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Hadronic Production of a Higgs Boson in Association with a Jet at Next-to-Next-to-Leading Order

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Dedicated to Shuo and my parents.

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Abstract

In this thesis the production of a Higgs boson in association with a hadronic jet at the Large Hadron Collider is studied using the effective interaction between the Higgs boson to gluons induced by a heavy quark. The Leading Order (LO), Next-to-Leading Order (NLO) and Next-to-Next-to-Leading Order (NNLO) perturbative QCD corrections are studied for all of the parton channels. The infrared (IR) divergent behaviour of the various contributions to the partonic cross section is regulated using the antenna subtraction formalism. This method has previously been used at NNLO in the calculation of three jets production in the e^+e^- annihilation and for the gluonic dijet production via proton collision. The research presented in this thesis extends the antenna formalism to include scattering processes in which the initial state parton changes its identity. All contributions to the $pp \to H+$jet processes are calculated at LO, NLO and NNLO and numerically tested to demonstrate the convergence between the matrix elements and the antenna subtraction terms in the various unresolved limits. As an example of the phenomenological impact of this work, numerical results for the total and differential Higgs plus one jet cross sections are presented for the purely gluonic subprocesses.

Declaration

The work in this thesis is based on research carried out at the Institute for Particle Physics Phenomenology, Department of Physics, University of Durham, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification. All work is my own unless referenced to the contrary in the text. Chapter 10 is based upon research done in collaboration with Thomas Gehrmann, Nigel Glover and Matthieu Jaquier, and has been published as:

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Chapter 1

Introduction to QCD and the Hadron Collider Environment

The main part of the research in this thesis focuses on the study of higher order corrections in QCD. In this chapter, I first introduce QCD at the level of the Lagrangian density from which the basic interactions of QCD particles can be derived in perturbation theory. To use these basic couplings to calculate physical observables, the idea of renormalization is motivated to remove ultraviolet singularities and fix the physical coupling constants. The physical cross section from parton model is defined and the higher order QCD corrections are made explicit in the perturbative expansion of the cross sections. At the Large Hadron Collider, protons are collided at very high energies. The improved parton model describes these interactions and is introduced by considering collinear factorization which links non-perturbative behaviour in the proton to the perturbative interactions of incoming partons. Both the renormalization and factorization processes preserve the properties of a self-similar system where one needs unphysical scales to describe the intermediate stages of the calculation. The infrared (IR) singularities, which appear both implicitly and explicitly during perturbative calculations, are one of the main challenges faced in this thesis and are briefly introduced. Infrared safe observables are introduced to help restrict the IR divergences in predictable structures which will be discussed intensively in chapter 3 and 4.

1.1 QCD Lagrangian

Quantum Chromodynamics (QCD) is the theory of the strong interaction and is based on a non-abelian Yang-Mills quantum field theory. The Lagrangian density for Quantum Chromodynamics (QCD) is gauge invariant under $SU(3)$ transformations and contains four fundamental parts:

$$\mathcal{L}_{QCD} = \mathcal{L}_{quark} + \mathcal{L}_{gluon} + \mathcal{L}_{g.f.} + \mathcal{L}_{ghost}. \tag{1.1.1}$$

The $\mathcal{L}_{quark}$ and $\mathcal{L}_{gluon}$ terms describe the kinetic energy and interactions of the quarks and gluons,

$$\mathcal{L}_{quark} = \sum_q \bar{\psi}_q^i (i\slashed{D}_{ij} - m_q \delta_{ij}) \psi_q^j \tag{1.1.2}$$

$$\mathcal{L}_{gluon} = -\frac{1}{4} \text{tr}\left(\mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} \right), \tag{1.1.3}$$

where the summation of q is over the quark flavours, m_q is corresponding mass of each flavour and ψ_q^i is the quark field. The i, j labels represent the fundamental representation indices of $SU(3)$, $i, j = 1, 2, 3$. The covariant derivative and gluon field strength tensor are defined as

$$\slashed{D}_{ij} = \gamma^\mu (\mathbf{I}\partial_\mu + ig\mathbf{A}_\mu)_{ij}, \tag{1.1.4}$$

$$\mathbf{F}_{\mu\nu} \equiv \frac{i}{g}[\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - ig[\mathbf{A}_\mu, \mathbf{A}_\nu], \tag{1.1.5}$$

where g is the QCD gauge coupling strength, $\mathbf{I}$ is the three by three unit matrix and $\mathbf{A}$ contains each of the $SU(3)$ group generators in the adjoint representation,

$$\mathbf{A}_\mu = A_\mu^a T^a. \tag{1.1.6}$$

In the Lie Algebra of $SU(3)$, there are eight independent group generators in the adjoint representation thus $a = 1, \cdots, 8$. Each T^a matrix is a three by three traceless Hermitian matrix with the structure constant f^{abc} of $SU(3)$ defined from the commutators of T^a,

$$[T^a, T^b] = i\sqrt{2}f^{abc}T^c. \tag{1.1.7}$$

The QCD Lagrangian is invariant under local gauge transformations. This means that the physical properties of quarks are universal in different locations of space

time. Under a general local gauge transformation of the quark field,

$$\psi_q^i \to \Omega(x)\psi_q^i, \tag{1.1.8}$$

$$\bar{\psi}_q^i \to \bar{\psi}_q^i \Omega^{-1}(x). \tag{1.1.9}$$

To guarantee that $\mathcal{L}_{quark}$ and $\mathcal{L}_{gluon}$ in Eqs. (1.1.2) and (1.1.3) are unchanged under such a local transformation, the covariant derivative must transform in the following way

$$\mathbf{D}_\mu \to \Omega(x)\mathbf{D}_\mu\Omega^{-1}(x). \tag{1.1.10}$$

From Eq. (1.1.4), the local gauge transformation of $\mathbf{D}_\mu$ is satisfied by the gluon field transformation,

$$\mathbf{A}_\mu \to \mathbf{A}_\mu^\Omega = \frac{i}{g}\Omega(x)(\mathbf{I}\partial_\mu\Omega^{-1}(x)) + \Omega(x)\mathbf{A}_\mu\Omega^{-1}(x). \tag{1.1.11}$$

The above transformation rules out the possibility of having a gluon mass term $m_g\mathbf{A}^\mu\mathbf{A}_\mu$ which is not locally gauge invariant.

Although the QCD Lagrangian is locally gauge invariant, it is not obviously renormalizable as the path integral would sum over all the possible gauge transformations. The solution to this problem is to introduce the gauge fixing terms, $\mathcal{L}_{g.f.} + \mathcal{L}_{ghost}$. The gauge symmetry is broken by the gauge fixing terms. However, the physical observables are independent from the choice of a specific gauge and the full QCD field theory can be made renormalizable. Specifically, by introducing the gauge fixing functional $\mathcal{G}[\mathbf{A}_\mu^\Omega(x)]$, the gauge fixing condition can be inserted into the functional integral through the Dirac function,

$$1 = \int \mathrm{d}\Omega\delta(\mathcal{G}[\mathbf{A}_\mu^\Omega])\mathtt{det}\left(\frac{\delta\mathcal{G}[\mathbf{A}_\mu^\Omega]}{\delta\Omega}\right). \tag{1.1.12}$$

The gauge fixing condition $\delta(\mathcal{G}[\mathbf{A}_\mu^\Omega])$ in Eq. (1.1.12) leads to the gauge fixing term,

$$\mathcal{L}_{g.f.} = -\frac{1}{\xi}tr(\mathcal{G}[\mathbf{A}_\mu]^2), \tag{1.1.13}$$

where ξ is a gauge parameter defining different gauge choices. For example, $\xi = 0$ (Landau gauge), $\xi = 1$ (Feynman gauge) and $\xi = \infty$ (Unitary gauge). There are two classes of popular choices of the gauge fixing functional. For the Lorentz gauge $\mathcal{G}[\mathbf{A}_\mu^\Omega] = \partial_\mu\mathbf{A}^\mu$, while for the axial gauge $\mathcal{G}[\mathbf{A}_\mu^\Omega] = n_\mu\mathbf{A}^\mu$ where n_μ is a arbitrary

space-time vector. The advantage of the Lorentz gauge is the simple formalism of gauge boson propagator while the trade-off for the axial gauge is that ghost fields are not required. The corresponding gauge boson propagators are

$$\Delta_{ab}^{\mu\nu}(p) = -\frac{i\delta_{ab}}{p^2 + i\epsilon}\left[\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]_{\texttt{Lorentz Gauge}}, \tag{1.1.14}$$

$$\Delta_{ab}^{\mu\nu}(p) = -\frac{i\delta_{ab}}{p^2 + i\epsilon}\left[\eta^{\mu\nu} - \frac{p^\mu n^\nu + n^\mu p^\nu}{p\cdot n} + \frac{(n\cdot n + \xi p\cdot p)\,p^\mu p^\nu}{(p\cdot n)^2}\right]_{\texttt{Axial Gauge}}, \tag{1.1.15}$$

where p is the propagating momentum of the gauge boson and n_μ is a reference vector. Typically, in the light-like gauge, $n\cdot n = 0$. More details about the above propagators are discussed in section 5.1.1.

Similarly, the determinant of the functional derivative matrix $\texttt{det}(\delta\mathcal{G}[\mathbf{A}_\mu^\Omega]/\delta\Omega)$ in Eq. (1.1.12) can be replaced by a Gaussian path integral over Grassmann variables. The exponent of the integral contributes to the QCD Lagrangian as,

$$\mathcal{L}_{ghost} = -\text{Tr}(\bar{\mathbf{C}}\partial^\mu \mathbf{D}_\mu \mathbf{C}), \tag{1.1.16}$$

where $\mathbf{C} = C^a T^a$ is the ghost field in adjoint representation and behaves fermionically. From the definition of $\mathbf{D}_\mu$ in (1.1.4) and $\text{Tr}(T^a T^b) = \delta_{ab}$, $\mathcal{L}_{ghost}$ can be rewritten in the scalar form

$$\mathcal{L}_{ghost} = -\bar{C}^a(\partial^2 C^a - g\sqrt{2}f^{abc}\partial^\mu A_\mu^b C^c). \tag{1.1.17}$$

In QED, $f^{abc} = 0$ such that the ghost field will not couple to the gauge field. The first term in Eq. (1.1.17) will be integrated out during path integral and will not affect the physical observables.

1.2 Renormalization of QCD

The quark and gluon fields, QCD gauge coupling strength and gauge-fixing parameters introduced in the QCD Lagrangian in section 1.1 characterize the QCD theory but are not yet sufficient to fix physical observables. In perturbative QCD, the self interactions of the fields reveals the detailed structure of quantum fluctuations in the vacuum. Starting from the second order of the perturbative expansion, the self interactions introduce loop calculations in Feynman diagrams. After applying

Feynman rules and simplifying intermediate tensor integrals [1], one typically finds the following type of integrals,

$$I_2 = \int \frac{\mathrm{d}^d l}{(2\pi)^4} \frac{1}{l^2(l-p)^2}, \tag{1.2.18}$$

where d is the space-time dimension and l is the internal loop momentum. This integral is divergent when $l \to \infty$ and cause ultraviolet (UV) singularities to appear during calculations of physical observables.

The aim of renormalization is to remove the UV singularities by redefinition of the fields and parameters without generating new terms in the Lagrangian. For the QCD Lagrangian as mentioned in section 1.1, one can rescale the bare Lagrangian according to the quantum fluctuations. The UV singularities in the bare Lagrangian can be absorbed into the rescaling factors and the physical observables calculated from the renormalized Lagrangian are then UV finite [2–4]. Specifically,

$$\psi_q = Z_2^{1/2} \psi_{ren,q}, \qquad A_\mu^a = Z_3^{1/2} A_{ren,\mu}^a, \qquad C^a = Z_\eta^{1/2} C_{ren}^a,$$
$$g^2 = Z_{\alpha_s} g_{ren}^2, \qquad \xi = Z_3 \xi_{ren}, \tag{1.2.19}$$

where all Z_i above are UV divergent and have perturbative expansions, $Z_i = 1 + \delta Z_i$, $\delta Z_i = \mathcal{O}(g^2)$. The variables on the left hand side of Eq. (1.2.19) are defined in the bare Lagrangian. Inserting Eq. (1.2.19), and expanding in powers of the coupling constant, the bare Lagrangian splits into two parts, a renormalized Lagrangian and a counter term:

$$\mathcal{L}_{QCD} = \mathcal{L}_{ren,QCD} + \mathcal{L}_{c.t.}. \tag{1.2.20}$$

This means that an UV finite physical observable calculated from $\mathcal{L}_{ren,QCD}$ can be calculated equivalently using the bare Lagrangian minus the correction of divergent counter terms $\mathcal{L}_{c.t.}$ produced by the δZ_i.

To fix the $\mathcal{L}_{c.t.}$ one needs to choose specific regularization and renormalization schemes. First to quantify the UV divergent integrals as mentioned in equation (1.2.18), a small parameter ϵ is introduced to regulate the divergent behaviour of the integral. The regulator is introduced by continuing the four dimensional integral into a d dimensional integral with $d = 4 - 2\epsilon$ [4]. By taking $\epsilon \to 0$ after integration, one recovers the integral in real world space time while the divergence of the integral

in d dimension can be analytically expressed by inverse powers of ϵ. In this thesis we assume the integrals for both internal (loop momentum integral) and external (phase space integral) particles are in d dimensions. This regularization scheme is usually referred as conventional dimensional regularization (CDR) [2, 5].

Using the regulator to evaluate the divergent loop integral, one can explicitly construct the rescaling parameters Z_i and check analytically that the UV divergences are removed by $\mathcal{L}_{c.t.}$ in all orders of perturbative calculations. However, in a truncated fixed order calculation, different finite shifts in the $\mathcal{L}_{c.t.}$ for a specific order would cause different shifts of divergences in other orders. A renormalization scheme is required to specify the finite shift at each order of $\mathcal{L}_{c.t.}$. The difference between any two schemes leads to a finite change in the theoretical prediction. However the value of coupling constants are also scheme dependent and the physical predictions are independent from the scheme choice. In the minimal subtraction (MS) scheme, the $\mathcal{L}_{c.t.}$ has no finite contribution at each order. In this thesis I choose to use the modified minimal subtraction ($\overline{MS}$). For each $1/\epsilon$ pole in the UV divergence the associated coefficient (in CDR regularization scheme) is,

$$\frac{\Gamma(1+\epsilon)}{\epsilon}(4\pi)^\epsilon = \frac{1}{\epsilon} + \ln(4\pi) - \gamma + \mathcal{O}(\epsilon), \qquad (1.2.21)$$

where γ is the Euler-Mascheroni constant. The same finite contribution can be retained by rescaling the regulator $\epsilon \to \bar{\epsilon}$ that

$$\frac{1}{\bar{\epsilon}} = (4\pi)^\epsilon e^{-\epsilon\gamma}\frac{1}{\epsilon} = \bar{C}(\epsilon)\frac{1}{\epsilon}. \qquad (1.2.22)$$

In $\overline{MS}$ scheme, the finite contribution of the UV pole in equation (1.2.21) are included in the $\mathcal{L}_{c.t.}$ term in all perturbative orders by rescaling the regulator as in equation (1.2.22).

The idea of rescaling the parameters of a quantum field theory is profound and links the mathematical properties of self-similar systems. A self-similar system is a system that appears similar at different resolution scales. In other words, a self-similar system can evolve from one scale to another while keeping the properties of the system. In QCD theory the field strengths and couplings can evolve through repeated emission and absorption. Before and after the rescaling in Eq. (1.2.19), QCD maintains the same dynamical properties and the rescaled parameters are

defined to remove the divergence from self-interactions in the vacuum corrections. This means that the renormalizable QCD theory describes a self-similar physics system and the rescaling parameters can be determined by the self-repeating pattern allowed by the physics system.

For example, the QCD gauge coupling strength $g(Q^2)$ describes the quark-anti-quark-gluon interaction vertex at a certain energy scale Q^2 (renormalization scale). The same coupling constant can be computed as a function of $g(\mu^2)$ at a lower energy scale μ^2, by iterating the quark-anti-quark-gluon interaction vertex an arbitrary number of times. As the physical observables are related to the squared matrix elements, here we define the strong coupling parameter related to $g^2(\mu^2)$ that

$$\alpha_s(\mu^2) = \frac{g^2(\mu^2)}{4\pi}. \tag{1.2.23}$$

From the Feynman rules we see that $\alpha_s(Q^2)$ can be expressed as a power series in $\alpha_s(\mu^2)$ such that

$$\alpha_s(Q^2) = \frac{g^2(Q^2)}{4\pi} = \alpha_s(\mu^2) + c_2(\mu^2)\alpha_s^2(\mu^2) + c_3(\mu^2)\alpha_s^3(\mu^2) + \cdots . \tag{1.2.24}$$

The $c_i(\mu^2)$ parameters above can be determined by the actual calculations in higher orders of quark-anti-quark-gluon interaction vertex. Taking the partial derivative of μ^2 in equation (1.2.24), one finds,

$$0 = \frac{\partial \alpha_s(\mu^2)}{\partial \mu^2} + \frac{\partial c_2(\mu^2)}{\partial \mu^2}\alpha_s^2(\mu^2) + 2c_2(\mu^2)\alpha_s(\mu^2)\frac{\partial \alpha_s(\mu^2)}{\partial \mu^2} + \cdots \tag{1.2.25}$$

By inserting the actual value of $c_i(\mu^2)$ and scaling by μ^2 one obtains the QCD beta-function $\beta(\alpha_s(\mu^2))$,

$$\mu^2\frac{\partial \alpha_s(\mu^2)}{\partial \mu^2} \equiv \beta(\alpha_s(\mu^2)) = -\beta_0\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2 - \beta_1\left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^3 - \mathcal{O}(\alpha_s(\mu^2)^4), \tag{1.2.26}$$

where the first two terms in the perturbative expansion are,

$$\beta_0 = \frac{11N - 2N_F}{6}, \qquad \beta_1 = \frac{17}{6}N^2 - \frac{13}{12}NN_F + \frac{N_F}{4N}, \tag{1.2.27}$$

where N being the number of colours and N_F being the number of light quark flavours. The beta-function describes the changes of a strong coupling parameter $\alpha_s(\mu^2)$ when altering the renormalization scale. The physical properties of a renormalizable QCD theory should be independent of the choice of the renormalization

scale. However, in a truncated calculation of perturbative QCD observables, the missing higher order terms lead to a residual dependence on μ^2. Nevertheless, such dependence in perturbative calculation decreases as more and more higher order terms is included. When the renormalization scale is varied from a central choice the range in which an observable varies is referred to the renormalization scale uncertainty.

1.3 QCD cross sections

The cross section σ is a physical observable that describes the probability per unit flux for a scattering process to occur. In a scattering process with two colliding particles with momentum p_1, p_2 and m final particles with momentum $p_3, \cdots, p_{m+2}$, the differential cross section $\mathrm{d}\sigma(p_1, p_2)$ describes the probability per unit flux of finding such event as the product of the momentum phase space measure $\mathrm{d}\Phi_n(p_3, \ldots, p_{n+2}; p_1, p_2)$ and a transition probability density calculated from the scattering matrix elements $|\mathcal{M}_{n+2}|^2$. In order to map the final state produced in the scattering to a physical observable, a selector that acts on the final state momentum is required in order to test whether the final state particles contribute to that particular observable, typically a number of "jets". Formally for an m-"jet" observable, the differential cross section can be expressed as a sum over n-particle final states that contribute to the m-"jet" observable,

$$\mathrm{d}\sigma^m(p_1, p_2) = \frac{1}{2s} \sum_{n \geq m} \mathrm{d}\Phi_n(p_3, \ldots, p_{n+2}; p_1, p_2) \frac{1}{s_n} |\mathcal{M}_{n+2}|^2 J_m^{(n)}(\{p\}_n), \qquad (1.3.28)$$

where s_n is the symmetry factor for the final state particles and $J_m^{(n)}$ is a selector "jet" function to check if the momentum set $\{p\}_n = p_3, \ldots, p_{n+2}$ is separate or close enough to form the required m final state "jets". In practice, the selector function need not identify jets, but should project the final state particles onto the particular observable being studied.

In perturbation theory, the scattering amplitude $\mathcal{M}$ can be calculated according to the order of the coupling parameter α so that,

$$\mathcal{M}_n = \mathcal{M}_n^0 + \frac{\alpha}{2\pi} \mathcal{M}_n^1 + \frac{\alpha^2}{(2\pi)^2} \mathcal{M}_n^2 + \mathcal{O}(\alpha^3), \qquad (1.3.29)$$

where $\mathcal{M}_n^0$, $\mathcal{M}_n^1$ and $\mathcal{M}_n^2$ corresponding to scattering amplitudes at tree, one-loop and two-loop level. Inserting (1.3.29) into (1.3.28) and noting that in general $|\mathcal{M}_{n+1}|^2$ is one α order higher than $|\mathcal{M}_n|^2$, one finds the differential cross section as an expansion in α such that (omitting label m)

$$\mathrm{d}\sigma(p_1, p_2) = \mathrm{d}\sigma_{LO}(p_1, p_2) + \frac{\alpha}{2\pi}\mathrm{d}\sigma_{NLO}(p_1, p_2) + \frac{\alpha^2}{(2\pi)^2}\mathrm{d}\sigma_{NNLO}(p_1, p_2) + \cdots , \quad (1.3.30)$$

where

$$\mathrm{d}\sigma_{LO}(p_1, p_2) = \frac{1}{2s}\mathrm{d}\Phi_m(p_3, \ldots, p_{m+2}; p_1, p_2)\frac{1}{s_m} B_{m+2}\, J_m^{(m)}(\{p\}_m),$$

$$\mathrm{d}\sigma_{NLO}(p_1, p_2) = \frac{1}{2s}\mathrm{d}\Phi_{m+1}(p_3, \ldots, p_{m+3}; p_1, p_2)\frac{1}{s_{m+1}} R_{m+3}\, J_m^{(m+1)}(\{p\}_{m+1})$$

$$+ \frac{1}{2s}\mathrm{d}\Phi_m(p_3, \ldots, p_{m+2}; p_1, p_2)\frac{1}{s_m} V_{m+2}\, J_m^{(m)}(\{p\}_m),$$

$$\mathrm{d}\sigma_{NNLO}(p_1, p_2) = \frac{1}{2s}\mathrm{d}\Phi_{m+2}(p_3, \ldots, p_{m+4}; p_1, p_2)\frac{1}{s_{m+2}} RR_{m+4}\, J_m^{(m+2)}(\{p\}_{m+2})$$

$$+ \frac{1}{2s}\mathrm{d}\Phi_{m+1}(p_3, \ldots, p_{m+3}; p_1, p_2)\frac{1}{s_{m+1}} RV_{m+3}\, J_m^{(m+1)}(\{p\}_{m+1})$$

$$+ \frac{1}{2s}\mathrm{d}\Phi_m(p_3, \ldots, p_{m+2}; p_1, p_2)\frac{1}{s_m} VV_{m+2}\, J_m^{(m)}(\{p\}_m). \quad (1.3.31)$$

The matrix elements at tree, one-loop and two-loop level are given by,

$$B_{m+2} = \mathcal{M}_{m+2}^0 \mathcal{M}_{m+2}^{0\dagger},$$

$$R_{m+3} = \mathcal{M}_{m+3}^0 \mathcal{M}_{m+3}^{0\dagger},$$

$$V_{m+2} = \mathcal{M}_{m+2}^0 \mathcal{M}_{m+2}^{1\dagger} + \mathcal{M}_{m+2}^1 \mathcal{M}_{m+2}^{0\dagger},$$

$$RR_{m+4} = \mathcal{M}_{m+4}^0 \mathcal{M}_{m+4}^{0\dagger},$$

$$RV_{m+3} = \mathcal{M}_{m+3}^0 \mathcal{M}_{m+3}^{1\dagger} + \mathcal{M}_{m+3}^1 \mathcal{M}_{m+3}^{0\dagger},$$

$$VV_{m+2} = \mathcal{M}_{m+2}^0 \mathcal{M}_{m+2}^{2\dagger} + \mathcal{M}_{m+2}^2 \mathcal{M}_{m+2}^{0\dagger} + \mathcal{M}_{m+2}^1 \mathcal{M}_{m+2}^{1\dagger}. \quad (1.3.32)$$

The fixed order differential cross sections beyond the leading order have contributions from scattering matrix elements with more particles than observed in the final states. The selector function $J_m^{(m+1)}$, ensures that only the regions of real radiation phase space with single unresolved (soft or collinear) particles will contribute to the next-to-leading order (NLO) differential cross section. Similarly, only the double unresolved regions of phase space are selected by the $J_m^{(m+2)}$ function in the double

real contribution at next-to-next-to-leading order (NNLO) while single unresolved regions of phase space are selected by $J_m^{(m+1)}$ for the real-virtual contribution.

For the convenience of reference, we name each line in Eq. (1.3.31) that

$$\mathrm{d}\sigma_{LO}(p_1, p_2) \equiv \mathrm{d}\hat{\sigma}^B(p_1, p_2),$$

$$\mathrm{d}\sigma_{NLO}(p_1, p_2) \equiv \mathrm{d}\hat{\sigma}_{NLO}^R(p_1, p_2) + \mathrm{d}\hat{\sigma}_{NLO}^V(p_1, p_2),$$

$$\mathrm{d}\sigma_{NNLO}(p_1, p_2) \equiv \mathrm{d}\hat{\sigma}_{NNLO}^{RR}(p_1, p_2) + \mathrm{d}\hat{\sigma}_{NNLO}^{RV}(p_1, p_2) + \mathrm{d}\hat{\sigma}_{NNLO}^{VV}(p_1, p_2). \quad (1.3.33)$$

In the Large Hadron Collider, the typical scattering process is the head on collision of two protons producing many new hadrons in the final states. Although the hard scattering processes at the center of the collision can be perturbatively calculated due to the asymptotic freedom of the strong coupling parameter, non-perturbative effects must be considered when extracting quarks and gluons from initial state protons and also when final state quarks and gluons undergo the hadronization process.

The initial state quarks and gluons from incoming protons can be described by the parton model. In general, for an incoming hadron with momentum much higher than the confinement scale, the parton distribution function (PDF) $f_a(\xi)$ describes the the probability of finding a parton of type a carrying a fraction ξ of the parent hadron momentum p_H. The differential cross section for incoming hadrons is then given by,

$$\mathrm{d}\sigma(p_{H_1}, p_{H_2}) = \sum_{i,j} \int \frac{\mathrm{d}\xi_1}{\xi_1} \frac{\mathrm{d}\xi_2}{\xi_2} f_i(\xi_1) f_j(\xi_2) \mathrm{d}\sigma_{ij}(\xi_1 p_{H_1}, \xi_2 p_{H_2}), \quad (1.3.34)$$

where the sum i, j is over all parton species and $\mathrm{d}\sigma_{ij}(\xi_1 p_{H_1}, \xi_2 p_{H_2})$ is the partonic differential cross section that can be calculated perturbatively as in Eq. (1.3.30). The assumption in the naive parton model is that the emitted partons are considered to be free particles and do not interact with the rest of the hadrons. More details about parton distribution functions and factorization are given in section 1.4.

Phenomenological models inspired by QCD have been developed to describe the final state hadronization process and usually include parameters that are determined by fitting experimental data. More details about how to compare parton level final states with hadron level states is discussed in section 1.6.

1.4 Factorization and QCD improved parton model

For higher order calculations in perturbative QCD, initial state partons could radiate other partons before entering the perturbative scattering in factorized cross sections. This means that the parton distribution functions can be written as a series expansion in the strong coupling parameter with different orders in the expansion representing multiple emissions. If the radiated parton is collinear with the incoming hadron, then the (singular) radiation behaviour modifies the fraction of the momentum which is actually carried by the partons entering the perturbative scattering. If the radiated parton has a large transverse momentum that can be distinguished as a final state particle, then the splitting should be included in the perturbative scattering matrix elements. A momentum scale μ_F is introduced as a cutoff to distinguish whether the initiate state radiation should be absorbed into a redefinition of the PDF or the radiation should be considered to be part of the perturbative hard scattering process. The process of separating initial state radiation into hadronic and partonic contributions is called factorization. The hadronic cross section is independent of the choice of μ_F when including all orders of perturbative corrections.

The redefinition of the PDF order by order in the strong coupling parameter is another self-similar process. In analogy to renormalization, the bare PDF (now labeled by $f_i^0(\xi)$) in the naive parton model is related to the physical PDF ($f_a(\xi, \mu_F)$) by a convolution with a factorization kernel $\mathbf{\Gamma}^{-1}$ so that [6, 7]

$$f^0 = f \otimes \mathbf{\Gamma}^{-1},$$
$$f_i^0(z) = \int \mathrm{d}x \mathrm{d}y f_j(x, \mu_F) \Gamma_{ji}^{-1}(y, \mu_F) \delta(z - xy). \qquad (1.4.35)$$

where $f_j(x, \mu_F)$ is the physical PDF depending on factorization scale μ_F and $\Gamma_{ji}^{-1}(y, \mu_F)$ is expressed as a power series in $\alpha_s(\mu_F)$ (the bold $\mathbf{\Gamma}^{-1}$ kernel contains the colour factors associated with T^a matrix in splitting vertex). The inverse of the convolution can also be defined,

$$f = f^0 \otimes \mathbf{\Gamma}, \qquad (1.4.36)$$

and the expansion of $\mathbf{\Gamma}^{-1}$ and $\mathbf{\Gamma}$ is symbolically given by,

$$\mathbf{\Gamma} = \mathbf{I} + \frac{\alpha_s}{2\pi}\mathbf{\Gamma}^1 + \left(\frac{\alpha_s}{2\pi}\right)^2 \mathbf{\Gamma}^2 + \mathcal{O}(\alpha_s^3), \tag{1.4.37}$$

$$\mathbf{\Gamma}^{-1} = \mathbf{I} - \frac{\alpha_s}{2\pi}\mathbf{\Gamma}^1 - \left(\frac{\alpha_s}{2\pi}\right)^2 \left[\mathbf{\Gamma}^2 - [\mathbf{\Gamma}^1 \otimes \mathbf{\Gamma}^1]\right] + \mathcal{O}(\alpha_s^3), \tag{1.4.38}$$

$$= \mathbf{I} + \delta_{\mathbf{\Gamma}}. \tag{1.4.39}$$

In full notation we have,

$$\mathbf{\Gamma}_{ij}(x,\mu_F) = \delta_{ij}\delta(1-x) + \frac{\alpha_s(\mu_F^2)}{2\pi}\bar{C}(\epsilon)\mathbf{\Gamma}_{ij}^1(x) + \left(\frac{\alpha_s(\mu_F^2)}{2\pi}\right)^2 \bar{C}^2(\epsilon)\mathbf{\Gamma}_{ij}^2(x) + \mathcal{O}(\alpha_s^3(\mu_F^2)),$$

$$\mathbf{\Gamma}_{ij}^{-1}(x,\mu_F) = \delta_{ij}\delta(1-x) - \frac{\alpha_s(\mu_F^2)}{2\pi}\bar{C}(\epsilon)\mathbf{\Gamma}_{ij}^1(x)$$

$$- \left(\frac{\alpha_s(\mu_F^2)}{2\pi}\right)^2 \bar{C}^2(\epsilon)\left[\mathbf{\Gamma}_{ij}^2(x) - [\mathbf{\Gamma}_{aj}^1 \otimes \mathbf{\Gamma}_{ia}^1](x)\right] + \mathcal{O}(\alpha_s^3(\mu_F^2)). \tag{1.4.40}$$

The details of $\mathbf{\Gamma}_{ij}^1$ and $\mathbf{\Gamma}_{ij}^2$ are discussed in section 3.4.2 and 4.6.2.

Now we replace the bare PDF in the differential cross section of proton proton collision in the native parton model by the physical PDF. Eq. (1.3.34) can be rewritten symbolically as

$$\mathrm{d}\sigma = f^0 \cdot \mathrm{d}\sigma \cdot f^{0'}$$

$$= f \otimes \mathbf{\Gamma}^{-1} \cdot \mathrm{d}\sigma \cdot \mathbf{\Gamma}^{-1} \otimes f'. \tag{1.4.41}$$

Now can now identify the factorized partonic differential cross section,

$$\mathrm{d}\hat{\sigma} = \mathbf{\Gamma}^{-1} \cdot \mathrm{d}\sigma \cdot \mathbf{\Gamma}^{-1}. \tag{1.4.42}$$

so that the hadronic differential cross section in the QCD improved parton model is given by,

$$\mathrm{d}\hat{\sigma}(p_{H_1}, p_{H_2}) = \sum_{i,j} \int \frac{\mathrm{d}\xi_1}{\xi_1}\frac{\mathrm{d}\xi_2}{\xi_2} f_i(\xi_1, \mu_F) f_j(\xi_2, \mu_F)\mathrm{d}\hat{\sigma}_{ij}(\xi_1 p_{H_1}, \xi_2 p_{H_2}), \tag{1.4.43}$$

In analogy to the renormalization of the QCD Lagrangian, by inserting the value of $\mathbf{\Gamma}^{-1}$ in Eq. (1.4.39), the factorized differential cross section can be split into two parts - the original differential cross section calculated from perturbative QCD theory plus a counter term which contains $\delta_{\mathbf{\Gamma}}$,

$$\mathrm{d}\hat{\sigma} = \mathrm{d}\sigma + \mathrm{d}\hat{\sigma}_{c.t.}, \tag{1.4.44}$$

where

$$d\hat{\sigma}_{c.t.} = \delta_{\boldsymbol{\Gamma}} \cdot d\sigma \cdot \mathbf{I} \; + \; \mathbf{I} \cdot d\sigma \cdot \delta_{\boldsymbol{\Gamma}} \; + \; \delta_{\boldsymbol{\Gamma}} \cdot d\sigma \cdot \delta_{\boldsymbol{\Gamma}}. \tag{1.4.45}$$

By using the calculated value of $\delta_{\boldsymbol{\Gamma}}$ in the expansion of α_s, $d\hat{\sigma}_{c.t.}$ can be arranged also in expansion of α_s that

$$d\hat{\sigma}_{c.t.} = \frac{\alpha_s}{2\pi} d\hat{\sigma}_{NLO}^{MF} + \left(\frac{\alpha_s}{2\pi}\right)^2 d\hat{\sigma}_{NNLO}^{MF} + \mathcal{O}(\alpha_s^3). \tag{1.4.46}$$

where the mass factorization counter terms at each order of α_s are given by,

$$d\hat{\sigma}_{NLO}^{MF} = - \bar{C}(\epsilon)\left(\boldsymbol{\Gamma}^1 \cdot d\sigma_{LO} \cdot \mathbf{I} + \mathbf{I} \cdot d\sigma_{LO} \cdot \boldsymbol{\Gamma}^1\right), \tag{1.4.47}$$

$$d\hat{\sigma}_{NNLO}^{MF} = - \bar{C}(\epsilon)\left(\boldsymbol{\Gamma}^1 \cdot d\sigma_{NLO} \cdot \mathbf{I} + \mathbf{I} \cdot d\sigma_{NLO} \cdot \boldsymbol{\Gamma}^1\right)$$
$$- \bar{C}^2(\epsilon)\left(\boldsymbol{\Gamma}^2 \cdot d\sigma_{LO} \cdot \mathbf{I} + \mathbf{I} \cdot d\sigma_{LO} \cdot \boldsymbol{\Gamma}^2 - \boldsymbol{\Gamma}^1 \cdot d\sigma_{LO} \cdot \boldsymbol{\Gamma}^1\right.$$
$$\left. - [\boldsymbol{\Gamma}^1 \otimes \boldsymbol{\Gamma}^1] \cdot d\sigma_{LO} \cdot \mathbf{I} - \mathbf{I} \cdot d\sigma_{LO} \cdot [\boldsymbol{\Gamma}^1 \otimes \boldsymbol{\Gamma}^1]\right). \tag{1.4.48}$$

Comparing above equation with the perturbative expansion of differential cross section in Eq. (1.3.30) and combining terms according to the order of α_s, we find the factorized differential cross section to be,

$$d\hat{\sigma}_{LO} = d\sigma_{LO},$$
$$d\hat{\sigma}_{NLO} = d\sigma_{NLO} + d\hat{\sigma}_{NLO}^{MF},$$
$$d\hat{\sigma}_{NNLO} = d\sigma_{NNLO} + d\hat{\sigma}_{NNLO}^{MF}. \tag{1.4.49}$$

The utilitization of the mass factorization terms is intensively discussed in chapter 3 and 4. In particular, the $\boldsymbol{\Gamma}_{ij}^1$ and $\boldsymbol{\Gamma}_{ij}^2$ functions are not infrared safe and are combined with divergent subtraction terms to define a infrared safe differential cross section in chapters 6, 7, 8 and 9.

