the singularities suggests we formulate all of these ideas as distributions and finally we consider the current equation encompassing them.

This work provides extensions of the theta lifts considered by Borcherds (1998), Bruinier (2002), Bruinier and Funke (2004), Hövel (2012) and Bringmann, Kane and Viazovska (2013).

Declaration

The work in this thesis is based on research carried out in the Pure Mathematics Group at the Department of Mathematical Sciences, University of Durham. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

1.1 Motivation

Number theory studies the properties of the integers and is one of the most natural, fascinating and oldest areas of mathematics. Many of the problems (unlike lots of areas of mathematics) can be simply stated and understood. This is part of what makes this subject intrinsically appealing to professional mathematicians and the layperson alike, as well as the obvious importance of investigating some of the most fundamental objects in mathematics.

Questions in number theory have been investigated by many of the most renowned mathematicians in history from Euclid to Euler. Gauss famously considered mathematics to be the "queen of the sciences" and number theory to be the "queen of mathematics". This subject was for many years, considered to be entirely abstract, with Leonard Dickson commenting "Thank God that number theory is unsullied by any application". However in recent years many uses have been found. These include contributions in theoretical physics, combinatorics, chemistry and computer science. Arguably the most significant of these is in modern cryptography, with the security of most online communications relying on ideas from number theory.

Some examples of number theoretic questions include: are there infinitely many primes? how many integer solutions are there to certain polynomial equations? and how many ways can we write a number as a sum of positive integers? The Riemann hypothesis and the Birch and Swinnerton-Dyer conjecture are further examples, and they form two of the seven Millenium Prize Problems (a list of some of the most important and difficult mathematical problems, each with an attached $1 million prize).

The simplicity of stating problems in number theory however belies the sophistication of the mathematical objects often needed to find solutions. One of the most significant of these tools, is modular forms. These form one of the largest areas of research in modern number theory. These powerful and striking objects have many "symmetries", important relationships to elliptic curves and often beguiling Fourier coefficients.

This thesis is concerned with the properties of modular objects and the links between them. As detailed in Section 1.3, the topics we investigate will include theta functions, theta lifts, half-integral weight harmonic weak Maass forms and locally harmonic weak Maass forms. A very small selection of famous headlines in these areas, include results on:

- Solutions to the "kissing number problem" in 8 and 24 dimensions [CS99].
- The Birch and Swinnerton-Dyer conjecture (in the case of rank 1) using Heegner points and weight 3/2 modular forms [GZ86, GKZ87].
- A recently claimed proof of the umbral moonshine conjecture [DGO15].
- A resolution (conditional on parts of Birch and Swinnerton-Dyer conjecture) to the "congruent number problem", using the Shimura correspondence [Shi73, Tun83, Kob93].
- Representation numbers, using the classical theory of theta functions [DS05].
- An explicit finite formula for the partition function, using a theta lift between harmonic weak Maass forms [BO13].
- And of course the proof of Fermat's last theorem [Wil95].

1.2 Literature Overview

In this section we succinctly discuss the literature and history related to this thesis. From now on (and throughout this work) we will assume knowledge of classical elliptic modular forms. If not, suggested introductory texts are [Kob93, DS05, BvdGHZ08, Kil08]. Formal definitions of many of the terms in this introduction can be found later in Chapter 2.

The theory of modular forms of half-integral weight really began with Shimura, in a famous paper [Shi73]. He laid the foundations by constructing a family of maps from half-integral weight cusp forms and even weight holomorphic modular forms. The significant results of Waldspurger [Wal81] and Kohnen and Zagier [KZ81], used these ideas to show there is a coefficient of a half-integral weight modular form that agrees with the central value of the L-function of an even weight modular form. Tunnel [Tun83] then applied these results to the aforementioned congruent number problem.

The Shimura correspondence was first realised as a theta lift by Niwa [Niw75]. Theta lifts provide important relationships between automorphic forms of different groups and often allow us to construct concrete examples. Shintani [Shi75] described a closely related theta lift. This mapped forms “in the opposite direction” to the Shimura correspondence and formed an adjoint lift. Much later Borcherds [Bor98] defined a notable singular theta lift of vector-valued forms. This encompassed the Shimura lift, as well as many other examples, such as the Gritsenko [Gri88] and Doi-Naganuma lifts [DN70]. Borcherds used a regularisation (Harvey and Moore [HM96]) of the theta integral to enlarge the inputs of his lift to weakly holomorphic modular forms. The Borcherds lift also gave rise to remarkable product expansions of some automorphic forms.

These theta lifts can also be viewed in a more general framework. They are examples of the theta correspondence between automorphic forms on two groups which form a dual reductive pair. This is in the sense of Howe duality [How79]. In the case of Borcherds’s singular theta correspondence the dual pair is the orthogonal group and the modular group.

More recently Bruinier [Bru02] extended the constructs from [Bor98] to some non-holomorphic Poincaré series. This led to the thorough introduction of harmonic weak Maass forms by Bruinier and Funke [BF04]. Harmonic weak Maass forms are natural generalisations of classical modular forms. This coincided with the work of Zwegers [Zwe02]. Zwegers showed that Ramanujan’s famous mock theta functions (from his famous death bed letter to Hardy) were holomorphic parts of harmonic weak Maass forms. These developments were the catalyst for a lot of recent exciting results and applications. In [BF04] they also introduce a new Borcherds lift for arbitrary signature and lift harmonic weak Maass forms. They prove this lift is adjoint to the Kudla-Millson lift [KM90].

Locally harmonic weak Maass forms were first formally defined by Bringmann, Kane and Kohnen [BKK12]. These forms are similar to harmonic weak Maass forms but may also exhibit singularities. Some examples of these objects have been constructed using theta lifts. Hövel [Höv12] describes a twisted singular theta lift of vector-valued weight $1/2$ harmonic weak Maass forms. He effectively works in a space of signature $(2, 1)$ and his lift generates weight 0 locally harmonic Maass forms. He links his lift to the Shimura lift. Bringmann, Kane and Viazovska [BKV13] also consider a very similar lift for higher weights. Specifically, they lift some scalar-valued non-twisted Poincaré series of full level and weight $3/2 - k$. Here $k \in \mathbb{Z}$ is restricted to be even and $k > 0$.

Our work fits into the literature in the following respects. We will form a regularised twisted singular theta lift working in signature $(2,1)$. In our case the theta correspondence is for the dual pair: $\mathrm{O}(2,1)$ and the modular group. We will use the same kernel function as [BKV13] (see also [BKZ14]). Let $k \in \mathbb{Z}, k \geq 1$. We will then lift vector-valued harmonic weak Maass forms of weight $3/2-k$ to recover locally harmonic Maass forms of weight $2-2k$ for the group $\Gamma_0(N)$. In particular, we will directly extend the works in [Bor98,Bru02,BF04,Höv12,BKV13]. We relate our lift to the Shimura and Shintani lifts, via a differential operator, and obtain a commutative diagram.

1.3 Thesis Overview

In this section we summarise the *main results* of this thesis. This also provides an overview of the content and key ideas in each chapter. We will often omit technical details.

Chapter 2, Background

We introduce the basic mathematical objects we will need throughout. We fix $\tau = u + iv \in \mathbb{H}$. We first comprehensively discuss *quadratic spaces* and *lattices*. We denote (V,Q) for a rational non-degenerate quadratic space of signature (b^+, b^-), L for a even lattice $L \subset V$, L' for the dual lattice and L'/L for the (abelian) discriminant group. We then compactly derive the well known *Weil representation* on the *metaplectic group*.

In this thesis we need to deal with Siegel theta functions on $L \subset V$ and various half-integral weight forms. So we will use vector-valued forms transforming under the action of $\tilde{\Gamma} \subset \mathrm{Mp}_2(\mathbb{R})$ (the metaplectic group) with respect to the Weil representation ρ_L where $\mathrm{Mp}_2(\mathbb{R})$ is a double cover of $\mathrm{SL}_2(\mathbb{R})$. We let $\kappa \in \frac{1}{2}\mathbb{Z}$. Then we will call a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ an *automorphic form* A_{κ,ρ_L} if $f|_{\kappa,\rho_L}\tilde{\gamma} = f$ for all $\tilde{\gamma} \in \tilde{\Gamma}$.

The functions that we will lift are generalisations of modular forms M_{κ,ρ_L} called *harmonic weak Maass forms* (see [BF04]) which we denote as H_{κ,ρ_L}. Instead of requesting holomorphicity we request the weaker condition that f vanishes under a Laplacian operator Δ_κ. We also have a growth condition on the cusps. We know $f \in H_{\kappa,\rho_L}$ have Fourier expansions of the form

$$\sum_{h \in L'/L} \sum_{n \gg -\infty} c^+(n,h) e(n\tau) \mathfrak{e}_h + \sum_{h \in L'/L} \sum_{n<0} c^-(n,h) \Gamma(1-\kappa, 4\pi|n|v) e(n\tau) \mathfrak{e}_h$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function and $\mathfrak{e}_h$ is the standard basis element of $\mathbb{C}[L'/L]$ corresponding to $h \in L'/L$. Crucially there is a *differential operator* $\xi_\kappa = 2iv^\kappa \overline{\frac{\partial}{\partial \overline{\tau}}}$. This maps surjectively to the cusp forms:

$$\xi_\kappa : H_{\kappa,\rho_L} \to S_{2-\kappa,\overline{\rho}_L}.$$

We also introduce (scalar-valued) *locally harmonic weak Maass forms* LH_κ ($\kappa \in 2\mathbb{Z}$). These forms mirror H_κ but are only harmonic within connected components, away from a measure zero exceptional set E. They have polynomial growth at the cusps. Finally we use the Weil representation to construct some *Siegel theta functions*. We denote $\mathrm{Gr}(V(\mathbb{R}))$ for the *Grassmannian* of $V(\mathbb{R})$, which is the set of negative definite b^--dimensional subspaces in $V(\mathbb{R})$. If $b^+ = 2$ or $b^- = 2$ then $\mathrm{Gr}(V(\mathbb{R}))$ can be given a complex structure. We then define for $z \in \mathrm{Gr}(V(\mathbb{R}))$,

$$\Theta_L(\tau, z) := v^{b^-/2} \sum_{h \in L'/L} \sum_{\lambda \in L+h} e(Q(\lambda)u + Q_z(\lambda)iv)\mathfrak{e}_h \tag{1.3.1}$$

where $Q_z(\lambda) = Q(\lambda_{z^\perp}) - Q(\lambda_z) \geq 0$ is the majorant. There is also a more general definition involving an additional polynomial term. We have two significant properties:

1. $\Theta_L(\tau, z)$ has weight $\frac{b^+ - b^-}{2}$ in τ for $\tilde{\Gamma}$.
2. $\Theta_L(\tau, z)$ is invariant in z under the action of the orthogonal group $\mathrm{O}(L)$.

Chapter 3, The Setting

We first fix (V, Q) to have signature $(2, 1)$. This will form the setting for the rest of our work. We also fix $N \in \mathbb{N}$. Then (V, Q) has an explicit realisation as

$$V := \{\lambda \in \mathrm{M}_2(\mathbb{Q}) \mid \mathrm{tr}(\lambda) = 0\}$$

with quadratic form $Q(\lambda) := -N \det(\lambda)$. We fix L as the lattice

$$L := \left\{ \left(\begin{smallmatrix} b & -a/N \\ c & -b \end{smallmatrix} \right) \middle| \ a, b, c \in \mathbb{Z} \right\}$$

which is even and of level $4N$ and we have $L'/L \cong \mathbb{Z}/2N\mathbb{Z}$. This lattice is of particular interest as each $\lambda \in L'$ corresponds to an integral binary quadratic form.

We notice that $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ acting via conjugation on $\lambda \in V$ is isometric i.e. $Q(\gamma.\lambda) = Q(\lambda)$. This leads to the significant accidental isomorphism $\mathrm{PSL}_2(\mathbb{Q}) \cong \mathrm{SO}^+(V)$. We have that $\Gamma_0(N)$ acts trivially on L'/L. The idea is to then consider $\Gamma_0(N)$ acting via *conjugation* on V and z (as opposed to $\mathrm{O}(L)$ from earlier). In signature $(2, 1)$ we can also realise $\mathrm{Gr}(V(\mathbb{R}))$ as the *real hyperbolic space* of dimension 2 which we also identify with $\mathbb{H}$. We fix $z = x + iy \in \mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}))$ and notice $\Gamma_0(N)$ then naturally acts on $z \in \mathbb{H}$ via fractional linear transformations.

We define a genus character $\chi_D(\lambda), \lambda \in L'$ (see [GKZ87]) where D is a fundamental discriminant. We also set $r \in \mathbb{Z}$ such that $D \equiv r^2 \pmod{4N}$. We will use $\chi_D(\lambda)$ to *twist* the Siegel theta functions. We show they will then transform in τ with respect to a twisted Weil representation ρ. If $D > 0$ then $\rho = \rho_L$ and if $D < 0$ then $\rho = \overline{\rho_L}$.

We fix $k \in \mathbb{Z}, k \geq 1$. We then define two Siegel kernel functions that will generate the singular theta lift and the Shimura lift. These are both adapted from (1.3.1) with some well chosen polynomial terms and by twisting. We denote $\Theta_{D,r,k}(\tau, z)$ for the *kernel function* and $\Theta^*_{D,r,k}(\tau, z)$ for the *Shintani kernel function*. These then have the following transformation properties in both variables (Theorems 3.6.8, 3.6.11).

1. $\Theta_{D,r,k}(\tau, z)$ has weight $k - 3/2$ in τ for $\tilde{\Gamma}$. It has weight $2 - 2k$ in z for $\Gamma_0(N)$.
2. $\Theta^*_{D,r,k}(\tau, z)$ has weight $k + 1/2$ in τ for $\tilde{\Gamma}$. It has weight $2k$ in $-\overline{z}$ for $\Gamma_0(N)$.

Chapter 4, The Singular Theta Lift

We finally consider the main item of our work, the *singular theta lift*. We fix $f \in H_{3/2-k,\overline{\rho}}$ from now on. We lift f by pairing it against the kernel function in a regularised Petersson scalar product as follows:

$$\Phi_{D,r,k}(z, f) := \int_{\tau \in \mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), \overline{\Theta_{D,r,k}(\tau, z)} \right\rangle \frac{dudv}{v^2}.$$

We discussed in detail in Section 1.2 how this fits in with previous work and in particular this is an extension of the Borcherds lift [Bor98]. The asymptotic behaviour of f means this integral could diverge in general, hence we have used a regularisation. This regularisation is a slightly weaker version of the method introduced by Harvey, Moore and Borcherds [HM96, Bor98]. There are then three main results in the chapter.

Theorem 1.3.1 (Theorems 4.1.3, 4.2.2, 4.3.7)**.**

1. *The regularised integral $\Phi_{D,r,k}(z, f)$ converges pointwise for any $z \in \mathbb{H}$.*
2. *$\Phi_{D,r,k}(z, f)$ has weight $2 - 2k$ for $\Gamma_0(N)$ and is a smooth function on $\mathbb{H}\backslash Z_{D,r}(f)$ with singularities along $Z_{D,r}(f)$.*
3. *For $z \in \mathbb{H}\backslash Z_{D,r}(f)$ then $\Delta_{2-2k}\Phi_{D,r,k}(z, f) = 0$.*

The first part tells us our definition makes sense and the integral converges everywhere. For the second part, the weight in z is immediately clear from the definition and the transformation properties of $\Theta_{D,r,k}(\tau, z)$. The set $Z_{D,r}(f)$ is a finite linear combination of (twisted) *geodesic cycles* associated to the principal part of f. In the upper half-plane model they can be visualised as vertical half-lines and semi-circles perpendicular to the real line. These geodesics divide $\mathbb{H}$ into connected components called *Weyl chambers*. The third part tells us that our lift is locally harmonic and real-analytic within the components.

Let D_λ be the geodesic associated to a $\lambda \in L, Q(\lambda) > 0$. We then find very explicit *wall crossing formulas* (Theorem 4.2.4) which tell us the nature of the singularities along $Z_{D,r}(f)$. In particular, if we cross a geodesic D_λ then we have a polynomial jump, given by $q_z(\lambda)^{k-1} =$

$(cNz^2 - bz + a)^{k-1}$. The value on a singularity is the average of the values of the adjacent Weyl chambers at that point. These singularities are of a similar nature to those found in the Heaviside step function.

Chapter 5, Partial Poisson Summation

The main aim is to rewrite $\Theta_{D,r,k}(\tau, z)$ as a *Poincaré series*, (Theorem 5.4.5). This will allow us to then find the Fourier expansion of $\Theta_{D,r,k}(\tau, z)$ in the next chapter, using the Rankin-Selberg unfolding trick.

Using the mixed model we do this by finding a Fourier transform and then applying *partial Poisson summation* to rewrite $\Theta_{D,r,k}(\tau, z)$ in terms of some theta functions $\Xi_k(\tau)$. This will require a lot of technical work with some careful calculations. The rewritten form also allows us to look at the asymptotic behaviour of $\Theta_{D,r,k}(\tau, z)$ as $y \to \infty$, (Proposition 5.4.6).

Chapter 6, The Fourier Expansion

The main aim is to find the *Fourier expansion* of $\Phi_{D,r,k}(z, f)$. To do this we first solve some tricky integrals and evaluate an "additional piece". A very simplified version (we omit constants and set $D = 1, r = 1, k \geq 2$) of the Fourier expansion is as follows.

Theorem 1.3.2 (Theorem 6.3.10). *For $y > C$, where $C > 0$ is "the maximum height" of the semi-circle geodesics, then:*

$$\begin{aligned}\Phi_k(z, f) = c^+(0,0)\zeta(k) &+ \sum_{m\geq 1} c^+\left(-\frac{m^2}{4N}, \frac{m}{2N}\right) B_k\left(mz + \lfloor mx \rfloor\right) \\ &+ \sum_{m\geq 1}\sum_{n\geq 1} c^-\left(-\frac{m^2}{4N}, \frac{m}{2N}\right)\left[e(nmz) + e(-nmz)\Gamma(2k-1, 4\pi nmy)\right] n^{-k}.\end{aligned}$$

Here $B_k(x)$ is the kth Bernoulli polynomial and $\zeta(k)$ is the Riemann zeta function. We can in fact also write the bottom part in terms of polylogarithms. We then make some observations. The vertical half-line singularities are encompassed by the first periodic Bernoulli polynomial in this expansion. The lift is trivial (just a constant) when f is a modular form. When f is a cusp form it vanishes.

We then use this expansion to show $\Phi_{D,r,k}(z, f) = \mathcal{O}(y^k)$ as $y \to \infty$ (for $k \geq 2$), i.e. polynomial growth, (Proposition 6.4.1). We also show using the Atkin-Lehner involutions that we have similar expansions at the other cusps of $\Gamma_0(N)$, (Theorem 6.3.12). Putting all this together with Theorem 1.3.1 means:

Theorem 1.3.3 (Theorem 6.4.2). *We have that $\Phi_{D,r,k}(z, f)$ is a locally harmonic Maass form for the group $\Gamma_0(N)$ with exceptional set $Z_{D,r}(f)$:*

$$\Phi_{D,r,k}(z, f) : H_{3/2-k,\overline{\rho}} \to LH_{2-2k}(\Gamma_0(N)).$$

Chapter 7, The Shimura Lift

There is two main parts. We now fix $g \in S_{k+1/2,\rho}$ a cusp form. In the first part we define our version of the (twisted) *Shimura lift*,

$$\Phi^*_{D,r,k}(z,g) := \int_{\tau\in\mathcal{F}}^{reg} \left\langle g(\tau), \Theta^*_{D,r,k}(\tau,z)\right\rangle v^{k+1/2}\frac{dudv}{v^2}.$$

This is a similar definition to the singular theta lift. But here instead we let our input be cusp forms and we use the (twisted) Shintani theta function $\Theta^*_{D,r,k}(\tau,z)$ as a kernel. As g decays exponentially it is immediately clear that the integral converges and defines a smooth real-analytic form of weight $2k$. In Section 1.2 we discussed the importance of this lift. We then have the following key theorem which links the two lifts.

Theorem 1.3.4 (Theorem 7.2.2)**.** *For $f \in H_{3/2-k,\overline{\rho}}$ and $z \in \mathbb{H}\backslash Z_{D,r}(f)$ then*

$$\Phi^*_{D,r,k}(z,\xi_{3/2-k}(f)) = \frac{1}{2}\xi_{2-2k,z}(\Phi_{D,r,k}(z,f)).$$

This link allows us to give new proofs of many of the properties of the Shimura lift. These properties are already well known. Firstly we can find the Fourier expansion by applying ξ_{2-2k} to Theorem 1.3.2. The operator ξ_{2-2k} clearly kills holomorphic terms so only the $e(-nmz)\Gamma(2k-1,4\pi nmy)$ terms will survive. In particular, let $g(\tau)=\sum_{h\in L'/L}\sum_{n>0}a(n,h)e(n\tau)\mathfrak{e}_h$ and assume $D=1, r=1, k\geq 2$. Then (omitting constants) the Fourier expansion of the Shimura lift is as follows:

Theorem 1.3.5 (Theorem 7.3.5)**.** *We have that*

$$\Phi^*_k(z,g) = \sum_{m\geq 1}\sum_{\substack{d\geq 1\\ d|m}} d^{k-1}a\left(\frac{m^2}{4Nd^2},\frac{m}{2Nd}\right)e(mz).$$

The link in Theorem 1.3.4 only held for $z\in\mathbb{H}\backslash Z_{D,r}(f)$. However ξ_{2-2k} kills the holomorphic polynomial singularities so we are able to smoothly continue the Fourier expansion in Theorem 1.3.5 to hold for all $z\in\mathbb{H}$. This expansion is clearly holomorphic. We can also use the Atkin-Lehner involutions to find the expansion at other cusps. Using these facts we then show that the Shimura lift maps cusp forms to *cusp forms* (normally).

Theorem 1.3.6 (Theorem 7.3.8)**.** *If $k=1, D\neq 1$ or $k\geq 2$ then:*

$$\Phi^*_{D,r,k}: S_{k+1/2,\rho}\to S_{2k}(\Gamma_0(N)).$$

If $k=1, D=1$ then $\Phi^*_{D,r,k}: S_{k+1/2,\rho}\to M_{2k}(\Gamma_0(N))$.

The results of this part can be summarised with the following commutative diagram:

$$\begin{tikzcd} H_{3/2-k,\overline{\rho}} \arrow[r, "\Phi_k"] \arrow[d, "\xi_{3/2-k}"'] & LH_{2-2k}(\Gamma_0(N)) \arrow[d, "\xi_{2-2k}"] \\ S_{k+1/2,\rho} \arrow[r, "\Phi_k^*"'] & S_{2k}(\Gamma_0(N)) \end{tikzcd}$$

Figure 1.1: Commutative Diagram

Distributions

The second and final part in Chapter 7 is to consider these ideas as *distributions*. We are motivated to introduce this concept because of the nature of the singularities. We follow the ideas in the classical theory. In particular we fix a space of *test functions* $g(z) \in A^c_\kappa(\Gamma_0(N)$ which are smooth and rapidly decay. We fix $h(z) \in LH_\kappa(\Gamma_0(N))$. Then we define the distribution associated to $h(z)$ as

$$[h]\,(g) := (g,h)_\kappa = \int_{\Gamma_0(N)\backslash\mathbb{H}} g(z)\overline{h(z)}y^\kappa \frac{dxdy}{y^2}$$

and define a "distributional derivative" as $\xi_\kappa[h](g) := -(h, \xi_{2-\kappa}(g))_\kappa$ for $g \in A^c_{2-\kappa}(\Gamma_0(N)$. This concept of the derivative makes sense even on the singularities. We then consider what happens when we apply these ideas to the singular theta lift.

Theorem 1.3.7 (Theorem 7.4.5, The Current Equation)**.** *We have that*

$$\xi_{2-2k}\left[\Phi_{D,r,k}(z,f)\right](g) = \left[\xi_{2-2k}(\Phi_{D,r,k}(z,f))\right](g) - \int_{Z'_{D,r}(f)} g(z)q_z(\lambda)^{k-1}dz.$$

So the distributional derivative of a locally harmonic Maass form matches the classical derivative but also sees the singularities. We obtain as an immediate corollary (Corollary 7.4.7) that

$$\xi_{2-2k}\left[\Phi_{D,r,k}(z,f)\right](g) = 2\left[\Phi^*_{D,r,k}(z,\xi_{3/2-k}(f))\right](g) - \int_{Z'_{D,r}(f)} g(z)q_z(\lambda)^{k-1}dz. \qquad (1.3.2)$$

This is an improved version of Theorem 1.3.4. We have one more useful corollary. We let $g \in S_{2k}(\Gamma_0(N))$ i.e the test functions are holomorphic as well. Then g vanish under the ξ_{2k} operator. So the left hand side of (1.3.2) vanishes. This then tells us the integral of a cusp form against the Shimura lift is equal to some period integral (Corollary 7.4.8). We also formulate this in terms of the Shintani lift.

Chapter 2

Background

In this chapter we discuss the basic ideas and notation we will use in this thesis. This is preliminary material most of which has been described before. We will be working in a rational vector space equipped with a quadratic form of signature (b^+, b^-) (in Chapter 3 onwards this will be fixed to be $(2,1)$), so Section 2.1 discusses these. We will need various spaces of automorphic forms which are discussed in Section 2.5. These have certain transformation properties with respect to the Weil representation (on the metaplectic group) associated to our lattice. This representation is derived in the Section 2.3. We make this representation explicit in the case of the dual reductive pair $(\mathrm{O}(V(\mathbb{R})), \mathrm{SL}_2(\mathbb{R}))$ in Section 2.4. In Section 2.6 we define Siegel theta functions, which we integrate against later. These are defined over two variables and naturally have $(\mathrm{O}(V(\mathbb{R})), \mathrm{Mp}_2(\mathbb{R}))$ acting on them. In fact we consider the action of a subgroup of $\mathrm{GSpin}(V)$. This group is defined in Section 2.2 where we also classify the Clifford algebras of $\mathbb{R}^{b^+,b^-}$.

In this chapter, to help completeness, I have attempted to comprehensively define all concepts but at the same time doing this as succinctly and compactly as possible. Most results used (which are often fairly well know and easy) will not be proven explicitly to save space and instead we normally give a reference for verification for the curious reader.

2.1 Quadratic Forms and Lattices

We start by recalling some very basic definitions and concepts about quadratic forms and lattices. These will form the environment in which we will be working throughout the thesis. The material here is condensed and amalgamated from [O'M00, Chapter 4], [Ser73, Chapter 4], [BvdGHZ08, Chapter 2], [Ger08], [Kit93] and [Sch85].

Quadratic Forms

Following the treatment in [Ger08, Chapter 2], we set R to be an integral domain, R^* the group of invertible elements (the units) in R, F its field of fractions, $F^* = F - \{0\}$ a multiplicative group and finally M a finite free R-module.

Definition 2.1.1. *A* ***bilinear form*** *on M, is a mapping $(\cdot,\cdot) : M \times M \to F$ that is R-linear in both variables. We call a bilinear form* ***symmetric*** *if $(x, y) = (y, x)$ for all $x, y \in M$ and* ***alternating*** *if $(x, x) = 0$ for all $x \in M$. Two elements $x, y \in M$ are* ***orthogonal*** *if $(x, y) = 0$ (we sometimes denote this as $x \perp y$). For a subset $A \subset M$ we denote $A^\perp$ for the* ***orthogonal complement*** *where*

$$A^\perp := \{x \in M \mid x \perp y \text{ for all } y \in A\}.$$

Finally a bilinear form is ***non-degenerate*** *if $M^\perp = \{0\}$ and is called* ***symplectic*** *if it is both alternating and non-degenerate.*

Definition 2.1.2. *A* ***quadratic form*** *on M, is a mapping $Q : M \to F$ such that*

1. *$Q(rx) = r^2 Q(x)$ for all $r \in R, x \in M$.*
2. *$(x, y) := Q(x + y) - Q(x) - Q(y)$ is a bilinear form.*

We note this associated bilinear form $(\cdot,\cdot)$ is symmetric. From now on we will assume R is not of characteristic 2. Then we can put $Q(x) = \frac{1}{2}(x, x)$ and we have a bijective correspondence between symmetric bilinear forms and quadratic forms. We call the pair (M, Q) a **quadratic R-module** over R. If R is a field, i.e. $R = F$ and so M is a vector space over F, then we call the pair (M, Q) a **quadratic R-space**. If R is not a field but is a principal ideal domain then we call the pair (M, Q) a **quadratic R-lattice**. From now on we set (M, Q) as a R-quadratic module. There are two simple examples that we will also need later.

Example 2.1.3. Let $b^+, b^- \in \mathbb{Z}$ be non-negative. We denote by $\mathbb{R}^{b^+,b^-}$ a quadratic $\mathbb{R}$-space with $M = \mathbb{R}^{b^+ + b^-}$ which for elements $x = (x_1, x_2, \ldots, x_{b^+ + b^-}) \in \mathbb{R}^{b^+ + b^-}$ has an attached quadratic form

$$Q(x) := x_1^2 + \cdots + x_{b^+}^2 - x_{b^+ + 1}^2 \cdots - x_{b^+ + b^-}^2.$$

Example 2.1.4. For $a, b, c \in \mathbb{R}, x, y \in M$ we define a **binary quadratic form** $[a, b, c]$ to be a quadratic R-space in two variables where

$$Q(x, y) := [a, b, c](x, y) = ax^2 + bxy + cy^2.$$

If $a, b, c \in \mathbb{Z}$ we call this an **integral binary quadratic form**.

Remark 2.1.5. We will refrain from using the common notation x^2 to denote (x, x) as x is a vector and so “squaring x” is confusing.

Definition 2.1.6. *For a quadratic R-module a non-zero element $x \in M$ is called **isotropic** if $Q(x) = 0$, otherwise x is called **anisotropic** i.e. $Q(x) \neq 0$. A quadratic R-module is isotropic if it contains an isotropic element, otherwise it is called anisotropic. A quadratic R-module is **totally isotropic** if $M \neq 0$ and every element of M is isotropic.*

Definition 2.1.7. *For a basis $\{b_i\}_{i=1}^n$ of M there is a symmetric matrix $T = (b_{ij}) \in M_n(F)$ with $b_{ij} = (b_i, b_j)$, which we call the **Gram matrix** with respect to that basis. We let the **discriminant** of M be the class of* $\det(T)$ *in $F^*/(R^*)^2 \cup \{0\}$.*

Setting v, w as the column vector of coordinates of $x, y \in M$ respectively in the basis $\{b_i\}_{i=1}^n$, then $Q(x) = \frac{1}{2}v^T T v$ and also $(x, y) = v^T T w$. Every quadratic R-space (M, Q) has an **orthogonal basis** (see for example [O'M00, Theorem 42.1], [Ser73, Theorem 4.1]) in which case the associated Gram matrix is diagonal, and so a quadratic R-space is non-degenerate if and only if $\det(T) \neq 0$.

Definition 2.1.8. *Let (M', Q') be another quadratic R-module. An **isometry** is an injective R-linear map $\sigma : M \to M'$ such that $Q'(\sigma(x)) = Q(x)$ for all $x \in M$. If σ is also surjective then M and M' are called **isometric**.*

Definition 2.1.9. *The **orthogonal group** and **special orthogonal group** of M are*

$$\mathrm{O}(M) \coloneqq \{\sigma : M \to M \mid \sigma \textit{ is an isometry}\},$$
$$\mathrm{SO}(M) \coloneqq \{\sigma \in \mathrm{O}(M) \mid \det(\sigma) = 1\}.$$

When discussing the Grassmannian space later (Definition 2.6.2) we will make use of the following result (often called "Witt's extension theorem").

Theorem 2.1.10 (Witt, [Ser73, Theorem 4.1.3])**.** *Let $(M, Q), (M', Q')$ be isometric non-degenerate quadratic R-modules. Then, for any subspace $U \subset M'$, any injective isometry $\sigma : U \to M$ extends to an isometry $\sigma : M \to M'$.*

Proposition 2.1.11 ([Ger08, Theorem 2.40], [Ser73, Section 4.2.3])**.** *For a non-degenerate real quadratic $\mathbb{R}$-space (V, Q), there exists unique non-negative $b^+, b^- \in \mathbb{Z}$ such that (V, Q) is isometric to $\mathbb{R}^{b^+, b^-}$ (see Example 2.1.3).*

Definition 2.1.12. *Non-degenerate quadratic $\mathbb{R}$-spaces (V, Q) are characterised by (b^+, b^-) and we call this the **signature** of (V, Q). If $b^- = 0$ then we call (V, Q) **positive definite**. If $b^+ = 0$ then we call (V, Q) **negative definite**. If b^+, b^- are both non-zero then we call (V, Q) **indefinite**.*

In the case that (V, Q) is a non-degenerate quadratic $\mathbb{R}$-space with signature (b^+, b^-) then $\mathrm{O}(V)$ and $\mathrm{SO}(V)$ will often be denoted as $\mathrm{O}(b^+, b^-)$ and $\mathrm{SO}(b^+, b^-)$.

Lattices

We now look at some properties of lattices. From now on let $R \subsetneq F$ be a principle ideal domain and let V be an F-space.

Definition 2.1.13. *We call L an R-**lattice** if $L \subset V$ is an R-submodule, i.e. $L = 0$ or there exists a linear independent subset $\{b_i\}_{i=1}^m$ of V such that $L = Rb_1 + \cdots + Rb_m$. The **rank** of L is the dimension of the finite free R-module L i.e. m. A lattice is of **maximal rank** if the rank of L is the same as the dimension of V. We call an element $x \in L$ **primitive** if $x \neq 0$ and x can be included in a basis of L.*

Now attaching a quadratic form Q to V we have a quadratic F-space (V, Q) and a quadratic R-lattice (L, Q).

Definition 2.1.14. *We say (L, Q) is **unimodular** if its Gram matrix is unimodular. We define the **scale** of (L, Q), sL, as the fractional R-ideal (an R-submodule of F) generated by the set $\{(x, y) \mid x, y \in L\}$. We say (L, Q) is called **integral** if $sL \subseteq R$.*

Definition 2.1.15. *The **dual lattice** of (L, Q) is*

$$L' := \{x \in V \mid (x, y) \subseteq R \text{ for all } y \in L\}.$$

We can check that this is in fact an R-lattice. For a basis $\{b_i\}_{i=1}^m$ of L we have a **dual basis** $\left\{b_i'\right\}_{i=1}^m$ such that $(b_i, b_j') = \delta_{ij}$. Then L' is the lattice generated by $\left\{b_i'\right\}_{i=1}^m$. We observe L is integral if and only if $L \subseteq L'$ and also that L is unimodular if and only if $L = L'$, [Kit93, Proposition 5.2.1], [Ger08, Proposition 6.25]. We also have $(L')' = L$. From now on we restrict ourselves to the rational case we need later and so let (V, Q) be a non-degenerate quadratic $\mathbb{Q}$-space.

Definition 2.1.16. *Let (V, Q) be a non-degenerate quadratic $\mathbb{Q}$-space. In this work we will call L a **lattice**, if L is a $\mathbb{Z}$-lattice with $L \subset V$.*

We denote by L^- for the $\mathbb{Z}$-lattice with the quadratic form $-Q$. The discriminant of a lattice is equal to $\det(T)$ as $(R^*)^2 = 1$. A lattice is then integral if $(x, y) \in \mathbb{Z}$ for all $x, y \in L$. A lattice is unimodular if the discriminant of L is 1 or -1. An element $x \in L, x \neq 0$ is primitive if $\mathbb{Q}x \cap L = \mathbb{Z}x$. We say L is **even** if $(x, x) \in 2\mathbb{Z}$ for all $x \in L$, otherwise it is called **odd**. The dual lattice in this case is

$$L' := \{x \in V \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$

and we call the following the **level** of L:

$$\min \{n \in \mathbb{N} \mid nQ(x) \in \mathbb{Z} \text{ for all } x \in L'\}.$$

We will say that the signature of a quadratic $\mathbb{Q}$-space (V, Q) is the signature of the associated real quadratic space $V(\mathbb{R}) := V \otimes_{\mathbb{Q}} \mathbb{R}$ that is equipped with the quadratic form $Q(x \otimes r) := r^2 Q(x)$ for $x \in V, r \in \mathbb{R}$. We will say that the signature of a lattice $L \subset V$ is the signature of V.

Lemma 2.1.17 ([Str13, Section 2], [CS99, Section 15.7]). *For a lattice L, if $b^+ + b^-$ is odd then the level is divisible by 4.*

Definition 2.1.18. *The **discriminant group** of an integral lattice L is the quotient L'/L.*

Lemma 2.1.19 ([Sch85, Lemma 3.3]). *The discriminant group is a finite abelian group of order $|r|$, where r is the discriminant of L.*

Lemma 2.1.20. *Let L be an even lattice. Then L'/L can be equipped with a well-defined map $\overline{Q}$, where*

$$\overline{Q} : L'/L \to \mathbb{Q}/\mathbb{Z}, \quad x + L \mapsto \overline{Q}(x + L) := Q(x) \pmod 1.$$

*We call L'/L, with associated quadratic form $\overline{Q}$, the **discriminant form** of L.*

2.2 The Clifford Algebra and Spin Groups

We now briefly discuss the Clifford algebra, the general spin group and the spin group associated to a quadratic R-module. We can form surjective homomorphisms from these groups to $\mathrm{SO}(M)$ and $\mathrm{SO}^+(M)$. We will use these ideas later when we investigate the action of these groups on Siegel theta functions (2.6.12). To define these groups we need to look at the Clifford algebra which is somehow the "freest" algebra containing (M, Q) that is compatible with the quadratic form. Most of the results here are taken from [Sch85, Chapter 9], [BvdGHZ08, Chapter 2], [Kit93, Chapter 1], [Por95, Chapter 15], [Har90] and [O'M00, Chapter 5].

Clifford Algebras

Let (M, Q) be a quadratic R-module, (R is not of characteristic 2 and contains unity 1).

Definition 2.2.1. *The **tensor algebra** $T(M)$, a $\mathbb{Z}$-graded R-algebra, is defined as the $\mathbb{Z}$-graded R-module*

$$T(M) := \bigoplus_{n=0}^{\infty} M^{\otimes n} = R \oplus M \oplus (M \otimes_R M) \oplus \dots$$

with a product of two elements on $T(M)$, $x = x_1 \otimes \cdots \otimes x_m \in \otimes^m M$ and $y = y_1 \otimes \cdots \otimes y_n \in \otimes^n M$, $(x_i, y_j \in M)$ given by $x \otimes y := x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \in \otimes^{m+n} M$.

To form an algebra compatible with the quadratic form we take a quotient of this.

Definition 2.2.2. *Let $I(M) \subset T(M)$ be the two sided ideal generated by the set $\{x \otimes x - Q(x) \mid x \in M\}$. Then the* ***Clifford algebra*** *$C(M)$ is defined as*

$$C(M) \coloneqq T(M)/I(M).$$

We note the Clifford algebra could also be defined by a universal property, see [O'M00, Section 54]. We will denote $x_1 \otimes \cdots \otimes x_m \in C(M)$ as $x_1 \cdots x_m$. We have that R and M are embedded in $C(M)$, and for $x, y \in M$, $x^2 = Q(x)$ and $xy + yx = (x, y)$ by construction.

Lemma 2.2.3 ([Kit93, Corollary 1.4.1])**.** *If $(M, Q), (M', Q)$ are isometric quadratic R-modules then $C(M)$ and $C(M')$ are isomorphic.*

Lemma 2.2.4 ([Sch85, Corollary 2.7], [Kit93, Theorem 1.4.1])**.** *Let $\{b_i\}_{i=1}^n$ be an orthogonal basis of M. Then $\{b_1^{\epsilon_1} \cdots b_n^{\epsilon_n} \mid e_i = 0, 1\}$ is a basis of $C(M)$ and so $C(M)$ is free R-module of rank 2^n.*

We now look at automorphisms on $C(M)$. The tensor algebra $T(M)$ has an anti-automorphism which descends to an anti-automorphism on $C(M)$ denoted by the transpose $^t : C(M) \to C(M)$ where

$$(x_1 \otimes x_2 \otimes \ldots x_n)^t \coloneqq (x_n \otimes x_{n-1} \otimes \ldots x_1).$$

We also have another automorphism. Let $x \in M$. The map $x \mapsto -x$ induces an automorphism of $C(M)$ which we denote as $J : C(M) \to C(M)$.

Definition 2.2.5. *The* ***Clifford norm*** *is a map $\mathrm{N} : C(M) \to C(M)$ defined by*

$$\mathrm{N}(x) \coloneqq x^t x.$$

As the transpose reduces to the identity map on R or M, we have $Q(x) = \mathrm{N}(x)$ for $x \in M$. Therefore this norm extends the quadratic form. Using $J : C(M) \to C(M)$, we have the following decomposition of $C(M)$.

Definition 2.2.6. *We define the* ***even and odd Clifford algebras****, by $C_0(M)$ and $C_1(M)$,*

$$C_0(M) \coloneqq \{x \in C(M) \mid J(x) = x\},$$
$$C_1(M) \coloneqq \{x \in C(M) \mid J(x) = -x\}.$$

Then $C(M) = C_0(M) \oplus C_1(M)$. We have that $C_0(M)$ and $C_1(M)$ are R-subalgebras, generated by an even number and odd number of basis vectors b_i respectively. We know that non-degenerate $\mathbb{R}$-quadratic spaces (V, Q) are isometric to $\mathbb{R}^{b^+, b^-}$ (Example 2.1.3). We denote the Clifford algebra of $\mathbb{R}^{b^+, b^-}$ by C^{b^+, b^-}. We denote $n \times n$ matrices with entries in R as $M_n(R)$.

Theorem 2.2.7 ([Har90, Theorem 11.3], [Por95, Chapter 15])**.** *Let $b^+ + b^- = n$. There is a complete classification of C^{b^+, b^-}. In particular C^{b^+, b^-} are isomorphic to the following matrix algebras:*

$b^+ - b^-$ (mod 8)	C^{b^+,b^-}
0,6	$\mathrm{M}_{2^{n/2}}(\mathbb{R})$
2,4	$\mathrm{M}_{2^{(n-2)/2}}(\mathbb{H})$
1,5	$\mathrm{M}_{2^{(n-1)/2}}(\mathbb{C})$
3	$\mathrm{M}_{2^{(n-3)/2}}(\mathbb{H}) \oplus \mathrm{M}_{2^{(n-3)/2}}(\mathbb{H})$
7	$\mathrm{M}_{2^{(n-1)/2}}(\mathbb{R}) \oplus \mathrm{M}_{2^{(n-1)/2}}(\mathbb{R})$

Lemma 2.2.8 ([Har90, Theorem 9.38], [Por95, Corollary 15.35])**.** *We have that*

$$C_0^{b^+,b^-+1} \cong C^{b^+,b^-} \quad \textit{and} \quad C_0^{b^++1,b^-} \cong C^{b^-,b^+}.$$

Example 2.2.9. We have that $C^{0,1} \cong \mathbb{C}, C^{1,1} \cong \mathrm{M}_2(\mathbb{R})$. In the case of signature $(2,1)$, (which we will use later) then $C^{2,1} \cong M_2(\mathbb{R} \oplus \mathbb{R})$ and $C_0^{2,1} \cong \mathrm{M}_2(\mathbb{R})$.

2.2.1 Spin Groups

Definition 2.2.10. *The **general spin group*** $\mathrm{GSpin}(M)$ *and the **Spin group*** $\mathrm{Spin}(M)$*, are contained within the **Clifford group**,* $\mathrm{CG}(M)$*. We define these as*

$$\begin{aligned}
\mathrm{CG}(M) &:= \left\{x \in C(M) \mid x \textit{ invertible and } xMJ(x)^{-1} = M\right\}, \\
\mathrm{GSpin}(M) &:= \mathrm{CG}(M) \cap C_0(M), \\
\mathrm{Spin}(M) &:= \{x \in \mathrm{GSpin}(M) \mid \mathrm{N}(x) = 1\}.
\end{aligned}$$

For the final results, which we take from [BvdGHZ08, Section 2.3], we will let $R = F$ be a field. For each element $x \in \mathrm{CG}(M)$, we can use our definition of $\mathrm{CG}(M)$ to define an automorphism $g_x \in \mathrm{GL}(M)$ of M, where

$$g_x(m) := xmJ(x)^{-1}.$$

for $m \in M$. In fact this automorphism g_x is an isometry. So we have a homomorphism

$$\mathfrak{g} : \mathrm{CG}(M) \to \mathrm{O}(M), \tag{2.2.1}$$

defined by $x \mapsto g_x$ and usefully, this is surjective. For $\mathrm{GSpin}(M)$ this homomorphism is surjective onto $\mathrm{SO}(M)$. We also note for later that, for elements of $x \in \mathrm{GSpin}(M)$, $J(x) = x$. So the automorphism generated is just defined by conjugation i.e. $g_x(m) = xm(x)^{-1}$ for $m \in M$. We have the well known exact sequence

$$1 \longrightarrow \{-1,1\} \longrightarrow \mathrm{Spin}(M) \xrightarrow{\mathfrak{g}} \mathrm{SO}(M) \longrightarrow R^*/(R^*)^2. \tag{2.2.2}$$

Definition 2.2.11. *The image of* $\mathrm{Spin}(M)$ *under* $\mathfrak{g}$ *in* $\mathrm{O}(M)$ *is denoted as* $\mathrm{SO}^+(M)$*.*

We recall that, in the case $\mathrm{O}(M) \cong \mathrm{O}(b^+,b^-)$, $\mathrm{O}(M)$ has four connected components as a Lie Group. We will call $\mathrm{SO}^+(M)$ the **connected component of the identity of** $\mathrm{O}(M)$. The following result is useful in our case later.

Lemma 2.2.12 ([BvdGHZ08, Lemma 2.14])**.** *When* $\dim(M) \leq 4$

$$\mathrm{GSpin}(M) := \left\{x \in C(M)^0 \mid \mathrm{N}(x) \in R^*\right\} \quad \textit{and} \quad \mathrm{Spin}(M) := \left\{x \in C(M)^0 \mid \mathrm{N}(x) = 1\right\}.$$

2.3 The Weil Representation

Shortly we will construct various types of half-integral weight vector-valued automorphic forms. When defining these forms we want to use a double cover of $\mathrm{SL}_2(\mathbb{R})$, the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$. Vector-valued forms are defined using an associated representation and there is a well known and natural representation of the metaplectic group called the Weil representation. There is a particularity nice explicit description of the Weil representation called the Schrödinger model. We will also use this to construct our Siegel theta functions. In this section we describe these concepts. The results here are based on [Pra93, Kud96, LV80, Gel93, Li08].

Let F be a local field not of characteristic 2, S be a complex vector space and W be a finite dimensional vector space over F equipped with a symplectic bilinear form $\langle , \rangle$.

Remark 2.3.1. We can also let F be a finite field or a global field and form analogous constructions. However we will only need the local case, in particular in Section 2.4 we consider the case when $F = \mathbb{R}$.

Definition 2.3.2. *We call the pair* $(W, \langle , \rangle)$ *a* ***symplectic*** F***-vector space****. We call a subspace of* W ***Lagrangian*** *if it is a maximal totally isotropic subspace of* W*. We let the* ***symplectic group*** $\mathrm{Sp}(W)$ *be the group of* F*-linear automorphisms that preserve the symplectic form i.e. for* $x, y \in W$

$$\mathrm{Sp}(W) := \{g \in \mathrm{GL}(W) \mid \langle gx, gy\rangle = \langle x, y\rangle\}.$$

We note that, by necessity, W has even dimension, $2n$. The Weil representation we are looking for is a projective representation of $\mathrm{Sp}(W)$.

Lemma 2.3.3 ([LV80, Lemma 1.1.4])**.** *Let* W_1 *be a Lagrangian subspace of* W*. There exists another Lagrangian subspace* W_2*, such that we have a decomposition* $W = W_1 \oplus W_2$*. Such a decomposition is a* ***complete polarisation*** *of* W*.*

There is a basis $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^n$ of $(W, \langle , \rangle)$ such that $e_i \in W_1, f_i \in W_2$ and $\langle e_i, f_j\rangle = \delta_{ij}$ (where $\delta_{i,j}$ is the Kronecker delta) called the **symplectic basis**. Using this we can represent $\mathrm{Sp}(W)$ as the well known matrix group

$$\mathrm{Sp}_{2n}(F) := \left\{M \in \mathrm{GL}_{2n}(F) \mid M^t \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right) M = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)\right\}.$$

Lemma 2.3.4 ([Ste62])**.** *We have that* $\mathrm{Sp}_{2n}(F)$ *is generated by*

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \quad \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.3.1}$$

where A *and* B *range through invertible matrices and symmetric matrices respectively.*

Definition 2.3.5. *The* ***Heisenberg group*** $H(W) \coloneqq W \oplus F$*, associated to* $(W, \langle , \rangle)$*, is the set of all pairs* $\{(w, r) \mid w \in W, r \in F\}$ *where, for two elements* $(w_1, r_1), (w_2, r_2) \in H(W)$*, the group operation is given by*

$$(w_1, r_1) \cdot (w_2, r_2) \coloneqq \left(w_1 + w_2, r_1 + r_2 + \frac{\langle w_1, w_2 \rangle}{2} \right).$$

The centre of $H(W)$ is $\{0\} \times F \cong F$. Let ψ be a non-trivial unitary additive character $\psi : F \to \mathbb{C}^*$, where $\mathbb{C}^* \coloneqq \{z \in \mathbb{C} \mid |z| = 1\}$. This is a character on the centre of $H(W)$. For any irreducible representation (ρ, S), $\rho : H(W) \to \mathrm{GL}(S)$, we call $(\rho|_{\{0\} \times R}, S)$ the **central character** of ρ. We now observe that these characters in fact classify the irreducible representations of $H(W)$. We will say a representation of $H(W)$ over S is smooth if every vector in the representation space is fixed by a compact open subgroup of $H(W)$.

Theorem 2.3.6 (Stone-von Neumann, [MVW87, Section 2.1])**.** *There exists a smooth irreducible representation* (ρ_ψ, S) *of* $H(W)$ *with central character* ψ *i.e.* $\rho_\psi((0, r)) = \psi(r) \cdot \mathrm{Id}_S$ *for all* $r \in F$*. This representation is unique up to isomorphism.*

We notice $g \in \mathrm{Sp}(W)$ acts naturally on $h = (w, r) \in H(W)$ by $g \cdot (w, r) \coloneqq (gw, r) = gh$ and is trivial on the centre of $H(W)$. We then let $(\rho^g_\psi(h), S) \coloneqq (\rho_\psi(gh), S)$. This is also a smooth irreducible representation with central character ψ and so by the Stone-von Neumann theorem it is isomorphic to (ρ_ψ, S). Therefore we have an operator $M_\psi(g) \in \mathrm{GL}(S)$ such that

$$\rho_\psi(gh) = M_\psi(g) \rho_\psi(h) M_\psi(g)^{-1}. \tag{2.3.2}$$

Then, using Schur's Lemma, we know this is uniquely determined up to a non-zero scalar in $\mathbb{C}^*$. Letting $[M_\psi(g)]$ denote the class of $M_\psi(g)$ up to scalars we have:

Definition 2.3.7. *The* ***Weil representation of the symplectic group*** $\mathrm{Sp}(W)$ *is the projective representation* $\overline{\rho_\psi} : \mathrm{Sp}(W) \to \mathrm{GL}(S)/\mathbb{C}^*$ *defined by the map* $\overline{\rho_\psi} : g \mapsto [M_\psi(g)]$*. The set of pairs* $(g, M_\psi(g)) \in \mathrm{Sp}(W) \times \mathrm{GL}(S)$ *such that* (2.3.2) *holds, defines a group, which we denote as* $\mathrm{Mp}_\psi(W)$*. We let the* ***Weil representation of*** $\mathrm{Mp}_\psi(W)$ *be the ordinary representation* $(\widetilde{\rho_\psi}, S)$*, defined by the projection map* $\widetilde{\rho_\psi} : (g, M_\psi(g)) \mapsto M_\psi(g)$*.*

Proposition 2.3.8 ([Pra93, Theorem 2.1], [Gel93, Section 1.7], [Kud96, Section 1.4])**.** *There exists a unique subgroup of* $\mathrm{Mp}_\psi(W)$*, which we denote as* $\mathrm{Mp}(W)$*, that is a central extension of* $\mathrm{Sp}(W)$ *and isomorphic to the two fold cover of* $\mathrm{Sp}(W)$*. This group is independent of the central character* ψ*. It can also be defined as* $\mathrm{Mp}(W) \coloneqq \mathrm{Sp}(W) \times \mathbb{C}^*$*, and then* $\mathrm{Mp}_\psi(W) = \mathrm{Mp}(W) \times_{\mathbb{Z}/2} \mathbb{C}^*$*.*

Definition 2.3.9. *We call* $\mathrm{Mp}(W)$ *the* ***metaplectic group*** *and we call the restriction of* $(\widetilde{\rho_\psi}, S)$ *to* $\mathrm{Mp}(W)$ *the* ***Weil representation of the metaplectic group****.*

2.3.1 The Schrödinger Model

A nice and commonly used realisation of these representations is the Schrödinger model, which we now describe. This will allow to obtain some explicit formulas (Proposition 2.3.14). This model is realised on the space of Bruhat-Schwartz functions.

Definition 2.3.10. *If* W *is Archimedean, a smooth function* $f : W \to \mathbb{C}$ *is called a* ***Schwartz function*** *if*

$$\sup_{x \in W} |x^\alpha \partial_\beta f(x)| < \infty$$

for all multi-indices α, β*. We will call a function* $f : W \to \mathbb{C}$ *a* ***Schwartz-Bruhat function****, if it is a Schwartz function in the case* F *is Archimedean and is a locally constant and compactly supported function in the case* F *is non-Archimedean. We denote the space of these as* $\mathcal{S}(W)$*.*

So the Schwartz functions are smooth functions all of whose derivatives decay faster, as $|x| \to \infty$, than any inverse power of x. We have the important property that the Fourier transform on $\mathcal{S}(W)$ is in fact an isomorphism, see for example [Lea10, Section 4.4]. Now let $W = W_1 \oplus W_2$ be a complete polarisation.

Definition 2.3.11. *The* ***Schrödinger representation*** $(\rho_\psi^{\mathrm{Sch}}, \mathcal{S}(W_1))$ *is a representation of* $\mathcal{S}(W_1)$ *described for* $f \in \mathcal{S}(W_1), x \in W_1, w_1 \in W_1, w_2 \in W_2, r \in F$ *by*

$$\rho_\psi^{\mathrm{Sch}}(w_1 + w_2, r) f(x) := \psi\left(r + \frac{\langle w_1, w_2\rangle}{2} + \langle x, w_2\rangle\right) f(x + w_1).$$

Lemma 2.3.12 ([MVW87, Chapter 2], [Kud96, Chapter 1])**.** *The Schrödinger representation is a smooth irreducible representation of* $H(W)$ *with central character* ψ*.*

Using the Schrödinger representation we can now obtain the Schrödinger model.

Definition 2.3.13. *The* ***Schrödinger model*** *is an explicit description of the Weil representations on* $\mathrm{Sp}(W), \mathrm{Mp}_\psi(W)$ *and* $\mathrm{Mp}(W)$*. It is defined by the operator* $M_\psi^{\mathrm{Sch}}(g) \in \mathrm{GL}(\mathcal{S}(W_1))$ *where*

$$\rho_\psi^{\mathrm{Sch}}(gh) = M_\psi^{\mathrm{Sch}}(g) \rho_\psi^{\mathrm{Sch}}(h) M_\psi^{\mathrm{Sch}}(g)^{-1}.$$

This was as we had in (2.3.2). The Schrödinger model can be written as follows.