The $d\hat{\sigma}_{NNLO}^{MF}$ term in equation (1.4.48) can further be arranged into two parts according to the number of final state particles. Inserting $d\sigma_{NLO}$ according to equation (1.3.33), the NNLO mass factorization counter term $d\hat{\sigma}_{NNLO}^{MF}$ is

$$d\hat{\sigma}_{NNLO}^{MF} = d\hat{\sigma}_{NNLO}^{MF,1} + d\hat{\sigma}_{NNLO}^{MF,2}, \tag{1.4.50}$$

where

$$d\hat{\sigma}_{NNLO}^{MF,1} = - \bar{C}(\epsilon)\left(\boldsymbol{\Gamma}^1 \cdot d\sigma_{NLO}^R \cdot \mathbf{I} + \mathbf{I} \cdot d\sigma_{NLO}^R \cdot \boldsymbol{\Gamma}^1\right) \tag{1.4.51}$$

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2} = & -\bar{C}(\epsilon)\bigg(\boldsymbol{\Gamma}^1 \cdot \mathrm{d}\sigma_{NLO}^V \cdot \mathbf{I} + \mathbf{I} \cdot \mathrm{d}\sigma_{NLO}^V \cdot \boldsymbol{\Gamma}^1\bigg) \\
& -\bar{C}^2(\epsilon)\bigg(\boldsymbol{\Gamma}^2 \cdot \mathrm{d}\sigma_{LO} \cdot \mathbf{I} + \mathbf{I} \cdot \mathrm{d}\sigma_{LO} \cdot \boldsymbol{\Gamma}^2 - \boldsymbol{\Gamma}^1 \cdot \mathrm{d}\sigma_{LO} \cdot \boldsymbol{\Gamma}^1 \\
& \qquad - [\boldsymbol{\Gamma}^1 \otimes \boldsymbol{\Gamma}^1] \cdot \mathrm{d}\sigma_{LO} \cdot \mathbf{I} - \mathbf{I} \cdot \mathrm{d}\sigma_{LO} \cdot [\boldsymbol{\Gamma}^1 \otimes \boldsymbol{\Gamma}^1]\bigg). \qquad (1.4.52)
\end{aligned}
$$

1.5 Infrared singularity cancellation

Renormalized quantum field theory has another type of divergence when two partons becomes collinear or one parton becomes soft (and also iteration of these two behaviour). These divergences are generically called infrared singularities. From Eq. (1.3.31), in the higher order contributions in perturbative cross sections, higher multiplicity processes from scattering matrix elements can contribute to lower multiplicity cross sections depending on the selector functions. As a scattering process involving unresolved particles (collinear or soft) may be indistinguishable from the one with lower multiplicity, the infrared singular regions of phase space will satisfy the selector functions and be considered as a contribution to the lower multiplicity observable. Similarly, during factorization process, if the radiation from initial state parton is collinear with the incoming hadron, the collinear divergence is simply absorbed in the splitting kernel of bare PDF and the physical PDF is infrared safe. Nevertheless, the mass factorization counter terms containing infrared singularities will contribute to the factorized cross section. Another source of the infrared singularity comes from loop integrals.

On the experiment side, physical cross sections are by definition finite. The studies by Bloch and Nordseick [8], and Kinoshita [9], Lee and Nauenberg [10] (KLN) showed that, although different contributions in renormalized quantum field theory may be separately divergent, all infrared singularities will cancel when summed over all degenerate states leading to a finite prediction for the cross section.

E_T(GeV)

Run : 138919
Event : 32253996
Dijet Mass : 2.130 TeV

Run : 138919
Event : 32253996
Dijet Mass : 2.130 TeV

Jet 1 P_T : 585 GeV

Jet 2 P_T : 557 GeV

Jet 1 P_T : 585 GeV

Jet 2 P_T : 557 GeV
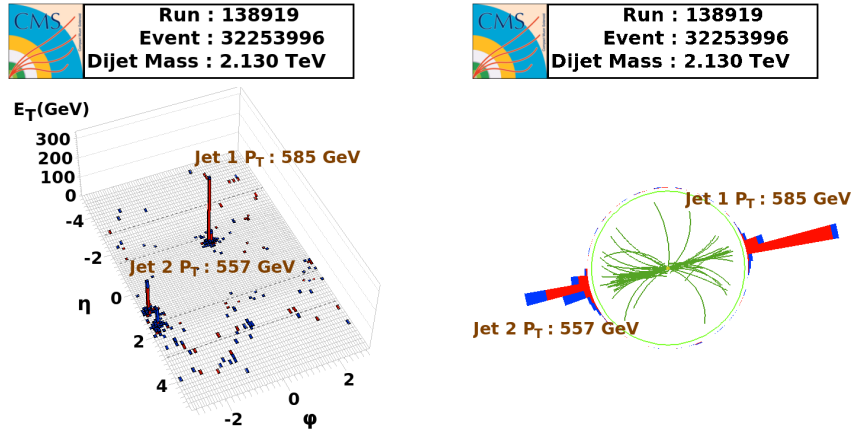
Figure 1.1: A di-jet event from CMS

1.6 Jet observables in the hadronic final state

Jet observables are an important concept for bridging between the partons produced in the hard scattering with the spray of hadrons produced via the hadronization process that is observed in the detector. The hadronic jet direction and energy are constructed by summing over the hadrons assigned to the jet. Likewise, one can construct a partonic jet from the partons produced in the hard scattering. After defining hadron-level jets from experimental data and parton-level jets from partonic event generators, one can compare the jet distributions observed in experiment with the corresponding theoretical predictions up to the uncertainties coming from, for example, the hadronization process or the contamination of the jet from the under-lying event.

A di-jet event from CMS experiment is shown in figure 1.1. [11]

A Jet Algorithm provides the criteria to define a jet. In a parton-level event generator, the jet algorithm is applied via the selector functions as mentioned in section 1.3. From experimental side, the jet algorithm is applied during the data processing stage after identifying all particles related to a single event. To define a jet one needs to set a jet resolution parameter and choose a combination scheme. The popular choice for a class of combination schemes at LHC is sequential re-combination [12, 13], for example, the k_T algorithm [12]. In general for sequential recombination, a distance measure is calculated for every pair of particles in the final state. Starting from the pairs with the smallest distance measure, final state

momenta are combined together to form a composite particle as a prototype jet. Repeating the process of calculating the distance measure with prototype jet and combining final state objects until all prototype jets have a distance measure larger than the resolution parameter leads to a set of well separated jets. Experimentally, the jet is formed from the many hadrons that are produced in a single event. Theoretically, the jet is created by clustering together the (relatively few) partons that are produced in the event. Since QCD radiation produces infrared singularities whenever a soft gluon is emitted or when a parton splits into two collinear partons, it is very important that the jet algorithm does not spoil the cancellation of infrared singularities between real and virtual contributions. In an infrared safe jet algorithm, soft and/or collinear emissions do not change the observed jet. For example, collinear emissions should automatically be recombined by the jet algorithm. Similarly, soft emissions within the jet should not alter the jet energy, while soft emission outside the jet should not generate additional jets. When the jet algorithm is insensitive to soft and collinear radiation in this way, it is called infrared safe.

Chapter 2

Introduction to the Higgs Boson and Higgs Phenomenology at the LHC

In this chapter, I introduce the basic ideas of electroweak symmetry breaking and the various Higgs boson interactions that are precisely predicted within the Standard Model. Direct evidence of the Higgs boson is the ultimate test of our ideas of electroweak symmetry breaking (EWSB). In 2012, CERNs Large Hadron Collider (LHC) finally revealed experimental evidence, that through comparison with precise predictions of the Higgs boson production and decay channels, is currently believed to be compatible with being the Higgs boson of the Standard Model.

Of course, in a hadron collider like the Large Hadron Collider, the Higgs boson production and decay processes are accompanied by large QCD corrections and there are many sources of Higgs-like events that have nothing directly to do with the Higgs boson - the background. Inspired by many phenomenological studies and the need for a precise comparison with LHC data, higher order QCD corrections at NNLO and boosted observables are necessary to study the precise properties of the Higgs boson. The unique combination of NNLO QCD corrections and Higgs boson phenomenology present in the $pp \rightarrow H$+jet process makes this project central to the successful exploitation of the LHC.

2.1   Electroweak symmetry breaking

Electroweak symmetry breaking was studied in the 1960's and is the foundation of the Standard Model [14–18]. The essential feature is to introduce a scalar (Higgs) field with a non-zero vacuum expectation value (vev) that gives mass to both vector bosons and fermions spontaneously breaking the electroweak symmetry. The physical manifestation of the Higgs field is the Higgs boson that observed by the ATLAS and CMS experiments at the LHC in 2012 [19–22]. In this section I introduce the main features of spontaneous symmetry breaking.

2.1.1   Spontaneous symmetry breaking in $U(1)$ gauge theory

The main aspects of a spontaneously broken gauge theory can be illustrated via the simple example of an Abelian $U(1)$ gauge group. I follow closely the notation in [23] and the Lagrangian density of a scalar field with $U(1)$ gauge symmetry is

$$\mathcal{L} = (D_\mu \psi)^\dagger (D^\mu \psi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\psi), \tag{2.1.1}$$

where

$$D_\mu \psi = (\partial_\mu - ig A_\mu)\psi,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.1.2}$$

$V(\psi)$ is the potential of the ψ field and here we choose a potential with non-zero vacuum expectation value (vev) that (the Mexican Hat Potential)

$$V(\psi) = -\mu^2 \psi^\dagger \psi + \lambda (\psi^\dagger \psi)^2. \qquad (\mu^2, \lambda > 0) \tag{2.1.3}$$

The Lagrangian density is invariant under local gauge transformations,

$$\psi(x) \rightarrow e^{-i\alpha(x)} \psi(x),$$
$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x). \tag{2.1.4}$$

Note that a term proportional to $A_\mu A^\mu$ is not allowed because such a term is not gauge invariant. Because of the gauge symmetry, $A_\mu(x)$ is required to be a massless gauge field.

The potential $V(\psi)$ reaches its minimum when the strength of the scalar field is

$$|\psi| = \sqrt{\frac{\mu^2}{2\lambda}}, \qquad \frac{\delta V(\psi)}{\delta|\psi|} = 0. \qquad (2.1.5)$$

This means that the scalar field ψ has a vev,

$$|\langle 0|\psi|0\rangle| = \frac{v}{\sqrt{2}}, \qquad v = \sqrt{\frac{\mu^2}{\lambda}}. \qquad (2.1.6)$$

For a physical system which is close to the bottom of the potential, the scalar field can be expressed as a perturbative expansion close to its vev such that,

$$\psi(x) = \frac{1}{\sqrt{2}}[v + \eta(x) + i\xi(x)], \qquad (2.1.7)$$

where the $\eta(x)$ and $\xi(x)$ fields have small values compared to v and have zero vev so that

$$|\langle 0|\eta|0\rangle| = 0, \qquad |\langle 0|\xi|0\rangle| = 0. \qquad (2.1.8)$$

If we insert (2.1.7) into (2.1.1) then there is no term proportional to $|\xi(x)|^2$. This means that the $\xi(x)$ field corresponds to a massless Goldstone boson [24,25]. Eq. (2.1.7) can be expressed as,

$$\psi(x) = e^{i\frac{\xi(x)}{v}} \frac{1}{\sqrt{2}}[v + \eta(x)], \qquad (2.1.9)$$

which means that we can gauge transform away the $\xi(x)$ field using Eq. (2.1.4) that

$$\psi'(x) = e^{-i\frac{\xi(x)}{v}}\psi(x) = \frac{1}{\sqrt{2}}[v + \eta(x)],$$

$$B_\mu(x) = A_\mu(x) - \frac{1}{gv}\partial_\mu\xi(x). \qquad (2.1.10)$$

The Lagrangian density after such a gauge transformation is

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}\Big((\partial_\mu - igB_\mu)(v+\eta)\Big)^\dagger \Big((\partial_\mu - igB_\mu)(v+\eta)\Big) \\
& - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4 \\
=& \frac{1}{2}(\partial_\mu\eta)^2 - \mu^2\eta^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}(gv)^2 B_\mu B^\mu \\
& + \mathcal{O}(\eta^3, B_\mu B^\mu \eta).
\end{aligned} \qquad (2.1.11)$$

Apart from the new interaction terms of order $\mathcal{O}(\eta^3, B_\mu B^\mu \eta)$, the Lagrangian density after gauge transformation (2.1.10) contains the $B_\mu(x)$ field representing a massive

vector boson and the massive scalar Higgs field $\eta(x)$. The unphysical massless Goldstone boson field $\xi(x)$ is entirely removed from $\mathcal{L}$ by the gauge transformation and the corresponding gauge choice is known as the unitary gauge. The original scalar field ξ (carrying a single degree of freedom) is 'eaten up' by the massless gauge field A_μ (with only two polarization states) to produce the new massive gauge field B_μ (with three polarization states). Because of the presence of the mass term for the B_μ field, the original $U(1)$ symmetry is spontaneously broken close to the minimum of the potential $V(\psi)$. This example of spontaneous symmetry breaking is also known as the Abelian Higgs mechanism. The non-Abelian case relevant to the Standard Model will be introduced in the next section.

2.1.2   Higgs mechanism in the Standard Model

The Standard Model is a spontaneously broken Yang-Mills theory with the $SU(2)_L \times U(1)_Y$ non-Abelian symmetry. The Higgs mechanism is implemented through a complex scalar $SU(2)$ doublet, with hypercharge $Y_\Phi = 1$,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}, \tag{2.1.12}$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ are properly normalized real scalar fields. The Lagrangian for the scalar field is

$$\mathcal{L}_\Phi = (\mathbf{D}_\mu \Phi)^\dagger (\mathbf{D}^\mu \Phi) - V(\Phi) + \mathcal{L}_{\text{Yukawa}}. \tag{2.1.13}$$

The form of the covariant derivative is dictated by the $SU(2) \times U(1)$ gauge symmetry,

$$\mathbf{D}_\mu = \mathbf{I}\partial_\mu + \frac{ig'}{2}\mathbf{I}B_\mu + \frac{ig}{2}\mathbf{W}_\mu, \tag{2.1.14}$$

where g and g' are the couplings of the $SU(2)$ and $U(1)$ gauge groups respectively, $\mathbf{I}$ is the two by two unit matrix and $\mathbf{W}_\mu$ contains the Pauli matrices τ^a (group generators of SU(2)) such that

$$\mathbf{W}_\mu = W_\mu^a \tau^a. \qquad (a = 1, 2, 3) \tag{2.1.15}$$

The Higgs potential $V(\Phi)$ is given by,

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2. \qquad (\mu^2, \lambda > 0), \tag{2.1.16}$$

which, as in the Abelian case, has a non-zero vev.

The Yukawa interaction $\mathcal{L}_{\text{Yukawa}}$ is introduced to generate the fermion mass via gauge invariant couplings to the scalar field Φ such that

$$\mathcal{L}_{\text{Yukawa}} = \Gamma_u^{ij} \bar{Q}_L^i \tilde{\Phi} u_R^j + \Gamma_d^{ij} \bar{Q}_L^i \Phi d_R^j + \Gamma_e^{ij} \bar{L}_L^i \Phi l_R^j + h.c. \qquad (2.1.17)$$

where $\tilde{\Phi} = i\tau^2 \Phi^*$ is the conjugate $SU(2)$ doublet with hypercharge -1, and $\Gamma_{u,d,l}$ are matrices of free parameters which are determined by experiment from the masses of fermions. Q_L^i and L_L^i (i=1,2,3 for generation index) are quark and lepton left handed doublets of $SU(2)_L$, and u_R^i, d_R^i and l_R^i are the right handed $SU(2)_L$ singlets.

Just as in the Abelian example in section 2.1.1, we expand the scalar field Φ about the vev of the Higgs potential,

$$|\langle 0 | \psi_3 | 0 \rangle| = v,$$

such that,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i\xi_2 \\ v + h + i\xi_3 \end{pmatrix}, \qquad (2.1.18)$$

where $v = \sqrt{\mu^2/\lambda}$ and ξ^1, ξ^2, ξ^2, h are real scalar field with zero vev. Choosing the unitary gauge to transform away the three component (ξ_1, ξ_2, ξ_3) of the scalar field, we can rewrite the SU(2) doublet in equation (2.1.12) as

$$\Phi \equiv \frac{1}{\sqrt{2}} \exp\left(\frac{i\xi^a \tau^a}{v} \right) \begin{pmatrix} 0 \\ v + h \end{pmatrix} \longrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}. \qquad (2.1.19)$$

The gauge transformation in Eq. (2.1.19) also changes the gauge fields $W_\mu^1, W_\mu^2, W_\mu^3$ and B_μ just as in Eq. (2.1.10). However, it is the beauty of the Higgs mechanism that after the gauge transformation, the $W_\mu^1, W_\mu^2, W_\mu^3$ and B_μ fields will describe massive bosons with the mass spectrum observed in experiment.

The quadratic terms for the gauge bosons in Eq. (2.1.13) come from $(\mathbf{D}_\mu \Phi)^\dagger (\mathbf{D}^\mu \Phi)$,

$$\begin{aligned} \mathcal{L}_{mass} &= \frac{v^2}{2} \begin{pmatrix} 0 & 1 \end{pmatrix} \left(-\frac{ig'}{2} \mathbf{I} B_\mu - \frac{ig}{2} \mathbf{W}_\mu \right) \left(\frac{ig'}{2} \mathbf{I} B_\mu + \frac{ig}{2} \mathbf{W}_\mu \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{v^2}{8} \left[g^2 (W_\mu^1)^2 + g^2 (W_\mu^2)^2 + (-gW_\mu^3 + g'B_\mu)^2 \right]. \end{aligned} \qquad (2.1.20)$$

If one replaces W_μ^1, W_μ^2, W_μ^3 and B_μ by normalized fields,

$$W_\mu^1 = \frac{1}{\sqrt{2}}(W_\mu^+ + W_\mu^-),$$

$$W_\mu^2 = -\frac{i}{\sqrt{2}}(W_\mu^+ - W_\mu^-),$$

$$W_\mu^3 = \frac{1}{\sqrt{g^2 + g'^2}}(gZ_\mu + g'A_\mu),$$

$$B_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(-g'Z_\mu + gA_\mu), \tag{2.1.21}$$

then

$$\mathcal{L}_{mass} = \frac{g^2 v^2}{4} W_\mu^- W^{+\mu} + \frac{\left(g^2 + g'^2\right) v^2}{8} Z^\mu Z_\mu. \tag{2.1.22}$$

From equation (2.1.22), we identify the three massive vector bosons as,

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm iW_\mu^2), \qquad\qquad m_W = \frac{gv}{2},$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gW_\mu^3 - g'B_\mu), \qquad m_Z = \frac{\sqrt{g^2 + g'^2}\,v}{2}, \tag{2.1.23}$$

while the massless photon is,

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(gW_\mu^3 + g'B_\mu), \qquad m_A = 0. \qquad\qquad . \tag{2.1.24}$$

In the unitary gauge, the Yukawa interaction is given by,

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &\sim -y_f \frac{1}{\sqrt{2}}\left[(v + h)\bar{f}_R f_L + (v + h)\bar{f}_L f_R\right] \\
&= -\frac{y_f}{\sqrt{2}}(v + h)\bar{f}f \\
&= -\left(\frac{y_f v}{\sqrt{2}}\right)\bar{f}f - \frac{y_f}{\sqrt{2}}h\bar{f}f, \tag{2.1.25}
\end{aligned}$$

showing that a fermion mass,

$$m_f = \frac{y_f v}{\sqrt{2}}, \tag{2.1.26}$$

is generated by the Higgs mechanism. Furthermore, the coupling of the Higgs boson to the fermions is proportional to the fermion mass.
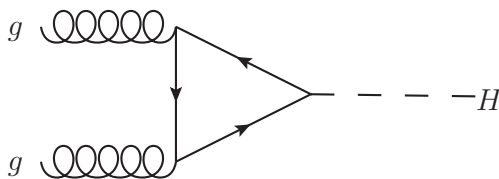
Figure 2.1: Higgs boson-gluon coupling through a quark loop in Standard Model at LO.

2.2 Higgs-Gluon-Gluon effective coupling

The coupling to quarks also enables the Higgs boson to couple to gluons via a quark loop, where the top quark gives the dominant contribution. The leading order contribution to the Higgs boson-gluon coupling is thus at the one-loop level. Because the top quark mass is assumed to be much larger than the Higgs boson mass, the top quark loop can be integrated out yielding an effective Lagrangian with new higher-dimensional interactions [26–28].

In the effective theory, the Higgs boson couples to the gluon field strength such that,

$$\mathcal{L}_H^{int} = \frac{\mathcal{C}}{2} H tr G_{\mu\nu} G^{\mu\nu} + \cdots , \qquad (2.2.27)$$

where the terms in $\cdots$ are proportional to $1/m_t^2$ and vanish in the $m_t \to \infty$ limit. The Wilson coefficient $\mathcal{C}$ can be expressed in the expansion of α_s and can be determined order by order from the fixed order calculation in the full theory. For example at leading order, the Feynman rule for the effective coupling in Eq. (2.2.27) is

$$-i\delta^{ab}\mathcal{C}\left(p_1 \cdot p_2 \eta^{\mu\nu} - p_1^\nu p_2^\mu\right), \qquad (2.2.28)$$

where p_i is the momentum of the gluons. In a light-like axial gauge where the reference momenta are $n_1 = p_2$ and $n_2 = p_1$, the scattering amplitude for $H \to gg$ is

$$\mathcal{M}^{Eff}(H \to gg) = -i\sum_{\lambda=\pm}\delta^{ab}\mathcal{C}p_1 \cdot p_2\varepsilon^\lambda(p_1, n_1) \cdot \varepsilon^{-\lambda}(p_2, n_2) = i2\delta^{ab}\mathcal{C}p_1 \cdot p_2. \quad (2.2.29)$$

Here $\varepsilon_\mu^\lambda(p_i, n_i)$ is the polarization vector with helicity λ and reference momentum n_i. From the full theory, we can calculate the triangle loop in Figure 2.1. In the

$m_t \to \infty$ limit, the amplitude is

$$\mathcal{M}_{m_t \to \infty}^{FT}(H \to gg) = -i \sum_{\lambda=\pm} \delta^{ab} \frac{\alpha_s}{6\pi v} p_1 \cdot p_2 \varepsilon^\lambda(p_1, n_1) \cdot (\varepsilon^\lambda(p_2, n_2))^* + \cdots \quad (2.2.30)$$

where the terms $\cdots$ are suppressed in the heavy top quark approximation. By comparing Eqs. (2.2.29) and (2.2.30), we can conclude that at leading order, the coefficient $\mathcal{C}$ is given by,

$$\mathcal{C}_{LO} = \frac{\alpha_s}{6\pi v}. \quad (2.2.31)$$

For higher order terms in $\mathcal{C}$, one needs to calculate $\mathcal{M}_{m_t \to \infty}^{FT}(H \to gg)$ with more loops. The NLO ($\mathcal{O}(\alpha_s^2)$) and NNLO ($\mathcal{O}(\alpha_s^3)$) results have been calculated in [29] and [30, 31].

Finite top and bottom mass effects typically induce contributions to the $gg \to H$ cross section of only a few percent. When more partons are produced in association with the Higgs boson, the finite top and bottom mass effect become more important for Higgs boson production with increasing transverse momentum. From a comparison of NLO Higgs boson plus jet final states between the effective theory and the full theory, the finite top mass effects lead to deviations of more than 5% when $p_{T,H} \gtrsim 200$ GeV [32, 33].

2.3 Higgs boson decay channels

The Higgs boson has a mean lifetime of $1.6 \times 10^{-22} s$ [34] which means that the Higgs boson decays to other particles before it interacts with the detector. Through the Lagrangian density (2.1.13), we can study the different processes through which the Higgs boson decays. The Feynman diagrams for the possible Higgs boson decay channels are listed in Fig. 2.2. The branching ratios for each decay depend on the mass of the Higgs boson and are presented in Fig. 2.3 [35].

One type of Higgs boson decay is when the Higgs boson splits into a fermion-antifermion pair. As the interaction strength is proportional to the mass of the fermion, the Higgs boson is more likely to decay into heavy fermions than light fermions [34–36]. Following this logic, the decay channel with the largest branching ratio should be into a top-antitop quark pair. However as the top quark is heavier

(a) H to fermion pair decay   (b) H to W pair decay   (c) H to Z pair decay

(d) H to gluon pair decay   (e) H to $\gamma\gamma(Z)$ q-loop   (f) H to $\gamma\gamma(Z)$ W-loop
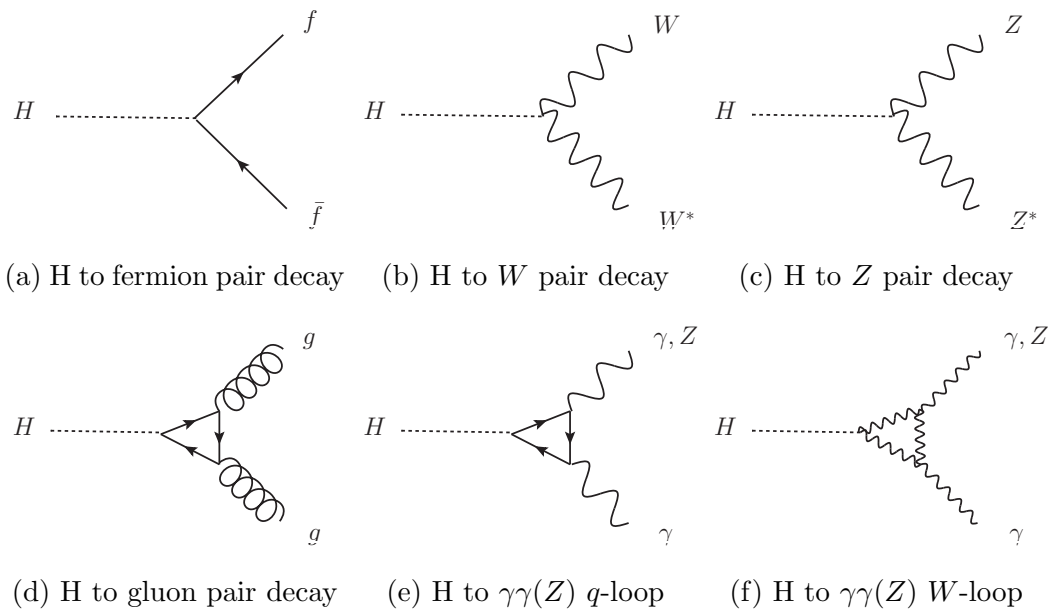
Figure 2.2: Feynman diagrams for the Standard Model Higgs boson decay channels.

than the Higgs boson this decay mode is forbidden. The dominant decay channel is then into bottom-antibottom quark pair with a branching ratio of about 56% [34–36].

Another type of Higgs boson decay is when the Higgs boson splits into a pair of vector bosons. Although both W and Z bosons are heavier than half of the Higgs boson mass, the decay mode proceeds via an off-shell vector boson. The most likely decay channel is $H \rightarrow WW^*$ which has branching ratio of around 23% [34–36]. The $H \rightarrow ZZ^*$ decay is much smaller because of the larger Z boson mass which pushes the off-shell Z further off-shell. This branching ratio is $\sim 3\%$.

The last type of decay channel is the decay into gauge bosons (gluons, photons, Z) via a loop of virtual quarks or massive gauge bosons. The decay channel with the highest branching ratio (8.5%) within this type is Higgs boson decay to gluon pair [34–36]. The gold-plated $H \rightarrow \gamma\gamma$ decay has a branching ratio of around 0.23%.

2.4   Higgs boson production channels

In a hadron collider like LHC, there are four main production processes for Higgs bosons: gluon fusion (GF), vector boson fusion (VBF), associated VH production and $t\bar{t}H$ production. The corresponding Feynman diagrams are shown in Figure
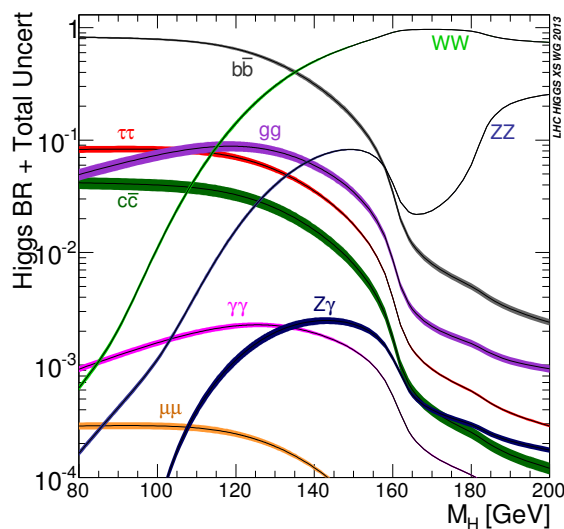
Figure 2.3: SM Higgs boson decay branching ratios as a function of m_H. From Ref. [35]

2.4.

Gluon fusion to Higgs boson has the largest production cross section at the LHC (about 85%) [34]. Using the effective Lagrangian, the fully differential NLO [37–39] and NNLO [6, 40–46] studies have shown that the QCD radiative corrections to this production channels are large. The state-of-the-art is now N3LO which reduces the theoretical prediction uncertainties to 5% [47–49]. The effects of a finite top mass have also been calculated recently at NNLO [32, 33, 50–55] showing that the deviation from the effective theory is small (about 2%).

The VBF channel cross section is about one tenth the size of that for gluon fusion [34]. This channel has a distinctive experimental signature as the two incoming quarks tend to be scattered by a relatively small angle, leading to two very energetic jets pointing close to the beam line in opposite halves of the detector.

The cross section for associated VH production is about $5 - 6\%$ of the inclusive Higgs boson cross section [34]. The $t\bar{t}H$ production rate is about 1% of the inclusive Higgs boson cross section [34]. However this channel is important as it directly probes the Higgs boson coupling to top quarks. The Standard Model Higgs boson production cross sections calculated as a function of Higgs boson mass are shown in Fig. 2.5 [34].

(a) gluon fusion to H

(b) VBF to H

(c) VH associated production
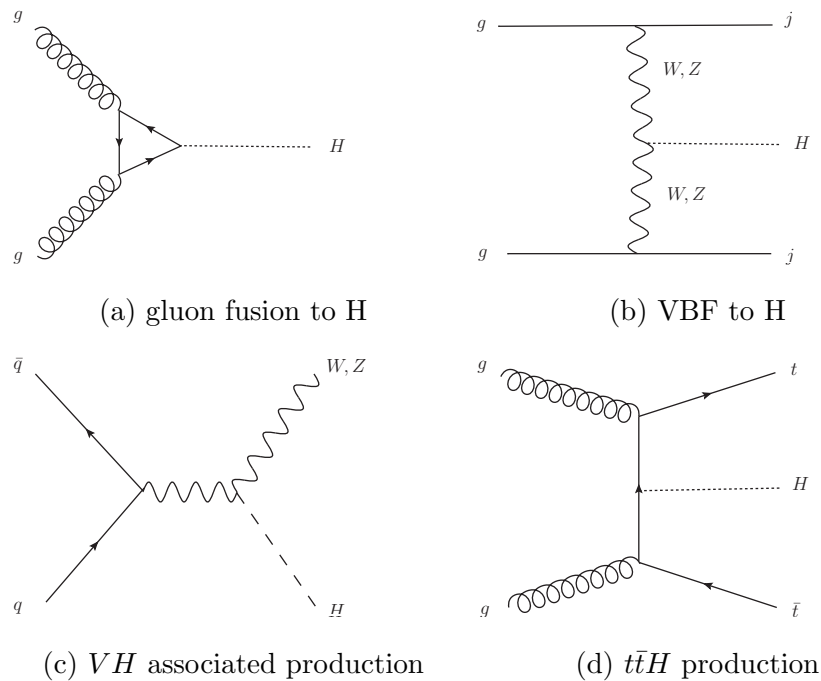
(d) $t\bar{t}H$ production

Figure 2.4: Feynman diagrams for Standard Model Higgs boson production channels.

2.5 Discovery of the Higgs boson

To prove the existence of the Higgs boson in the collider environment is a tremendous challenge. First, one needs a powerful accelerator like the LHC to provide sufficient energy for a Higgs boson to be created through one of the production channels. Second, the production of a Higgs boson has very small rate compared to other interactions allowed in the Standard Model. One therefore needs to repeat the collision processes at a high frequency. Third, the Higgs boson decays before it reaches the detectors. The various decay channels pose different challenges to identify a signal superimposed on a the large non-resonant background. Although the $H \to b\bar{b}$ mode has the largest decay branching ratio, the enormous QCD background from di-jet production makes detection in this decay model impossible. The same holds for the gg and $c\bar{c}$ decay modes. Final states involving charged leptons are much easier to observe, since the charged leptons in the final states are easily detected through tracking systems and electromagnetic calorimeters. Final state photons are also well measured in electromagnetic calorimeters. The mass resolution of the LHC detectors for lepton and photon pairs is excellent. Neutrinos do not interact in the
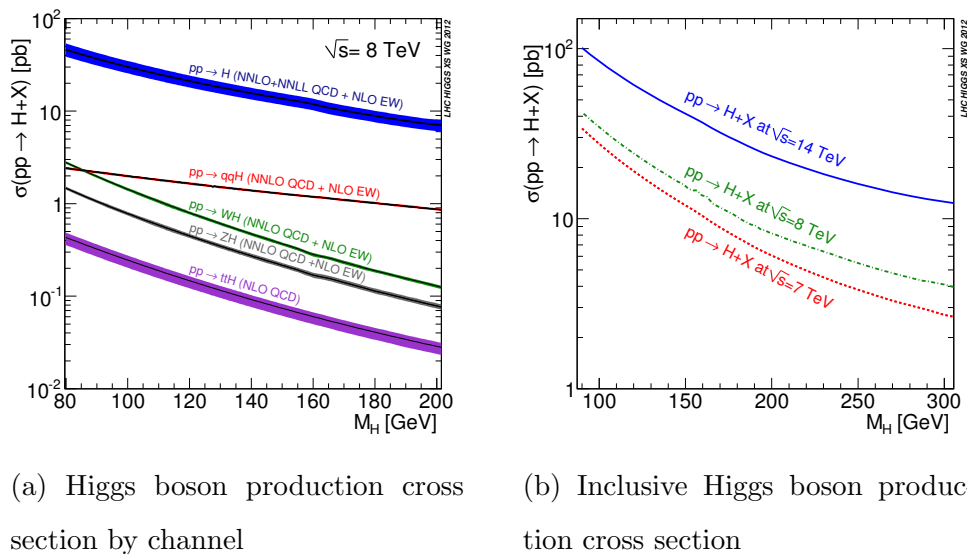
(a) Higgs boson production cross section by channel

(b) Inclusive Higgs boson production cross section

Figure 2.5: Standard Model Higgs boson production cross sections. From Ref. [34].

detector and can be reconstructed as missing energy with a 4-π detector. The easily observable decays have quite small branching ratios [34–36]. The branching ratio for $H \to W^+W^- \to l^+l^-\nu\nu$ is about 1.1%. The $H \to \gamma\gamma$ channel has a small branching ratio about 0.23% while the very clean $H \to ZZ^* \to 4l$ decay has a branching ratio of $\sim 0.013\%$.

The ATLAS and CMS detectors at the LHC independently searched for the Higgs boson signals and announced the discovery of a new type of boson on the 4th of July 2012 [56]. The new boson has a mass of $125.02 \pm 0.27(stat) \pm 0.15(syst)$ GeV according to CMS experiment [57] and $125.36 \pm 0.37(stat) \pm 0.18(syst)$ GeV according to ATLAS experiment [58]. The discovery channels were firstly the $H \to \gamma\gamma$ and $H \to ZZ^*$ channels [56] followed by the $H \to W^+W^-$ channel was later included [20, 59]. Studies of the $H \to \tau\bar{\tau}$ and $H \to b\bar{b}$ channels shows strong evidence for a direct coupling between the new boson and the down-type fermions. However the excesses of events over the expected background from other Standard Model processes have not reached 5 standard deviations yet [60–62].

The next key question is whether or not the newly discovered boson is the Standard Model Higgs boson. From the discovery itself, the new boson production and decay rates agree with the predictions of a Standard Model Higgs boson. From studies to determine the spin and parity quantum numbers, a spin-0 and positive parity

is tentatively confirmed [59,63–66] while spin-2 or negative parity are excluded with high confidence level [63,65].

Studies show that an extended Standard Model with two Higgs doublets [67] is disfavored by the ATLAS experiment when searching for the CP-odd Higgs boson [66]. The Standard Model CP-even Higgs boson is supported by studying the angular distributions of four lepton final states in $H \to ZZ^* \to 4l$ channel by CMS experiment [68]. In the search for neutral Higgs bosons predicted by the Minimal Supersymmetric Standard Model, no excess from background is observed in the tau-lepton-pair invariant mass spectrum by both the ATLAS and CMS [69,70]. All these studies provide strong evidence that the new boson discovered in the ATLAS and CMS detectors at the LHC is indeed the Standard Model Higgs boson.

2.6 Motivation for Higgs boson plus jets events at high precision

With the discovery of the Standard Model Higgs boson, the Higgs boson related research focuses on the detailed testing of Higgs boson couplings and dynamic properties. As mentioned in previous sections, both the dominant Higgs boson production $(gg \to H)$ and Higgs boson decay $(H \to b\bar{b})$ channels have large QCD backgrounds. The VBF production channel also have associated energetic forward jets. To better discriminate the Higgs boson signal from the QCD background and to study the dynamic properties of the Higgs boson, it is important to improve our predictions of exclusive Higgs boson plus jets events.

2.6.1 Higgs bosons with boosted kinematics

When a Higgs boson is produced in association with other particles, then from momentum conservation, it will recoil against the associated particles and could have large transverse momentum (boosted Higgs boson). The decay particles from the boosted Higgs boson would have a much narrower separation angle than in normal exclusive Higgs boson production where the Higgs boson is produced at rest

and the decay particles would be back-to-back.

If the boosted Higgs boson decays into $b\bar{b}$ pair, then after the hadronization process will in principle lead to two b-jets. The b-jets may be identified, or b-tagged, because if the originating particle is a neutral B-meson, the origin of the hadrons making the jet may be displaced from the beam line. This b-tagging is a powerful tool for identifying the Higgs boson signal [71, 72].

However, if the two bottom jets merge into a single "fat jet", then the invariant mass of the "fat jet" is directly related to the mass of the Higgs boson. The unique features of events with a boosted Higgs boson could lead to improvements in the detailed measurement of Higgs bosons properties, and specifically the coupling of the Higgs boson to bottom quarks.

QCD backgrounds are reduced drastically if one selects events with pairs of b-jets with large transverse momentum [73, 74]. Experimentally , the "fat jet" substructure from $H \to b\bar{b}$ is very different from QCD "fat jets" [74]. Selecting events with boosted objects is commonly used in searches for the associated Higgs boson production channels [75, 76]. Boosted Higgs boson from VH production channel can be used to study possible BSM properties [77]. Different BSM models related to $Hf\bar{f}$ couplings can be distinguished from boosted events shapes [78]. From these studies, Higgs bosons with boosted kinematics have the potential to reveal more details about the decaying particle. More accurate prediction of exclusive proton proton collision to Higgs boson plus jets at NNLO will further improve our understanding of the Higgs bosons properties.

2.6.2   Higgs boson differential cross section

Differential cross sections for boosted Higgs bosons (p_T distributions, pseudorapidity distributions etc. of the boosted object) provide information about both the rate as well as the kinematic properties of the event for detailed comparison between experiment and theory.

From the theory side, the boosted Higgs boson in the gluon fusion channel as been implemented in many NLO Monte Carlo event generators [79–82]. Studies at NNLO for pure gluon channel are also available [83, 84]. In the associated Higgs

boson production channels, the Higgs boson is boosted through recoil against the vector boson and fully exclusive studies are available at NNLO [85, 86]. In the $t\bar{t}H$ production channel, the Higgs boson is boosted through its recoil against the top quark pair. The simulation at NLO accuracy was first studied in [87, 88] and is now implemented in many NLO Monte Carlo event generators [89–91].

On the experimental side, as more and more data relating to boosted Higgs boson becomes available, comparisons of differential cross sections with simulation results become possible. Although with limited experimental statistics of LHC RUN 1 data, the differential cross sections have been studied in the $H \to \gamma\gamma$ and $H \to ZZ^* \to 4l$ decay channels [92, 93]. With more data from LHC RUN 2, studies of Higgs boson differential cross sections will become more and more important for revealing detailed Higgs boson properties.

2.6.3   Jet-bin analysis

In a proton proton collider like LHC, events are usually classified according to the number of jets in the event. These jets could come from a variety of sources, either from the underlying event, or from QCD radiation from the hard scattering. Experimental plots usually collect events with different numbers of jets into jet-bins. In this thesis, we are mainly concerned with the analysis of $H + n$ jets $(n = 0, 1, 2 \cdots)$, with corresponding cross sections σ_n, such that

$$\sigma^{tot} = \sum_{n=0} \sigma_n.$$

The jet-bin plot from ATLAS for $H \to \gamma\gamma$ and $H \to ZZ^* \to 4l$ is shown in Fig. 2.6 [93] and 2.7 [92]. Although the Higgs boson decays themselves do not produce additional jets, the signal events contain different numbers of jets. For each jet-bin in Fig. 2.6, the signal to background ratio varies. Figure 2.7, shows the different statistic and systematic uncertainties for each bin, and these uncertainties in general increase with higher jet multiplicities. From the theoretical prediction at NLO shown in the same plot, we can see that the different Higgs boson production channels contribute different proportions to each jet-bin. We also see that the deviation between experiment and theoretical prediction increases with the jet
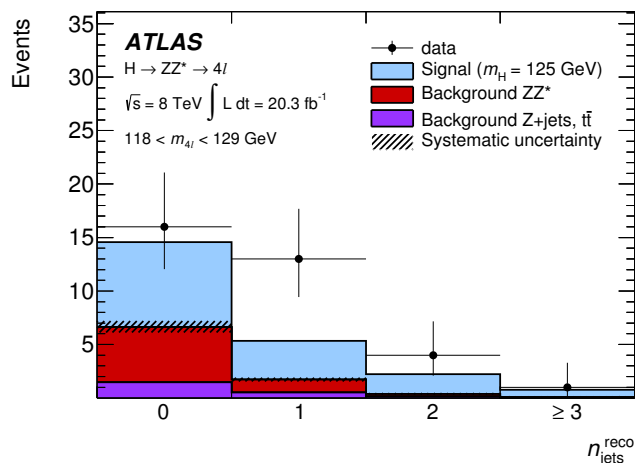
Figure 2.6: Four lepton final states associated with n jets.
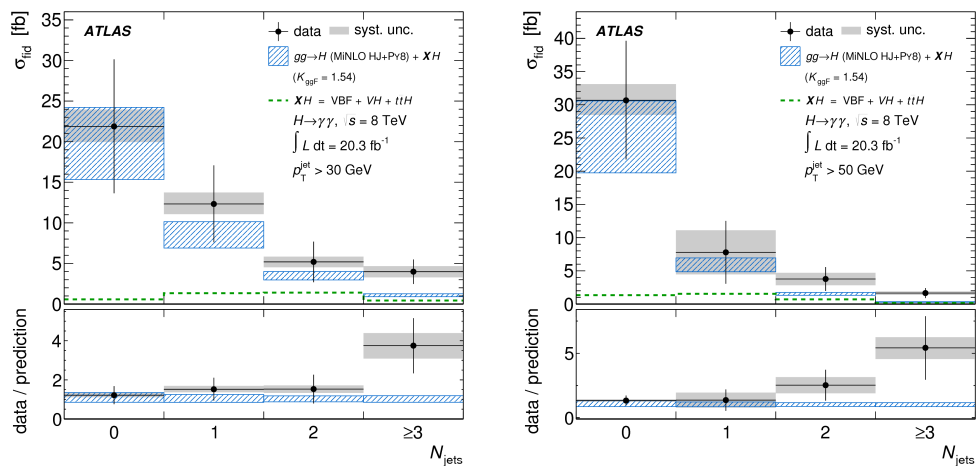
Figure 2.7: $pp \rightarrow H \rightarrow \gamma\gamma$ associated with n jets with different jet cut p_T^{jet}

multiplicity. To study the detailed properties of the Higgs boson, these comparisons need to be improved both experimentally and theoretically. From the theory side, more accurate predictions for exclusive Higgs boson plus jets beyond NLO are needed to reduce theoretical uncertainty to match the anticipated experimental uncertainty. As exclusive Higgs boson production from gluon fusion is now predicted at N3LO accuracy [47–49], the next step is to increase the accuracy of exclusive gluon fusion to Higgs boson plus one jet at NNLO accuracy [83, 84].

As introduced in section 1.6, jet observables are selected according to various parameters. The jet cut p_T^{cut} is an important parameter to define a jet with transverse
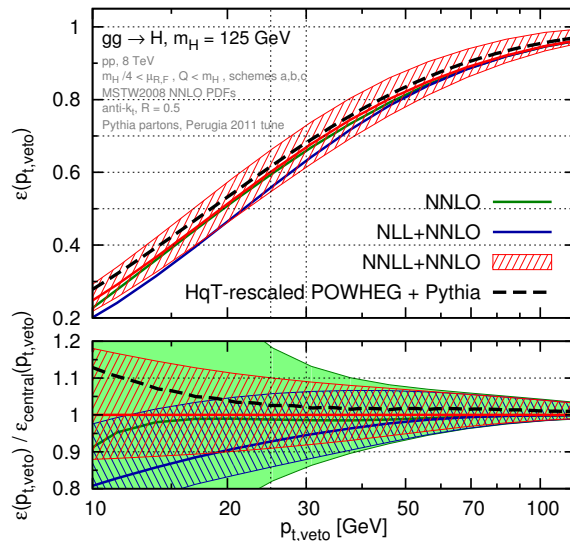
Figure 2.8: Jet veto efficiency in gluon fusion to Higgs boson channel

momentum $p_T^{jet} > p_T^{cut}$. As we see in Figure 2.7, changing the jet cut from 30 GeV to 50 GeV has a big effect on both the data and theoretical prediction.

The jet veto efficiency $(\varepsilon(p_{T,veto}))$ is an important observable defined as the fraction of the measured cross section that does not contain a jet above the jet cut p_T^{jet} [94–97]. In a jet veto analysis, the prediction for the $0-$jet-bin is defined to be $\sigma_0 = \sigma_{tot} - \sigma_{\geq 1}$. For the various jet cuts used in Higgs boson production channels, the prediction for the jet veto efficiency is shown in Fig. 2.8 for massless active quarks [94]. The exclusive Higgs boson production is calculated at NNLO accuracy in Fig. 2.8 while $\sigma_{\geq 1}$ is at NLO. To further reduce the theoretical uncertainty, NNLO calculations for $\sigma_{\geq 1}$ are needed.

2.6.4   Higgs boson plus jet at high precision

The main goal of this thesis is to make predictions for the specific process in which proton proton collisions lead to a Higgs boson plus one jet final state at next-to-next-to-leading order in the perturbative expansion. To do this, we will study the infrared singularities associated with the various real and virtual contributions at NLO, as well as the double-real, real-virtual and double-virtual contributions at NNLO. We will describe the numerical implementation of the various contributions. Although the infrared singularities ultimately cancel, each different final state multiplicity, is

separately divergent. A key part of the work described here is the introduction of subtraction terms that will produce an infrared finite integrand for each multiplicity. Clearly the subtraction terms themselves will mimic the explicit divergences present in the matrix elements as well as the implicit divergences in the single and double unresolved regions of phase space. We will employ a particular technique known as antenna subtraction to derive these subtraction terms analytically. The antenna subtraction method has been successfully applied to the $e^+e^- \to 3$ jets and $pp \to gg$ at NNLO [98–102]. The challenge of applying such method to $pp \to H+$ jet is to address the singular configurations involving different types of initial state partons and phase space mappings.

The thesis is organized as follows: Chapter 3 describes the structure and divergent behaviour of a general QCD process at NLO and the antenna subtraction method for NLO calculations. Chapter 4 continues the discussion to NNLO which is the main topic of this thesis. The tree and one-loop level matrix elements involved in $pp \to H+$jet at NNLO are discussed and recast in numerically stable form in chapter 5. The explicit antenna subtraction terms for $pp \to H+$jet at NLO and NNLO are intensively studied in chapters 7, 8 and 9 for gluon-gluon, quark-gluon and quark-quark initiated processes. In chapter 9, the numerical results for $pp \to H+$jet in the all-gluon channel are presented. In chapter 10, we draw our conclusions. Appendix A, B and C contains more examples of the explicit antenna subtraction terms used in chapter 7, 8 and 9. Appendix D provides a short review of mass factorization terms at NLO.

Chapter 3

NLO Corrections to QCD Scattering Processes

3.1 Colour ordered QCD amplitudes and matrix elements at NLO

As QCD is a non-abelian theory, it is convenient to organise QCD amplitudes according to the colour ordering of gluons and quarks. The colour ordering provides information on the real radiation singularities present in unresolved limits as well as the explicit pole structure present in loop amplitudes - all of these infrared singularities are produced by partons that are adjacent in the colour ordering. For recursive calculation methods, colour ordered amplitudes form the building blocks of the calculation and are easily extended to higher parton multiplicities.

One can extract the colour coefficients of each vertex allowed by $SU(N)$ QCD and use that to define colour ordered QCD amplitudes. The generators of $SU(N)$ are traceless hermitian $N \times N$ matrices T^a where $a = 1, \cdots, N^2 - 1$.

In the quark-gluon-quark vertex, the colour factor T^a_{ij} links a gluon with adjoint colour index a to the quark pair in the fundamental representation with colour indices i and j. In the three-gluon vertex, the structure constant f^{abc} defined by

$$[T^a, T^b] = i\sqrt{2}f^{abc}T^c, \qquad (3.1.1)$$

links gluons with adjoint colour labels a, b and c. In the four-gluon vertex (with

colour indices a, b, c, d), a pair of structure constant is used of the type $f^{abe}f^{ecd}$ where e is a (summed over) internal colour index. The full four-gluon vertex is symmetric under the interchange of the external gluons.

We can systematically replace the structure constants appearing in the Feynman rules for the three- and four-gluon vertex by group generators using the following identity:

$$f^{abc} = -\frac{i}{\sqrt{2}}(Tr(T^a T^b T^c) - Tr(T^a T^c T^b)). \tag{3.1.2}$$

Similarly, for amplitudes with two or more quark pairs, we can use the Fierz Ientity to disentangle the non-abelian and the abelian part of the colour flow,

$$T^a_{q\bar{q}} T^a_{Q\bar{Q}} = \delta_{q\bar{Q}}\delta_{Q\bar{q}} - \frac{1}{N}\delta_{q\bar{q}}\delta_{Q\bar{Q}}. \tag{3.1.3}$$

The left-hand side of equation (3.1.3) represents the colour structure of an intermediate gluon (a) connecting two quark pairs ($q\bar{q}$ and $Q\bar{Q}$). The first term in the right-hand side is the non-abelian colour flow, while the second term indicates a "photon-like" connection between the two quark pairs (abelian part). For any given QCD scattering amplitude, we can decompose the colour parts of the basic vertices using Eqs. (3.1.2) and (3.1.3) then collect terms according to the ordered products of $SU(3)$ generators. In this way we can decompose an amplitude into sets of partial amplitudes which are grouped by colour-related coefficients - these are the colour ordered amplitudes.

For example, by repeated use of Eq. (3.1.2), the pure gluon tree level scattering amplitude for n external on-shell gluons with momenta k_i, helicity λ_i and colour a_i can be represented by

$$M_n^0(k_i, \lambda_i, a_i) = g^{n-2} \sum_{P(2,\cdots,n)} Tr(T^{a_1} \cdots T^{a_n})\mathcal{M}_n^0(g_1^{\lambda_1}, \cdots, g_n^{\lambda_n}). \tag{3.1.4}$$

The sum of $P(2, \cdots, n)$ is the sum over all permutations of $2, \cdots, n$. In the RHS, we use the obvious notation $g_i^{\lambda_i}$ to denote a gluon with momentum k_i, helicity λ_i and colour index a_i. $\mathcal{M}_n^0(g_1^{\lambda_1}, \cdots, g_n^{\lambda_n})$ is the colour ordered amplitude.

Similarly, the tree level amplitude for one quark pair (with momenta k_1 and k_2 and colour indices q and $\bar{q}$) and many gluons has the general form,

$$M_n^0(k_i, \lambda_i, a_i) = g^{n-2} \sum_{P(3,\cdots,n)} (T^{a_3} \cdots T^{a_n})_{q\bar{q}}\mathcal{M}_n^0(q_1^{\lambda_1}, g_3^{\lambda_3} \cdots, g_n^{\lambda_n}, \bar{q}_2^{\lambda_2}) \tag{3.1.5}$$

where the sum $P(3, \cdots, n)$ runs over the permutations of the gluons with momenta $k_3, \ldots, k_n$ and colour indices $a_3, \ldots, a_n$. For two and more quark pairs, more quark indices are involved, leading to products of colour-strings such as,

$$(T^{a_i} \cdots T^{a_m})_{q\bar{Q}} (T^{a_{m+1}} \cdots T^{a_n})_{Q\bar{q}}.$$

At one-loop, the colour ordered amplitudes follows a similar pattern. Let us consider the pure one-loop gluon amplitude as an example. Because of the loop, there is an additional internal gluon (relative to tree-level) whose colour is summed over. This produces two types of colour structure, one where the extra trace acts on the unit matrix yielding a factor of N, and another one where two separate traces are produced (that is formally sub-leading in powers of N). A third structure proportional to the number of quark flavours is produced when there is an internal quark loop. In general,

$$
\begin{aligned}
&M_n^1(k_i, \lambda_i, a_i) \\
&= g^n \Bigg[\sum_{P(2, \cdots, n)} Tr(T^{a_1} \cdots T^{a_n}) \Big(N \mathcal{M}_{n;1}^1(g_1^{\lambda_1}, \cdots, g_n^{\lambda_n}) + N_F \hat{\mathcal{M}}_{n;1}^1(g_1^{\lambda_1}, \cdots, g_n^{\lambda_n}) \Big) \\
&\quad + \sum_{c=2}^{[n/2]+1} \sum_{P^{nc}(\sigma)} Tr(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c)}}) Tr(T^{a_{\sigma(c+1)}} \cdots T^{a_{\sigma(n)}}) \mathcal{M}_{n;c}^1(g_{\sigma(1)}^{\lambda_{\sigma(1)}}, \cdots, g_{\sigma(n)}^{\lambda_{\sigma(n)}}) \Bigg],
\end{aligned}
$$

$$(3.1.6)$$

where the $\sum_{c=2}^{[n/2]+1}$ sum is for all the possible separations of gluons and the sum of $P^{nc}(\sigma)$ is for all non-cyclic permutations of the inner string indices $(\sigma(1) \cdots \sigma(c)) \oplus (\sigma(c+1) \cdots \sigma(n))$.

When squaring the matrix elements, the colour indices for each quark and gluon appear in pairs. Using the Fierz identity (3.1.3), one can simply eliminate the generators appearing in the colour factors and eventually simplify the colour factors into scalar coefficients that are polynomials in N. At leading colour, that is retaining only the highest powers of N, the squared matrix elements are simply proportional to sums of products of colour-ordered amplitudes with the same colour ordering. The sub-leading colour contributions generally depend on the particular process, both in terms of the type and number of particles involved in the scattering.

For example, the colour leading contribution to the pure gluon (plus Higgs boson) matrix elements is given by (following the notation in Eq. (1.3.32)),

$$|M_n^0(k_i)|^2 \propto \sum_{P(2,\cdots,n)} \mathcal{M}_H^0(g_1,\cdots,g_n)\mathcal{M}_H^{0\dagger}(g_1,\cdots,g_n), \tag{3.1.7}$$

$$|M_n^1(k_i)|^2 \propto 2\Re\{ \sum_{P(2,\cdots,n)} \mathcal{M}_H^0(g_1,\cdots,g_n)\mathcal{M}_H^{1\dagger}(g_1,\cdots,g_n)\}. \tag{3.1.8}$$

The matrix elements on the right-hand side are implicitly summed over the helicity of the gluons.

3.1.1   Matrix elements with identical quark pairs

In spin averaged cross-section calculations, one cannot distinguish between the two quarks (or two antiquarks) if the quarks (antiquarks) share the same flavour. In this case, one needs to consider additional squared matrix element structures. Consider an scattering amplitude with at least two quark pairs and many other partons (plus one Higgs boson): $\mathcal{M}_{q\bar{q}Q\bar{Q}\cdots(H)}$. For identical quark (antiquark) flavours, $q = Q$ ($\bar{q} = \bar{Q}$), one needs to consider the amplitude where either the quarks (or antiquarks) are exchanged: $\mathcal{M}_{q\bar{Q}Q\bar{q}\cdots(H)}$. When the helicities of the two quarks are different, the two quarks are not identical particles. In this case, $\mathcal{M}_{q\bar{q}Q\bar{Q}\cdots(H)}$ and $\mathcal{M}_{q\bar{Q}Q\bar{q}\cdots(H)}$ can be treated as independent amplitudes just as in the different flavour case. However, when the helicity of the two quarks (and antiquarks) is the same, then one needs to construct a new amplitude that satisfies anti-symmetry property when exchanging identical fermions:

$$\mathcal{M}_{q_1^{\lambda_q}\bar{q}_2^{-\lambda_q}q_3^{\lambda_q}\bar{q}_4^{-\lambda_q}\cdots(H)} = -\mathcal{M}_{q_1^{\lambda_q}\bar{q}_4^{-\lambda_q}q_3^{\lambda_q}\bar{q}_2^{-\lambda_q}\cdots(H)}. \tag{3.1.9}$$

One can use the known amplitudes with non-identical quarks to construct $M^0_{q\bar{q}q\bar{q}\cdots(H)}$ which satisfies equation (3.1.9):

$$\mathcal{M}_{q_1^{\lambda_q}\bar{q}_2^{-\lambda_q}q_3^{\lambda_q}\bar{q}_4^{-\lambda_q}\cdots(H)} = \mathcal{M}_{q_1^{\lambda_q}\bar{q}_2^{-\lambda_q}Q_3^{\lambda_a}\bar{Q}_4^{-\lambda_q}\cdots(H)} - \mathcal{M}_{q_1^{\lambda_q}\bar{Q}_4^{-\lambda_q}Q_3^{\lambda_q}\bar{q}_2^{-\lambda_q}\cdots(H)}. \tag{3.1.10}$$

For a general squared matrix element with at least two quark pairs of the same flavour $|M_{q\bar{q}q\bar{q}\cdots(H)}|^2$, the relation to the non-identical flavour case is as follows:

$$|M_{q\bar{q}q\bar{q}\cdots(H)}|^2 = \sum_{\lambda_q \neq \lambda_Q} \left[|\mathcal{M}_{q^{\lambda_q}\bar{q}^{-\lambda_q}Q^{\lambda_Q}\bar{Q}^{-\lambda_Q}\cdots(H)}|^2 + |\mathcal{M}_{q^{\lambda_q}\bar{Q}^{-\lambda_Q}Q^{\lambda_Q}\bar{q}^{-\lambda_q}\cdots(H)}|^2 \right]$$

$$+ \sum_{\lambda_q = \lambda_Q} \left[|\mathcal{M}_{q^{\lambda_q} \bar{q}^{-\lambda_q} Q^{\lambda_Q} \bar{Q}^{-\lambda_Q} \cdots (H)} - \mathcal{M}_{q^{\lambda_q} \bar{Q}^{-\lambda_Q} Q^{\lambda_Q} \bar{q}^{-\lambda_q} \cdots (H)}|^2 \right]$$

$$= |M_{q\bar{q}Q\bar{Q}\cdots(H)}|^2 + |M_{q\bar{Q}Q\bar{q}\cdots(H)}|^2 - 2\Re\{\mathcal{M}_{q\bar{q}Q\bar{Q}p\cdots p(H)} \mathcal{M}^{\dagger}_{q\bar{Q}Q\bar{q}\cdots(H)}\}_{\lambda_q = \lambda_Q},$$

$$(3.1.11)$$

where $|M_{q\bar{q}Q\bar{Q}\cdots(H)}|^2$ and $|M_{q\bar{Q}Q\bar{q}\cdots(H)}|^2$ are matrix elements summed with all possible spin combinations, and $2\Re\{\mathcal{M}_{q\bar{q}Q\bar{Q}p\cdots p(H)} \mathcal{M}^{\dagger}_{q\bar{Q}Q\bar{q}\cdots(H)}\}_{\lambda_q = \lambda_Q}$ is a new structure with only identical spin sum.

3.1.2   Example for Higgs boson plus one jet at LO

To compute the LO corrections to Higgs boson plus one jet, we need the tree level amplitudes for a Higgs boson plus three partons. We assume that the top quark is sufficiently massive so that the Higgs boson couples to gluons via the effective vertex. In terms of colour ordered amplitudes, we have the tree amplitudes:

$$M^0_{gggH} = g \frac{\mathcal{C}}{2} \sum_{P(j,k)} Tr(T^{a_i} T^{a_j} T^{a_k}) \mathcal{M}^0_H(i,j,k), \qquad (3.1.12)$$

$$M^0_{qg\bar{q}H} = g \frac{\mathcal{C}}{2} (T^{a_i})_{q\bar{q}} \mathcal{M}^0_H(q,i,\bar{q}), \qquad (3.1.13)$$

where we denote the gluon of momentum k_i and colour a_i simply by the label i. $\mathcal{C}$ is the effective coupling in Eq. (2.2.27).

The squared matrix elements, summed over helicities and colours, involving a Higgs boson and three partons at tree level are:

$$\sum |M^0_{gggH}|^2 \equiv g^2 \frac{\mathcal{C}^2}{2} N(N^2 - 1) A^0_{3gH}(i,j,k), \qquad (3.1.14)$$

$$\sum |M^0_{qg\bar{q}H}|^2 \equiv g^2 \frac{\mathcal{C}^2}{4} (N^2 - 1) B^0_{1gH}(q,i,\bar{q}), \qquad (3.1.15)$$

where for $X = A, B$

$$X^0_H(i,j,k) = \mathcal{M}^0_H(i,j,k) \mathcal{M}^{0\dagger}_H(i,j,k). \qquad (3.1.16)$$

3.1.3   Example for Higgs boson plus one jet at NLO

To compute the NLO corrections to Higgs boson plus one jet, we need the tree level amplitudes for a Higgs boson plus four partons and the one-loop amplitudes Higgs

boson plus three partons. We assume that the top quark is sufficiently massive so that the Higgs boson couples to gluons via the effective vertex. In terms of colour ordered amplitudes, we have the tree amplitudes:

$$M_{ggggH}^0 = g^2 \frac{\mathcal{C}}{2} \sum_{P(j,k,l)} Tr(T^{a_i} T^{a_j} T^{a_k} T^{a_l}) \mathcal{M}_H^0(i,j,k,l), \tag{3.1.17}$$

$$M_{qgg\bar{q}H}^0 = g^2 \frac{\mathcal{C}}{2} \sum_{P(i,j)} (T^{a_i} T^{a_j})_{q\bar{q}} \mathcal{M}_H^0(q,i,j,\bar{q}), \tag{3.1.18}$$

$$M_{q\bar{q}Q\bar{Q}H}^0 = g^2 \frac{\mathcal{C}}{2} \left(\delta_{q\bar{Q}} \delta_{Q\bar{q}} \mathcal{M}_H^0(q,\bar{Q},Q,\bar{q}) - \frac{1}{N} \delta_{q\bar{q}} \delta_{Q\bar{Q}} \mathcal{M}_H^0(q,\bar{q},Q,\bar{Q}) \right), \tag{3.1.19}$$

where we denote the gluon of momentum k_i and colour a_i simply by the label i. $\mathcal{C}$ is the effective coupling in Eq. (2.2.27). Note that in this special case we have

$$\mathcal{M}_H^0(q,\bar{Q},Q,\bar{q}) = \mathcal{M}_H^0(q,\bar{q},Q,\bar{Q}). \tag{3.1.20}$$

For the one-loop amplitudes we have:

$$M_{gggH}^1 = g^2 \frac{\mathcal{C}}{2} \sum_{P(j,k)} Tr(T^{a_i} T^{a_j} T^{a_k}) \left[N \mathcal{M}_{3;1H}^1(i,j,k) + N_F \widehat{\mathcal{M}}_{3;1H}^1(i,j,k) \right], \tag{3.1.21}$$

$$M_{qg\bar{q}H}^1 = g^2 \frac{\mathcal{C}}{2} T_{q\bar{q}}^{a_i} \left[N \mathcal{M}_{3;1H}^1(q,i,\bar{q}) - \frac{1}{N} \widetilde{\mathcal{M}}_{3;1H}^1(q,i,\bar{q}) + N_F \widehat{\mathcal{M}}_{3;1H}^1(q,i,\bar{q}) \right]. \tag{3.1.22}$$

Unusually, there is no double trace (or double colour string) one-loop contribution because individual group generators are traceless. In general $\widehat{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ represent contributions that are proportional to N_F or $1/N$ respectively: $\widehat{\mathcal{M}}^1$ is the contribution from internal quark loops as explained in (3.1.6), $\widetilde{\mathcal{M}}^1$ is the sub-leading colour contribution from the gluon loop.

The squared matrix elements, summed over helicities and colours, involving a Higgs boson plus four partons at tree level are:

$$\sum |M_{ggggH}^0|^2 = g^4 \frac{\mathcal{C}^2}{2} N^2(N^2-1) \left[A_{4gH}^0(i,j,k,l) + A_{4gH}^0(i,j,l,k) + A_{4gH}^0(i,k,j,l) \right], \tag{3.1.23}$$

$$\sum |M_{qgg\bar{q}H}^0|^2 = g^4 \frac{\mathcal{C}^2}{4} N(N^2-1) \left[B_{2gH}^0(q,i,j,\bar{q}) + B_{2gH}^0(q,j,i,\bar{q}) - \frac{1}{N^2} \widetilde{B}_{2gH}^0(q,\tilde{i},\tilde{j},\bar{q}) \right], \tag{3.1.24}$$

$$\sum |M_{q\bar{q}Q\bar{Q}H}^0|^2 = g^4 \frac{\mathcal{C}^2}{4} (N^2-1) C_{0gH}^0(q,\bar{Q},Q,\bar{q}), \tag{3.1.25}$$

$$\sum |M^0_{q\bar{q}q\bar{q}H}|^2 = g^4 \frac{\mathcal{C}^2}{4}(N^2-1)\left[C^0_{0gH}(q,\bar{Q},Q,\bar{q}) + C^0_{0gH}(q,\bar{q},Q,\bar{Q}) - \frac{1}{N}D^0_{0gH}(q,\bar{q},q,\bar{q})\right],$$

(3.1.26)

where for $X = A, B, C$

$$X^0_H(i,j,k,l) = \mathcal{M}^0_H(i,j,k,l)\mathcal{M}^{0\dagger}_H(i,j,k,l),$$

(3.1.27)

and

$$\widetilde{B}^0_{2gH}(q,\tilde{i},\tilde{j},\bar{q}) = [\mathcal{M}^0_H(q,i,j,\bar{q}) + \mathcal{M}^0_H(q,j,i,\bar{q})][\mathcal{M}^0_H(q,i,j,\bar{q}) + \mathcal{M}^0_H(q,j,i,\bar{q})]^\dagger$$

(3.1.28)

$$D^0_{0gH}(i_q,j_{\bar{q}},k_q,l_{\bar{q}}) = -2\Re\{\mathcal{M}^0_H(i_q,j_{\bar{Q}},k_Q,l_{\bar{q}})\mathcal{M}^{0\,\dagger}_H(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\}_{\lambda_q=\lambda_Q}.$$

(3.1.29)

The spin sum in equation (3.1.29) follows the discussion in section 3.1.1. In general, we use the notation A for squared matrix elements involving 0 quark pairs, B for those involving one quark pair, C and D for those involving two quark pairs. In the sub-leading colour contribution $\widetilde{B}$, the gluons are "photon"-like in the sense that there is no singularity when two "photon"-like gluons are collinear.

Similarly, the interference of loop and tree amplitudes, summed over helicities and colours, involving a Higgs boson plus three partons at one-loop are:

$$\sum |M^1_{gggH}|^2 = g^4 \frac{\mathcal{C}^2}{2}N^2(N^2-1)\left[A^1_{3gH}(i,j,k) + \frac{N_F}{N}\widehat{A}^1_{3gH}(i,j,k)\right],$$

(3.1.30)

$$\sum |M^1_{q\bar{q}gH}|^2 = g^4 \frac{\mathcal{C}^2}{4}N(N^2-1)\left[B^1_{1gH}(q,i,\bar{q}) - \frac{1}{N^2}\widetilde{B}^1_{1gH}(q,i,\bar{q}) + \frac{N_F}{N}\hat{B}^1_{1gH}(q,i,\bar{q})\right],$$

(3.1.31)

where for $X = A, \widehat{A}, B, \widetilde{B}$ and $\widehat{B}$,

$$X^1_H(i,j,k) = 2\Re\{\mathcal{M}^0_H(i,j,k)\mathcal{M}^{1\dagger}_H(i,j,k)\}.$$

(3.1.32)

3.2 IR behaviour of real contribution at NLO

The expressions for scattering matrix elements are functions with Lorentz invariant variables which, for massless particles, have the following general form,

$$s_{ij} = (p_1 + p_j)^2 = 2p_i \cdot p_j.$$

(3.2.33)

In the soft limit of parton i or j $(p_{i(j)} \to 0)$, then s_{ij} vanishes. Similarly, in the collinear limit of parton i with j, $s_{ij} = 2E_i E_j (1 - cos(\theta))$, where θ is the relative splitting angle between the two partons, s_{ij} also vanishes. In both cases, one or more partons are unresolved.