Proposition 2.3.14 ([Wal13, Section 2.1.4], [Pra93, Section 2], [Kud96, Chapter 1]). *Let $f \in \mathcal{S}(W_1)$. Then up to a scalar*

$$M_\psi^{\mathrm{Sch}}\left[\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}\right] f(x) := |\det(A)|^{1/2} f(A^t x),$$

$$M_\psi^{\mathrm{Sch}}\left[\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}\right] f(x) := \psi\left(\frac{\langle Bx, x\rangle}{2}\right) f(x),$$

$$M_\psi^{\mathrm{Sch}}\left[\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\right] f(x) := \int_{W_1} f(y)\psi\left(\langle x, y\rangle\right) dy,$$

where dy is the Haar measure such that this Fourier transform is self dual. With choice of scalar as above, M_ψ^{Sch} generates unitary Weil representations on $\mathcal{S}(V)$.

2.4 The Weil Representation over $\mathbb{R}$

We show later that the theta functions we use can be defined naturally from the Weil representation on a certain pair of subgroups of $\mathrm{Sp}(W)$ called a dual reductive pair, which gives rise to a local theta correspondence between automorphic forms on these groups. The automorphic forms we use later transform under this representation. We will consider and define these ideas in the case of the dual reductive pair $(\mathrm{O}(V(\mathbb{R})), \mathrm{SL}_2(\mathbb{R}))$ which we make explicit here. Some references are [BF04, Section 2], [Bor98], [Bru02, Section 1.1] and [Shi75].

2.4.1 Dual Reductive Pairs

Definition 2.4.1. *A **dual reductive pair** is a pair of reductive subgroups $G, G' \subset \mathrm{Sp}(W)$ such that G is the centraliser of G' in $\mathrm{Sp}(W)$ and G' is the centraliser of G in $\mathrm{Sp}(W)$.*

We consider the groups $\tilde{G}$ and $\tilde{G}'$ which are the inverse images of G and G' respectively in $\mathrm{Mp}(W)$.

Lemma 2.4.2 ([MVW87, Lemma 2.5]). *The centraliser of $\tilde{G}$ in $\mathrm{Mp}(W)$ is $\tilde{G}'$, the centraliser of $\tilde{G}'$ in $\mathrm{Mp}(W)$ is $\tilde{G}$ and there is homomorphism*

$$j : \tilde{G} \times \tilde{G}' \to \mathrm{Mp}(W).$$

Consider the pullback of the Weil representation of the metaplectic group to $\tilde{G} \times \tilde{G}'$. The Howe duality principle then roughly says that, this pullback decomposes into two irreducible representations π and $\tilde{\pi}$ of $\tilde{G}$ and $\tilde{G}'$ respectively and then $\tilde{\pi}$ is determined by π. This bijection is the so-called (local) theta correspondence. This can often be realised as map between automorphic forms explicitly using theta lifts. There is a large amount of material that could be discussed here which we do not have space to detail and refer the reader to [Kud96,Pra93,Wal13,Gel93].

Instead we focus on using these results in the specific case we need.

We let (V, Q) be a rational non-degenerate quadratic vector space, $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ with signature (b^+, b^-) and let $(W, \langle , \rangle)$ be a real non-degenerate symplectic space of dimension $2n$. We let $\mathbb{W} := V \otimes_{\mathbb{R}} W$. We form a symplectic space by equipping $\mathbb{W}$ with the quadratic form

$$\langle\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle\rangle := \frac{1}{2}(v_1, v_2) \langle w_1, w_2 \rangle$$

for $v_1 \otimes w_1, v_2 \otimes w_2 \in \mathbb{W}$. We define the natural right action for $g \in \mathrm{O}(V)$ on $v \otimes w \in \mathbb{W}$ as $(v \otimes w).g := g^{-1}v \otimes w$. We define the natural right action for $g' \in \mathrm{Sp}(W)$ on $v \otimes w \in \mathbb{W}$ as $(v \otimes w).g := v \otimes wg$. Then $\mathrm{O}(V(\mathbb{R}))$ and $\mathrm{Sp}(W)$ form a dual reductive pair in $\mathrm{Sp}(\mathbb{W})$. There is a standard polarization where $\mathbb{W} = V(\mathbb{R})^n \oplus V(\mathbb{R})^n$ and this allows us to form, as before, a Schrödinger representation and Schrödinger model of $\mathrm{Mp}(\mathbb{W})$ acting on $\mathcal{S}(V(\mathbb{R})^n)$. We can then restrict this to the dual pair $(\mathrm{O}(V(\mathbb{R})), \mathrm{Sp}(W))$ and obtain formulas for their action on $\mathcal{S}(V(\mathbb{R})^n)$ (see for example [Kud96, Section 2.4], [Wal13, Section 2.2.1]). We will shortly make this explicit in our case ($n = 1$) in equations (2.4.5). The singular theta lift in [Bor98] then realises the (singular) theta correspondence for $(\mathrm{O}(V(\mathbb{R})), \mathrm{Sp}(W))$ when $n = 1$.

2.4.2 The Metaplectic Group over $\mathbb{R}$

From now on we let W be of dimension 2. In general we would like to think of fractional weight modular forms in terms of central extensions of $\mathrm{SL}_2(\mathbb{Z})$. In our case, for half-integral forms, this extension can just be a double cover. We have the well known and helpful isomorphism that $\mathrm{Sp}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{R})$ (see for example [Jac85, Section 6.9]). So it suffices to consider the unique double cover of $\mathrm{Sp}_2(\mathbb{R})$ i.e. the metaplectic group $\mathrm{Mp}(W)$ discussed in the previous section. First we list a few basic definitions that we will need throughout this work.

Definition 2.4.3. *Let $\mathbb{H}$ be the **complex upper half plane**, $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. For $\omega \in \mathbb{C}$ we denote $\sqrt{\omega}$ as the **principal root** so that $\arg(\sqrt{\omega}) \in (-\pi/2, \pi/2]$ and denote $e(\omega) := e^{2\pi i \omega}$. We denote the **special linear group** as $\mathrm{SL}_2(\mathbb{R})$ which consists of all real 2×2 matrices with determinant 1. We let $\tau = u + iv \in \mathbb{H}$ and elements $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ act on $\mathbb{H}$ via **linear fractional transformations**, $g\tau := \frac{a\tau+b}{c\tau+d}$. Set $j(g, \tau) = c\tau + d$. We define the matrix $g_\tau \in \mathrm{SL}_2(\mathbb{R})$ as*

$$g_\tau := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix} = \begin{pmatrix} \sqrt{u} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}.$$

We remember that $j(gg', \tau) = j(g, g'\tau)j(g'\tau)$ for $g, g' \in \mathrm{SL}_2(\mathbb{R})$ (see [DS05, Lemma 1.2.2]). We also see that g_τ is a matrix of determinant 1 and $g_\tau i = \tau$. Later we will make use of the following subgroups.

Definition 2.4.4. *Let $N \in \mathbb{Z}, N > 0$. We define the **modular group** to be $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. We define the following groups:*

$$\Gamma(N) := \left\{ \gamma \in \Gamma \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}, \tag{2.4.1}$$

$$\Gamma_0(N) := \left\{ \gamma \in \Gamma \;\middle|\; c \equiv 0 \pmod N \right\}, \tag{2.4.2}$$

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}. \tag{2.4.3}$$

We observe that $\Gamma(N) \subset \Gamma_0(N) \subset \Gamma$ and $\Gamma_\infty \subset \Gamma_0(N) \subset \Gamma$.

Lemma 2.4.5 ([DS05, Chapter 1.2])**.** *We have that $\Gamma/\Gamma(N)$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$.*

We have an explicit realisation of the group $\mathrm{Mp}(W)$, which we denote as $\mathrm{Mp}_2(\mathbb{R})$, through the two choices of square roots of $c\tau + d$. We take this definition from [Ray06, Definition 3.3.2], [Bor98, Section 2], [Bru02, Section 1.1].

Definition 2.4.6. *Let $\tau \in \mathbb{H}$. We define an element of $\mathbf{Mp_2}(\mathbb{R})$ to be a pair (γ, ϕ_γ) where $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $\phi_\gamma : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that $\phi_\gamma(\tau)^2 = j(\gamma, \tau)$. The multiplication of two elements in $\mathrm{Mp}_2(\mathbb{R})$ is defined as*

$$(\gamma, \phi_\gamma(\tau))(\gamma', \phi_{\gamma'}(\tau)) := (\gamma\gamma', \phi_\gamma(\gamma'\tau)\phi_{\gamma'}(\tau)).$$

We also define $\tilde{\Gamma}$, $\tilde{\Gamma}(N)$ and $\tilde{\mathrm{SO}}(2)$ to be the inverse images of $\Gamma, \Gamma(N)$ and $\mathrm{SO}(2)$ respectively under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$.

Lemma 2.4.7 ([Ray06, Lemma 3.3.3], [Bum97, Proposition 1.2.3])**.** *The generators of $\tilde{\Gamma}$ are*

$$T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

The centre of $\tilde{\Gamma}$ is cyclic of order 4 and is generated by

$$S^2 = (ST)^3 = Z := \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right).$$

We define the group

$$\tilde{\Gamma}_\infty := \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, 1 \right) \;\middle|\; n \in \mathbb{Z} \right\} \subset \tilde{\Gamma}.$$

Definition 2.4.8. *The **Legendre symbol** $\left(\frac{a}{b}\right)$ is defined for $a \in \mathbb{Z}$ and b an odd prime as*

$$\left(\frac{a}{b}\right) = a^{(p-1)/2} \pmod p$$

(or it can be stated in terms of quadratic residues). We extend this definition to $\left(\frac{a}{b}\right)$ *for all* $a, b \in \mathbb{Z}$ *and call this the **Kronecker symbol**. This is multiplicative in both* a *and* b *so it is extended by setting the following:* $\left(\frac{a}{1}\right) = 1$; $\left(\frac{a}{-1}\right) = 1$ *if* $a \geq 0$ *and* -1 *otherwise;* $\left(\frac{a}{0}\right) = 1$ *if* $a = \pm 1$ *and* 0 *otherwise; and finally* $\left(\frac{a}{2}\right) = 1$ *if* $a \equiv \pm 1 \pmod 8$, 0 *if* a *is even and* -1 *otherwise.*

When considering what the components of our vector-valued forms look like we will need the following section map. We remember (Lemma 2.1.17) that if $b^+ + b^-$ is odd then the level is divisible by 4. Then in this case we know there exists a section (see [Ste07, Chapter 3]) $s : \Gamma(4) \to \tilde{\Gamma}(4)$ so that for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(4)$

$$s : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{c}{d}\right) \sqrt{c\tau + d} \right). \tag{2.4.4}$$

2.4.3 The Weil Representation over $\mathrm{Mp}_2(\mathbb{R})$

We return to our dual reductive pair $(\mathrm{O}(V(\mathbb{R})), \mathrm{SL}_2(\mathbb{R}))$. We consider the Schrödinger model in this case. Recall that we will use these explicit equations to define our Siegel theta functions and vector-valued forms. In this Archimedean case the Schwartz-Bruhat functions are just Schwartz functions, $\mathcal{S}(V(\mathbb{R}))$.

Definition 2.4.9. *We let the **Fourier transform**,* $\hat{f}$, *of an integrable function* $f(x) : V(\mathbb{R}) \to \mathbb{C}$ *be defined as*

$$\hat{f}(\xi) := \int_{V(\mathbb{R})} f(x) e((x, \xi)) dx.$$

We note there are several ways of defining this. Our version agrees with the definitions in [Bor98] and [BF04]. We will make use of the following later.

Lemma 2.4.10 (Poisson Summation Formula, [ABST13, Theorem 2.1], [Bor98, Section 4])**.** *For any lattice* $L \subset V(\mathbb{R})$, $f \in \mathcal{S}(V(\mathbb{R})$

$$\sqrt{|L'/L|} \sum_{\lambda \in L} f(\lambda) = \sum_{\lambda \in L'} \hat{f}(\lambda).$$

We fix our central character as the standard one i.e. $\psi : \mathbb{R} \to \mathbb{C}^*$ is set as $\psi(x) := e^{2\pi i x}$. We then denote M_ψ^{Sch} as M^{Sch}. The Schrödinger model of the Weil representation for the case $(\mathrm{O}(V(\mathbb{R})), \mathrm{SL}_2(\mathbb{R}))$ is then described below.

Lemma 2.4.11 ([BF04, Section 2], [Kud96, Section 2.4])**.** *Let* $f \in \mathcal{S}(V(\mathbb{R}))$ *and* $g \in \mathrm{O}(V(\mathbb{R}))$. *Recall the generators from* (2.3.1). *In this case* $a > 0, a, b \in \mathbb{R}$. *We can then represent the*

Schrödinger model as follows:

$$M^{\mathrm{Sch}}\,[g]\,f(x) := f(g^{-1}x), \tag{2.4.5a}$$

$$M^{\mathrm{Sch}}\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right] f(x) := a^{(b^{+}+b^{-})/2} f(ax), \tag{2.4.5b}$$

$$M^{\mathrm{Sch}}\left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right] f(x) := e^{\pi i b(x,x)} f(x), \tag{2.4.5c}$$

$$M^{\mathrm{Sch}}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] f(x) := e((b^{-}-b^{+})/8)\hat{f}(-x). \tag{2.4.5d}$$

These equations can be easily derived from Proposition 2.3.14. We now set L to be an even lattice of level N, $L \subset V$.

Definition 2.4.12. *For $f \in \mathcal{S}(V(\mathbb{R})), h \in L'/L$ and $\tilde{g} = (g, \phi_g) \in \mathrm{Mp}_2(\mathbb{R})$ we let*

$$\theta_L(\tilde{g}, f, h) := \sum_{\lambda \in L+h} M^{\mathrm{Sch}}\,[\tilde{g}]\,f(\lambda)$$

*be a **theta function**.*

Lemma 2.4.13 ([BF04, Equations (2.2),(2.3)], [Shi75, Section 1])**.** *We have that*

$$\theta_L(T\tilde{g}, f, h) = e(Q(h))\theta_L(\tilde{g}, f, h),$$
$$\theta_L(S\tilde{g}, f, h) = \frac{e((b^{-}-b^{+})/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h,h'))\theta_L(\tilde{g}, f, h').$$

Remark 2.4.14. We prove a twisted version of these equations later in Proposition 3.2.6.

Definition 2.4.15. *The $\mathbb{C}$-group algebra $\mathbb{C}[L'/L]$ consists of formal linear combinations $\sum_{h\in L'/L} \lambda_h \mathfrak{e}_h$ where $\lambda_h \in \mathbb{C}$ and $\mathfrak{e}_h$ is the standard basis element corresponding to $h \in L'/L$. Multiplication is such that $\mathfrak{e}_h \cdot \mathfrak{e}_{h'} = \mathfrak{e}_{h+h'}$ for $h, h' \in L'/L$. We define a Hermitian scalar product on $\mathbb{C}[L'/L]$ by letting $\langle \mathfrak{e}_h, \mathfrak{e}_{h'} \rangle := \delta_{h,h'}$ and extending this to $\mathbb{C}[L'/L]$ by sesquilinearity i.e.*

$$\left\langle \sum_{h\in L'/L} \lambda_h \mathfrak{e}_h, \sum_{h'\in L'/L} \mu_{h'} \mathfrak{e}_{h'} \right\rangle = \sum_{h\in L'/L} \lambda_h \overline{\mu_h}.$$

Finally, for a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ we denote the components as f_h, such that $f = \sum_{h\in L'/L} f_h \mathfrak{e}_h$.

T and S were the generators of $\tilde{\Gamma}$. Using Lemma 2.4.13 we are then able to generate a unitary (the inner product is preserved, see for example [Boy15, Definition 2.7]) representation of $\tilde{\Gamma}$ on $\mathbb{C}[L'/L]$. See also [Shi75, Section 1], [Bor98, Section 4], [Völ13, Section 5.1].

Definition 2.4.16. *Let* $\mathrm{U}(\mathbb{C}[L'/L])$ *be the unitary group on* $\mathbb{C}[L'/L]$. *Then we define a representation via the generators, as before:*

$$\begin{aligned}\rho_L(T)(\mathfrak{e}_h) &:= e(Q(h))\mathfrak{e}_h,\\ \rho_L(S)(\mathfrak{e}_h) &:= \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-(h,h'))\mathfrak{e}_{h'}.\end{aligned}$$

We call $\rho_L : \tilde{\Gamma} \to \mathrm{U}(\mathbb{C}[L'/L])$ *the* ***Weil representation on*** $\mathbb{C}[L'/L]$.

We denote $\overline{\rho}_L$ for the complex conjugate representation of ρ_L and note that $\rho_{L^-} = \overline{\rho}_L$. We also have

$$\rho_L(Z)(\mathfrak{e}_h) = e\left(\frac{b^- - b^+}{4}\right)\mathfrak{e}_{-h}. \tag{2.4.6}$$

Then (2.4.6) implies that $\rho_L(\gamma, -\phi_\gamma)(\mathfrak{e}_h) = (-1)^{b^- + b^+}\rho_L(\gamma, \phi_\gamma)(\mathfrak{e}_h)$ for $(\gamma, \phi_\gamma) \in \tilde{\Gamma}$. So if $b^+ + b^-$ is even then the Weil representation just factors through (See for example [Boy15, Definition 2.6]) Γ.

Lemma 2.4.17 ([BS10, Section 2], [Bor00, Section 5], [Zem12, Theorem 3.2])**.** *If* $b^+ + b^-$ *is even then the Weil representation* ρ_L *is trivial on* $\Gamma(N)$. *If* $b^+ + b^-$ *is odd then the Weil representation* ρ_L *is trivial on* $s(\Gamma(N))$.

So the Weil representation factors through the group $\Gamma/\Gamma(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ if $b^+ + b^-$ is even and factors through $\tilde{\Gamma}/s(\Gamma(N))$ (a double cover of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$) if $b^+ + b^-$ is odd.

2.5 Automorphic Forms

Throughout this thesis we deal with several types of automorphic objects. In particular our input in the lift will be some harmonic weak Maass forms and the output will be a locally harmonic Maass form. The Siegel theta functions that we use as kernels also transform with weight in two variables. These automorphic forms will normally be vector-valued forms with respect to the Weil representation as discussed in the previous section. We discuss their properties. This is very standard material the main reference being [BF04, Section 3] but is also discussed in the introductions of [Bru02, Section 1.1], [BO10, Section 2.2], [BO13, Section 2.2] and [BFI15, Section 2.2].

We fix the following throughout this chapter: L an even lattice of level N in a rational non-degenerate quadratic space (V, Q) which has signature (b^+, b^-); $k \in \frac{1}{2}\mathbb{Z}$; and $\tau = u + iv \in \mathbb{H}$.

2.5.1 Vector-Valued Forms

We will specialise to the Weil representation shortly but first in greater generality we define half-integral weight vector-valued forms with respect to any representation. This definition is taken from [Ray06, Section 3.3] (also see [Bor00, Section 2], [Bor99, Section 2]).

Definition 2.5.1. *Let ρ be a representation of $\mathrm{Mp}_2(\mathbb{R})$ on a complex vector space V of finite dimension. A* ***vector-valued modular form*** *of weight k with respect to ρ for $\tilde{\Gamma}$ is a function $f : \mathbb{H} \to V$ such that*

1. *$f(\tilde{\gamma}\tau) = \phi_\gamma(\tau)^{2k}\rho(\tilde{\gamma})f(\tau)$ for all $\tilde{\gamma} = (\gamma, \phi_\gamma) \in \tilde{\Gamma}$,*
2. *f is holomorphic on $\mathbb{H}$,*
3. *f is holomorphic at ∞.*

As in the classical case, we will need a slash operator for the Weil representation on $\mathrm{Mp}_2(\mathbb{R})$.

Definition 2.5.2. *We denote the* ***Petersson slash operator*** *as $|_{k,\rho_L}$. For functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and $\tilde{\gamma} = (\gamma, \phi_\gamma) \in \tilde{\Gamma}$ we set*

$$(f|_{k,\rho_L}\tilde{\gamma})(\tau) := \phi_\gamma(\tau)^{-2k}\rho_L(\tilde{\gamma})^{-1}f(\gamma\tau).$$

Definition 2.5.3. *Let $\Gamma' \subset \Gamma$ a finite index subgroup. Then a (vector-valued)* ***modular form****, of weight k with respect to ρ_L for $\tilde{\Gamma}'$, is a function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ such that*

1. *$(f|_{k,\rho_L}\tilde{\gamma}) = f$ for all $\tilde{\gamma} \in \tilde{\Gamma}'$,*
2. *f is holomorphic on $\mathbb{H}$,*
3. *for any cusp $s \in \mathbb{Q} \cup \{\infty\}$ of $\tilde{\Gamma}'$ and taking $(\gamma, \phi_\gamma) \in \tilde{\Gamma}$ with $\gamma\infty = s$, then $(f|_{k,\rho_L}\tilde{\gamma})$ is holomorphic at ∞.*

If f is a function that only satisfies the first condition then, for our purposes, we call this a (vector-valued) ***automorphic form****. If f is not holomorphic, but merely meromorphic at the cusps then we call this a (vector-valued)* ***weakly holomorphic modular form****. If f is holomorphic and vanishes at the cusps we call this a (vector-valued)* ***cusp form****.*

We also consider some further generalisation of modular forms, where instead of asking for holomorphicity we just require our automorphic forms to vanish under the action of a Laplacian operator. The main discussion can be found in [BF04, Chapter 3]. These will form the input of our lift.

Definition 2.5.4. *The weight k* ***hyperbolic Laplacian****, $\Delta_{k,\tau}$ is defined as:*

$$\Delta_k = \Delta_{k,\tau} := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Definition 2.5.5. *Let $\Gamma' \subset \Gamma$ a finite index subgroup. Then a (vector-valued)* ***weak Maass form*** *of weight k with respect to ρ_L for $\tilde{\Gamma}'$ with eigenvalue $\lambda \in \mathbb{C}$, is a twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ such that*

1. *$(f|_{k,\rho_L}\tilde{\gamma}) = f$ for all $\tilde{\gamma} \in \tilde{\Gamma}$,*

2. *$\Delta_k f = \lambda f$,*
3. *for any cusp $s \in \mathbb{Q} \cup \{\infty\}$ of $\tilde{\Gamma}'$ and taking $(\gamma, \phi_\gamma) \in \tilde{\Gamma}$ with $\gamma\infty = s$, then there exists a $C > 0$ so that $(f|_{k,\rho_L}\tilde{\gamma})(\tau) = \mathcal{O}(e^{Cv})$ as $v \to \infty$.*

A (vector-valued) ***harmonic weak Maass form*** *is a weak Maass form with eigenvalue $\lambda = 0$. We also form a subspace of harmonic weak Maass forms by altering 3. We instead request that there exists an $\epsilon > 0$ and a Fourier polynomial*

$$P_f := \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ -\infty \ll n \leq 0}} c^+(n,h) e(n\tau) \mathfrak{e}_h$$

so that $f(\tau) - P_f(\tau) = \mathcal{O}(e^{-\epsilon v})$ as $v \to \infty$ (and analogously for all cusps). This is a stricter condition. This Fourier polynomial is uniquely determined by f and we call it the ***principal part*** *of f.*

We remember that every harmonic (and holomorphic) function is real analytic [ABR01, Theorem 1.28]. It is easy to check that each holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is harmonic with respect to Δ_k. We denote, $A_{k,\rho_L}(\tilde{\Gamma}'), \mathcal{H}_{k,\rho_L}(\tilde{\Gamma}'), H_{k,\rho_L}(\tilde{\Gamma}'), M^!_{k,\rho_L}(\tilde{\Gamma}'), M_{k,\rho_L}(\tilde{\Gamma}'), S_{k,\rho_L}(\tilde{\Gamma}')$ for the $\mathbb{C}$-vector spaces of weight k automorphic, harmonic weak Maass, the subspace of harmonic weak Maass, weakly holomorphic modular, modular and cusp forms respectively. We have

$$A_{k,\rho_L}(\tilde{\Gamma}') \supset \mathcal{H}_{k,\rho_L}(\tilde{\Gamma}') \supset H_{k,\rho_L}(\tilde{\Gamma}') \supset M^!_{k,\rho_L}(\tilde{\Gamma}') \supset M_{k,\rho_L}(\tilde{\Gamma}') \supset S_{k,\rho_L}(\tilde{\Gamma}').$$

We usually only deal with the full group $\tilde{\Gamma}$, in which case we drop the $\tilde{\Gamma}'$ from the notation. There is some inconsistency in the literature here. The space H_{k,ρ_L} is often called the space of harmonic weak Maass forms, as in [BO13] but we follow the more common naming convention as in [BF04, BO10]. We now consider the Fourier expansions of these forms.

Lemma 2.5.6. *Every holomorphic function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ that is invariant under the $|_{k,\rho_L}$-operator has a Fourier expansion of the form*

$$f(\tau) = \sum_{h \in L'/L} \sum_{n \in \mathbb{Z}+Q(h)} a(n,h) e(n\tau) \mathfrak{e}_h, \tag{2.5.1}$$

where $a(n,h)$ denotes the Fourier coefficients $a(n,h) = \int_0^1 \langle f(\tau), \mathfrak{e}_h \rangle e(-n\tau) du$.

Proof. It suffices to notice $e(-Q(h)\tau) f_h(\tau)$ is 1 periodic, see [Bru02, Section 1.1]. □

If we request that f is also holomorphic at ∞ i.e. $f \in M_{k,\rho_L}$ then the $a(n,h)$ terms in the Fourier expansion (2.5.1) vanish for $n < 0$. If f is meromorphic at ∞ i.e. $f \in M^!_{k,\rho_L}$ then there are only finitely many $a(n,h)$ terms with $n < 0$. If f vanishes at ∞ i.e. $f \in S_{k,\rho_L}$ then the $a(n,h)$ terms vanish for $n \leq 0$. For the weak Maass forms we do not have holomorphicity so (2.5.1) does not hold.

Definition 2.5.7. *The* ***incomplete gamma function****,* $\Gamma(a,x)$ *for* $a, x \in \mathbb{C}, \mathrm{Re}(a) > 0$ *is defined as*

$$\Gamma(a,x) := \int_x^\infty e^{-t} t^{a-1} dt,$$

which can then be holomorphically continued to all $a \in \mathbb{C}, x \neq 0$*. We also define the* ***gamma function****,* $\Gamma(a) := \Gamma(a, 0)$*.*

Proposition 2.5.8 ([BF04, Equation 3.2])**.** *Let* $k \neq 1$*. Any* $f \in \mathcal{H}_{k,\rho_L}$ *has a unique decomposition* $f = f^+ + f^-$*, where*

$$f^+ := \sum_{h\in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n \gg -\infty}} c^+(n,h) e(n\tau) \mathfrak{e}_h,$$

$$f^- := \sum_{h\in L'/L} c^-(0,h) v^{1-k} \mathfrak{e}_h + \sum_{h\in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n \ll \infty \\ n \neq 0}} c^-(n,h) \Gamma(1-k, -4\pi n v) e(n\tau) \mathfrak{e}_h.$$

This is easily seen by noticing the Fourier expansion must vanish under the Laplacian operator Δ_k. For the case $k = 1$ we simply replace the $c^-(0,h)v^{1-k}$ term with $c^-(0,h)\log(v)$. We call f^+ the **holomorphic part** and f^- the **non-holomorphic part** of f. The next explicit decomposition we will use extensively when finding the Fourier expansion of our lift.

Proposition 2.5.9. *Any* $f \in H_{k,\rho_L}$ *has a unique decomposition*

$$f^+ := \sum_{h\in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n \gg -\infty}} c^+(n,h) e(n\tau) \mathfrak{e}_h, \tag{2.5.2a}$$

$$f^- := \sum_{h\in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n < 0}} c^-(n,h) \Gamma(1-k, 4\pi |n| v) e(n\tau) \mathfrak{e}_h, \tag{2.5.2b}$$

and for $k \geq 2$*, the* f^- *part vanishes.*

Proof. This follows from Theorem 2.5.18 and the fact that there are no negative weight cusp forms. See also [BF04, Section 3]. □

We can interpret a weakly holomorphic modular form $M^!_{k,\rho_L}$ as a harmonic weak Maass form H_{k,ρ_L} which has no non-holomorphic part, f^-. Observe that there are only finitely many $c^+(n,h)$ terms for $n \leq 0$. This fact will be important later. We also have the following property.

Lemma 2.5.10. *For* $f \in \mathcal{H}_{k,\rho_L}$

$$c^\pm(n,h) = (-1)^{k+\frac{b^- - b^+}{2}} c^\pm(n,-h) \tag{2.5.3}$$

Proof. Apply the action of Z on f and use (2.4.6). □

We now consider what these spaces look like. Using Lemma 2.5.10 we observe that any $f \in \mathcal{H}_{k,\rho_L}$ vanishes if $2k \not\equiv b^+ - b^- \pmod 2$. So when $b^+ + b^-$ is even (or respectively, odd) there exists only non-trivial modular forms of integral weight (respectively half-integral weight) under this representation.

Let $f \in M_{k,\rho_L}$. Lemma 2.4.17 told us that the Weil representation is trivial on $\Gamma(N)$ (respectively $s(\Gamma(N))$) if $b^+ + b^-$ is even (respectively odd). This means that if $b^+ + b^-$ is even the components f_h are classical scalar-valued integral weight modular forms on the congruence subgroup $\Gamma(N)$, (see also [Völ13, Proposition 5.3.5]). If $b^+ + b^-$ is odd then the components f_h are classical scalar-valued half-integral weight modular forms on the congruence subgroup $\Gamma(N)$ (remember here N is divisible by 4) with the automorphy factor defined by the section map (2.4.4) i.e. $\left(\frac{c}{d}\right)\sqrt{c\tau+d}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(N)$. This is compatible with the definitions in [Kob93, Section 4.1] and [Ono04, Chapter 1]. This is useful, in that we merely need to consider level 1 vector-valued forms on $\tilde{\Gamma}$ to obtain some level N scalar-valued forms. We now have three well known examples:

Example 2.5.11. If $L = \mathbb{Z}^2$ with associated quadratic form $Q(\lambda) = \frac{1}{2}(\lambda_1^2 - \lambda_2^2)$. Then the signature is $(1,1)$, $L' = L$ and so L'/L is trivial i.e. $\mathbb{C}[L'/L] \cong \mathbb{C}$ and the Weil representation is trivial. So we just have the classical slash operator (Definition 2.5.29) and vector-valued forms certainly form a generalisation of classical scalar-valued forms for the group Γ.

Example 2.5.12. Let $m \in \mathbb{Z}, m > 0$ and let L be the 1-dimensional lattice $\mathbb{Z}$ with associated quadratic form $Q(\lambda) := m\lambda^2$ for all $\lambda \in L$. Then the space of Jacobi forms $J_{k,m}$ of weight k and index m is isomorphic to $M_{k-1/2,\overline{\rho}_L}$. See [EZ85, Theorem 5.1].

Example 2.5.13. Let $k \in 2\mathbb{Z} + 1/2$ and let p be a prime. Then the space $M_k^{+,!}(\Gamma_0(4p))$ of scalar valued weakly holomorphic forms satisfying the Kohnen plus space condition is isomorphic to $M^!_{k,\rho_L}$ for some lattice L. This is described in more detail in Example 3.1.4.

2.5.2 Differential Operators

There are also some natural differential operators on these spaces. We will use these to form the link from the Shimura lift to the singular theta lift that we construct.

Definition 2.5.14. *For any smooth function $f(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ and $k \in \frac{1}{2}\mathbb{Z}$, we define the **Maass raising and lowering operators** on f as*

$$\begin{aligned} R_k &:= 2i\frac{\partial}{\partial \tau} + kv^{-1} = i\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) + kv^{-1}, \\ L_k &:= -2iv^2\frac{\partial}{\partial \overline{\tau}} = -iv^2\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \end{aligned}$$

Definition 2.5.15. *For $f \in \mathcal{H}_{k,\rho_L}$ we let the* ***anti-linear differential operator*** *be*

$$\xi_k(f)(\tau) := v^{k-2}\overline{L_k f(\tau)} = R_{-k} v^k \overline{f(\tau)}.$$

We can check that $-\Delta_k = L_{k+2}R_k + k = R_{2-k}L_k = \xi_{2-k}\xi_k$. R_k and L_k are called the raising and lowering operators because of the following property.

Lemma 2.5.16 ([Bum97, Lemma 2.1.1])**.** *For any smooth function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and $\tilde{\gamma} \in \tilde{\Gamma}$,*

$$(R_k f)|_{k+2}\tilde{\gamma} = R_k(f|_k\tilde{\gamma}) \quad \textit{and} \quad (L_k f)|_{k-2}\tilde{\gamma} = L_k(f|_k\tilde{\gamma}).$$

The anti-linear differential operator ξ_k (as does L_k) annihilates the holomorphic part of a harmonic weak Maass form. Explicitly:

Proposition 2.5.17. *Let $f \in H_{k,\rho_L}$ with Fourier expansion as in* (2.5.2)*. Then*

$$\xi_k(f)(\tau) = -\sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(h) \\ n > 0}} (4\pi n)^{1-k} \overline{c^-(-n,h)} e(n\tau)\mathfrak{e}_h. \tag{2.5.4}$$

Proof. A straightforward direct calculation. See also [BO10, Section 2.2]. □

We observe this only depends on f^-, the non-holomorphic part of f, and hence vanishes for $f \in M^!_{k,\rho_L}$. This leads to the following significant relation.

Theorem 2.5.18 ([BF04, Proposition 3.2, Theorem 3.7])**.** *The assignment $f \mapsto \xi_k(f)$ defines a surjective map*

$$\xi_k : \mathcal{H}_{k,\rho_L} \to M^!_{2-k,\overline{\rho}_L}.$$

The map has kernel $M^!_{k,\rho_L}$. The assignment also defines a surjective map

$$\xi_k : H_{k,\rho_L} \to S_{2-k,\overline{\rho}_L}.$$

As in the classical case (Definition 2.5.32) we have the following inner product.

Definition 2.5.19. *Let $f, g \in M_{k,\rho_L}$, where one of f, g is a cusp form. Let $\mathcal{F}$ be the standard* ***fundamental domain***

$$\mathcal{F} := \{\tau \in \mathbb{H} \mid |u| \leq 1/2, |\tau| \geq 1\}.$$

Then we call

$$(f, g)_{k,\rho_L} := \int_{\mathcal{F}} \langle f(\tau), g(\tau) \rangle v^k \frac{dudv}{v^2}$$

the ***Petersson scalar product****.*

This definition makes sense as we notice $\langle f(\tau), g(\tau) \rangle v^k \frac{dudv}{v^2}$ is $\tilde{\Gamma}$ invariant [DS05, Section 5.4] and bounded on $\mathbb{H}$. We have in fact formed a Hermitian non-degenerate inner product (see for example [Kil08, Chapter 4]). We have a regularised version of this pairing. This will be used to define our lift by pairing a harmonic weak Maass form against a kernel theta function.

Definition 2.5.20. *Let $f, g \in A_{k,\rho_L}$. Let $\mathcal{F}_t$ be the* ***truncated fundamental domain***

$$\mathcal{F}_t := \{\tau \in \mathcal{F} \mid \mathrm{Im}(\tau) \leq t\}.$$

Then we call

$$(f,g)^{\mathrm{reg}}_{k,\rho_L} := \int_{\mathcal{F}}^{\mathrm{reg}} \langle f(\tau), g(\tau)\rangle\, v^k \frac{dudv}{v^2} := \lim_{t\to\infty} \int_{\tau\in\mathcal{F}_t} \langle f(\tau), g(\tau)\rangle\, v^k \frac{dudv}{v^2}$$

the ***regularised Petersson scalar product****, whenever this limit exists.*

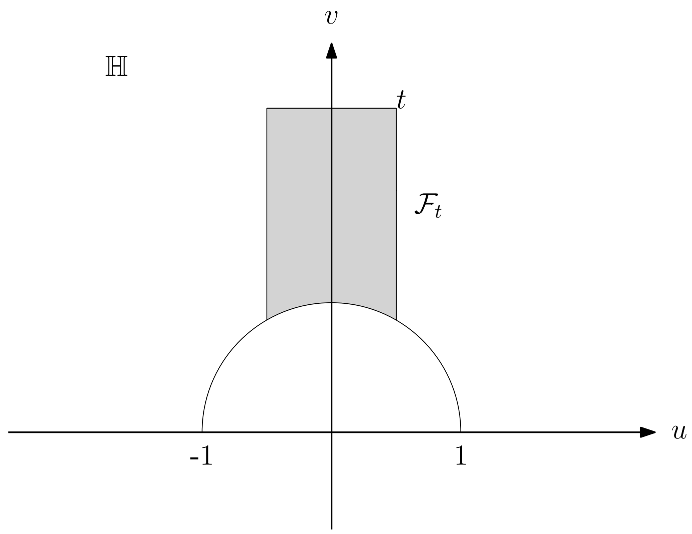

Figure 2.1: The truncated fundamental domain

This regularisation is a method of Harvey, Moore and Borcherds [HM96,Bor98,Bru02]. Further discussion can be found in Section 4.1. This leads to following useful pairing.

Definition 2.5.21. *For $f \in H_{k,\rho_L}, g \in M_{2-k,\overline{\rho}_L}$ let*

$$\{g, f\} := (g, \xi_k(f))_{2-k,L^-}. \tag{2.5.5}$$

For a fixed $g \in M_{2-k,\overline{\rho}_L}$, this is determined by f^-, (recall that $\xi_k(f) = \xi_k(f^-)$). We immediately observe that $\{g, f\} = 0$ if $f \in M^!_{k,\rho_L}$. This pairing is also determined by P_f, the principal part of f.

Proposition 2.5.22 ([BF04, Proposition 3.5])**.** *Let $f \in H_{k,\rho_L}$ and $g \in M_{2-k,\overline{\rho_L}}$ with Fourier expansions as in* (2.5.2) *and* (2.5.1)*. Then*

$$\{g, f\} = \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Z}-Q(h)\\ n\leq 0}} c^+(n,h)a(-n,h).$$

We saw that the Petersson scalar product was non-degenerate and that $\xi_k : H_{k,\rho_L} \mapsto S_{2-k,\overline{\rho}_L}$ was surjective, so then:

Theorem 2.5.23 ([BF04, Theorem 3.6]). *The pairing between $H_{k,\rho_L}/M^!_{k,\rho_L}$ and $S_{2-k,\overline{\rho}_L}$ induced by* (2.5.5) *is non-degenerate.*

We consider $f \in H_{k,\rho_L}$ when the principal part P_f vanishes (or is constant) i.e. when $c^+(n,h) = 0$ for $n \leq 0$ (or $n < 0$ respectively). Then we see that $\{g, f\} = 0$ for all $g \in S_{2-k,\overline{\rho}_L}$. Theorem 2.5.23 then tells us that $f^- \equiv 0$. So in fact $f \in S_{k,\rho_L}$ (or M_{k,ρ_L} respectively). Alternatively, if $f^- \not\equiv 0$, then P_f is non-constant.

Proposition 2.5.24 ([BF04, Proposition 3.11]). *Let P be a Fourier polynomial*

$$P(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n<0}} c^+(n,h) e(n\tau)\mathfrak{e}_h$$

with $c^+(n,h)$ satisfying (2.5.3). *Then there exists a $f \in H_{k,\rho_L}$ with principal part $P_f = P + \mathfrak{c}$ for some T-invariant constant $\mathfrak{c} \in \mathbb{C}[L'/L]$. In fact, if $k < 0$ then f is uniquely determined.*

So certainly for $k \leq 0, k \geq 2$ the principal part P_f of any $f \in H_{k,\rho_L}$ uniquely determines f^-.

Conversly f^- does not uniquely determine P_f. We do however have the weaker statement that f^- does uniquely determine some weighted sums. The following is immediately clear from the fact that $\xi_k(f)$ only depended on f^-.

Corollary 2.5.25. *Let $g \in M_{2-k,\overline{\rho_L}}$ be fixed, with Fourier expansion as in* (2.5.1). *Then for any $f \in H_{k,\rho_L}$, with Fourier expansion as in* (2.5.2), *the pairing*

$$\{g, f\} = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}-Q(h) \\ n \leq 0}} c^+(n,h) a(-n,h)$$

is uniquely determined by f^-.

Let $f \in H_{k,\rho_L}$, with expansion (2.5.2). We denote $n_0 \in \mathbb{Z}+Q(h')$, $(h' \in L'/L)$ for the smallest (possibly negative) number such that $c^+(n,h) = 0$ for all $n < -n_0, (n \in \mathbb{Z} + \operatorname{sgn}(D)Q(h)$ and $h \in L'/L)$. We can then improve (2.5.2) by replacing (2.5.2a) with

$$f^+ = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(h) \\ n \geq -n_0}} c^+(n,h) e(n\tau)\mathfrak{e}_h. \tag{2.5.6}$$

If $n_0 < 0$ (or $n_0 \leq 0$ respectively) then f^- vanishes and we know that f is a cusp form (or a modular form respectively). Using this, we have the following growth properties that we will need when checking various equations converge.

Lemma 2.5.26 ([BF04, Lemma 3.3], [Höv12, Theorem 1.48]). *Let $f \in H_{k,\rho_L}$ with Fourier expansion* (2.5.6). *Then f^- decays exponentially fast as $v \to \infty$ and*

$$f(\tau) = \mathcal{O}(f^+(\tau)) = \mathcal{O}(e^{2\pi n_0 v})$$

as $v \to \infty$, uniformly in u. If $n_0 < 0$ (so f is a cusp form and $k \geq 0$) then

$$f(\tau) = f^+(\tau) = \mathcal{O}\left(\frac{1}{v^{k/2}}\right)$$

as $v \to 0$, uniformly in u. If $n_0 \geq 0$ then

$$f(\tau) = \mathcal{O}\left(f^+(\tau)\right) = \begin{cases} \mathcal{O}\left(\frac{1}{v^k} e^{2\pi n_0 \frac{1}{v}}\right) & \text{if } k \geq 0, \\ \mathcal{O}\left(e^{2\pi n_0 \frac{1}{v}}\right) & \text{if } k < 0, \end{cases}$$

as $v \to 0$, uniformly in u.

Lemma 2.5.27 ([BF04, Lemma 3.4], [Höv12, Lemma 1.49])**.** *Let $f \in H_{k,\rho_L}$ with Fourier expansion* (2.5.6)*. Then*

$$c^+(n,h) = \begin{cases} \mathcal{O}\left(n^{k/2} e^{4\pi\sqrt{n_0}\sqrt{n}}\right) & \text{if } n_0 > 0, k \geq 0, \\ \mathcal{O}\left(e^{4\pi\sqrt{n_0}\sqrt{n}}\right) & \text{if } n_0 \geq 0, k < 0, \\ \mathcal{O}\left(n^k\right) & \text{if } n_0 = 0, k \geq 0, \\ \mathcal{O}\left(n^{k/2}\right) & \text{if } n_0 < 0, k \geq 0, \end{cases}$$

as $n \to \infty$. For f^- we have

$$c^-(n,h) = \mathcal{O}(|n|^{k/2})$$

as $n \to -\infty$.

Lemma 2.5.28 ([BF04, Section 3])**.** *We have that*

$$\Gamma(1-k, -2x) = \begin{cases} \mathcal{O}\left(\frac{1}{(-2x)^k} e^{2x}\right) & \text{as } x \to \infty, \\ \mathcal{O}\left(\frac{1}{(2|x|)^k} e^{-2|x|}\right) & \text{as } x \to -\infty. \end{cases}$$

2.5.3 Scalar-Valued Forms

Once we have performed our lifts we will be dealing with more traditional scalar-valued automorphic objects.

Definition 2.5.29. *Let $k \in \mathbb{Z}$. We denote the* ***Petersson slash operator*** *as $|_k$. For functions $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$ we set*

$$(f|_k\gamma)(\tau) \coloneqq j(\gamma,\tau)^{-k} f(\gamma\tau).$$

Let $k \in \mathbb{Z}$ and $\Gamma' \subset \Gamma$ a finite index subgroup. We then define scalar-valued forms $f : \mathbb{H} \to \mathbb{C}$ of weight k for Γ', by replacing $|_{k,\rho_L}$ with $|_k$ and $\tilde{\Gamma}' \subset \tilde{\Gamma}$ with $\Gamma' \subset \Gamma$ throughout Definition 2.5.3. We denote, $A_k(\Gamma'), M^!_k(\Gamma'), M_k(\Gamma'), S_k(\Gamma')$ for the spaces of weight k (scalar-valued) automorphic, weakly holomorphic modular, modular and cusp forms respectively. We remember there are no modular forms of negative weight.

One of the main aims of this thesis is to show that we obtain from our lift a locally harmonic weak Maass form. We form our definition by adapting [BKK12, BK14, BKV13] (other similar definitions could also be usefully considered).

For a (not necessarily continuous) function $f : \mathbb{H} \to \mathbb{C}$ and a nowhere dense (see [Rud91, Definition 2.1]) **exceptional set** $E \subset \mathbb{H}$, we will denote f_W as the restriction of f to a connected component $W \subset \mathbb{H}\backslash E$. For a point $\tau \in \mathbb{H}$ we denote W_τ^E for the (not necessarily finite) set of connected components that contain τ in their closure i.e.

$$W_\tau^E := \left\{ W \subset \mathbb{H}\backslash E \mid \tau \in \overline{W} \right\}.$$

Finally let

$$\mathcal{A}_E(f)(\tau) := \frac{1}{\# W_\tau^E} \sum_{W \in W_\tau^E} \lim_{\substack{w \in W \\ w \to \tau}} f(w)$$

be the average value of f on the connected components in which τ lies (when this limit exists).

Definition 2.5.30. *Let $k \in 2\mathbb{Z}, k \leq 0$, $\Gamma' \subset \Gamma$ be a finite index subgroup and E be a Γ'-invariant exceptional set $E \subset \mathbb{H}$. We will call a function $f : \mathbb{H} \to \mathbb{C}$ a (scalar-valued)* ***locally harmonic weak Maass form****, of weight k for Γ' and E, if*

1. *$(f|_k\gamma) = f$ for all $\gamma \in \Gamma'$.*
2. *For all $\tau \in \mathbb{H}\backslash E$ there is a neighbourhood $U \subset \mathbb{H}$ of τ in which f is real analytic and $\Delta_k f = f$.*
3. *For all $\tau \in \mathbb{H}$ we have that: W_τ^E is a finite set, the limit defining $\mathcal{A}_E(f)(\tau)$ exists and $f = \mathcal{A}_E(f)$.*
4. *For any cusp $s \in \mathbb{Q} \cup \{\infty\}$ of Γ' and taking $\gamma \in \Gamma'$ with $\gamma\infty = s$, then there exists a $C > 0$ so that $(f|_k\gamma)(\tau) = \mathcal{O}(v^C)$ as $v \to \infty$.*

Remark 2.5.31. These are similar to harmonic weak Maass forms but only harmonic within connected components which are divided by singularities (the exceptional set). In this case where f is real analytic on the connected components we will call the components **Weyl chambers** as in [Bor98, Section 6], which we also discuss in Section 4.2.1. These singularities are nice in that the value on the singularity is the average of the values in the adjacent connected components. We also note that we have restricted this definition to forms with polynomial growth at the cusps and also of non-positive weight.

We will denote the space of locally harmonic weak Maass forms as $LH_k(\Gamma')$. We will also use a scalar-valued version of the Petersson scalar product. Let $\Gamma' \subset \Gamma$ be a finite index subgroup. Then

$$\mathcal{F}_t(\Gamma') := \bigcup_{\gamma \in \Gamma' \backslash \Gamma} \gamma.\mathcal{F}$$

is the truncated fundamental domain for Γ'.

Definition 2.5.32. *Let $k \in \mathbb{Z}$ and $f, g \in A_k(\Gamma')$. Then we call*

$$\begin{aligned}(f,g)_k^{\mathrm{reg}} &:= \int_{\mathcal{F}(\Gamma')}^{\mathrm{reg}} f(\tau)\overline{g(\tau)} v^k \frac{dudv}{v^2} \\ &:= \lim_{t \to \infty} \int_{\tau \in \mathcal{F}_t(\Gamma')} f(\tau)\overline{g(\tau)} v^k \frac{dudv}{v^2}\end{aligned}$$

the (scalar-valued) ***regularised Petersson scalar product*** *whenever the limit exists.*

2.5.4 Atkin-Lehner Involutions

When finding the Fourier expansion of our lift at different cusps we will need to consider Atkin-Lehner involutions. For further details see [Sch04, Chapter 4], [BO13, Section 4.3], [Kna92, Section 9.7] and [AL70].

Definition 2.5.33. *We let $N \in \mathbb{Z}, n > 0$, m be an* ***exact divisor*** *of N (when $m|N$ and $\gcd(N/m, m) = 1$) and let $\left(\begin{smallmatrix} a & b \\ Nc/m & dm \end{smallmatrix}\right) \in \Gamma_0(N/m)$. Then the* ***Atkin-Lehner involutions on*** *$\Gamma_0(N)$ are given by*

$$W_m^N := \begin{pmatrix} a & b \\ Nc/m & dm \end{pmatrix} \begin{pmatrix} \sqrt{m} & 0 \\ 0 & 1/\sqrt{m} \end{pmatrix}.$$

W_m^N have determinant 1 and are uniquely determined up to elements of $\Gamma_0(N)$. It is well known that they form cosets of $\Gamma_0(N)$ in its normaliser in Γ, and that, if m' is another exact divisor, then

$$W_m^N W_{m'}^N = W_{mm'/\gcd(m,m')}^N \pmod{\Gamma_0(N)}.$$

In particular, $(W_m^N)^2 = 1 \pmod{\Gamma_0(N)}$. We can use these to form an involution on $A_k(\Gamma_0(N))$.

Definition 2.5.34. *An* ***Atkin-Lehner involution on*** *$A_k(\Gamma_0(N))$ is defined for $f \in A_k(\Gamma_0(N))$ by $f \mapsto f|_k W_m^N$.*

As before, we have $f|_k W_m^N|_k W_{m'}^N = f|_k W_{mm'/\gcd(m,m')}^N$ and $f|_k (W_m^N)^2 = f$.

2.6 Siegel Theta Functions

In this section we discuss the general theory of Siegel theta functions. These are defined on a Grassmannian, which we discuss as well. The kernel functions we will define are Siegel theta functions. The main references are [Bor98, Section 4], [BF04, Section 2] and [Bru02, Section 2.1].

2.6.1 Theta Functions

We now consider a general construction of some theta functions, following [Shi75] and [BF04, Section 2].

Lemma 2.6.1 ([Ray06, Lemma 3.2.1]). *The stabiliser of $i \in \mathbb{H}$ is* $\mathrm{SO}(2)$ *and* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ *is homeomorphic to* $\mathbb{H}$.

We have a character $\sigma_{1/2} : \tilde{\mathrm{SO}}(2) \to \mathbb{C}$. For an element $g_\theta = \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$, where $\theta \in (-\pi, \pi]$, we let

$$\sigma_{1/2}\left(\left(g_\theta, \pm\sqrt{j(g_\theta, \tau)}\right)\right) \coloneqq \pm\sqrt{j(g_\theta, i)}^{-1} = \pm e^{i\theta/2}.$$

For $f \in \mathcal{S}(V(\mathbb{R})), \tilde{g} \in \mathrm{Mp}_2(\mathbb{R})$ and $h \in L'/L$ we had a theta function $\theta_L(\tilde{g}, f, h)$ (Definition 2.4.12) and we associate to this

$$\Theta_L(\tilde{g}, f) \coloneqq \sum_{h \in L'/L} \theta_L(\tilde{g}, f, h)\mathfrak{e}_h.$$

Then, for $\tilde{\gamma} \in \tilde{\Gamma}$, this by definition satisfies

$$\Theta_L(\tilde{\gamma}\tilde{g}, f) = \rho_L(\tilde{\gamma})\Theta_L(\tilde{g}, f).$$

We also form a function defined on the upper half plane. If f is such that

$$M^{\mathrm{Sch}}\left[g_\theta\right] f(\lambda) = \sigma_{1/2}\left(\tilde{g}_\theta\right)^r f(\lambda)$$

for any $\tilde{g}_\theta \in \tilde{\mathrm{SO}}(2)$ and some fixed $r \in \mathbb{Z}$, we will say (following [Cip83, (1.7)]) that f satisfies the **first spherical property** for $r/2$. If this is the case then we let

$$\Theta_L(\tau, f) \coloneqq \sum_{h \in L'/L} \sum_{\lambda \in L+h} j(g_\tau, i)^{r/2} M^{\mathrm{Sch}}\left[g_\tau\right] f(\lambda)\mathfrak{e}_h = j(g_\tau, i)^{r/2}\Theta_L(g_\tau, f). \tag{2.6.1}$$

We notice both $g_{\gamma\tau}$ and γg_τ map i to $\gamma\tau$, so $g_{\gamma\tau} = \gamma g_\tau g_\theta$ for some $g_\theta \in \mathrm{SO}(2)$ (Lemma 2.6.1). Further $j(\gamma g_\tau g_\theta, i) = j(\gamma, \tau)j(g_\tau, i)j(g_\theta, i)$, so we can then check that our function transforms as we would hope under (γ, ϕ_γ) and is an element of $A_{r/2,\rho_L}$, i.e.

$$\Theta_L(\gamma\tau, f) = \phi_\gamma(\tau)^r \rho_L(\gamma, \phi_\gamma)\Theta_L(\tau, f). \tag{2.6.2}$$

2.6.2 The Grassmannian

Definition 2.6.2. *We define the* ***Grassmannian****,* $\mathrm{Gr}(V(\mathbb{R}))$ *as*

$$\mathrm{Gr}(V(\mathbb{R})) \coloneqq \left\{z \subset V(\mathbb{R}) \mid \dim\ z = b^- \text{ and } Q|_z < 0\right\}.$$

This is the set of negative definite b^--dimensional subspaces in $V(\mathbb{R})$.

Remark 2.6.3. This definition is in the form of [BF04, BFI15], whereas in most other sources (e.g. [Bor98, Bru02, BO10]) the Grassmannian is the set of positive definite b^+-dimensional subspaces. This often means when we work in signature (b^+, b^-) the equivalent theory in these other sources is described in a space of signature (b^-, b^+).

From basic algebra we know that if G is a group acting transitively on a set X and K_{x_0} is the stabiliser of a point $x_0 \in X$, then G/K_{x_0} is in bijection with X by the mapping $gK_{x_0} \mapsto gx_0$ (see for example [Coh03, Thorem 2.1.3]). Using Witt's extension theorem 2.1.10, we see that $\mathrm{O}(V(\mathbb{R})) \cong \mathrm{O}(b^+, b^-)$ acts transitively on $\mathrm{Gr}(V(\mathbb{R}))$. Fix an element $v_0 \in \mathrm{Gr}(V(\mathbb{R}))$ and consider its stabiliser $K_{v_0} \subset \mathrm{O}(b^+, b^-)$. We have $K_{v_0} \cong \mathrm{O}(b^+) \times \mathrm{O}(b^-)$, as K_{v_0} preserves the planes v_0 and $v_0^\perp$. So,

$$\mathrm{Gr}(V(\mathbb{R})) \cong \mathrm{O}(b^+, b^-)/\mathrm{O}(b^+) \times \mathrm{O}(b^-).$$

Similarly, we can write

$$\mathrm{Gr}(V(\mathbb{R})) \cong \mathrm{SO}^+(b^+, b^-)/\mathrm{SO}(b^+) \times \mathrm{SO}(b^-). \tag{2.6.3}$$

We can ask when we can associate a complex structure to the Grassmannian. We very briefly sketch the theory here. These definitions are taken from [Mil05, Chapter 1] and [Huy05] which have more rigorous detail.