In computing the cross section, one needs to integrate over all possible momentum configurations of final state particles and this means that unresolved phase space regions should always be included. In the general case, matrix elements that contain inverse powers of the vanishing Lorentz invariants will become divergent in these unresolved regions. The divergent behaviour can be factorized by universal functions (that contain the singular factors) multiplying reduced matrix elements involving a smaller number of resolved partons [103, 104].

In colour ordered matrix elements, the divergent behaviour involves the unresolved parton and its colour adjacent (or colour connected) neighbors [105]. Unresolved gluons can be either soft or collinear with a quark or another gluon. Quarks may become collinear with a gluon or an antiquark of the same flavour. There is no soft limit for quarks because they are fermions and couple via conserved currents. The soft quark current causes the full matrix element to vanish..

At tree-level, in the limit where a single parton j becomes soft, the colour ordered matrix element $|M^0(\cdots, i, j, k, \cdots)|^2$ where j is colour sandwiched by partons i and k can be factorized as

$$|M^0(\cdots, i, j_g, k, \cdots)|^2 \xrightarrow{j\ soft} S_{ijk}|M^0(\cdots, i, k, \cdots)|^2, \qquad (3.2.34)$$

where S_{ijk} function is the soft Eikonal factor,

$$S_{ijk} = \frac{2s_{ik}}{s_{ij}s_{jk}}. \qquad (3.2.35)$$

The partonic identities of i and k could be either gluons, quarks or mixed.

Similarly, when $i//j$ the matrix element satisfies the following factorization in the spin averaged collinear limit,

$$|M^0(\cdots, i, j, \cdots)|^2 \xrightarrow{i//j} \frac{1}{s_{ij}}P_{ij \to K}(z)|M^0(\cdots, K, \cdots)|^2. \qquad (3.2.36)$$

Here $P_{ij \to K}(z)$ function is the collinear splitting function where z represents the fraction of the almost light-like composite momentum $p_K = p_i + p_j$ carried by parton

i. The collinear splitting functions depend on the partonic identities of i and j. In $d = 4 - 2\epsilon$ dimensions the independent functions are

$$P_{qg \to Q} = P_{\bar{q}g \to \bar{Q}} = \frac{1 + (1 - z)^2 - \epsilon z^2}{z},$$

$$P_{q\bar{q} \to G} = P_{\bar{q}q \to G} = \frac{z^2 + (1 - z)^2 - \epsilon}{1 - \epsilon},$$

$$P_{gg \to G} = 2 \left(\frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right). \tag{3.2.37}$$

The splitting functions in Eq. (3.2.37) are summed over the spins of the final state partons and averaged over the spin of the parent parton. In general, the full splitting functions do depend on the spin of the parent parton. In the case where a parent quark splits into a quark and a gluon, there is no helicity correlation and the spin averaged splitting function completely describes the factorization of the corresponding matrix elements. However, in the case where a parent gluon splits into either a quark pair or two gluons, the full splitting functions are tensorial and related to the Lorentz index of the gluon. The factorization of the matrix element including all spin-dependent effects is:

$$|M^0(\cdots, i, j, \cdots)|^2 \xrightarrow{i//j} \frac{1}{s_{ij}} P^{\mu\nu}_{ij \to K_g}(z, k_\perp) |M^0_{\mu\nu}(\cdots, K, \cdots)|^2,$$

$$= \frac{1}{s_{ij}} P_{ij \to K_g}(z) |M^0(\cdots, K, \cdots)|^2 + ang. \tag{3.2.38}$$

The full splitting functions are given by [128]:

$$P^{\mu\nu}_{q\bar{q} \to G}(z, k_\perp) = -\eta^{\mu\nu} + 4z(1 - z) \frac{k^\mu_\perp k^\nu_\perp}{k^2_\perp},$$

$$P^{\mu\nu}_{gg \to G}(z, k_\perp) = -2 \left[\eta^{\mu\nu} \left(\frac{z}{1 - z} + \frac{1 - z}{z} \right) + 2z(1 - \epsilon)(1 - z) \frac{k^\mu_\perp k^\nu_\perp}{k^2_\perp} \right], \tag{3.2.39}$$

where $k^\mu_\perp$ is the component of momentum perpendicular to the collinear splitting axis.

The angular term in Eq.(3.2.38) depends on the azimuthal angle (ϕ) of the two splitting partons relative to the parent parton direction and is proportional to $\cos(2\phi)$. These angular terms in principle vanish during the final-state phase space integration if one performs an integration with an infinitely large number of events. In my implementation, we consider pairs of phase space points related by a rotation of $\Delta\phi = \pi/2$. The angular terms for the corresponding matrix elements will

cancel exactly. The extension to triple collinear limits (when three partons become simultaneously in the same direction) is similar. To make sure that the angular terms cancel exactly during the final-state phase space integration, we combine of pairs of phase space points related by a rotation of $\Delta\phi = \pi/2$ around the triple collinear axis.

3.3 Antenna subtraction term for real level at NLO

When calculating NLO cross sections, the divergent behaviour of the real matrix elements in the unresolved phase space will spoil the numerical evaluation of the real radiation contribution. One method to solve this problem is to remove the divergent behaviour from the matrix elements by adding compensating divergent subtraction terms. The subtraction terms are finite, but, after integration in d dimensions over the unresolved phase space regions, the integrated subtraction terms can be expressed as a Laurent expansion in ϵ. The implicit divergent behaviour is thus rendered explicit and cancels with the explicit poles from the virtual matrix elements. In this section, we introduce the antenna subtraction method for NLO cross sections.

3.3.1 Phase space mapping

In general, the singularities can be isolated in a particular region of phase space. By factorizing the full Lorentz invariant phase space into a sub-space that depends on the unresolved momenta and one that involved the remaining hard particles, one can mimic the factorization of matrix elements. At NLO, the single unresolved limit requires a mapping of $n + 1$ parton phase space to n parton phase space. In the colour ordered matrix element, the unresolved behaviour only relates to the colour adjacent partons, and therefore a three-particle sub-phase space involving the unresolved particle and the two hard radiators is sufficient. Depending on whether the colour adjacent partons are in the initial or final state, we need three different

mappings: a final-final (FF) mapping where two hard radiators are in the final state; initial-final (IF) mapping where one hard radiator is in the initial state and one is in the final state; initial-initial (II) mapping where both hard radiators are in the initial state.

Phase space mapping involves two parts. First is the momentum mapping itself, where the momentum in the three parton sub-phase space is mapped on to two hard radiators in the reduced sub-phase space. The on-shell condition of each hard radiator are required while the momentum conservation are preserved. Part two is the sub-phase space factorization. The sub-phase space is factorized from the full Lorentz invariant phase space and contains all dependence on the momentum of the unresolved particle.

Final-Final Mapping

In the final-final momentum mapping $\{i, j, k\} \to \{I, K\}$ [106, 107],

$$
\begin{aligned}
p_I^\mu \equiv p_{\widetilde{(ij)}}^\mu &= x\, p_i^\mu + r\, p_j^\mu + z\, p_k^\mu \\
p_K^\mu \equiv p_{\widetilde{(jk)}}^\mu &= (1-x)\, p_i^\mu + (1-r)\, p_j^\mu + (1-z)\, p_k^\mu
\end{aligned}
\tag{3.3.40}
$$

where,

$$
\begin{aligned}
x &= \frac{1}{2(s_{ij} + s_{ik})}\Big[(1+\rho)\, s_{ijk} - 2\, r\, s_{jk}\Big], \\
z &= \frac{1}{2(s_{jk} + s_{ik})}\Big[(1-\rho)\, s_{ijk} - 2\, r\, s_{ij}\Big],
\end{aligned}
$$

$$
\rho^2 = 1 + \frac{4\, r(1-r)\, s_{ij} s_{jk}}{s_{ijk} s_{ik}}.
\tag{3.3.41}
$$

The mapping (3.3.40) preserves momentum conservation $p_{\widetilde{(ij)}} + p_{\widetilde{(jk)}} = p_i + p_j + p_k$ and has three free parameters. The two on-shell conditions for p_I and p_K fix x and z, and there is still one free parameter. For the convenience of integration, here we choose $r = s_{jk}/(s_{ij} + s_{jk})$. In the single unresolved limits where p_j^μ becomes soft, $p_I^\mu \to p_i^\mu$ and $p_K^\mu \to p_k^\mu$. In the collinear limit when $p_j^\mu // p_i^\mu$, $p_I^\mu \to (p_i^\mu + p_j^\mu)$ and $p_K^\mu \to p_k^\mu$.

The full phase space as mentioned in section 1.3 is

$$d\Phi_{n+1}(p_3, \ldots, p_{n+3}; p_1, p_2) = (2\pi)^d \delta\left(p_1 + p_2 - \sum_{l=3}^{n+3} p_l\right) \prod_{l=3}^{n+3} [dp_l], \qquad (3.3.42)$$

where $[dp] = d^d p \, \delta^+(p^2)/(2\pi)^{d-1}$ is the phase space measure [108]. In the final-final sub-phase space factorization, inserting the following identity

$$1 = \int d^{d-1}p_I d^{d-1}p_K \delta(p_I + p_K - p_i - p_k - p_j)\delta(p_I^2)\delta(p_K^2),$$

$$1 = \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \delta(1 - x_1)\delta(1 - x_2), \qquad (3.3.43)$$

one have [7, 108]

$$\begin{aligned} d\Phi_{n+1}(p_3, \ldots, p_{n+3}; p_1, p_2) &= d\Phi_n(p_3, \ldots, p_I, p_K, \ldots, p_{m+3}; p_1, p_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2} \\ &\times \delta(1 - x_1)\delta(1 - x_2)d\Phi_{X_{ijk}}^{FF}(p_i, p_j, p_k; p_I, p_K). \end{aligned}$$

$$(3.3.44)$$

The three to two final-final antenna phase space $d\Phi_{X_{ijk}}^{FF}$ is proportional to the three-particle phase space [109],

$$d\Phi_{X_{ijk}}^{FF}(p_i, p_j, p_k; p_I, p_K) = \frac{1}{P_2} \cdot d\Phi_3(p_i, p_j, p_k; p_I, p_K), \qquad (3.3.45)$$

where the $d-$dimensional two particle phase space P_2 is a constant given by,

$$P_2 = 2^{-3+2\epsilon}\pi^{-1+\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}(p_i + p_j + p_k)^{-\epsilon}. \qquad (3.3.46)$$

The composite momentum p_I and p_K only appears in the reduced phase space such that the phase space of $d\Phi_n$ is independent from $d\Phi_{X_{ijk}}^{FF}$ in the integration. The $x_i(i = 1, 2)$ parameter is introduced to keep a unified form for the reduced phase space as in the IF and II mapping.

Initial-Final Mapping

In the initial-final momentum mapping, the initial state momentum is rescaled by the mapping. Using hat notation for initial state and bar notation for rescaling we have $\{\hat{i}, j, k\} \rightarrow \{\hat{I}, K\} \equiv \{\hat{\bar{i}}, K\}$ mapping [108],

$$p_{\hat{I}}^\mu \equiv \bar{p}_i^\mu = \hat{x}_i \, p_i^\mu,$$

$$p_K^\mu \equiv p_{\widetilde{(jk)}}^\mu = p_j^\mu + p_k^\mu - (1 - \hat{x}_i)\, p_i^\mu. \tag{3.3.47}$$

With the on-shell condition $p_{\hat{I}}^2 = p_K^2 = 0$ the $\hat{x}_i$ parameter is fixed to be,

$$\hat{x}_i = \frac{s_{ij} + s_{ik} + s_{jk}}{s_{ij} + s_{ik}}. \tag{3.3.48}$$

For single unresolved limits where p_j^μ become soft, $p_{\hat{I}}^\mu \to p_i^\mu$ and $p_K^\mu \to p_k^\mu$. In the collinear limit when $p_j^\mu // p_i^\mu$, $p_{\hat{I}}^\mu \to (p_i^\mu + p_j^\mu)$ and $p_K^\mu \to p_k^\mu$.

Inserting the following identity

$$1 = \int \mathrm{d}^d q\, \delta\left(q + p_i - p_j - p_k\right) \tag{3.3.49}$$

$$1 = \frac{Q^2}{2\pi} \int \frac{\mathrm{d}x_i}{x_i} \int [\mathrm{d}p_K]\, (2\pi)^d \delta\left(q + x_i p_i - p_K\right) \tag{3.3.50}$$

into the full phase space (3.3.42) and integrating over q, where $Q^2 = -q^2$ and $q = p_j + p_k - p_i$. The full phase space can be rewritten as (for $i = 1$) [7, 108]

$$\begin{aligned}
\mathrm{d}\Phi_{n+1}(p_3, \ldots, k_{n+3}; p_1, p_2) &= \mathrm{d}\Phi_n(p_3, \ldots, p_K, \ldots, p_{n+3}; x_1 p_1, p_2) \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \\
&\times \; \delta(x_1 - \hat{x}_1)\delta(1 - x_2)\mathrm{d}\Phi_{X_{\hat{i}jk}}^{IF}(p_j, p_k; p_i, q),
\end{aligned} \tag{3.3.51}$$

where the three to two initial-final antenna phase space is defined as

$$\mathrm{d}\Phi_{X_{\hat{i}jk}}^{IF}(p_j, p_k; p_i, q) \equiv \frac{Q^2}{2\pi} \mathrm{d}\Phi_2(p_j, p_k; p_i, q). \tag{3.3.52}$$

Initial-Initial Mapping

Finally for the initial-initial momentum mapping $\{\hat{i}, j, \hat{k}\} \to \{\hat{I}, \hat{K}\} \equiv \{\hat{\hat{i}}, \hat{\hat{k}}\}$, both initial states are rescaled. As $q = p_i + p_k - p_j$ in general is not on beam, a Lorentz boost for all final state partons is needed to preserve momentum conservation. The initial-initial mapping changes the momentum of all partons $\{\hat{i}, j, \hat{k}, \ldots, l, m, \ldots\} \to \{\hat{I}, \hat{K}, \ldots, \tilde{l}, \tilde{m}, \ldots\}$ that [108],

$$\begin{aligned}
p_{\hat{I}}^\mu \equiv \bar{p}_i^\mu &= \hat{x}_i\, p_i^\mu, \\
p_{\hat{K}}^\mu \equiv \bar{p}_k^\mu &= \hat{x}_k\, p_k^\mu, \\
\tilde{p}_\ell^\mu &= p_\ell^\mu - \frac{2p_\ell \cdot (q + \tilde{q})}{(q + \tilde{q})^2}\left(q^\mu + \tilde{q}^\mu\right) + \frac{2p_\ell \cdot q}{q^2}\, \tilde{q}^\mu,
\end{aligned} \tag{3.3.53}$$

where $\ell \neq j$ and

$$q^\mu = p_i^\mu + p_k^\mu - p_j^\mu, \qquad\qquad \tilde{q}^\mu = \bar{p}_i^\mu + \bar{p}_k^\mu,$$

$$\hat{x}_i = \sqrt{\frac{s_{ik} + s_{jk}}{s_{ik} + s_{ij}}} \sqrt{\frac{s_{ik} + s_{ij} + s_{jk}}{s_{ik}}},$$

$$\hat{x}_k = \sqrt{\frac{s_{ik} + s_{ij}}{s_{ik} + s_{jk}}} \sqrt{\frac{s_{ik} + s_{jk} + s_{ij}}{s_{ik}}}. \tag{3.3.54}$$

This mapping contains a transverse Lorentz boost $\Lambda(p_\ell, \tilde{p}_\ell)$ $(\Lambda(p_\ell, \tilde{p}_\ell)p_\ell = \tilde{p}_\ell)$ that preserves the Lorentz invariants $p_\ell^2 = \tilde{p}_\ell^2 = 0$ and momentum conservation. In the single unresolved limits where p_j^μ becomes soft or collinear with p_i^μ or p_k^μ, the intermediate momentum q^μ become proportional to $\tilde{q}^\mu$. In this case, for each $l \neq j$, $\tilde{p}_l^\mu \to p_l^\mu$ and $p_{\hat{I}}^\mu$, $p_{\hat{K}}^\mu$ become the corresponding composite momentum. Further details of the composite momentum in the single unresolved limits can be found in [108, 113].

In the initial-initial sub-phase space factorization (for $i = 1, k = 2$), inserting the following identity

$$1 = \int \mathrm{d}^d q \mathrm{d}^d \tilde{q}\, \delta\left(p_1 + p_2 - p_j - q\right)\, \delta\left(x_1 p_1 + x_2 p_2 - \tilde{q}\right), \tag{3.3.55}$$

$$1 = \int \prod_{\ell \neq j} \delta(\tilde{p}_\ell - \Lambda(p_\ell, \tilde{p}_\ell)p_\ell)[\mathrm{d}\tilde{p}_\ell], \tag{3.3.56}$$

$$1 = \int \mathrm{d}x_1 \mathrm{d}x_2 \delta(x_1 - \hat{x}_1)\delta(x_2 - \hat{x}_2), \tag{3.3.57}$$

into the full phase space (3.3.42) and integrating over $q, \tilde{q}$ and p_ℓ. One will have the factorized phase space [7, 108]

$$\begin{aligned}
\mathrm{d}\Phi_{n+1}(p_3, \ldots, p_{n+3}; p_1, p_2) &= \mathrm{d}\Phi_n(\tilde{p}_3, \ldots, \tilde{p}_{n+3}; x_1 p_1, x_2 p_2)\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2} \\
&\quad \times \delta(x_1 - \hat{x}_1)\, \delta(x_2 - \hat{x}_2)\mathrm{d}\Phi_{X_{\hat{1}j\hat{2}}}^{II},
\end{aligned} \tag{3.3.58}$$

where the three to two initial-initial antenna phase space is defined as

$$\mathrm{d}\Phi_{X_{\hat{1}j\hat{2}}}^{II} \equiv x_1 x_2 [\mathrm{d}p_j]. \tag{3.3.59}$$

3.3.2 Antenna functions for real level at NLO

The essential ingredient of the antenna subtraction method are antenna functions which mimic the various divergent behaviour of the colour ordered matrix elements.

maple	fortran/form	latex	comment
A30FF	FullA30FF	A_3^0	Eq. (5.5) of [109].
D30FF	FullD30FF	D_3^0	Eq. (6.8) of [109].
			$D_3^0 = d_3^0(i_1, i_2, i_3) + d_3^0(i_1, i_3, i_2)$.
d30FF	d30FF	d_3^0	Eq. (6.13) of [109]. Only has i_2 soft limit.
E30FF	FullE30FF	E_3^0	Eq. (6.14) of [109].
F30FF	FullF30FF	F_3^0	Eq. (7.9) of [109].
			$F_3^0 = f_3^0(i_1, i_2, i_3) + f_3^0(i_2, i_3, i_1) + f_3^0(i_3, i_1, i_2)$.
f30FF	f30FF	f_3^0	Eq. (7.13) of [109]. Only has i_2 soft limit.
G30FF	FullG30FF	G_3^0	Eq. (7.14) of [109]. Only has $i_2 \| i_3$ collinear limit.

Table 3.1: X_3^0 antenna functions for final-final state. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

These antenna functions are obtained from physical matrix elements and therefore naturally describe the singular limits of matrix elements.

Antenna functions each have two hard radiators and one or two unresolved partons. According to the type of the hard radiators there are three classes and each class of functions are calculated from different physical colour ordered matrix elements: for quark-anti-quark hard radiators, antenna functions are derived from the matrix elements for virtual photon decay into quark-anti-quark pair plus additional (unresolved) partons radiated from the quark pair [110]; for quark-gluon hard radiators, the antenna functions are derived from the matrix elements for heavy neutralino decay into gluino and gluon plus (unresolved) partons radiated from the gluino-gluon pair [111]; for gluon-gluon hard radiators, antenna functions are derived from the matrix elements for Higgs boson decay into two gluons (via effective vertex) plus (unresolved) partons radiated from the gluon-gluon pair [112].

At NLO, a single unresolved parton j is colour adjacent to the hard radiator partons i and k. To describe the unresolved behaviour of parton j, we construct the

maple	fortran/form	latex	comment
gA30IF	FullgA30IF	$A_{3,g\to q}^0$	Eq. (4.25) of [108]. Mixed flavour changing.
			$A_{3,g}^0 = a_{3,g\to q}^0(i_1, \hat{i}_2, i_3) + a_{3,g\to q}^0(i_3, \hat{i}_2, i_1)$.
ga30IFgtoq	ga30IFGTOQ	$a_{3,g\to q}^0$	Only contains $i_1\|\|i_2$ collinear limit.
			Flavour changing $g \to q$.
gD30IF	FullgD30IF	$D_{3,g}^0$	Mixed flavour changing.
			$D_{3,g}^0 = d_{3,g\to q}^0(i_1, \hat{i}_2, i_3) + d_{3,g}^0(i_1, i_3, \hat{i}_2)$.
gd30IF	gd30IF	$d_{3,g}^0$	Eq. (4.29) of [108]. i_3 initial state i_2 soft.
gd30IFgtoq	gd30IFGTOQ	$d_{3,g\to q}^0$	Eq. (4.28) of [108]. Flavour changing $g \to q$.
gF30IF	FullgF30IF	$F_{3,g}^0$	Eq. (4.32) of [108].
			$F_{3,g}^0 = f_{3,g}^0(\hat{i}_1, i_2, i_3) + f_{3,g}^0(\hat{i}_1, i_3, i_2)$.
gf30IF	gf30IF	$f_{3,g}^0$	Only contains $i_1\|\|i_2$ collinear limit.
gG30IF	FullgG30IF	$G_{3,g}^0$	Eq. (4.33) of [108].
qA30IF	FullqA30IF	$A_{3,q}^0$	Eq. (4.15) of [108].
qD30IF	FullqD30IF	$D_{3,q}^0$	Eq. (4.17) of [108].
			$D_{3,q}^0 = d_{3,q}^0(\hat{i}_1, i_2, i_3) + d_{3,q}^0(\hat{i}_1, i_3, i_2)$.
qd30IF	qd30IF	$d_{3,q}^0$	Only contains $i_1\|\|i_2$ collinear limit.
qE30IF	FullqE30IF	$E_{3,q}^0$	Eq. (4.18) of [108].
qpE30IF	FullqpE30IF	$E_{3,q'\to g}^0$	Eq. (4.19) of [108]. Flavour changing $q' \to g$.
qpG30IF	FullqpG30IF	$G_{3,q'\to g}^0$	Eq. (4.23) of [108]. Flavour changing $q' \to g$.

Table 3.2: X_3^0 antenna functions for initial-final state. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

maple	fortran/form	latex	comment
ggD30II	FullggD30II	$D^0_{3,gg}$	Mixed flavour changing. $D^0_{3,gg} = d^0_{3,gg \to qg}(i_1, \hat{\hat{i}}_2, \hat{\hat{i}}_3) + d^0_{3,gg \to qg}(i_1, \hat{\hat{i}}_3, \hat{\hat{i}}_2)$.
ggd30II	ggd30II	$d^0_{3,gg \to qg}$	Eq. (5.22) of [108]. Only contains $i_1 \| i_2$ collinear limit. Flavour changing $g \to q$.
ggF30II	FullggF30II	$F^0_{3,gg}$	Crossing of F^0_3.
gqA30II	FullgqA30II	$A^0_{3,gq \to qq}$	Crossing of A^0_3. Flavour changing $g \to q$.
gqD30II	FullgqD30II	$D^0_{3,gq}$	Crossing of D^0_3.
gqG30II	FullgqG30II	$G^0_{3,gq \to gg}$	Crossing of G^0_3. Flavour changing $q \to g$.
qgA30II	FullqgA30II	$A^0_{3,qg \to qq}$	Crossing of A^0_3. Flavour changing $g \to q$.
qgD30II	FullqgD30II	$D^0_{3,qg}$	Crossing of D^0_3.
qgG30II	FullqgG30II	$G^0_{3,qg \to gg}$	Crossing of G^0_3. Flavour changing $q \to g$.
qpqpE30II	FullqpqpE30II	$E^0_{3,q\bar{q}}$	Crossing of E^0_3.
qqA30II	FullqqA30II	$A^0_{3,q\bar{q}}$	Crossing of A^0_3.
qqG30II	FullqqG30II	$G^0_{3,q\bar{q}}$	Crossing of G^0_3.
qqpE30II	FullqqpE30II	$E^0_{3,qq' \to qg}$	Crossing of E^0_3. Flavour changing $q' \to g$.

Table 3.3: X^0_3 antenna functions for initial-initial state. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

following antenna function

$$X_3^0(i,j,k) = \mathcal{S}_{ijk/IK} \frac{M_3^0(i,j,k)}{M_2^0(I,K)}. \tag{3.3.60}$$

The $X_3^0(i,j,k)$ antenna function is defined as the ratio of the colour ordered matrix element $M_3^0(i,j,k)$ to the underlying two-parton process $M_2^0(I,K)$. Partons i and k are hard radiators in j unresolved limits and I, K are the mapped momentum introduced in section 3.3.1. $\mathcal{S}_{ijk/IK}$ is a symmetry factor. The antenna functions for the various parton types and initial or final state configurations are summarised in table 3.1-3.3.

To help with the book keeping, we have established a maple scripting language in which the various antenna functions, matrix elements and subtraction terms are uniquely identified. Maple scripts converts input ".map" files, that are written in condensed (but easily understood) analytical formulae, into Fortran, FORM and LaTex codes for further numerical and analytical testing. These ".map" files are the link between the human-intensive construction of subtraction terms, and their numerical and analytic validation. They also guarantee that what appears in latex form, is the same as what appears in the fortran or form routines and there is a one to one correspondence between a name in the maple script and a function in the Fortran or FORM libraries where the details are included in table 3.1-3.3. The maple script also automatically deals with the momentum mappings associated with each antenna function and with the reduced matrix elements freeing us to concentrate on the physical information inside the antenna subtraction terms.

In general, $X_3^0(i,j,k)$, analytically describes the divergent behaviour of the matrix elements in the single soft $(j \to 0)$ and single collinear limits $(i//j$ and $j//k)$. In some special cases, the $M_3^0(i,j,k)$ matrix element used to define X_3^0 also has colour connections between i and k. In these cases, the antenna function is also divergent in the i soft, k soft or $i//k$ collinear limits. However, the phase space mapping (e.g. FF mapping) will not factorise onto the correct momentum required for the reduced matrix elements. To avoid this problem, in the few cases where i and k are colour connected, such as the gluon-gluon antenna F_3^0 and quark-gluon antenna D_3^0, we further decompose the full X_3^0 antenna function into sub-antenna functions, f_3^0 and d_3^0, which eliminate the colour connection between parton i and k [108, 113].

If an antenna function contains unresolved limits that correspond to having different parton types in the initial state, then there is also an ambiguity in the type of initial state parton in the reduced matrix elements. By taking advantage of the well defined colour connection in the subantenna functions we can remove the ambiguity in initial-state-identity-changing unresolved limits.

3.3.3 Antenna subtraction terms $d\hat{\sigma}_{NLO}^S$

$d\hat{\sigma}_{NLO}^S$ is designed to mimic the divergent behaviour of the matrix elements in the NLO real contribution, $d\hat{\sigma}^R$. In the single unresolved limit of parton j, the colour ordered matrix behave as the factorized divergent functions times the reduced matrix element with parton j pinched out (equation (3.2.34)(3.2.36)). In the antenna subtraction framework, $X_3^0(i,j,k)$ antenna functions are used to mimic the unresolved singularities of the matrix elements. The reduced matrix elements associated with antenna functions involve mapped momentum produced by the $\{i,j,k\} \rightarrow \{I,K\}$ mapping. In general, for a colour ordered matrix element $|M^0(\cdots,i,j,k,\cdots)|^2$ with $(n+1)$ final state partons with momenta in the set $\{p\}_{n+1}$, the NLO real contribution is

$$
d\hat{\sigma}_{NLO}^R = \mathcal{N}_{NLO}^R \frac{\bar{C}(\epsilon)}{C(\epsilon)} d\Phi_{n+1}(p_3,\cdots,p_{n+3};p_1,p_2) \frac{1}{s_{n+1}} |M_{n+3}^0(\cdots,i,j,k,\cdots)|^2
$$
$$
\times J_{n+1}^{(n)}(\{p\}_{n+1}). \tag{3.3.61}
$$

while the NLO antenna subtraction term has the structure

$$
d\hat{\sigma}_{NLO}^S = \mathcal{N}_{NLO}^R \frac{\bar{C}(\epsilon)}{C(\epsilon)} d\Phi_{n+1}(p_3,\cdots,p_{n+3};p_1,p_2) \frac{1}{s_{n+1}}
$$
$$
\times \left\{ \sum_j X_3^0(i,j,k) |M_{n+2}^0(\cdots,I,K,\cdots)|^2 \right\} J_n^{(n)}(\{p\}_n). \tag{3.3.62}
$$

Here $\mathcal{N}_{NLO}^R$ is an overall constant containing the colour factors, momentum flux and coupling constants while $J_n^{(m)}(\{p\}_n)$ applies the jet algorithm to obtain m jets from the n partons with momenta in the set $\{p\}_n$. The mapping applied to each term in the sum over j could be either FF, IF or II type depending on whether the partons adjacent to j, i and k, are in the initial or final state. Note that the reduced matrix element $|M^0(\cdots,I,K,\cdots)|^2$ will diverge if additional particles are unresolved, but this is prevented by requiring that m-jets are observed.

The difference between $d\hat{\sigma}^R_{NLO}$ and $d\hat{\sigma}^S_{NLO}$ is now a finite contribution in all phase space region, and the integration of the difference:

$$\int_{d\Phi_{n+1}} \left(d\hat{\sigma}^R_{NLO} - d\hat{\sigma}^S_{NLO} \right) \tag{3.3.63}$$

is mathematically well defined.

3.4 Divergent behaviour of virtual contributions at NLO

3.4.1 Pole structure of one-loop matrix elements

The NLO virtual contribution contains explicit divergences coming from the integration of the loop momentum. In order to express the poles of one-loop matrix elements in a way such that the cancellation of poles against the integrated real radiation subtraction term can be analytically made, it is convenient to use the language introduced by Catani [114, 115]. The $\mathbf{I}^{(1)}$-operator [114, 115] is an operator in colour space such that the renormalised one-loop amplitude with Laurent expansion in ϵ can be written as

$$\mathcal{M}^1(\epsilon) = \mathbf{I}^{(1)}(\epsilon)\mathcal{M}^0 + \mathcal{M}^{1,fin}, \tag{3.4.64}$$

where $\mathcal{M}^{1,fin}$ is a process dependent function that is finite in the $\epsilon \to 0$ limit.

For the convenience of matching the integrated real radiation from colour ordered matrix elements, the $\mathbf{I}^{(1)}$-operator can be straightfowwardly recast to act on renormalised colour ordered one-loop matrix elements [109].

Depending on the parton type and internal loop there are six independent $\mathbf{I}^{(1)}$-operators given by

$$\begin{aligned}
\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon, s_{q\bar{q}}) &= -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right] \Re(-s_{q\bar{q}})^{-\epsilon}\,, \\
\mathbf{I}^{(1)}_{qg}(\epsilon, s_{qg}) &= -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \left[\frac{1}{\epsilon^2} + \frac{5}{3\epsilon} \right] \Re(-s_{qg})^{-\epsilon}\,, \\
\mathbf{I}^{(1)}_{gg}(\epsilon, s_{gg}) &= -\frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \left[\frac{1}{\epsilon^2} + \frac{11}{6\epsilon} \right] \Re(-s_{gg})^{-\epsilon}\,, \\
\mathbf{I}^{(1)}_{q\bar{q},F}(\epsilon, s_{q\bar{q}}) &= 0\,,
\end{aligned}$$

$$\mathbf{I}_{qg,F}^{(1)}(\epsilon, s_{qg}) = \frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \frac{1}{6\epsilon} \Re(-s_{qg})^{-\epsilon} ,$$

$$\mathbf{I}_{gg,F}^{(1)}(\epsilon, s_{gg}) = \frac{e^{\epsilon\gamma}}{2\Gamma(1-\epsilon)} \frac{1}{3\epsilon} \Re(-s_{gg})^{-\epsilon} . \tag{3.4.65}$$

The antiquark-gluon operators are obtained by charge conjugation:

$$\mathbf{I}_{g\bar{q}}^{(1)}(\epsilon, s_{g\bar{q}}) = \mathbf{I}_{qg}^{(1)}(\epsilon, s_{g\bar{q}}) \qquad \text{and} \qquad \mathbf{I}_{g\bar{q},F}^{(1)}(\epsilon, s_{g\bar{q}}) = \mathbf{I}_{qg,F}^{(1)}(\epsilon, s_{g\bar{q}}) .$$

The three $\mathbf{I}_{ij}^{(1)}$ operators (without F label) describe the pole structure of colour adjacent partons i and j in a colour ordered one-loop matrix element having a gluon loop while the three $\mathbf{I}_{ij}^{(1)}$ operators (with the F label) are for the corresponding matrix elements with a fermion loop. The dimensional regularization parameter ϵ exhibits the divergent behaviour of the loop integral in explicit form.

The $\mathbf{I}_{ij}^{(1)}$ operators depend only on the colour connected partons i and j and thus can be used to describe the pole structure of one-loop matrix elements in a dipole formalism. In general, for a colour leading one-loop matrix element $|M_n^1(\cdots, i, j, \cdots)|^2$ with n partons, the explicit pole structure is

$$|M_n^1(\cdots, i, j, \cdots)|^2 \xrightarrow{pole} \sum_{ij} 2\mathbf{I}_{ij}^{(1)}(\epsilon, s_{ij})|M_n^0(\cdots, i, j, \cdots)|^2$$

$$\equiv 2\mathbf{I}_n^{(1)}(\epsilon, 1, \cdots, n)|M_n^0(\cdots, i, j, \cdots)|^2 \tag{3.4.66}$$

3.4.2   Mass factorization terms at NLO

Apart from the pole structure of renormalised one-loop matrix element, for a proton proton collision process, there is an explicit divergent contribution from the initial colour particle radiation. As mentioned in section 1.4, the specific expression for NLO mass factorization terms in equation (1.4.47) have the general structure

$$d\hat{\sigma}_{ij,NLO}^{MF}(\xi_1 H_1, \xi_2 H_2) = -\int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Big[\delta(1-x_2)\mathbf{\Gamma}_{ki}^1(x_1)d\hat{\sigma}_{kj}^B$$

$$+ \delta(1-x_1)\mathbf{\Gamma}_{lj}^1(x_2)d\hat{\sigma}_{il}^B \Big] (x_1\xi_1 H_1, x_2\xi_2 H_2) \tag{3.4.67}$$

where $\mathbf{\Gamma}_{ij}^1(x)$ contains an explicit divergence given by,

$$\mathbf{\Gamma}_{ij}^1(x) \sim \Gamma_{ij}^1(x) = -\frac{1}{\epsilon} p_{ij}^{(0)}(x). \tag{3.4.68}$$

The relation between $\mathbf{\Gamma}^1_{ij}(x)$ and $\Gamma^1_{ij}(x)$ and explicit formulas for $p^{(0)}_{ij}(x)$ are given in Appendix F.1. The repeating indices k and l imply a sum over parton types (gluon, quarks and anti-quarks), while $\mathrm{d}\hat{\sigma}^B_{kj}$ is the Born level differential cross section initiated by partons k and j.