Definition 2.6.4. *A real smooth manifold M has a* ***complex structure*** *if it has a holomorphic atlas of charts. An* ***almost complex structure*** *on a smooth real manifold M is a smooth tensor field $J : TM \to TM$ such that $J^2 = -Id$, where TM denotes the tangent bundle, the collection of tangent space $T_m M$. A manifold with complex structure always admits an almost complex structure (see [Huy05, Proposition 2.6.2]). We call a Riemannian manifold M, (a smooth real manifold with a Riemannian metric g) a* ***Riemannian symmetric space*** *if for any point $m \in M$ there exists an involution s_m such that $s_m^2 = Id$ and m is the only fixed point of s_m in some neighbourhood of m. We call a Riemannian manifold* ***Hermitian*** *if it has a complex structure and $g(JX, JY) = g(X, Y)$ for all vector fields X, Y. We call a Riemannian symmetric space* ***irreducible*** *if it is not a product of Riemannian symmetric spaces of lower dimensions.*

Lemma 2.6.5 ([Fio13, Section 2.3.3])**.** *K_{v_0} is a maximal compact subgroup of* $\mathrm{O}(b^+, b^-)$.

This is useful as Cartan tells us that non-compact simple Lie groups modulo a maximal compact subgroup correspond to irreducible simply connected Riemannian symmetric spaces of negative curvature.

We can determine when these space are Hermitian as we also know that non-compact simple Lie groups with trivial centre modulo a maximal compact subgroup with positive dimension centre correspond to non-compact irreducible Hermitian Riemannian symmetric spaces, (see [Hel62, Theorem 8.6.1]). Finally $\mathrm{SO}(b^+) \times \mathrm{SO}(b^-)$ has positive dimension centre only if one or both of $b^+, b^- = 2$ (see [BJ06, Section 1.5]). Specifically we know $\mathrm{SO}(2) \cong U(1)$ which is abelian. When one of $b^+, b^- = 2$ this complex structure can can be written explicitly using the tube domain model, see [Bru02, Section 3.2], [Ray06, Lemma 5.2.2]. In our case of signature $(2, 1)$ we will be able to associate the Grassmannian to the upper-half plane.

We now define another quadratic form on $V(\mathbb{R})$ associated to a given element $z \in \mathrm{Gr}(V(\mathbb{R}))$, called the majorant. This form will always be positive definite and so ensures that our Siegel theta functions will converge. Let $z \in \mathrm{Gr}(V(\mathbb{R}))$ and $\lambda \in V(\mathbb{R})$. We denote $z^\perp$ for the orthogonal complement of $z \in V(\mathbb{R})$, which is a b^+-dimensional positive definite subspace. Then $V(\mathbb{R}) = z \oplus z^\perp$ and we have a unique decomposition $\lambda = \lambda_z + \lambda_{z^\perp}$.

Definition 2.6.6. *We define the* ***majorant***

$$Q_z(\lambda) := Q(\lambda_{z^\perp}) - Q(\lambda_z).$$

2.6.3 Siegel Theta Functions

We now describe some Siegel theta functions and a few of their properties. We will construct these as before by using an Schwartz function at the base point. We first discuss some polynomials. Remember $\mathbb{R}^{b^+,b^-}$ was the quadratic $\mathbb{R}$-space Example 2.1.3 and elements $x \in \mathbb{R}^{b^++b^-}$ in the vector space were denoted as $x = (x_1, x_2, \ldots, x_{b^++b^-})$.

Definition 2.6.7. *We call a polynomial p on $\mathbb{R}^{b^+,b^-}$* ***homogeneous of degree*** (m^+, m^-) *if p is homogeneous of degree m^+ (and m^-) in the b^+ (and b^-) variables respectively. We also let*

$$\Delta := \sum_{n=1}^{b^++b^-} \frac{\partial^2}{\partial x_n^2}, \tag{2.6.4}$$

be an operator on $\mathbb{R}^{b^+,b^-}$. We say a polynomial p, is ***harmonic*** *when $\Delta(p) = 0$.*

We note Δ is just the traditional Laplace operator on $\mathbb{R}^{b^++b^-}$ but we will let it act on the quadratic space $\mathbb{R}^{b^+,b^-}$. Let $c \in \mathbb{C}$ and $x \in \mathbb{R}^{b^++b^-}$. Using Borcherds' [Bor98] notation we will write $\exp(c\Delta)(p)(x)$ to denote $\sum_{j=0}^{\infty} \frac{c^j}{j!} \Delta^j(p)(x)$. We observe that $\exp(c\Delta)(p)(dx) = d^{m^++m^-} \exp(c\Delta/d^2)(p)(x)$ for $d \in \mathbb{C}$ and if p is harmonic, then $\exp(c\Delta)(p)(x) = p(x)$.

Definition 2.6.8. *Let $\lambda \in L$ and $z \in \mathrm{Gr}(V(\mathbb{R}))$. We call*

$$\varphi_0(\lambda, z) := e(Q_z(\lambda)i)$$

the ***Gaussian****. Let $\sigma : V(\mathbb{R}) \to \mathbb{R}^{b^+,b^-}$ be an isometry and p be a homogeneous polynomial on $\mathbb{R}^{b^+,b^-}$ of degree (m^+, m^-), then we also define:*

$$\varphi_0(\lambda, z, \sigma, p) := \exp\left(-\Delta/8\pi\right)(p)(\sigma(\lambda))e\left(Q_z(\lambda)i\right).$$

Both forms of the Gaussian are Schwartz functions, i.e. elements of $\mathcal{S}(V(\mathbb{R}))$. We will now check that $\varphi_0(\lambda,z,\sigma,p)$ is in fact an eigenfunction. We set for the moment $r = b^+ - b^- + 2m^+ - 2m^-$.

Lemma 2.6.9. *We have that $\varphi_0(\lambda,z,\sigma,p)$ satisfies the first spherical property for $r/2$.*

Proof. We want to show that, for all $\tilde{g}_\theta \in \widetilde{\mathrm{SO}}(2)$,

$$M^{\mathrm{Sch}}\left[g_\theta\right]\varphi_0(\lambda,z,\sigma,p) = \sigma_{1/2}\left(\tilde{g}_\theta\right)^{b^+-b^-+2m^+-2m^-}\varphi_0(\lambda,z,\sigma,p).$$

As before we write an element $g_\theta = \left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) \in \mathrm{SO}(2)$ where $a = \cos(\theta), b = \sin(\theta)$ and $\theta \in (-\pi,\pi]$. We can easily check that these decompose into the following useful forms. For $b<0$

$$\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) = \left(\begin{smallmatrix} -1/b & 0 \\ 0 & -b\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -ab \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a/b \\ 0 & 1\end{smallmatrix}\right),$$

and for $b>0$

$$\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)^2\left(\begin{smallmatrix} 1/b & 0 \\ 0 & b\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -ab \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a/b \\ 0 & 1\end{smallmatrix}\right).$$

We first consider the $b<0$ case. Then using the equations (2.4.5)

$$\begin{aligned}
&M^{\mathrm{Sch}}\left[\left(\begin{smallmatrix} 1 & -a/b \\ 0 & 1\end{smallmatrix}\right)\right]\varphi_0(\lambda,z,\sigma,p)\\
&= \exp\left(-\Delta/8\pi\right)(p)(\sigma(\lambda))e\left(Q_z(\lambda)i - Q(\lambda)a/b\right)\\
&= \exp\left(-\Delta/8\pi\right)(p)(\sigma(\lambda))e\left(Q(\lambda_{z^\perp})(i-a/b) + Q(\lambda_z)(-i-a/b)\right)
\end{aligned}$$

Denote this last equation as $g(\lambda)$. We now need the Fourier transform $\hat{g}(\xi)$ of $g(\lambda)$. Using [Bor98, Corollary 3.5], setting $\tau = (i-a/b)$, we see that

$$\begin{aligned}
\hat{g}(\xi) &= (b(a+bi))^{b^+/2+m^+}(b(a-bi))^{b^-/2+m^-}i^{(-b^+-3b^-)/2}\\
&\quad\times\exp\left(-\Delta/8\pi b^2\right)(p)(\sigma(\xi))e\left(Q(\xi_{z^\perp})(a+bi)b + Q(\xi_z)(a-bi)b\right).
\end{aligned}$$

Using this, we have

$$\begin{aligned}
&M^{\mathrm{Sch}}\left[\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a/b \\ 0 & 1\end{smallmatrix}\right)\right]\varphi_0(\lambda,z,\sigma,p)\\
&= (b(a+bi))^{b^+/2+m^+}(b(a-bi))^{b^-/2+m^-}i^{-b^+-b^-+2(m^++m^-)}\\
&\quad\times\exp\left(-\Delta/8\pi b^2\right)(p)(\sigma(\lambda))e\left(Q(\lambda_{z^\perp})(a+bi)b + Q(\lambda_z)(a-bi)b\right).
\end{aligned}$$

Continuing

$$\begin{aligned}
&M^{\mathrm{Sch}}\left[\left(\begin{smallmatrix} 1 & -ab \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -a/b \\ 0 & 1\end{smallmatrix}\right)\right]\varphi_0(\lambda,z,\sigma,p)\\
&= (b(a+bi))^{b^+/2+m^+}(b(a-bi))^{b^-/2+m^-}i^{-b^+-b^-+2(m^++m^-)}\\
&\quad\times\exp\left(-\Delta/8\pi b^2\right)(p)(\sigma(\lambda))e\left(Q(\lambda_{z^\perp})b^2 i - Q(\lambda_z)b^2 i\right)
\end{aligned}$$

and finally

$$\begin{aligned}
&M^{\mathrm{Sch}}\left[\begin{pmatrix} -1/b & 0 \\ 0 & -b \end{pmatrix}\begin{pmatrix} 1 & -ab \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -a/b \\ 0 & 1 \end{pmatrix}\right]\varphi_0(\lambda, z, \sigma, p)\\
&= b^{-(b^+ + b^-)/2}(b(a+bi))^{b^+/2+m^+}(b(a-bi))^{b^-/2+m^-}i^{2(m^+ + m^-)}\\
&\quad\times \exp\left(-\Delta/8\pi b^2\right)(p)(\sigma(-\lambda/b))e\left(Q(\lambda_{z^\perp})i - Q(\lambda_z)i\right)\\
&= (a+bi)^{b^+/2+m^+}(a-bi)^{b^-/2+m^-}\varphi_0(\lambda, z, \sigma, p)\\
&= \sigma_{1/2}\left(\tilde{g}_\theta\right)^{b^+ + 2m^+ - b^- - 2m^-}\varphi_0(\lambda, z, \sigma, p).
\end{aligned}$$

For the $b > 0$ case it suffices to note that $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ hence $M^{\mathrm{Sch}}\left[\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right] f(x) = i^{b^+ + b^-} f(-x)$. Then we can easily show the result by adapting the $b < 0$ case. □

We then follow the earlier construction (2.6.1) to form a theta function for an element $\tau = u + iv \in \mathbb{H}$.

$$\begin{aligned}
&\Theta_L(\tau, \varphi_0(\lambda, z, \sigma, p))\\
&= \sum_{h \in L'/L}\sum_{\lambda \in L+h} j(g_\tau, i)^{r/2} M^{\mathrm{Sch}}\left[g_\tau\right]\varphi_0(\lambda, z, \sigma, p)\mathfrak{e}_h\\
&= v^{-r/4}\sum_{h \in L'/L}\sum_{\lambda \in L+h} M^{\mathrm{Sch}}\left[\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{v} & 0 \\ 0 & 1/\sqrt{v} \end{pmatrix}\right]\varphi_0(\lambda, z, \sigma, p)\mathfrak{e}_h\\
&= v^{-r/4}\sum_{h \in L'/L}\sum_{\lambda \in L+h} M^{\mathrm{Sch}}\left[\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right] v^{(b^+ + b^-)/4}\varphi_0(\sqrt{v}\lambda, z, \sigma, p)\mathfrak{e}_h\\
&= v^{b^-/2+m^-}\sum_{h \in L'/L}\sum_{\lambda \in L+h}\exp\left(-\Delta/8\pi v\right)(p)(\sigma(\lambda))e\left(Q(\lambda)u + Q_z(\lambda)iv\right)\mathfrak{e}_h. \qquad (2.6.5)
\end{aligned}$$

Remark 2.6.10. Cipra [Cip83, Theorem 1.9] tells us what Schwartz functions that satisfy the first spherical property look like. Essentially the only functions of this type are (possibly infinite) sums of Gaussians involving Hermite polynomials. We keep the discussion brief here. Definitions and details of Hermite polynomials can be found in Section 5.2.1. We observe that we can obtain Hermite polynomials from the term $\exp\left(-\Delta/8\pi\right)(p)$. For example if we let p be of the form $p(x_1, x_2 \dots x_{b^+ + b^-}) = x_n^\kappa$ for some $\kappa \in \mathbb{Z}, \kappa \geq 0$ then $\exp\left(-\Delta/8\pi\right)(p)$ is up to constants the κth Hermite polynomial, (see Lemma 5.3.4).

Equation (2.6.5) then motivates the following definition.

Definition 2.6.11. *Let $z \in \mathrm{Gr}(V(\mathbb{R})), h \in L'/L$. Let $\sigma : V(\mathbb{R}) \to \mathbb{R}^{b^+, b^-}$ be an isometry and let p be a homogeneous polynomial on $\mathbb{R}^{b^+, b^-}$ of degree (m^+, m^-). Then the **Siegel theta function** is*

$$\begin{aligned}
\vartheta_{L+h}(\tau, z, \sigma, p) &:= v^{b^-/2+m^-}\sum_{\lambda \in L+h}\exp(-\Delta/8\pi v)p(\sigma(\lambda))e\left(\frac{(\lambda_z, \lambda_z)}{2}\overline{\tau} + \frac{(\lambda_{z^\perp}, \lambda_{z^\perp})}{2}\tau\right)\\
&= v^{b^-/2+m^-}\sum_{\lambda \in L+h}\exp(-\Delta/8\pi v)p(\sigma(\lambda))e\left(Q(\lambda)u + Q_z(\lambda)iv\right)\\
&= \sum_{\lambda \in L+h}\varphi_0(\lambda, \tau, z, \sigma, p)
\end{aligned}$$

where $\varphi_0(\lambda,\tau,z,\sigma,p) := v^{b^-/2+m^-}\exp(-\Delta/8\pi v)p(\sigma(\lambda))e\left(Q(\lambda)u + Q_z(\lambda)iv\right)$. *The* $\mathbb{C}[L'/L]$-***valued Siegel theta function*** *is then*

$$\vec{\vartheta}_L(\tau,z,\sigma,p) := \sum_{h\in L'/L}\vartheta_{L+h}(\tau,z,\sigma,p)\mathfrak{e}_h = \Theta_L(\tau,\varphi_0(\lambda,z,\sigma,p)).$$

This definition makes sense, because the term $e(Q_z(\lambda)iv)$ rapidly decays, ensuring the components are absolutely and locally uniformly convergent in τ and z (see for example [DS05, Section 4.9]). We know $\vec{\vartheta}_L(\tau,z,\sigma,p)$ is a real analytic function. We observe as $v\to\infty$ then $\vartheta_{L+h}(\tau,z,\sigma,p) = \mathcal{O}(v^{b^-/2+m^-})$, uniformly in u. We will also need a more general version. This definition is taken from [Bor98, Section 4].

Definition 2.6.12. *Let* $z\in \mathrm{Gr}(V(\mathbb{R})), h\in L'/L$. *Let* $\sigma: V(\mathbb{R})\to\mathbb{R}^{b^+,b^-}$ *be an isometry, let* p *be a homogeneous polynomial on* $\mathbb{R}^{b^+,b^-}$ *of degree* (m^+,m^-) *and let* $\alpha,\beta\in V(\mathbb{R})$. *Then*

$$\vartheta_{L+h}(\tau,z,\sigma,p,\alpha,\beta) :=$$
$$v^{\frac{b^-}{2}+m^-}\sum_{\lambda\in L+h}\exp(-\Delta/8\pi v)p(\sigma(\lambda+\beta))e\left(Q(\lambda+\beta)u + Q_z(\lambda+\beta)iv - \left(\lambda+\frac{\beta}{2},\alpha\right)\right).$$

and then

$$\vec{\vartheta}_L(\tau,z,\sigma,p,\alpha,\beta) := \sum_{h\in L'/L}\vartheta_{L+h}(\tau,z,\sigma,p,\alpha,\beta)\mathfrak{e}_h.$$

2.6.4 The Action of the Dual Pair

We look at the action of the dual pair $(\mathrm{O}(V(\mathbb{R})),\mathrm{SL}_2(\mathbb{R}))$ via the Weil representation on these theta functions. This corresponds with the natural actions on τ and z. We still set $r = b^+ - b^- + 2m^+ - 2m^-$ throughout.

We consider the theta function $\vec{\vartheta}_L(\tau,z,\sigma,p)$. Let $(\gamma,\phi_\gamma)\in\mathrm{Mp}_2(\mathbb{R})$ and let $g_\theta\in\mathrm{SO}_2(\mathbb{R})$. We remember $g_{\gamma\tau} = \gamma\gamma g_\tau g_\theta$ and $j(\gamma g_\tau g_\theta,i) = j(\gamma,\tau)j(g_\tau,i)j(g_\theta,i)$. Then using the definition (2.6.1) we see that

$$\begin{aligned}
M^{\mathrm{Sch}}[\gamma]\varphi_0(\lambda,\tau,z,\sigma,p) &= M^{\mathrm{Sch}}\left[g_{\gamma\tau}g_\theta^{-1}g_\tau^{-1}\right]j(g_\tau,i)^{r/2}M^{\mathrm{Sch}}[g_\tau]\varphi_0(\lambda,z,\sigma,p)\\
&= M^{\mathrm{Sch}}[g_{\gamma\tau}]j(g_\tau,i)^{r/2}j(g_\theta,i)^{r/2}\varphi_0(\lambda,z,\sigma,p)\\
&= j(\gamma,\tau)^{-r/2}j(g_{\gamma\tau})^{r/2}M^{\mathrm{Sch}}[g_{\gamma\tau}]\varphi_0(\lambda,z,\sigma,p)\\
&= j(\gamma,\tau)^{-r/2}\varphi_0(\lambda,\gamma\tau,z,\sigma,p).
\end{aligned}$$

So this does indeed give rise to an action on τ. Using equation (2.6.2) and Lemma 2.6.9 we clearly have:

Corollary 2.6.13. *The Siegel theta function* $\vec{\vartheta}_L(\tau,z,\sigma,p)\in A_{r/2,\rho_L}$ *is an automorphic form of weight* $(\frac{b^+-b^-}{2}+m^+-m^-)$. *That is, for* $(\gamma,\phi_\gamma)\in\tilde{\Gamma}$,

$$\vec{\vartheta}_L(\gamma\tau,z,\sigma,p) = \phi_\gamma(\tau)^r\rho_L(\gamma,\phi_\gamma)\vec{\vartheta}_L(\tau,z,\sigma,p).$$

So $\vec{\vartheta_L}(\tau, z, \sigma, 1)$ has for example weight $(b^+ - b^-)/2$ in τ under $\tilde{\Gamma}$. For the general version:

Theorem 2.6.14 ([Bor98, Theorem 4.1])**.** *For any* $(\gamma, \phi_\gamma) \in \tilde{\Gamma}, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$\vec{\vartheta_L}(\gamma\tau, z, \sigma, p, a\alpha + b\beta, c\alpha + d\beta) = \phi_\gamma(\tau)^r \rho_L(\gamma, \phi_\gamma)\vec{\vartheta_L}(\tau, z, \sigma, p, \alpha, \beta).$$

Proof. This is shown in [Bor98, Theorem 4.1] using essentially the same methods, i.e. by finding a Fourier transform and applying the Poisson summation formula. □

We consider the action of the orthogonal group (we restrict to the case when $p = 1$). Let $z \in \mathrm{Gr}(V(\mathbb{R}))$ and let $g \in \mathrm{O}(V(\mathbb{R}))$. We then have using (2.4.5)

$$M^{\mathrm{Sch}}\,[g]\,\varphi_0(\lambda, \tau, z) = \varphi_0(g^{-1}\lambda, \tau, z) = \varphi_0(\lambda, \tau, gz).$$

This can be observed by letting g act on all elements $\lambda \in L$. We also recall the quadratic form is invariant under g and notice that

$$g(g^{-1}\lambda)_z = \lambda_{(gz)} \quad \text{and} \quad g(g^{-1}\lambda)_{z^\perp} = \lambda_{(gz)^\perp}.$$

Clearly then $\vec{\vartheta_L}(\tau, z)$ will be invariant under the action of a subgroup of $\mathrm{O}(L)$ on z that acts trivially on L'/L. We denote this group as $\mathrm{O}_d(L)$. In fact, we have the natural surjective homomorphism from $\mathrm{SO}(L)$ to $\mathrm{Aut}(L'/L)$. So we will actually consider the the action of a subgroup of $\mathrm{SO}(L)$ that acts trivially on L'/L. Which we call the **discriminant kernel** and denote as $\mathrm{SO}_d(L)$.

Our notation differs from [BO10] and is more similar to [Bru02]. In the case of signature $(2, 1)$ we have mentioned that z can be thought of as an element of the upper-half plane. In this case we can use a similar construction as in Section 2.6.1 to form a Siegel theta function (with a polynomial that satisfies the second spherical condition see Section 4.3) that will actually transform with a certain weight under $\mathrm{SO}_d(L)$. Section 2.2 told us that we have a surjective homomorphism, $\mathfrak{g}$ (see (2.2.1)) from $\mathrm{GSpin}(L)$ to $\mathrm{SO}(L)$. This means we can (and will) look at an action on z in terms of a subgroup of $\mathrm{GSpin}(L)$ which acts via conjugation on V. This discussion explains the singular theta correspondence in our context. We will lift from forms for the group $\mathrm{Mp}_2(\mathbb{R})$ to a subgroup of $\mathrm{O}(V(\mathbb{R})) \cong \mathrm{O}(b^+, b^-)$. More specifically, from $\tilde{\Gamma}$ to a group G in $\mathrm{GSpin}(L)$ whose image under $\mathfrak{g}$ is $\mathrm{SO}_d(L)$.

Chapter 3

The Setting

We will first detail in Section 3.1 the properties of a specific lattice L and the quadratic space (V, Q) of signature $(2, 1)$ that it lies in. The rest of our work will be based in this setting. This lattice L also has an associated character which allows us to twist the Weil representation. We discuss this in Section 3.2. We then describe in detail what the Grassmannian, cusps and the modular curve look like in this setting. We also define some twisted cycles on this Grassmannian. We discuss objects in roughly the same order as we defined them in Chapter 2.

3.1 A Lattice of Signature $(2, 1)$

We first describe a realisation of our space and its properties. Throughout the rest of this document we fix V to be a quadratic $\mathbb{Q}$-space of dimension 3 with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ of signature $(2, 1)$. We also fix $N \in \mathbb{N}$. Our setting is also used in [FM11, BO10, BFI15]. The settings in [BO13, AE13, Höv12] are also similar. However there the quadratic space is $(V, -Q)$ which has signature $(1, 2)$, see also Remark 2.6.3. All of these papers form a good reference for what follows.

We will work in a well known explicit realisation of this. In particular, we let V be traceless 2×2 matrices, i.e.

$$V := \{\lambda \in \mathrm{M}_2(\mathbb{Q}) \mid \mathrm{tr}(\lambda) = 0\}. \tag{3.1.1}$$

Let $\lambda, \mu \in V$. We form a quadratic space of signature $(2, 1)$ by setting $Q(\lambda) := -N \det(\lambda)$ and $(\lambda, \mu) := N\mathrm{tr}(\lambda\mu)$. It is easily checked the quadratic $\mathbb{Q}$-space (V, Q) is isotropic and indefinite of signature $(2, 1)$. In particular we have the following orthonormal basis of $V(\mathbb{R})$

$$e_1 := \frac{1}{\sqrt{2N}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 := \frac{1}{\sqrt{2N}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e_3 := \frac{1}{\sqrt{2N}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1.2}$$

We fix a lattice L as:

$$L := \left\{ \begin{pmatrix} b & -a/N \\ c & -b \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}.$$

The following properties are simple to check.

Lemma 3.1.1. *The lattice L is even (and therefore integral), has level $4N$ and discriminant $2N$. The discriminant group L'/L can be identified with: $\mathbb{Z}/2N\mathbb{Z}$ with discriminant form $x \mapsto x^2/4N$. The lattice L has dual lattice:*

$$L' := \left\{ \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}. \tag{3.1.3}$$

Remark 3.1.2. The elements $\lambda \in L'$ correspond to integral binary quadratic forms, Example 2.1.4. In particular, each $\lambda \in L'$ corresponds to a form $[a, b, Nc]$. Also the discriminant D' of $[a, b, Nc]$ is $D' = b^2 - 4Nac = 4NQ(\lambda)$.

Remember we have the identifications from Example 2.2.9

$$C(V(\mathbb{R})) \cong \mathrm{M}_2(\mathbb{R} \oplus \mathbb{R}), \quad C^0(V(\mathbb{R})) \cong \mathrm{M}_2(\mathbb{R}) \quad \text{and} \quad C^0(V) \cong \mathrm{M}_2(\mathbb{Q}).$$

We let $\mathrm{GL}_2(\mathbb{Q})$ act on V by conjugation, i.e. $\gamma.\lambda = \gamma\lambda\gamma^{-1}$ for $\gamma \in \mathrm{GL}_2(\mathbb{Q}), \lambda \in V$ and notice this action is isometric, i.e. $Q(\gamma.\lambda) = Q(\lambda)$. From Section 2.2.1 we have that elements $x \in \mathrm{GSpin}(V)$ act on V isometrically via conjugation. This gives rise to the isomorphisms

$$\mathrm{GL}_2(\mathbb{Q}) \cong \mathrm{GSpin}(V) \quad \text{and} \quad \mathrm{SL}_2(\mathbb{Q}) \cong \mathrm{Spin}(V)$$

noting that the Clifford norm on $\mathrm{M}_2(\mathbb{Q})$ is the determinant, (see Lemma 2.2.12). Using the exact sequence in (2.2.2), we have

$$\mathrm{PSL}_2(\mathbb{Q}) \cong \mathrm{SO}^+(V). \tag{3.1.4}$$

This accidental isomorphism is what makes signature $(2,1)$ particularly interesting with regards to modular forms.

We also want to find the group that takes L to itself and acts trivially on the discriminant group L'/L when acting via conjugation. We will see in Section 3.3 that there are two components in our model of the Grassmanian, and we fix an orientation by choosing one. We let $\Gamma(L) := \mathrm{SO}_d(L) \cap \mathrm{SO}^+(L)$. Then following Section 2.6.4 and Definition 2.2.11, it suffices to find a subgroup of $\mathrm{Spin}(L) \subset \mathrm{Spin}(V) \cong \mathrm{SL}_2(\mathbb{Q})$, whose image under $\mathfrak{g}$ is $\Gamma(L)$. It is easily checked that all W_m^N (where m is an exact divisor of N) map L to itself. So the image of W_m^N under $\mathfrak{g}$ is an element of $\mathrm{SO}^+(L)$. In fact we have the following:

Proposition 3.1.3 ([BO10, Proposition 2.2])**.** *The image of all W_m^N under $\mathfrak{g}$ is $\mathrm{SO}^+(L)$. The image of $\Gamma_0(N)$ under $\mathfrak{g}$ is $\Gamma(L)$.*

Example 3.1.4. Continuing from Example 2.5.13 (see also [BFI15, Example 2.2]) if $f(\tau) \in M^!_{k,\rho_L}$ then the scalar valued components f_h are certainly scalar-valued half-integral weight weakly holomorphic forms for $\Gamma(4N)$. (using the discussion in Section 2.5.1). The function

$$\sum_{h \in L'/L} f_h(4N\tau) = \sum_{h \in L'/L} \sum_{n \in \mathbb{Z}+Q(h)} c(n,h)e(4Nn\tau) = \sum_{m \in \mathbb{Z}} c(m + h^2/4N, h)e((4Nm + h^2)\tau)$$

is a form that satisfies the Kohnen plus space condition. I.e. the n-th fourier coefficient vanishes unless it is a square modulo $4N$. [EZ85, Theorem 5.4] tells us that this in fact forms an isomorphism from $M^!_{k,\rho_L}$ to $M^{+,!}_k(\Gamma_0(N))$ in the case where $k \in 2\mathbb{Z}+\frac{1}{2}$ and N prime. Later we define theta lifts of vector-valued forms using the lattice L. So in this case the isomorphism $M^!_{k,\rho_L} \cong M^{+,!}_k(\Gamma_0(N))$ tells us that we are also lifting scalar-valued forms in the Kohnen plus space.

3.2 The Twisted Weil Representation

The elements of L' corresponded to integral binary quadratic forms. This allows us to define a genus character on the lattice. We can then twist the Siegel theta functions from earlier with this character. These will then transform with respect to the twisted Weil representation. We still always set $\lambda = \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right) \in L'$ in this section. The main sources for the definition and properties of this character are [GKZ87, Section 1.2] and [Sko90b, Chapter 1] and there is some discussion of the twisted Weil representation in [AE13, Section 3] and [BO10, Section 4].

Remark 3.2.1. In particular, in [AE13] they use an intertwining linear map to modify any non-twisted theta function (on the lattice L) to a twisted theta function, which they show then transforms with respect to the twisted Weil representation. Our approach is different. We will form our twisted theta functions (in Section 3.6) using Gaussians at the base point, mirroring the construction in Section 2.6.1. This will allow us to see their weight immediately, mirroring Corollary 2.6.13. To do this we will need twisted versions of the formulas in Lemma 2.4.13 which we obtain in Proposition 3.2.6. Both approaches have essentially the same result and so which one to use is a matter of taste.

Definition 3.2.2. *We call an integer a* ***fundamental discriminant*** *if it is equal to* 1 *or the discriminant of a quadratic field. We set $D_N \in \mathbb{Z}$ a fundamental discriminant, and $r \in \mathbb{Z}$ such that $D_N \equiv r^2 \pmod{4N}$. From now on we simply denote D_N as D. Let n be any integer that is coprime to D and representable by a binary quadratic form $[N_1a, b, N_2c]$ with $N_1N_2 = N$ and $N_1, N_2 > 0$ i.e. $n = [N_1a, b, N_2c](x, y)$ for some $x, y \in \mathbb{Z}$. If D is such that*

1. $4NQ(\lambda)/D \equiv s^2 \pmod{4N}$ *for some* $s \in \mathbb{Z}$,
2. $\gcd(a, b, c, D) = 1$,

then we define the ***generalised genus character*** *as* $\chi_D(\lambda) := \left(\frac{D}{n}\right)$ *otherwise* $\chi_D(\lambda) := 0$.

This makes sense as [GKZ87, Proposition 1.2.1] tells us there always exists such an n.

Remark 3.2.3. It is common in the literature to denote the fundamental discriminant as Δ. We have used D to denote this so as to avoid confusion with the operator in (2.6.4) and the weight k hyperbolic Laplacian operator in Definition 2.5.4.

The key properties we need concerning this character are summarised below:

Proposition 3.2.4 ([GKZ87, Proposition 1.2.1]). *The character χ_D is independent of the choice of N_1, N_2 and n. The character χ_D is invariant under the action of $\Gamma_0(N)$ and the Atkin-Lehner involutions W_m^N. The character $\chi_D(\lambda)$ only depends on $\lambda \in L'$ modulo DL. Finally if we have the factorisations $D = D_1 D_2$ into discriminants and $N = N_1 N_2$ into positive factors so that* $\gcd(D_1, N_1 a) = \gcd(D_2, N_2 c) = 1$ *then*

$$\chi_D(\lambda) = \left(\frac{D_1}{N_1 a}\right)\left(\frac{D_2}{N_2 c}\right).$$

In particular, $\chi_D(-\lambda) = \operatorname{sgn}(D)(\lambda)$. We define a twisted version of Definition 2.4.12. Let D be a fundamental discriminant and then consider the scaled lattice DL. The scaled lattice DL has associated quadratic and bilinear forms $Q_D(\lambda) := Q(\lambda)/|D|$ and $(\lambda,\mu)_D := (\lambda,\mu)/|D|$ for $\lambda, \mu \in V$. Using the same ideas as in Lemma 3.1.1 we can then quickly see that DL is even, with dual lattice L' and discriminant $2N|D|^3 = |D|^3|L'/L|$. We will denote $\operatorname{sgn}(s) = s/|s|$ for all $s \in \mathbb{R}, s \neq 0$.

Definition 3.2.5. *Let $f \in \mathcal{S}(V(\mathbb{R})), h \in L'/L, \tilde{g} = (g, \phi_g) \in \mathrm{Mp}_2(\mathbb{R})$ and let D be a fundamental discriminant as before. Then we call*

$$\theta_{L,D,r}(\tilde{g}, f, h) := \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \sum_{\lambda \in DL + h'} \chi_D(\lambda) M^{\mathrm{Sch}}\left[\tilde{g}\right] f(\lambda)$$

the ***twisted theta function****.*

We now show a twisted version of Lemma 2.4.13.

Proposition 3.2.6. *We have that*

$$\theta_{L,D,r}(T\tilde{g}, f, h) = e(\operatorname{sgn}(D) \cdot Q(h))\theta_{L,D,r}(\tilde{g}, f, h),$$

$$\theta_{L,D,r}(S\tilde{g}, f, h) = \frac{e(-\operatorname{sgn}(D)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-\operatorname{sgn}(D)(h, h'))\theta_{L,D,r}(\tilde{g}, f, h').$$

Proof. We first show this for $T = \left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), 1\right)$. We use the formulas in Lemma 2.4.5 for the lattice DL.

$$\begin{aligned}\theta_{L,D,r}(T\tilde{g}, f, h) &= \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in DL+h'} \chi_D(\lambda) M^{\mathrm{Sch}}[T]\, M^{\mathrm{Sch}}[\tilde{g}]\, f(\lambda)\\ &= \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in L} e(Q_D(D\lambda+h'))\chi_D(D\lambda+h') M^{\mathrm{Sch}}[\tilde{g}]\, f(D\lambda+h').\end{aligned}$$

We observe that $Q_D(D\lambda) \in \mathbb{Z}$ for $\lambda \in L$ and $Q_D(h') = DQ(h)/|D| \pmod 1$, therefore

$$\begin{aligned}&= \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in L} e(DQ(h)/|D|)\chi_D(D\lambda+h') M^{\mathrm{Sch}}[\tilde{g}]\, f(D\lambda+h')\\ &= e(\mathrm{sgn}(D)\cdot Q(h))\theta_{L,D,r}(\tilde{g}, f, h).\end{aligned}$$

Using the invariance of χ_D modulo DL we have for $S = \left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right), \sqrt{\tau}\right)$ the following:

$$\begin{aligned}\theta_{L,D,r}(S\tilde{g}, f, h) &= \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in DL+h'} \chi_D(\lambda) M^{\mathrm{Sch}}[S]\, M^{\mathrm{Sch}}[\tilde{g}]\, f(\lambda)\\ &= e((b^- - b^+)/8) \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in DL+h'} \chi_D(\lambda) \widehat{M^{\mathrm{Sch}}[\tilde{g}]\, f}(-\lambda)\\ &= e(-1/8) \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in DL} \chi_D(h') \widehat{M^{\mathrm{Sch}}[\tilde{g}]\, f}(-\lambda-h').\end{aligned}$$

Using the Poisson summation formula (Lemma 2.4.10) and the fact that $\hat{f}(-\lambda - h') = e(-(\lambda, h')_D)f(\lambda)$ for $\lambda \in DL$ (see eg. [Bor98, Lemma 3.1]) we obtain

$$\begin{aligned}&= \frac{e(-1/8)}{\sqrt{|L'/DL|}} \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\lambda\in L'} \chi_D(h') e(-(h',\lambda)_D) M^{\mathrm{Sch}}[\tilde{g}]\, f(\lambda)\\ &= \frac{e(-1/8)}{\sqrt{|D|^3|L'/L|}} \sum_{\substack{h'\in L'/DL\\ h'\equiv rh(L)\\ Q(h')\equiv DQ(h)(D)}} \sum_{\substack{\lambda\in DL+h''\\ h''\in L'/DL}} \chi_D(h') e(-(h',h'')/|D|) M^{\mathrm{Sch}}[\tilde{g}]\, f(\lambda)\end{aligned}$$

and then using [BO10, Proposition 4.2] we finally have

$$\begin{aligned}&= \frac{\epsilon_D |D|^{3/2} e(-1/8)}{\sqrt{|D|^3|L'/L|}} \sum_{\substack{h'\in L'/L\\ h''\equiv rh'(L)\\ Q(h'')\equiv DQ(h')(D)}} \sum_{\substack{\lambda\in DL+h''\\ h''\in L'/DL}} \chi_D(h'') e(-\mathrm{sgn}(D)(h,h')) M^{\mathrm{Sch}}[\tilde{g}]\, f(\lambda)\\ &= \frac{e(-\mathrm{sgn}(D)/8)}{\sqrt{|L'/L|}} \sum_{h'\in L'/L} e(-\mathrm{sgn}(D)(h,h'))\theta_{L,D,r}(\tilde{g}, f, h').\end{aligned}$$

□

This leads to the following definition, mirroring Definition 2.4.16.

Definition 3.2.7. *Let* $\mathrm{U}(\mathbb{C}[L'/L])$ *be the unitary group on* $\mathbb{C}[L'/L]$*. Then we define a representation via the generators:*

$$\tilde{\rho}_L(T)(\mathfrak{e}_h) := e(\mathrm{sgn}(D)Q(h))\mathfrak{e}_h,$$
$$\tilde{\rho}_L(S)(\mathfrak{e}_h) := \frac{e(-\mathrm{sgn}(D)/8)}{\sqrt{|L'/L|}} \sum_{h' \in L'/L} e(-\mathrm{sgn}(D)(h,h'))\mathfrak{e}_{h'}.$$

We call $\tilde{\rho}_L : \tilde{\Gamma} \to \mathrm{U}(\mathbb{C}[L'/L])$ *the* ***twisted Weil representation on*** $\mathbb{C}[L'/L]$*.*

Comparing with Definition 2.4.16, we observe that $\tilde{\rho}_L$ is just ρ_L if $D > 0$ and $\overline{\rho_L}$ if $D < 0$.

3.3 The Grassmannian in Signature $(2, 1)$

We will now describe in more detail the Grassmannian of $V(\mathbb{R})$, a real hyperbolic 2-space. This is also discussed in [FM11, Section 2.1], [Bru02, Chapter 3] and [BFI15, Section 2.1]. We remember that $\mathrm{Gr}(V(\mathbb{R}) = \{z \subset V(\mathbb{R}) \mid \dim z = 1 \text{ and } Q|_z < 0\}$, so $\mathrm{Gr}(V(\mathbb{R}))$ is the set of negative lines. The set of norm -1 vectors form a two-sheet hyperboloid. I.e. $(V(\mathbb{R}), Q)$ is isometric to $\mathbb{R}^{2,1}$, so being of norm -1 dictates that $x_1^2 + x_2^2 - x_3^2 = -1$. We take one component of this two-sheet hyperboloid. Each vector on this component then represents an element of $\mathrm{Gr}(V(\mathbb{R}))$. Explicitly:

Definition 3.3.1. *We fix an isotropic vector* $l \in V$*. Then we call*

$$V_{-1} := \{v_{-1} \in V(\mathbb{R}) \mid (v_{-1}, v_{-1}) = -1,\ (v_{-1}, l) < 0\}$$

the ***hyperboloid model****, where we form a bijection* $\sigma_{V_{-1}}^{\mathrm{Gr}} : V_{-1} \to \mathrm{Gr}(V(\mathbb{R}))$ *via the map* $v_{-1} \mapsto \mathbb{R}v_{-1}$*.*

If instead we took $-l$ as our isotropic vector, we would obtain the other component of the two-sheet hyperboloid. We denote the bijection in the other direction, as $\sigma_{\mathrm{Gr}}^{V_{-1}} : \mathrm{Gr}(V(\mathbb{R})) \to V_{-1}$ and it is given by the map $z \mapsto l_z/|l_z|$. We see that $l_z/|l_z|$ is a norm -1 vector which lies in z and also that $l_z \neq 0$.

Section 2.6.2 and (2.6.3) told us that $\mathrm{Gr}(V(\mathbb{R})) \cong \mathrm{SO}^+(2,1)/\mathrm{SO}(2) \times \mathrm{SO}(1)$ and this is Hermitian. In fact using, (3.1.4) and Lemma 2.6.1, we have $\mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}^+(2,1)$ and $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}$. We conclude that the Grassmannian is isomorphic to $\mathbb{H}$. Explicitly:

Definition 3.3.2. *We let* $z' = x + iy \in \mathbb{H}$ *be the* ***upper-half plane model****, where we form a bijection* $\sigma_{\mathbb{H}}^{\mathrm{Gr}} : \mathbb{H} \to \mathrm{Gr}(V(\mathbb{R}))$ *via the map*

$$z' \mapsto \mathbb{R}(g_{z'}.e_3) = \mathbb{R}\begin{pmatrix} -x & x^2 + y^2 \\ -1 & x \end{pmatrix}.$$

This map is clearly injective. It is surjective as each element of $\mathrm{Gr}(V(\mathbb{R}))$ can be written as $\mathbb{R}\left(\begin{smallmatrix}-b & a\\ -1 & b\end{smallmatrix}\right), a > b^2, a, b \in \mathbb{R}$. We then have a bijection from $\sigma_{\mathbb{H}}^{V_{-1}} : \mathbb{H} \to V_{-1}$ via the map $z' \mapsto \sigma_{\mathrm{Gr}}^{V_{-1}}\left(\sigma_{\mathbb{H}}^{\mathrm{Gr}}(z')\right)$. Explicitly

$$\sigma_{\mathbb{H}}^{V_{-1}}(z') := \frac{l_{\mathbb{R}(g_{z'}.e_3)}}{|l_{\mathbb{R}(g_{z'}.e_3)}|} = \frac{-g_{z'}.e_3(l, g_z.e_3)}{|(l, g_{z'}.e_3)|} = \frac{1}{\sqrt{2Ny}}\,\mathrm{sgn}\left(\left(-l, \left(\begin{smallmatrix}-x & x^2+y^2\\ -1 & x\end{smallmatrix}\right)\right)\right)\left(\begin{smallmatrix}-x & x^2+y^2\\ -1 & x\end{smallmatrix}\right)$$

where we have used that $(g_{z'}.e_3, g_{z'}.e_3) = -1$. If we fix l such that the sgn term equals 1 (or -1), then this is a positive (or negative) orientation of $\mathrm{Gr}(V(\mathbb{R}))$. In summary we have $\mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}) \cong V_{-1}$ and the following bijective diagram:

$$\begin{tikzcd}
& \mathbb{H} \arrow[dl, bend right, "\sigma_{\mathbb{H}}^{V_{-1}}"'] \arrow[dr, bend left, "\sigma_{\mathbb{H}}^{\mathrm{Gr}}"] & \\
V_{-1} \arrow[rr, bend right, "\sigma_{V_{-1}}^{\mathrm{Gr}}"'] & & \mathrm{Gr} \arrow[ll, bend right, "\sigma_{\mathrm{Gr}}^{V_{-1}}"']
\end{tikzcd}$$

Figure 3.1: Bijective Diagram

As is standard, we will often abuse notation, and set $z = x + iy \in \mathbb{H}$ but also denote by z its identifications in V_{-1} and $\mathrm{Gr}(V(\mathbb{R}))$. We had that $\mathrm{SL}_2(\mathbb{Q})$ acts on V via conjugation and, as we would expect, this intertwines with the natural action on $\mathbb{H}$. We can check with some simple linear algebra that $\sigma_{\mathbb{H}}^{V_{-1}}(\gamma z) = \gamma.(\sigma_{\mathbb{H}}^{V_{-1}}(z))$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Q}), z = x + iy \in \mathbb{H}$.

We now look for an oriented basis for $V(\mathbb{R})$ compatible with these models. We fix a positive orientation of $\mathrm{Gr}(V(\mathbb{R}))$ from here on. In (3.1.2) we had an orthonormal basis e_1, e_2, e_3 of $V(\mathbb{R})$. We notice for the base point $z = i$ that $e_3 = \sigma_{\mathbb{H}}^{V_{-1}}(i)$. We then define a basis $b_1(z) := g_z.e_1$, $b_2(z) := g_z.e_2$ and $b_3(z) := g_z.e_3$ for any $z = x + iy \in \mathbb{H}$. We will have $b_3(z) = \sigma_{\mathbb{H}}^{V_{-1}}(z)$. Explicitly

$$b_1(z) = \frac{1}{\sqrt{2Ny}}\begin{pmatrix} x & -x^2+y^2 \\ 1 & -x \end{pmatrix}, \tag{3.3.1a}$$

$$b_2(z) = \frac{1}{\sqrt{2Ny}}\begin{pmatrix} y & -2xy \\ 0 & -y \end{pmatrix}, \tag{3.3.1b}$$

$$b_3(z) = \frac{1}{\sqrt{2Ny}}\begin{pmatrix} -x & z\overline{z} \\ -1 & x \end{pmatrix}. \tag{3.3.1c}$$

It can be verified $(b_1(z), b_1(z)) = 1, (b_2(z), b_2(z)) = 1, (b_3(z), b_3(z)) = -1$ and $(b_i(z), b_j(z)) = 0$ for $i \neq j$. If $\lambda \in V(\mathbb{R})$ then $\lambda = \sum \lambda_i(z) b_i(z)$ where $\lambda_i(z) := \frac{(\lambda, b_i(z))}{(b_i(z), b_i(z))}$. We have

$$\lambda_1(z) = \frac{1}{\sqrt{2N}y}(cN(-x^2+y^2) + bx - a), \tag{3.3.2a}$$

$$\lambda_2(z) = \frac{1}{\sqrt{2N}y}(-2cNxy + by), \tag{3.3.2b}$$

$$\lambda_3(z) = \frac{-1}{\sqrt{2N}y}(cN|z|^2 - bx + a). \tag{3.3.2c}$$

Also $(\lambda, \lambda) = \lambda_1(z)^2 + \lambda_2(z)^2 - \lambda_3(z)^2$, $\lambda_{z^\perp} = \lambda_1(z) b_1(z) + \lambda_2(z) b_2(z)$ and $\lambda_z = \lambda_3(z) b_3(z)$.

3.4 The Modular Curve

We now consider the modular curve and its associated cusps. We will find Fourier expansions at these cusps. Good references include [DS05, Section 2.4], [BF06, Section 2] and [FM11, Section 2.1].

We identify $\Gamma_0(N)\backslash \mathrm{Gr}(V(\mathbb{R}))$ with the **modular curve** $Y_0(N) := \Gamma_0(N)\backslash\mathbb{H}$. We compactify this by adjoining finitely many **cusps** i.e. $X_0(N) := Y_0(N) \cup \Gamma_0(N)\backslash (\mathbb{Q}\cup\{\infty\})$. The elements of $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q}\cup\{\infty\}$ are pairs (m/n) where $\gcd(m,n) = 1$. Matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ act on (m/n) by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(m/n) = (am+bn)/(cm+dn)$.

When N is square-free there are $\sum_{d|N}$ cusps. It is well known [DS05, Section 3.8], [Sch04, Section 4] that the cusps of $Y_0(N)$ can be represented by $1/d$, where d are the divisors of N. Any two cusps $(m/n), (m'/n')$ are $\Gamma_0(N)$ equivalent when $\gcd(n,N) = \gcd(n',N)$. Letting d' be another divisor of N, then $W_{d'}^N$ maps the cusp $1/d$ to $\gcd(d',d)/(d'd)$. We observe that all the cusps can be represented as $W_d^N\infty$, with d running over the divisors of N.

We denote by $\mathrm{Iso}(V)$ the set of isotropic lines in V. Then the map $\sigma_{\mathbb{P}^1(\mathbb{Q})}^{\mathrm{Iso}(V)} : \mathbb{P}^1(\mathbb{Q}) \to \mathrm{Iso}(V)$, given by the identification $(m/n) \mapsto \mathrm{Span}\begin{pmatrix} -mn & m^2 \\ -n^2 & mn \end{pmatrix}$, is clearly bijective. We can check that the actions intertwine i.e. $\sigma_{\mathbb{P}^1(\mathbb{Q})}^{\mathrm{Iso}(V)}(\gamma(m/n)) = \gamma.\sigma_{\mathbb{P}^1(\mathbb{Q})}^{\mathrm{Iso}(V)}(m/n)$ for $(m/n) \in \mathbb{P}^1(\mathbb{Q}), \gamma \in \mathrm{SL}_2(\mathbb{Q})$. This means we can view the cusps attached to $Y_0(N)$ as $\Gamma_0(N)$-classes of isotropic lines in V. Each of these lines can be uniquely represented by a primitive isotropic vector in L up to sign. We have fixed an orientation earlier so we choose our primitive vectors l' so that $\mathrm{sgn}((-l', g_z.e_3) = 1$. Then the cusps ∞ and 0 correspond to $\Gamma_0(N)$-classes of $l_\infty := \left(\begin{smallmatrix} 0 & 1/N \\ 0 & 0 \end{smallmatrix}\right)$ and $l_0 := \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$. We observe that any other cusp l' given by $\gamma.l_\infty = l'$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ is still positively oriented. Explicitly: $(-\gamma.l_\infty, g_z.e_3) = ((cx-a)^2 + (cy)^2)/\sqrt{2N}y > 0$.

3.5 Twisted Special Cycles

We describe some geodesics in $\mathrm{Gr}(V(\mathbb{R}))$ and cycles in $Y_0(N)$, which we also associate to an $f \in H_{k,\tilde{\rho}_{L^-}}$. This will later allow us to describe the singularities of our lift, which will lie along these cycles. Similar discussions can be found in [FM11, Section 3.1], [BF04, Section 2], [BO10, Section 5], [FM02, Section 3], [BF06, Section 2] and [Fun02]. We fix a vector $\lambda \in V$ with positive norm $Q(\lambda) > 0$ and $\lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix}$.

Definition 3.5.1. *We denote*

$$D_\lambda := \{z \in \mathrm{Gr}(V(\mathbb{R})) \mid z \perp \lambda\}.$$

for a ***geodesic*** *in* $\mathrm{Gr}(V(\mathbb{R}))$. *In the upper half plane model we can easily see this is*

$$D_\lambda \cong \left\{z \in \mathbb{H} \mid cN|z|^2 - bx + a = 0\right\}. \tag{3.5.1}$$

We note $D_{\gamma.\lambda} = \gamma.D_\lambda$ for $\gamma \in \mathrm{SL}_2(\mathbb{R})$. If $\lambda \perp l_\infty$ and $Q(\lambda) > 0$ this is equivalent to when $c = 0, b \neq 0$. So $z \in D_\lambda$ is equivalent to when $x = a/b$ i.e. D_λ defines vertical half-line in $\mathbb{H}$. The other case is when $\lambda \not\perp l_\infty$. This is equivalent to having $c \neq 0$, and after completing the square in x we have that

$$y = \sqrt{-\left(x - \frac{b}{2Nc}\right)^2 + \frac{Q(\lambda)}{Nc^2}}$$

i.e. D_λ defines a semi-circle with centre at $(x = b/2Nc, y = 0)$ and radius $\sqrt{Q(\lambda)/Nc^2}$. Clearly any two geodesics intersect at most once.

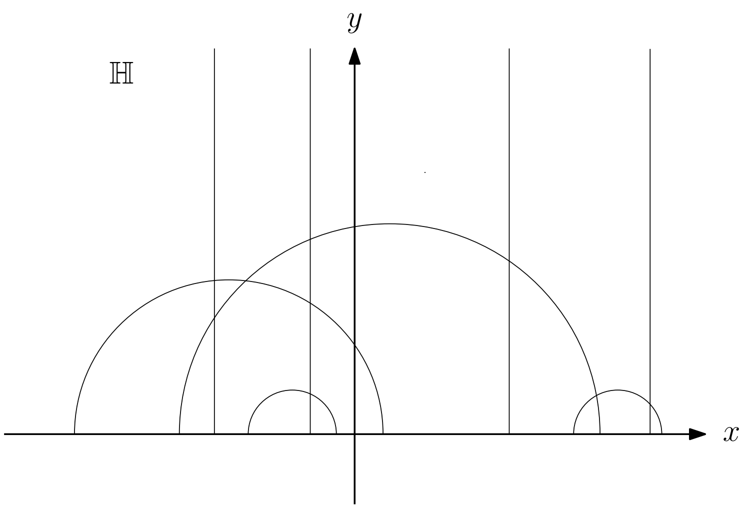

Figure 3.2: Geodesics on the upper-half plane

Any $\lambda^\perp$ in $V(\mathbb{R})$ can be spanned with two positively oriented isotropic vectors in $V(\mathbb{R})$ (not necessarily in V or primitive), which it is standard to denote as l_λ, l'_λ. The corresponding isotropic lines are uniquely determined and lie in $\mathrm{Iso}(V(\mathbb{R})) \cong \mathbb{R} \cup \{\infty\}$, so the end points of D_λ correspond to $\mathrm{Span}(l_\lambda)$ and $\mathrm{Span}(l'_\lambda)$ under the $\sigma^{\mathrm{Iso}(V(\mathbb{R}))}_{\mathbb{P}^1(\mathbb{Q})}$ map. So our first case, $\lambda \perp l_\infty$, is when D_λ joins the cusps ∞ and a/b. In this case $Q(\lambda)/N \in (\mathbb{Q}^\times)^2$. The second case, $\lambda \not\perp l_\infty$, is when D_λ joins the "cusps" $b/2Nc \pm \sqrt{Q(\lambda)/Nc^2}$ which are clearly "rational cusps" if and only if $Q(\lambda)/N \in (\mathbb{Q}^\times)^2$.

We will also need an orientation of the geodesics. In Section 3.3 we fixed an oriented basis $\{b_1(z), b_2(z), b_3(z)\}$ for $V(\mathbb{R})$ and $\mathrm{Gr}(V(\mathbb{R}))$. we orient D_λ such that $l_\lambda, \lambda, l'_\lambda$ is also a positively oriented basis of V. This ensures $D_{\gamma.\lambda}$ and $\gamma.D_\lambda$ both have the same orientation. Using the upper half-plane model the geodesics were defined by $a, b, c \in \mathbb{Z}$. In the case $c = 0, b \neq 0$ (a vertical half-line), this is oriented towards $i\infty$ if $b > 0$ and in reverse if $b < 0$. When $c \neq 0$ the semi-circle is oriented clockwise if $c > 0$ and anti-clockwise if $c < 0$.

Definition 3.5.2. *We let the **stabiliser** of λ in $\Gamma_0(N)$ be*

$$\Gamma_\lambda := \{\gamma \in \Gamma_0(N) \mid \gamma.\lambda = \lambda\}.$$

*We let $\overline{\Gamma}_\lambda$ be the image of the stabiliser in $\mathrm{PSL}_2(\mathbb{Z})$. We denote by $Z(\lambda)$, the image of the quotient $\Gamma_\lambda \backslash D_\lambda$ in the modular curve $Y_0(N)$. We call $Z(\lambda)$ a **cycle**.*

In other words, cycles are "geodesics in the modular curve". We classify these into two types. The cycle $Z(\lambda)$ is an infinite (or closed) geodesic in $Y_0(N)$, this is exactly when $\overline{\Gamma}_\lambda$ is trivial (or infinite cyclic). This is exactly when $\lambda^\perp \subset V$ is isotropic (or anisotropic). Using our earlier discussion $\lambda^\perp \subset V$ is isotropic (or anisotropic) when the isotropic vectors l_λ, l'_λ which span $\lambda^\perp$ are in V (or $V(\mathbb{R})\backslash V$). This happened when $Q(\lambda)/N \in (\mathbb{Q}^\times)^2$ (or $Q(\lambda)/N \notin (\mathbb{Q}^\times)^2$). So if D_λ is infinite then it joins two "rational" cusps (or closed when it joins two "irrational" cusps).

We will see that the singularities of the lift will depend on a collection of vectors of certain length and in a certain coset. We form some notation for this. Set $\mathcal{D} \in \mathbb{Z}$ and write

$$L_\mathcal{D} := \{\lambda \in L' \mid Q(\lambda) = \mathcal{D}/4N\}.$$

Such vectors only exist if $\mathcal{D} \equiv s^2 \pmod{4N}$ for some $s \in \mathbb{Z}$. Then, for $h \in L'/L$, we write

$$L_{\mathcal{D},h} := \{\lambda \in L' \mid Q(\lambda) = \mathcal{D}/(4N),\ \lambda \equiv h \pmod{L}\}. \tag{3.5.2}$$

Again such vectors only exist if $\mathcal{D} \equiv h^2 \pmod{4N}$. The group $\Gamma_0(N)$ acts by conjugation on $L_{\mathcal{D},h}$ and it is well known, by reduction theory on binary quadratic forms, that there are only finitely many orbits of $L_{\mathcal{D},h}$ (as long as $\mathcal{D} \neq 0$).

Definition 3.5.3. *Let D be a fundamental discriminant with $r \in \mathbb{Z}$ such that $D \equiv r^2 \pmod{4N}$ and let $h \in L'/L$. Let $m \in \mathbb{Z} - \mathrm{sgn}(D)Q(h)$ with $m < 0$. We define $d := 4N\mathrm{sgn}(D)m$ and we have the following formal linear combination on $Y_0(N)$:*

$$Z'_{D,r}(m,h) := \sum_{\lambda \in L_{-dD,rh}/\Gamma_0(N)} \chi_D(\lambda) Z(\lambda).$$

We call $Z'_{D,r}(m,h)$ a ***twisted special cycle****. We write $Z_{D,r}(m,h)$ for its image in $\mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}))$. Let $k \in \frac{1}{2}\mathbb{Z}$ and let $f \in H_{k,\tilde{\rho}_L^-}$ with expansion as in (2.5.2). Then we call*

$$Z'_{D,r}(f) := \sum_{h \in L'/L} \sum_{\substack{m \in \mathbb{Z} - \mathrm{sgn}(D)Q(h) \\ m<0}} c^+(m,h) Z'_{D,r}(m,h)$$

an ***associated twisted special cycle****. We denote $Z_{D,r}(f)$ for its image in $\mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}))$.*

We remember that χ_D is invariant under the action of $\Gamma_0(N)$. We can think of a twisted special cycle roughly as being a finite linear combination of twisted cycles, associated to vectors of norm $-m|D| > 0$ and twisted coset rh. The associated twisted special cycle depends only on the principal part of f that has only finitely many $c^+(m,h)$ terms for $m < 0$, so again this is a finite collection of cycles. We also note that $Z_{D,r}(f) \subset \mathbb{H}$ is a nowhere dense set.

Remark 3.5.4. There always exists a vector $\lambda \in L_{-dD,rh}$ such that $\chi_D(\lambda) \neq 0$ (in particular $Q(\lambda) = -m|D|$ so in Definition 3.2.2, we certainly have $4NQ(\lambda)/D = -4N\mathrm{sgn}(D)m \equiv 0 \pmod{4N}$). This implies $Z_{D,r}(f)$ is the empty set if and only if the principal part of f is constant. When the principal part of f is not constant we know that n_0 (from (2.5.6)) is positive, $n_0 > 0$. For each $-n_0 \leq n < 0$ there is some associated geodesics D_λ, where $Q(\lambda) = -n|D|$. In the case $\lambda \perp l_\infty$, the geodesics D_λ are semi-circles with radius $\sqrt{-n|D|/Nc^2} \leq \sqrt{n_0|D|/N}$. We conclude that, above this "height" we will only find vertical half-line geodesics associated to $Z_{D,r}(f)$. Within these semi-circles we have bounded connected components and above them we have unbounded connected components (which are regions between two vertical half-lines).

3.6 Twisted Siegel Theta Functions

In this section we describe some twisted Siegel theta functions. We work with the lattice L from earlier, which allows us to twist these theta functions with the genus character. Following Remark 3.2.1 we construct these in the same manner that we did in Section 2.6. We discuss the transformation properties of the twisted Siegel theta functions. We then define two examples with some well chosen polynomials. These examples will be the kernel functions we use to define the singular theta lift and the Shimura lift. We will then show the transformation properties of the kernel functions in both variables, τ and z. We fix $L, (V,Q), D, r$ and

$\lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix} \in L'$, as before. We also let $\tau = u+iv \in \mathbb{H}$ and $z = x+iy \in \mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}))$.

Mirroring the construction in (2.6.1) we work with respect to the lattice DL and quadratic form Q_D. We fix an $f \in \mathcal{S}(V(\mathbb{R}))$ which satisfies the first spherical property i.e.

$$M^{\mathrm{Sch}}\left[g_\theta\right] f(\lambda) = \sigma_{1/2}\left(\tilde{g}_\theta\right)^{r'} f(\lambda) \tag{3.6.1}$$

for $\tilde{g}_\theta \in \widetilde{\mathrm{SO}}(2)$ and some fixed $r' \in \mathbb{Z}$. We then let

$$\Theta_{L,D,r}(\tau, f) \coloneqq \sum_{h \in L'/L} \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \sum_{\lambda \in DL+h'} j(g_\tau, i)^{r'/2} \chi_D(\lambda) M^{\mathrm{Sch}}\left[g_\tau\right] f(\lambda) \mathfrak{e}_h. \tag{3.6.2}$$

Let $\sigma : V(\mathbb{R}) \to \mathbb{R}^{2,1}$ be an isometry for the quadratic form Q. We then define $\sigma_D \coloneqq |D|^{-1/2}\sigma$, which is an isometry $\sigma_D : V(\mathbb{R}) \to \mathbb{R}^{2,1}$ for the quadratic form Q_D. We also set p a homogeneous polynomial on $\mathbb{R}^{2,1}$ of degree (m^+, m^-) and set $r' = 2 - 1 + 2m^+ - 2m^-$. We define the following functions:

$$\varphi_0(\lambda, z, \sigma_D, p, D) \coloneqq |D|^{(m^+ + m^-)/2} \exp\left(-\Delta/8\pi\right)(p)(\sigma_D(\lambda)) e\left(\frac{Q_z(\lambda)}{|D|} i\right),$$
$$\varphi_0(\lambda, \tau, z, \sigma, p, D) \coloneqq v^{1/2+m^-} \chi_D(\lambda) \exp\left(-|D|\Delta/8\pi v\right)(p)(\sigma(\lambda)) e\left(\frac{Q(\lambda)}{|D|} u + \frac{Q_z(\lambda)}{|D|} iv\right).$$

Using Lemma 2.6.9 for the lattice DL with quadratic form Q_D we see that $\varphi_0(\lambda, z, \sigma_D, p, D)$ satisfies (3.6.1). We then use the fact that

$$\exp\left(-\Delta/8\pi\right)(p)(\sigma_D(\sqrt{v}\lambda)) = \left(\frac{v}{|D|}\right)^{(m^+ + m^-)/2} \exp\left(-|D|\Delta/8\pi v\right)(p)(\sigma(\lambda))$$

to obtain the following:

$$\begin{aligned}
&\Theta_{L,D,r}(\tau, \varphi_0(\lambda, z, \sigma_D, p, D) \\
&= \sum_{h \in L'/L} \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \sum_{\lambda \in DL+h'} j(g_\tau, i)^{r'/2} \chi_D(\lambda) M^{\mathrm{Sch}}\left[g_\tau\right] \varphi_0(\lambda, z, \sigma_D, p, D) \mathfrak{e}_h \\
&= \sum_{h \in L'/L} \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \sum_{\lambda \in DL+h'} \varphi_0(\lambda, \tau, z, \sigma, p, D) \mathfrak{e}_h \\
&= \sum_{h \in L'/L} \sum_{\lambda \in L} \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \varphi_0(D\lambda + h', \tau, z, \sigma, p, D) \mathfrak{e}_h.
\end{aligned}$$

We then make the "substitution" $h' \mapsto h' - D\lambda$ to get

$$= \sum_{h \in L'/L} \sum_{\substack{h' \in L' \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \varphi(h', \tau, z, \sigma, p, D) \mathfrak{e}_h.$$

This motivates the following definition.

Definition 3.6.1. *Let $h \in L'/L$, let $\sigma : V(\mathbb{R}) \to \mathbb{R}^{2,1}$ be an isometry and let p be a homogeneous polynomial on $\mathbb{R}^{2,1}$ of degree (m^+, m^-). Then the* ***twisted Siegel theta function*** *is*

$$\vartheta_{L+h,D,r}(\tau, z, \sigma, p) := \sum_{\substack{\lambda \in L+h \\ Q(\lambda) \equiv DQ(h)(D)}} \varphi_0(\lambda, \tau, z, \sigma, p, D).$$

*The $\mathbb{C}[L'/L]$**-valued twisted Siegel theta function** is*

$$\vec{\vartheta}_{L,D,r}(\tau, z, \sigma, p) := \sum_{h \in L'/L} \vartheta_{L+h,D,r}(\tau, z, \sigma, p)\mathfrak{e}_h = \Theta_{L,D,r}(\tau, \varphi_0(\lambda, z, \sigma, p, D)).$$

We observe as $v \to \infty$ then $\vartheta_{L+h}(\tau, z, \sigma, p) = \mathcal{O}(v^{1/2+m^-})$, uniformly in u. The next lemma mirrors Corollary 2.6.13.

Lemma 3.6.2. *The twisted Siegel theta function $\vec{\vartheta_L}(\tau, z, \sigma, p) \in A_{r/2,\tilde{\rho}_L}$ is an automorphic form of weight $(1/2 + m^+ - m^-)$. That is, for $(\gamma, \phi_\gamma) \in \tilde{\Gamma}$*

$$\vec{\vartheta_L}(\gamma\tau, z, \sigma, p) = \phi_\gamma(\tau)^{r'} \tilde{\rho}_L(\gamma, \phi_\gamma)\vec{\vartheta_L}(\tau, z, \sigma, p).$$

Proof. We remember $g_{\gamma\tau} = \gamma g_\tau g_\theta$ for some $g_\theta \in \mathrm{SO}(2)$ and $j(\gamma g_\tau g_\theta, i) = j(\gamma, \tau)j(g_\tau, i)j(g_\theta, i)$. Then, using Proposition 3.2.6 and the fact that $\varphi_0(\lambda, z, \sigma, p, D)$ satisfies (3.6.1), we see for each coset

$$\begin{aligned} &\varphi_0(\lambda, \gamma.\tau, z, \sigma, p, D) \\ =& \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \sum_{\lambda \in DL+h'} j((\gamma, \tau)j(g_\tau, i)j(g_\theta, i))^{r'/2} \chi_D(\lambda) M^{\mathrm{Sch}} [\gamma g_\tau g_\theta] \varphi_0(\lambda, z, \sigma, p, D)\mathfrak{e}_h \\ =& j(\gamma, \tau)^{r'/2} \tilde{\rho}_L(\gamma, \phi_\gamma)\varphi_0(\lambda, \tau, z, \sigma, p, D). \end{aligned}$$

□

3.6.1 Kernel Functions

We will use two examples of these twisted Siegel theta functions to form the kernels of our lift. We first consider their polynomials.

We first define the vectors $v(z) := -g_z.e_3$ and $v(z^\perp) := g_z.(e_1 + ie_2)$. We notice $v(z)$ spans $z \in \mathrm{Gr}(V(\mathbb{R}))$ and $b_1(z), b_2(z)$ span $z^\perp$, so $v(z^\perp) \in z^\perp \in V(\mathbb{C})$. Further we have $(v(z^\perp), v(z^\perp)) = 0$ and $(v(z^\perp), \overline{v(z^\perp)}) = 2 > 0$. We then consider the two polynomials

$$\begin{aligned} (\lambda, v(z)) &= -(\lambda, b_3(z)) = -(\lambda, g_z.e_3) = \lambda_3(z), \\ y(\lambda, v(z^\perp)) &= y(\lambda, b_1(z) + ib_2(z)) = y(\lambda, g_z.(e_1 + ie_2)) = y(\lambda_1(z) + i\lambda_2(z)). \end{aligned}$$

We will often denote these as $p_z(\lambda)$ and $q_z(\lambda)$ respectively. For $\lambda := \left(\begin{smallmatrix} b/2N & -a/N \\ c & -b/2N \end{smallmatrix}\right)$ we have

$$p_z(\lambda) := (\lambda, v(z)) = \frac{-1}{\sqrt{2Ny}}(cN|z|^2 - bx + a), \tag{3.6.3}$$

$$q_z(\lambda) := y(\lambda, v(z^\perp)) = \frac{-1}{\sqrt{2N}}\left(cNz^2 - bz + a\right). \tag{3.6.4}$$

Lemma 3.6.3. *Let $k \in \mathbb{Z}, k > 0$. The polynomials $(\lambda, v(z))(\lambda, v(z^\perp)y)^{k-1}$ and $(\lambda, v(z^\perp)/y)^k$ are harmonic and of degree $(k-1, 1)$ and $(k, 0)$.*

Proof. We remember $Q(\lambda) = \frac{1}{2}\left(\lambda_1(z)^2 + \lambda_2(z)^2 - \lambda_3(z)^2\right)$ so we have an explicit isometry $\sigma_z : V(\mathbb{R}) \to \mathbb{R}^{2,1}$ given by

$$\sigma_z(\lambda) := \left(\frac{1}{\sqrt{2}}\lambda_1(z), \frac{1}{\sqrt{2}}\lambda_2(z), \frac{1}{\sqrt{2}}\lambda_3(z)\right). \tag{3.6.5}$$

Thus, if $p(\sigma_z(\lambda)) = (\lambda, v(z))$ and $q(\sigma_z(\lambda)) = y(\lambda, v(z^\perp)))$ then $p(x_1, x_2, x_3) = \sqrt{2}x_3$ and $q(x_1, x_2, x_3) = y\sqrt{2}(x_1 + ix_2)$ i.e. they are polynomials of degrees $(0, 1)$ and $(1, 0)$ in $\mathbb{R}^{2,1}$. We set $p_k(x_1, x_2, x_3) := (\sqrt{2}x_3)(y\sqrt{2}(x_1 + ix_2))^{k-1}$ and $p_k^*(x_1, x_2, x_3) := (\sqrt{2}(x_1 + ix_2)/y)^k$. These are both are harmonic as it is easily verified, for example, that

$$\frac{\partial^2 p_k}{\partial x_1^2} + \frac{\partial^2 p_k}{\partial x_2^2} + \frac{\partial^2 p_k}{\partial x_3^2} = 0.$$

□

We now define the kernel function.

Definition 3.6.4. *Let $h \in L'/L$ and let $k \in \mathbb{Z}, k > 0$. Then the **kernel function** is*

$$\theta_{D,r,h,k}(\tau, z) := v^{3/2} \sum_{\substack{\lambda \in L + rh \\ Q(\lambda) \equiv DQ(h)(D)}} \chi_D(\lambda)(\lambda, v(z))(\lambda, v(z^\perp)y)^{k-1} e\left(\frac{Q(\lambda)}{|D|}u + \frac{Q_z(\lambda)}{|D|}iv\right).$$

*The $\mathbb{C}[L'/L]$**-valued kernel function** is*

$$\Theta_{D,r,k}(\tau, z) := \sum_{h \in L'/L} \theta_{D,r,h,k}(\tau, z)\mathfrak{e}_h.$$

We can also rewrite this definition in the following useful forms:

$$\begin{aligned}
\theta_{D,r,h,k}(\tau, z) &= \vartheta_{L+h,D,r}(\tau, z, \sigma_z, p_k) \\
&= \sum_{\substack{\lambda \in L + rh \\ Q(\lambda) \equiv DQ(h)(D)}} \varphi_0(\lambda, \tau, z, \sigma_z, p_k, D) \\
&= v^{3/2} \sum_{\substack{\lambda \in L + rh \\ Q(\lambda) \equiv DQ(h)(D)}} \frac{-\chi_D(\lambda)(cN|z|^2 - bx + a)}{\sqrt{2Ny}} \left(\frac{-\left(cNz^2 - bz + a\right)}{\sqrt{2N}}\right)^{k-1} \\
&\quad \times e\left(\left(\frac{b^2}{4N} - ac\right)\frac{u}{|D|} + \left(\frac{b^2}{4N} - ac + \frac{(cN|z|^2 - bx + a)^2}{2Ny^2}\right)\frac{iv}{|D|}\right).
\end{aligned} \tag{3.6.6}$$

Definition 2.6.11 tells us this is a real analytic function in τ and z. There exists a $C > 0$ such that $\Theta_{D,r,k}(\tau, z) = \mathcal{O}(e^{-Cv})$ as $v \to \infty$, uniformly in u. This exponential decay is a better asymptotic than the polynomial growth given in [Höv12] and Definition 2.6.11. This is because in our case the polynomial kills the term when $\lambda = 0$.

We will often just denote $\theta_{D,r,h,k}(\tau, z)$ as $\theta_{h,k}(\tau, z)$ and $\Theta_{D,r,k}(\tau, z)$ as $\Theta_k(\tau, z)$ when the context is clear. We also have

$$\overline{\Theta_{D,r,k}(\tau, z)} = \Theta_{D,r,k}(-\overline{\tau}, -\overline{z}). \tag{3.6.7}$$

Remark 3.6.5. Let us restrict to the case $k \in 2\mathbb{Z}, k > 0, N = 1$ and $D = 1$. Then the kernel function agrees, up to the constant i, with the scalar-valued theta function "$\Theta_1(\tau, z)$" discussed in [BKZ14, Section 1]. In the scalar-valued case we need k to be even, as otherwise the sum over λ and $-\lambda$ cancel and the theta function is zero. The theta function "$\Theta^*(z, \tau)$" as denoted in [BKV13, (1.6)] also matches our definition. In particular, $i\overline{\Theta^*(-\overline{z}, \tau)} = i\Theta^*(z, -\overline{\tau}) = v^{k-3/2}\Theta_1(\tau, z)$. In the case $k = 1$ our kernel function $v^{k-3/2}\overline{\Theta_{D,r,k}(\tau, -\overline{z})}$ matches the theta function defined in [Höv12, Definition 2.5] (noting that in [Höv12] the lattice is of signature $(1, 2)$). The kernel function (in the case $k = 1$) is also used in [AGOR14, Section 3] to define an adjoint lift to the one considered in [Höv12].

Definition 3.6.6. *Let $h \in L'/L$ and $k \in \mathbb{Z}, k > 0$. Then the* ***Shintani kernel function*** *is*

$$\begin{aligned}\theta^*_{D,r,h,k}(\tau, z) &:= v^{1/2} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D)}} \chi_D(\lambda) \left(\frac{(\lambda, v(z^\perp))}{y}\right)^k e\left(\frac{Q(\lambda)}{|D|}u + \frac{Q_z(\lambda)}{|D|}iv\right) \\ &= \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D)}} \varphi_0(\lambda, \tau, z, \sigma_z, p^*_k, D)\end{aligned}$$

The $\mathbb{C}[L'/L]$***-valued Shintani kernel function*** is

$$\Theta^*_{D,r,k}(\tau, z) := \sum_{h \in L'/L} \theta^*_{D,r,h,k}(\tau, z)\mathfrak{e}_h.$$

There exists a $C > 0$ such that $\Theta^*_{D,r,k}(\tau, z) = \mathcal{O}(e^{-Cv})$ as $v \to \infty$ uniformly in u. We have

$$\overline{\Theta^*_{D,r,k}(\tau, z)} = \Theta^*_{D,r,k}(-\overline{\tau}, -\overline{z}).$$

Remark 3.6.7. Let us restrict to the case $k \in 2\mathbb{Z}, k > 0, N = 1$ and $D = 1$. Then the Shintani kernel function agrees, up to constant with the theta function "$\Theta_2(\tau, z)$" defined in [BKZ14, Section 1]. It also agrees with the theta function "$\Theta(z, \tau)$" as denoted in [BKV13, (1.6)]. This kernel function actually goes back to [Shi75] (see also [Niw75] and [Cip83]).

3.6.2 Transformation Properties

Next, we examine the transformation properties of these kernel functions in both variables τ and z. We will integrate in τ so our singular theta lift will be a function in z. In the z variable we have weight $2 - 2k$.

Theorem 3.6.8. *For $\tau \in \mathbb{H}, \Theta_k(\tau, z) \in A_{k-3/2,\tilde{\rho}_L}$ and $\Theta_k^*(\tau, z) \in A_{k+1/2,\tilde{\rho}_L}$.*

Proof. Easily seen using Lemma 3.6.3 and Lemma 3.6.2. □

When considering the z variable we will need two short lemmas.

Lemma 3.6.9. *The polynomial $q_z(\lambda)$ transforms under the action of $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $q_{\gamma.z}(\gamma.\lambda) = j(\gamma, z)^{-2} q_z(\lambda)$.*

Proof. We set $f_z(\lambda) = -\sqrt{2} q_z(\lambda)/\sqrt{N} = \frac{1}{N}\left(cNz^2 - bz + a\right)$ and

$$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \lambda = \begin{pmatrix} b/2N & -a/N \\ c & -b/2N \end{pmatrix}.$$

We notice $\alpha^{-1}\gamma\alpha = \gamma^{-1}$ for all elements $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and we can write

$$f_z(\lambda) = \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} c & -b/2N \\ -b/2N & a/N \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z & 1 \end{pmatrix} \alpha\lambda \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

Then applying the action of γ to both z and λ

$$\begin{aligned} f_{\gamma.z}(\gamma.\lambda) &= \begin{pmatrix} \gamma.z & 1 \end{pmatrix} \alpha \left(\gamma\lambda\gamma^{-1}\right) \begin{pmatrix} \gamma.z \\ 1 \end{pmatrix} \\ &= j(\gamma, z)^{-1} \begin{pmatrix} z & 1 \end{pmatrix} \gamma^T \alpha\gamma\lambda\gamma^{-1} j(\gamma, z)^{-1} \gamma \begin{pmatrix} z \\ 1 \end{pmatrix} \\ &= j(\gamma, z)^{-2} \begin{pmatrix} z & 1 \end{pmatrix} \alpha(\alpha^{-1}\gamma^T\alpha)\gamma\lambda \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\gamma, z)^{-2} f_z(\lambda). \end{aligned}$$ □

Lemma 3.6.10. *The polynomial $g_z(\lambda) = q_z(\lambda)/\operatorname{Im}(z)^2$ transforms under the action of $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $g_{-\overline{\gamma.z}}(\gamma.\lambda) = j(\gamma, z)^2 g_{-\overline{z}}(\lambda)$.*

Proof. Recall that $\operatorname{Im}(\gamma.z) = \frac{\operatorname{Im}(z)}{j(\gamma,z)j(\gamma,\overline{z})}$ for all elements $\gamma \in \mathrm{SL}_2(\mathbb{R})$. Combining this with Lemma 3.6.9 we have

$$g_{-\overline{\gamma.z}}(\gamma.\lambda) = \frac{q_{(-\gamma).\overline{z}}(-\gamma.\lambda)}{\operatorname{Im}((-\gamma).\overline{z})^2} = \frac{q_{-\overline{z}}(\lambda)}{\operatorname{Im}(-\overline{z})^2} \frac{j(-\gamma, \overline{z})^2 j(-\gamma, z)^2}{j(-\gamma, \overline{z})^2} = j(\gamma, z)^2 g_{-\overline{z}}(\lambda).$$ □

Theorem 3.6.11. *The kernel functions $\Theta_k(\tau, z), \Theta_k^*(\tau, -\overline{z})$ transform in the variable $z \in \mathrm{Gr}(V(\mathbb{R}))$ under the action of $\Gamma_0(N)$ with weight $2 - 2k$ and $2k$ respectively. That is, for all $\gamma \in \Gamma_0(N)$ we have*

$$\begin{aligned} \Theta_k(\tau, z)|_{2-2k}\gamma &= j(\gamma, z)^{2k-2}\Theta_k(\tau, \gamma.z) = \Theta_k(\tau, z), \\ \Theta_k^*(\tau, -\overline{z})|_{2k}\gamma &= j(\gamma, z)^{-2k}\Theta_k^*(\tau, -\overline{\gamma.z}) = \Theta_k^*(\tau, -\overline{z}). \end{aligned}$$

Proof. We fix $z \in Gr(L), \gamma \in \Gamma_0(N)$ and a coset $h \in L^{'}/L$ and show the transformation property for the scalar-valued function $\theta_{h,k}(\tau, z)$. We can safely apply the action of γ on the $\lambda \in L + rh$ terms leaving $\theta_{h,k}(\tau, z)$ unchanged. This is because $\Gamma_0(N)$ acts trivially on L'/L (using Proposition 3.1.3) and we sum over all $\lambda \in L + rh$. Then, applying the action of γ on z as well, we have

$$\begin{aligned}\theta_{h,k}(\tau, \gamma.z) = v^{3/2} & \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D)}} \chi_D(\gamma.\lambda) \cdot p_{\gamma.z}(\gamma.\lambda) \cdot (q_{\gamma.z}(\gamma.\lambda))^{k-1} \\ & \times e\left(\frac{Q(\gamma.\lambda)}{|D|} u + \frac{Q_{(\gamma.z)}(\gamma.\lambda)}{|D|} iv\right).\end{aligned}$$

We then note that the quadratic form is invariant under the action of $\Gamma_0(N)$ and

$$\begin{aligned}(\gamma.\lambda)_{(\gamma.z)} &= \gamma.(\lambda_z), \\ (\gamma.\lambda)_{(\gamma.z)^\perp} &= \gamma.(\lambda_{z^\perp}), \\ (\gamma.\lambda, v(\gamma.z)) &= -(\gamma.\lambda, \sigma_{\mathbb{H}}^{V_{-1}}(\gamma.z)) = -(\gamma.\lambda, \gamma.\sigma_{\mathbb{H}}^{V_{-1}}(z)) = (\lambda, v(z)).\end{aligned}$$

Finally Proposition 3.2.4 tells us that $\chi_D(\gamma.\lambda) = \chi_D(\lambda)$. Combining these facts with Lemma 3.6.9, we obtain $j(\gamma, z)^{2-2k}\theta_{h,k}(\tau, z)$ as hoped. The proof for the Shintani kernel function follows similarly using Lemma 3.6.10. $\square$

Much later we will need to find the Fourier expansion and asymptotic behaviour at cusps other than l_∞. We will use the following.

Proposition 3.6.12. *The kernel function $\Theta_k(\tau, z)$ transforms in the variable $z \in \mathrm{Gr}(V(\mathbb{R}))$ under the action of Atkin-Lehner involutions such that*

$$\Theta_k(\tau, W_m^N.z) = j(W_m^N, z)^{2-2k} \sum_{h \in L'/L} \theta_{W_m^N.h,k}(\tau, z)\mathfrak{e}_h$$

for all W_m^N where m is an exact divisor of N.

Proof. This follows in the same way as Theorem 3.6.11. Noting that $W_m^N \subset \mathrm{SL}_2(\mathbb{R})$, and remembering the genus character was also invariant under Atkin-Lehner involutions (Proposition 3.2.4). However, there is a difference when applying the action of W_m^N to the $\lambda \in L + rh$ terms. We know W_m^N maps L to itself but we may not necessarily stay in the same coset, so to leave our function unchanged we move to the coset $(W_m^N)^{-1}.h = W_m^N.h$, (remembering $(W_m^N)^2 = 1 \pmod{\Gamma_0(N)}$ from Section 2.5.4). $\square$

Chapter 4

The Singular Theta Lift

After this groundwork, we can now finally construct the main item of our work: a regularised twisted singular theta lift. This is defined by integrating weak harmonic Maass forms against the kernel function, over the τ variable. This Petersson scalar product needs to be regularised. To do this, we (essentially) use a standard method of Harvey, Moore and Borcherds [HM96, Bor98]. Our lift is then an extension of the Borcherds lift [Bor98].

There are three sections in this chapter. In Section 4.1 we first discuss the definition of our lift and how it relates to other lifts. We then check this integral does indeed converge.

We then discuss the properties of this lift. In particular in Section 4.2 we show that we obtain some weight $2-2k$ forms for the group $\Gamma_0(N)$. These forms have some well described singularities along certain geodesics associated to the input f. The geodesics divide the upper-half plane into connected components. We provide wall crossing formulas, displaying how the function changes as we move between these components.

Finally in Section 4.3 we find that the lift is harmonic (and therefore real analytic and smooth) within these connected components.

Further discussion of similar regularised lifts can be found in many of the previous references such as [Bor98, BF04, BO10, BKV13, Bru02].

Throughout the *rest of this thesis* we will fix the following notation, unless stated otherwise. We fix L as in (3.1.3), V as in (3.1.1) and $N \in \mathbb{N}$. We also fix $D \in \mathbb{Z}$, a fundamental discriminant, $r \in \mathbb{Z}$ with $D \equiv r^2 \pmod{4N}$. We fix $z \in \mathrm{Gr}(V(\mathbb{R}))$ which is identified with $z = x + iy \in \mathbb{H}$ and fix $\tau = u + iv \in \mathbb{H}$ and $k \in \mathbb{Z}, k > 0$. We will also denote $\rho := \tilde{\rho}_L$.

4.1 Definition

Definition 4.1.1. *Let $k \in \mathbb{Z}, k \geq 1$ and $f \in H_{3/2-k,\overline{\rho}}$. Then we will call*

$$\Phi_{D,r,k}(z,f) := \left(f(\tau), v^{k-3/2}\overline{\Theta_{D,r,k}(\tau,z)}\right)^{\mathrm{reg}}_{3/2-k,\overline{\rho}} = \int^{\mathrm{reg}}_{\tau\in\mathcal{F}} \left\langle f(\tau), \overline{\Theta_{D,r,k}(\tau,z)}\right\rangle \frac{dudv}{v^2}$$

the ***singular theta lift****.*

We check this definition makes sense. We have

$$v^{k-3/2}\overline{\theta_{D,r,h,k}(\tau,z)} = v^k \sum_{\substack{\lambda\in L+rh \\ Q(\lambda)\equiv DQ(h)(D)}} \chi_D(\lambda)\overline{p_z(\lambda)q_z(\lambda)}^{k-1} e\left(\frac{-Q(\lambda)}{|D|}u + \frac{Q_z(\lambda)}{|D|}iv\right). \tag{4.1.1}$$

Thus, using 2.6.11 we see that (4.1.1) is a twisted Siegel theta function for the lattice L^- with quadratic form $-Q$. This space is of signature $(1,2)$. We have a harmonic homogeneous polynomial of degree $(1,k-1)$. So (4.1.1) transforms with respect to $\overline{\rho}$ with weight $3/2-k$, using Lemma 2.6.13.

Lemma 2.5.26 tells us that f can increase exponentially as $v \to \infty$. So the integral could diverge in general, hence the regularisation. Our first task is to check this regularised Petersson scalar product does indeed converge, which we do shortly in Theorem 4.1.3. As ever, we often drop D and r from the notation and often refer to $\Phi_{D,r,k}(z,f)$ as just "the lift".

Remark 4.1.2. This choice of regularisation essentially just prescribes that we integrate first over u, and then over v. The original regularisation of Harvey, Moore and Borcherds would have been defined as, the constant term of the Laurent expansion in $s \in \mathbb{C}$ of

$$\Phi_{D,r,k}(z,f) := \left(f(\tau), v^{k-3/2-s}\overline{\Theta_{D,r,k}(\tau,z)}\right)^{\mathrm{reg}}_{3/2-k,\overline{\rho}}$$

at $s=0$. This is a stronger regularisation, but it coincides with our version when the constant term vanishes. In particular, as we have a polynomial term attached to our kernel function, the integral vanishes when $\lambda \in L', \lambda = 0$. See also [Bru02, Proposition 2.11].

The singular theta lift has an input of harmonic weak Maass forms and is also twisted. It forms an extension in the case of signature $(2,1)$ of the original Borcherds lift. Borcherds lift was not twisted and he only considered weakly holomorphic modular forms. The Borcherds lift encompassed many other lifts such as the Shimura lift [Shi73] (which we discuss in more detail in Chapter 7), the Gritsenko lift [Gri88] and the Doi-Naganuma lift [DN70], see [Bor98, Section 14]. [BF04,Bru02] also constructed non-twisted lifts for some harmonic weak Maass forms.

In the case $k = 1$ then $\Phi_{D,r,k}(z, f)$ exactly coincides with the theta lift "$\Phi_{D,r}(z, f)$" defined in [Höv12, Definition 3.1]. This is because Remark 3.6.5 told us that $v^{k-3/2}\overline{\Theta_{D,r,k}(\tau, z)}$ agrees with the theta function given in [Höv12, Definition 2.5]. Remark 3.6.5 also told us that thet theta function "$v^{3/2-k}\overline{\theta^*(-\overline{z}, \tau)}$" from [BKV13, (1.6)] is essentially $\Theta_{D,r,k}(\tau, z)$ when $N = 1, D = 1$. So the theta lift "$\Phi^*_{1-k}(H)(z)$" constructed there is essentially a scalar-valued version of $\Phi_{D,r,k}(z, f)$ in the non-twisted $D = 1$, level $N = 1$ case. They then apply this lift to some Poincaré series.

We now show that the regularised integral does indeed converge. This includes the points lying on the singularities, which we discuss in Section 4.2. We follow the ideas in [Bru02, Proposition 2.8], [Bor98, Section 6] and [BF04, Proposition 5.6].

Theorem 4.1.3. *The regularised Petersson scalar product $\Phi_{D,r,k}(z, f)$ converges pointwise for any $z = x + iy \in \mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}))$.*

Proof. For any fixed $z \in \mathbb{H} \cong \mathrm{Gr}(V(\mathbb{R}))$, we consider whether the following rectangular integral converges:

$$\int_{v=1}^{\infty}\int_{u=-1/2}^{1/2}\left\langle f^+(\tau), \overline{\Theta_{D,r,k}(\tau, z)}\right\rangle \frac{dudv}{v^2}. \tag{4.1.2}$$

We see, by a standard easy argument, that this suffices. In particular the non-holomorphic part of f decays exponentially fast (Lemma 2.5.26) and $\Theta_{D,r,k}(\tau, z)$ also decays exponentially fast (using the discussion following Definition 3.6.4), as $v \to \infty$. We combine this with the fact that the integral

$$\int_{\tau\in\mathcal{F}_1}\left\langle f^+(\tau), \overline{\Theta_{D,r,k}(\tau, z)}\right\rangle \frac{dudv}{v^2}$$

converges absolutely over the compact region $\mathcal{F}_1$.

We then continue, by plugging in some explicit expansions into (4.1.2). We use the expansions given in (2.5.2) and Definition 3.6.4 to obtain

$$\sum_{h\in L'/L}\int_{v=1}^{\infty}\int_{u=-1/2}^{1/2}\sum_{\substack{n\in\mathbb{Z}-\mathrm{sgn}(D)Q(h)\\ n\gg-\infty}}\sum_{\substack{\lambda\in L+rh\\ Q(\lambda)\equiv DQ(h)(D)}} c^+(n,h)\chi_D(\lambda)$$
$$\times\, p_z(\lambda)q_z(\lambda)^{k-1}e\left(\left(n+\frac{Q(\lambda)}{|D|}\right)u\right)e\left(\left(n+\frac{Q_z(\lambda)}{|D|}\right)iv\right)v^{-1/2}dudv.$$

The next step is to carry out the integration over u (a compact region). When $m \in \mathbb{Z}, m \neq 0$, then $\int_{-1/2}^{1/2} e\,(mu)\,du = 0$ so the integral vanishes unless $n = -\frac{Q(\lambda)}{|D|}$ i.e. we pick out the 0-th Fourier coefficient. We also note $p_z(0) = 0$ so we can exclude the case $\lambda = 0$. We are left with

$$\int_{v=1}^{\infty}\sum_{h\in L'/L}\sum_{\substack{\lambda\in L+rh\\ Q(\lambda)\equiv DQ(h)(D)\\ \lambda\neq0}} c^+\left(\frac{-Q(\lambda)}{|D|}, h\right)\chi_D(\lambda)p_z(\lambda)q_z(\lambda)^{k-1}e\left(\frac{-2Q(\lambda_z)}{|D|}iv\right)v^{-1/2}dv. \tag{4.1.3}$$

Here we note that $-2Q(\lambda_z) = \lambda_3(z)^2 = p_z(\lambda)^2$ and so we need to remember throughout this proof that terms with $Q(\lambda_z) = 0$ vanish. We swap the integral over v and the summations as we notice that for $v \in [1, \infty)$ the series certainly converges absolutely.

Noting that $v^{-1/2} \leq 1$ for $1 \leq v \leq \infty$, it then suffices to check if the following converges:

$$\sum_{h \in L'/L} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D) \\ \lambda \neq 0}} \left| c^+ \left(\frac{-Q(\lambda)}{|D|}, h \right) \right| |p_z(\lambda)| \, |q_z(\lambda)|^{k-1} \int_{v=1}^{\infty} e\left(\frac{-2Q(\lambda_z)}{|D|} iv \right) dv$$

$$= \sum_{h \in L'/L} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D) \\ \lambda \neq 0}} \left| c^+ \left(\frac{-Q(\lambda)}{|D|}, h \right) \right| |p_z(\lambda)| \, |q_z(\lambda)|^{k-1} \frac{|D|}{4\pi Q(\lambda_z)} e\left(\frac{-2Q(\lambda_z) i}{|D|} \right)$$

and we know for all $\lambda \in L'$ that $|p_z(\lambda)| = \sqrt{-2Q(\lambda_z)}$ and $|q_z(\lambda)| = y\sqrt{2Q(\lambda_{z^\perp})}$ so we have

$$= \frac{|D|}{2\sqrt{2}\pi} \sum_{h \in L'/L} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D) \\ \lambda \neq 0}} \left| c^+ \left(\frac{-Q(\lambda)}{|D|}, h \right) \right| \frac{(y\sqrt{2Q(\lambda_{z^\perp})})^{k-1}}{\sqrt{-Q(\lambda_z)}} e\left(\frac{-2Q(\lambda_z) i}{|D|} \right). \quad (4.1.4)$$

We now split the sum into three parts when $Q(\lambda) = 0, Q(\lambda) < 0$ and $Q(\lambda) > 0$ and check each one converges.

Case $Q(\lambda) = 0$:

If $Q(\lambda) = 0$ then $Q(\lambda_{z^\perp}) = -Q(\lambda_z) = Q_z(\lambda)/2$. Then using 4.1.4 we obtain

$$\frac{|D|}{2\pi} \sum_{\substack{h \in L'/L \\ Q(h) \in \mathbb{Z}}} \left| c^+(0, h) \right| \sum_{\substack{\lambda \in L+rh \\ \lambda \neq 0 \\ Q(\lambda) = 0}} \frac{(y\sqrt{Q_z(\lambda)})^{k-1}}{\sqrt{Q_z(\lambda)}} e\left(\frac{Q_z(\lambda) i}{|D|} \right).$$

It is clear this converges as the sum over $\lambda \in L + rh$ is a subseries of a convergent theta series, for the positive definite quadratic form $Q_z(\lambda)$.

Case $Q(\lambda) < 0$:

Again we bound (4.1.4) with a function in terms of $Q_z(\lambda)$. We have from Lemma 2.5.27 that there exists a constant $C > 0$ such that $\left| c^+ \left(\frac{-Q(\lambda)}{|D|}, h \right) \right| \leq Ce^{C\sqrt{-Q(\lambda)}}$. We also observe in this case that $Q_z(\lambda) \geq -Q(\lambda), Q_z(\lambda) > Q(\lambda_{z^\perp})$ and $-2Q(\lambda_z) = -Q(\lambda_z) - Q(\lambda) + Q(\lambda_{z^\perp}) > Q_z(\lambda)$. Combining these facts we see that (4.1.4) is less than

$$\frac{C|D|}{2\pi} \sum_{h \in L'/L} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D) \\ Q(\lambda) < 0}} \frac{(y\sqrt{2Q_z(\lambda)})^{k-1}}{\sqrt{Q_z(\lambda)}} e^{C\sqrt{Q_z(\lambda)} - \frac{2\pi Q_z(\lambda)}{|D|}}.$$

This once again converges courtesy of being a subseries of a convergent theta series, for the positive definite quadratic form $Q_z(\lambda)$.

Case $Q(\lambda) > 0$:

We remember from (2.5.6) that there exists an $n_0 > 0$ such that $c^+(m,h) = 0$ for all $m < -n_0, h \in L'/L$, i.e. there are only finitely many $c^+(m,h) \neq 0$ with $m < 0$. So we see (4.1.4) vanishes except for finitely many m where $-n_0 \leq m < 0$. We have $-Q(\lambda) = |D|m$. We use (4.1.4). For each m it then suffices to check that

$$\frac{|D|}{2\sqrt{2}\pi} \sum_{\substack{\lambda \in L+rh \\ -Q(\lambda)=|D|m \\ Q(\lambda_z)\neq 0}} \frac{(y\sqrt{2Q(\lambda_{z^\perp})})^{k-1}}{\sqrt{-Q(\lambda_z)}} e\left(\frac{-2Q(\lambda_z)i}{|D|}\right) \tag{4.1.5}$$

converges (the sums over $m \in \mathbb{Z} - \operatorname{sgn}(D), -n_0 \leq m < 0$ and $h \in L'/L$ are finite). We know from [Bru02, p.50] that for any $C \geq 0$ and any compact $U \subset Gr(L)$ the set

$$\{\lambda \in L' \mid -Q(\lambda) = |D|m, \ \exists z' \in U \text{ with } -Q(\lambda_{z'}) \leq C\} \tag{4.1.6}$$

is finite. I.e. there are only finitely many small $-Q(\lambda_z) \geq 0$ terms. Let $\lambda \in L'$ be such that $-Q(\lambda) = |D|m$ and $Q(\lambda_z) \neq 0$. Then there exists an $\epsilon > 0$ such that $-Q(\lambda_z) > \epsilon$ for all λ. This means we have $Q_z(\lambda) \geq Q(\lambda_{z^\perp}), -Q(\lambda_z) > \epsilon$ and $-2Q(\lambda_z) = -Q(\lambda)+Q_z(\lambda) = |D|m+Q_z(\lambda)$. These facts tell us that (4.1.5) is less than

$$\frac{|D|}{2\sqrt{2\epsilon}\pi} e^{-2\pi m} \sum_{\substack{\lambda \in L+rh \\ -Q(\lambda)=|D|m \\ Q(\lambda_z)\neq 0}} (y\sqrt{2Q_z(\lambda)})^{k-1} e\left(\frac{Q_z(\lambda)i}{|D|}\right).$$

Which once again converges courtesy of being a subseries of a convergent theta series, for the positive definite quadratic form $Q_z(\lambda)$. □

4.2 The Singularities

We investigate the properties of the lift $\Phi_{D,r,k}(z,f)$ further. In particular, in this section we observe its weight in z and describe its singularities. We show the lift is a smooth function away from these singularities. The singularities lie on the twisted special cycles associated to f, as discussed in Definition 3.5.3. We have already seen in Theorem 4.1.3 that our lift $\Phi_{D,r,k}(z,f)$ converges pointwise so it will take meaningful values on these singularities. These types of singularities are seen in [Bor98, Section 6] and they divide $\mathrm{Gr}(V(\mathbb{R})) \cong \mathbb{H}$ into Weyl chambers with wall crossing formulas.

We first look at the concept of a jump (step) singularity on a geodesic D_λ.

Definition 4.2.1. *As in Definition 3.5.1 we fix $\lambda \in V, Q(\lambda) > 0$ and $D_\lambda \subset \mathbb{H}$ (the associated oriented geodesic). For each fixed point $z_0 \in D_\lambda$, we associate an open subset $U \subset \mathbb{H}$ which surrounds z_0. For any point $\omega \in D_\lambda$ we denote U_ω for an open subset $U_\omega \subset U$ (if it exists)* *which surrounds ω. Let $g : U\backslash D_\lambda \to \mathbb{C}$ be a smooth function. We also define a function $\tilde{g} : U_w\backslash D_\lambda \to \mathbb{C}$ as*

$$\tilde{g} := \begin{cases} g(z) & \text{if } (\lambda, v(z)) > 0, \\ g(z) + c & \text{if } (\lambda, v(z)) < 0. \end{cases}$$

where $c \in \mathbb{C}$ is a constant. Then we will say that g has a ***jump singularity*** *along D_λ, in U, of size $c \in \mathbb{C}$, if for any point $\omega \in D_\lambda$, there exists a U_ω such that $\tilde{g}$ has a continuation to a smooth function on U_ω.*

Let $\lambda \in V, Q(\lambda) > 0$. Then the function $\frac{(\lambda,v(z))}{|(\lambda,v(z))|}$ (which we define to equal 0 when $(\lambda, v(z)) = 0$) is a locally constant function on $\mathbb{H}\backslash D_\lambda$ that has a jump singularity of size 2 along D_λ (given our orientation in Section 3.5). In particular, $\frac{(\lambda,v(z))}{|(\lambda,v(z))|}$ is clearly equal to $+1$ (or -1) if $(\lambda, v(z)) > 0$ (or $(\lambda, v(z)) < 0$). In fact this function has a a jump singularity of size 2 irrespective of its value along the singularity (when $(\lambda, v(z)) = 0$).

We are now ready to state and prove the main theorem in this section. We will observe the role that the twisted cycles $Z'_{D,r}(f)$ play. In particular, the singularities lie on $Z_{D,r}(f)$. We describe the singularities explicitly. We also show the lift to be a smooth function on $\mathbb{H}\backslash Z_{D,r}(f)$. Proofs in other similar cases can be found in [Bor98, Theorem 6.2], [Bru02, Theorem 2.12] and [BF04, Proposition 5.6].

Theorem 4.2.2. *For $f \in H_{3/2-k,\overline{\rho}}$ with Fourier expansion as in 2.5.2 then*

1. *$\Phi_{D,r,k}(z, f)$ has weight $2 - 2k$ for $\Gamma_0(N)$ i.e. $(\Phi_{D,r,k}(z, f)|_{2-2k}\gamma) = \Phi_{D,r,k}(z, f)$ for all $\gamma \in \Gamma_0(N)$.*
2. *$\Phi_{D,r,k}(z, f)$ is a smooth function on $\mathbb{H}\backslash Z_{D,r}(f)$.*
3. *$\Phi_{D,r,k}(z, f)$ has singularities along $Z_{D,r}(f)$. More precisely for a point $z_0 \in \mathbb{H}$ exists an open neighbourhood $U \subset \mathbb{H}$ (with compact closure $\overline{U} \subset \mathbb{H}$) so that the function*

$$\Phi_{D,r,k}(z, f) - \sqrt{\frac{|D|}{2}} \sum_{h\in L'/L} \sum_{\substack{m\in\mathbb{Z}-\mathrm{sgn}(D)Q(h) \\ m<0}} c^+(m, h) \sum_{\substack{\lambda\in L_{-dD,rh} \\ \lambda\perp z_0}} \chi_D(\lambda)\frac{(\lambda, v(z))}{|(\lambda, v(z))|} q_z(\lambda)^{k-1}$$

(where for $z \in U, (\lambda, v(z)) = 0$ we let the term on the right hand side vanish) can be continued to a smooth function on U.

Remarks 4.2.3. We make a few observations before proceeding with the proof. We remember $Z_{D,r}(f)$ was the image of $Z'_{D,r}(f)$ in $\mathbb{H}$ and $Z'_{D,r}(f)$ consisted of only *finitely* many twisted cycles (on $Y_0(N)$). For any given point $z_0 \in \mathbb{H}$ we will see the sum $\lambda \in L_{-dD,rh}, \lambda \perp z_0$ is finite i.e. each z_0 lies on finitely may geodesics. The singularities have "polynomial jumps". This is because we had $q_z(\lambda) = \frac{-1}{\sqrt{2N}}\left(cNz^2 - bz + a\right)$ which is simply a (holomorphic) polynomial in z and $\frac{(\lambda,v(z))}{|(\lambda,v(z))|}$ is equal to either $+1$ or -1 (when $\lambda \not\perp z$).

Roughly, the final part of our theorem then says our function is not smooth only when $z_0 \in \mathbb{H}$ happens to lie on a geodesic (or finitely many geodesics) associated to f. The lift can be made smooth by adding a polynomial (or finitely many polynomials) for points on one side of the geodesic, subtracting the same polynomial on the other side and making no contribution if a point lies on the geodesic. The value of $\Phi_{D,r,k}(z,f)$ along some geodesic is the average of the values of $\Phi_{D,r,k}(z,f)$ in the adjacent connected components. This corresponds to condition 3 in our definition of a locally harmonic weak Maass form, Definition 2.5.30.

We will shortly see in Theorem 4.3.7 that $\Phi_{D,r,k}(z,f)$ is not just smooth, but actually harmonic, and therefore real analytic for $z \in \mathbb{H}\backslash Z_{D,r}(f)$.

We observe that the singularities depend only on P_f the principal part of f, as we only have $c^+(m,h)$ coefficients where $m < 0$. In fact we had in Section 3.5 that $Z_{D,r}(f)$ is the empty set if and only if the principal part of f is constant. So $\Phi_{D,r,k}(z,f)$ will have no singularities to consider and be smooth for all $z \in \mathbb{H}$ if $f \in M_{3/2-k,\overline{\rho}}$. However when the principal part is non-constant (which we remember certainly happens if $f^- \not\equiv 0$) then the jump singularities ensure that $\Phi_{D,r,k}$ is non-constant. So when $f^- \not\equiv 0$ we do not lift to the 0 function and our lift is not trivial.

Proof. The first statement is clear as we showed in Theorem 3.6.11 that the kernel function is of weight $2-2k$ in the z variable for the group $\Gamma_0(N)$ (we have integrated in the τ variable).

Following the notation from [Bru02, Theorem 2.12] we denote $f \approx g$ if $f-g$ can be continued to a smooth function on $\mathbb{H}$ in which case we say f has a singularity of type g. If $f-g$ can only be continued to a smooth function locally i.e. on $U \subset \mathbb{H}$ then we write $f \approx_U g$.

Following the early arguments in Theorem 4.1.3 we see that the integral of the non-holomorphic part f^- and the integral over the compact region $\mathcal{F}_1$ do not contribute to the singularities. These parts converge absolutely and therefore define a real analytic (and therefore smooth) function on $\mathbb{H}$. It then suffices, using (4.1.3), to consider

$$\Phi_{D,r,k}(z,f) \approx \int_{v=1}^{\infty} \sum_{h\in L'/L} \sum_{\substack{\lambda\in L+rh \\ Q(\lambda)\equiv DQ(h)(D) \\ \lambda\neq 0}} c^+\left(\frac{-Q(\lambda)}{|D|},h\right)\chi_D(\lambda)p_z(\lambda)q_z(\lambda)^{k-1}e\left(\frac{-2Q(\lambda_z)}{|D|}iv\right)v^{-1/2}dv.$$

A function is smooth (a C^∞-function) if all orders of its derivatives exist and are continuous. So we would like to show that, for all orders of partial derivatives of $z = x + iy \in \mathbb{H}$, then the integral over $\mathcal{F}_t$ converges locally uniformly and absolutely as $t \to \infty$.

We remember that $Q(\lambda_z) \leq 0$ for all $\lambda \in L$. In Theorem 4.1.3 we have seen that for a fixed $z = x + iy \in \mathbb{H}$ then this integral is absolutely convergent (it was bounded by some positive definite theta series in $Q_z(\lambda)$). We consider the cases of $Q(\lambda) < 0$, $Q(\lambda) = 0$.

We can then easily adapt the arguments in Theorem 4.1.3 to show local uniform and absolute convergence of this integral. We can do this, if for any point $z_0 \in \mathbb{H}$ there exists an open subset $U \subset \mathbb{H}$ (with compact closure $\overline{U} \subset \mathbb{H}$) and a constant $\epsilon > 0$ such that $Q(\lambda_z) < \epsilon$ for all $\lambda \in L', \lambda \neq 0, Q(\lambda) \leq 0$ and $z \in U$. I.e. $Q(\lambda_z) \neq 0$. This is easily seen to be true, as in [Höv12, Equation 3.17]. The partial derivatives of this term also clearly converge locally uniformly and absolutely. This is because after differentiating we will still have a similar series, where the identical exponential term guarantees convergence.

We fix a $z_0 \in \mathbb{H}$. We have just seen that only the terms where $Q(\lambda) > 0$ contribute to the singularities. It then suffices, using the discussion preceding (4.1.5), to consider

$$\Phi_{D,r,k}(z,f) \approx \sum_{h \in L'/L} \sum_{\substack{m \in \mathbb{Z} - \operatorname{sgn}(D)Q(h) \\ m<0}} c^+(m,h) \\ \times \int_{v=1}^{\infty} \sum_{\lambda \in L_{-dD,rh}} \chi_D(\lambda) p_z(\lambda) q_z(\lambda)^{k-1} e\left(\frac{-2Q(\lambda_z)}{|D|} iv\right) v^{-1/2} dv.$$

Where we observe that using (3.5.2) the set $L_{-dD,rh}$ is exactly the vectors $\lambda \in L + rh$ where $-Q(\lambda) = |D|m$. We now split the sum over $\lambda \in L_{-dD,rh}$ into two sums. One over $\lambda \perp z_0$ and one over $\lambda \not\perp z_0$. We first consider when $\lambda \not\perp z_0$.

As before, we adapt the proof from Theorem 4.1.3. Once again this will suffice if for $z_0 \in \mathbb{H}$ there exists an open subset $U \subset \mathbb{H}$ (with compact closure $\overline{U} \subset \mathbb{H}$) and a constant $\epsilon > 0$ such that $Q(\lambda_z) < \epsilon$ for all $\lambda \in L_{-dD,rh}, \lambda \not\perp z_0$ and $z \in U$. This is true using (4.1.6) and noting that $\lambda \not\perp z_0$ means we can choose a neighbourhood U of z_0 small enough such that $Q(\lambda_z) \neq 0$.

Finally we look at the sum over $\lambda \in L_{-dD,rh}$ where $\lambda \perp z_0$. We first notice that $\lambda \perp z_0$ means that $Q(\lambda_{z_0}) = 0$. We can then use (4.1.6) to see we actually have a finite sum over $\lambda \in L_{-dD,rh}, \lambda \perp z_0$. I.e. any given $z_0 \in \mathbb{H}$ lies on only finitely many geodesics associated to f and only contributes finitely many terms to the singularities. We now look at the remaining integral. We have

$$\int_{v=1}^{\infty} p_z(\lambda)q_z(\lambda)^{k-1}e\left(\frac{-2Q(\lambda_z)}{|D|}iv\right)v^{-1/2} = \sqrt{\frac{|D|}{2\pi}}\frac{p_z(\lambda)q_z(\lambda)^{k-1}}{|2Q(\lambda_z)|}\Gamma\left(\frac{1}{2},\frac{-4\pi Q(\lambda_z)}{|D|}\right).$$

When $\lambda_z = 0$ this has a singularity of type

$$\sqrt{\frac{|D|}{2}}\frac{(\lambda, v(z))}{|(\lambda, v(z))|}f_z(\lambda)^{k-1}$$

as we know that $\Gamma(1/2, -4\pi Q(\lambda_z)/|D|) = \Gamma(1/2) + \mathcal{O}(|Q(\lambda_z)|)$ as $\lambda_z \to 0$. The integral vanishes when $-2Q(\lambda_z) = p_z(\lambda) = (\lambda, v(z)) = 0$ so this is the zero contribution to the singularities when $(\lambda, v(z)) = 0$. So finally we have the required result

$$\Phi_{D,r,k}(z,f) \approx_U \sqrt{\frac{|D|}{2}} \sum_{h\in L'/L} \sum_{\substack{m\in\mathbb{Z}-\operatorname{sgn}(D)Q(h)\\ m<0}} c^+(m,h) \sum_{\substack{\lambda\in L_{-dD,rh}\\ \lambda\perp z_0}} \chi_D(\lambda)\frac{(\lambda, v(z))}{|(\lambda, v(z))|}f_z(\lambda)^{k-1}.$$

We remember $(\lambda, v(z))/|(\lambda, v(z))|$ had a jump singularity of size 2 along D_λ for a given $\lambda \in L', Q(\lambda) > 0$. □

4.2.1 The Wall Crossing Formula

We have seen in Theorem 4.2.2 that $\Phi_{D,r,k}(z,f)$ is smooth on $\mathbb{H}$, away from geodesics $Z'_{D,r}(f)$ associated to f. These geodesics divide $D \cong \mathbb{H}$ into connected components. Theorem 4.3.7 will in fact tell us that $\Phi_{D,r,k}(z,f)$ is harmonic and therefore real analytic on these connected components. We remember from Section 2.5.3 we called these real analytic connected components Weyl chambers. We would then like to find the "wall crossing formula" in our case. This will tell us how the function changes as we move between Weyl chambers.