3.5 Antenna subtraction term for virtual level at NLO

The counter terms introduced to render the real radiation contribution finite, $\mathrm{d}\hat{\sigma}^S_{NLO}$, are unphysical and need to be removed in the full NLO calculation. At the virtual level, one simply adds the integrated version of $\int_1 \mathrm{d}\hat{\sigma}^S_{NLO}$ which contains both finite contributions and explicit pole structures that must cancel with the explicit singularities present in the virtual and mass factorization contributions:

$$
\begin{aligned}
\hat{\sigma}_{NLO} &= \int_{\mathrm{d}\Phi_{n+1}} \mathrm{d}\hat{\sigma}^R_{NLO} + \int_{\mathrm{d}\Phi_n} \left(\mathrm{d}\hat{\sigma}^V_{NLO} + \mathrm{d}\hat{\sigma}^{MF}_{NLO} \right) \\
&= \int_{\mathrm{d}\Phi_{n+1}} \left(\mathrm{d}\hat{\sigma}^R_{NLO} - \mathrm{d}\hat{\sigma}^S_{NLO} \right) + \int_{\mathrm{d}\Phi_n} \left(\mathrm{d}\hat{\sigma}^V_{NLO} - \mathrm{d}\hat{\sigma}^T_{NLO} \right).
\end{aligned} \quad (3.5.69)
$$

Each bracket above is free of infrared divergence and the virtual subtraction term in the antenna subtraction framework is,

$$
\mathrm{d}\hat{\sigma}^T_{NLO} = -\int_1 \mathrm{d}\hat{\sigma}^S_{NLO} - \mathrm{d}\hat{\sigma}^{MF}_{NLO}. \quad (3.5.70)
$$

3.5.1 Integration of the antenna subtraction terms for real contribution

The general form of NLO antenna subtraction terms is given in Eq. (3.3.62). The full phase space $\mathrm{d}\Phi_{n+1}$ is factorized into an antenna phase space that involves the unresolved parton and a reduced phase space that described the re-mapped hard radiators. The phase space factorization is described in Eqs. (3.3.44),(3.3.51) and (3.3.58) for the FF, IF and II mappings. Integration of the real subtraction term $\int_1 \mathrm{d}\hat{\sigma}^S_{NLO}$ is performed by integrating the antenna function over the antenna phase

maple	fortran/form	latex	comment
calA30FF	A30FFint	$\mathcal{A}_3^0$	Eq. (5.6) of [109].
calD30FF	D30FFint	$\mathcal{D}_3^0$	Eq. (6.9) of [109].
calE30FF	E30FFint	$\mathcal{E}_3^0$	Eq. (6.14) of [109].
calF30FF	F30FFint	$\mathcal{F}_3^0$	Eq. (7.9) of [109].
calG30FF	G30FFint	$\mathcal{G}_3^0$	Eq. (7.15) of [109].

Table 3.4: $\mathcal{X}_3^0$ antenna functions for final-final state. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

space. The general form can be arranged as:

$$
\int_1 \mathrm{d}\hat\sigma_{NLO}^S = \mathcal{N}_{NLO}^R \bar{C}(\epsilon) \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \frac{1}{s_{n+1}} \mathrm{d}\Phi_n \left\{ \sum_{I,K} \mathcal{X}_3^0(I,K) |M_{n+2}^0(\cdots, I, K, \cdots)|^2 \right\}
$$
$$
\times J_n^{(n)}(\{p\}_n), \tag{3.5.71}
$$

where $\mathcal{X}_3^0(I,K)$ depends on the primary mapping $\{p_i, p_j, p_k\} \to \{p_I, p_K\}$ that,

$$
\mathcal{X}_3^0(I,K) \xrightarrow{FF} \mathcal{X}_3^0(s_{IK}, x_1, x_2) = \frac{1}{C(\epsilon)} \int \delta(1-x_1)\delta(1-x_2)\mathrm{d}\Phi_{X_3} X_3^0(i,j,k),
$$
$$
\mathcal{X}_3^0(I,K) \xrightarrow{IF} \mathcal{X}_3^0(s_{\bar{1}K}, x_1, x_2) = \frac{1}{C(\epsilon)} \int \delta(x_1-\hat{x}_1)\delta(1-x_2)\frac{Q^2}{2\pi}\mathrm{d}\Phi_2 X_3^0(1,j,k),
$$
$$
\mathcal{X}_3^0(I,K) \xrightarrow{II} \mathcal{X}_3^0(s_{\bar{1}\bar{2}}, x_1, x_2) = \frac{1}{C(\epsilon)} \int \delta(x_1-\hat{x}_1)\delta(x_2-\hat{x}_2)x_1 x_2 [\mathrm{d}p_j] X_3^0(1,j,2),
$$
$$
\tag{3.5.72}
$$

Note that initial state labels 1 and 2 are exchangeable in these formulas. The integrated antenna for the allowed combinations of hard parton radiators are summarised in Tables 3.4–3.6.

3.5.2   Antenna subtraction terms $\mathrm{d}\hat\sigma_{NLO}^T$

The virtual subtraction term is a combination of the integrated real subtraction term and the mass factorization contribution. It is convenient to combine integrated antenna functions and tree-level mass factorization terms to form the dipole functions $\boldsymbol{J}_2^{(1)}(I,K)$ [116]. Dipole functions associated with born level matrix elements are then used to cancel the explicit poles for each colour connected parton

maple	fortran/form	latex	comment
calgA30IF	gA30IFint	$\mathcal{A}^0_{3,g\to q}$	Eq. (4.26) of [108]. Flavour changing $g \to q$.
calgD30IF	calgD30IF	$\mathcal{D}^0_{3,g}$	
calgD30gqIF	gD30gqIFint	$\mathcal{D}^0_{3,g\to g}$	Eq. (4.31) of [108].
calgD30qgIF	gD30qgIFint	$\mathcal{D}^0_{3,g\to q}$	Eq. (4.30) of [108]. Flavour changing $g \to q$.
calgF30IF	gF30IFint	$\mathcal{F}^0_{3,g}$	Eq. (4.34) of [108].
calgG30IF	gG30IFint	$\mathcal{G}^0_{3,g}$	Eq. (4.35) of [108].
calqA30IF	qA30IFint	$\mathcal{A}^0_{3,q}$	Eq. (4.16) of [108].
calqD30IF	qD30IFint	$\mathcal{D}^0_{3,q}$	Eq. (4.20) of [108].
calqE30IF	qE30IFint	$\mathcal{E}^0_{3,q}$	Eq. (4.21) of [108].
calqpE30IF	qpE30IFint	$\mathcal{E}^0_{3,q'\to g}$	Eq. (4.22) of [108]. Flavour changing $q \to g$.
calqpG30IF	qpG30IFint	$\mathcal{G}^0_{3,q'\to g}$	Eq. (4.24) of [108]. Flavour changing $q \to g$.

Table 3.5: $\mathcal{X}^0_3$ antenna functions for initial-final state. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

maple	fortran/form	latex	comment
`calggD301II`	`ggD301IIint`	$\mathcal{D}^0_{3,gg\to qg}$	Eq. (5.26) of [108] with $x_1 \leftrightarrow x_2$. Flavour changing $g \to q$ in P_1.
`calggD302II`	`ggD302IIint`	$\mathcal{D}^0_{3,gg\to gq}$	Eq. (5.26) of [108]. Flavour changing $g \to q$ in P_2.
`calggD30II`	`ggD30IIint`	$\mathcal{D}^0_{3,really}$	ggD30IIint
`calggF30II`	`ggF30IIint`	$\mathcal{F}^0_{3,gg}$	Eq. (5.29) of [108].
`calgqA30II`	`gqA30IIint`	$\mathcal{A}^0_{3,gq\to qq}$	Eq. (5.23) of [108] with $x_1 \leftrightarrow x_2$. Flavour changing $g \to q$ in P_1.
`calgqD30II`	`gqD30IIint`	$\mathcal{D}^0_{3,gq}$	Eq. (5.25) of [108] with $x_1 \leftrightarrow x_2$.
`calgqG30II`	`gqG30IIint`	$\mathcal{G}^0_{3,gq\to gg}$	Eq. (5.31) of [108]. Flavour changing $q \to g$ in P_1.
`calqgA30II`	`qgA30IIint`	$\mathcal{A}^0_{3,qg\to qq}$	Eq. (5.23) of [108]. Flavour changing $g \to q$ in P_2.
`calqgD30II`	`qgD30IIint`	$\mathcal{D}^0_{3,qg}$	Eq. (5.25) of [108].
`calqgG30II`	`qgG30IIint`	$\mathcal{G}^0_{3,qg\to gg}$	Eq. (5.31) of [108] with $x_1 \leftrightarrow x_2$. Flavour changing $q \to g$ in P_2.
`calqpqE30II`	`qpqE30IIint`	$\mathcal{E}^0_{3,q'q\to gq}$	Eq. (5.28) of [108] with $x_1 \leftrightarrow x_2$. Flavour changing $q \to g$ in P_1.
`calqpqpE30II`	`qpqpE30IIint`	$\mathcal{E}^0_{3,qq}$	Eq. (5.27) of [108].
`calqqA30II`	`qqA30IIint`	$\mathcal{A}^0_{3,qq}$	Eq. (5.24) of [108].
`calqqG30II`	`qqG30IIint`	$\mathcal{G}^0_{3,qq}$	Eq. (5.30) of [108].
`calqqpE30II`	`qqpE30IIint`	$\mathcal{E}^0_{3,qq'\to qg}$	Eq. (5.28) of [108]. Flavour changing $q \to g$ in P_2.

Table 3.6: $\mathcal{X}^0_3$ antenna functions for initial-initial state. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

pair in the virtual contribution. In general, for a one-loop colour ordered matrix element $|M^1(\cdots, I, K, \cdots)|^2$ with n final state partons, the NLO virtual contribution is given by

$$
d\hat{\sigma}^V_{NLO} = \mathcal{N}^R_{NLO} \bar{C}(\epsilon) d\Phi_{n+1}(p_3, \cdots, p_{n+2}; p_1, p_2) \frac{1}{s_n} |M^1_{n+2}(\cdots, I, K, \cdots)|^2
$$
$$
\times J_n^{(n)}(\{p\}_n). \tag{3.5.73}
$$

while the virtual antenna subtraction term has the structure

$$
d\hat{\sigma}^T_{NLO} = -\mathcal{N}^R_{NLO} \bar{C}(\epsilon) \frac{dx_1}{x_1} \frac{dx_2}{x_2} d\Phi_{n+1}(p_3, \cdots, p_{n+2}; p_1, p_2) \frac{1}{s_n}
$$
$$
\times \left\{ \sum_{I,K} \boldsymbol{J}_2^{(1)}(I, K) \right\} |M^0_{n+2}(\cdots, I, K, \cdots)|^2 J_n^{(n)}(\{p\}_n). \tag{3.5.74}
$$

The dipole functions $\boldsymbol{J}_2^{(1)}(I, K)$ in Eq. (3.5.74) depend on the parton types of the hard radiators I and K in the radiation from the NLO real contribution, and is defined to all orders in ϵ.

There are two types of dipole functions - flavour preserving and flavour changing - depending on whether or not the identity of the hard radiators change. In practice, this occurs for IF or II antennas when an initial state quark is changed to a gluon, or vice versa.

For identity preserving dipole functions, there are colour leading (N) $\boldsymbol{J}_2^{(1)}$ functions and the colour sub-leading functions $\hat{\boldsymbol{J}}_2^{(1)}$ that depends on the number of quark flavours N_F. In the final-final hard radiator case, the dipole functions only contains the integrated X_3^0 antenna functions and are summarised in Table 3.7 for different parton types [116].

In the initial-final and initial-initial hard radiator case, the dipole functions can contain both the integrated antenna functions and a mass factorization contribution. However, for some parton types, some of the N_F contributions of the integrated antenna functions and/or the tree-level splitting functions vanish. In particular, the $\hat{\boldsymbol{J}}_2^{(1)}(q, \bar{q})$ functions have no contributions from either the integrated antenna functions or the mass factorization splitting functions and are zero.

For identity changing dipole functions, one must insert the d-dimensional spin averaging factor associated with the mass factorization splitting function when a

Final-Final Integrated Antennae		
Matrix element, M_{n+3}^0	Integrated dipole, $\boldsymbol{J}_2^{(1)}$ and $\hat{\boldsymbol{J}}_2^{(1)}$	Reduced matrix element, M_{n+2}^0
Flavour Preserving		
$(\cdots; i_q, j_g, k_{\bar{q}}; \cdots)$	$J_{2,QQ}^{1,FF}(s_{IK}) = \mathcal{A}_3^0(s_{IK})$	$(\cdots; I_q, K_{\bar{q}}; \cdots)$
$(\cdots; i_q, j_g, k_{\bar{q}}; \cdots)$	$\hat{J}_{2,QQ}^{1,FF}(s_{IK}) = 0$	$(\cdots; I_q, K_{\bar{q}}; \cdots)$
$(\cdots; i_q, j_g, k_g, \cdots)$	$J_{2,QG}^{1,FF}(s_{IK}) = \frac{1}{2}\mathcal{D}_3^0(s_{IK})$	$(\cdots; I_q, K_g, \cdots)$
$(\cdots; i_q, j_{\bar{q}'}; k_{q'}, \cdots)$	$\hat{J}_{2,QG}^{1,FF}(s_{IK}) = \frac{1}{2}\mathcal{E}_3^0(s_{IK})$	$(\cdots; I_q, K_g, \cdots)$
$(\cdots, i_g, j_g, k_g, \cdots)$	$J_{2,GG}^{1,FF}(s_{IK}) = \frac{1}{3}\mathcal{F}_3^0(s_{IK})$	$(\cdots, I_g, K_g, \cdots)$
$(\cdots, i_g, j_{\bar{q}'}; k_{q'}, \cdots)$	$\hat{J}_{2,GG}^{1,FF}(s_{IK}) = \mathcal{G}_3^0(s_{IK})$	$(\cdots, I_g, K_g, \cdots)$

Table 3.7: The correspondence between the real radiation matrix elements, M_{n+3}^0 and the integrated NLO dipoles $\boldsymbol{J}_2^{(1)}$ and reduced matrix elements, M_{n+2}^0 for various particle assignments and colour structures for the final-final configuration.

initial state quark (gluon) changes to a gluon (quark) after the initial state radiation. Explicitly one has,

$$S_{g \to q} = \frac{S_g}{S_q} = 1 - \epsilon, \tag{3.5.75}$$

$$S_{q \to g} = \frac{S_q}{S_g} = \frac{1}{1 - \epsilon}. \tag{3.5.76}$$

For various partons types, complete summaries of the dipole functions with initial-final and initial-initial hard radiators are given in Table 3.8 and Table 3.9.

The explicit pole cancellation is straightforward using the $\boldsymbol{J}_2^{(1)}(I, K)$ functions. Comparing the definition of $\boldsymbol{J}_2^{(1)}(I, K)$ functions with equation (3.4.65), one can check that the identity preserving dipole functions satisfy,

$$\boldsymbol{J}_2^{(1)}(I, K) \xrightarrow{pole} 2\mathbf{I}_{IK}^{(1)}(\epsilon, s_{IK}), \tag{3.5.77}$$

while the identity changing dipole functions have no pole structure,

$$\boldsymbol{J}_{2,a \to b}^{(1)}(I, K) \xrightarrow{pole} 0. \tag{3.5.78}$$

From equation (3.5.73) and (3.5.74) while inserting (3.4.66) and (3.5.79) one can find that

$$\mathrm{d}\hat{\sigma}_{NLO}^V \xrightarrow{pole} \mathrm{d}\hat{\sigma}_{NLO}^T. \tag{3.5.79}$$

Initial-Final Integrated Antennae		
Matrix element, M_{n+3}^0	Integrated dipole, $\boldsymbol{J}_2^{(1)}$ and $\hat{\boldsymbol{J}}_2^{(1)}$	Reduced matrix element, M_{n+2}^0
Flavour Preserving		
$(\cdots; \hat{1}_q, i_g, j_{\bar{q}}; \cdots)$	$J_{2,QQ}^{1,IF}(s_{\bar{1}J}) = \mathcal{A}_{3,q}^0(s_{\bar{1}J}) - \Gamma_{qq}^{(1)}(x_1)\delta_2$	$(\cdots; \hat{\bar{1}}_q, J_{\bar{q}}; \cdots)$
$(\cdots; \hat{1}_q, i_g, j_{\bar{q}}; \cdots)$	$\hat{J}_{2,QQ}^{1,IF}(s_{\bar{1}J}) = 0$	$(\cdots; \hat{\bar{1}}_q, J_{\bar{q}}; \cdots)$
$(\cdots; \hat{1}_q, i_g, j_g, \cdots)$	$J_{2,QG}^{1,IF}(s_{\bar{1}J}) = \frac{1}{2}\mathcal{D}_{3,q}^0(s_{\bar{1}J}) - \Gamma_{qq}^{(1)}(x_1)\delta_2$	$(\cdots; \hat{\bar{1}}_q, J_g, \cdots)$
$(\cdots; \hat{1}_q, i_{\bar{q}'}; j_{q'}, \cdots)$	$\hat{J}_{2,QG}^{1,IF}(s_{\bar{1}J}) = \frac{1}{2}\mathcal{E}_{3,q}^0(s_{\bar{1}J})$	$(\cdots; \hat{\bar{1}}_q, J_g, \cdots)$
$(\cdots; i_q, j_g, \hat{1}_g, \cdots)$	$J_{2,GQ}^{1,IF}(s_{\bar{1}J}) = \mathcal{D}_{3,g\to g}^0(s_{\bar{1}J}) - \frac{1}{2}\Gamma_{gg}^{(1)}(x_1)\delta_2$	$(\cdots; J_q, \hat{\bar{1}}_g, \cdots)$
$(\cdots; i_q, j_g, \hat{1}_g, \cdots)$	$\hat{J}_{2,GQ}^{1,IF}(s_{\bar{1}J}) = \qquad\qquad - \frac{1}{2}\hat{\Gamma}_{gg}^{(1)}(x_1)\delta_2$	$(\cdots; J_q, \hat{\bar{1}}_g, \cdots)$
$(\cdots, \hat{1}_g, i_g, j_g, \cdots)$	$J_{2,GG}^{1,IF}(s_{\bar{1}J}) = \frac{1}{2}\mathcal{F}_{3,g}^0(s_{\bar{1}J}) - \frac{1}{2}\Gamma_{gg}^{(1)}(x_1)\delta_2$	$(\cdots, \hat{\bar{1}}_g, J_g, \cdots)$
$(\cdots, \hat{1}_g, i_{\bar{q}'}; j_{q'}, \cdots)$	$\hat{J}_{2,GG}^{1,IF}(s_{\bar{1}J}) = \frac{1}{2}\mathcal{G}_{3,g}^0(s_{\bar{1}J}) - \frac{1}{2}\hat{\Gamma}_{gg}^{(1)}(x_1)\delta_2$	$(\cdots, \hat{\bar{1}}_g, J_g, \cdots)$
Flavour Changing		
$(\cdots; i_q, \hat{1}_g, j_{\bar{q}}; \cdots)$	$J_{2,QQ,g\to q}^{1,IF}(s_{\bar{1}J}) = -\frac{1}{2}\mathcal{A}_{3,g\to q}^0(s_{\bar{1}J})$ $\qquad\qquad - S_{g\to q}\Gamma_{qg}^{(1)}(x_1)\delta_2$	$(\cdots; \hat{\bar{1}}_{\bar{q}}, J_{\bar{q}}; \cdots)$
$(\cdots; i_q, \hat{1}_g, j_g, \cdots)$	$J_{2,QG,g\to q}^{1,IF}(s_{\bar{1}J}) = -\mathcal{D}_{3,g\to q}^0(s_{\bar{1}J})$ $\qquad\qquad - S_{g\to q}\Gamma_{qg}^{(1)}(x_1)\delta_2$	$(\cdots; \hat{\bar{1}}_q, J_g, \cdots)$
$(\cdots; i_{q'}, \hat{1}_{q'}; j_q, \cdots)$	$J_{2,GQ,q'\to g}^{1,IF}(s_{\bar{1}J}) = -\mathcal{E}_{3,q'\to g}^0(s_{\bar{1}J})$ $\qquad\qquad - S_{q\to g}\Gamma_{gq}^{(1)}(x_1)\delta_2$	$(\cdots; J_q, \hat{\bar{1}}_g, \cdots)$
$(\cdots, i_g; \hat{1}_{q'}, j_{q'}, \cdots)$	$J_{2,GG,q'\to g}^{1,IF}(s_{\bar{1}J}) = -\mathcal{G}_{3,q'\to g}^0(s_{\bar{1}J})$ $\qquad\qquad - S_{q\to g}\Gamma_{gq}^{(1)}(x_1)\delta_2$	$(\cdots, J_g, \hat{\bar{1}}_g, \cdots)$

Table 3.8: The correspondence between the real radiation matrix elements, M_{n+3}^0 and the integrated NLO dipoles $\boldsymbol{J}_2^{(1)}$ and reduced matrix elements, M_{n+2}^0 for various particle assignments and colour structures for the initial-final configuration. For brevity $\delta(1 - x_i) = \delta_i$ for $i = 1, 2$.

Initial-Initial Integrated Antennae		
Matrix element, M_{n+3}^0	Integrated dipole, $\boldsymbol{J}_2^{(1)}$ and $\hat{\boldsymbol{J}}_2^{(1)}$	Reduced matrix element, M_{n+2}^0
Flavour Preserving		
$(\cdots;\hat{1}_q,i_g,\hat{2}_{\bar{q}},\cdots)$	$J_{2,QQ}^{1,II}(s_{\bar{1}\bar{2}}) = \mathcal{A}_{3,qq}^0(s_{\bar{1}\bar{2}})$ $\qquad - \Gamma_{qq}^{(1)}(x_1)\delta_2 - \Gamma_{qq}^{(1)}(x_2)\delta_1$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_{\bar{q}};\cdots)$
$(\cdots;\hat{1}_q,i_g,\hat{2}_{\bar{q}},\cdots)$	$\hat{J}_{2,QQ}^{1,II}(s_{\bar{1}\bar{2}}) = 0$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_{\bar{q}};\cdots)$
$(\cdots;\hat{1}_q,i_g,\hat{2}_g,\cdots)$	$J_{2,QG}^{1,II}(s_{\bar{1}\bar{2}}) = \mathcal{D}_{3,qg}^0(s_{\bar{1}\bar{2}})$ $\qquad - \Gamma_{qq}^{(1)}(x_1)\delta_2 - \frac{1}{2}\Gamma_{gg}^{(1)}(x_2)\delta_1$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_g,\cdots)$
$(\cdots;\hat{1}_q,i_g,\hat{2}_g,\cdots)$	$\hat{J}_{2,QG}^{1,II}(s_{\bar{1}\bar{2}}) = -\frac{1}{2}\hat{\Gamma}_{gg}^{(1)}(x_2)\delta_1$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_g,\cdots)$
$(\cdots,\hat{1}_g,i_g,\hat{2}_g,\cdots)$	$J_{2,GG}^{1,II}(s_{\bar{1}\bar{2}}) = \mathcal{F}_{3,gg}^0(s_{\bar{1}\bar{2}})$ $\qquad - \frac{1}{2}\Gamma_{gg}^{(1)}(x_1)\delta_2 - \frac{1}{2}\Gamma_{gg}^{(1)}(x_2)\delta_1$	$(\cdots,\hat{\bar{1}}_g,\hat{\bar{2}}_g,\cdots)$
$(\cdots,\hat{1}_g,i_g,\hat{2}_g,\cdots)$	$\hat{J}_{2,GG}^{1,II}(s_{\bar{1}\bar{2}}) = -\frac{1}{2}\hat{\Gamma}_{gg}^{(1)}(x_1)\delta_2 - \frac{1}{2}\hat{\Gamma}_{gg}^{(1)}(x_2)\delta_1$	$(\cdots,\hat{\bar{1}}_g,\hat{\bar{2}}_g,\cdots)$
Flavour Changing		
$(\cdots;\hat{1}_q,\hat{2}_g,i_{\bar{q}};\cdots)$	$J_{2,QQ,qg\to qq}^{1,II}(s_{\bar{1}\bar{2}}) = -\mathcal{A}_{3,qg\to qq}^0(s_{\bar{1}\bar{2}})$ $\qquad - S_{g\to q}\Gamma_{qg}^{(1)}(x_2)\delta_1$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_{\bar{q}};\cdots)$
$(\cdots;i_q,\hat{1}_g,\hat{2}_g,\cdots)$	$J_{2,QG,gg\to qg}^{1,II}(s_{\bar{1}\bar{2}}) = -\mathcal{D}_{3,gg\to qg}^0(s_{\bar{1}\bar{2}})$ $\qquad - S_{g\to q}\Gamma_{qg}^{(1)}(x_1)\delta_2$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_g,\cdots)$
$(\cdots;\hat{1}_q,\hat{2}_{\bar{q}'};i_{q'},\cdots)$	$J_{2,QG,qq'\to qg}^{1,II}(s_{\bar{1}\bar{2}}) = -\mathcal{E}_{3,qq'\to qg}^0(s_{\bar{1}\bar{2}})$ $\qquad - S_{q\to g}\Gamma_{gq}^{(1)}(x_2)\delta_1$	$(\cdots;\hat{\bar{1}}_q,\hat{\bar{2}}_g,\cdots)$
$(\cdots,\hat{1}_g,\hat{2}_{\bar{q}'};i_{q'},\cdots)$	$J_{2,GG,gq'\to gg}^{1,II}(s_{\bar{1}\bar{2}}) = -\mathcal{G}_{3,gq\to gg}^0(s_{\bar{1}\bar{2}})$ $\qquad - S_{q\to g}\Gamma_{gq}^{(1)}(x_2)\delta_1$	$(\cdots,\hat{\bar{1}}_g,\hat{\bar{2}}_g,\cdots)$

Table 3.9: The correspondence between the real radiation matrix elements, M_{n+3}^0 and the integrated NLO dipoles $\boldsymbol{J}_2^{(1)}$ and reduced matrix elements, M_{n+2}^0 for various particle assignments and colour structures for the initial-initial configuration. For brevity $\delta(1-x_1) = \delta_1$, $\delta(1-x_2) = \delta_2$.

Chapter 4

NNLO Corrections to QCD Scattering Processes

The NNLO calculation of total cross section in general contains three contributions: double real (RR), real-virtual (RV) and double virtual (VV). Each level has a different final state parton multiplicity and produces an infrared divergent behaviour. To regulate the implicit infrared divergences present in each contribution we use subtraction terms to produce a well defined function that is finite over the full integration region so that,

$$
\begin{aligned}
\hat{\sigma}_{NNLO} \;=\; & \int_{\mathrm{d}\Phi_{n+2}} \left(\mathrm{d}\hat{\sigma}_{NNLO}^{RR} - \mathrm{d}\hat{\sigma}_{NNLO}^{S}\right) + \int_{\mathrm{d}\Phi_{n+2}} \mathrm{d}\hat{\sigma}_{NNLO}^{S} \\
& + \int_{\mathrm{d}\Phi_{n+1}} \left(\mathrm{d}\hat{\sigma}_{NNLO}^{RV} - \mathrm{d}\hat{\sigma}_{NNLO}^{VS}\right) + \int_{\mathrm{d}\Phi_{n+1}} \mathrm{d}\hat{\sigma}_{NNLO}^{VS} + \int_{\mathrm{d}\Phi_{n+1}} \mathrm{d}\hat{\sigma}_{NNLO}^{MF,1} \\
& + \int_{\mathrm{d}\Phi_{n}} \mathrm{d}\hat{\sigma}_{NNLO}^{VV} + \int_{\mathrm{d}\Phi_{n}} \mathrm{d}\hat{\sigma}_{NNLO}^{MF,2}.
\end{aligned} \tag{4.0.1}
$$

Here, $\mathrm{d}\hat{\sigma}_{NNLO}^{S}$ term is the double real subtraction term that mimics $\mathrm{d}\hat{\sigma}_{NNLO}^{RR}$ in all singular limits. Likewise, the $\mathrm{d}\hat{\sigma}_{NNLO}^{VS}$ term is the subtraction term that removes the implicit singular divergences from $\mathrm{d}\hat{\sigma}_{NNLO}^{RV}$. To make sure one doesn't introduce any unphysical contributions, the subtracted terms are added back in the integrated form as $\int_{\mathrm{d}\Phi_{n+2}} \mathrm{d}\hat{\sigma}_{NNLO}^{S}$ and $\int_{\mathrm{d}\Phi_{n+1}} \mathrm{d}\hat{\sigma}_{NNLO}^{VS}$.

It is convenient to rearrange the integrated subtraction terms such that all the explicit divergences in ϵ cancel within the same phase space integral:

$$
\hat{\sigma}_{NNLO} \;=\; \int_{\mathrm{d}\Phi_{n+2}} \left[\mathrm{d}\hat{\sigma}_{NNLO}^{RR} - \mathrm{d}\hat{\sigma}_{NNLO}^{S}\right]
$$

$$+ \int_{\mathrm{d}\Phi_{n+1}} \left[\mathrm{d}\hat{\sigma}^{RV}_{NNLO} - \mathrm{d}\hat{\sigma}^{T}_{NNLO} \right]$$

$$+ \int_{\mathrm{d}\Phi_n} \left[\mathrm{d}\hat{\sigma}^{VV}_{NNLO} - \mathrm{d}\hat{\sigma}^{U}_{NNLO} \right], \qquad (4.0.2)$$

where the combination in each of the square brackets is free from both explicit and implicit divergences, i.e., is (a) finite and (b) well behaved in the unresolved regions of phase space. More precisely,

$$\mathrm{d}\hat{\sigma}^{T}_{NNLO} \quad = \quad \mathrm{d}\hat{\sigma}^{VS}_{NNLO} - \int_1 \mathrm{d}\hat{\sigma}^{S,1}_{NNLO} - \mathrm{d}\hat{\sigma}^{MF,1}_{NNLO}, \qquad (4.0.3)$$

$$\mathrm{d}\hat{\sigma}^{U}_{NNLO} \quad = \quad - \int_1 \mathrm{d}\hat{\sigma}^{VS}_{NNLO} - \int_2 \mathrm{d}\hat{\sigma}^{S,2}_{NNLO} - \mathrm{d}\hat{\sigma}^{MF,2}_{NNLO}. \qquad (4.0.4)$$

Here we have decomposed $\mathrm{d}\hat{\sigma}^{S}_{NNLO}$ into two parts, one part that is integrated over single unresolved phase space region and another part that is integrated over the double unresolved phase space region.

$$\int_{\mathrm{d}\Phi_{n+2}} \mathrm{d}\hat{\sigma}^{S}_{NNLO} = \int_{\mathrm{d}\Phi_{n+1}} \int_1 \mathrm{d}\hat{\sigma}^{S,1}_{NNLO} + \int_{\mathrm{d}\Phi_n} \int_2 \mathrm{d}\hat{\sigma}^{S,2}_{NNLO}. \qquad (4.0.5)$$

This means that the integrated double real subtraction term $(\int \mathrm{d}\hat{\sigma}^{S}_{NNLO})$ contributes to both real-virtual $(\mathrm{d}\hat{\sigma}^{T}_{NNLO})$ and double virtual $(\mathrm{d}\hat{\sigma}^{U}_{NNLO})$ subtraction terms.

In this chapter, the NNLO contribution from the scattering matrix elements will be introduced in the specific context of $pp \to H+$ jet. We will then discuss the IR singular behaviour of the relevant matrix elements at NNLO before reviewing the NNLO antenna subtraction method [7, 98, 109, 113, 116, 117].

4.1   Colour ordered amplitudes and matrix elements at NNLO

4.1.1   Colour ordered amplitudes and matrix elements at two-loops

At NNLO, the scattering matrix elements contributes to the cross section at three levels: double real, real-virtual and double virtual. The colour ordered tree and one-loop matrix elements needed for the double real and real-virtual contributions have

been introduced in section 3.1 while the new ingredient for NNLO is the two-loop matrix element. The two-loop amplitudes can also be arranged according to the colour connections of the partons [117–119]. The colour leading contribution to the squared matrix elements for the n-gluon plus Higgs boson two-loop process has the generic structure:

$$|M_n^2(g_1, \cdots, g_n)|^2 \propto 2\Re\{\mathcal{M}_H^0(g_1, \cdots, g_n)\mathcal{M}_H^{2\dagger}(g_1, \cdots, g_n)\}$$
$$+ \mathcal{M}_H^1(g_1, \cdots, g_n)\mathcal{M}_H^{1\dagger}(g_1, \cdots, g_n). \qquad (4.1.6)$$

4.1.2   Example for Higgs boson plus one jet at NNLO

Focusing on the $pp \to H+$ jet process at NNLO, we need the scattering amplitudes for

   (a) Higgs boson plus five partons at tree level,

   (b) Higgs boson plus four partons at one-loop level,

   (c) Higgs boson plus three partons at two-loop level.