We follow [Bor98, Section 6] and [Bru02, Section 3.1]. Let $W \subset \mathbb{H}$ be a Weyl chamber and let $\lambda \in L'$. Then we say $(\lambda, W) < 0$ if $(\lambda, w) < 0$ for all $w \in W \subset \mathbb{H}$. We will denote $\Phi_{W_1}(z)$ and $\Phi_{W_2}(z)$ for the restrictions of $\Phi_{D,r,k}(z,f)$ to two adjacent Weyl chambers W_1 and W_2. The restrictions $\Phi_{W_1}(z)$ and $\Phi_{W_2}(z)$ can both be extended to real analytic functions on $\overline{W_1} \cup \overline{W_2}$ and we denote $W_{12} := \overline{W_1} \cap \overline{W_2}$ for the "wall" dividing W_1, W_2. This next proof follows [Bor98, Corollary 6.3].

Theorem 4.2.4 (The wall crossing formula)**.** *The difference $\Phi_{W_1}(z) - \Phi_{W_2}(z)$ is given by*

$$2\sqrt{2|D|} \sum_{h\in L'/L} \sum_{\substack{m\in\mathbb{Z}+\operatorname{sgn}(D)Q(h)\\ m<0}} c^+(m,h) \sum_{\substack{\lambda\in L_{-dD,rh}\\ \lambda\perp W_{12}\\ (\lambda,W_1)<0}} \chi_D(\lambda)q_z(\lambda)^{k-1}.$$

Proof. Using Theorem 4.2.2 we know that $\Phi_{D,r,k}(z,f)$ has a singularity of type

$$\sqrt{\frac{|D|}{2}} \sum_{h\in L'/L} \sum_{\substack{m\in\mathbb{Z}-\operatorname{sgn}(D)Q(h)\\ m<0}} c^+(m,h) \sum_{\substack{\lambda\in L_{-dD,rh}\\ \lambda\perp W_{12}}} \chi_D(\lambda)\frac{(\lambda, v(z))}{|(\lambda, v(z))|}q_z(\lambda)^{k-1} \tag{4.2.1}$$

along W_{12}. We consider the sum over λ and $-\lambda$. These two sums are the same. We see this by observing that $\chi_D(-\lambda) = (-1)^{(1-\mathrm{sgn}(D))/2}\chi_D(\lambda)$ (using Proposition 3.2.4), $p_z(-\lambda)q_z(-\lambda)^{k-1} = (-1)^k p_z(\lambda)q_z(\lambda)^{k-1}$ and $c^+(m,h) = (-1)^{3/2-k+(\mathrm{sgn}(D))/2}c^+(m,-h)$ (using 2.5.10). We can then rewrite (4.2.1) as a sum over elements with $(\lambda, W_1) < 0$. We pick up a factor of 2 and also another factor of 2 from the jump of size 2 arising from $(\lambda, v(z))/|(\lambda, v(z))|$. □

4.3 Locally Harmonic

The main aim of this section is to show that the singular theta lift is harmonic away from the singularities $Z_{D,r}(f)$ i.e. locally harmonic. To do this we use a few lemmas which essentially just involve some yoga with the hyperbolic Laplacian operator. We follow the ideas in [Bru02, Section 4.1].

We will first show a simple and useful lemma linking the Laplacian operator of weight κ and conjugation of a smooth function.

Lemma 4.3.1. *For $f : \mathbb{H} \to \mathbb{C}[L'/L]$ a smooth function and $\kappa \in \frac{1}{2}\mathbb{Z}$ then*

$$\Delta_{-\kappa}(v^\kappa\overline{f(\tau)}) = v^\kappa\overline{\Delta_\kappa f(\tau)} + \kappa v^\kappa\overline{f(\tau)}.$$

Proof. We first note that

$$\frac{\partial}{\partial\overline{\tau}}v^\kappa = \frac{i\kappa v^{\kappa-1}}{2} \quad \text{and} \quad \frac{\partial}{\partial\tau}v^\kappa = -\frac{i\kappa v^{\kappa-1}}{2}.$$

Then simply using the product rule we can see that

$$\begin{aligned}
&\Delta_{-\kappa}(v^\kappa\overline{f(\tau)})\\
&= \left(-4v^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}} - 2i\kappa v\frac{\partial}{\partial\overline{\tau}}\right)\left(v^\kappa\overline{f(\tau)}\right)\\
&= -4v^2\left[v^\kappa\left(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right) + \overline{f(\tau)}\left(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}v^\kappa\right) + \left(\frac{\partial}{\partial\tau}\overline{f(\tau)}\right)\left(\frac{\partial}{\partial\overline{\tau}}v^\kappa\right) + \left(\frac{\partial}{\partial\tau}v^\kappa\right)\left(\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right)\right]\\
&\quad - 2i\kappa v\left[v^\kappa\left(\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right) + \overline{f(\tau)}\left(\frac{\partial}{\partial\overline{\tau}}v^\kappa\right)\right]\\
&= -4v^2\left[v^\kappa\left(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right) + \overline{f(\tau)}\frac{\kappa(\kappa-1)}{4}v^{\kappa-2} + \left(\frac{\partial}{\partial\tau}\overline{f(\tau)}\right)\frac{i\kappa v^{\kappa-1}}{2} - \frac{i\kappa v^{\kappa-1}}{2}\left(\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right)\right]\\
&\quad - 2i\kappa v\left[v^\kappa\left(\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right) + \overline{f(\tau)}\frac{i\kappa v^{\kappa-1}}{2}\right]\\
&= v^\kappa\left[-4v^2\left(\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}\overline{f(\tau)}\right) + 2iv\kappa\left(\frac{\partial}{\partial\tau}\overline{f(\tau)}\right)\right] + \kappa v^\kappa\overline{f(\tau)}\\
&= v^\kappa\overline{\Delta_\kappa f(\tau)} + \kappa v^\kappa\overline{f(\tau)}.
\end{aligned}$$

□

In particular, if $f \in H_{\kappa,\overline{\rho}}$, i.e. f is harmonic, then $\Delta_{-\kappa}(v^\kappa\overline{f(\tau)}) = \kappa v^\kappa\overline{f(\tau)}$.

The next key proposition links the action of the hyperbolic Laplacian operator on both variables of the kernel function. We first makes some observations concerning our kernel functions from Section 3.6 which will allows us to show this proposition.

The Second Spherical Property

In the case of signature $(2,1)$ we were able to identify $\mathrm{Gr}(V(\mathbb{R}))$ with $\mathbb{H}$. This means we can consider an alternative way of constructing the theta functions from Section 3.6. We follow the construction in [Cip83]. We do this by working at the base point of z. We make this more precise. We will say $f \in \mathcal{S}(V(\mathbb{R}))$ satisfies the **second spherical property** for $2m, m \in \mathbb{Z}$ if

$$M^{\mathrm{Sch}}\left[g_\theta\right] f(\lambda) = f(g_\theta^{-1}.\lambda) = \sigma_{1/2}(g_\theta)^{4m} f(\lambda)$$

for any $g_\theta \in \mathrm{SO}(2)$. Here we are thinking of g_θ as an element of the orthogonal group in our dual pair $(\mathrm{O}(V(\mathbb{R})), \mathrm{Mp}_2(\mathbb{R}))$, using the accidental isomorphism (3.1.4). So we recall that g_θ acts via conjugation on $\lambda \in V(\mathbb{R})$.

Then if $f \in \mathcal{S}(V(\mathbb{R}))$ satisfies the first spherical property for $r'/2$ and the second spherical property for $2m$ we can construct the following:

$$\Theta_{L,D,r}(\tau, z, f) := \sum_{h \in L'/L} \sum_{\substack{h' \in L'/DL \\ h' \equiv rh(L) \\ Q(h') \equiv DQ(h)(D)}} \sum_{\lambda \in DL + h'} \times j(g_\tau, i)^{r'/2} j(g_z, i)^{2m} \chi_D(\lambda) M^{\mathrm{Sch}}\left[g_\tau\right] M^{\mathrm{Sch}}\left[g_z\right] f(\lambda) \mathfrak{e}_h.$$

Which is a another form of (3.6.2). Here we are thinking of g_z as an element of the orthogonal group in our dual pair. We can then show that $\Theta_{L,D,r}(\tau, z, f)$ transforms as a scalar-valued form with weight $2m$ in z. We do not detail this here. The proof is essentially a repeat of the ideas in Theorem 3.6.11 and Lemma 3.6.2. Recall from earlier that for $\lambda \in V(\mathbb{R})$ we had the decomposition $\lambda = \sum \lambda_i(z) b_i(z)$, see also (3.3.2).

Lemma 4.3.2. *If we let $f \in \mathcal{S}(V(\mathbb{R}))$ be*

$$f(\lambda) = (\lambda_1(i) + i\lambda_2(i))^{k-1} \lambda_3(i) e\left(\frac{Q_i(\lambda)}{|D|} i\right).$$

Then f satisfies the first spherical property for $k - 3/2$, and the second spherical property for $2 - 2k$.

Proof. We know $f(\lambda)$ satisfies the first spherical property for $k - 3/2$ using Lemma 2.6.9. For the second part we set $g_\theta \in \mathrm{SO}(2)$. We then recall that

$$(\gamma.\lambda)_{(\gamma.z)} = \gamma.(\lambda_z) \quad \text{and} \quad (\gamma.\lambda)_{(\gamma.z)^\perp} = \gamma.(\lambda_{z^\perp}).$$

for $\gamma \in \mathrm{SL}_2(\mathbb{R})$. So using this in combination with Lemma 3.6.9 when $z = i$ we see that

$$M^{\mathrm{Sch}}\left[g_\theta\right] f(\lambda) = j(g_\theta, i)^{2-2k} f(\lambda) = \sigma_{1/2}(g_\theta)^{4-4k} f(\lambda). \qquad \square$$

Our final observation in this part is that

$$M^{\mathrm{Sch}}\left[g_z\right] \left[(\lambda_1(i) + i\lambda_2(i))^{k-1} \lambda_3(i) e\left(\frac{Q_i(\lambda)}{|D|} i \right) \right] = (\lambda_1(z) + i\lambda_2(z))^{k-1} \lambda_3(z) e\left(\frac{Q_z(\lambda)}{|D|} i \right).$$

It is then clear from Section 3.6 that $\Theta_{D,r,k}(\tau, z)$ is of the form $\Theta_{L,D,r}(\tau, z, f)$, where f satisfies the first spherical property for $k - 3/2$ and the second spherical property for $2 - 2k$. We are now in a position to show the aforementioned key proposition.

Proposition 4.3.3. *We have that*

$$4\Delta_{k-3/2,\tau} \Theta_k(\tau, z) = \Delta_{2-2k,z} \Theta_k(\tau, z) + (6 - 4k)\Theta_k(\tau, z).$$

Proof. We use [Cip83, Proposition 2.13]. In our setup this result holds for theta functions defined on lattices of signature $(1, 2)$, so we first consider $v^{k-3/2}\overline{\Theta_k(\tau, -\overline{z})}$. The discussion in the previous paragraph tells us that $v^{k-3/2}\overline{\Theta_k(\tau, -\overline{z})}$ matches the form given in [Cip83, Proposition 2.13] and arose from a Schwartz function with first spherical property for $3/2 - k$ and second spherical property for $2 - 2k$. So we have

$$4\Delta_{3/2-k,\tau}(v^{k-3/2}\overline{\Theta_k(\tau, -\overline{z})}) = \Delta_{2-2k,z}(v^{k-3/2}\overline{\Theta_k(\tau, -\overline{z})}). \tag{4.3.1}$$

We then use Lemma 4.3.1 to obtain

$$4v^{k-3/2}\overline{\Delta_{k-3/2,\tau}(\Theta_k(\tau, -\overline{z}))} + (4k - 6)v^{k-3/2}\overline{\Theta_k(\tau, -\overline{z})} = v^{k-3/2}\Delta_{2-2k,z}(\overline{\Theta_k(\tau, -\overline{z})}).$$

Using (3.6.7) we then have

$$4\Delta_{k-3/2,-\overline{\tau}}(\Theta_k(-\overline{\tau}, z)) = \Delta_{2-2k,z}(\Theta_k(-\overline{\tau}, z)) + (6 - 4k)\Theta_k(-\overline{\tau}, z)).$$

We then see the stated result by letting $\tau \mapsto -\overline{\tau}$. The genus character clearly just goes for the ride throughout these differential calculations. $\square$

Remarks 4.3.4. This proof was essentially just a proof on the level of the Schwartz functions. The key ideas in Cipra's result go back to [Shi75] and it is essentially a derivation from [Shi75, Proposition 1.7]. This result relies on linking the Schwartz functions and the Weil representation acting on them with the Casimir elements of the universal enveloping algebras of the Lie algebras of the dual pair. These Casimir elements then correspond to the hyperbolic Laplacian operator in each variable τ and z.

In the case of general signature we cannot necessarily identify $\mathrm{Gr}(V(\mathbb{R}))$ with $\mathbb{H}$. Therefore taking the hyperbolic Laplacian in z would not make sense. However when the Grassmannian is Hermitian we are able to give complex structure to $\mathrm{Gr}(V(\mathbb{R}))$. For example in the case when $\mathrm{Gr}(V(\mathbb{R})) \cong \mathbb{H}_l$ (the generalised upper half plane, see [Bru02, Section 3.2]) we can adapt [Bru02, Proposition 4.5] and [Shi75, Proposition 1.7] to show a similar link for Siegel theta functions to the invariant Laplacian operator on $\mathbb{H}_l$.

We see that (4.3.1) agrees with [Höv12, Proposition 3.10] in the case $k = 1$.

There is an alternative method of proof of Proposition 4.3.3. We could simply use the explicit form of $\Theta_k(\tau, z)$ given in (3.6.6) and then carry out some easy but long and tedious partial differential calculations in x, y, u and v.

To show our function is locally harmonic we will use the adjointness of the hyperbolic Laplacian operator in the Petersson scalar product. This is an easy and well known consequence of Stokes' theorem.

Lemma 4.3.5 ([Bru02, Lemma 4.3])**.** *Let $\kappa \in \frac{1}{2}\mathbb{Z}$. Let $f, g \in A_{k,\overline{\rho}}$ be smooth functions. Then*

$$\begin{aligned}
&\int_{\mathcal{F}_t} \langle f, \Delta_\kappa g\rangle\, v^{\kappa-2} du dv - \int_{\mathcal{F}_t} \langle \Delta_\kappa f, g\rangle\, v^{\kappa-2} du dv \\
&= \int_{-1/2}^{1/2} \left[\langle L_\kappa f, g\rangle\, v^{\kappa-2}\right]_{v=t} du - \int_{-1/2}^{1/2} \left[\langle f, L_\kappa g\rangle\, v^{\kappa-2}\right]_{v=t} du.
\end{aligned}$$

We next show that the boundary terms in our case will vanish.

Lemma 4.3.6. *For all $f \in H_{3/2-k,\overline{\rho}}$ and $z \in \mathbb{H}$ we have*

$$\begin{aligned}
&\lim_{t\to\infty} \int_{-1/2}^{1/2} \left[\left\langle L_{k-3/2,\tau}(\Theta_{D,r,k}(\tau, z)), \overline{f(\tau)} \right\rangle v^{-2}\right]_{v=t} du = 0, \\
&\lim_{t\to\infty} \int_{-1/2}^{1/2} \left[\left\langle \Theta_{D,r,k}(\tau, z), L_{k-3/2,\tau}(v^{3/2-k}\overline{f(\tau)}), \right\rangle v^{k-7/2}\right]_{v=t} du = 0.
\end{aligned}$$

Proof.

The First Integral

We remember (Definition 2.5.14) that $L_{k-3/2} = -2iv^2 \frac{\partial}{\partial \overline{\tau}}$. We then use our explicit expansion of $\Theta_k(\tau, z)$ from Definition 3.6.4 to see that

$$\begin{aligned}
&L_{k-3/2,\tau}(\Theta_{D,r,h,k}(\tau, z)) \\
&= \left(\frac{4\pi Q(\lambda_z) v^2}{|D|} + \frac{3v}{2}\right) v^{3/2} \sum_{\substack{\lambda \in L + rh \\ Q(\lambda) \equiv DQ(h)(D)}} \chi_D(\lambda) p_z(\lambda) q_z(\lambda)^{k-1} e\left(\frac{Q(\lambda)}{|D|} u + \frac{Q_z(\lambda)}{|D|} iv\right).
\end{aligned}$$

As usual we remember f^-, the non-holomorphic part of f, decays exponentially fast as $t \to \infty$ and $\Theta_{D,r,k}(\tau, z)$ also decays exponentially so this part of the integral (over the compact region

$-1/2 \leq u \leq 1/2$) vanishes. We consider f^+ and plug in the explicit expansion for $f \in H_{3/2-k,\overline{\rho}}$ given in (2.5.2). We then use (4.1.3) (the integral over u picks out the 0-th Fourier coefficient) to obtain

$$\int_{-1/2}^{1/2} \left\langle L_{k-3/2,\tau}(\Theta_{D,r,k}(\tau,z)), \overline{f^+(\tau)} \right\rangle v^{-2} du = \left(\frac{4\pi Q(\lambda_z) v}{|D|} + \frac{3}{2} \right) v^{1/2}$$
$$\times \sum_{h \in L'/L} \sum_{\substack{\lambda \in L+rh \\ Q(\lambda) \equiv DQ(h)(D) \\ \lambda \neq 0}} c^+ \left(\frac{-Q(\lambda)}{|D|}, h \right) \chi_D(\lambda) p_z(\lambda) q_z(\lambda)^{k-1} e\left(-\frac{2Q(\lambda_z)}{|D|} iv \right).$$

If $Q(\lambda_z) = 0$ then $p_z(\lambda) = 0$ and those terms simply vanish. For the rest of the terms $Q_z(\lambda_z) < 0$. We can use the same cases as in Theorem 4.1.3 to see this is dominated by a theta series in the positive definite form $Q_z(\lambda)$. So then for all $z \in \mathbb{H}$ this sum is certainly uniformly and absolutely convergent for $v \in [1,\infty)$. This means if we take the limit as $v \to \infty$ we can swap this with the summation and observe all the summands vanish in this limit. Therefore the integral as $t \to \infty$ vanishes.

The Second Integral

We first use (2.5.2) to find

$$L_{k-3/2,\tau}(v^{3/2-k}\overline{f(\tau)}) = \left(v^{5/2-k}(3/2-k) - v^2 4\pi n \right) \overline{f(\tau)}$$
$$- v^{k+1/2} \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z} - \operatorname{sgn}(D)Q(h) \\ n<0}} \overline{c^-(n,h)} (4\pi|n|)^{k-1/2} e(-n\tau) \mathfrak{e}_h.$$

We consider the first term. As f^- decays exponentially fast and we have

$$\int_{-1/2}^{1/2} \left\langle \Theta_{D,r,k}(\tau,z)), (v^{5/2-k}(3/2-k) - v^2 4\pi n) \overline{f^+(\tau)} \right\rangle v^{k-7/2} du$$

which is essentially the same as the 3/2 part of (4.3.2) (up to some powers of v) and so will also vanish as $t \to \infty$. For the second term this is in the form of a Fourier expansion of a cusp form with no constant term (noting that $c^-(n,h) = \mathcal{O}(|n|^{k/2})$ as $n \to -\infty$, i.e. the coefficients still only grow polynomially, Lemma 2.5.27) which we also know decays exponentially fast as $v \to \infty$. So again this term vanishes as $t \to \infty$. □

We are now able to state the main theorem of this section. We show that away from the singularities, our lift is harmonic and therefore real analytic. This does not hold on the singularities. On the singularities our function is discontinuous and so is not naturally differentiable and we also do not have local uniform convergence.

Theorem 4.3.7. *For $f \in H_{3/2-k,\overline{\rho}}$ and $z \in \mathbb{H} \backslash Z_{D,r}(f)$ then*

$$\Delta_{2-2k,z} \Phi_{D,r,k}(z,f) = 0$$

and $\Phi_{D,r,k}(z,f)$ is also real analytic on $\mathbb{H} \backslash Z_{D,r}(f)$.

Proof. We know from Theorem 4.2.2 that, the regularised integral $\Phi_{D,r,k}(z,f)$, converged as $t\to\infty$, locally uniformly for $z\in\mathbb{H}\backslash Z_{D,r}(f)$. So for these points we can swap the partial derivatives with the integral and take the Laplacian operator inside to obtain

$$\Delta_{2-2k,z}\Phi_{D,r,k}(z,f)=\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle f(\tau),\overline{\Delta_{2-2k,z}\Theta_{D,r,k}(\tau,z)}\right\rangle\frac{dudv}{v^2}.$$

Then using Proposition 4.3.3 we have

$$\begin{aligned}\Delta_{2-2k,z}\Phi_{D,r,k}(z,f)&=4\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle f(\tau),\overline{\Delta_{k-3/2,\tau}\Theta_{D,r,k}(\tau,z)}+(k-3/2)\overline{\Theta_{D,r,k}(\tau,z)}\right\rangle\frac{dudv}{v^2}\\&=4\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\Delta_{k-3/2,\tau}\Theta_{D,r,k}(\tau,z),v^{3/2-k}\overline{f(\tau)}\right\rangle v^{k-3/2}\frac{dudv}{v^2}\\&\quad+4(k-3/2)\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\Theta_{D,r,k}(\tau,z),\overline{f(\tau)}\right\rangle\frac{dudv}{v^2}\end{aligned}$$

and using Lemma 4.3.5 we then obtain

$$\begin{aligned}&=4\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\Theta_{D,r,k}(\tau,z),\Delta_{k-3/2,\tau}(v^{3/2-k}\overline{f(\tau)})\right\rangle v^{k-3/2}\frac{dudv}{v^2}\\&\quad+4(k-3/2)\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\Theta_{D,r,k}(\tau,z),\overline{f(\tau)}\right\rangle\frac{dudv}{v^2}\\&\quad-4\lim_{t\to\infty}\int_{-1/2}^{1/2}\left[\left\langle L_{k-3/2,\tau}(\Theta_{D,r,k}(\tau,z)),\overline{f(\tau)}\right\rangle v^{-2}\right]_{v=t}du\\&\quad+4\lim_{t\to\infty}\int_{-1/2}^{1/2}\left[\left\langle\Theta_{D,r,k}(\tau,z),L_{k-3/2,\tau}(v^{3/2-k}\overline{f(\tau)}),\right\rangle v^{k-7/2}\right]_{v=t}du.\end{aligned}$$

We know the last two terms disappear using Lemma 4.3.6. Using Lemma 4.3.1 we are then left with

$$=4\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\Theta_{D,r,k}(\tau,z),\overline{\Delta_{3/2-k,\tau}(f(\tau))}\right\rangle\frac{dudv}{v^2}.$$

However $f\in H_{3/2-k,\overline{\rho}}$, which means that $\Delta_{3/2-k,\tau}(f(\tau))$ vanishes as well and we have the stated result. $\square$

Chapter 5

Partial Poisson Summation

In the next chapter we will obtain a Fourier expansion of our lift in the z variable, at a cusp. This requires the evaluation of the integral in Definition 4.1.1 that forms our lift. This will be done using the Rankin-Selberg method, which is detailed explicitly in Section 6.3.10. This unfolding trick will first rely on us rewriting $\Theta_k(\tau, z)$ in terms of Poincaré series. To do this we first need to discuss a sublattice K. Roughly our aim is to rewrite the kernel function such that $\Theta_k(\tau, z) = \sum_{\tilde{\gamma} \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}} f(\tau)|_{k-3/2, \tilde{\rho}_K} \tilde{\gamma}$ in terms of theta functions on this sublattice. We do this by applying a partial Poisson summation to the kernel function.

5.1 A Sublattice

We first discuss a sublattice $K_l \subset L$ and its properties. These ideas here have been seen before. We take our results from [BO10, Section 4.1], [Bru02, Section 2.1], [Bor98, Section 5] and [BFI15, Section 2].

Remember from Section 3.4 the cusps of the modular curve $Y_0(N) = \Gamma_0(N) \backslash \mathbb{H}$ correspond to $\Gamma_0(N)$ isotropic lines in L. We took primitive isotropic vectors $l \in L$, with a fixed orientation, to represent each cusp.

Definition 5.1.1. *We define a space $W_l := l^\perp / l$, equipped with the same quadratic form. We also define a sublattice,*

$$K_l := (L \cap l^\perp)/(L \cap l).$$

W_l is a one dimensional positive definite vector space of signature $(1, 0)$ and K_l is an even lattice in W_l. The dual lattice is given by

$$K_l' = (L' \cap l^\perp)/(L' \cap l).$$

Alternatively, we also know there exists a $l' \in L'$ such that $(l, l') = 1$. Using this we can write the sublattice as

$$K_l = L \cap l'^{\perp} \cap l^{\perp}.$$

For $\lambda \in V(\mathbb{R})$, we denote λ_K for the orthogonal projection onto $K \otimes \mathbb{R}$. This is given by

$$\lambda_K = \lambda - (\lambda, l')l - (\lambda, l)l' + (\lambda, l)(l', l')l. \tag{5.1.1}$$

If $\lambda \in L'$, then $\lambda_K \in K_l'$. There exists a unique $\mathcal{N}$ such that $(l, L) = \mathcal{N}\mathbb{Z}$. We then have $|L'/L| = \mathcal{N}^2|K_l'/K_l|$. Let $\lambda' \in L$, such that $(\lambda', l) = \mathcal{N}$. Then

$$L = K_l \oplus \mathbb{Z}\lambda' \oplus \mathbb{Z}l.$$

We will now assume that $(l, L) = \mathbb{Z}$ and restrict ourselves to this case. We remember the cusp ∞ corresponds to $l_\infty = \left(\begin{smallmatrix} 0 & 1/N \\ 0 & 0 \end{smallmatrix}\right)$. We will find the Fourier expansion at this cusp. We can easily check that, $(l_\infty, L) = \mathbb{Z}$.

Remark 5.1.2. In this text we will extract the Fourier expansion at other cusps (in the case of N square-free) from the l_∞ case by using the Atkin-Lehner involutions, see Theorem 6.3.12. It is also true that if N is square-free, our assumption is not a restriction i.e. all the cusps (primitive isotropic lines l), satisfy $(l, L) = \mathbb{Z}$. So the results of this section hold for any cusp l (if N square-free) and we could alternatively find the Fourier expansion at a cusp l by altering the calculations in Theorem 6.3.10 by modifying the identities in (6.3.1) for the l case (as opposed to l_∞). When N is not necessarily square-free, we have $|L'/L| = \mathcal{N}^2|K_l'/K_l|$. We do not consider this case (which generates even longer calculations) but the results of this section can indeed be generalised for all cusps for any N. In particular, we have to be more careful with the cosets of K in Theorem 5.4.3, see [Bor98, Section 5 (Theorem 5.2)] and [Bru02, Section 2.1 (Theorem 2.4)]. We could then of course find the Fourier expansion for any cusp. This would allow us to extend the results found in Proposition 5.4.6, Thoerem 6.3.12, Theorem 6.4.2 and Theorem 7.3.8 to non-square-free N.

With this assumption, we can choose $l' \in L'$, such that l' is isotropic as well. We have

$$L = K_l \oplus \mathbb{Z}l' \oplus \mathbb{Z}l, \tag{5.1.2a}$$

$$V(\mathbb{R}) = (K_l \otimes_{\mathbb{Z}} \mathbb{R}) \oplus \mathbb{R}l' \oplus \mathbb{R}l. \tag{5.1.2b}$$

Crucially, $K_l'/K_l \cong L'/L$. The lattice K is of signature $(1, 0)$ and the Grassmannian of $K \otimes \mathbb{R}$ consists of only one point. Let $z \in \mathrm{Gr}(V(\mathbb{R}))$. Then $z^\perp$ was a two dimensional positive definite space. We denote $w^\perp$, for the orthogonal complement of $l_{z^\perp}$ in $z^\perp$. We denote the component of any $\lambda \in V(\mathbb{R})$ in $w^\perp$, as $\lambda_{w^\perp}$. We also can write w to denote the orthogonal complement of l_z in z. However, in the case of signature $(2, 1)$ this is empty. We clearly have

$$V(\mathbb{R}) = z \oplus z^\perp = \mathbb{R}l_z \oplus \mathbb{R}l_{z^\perp} \oplus w^\perp.$$

From now on we will denote K_l and K'_l, as K and K' respectively.

5.1.1 Vectors

The following vectors will show up in our proofs. Using the basis given in (3.3.1) we saw $z^\perp$ was spanned by $b_1(z)$ and $b_2(z)$. We have $l_{z^\perp} = l_1(z)b_1(z) + l_2(z)b_2(z)$ and we then define a vector

$$\mathfrak{w}^\perp := (l, b_2(z))b_1(z) - (l, b_1(z))b_2(z).$$

We observe that $(l_{z^\perp}, \mathfrak{w}^\perp) = 0$, thus $\mathfrak{w}^\perp$ spans the one dimensional space $w^\perp$. Further $(l, \mathfrak{w}^\perp) = (l_z, \mathfrak{w}^\perp) = 0$ and $(\mathfrak{w}^\perp, \mathfrak{w}^\perp) = l_1(z)^2 + l_2(z)^2 = l_3(z)^2 = Q_z(l)$ (as l is isotropic). This means we have

$$\lambda_{w^\perp} = \frac{(\lambda, \mathfrak{w}^\perp)}{Q_z(l)}\mathfrak{w}^\perp = \frac{\lambda_1(z)l_2(z) - \lambda_2(z)l_1(z)}{Q_z(l)}\mathfrak{w}^\perp.$$

Using this discussion we see that $w^\perp \subset V(\mathbb{R}) \cap l^\perp$ and $V(\mathbb{R}) \cap l^\perp = (K \otimes_{\mathbb{Z}} \mathbb{R}) + \mathbb{R}l$, using (5.1.2). In general $\lambda, \lambda_K, \lambda_{w^\perp}$ are not the same vector. However if $\lambda \in V(\mathbb{R}) \cap l^\perp$, then $(\lambda, \lambda) = (\lambda_K, \lambda_K) = (\lambda_{w^\perp}, \lambda_{w^\perp})$. This follows by noting if $\lambda = \lambda_K + dl, d \in \mathbb{R}$, then $(\lambda, \lambda) = (\lambda_K, \lambda_K)$. Finally, if $\lambda \in V(\mathbb{R}) \cap l^\perp$, then $Q(\lambda) = (\lambda, \lambda)/2 = (\lambda_{w^\perp}, \lambda_{w^\perp})/2 = (\lambda, \mathfrak{w}^\perp)^2/(2Q_z(l))$.

We will also use a vector $\mu(z)$, where

$$\mu(z) := -l' + \frac{l_z}{2(l_z, l_z)} + \frac{l_{z^\perp}}{2(l_{z^\perp}, l_{z^\perp})}. \tag{5.1.3}$$

From now on we will denote $\mu(z)$ as μ.

Lemma 5.1.3. *We have that*

1. $\mu \in V(\mathbb{R} \cap l^\perp) = (K \otimes_{\mathbb{Z}} \mathbb{R}) \oplus \mathbb{R}l'$,
2. $\mu = \mu_K + (\mu, l')l$,
3. $(\mu, l) = (\mu_K, l) = 0$,
4. $(\mu, \mu) = (\mu_K, \mu_K) = (\mu_{w^\perp}, \mu_{w^\perp})$,
5. $\mu_{w^\perp} = (\mu_K)_{w^\perp} = -l'_{w^\perp}$,
6. $(\mu, \mathfrak{w}^\perp) = (\mu_K, \mathfrak{w}^\perp) = (-l', \mathfrak{w}^\perp)$,
7. $(\mu, \mu)/2 = (\mu_K, \mu_K)/2 = -(l', l_{z^\perp} - l_z)/(2Q_z(l))$,
8. *For* $\lambda \in K \otimes \mathbb{R}$ *then* $(\lambda, \mu) = (\lambda, \mu_K) = (\lambda, l_{z^\perp} - l_z)/(2Q_z(l))$.

Proof. We easily check that $(\mu, l) = 0$ so $\mu \in V(\mathbb{R} \cap l^{\perp})$ and so (as we just discussed) $(\mu, \mu) = (\mu_K, \mu_K) = (\mu_{w^\perp}, \mu_{w^\perp})$. We know $\mu_K \in K \otimes \mathbb{R}$ and using (5.1.1) we see that $\mu = \mu_K + (\mu, l')l$. This tells us that $(\mu_K, l) = 0$ as well. It is then immediate that if $\lambda \in K \otimes \mathbb{R}$, then $(\lambda, \mu) = (\lambda, \mu_K)$. We have seen that the projection map onto $w^\perp$ vanishes on $l_z, l_z^\perp$ and l. This means $\mu_{w^\perp} = (\mu_K)_{w^\perp} = -l'_{w^\perp}$. This fact then implies that $(\mu, \mathfrak{w}^\perp) = (\mu_K, \mathfrak{w}^\perp) = (-l', \mathfrak{w}^\perp)$. For identity 7, we note $(l_z, l_z) = -(l_{z^\perp}, l_{z^\perp}) = -Q_z(l)$. The final identity follows by recalling that if $\lambda \in K \otimes \mathbb{R}$, then $(\lambda, l') = 0$. $\square$

5.2 The Mixed Model

We remember in section 2.3.1 that the Schrödinger representation (Definition 2.3.11) and the corresponding Schrödinger model depended on the choice of polarisation $W = W_1 \oplus W_2$. However, the Stone-von Neumann theorem (Theorem 2.3.6) told us that the Schrödinger representation is unique up to isomorphism and there was an intertwining operator (2.3.2) that is unique up to a scalar.

Definition 5.2.1. *Let $W = W_1 \oplus W_2 = W_1' \oplus W_2'$ be two complete polarisations of W and let $f \in \mathcal{S}(W_1)$. We will call the operator $\mathfrak{F} : \mathcal{S}(W_1) \to \mathcal{S}(W_1')$ defined by*

$$(\mathfrak{F}f)(x) := \int_{W_1/W_1 \cap W_1'} f(y)\psi\left(\frac{\langle x, y\rangle}{2}\right) dy$$

*the **partial Fourier transform**. Here dy is a positive W_1-invariant measure on $W_1/W_1 \cap W_1'$.*

Proposition 5.2.2 ([LV80, Proposition 1.4.7])**.** *The partial Fourier transform $\mathfrak{F}$ is an intertwiner of the Schrödinger models for W_1 and W_1'. I.e.*

$$\mathfrak{F} \circ M^{\mathrm{Sch}}_{\psi, W_1}(g) = M^{\mathrm{Sch}}_{\psi, W_1'}(g) \circ \mathfrak{F}.$$

We make this explicit, in the case of the dual pair $(\mathrm{O}(V(\mathbb{R})), \mathrm{SL}_2(\mathbb{R}))$. We explicitly realised this case using the equations in (2.4.5). These acted on the Schwartz functions $\mathcal{S}(V(\mathbb{R}))$. We used the character $\psi = e^{2\pi i x}$ and had a complete polarisation $\mathbb{W} = V(\mathbb{R}) \oplus V(\mathbb{R})$. We see that $\mathbb{R}l$ and $\mathbb{R}l'$ form totally isotropic subspaces of $V(\mathbb{R})$ with $V(\mathbb{R}) = \mathbb{R}l \oplus (K \otimes_{\mathbb{Z}} \mathbb{R}) \oplus \mathbb{R}l'$ and we will call the corresponding Schrödinger model on $\mathcal{S}(\mathbb{R}l) \otimes \mathcal{S}(K \otimes_{\mathbb{Z}} \mathbb{R}) \otimes \mathcal{S}(\mathbb{R}l')$ the **mixed model**. Following [FM13, Section 4.2.1] we can move between the two models by the partial Fourier transform:

$$\hat{f}(\xi l + \lambda + cl') := \int_{\mathbb{R}} f(xl + \lambda + cl')e^{2\pi i \xi x} dx.$$

where $c, x \in \mathbb{R}, \lambda \in K \otimes_{\mathbb{Z}} \mathbb{R}$ and $f \in \mathcal{S}(V(\mathbb{R}))$. The map $\hat{f}$, when restricted to $\mathcal{S}(K \otimes_{\mathbb{Z}} \mathbb{R})$ is indeed an intertwiner with $\mathcal{S}(V(\mathbb{R}))$. Naturally we still have actions under the dual pair in the mixed model, these are explicitly given in [FM13, Lemma 4.1]. This motivates why we use the decomposition (5.1.2), to rewrite our kernel function and then use partial Poisson summation on the x variable (denoted as d later).

5.2.1 Fourier Transforms

We first need a few lemmas to evaluate the complicated partial Fourier transform that will appear in the next part. In particular, we will need Lemma 5.2.10. To do this we will use the Hermite polynomials, which we discuss as well.

Definition 5.2.3. *Let $x \in \mathbb{R}, n \in \mathbb{Z}, n \geq 0$. Then the nth* ***Hermite polynomial*** *is defined as*

$$H_n(x) \coloneqq (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2} \right).$$

Example 5.2.4. The first four Hermite polynomials are

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x.$$

Lemma 5.2.5 ([EMOT81b, Section 10.13])**.** *We have that*

$$H_n(-x) = (-1)^n H_n(x), \tag{5.2.1a}$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{5.2.1b}$$

$$H_n^{'}(x) = 2nH_{n-1}(x), \tag{5.2.1c}$$

$$H_n(x) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!}, \tag{5.2.1d}$$

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_k(x)(2y)^{n-k}. \tag{5.2.1e}$$

We defined a Fourier transform in Definition 2.4.9. So over $\mathbb{R}$ we let

$$\hat{f}(\xi) \coloneqq \int_{\mathbb{R}} f(x) e^{2\pi i \xi x} dx.$$

We remember this agrees with the versions used in [Bor98,BO10,Bru02,BO10] but not [FM13]. We then have the following basic properties.

Lemma 5.2.6 ([Kam07, Chapter 3, Appendix 3])**.** *The Fourier transform of:*

1. *Linearity: $c_1 f_1(x) + c_2 f_2(x)$ where $c_1, c_2 \in \mathbb{C}$ is $c_1 \hat{f}_1(\xi) + c_2 \hat{f}_2(\xi)$,*

2. *Reflection: $f(-x)$ is $\hat{f}(-\xi)$,*

3. *Conjugation: $\overline{f(x)}$ is $\overline{\hat{f}(-\xi)}$,*

4. *Translation: $f(x-a)$ is $e^{2\pi i a \xi} \hat{f}(\xi)$,*

5. *Modulation: $f(x)e^{2\pi i a x}$ is $\hat{f}(\xi + a)$,*

6. *Power Scaling: $xf(x)$ is $\frac{d}{d\xi}\hat{f}(\xi)/2\pi i$,*

7. *Derivative: $\frac{d}{dx} f(x)$ is $-2\pi i \xi \hat{f}(\xi)$,*

8. *Dilation:* $f(ax)$ *is* $|a|^{-1}\hat{f}(\xi/a)$

9. $e^{-\pi x^2}$ *is* $e^{-\pi\xi^2}$,

10. $H_n(\sqrt{2\pi}x)e^{-\pi x^2}$ *is* $i^n H_n(\sqrt{2\pi}\xi)e^{-\pi\xi^2}$.

Lemma 5.2.7. *The Fourier transform of* $x^n e^{-\pi x^2}$ *is*

$$\left(\frac{i}{2\sqrt{\pi}}\right)^n H_n(\sqrt{\pi}\xi)e^{-\pi\xi^2}.$$

Proof. This follows from [FM13, Lemma 4.5] noting that $\tilde{H}_n(x) := (2\pi)^{-n/2}H_n(\sqrt{2\pi}x)$ and that the inverse Fourier transform is used here. This shows the Fourier transform of $(-\sqrt{2}ix)^n e^{-\pi x^2}$ is $(2\pi)^{-n/2}H_n(\sqrt{\pi}\xi)e^{-\pi\xi^2}$ and we then multiply by $(-\sqrt{2}i)^{-n}$. □

We now find the specific Fourier transform we will need in our case. To do this we let

$$f(x) := \left(E - \frac{DB}{2A} + \frac{Dx}{\sqrt{-2Ai}}\right)^{k-1} e\left(\frac{2Aix^2 - B^2 + 4AC}{4A}\right)$$

where $A, B, C, D, E \in \mathbb{C}, \operatorname{Im}(A) > 0, x \in \mathbb{R}$ and $k \in \mathbb{Z}, k > 0$. Then we can check that

$$f\left(\frac{-2Aix - Bi}{\sqrt{-2Ai}}\right) = (E + Dx)^{k-1}e(Ax^2 + Bx + C).$$

We calculate the Fourier transform of $f(x)$.

Lemma 5.2.8. *We have that*

$$\hat{f}(\xi) = \left(\frac{Di}{\sqrt{-8\pi Ai}}\right)^{k-1} H_{k-1}\left(\sqrt{\pi}\xi + \left(E - \frac{DB}{2A}\right)\frac{\sqrt{-2\pi Ai}}{Di}\right) e\left(\frac{2Ai\xi^2 - B^2 + 4AC}{4A}\right).$$

Proof. Using the binomial theorem we have

$$\begin{aligned} f(x) &= \left(E - \frac{DB}{2A} + \frac{Dx}{\sqrt{-2Ai}}\right)^{k-1} e\left(\frac{2Aix^2 + 4AC - B^2}{4A}\right) \\ &= \sum_{n=0}^{k-1}\binom{k-1}{n}\left(E - \frac{DB}{2A}\right)^{k-1-n}\left(\frac{Dx}{\sqrt{-2Ai}}\right)^n e\left(\frac{2Aix^2 - B^2 + 4AC}{4A}\right). \end{aligned}$$

Then using Lemma 5.2.7 we have

$$\begin{aligned} \hat{f}(\xi) &= \sum_{n=0}^{k-1}\binom{k-1}{n}\left(E - \frac{DB}{2A}\right)^{k-1-n}\left(\frac{Di}{\sqrt{-8\pi Ai}}\right)^n H_n(\sqrt{\pi}\xi)e\left(\frac{2Ai\xi^2 - B^2 + 4AC}{4A}\right) \\ &= \sum_{n=0}^{k-1}\binom{k-1}{n}\left(\frac{\sqrt{-8\pi Ai}}{Di}\left(E - \frac{DB}{2A}\right)\right)^{k-1-n}\left(\frac{Di}{\sqrt{-8\pi Ai}}\right)^{k-1} \\ &\quad \times H_n(\sqrt{\pi}\xi)e\left(\frac{2Ai\xi^2 - B^2 + 4AC}{4A}\right) \end{aligned}$$

and using the final property of the Hermite polynomials, Lemma 5.2.5, this is

$$\hat{f}(\xi) = \left(\frac{Di}{\sqrt{-8\pi Ai}}\right)^{k-1} H_{k-1}\left(\sqrt{\pi}\xi + \left(E - \frac{DB}{2A}\right)\frac{\sqrt{-2\pi Ai}}{Di}\right) e\left(\frac{2Ai\xi^2 - B^2 + 4AC}{4A}\right). \square$$

Lemma 5.2.9. *The Fourier transform of* $(E+Dx)^{k-1}e(Ax^2+Bx+C)$ *is*

$$\left(\frac{i}{2A}\right)^{k/2}\left(\frac{Di}{2\sqrt{\pi}}\right)^{k-1}H_{k-1}\left(i\sqrt{-2\pi Ai}\left(\frac{\xi+B}{2A}-\frac{E}{D}\right)\right)e\left(C-\frac{(\xi+B)^2}{4A}\right).$$

Proof. We remember $f\left(\frac{-2Aix-Bi}{\sqrt{-2Ai}}\right)=(E+Dx)^{k-1}e(Ax^2+Bx+C)$ and so using dilation, Lemma 5.2.6 part 8 in combination with Lemma 5.2.8 we have

$$\begin{aligned}\hat{f}\left(\xi\sqrt{-2Ai}\right)&=\frac{1}{\sqrt{-2Ai}}\left(\frac{Di}{\sqrt{-8\pi Ai}}\right)^{k-1}H_{k-1}\left(\frac{\sqrt{\pi}\xi}{\sqrt{-2Ai}}+\left(E-\frac{DB}{2A}\right)\frac{\sqrt{-2\pi Ai}}{Di}\right)\\&\quad\times e\left(\frac{-\xi^2-B^2+4AC}{4A}\right).\end{aligned}$$

Next using translation, Lemma 5.2.6 part 4. we obtain

$$\begin{aligned}\hat{f}\left(\frac{-2Ai\xi-Bi}{\sqrt{-2Ai}}\right)&=\frac{1}{\sqrt{-2Ai}}\left(\frac{Di}{\sqrt{-8\pi Ai}}\right)^{k-1}H_{k-1}\left(\frac{\sqrt{\pi}\xi}{\sqrt{-2Ai}}+\left(E-\frac{DB}{2A}\right)\frac{\sqrt{-2\pi Ai}}{Di}\right)\\&\quad\times e\left(\frac{-\xi^2-B^2+4AC}{4A}\right)e\left(\frac{-B\xi}{2A}\right)\\&=\left(\frac{i}{2A}\right)^{k/2}\left(\frac{Di}{2\sqrt{\pi}}\right)^{k-1}H_{k-1}\left(i\sqrt{-2\pi Ai}\left(\frac{\xi+B}{2A}-\frac{E}{D}\right)\right)\\&\quad\times e\left(C-\frac{(\xi+B)^2}{4A}\right).\end{aligned}$$

□

The next final lemma is the Fourier transform that we will actually use in Section 5.4.

Lemma 5.2.10. *Let* $F,G\in\mathbb{C}$*. Then the Fourier transform of* $(G+Fx)(E+Dx)^{k-1}e(Ax^2+Bx+C)$ *is*

$$\begin{aligned}&\left(\frac{i}{2A}\right)^{k/2}\left(\frac{Di}{2\sqrt{\pi}}\right)^{k-1}\sum_j\left(G-F\left(\frac{\xi+B}{2A}\right)\right)^{1-j}\left(\frac{F(k-1)}{i\sqrt{-2\pi Ai}}\right)^j\\&\qquad\times H_{k-1-j}\left(i\sqrt{-2\pi Ai}\left(\frac{\xi+B}{2A}-\frac{E}{D}\right)\right)e\left(C-\frac{(\xi+B)^2}{4A}\right).\end{aligned}\tag{5.2.2}$$

Remark 5.2.11. Here the sum over j is $0\le j\le\max(k-1,1)$. This convention holds throughout the text (to keep our notation compact).

Proof. We start with 5.2.9 and then use power scaling, Lemma 5.2.6 part 6 we get

$$\begin{aligned}&\left(\frac{i}{2A}\right)^{k/2}\left(\frac{Di}{2\sqrt{\pi}}\right)^{k-1}\left[\left(G-F\left(\frac{\xi+B}{2A}\right)\right)H_{k-1}\left(i\sqrt{-2\pi Ai}\left(\frac{\xi+B}{2A}-\frac{E}{D}\right)\right)\right.\\&\left.+\frac{F(k-1)}{i\sqrt{-2\pi Ai}}H_{k-2}\left(i\sqrt{-2\pi Ai}\left(\frac{\xi+B}{2A}-\frac{E}{D}\right)\right)\right]e\left(C-\frac{(\xi+B)^2}{4A}\right).\end{aligned}$$

Which can be rewritten as

$$\begin{aligned}&\left(\frac{i}{2A}\right)^{k/2}\left(\frac{Di}{2\sqrt{\pi}}\right)^{k-1}\sum_j\left(G-F\left(\frac{\xi+B}{2A}\right)\right)^{1-j}\left(\frac{F(k-1)}{i\sqrt{-2\pi Ai}}\right)^j\\&\times H_{k-1-j}\left(i\sqrt{-2\pi Ai}\left(\frac{\xi+B}{2A}-\frac{E}{D}\right)\right)e\left(C-\frac{(\xi+B)^2}{4A}\right).\end{aligned}$$

□

5.3 Theta Functions on the Sublattice

Once we have written our kernel function in terms of elements on the sublattice K we will then write it as a sum of some specific Siegel theta functions defined on K. In this section we define these theta functions. We show they have a certain transformation property in τ.

The next definition mirrors Definition 2.6.12. However it is different in that we twist and we also fix the lattice K, a vector μ_K and also an isometry σ'. In particular, $\sigma' : V(\mathbb{R}) \mapsto \mathbb{R}^{2,1}$ is defined as $\sigma'(\lambda) = \sigma(\lambda_{w^\perp})$ where $\sigma : V(\mathbb{R}) \mapsto \mathbb{R}^{2,1}$ is an isometry of $V(\mathbb{R})$. We notice that σ' vanishes on l_z and $l_{z^\perp}$. Definition 5.3.1 is a more general, polynomial version, of the function defined in [BO10, Equation 4.5].

Definition 5.3.1. *Let $\alpha, \beta \in \mathbb{Z}, h \in K'/K, \mu_K \in K \otimes_{\mathbb{Z}} \mathbb{R}$ and let p be a homogeneous polynomial on $\mathbb{R}^{1,0}$ of degree $(m^+, 0)$. Then we define*

$$\xi_h(\tau, \mu_K, \sigma', p, \alpha, \beta) \coloneqq \sum_{\lambda \in K + rh} \sum_{\substack{t(D) \\ Q(\lambda - \beta l' + tl) \equiv DQ(h)(D)}} \exp\left(-|D|\Delta/8\pi v\right) p(\sigma'(\lambda + \beta\mu_K))$$
$$\times \chi_D(\lambda - \beta l' + tl) e\left(\frac{-\alpha t}{|D|}\right) e\left(\frac{Q(\lambda + \beta\mu_K)\tau}{|D|} - \frac{(\lambda + \beta\mu_K/2, \alpha\mu_K)}{|D|}\right)$$

and a $\mathbb{C}[K'/K]$-valued version

$$\Xi(\tau, \mu_K, \sigma', p, \alpha, \beta) \coloneqq \sum_{h \in K'/K} \xi_h(\tau, \mu_K, \sigma', p, \alpha, \beta)\mathfrak{e}_h.$$

Remember the Grassmannian of K consisted of one element so we drop the variable z from the notation. We note that using the identities in Lemma 5.1.3 that we could replace μ_K with μ throughout this definition, which is commonly done in the literature. However this results in an abuse of notation as μ is not an element of $K \otimes \mathbb{R}$.

As ever we look at the transformation behaviour.

Theorem 5.3.2. *For any $(\gamma, \phi_\gamma) \in \tilde{\Gamma}, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then*

$$\Xi(\gamma\tau, \mu_K, \sigma', p, a\alpha + b\beta, c\alpha + d\beta) = \phi_\gamma(\tau)^{1+2m^+} \tilde{\rho}_K(\gamma, \phi_\gamma)\Xi(\tau, \mu_K, \sigma', p, \alpha, \beta).$$

Proof. To check the transformation behaviour under $(\gamma, \phi_\gamma) \in \tilde{\Gamma}$ it is enough to check the generators S, T for each component. We remember $|K'/K| = |L'/L| = 2N$. Then using Definition 3.2.7 we would like to show for T that

$$\xi_h(\tau + 1, \mu_K, \sigma', p, \alpha + \beta, \beta) = e(\mathrm{sgn}(D))Q(h))\xi_h(\tau, \mu_K, \sigma', p, \alpha, \beta). \tag{5.3.1}$$

Plugging $\xi_h(\tau+1,\mu_K,\sigma',p,\alpha+\beta,\beta)$ into Definition 5.3.1 we get

$$\sum_{\lambda\in K+rh}\sum_{\substack{t(D)\\ Q(\lambda-\beta l'+tl)\equiv DQ(h)(D)}}\chi_D(\lambda-\beta l'+tl)\exp\left(-\frac{|D|\Delta}{8\pi v}\right)p(\sigma'(\lambda+\beta\mu_K))$$
$$\times\, e\left(-\frac{\alpha t}{|D|}\right)e\left(\frac{Q(\lambda+\beta\mu_K)}{|D|}\tau-\frac{(\lambda+\beta\mu_K/2,\alpha\mu_K)}{|D|}\right)$$
$$\times\, e\left(-\frac{\beta t}{|D|}+\frac{Q(\lambda+\beta\mu_K)}{|D|}-\frac{(\lambda+\beta\mu_K/2,\alpha\mu_K)}{|D|}\right).$$

However we then notice

$$-\beta t+\frac{1}{2}(\lambda+\beta\mu_K,\lambda+\beta\mu_K)-(\lambda+\beta\mu_K/2,\beta\mu_K)=Q(\lambda-\beta l'+tl)\equiv DQ(h)(D),$$

and so (5.3.1) is clear. For S we would like to show

$$\xi_h\left(-\frac{1}{\tau},\mu_K,\sigma',p,-\beta,\alpha\right)=\tau^{1/2+m^+}\frac{e(-\mathrm{sgn}(D)/8)}{\sqrt{2N}}$$
$$\times\sum_{h'\in K'/K}e(-\mathrm{sgn}(D)(h,h')\xi_{h'}(\tau,\mu_K,\sigma',p,\alpha,\beta).\quad(5.3.2)$$

We rewrite $\xi_h(\tau,\mu_K,\sigma',p,\alpha,\beta)$ in terms of the Siegel theta function $\vartheta_K(\tau,\sigma',p,\alpha,\beta)$ (Definition 2.6.12)

$$\xi_h(\tau,\mu_K,\sigma',p,\alpha,\beta)=|D|^{m^+}\sum_{\substack{\lambda'\in K'/DK\\ \lambda'\equiv rh(K)}}\sum_{\substack{t(D)\\ Q(\lambda'-\beta l'+tl)\equiv DQ(h)(D)}}\chi_D(\lambda'-\beta l'+tl)$$
$$\times\, e\left(-\frac{\alpha t}{|D|}-\frac{(\lambda',\alpha\mu_K)}{2|D|}\right)\vartheta_K\left(|D|\tau,\sigma',p,\alpha\mu_K,\frac{\lambda'+\beta\mu_K}{|D|}\right).$$

This can be checked by inserting the definition of ϑ_K and then making the "substitution" $\lambda'\mapsto\lambda'-|D|\lambda$ and remembering the invariance of χ_D modulo DL. So we have

$$\xi_h\left(-\frac{1}{\tau},\mu_K,\sigma',p,-\beta,\alpha\right)=|D|^{m^+}\sum_{\substack{\lambda'\in K'/DK\\ \lambda'\equiv rh(K)}}\sum_{\substack{t(D)\\ Q(\lambda'-\alpha l'+tl)\equiv DQ(h)(D)}}\chi_D(\lambda'-\alpha l'+tl)$$
$$\times\, e\left(\frac{\beta t}{|D|}+\frac{(\lambda',\beta\mu_K)}{2|D|}\right)\vartheta_K\left(-\frac{|D|}{\tau},\sigma',p,-\beta\mu_K,\frac{\lambda'+\alpha\mu_K}{|D|}\right)$$

and we then use the transformation property of ϑ_K for S, Theorem 2.6.14 to see that

$$\xi_h\left(-\frac{1}{\tau},\mu_K,\sigma',p,-\beta,\alpha\right)=\frac{\tau^{1/2+m^+}e(-1/8)}{(\sqrt{|D||K'/K|})}\sum_{\lambda\in K'}\exp(-|D|\Delta/8\pi v)p(\sigma'(\lambda+\beta\mu_K))$$
$$\times\, g_h(\lambda,-\beta,-\alpha)e\left(\frac{Q(\lambda+\beta\mu_K)}{|D|}\tau-\frac{(\lambda+\beta\mu_K/2,\alpha\mu_K)}{|D|}\right)$$

where for $h\in K'/K,\lambda\in K'/DK$ and $a,b\in\mathbb{Z}/D\mathbb{Z}$ we let

$$g_h(\lambda,a,b):=\sum_{\substack{\lambda'\in K'/DK\\ \lambda'\equiv rh(K)\\ t(D)\\ Q(\lambda'+al'+bl)\equiv DQ(h)(D)}}\chi_D(\lambda'+bl'+tl)e\left(-\frac{1}{|D|}((\lambda,\lambda')+at)\right).$$

Then by [BO10, Proposition 4.5] we easily obtain

$$\xi_h\left(-\frac{1}{\tau},\mu_K,\sigma',p,-\beta,\alpha\right)=\frac{\tau^{1/2+m^+}e(-1/8)\epsilon_D}{\sqrt{|2N|}}\\ \times\sum_{h'\in K'/K}e(-\mathrm{sgn}(D)(h,h'))\xi_{h'}\left(\tau,\mu_K,\sigma',p,\alpha,\beta\right)$$

where $\epsilon_D=1$ if $D>0$ or $\epsilon_D=i$ if $D<0$. We note $\epsilon_D=e((1-\mathrm{sgn}(D))/8)$ so we have shown (5.3.2). $\square$

We now give another example that we will need. This involves Hermite polynomials.

Definition 5.3.3. *Let $\alpha,\beta\in\mathbb{Z},h\in K'/K,\mu_K\in K\otimes_{\mathbb{Z}}\mathbb{R}$ and let $\kappa\in\mathbb{Z},\kappa\geq 0$ then we define*

$$\xi_{\kappa,h}(\tau,\mu_K,\alpha,\beta):=v^{-\kappa/2}\sum_{\substack{\lambda\in K+rh\\ t(D)\\ Q(\lambda-\beta l'+tl)\equiv DQ(h)(D)}}H_\kappa\left(\frac{\sqrt{\pi}(\alpha-\beta\overline{\tau}-2|D|v(\lambda+\beta\mu_K,\mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right)\\ \times\chi_D(\lambda-\beta l'+tl)e\left(\frac{-\alpha t}{|D|}\right)e\left(\frac{Q(\lambda+\beta\mu_K)\tau}{|D|}-\frac{(\lambda+\beta\mu_K/2,\alpha\mu_K)}{|D|}\right)$$

and a $\mathbb{C}[K'/K]$-valued version

$$\Xi_\kappa(\tau,\mu_K,\alpha,\beta):=\sum_{h\in K'/K}\xi_{\kappa,h}(\tau,\mu_K,\alpha,\beta)\mathfrak{e}_h.$$

Lemma 5.3.4. *For any $(\gamma,\phi_\gamma)\in\tilde{\Gamma},\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ then*

$$\Xi_\kappa(\gamma\tau,\mu_K,a\alpha+b\beta,c\alpha+d\beta)=\phi_\gamma(\tau)^{1+2\kappa}\tilde{\rho}_K(\gamma,\phi_\gamma)\Xi_\kappa(\tau,\mu_K,\alpha,\beta).$$

Proof. Using part e of Lemma 5.2.1 we can write

$$H_\kappa\left(\frac{\sqrt{\pi}(\alpha-\beta\overline{\tau}-2v(\lambda+\beta\mu_K,\mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right)\\ =\sum_{m=0}^{\kappa}\binom{\kappa}{m}H_m\left(\frac{-\sqrt{2\pi v}(\lambda+\beta\mu_K,\mathfrak{w}^\perp)}{\sqrt{|D|Q_z(l)}}\right)\left(\frac{\sqrt{2\pi}(\alpha-\beta\overline{\tau})}{\sqrt{|D|vQ_z(l)}}\right)^{\kappa-m}.$$

We mentioned in Section 2.6 that we could think of Hermite polynomials in terms of the $\exp\left(-\Delta/8\pi\right)(p)$ polynomial. Explicitly, $K\otimes\mathbb{R}$ is a vector space of signature $(1,0)$ isometric to $\mathbb{R}$ with basis x_1. We then let $p(x_1)=x_1^m$. Then for $\lambda\in K$ and $a\in\mathbb{C}$

$$\left(\frac{8\pi v}{|D|}\right)^{m/2}\exp\left(-|D|\Delta/8\pi v\right)(p)(ax_1)=m!\sum_{s=0}^{\lfloor m/2\rfloor}\frac{(-1)^s}{s!(m-2s)!}\left(\frac{\sqrt{8\pi v}ax_1}{\sqrt{|D|}}\right)^{m-2s}\\ =H_m\left(\frac{\sqrt{2\pi v}ax_1}{\sqrt{|D|}}\right)$$

using part d of Lemma 5.2.1. So setting $\sigma'(\lambda)=-(\lambda,\mathfrak{w}^\perp)/(Q_z(l)a)$ we have

$$\xi_{\kappa,h}(\tau,\mu_K,\alpha,\beta)=\sum_{m=0}^{\kappa}\binom{\kappa}{m}\left(\frac{8\pi}{|D|}\right)^{m/2}\left(\frac{\sqrt{2\pi}(\alpha-\beta\overline{\tau})}{v\sqrt{|D|Q_z(l)}}\right)^{\kappa-m}\xi_h(\tau,\mu_K,\sigma',p,\alpha,\beta).$$

As in the previous proof we now consider T and S. For T we see that the term $(\alpha - \beta\overline{\tau}/v)$ is invariant under the maps $\tau \mapsto \tau + 1, \alpha \mapsto \alpha + \beta$. For S we have the maps $\tau \mapsto -1/\tau, \alpha \mapsto -\beta, \beta \mapsto \alpha$ and plugging these in we get $(\alpha - \beta\overline{\tau}/v) \mapsto \tau(\alpha - \beta\overline{\tau}/v)$ so this term essential "raises the weight" by 1. Each $\xi_h(\tau, \mu_K, \sigma', p, \alpha, \beta)$ piece transforms with weight $1/2 + m$ by Theorem 5.3.2 so we have total weight $1/2 + m + \kappa - m = 1/2 + \kappa$ as required. □

5.3.1 Properties of $\Xi_\kappa(\tau, \mu_K, -n, 0)$

The Poincaré series we will obtain in Theorem 5.4.5 are of the form $\Xi_{k-1-j}(\tau, \mu_K, -n, 0)$ where $n \in \mathbb{Z}, n \geq 1$. We will use these in Theorem 6.3.10 to find our Fourier expansion. We consider some of the properties of these functions. We first have the following more explicit form. We simplify the sum over t in Definition 5.3.3:

Lemma 5.3.5. *For $\kappa \in \mathbb{Z}, \kappa \geq 0$ then*

$$\xi_{\kappa,h}(\tau, \mu_K, -n, 0) = \left(\frac{D}{n}\right) \epsilon_D |D|^{1/2} v^{-\kappa/2} \times \sum_{\substack{\lambda \in K + rh \\ Q(\lambda) \equiv DQ(h)(D)}} H_\kappa\left(\frac{\sqrt{\pi}(n - 2|D|v(\lambda, \mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right) e\left(\frac{Q(\lambda)\tau}{|D|} - \frac{(\lambda, n\mu_K)}{|D|}\right).$$

Proof. We start by using the definition given in 5.3.1 to write

$$\xi_{\kappa,h}(\tau, \mu_K, -n, 0) = v^{-\kappa/2} \sum_{\substack{\lambda \in K + rh \\ Q(\lambda) \equiv DQ(h)(D)}} \sum_{t(D)} \chi_D(\lambda + tl) \times H_\kappa\left(\frac{\sqrt{\pi}(-n - 2|D|v(\lambda, \mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right) e\left(\frac{nt}{|D|}\right) e\left(\frac{Q(\lambda)\tau}{|D|} + \frac{(\lambda, n\mu_K)}{|D|}\right).$$

We know that if $\lambda \in K + rh, Q(\lambda) \equiv DQ(h)(D)$ then $\chi_D(\lambda + tl) = \left(\frac{D}{n}\right)$ so we obtain the stated result by using the following Gauss sum (see eg. [BO10, Equation 4.7])

$$\sum_{t(D)} \left(\frac{D}{t}\right) e\left(\frac{nt}{D}\right) = \left(\frac{D}{n}\right) \epsilon_D |D|^{1/2}.$$ □

We remember from Theorem 5.3.2 that ϵ_D was defined to equal 1 if $D > 0$ or i if $D < 0$. We then note using Definition 2.4.8 that $\left(\frac{D}{0}\right) = 0$ if $D \neq 1$ and if $D = 1$ then $\left(\frac{D}{0}\right) = 1$. This means that $\xi_{\kappa,h}(\tau, 0, 0, 0) = 0$ unless $D = 1$.

We will now look at the asymptotic behaviour. Before we show this, we first introduce the polylogarithms. These will also crop up in the proof of Theorem 6.3.10.

Definition 5.3.6. *Let $\kappa \in \mathbb{Z}$ and $z \in \mathbb{C}, |z| < 1$. Then the **polylogarithm** is defined as*

$$\mathrm{Li}_\kappa(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\kappa},$$

which can be analytically continued to all $z \in \mathbb{C}$.

Lemma 5.3.7 ([MR14, Section 2.1.1.7] [Woo92])**.** *We have that*

1. $\mathrm{Li}_1(z) = -\ln(1-z)$,
2. $z\dfrac{\partial \mathrm{Li}_\kappa(z)}{\partial z} = \mathrm{Li}_{\kappa-1}(z)$,
3. $\dfrac{\partial \mathrm{Li}_\kappa(e^z)}{\partial z} = \mathrm{Li}_{\kappa-1}(e^z)$,
4. $\mathrm{Li}_\kappa(e^z) = \mathcal{O}(z^{\kappa-1})$ *as* $|z| \to 0$*, for* $\kappa \leq 0$,
5. $\mathrm{Li}_\kappa(z) = \mathcal{O}(z)$ *as* $|z| \to 0$.

We can now consider the growth of $\Xi(\tau, \mu_K, -n, 0, \kappa)$. This lemma will once again be needed in Theorem 6.3.10 to show convergence of certain integrals.

Lemma 5.3.8. *The function* $\Xi_\kappa(\tau, \mu_K, -n, 0) = \mathcal{O}(1)$ *as* $v \to \infty$*, uniformly in* u*. The function* $\Xi_\kappa(\tau, \mu_K, -n, 0) = \mathcal{O}(v^{-2\kappa-1})$ *as* $v \to 0$*, uniformly in* u*.*

Proof. For the first part we remember $\Xi_\kappa(\tau, \mu_K, -n, 0)$ is just a positive definite theta series (with a polynomial term) on the lattice K. We also observe that $v^{-\kappa/2}H_\kappa(\sqrt{v}) = \mathcal{O}(1)$ as $v \to \infty$ i.e. we only have non-positive powers of v in our series. As $v \to \infty$ everything decays except possibly a constant term. For the second part, we see that, for $0 \leq v \leq 1$, then certainly

$$|\xi_{\kappa,h}(\tau, \mu_K, -n, 0)| \leq C_\kappa v^{-\kappa} |(\lambda, \mathfrak{w}^\perp)|^\kappa \sum_{\substack{\lambda \in K+rh \\ Q(\lambda) D Q(h)(D)}} e^{-Q(\lambda)v/|D|}$$

for some constant $C_\kappa > 0$. We let $b \in K'$ be a basis for the 1-dimensional positive definite lattice K'. It then suffices to consider

$$|\xi_{\kappa,h}(\tau, \mu_K, -n, 0)| \leq C_\kappa' v^{-\kappa} \sum_{m \in \mathbb{Z}} |m|^\kappa e^{-m^2 Q(b)v/|D|}$$

for some constant $C_\kappa' > 0$. We then see that

$$\begin{aligned} \frac{1}{2}\sum_{m \in \mathbb{Z}} |m|^\kappa e^{-m^2 Q(b)v/|D|} &= \sum_{m \geq 1} m^\kappa \left(e^{-Q(b)v/|D|}\right)^{m^2} \\ &\leq \sum_{m \geq 1} m^\kappa \left(e^{-Q(b)v/|D|}\right)^{m} = \mathrm{Li}_{-\kappa}\left(e^{-Q(b)v/|D|}\right). \end{aligned}$$

So putting this all together with Lemma 5.3.7 part 4 we obtain the stated result. This estimate is not optimal but will suffice for our purposes. $\square$

In the case when $n = 0$, we have the following nice lemma which tells us the effect of the raising and lowering operators. We will particularity need this lemma in Section 6.1. We also remark that in general $\Xi_\kappa(\tau, 0, 0, 0)$ is neither harmonic nor holomorphic.

Lemma 5.3.9. *Let $\kappa \in \mathbb{Z}, \kappa \geq 2$ and $D = 1$. Then*

$$R_{k-3/2}\Xi_{\kappa-2}(\tau,0,0,0) = -\frac{1}{4}\Xi_{\kappa}(\tau,0,0,0),$$
$$L_{\kappa+1/2}\Xi_{\kappa}(\tau,0,0,0) = \kappa(\kappa-1)\Xi_{\kappa-2}(\tau,0,0,0),$$
$$L_{\kappa+1/2}R_{\kappa-3/2}\Xi_{\kappa-2}(\tau,0,0,0) = -\frac{\kappa(\kappa-1)}{4}\Xi_{\kappa-2}(\tau,0,0,0).$$

Proof. We assume $D = 1$ otherwise $\Xi_{\kappa-2}(\tau,0,0,0)$ vanishes, see Lemma 5.3.5. Lemma 5.3.4 told us that $\Xi_{\kappa-2}(\tau,0,0,0)$ is of weight $\kappa - 3/2$. We then use Definition 5.3.3 to write out explicitly

$$\xi_{\kappa-2,h}(\tau,0,0,0) = v^{-(\kappa-2)/2} \sum_{\lambda\in K+h} H_{\kappa-2}\left(\frac{-\sqrt{2\pi v}(\lambda,\mathfrak{w}^{\perp})}{\sqrt{Q_z(l)}}\right) e(Q(\lambda)\tau).$$

We set $a = -\sqrt{2\pi}(\lambda,\mathfrak{w}^{\perp})/\sqrt{Q_z(l)}$ and then we have that $a^2 = 4\pi Q(\lambda)$, (Section 5.1.1). We also remember that

$$R_{\kappa-3/2} = i\frac{\partial}{\partial u} + \frac{\partial}{\partial v} + \frac{\kappa-3/2}{v} \quad \text{and} \quad L_{\kappa+1/2} = v^2\left(-i\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right).$$

For the first part, it then suffices to consider

$$\begin{aligned}
&R_{\kappa-3/2}\left[v^{-(\kappa-2)/2}H_{\kappa-2}(a\sqrt{v})e^{ia^2\tau/2}\right]\\
&= \left(\left(\frac{\kappa-1}{2v} - a^2\right)H_{\kappa-2}(a\sqrt{v}) + \frac{(\kappa-2)a}{\sqrt{v}}H_{\kappa-3}(a\sqrt{v})\right)v^{1-\kappa/2}e^{ia^2\tau/2}\\
&= \left((\kappa-1)H_{\kappa-2}(a\sqrt{v}) - a\sqrt{v}H_{\kappa-1}(a\sqrt{v})\right)v^{-\kappa/2}\frac{e^{ia^2\tau/2}}{2}\\
&= -H_{\kappa}(a\sqrt{v})v^{-k/2}\frac{e^{ia^2\tau/2}}{4}
\end{aligned}$$

where we have simplified using the properties of the Hermite polynomials stated in Lemma 5.2.5. For the second part we note that

$$\begin{aligned}
\frac{\partial}{\partial v}\left(H_{\kappa}(a\sqrt{v})v^{-\kappa/2}\right) &= -\frac{\kappa v^{-(\kappa+2)/2}}{2}\left(H_{\kappa}(a\sqrt{v}) - 2a\sqrt{v}H_{\kappa-1}(a\sqrt{v})\right)\\
&= \kappa(\kappa-1)v^{-(\kappa+2)/2}H_{k-2}(a\sqrt{v})
\end{aligned}$$

so

$$L_{\kappa+1/2}\left[H_{\kappa}(a\sqrt{v})v^{-\kappa/2}e^{ia^2\tau/2}\right] = \kappa(\kappa-1)v^{-(\kappa-2)/2}H_{\kappa-2}(a\sqrt{v})e^{ia^2\tau/2}.$$

Combining parts one and two gives the third part. □

5.4 The Poincaré Series

The next few lemmas involve rewriting our kernel function in various forms with the aim of writing it as a Poincaré series in Theorem 5.4.5. The first lemma will help to simplify some of the terms we will obtain when we apply our partial Fourier transform.

Lemma 5.4.1. *Let* $A = Q(l_{z^\perp})(\tau - \overline{\tau}), B = (\lambda, l_z)\overline{\tau} + (\lambda, l_{z^\perp})\tau, C = Q(\lambda_z)\overline{\tau} + Q(\lambda_{z^\perp})\tau, D = q_z(l), E = q_z(\lambda), F = p_z(l)$ *and* $G = p_z(\lambda)$*. Then*

$$\begin{aligned} C - \frac{(d+B)^2}{4A} &= \tau Q(\lambda_{w^\perp}) - \frac{d(\lambda, l_{z^\perp} - l_z)}{2Q_z(l)} - \frac{|d + (\lambda, l)\tau|^2}{4ivQ_z(l)}, \\ i\sqrt{-2\pi A i}\left(\frac{d+B}{2A} - \frac{E}{D}\right) &= \frac{\sqrt{\pi}(d + (\lambda, l)\overline{\tau} - 2v(\lambda, \mathfrak{w}^\perp))}{\sqrt{2vQ_z(l)}}, \\ G - F\left(\frac{d+B}{2A}\right) &= \frac{i(d + (\lambda, l)\tau)}{2v\sqrt{Q_z(l)}}. \end{aligned}$$

Proof. Remember $Q(l_{z^\perp}) = -Q(l_z)$and $Q_z(l) = 2Q(l_{z^\perp}) = q_z(l)\overline{q_z(l)}y^{-2} = (l, l_{z^\perp}) = p_z(l)^2$ (as l is isotropic). We have fixed an orientation of isotropic vectors such that $\operatorname{sgn}(-l, b_3(z)) = 1$ so $p_z(l) > 0$ and $p_z(l) = \sqrt{Q_z(l)}$. For the first part we plug in the terms and obtain

$$C - \frac{(d+B)^2}{4A} = \frac{-d^2 - 2d(\overline{\tau}(\lambda, l_z) + \tau(\lambda, l_{z^\perp})) - (\overline{\tau}(\lambda, l_z) + \tau(\lambda, l_{z^\perp}))^2}{4ivQ_z(l)} + \overline{\tau}Q(\lambda_z) + \tau Q(\lambda_{z^\perp})$$

and then adapting [Bru02, Lemma 2.3] we get our stated result. For the second part we have

$$\begin{aligned} i\sqrt{-2\pi A i}\left(\frac{d+B}{2A} - \frac{E}{D}\right) &= \frac{i\sqrt{2\pi v Q_z(l)}}{Q_z(l)}\left(Q_z(l)\left(\frac{d+B}{2A}\right) - \frac{q_z(\lambda)\overline{q_z(l)}}{y^2}\right) \\ &= \frac{i\sqrt{2\pi v}}{\sqrt{Q_z(l)}}\left(\left(\frac{d+B}{2iv}\right) - (\lambda, l_{z^\perp}) + i(\lambda, \mathfrak{w}^\perp)\right) \\ &= \frac{i\sqrt{2\pi v}}{\sqrt{Q_z(l)}}\left(\frac{d + \tau(\lambda, l_{z^\perp}) + \overline{\tau}(\lambda, l_z) - 2iv(\lambda, l_{z^\perp})}{2iv} + i(\lambda, \mathfrak{w}^\perp)\right) \\ &= \frac{\sqrt{\pi}}{\sqrt{2vQ_z(l)}}(d + (\lambda, l)\overline{\tau} - 2v(\lambda, \mathfrak{w}^\perp)). \end{aligned}$$

For the third part we have

$$\begin{aligned} G - F\left(\frac{d+B}{2A}\right) &= \frac{1}{p_z(l)}\left(p_z(\lambda)p_z(l) - (p_z(l))^2\left(\frac{d+B}{2A}\right)\right) \\ &= \frac{-1}{\sqrt{Q_z(l)}}\left((\lambda, l_z) + \frac{d+B}{2iv}\right) \\ &= \frac{-1}{\sqrt{Q_z(l)}}\left(\frac{2iv(\lambda, l_z) + d + \tau(\lambda, l_{z^\perp}) + \overline{\tau}(\lambda, l_z)}{2iv}\right) \\ &= \frac{i}{2v\sqrt{Q_z(l)}}(d + (\lambda, l)\tau). \end{aligned}$$ □

The next lemma consists of applying our partial Poisson summation and simplifying as much as possible. This is the same idea as in [Bor98, Lemma 5.1], [BO10, Lemma 4.6] and [Bru02, Lemma 2.3]. We will use the following constant

$$c_{z,k,j} = \frac{i}{2\sqrt{2|D|}Q_z(l)}\left(\frac{q_z(l)i\sqrt{|D|}}{2\sqrt{2\pi Q_z(l)}}\right)^{k-1}\left(\frac{(1-k)\sqrt{2|D|Q_z(l)}}{\sqrt{\pi}}\right)^j. \tag{5.4.1}$$

Lemma 5.4.2. *We have that*

$$\theta_{h,k}(\tau,z) = \sum_{\lambda\in L/\mathbb{Z}Dl+rh} \sum_{\substack{t(D)\\ Q(\lambda+tl)\equiv DQ(h)(D)}} \sum_{d\in\mathbb{Z}} \sum_j \chi_D(\lambda+tl) c_{z,k,j} v^{-(k-1-j)/2}$$
$$\times (d+(\lambda,l)\tau)^{1-j} H_{k-1-j}\left(\frac{\sqrt{\pi}(d+(\lambda,l)\overline{\tau}-2v(\lambda,\mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right)$$
$$\times e\left(-\frac{dt}{|D|}\right) e\left(\frac{\tau Q(\lambda_{w^\perp})}{|D|} - \frac{d(\lambda, l_{z^\perp}-l_z)}{2|D|Q_z(l)} - \frac{|d+(\lambda,l)\tau|^2}{4|D|ivQ_z(l)}\right).$$

Proof. As is standard, we rewrite the sum over $\lambda \in rh+L$ in the definition of $\theta_{h,k}(\tau,z)$ as a sum over $\lambda' + d|D|l$. This is where λ' runs over $rh + L/\mathbb{Z}Dl$ and d runs over $\mathbb{Z}$. Noting that $\chi_D(\lambda+d|D|l) = \chi_D(\lambda)$ and $Q(\lambda+d|D|l) \equiv Q(\lambda)(D)$ where $\lambda \in L'$ and $d\in\mathbb{Z}$ we then obtain

$$\theta_{h,k}(\tau,z) = v^{3/2} \sum_{\substack{\lambda\in L/\mathbb{Z}Dl+rh\\ Q(\lambda)\equiv DQ(h)(D)}} \chi_D(\lambda) \sum_{d\in\mathbb{Z}} p_z(\lambda+d|D|l) q_z(\lambda+d|D|l)^{k-1}$$
$$\times e\left(\frac{Q(\lambda+d|D|l)}{|D|}u + \frac{Q_z(\lambda+d|D|l)}{|D|}iv\right)$$
$$= v^{3/2} \sum_{\substack{\lambda\in L/\mathbb{Z}Dl+rh\\ Q(\lambda)\equiv DQ(h)(D)}} \chi_D(\lambda) \sum_{d\in\mathbb{Z}} g(|D|\tau, z, \frac{\lambda}{|D|}, k, d)$$

where

$$g(\tau,z,\lambda,k,d) := |D|^k p_z(\lambda+dl) q_z(\lambda+dl)^{k-1} e\left(Q(\lambda+dl)u + Q_z(\lambda+dl)iv\right).$$

We notice

$$Q(\lambda+dl)u + Q_z(\lambda+dl)iv = Ad^2 + Bd + C$$

where $A = Q(l_{z^\perp})(\tau-\overline{\tau}) = Q_z(l)iv, B = (\lambda,l_z)\overline{\tau} + (\lambda,l_{z^\perp})\tau$ and $C = Q(\lambda_z)\overline{\tau} + Q(\lambda_{z^\perp})\tau = Q(\lambda)u + Q_z(\lambda)iv$. We also set $D' = q_z(l), E = q_z(\lambda), F = p_z(l)$ and $G = p_z(\lambda)$. Then we find the partial Fourier transform of $g(\tau,z,\lambda,k,d)$ in d by using Lemma 5.2.10. Combining this with the simplifications given in Lemma 5.4.1 we see that $\hat{g}(\tau,z,\lambda,k,d)$ is equal to

$$|D|^k \left(\frac{1}{2vQ_z(l)}\right)^{k/2} \left(\frac{q_z(l)i}{2\sqrt{\pi}}\right)^{k-1} \sum_j \left(\frac{i(d+(\lambda,l)\tau)}{2v\sqrt{Q_z(l)}}\right)^{1-j} \left(\frac{(k-1)}{i\sqrt{2\pi v}}\right)^j$$
$$\times H_{k-1-j}\left(\frac{\sqrt{\pi}(d+(\lambda,l)\overline{\tau}-2v(\lambda,\mathfrak{w}^\perp))}{\sqrt{2vQ_z(l)}}\right) e\left(\tau Q(\lambda_{w^\perp}) - \frac{d(\lambda,l_{z^\perp}-l_z)}{2Q_z(l)} - \frac{|d+(\lambda,l)\tau|^2}{4ivQ_z(l)}\right).$$

Using the Poisson summation formula on the variable d we can simply replace our old term $g(|D|\tau, z, \frac{\lambda}{|D|}, k, d)$ with $\hat{g}(|D|\tau, z, \frac{\lambda}{|D|}, k, d)$ in (5.4.2). We find that $\theta_{h,k}(\tau,z)$ is equal to

$$\sum_{\substack{\lambda\in L/\mathbb{Z}Dl+rh\\ Q(\lambda)\equiv DQ(h)(D)}} \sum_{d\in\mathbb{Z}} \sum_j \chi_D(\lambda) c_{z,k,j} v^{-(k-1-j)/2} (d+(\lambda,l)\tau)^{1-j}$$
$$\times H_{k-1-j}\left(\frac{\sqrt{\pi}(d+(\lambda,l)\overline{\tau}-2v(\lambda,\mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right) e\left(\frac{\tau Q(\lambda_{w^\perp})}{|D|} - \frac{d(\lambda,l_{z^\perp}-l_z)}{2|D|Q_z(l)} - \frac{|d+(\lambda,l)\tau|^2}{4|D|ivQ_z(l)}\right).$$

Finally we rewrite the sum over $\lambda \in L/\mathbb{Z}Dl + rh$ as a sum over $\lambda' + tl$ where λ' and t run over $L/\mathbb{Z}l + rh$ and $\mathbb{Z}/D\mathbb{Z}$ respectively. We get the result stated at the start of the lemma. Also observe that $(l, l_{z^\perp} - l_z)/2Q_z(l) = 1$ and $(l, \mathfrak{w}^\perp) = 0$. $\square$

We now rewrite the previous lemma in terms of the theta function (Definition 5.3.3) on the sublattice. This is the same idea as in [Bor98, Theorem 5.2], [BO10, Lemma 4.7] and [Bru02, Theorem 2.4].

Theorem 5.4.3. *For $h \in L'/L \cong K'/K$*

$$\theta_{h,k}(\tau, z) = \sum_{c,d\in\mathbb{Z}} \sum_{j} c_{z,k,j}(c\tau + d)^{1-j} e\left(-\frac{|d + c\tau|^2}{4|D|ivQ_z(l)}\right) \xi_{k-1-j,h}(\tau, \mu_K, d, -c).$$

Proof. We use will use the fact that $L/\mathbb{Z}l + rh \cong K + \mathbb{Z}l' + rh$ to rewrite Lemma 5.4.2 in terms of $\lambda \in K + rh$. We do this by making the "substitution" $\lambda \mapsto \lambda + cl'$, where now $\lambda \in K \otimes \mathbb{R}$ and $c \in \mathbb{Z}$. We have that $(l, l') = 1$ and $(\lambda, l) = 0$. Combining these facts with several of the identities in Lemma 5.1.3 we obtain

$$\begin{aligned}\theta_{h,k}(\tau, z) = &\sum_{\lambda\in K+rh} \sum_{c,d\in\mathbb{Z}} \sum_{\substack{t(D)\\ Q(\lambda+cl'+tl)\equiv DQ(h)(D)}} \sum_{j} \chi_D(\lambda + cl' + tl)c_{z,k,j}v^{-(k-1-j)/2} \\ &\times (d + c\tau)^{1-j}\, H_{k-1-j}\left(\frac{\sqrt{\pi}(d + c\overline{\tau} - 2v(\lambda - c\mu_K, \mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right) \\ &\times e\left(-\frac{dt}{|D|}\right) e\left(\frac{\tau}{|D|}Q((\lambda - c\mu_K)_{w^\perp}) - \frac{d}{|D|}(\lambda - c\mu_K/2, \mu_K) - \frac{|d + c\tau|^2}{4|D|ivQ_z(l)}\right).\end{aligned} \tag{5.4.3}$$

Inserting the definition of $\xi_h(\tau, \mu_K, \alpha, \beta, n)$ gives the result. We recall that $L'/L \cong K'/K$ and note $h, h' \in L'/L$ are equal exactly when $k', k \in K'/K$ are equal. So $\xi_h(\tau, \mu_K, \alpha, \beta, n)$ is well defined for $h \in L'/L$. $\square$

Remark 5.4.4. We observe that using the identities in Lemma 5.1.3 we could in fact replace μ_K with μ throughout this theorem.

Finally we write the $\mathbb{C}[L'/L]$-valued theta function $\Theta_{D,r,h}(\tau, z)$ in terms of the $\mathbb{C}[K'/K]$-valued theta function $\Xi(\tau, \mu_K, \alpha, \beta, n)$. This is in a form that we can unfold later.

Theorem 5.4.5. *We have that*

$$\begin{aligned}\Theta_k(\tau, z) = &\frac{1}{2}\sum_{n\geq1} \sum_{\tilde{\gamma}\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} \sum_{j} c_{z,k,j}(-n)^{1-j} \\ &\times \left[e\left(-\frac{n^2}{4|D|i\,\mathrm{Im}(\tau)Q_z(l)}\right) \Xi_{k-1-j}(\tau, \mu_K, -n, 0)\right]\Big|_{k-3/2,\tilde{\rho}_K}\tilde{\gamma}\end{aligned}$$

and if $k \geq 2$ we also have the additional term

$$c_{z,k,1}\Xi_{k-2}(\tau, 0, 0, 0).$$

Proof. Theorem 5.4.3 told us that

$$\Theta_k(\tau,z)=\sum_{c,d\in\mathbb{Z}}\sum_j c_{z,k,j}(c\tau+d)^{1-j}e\left(-\frac{|d+c\tau|^2}{4|D|ivQ_z(l)}\right)\Xi_{k-1-j}(\tau,\mu_K,d,-c).$$

Then, remembering to include the $c=0,d=0$ term which vanishes unless $k\geq 2, j=1$, we have

$$\Theta_k(\tau,z)=c_{z,k,1}\Xi_{k-2}(\tau,0,0,0)+\sum_{n\geq 1}\sum_{\substack{c,d\in\mathbb{Z}\\(c,d)=1}}\sum_j c_{z,k,j}(-n)^{1-j}(c\tau+d)^{1-j}$$
$$\times e\left(-\frac{n^2|d+c\tau|^2}{4|D|ivQ_z(l)}\right)\Xi_{k-1-j}(\tau,\mu_K,-nd,nc).$$

We know two elements $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\left(\begin{smallmatrix}a'&b'\\c'&d'\end{smallmatrix}\right)\in\Gamma$ are equal in $\Gamma_\infty\backslash\Gamma$ if and only if $c=c'$ and $d=d'$, see for example [Sad12, Lemma 12]. We now rewrite the sum over coprime integers as a sum over $\tilde\gamma=(\gamma,\phi_\gamma)\in\tilde\Gamma_\infty\backslash\tilde\Gamma$ where $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$. This also introduces a factor of $1/2$ due to the two possibilities (γ,ϕ_γ) and $(\gamma,-\phi_\gamma)$. We note that $\phi_\gamma(\tau)^2=c\tau+d$ and

$$\Xi_{k-1-j}(\tau,\mu_K,-nd,nc)=\phi_\gamma(\tau)^{1-2k+2j}\tilde\rho_K^{-1}(\gamma,\phi_\gamma)\Xi_{k-1-j}(\gamma\tau,\mu_K,-n,0).$$

We also recall from Lemma 3.6.10 that $\mathrm{Im}(\gamma\tau)=\mathrm{Im}(\tau)/j(\gamma,\tau)\overline{j(\gamma,\tau)}$. We then finally obtain

$$\Theta_k(\tau,z)=c_{z,k,1}\Xi_{k-2}(\tau,0,0,0)+\frac{1}{2}\sum_{n\geq 1}\sum_{\tilde\gamma\in\tilde\Gamma_\infty\backslash\tilde\Gamma}\sum_j c_{z,k,j}(-n)^{1-j}$$
$$\times\left[e\left(-\frac{n^2}{4|D|i\,\mathrm{Im}(\tau)Q_z(l)}\right)\Xi_{k-1-j}(\tau,\mu_K,-n,0)\right]\Bigg|_{k-3/2,\tilde\rho_K}\tilde\gamma.\qquad\square$$

5.4.1 Asymptotics

We can also look at the asymptotic behaviour as $y\to\infty$. We first find all our terms explicitly, in the case of the cusp l_∞ (we will also need a lot these in Theorem 6.3.10 where we find the Fourier expansion at this cusp). In this case we have

$$l=l_\infty=\begin{pmatrix}0&1/N\\0&0\end{pmatrix}\quad\text{and}\quad l'=-l_0=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Then using (3.3.1) and (3.3.2) we see that

$$\begin{aligned}
l_z&=l_3(z)b_3(z)=\frac{1}{\sqrt{2N}y}b_3(z)=\frac{1}{2Ny^2}\left(\begin{smallmatrix}-x&x^2+y^2\\-1&x\end{smallmatrix}\right),\\
l_{z^\perp}&=l_1(z)b_1(z)=\frac{1}{\sqrt{2N}y}b_1(z)=\frac{1}{2Ny^2}\left(\begin{smallmatrix}x&-x^2+y^2\\1&-x\end{smallmatrix}\right),\\
\mathfrak{w}^\perp&=-\frac{1}{\sqrt{2N}y}b_2(z)=-\frac{1}{2Ny^2}\left(\begin{smallmatrix}y&-2xy\\0&-y\end{smallmatrix}\right),\\
\mu&=\left(\begin{smallmatrix}x&-x^2\\0&-x\end{smallmatrix}\right),\\
\mu_K&=\left(\begin{smallmatrix}x&0\\0&-x\end{smallmatrix}\right),
\end{aligned}$$

and then

$$\begin{aligned}
2Q(l_z) &= (l_z, l_z) = -\frac{1}{2Ny^2},\\
2Q(l_{z^\perp}) &= (l_{z^\perp}, l_{z^\perp}) = Q_z(l) = \frac{1}{2Ny^2},\\
q_z(l) &= y(l_1(z)) = \frac{1}{\sqrt{2N}},\\
c_{z,k,j} &= \frac{iNy^2}{\sqrt{2|D|}}\left(\frac{iy\sqrt{|D|}}{2\sqrt{2\pi}}\right)^{k-1}\left(\frac{(1-k)\sqrt{|D|}}{y\sqrt{N\pi}}\right)^{j}.
\end{aligned}$$

We also clearly have

$$\begin{aligned}
K &= L\cap l^\perp \cap l'^\perp = \mathbb{Z}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\
K' &= L'\cap l^\perp \cap l'^\perp = \frac{1}{2N}\mathbb{Z}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.
\end{aligned}$$

We say $\lambda > 0$ for $\lambda \in K\otimes\mathbb{R}$ if $\lambda = C\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ for some $C > 0$. We also, as is standard (see [Bor98, Section 13] [BO10, Section 5]), associate the upper half plane with an open subset of $K\otimes\mathbb{C}$ by mapping $z' \in \mathbb{H}$ to $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\otimes z'$. Using this identification we then clearly have for $\lambda \in K\otimes\mathbb{R}$ that

$$(\lambda, \mu_K) = (\lambda, x) \quad\text{and}\quad (\lambda, \mathfrak{w}^\perp) = -\frac{1}{2Ny^2}(\lambda, y).$$

Finally we can also identify K' with the lattice $\frac{1}{2N}\mathbb{Z}$, by letting an element $\frac{m}{2N}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) \in K'$ be identified with $\frac{m}{2N}$ for $m \in \mathbb{Z}$. This means we can switch from a sum over $\lambda \in K'$ to a sum over $m \in \mathbb{Z}$. In which case

$$Q(\lambda) = \frac{m^2}{4N} \quad\text{and}\quad (\lambda, z') = mz'.$$

for $\lambda \in K'$ and $z' \in \mathbb{H}$. In the next lemma we will let $c_{k,1} = c_{z,k,1}y^{-k}$. We have already seen that the theta kernel decays exponentially as $v \to \infty$. We now look at the growth in the other variable, i.e. as $y \to \infty$.

We will consider other cusps of $\Gamma_0(N)$, and not just l_∞. When N is square-free we can use the Atkin-Lehner involutions to adapt the l_∞ case. Section 3.4 told us that for any $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}\cup\{\infty\}$ there exists a W_m^N such that $W_m^N\infty = s$, where m is an exact divisor of N. For any function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ with components f_h we will denote

$$f_{W_m^N} := \sum_{h\in L'/L} f_{W_m^N.h}\mathfrak{e}_h.$$

We remember $(W_m^N)^2 = 1 \pmod{\Gamma_0(N)}$ and can write $f_{W_m^N} = \sum_{h\in L'/L} f_h\mathfrak{e}_{W_m^N.h}$.

Proposition 5.4.6. *Let N be square-free. Let $s \in \mathbb{P}^1(\mathbb{Q})$ be a cusp for $\Gamma_0(N)$ and W_m^N the Atkin-Lehner involution such that $W_m^N\infty = s$. Then there is a constant $C > 0$ such that as $y \to \infty$ we have*

$$\left(\Theta_k(\tau)\big|_{2-2k} W_m^N\right)(z) = y^k c_{k,1}\Xi_{k-2,W_m^N}(\tau,0,0,0) + \mathcal{O}(e^{-Cy^2})$$

unless $k = 1$, in which case

$$\left(\Theta_1(\tau)\big|_{2-2k} W_m^N\right)(z) = \mathcal{O}(e^{-Cy^2}).$$

Proof. Using (5.4.3) we see the term

$$e\left(-\frac{Ny^2|d+c\tau|^2}{2|D|iv}\right)$$

means that $\theta_{h,k}(\tau,z)$ decays exponentially as $y \to \infty$ (uniformly in x) except the case when $c = 0, d = 0$. In the case $c = 0, d = 0$ we observe $\theta_{h,k}(\tau,z)$ simply vanishes unless $k \geq 2$ and $j = 1$. In the remaining cases Theorem 5.4.3 tells us we have

$$c_{z,k,1}\xi_{k-2,h}(\tau,0,0,0)$$

left to consider. We can check using the explicit terms given earlier that $\xi_{k-2,h}(\tau,0,0,0)$ does not depend on y and so

$$\Theta_k(\tau,z) = c_{k,1}y^k\Xi_{k-2}(\tau,0,0,0) + \mathcal{O}(e^{-Cy^2})$$

as $y \to \infty$.

We now let $s \in \mathbb{P}^1(\mathbb{Q})$ be any cusp of $\Gamma_0(N)$. Following our earlier discussion, we know $W_m^N\infty = s$ for some m an exact divisor of N. Then the Fourier expansion of $\Theta_k(\tau,z)$ at the cusp s is given by the Fourier expansion of

$$\left(\Theta_k(\tau)\big|_{2-2k} W_m^N\right)(z) = j(W_m^N,z)^{2k-2}\Theta_k(\tau, W_m^N.z)$$

at the cusp ∞ (see for example [DS05, Section 1.2]). Using Proposition 3.6.12 we then see

$$j(W_m^N,z)^{2k-2}\Theta_k(\tau,W_m^N.z) = \sum_{h\in L'/L}\theta_{W_m^N.h,k}(\tau,z)\mathfrak{e}_h.$$

So it is then clear

$$\left(\Theta_k(\tau)\big|_{2-2k} W_m^N\right)(z) = c_{k,1}y^k\Xi_{k-2,W_m^N}(\tau,0,0,0) + \mathcal{O}(e^{-Cy^2})$$

as $y \to \infty$. $\square$

Chapter 6

The Fourier Expansion

This part is dedicated to investigating the Fourier expansion of our lift $\Phi_{D,r,k}(z, f)$ that we defined in Chapter 4. Before we get to the main result, Theorem 6.3.10, we first need some groundwork that is useful in its own right. We discuss an evaluation of a pairing adapted from Definition 2.5.21 that allows us to find the integral over the fundamental domain of a harmonic weak Maass form against a modular form. We use this to find the expansion associated to the additional term which appears in Theorem 5.4.5 (when $k \geq 2$). We also solve some tricky integrals that will crop up in our proof. Finally using our Fourier expansion at the cusp l_∞ we can consider the expansion at other cusps and the asymptotic behaviour showing that we have obtained a locally harmonic weak Maass form as in Definition 2.5.30.

The key references, where similar Fourier expansions are computed are [Bor98, BO10, Bru02].

6.1 The Additional Term

In this section the aim is to evaluate the integral

$$\int_{\tau \in \mathcal{F}}^{\text{reg}} \left\langle f(\tau), \overline{\Xi_{k-2}(\tau, 0, 0, 0)} \right\rangle \frac{dudv}{v^2}.$$

This integral will arise in the proof of Theorem 6.3.10 and it originates from the additional term that appears in Theorem 5.4.5 (when $k \geq 2$). Analogous integrals are considered in [Bor98, Section 9].

We first quickly state another convenient version of Lemma 4.3.5 which we will use here. We also correct a sign error and note this result is stated for any even lattice and any rational non-degenerate quadratic space (V, Q) of signature (b^+, b^-) as in Section 2.5.1.

Lemma 6.1.1 ([Bru02, Lemma 4.2])**.** *Let $\kappa \in \frac{1}{2}\mathbb{Z}$ and $f, g \in A_{\kappa,\rho_L}$ be smooth functions. Then*

$$\int_{\mathcal{F}_t} \langle f, L_{\kappa+2}g\rangle v^{\kappa-2} dudv + \int_{\mathcal{F}_t} \langle R_\kappa f, g\rangle v^\kappa dudv = \int_{-1/2}^{1/2} \left[\langle f, g\rangle v^\kappa\right]_{v=t} du.$$

This next proposition is a useful manipulation of 2.5.21 that gives us an explicit way to find the integral of any harmonic weak Maass form $f \in H_{\kappa,\rho_L-}$ against a modular form $g \in M_{-\kappa,\rho_L}$ (the Petersson scalar product).

Proposition 6.1.2. *For $f \in H_{\kappa,\rho_L-}$ and $g \in M_{-\kappa,\rho_L}$ with Fourier expansions as in* (2.5.2) *and* (2.5.1)*, $\kappa \in \frac{1}{2}\mathbb{Z}$ and extending our definition of 2.5.21 ($R_{-\kappa}(g)$ is not necessarily an element of $M_{2-\kappa,\rho_L}$) we have*

$$\{R_{-\kappa}(g), f\} \coloneqq (R_{-\kappa}(g), \xi_\kappa(f))^{\mathrm{reg}}_{2-\kappa,\rho_L} = 0$$

and

$$\int^{\mathrm{reg}}_{\tau\in\mathcal{F}} \left\langle f(\tau), \overline{g(\tau)}\right\rangle \frac{dudv}{v^2} = \frac{1}{\kappa} \sum_{h\in L'/L} \sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n\le 0}} c^+(n,h)a(-n,h)(4\pi n).$$

Proof. For the first statement we see that

$$\begin{aligned}\{R_{-\kappa}(g), f\} &= (R_{-\kappa}(g), \xi_\kappa(f))^{\mathrm{reg}}_{2-\kappa,L}\\ &= \lim_{t\to\infty} \int_{\tau\in\mathcal{F}_t} \langle R_{-\kappa}(g), \xi_\kappa(f)\rangle v^{-\kappa} dudv.\end{aligned}$$

Then using Lemma 6.1.1 we have

$$\begin{aligned}&= -\lim_{t\to\infty} \int_{\tau\in\mathcal{F}_t} \langle g, L_{2-\kappa}\xi_\kappa(f)\rangle v^{-\kappa} \frac{dudv}{v^2}\\ &\quad + \lim_{t\to\infty} \int_{-1/2}^{1/2} \left[\langle g, \xi_\kappa(f)\rangle v^{-\kappa}\right]_{v=t} du.\end{aligned}$$

However we then note the second term (the integral over the boundary) disappears as g is a modular form and $\xi_\kappa(f)$ is an exponentially decaying cusp form as $t \to \infty$. The first term also disappears as we remember f is harmonic and using Definition 2.5.15 then

$$L_{2-\kappa}\xi_\kappa(f) = v^\kappa \overline{\xi_{2-\kappa}\xi_\kappa(f)} = -v^\kappa \overline{\Delta_\kappa(f)} = 0.$$

To show the second part of the proposition we follow similar lines but when using 6.1.1 we move to the other side.

$$\begin{aligned}\{R_{-\kappa}(g), f\} &= (R_{-\kappa}(g), \xi_\kappa(f))^{\mathrm{reg}}_{2-\kappa,L}\\ &= \lim_{t\to\infty} \int_{\tau\in\mathcal{F}_t} \langle R_{-\kappa}(g), \xi_\kappa(f)\rangle v^{-\kappa} dudv\\ &= \lim_{t\to\infty} \overline{\int_{\tau\in\mathcal{F}_t} \left\langle R_{-\kappa}(v^\kappa \overline{f}), R_{-\kappa}(g)\right\rangle v^{-\kappa} dudv}.\end{aligned}$$

So using Lemma 6.1.1 this time we obtain

$$= -\lim_{t\to\infty}\overline{\int_{\tau\in\mathcal{F}_t}\left\langle v^\kappa\overline{f}, L_{2-\kappa}R_{-\kappa}(g)\right\rangle v^{-\kappa}\frac{dudv}{v^2}} + \lim_{t\to\infty}\overline{\int_{-1/2}^{1/2}\left[\left\langle\overline{f}, R_{-\kappa}(g)\right\rangle\right]_{v=t}du}$$
$$= -\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle L_{2-\kappa}R_{-\kappa}(g), \overline{f}\right\rangle\frac{dudv}{v^2} + \lim_{t\to\infty}\int_{-1/2}^{1/2}\left[\left\langle f, \overline{R_{-\kappa}(g)}\right\rangle\right]_{v=t}du.$$

Then crucially as g is harmonic (it is holomorphic) we know (Definition 2.5.15) that $-\Delta_{-\kappa}(g)$ $= L_{2-\kappa}R_{-\kappa}(g) - \kappa(g) = 0$ so

$$L_{2-\kappa}R_{-\kappa}(g) = \kappa(g).$$

So this combined with the first part gives

$$\int_{\tau\in\mathcal{F}}^{\mathrm{reg}}\left\langle g, \overline{f}\right\rangle\frac{dudv}{v^2} = \frac{1}{\kappa}\lim_{t\to\infty}\int_{-1/2}^{1/2}\left\langle f(u+it), \overline{R_{-\kappa}(g(u+it))}\right\rangle du. \tag{6.1.1}$$

We then note using the explicit Fourier expansion (2.5.1) that for one component then

$$R_{-\kappa}(g(\tau)) = \sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n\geq 0}}\left(-4\pi n - \frac{\kappa}{v}\right)a(n,h)e(n\tau)\mathfrak{e}_h$$

and so remembering the integral over u picks out the 0-th Fourier coefficient we obtain our stated result:

$$\begin{aligned}&\frac{1}{\kappa}\lim_{t\to\infty}\int_{-1/2}^{1/2}\left\langle f(u+it), \overline{R_{-\kappa}(g(u+it))}\right\rangle du\\ =&\frac{1}{\kappa}\lim_{t\to\infty}\sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n\leq 0}}c^+(n,h)\left(4\pi n - \frac{\kappa}{t}\right)a(-n,h) + \mathcal{O}(e^{-\epsilon t})\\ =&\frac{1}{\kappa}\sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+Q(h)\\ n\leq 0}}c^+(n,h)a(-n,h)(4\pi n)\end{aligned}$$

for some $\epsilon > 0$. □

We would hope to use Proposition 6.1.2 in the case when $g = \Xi(\tau,0,0,0,k-2)$. However we remember this function is in general neither harmonic nor holomorphic. We need a modification of Proposition 6.1.2. For the specific case of $\Xi(\tau,0,0,0,k-2)$, we can get round these problems. Specifically we remember from Section 5.3.1 that $\Xi(\tau,0,0,0,k-2)$ still has the same asymptotic growth as a modular form and although it is not harmonic when we apply the raising and lowering operators we still get back $\Xi(\tau,0,0,0,k-2)$ up to some constants, (see Lemma 5.3.9).

Lemma 6.1.3. *Let $f \in H_{3/2-k,\overline{\rho}}$ with Fourier expansion as in* (6.3.3) *and let $k \geq 2, D = 1$. Then*

$$\int_{\tau\in\mathcal{F}}^{\mathrm{reg}}\left\langle f(\tau), \overline{\Xi_{k-2}(\tau,0,0,0)}\right\rangle\frac{dudv}{v^2}$$
$$= \frac{1}{k(k-1)}\sum_{h\in K'/K}\sum_{\lambda\in K+h}c^+(-Q(\lambda),h)\left(\frac{-2\sqrt{2\pi}(\lambda,\mathfrak{w}^\perp)}{\sqrt{Q_z(l)}}\right)^k.$$

Proof. The first part of Proposition 6.1.2 still holds in the case that $f \in H_{3/2-k,\overline{\rho}}$ and $g = \Xi_{k-2}(\tau,0,0,0)$ i.e.

$$(R_{k-3/2}(g), \xi_{3/2-k}(f))^{\mathrm{reg}}_{1/2-k,L} = 0.$$

We see this is true because the arguments still hold the same in this case up to the point where we check the integral over the boundary vanishes. Then g is no longer a modular form. However we know from Lemma 5.3.8 that g has the same growth behaviour as $v \to \infty$ so this term does indeed still vanish.

Now we can adapt the second part of Proposition 6.1.2 and obtain that

$$\int^{\mathrm{reg}}_{\tau\in\mathcal{F}} \left\langle f(\tau), \overline{g(\tau)} \right\rangle \frac{dudv}{v^2} = -\frac{4}{k(k-1)} \lim_{t\to\infty} \int_{-1/2}^{1/2} \left\langle f(u+it), \overline{R_{k-3/2}(g(u+it))} \right\rangle du.$$

This is clear from (6.1.1) and remembering from Lemma 5.3.9 that if $g = \Xi_{k-2}(\tau,0,0,0)$ then $L_{k+1/2}R_{k-3/2}(g) = -\frac{k(k-1)}{4}(g)$. We also know from Lemma 5.3.9 that

$$\begin{aligned} R_{k-3/2}(\xi_{k-2,h}(\tau,0,0,0)) &= -\frac{1}{4}\xi_{k,h}(\tau,0,0,0) \\ &= -\frac{v^{-k/2}}{4} \sum_{\lambda\in K+rh} H_k\left(\frac{-\sqrt{2\pi v}(\lambda,\mathfrak{w}^{\perp})}{\sqrt{Q_z(l)}}\right) e(Q(\lambda)\tau). \end{aligned}$$

The integral over u picks out the 0-th Fourier coefficient and $f^{-}(u+it)$ decays exponentially as $t \to \infty$ so it remains to consider

$$\begin{aligned} &-\frac{4}{k(k-1)} \lim_{t\to\infty} \int_{-1/2}^{1/2} \left\langle f^{+}(u+it), \overline{R_{k-3/2}(g(u+it))} \right\rangle du \\ &= \frac{1}{k(k-1)} \lim_{t\to\infty} \sum_{h\in K'/K} \sum_{\lambda\in K+h} c(-Q(\lambda),h,t) t^{-k/2} H_k\left(\frac{-\sqrt{2\pi t}(\lambda,\mathfrak{w}^{\perp})}{\sqrt{Q_z(l)}}\right). \end{aligned}$$

Here we only need to consider the finite terms $c^{+}(-Q(\lambda),h)$ as we remember K is a positive definite lattice, so $-Q(\lambda) \leq 0$. We then use part d of Lemma 5.2.5 to obtain

$$\begin{aligned} &= \frac{1}{k(k-1)} \lim_{t\to\infty} \sum_{h\in K'/K} \sum_{\lambda\in K+h} c^{+}(-Q(\lambda),h) t^{-k/2} k! \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{(-1)^n}{n!(k-2n)!} \left(\frac{-2\sqrt{2\pi t}(\lambda,\mathfrak{w}^{\perp})}{\sqrt{Q_z(l)}}\right)^{k-2n} \\ &= \frac{1}{k(k-1)} \sum_{h\in K'/K} \sum_{\lambda\in K+h} c^{+}(-Q(\lambda),h) \left(\frac{-2\sqrt{2\pi}(\lambda,\mathfrak{w}^{\perp})}{\sqrt{Q_z(l)}}\right)^{k}. \qquad \square \end{aligned}$$

6.2 Integrals

In this section we solve and simplify some difficult integrals. The integrals in Lemma 6.2.2 and Lemma 6.2.4 will be critical to the proof of Theorem 6.3.10. The first of these lemmas (Lemma 6.2.1) is already known. Specifically it can be found in even and odd cases in a complicated hypergeometric form in [GR15, (7.376.2), (7.376.3)]. We give a simpler exposition.