Following the convention introduced in section 3.1 and using colour ordered amplitudes, we have the following tree amplitudes:

$$M_{ggggH}^0 = g^3\frac{\mathcal{C}}{2} \sum_{P(j,k,l,m)} Tr(T^{a_i}T^{a_j}T^{a_k}T^{a_l}T^{a_m})\mathcal{M}_H^0(i,j,k,l,m), \qquad (4.1.7)$$

$$M_{qggg\bar{q}H}^0 = g^3\frac{\mathcal{C}}{2} \sum_{P(i,j,k)} (T^{a_i}T^{a_j}T^{a_k})_{q\bar{q}}\mathcal{M}_H^0(q,i,j,k,\bar{q}), \qquad (4.1.8)$$

$$M_{q\bar{q}Q\bar{Q}gH}^0 = g^3\frac{\mathcal{C}}{2} \left(T_{q\bar{Q}}^g\delta_{Q\bar{q}}\mathcal{M}_H^0(q,g,\bar{Q},Q,\bar{q}) - \frac{1}{N}T_{q\bar{q}}^g\delta_{Q\bar{Q}}\mathcal{M}_H^0(q,g,\bar{q},Q,\bar{Q}) \right)$$
$$+ g^3\frac{\mathcal{C}}{2} \left(T_{Q\bar{q}}^g\delta_{q\bar{Q}}\mathcal{M}_H^0(q,\bar{Q},Q,g,\bar{q}) - \frac{1}{N}T_{Q\bar{Q}}^g\delta_{q\bar{q}}\mathcal{M}_H^0(q,\bar{q},Q,g,\bar{Q}) \right). \qquad (4.1.9)$$

For the one-loop amplitudes we have [120–123]:

$$M_{ggggH}^1 = g^3\frac{\mathcal{C}}{2}\left\{ \sum_{P(j,k,l)} Tr(T^{a_i}T^{a_j}T^{a_k}T^{a_l})\left[\mathcal{M}_{4;1H}^1(i,j,k,l) + \frac{N_F}{N}\widehat{\mathcal{M}}_{4;1H}^1(i,j,k,l)\right] \right.$$
$$+ Tr(T^{a_i}T^{a_j})Tr(T^{a_k}T^{a_l})\mathcal{M}_{4;3H}^1(i,j,k,l)$$
$$+ Tr(T^{a_i}T^{a_k})Tr(T^{a_j}T^{a_l})\mathcal{M}_{4;3H}^1(i,k,j,l)$$

$$+ Tr(T^{a_i}T^{a_l})Tr(T^{a_j}T^{a_k})\mathcal{M}^1_{4;3H}(i,l,j,k)\Big\}, \qquad (4.1.10)$$

$$M^1_{qgg\bar{q}H} = g^3 \frac{C}{2}\Big\{ \sum_{P(i,j)} (T^{a_i}T^{a_j})_{q\bar{q}}\Big[\mathcal{M}^1_{4;1H}(q,i,j,\bar{q}) - \frac{1}{N^2}\widetilde{\mathcal{M}}^1_{4;1H}(q,i,j,\bar{q})$$

$$+ \frac{N_F}{N}\widehat{\mathcal{M}}^1_{4;1H}(q,i,j,\bar{q})\Big] + Tr(T^{a_i}T^{a_j})\delta_{q\bar{q}}\mathcal{M}^1_{4;3H}(q,i,j,\bar{q})\Big\}$$

$$(4.1.11)$$

$$M^1_{q\bar{q}Q\bar{Q}H} = g^3 \frac{C}{2}\delta_{q\bar{Q}}\delta_{\bar{q}Q}\Big[\mathcal{M}^1_{4;1H}(q,\bar{q},Q,\bar{Q}) - \frac{1}{N^2}\widetilde{\mathcal{M}}^1_{4;1H}(q,\bar{q},Q,\bar{Q})$$

$$+ \frac{N_F}{N}\widehat{\mathcal{M}}^1_{4;1H}(q,\bar{q},Q,\bar{Q})\Big]$$

$$+ g^3 \frac{C}{2}\delta_{q\bar{q}}\delta_{Q\bar{Q}}\Big[\mathcal{M}^1_{4;2H}(q,\bar{q},Q,\bar{Q}) + \frac{1}{N^2}\widetilde{\mathcal{M}}^1_{4;2H}(q,\bar{q},Q,\bar{Q})$$

$$+ \frac{N_F}{N}\widehat{\mathcal{M}}^1_{4;2H}(q,\bar{q},Q,\bar{Q})\Big]. \qquad (4.1.12)$$

For the two-loop amplitudes we have [119]:

$$M^2_{gggH} = g^3 \frac{C}{2} \sum_{P(j,k)} Tr(T^{a_i}T^{a_j}T^{a_k})\Big[N^2\mathcal{M}^2_{3;1H}(i,j,k) + \widetilde{\mathcal{M}}^2_{3;1H}(i,j,k) + \frac{1}{N^2}\widetilde{\widetilde{\mathcal{M}}}^2_{3;1H}(i,j,k)$$

$$+ NN_F\widehat{\mathcal{M}}^2_{3;1H}(i,j,k) + \frac{N_F}{N}\widetilde{\widehat{\mathcal{M}}}^2_{3;1H}(i,j,k) + N_F^2\widehat{\widehat{\mathcal{M}}}^2_{3;1H}(i,j,k)\Big],$$

$$(4.1.13)$$

$$M^2_{qg\bar{q}H} = g^3 \frac{C}{2} T^{a_i}_{q\bar{q}}\Big[N^2\mathcal{M}^2_{3;1H}(q,i,\bar{q}) + \widetilde{\mathcal{M}}^2_{3;1H}(q,i,\bar{q}) + \frac{1}{N^2}\widetilde{\widetilde{\mathcal{M}}}^2_{3;1H}(q,i,\bar{q})$$

$$+ NN_F\widehat{\mathcal{M}}^2_{3;1H}(q,i,\bar{q}) + \frac{N_F}{N}\widetilde{\widehat{\mathcal{M}}}^2_{3;1H}(q,i,\bar{q}) + N_F^2\widehat{\widehat{\mathcal{M}}}^2_{3;1H}(q,i,\bar{q})\Big].$$

$$(4.1.14)$$

The squared matrix elements, summed over helicities and colours, for Higgs boson plus five partons at tree level are:

$$|M^0_{ggggg H}|^2 = g^6 \frac{C^2}{2} N^3(N^2-1) \sum_{P(k,l,m)} \Big[A^0_{5gH}(i,j,k,l,m) + A^0_{5gH}(i,k,j,l,m)\Big],$$

$$(4.1.15)$$

$$|M^0_{qggg\bar{q}H}|^2 = g^6 \frac{C^2}{4} N^2(N^2-1)\Big\{ \sum_{(i,j,k)} \Big[B^0_{3gH}(q,i,j,k,\bar{q}) - \frac{1}{N^2}\widetilde{B}^0_{3gH}(q,\tilde{i},j,k,\bar{q})\Big]$$

$$+ \frac{(N^2+1)}{N^4}\Big[\widetilde{\widetilde{B}}^0_{3gH}(q,\tilde{i},\tilde{j},\tilde{k},\bar{q})\Big]\Big\}, \qquad (4.1.16)$$

$$|M^0_{q\bar{q}Q\bar{Q}gH}|^2 = g^6 \frac{C^2}{4}\Big\{ N(N^2-1)\Big[C^0_{1gH}(q,g,\bar{Q},Q,\bar{q}) + C^0_{1gH}(q,\bar{Q},Q,g,\bar{q})\Big]$$

$$
+ \frac{(N^2-1)}{N} \left[\widetilde{C}^0_{1gH}(q,g,\bar{q},Q,\bar{Q}) + \widetilde{C}^0_{1gH}(q,\bar{q},Q,g,\bar{Q}) - \widetilde{\widetilde{C}}^0_{1gH}(q,\bar{q},Q,\bar{Q},g) \right] \bigg\},
$$

(4.1.17)

$$
\begin{aligned}
|M^0_{q\bar{q}q\bar{q}gH}|^2 = g^6 \frac{\mathcal{C}^2}{4} \bigg\{ & N(N^2-1) \sum_{P(\bar{Q},\bar{q})} \left[C^0_{1gH}(q,g,\bar{Q},Q,\bar{q}) + C^0_{1gH}(q,\bar{Q},Q,g,\bar{q}) \right] \\
& + \frac{(N^2-1)}{N} \sum_{P(\bar{Q},\bar{q})} \left[\widetilde{C}^0_{1gH}(q,g,\bar{q},Q,\bar{Q}) + \widetilde{C}^0_{1gH}(q,\bar{q},Q,g,\bar{Q}) \right. \\
& \hspace{5em} \left. - \widetilde{\widetilde{C}}^0_{1gH}(q,\bar{q},Q,\bar{Q},g) \right] \\
& - (N^2-1) \left[D^0_{1gH}(q,q,\bar{q},\bar{q},g) - \widetilde{D}^0_{1gH}(q,q,\bar{q},\bar{q},g) \right] \\
& + \frac{(N^2-1)}{N^2} \widetilde{D}^0_{1gH}(q,q,\bar{q},\bar{q},g) \bigg\},
\end{aligned}
$$

(4.1.18)

where, for $X = A, B, C, \widetilde{C}$,

$$
X^0_H(i,j,k,l,m) = \mathcal{M}^0_H(i,j,k,l,m) \mathcal{M}^{0\,\dagger}_H(i,j,k,l,m), \tag{4.1.19}
$$

$$
\begin{aligned}
\widetilde{B}^0_{3gH}(q,\tilde{i},j,k,\bar{q}) = & \left[\mathcal{M}^0_H(q,i,j,k,\bar{q}) + \mathcal{M}^0_H(q,j,i,k,\bar{q}) + \mathcal{M}^0_H(q,j,k,i,\bar{q}) \right] \\
& \times \left[\mathcal{M}^0_H(q,i,j,k,\bar{q}) + \mathcal{M}^0_H(q,j,i,k,\bar{q}) + \mathcal{M}^0_H(q,j,k,i,\bar{q}) \right]^\dagger,
\end{aligned}
$$

(4.1.20)

$$
\widetilde{\widetilde{B}}^0_{3gH}(q,\tilde{i},\tilde{j},\tilde{k},\bar{q}) = \left[\sum_{P(i,j,k)} \mathcal{M}^0_H(q,i,j,k,\bar{q}) \right] \left[\sum_{P(i,j,k)} \mathcal{M}^0_H(q,i,j,k,\bar{q}) \right]^\dagger, \tag{4.1.21}
$$

$$
\begin{aligned}
\widetilde{\widetilde{C}}^0_{1gH}(i_q,k_{\bar{q}},l_Q,m_{\bar{Q}},j) = & 2\Re\{[\mathcal{M}^0_H(i_q,j,m_{\bar{Q}},l_Q,k_{\bar{q}}) + \mathcal{M}^0_H(i_q,m_{\bar{Q}},l_Q,j,k_{\bar{q}})] \\
& \times [\mathcal{M}^0_H(i_q,j,k_{\bar{q}},l_Q,m_{\bar{Q}}) + \mathcal{M}^0_H(i_q,k_{\bar{q}},l_Q,j,m_{\bar{Q}})]^\dagger\} \\
= & 2|\mathcal{M}^0_H(i_q,j,m_{\bar{Q}},l_Q,k_{\bar{q}}) + \mathcal{M}^0_H(i_q,m_{\bar{Q}},l_Q,j,k_{\bar{q}})|^2, \tag{4.1.22}
\end{aligned}
$$

$$
\begin{aligned}
D^0_{1gH}(i_q,j_q,k_{\bar{q}},l_{\bar{q}},m_g) = & -2\Re\{\mathcal{M}^0_H(i_q,m,l_{\bar{Q}},j_Q,k_{\bar{q}}) \mathcal{M}^{0\,\dagger}_H(i_q,m,l_{\bar{q}},j_Q,k_{\bar{Q}}) \\
& + \mathcal{M}^0_H(i_q,l_{\bar{Q}},j_Q,m,k_{\bar{q}}) \mathcal{M}^{0\,\dagger}_H(i_q,l_{\bar{q}},j_Q,m,k_{\bar{Q}}) \\
& + \mathcal{M}^0_H(i_q,m,k_{\bar{q}},j_Q,l_{\bar{Q}}) \mathcal{M}^{0\,\dagger}_H(i_q,m,k_{\bar{Q}},j_Q,l_{\bar{q}}) \\
& + \mathcal{M}^0_H(i_q,k_{\bar{q}},j_Q,m,l_{\bar{Q}}) \mathcal{M}^{0\,\dagger}_H(i_q,k_{\bar{Q}},j_Q,m,l_{\bar{q}})\}_{\lambda_q=\lambda_Q}
\end{aligned}
$$

(4.1.23)

$$
\widetilde{D}^0_{1gH}(i_q,j_q,k_{\bar{q}},l_{\bar{q}},m_g) = -2\Re\{[\mathcal{M}^0_H(i_q,m,l_{\bar{Q}},j_Q,k_{\bar{q}}) + \mathcal{M}^0_H(i_q,l_{\bar{Q}},j_Q,m,k_{\bar{q}})]
$$

$$\times [\mathcal{M}_H^0(i_q, m, l_{\bar{q}}, j_Q, k_{\bar{Q}}) + \mathcal{M}_H^0(i_q, l_{\bar{q}}, j_Q, m, k_{\bar{Q}})]^\dagger\}_{\lambda_q = \lambda_Q},$$

(4.1.24)

where the identical spin sum $(\lambda_q = \lambda_Q)$ follows the discussion in section 3.1.1.

The squared matrix elements, summed over helicities and colours, for Higgs boson plus four parton at one-loop level are,

$$|M_{ggggH}^1|^2 = g^6 \frac{\mathcal{C}^2}{2} \left\{ N^3(N^2 - 1)\left[A_{4gH}^1(i,j,k,l) + A_{4gH}^1(i,j,l,k) + A_{4gH}^1(i,k,j,l) \right] \right.$$
$$\left. + N^2 N_F(N^2 - 1)\left[\widehat{A}_{4gH}^1(i,j,k,l) + \widehat{A}_{4gH}^1(i,j,l,k) + \widehat{A}_{4gH}^1(i,k,j,l) \right] \right\},$$

(4.1.25)

$$|M_{qgg\bar{q}H}^1|^2 = g^6 \frac{\mathcal{C}^2}{4} \left\{ N^2(N^2 - 1) \sum_{P(i,j)} \left[B_{2gH}^1(q,i,j,\bar{q}) - \frac{1}{N^2}\widetilde{B}_{2gH}^1(q,i,j,\bar{q}) \right. \right.$$
$$\left. + \frac{N_F}{N}\widehat{B}_{2gH}^1(q,i,j,\bar{q}) \right]$$
$$- (N^2 - 1)\left[\widetilde{\widetilde{B}}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) - \frac{1}{N^2}\widetilde{\widetilde{B}}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) + \frac{N_F}{N}\widehat{\widetilde{B}}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) \right]$$
$$\left. + (N^2 - 1)\left[\widehat{\widehat{B}}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) \right] \right\},$$

(4.1.26)

$$|M_{q\bar{q}Q\bar{Q}H}^1|^2 = g^6 \frac{\mathcal{C}^2}{4} N(N^2 - 1)\left[C_{0gH}^1(q,\bar{Q},Q,\bar{q}) - \frac{1}{N^2}\widetilde{C}_{0gH}^1(q,\bar{Q},Q,\bar{q}) \right.$$
$$\left. + \frac{N_F}{N}\widehat{C}_{0gH}^1(q,\bar{Q},Q,\bar{q}) \right],$$

(4.1.27)

$$|M_{q\bar{q}q\bar{q}H}^1|^2 = g^6 \frac{\mathcal{C}^2}{4} \left\{ \vphantom{\sum_{P(\bar{Q},\bar{q})}} \right.$$
$$+ N(N^2 - 1) \sum_{P(\bar{Q},\bar{q})} \left[C_{0gH}^1(q,\bar{Q},Q,\bar{q}) - \frac{1}{N^2}\widetilde{C}_{0gH}^1(q,\bar{Q},Q,\bar{q}) \right.$$
$$\left. + \frac{N_F}{N}\widehat{C}_{0gH}^1(q,\bar{Q},Q,\bar{q}) \right]$$
$$\left. - (N^2 - 1)\left[D_{0gH}^1(q,\bar{q},q,\bar{q}) + \frac{1}{N^2}\widetilde{D}_{0gH}^1(q,\bar{q},q,\bar{q}) - \frac{N_F}{N}\widehat{D}_{0gH}^1(q,\bar{q},q,\bar{q}) \right] \right\},$$

(4.1.28)

where for $X = A, B, C$,

$$X_H^1(i,j,k,l) = 2\Re\{\mathcal{M}_{4;1H}^1(i,j,k,l)\mathcal{M}_H^{0\,\dagger}(i,j,k,l)\}$$

(4.1.29)

$$\widetilde{X}_H^1(i,j,k,l) = 2\Re\{\widetilde{\mathcal{M}}_{4;1H}^1(i,j,k,l)\mathcal{M}_H^{0\,\dagger}(i,j,k,l)\}$$

(4.1.30)

$$\widehat{X}_H^1(i,j,k,l) = 2\Re\{\hat{\mathcal{M}}_{4;1H}^1(i,j,k,l)\mathcal{M}_H^{0\,\dagger}(i,j,k,l)\}, \tag{4.1.31}$$

and

$$\widetilde{\widetilde{B}}{}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) = 2\Re\{[\mathcal{M}_{4;1H}^1(q,i,j,\bar{q}) + \mathcal{M}_{4;1H}^1(q,j,i,\bar{q})]$$
$$\times [\mathcal{M}_H^0(q,i,j,\bar{q}) + \mathcal{M}_H^0(q,j,i,\bar{q})]^\dagger\}, \tag{4.1.32}$$

$$\widetilde{\widetilde{\widetilde{B}}}{}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) = 2\Re\{[\widetilde{\mathcal{M}}_{4;1H}^1(q,i,j,\bar{q}) + \widetilde{\mathcal{M}}_{4;1H}^1(q,j,i,\bar{q})]$$
$$\times [\mathcal{M}_H^0(q,i,j,\bar{q}) + \mathcal{M}_H^0(q,j,i,\bar{q})]^\dagger\}, \tag{4.1.33}$$

$$\widehat{\widetilde{\widetilde{B}}}{}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) = 2\Re\{[\widehat{\mathcal{M}}_{4;1H}^1(q,i,j,\bar{q}) + \widehat{\mathcal{M}}_{4;1H}^1(q,j,i,\bar{q})]$$
$$\times [\mathcal{M}_H^0(q,i,j,\bar{q}) + \mathcal{M}_H^0(q,j,i,\bar{q})]^\dagger\}, \tag{4.1.34}$$

$$\widehat{\widehat{B}}{}_{2gH}^1(q,\tilde{i},\tilde{j},\bar{q}) = 2\Re\{\mathcal{M}_{4;3H}^1(q,i,j,\bar{q})[\mathcal{M}_H^0(q,i,j,\bar{q}) + \mathcal{M}_H^0(q,j,i,\bar{q})]^\dagger\}, \tag{4.1.35}$$

$$D_{0gH}^1(i_q,j_{\bar{q}},k_q,l_{\bar{q}}) = 2\Re\{\mathcal{M}_{4;2H}^1(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\mathcal{M}_H^{0\,\dagger}(i_q,l_{\bar{q}},k_Q,j_{\bar{Q}})$$
$$+ \mathcal{M}_{4;2H}^1(i_q,l_{\bar{q}},k_Q,j_{\bar{Q}})\mathcal{M}_H^{0\,\dagger}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\}, \tag{4.1.36}$$

$$\widetilde{D}_{0gH}^1(i_q,j_{\bar{q}},k_q,l_{\bar{q}}) = 2\Re\{\widetilde{\mathcal{M}}_{4;2H}^1(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\mathcal{M}_H^{0\,\dagger}(i_q,l_{\bar{q}},k_Q,j_{\bar{Q}})$$
$$+ \widetilde{\mathcal{M}}_{4;2H}^1(i_q,l_{\bar{q}},k_Q,j_{\bar{Q}})\mathcal{M}_H^{0\,\dagger}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\}, \tag{4.1.37}$$

$$\widehat{D}_{0gH}^1(i_q,j_{\bar{q}},k_q,l_{\bar{q}}) = 2\Re\{\widehat{\mathcal{M}}_{4;2H}^1(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\mathcal{M}_H^{0\,\dagger}(i_q,l_{\bar{q}},k_Q,j_{\bar{Q}})$$
$$+ \widehat{\mathcal{M}}_{4;2H}^1(i_q,l_{\bar{q}},k_Q,j_{\bar{Q}})\mathcal{M}_H^{0\,\dagger}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}})\}. \tag{4.1.38}$$

The colour ordered amplitudes in $|M_{q\bar{q}Q\bar{Q}H}^1|^2$ and $|M_{q\bar{q}q\bar{q}H}^1|^2$ are related to the functions for Higgs boson plus four partons given in [121] by

$$\mathcal{M}_{4;1H}^1(i_q,l_{\bar{Q}},k_Q,j_{\bar{q}}) = \mathcal{M}_H^{1,lc}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}), \tag{4.1.39}$$

$$\widetilde{\mathcal{M}}_{4;1H}^1(i_q,l_{\bar{Q}},k_Q,j_{\bar{q}}) = 2\mathcal{M}_H^{1,lc}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}) + 2\mathcal{M}_H^{1,lc}(i_q,j_{\bar{q}},l_Q,k_{\bar{Q}})$$
$$+ \mathcal{M}_H^{1,slc}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}), \tag{4.1.40}$$

$$\widehat{\mathcal{M}}_{4;1H}^1(i_q,l_{\bar{Q}},k_Q,j_{\bar{q}}) = \mathcal{M}_H^{1,f}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}), \tag{4.1.41}$$

$$\mathcal{M}_{4;2H}^1(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}) = \mathcal{M}_H^{1,lc}(i_q,j_{\bar{q}},l_Q,k_{\bar{Q}}), \tag{4.1.42}$$

$$\widetilde{\mathcal{M}}_{4;2H}^1(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}) = \mathcal{M}_H^{1,lc}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}) + \mathcal{M}_H^{1,lc}(i_q,j_{\bar{q}},l_Q,k_{\bar{Q}})$$
$$+ \mathcal{M}_H^{1,slc}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}), \tag{4.1.43}$$

$$\widehat{\mathcal{M}}_{4;2H}^1(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}) = \mathcal{M}_H^{1,f}(i_q,j_{\bar{q}},k_Q,l_{\bar{Q}}). \tag{4.1.44}$$

For Higgs boson plus three partons at the two-loop level, the squared matrix elements, summed over helicities and colours are,

$$
|M_{gggH}^2|^2 = g^6 \frac{\mathcal{C}^2}{2} N^3 (N^2 - 1) \Bigg[A_{3gH}^2(i,j,k) + \frac{N_F}{N} \widehat{A}_{3gH}^2(i,j,l) + \frac{N_F^2}{N^2} \widehat{\widehat{A}}_{3gH}^2(i,j,l)
$$
$$
+ \frac{N_F}{N^3} \widetilde{\widehat{A}}_{3gH}^2(i,k,j) + \frac{1}{N^2} \widetilde{A}_{3gH}^2(i,k,j) + \frac{1}{N^4} \widetilde{\widetilde{A}}_{3gH}^2(i,k,j) \Bigg], \quad (4.1.45)
$$

$$
|M_{qg\bar{q}H}^2|^2 = g^6 \frac{\mathcal{C}^2}{4} N^2 (N^2 - 1) \Bigg[B_{1gH}^2(q,i,\bar{q}) + \frac{N_F}{N} \widehat{B}_{1gH}^2(q,j,\bar{q}) + \frac{N_F^2}{N^2} \widehat{\widehat{B}}_{1gH}^2(q,j,\bar{q})
$$
$$
+ \frac{N_F}{N^3} \widetilde{\widehat{B}}_{3gH}^2(q,k,\bar{q}) + \frac{1}{N^2} \widetilde{B}_{1gH}^2(q,k,\bar{q}) + \frac{1}{N^4} \widetilde{\widetilde{B}}_{1gH}^2(q,k,\bar{q}) \Bigg]. \quad (4.1.46)
$$

Explicit expressions for Eqs. (4.1.45) and (4.1.46) are given in [119].

4.2 IR behaviour of the double real contribution at NNLO

4.2.1 Classification

The double real contributions come from the tree level scattering matrix elements which have two additional radiation of partons compared to the Born process:

$$
d\hat{\sigma}_{NNLO}^{RR} = \mathcal{N}_{NNLO}^{RR} d\Phi_{n+2}(p_3, \cdots, p_{n+4}; p_1, p_2) \frac{1}{s_{n+2}} |M_{n+4}^0(\cdots, i, j, k, l, \cdots)|^2
$$
$$
\times J_{n+2}^{(n)}(\{p\}_{n+2}). \quad (4.2.47)
$$

The $\mathcal{N}_{NNLO}^{RR}$ factor is a normalisation factor related to strong coupling parameter α_s, momentum flux s and colour factor N. The jet function $J_n^{(n+2)}$ selects the regions of phase space where the $(n+2)-$ partons contribute to $n-$jet final states. This means that both single and double unresolved behaviour are allowed. Depending on the colour connection of the double unresolved partons, simple iteration of single unresolved limits is not enough for describing the divergent behaviour. The following configurations need to be considered separately:

(1) Zero unresolved partons but n jets observed.

(2) One unresolved parton and n jets observed.

(3) Two unresolved partons which are separated by at least two hard partons in the colour string (colour-unconnected).

(4) Two unresolved partons which are separated by only one resolved parton in the colour string (almost colour-unconnected).

(5) Two unresolved partons which are colour-connected in the colour string.

Configurations (1) and (2) are similar to those occuring in the NLO real contribution introduced in 3.2. The divergent behaviour in configurations (3) and (4) can be described by the iteration of single unresolved limits. In configuration (4) the phase space mapping is more involved as the common hard radiator between the two unresolved partons causes the overlapping of two $3 \to 2$ mappings. The order of the two $3 \to 2$ mappings could cause a mismatch in the single soft limit and the details will be discussed in section 4.3.3. Nevertheless, the divergent functions needed in configuration (3) and (4) are simple iterations of the functions in single unresolved limits.

In the colour-connected double unresolved limits (5), the unresolved partons may be either soft and/or collinear, and a new class of universal divergent functions contributes in the various types of limits.

4.2.2 Factorization of colour-connected double unresolved limits

Double soft unresolved partons

If partons a, i, j, b were colour connected (as part of a colour string) and parton i and j became simultaneously soft, the matrix element containing those partons can be described by double soft factorized functions times the reduced matrix elements with parton i and j pinched out:

$$|M^0(\cdots, a, i, j, b, \cdots)|^2 \xrightarrow{i,j \ soft} S_{aijb}|M^0(\cdots, a, b, \cdots)|^2. \qquad (4.2.48)$$

The partons i and j could be a pair of gluons or a quark pair produced by a gluon splitting. The double soft function S_{aijb} depends on the identities of the unresolved partons [109, 124, 125] but not on the identities of hard radiator partons a and b.

Triple collinear unresolved partons

When three colour-connected partons become simultaneously collinear, the matrix elements also factorise into a triple collinear splitting function multiplied by a reduced matrix element [109, 125–128],

$$|M^0(\cdots,i,j,k,\cdots)|^2 \xrightarrow{i//j//k} P_{ijk\to A}(z_1, z_2, z_3)|M^0(\cdots,A,\cdots)|^2, \qquad (4.2.49)$$

where A is the composite parton produced by the three triple collinear partons, $P_{ijk\to A}$ is the triple collinear function and z_i, z_j, z_k are the fractions of composite momentum p_A from parton i, j or k. The leading colour triple collinear limits come from the configurations $g//g//g$, $q//g//g$, $q//\bar{q}//g$, $q//\bar{q}//Q$ and $q//\bar{q}//q$ while the sub-leading colour triple collinear limits are produced by $q//\tilde{g}//\tilde{g}$ and $q//g//\bar{q}$ configurations. Here $\tilde{g}$ again represents a "photon"-like gluon and in the $q//\tilde{g}//\tilde{g}$ limit, both $\tilde{g}$ are colour connected to the quark. The $q//g//\bar{q}$ configuration occurs when the gluon emission is from within a quark pair rather than adjacent to a quark pair as in the colour-leading case, $q//\bar{q}//g$. Usually, we denotes the sub-leading triple collinear functions by $\widetilde{P}_{ijk\to A}(z_1, z_2, z_3)$.

Soft collinear unresolved partons

If partons a, i, j, k are colour connected and parton i becomes soft while simultaneously j is collinear with k, the matrix element can be described by a universal soft-collinear function $S_{a,ijk}$ multiplying the corresponding single collinear splitting function [109, 124, 125]:

$$|M^0(\cdots,a,i,j,k,\cdots)|^2 \xrightarrow{i\ soft,\ j//k} S_{a,ijk}\frac{1}{s_{jk}}P_{jk\to K}(z)|M^0(\cdots,a,K,\cdots)|^2, \qquad (4.2.50)$$

where z is the momentum fraction of p_j inside p_K and

$$S_{a,ijk} = \frac{(s_{aj}+s_{ak})}{s_{ai}s_{ij}}\left(z+\frac{s_{ij}+zs_{jk}}{s_{ijk}}\right). \qquad (4.2.51)$$

In the p_i soft limit, $S_{a,ijk} \to S_{aij}$. The limit when $p_j \to 0$ and $p_i//p_k$ is a special case of the triple collinear limit when $z_j \to 0$.

4.3   Antenna subtraction term for the double real level at NNLO

In the single or double unresolved phase space, the double real contribution of the NNLO cross section is divergent. Using the same idea as in NLO calculations, in the antenna subtraction framework, we define antenna functions together with the associated with phase space mapping that can compensate these implicit divergences. The subtraction terms are subsequently added back in integrated form to the real-virtual and double virtual subtraction terms.

4.3.1   Phase space mapping

One needs appropriate phase space mappings to prepare the momentum sets for the subtraction terms and reduced matrix elements for each of the different unresolved behaviour discussed in section 4.2.1. For single unresolved, colour-unconnected double unresolved and almost colour-unconnected double unresolved behaviour, the phase space mappings are simply (iterations of) the $3 \to 2$ mapping introduced in section 3.3.1.

However, for colour-connected double unresolved behaviour the two unresolved partons and their colour-adjacent hard-radiation partons are all involved in the double unresolved limits and this requires $4 \to 2$ mapping. Just as for the $3 \to 2$ mappings, the two hard radiators could be either in final-final (FF), initial-final (IF) or initial-initial (II) states. The momentum mapping should keep the hard radiators on-shell and preserve momentum conservation.

Final-Final Mapping

The final-final momentum mapping $\{i, j, k, l\} \to \{I, L\}$ [113, 129] is defined as

$$
\begin{aligned}
p_I^\mu &\equiv p_{\widetilde{(ijk)}}^\mu = x_1 p_i^\mu + x_2 p_j^\mu + x_3 p_k^\mu + x_4 p_l^\mu, \\
p_L^\mu &\equiv p_{\widetilde{(jkl)}}^\mu = (1 - x_1)p_i^\mu + (1 - x_2)p_j^\mu + (1 - x_3)p_k^\mu + (1 - x_4)p_l^\mu,
\end{aligned} \tag{4.3.52}
$$

where, in a similar manner to the $3 \to 2$ FF mapping, the on-shell conditions fix the values of x_1 and x_4, and leaving a choice for x_2 and x_3. We use,

$$
\begin{aligned}
x_1 = {} & \frac{1}{2(s_{ij} + s_{ik} + s_{il})}\Bigg[(1+\rho)s_{ijkl} - x_2(s_{jk} + 2s_{jl}) \\
& - x_3(s_{jk} + 2s_{kl}) + (x_2 - x_3)\left(\frac{s_{ij}s_{kl} - s_{ik}s_{jl}}{s_{il}}\right)\Bigg], \\
x_2 = {} & \frac{s_{jk} + s_{jl}}{s_{ij} + s_{jk} + s_{jl}}, \\
x_3 = {} & \frac{s_{kl}}{s_{ik} + s_{jk} + s_{kl}}, \\
x_4 = {} & \frac{1}{2(s_{il} + s_{jl} + s_{kl})}\Bigg[(1+\rho)s_{ijkl} - x_2(s_{ij} + 2s_{jl}) \\
& - x_3(s_{jk} + 2s_{ik}) - (x_2 - x_3)\left(\frac{s_{ij}s_{kl} - s_{ik}s_{jl}}{s_{il}}\right)\Bigg], \quad (4.3.53)
\end{aligned}
$$

where ρ is given by,

$$
\begin{aligned}
\rho = {} & \Bigg[1 + \frac{(x_2 - x_3)^2}{s_{il}^2 s_{ijkl}^2}\lambda(s_{ij}s_{kl}, s_{il}s_{jk}, s_{ik}s_{jl}) \\
& + \frac{1}{s_{il}s_{ijkl}}\Bigg(2(x_2(1-x_3) + x_3(1-x_2))(s_{ij}s_{kl} + s_{ik}s_{jl} - s_{jk}s_{il}) \\
& + 4x_2(1-x_2)s_{ij}s_{jl} + 4x_3(1-x_3)s_{ik}s_{kl}\Bigg)\Bigg]^{\frac{1}{2}}, \quad (4.3.54)
\end{aligned}
$$

where

$$
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz). \quad (4.3.55)
$$

For various double unresolved limits of p_j^μ and p_k^μ, p_I^μ and p_L^μ become the corresponding composite momentum. For example, for double soft limits when $p_j^\mu, p_k^\mu \to 0$, $p_I^\mu \to p_i^\mu$ and $p_L^\mu \to p_l^\mu$. Further details of the behaviors in the double unresolved limits can be found in [113].

The full phase space for 2 to $n-$final state partons for the double real contribution is

$$
\mathrm{d}\Phi_{n+2}(p_3, \ldots, p_{n+4}; p_1, p_2) = (2\pi)^d \delta\left(p_1 + p_2 - \sum_{l=3}^{n+4} p_l\right)\prod_{l=3}^{n+4}[\mathrm{d}p_l]. \quad (4.3.56)
$$

Using similar identities as in Eq. (3.3.43), the full phase space can be factorized into the product of an antenna phase space and the reduced phase space:

$$
\mathrm{d}\Phi_{n+2}(p_3, \ldots, p_{n+4}; p_1, p_2) = \mathrm{d}\Phi_n(p_3, \ldots, p_I, p_L, \ldots, p_{m+4}; p_1, p_2)\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}
$$

$$\times \quad \delta(1-x_1)\delta(1-x_2)\mathrm{d}\Phi^{FF}_{X_{ijkl}}(p_i,p_j,p_k,p_l;p_I,p_L).$$

$$(4.3.57)$$

Similarly, the four to two final-final antenna phase space $\mathrm{d}\Phi^{FF}_{X_{ijkl}}$ is proportional to the four-particle phase space [109],

$$\mathrm{d}\Phi^{FF}_{X_{ijkl}}(p_i,p_j,p_k,p_l;p_I,p_L) = \frac{1}{P_2} \cdot \mathrm{d}\Phi_{ijkl}(p_i,p_j,p_k,p_l;p_I,p_L). \qquad (4.3.58)$$

Initial-Final Mapping

In the initial-final momentum mapping, a technique very similar to that used for the $3 \to 2$ IF mapping can be used to derive a $4 \to 2$ mapping such that for $\{\hat{i},j,k,l\} \to \{\hat{I},L\} \equiv \{\hat{\bar{i}},L\}$ [108],

$$
\begin{aligned}
p^\mu_{\hat{I}} &\equiv \bar{p}^\mu_i = \hat{x}_i\, p^\mu_i, \\
p^\mu_L &\equiv p^\mu_{\widetilde{(jkl)}} = p^\mu_j + p^\mu_k + p^\mu_l - (1-\hat{x}_i)\,p^\mu_i.
\end{aligned}
\qquad (4.3.59)
$$

With on-shell condition $p^2_{\hat{I}} = p^2_L = 0$ the $\hat{x}_i$ parameter is fixed to be,

$$\hat{x}_i = \frac{s_{ij}+s_{ik}+s_{il}+s_{jk}+s_{jl}+s_{kl}}{s_{ij}+s_{ik}+s_{il}}. \qquad (4.3.60)$$

For various double unresolved limits of p^μ_j and p^μ_k, $p^\mu_{\hat{I}}$ and p^μ_L become the corresponding composite momentum. For example, in double soft limits $p^\mu_j,p^\mu_k \to 0$, $p^\mu_{\hat{I}} \to p^\mu_i$ and $p^\mu_L \to p^\mu_l$. Further details of the composite momentum in the double unresolved limits can be found in [108, 113].