Lemma 6.2.1. *Let $n \in \mathbb{Z}, n \geq 0$ and let $r \in \mathbb{Z}, r \geq n$. Then*

$$\int_{t=0}^{\infty} t^r H_n(t) e^{-t^2} dt = \frac{r!}{2(r-n)!} \Gamma\left(\frac{r-n+1}{2}\right).$$

Proof. We first notice that (for $n \geq 1$)

$$\frac{d\left(H_n(t) e^{-t^2}\right)}{dt} = \left(2H_{n-1}(t) - 2tH_n(t)\right) e^{-t^2} = -H_{n+1}(t)e^{-t^2}.$$

Then using integration by parts

$$\begin{aligned}\int_{t=0}^{\infty} t^r H_n(t) e^{-t^2} dt &= -\left[t^r H_{n-1}(t) e^{-t^2}\right]_0^{\infty} + r\int_{t=0}^{\infty} t^{r-1} H_{n-1}(t) e^{-t^2} dt \\ &= r\int_{t=0}^{\infty} t^{r-1} H_{n-1}(t) e^{-t^2} dt.\end{aligned}$$

Repeating this n times (remembering $H_0(t) = 1$) we obtain

$$\frac{r!}{(r-n)!}\int_{t=0}^{\infty} t^{r-n} e^{-t^2} dt = \frac{r!}{2(r-n)!}\int_{t=0}^{\infty} t^{(r-n-1)/2} e^{-t} dt = \frac{r!}{2(r-n)!}\Gamma\left(\frac{r-n+1}{2}\right)$$

where we have used Definition 2.5.7. □

Lemma 6.2.2. *Let $\kappa \in \mathbb{Z}, \kappa \geq 0$ and let $\alpha, \beta \in \mathbb{R}, \alpha > 0$. Then*

$$\int_{v=0}^{\infty} \sum_j v^{-\kappa/2} H_{\kappa-j}\left(-\frac{\alpha}{\sqrt{v}} + \beta\sqrt{v}\right)\left(\frac{\kappa\sqrt{v}}{\alpha}\right)^j e^{-\alpha^2/v}\frac{dv}{v^2} = \frac{e^{-2\alpha\beta}\Gamma(\kappa+1, -2\alpha\beta)}{(-\alpha)^{\kappa+2}}.$$

Proof. Making the substitution $v = \alpha^2/t^2$ we obtain

$$\frac{2}{(-\alpha)^{\kappa+2}}\int_{t=0}^{\infty}\sum_j t^{\kappa+1} H_{\kappa-j}\left(t - \frac{\alpha\beta}{t}\right)\left(-\frac{\kappa}{t}\right)^j e^{-t^2} dt$$

and then using part e of Lemma 5.2.1

$$= \frac{2(2\beta)^{\kappa}}{\alpha^2}\sum_j\sum_{n=0}^{\kappa-j}\binom{\kappa-j}{n}\left(\frac{\kappa}{2\alpha\beta}\right)^j(-2\alpha\beta)^{-n}\int_{t=0}^{\infty} t^{n+1} H_n(t) e^{-t^2} dt.$$

Applying Lemma 6.2.1 for $r = n+1$

$$= \frac{2(2\beta)^{\kappa}}{2\alpha^2}\sum_j\sum_{n=0}^{\kappa-j}\binom{\kappa-j}{n}\left(\frac{\kappa}{2\alpha\beta}\right)^j(-2\alpha\beta)^{-n}(n+1)!$$

and considering this sum in the two cases of $j = 0, 1$

$$\begin{aligned}&= \frac{(2\beta)^{\kappa}}{\alpha^2}\left[\sum_{n=0}^{\kappa}\frac{(n+1)\kappa!}{(\kappa-n)!}(-2\alpha\beta)^{-n} - \sum_{n=1}^{\kappa}\frac{n\kappa!}{(\kappa-n)!}(-2\alpha\beta)^{-n}\right] \\ &= \frac{(2\beta)^{\kappa}\kappa!}{\alpha^2}\sum_{n=0}^{\kappa}\frac{(-2\alpha\beta)^{-n}}{(\kappa-n)!} \\ &= (-\alpha)^{-\kappa-2}\kappa!\sum_{n=0}^{\kappa}\frac{(-2\alpha\beta)^n}{n!}.\end{aligned}$$

Finally using [JD08, Section 11.1.9] we know that

$$\Gamma(\kappa+1, -2\alpha\beta) = e^{2\alpha\beta}\kappa!\sum_{n=0}^{\kappa}\frac{(-2\alpha\beta)^n}{n!} \tag{6.2.1}$$

so we have the stated result. □

The next lemma will make use of the modified K-Bessel function of the second kind, as defined in [Ste84, Section 9.6] or [EMOT54, Appendix]. We will denote these as $K_t(\cdot)$.

Lemma 6.2.3. *Let $\kappa \in \mathbb{Z}, \kappa \geq 0$ with $\alpha, \beta \in \mathbb{R}, \alpha, \beta \neq 0$ and let $n \in \mathbb{Z}, \kappa \geq n \geq 0$. Then*

$$\int_{v=0}^{\infty} \Gamma(\kappa + 1/2, \beta^2 v) v^{-n} e^{-\alpha^2/v} \frac{dv}{v^2}$$
$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!} \frac{2|\beta|^{2\kappa+1}}{|\alpha|^{2r+2}} \left(\frac{|\alpha|}{|\beta|}\right)^{\kappa+1/2-n+r} K_{\kappa+1/2-n+r}\left(2|\alpha\beta|\right).$$

Proof. In the case when $n = 0$ we find

$$\int_{v=0}^{\infty} \Gamma(\kappa + 1/2, \beta^2 v) e^{-\alpha^2/v} \frac{dv}{v^2} = \alpha^{-2} |\beta|^{2\kappa+1} \int_{v=0}^{\infty} v^{\kappa-1/2} e^{-\beta^2 v - \alpha^2/v} dv$$

with a simple integration by parts (see [Bru02, Proposition 3.1]). Then we have the following identity [EMOT54, 6.3.17]

$$\int_{v=0}^{\infty} e^{(-\beta^2 v - \alpha^2/v)} v^{t-1} dv = 2 \left(\frac{|\alpha|}{|\beta|}\right)^t K_t(2|\alpha\beta|).$$

So we have

$$\int_{v=0}^{\infty} \Gamma(\kappa + 1/2, \beta^2 v) e^{-\alpha^2/v} \frac{dv}{v^2} = \frac{2|\beta|^{2\kappa+1}}{\alpha^2} \left(\frac{|\alpha|}{|\beta|}\right)^{\kappa+1/2} K_{\kappa+1/2}(2|\alpha\beta|).$$

Now using integration by parts and remembering

$$\int_{v=0}^{\infty} e^{-\alpha^2/v} \frac{dv}{v^2} = \left[\frac{e^{-\alpha^2/v}}{\alpha^2}\right]_{v=0}^{\infty}$$

we see that

$$\int_{v=0}^{\infty} \Gamma(\kappa + 1/2, \beta^2 v) v^{-n} e^{-\alpha^2/v} \frac{dv}{v^2}$$
$$= \left[\Gamma(\kappa + 1/2, \beta^2 v) v^{-n} \frac{e^{-\alpha^2/v}}{\alpha^2}\right]_{v=0}^{\infty} - \int_{v=0}^{\infty} \frac{d}{dv} \left(\Gamma(\kappa + 1/2, \beta^2 v) v^{-n}\right) \frac{e^{-\alpha^2/v}}{\alpha^2} dv.$$

The first term vanishes and we know $\frac{\partial \Gamma(s,x)}{\partial x} = -x^{s-1} e^{-x}$ [Ste84, 6.5.25] so continuing

$$= \int_{v=0}^{\infty} \Gamma(\kappa + 1/2, \beta^2 v) n v^{-n+1} \frac{e^{-\alpha^2/v}}{\alpha^2} \frac{dv}{v^2} - \int_{v=0}^{\infty} (-\beta^2 e^{-\beta^2 v} (\beta^2 v)^{\kappa-1/2}) v^{-n} \frac{e^{-\alpha^2/v}}{\alpha} dv$$
$$= n \int_{v=0}^{\infty} \Gamma(\kappa + 1/2, \beta^2 v) v^{-n+1} \frac{e^{-\alpha^2/v}}{\alpha^2} \frac{dv}{v^2} + \frac{2|\beta|^{2\kappa+1}}{\alpha^2} \left(\frac{|\alpha|}{|\beta|}\right)^{(\kappa+1/2-n)} K_{\kappa+1/2-n}\left(2|\alpha\beta|\right)$$

and repeated application of this reveals

$$= \sum_{r=0}^{n} \frac{(n)!}{(n-r)!} \frac{2|\beta|^{2\kappa+1}}{|\alpha|^{2r+2}} \left(\frac{|\alpha|}{|\beta|}\right)^{(\kappa+1/2-n+r)} K_{\kappa+1/2-n+r}\left(2|\alpha\beta|\right). \qquad \square$$

Lemma 6.2.4. *Let $\kappa \in \mathbb{Z}, \kappa \geq 0$ and let $\alpha, \beta \in \mathbb{R}, \alpha > 0, \beta \neq 0$. Then*

$$\int_{v=0}^{\infty} \sum_{j} v^{-\kappa/2} H_{\kappa-j}\left(-\frac{\alpha}{\sqrt{v}} + \beta\sqrt{v}\right)\left(\frac{\kappa\sqrt{v}}{\alpha}\right)^{j} e^{-\alpha^2/v}\Gamma\left(\kappa+1/2, \beta^2 v\right)\frac{dv}{v^2}$$
$$= \begin{cases} (-1)^{\kappa}\dfrac{(2\kappa)!\sqrt{\pi}}{4^{\kappa}\alpha^{\kappa+2}}e^{-2\alpha\beta} & \text{if } \beta > 0, \\ (-1)^{\kappa}\dfrac{\sqrt{\pi}}{4^{\kappa}\alpha^{\kappa+2}}e^{-2\alpha\beta}\Gamma(2\kappa+1, -4\alpha\beta) & \text{if } \beta < 0. \end{cases}$$

Proof. We will use the fact that our Fourier expansion is harmonic (and therefore its coefficients satisfy a certain differential equation) to help us here. This will rely on knowledge of our Fourier expansion that we only show in the next section but to simplify the exposition in Theorem 6.3.10 we will evaluate the integral here. We reformulate the stated integral as

$$I(y) := y^{k+1}\int_{v=0}^{\infty}\sum_{j} v^{-(k-1)/2}\left(\frac{(k-1)\sqrt{v}}{ay}\right)^{j}$$
$$\times H_{k-1-j}\left(-\frac{ay}{\sqrt{v}} + \beta\sqrt{v}\right)e^{-(ay)^2/v}\Gamma\left(k-1/2, \beta^2 v\right)\frac{dv}{v^2}$$

where $\kappa = k-1$, $\alpha = ay$ $(a, y > 0)$ and we have multiplied by y^{k+1}. We have written it in this way so that it exactly matches the Fourier expansion form which we will obtain in (6.3.16) after unfolding and simplifying (associating β with $\operatorname{sgn}(D)m\tilde{\beta}$). We observe that a, β do not depend on x or y. This is useful as we know that our lift and therefore (6.3.16) must vanish under the Laplacian operator $\Delta_{2-2k,z}$ using Theorem 4.3.7. In particular, gathering all the terms dependent on x and y, we see that $I(y)$ must satisfy the following differential equation

$$\Delta_{2-2k,z}\left(I(y)e^{2a\beta i x}\right) = 0.$$

Applying the Laplacian we then have

$$yI''(y) + (2-2k)I'(y) + \left((2-2k)2a\beta - 4a^2\beta^2 y\right)I(y) = 0.$$

We remember $a\beta \neq 0$ and so easily check that $e^{-2a\beta y}$ is a solution, as is $e^{-2aby}\Gamma(2k-1, -4aby)$. This means we have a solution of the form

$$e^{-2a\beta y}\left(c_1(a,\beta,k) + c_2(a,\beta,k)\Gamma(2k-1, -4a\beta y)\right). \tag{6.2.2}$$

We now determine the two constants. This will briefly involve some long messy sums but we have failed to spot a nicer approach. Using Lemma 5.2.5 parts d and e we expand out the

Hermite polynomial terms of $I(y)$ to obtain

$$\begin{aligned}
&\sum_j \left(\frac{k-1}{a}\right)^j \sum_{c=0}^{k-1-j} \binom{k-1-j}{c} (2\beta)^{k-1-j-c} \\
&\times y^{k+1-j} \int_{v=0}^{\infty} v^{-c/2} H_c\left(-\frac{ay}{\sqrt{v}}\right) e^{-(ay)^2/v} \Gamma\left(k-1/2, \beta^2 v\right) \frac{dv}{v^2} \\
=& \sum_j \left(\frac{k-1}{a}\right)^j \sum_{c=0}^{k-1-j} \sum_{d=0}^{\lfloor c/2 \rfloor} \binom{k-1-j}{c} \frac{c!(-1)^{d+c}(2a)^{c-2d}\,(2\beta)^{k-1-j-c}}{d!(c-2d)!} \\
&\times y^{k+1-j+c-2d} \int_{v=0}^{\infty} v^{-c+d} e^{-(ay)^2/v} \Gamma\left(k-1/2, \beta^2 v\right) \frac{dv}{v^2}.
\end{aligned}$$

We then use Lemma 6.2.3 for $\kappa = k-1, n = c-d, \alpha = ay, \beta = \beta$ to write this as a sum of K-Bessel functions:

$$\begin{aligned}
=& \sum_j \left(\frac{k-1}{a}\right)^j \sum_{c=0}^{k-1-j} \sum_{d=0}^{\lfloor c/2 \rfloor} \binom{k-1-j}{c} \frac{c!(-1)^{d+c}(2a)^{c-2d}\,(2\beta)^{k-1-j-c}}{d!(c-2d)!} \\
&\times \sum_{r=0}^{c-d} \frac{(c-d)!}{(c-d-r)!} \frac{2|\beta|^{2k-1}}{a^{2r+2}} \left(\frac{a}{|\beta|}\right)^{k-1/2-c+d+r} K_{k-1/2-c+d+r}\left(2a|\beta|y\right) y^{2k-3/2-j+d-r}.
\end{aligned}$$

At this point we consider the identity [EMOT81b, 7.2.6.40]

$$K_{\kappa+1/2}(z) = \sqrt{\pi/2z}\, e^{-z} \sum_{0 \le s \le \kappa} (2z)^{-s} \frac{(\kappa+s)!}{s!(\kappa-s)!}$$

for κ a non-negative integer. This tells us then that $I(y)$ is certainly of the form

$$\sum_{t=0}^{2k-2} c(a,\beta,t,k) y^t e^{-2a|\beta|y} \tag{6.2.3}$$

for some constants $c(a,\beta,t,k)$ which do not depend on y. We take the limit as $y \to \infty$ in (6.2.3). This then converges to 0 and comparing with (6.2.2) and remembering the asymptotic behaviour of the incomplete gamma function from Lemma 2.5.28 we see that if $\beta > 0$ then $c_2(a,\beta,k)$ must equal 0 and if $\beta < 0$ then $c_1(a,\beta,k)$ must equal 0.

It remains to determine the other constant. To do this we find the $t = 0$ term in (6.2.3) and take the limit as $y \to 0$. Some staring reveals that we have $t = 0$ in the cases when $0 \le j \le 1, c = k-1-j, 0 \le d \le \lfloor (k-1-j)/2 \rfloor, r = k-1-j-d$ and $s = k-1$. Plugging these all in we find the $t = 0$ term is

$$\frac{(2k-2)!\sqrt{\pi}}{a^2} \left(\frac{-1}{2a}\right)^{k-1} e^{-2a|\beta|y} \sum_j (-2)^{-j} \sum_{d=0}^{\lfloor (k-1-j)/2 \rfloor} \binom{k-1-j-d}{d} (-4)^{-d}.$$

We then use the following identity which we take from [Gou72, 1.72]

$$\sum_{d=0}^{\lfloor \kappa/2 \rfloor} (-1)^d \binom{\kappa-d}{d} 2^{\kappa-2d} = \kappa+1$$

to see that the $t = 0$ term is

$$\begin{aligned}
&\frac{(2k-2)!\sqrt{\pi}}{2^{2k-2}a^{k+1}} (-1)^{k-1} e^{-2a|\beta|y} \sum_j (-1)^j (k-j) \\
= \; &(-1)^{k-1} \frac{(2k-2)!\sqrt{\pi}}{4^{k-1}a^{k+1}} e^{-2a|\beta|y}.
\end{aligned}$$

We consider (6.2.3) and (6.2.2) as we take the limit, $y \to 0$. We see that if $\beta > 0$, then

$$c_1(a,\beta,k) = (-1)^{k-1} \frac{(2k-2)!\sqrt{\pi}}{4^{k-1}a^{k+1}}$$

and if $\beta < 0$, then

$$c_2(a,\beta,k) = (-1)^{k-1} \frac{(2k-2)!\sqrt{\pi}}{2^{2k-2}a^{k+1}} \frac{1}{\Gamma(2k-1,0)} = (-1)^{k-1} \frac{\sqrt{\pi}}{4^{k-1}a^{k+1}}.$$

Using (6.2.2) we then know $I(y)$ and we have the stated result (remembering to divide by y^{k+1}). □

6.3 The Fourier Expansion

6.3.1 Objects

In our expansion a few mathematical objects will show up. We summarise these here.

Definition 6.3.1. *Let $x \in \mathbb{R}$ and let $A \subset \mathbb{R}$. Then we define our* ***indicator function*** *as*

$$\mathrm{I}_A(x) := \begin{cases} \frac{1}{2} & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 6.3.2. *Let $x \in \mathbb{R}, n \in \mathbb{Z}, n \geq 0$. Then the nth* ***periodic Bernoulli polynomial*** *is defined as*

$$\mathbb{B}_n(x) := -n! \sum_{m \neq 0} \frac{e(mx)}{(2\pi i m)^n}$$

and setting $\mathbb{B}_0(x) = 1$.

Lemma 6.3.3 ([EMOT81a, Section 1.13], [Bor98, Section 10])**.** *We have that*

1. $\mathbb{B}_n(-x) = (-1)^n \mathbb{B}_n(x)$,
2. $\mathbb{B}_n(x+1) = \mathbb{B}_n(x)$,
3. $\mathbb{B}_n^{'}(x) = n\mathbb{B}_{n-1}(x)$ *for* $x \notin \mathbb{Z}$ *or* $n \neq 1, 2$,
4. $\mathbb{B}_1(x) = x - 1/2$ *for* $0 < x < 1$ *and* $\mathbb{B}_1(0) = 0$.

Remark 6.3.4. We see that $\mathbb{B}_1(x)$ is a discontinuous sawtooth function with singularities on $x \in \mathbb{Z}$ and the value on the singularity (i.e. 0) is the average of the limits from either side (i.e. $-1/2$ and $1/2$). These functions will encompass the vertical half-line singularities of our lift.

Definition 6.3.5. *Let $x \in \mathbb{R}, n \in \mathbb{Z}, n \geq 0$. Then the nth **Bernoulli polynomial** is defined by the generating function*

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

*The nth **Bernoulli number** is defined as $B_n \coloneqq B_n(0)$.*

Example 6.3.6. The first four Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - 1/2, \quad B_2(x) = x^2 - x + 1/6, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

We denote the **fractional part** of $x \in \mathbb{R}$ as $\langle x \rangle \coloneqq x - \lfloor x \rfloor$.

Lemma 6.3.7 ([EMOT81a, Section 1.13], [Bor98, Section 10], [GR15, Section 9.62])**.** *We have that*

1. $B_n(1-x) = (-1)^n B_n(x)$,
2. $B_n(x+1) = B_n(x) + nx^{n-1}$,
3. $B_n'(x) = nB_{n-1}(x)$,
4. $B_n(1) = B_n(0) = B_n$ *for* $n \neq 1$,
5. $B_n(x+y) = \sum_{\kappa=0}^{n} \binom{n}{\kappa} B_\kappa(x) y^{n-\kappa}$,
6. $B_n(\langle x \rangle) = \mathbb{B}_n(x)$ *for* $n \neq 1$,
7. $B_1(\langle x \rangle) + \mathrm{I}_{\mathbb{Z}}(x) = \mathbb{B}_1(x)$.

We notice using property 4 that $\mathbb{B}_n(x)$ are continuous functions for $n \neq 1$. The constant term of our expansion in Theorem 6.3.10 will invole the following function:

Definition 6.3.8. *For $s \in \mathbb{C}, \mathrm{Re}(s) > 1$ we let the **Dirichlet L-function** associated to the quadratic character $\left(\frac{D}{\cdot}\right)$ be defined as*

$$L\left(s, \left(\frac{D}{\cdot}\right)\right) \coloneqq \sum_{n \geq 1} \left(\frac{D}{n}\right) \frac{1}{n^s},$$

which can be meromorphically continued to all $s \in \mathbb{C}$.

We recall that we have already defined polylogarithms in Definition 5.3.6. To help make our expansion even more compact we also introduce the following functions.

Definition 6.3.9. *Let $\kappa \in \mathbb{Z}, b \in \mathbb{R}, b > 0$ and let $z \in \mathbb{C}, |z| < 1$. Then the **shifted incomplete polylogarithm** is defined as*

$$\mathrm{Li}_{\kappa,r}(b,z) := \sum_{n=1}^{\infty} \frac{z^n}{n^{\kappa+r}} \frac{\Gamma(\kappa, nb)}{\Gamma(\kappa)},$$

*which can be analytically continued to all $z \in \mathbb{C}$. When $r = 0$ we call this the **incomplete polylogarithm**.*

We note using (6.2.1) that the shifted incomplete polylogarithm is just a finite sum of polylogarithms:

$$\mathrm{Li}_{\kappa,r}(b,z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{\kappa+r}} e^{-nb} \sum_{m=0}^{\kappa-1} \frac{nb^m}{m!} = \sum_{m=0}^{\kappa-1} \frac{b^m}{m!} \sum_{n=1}^{\infty} \frac{\left(ze^{-b}\right)^n}{n^{\kappa+r-m}} = \sum_{m=0}^{\kappa-1} \frac{b^m}{m!} \mathrm{Li}_{\kappa+r-m}\left(ze^{-b}\right).$$

We also remember from Section 5.4.1 for the cusp l_∞ we had the following identifications

$$Q_z(l) = \frac{1}{2Ny^2}, \tag{6.3.1a}$$

$$(\lambda, \mu_K) = (\lambda, x), \tag{6.3.1b}$$

$$(\lambda, \mathfrak{w}^\perp) = -\frac{1}{2Ny^2}(\lambda, y), \tag{6.3.1c}$$

$$c_{z,k,j} = \frac{iNy^2}{\sqrt{2|D|}} \left(\frac{iy\sqrt{|D|}}{2\sqrt{2\pi}}\right)^{k-1} \left(\frac{(1-k)\sqrt{|D|}}{y\sqrt{N\pi}}\right)^j, \tag{6.3.1d}$$

and denote $c_{k,j} := y^{-k-1+j} c_{z,k,j}$. Using (2.5.6) and (2.5.2) we will as ever write an element $f(\tau) \in H_{3/2-k,\bar{\rho}}$ as follows:

$$f^+ = \sum_{h \in L'/L} \sum_{\substack{m \in \mathbb{Z} - \mathrm{sgn}(D)Q(h) \\ m \geq -n_0}} c^+(m,h) e(m\tau) \mathfrak{e}_h, \tag{6.3.2a}$$

$$f^- = \sum_{h \in L'/L} \sum_{\substack{m \in \mathbb{Z} - \mathrm{sgn}(D)Q(h) \\ m < 0}} c^-(m,h) \Gamma(k-1/2, -4\pi m v) e(m\tau) \mathfrak{e}_h, \tag{6.3.2b}$$

Or alternatively

$$f(\tau) = \sum_{h \in L'/L} \sum_{m \in \mathbb{Z} - \mathrm{sgn}(D)Q(h)} c(m,h,v) e(m\tau) \mathfrak{e}_h, \tag{6.3.3}$$

where

$$c(m,h,v) = c^+(m,h) + c^-(m,h)\Gamma(k-1/2, -4\pi m v). \tag{6.3.4}$$

6.3.2 The Proof

We can now state the main result of this chapter. This Fourier expansion is at the cusp l_∞ and for other cusps of $\Gamma_0(N)$ we also have similar Fourier expansions, see Theorem 6.3.12. This is (unavoidably) a very long and technical proof (similar in nature to those found in [Bor98, Sections 7 and 14] and [Bru02, Chapters 2 and 3]), so we have divided parts of it into paragraphs to hopefully make it more manageable.

Theorem 6.3.10. *Let $f \in H_{3/2-k,\overline{\rho}}$ be a harmonic weak Maass form with expansion as given in* (6.3.2)*. If $n_0 < 0$ then $\Phi_{D,r,k}(z,f) \equiv 0$. If $n_0 \geq 0$, let $z = x+iy \in \mathbb{H}$ where $y > \sqrt{-|D|n_0/N}$. Then the Fourier expansion of $\Phi_{D,r,k}(z,f)$ is as follows:*

$$\frac{\epsilon_D|D|\sqrt{2}}{i\pi}\left(\frac{|D|}{i\pi 2\sqrt{2N}}\right)^{k-1} c^+(0,0)L\left(k,\left(\tfrac{D}{\cdot}\right)\right) \tag{6.3.5a}$$

$$-\frac{2\sqrt{2}\epsilon_D\sqrt{D}}{k}\left(\frac{D}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right) \tag{6.3.5b}$$

$$\times\left[B_k\left(\langle mx+b/D\rangle + imy\right) + \frac{k\mathrm{I}_{\mathbb{Z}}(mx+b/D)}{(imy)^{1-k}}\right] \tag{6.3.5c}$$

$$+\frac{\sqrt{2}\epsilon_D\sqrt{D}(2k-2)!}{i\sqrt{\pi}}\left(\frac{D}{8\pi i\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right) \tag{6.3.5d}$$

$$\times\left[\mathrm{Li}_k\left(e((mz+b/D))\right) + (-1)^k\mathrm{sgn}(D)\mathrm{Li}_{2k-1,1-k}\left(4\pi my, e(-(mz-b/D))\right)\right]. \tag{6.3.5e}$$

In the case $k=1$ (6.3.5c) *is replaced with $\mathbb{B}_1(mx+b/D)$. The constant term* (6.3.5a) *vanishes if k is odd and $D>0$ (or if k is even and $D<0$).*

Remarks 6.3.11. This is a very large equation, which we have failed to simplify further. We will often refer to: (6.3.5a) as the "constant term"; (6.3.5b)(6.3.5c) as the "c^+ terms"; and (6.3.5d)(6.3.5e) as the "c^- terms".

We observe during this proof that this expansion actually converges for all $z = x+iy \in \mathbb{H}$. However we will also show this form only truly represents $\Phi_{D,r,k}(z,f)$ in the unbounded Weyl chambers. In particular, this expansion only encapsulates the vertical half-line singularities, hence the restriction to the case $y > \sqrt{|D|n_0/N}$. We recall that, above that height there are no semi-circle singularities associated to f. We can see the jumps generated by the vertical half-line singularities are represented by the first periodic Bernoulli polynomial.

The Fourier expansion in the bounded Weyl chambers can be adapted from (6.3.5) with the addition of some appropriate holomorphic polynomials. We explain this in detail at the very end of this proof.

In Theorem 7.3.3 we will consider the effect of applying ξ_{2-2k} to this expansion (for $z \in \mathbb{H}\backslash Z_{D,r}(f)$). This operator kills holomorphic pieces. In particular (for $k \geq 2$) only the shifted incomplete polylogarithm term survives.

For the other cusps of $\Gamma_0(N)$ we have similar Fourier expansions, see Remark 5.1.2. We are able to make this explicit in Theorem 6.3.12 in the case of N square-free. We do this by adapting the l_∞ case with some Atkin-Lehner involutions.

Finally we consider when our lift is trivial. Remarks 4.2.3 told us that if f had non-constant principal part P_f then $\Phi_{D,r,k}(z,f)$ was a non-constant function. I.e. the lift was not trivial. However during the course of the proof we will see that (6.3.5) actually holds true for all $y>0$ if $n_0 \leq 0$. It is then immediately clear then that if $f \in S_{3/2-k,\overline{\rho}}$ (i.e. $P_f \equiv 0$, $n_0 < 0, f^- \equiv 0$) then our lift is trivial and vanishes. We also look at the case when $f \in M_{3/2-k,\overline{\rho}}$. Then P_f is a constant, $f^- \equiv 0$ and $n_0 = 0$. So we observe that our lift is still trivial. We just obtain a constant function, given by (6.3.5a).

Proof. Using Definition 4.1.1, we need to consider the integral

$$\Phi_{D,r,k}(z,f) = \int_{\tau\in\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), \overline{\Theta_{D,r,k}(\tau,z)} \right\rangle \frac{dudv}{v^2}.$$

Then using 5.4.5, we write this in terms of the Poincaré series $\Xi(\tau,\mu_K,n,0,k-1-j)$, to get

$$\Phi_{\Delta,r,k}(z,f) = c_{z,k,1} \int_{\tau\in\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), \overline{\Xi_{k-2}(\tau,0,0,0)} \right\rangle \frac{dudv}{v^2} + \frac{1}{2} \int_{\tau\in\mathcal{F}}^{\mathrm{reg}} \sum_{n\geq 1} \sum_{\tilde{\gamma}\in\tilde{\Gamma}_\infty\backslash\tilde{\Gamma}} \sum_{j} c_{z,k,j}(-n)^{1-j}$$
$$\times \left\langle f(\tau), \left[e\left(-\frac{n^2}{4|D| i \operatorname{Im}(\tau) Q_z(l)}\right) \overline{\Xi_{k-1-j}(\tau,\mu_K,-n,0)} \right] \Big|_{k-3/2,\rho} \tilde{\gamma} \right\rangle \frac{dudv}{v^2}.$$

The Additional Term

In this paragraph we look at the first term given above. This disappears unless $k \geq 2$ (using Theorem 5.4.5) and $D = 1$ (using Lemma 5.3.5). If it does not vanish we use the integral calculated earlier in Lemma 6.1.3 to obtain

$$\frac{c_{z,k,1}}{k(k-1)} \sum_{h\in L'/L} \sum_{\lambda\in K+h} c^+(-Q(\lambda),h) \left(\frac{-2\sqrt{2}\pi(\lambda,\mathfrak{w}^\perp)}{\sqrt{Q_z(l)}} \right)^k$$

we then simplify the sums over the cosets $h \in K'/K$ and $\lambda \in K+h$ to one sum over K' and use the identities in (6.3.1) to get

$$-\frac{2\sqrt{N}}{k} \left(\frac{i}{\sqrt{2N}} \right)^k \sum_{\lambda\in K'} (\lambda,y)^k c^+(-Q(\lambda),\lambda).$$

We see that $c^+(-Q(\lambda),\lambda) = (-1)^k c^+(-Q(\lambda),-\lambda)$ using Lemma 2.5.3. So the sum over $\lambda < 0$ is the same as the sum over $\lambda > 0$. The $\lambda = 0$ term vanishes. We also identify K' with $\frac{m}{2N}$ for $m \in \mathbb{Z}$ as discussed in Section 5.4.1. Putting all this together we have

$$\frac{2\sqrt{2}y}{ik} \left(\frac{yi}{\sqrt{2N}} \right)^{k-1} \sum_{m\geq 1} m^k c^+\left(-\frac{m^2}{4N}, \frac{m}{2N}\right). \tag{6.3.6}$$

Unfolding

For the remaining main term we first rewrite this as a sum over $\gamma \in \Gamma_\infty \backslash \Gamma$ (as opposed to $\tilde{\gamma} \in \tilde{\Gamma}_\infty \backslash \tilde{\Gamma}$) and so gain a factor of 2 outside. We then split this sum into two parts. The first over $\gamma = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the second over $\gamma \in \Gamma_\infty \backslash \Gamma, \gamma \neq \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Remembering $f \in H_{3/2-k,\overline{\rho}}$ i.e. the opposite of our slash termed $(k-3/2, \rho)$, we then obtain:

$$2 \int_{\tau \in \mathcal{F}}^{\mathrm{reg}} \sum_{n \geq 1} \sum_j c_{z,k,j} (-n)^{1-j} e\left(-\frac{n^2}{4|D| i \operatorname{Im}(\tau) Q_z(l)}\right) \left\langle f(\tau), \overline{\Xi_{k-1-j}(\tau, \mu_K, -n, 0)} \right\rangle \frac{dudv}{v^2} \tag{6.3.7}$$

$$+ \int_{\tau \in \mathcal{F}}^{\mathrm{reg}} \sum_{n \geq 1} \sum_{\substack{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma \\ c \neq 0}} \sum_j c_{z,k,j} (-n)^{1-j} e\left(-\frac{n^2}{4|D| i \operatorname{Im}(\gamma\tau) Q_z(l)}\right) \tag{6.3.8}$$

$$\times \left\langle f(\gamma\tau), \overline{\Xi_{k-1-j}(\gamma\tau, \mu_K, -n, 0)} \right\rangle \frac{dudv}{v^2} \tag{6.3.9}$$

The second additional factor of 2 outside the first term arises as the two elements $\gamma = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ each generate the same term.

Theorem 4.1.3 told us that the regularised integral converges for all $z \in \mathbb{H}$. However we now need to check that each of these two integrals also converge individually.

We first look at the second integral (6.3.8), (6.3.9). We observe that $\operatorname{Im}(\gamma\tau) = \frac{v}{|c\tau+d|^2}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have $c \neq 0$ so to check the convergence of this integral as $v \to \infty$, we need to look at the terms $f(\tau), \Xi(\tau, \mu_K, -n, 0, k-1-j)$ and $e\left(-\frac{n^2}{4|D| i v Q_z(l)}\right)$ as $v \to 0$. We use Lemma 2.5.26 and Lemma 5.3.8. We see immediately that if $n_0 < 0$ this integral converges. In the case $n_0 \geq 0$ it converges if

$$2\pi n_0 - \frac{n^2 \pi}{|D| Q_z(l)} < 0.$$

So using the identification in (6.3.1a) (and observing that $n \geq 1$) this is when

$$y > \sqrt{\frac{2|D| n_0}{N}}.$$

This is always true if $n_0 = 0$. We will assume from now on that either $n_0 \leq 0$ or $y > \sqrt{2|D|n_0/N}$. In this case the integral converges absolutely and the regularisation is not necessary. The first integral (6.3.7) also converges for $n_0 \leq 0$ or $y > \sqrt{2|D|n_0/N}$. This is clear as the regularised integral and the second integral converged in this case.

Then using the Rankin-Selberg method (see for example [Bor98, Bru02], [Sad12, Proposition 13]) the second integral is:

$$2 \int_{\tau \in \mathcal{G}} \sum_{n \geq 1} \sum_j c_{z,k,j} (-n)^{1-j} e\left(-\frac{n^2}{4|D| i \operatorname{Im}(\tau) Q_z(l)}\right) \left\langle f(\tau), \overline{\Xi_{k-1-j}(\tau, \mu_K, -n, 0)} \right\rangle \frac{dudv}{v^2}$$

where

$$\mathcal{G} := \{\tau \in \mathbb{H} \mid |u| \leq 1/2, |\tau| \leq 1\}.$$

We can check this converges absolutely for $n_0 \leq 0$ or $y > \sqrt{2|D|n_0/N}$ using the same arguments as before (Lemmas 2.5.26 and Lemmas 5.3.8). We can then recombine our two integrals from above and we now need to evaluate

$$2\int_{v=0}^{\infty}\int_{u=-1/2}^{1/2}\sum_{n\geq 1}\sum_{j} c_{z,k,j}(-n)^{1-j} e\left(-\frac{n^2}{4|D|ivQ_z(l)}\right)\left\langle f(\tau), \overline{\Xi_{k-1-j}(\tau,\mu_K,-n,0)}\right\rangle \frac{dudv}{v^2}.$$

At this point we insert the expansion for $f(\tau)$ and $\Xi_{k-1-j}(\tau,\mu,-n,0)$ given in (6.3.3) and Lemma 5.3.5. We remember that $L'/L \cong K'/K$.

$$\begin{aligned}
&2\epsilon_D|D|^{1/2}\int_{v=0}^{\infty}\int_{u=-1/2}^{1/2}\sum_{n\geq 1}\sum_{j}\sum_{h\in K'/K}\sum_{m\in\mathbb{Z}-\operatorname{sgn}(D)Q(h)}\sum_{\substack{\lambda\in K+rh\\ Q(\lambda)\equiv DQ(h)(D)}} c_{z,k,j}(-n)^{1-j}\\
&\times c(m,h,v)e(m\tau)\left(\frac{D}{n}\right)v^{-(k-1-j)/2}H_{k-1-j}\left(\frac{\sqrt{\pi}(-n-2v(\lambda,\mathfrak{w}^{\perp}))}{\sqrt{2|D|vQ_z(l)}}\right)\\
&\times e\left(\frac{Q(\lambda)\tau}{|D|}\right)e\left(\frac{n}{|D|}(\lambda,\mu_K)\right)e\left(-\frac{n^2}{4|D|ivQ_z(l)}\right)\frac{dudv}{v^2}.
\end{aligned}$$

The next step is to carry out the integration over u (a compact region). We remember that this integral picks out the 0-th Fourier coefficient (for example as in Theorem 4.1.3)

$$\begin{aligned}
&2\epsilon_D|D|^{1/2}\int_{v=0}^{\infty}\sum_{j}\sum_{h\in K'/K}\sum_{\substack{\lambda\in K+rh\\ Q(\lambda)\equiv DQ(h)(D)}}\sum_{n\geq 1} c_{z,k,j}(-n)^{1-j}\\
&\times c\left(-\frac{Q(\lambda)}{|D|},h,v\right)\left(\frac{D}{n}\right)v^{-(k-1-j)/2}H_{k-1-j}\left(\frac{\sqrt{\pi}(-n-2v(\lambda,\mathfrak{w}^{\perp}))}{\sqrt{2|D|vQ_z(l)}}\right)\\
&\times e\left(\frac{n}{|D|}(\lambda,\mu_K)\right)e\left(-\frac{n^2}{4|D|ivQ_z(l)}\right)\frac{dv}{v^2}.
\end{aligned}$$

Next we simplify the sums over the cosets and K to one sum over K'. We notice taking a sum over $\lambda \in K, h \in K'/K$ such that $Q(\lambda) \equiv \Delta Q(h)(\Delta)$ and $\lambda \equiv rh(K)$ is equivalent to taking a sum over $\lambda' \in K'$ with $\lambda = \Delta\lambda'$ and $r\lambda' \equiv h(K)$ (in particular we remember $D \equiv r^2(4N)$). This leaves

$$\begin{aligned}
&2\epsilon_D|D|^{1/2}\int_{v=0}^{\infty}\sum_{j}\sum_{\lambda\in K'}\sum_{n\geq 1}\left(\frac{D}{n}\right)c_{z,k,j}(-n)^{1-j}v^{-(k-1-j)/2}c\left(-|D|Q(\lambda),r\lambda,v\right)\\
&\times H_{k-1-j}\left(\frac{\sqrt{\pi}(-n-2Dv(\lambda,\mathfrak{w}^{\perp}))}{\sqrt{2|D|vQ_z(l)}}\right)e\left(\operatorname{sgn}(D)n(\lambda,\mu_K)\right)e\left(-\frac{n^2}{4|D|ivQ_z(l)}\right)\frac{dv}{v^2}.
\end{aligned}\tag{6.3.10}$$

At this point we consider the two parts given by 6.3.4

$$c(m,h,v) = c^{+}(m,h) + c^{-}(m,h)\Gamma(k-1/2,-4\pi mv).\tag{6.3.11}$$

The c^+ Terms

We remember our lattice is positive definite i.e. $-|D|Q(\lambda)$ is negative and so we will have only finitely many $c^+(-|D|Q(\lambda), r\lambda)$ terms that do not vanish. We set

$$\alpha := \frac{n\sqrt{\pi}}{\sqrt{2|D|Q_z(l)}} > 0 \quad \beta := -\frac{\sqrt{2\pi}D(\lambda, \mathfrak{w}^\perp)}{\sqrt{|D|Q_z(l)}} \tag{6.3.12}$$

We also see that using (5.4.1) then $(-n)^{-j}v^{j/2}c_{z,k,j} = c_{z,k,0}\left(\frac{(k-1)\sqrt{v}}{\alpha}\right)^j$ and

$$\alpha\beta = -\frac{\operatorname{sgn}(D)\pi n(\lambda, \mathfrak{w}^\perp)}{Q_z(l)}.$$

We now carry out the integration over v. We can swap the integration and summation as the sum over m is finite and we can check through this proof that the absolute value of (6.3.10) converges. In this case the integral over v (and the sum over j) we need to consider is

$$\int_{v=0}^{\infty} \sum_j v^{-(k-1)/2} H_{k-1-j}\left(-\frac{\alpha}{\sqrt{v}} + \beta\sqrt{v}\right)\left(\frac{(k-1)\sqrt{v}}{\alpha}\right)^j e^{-\frac{\alpha^2}{v}}\frac{dv}{v^2}.$$

Using Lemma 6.2.2 (for $\kappa = k-1$) we then have

$$\begin{aligned} &-2\epsilon_D|D|^{1/2}\left(-\frac{\sqrt{2|D|Q_z(l)}}{\sqrt{\pi}}\right)^{k+1} c_{z,k,0}\sum_{\lambda\in K'}\sum_{n\geq 1}\left(\frac{D}{n}\right)\frac{c^+(-|D|Q(\lambda), r\lambda)}{n^k} \\ &\qquad \times e\left(\operatorname{sgn}(D)n\left((\lambda,\mu_K) - \frac{i(\lambda,\mathfrak{w}^\perp)}{Q_z(l)}\right)\right)\Gamma\left(k, \frac{\operatorname{sgn}(D)2\pi n(\lambda,\mathfrak{w}^\perp)}{Q_z(l)}\right). \end{aligned} \tag{6.3.13}$$

We then observe that we change the sign of λ and n and the terms in the sum remain unchanged. We see this by noting that

$$c^+(m,h) = (-1)^k\operatorname{sgn}(D)c^+(m,-h)$$

as f transforms with respect to $\overline{\rho}$ using Lemma 2.5.10 and that $\left(\frac{D}{-n}\right) = \operatorname{sgn}(D)\left(\frac{D}{n}\right)$ using Proposition 3.2.4. This means we can replace the sums over $n \geq 1$ and $\lambda \in K'$ with sums over $n \in \mathbb{Z}, n \neq 0$ and $\lambda \in K', \lambda > 0$ respectively. (We treat the case when $\lambda = 0$ separately, in the constant term paragraph). This leaves

$$\begin{aligned} &-2\epsilon_D|D|^{1/2}\left(-\frac{\sqrt{2|D|Q_z(l)}}{\sqrt{\pi}}\right)^{k+1} c_{z,k,0}\sum_{\substack{\lambda\in K'\\ \lambda>0}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\left(\frac{D}{n}\right)\frac{c^+(-|D|Q(\lambda), r\lambda)}{n^k} \\ &\qquad \times e\left(\operatorname{sgn}(D)n\left((\lambda,\mu_K) - \frac{i(\lambda,\mathfrak{w}^\perp)}{Q_z(l)}\right)\right)\Gamma\left(k, \frac{\operatorname{sgn}(D)2\pi n(\lambda,\mathfrak{w}^\perp)}{Q_z(l)}\right). \end{aligned}$$

We remember using [BO10, Equation 4.7] that

$$\sum_{b(D)}\left(\frac{D}{b}\right)e\left(\frac{nb}{|D|}\right) = \left(\frac{D}{n}\right)\epsilon_D|D|^{1/2} \tag{6.3.14}$$

so we can rewrite to

$$-2\left(-\frac{\sqrt{2|D|Q_z(l)}}{\sqrt{\pi}}\right)^{k+1} c_{z,k,0}\sum_{\substack{\lambda\in K'\\ \lambda>0}}\sum_{n\in\mathbb{Z}\setminus\{0\}}\sum_{b(D)}\left(\frac{D}{b}\right)\frac{c^+\left(-|D|Q(\lambda),r\lambda\right)}{n^k}$$
$$\times\, e\left(\operatorname{sgn}(D)n\left((\lambda,\mu_K)-\frac{i(\lambda,\mathfrak{w}^\perp)}{Q_z(l)}+\frac{b}{D}\right)\right)\Gamma\left(k,\frac{\operatorname{sgn}(D)2\pi n(\lambda,\mathfrak{w}^\perp)}{Q_z(l)}\right).$$

We had some identifications in (6.3.1) for the specific cusp l_∞. We also identifying K' with $\frac{m}{2N}$ for $m\in\mathbb{Z}$ as discussed in Section 5.4.1. This allows us to simplify to

$$\frac{\sqrt{2|D|}}{i\pi}\left(\frac{-i|D|}{2\pi\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\sum_{b(D)}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\, n^{-k}e\left(\operatorname{sgn}(D)n\left(mz+\frac{b}{D}\right)\right)\Gamma\left(k,-\operatorname{sgn}(D)2\pi nmy\right).$$

The c^+ Terms, Bernoulli Form

We can reformulate this more appealingly in terms of Bernoulli polynomials. Using the decomposition of the incomplete gamma function given in (**??**) we have

$$\frac{\sqrt{2|D|}}{i\pi}\left(\frac{-i|D|}{2\pi\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{n\in\mathbb{Z}\setminus\{0\}}\sum_{b(D)}\sum_{s=0}^{k-1}\frac{(k-1)!}{(k-1-s)!}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\, n^{-k}e\left(\operatorname{sgn}(D)n\left(mx+\frac{b}{D}\right)\right)\left(-\operatorname{sgn}(D)2\pi nmy\right)^{k-1-s}.$$

We then remember the periodic Bernoulli polynomials we defined earlier in Definition 6.3.2 which allows us to write this as

$$-\frac{2\sqrt{2|D|}}{k}\left(\frac{|D|}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\sum_{s=0}^{k-1}\binom{k}{s+1}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\, \mathbb{B}_{s+1}\left(\operatorname{sgn}(D)\left(mx+\frac{b}{D}\right)\right)\left(\operatorname{sgn}(D)imy\right)^{k-1-s}$$

This finite sum certainly converges for $z\in\mathbb{H}$. Using property 3 of Lemma 6.3.3 we obtain

$$\frac{-\epsilon_D 2\sqrt{2D}}{k}\left(\frac{D}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\sum_{s=1}^{k}\binom{k}{s}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)\mathbb{B}_s\left(mx+\frac{b}{D}\right)(imy)^{k-s}.$$

This gives the stated result in the case $k=1$. We know $\sum_{b(D)}\left(\frac{D}{b}\right)$ vanishes if $D\neq 1$ and $\mathbb{B}_0(x)=1$ so this allows us to rewrite the additional term (6.3.6) (in the case $k\geq 2$) from earlier as

$$-\frac{\epsilon_D 2\sqrt{2D}}{k}\left(\frac{D}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\binom{k}{0}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)\mathbb{B}_0\left(mx+\frac{b}{D}\right)(imy)^{k}$$

so these two nicely combine and we have:

$$\frac{-\epsilon_D 2\sqrt{2D}}{k}\left(\frac{D}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\sum_{s=0}^{k}\binom{k}{s}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)\mathbb{B}_s\left(mx+\frac{b}{D}\right)(imy)^{k-s}. \tag{6.3.15}$$

The c^+ Terms, Alternative Bernoulli Form

This is a nice enough version but we can go further by using properties 5, 6 and 7 of Lemma 6.3.7 to obtain

$$\frac{-\epsilon_D 2\sqrt{2D}}{k}\left(\frac{D}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\left(\frac{\mathrm{I}_{\mathbb{Z}}(mx+b/D)k}{(imy)^{1-k}}+\sum_{s=0}^{k}\binom{k}{s}B_s\left(\langle mx+b/D\rangle\right)(imy)^{k-s}\right)$$

and so finally we have the stated form:

$$\frac{-\epsilon_D 2\sqrt{2D}}{k}\left(\frac{D}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\left(B_k\left(\langle mx+b/D\rangle+imy\right)+\frac{\mathrm{I}_{\mathbb{Z}}(mx+b/D)k}{(imy)^{1-k}}\right).$$

The Constant Term

We remember we had still had to consider the case when $\lambda=0$ from (6.3.13). This is equal to

$$-2(k-1)!\epsilon_D|D|^{1/2}\left(-\frac{\sqrt{2|D|Q_z(l)}}{\sqrt{\pi}}\right)^{k+1}c_{z,k,0}c^+(0,0)\sum_{n\geq 1}\left(\frac{D}{n}\right)\frac{1}{n^k}$$

and using the identifications in (6.3.1) and Definition 6.3.8 we obtain

$$\frac{\epsilon_D|D|\sqrt{2}}{i\pi}\left(\frac{|D|}{i\pi 2\sqrt{2N}}\right)^{k-1}c^+(0,0)L\left(k,\left(\tfrac{D}{\cdot}\right)\right).$$

This is a constant term that does not depend on z. We also see using Lemma 2.5.10 it disappears if $\frac{1}{2}(1-2k-\operatorname{sgn}(D))$ is odd as then $c^+(0,0)=0$. In particular it disappears if $k=1$ and $D>0$ so we never have to consider the case when $L\left(k,\left(\frac{D}{\cdot}\right)\right)$ has a pole.

The c^- Terms

Next we consider our expansion associated to the non-holomorphic terms. Our lattice K is positive definite so we will have infinitely many terms of the form $c^-(-|D|Q(\lambda),r\lambda)$. We had using (6.3.11) the following form

$$2\epsilon_D|D|^{1/2}\int_{v=0}^{\infty}\sum_j\sum_{\lambda\in K'}\sum_{n\geq 1}\left(\frac{D}{n}\right)c_{z,k,j}(-n)^{1-j}v^{-(k-1-j)/2}$$
$$\times c^-\left(-|D|Q(\lambda),r\lambda\right)\Gamma\left(k-1/2,4\pi|D|Q(\lambda)v\right)$$
$$\times H_{k-1-j}\left(\frac{\sqrt{\pi}(-n-2Dv(\lambda,\mathfrak{w}^\perp))}{\sqrt{2|D|vQ_z(l)}}\right)e\left(\operatorname{sgn}(D)n(\lambda,\mu_K)\right)e\left(-\frac{n^2}{4|D|ivQ_z(l)}\right)\frac{dv}{v^2}.$$

We note we have no $c^-(0,0)$ term so this vanishes for $\lambda=0$. We then swap the integration over v with the summations. We see through this proof that the absolute value of (6.3.10) converges. Also note that $\Gamma(k-1/2,4\pi|D|Q(\lambda)v)$ decays exponentially as $v\to\infty$ and the coefficients $c^-(m,h)$ only grow polynomially fast. We then use our identifications in (6.3.1) and then setting $a = \alpha/y > 0$ and $\beta \neq 0$ as before in (6.3.12) we have

$$-2\epsilon_D|D|^{1/2}c_{k,0}\sum_j \sum_{\substack{\lambda\in K'\\ \lambda\neq 0}}\sum_{n\geq 1}\left(\frac{D}{n}\right)\int_{v=0}^{\infty} v^{-(k-1)/2}y^{k+1}n\left(\frac{(k-1)\sqrt{v}}{ay}\right)^j c^-\left(-|D|Q(\lambda), r\lambda\right)$$
$$\times\Gamma\left(k-1/2,\beta^2 v\right)H_{k-1-j}\left(-\frac{ay}{\sqrt{v}}+\beta\sqrt{v}\right)e\left(\mathrm{sgn}(D)n(\lambda,x)\right)e^{-\frac{(ay)^2}{v}}\frac{dv}{v^2}.$$

Next we identify K' with $\frac{m}{2N}$ for $m\in\mathbb{Z}$ and $\tilde{\beta}=\beta/\mathrm{sgn}(D)m > 0$ to obtain

$$-2\epsilon_D|D|^{1/2}c_{k,0}\sum_j\sum_{m\in\mathbb{Z}\setminus\{0\}}\sum_{n\geq 1}\left(\frac{D}{n}\right)c^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)e^{\mathrm{sgn}(D)2am\tilde{\beta}ix}ny^{k+1}$$
$$\times\int_{v=0}^{\infty}\left(\frac{(k-1)\sqrt{v}}{ay}\right)^j v^{-\frac{(k-1)}{2}}\Gamma\left(k-1/2,(m\tilde{\beta})^2 v\right)H_{k-1-j}\left(-\frac{ay}{\sqrt{v}}+\mathrm{sgn}(D)m\tilde{\beta}\sqrt{v}\right)e^{-\frac{(ay)^2}{v}}\frac{dv}{v^2}. \tag{6.3.16}$$

We solved this integral in v (and the sum over j) in Lemma 6.2.4. Carefully considering the cases when m and $\mathrm{sgn}(D)$ are positive and negative we can now switch to a sum over $m\geq 1$ and remember

$$c^-(m,h) = (-1)^k\mathrm{sgn}(D)c^-(m,-h).$$

So we obtain for $D>0$

$$-\frac{2\epsilon_D\sqrt{|D|\pi}}{a^2}c_{k,0}\left(\frac{-1}{4ay}\right)^{k-1}\sum_{m\geq 1}\sum_{n\geq 1}\left(\frac{D}{n}\right)nc^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\left[(2k-2)!e^{2am\tilde{\beta}iz}+(-1)^k e^{-2am\tilde{\beta}iz}\Gamma(2k-1,4am\tilde{\beta}y)\right]$$

and for $D<0$

$$-\frac{2\epsilon_D\sqrt{|D|\pi}}{a^2}c_{k,0}\left(\frac{-1}{4ay}\right)^{k-1}\sum_{m\geq 1}\sum_{n\geq 1}\left(\frac{D}{n}\right)nc^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\left[(-1)^{k+1}(2k-2)!e^{2am\tilde{\beta}iz}+e^{-2am\tilde{\beta}iz}\Gamma(2k-1,4am\tilde{\beta}y)\right].$$

So combining these two cases and then plugging in all the identities (6.3.1) once again

$$\frac{\sqrt{2}\epsilon_D D}{i\sqrt{\pi}}\left(\frac{D}{8\pi i\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{n\geq 1}\left(\frac{D}{n}\right)c^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times n^{-k}\left[\mathrm{sgn}(D)(2k-2)!e(nmz)+(-1)^k e(-nmz)\Gamma(2k-1,4\pi nmy)\right].$$

Using (6.3.14) we know

$$\sum_{b(D)}\left(\frac{D}{b}\right)e\left(\frac{nb}{D}\right)=\mathrm{sgn}(D)\left(\frac{D}{n}\right)\sqrt{D},$$

and we then use this to finally obtain

$$\frac{\sqrt{2}\epsilon_D\sqrt{D}}{i\sqrt{\pi}}\left(\frac{D}{8\pi i\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{n\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)n^{-k}$$
$$\times\left[(2k-2)!e(n(mz+b/D))+(-1)^k\mathrm{sgn}(D)e(-n(mz-b/D))\Gamma(2k-1,4\pi nmy)\right]. \tag{6.3.17}$$

The c^- Terms, Polylogarithm Form

We can go further than (6.3.17). We reformulate this more compactly (and remove an infinite sum) by writing this in terms of polylogarithms. In particular using Definitions 5.3.6 and 6.3.9 we have

$$\frac{\sqrt{2}\epsilon_D\sqrt{D}(2k-2)!}{i\sqrt{\pi}}\left(\frac{D}{8\pi i\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^-\left(-\frac{|D|m^2}{4N},\frac{rm}{2N}\right)$$
$$\times\left[\mathrm{Li}_k\left(e((mz+b/D))\right)+(-1)^k\mathrm{sgn}(D)\mathrm{Li}_{2k-1,-k+1}\left(4\pi my, e(-(mz-b/D))\right)\right].$$

We check this series converges absolutely for $z\in\mathbb{H}$ as Lemma 2.5.27 told us the $c^-(m,h)$ coefficients only grow polynomially but the polylogarithm terms decay exponentially (noting $n,m\neq 0$), see Lemma 5.3.7.

The Expansion in the Weyl Chambers

So far we have seen that our expansion is valid only when $n_0<0$ or $y>\sqrt{2|D|n_0/N}$. However we have also observed that the expansion given in (6.3.5) converges absolutely for all $z\in\mathbb{H}$ and so can be continued to hold for the entire upper-half plane. Using Remark 3.5.4 we know that there are only vertical half-line singularities for $y>\sqrt{|D|n_0/N}$. Theorem 4.2.2 told us that our singularities arose from the $c^+(m,h)$ terms where $m<0$. We then observe that the vertical half-line singularities are fully encapsulated by the periodic Bernoulli polynomials in our expansion. In particular the $\mathbb{B}_1(mx+b/D)$ term in (6.3.15), see also Remark 6.3.4.

We saw in Theorem 4.3.7 that $\Phi_{D,r,k}(z,f)$ was real-analytic for all $z\in\mathbb{H}\backslash Z_{D,r}(f)$. The Fourier expansion we have found is also real-analytic outside the vertical half-lines so will equal $\Phi_{D,r,k}(z,f)$ within the unbounded Weyl chambers.