The initial-final antenna phase space can be rewritten using identities similar to Eq. (3.3.50) but with $q = p_j + p_k + p_l - p_i$, as (for $i = 1$),

$$
\begin{aligned}
\mathrm{d}\Phi_{n+2}(p_3,\ldots,k_{n+4};p_1,p_2) &= \mathrm{d}\Phi_n(p_3,\ldots,p_L,\ldots,p_{n+4};x_1 p_1,p_2)\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2} \\
&\times \quad \delta(x_1-\hat{x}_1)\delta(1-x_2)\mathrm{d}\Phi^{IF}_{X_{\hat{1}jkl}}(p_j,p_k,p_l;p_1,q),
\end{aligned}
$$

$$(4.3.61)$$

where the four to two initial-final antenna phase space is defined as

$$\mathrm{d}\Phi^{IF}_{X_{\hat{1}jkl}}(p_j,p_k,p_l;p_1,q) \equiv \frac{Q^2}{2\pi}\mathrm{d}\Phi_3(p_j,p_k,p_l;p_1,q). \qquad (4.3.62)$$

Initial-Initial Mapping

Finally for the initial-initial momentum mapping $\{\hat{i}, j, k, \hat{l}\} \rightarrow \{\hat{I}, \hat{L}\} \equiv \{\hat{\bar{i}}, \hat{\bar{l}}\}$, a Lorentz boost similar to that employed in the $3 \rightarrow 2$ mapping is used but with extended values of q and $\tilde{q}$ [108],

$$
\begin{aligned}
p_{\hat{I}}^{\mu} \equiv \bar{p}_i^{\mu} &= \hat{x}_i\, p_i^{\mu}, \\
p_{\hat{L}}^{\mu} \equiv \bar{p}_l^{\mu} &= \hat{x}_l\, p_l^{\mu}, \\
\tilde{p}_m^{\mu} &= p_m^{\mu} - \frac{2p_m \cdot (q + \tilde{q})}{(q + \tilde{q})^2}\,(q^{\mu} + \tilde{q}^{\mu}) + \frac{2p_m \cdot q}{q^2}\,\tilde{q}^{\mu},
\end{aligned}
\tag{4.3.63}
$$

where $m \neq j, k$ and

$$
\begin{aligned}
q^{\mu} &= p_i^{\mu} + p_l^{\mu} - p_j^{\mu} - p_k^{\mu}, \qquad \tilde{q}^{\mu} = \bar{p}_i^{\mu} + \bar{p}_l^{\mu}, \\
\hat{x}_i &= \left(\frac{(s_{il} - s_{lj} - s_{lk})(s_{il} - s_{ij} - s_{ik} - s_{lj} - s_{lk} + s_{jk})}{s_{il}(s_{il} - s_{ij} - s_{ik})} \right)^{\frac{1}{2}}, \\
\hat{x}_l &= \left(\frac{(s_{il} - s_{ij} - s_{ik})(s_{il} - s_{ij} - s_{ik} - s_{lj} - s_{lk} + s_{jk})}{s_{il}(s_{il} - s_{lj} - s_{lk})} \right)^{\frac{1}{2}}.
\end{aligned}
\tag{4.3.64}
$$

In the double unresolved limits where $(p_j^{\mu} + p_k^{\mu})$ is along the beam line, the intermediate momentum q^{μ} become proportional to $\tilde{q}^{\mu}$. In this case, for each $l \neq j, k$, $\tilde{p}_l^{\mu} \rightarrow p_l^{\mu}$ and $p_{\hat{I}}^{\mu}, p_{\hat{K}}^{\mu}$ become the corresponding composite momentum. Further details of the composite momentum in the double unresolved limits can be found in [108, 113].

For the initial-initial antenna phase space (with $i = 1$ and $l = 2$), using similar identities to Eq. (3.3.57), we find that,

$$
\begin{aligned}
d\Phi_{n+2}(p_3, \ldots, p_{n+4}; p_1, p_2) &= d\Phi_n(\tilde{p}_3, \ldots, \tilde{p}_{n+4}; x_1 p_1, x_2 p_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2} \\
&\quad \times \delta(x_1 - \hat{x}_1)\,\delta(x_2 - \hat{x}_2)\, d\Phi_{X_{\hat{1}jk\hat{2}}}^{II},
\end{aligned}
\tag{4.3.65}
$$

where the four to two initial-initial antenna phase space is defined as

$$
d\Phi_{X_{\hat{1}jk\hat{2}}}^{II} \equiv x_1 x_2 [dp_j][dp_k].
\tag{4.3.66}
$$

4.3.2 Antenna functions for double real level at NNLO

The antenna functions used in the double real contribution must remove the various single and double unresolved singular behaviour from the matrix elements. The

$X_3^0(i, j, k)$ antenna functions introduced in section 3.3.2 are useful to remove the single unresolved limits while iterated structures of the type $X_3^0(i, j, k)X_3^0(l, m, n)$ are capable of removing the almost colour-unconnected and colour-unconnected double unresolved limits.

To remove the colour-connected double unresolved limits, new antenna functions with four colour-connected partons for various types of hard radiators are introduced in [110–112]. As with the construction of X_3^0, the X_4^0 functions are derived from matrix elements with four partons normalised by the underlying two-parton process:

$$X_4^0(i, j, k, l) = \mathcal{S}_{ijkl/IL} \frac{M_4^0(i, j, k, l)}{M_2^0(I, L)}. \tag{4.3.67}$$

Partons i and l are hard radiators in the j and k double unresolved limits, and I, L are the mapped momentum through $4 \to 2$ mappings introduced in section 4.3.1. Depending on the parton types of i, j, k, l, $X_4^0(i, j, k, l)$ in general contains several divergent behaviour such as double soft (i.e. $j, k \to 0$), soft collinear (i.e. $j \to 0$, $k//l$), double collinear (i.e. $i//j$, $k//l$) or triple collinear (i.e. $i//j//k$) limits. Besides double unresolved limits, X_4^0 is also divergent in single unresolved limits which naturally reside in the $M_4^0(i, j, k, l)$ matrix elements. From the definition of X_3^0 in Eq. (3.3.60), we see that in single unresolved limits $X_4^0(i, j, k, l)$ is simply equivalent to a single unresolved divergent function multiplying the reduced X_3^0 antenna function, e.g., for single soft j,

$$X_4^0(i, j, k, l) \xrightarrow{j \ soft} \frac{2s_{ik}}{s_{ij}s_{jk}} X_3^0(i, k, l). \tag{4.3.68}$$

In some special cases, the $M_4^0(i, j, k, l)$ matrix elements that form the core of X_4^0 have colour connections between the two hard radiators i and l. This feature generally means that the X_4^0 antenna contains limits when either i or l are soft. This feature also occurred in some X_3^0 antennae at NLO where it was possible to define sub-antennae that did not have these limits, see section 3.3.2. Several studies have discussed the splitting of full X_4^0 antenna functions into well behaved sub antenna functions [98, 113]. The X_4^0 antenna functions for the various parton types are summarised in Table 4.1.

Similarly, it is also possible to have an alternative colour connection such that partons j, k are both colour connected to i, l. In general, we name this type

of four parton antenna functions as $\widetilde{X}_4^0(i,j,k,l)$ [116]. The $\widetilde{X}_4^0$ antennae do not contain colour connected double soft limits but are vital to mimic triple collinear (i.e. $j//i//k$), soft collinear (i.e. $k \to 0$, $j//l$) and double collinear (i.e. $i//k$, $j//l$) limits.

4.3.3 Antenna subtraction terms $d\hat{\sigma}_{NNLO}^S$

$d\hat{\sigma}_{NNLO}^S$ is designed to mimic all the divergent behaviour of the matrix elements in the double real contributions of NNLO (in Eq. (4.2.47)). From the classification of the five configurations of double real contributions in section 4.2.1, the $d\hat{\sigma}_{NNLO}^S$ can be decomposed according to the four unresolved configurations [98, 109, 113, 130]:

$$d\hat{\sigma}_{NNLO}^S = d\hat{\sigma}_{NNLO}^{S,a} + d\hat{\sigma}_{NNLO}^{S,b} + d\hat{\sigma}_{NNLO}^{S,c} + d\hat{\sigma}_{NNLO}^{S,d}, \qquad (4.3.69)$$

where

(a) $d\hat{\sigma}_{NNLO}^{S,a}$ is to remove single unresolved limits as in configuration (1);

(b) $d\hat{\sigma}_{NNLO}^{S,b}$ is to remove colour-connected double unresolved limits as in configuration (5);

(c) $d\hat{\sigma}_{NNLO}^{S,c}$ is to remove almost colour-unconnected double unresolved limits as in configuration (4);

(d) $d\hat{\sigma}_{NNLO}^{S,d}$ is to remove colour-unconnected double unresolved limits as in configuration (3).

In the following sections, each contribution to $d\hat{\sigma}_{NNLO}^S$ will be discussed first in the general case of multiple number of partons. Secondly some special considerations in the case of five partons relevant for $pp \to H+$ jet will be introduced.

Single unresolved subtraction term, $d\hat{\sigma}_{NNLO}^{S,a}$

To remove the single unresolved divergence from the double real matrix elements, the antenna subtraction structure for the real contribution in the NLO calculation would be sufficient,

$$d\hat{\sigma}_{NNLO}^{S,a} = \mathcal{N}_{NNLO}^{RR} d\Phi_{n+2}(p_3, \cdots, p_{n+4}; p_1, p_2) \frac{1}{s_{n+2}}$$

maple	fortran/form	latex	comment
A40	FullA40	A_4^0	Eqs. (5.27) and (5.29) of [109].
At40	FullAt40	$\tilde{A}_4^0$	Eqs. (5.28) and (5.30) of [109].
At40a	At40a	$\tilde{A}_4^{0,a}$	Eqs. (3.16) and (3.17) of [98] and Eq. (5.30) of [109].
B40	FullB40	B_4^0	Eqs. (5.37) and (5.38) of [109].
C40	FullC40	C_4^0	Eqs. (5.42) and (5.43) of [109].
D40	FullD40	D_4^0	Eqs. (6.43) and (6.44) of [109].
D40a	D40a	$D_4^{0,a}$	Eq. (3.23) of [98].
D40c	D40c	$D_4^{0,c}$	Eq. (3.23) of [98].
E40	FullE40	E_4^0	Eq. (6.48) of [109].
E40a	E40a	$E_4^{0,a}$	Eq. (3.18) of [98].
E40b	E40b	$E_4^{0,b}$	Eq. (3.19) of [98].
Et40	FullEt40	$\tilde{E}_4^0$	Eqs. (6.49) and (6.50) of [109].
F40	FullF40	F_4^0	Eqs. (7.43) and (7.44) of [109].
F40a	F40a	$F_4^{0,a}$	Eq. (4.41) of [113].
F40b	F40b	$F_4^{0,b}$	Eq. (4.41) of [113]. Now labelled $F_4^{0,b}(1,2,4,3)$.
G40	FullG40	G_4^0	Eqs. (7.48) and (7.50) of [109].
G40a	G40a	$G_4^{0,a}$	unpublished work by Pires, J.
G40b	G40b	$G_4^{0,b}$	unpublished work by Pires, J.
G40c	G40c	$G_4^{0,c}$	unpublished work by Pires, J.
Gt40	FullGt40	$\tilde{G}_4^0$	Eqs. (7.49) and (7.51) of [109].
H40	FullH40	H_4^0	Eq. (7.58) of [109].

Table 4.1: X_4^0 antenna functions. The nomenclature used in the .map input files as well as in the numerical fortran codes are indicated.

$$\times \left\{ \sum_{j} X_3^0(i,j,k) |M_{n+3}^0(\cdots, \widetilde{(ij)}, \widetilde{(jk)}, \cdots)|^2 \right\} J_{n+1}^{(n)}(\{p\}_{n+1}). \quad (4.3.70)$$

As there is one more parton in the final states in the NNLO calculation, the jet function requires only n-jets are identified from the $(n+1)$ final state partons. Therefore, the jet function actually allows extra unresolved contributions in the reduced matrix elements, which must be removed using other subtraction terms. Depending on the colour connection between the first unresolved parton j and a putative second unresolved parton (which could be any of the final state partons in the reduced matrix element), one needs the $\mathrm{d}\hat{\sigma}^{S,c}$ and $\mathrm{d}\hat{\sigma}^{S,d}$ subtraction terms to compensate the spurious divergent limits.

Colour-connected double unresolved subtraction term, $\mathrm{d}\hat{\sigma}_{NNLO}^{S,b}$

The X_4^0 ($\widetilde{X}_4^0$) antenna functions along with $4 \to 2$ mappings are designed to remove the double unresolved colour-connected limits, and are used to construct the first part of the $\mathrm{d}\hat{\sigma}_{NNLO}^{S,b}$ term,

$$\mathrm{d}\hat{\sigma}_{NNLO}^{S,b_1} = \mathcal{N}_{NNLO}^{RR} \mathrm{d}\Phi_{n+2}(p_3, \cdots, p_{n+4}; p_1, p_2) \frac{1}{s_{n+2}}$$
$$\times \left\{ \sum_{j,k} X_4^0(i,j,k,l) |M_{n+2}^0(\cdots, \widetilde{(ijk)}, \widetilde{(jkl)}, \cdots)|^2 \right\} J_n^{(n)}(\{p\}_n). \quad (4.3.71)$$

However, as discussed in section 4.3.2, the X_4^0 and $\widetilde{X}_4^0$ functions also contain spurious single unresolved limits. Accordingly the second part of $\mathrm{d}\hat{\sigma}_{NNLO}^{S,b}$ must remove these spurious divergences. For the hard radiators i and l, the corresponding subtraction terms that remove the single unresolved limits for j or k present in $X_4^0(i,j,k,l)$ are,

$$\mathrm{d}\hat{\sigma}_{NNLO}^{S,b_2} = -\mathcal{N}_{NNLO}^{RR} \mathrm{d}\Phi_{n+2}(p_3, \cdots, p_{n+4}; p_1, p_2) \frac{1}{s_{n+2}}$$
$$\times \sum_{j,k} \left\{ X_3^0(i,j,k) X_3^0(\widetilde{(ij)}, \widetilde{(jk)}, l) |M_{n+2}^0(\cdots, \widetilde{((ij)(jk))}, \widetilde{((jk)l)}, \cdots)|^2 J_n^{(n)}(\{p\}_n) \right.$$
$$\left. + X_3^0(j,k,l) X_3^0(i, \widetilde{(jk)}, \widetilde{(kl)}) |M_{n+2}^0(\cdots, \widetilde{(i(jk))}, \widetilde{((jk)(kl))}, \cdots)|^2 J_n^{(n)}(\{p\}_n) \right\}.$$

$$(4.3.72)$$

For the hard radiators j and l, the corresponding subtraction terms that remove the single unresolved limits for i or k present in $\widetilde{X}_4^0(j,i,k,l)$ are,

$$\mathrm{d}\hat{\sigma}_{NNLO}^{S,b_2} = -\mathcal{N}_{NNLO}^{RR} \mathrm{d}\Phi_{n+2}(p_3, \cdots, p_{n+4}; p_1, p_2) \frac{1}{s_{n+2}}$$

$$\times \sum_{j,k} \left\{ X_3^0(l,i,j) X_3^0(\widetilde{(ij)},k,\widetilde{(il)}) |M_{n+2}^0(\cdots,\widetilde{((ij)k)},\widetilde{(k(il))},\cdots)|^2 J_n^{(n)}(\{p\}_n) \right.$$

$$\left. + X_3^0(j,k,l) X_3^0(\widetilde{(jk)},i,\widetilde{(kl)}) |M_{n+2}^0(\cdots,\widetilde{((jk)i)},\widetilde{(i(kl))},\cdots)|^2 J_n^{(n)}(\{p\}_n) \right\}.$$

$$(4.3.73)$$

Note that there is potentially a secondary unresolved limit inside the second X_3^0 antenna function in Eq. (4.3.73). Just as for $d\hat{\sigma}_{NNLO}^{S,a}$, one needs subtraction terms in $d\hat{\sigma}^{S,c}$ that will compensate these spurious limits. Altogether, the full $d\hat{\sigma}_{NNLO}^{S,b}$ subtraction term is given by,

$$d\hat{\sigma}_{NNLO}^{S,b} = d\hat{\sigma}_{NNLO}^{S,b_1} + d\hat{\sigma}_{NNLO}^{S,b_2}. \qquad (4.3.74)$$

Almost colour-unconnected double unresolved subtraction term, $d\hat{\sigma}_{NNLO}^{S,c}$

To have almost colour-unconnected double unresolved limits, the scattering processes needs to involve at least five partons. Besides the actual double unresolved divergences from the matrix elements, both $d\hat{\sigma}_{NNLO}^{S,a}$ and $d\hat{\sigma}_{NNLO}^{S,b_2}$ would contribute to the same double unresolved limits.

An iterating pattern has been found for part of $d\hat{\sigma}_{NNLO}^{S,c}$ [116]. For every $\widetilde{X}_4^0(j,i,k,l)$ antenna present in $d\hat{\sigma}_{NNLO}^{S,b_1}$ which has hard radiator j and l, there is a block of nine terms in $d\hat{\sigma}_{NNLO}^{S,c}$ that correctly compensates the over subtraction of double-unresolved limits in the almost colour-unconnected configuration. For a general structure of colour connection without the unresolved parton i and k, $(\cdots,a,j,l,b,\cdots)$, the three possible unresolved regions can be diagrammatically shown as region I, II and III in Figure 4.1. The primary unresolved parton i could stay in any of the three regions while the secondary emission of parton k stays in region I. The block of nine terms are:

$$d\hat{\sigma}_{NNLO}^{S,c} = \mathcal{N}_{NNLO}^{RR} d\Phi_{n+2}(p_3,\cdots,p_{n+4};p_1,p_2) \frac{1}{s_{n+2}} \sum_{i,k} \left\{ \vphantom{\frac{1}{2}} \right.$$

$$+ \frac{1}{2} X_3^0(j,i,l) X_3^0(\widetilde{(ji)},k,\widetilde{(il)}) |M_{n+2}^0(\ldots,a,\widetilde{((ji)k)},\widetilde{(k(il))},b,\ldots)|^2$$

$$- \frac{1}{2} X_3^0(a,i,j) X_3^0(\widetilde{(ji)},k,l) |M_{n+2}^0(\ldots,\widetilde{(ai)},\widetilde{((ji)k)},\widetilde{(kl)},b,\ldots)|^2$$

$$- \frac{1}{2} X_3^0(l,i,b) X_3^0(j,k,\widetilde{(li)}) |M_{n+2}^0(\ldots,a,\widetilde{(jk)},\widetilde{(k(li))},\widetilde{(ib)},\ldots)|^2$$
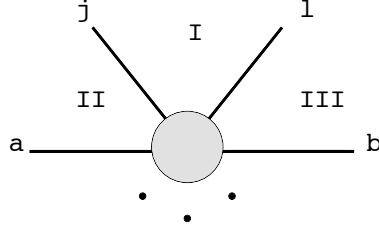
Figure 4.1: The three regions associated with the radiation of almost colour-unconnected partons.

$$-\frac{1}{2}\left[\left(S_{\widetilde{(ij)},i,\widetilde{(il)}}-S_{\widetilde{((ij)k)}i\widetilde{(k(il))}}\right)-\left(S_{ai\widetilde{(ij)}}-S_{ai\widetilde{((ij)k)}}\right)-\left(S_{bi\widetilde{(il)}}-S_{bi\widetilde{((il)k)}}\right)\right]$$
$$X_3^0(\widetilde{(ij)},k,\widetilde{(il)})\,|M_{n+2}^0(\ldots,a,\widetilde{((ij)k)},\widetilde{(k(il))},b,\ldots)|^2\bigg\}\,J_n^{(n)}(\{p\}_n). \quad (4.3.75)$$

The factor of half in Eq. (4.3.75) is from the symmetry of summing all colour orderings. The six large angle soft terms (terms with eikonal factors) are designed to remove the over subtracted divergences when parton i becomes soft. As the choice of the primary emission of parton i in Eq. (4.3.75) may not cancel the corresponding secondary emission of parton i in $d\hat{\sigma}_{NNLO}^{S,a}$ and $d\hat{\sigma}_{NNLO}^{S,b_2}$, the six large angle soft terms are grouped into three pairs with single and double mapped hard radiators to compensate the mismatched ordering of emission. Note that in the case of five parton scattering, the hard radiators a and b in Fig. 4.1 overlap.

Additional $d\hat{\sigma}_{NNLO}^{S,c}$ terms are needed on a case by case basis. Nevertheless, after integration over the phase space of the first antenna function, $\int_1 d\hat{\sigma}_{NNLO}^{S,c}$ should have all its divergences canceled at the real-virtual level. This means that the rather simple structure of subtraction terms present at the real-virtual level gives a hint to the terms that must appear in $d\hat{\sigma}_{NNLO}^{S,c}$. A second iterating pattern has been found from such a comparison with real-virtual terms. In general, for double real subtraction terms that share the same primary X_3^0 function that contains identity changing limits (idc), one can regroup those terms into blocks which share exactly the same primary antenna function and mapping. Inside each block like this, an NLO structure for real subtraction terms can be found in the secondary X_3^0 function and the reduced matrix element. This structure guarantees that the secondary

unresolved limits all cancel within each block,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{S,idc} = \mathcal{N}_{NNLO}^{RR}\mathrm{d}\Phi_{n+2}(p_3,\cdots,p_{n+4};p_1,p_2)\frac{1}{s_{n+2}}\Bigg\{ & \\
+X_3^{0,idc}(i,j,k)|M_{n+3}^0(\ldots,\widetilde{(ij)},\widetilde{(jk)}\ldots)|^2 & \\
-X_3^{0,idc}(i,j,k)\sum_{M\in\{p\}_{n+1}}X_3^0(L,M,N)|M_{n+2}^0(\ldots,\widetilde{(LM)},\widetilde{(MN)},\ldots)|^2\Bigg\}\, & J_n^{(n)}(\{p\}_n).
\end{aligned}
$$

$$(4.3.76)$$

The $X_3^{0,idc}M_{n+3}^0$ structure in the first line of Eq. (4.3.76) is produced by $\mathrm{d}\hat{\sigma}_{NNLO}^{S,a}$ while knowledge of the single unresolved structure of M_{n+3}^0 helps to predict the $X_3^{0,idc}X_3^0M_{n+2}^0$ structures in the second line that are naturally part of either $\mathrm{d}\hat{\sigma}_{NNLO}^{S,c}$ or $\mathrm{d}\hat{\sigma}_{NNLO}^{S,d}$. More details will be discussed in section 4.5.3.

Colour-unconnected double unresolved subtraction term, $\mathrm{d}\hat{\sigma}_{NNLO}^{S,d}$

In order to have colour-unconnected double unresolved limits, the scattering processes must have at least six partons. Besides the actual double unresolved divergences from the matrix elements, $\mathrm{d}\hat{\sigma}_{NNLO}^{S,a}$ also contributes to the same limits. In each single unresolved limit there would be one $X_3^0M_{n+3}^0J_{n+1}^{(n)}$ term, the colour-unconnected double unresolved limits are double counted. This means that the $\mathrm{d}\hat{\sigma}_{NNLO}^{S,d}$ terms would have an overall minus sign to cancel the over-subtraction. In general, $\mathrm{d}\hat{\sigma}_{NNLO}^{S,d}$ uses the iteration of two X_3^0 antenna functions and is summed over all the possible colour-unconnected partons,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{S,d} = -\mathcal{N}_{NNLO}^{RR}\mathrm{d}\Phi_{n+2}(p_3,\cdots,p_{n+4};p_1,p_2)\frac{1}{s_{n+2}}\Bigg\{ & \\
\sum_{j,m}X_3^0(i,j,k)X_3^0(l,m,n)|M_{n+2}^0(\ldots,I,K,\cdots,L,N,\ldots)|^2\Bigg\}\, & J_n^{(n)}(\{p\}_n).
\end{aligned}
$$

$$(4.3.77)$$

As the sub-phase space factorization for both antenna functions have no common momentum, the phase space integration of the antenna functions can be performed simultaneously. After integration, the $\int_2 \mathrm{d}\hat{\sigma}_{NNLO}^{S,d}$ terms are added back to $\mathrm{d}\hat{\sigma}_{NNLO}^U$.

4.4 Divergent behaviour of real-virtual contributions at NNLO

The real-virtual contribution of NNLO calculations contains two parts: the one-loop matrix elements and the real-virtual mass factorization counter term. Each of the two contributions has explicit divergences in terms of Laurent expansion in ϵ and implicit divergences in single unresolved phase space.

4.4.1 IR behaviour of real-virtual matrix elements

The real-virtual contribution has one additional parton compared to the Born level process,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{RV} =& \mathcal{N}_{NNLO}^{RV}\mathrm{d}\Phi_{n+1}(p_3,\cdots,p_{n+3};p_1,p_2)\frac{1}{s_{n+1}}|M_{n+3}^1(\cdots,i,j,k,\cdots)|^2 \\
& \times J_{n+1}^{(n)}(\{p\}_{n+1}).
\end{aligned}
\tag{4.4.78}
$$

The $\mathcal{N}_{NNLO}^{RV}$ factor is a normalisation factor related to strong coupling parameter α_s, momentum flux s and colour factor N. The relation between $\mathcal{N}_{NNLO}^{RV}$ and $\mathcal{N}_{NNLO}^{RR}$ is

$$
\frac{\mathcal{N}_{NNLO}^{RR}}{\mathcal{N}_{NNLO}^{RV}} = \frac{1}{C(\epsilon)}
\tag{4.4.79}
$$

The one-loop matrix elements in Eq. (4.4.78) contain the explicit divergences discussed in section 3.4.1 over the full phase space. In addition, the jet function $J_{n+1}^{(n)}$ allows contributions from single unresolved phase space region where the one-loop matrix elements would become implicitly divergent. The implicitly divergent behaviour of one-loop matrix elements follows a similar pattern as at tree level [131]. In the single soft limit the matrix element behaves as

$$
|M^1(\cdots,i,j_g,k,\cdots)|^2 \xrightarrow{j\ soft} S_{ijk}|M^1(\cdots,i,k,\cdots)|^2 + S_{ijk}^1|M^0(\cdots,i,k,\cdots)|^2;
\tag{4.4.80}
$$

and for single collinear limit one has

$$
|M^1(\cdots,i,j,\cdots)|^2 \xrightarrow{i//j} \frac{1}{s_{ij}}P_{ij\to K}(z)|M^1(\cdots,K,\cdots)|^2 + \frac{1}{s_{ij}}P_{ij\to K}^1(z)|M^0(\cdots,K,\cdots)|^2.
\tag{4.4.81}
$$

More precisely, S^1_{ijk} is the general one-loop soft function studied in [132] which also includes explicit poles up to $1/\epsilon^2$, and $P^1_{ij \to K}(z)$ is the general one-loop splitting function as in [133]. As at tree-level, these universal divergent functions only involve the unresolved parton and its colour adjacent neighbours.

4.4.2 Mass factorization terms at real-virtual level

The mass factorization terms contributing to the real-virtual level of the NNLO calculation have a very similar structure to the mass factorization terms in the real contribution of NLO discussed in section 3.4.2. Following the discussion of section 1.4, the expression for NNLO real-virtual level mass factorization terms in Eq. (1.4.51) has the general structure,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{MF,1}_{ij,NNLO}(\xi_1 H_1, \xi_2 H_2) = & -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon) \times \\
& \left[\delta(1-x_2)\mathbf{\Gamma}^1_{ki}(x_1)\left(\mathrm{d}\hat{\sigma}^R_{kj,NLO} - \mathrm{d}\hat{\sigma}^S_{kj,NLO} \right) \right. \\
& \left. + \delta(1-x_1)\mathbf{\Gamma}^1_{lj}(x_2)\left(\mathrm{d}\hat{\sigma}^R_{il,NLO} - \mathrm{d}\hat{\sigma}^S_{il,NLO} \right) \right](x_1\xi_1 H_1, x_2\xi_2 H_2).
\end{aligned}
$$

$$(4.4.82)$$

Note that here we made the replacement $\mathrm{d}\hat{\sigma}^R_{NLO} \to \mathrm{d}\hat{\sigma}^R_{NLO} - \mathrm{d}\hat{\sigma}^S_{NLO}$ in $\mathrm{d}\hat{\sigma}^{MF,1}_{ij,NNLO}$ and add $\int_1 \mathrm{d}\hat{\sigma}^S_{NLO}$ accordingly in $\mathrm{d}\hat{\sigma}^{MF,2}_{ij,NNLO}$.

The phase space integral in Eq. (4.4.82) is over real-virtual phase space at NNLO which has the same parton multiplicity as the real contribution at NLO. For $k = i$ or $l = j$, the terms in $\mathrm{d}\hat{\sigma}^{MF,1}_{ij,NNLO}$ are initial state identity preserving contributions (idp) $\mathrm{d}\hat{\sigma}^{MF,1idp}_{ij,NNLO}$ while for $k \neq i$ or $l \neq j$, the terms in $\mathrm{d}\hat{\sigma}^{MF,1}_{ij,NNLO}$ are initial state identity changing contributions (idc) $\mathrm{d}\hat{\sigma}^{MF,1idc}_{ij,NNLO}$. For the convenience of further use, one usually defines $\mathrm{d}\hat{\sigma}^{MF,1R}_{ij,NNLO}$ and $\mathrm{d}\hat{\sigma}^{MF,1S}_{ij,NNLO}$ to be,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{MF,1R}_{ij,NNLO}(\xi_1 H_1, \xi_2 H_2) = & -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon) \times \\
& \left[\delta(1-x_2)\mathbf{\Gamma}^1_{ki}(x_1)\mathrm{d}\hat{\sigma}^R_{kj,NLO} + \delta(1-x_1)\mathbf{\Gamma}^1_{lj}(x_2)\mathrm{d}\hat{\sigma}^R_{il,NLO} \right](x_1\xi_1 H_1, x_2\xi_2 H_2)
\end{aligned}
$$

$$(4.4.83)$$

$$
\mathrm{d}\hat{\sigma}^{MF,1S}_{ij,NNLO}(\xi_1 H_1, \xi_2 H_2) = -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon) \times
$$

$$\left[\delta(1-x_2)\mathbf{\Gamma}^1_{ki}(x_1)\mathrm{d}\hat{\sigma}^S_{kj,NLO} + \delta(1-x_1)\mathbf{\Gamma}^1_{lj}(x_2)\mathrm{d}\hat{\sigma}^S_{il,NLO}\right](x_1\xi_1 H_1, x_2\xi_2 H_2).$$

$$(4.4.84)$$

4.5 Antenna subtraction term for real-virtual level at NNLO

The real-virtual contributions at NNLO contain both explicit and unresolved divergences that the $\mathrm{d}\hat{\sigma}^{VS}_{NNLO}$ subtraction term must remove. The integrated form of $\mathrm{d}\hat{\sigma}^{VS}_{NNLO}$ will be added to $\mathrm{d}\hat{\sigma}^U_{NNLO}$.

4.5.1 Antenna functions for real-virtual level at NNLO

New loop-level antenna functions allowing single unresolved parton are needed to remove the single unresolved divergence behaviour of the one-loop matrix elements discussed in section 4.4.1. Using the same idea used to construct X_3^0 and X_4^0, one would expect to use normalised one-loop three parton matrix elements to define X_3^1 antenna functions. From the single unresolved behaviour of one-loop matrix elements in Eqs. (4.4.80) and (4.4.81), one can generalise the decomposition as

$$S_{ijk,IK}\,|\mathcal{M}^1_3(i,j,k)|^2 = X_3^1(i,j,k)|\mathcal{M}^0_2(I,K)|^2 + X_3^0(i,j,k)\,|\mathcal{M}^1_2(I,K)|^2, \quad (4.5.85)$$

such that

$$X_3^1(i,j,k) = S_{ijk/IK}\,\frac{|\mathcal{M}^1_3(i,j,k)|^2}{|\mathcal{M}^0_2(I,K)|^2} - X_3^0(i,j,k)\,\frac{|\mathcal{M}^1_2(I,K)|^2}{|\mathcal{M}^0_2(I,K)|^2}. \quad (4.5.86)$$

As in the X_3^0 case, parton i and k are hard radiators while in the soft j limit,

$$X_3^1(i,j,k) \xrightarrow{j\ soft} S^1_{ijk}, \quad (4.5.87)$$

and in the $i//j$ single collinear limit,

$$X_3^1(i,j,k) \xrightarrow{i//j} \frac{1}{s_{ij}}P^1_{ij\to I}(z). \quad (4.5.88)$$

Depending on the type of one-loop matrix elements with three partons, one finds X_3^1, $\tilde{X}_3^1$ and $\hat{X}_3^1$ functions for the leading colour, sub-leading colour and N_F contributions. Note that in general $X_3^1(i,j,k)$ is renormalized at the scale of total

maple	fortran/form	latex	comment
A31FF	FullA31FF	A_3^1	Eqs. (5.12) and (5.13) of [109].
Ah31FF	FullAh31FF	$\hat{A}_3^1$	Eqs. (5.16) and (5.17) of [109].
At31FF	FullAt31FF	$\tilde{A}_3^1$	Eqs. (5.14) and (5.15) of [109].
D31FF	FullD31FF	D_3^1	Eqs. (6.18) and (6.19) of [109].
			$D_3^1 = d_3^1(i_1, i_2, i_3) + d_3^1(i_1, i_3, i_2)$.
d31FF	d31FF	d_3^1	Only has i_2 soft limit.
Dh31FF	FullDh31FF	$\hat{D}_3^1$	Eqs. (6.20) and (6.21) of [109].
			$\hat{D}_3^1 = \hat{d}_3^1(i_1, i_2, i_3) + \hat{d}_3^1(i_1, i_3, i_2)$.
dh31FF	dh31FF	$\hat{d}_3^1$	Only has i_2 soft limit.
F31FF	FullF31FF	F_3^1	Eqs. (7.18) and (7.19) of [109].
			$F_3^1 = f_3^1(i_1, i_2, i_3) + f_3^1(i_2, i_3, i_1) + f_3^1(i_3, i_1, i_2)$.
f31FF	f31FF	f_3^1	Only has i_2 soft limit.
Fh31FF	FullFh31FF	$\hat{F}_3^1$	Eqs. (7.20) and (7.21) of [109].
			$\hat{F}_3^1 = \hat{f}_3^1(i_1, i_2, i_3) + \hat{f}_3^1(i_2, i_3, i_1) + \hat{f}_3^1(i_3, i_1, i_2)$.
fh31FF	fh31FF	$\hat{f}_3^1$	Only has i_2 soft limit.
E31FF	FullE31FF	E_3^1	Eqs. (6.28) and (6.29) of [109].
Eh31FF	FullEh31FF	$\hat{E}_3^1$	Eqs. (6.32) and (6.33) of [109].
Et31FF	FullEt31FF	$\tilde{E}_3^1$	Eqs. (6.30) and (6.31) of [109].
G31FF	FullG31FF	G_3^1	Eqs. (7.28) and (7.29) of [109].
Gh31FF	FullGh31FF	$\hat{G}_3^1$	Eqs. (7.32) and (7.33) of [109].
Gt31FF	FullGt31FF	$\tilde{G}_3^1$	Eqs. (7.30) and (7.31) of [109].