We find the expansion in the bounded Weyl chambers as well. Section 3.5 told us the vertical half-lines were defined by D_λ when $\lambda\perp l_\infty$. Theorem 4.2.2 told us that for a point $z_0\in\mathbb{H}$ there exists an open neighbourhood $U\subset\mathbb{H}$ so that subtracting

$$\sqrt{\frac{|D|}{2}}\sum_{h\in L'/L}\sum_{\substack{m\in\mathbb{Z}-\mathrm{sgn}(D)Q(h)\\ m<0}}c^+(m,h)\sum_{\substack{\lambda\in L_{-dD,rh}\\ \lambda\perp z_0}}\chi_D(\lambda)\frac{(\lambda,v(z))}{|(\lambda,v(z))|}q_z(\lambda)^{k-1},$$

allowed $\Phi_{D,r,k}(z,f)$ to be continued to a smooth function on U. Using the wall crossing formula (Theorem 4.2.4) we let

$$g_{W_{12}}(z)=2\sqrt{2|D|}\sum_{h\in L'/L}\sum_{\substack{m\in\mathbb{Z}-\mathrm{sgn}(D)Q(h)\\ m<0}}c^+(m,h)\sum_{\substack{\lambda\in L_{-dD,rh}\\ \lambda\not\perp l_\infty\\ \lambda\perp W_{12}\\ (\lambda,W_1)<0}}\chi_D(\lambda)q_z(\lambda)^{k-1}.$$

This function then tells us the finite number of polynomials we will need to add as we cross a wall W_{12} defined by a semi-circle between two Weyl chambers W_1, W_2. This function is 0 if W_{12} is a wall defined by a vertical half-line. Using the real analyticity of $\Phi_{D,r,k}(z,f)$ it is clear the expansion in a bounded Weyl chamber is given by the expansion (6.3.5) with the addition of the appropriate polynomials $g_{W_{12}}(z)$. The appropriate $g_{W_{12}}(z)$ being those defined by walls W_{12} which we crossed in order to reach that Weyl chamber from the unbounded Weyl chamber above. The value of a point lying on a semi-circle W_{12} is given by the Fourier expansion in the Weyl chamber W_1 plus the polynomial $(1/2)g_{W_{12}}(z)$ i.e. the average value of the surrounding Weyl chambers. □

6.3.3 The Fourier Expansion at other Cusps

Theorem 6.3.10 held only for the cusp l_∞. However by considering the Atkin-Lehner involutions from earlier we can now adapt this to find the expansion at other cusps in the case when N is square-free. This is the same idea as in Lemma 5.4.6. In particular we recall from Section 3.4 that crucially all the cusps of $\Gamma_0(N)$ can be represented by $W_m^N\infty$ where m are the divisors of N.

For $f \in H_{3/2-k,\overline{\rho}}$ with components f_h we denoted

$$f_{W_m^N} := \sum_{h \in L'/L} f_{W_m^N.h}\mathfrak{e}_h.$$

We also wrote $f_{W_m^N} = \sum_{h\in L'/L} f_h \mathfrak{e}_{W_m^N.h}$ and see that $f_{W_m^N} \in H_{3/2-k,\overline{\rho}}$. We then have the following useful theorem.

Theorem 6.3.12. *Let N be square-free, $f \in H_{3/2-k,\overline{\rho}}$ and $s \in \mathbb{P}^1(\mathbb{Q})$ be a cusp for $\Gamma_0(N)$. Then there exists an m, an exact divisor of N, such that the Fourier expansion of $\Phi_{D,r,k}(z,f)$ at the cusp s is given by*

$$\Phi_{D,r,k}\left(z, f_{W_m^N}\right).$$

Proof. For $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q}\cup\{\infty\}$ we know $W_m^N\infty = s$ for some exact divisor m of N. The Fourier expansion of $\Phi_{D,r,k}(z,f)$ at the cusp s is given by the Fourier series of

$$\left(\Phi_{D,r,k}\big|_{2-2k}W_m^N\right)(z,f) = j(W_m^N, z)^{2k-2}\Phi_{D,r,k}(W_m^N.z, f)$$

at the cusp ∞ (see for example [DS05, Section 1.2]) and remembering $\Phi_{D,r,k}(z,f)$ is of weight

$2-2k$ from Theorem 4.2.2. Using Proposition 3.6.12 we then have

$$\begin{aligned} j(W_m^N, z)^{2k-2}\Phi_{D,r,k}(W_m^N.z, f) &= j(W_m^N, z)^{2k-2}\int_{\tau\in\mathcal{F}}^{\mathrm{reg}}\left\langle f(\tau), \overline{\Theta_k(\tau, W_m^N.z)}\right\rangle \frac{dudv}{v^2} \\ &= \int_{\tau\in\mathcal{F}}^{\mathrm{reg}}\left\langle f(\tau), \overline{(\Theta_k(\tau,z))_{W_m^N}}\right\rangle \frac{dudv}{v^2} \\ &= \int_{\tau\in\mathcal{F}}^{\mathrm{reg}}\left\langle f_{W_m^N}(\tau), \overline{\Theta_k(\tau,z)}\right\rangle \frac{dudv}{v^2} \\ &= \Phi_{D,r,k}\left(z, f_{W_m^N}\right). \end{aligned}$$

□

Fix $f \in H_{3/2-k,\overline{\rho}}$ in the form (6.3.2). Then the expansion of the lift can be found at the cusp s using Theorem 6.3.10, where we find the expansion at ∞ for the coefficients $c^+(m, W_m^N.h), c^-(m, W_m^N.h)$ (as opposed to $c^+(m,h), c^-(m,h)$).

6.4 A Locally Harmonic Weak Maass Form

In this part we consider the asymptotic behaviour of our lift as $y \to \infty$. This is the final proposition that we will need to show that our lift is a locally harmonic weak Maass form, as in Definition 2.5.30.

Proposition 6.4.1. *Let $f \in H_{3/2-k,\overline{\rho}}$ and let $k \geq 2$. Then*

$$\Phi_{D,r,k}(z,f) = \mathcal{O}(y^k)$$

as $y \to \infty$, uniformly in x. In the case $k=1$ then

$$\begin{aligned} \lim_{y\to\infty}\Phi_{D,r,1}(z,f) = {} & \frac{\epsilon_D|D|\sqrt{2}}{i\pi}c^+(0,0)L\left(1, \left(\tfrac{D}{\cdot}\right)\right) \\ & - 2\sqrt{2}\epsilon_D\sqrt{D}\sum_{m\geq 1}\sum_{b(D)}\left(\frac{D}{b}\right)c^+\left(-\frac{|D|m^2}{4N}, \frac{rm}{2N}\right)\mathbb{B}_1\left(mx + b/D\right). \end{aligned}$$

Proof. We show this by using the expansion given in Theorem 6.3.10. This was in three parts. The first part (6.3.5a), was just a constant.

The second part (6.3.5b), (6.3.5c), consisted of a finite sum over $m \geq 1$ as we remember there are only finitely many non-zero $c^+(-n,h)$ for $n > 0$. Considering this part in the form given in (6.3.15) we note that $\mathbb{B}_s(mx+b/D)$ are actually bounded for any m, x and it is clear that this part grows $\mathcal{O}(y^k)$ as $y \to \infty$. In the case $k=1$ this is again just a constant (dependant on x) that does not depend on y.

We now consider the third part. We look at the form given in (6.3.17). This was the part associated to the $c^-(n,h)$ coefficients. We first note that using (6.2.1)

$$e\left(-n\left(mz - b/D\right)\right)\Gamma(2k-1, 4\pi nmy) = (2k-2)!e\left(n\left(m(-x+iy) - b/D\right)\right)\sum_{s=0}^{2k-2}\frac{(4\pi nmy)^s}{s!}.$$

Using this we will now bound the absolute value of (6.3.17). Up to constants it then suffices to consider

$$\left| \sum_{m\geq 1} c^{-}\left(-\frac{|D|m^2}{4N}, \frac{rm}{2N}\right) \sum_{n\geq 1} \sum_{s=0}^{2k-2} \frac{e^{-2\pi mny}}{n^k}(nmy)^s \right|$$

and using Lemma 2.5.27 we know the asymptotic behaviour of $c^{-}(n,h)$ so we have

$$\begin{aligned}
&\leq \text{constant} \cdot \sum_{s=0}^{2k-2} \sum_{m\geq 1} m^{1/2} \sum_{n\geq 1} \frac{e^{-2\pi mny}}{n^k}(nmy)^s \\
&\leq \text{constant} \cdot \sum_{s=0}^{2k-2} y^s \sum_{m\geq 1} m^{1+s} e^{-\pi my} \sum_{n\geq 1} n^s e^{-\pi ny} \\
&\leq \text{constant} \cdot \sum_{s=0}^{2k-2} y^s \mathrm{Li}_{-s-1}(e^{-\pi y}) \mathrm{Li}_{-s}(e^{-\pi y})
\end{aligned}$$

and then using Lemma 5.3.7 part 5 we know this decays exponentially fast as $y \to \infty$. □

We are finally in a position to put together all our main theorems and complete one of the main aims of this thesis. That is, to show the singular theta lift is a locally harmonic weak Maass form.

Theorem 6.4.2. *Let $f \in H_{3/2-k,\overline{\rho}}$ and let N be square-free. Then $\Phi_{D,r,k}(z,f)$ is a locally harmonic weak Maass form of weight $2-2k$ for the group $\Gamma_0(N)$ with exceptional set $Z_{D,r}(f)$. I.e.*

$$\Phi_{D,r,k}(z,f) \in LH_{2-2k}\left(\Gamma_0(N)\right).$$

Proof. We remember from Section 3.5 that $Z_{D,r}(f)$ was a nowhere dense $\Gamma_0(N)$-invariant set. We look at the four conditions in Definition 2.5.30.

1. We know from Theorem 4.2.2 that $\Phi_{D,r,k}(z,f)$ has weight $2-2k$ for the group $\Gamma_0(N)$.

2. Theorem 4.3.7 says $\Phi_{D,r,k}(z,f)$ is real analytic and harmonic outside the exceptional set $Z_{D,r}(f)$.

3. Theorem 4.2.2 (see also Section 4.2.1) tells us that the value on the singularities is the average of the values in the adjacent Weyl chambers.

4. For the cusp condition we note if N is square-free, then the Fourier expansion at a cusp $s = W_m^N \infty$ was given by the Fourier expansion at ∞ of $\Phi_{D,r,k}\left(z, f_{W_m^N}\right)$ using Theorem 6.3.12. Then using Proposition 6.4.1 for $f_{W_m^N}$ (where we note $c^{+}\left(n, W_m^N.h\right)$, $c^{-}\left(n, W_m^N.h\right)$ still grow at the same rate as $c^{+}(n,h), c^{-}(n,h)$) we can check that we have polynomial growth at all the cusps. □

Chapter 7

The Shimura Lift

In this final chapter we consider the relationship of our lift with the well known Shimura correspondence. The nature of the singularities found in Theorem 4.2.2 also leads us to consider these ideas as distributions. We first show that the kernel functions of each lift are related via the anti-linear differential operator and so link the two lifts with ξ_{2-2k} and obtain a commutative diagram. Using this link we can then derive new proofs of the properties of the Shimura lift including its holomorphicity and Fourier expansion. We then give our definition of a distribution associated to a locally harmonic weak Maass form and show that this then satisfies a current equation. Similar ideas can be found in [BF04, BKV13, Höv12].

7.1 Definition

We start by defining our version of the Shimura lift. As mentioned in the introduction this is a family of maps from forms of half-integral weight first defined by Shimura [Shi73]. In the introduction we also discuss the history, applications and significance of this lift. Niwa [Niw75] (see also [Shi75] and [Cip83]) later formulated this correspondence in terms of theta lifts (an example of a Borcherds lift [Bor98]). This lift was also seen in [SZ88, GKZ87, Sko90a, Sko90b] and examined for some Jacobi forms. We note however that vector-valued forms essentially correspond to some Jacobi (and skew-holomorphic Jacobi) forms, see also Example 2.5.12 and [Bru02, Example 1.3].

For our construction we will use the twisted vector-valued Shintani kernel function (Definition 3.6.6) to define a twisted Shimura lift. This is the same idea as in Definition 4.1.1.

Definition 7.1.1. *Let $\kappa \in \mathbb{Z}, \kappa \geq 1$. For $g \in S_{\kappa+1/2,\rho}$ we will call*

$$\Phi^*_{D,r,\kappa}(z,g) := \big(g(\tau), \Theta^*_{D,r,\kappa}(\tau,z)\big)_{\kappa+1/2,\rho}$$

$$= \int_{\tau \in \mathcal{F}} \left\langle g(\tau), \Theta^*_{D,r,\kappa}(\tau, z) \right\rangle v^{\kappa+1/2} \frac{dudv}{v^2}$$

*the twisted **Shimura lift**.*

This definition makes sense, as Theorem 3.6.8 told us that $\Theta^*_{D,r,\kappa}(\tau, z)$ is of weight $\kappa+1/2$ in τ. As f is a cusp form, which decays exponentially as $v \to \infty$, and the Shintani kernel function also decays exponentially we know immediately that the integral converges absolutely and defines a real analytic (and therefore smooth) function on $\mathbb{H}$. There is no need to regularise the integral.

Remark 7.1.2. The properties of the Shimura lift are very well known, see e.g. [Niw75]. However, during this chapter, we will pretend we are ignorant of these facts and rederive them from scratch. We will use the singular theta lift to do this.

Corollary 7.1.3. *For $\kappa \in \mathbb{Z}, \kappa \geq 1$ then*

$$\Phi^*_{D,r,\kappa} : S_{\kappa+1/2,\rho} \to A_{2\kappa}(\Gamma_0(N)).$$

Proof. Using Lemma 3.6.11 it is immediately clear the Shimura lift maps cusp forms to some even weight (real analytic) functions for the group $\Gamma_0(N)$. □

Remarks 7.1.4. In Theorem 7.3.8 we will show that actually the Shimura lift usually maps (vector-valued) cusp forms to holomorphic (scalar-valued) cusp forms.

In this work we have restricted the input to cusp forms and this will suffice for us. In particular, our original lift was defined for $f \in H_{3/2-k,\overline{\rho}}$ and we will soon see that we want to lift $\xi_{3/2-k}(f)$, which are elements of $S_{\kappa+1/2,\rho}$. However in general we can regularise the integral and extend the utility of this lift to much more general forms. For example weakly holomorphic forms as in [Bor98, Section 14] or weak Maass forms as in [BKV13] (after checking that the regularised Petersson scalar product in these cases does indeed converge).

We remember that taking the complex conjugate of a Siegel theta function and multiplying by an appropriate power of v essentially means we then work with swapped signature (b^-, b^+). So our final remark here is that the Shimura lift definition is different in this respect; for for the singular theta lift we first took the complex conjugate of the kernel function.

7.2 The Relationship

We now show that the Shimura lift and the singular theta lift obtained in this work are closely connected. The connection formed here goes back to the results of [BF04]. We need the following key lemma. This is adapted from [BKV13, Lemma 3.3].

Lemma 7.2.1. *We have that*

$$\xi_{k+1/2,\tau}(\Theta^*_{D,r,k}(\tau,z)) = -\frac{1}{2}\xi_{2-2k,z}\left(v^{k-3/2}\Theta_{D,r,k}(\tau,z)\right).$$

Proof. This is actually a proof on the level of the Schwartz functions. Using Section 3.6.1 the associated Schwartz functions for our two kernel functions were

$$\varphi_0\left(\lambda,\tau,z,\sigma_z,p_k^*,D\right) = v^{1/2}\chi_D(\lambda)\left(\frac{q_z(\lambda)}{y^2}\right)^k e\left(\frac{Q(\lambda)}{|D|}u + \frac{Q_z(\lambda)}{|D|}iv\right)$$
$$\varphi_0\left(\lambda,\tau,z,\sigma_z,p_k,D\right) = v^{3/2}\chi_D(\lambda)p_z(\lambda)q_z(\lambda)^{k-1}e\left(\frac{Q(\lambda)}{|D|}u + \frac{Q_z(\lambda)}{|D|}iv\right).$$

Section 3.6.1 told us that $\Theta^*_{D,r,k}(\tau,z)$ corresponded to $\Theta(z,\tau)$ in [BKV13, (1.2)] and that $v^{k-3/2}\overline{\Theta_{D,r,k}(\tau,z)}$ corresponded to $\Theta^*(-z,\tau)$ in [BKV13, (1.6)]. Putting this all together with the partial derivative calculations in [BKV13, Lemma 3.3] we obtain

$$2\xi_{k+1/2,\tau}\left[\varphi_0\left(\lambda,\tau,z,\sigma_z,p_k^*,D\right)\right] = -\xi_{2-2k,z}\left[v^{k-3/2}\varphi_0\left(\lambda,\tau,z,\sigma_z,p_k,D\right).\right] \tag{7.2.1}$$

The genus character clearly just goes for the ride during these differential calculations. Observe that our kernel Siegel theta functions converged absolutely and locally uniformly in (τ,z). This means we can interchange the partial differential operators with the summation of the theta series and (7.2.1) implies the stated result. □

Using this we then have the following important result. Once again this relies on Stokes' theorem and the result only holds away from the cycles $Z_{D,r}(f)$ (where the singular lift is not naturally differentiable) as then the boundary term does not necessarily disappear. This proof is adapted from [BKV13, Lemma 3.4].

Theorem 7.2.2. *Let $f \in H_{3/2-k,\overline{\rho}}$ and $z \in \mathbb{H}\backslash Z_{D,r}(f)$. Then*

$$\Phi^*_{D,r,k}(z,\xi_{3/2-k}(f)) = \frac{1}{2}\xi_{2-2k,z}(\Phi_{D,r,k}(z,f)).$$

Proof. We let $f \in H_{3/2-k,\overline{\rho}}$ and then using the Definition 7.1.1 we have

$$\begin{aligned}\Phi^*_{D,r,k}(z,\xi_{3/2-k,\tau}(f(\tau))) &= \lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\xi_{3/2-k,\tau}(f(\tau)),\Theta^*_{D,r,k}(\tau,z)\right\rangle v^{k+1/2}\frac{dudv}{v^2}\\ &= \lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle R_{k-3/2,\tau}(v^{3/2-k}\overline{f(\tau)}),\Theta^*_{D,r,k}(\tau,z)\right\rangle v^{k+1/2}\frac{dudv}{v^2}.\end{aligned}$$

Using Stokes' theorem, Lemma 6.1.1 we have that this is equal to

$$\begin{aligned}&= -\lim_{t\to\infty}\int_{\tau\in\mathcal{F}_t}\left\langle\overline{f(\tau)},L_{k+1/2,\tau}\left(\Theta^*_{D,r,k}(\tau,z)\right)\right\rangle\frac{dudv}{v^2}\\ &\quad+\lim_{t\to\infty}\int_{-1/2}^{1/2}\left[\left\langle\overline{f(\tau)},\Theta^*_{D,r,k}(\tau,z)\right\rangle\right]_{v=t}du.\end{aligned}$$

Looking at the first term and using Lemma 7.2.1 we obtain

$$\begin{aligned}
&= -\overline{\int_{\tau\in\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), \xi_{k+1/2,\tau}\left(\Theta^*_{D,r,k}(\tau,z)\right)\right\rangle v^{3/2-k}\frac{dudv}{v^2}} \\
&= \overline{\int_{\tau\in\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), iy^{2-2k}\frac{\partial}{\partial z}\overline{(\Theta_{D,r,k}(\tau,z))}\right\rangle \frac{dudv}{v^2}} \\
&= \frac{1}{2}\xi_{2-2k}\left(\int_{\tau\in\mathcal{F}}^{\mathrm{reg}} \left\langle f(\tau), \overline{\Theta_{D,r,k}(\tau,z)}\right\rangle \frac{dudv}{v^2}\right) \\
&= \frac{1}{2}\xi_{2-2k}(\Phi_{D,r,k}(z,f)).
\end{aligned}$$

Recalling from Theorem 4.2.2 that the regularised integral $\Phi_{D,r,k}(z,f)$ on the truncated fundamental domain $\mathcal{F}_t$ converged as $t\to\infty$ locally uniformly for $z\in\mathbb{H}\backslash Z_{D,r}(f)$. So for these points we can interchange the partial derivatives in z with the integral.

To show the stated result it remains to show the (complex conjugate) of the second term vanishes for $z\in\mathbb{H}\backslash Z_{D,r}(f)$. We know f^- decays exponentially, as does the Shintani kernel function. We look at the integral of the f^+ part and plug in the expansions given in (2.5.2) and Definition 3.6.6 to get

$$\begin{aligned}
&\lim_{t\to\infty}\int_{-1/2}^{1/2}\sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}+\mathrm{sgn}(D)Q(h)\\ n\gg-\infty}}\sum_{\substack{\lambda\in L+rh\\ Q(\lambda\equiv DQ(h)(D)}} c^+(n,h)\chi_D(\lambda) \\
&\times\left(\frac{q_z(\lambda)}{y^2}\right)^k e\left(\left(n+\frac{Q(\lambda)}{|D|}\right)u\right)e\left(\left(n+\frac{Q_z(\lambda)}{|D|}\right)it\right)t^{1/2}du.
\end{aligned}$$

This integral then picks out the 0th Fourier coefficient and remembering $q_z(\lambda)$ vanishes when $\lambda=0$ we have

$$= \lim_{t\to\infty}\sum_{\substack{\lambda\in L+rh\\ Q(\lambda\equiv DQ(h)(D)\\ \lambda\neq 0}} c^+\left(-\frac{Q(\lambda)}{|D|},h\right)\chi_D(\lambda)\left(\frac{q_z(\lambda)}{y^2}\right)^k e\left(\frac{-2Q(\lambda_z)}{|D|}it\right)t^{1/2}.$$

Using the same analysis in Theorem 4.1.3 (see also Lemma 4.3.6) we know this converges to 0 exponentially as $t\to\infty$ unless $Q(\lambda_z)=0$ (and $c^+\left(-\frac{Q(\lambda)}{|D|},h\right)\neq 0$ and $\chi_D(\lambda)\neq 0$). As before, this is exactly the case when $z\in Z_{D,r}(f)$. $\square$

7.3 Properties of the Shimura Lift

In this next part we use the link found in Theorem 7.2.2 to show various properties of the Shimura lift. We first have an easy corollary using this link. The fact that our lift was harmonic (away from the singularities) implies that the Shimura lift is holomorphic.

Corollary 7.3.1. *Let $g\in S_{k+1/2,\rho}$ and $z\in\mathbb{H}\backslash Z_{D,r}(f)$. Then*

$$\xi_{2k}\left(\Phi^*_{D,r,k}(z,g)\right)=0.$$

I.e. the Shimura lift is holomorphic away from the singularities.

Proof. Theorem 2.5.18 told us that the anti-linear operator mapped harmonic weak Maass forms surjectiovely to cusp forms. I.e. for any cusp form $g \in S_{k+1/2,\rho}$, there exists an $f \in H_{3/2-k,\overline{\rho}}$ such that $\xi_{3/2-k}(f) = g$. So we know using Theorem 7.2.2 that

$$\Phi^*_{D,r,k}(z,g) = \Phi^*_{D,r,k}(z,\xi_{3/2-k}(f)) = \frac{1}{2}\xi_{2-2k}(\Phi_{D,r,k}(z,f))$$

for some $f \in H_{3/2-k,\overline{\rho}}$. However for $z \in \mathbb{H}\backslash Z_{D,r}(f)$ then

$$\xi_{2k}\left(\Phi^*_{D,r,k}(z,g)\right) = \frac{1}{2}\xi_{2k}\xi_{2-2k}\left(\Phi_{D,r,k}(z,f)\right) = \frac{1}{2}D_{2-2k}\left(\Phi_{D,r,k}(z,f)\right) = 0$$

using Theorem 4.3.7. □

Remark 7.3.2. In fact we will show in Theorem 7.3.5 that the Shimura lift is holomorphic on all $z \in \mathbb{H}$. This is because ξ_{2-2k} kills the holomorphic polynomial singularities.

We next find the expansion of $\xi_{2-2k}\left(\Phi_{D,r,k}\right)$. We observe using Theorem 7.2.2 that this expansion is essentially the expansion of the Shimura lift. We make this explicit in Theorem 7.3.5. We will see that we can continue the expansion of $\xi_{2-2k}\left(\Phi_{D,r,k}\right)$ to hold for the entire upper-half plane, and not just for $z \in \mathbb{H}\backslash Z_{D,r}(f)$.

Theorem 7.3.3. *Let $f \in H_{3/2-k,\overline{\rho}}$ with expansion as in* (6.3.2) *and let $z \in \mathbb{H}\backslash Z_{D,r}(f)$. Then*

$$\xi_{2-2k}(\Phi_{D,r,k}(z,f)) = \frac{4\sqrt{2\pi}\overline{\epsilon_D}D}{i}\left(\frac{\pi D}{i\sqrt{2N}}\right)^{k-1} \times \sum_{m\geq 1}\sum_{\substack{d\geq 1\\ d|m}}\left(\frac{D}{d}\right)\frac{m^{2k-1}}{d^k}\overline{c^-\left(-\frac{m^2}{d^2}\frac{|D|}{4N},\frac{n}{d}\frac{r}{2N}\right)}e\left(mz\right)$$

and in the case $k=1, D=1$ we have an additional constant term

$$\frac{\sqrt{2}}{ik}\left(\frac{1}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1} m\cdot\overline{c^+\left(-\frac{m^2}{4N},\frac{rm}{2N}\right)}.$$

This expansion can be analytically continued to a holomorphic function on the entire upper-half plane.

Proof. We recall that $\Phi_{D,r,k}(z,f)$ is of weight $2-2k$ and we need to find

$$\xi_{2-2k}(\Phi_{D,r,k}(z,f)) = iy^{2-2k}\overline{\left(\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\Phi_{D,r,k}(z,f)\right)}$$

We deal with each part of our expansion (6.3.5) in turn.

Constant Term

We had a constant term (6.3.5a)

$$\frac{\epsilon_D|D|\sqrt{2}}{i\pi}\left(\frac{|D|}{i\pi 2\sqrt{2N}}\right)^{k-1}c^+(0,0)L\left(k,\left(\tfrac{D}{\cdot}\right)\right)$$

this does not depend on x or y so immediately vanishes under ξ_{2-2k}.

The c^+ terms

Looking at (6.3.5b), (6.3.5c) (a finite sum over m) we consider for $mx + b/D \notin \mathbb{Z}$ (away from the vertical half-line singularities) and $k \geq 2$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left[B_k\left(\langle mx + b/D\rangle + imy\right) + \frac{k\mathrm{I}_{\mathbb{Z}}(mx+b/D)}{(imy)^{1-k}}\right]$$

which we see, by using Lemma 6.3.7 part 3, vanishes. In the case of $k = 1$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left[\mathbb{B}_1(mx + b/D)\right] = \frac{m}{2}$$

and we remember $\sum_{b(D)}\left(\frac{D}{b}\right) = 0$ unless $D = 1$. This gives the constant term in the stated result.

The c^- terms

We use the from given in(6.3.17). We may swap the partial derivatives with the sums over m, n as we have absolute and locally uniform convergence of the series (noting $m, n \geq 1$) for $z \in \mathbb{H}$. Then

$$\begin{aligned}&\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left[e\left(n\left(mz + b/D\right)\right) + \frac{(-1)^k\mathrm{sgn}(D)}{(2k-2)!}e\left(-n\left(mz - b/D\right)\right)\Gamma(2k-1, 4\pi nmy)\right]\\ &= \frac{(-1)^k\mathrm{sgn}(D)}{i(2k-2)!}e\left(-n\left(m\overline{z} - b/D\right)\right)\left(4\pi nm\right)^{2k-1}y^{2k-2}.\end{aligned}$$

Using the fact that

$$\sum_{b(D)}\left(\frac{D}{b}\right)e\left(\frac{nb}{D}\right) = \mathrm{sgn}(D)\left(\frac{D}{n}\right)\sqrt{D}$$

we see that $\xi_{2-2k,z}$ of (6.3.5d) (6.3.5e) is

$$\frac{4\sqrt{2\pi}\overline{\epsilon_D}D}{i}\left(\frac{\pi D}{i\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{n\geq 1}\left(\frac{D}{n}\right)m^{2k-1}n^{k-1}\overline{c^-\left(-\frac{|D|m^2}{4N}, \frac{rm}{2N}\right)}e\left(nmz\right).$$

and then making the substitutions $m \mapsto \frac{m}{n}$ and $n \mapsto d$ we have

$$\frac{4\sqrt{2\pi}\overline{\epsilon_D}D}{i}\left(\frac{\pi D}{i\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1}\sum_{\substack{d\geq 1\\ d|m}}\left(\frac{D}{d}\right)\frac{m^{2k-1}}{d^k}\overline{c^-\left(-\frac{m^2}{d^2}\frac{|D|}{4N}, \frac{n}{d}\frac{r}{2N}\right)}e\left(mz\right). \tag{7.3.1}$$

Theorem 4.3.7 told us that $\Phi_{D,r,k}(z, f)$ was real analytic for $z \in \mathbb{H}\backslash Z_{D,r}(f)$. Theorems 6.3.10 and 4.2.2 told us that our Fourier expansion held everywhere, even on the singularities and for $y < \sqrt{-|D|n_0/N}$, if we added on appropriate polynomials when crossing walls. We remember, from (3.6.4), that these polynomials were of the form

$$q_z(\lambda) = y(\lambda, v(z^\perp)) = \frac{-1}{\sqrt{2N}}\left(cNz^2 - bz + a\right).$$

I.e. holomorphic. So they vanish when we apply the ξ_{2-2k} operator and we can smoothly continue $\xi_{2-2k}\left(\Phi_{D,r,k}(z, f)\right)$ to the entire upper-half plane with the expansion given in (7.3.1).

Corollary 7.3.1 said that $\xi_{2-2k}\left(\Phi_{D,r,k}(z,f)\right)$ was holomorphic for all $z \in \mathbb{H}\backslash Z_{D,r}(f)$ but clearly the expansion in (7.3.1) is holomorphic for all $z \in \mathbb{H}$ so this also provides a holomorphic continuation of $\xi_{2-2k}\left(\Phi_{D,r,k}(z,f)\right)$ to the entire upper-half plane. □

The previous theorem in conjunction with Theorem 7.2.2 then allows us to easily find the Fourier expansion of the Shimura lift.

Remark 7.3.4. An alternative way to find this expansion is to repeat the same analysis that we carried out for the singular theta lift. I.e. write the Shintani kernel function as a Poincaré series and then unfold and simplify the integral in Definition 7.1.1. This, as we have seen, would take considerable work. This method can be found in [Bor98, Section 14]

Theorem 7.3.5. *Let $g \in S_{k+1/2,\rho}$ have Fourier expansion of the form* (2.5.1)

$$g(\tau) = \sum_{h \in L'/L} \sum_{\substack{n \in \mathbb{Z}+\mathrm{sgn}(D)Q(h) \\ n>0}} a(n,h) e(n\tau)\mathfrak{e}_h \tag{7.3.2}$$

then the Fourier expansion of the Shimura lift is holomorphic and is given by

$$\Phi^*_{D,r,k}(z,g) = 2i\epsilon_D\sqrt{2N|D|}\left(\frac{\mathrm{sgn}(D)\sqrt{N}}{i\sqrt{2}}\right)^{k-1} \times \sum_{m\geq 1}\sum_{\substack{d\geq 1 \\ d|m}}\left(\frac{D}{d}\right)d^{k-1}a\left(\frac{m^2}{d^2}\frac{|D|}{4N}, \frac{m}{d}\frac{r}{2N}\right)e(mz)$$

and in the case when $k=1, D=1$ we have an additional constant term

$$\frac{1}{ik\sqrt{2}}\left(\frac{1}{\sqrt{2N}}\right)^{k-1}\sum_{m\geq 1} m\cdot\overline{c^+\left(-\frac{m^2}{4N},\frac{rm}{2N}\right)}.$$

where $c^+(-m,h)$ are the coefficients of the principal part of any $f \in H_{3/2-k,\bar{\rho}}$ such that $\xi_{3/2-k}(f) = g$.

Proof. As in Corollary 7.3.1 we know there exists an $f \in H_{3/2-k,\bar{\rho}}$ such that $\xi_{3/2-k}(f) = g$. This f must have a Fourier expansion of the form

$$f^+ = \sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}-\mathrm{sgn}(D)Q(h) \\ n\gg-\infty}} c^+(n,h)e(n\tau)\mathfrak{e}_h,$$

$$f^- = \sum_{h\in L'/L}\sum_{\substack{n\in\mathbb{Z}-\mathrm{sgn}(D)Q(h) \\ n<0}}\left[-\frac{\overline{a(-n,h)}}{(-4\pi n)^{k-1/2}}\right]\Gamma(k-1/2,-4\pi n v)e(n\tau)\mathfrak{e}_h,$$

to agree with (7.3.2) i.e. $c^-(n,h) = -\overline{a(-n,h)}(-4\pi n)^{1/2-k}$. We can check this by using 2.5.4. Then as

$$\Phi^*_{D,r,k}(z,g) = \frac{1}{2}\xi_{2-2k}\left(\Phi_{D,r,k}(z,f)\right)$$

for $z \in \mathbb{H} \backslash Z_{D,r}(f)$ by Theorem 7.2.2 we then plug in $c^-(n,h) = -\overline{a(-n,h)}(-4\pi n)^{1/2-k}$ into Theorem 7.3.3 to obtain

$$2i\epsilon_D \sqrt{2N|D|} \left(\frac{\operatorname{sgn}(D)\sqrt{N}}{i\sqrt{2}}\right)^{k-1} \sum_{m\geq 1} \sum_{\substack{d \geq 1 \\ d|m}} \left(\frac{D}{d}\right) d^{k-1} a\left(\frac{m^2}{d^2}\frac{|D|}{4N}, \frac{m}{d}\frac{r}{2N}\right) e\left(mz\right)$$

plus when $k=1$ and $D=1$ we have an additional constant term

$$\frac{1}{ik\sqrt{2}}\left(\frac{1}{\sqrt{2N}}\right)^{k-1} \sum_{m\geq 1} m \cdot \overline{c^+\left(-\frac{m^2}{4N}, \frac{rm}{2N}\right)}. \tag{7.3.3}$$

As in the previous proof this expansion is analytically continued to hold for all $z \in \mathbb{H}$ and is a holomorphic function. □

We can check this expansion agrees (up to constants) with the one given in [Höv12, Theorem 4.1] which is adapted from [SZ88, GKZ87, Sko90a, Sko90b].

Remark 7.3.6. In the case $k=1, D=1$ we notice the additional constant term depends only on the $c^+(n,h)$ coefficients of f (where $f \in H_{3/2-k,\overline{\rho}}$ such that $\xi_{3/2-k} f = g$). This implies that g *uniquely determines* this constant. This at first glance seems strange as f^- in general does not determine the $c^+(n,h)$ coefficients. This is actually an example of Corollary 2.5.25 using the unary theta function. A lot more detail of this can be found in [Höv12, Section 4.4] which we do not repeat here. However we do obtain the following corollary.

Corollary 7.3.7. *For $f \in M^!_{1/2,\overline{\rho}}$ with expansion as in* (6.3.2) *and $D=1$ i.e. $r^2 \equiv 1 \pmod{4N}$ then*

$$\sum_{m\geq 1} m \cdot \overline{c^+\left(-\frac{m^2}{4N}, \frac{rm}{2N}\right)} = 0.$$

Proof. We use Theorem 7.3.5 in the case when $k=1, D=1$ and $g \equiv 0$. Any $f \in M^!_{1/2,\overline{\rho}}$ satisfies $\xi_{1/2}(f) = g \equiv 0$. Then certainly $\Phi^*_{D,r,1}(z,g) \equiv 0$ using Definition 7.1.1 but we also know

$$\Phi^*_{D,r,1}(z,g) = \frac{1}{ik\sqrt{2}}\left(\frac{1}{\sqrt{2N}}\right)^{k-1} \sum_{m\geq 1} m \cdot \overline{c^+\left(-\frac{m^2}{4N}, \frac{rm}{2N}\right)}.$$

This implies the stated result. □

We are now in the position to show that the Shimura lift maps to cusp forms most of the time. We can use the Atkin-Lehner involutions to find the Fourier expansions at all the cusps.

Theorem 7.3.8. *Let $k \in \mathbb{Z}, k \geq 2$ or $k=1, D \neq 0$ and N square-free. Then $\Phi_{D,r,k}$ is a cusp form of weight $2k$ for the group $\Gamma_0(N)$ i.e.*

$$\Phi^*_{D,r,k} : S_{k+1/2,\rho} \to S_{2k}(\Gamma_0(N)).$$

In the case $k=1, D=1$ and N square-free, $\Phi_{D,r,k}$ is a modular form of weight $2k$ for the group $\Gamma_0(N)$.

Proof. Let $g \in S_{k+1/2,\rho}$ and $f \in H_{3/2-k,\overline{\rho}}$ such that $\xi_{3/2-k} f = g$. We know from Corollary 7.1.3 that $\Phi^*_{D,r,k}(z,g)$ is a weight $2k$ automorphic form for $\Gamma_0(N)$. We also know using Theorem 7.3.5 that the Shimura lift is holomorphic. It remains to check holomorphicity at all the cusps.

For the cusp ∞ this is clear as we only have a constant term in Theorem 7.3.5 when $k = 1, D = 1$. For any other cusp $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, we know $W^N_m \infty = s$ for some exact divisor m of N. Then

$$\left(\xi_{2-2k}\left(\Phi_{D,r,k}\right)\big|_{2k} W^N_m\right)(z,f) = \xi_{2-2k}\left(\Phi_{D,r,k}(z, f_{W^N_m}\right).$$

So we can find Fourier expansions of $\Phi^*_{D,r,k}(z,f)$ by just considering the expansion in Theorem 7.3.3 at ∞ for $f_{W^N_m}$. So again there is a constant term only when $k = 1, D = 1$.

In the case $k = 1, D = 1$ we have the same properties except the constant term of the expansions at the cusps does not vanish so we have a modular form. □

Remarks 7.3.9. In fact, in the case when N is not square-free, we can find analogous Fourier expansions at all the cusps, see Remarks 6.3.11 and 5.1.2. So this result can also be investigated for any N. It is also well known (see for example [Niw75] [Bor98, Section 14]) that the Shimura lift maps to cusp forms for all $k \geq 2$.

In the case $k = 1, D = 1$ we can still obtain cusp forms. This essentially happens when we lift a cusp form that is orthogonal to the unary theta function and the weighted sum (7.3.3) disappears. Again the details of this can be found in [Höv12, Theorem 4.11].

We end this section with the following commutative diagram which holds for $k \in \mathbb{Z}, k \geq 2$ (or $k = 1, D \neq 1$) and summarises these results nicely.

$$\begin{tikzcd}
H_{3/2-k,\overline{\rho}} \arrow[r, "\Phi_k"] \arrow[d, "\xi_{3/2-k}"'] & LH_{2-2k}(\Gamma_0(N)) \arrow[d, "\xi_{2-2k}"] \\
S_{k+1/2,\rho} \arrow[r, "\Phi_k^*"'] & S_{2k}(\Gamma_0(N))
\end{tikzcd}$$

Figure 7.1: Commutative Diagram

7.4 Locally Harmonic Maass Forms as Distributions

In this final section we will think about locally harmonic Maass forms as distributions (generalised functions). We are motivated to introduce this concept as the singularities in Definition 2.5.30 are of a similar nature to those found in the Heaviside step function.

We first form our definition and then look at the associated current equation. Finally we reinterpret the link between the two lifts from this cohomological point of view. The main reference is [BF04, Section 7]. Throughout this section we fix $\kappa \in 2\mathbb{Z}, \kappa \leq 0$.

We first briefly consider the classical theory of distributions, following [Gru09]. This will help motivate the definitions that we introduce soon. The idea is to first form a space of test functions. Letting Ω be an open subset of $\mathbb{R}^n$ then we let $\mathcal{D}(\Omega)$ be the space of functions on Ω which are smooth with compact support. We call this the space of test functions. Then a distribution on Ω is just a continuous linear functional on $\mathcal{D}(\Omega)$.

For example if h is a locally integrable function $h : \mathbb{R} \to \mathbb{R}$ we can let

$$[h]\,(g) \coloneqq \int_{\mathbb{R}} h(x)g(x)dx \tag{7.4.1}$$

for $g \in \mathcal{D}(\mathbb{R})$, be its corresponding distribution. We can then define the derivative of the distribution to be

$$d\,[h]\,(g) \coloneqq -\int_{\mathbb{R}} h(x)g'(x)dx.$$

This is the key idea as this concept of the derivative makes sense even on singularities. We also see that $d\,[h] = [h']$ if h is smooth. We examine an example: the Heaviside step function.

Definition 7.4.1. *For $x \in \mathbb{R}$ we let the **Heaviside step function** be defined as*

$$H(x) := \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

This has a jump singularity of size 1 in the sense of Definition 4.2.1. As a distribution we define this as

$$[H]\,(g) := \int_{\mathbb{R}} H(x)g(x)dx$$

where $g \in D(\mathbb{R})$. We also define the **delta distribution**. This is the linear functional

$$[\delta]\,(g) := g(0)$$

where $g \in D(\mathbb{R})$. This is the value of the test functions at the singularity of $H(x)$. There is no consistent interpretation of the delta distribution as a classical function. We see that the delta distribution is the derivative of the Heaviside step distribution. I.e.

$$d\,[H]\,(g) = -\int_{\mathbb{R}} H(x)g'(x)dx = -\int_0^\infty g'(x)dx = g(0) - g(\infty) = g(0) = [\delta]\,(g).$$

We follow these ideas in our case. From now we will let the space of **test functions** be smooth functions $g \in A_\kappa\left(\Gamma_0(N)\right)$ with rapid (exponential) decay at the cusps of $\Gamma_0(N)$. We will denote this space as $A_\kappa^c\left(\Gamma_0(N)\right)$. Mirroring (7.4.1) it seems natural to form our version of the definition of a distribution as follows:

Definition 7.4.2. *We let a locally harmonic Maass form $h \in LH_\kappa\left(\Gamma_0(N)\right)$ (with associated exceptional set E) define a **distribution** $[h]$ on $Y_0(N)$ where*

$$[h]\,(g) := (g, h)_\kappa = \int_{Y_0(N)} g(z)\overline{h(z)}y^\kappa \frac{dxdy}{y^2}$$

for $g \in A_\kappa^c\left(\Gamma_0(N)\right)$.

To confirm this definition make sense and converges we have the following simple lemma. We have use the (scalar-valued) Petersson scalar product from Definition 2.5.32 which we do not need to regularise in this case.

Lemma 7.4.3. *For any locally harmonic Maass form $h \in LH_\kappa\left(\Gamma_0(N)\right)$ with associated exceptional set E, then*

$$(g, h)_\kappa = \int_{Y_0(N)} g(z)\overline{h(z)}y^\kappa \frac{dxdy}{y^2}$$

converges for any $g \in A_\kappa^c\left(\Gamma_0(N)\right)$.

Proof. Using Definition 2.5.30 properties 2 and 3 we know that f is locally bounded and g is smooth. We consider convergence as $y \to \infty$. We have requested that g rapidly decays. A locally harmonic Maass form only has polynomial growth at all the cusps so we have convergence. $\square$

For our distributions, involving automorphic forms, it also seems natural to consider the differential operator ξ_κ (as opposed to the derivative in the classical case). We also want to be able to apply these concepts to Theorem 7.2.2.

Definition 7.4.4. *Let $h \in LH_\kappa\left(\Gamma_0(N)\right)$ and $[h]$ be its associated distribution. Then we let*

$$\xi_\kappa\left[h\right] := -\left(h, \xi_{2-\kappa}(g)\right)_\kappa$$

for $g \in A^c_{2-\kappa}\left(\Gamma_0(N)\right)$.

We can then consider ξ_κ (the derivative) of the distribution of any locally harmonic Maass form f. I.e. find the integral of h against a test function $\xi_{2-\kappa}(g)$. This generates a current equation in the sense of [BF04, Section 7]. We consider for simplicity the specific case where h is the singular theta lift.

Theorem 7.4.5. *Let $f \in H_{3/2-k,\overline{\rho}}$. Then*

$$\xi_{2-2k}\left[\Phi_{D,r,k}(z,f)\right](g) = \left[\xi_{2-2k}(\Phi_{D,r,k}(z,f))\right](g) - \sqrt{2|D|}\int_{Z'_{D,r}(f)} g(z) q_z(\lambda)^{k-1} dz$$

where $g \in A^c_{2k}\left(\Gamma_0(N)\right)$.

Proof. We start with the integral on the left hand side:

$$\begin{aligned}
\xi_{2-2k}\left[\Phi_{D,r,k}(z,f)\right] &= -\int_{Y_0(N)} \Phi_{D,r,k}(z,f)\overline{\xi_{2k}(g(z))}y^{2-2k}\frac{dxdy}{y^2} \\
&= -\overline{\int_{Y_0(N)} y^{2-2k}\overline{\Phi_{D,r,k}(z,f)}\cdot\overline{L_{2k}(g(z))}y^{2k-2}\frac{dxdy}{y^2}}.
\end{aligned}$$

We then use [Bru02, Lemma 4.2] to obtain:

$$\begin{aligned}
&= \int_{Y_0(N)} \overline{R_{2k-2}\left(y^{2-2k}\overline{\Phi_{D,r,k}(z,f)}\right)}g(z)y^{2k}\frac{dxdy}{y^2} \\
&\quad - \int_{Y_0(N)} d\left(g(z)\Phi_{D,r,k}(z,f)\right)dz \\
&= \left[\xi_{2-2k}(\Phi_{D,r,k}(z,f))\right] - \int_{Y_0(N)} d\left(g(z)\Phi_{D,r,k}(z,f)\right)dz.
\end{aligned}$$

So now we just have to look at the right hand term. We can decompose $\Phi_{D,r,k}(z,f)$ into its smooth and singular parts, both of which are of weight $2-2k$ for $\Gamma_0(N)$, see Theorem 4.2.2. For the smooth part $h(z)$ we know (see for example Lemma 6.1.1 and [Bru02, Lemma 4.2]) that

$$\lim_{t\to\infty}\int_{Y_0(N)} d\left(g(z)h(z)\right)dz = \int_{-1/2}^{1/2}\left[g(z)h(z)\right]_{y=t}dx.$$

This then simply vanishes as h only grows polynomially and g decays exponentially as $t \to \infty$. We now consider the singular part of $\Phi_{D,r,k}(z,f)$. Using Theorem 4.2.2 it suffices to consider

$$\begin{aligned}
&-\sqrt{\frac{|D|}{2}} \sum_{h\in L'/L} \sum_{\substack{m\in\mathbb{Z}-\operatorname{sgn}(D)Q(h)\\ m<0}} c^+(m,h) \\
&\times \int_{Y_0(N)} d\left(g(z)\sum_{\lambda\in L_{-dD,rh}} \chi_D(\lambda)\frac{(\lambda,v(z))}{|(\lambda,v(z))|}q_z(\lambda)^{k-1}\Gamma\left(\frac{1}{2},\frac{-4\pi Q(\lambda_z)}{|D|}\right)\right)dz. \\
=&-\sqrt{\frac{|D|}{2}} \sum_{h\in L'/L} \sum_{\substack{m\in\mathbb{Z}-\operatorname{sgn}(D)Q(h)\\ m<0}} c^+(m,h) \sum_{\lambda\in\Gamma_0(N)\backslash L_{-dD,rh}} \chi_D(\lambda) \\
&\times \sum_{\gamma\in\Gamma_\lambda\backslash\Gamma_0(N)} \int_{Y_0(N)} d\left(g(z)\frac{(\gamma^{-1}.\lambda,v(z))}{|(\gamma^{-1}.\lambda,v(z))|}q_z(\gamma^{-1}.\lambda)^{k-1}\Gamma\left(\frac{1}{2},\frac{-4\pi Q((\gamma^{-1}.\lambda)_z)}{|D|}\right)\right)dz.
\end{aligned} \tag{7.4.2}$$

Lemma 3.6.9 told us that $q_{\gamma.z}(\gamma.\lambda) = j(\gamma,z)^{-2}q_z(\lambda)$. We also know that $g(z)$ has weight $2k$ and

$$(\gamma.\lambda)_{(\gamma.z)} = \gamma.(\lambda_z) \quad \text{and} \quad (\gamma.\lambda, v(\gamma.z)) = (\lambda, v(z)).$$

So we see the last line (7.4.2) is equal to

$$\int_{\Gamma_\lambda\backslash\mathbb{H}} d\left(g(z)\frac{(\lambda,v(z))}{|(\lambda,v(z))|}q_z(\lambda)^{k-1}\Gamma\left(\frac{1}{2},\frac{-4\pi Q(\lambda_z)}{|D|}\right)\right)dz.$$

For $z \in \mathbb{H}$ and any cycle D_λ we let

$$\operatorname{dist}(z, D_\lambda) := \min\{|z-w| \mid w \in D_\lambda\}.$$

For any $\epsilon > 0$ let

$$U_\epsilon(E) := \{z \in \mathbb{H} \mid \operatorname{dist}(z, D_\lambda) < \epsilon\},$$

which defines an ϵ-neighbourhood around the cycle. We then use Stokes' theorem to obtain

$$\begin{aligned}
&\int_{\Gamma_\lambda\backslash\mathbb{H}} d\left(g(z)\frac{(\lambda,v(z))}{|(\lambda,v(z))|}q_z(\lambda)^{k-1}\Gamma\left(\frac{1}{2},\frac{-4\pi Q(\lambda_z)}{|D|}\right)\right)dz \\
&= \lim_{\epsilon\to 0}\int_{\partial(\Gamma_\lambda\backslash(\mathbb{H}\backslash U_\epsilon(\lambda)))} g(z)\frac{(\lambda,v(z))}{|(\lambda,v(z))|}q_z(\lambda)^{k-1}\Gamma\left(\frac{1}{2},\frac{-4\pi Q(\lambda_z)}{|D|}\right)dz \\
&= 2\sqrt{\pi}\int_{\Gamma_\lambda\backslash D_\lambda} g(z)q_z(\lambda)^{k-1}.
\end{aligned}$$

This is clear from the following facts. We oriented our cycles earlier. So approaching $(\lambda,v(z))/|(\lambda,v(z))|$ with a left orientation or right orientation generates -1 or 1 i.e. 2. The contributions from the $\Gamma_0(N)$-equivalent boundary pieces cancel and also as $Q(\lambda_z) \to 0$ the $\Gamma\left(\frac{1}{2},\frac{-4\pi Q(\lambda_z)}{|D|}\right)$ term approaches $\Gamma(1/2) = \sqrt{\pi}$. Putting all of this together gives the stated theorem. $\square$

Remark 7.4.6. In general, if $h \in LH_\kappa(\Gamma_0(N))$, then we see that $\xi_\kappa[h] = [\xi_\kappa(h)]$ if h is smooth. This was as we hoped. If h it is not smooth then we picked up an additional term. This is (roughly) the integral over the exceptional set of g multiplied by the size of the singularity. This term corresponds to the classical delta distribution. In Theorem 7.4.5 this was given by $\int_{Z'_{D,r}(f)} g(z)q_z(\lambda)^{k-1}dz$.

Using Theorem 7.2.2 in combination with Theorem 7.4.5 we then have the following immediate corollary.

Corollary 7.4.7. *Let $f \in H_{3/2-k,\overline{\rho}}$. Then*

$$\xi_{2-2k}[\Phi_{D,r,k}(z,f)](g) + \sqrt{2|D|}\int_{Z'_{D,r}(f)} g(z)q_z(\lambda)^{k-1}dz = 2\left[\Phi^*_{D,r,k}(z,\xi_{3/2-k}(f))\right](g)$$

where $g \in A^c_{2k}(\Gamma_0(N))$.

This is a better interpretation of Theorem 7.2.2. We still see the link between the two lifts. However thinking of the lifts as distributions also means that we see what happens at the singularities.

We now consider the case where $g \in S_{2k}(\Gamma_0(N))$. This is a smooth rapidly decaying weight $2k$ form so it is an element of $A^c_{2k}(\Gamma_0(N))$ but it is also holomorphic. In particular it vanishes under the ξ_{2k} operator so we have another easy corollary.

Corollary 7.4.8. *Let $f \in H_{3/2-k,\overline{\rho}}$. Then*

$$\left[\Phi^*_{D,r,k}(z,\xi_{3/2-k}(f))\right](g) = \sqrt{\frac{|D|}{2}}\int_{Z'_{D,r}(f)} g(z)q_z(\lambda)^{k-1}dz$$

where $g \in S_{2k}(\Gamma_0(N))$.

Proof. This is a simple application of Theorem 7.4.5 where we see that if g is a cusp form then $\xi_{2-2k}[\Phi_{D,r,k}(z,f)](g) = (\Phi_{D,r,k}(z,f),\xi_{2k}(g))_{2-2k} = 0$. $\square$

So this corollary tells us what happens when we integrate a cusp form against the Shimura lift. In particular it is equal to some period integral. See for example [FM11, Section 4] and [Shi75, Section 3].

Our final observation of this chapter is that we also have an interpretation of Corollary 7.4.8 in terms of the Shintani lift. To show this we will first need to define the Shintani lift, we keep the details brief. The Shintani lift as mentioned in the introduction is adjoint to the Shimura lift and "maps the other way". In particular we integrate in the z variable using the same Shintani kernel function to obtain a map from even weight cusp forms to half-integral weight cusp forms.

Definition 7.4.9. *Let $\kappa \in \mathbb{Z}, \kappa \geq 1$. For $g \in S_{2\kappa}(\Gamma_0(N))$ we will let*

$$\begin{aligned}\varphi^*_{D,r,k}(\tau, g) &:= (g(z), \overline{\Theta^*_{D,r,\kappa}(\tau, z)})_{2\kappa} \\ &= \int_{Y_0(N)} g(z)\Theta^*_{D,r,\kappa}(\tau, z)y^{2\kappa}\frac{dxdy}{y^2}\end{aligned}$$

be the twisted Shintani lift.

This definition makes sense as Theorem 3.6.11 told us that $\Theta^*_{D,r,k}(\tau, z)$ is of weight $2k$ in z. Shintani [Shi75, Section 1.7] tells us that $\Theta^*_{D,r,k}(\tau, z)$ has polynomial growth in z. Alternatively we can check this analogously to Proposition 5.4.6. The exponential decay of the cusp form g then ensures the (scalar-valued) Petersson scalar product converges absolutely and defines a real analytic (and therefore smooth) function on $\mathbb{H}$. It is clear from Theorem 3.6.8 that $\varphi^*_{D,r,k}(\tau, g)$ will have weight $k + 1/2$ i.e. an element of $A_{k+1/2,\rho}$. In fact [Shi75, Theorem 1] tells us this will be a cusp form as we would hope.

We then use Corollary 7.4.8 to easily find the integral of the Shintani lift against a cusp form ($\xi_{3/2-k}(f) \in S_{k+1/2,\rho}$). In particular it is equal to some period integrals.

Corollary 7.4.10. *Let $f \in H_{3/2-k,\overline{\rho}}$ and $g \in S_{2k}\left(\Gamma_0(N)\right)$. Then*

$$\left(\varphi^*_{D,r,k}(\tau, g), \xi_{3/2-k}(f)\right)_{k+1/2,\rho} = \sqrt{\frac{|D|}{2}}\int_{Z'_{D,r}(f)} g(z)q_z(\lambda)^{k-1}dz.$$

Proof. This clear from Corollary 7.4.8 after noticing

$$\begin{aligned}&\left[\Phi^*_{D,r,k}(z, \xi_{3/2-k}(f))\right](g) \\ =& \int_{Y_0(N)} g(z)\overline{\Phi^*_{D,r,k}(z, \xi_{3/2-k}(f))}y^{2k}\frac{dxdy}{y^2} \\ =& \int_{Y_0(N)} g(z)\int_{\tau\in\mathcal{F}} \left\langle\Theta^*_{D,r,k}(\tau, z), \xi_{3/2-k}(f(\tau))\right\rangle v^{k+1/2}\frac{dudv}{v^2}y^{2k}\frac{dxdy}{y^2} \\ =& \int_{\tau\in\mathcal{F}} \left\langle\int_{Y_0(N)} g(z)\Theta^*_{D,r,k}(\tau, z)y^{2k}\frac{dxdy}{y^2}, \xi_{3/2-k}(f(\tau))\right\rangle v^{k+1/2}\frac{dudv}{v^2} \\ =& \int_{\tau\in\mathcal{F}} \left\langle\varphi^*_{D,r,k}(\tau, g), \xi_{3/2-k}(f(\tau))\right\rangle v^{k+1/2}\frac{dudv}{v^2} \\ =& \left(\varphi^*_{D,r,k}(\tau, g), \xi_{3/2-k}(f)\right)_{k+1/2,\rho}.\end{aligned}$$

We can swap the integrals as both g and $\xi_{3/2-k}(f)$ decay exponentially and $\Theta^*_{D,r,k}(\tau, z))$ only has polynomial growth in both variables. $\square$

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