Table 4.2: X_3^1 antenna functions for final-final state

maple	fortran/form	latex	comment
gA31IF	FullgA31IF	$A_{3,g}^1$	Crossing of A_3^1. Mixed flavour changing.
gAh31IF	FullgAh31IF	$\hat{A}_{3,g}^1$	Crossing of $\hat{A}_3^1$. Mixed flavour changing.
gAt31IF	FullgAt31IF	$\tilde{A}_{3,g}^1$	Crossing of $\tilde{A}_3^1$. Mixed flavour changing.
gD31IF	FullgD31IF	$D_{3,g}^1$	Mixed flavour changing.
			$D_{3,g}^1 = d_{3,g\to q}^1(i_1, \hat{i}_2, i_3) + d_{3,g}^1(i_1, i_3, \hat{i}_2)$.
gd31IF	gd31IF	$d_{3,g}^1$	i_3 initial state i_2 soft.
gd31IFgtoq	gd31IFgtoq	$d_{3,g\to q}^1$	Only contains $i_1\|\|i_2$ collinear limit.
			Flavour changing $g \to q$.
gDh31IF	FullgDh31IF	$\hat{D}_{3,g}^1$	Mixed flavour changing.
			$\hat{D}_{3,g}^1 = \hat{d}_{3,g\to q}^1(i_1, \hat{i}_2, i_3) + \hat{d}_{3,g}^1(i_1, i_3, \hat{i}_2)$.
gdh31IF	gdh31IF	$\hat{d}_{3,g}^1$	i_3 initial state i_2 soft.
gdh31IFgtoq	gdh31IFgtoq	$\hat{d}_{3,g\to q}^1$	Only contains $i_1\|\|i_2$ collinear limit.
			Flavour changing $g \to q$.
gF31IF	FullgF31IF	$F_{3,g}^1$	Crossing of F_3^1.
			$F_{3,g}^1 = f_{3,g}^1(\hat{i}_1, i_2, i_3) + f_{3,g}^1(\hat{i}_1, i_3, i_2)$.
gf31IF	gf31IF	$f_{3,g}^1$	Only contains $i_1\|\|i_2$ collinear limit.
gFh31IF	FullgFh31IF	$\hat{F}_{3,g}^1$	Crossing of $\hat{F}_3^1$.
			$\hat{F}_{3,g}^1 = \hat{f}_{3,g}^1(\hat{i}_1, i_2, i_3) + \hat{f}_{3,g}^1(\hat{i}_1, i_3, i_2)$.
gfh31IF	gfh31IF	$\hat{f}_{3,g}^1$	Only contains $i_1\|\|i_2$ collinear limit.
gG31IF	FullgG31IF	$G_{3,g}^1$	Crossing of G_3^1.
gGh31IF	FullgGh31IF	$\hat{G}_{3,g}^1$	Crossing of $\hat{G}_3^1$.
gGt31IF	FullgGt31IF	$\tilde{G}_{3,g}^1$	Crossing of $\tilde{G}_3^1$.

Table 4.3: Gluon initiated initial-final state X_3^1 antenna functions

maple	fortran/form	latex	comment
qA31IF	FullqA31IF	$A^1_{3,q}$	Crossing of A^1_3.
qAh31IF	FullqAh31IF	$\hat{A}^1_{3,q}$	Crossing of $\hat{A}^1_3$.
qAt31IF	FullqAt31IF	$\tilde{A}^1_{3,q}$	Crossing of $\tilde{A}^1_3$.
qD31IF	FullqD31IF	$D^1_{3,q}$	Crossing of D^1_3.
			$D^1_{3,q} = d^1_{3,q}(\hat{i}_1, i_2, i_3) + d^1_{3,q}(\hat{i}_1, i_3, i_2)$.
qd31IF	qd31IF	$d^1_{3,q}$	Only contains $i_1 \| i_2$ collinear limit.
qDh31IF	FullqDh31IF	$\hat{D}^1_{3,q}$	Crossing of $\hat{D}^1_{3,q}$.
			$\hat{D}^1_{3,q} = \hat{d}^1_{3,q}(\hat{i}_1, i_2, i_3) + \hat{d}^1_{3,q}(\hat{i}_1, i_3, i_2)$.
qdh31IF	qdh31IF	$\hat{d}^1_{3,q}$	Only contains $i_1 \| i_2$ collinear limit.
qE31IF	FullqE31IF	$E^1_{3,q}$	Crossing of E^1_3.
qEh31IF	FullqEh31IF	$\hat{E}^1_{3,q}$	Crossing of $\hat{E}^1_3$.
qEt31IF	FullqEt31IF	$\tilde{E}^1_{3,q}$	Crossing of $\tilde{E}^1_3$.
qpE31IF	FullqpE31IF	$E^1_{3,q'}$	Crossing of E^1_3. Flavour changing $q' \to g$.
qpEh31IF	FullqpEh31IF	$\hat{E}^1_{3,q'}$	Crossing of $\hat{E}^1_3$. Flavour changing $q' \to g$.
qpEt31IF	FullqpEt31IF	$\tilde{E}^1_{3,q'}$	Crossing of $\tilde{E}^1_3$. Flavour changing $q' \to g$.
qpG31IF	FullqpG31IF	$G^1_{3,q'}$	Crossing of G^1_3. Flavour changing $q' \to g$.
qpGh31IF	FullqpGh31IF	$\hat{G}^1_{3,q'}$	Crossing of $\hat{G}^1_3$. Flavour changing $q' \to g$.
qpGt31IF	FullqpGt31IF	$\tilde{G}^1_{3,q'}$	Crossing of $\tilde{G}^1_3$. Flavour changing $q' \to g$.

Table 4.4: Quark initiated initial-final state X^1_3 antenna functions

maple	fortran/form	latex	comment
ggD31II	FullggD31II	$D^1_{3,gg}$	Mixed flavour changing. $D^1_{3,gg} = d^1_{3,gg\to qg}(i_1, \hat{\hat{i}}_2, \hat{\hat{i}}_3) + d^1_{3,gg\to qg}(i_1, \hat{\hat{i}}_3, \hat{\hat{i}}_2)$.
ggd31II	ggd31II	$d^1_{3,gg\to qg}$	Only contains $i_1 \| i_2$ collinear limit. Flavour changing $g \to q$.
ggDh31II	FullggDh31II	$\hat{D}^1_{3,gg}$	Mixed flavour changing. $\hat{D}^1_{3,gg} = \hat{d}^1_{3,gg\to qg}(i_1, \hat{\hat{i}}_2, \hat{\hat{i}}_3) + \hat{d}^1_{3,gg\to qg}(i_1, \hat{\hat{i}}_3, \hat{\hat{i}}_2)$.
ggdh31II	ggdh31II	$\hat{d}^1_{3,gg\to qg}$	Only contains $i_1 \| i_2$ collinear limit. Flavour changing $g \to q$.
ggF31II	FullggF31II	$F^1_{3,gg}$	Crossing of F^1_3.
ggFh31II	FullggFh31II	$\hat{F}^1_{3,gg}$	Crossing of $\hat{F}^1_3$.
qgA31II	FullqgA31II	$A^1_{3,qg\to qq}$	Crossing of A^1_3. Flavour changing $g \to q$.
qgAh31II	FullqgAh31II	$\hat{A}^1_{3,qg\to qq}$	Crossing of $\hat{A}^1_3$. Flavour changing $g \to q$.
qgAt31II	FullqgAt31II	$\tilde{A}^1_{3,qg\to qq}$	Crossing of $\tilde{A}^1_3$. Flavour changing $g \to q$.
qgD31II	FullqgD31II	$D^1_{3,qg}$	Crossing of D^1_3.
qgDh31II	FullqgDh31II	$\hat{D}^1_{3,qg}$	Crossing of $\hat{D}^1_3$.
qgG31II	FullgqG31II	$G^1_{3,qg\to gg}$	Crossing of G^1_3. Flavour changing $q \to g$.
qgGh31II	FullgqGh31II	$\hat{G}^1_{3,qg\to gg}$	Crossing of $\hat{G}^1_3$. Flavour changing $q \to g$.
qgGt31II	FullgqGt31II	$\tilde{G}^1_{3,qg\to gg}$	Crossing of $\tilde{G}^1_3$. Flavour changing $q \to g$.
gqA31II	FullqgA31II	$A^1_{3,gq\to qq}$	Crossing of A^1_3. Flavour changing $g \to q$.
gqAh31II	FullqgAh31II	$\hat{A}^1_{3,gq\to qq}$	Crossing of $\hat{A}^1_3$. Flavour changing $g \to q$.
gqAt31II	FullqgAt31II	$\tilde{A}^1_{3,gq\to qq}$	Crossing of $\tilde{A}^1_3$. Flavour changing $g \to q$.
gqD31II	FullqgD31II	$D^1_{3,gq}$	Crossing of D^1_3.
gqDh31II	FullqgDh31II	$\hat{D}^1_{3,gq}$	Crossing of $\hat{D}^1_3$.
gqG31II	FullgqG31II	$G^1_{3,gq\to gg}$	Crossing of G^1_3. Flavour changing $q \to g$.
gqGh31II	FullgqGh31II	$\hat{G}^1_{3,gq\to gg}$	Crossing of $\hat{G}^1_3$. Flavour changing $q \to g$.
gqGt31II	FullgqGt31II	$\tilde{G}^1_{3,gq\to gg}$	Crossing of $\tilde{G}^1_3$. Flavour changing $q \to g$.

Table 4.5: gluon-gluon, quark-gluon and gluon-quark initiated X^1_3 antenna functions

maple	fortran/form	latex	comment
qpqpE31II	FullqpqpE31II	$E^1_{3,q\bar{q}}$	Crossing of E^1_3.
qpqpEh31II	FullqpqpEh31II	$\hat{E}^1_{3,q\bar{q}}$	Crossing of $\hat{E}^1_3$.
qpqpEt31II	FullqpqpEt31II	$\tilde{E}^1_{3,q\bar{q}}$	Crossing of $\tilde{E}^1_3$.
qqA31II	FullqqA31II	$A^1_{3,q\bar{q}}$	Crossing of A^1_3.
qqAh31II	FullqqAh31II	$\hat{A}^1_{3,q\bar{q}}$	Crossing of $\hat{A}^1_3$.
qqAt31II	FullqqAt31II	$\tilde{A}^1_{3,q\bar{q}}$	Crossing of $\tilde{A}^1_3$.
qqG31II	FullqqG31II	$G^1_{3,q\bar{q}}$	Crossing of G^1_3.
qqGh31II	FullqqGh31II	$\hat{G}^1_{3,q\bar{q}}$	Crossing of $\hat{G}^1_3$.
qqGt31II	FullqqGt31II	$\tilde{G}^1_{3,q\bar{q}}$	Crossing of $\tilde{G}^1_3$.
qqpE31II	FullqqpE31II	$E^1_{3,qq'\to qg}$	Crossing of E^1_3. Flavour changing $q' \to g$.
qqpEh31II	FullqqpEh31II	$\hat{E}^1_{3,qq'\to qg}$	Crossing of $\hat{E}^1_3$. Flavour changing $q' \to g$.
qqpEt31II	FullqqpEt31II	$\tilde{E}^1_{3,qq'\to qg}$	Crossing of $\tilde{E}^1_3$. Flavour changing $q' \to g$.

Table 4.6: quark-quark initiated X^1_3 antenna functions

momentum flow, $s_{ijk} = s_{ij} + s_{jk} + s_{ik}$, while the one-loop matrix elements $d\hat{\sigma}^{RV}_{NNLO}$ are renormalized at μ_R^2. To ensure that the poles precisely cancels, we add the corresponding scale fixing terms

$$X^1_3(i,j,k) \to X^1_3(i,j,k) + \frac{\beta_{\mathcal{N}}}{\epsilon} C(\epsilon) X^0_3(i,j,k) \left(\left(\frac{|s_{ijk}|}{\mu_R^2} \right)^{-\epsilon} - 1 \right), \qquad (4.5.89)$$

where for X^1_3, $\tilde{X}^1_3$ and $\hat{X}^1_3$, $\beta_{\mathcal{N}}$ is b_0, 0 and $b_{0,F}$. The X^1_3 antenna functions depend on the parton types, and on whether which hard radiators are in the initial or final states. They are summarised in Tables 4.2-4.6.

4.5.2 Integrated eikonal factor in Initial-Final mapping

Full result

The large angle soft terms (LAST) are added to the double real subtraction term to regulate single soft limits. The structure of the LAST is the product of an eikonal factor, an X^0_3 antenna function and a reduced matrix element. At the real-virtual

level, the LAST need to be integrated over the soft phase space of the primary mapping and added back in to $d\hat{\sigma}_{NNLO}^T$. In $pp \to$ di-jet processes, the LAST could be written in a form where the first mapping is of final-final type [7] so that the eikonal factor is integrated over a final-final sub-phase space $d\Phi_{X_{ijk}}$. For the $pp \to H+$jet processes, there are not enough partons to ensure that the first mapping is of final-final type. In this case, we are forced to write the LAST in a form where the first mapping is of initial-final type. The eikonal factor is then integrated over the initial-final sub-phase space $d\Phi_2$. From Eq. (3.3.50) and (3.3.51), before integration over q, one has

$$d\Phi_2(p_j, p_k; p_i, q) \equiv d\Phi_{\hat{i}jk} = (2\pi)^d [dp_j][dp_k] d^d q \delta^d(q + \hat{p}_i - p_j - p_k) \delta^d(q + x_i \hat{p}_i - p_K).$$
(4.5.90)

We need to consider the integrated eikonal factor in two scenarios: (a) unboosted soft momentum and (b) boosted soft momentum.

When the unresolved momentum in the eikonal factor is unboosted (labeled by $\{j\}$), the integrated eikonal factor is,

$$\mathcal{S}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K}) = \frac{1}{C(\epsilon)} \int d\Phi_{\hat{i}jk} \frac{Q^2}{2\pi} S_{ajc}.$$
(4.5.91)

The full result of the above integration is [134],

$$\mathcal{S}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K}) = (Q^2)^{-\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\epsilon\gamma}}{\Gamma(1-2\epsilon)} \left(-\frac{2}{\epsilon}\right) x_i^{1+2\epsilon} (1-x_i)^{-1-2\epsilon} y_{ac,\hat{I}K}^{-\epsilon}(x_i)$$
(4.5.92)

where

$$y_{ac,\hat{I}K}(x) = \frac{s_{ac}|s_{\hat{I}K}|}{\left[s_{aK} + \left(\frac{1-x}{x}\right) s_{a\hat{I}}\right] \left[s_{cK} + \left(\frac{1-x}{x}\right) s_{c\hat{I}}\right]}.$$
(4.5.93)

$\{\hat{i}, j, k\}$ are mapped into $\{\hat{I}, K\}$ in the primary mapping which is fixed to be of initial-final type as described in [113]. $\{p_a, p_c\}$ are from the single or double mapped momentum set $\{p\}_{n+1}$ or $\{p\}_n$. By definition, $\{p_j, p_k\}$ are final state momenta while $\{p_a, p_c\}$ could be either initial or final state momenta as the eikonal factor is invariant under crossing.

When the unresolved momentum in the eikonal factor is boosted (labeled by $\{\tilde{j}\}$), the integrated eikonal factor is,

$$\tilde{\mathcal{S}}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K}) = \frac{1}{C(\epsilon)} \int d\Phi_{\hat{i}jk} \frac{Q^2}{2\pi} S_{a\tilde{j}c}.$$
(4.5.94)

In this case, $\{p_a, p_c\}$ are from the double mapped momentum set $\{p\}_n$ where the second mapping is of initial-initial type. $\tilde{p}_j$ is the p_j momentum boosted by the secondary initial-initial mapping, $\Lambda(p_j, \tilde{p}_j)p_j = \tilde{p}_j$. Since $S_{a\tilde{j}c}$ is composed of Lorentz invariants, we can have the inverse Lorentz boost $\Lambda^{-1}(p_j, \tilde{p}_j)$ such that

$$\mathcal{S}(s_{\underline{ac}}, s_{\hat{I}K}, y_{\underline{ac},\hat{I}K}) = \Lambda^{-1}(p_j, \tilde{p}_j)\tilde{\mathcal{S}}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K}) = \frac{1}{C(\epsilon)} \int d\Phi_{ijk} \frac{Q^2}{2\pi} S_{a\tilde{j}\underline{c}} \quad (4.5.95)$$

with $p_{\underline{a}} = \Lambda^{-1}(p_j, \tilde{p}_j)p_a$ and $p_{\underline{c}} = \Lambda^{-1}(p_j, \tilde{p}_j)p_c$.

The RHS has precisely the same form as Eq. (4.5.91) and can be replaced by Eq. (4.5.92) with $a \to \underline{a}$ and $c \to \underline{c}$,

$$\mathcal{S}(s_{\underline{ac}}, s_{\hat{I}K}, y_{\underline{ac},\hat{I}K}) = (Q^2)^{-\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\epsilon\gamma}}{\Gamma(1-2\epsilon)}(-\frac{2}{\epsilon})x_i^{1+2\epsilon}(1-x_i)^{-1-2\epsilon}y_{\underline{ac},\hat{I}K}^{-\epsilon}(x_i). \quad (4.5.96)$$

Furthermore, we can exploit the fact that the above equation only depends on Lorentz invariants so we can apply the same Lorentz boost such that

$$\tilde{\mathcal{S}}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K}) = \Lambda(p_j, \tilde{p}_j)\mathcal{S}(s_{\underline{ac}}, s_{\hat{I}K}, y_{\underline{ac},\hat{I}K}) = \mathcal{S}(s_{ac}, s_{\tilde{\hat{I}}\tilde{K}}, y_{ac,\tilde{\hat{I}}\tilde{K}}). \quad (4.5.97)$$

As p_K lies in the $\{p\}_{n+1}$ momentum set, $\tilde{p}_K$ is the corresponding boosted momentum *after* applying the initial-initial mapping to the $\{p\}_n$ set. $\tilde{p}_{\hat{I}}$ is the boosted momentum from $p_{\hat{I}}$ and does not exist in $\{p\}_n$ set and is given by,

$$\tilde{p}_{\hat{I}} = \Lambda(p_j, \tilde{p}_j)p_{\hat{I}}. \quad (4.5.98)$$

The denominator of $y_{ac,\hat{I}K}(x)$ could become zero in the case when $c\ (a) = K$ and $x = 1$. In this case we need to combine the divergent denominator present in $y_{ac,\hat{I}K}(x)$ with the regulating $(1-x)^{-1-2\epsilon}$ factor in Eq.(4.5.92). In general,

$$y_{aK,\hat{I}K}(x) = \frac{s_{aK}|s_{\hat{I}K}|}{\left[s_{aK} + \left(\frac{1-x}{x}\right)s_{a\hat{I}}\right]\left[s_{KK} + \left(\frac{1-x}{x}\right)s_{K\hat{I}}\right]} = \frac{xs_{aK}|s_{K\hat{I}}|}{(1-x)\left[s_{aK} + \left(\frac{1-x}{x}\right)s_{ai}\right]s_{K\hat{I}}}$$

$$= \frac{x}{(1-x)}y_{aK,\hat{I}}(x). \quad (4.5.99)$$

The full result is thus,

$$\mathcal{S}(s_{aK}, s_{\hat{I}K}, y_{aK,\hat{I}K}) \to \mathcal{S}(s_{aK}, s_{\hat{I}K}, y_{aK,\hat{I}}) =$$
$$(Q^2)^{-\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)e^{\epsilon\gamma}}{\Gamma(1-2\epsilon)}(-\frac{2}{\epsilon})x_i^{1+\epsilon}(1-x_i)^{-1-\epsilon}y_{aK,I}^{-\epsilon}(x_i),$$

$$(4.5.100)$$

where

$$y_{aK,\hat{I}}(x) = -\frac{s_{aK}}{\left[s_{aK} + \left(\frac{1-x}{x}\right)s_{a\hat{I}}\right]}. \tag{4.5.101}$$

Expanding in distributions

General case $K \neq c$ (a)

In the general case when $K \neq c$ (a), expanding Eq. (4.5.92) in distributions yields

$$\begin{aligned}
\mathcal{S}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K}) =& (|s_{\hat{I}K}|)^{-\epsilon} \bigg\{ \bigg[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln(y_{ac,\hat{I}K}(1)) + \frac{1}{2}\ln^2(y_{ac,\hat{I}K}(1)) - \frac{\pi^2}{12}\bigg]\delta(1-x) \\
&+ \bigg[\frac{2}{\epsilon}(1 - \mathcal{D}_0(x)) + 4\mathcal{D}_1(x) + 2\ln(y_{ac,\hat{I}K}(1))\mathcal{D}_0(x) - 4\ln(1-x) \\
&- \frac{4x}{(1-x)}\ln(x) + \frac{2x}{(1-x)}\ln(y_{ac,\hat{I}K}(x)) - \frac{2}{(1-x)}\ln(y_{ac,\hat{I}K}(1))\bigg] \\
&+ \mathcal{O}(\epsilon)\bigg\}.
\end{aligned} \tag{4.5.102}$$

The value of $y_{ac,\hat{I}K}(1)$ is positive by definition. If the value of $y_{ac,\hat{I}K}(x)$ is negative then $\ln(y_{ac,\hat{I}K}(x))$ would give an imaginary contribution according to,

$$\ln(y_{ac,\hat{I}K}(x)) = \ln(|y_{ac,\hat{I}K}(x)|) - i\pi. \tag{4.5.103}$$

However, we usually add $\mathcal{S}_{ac;\hat{I},K}$ terms in pairs opposite signs, the imaginary contributions tend to cancel each other. In the particular case when we pair $\mathcal{S}(s_{ac}, s_{\hat{I}K}, y_{ac,\hat{I}K})$ with $-\mathcal{S}(s_{ac}, s_{\tilde{I}\tilde{K}}, y_{ac,\tilde{I}\tilde{K}})$, the cancellation still holds as Lorentz invariance guarantees $s_{\hat{I}K} = s_{\tilde{I}\tilde{K}}$.

Special case $K = c$ (a)

In the special case when $K = c$, expanding Eq. (4.5.100) in distributions yields

$$\begin{aligned}
\mathcal{S}(s_{aK}, s_{\hat{I}K}, y_{aK,\hat{I}}) =& (|s_{\hat{I}K}|)^{-\epsilon} \bigg\{ \bigg[\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln(y_{aK,\hat{I}}(1)) + \ln^2(y_{aK,\hat{I}}(1)) - \frac{\pi^2}{6}\bigg]\delta(1-x) \\
&+ \bigg[\frac{2}{\epsilon}(1 - \mathcal{D}_0(x)) + 2\mathcal{D}_1(x) + 2\ln(y_{aK,\hat{I}}(1))\mathcal{D}_0(x) - 2\ln(1-x) \\
&- \frac{2x}{(1-x)}\ln(x) + \frac{2x}{(1-x)}\ln(y_{aK,\hat{I}}(x)) - \frac{2}{(1-x)}\ln(y_{aK,\hat{I}}(1))\bigg] \\
&+ \mathcal{O}(\epsilon)\bigg\}.
\end{aligned} \tag{4.5.104}$$

When imaginary contributions are produced because $y_{aK,\hat{I}}(1) = -1$ and $y_{aK,\hat{I}}(x)$ is negative, they undergo a pairwise cancellation as in the general case. In the case of $K = a$, as we simply exploit the a, c exchange symmetry present in Eq. (4.5.93), and use Eq. (4.5.104) with a replaced by c.

4.5.3 Antenna subtraction term $d\hat{\sigma}^T_{NNLO}$

The real-virtual antenna subtraction term is the bridge that links the double real and double virtual subtraction terms. All contributions in the double real subtraction term that can be integrated over a single unresolved phase space region are compensated at the real-virtual level while new terms introduced at the real-virtual level will be later integrated and added back with minus sign at the double virtual level. As discussed at the beginning of this chapter, $d\hat{\sigma}^T_{NNLO}$ has the following structure,

$$d\hat{\sigma}^T_{NNLO} \quad = \quad d\hat{\sigma}^{VS}_{NNLO} - \int_1 d\hat{\sigma}^{S,1}_{NNLO} - d\hat{\sigma}^{MF,1}_{NNLO}. \qquad (4.5.105)$$

Note that,

$$\int_1 d\hat{\sigma}^{S,1}_{NNLO} \quad = \quad \int_1 d\hat{\sigma}^{S,a}_{NNLO} + \int_1 d\hat{\sigma}^{S,b_2}_{NNLO} + \int_1 d\hat{\sigma}^{S,c}_{NNLO}. \qquad (4.5.106)$$

All the explicit and implicit divergences in $d\hat{\sigma}^{RV}_{NNLO}$ should be canceled by $d\hat{\sigma}^T_{NNLO}$. Although they arise from various sources, the terms in $d\hat{\sigma}^T_{NNLO}$ can be catalogued into four types.

(1) Terms that remove the explicit pole from $d\hat{\sigma}^{RV}_{NNLO}$

(2) Terms that remove the implicit divergence from $d\hat{\sigma}^{RV}_{NNLO}$

(3) Terms that remove the secondary emission from $\int_1 d\hat{\sigma}^{S,a}_{NNLO}$ and $d\hat{\sigma}^{VS,b}_{NNLO}$

(4) Terms that give a finite contribution $d\hat{\sigma}^{VS,d}_{NNLO}$ due to identity changing transitions

Terms that remove the explicit pole from $d\hat{\sigma}^{RV}_{NNLO}$

The explicit pole structure of $d\hat{\sigma}^{RV}_{NNLO}$ comes from the one-loop matrix elements which have the general form given in Eq. (3.4.66). The following combination of

integrated subtraction terms and mass factorization counter terms precisely cancels this pole structure,

$$
-\int_1 d\hat{\sigma}_{NNLO}^{S,a} - d\hat{\sigma}_{NNLO}^{MF,1R} = \mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2}
$$
$$
\times \sum_{i,j} \boldsymbol{J}_2^{(1)}(i,j) |M_{n+3}^0(\cdots, i, j, \cdots)|^2 J_{n+1}^{(n)}(\{p\}_{n+1}).
$$
(4.5.107)

Terms that remove the implicit divergence from $d\hat{\sigma}_{NNLO}^{RV}$

From the construction of X_3^1 in Eq. (4.5.86), one can use the following subtraction terms to remove the implicit divergence from $d\hat{\sigma}_{NNLO}^{RV}$ in single unresolved phase space regions,

$$
d\hat{\sigma}_{NNLO}^{VS,a} = \mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \delta(1-x_1)\delta(1-x_2)
$$
$$
\times \sum_j \Bigg[X_3^1(i,j,k) |M_{n+2}^0(\cdots, I, K \cdots)|^2
$$
$$
+ X_3^0(i,j,k) |M_{n+2}^1(\cdots, I, K, \cdots)|^2 \Bigg] J_n^{(n)}(\{p\}_n).
$$
(4.5.108)

As the explicit poles from one-loop matrix elements have already been removed by Eq. (4.5.107), the explicit poles introduced in Eq. (4.5.108) must be removed by additional terms. According to Eq. (4.5.86) the pole structure of X_3^1 is given by,

$$
X_3^1(i,j,k) \xrightarrow{pole} \Bigg[\boldsymbol{J}_2^{(1)}(i,j) + \boldsymbol{J}_2^{(1)}(j,k) + \boldsymbol{J}_2^{(1)}(i,k) - \boldsymbol{J}_2^{(1)}(I,K) \Bigg] X_3^0(i,j,k),
$$
(4.5.109)

where p_i, p_j and p_k lie in the momentum set $\{p\}_{n+1}$, while p_I and p_K are mapped momenta in the set $\{p\}_n$. To cancel above pole structure, we add

$$
-\int_1 d\hat{\sigma}_{NNLO}^{S,b_2} - d\hat{\sigma}_{NNLO}^{MF,1S} = -\mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2}
$$
$$
\times \sum_j \Bigg[\Bigg(\boldsymbol{J}_2^{(1)}(i,j) + \boldsymbol{J}_2^{(1)}(j,k) + \boldsymbol{J}_2^{(1)}(i,k) \Bigg) X_3^0(i,j,k) |M_{n+2}^0(\cdots, I, K \cdots)|^2 J_n^{(n)}(\{p\}_n) \Bigg].
$$
(4.5.110)

The rest pole structure in Eq. (4.5.108) are canceled by constructing new terms,

$$
d\hat{\sigma}_{NNLO}^{VS,b} = \mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2}
$$

$$\times \sum_{j} \Bigg[\boldsymbol{J}_2^{(1)}(I,K)|M_{n+2}^0(\cdots,I,K\cdots)|^2$$

$$- \sum_{L,M} \boldsymbol{J}_2^{(1)}(L,M)|M_{n+2}^0(\cdots,L,M,\cdots)|^2 \Bigg] X_3^0(i,j,k) J_n^{(n)}(\{p\}_n).$$

$$(4.5.111)$$

The sum of

$$d\hat{\sigma}_{NNLO}^{VS,a} + d\hat{\sigma}_{NNLO}^{VS,b} - \int_1 d\hat{\sigma}_{NNLO}^{S,b_2} - d\hat{\sigma}_{NNLO}^{MF,1S}, \qquad (4.5.112)$$

removes implicit divergences from $d\hat{\sigma}_{NNLO}^{RV}$ while introducing no new explicit poles.

Terms that remove the secondary emission from $\int_1 d\hat{\sigma}_{NNLO}^{S,a}$ and $d\hat{\sigma}_{NNLO}^{VS,b}$

In the various single unresolved limits at real-virtual level, the jet function in $\int_1 d\hat{\sigma}_{NNLO}^{S,a}$ allows the emission of one potentially unresolved parton. In constructing $d\hat{\sigma}_{NNLO}^{VS,b}$, the aim was to remove the explicit pole structure introduced in $d\hat{\sigma}_{NNLO}^{VS,a}$. Nevertheless the $X_3^0(i,j,k)$ antenna function in $d\hat{\sigma}_{NNLO}^{VS,b}$ (Eq. (4.5.111)) could become divergent in single unresolved region. To compensate this over subtraction we introduce a new term $d\hat{\sigma}_{NNLO}^{VS,c}$ together with $\int_1 d\hat{\sigma}_{NNLO}^{S,c}$ that regulate the phase space integration at the real-virtual level.

The relevant terms in the double real subtraction term are a nine term block of $d\hat{\sigma}_{NNLO}^{S,c}$ in Eq. (4.3.75) that includes three primary antenna functions X_3^0 and six eikonal factors S_{ijk}. The new contribution in $d\hat{\sigma}_{NNLO}^{VS,c}$ for each block of $\int_1 d\hat{\sigma}_{NNLO}^{S,c}$ are the three terms with the structure

$$\mathcal{X}_3^0(L,M)X_3^0(i,j,k)|M_{n+2}^0(\cdots,L,M,\cdots)|^2, \qquad (4.5.113)$$

where p_i, p_j and p_j are the momentum at real-virtual in final state $\{p\}_{n+1}$ set, and p_L and p_M are the mapped momentum in final state $\{p\}_n$ set. Following the notation in Eq. (4.3.75), for each block of $\int_1 d\hat{\sigma}_{NNLO}^{S,c}$ one has,

$$-\int_1 d\hat{\sigma}_{NNLO}^{S,c} + d\hat{\sigma}_{NNLO}^{VS,c} = -\frac{1}{2}\mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \Bigg\{$$

$$+ \left(\mathcal{X}_3^0(J,L) - \mathcal{X}_3^0(\widetilde{JK},\widetilde{KL}) \right) X_3^0(J,K,L) \, |M_{n+2}^0(\ldots,a,\widetilde{JK},\widetilde{KL},b,\ldots)|^2$$

$$- \left(\mathcal{X}_3^0(A,J) - \mathcal{X}_3^0(A,\widetilde{JK}) \right) X_3^0(J,K,L) \, |M_{n+2}^0(\ldots,A,\widetilde{JK},\widetilde{KL},b,\ldots)|^2$$

$$- \left(\mathcal{X}_3^0(L, B) - \mathcal{X}_3^0(\widetilde{KL}, B) \right) X_3^0(J, K, L) \, |M_{n+2}^0(\dots, a, \widetilde{JK}, \widetilde{KL}, B, \dots)|^2$$

$$- \Big[\big(\mathcal{S}(s_{JL}, s_{JL}, y_{JL,JL}) - \mathcal{S}(s_{\widetilde{(JK)(KL)}}, s_{JL}, y_{\widetilde{(JK)(KL)},JL}) \big)$$

$$- \big(\mathcal{S}(s_{aJ}, s_{JL}, y_{aJ,JL}) - \mathcal{S}(s_{a\widetilde{(JK)}}, s_{JL}, y_{a\widetilde{(JK)},JL}) \big)$$

$$- \big(\mathcal{S}(s_{bL}, s_{JL}, y_{bL,JL}) - \mathcal{S}(s_{b\widetilde{(KL)}}, s_{JL}, y_{b\widetilde{(KL)},JL}) \big) \Big] \times$$

$$X_3^0(J, K, L) \, |M_{n+2}^0(\dots, a, \widetilde{JK}, \widetilde{KL}, b, \dots)|^2 \Big\} \, J_n^{(n)}(\{p\}_n). \tag{4.5.114}$$

Note that the integrated eikonal factor could be integrated with either an IF or a FF mapping.

The explicit pole structure of in Eq. (4.5.114) cancels amongst itself in a pairwise manner. Together with Eqs. (4.5.107) and (4.5.112), Eq. (4.5.114) removes all the remaining divergences in single unresolved phase space regions. For the convenience of the construction of the double virtual subtraction terms, $d\hat{\sigma}_{NNLO}^{VS,c}$ is usually decomposed into two parts $d\hat{\sigma}_{NNLO}^{VS,c_1}$ and $d\hat{\sigma}_{NNLO}^{VS,c_2}$ such that,

$$d\hat{\sigma}_{NNLO}^{VS,c_1} = -\mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \Big\{$$

$$-\mathcal{X}_3^0(\widetilde{JK}, \widetilde{KL}) \, X_3^0(J, K, L) \, |M_{n+2}^0(\dots, a, \widetilde{JK}, \widetilde{KL}, b, \dots)|^2 \Big\} \, J_n^{(n)}(\{p\}_n). \tag{4.5.115}$$

$$d\hat{\sigma}_{NNLO}^{VS,c_2} = -\frac{1}{2}\mathcal{N}_{NNLO}^{RV} d\Phi_{n+1} \frac{1}{s_{n+1}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \Big\{$$

$$+\mathcal{X}_3^0(\widetilde{JK}, \widetilde{KL}) \, X_3^0(J, K, L) \, |M_{n+2}^0(\dots, a, \widetilde{JK}, \widetilde{KL}, b, \dots)|^2$$

$$+\mathcal{X}_3^0(A, \widetilde{JK}) \, X_3^0(J, K, L) \, |M_{n+2}^0(\dots, A, \widetilde{JK}, \widetilde{KL}, b, \dots)|^2$$

$$+\mathcal{X}_3^0(\widetilde{KL}, B) \, X_3^0(J, K, L) \, |M_{n+2}^0(\dots, a, \widetilde{JK}, \widetilde{KL}, B, \dots)|^2 \Big\} \, J_n^{(n)}(\{p\}_n). \tag{4.5.116}$$

Terms that give a finite contribution $d\hat{\sigma}_{NNLO}^{VS,d}$ due to identity changing transitions

So far, we have only discussed the contributions to $d\hat{\sigma}_{NNLO}^{VS}$ that preserve the identity of the initial state partons. The initial state identity changing terms coming from the double real subtraction terms and the real-virtual mass factorization terms combine to give a finite contribution to the total cross section. Adding Eq. (4.3.76) and the corresponding initial state identity changing mass factorization terms in Eqs. (4.4.83)

and (4.4.84), we find,

$$
-\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{S,idc} - \mathrm{d}\hat{\sigma}_{NNLO}^{MF,1idc} =
$$

$$
= -\mathcal{N}_{NNLO}^{RV}\mathrm{d}\Phi_{n+1}\frac{1}{s_{n+1}}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\Bigg\{
$$

$$
+ \boldsymbol{J}_{2,b\to a}^{(1)}(I,K)|M_{n+3}^0(\ldots,I,K,\ldots)|^2
$$

$$
- \boldsymbol{J}_{2,b\to a}^{(1)}(I,K)\sum_{M\in\{p\}_{n+1}} X_3^0(L,M,N)|M_{n+2}^0(\ldots,\widetilde{(LM)},\widetilde{(MN)},\ldots)|^2 \Bigg\} J_n^{(n)}(\{p\}_n),
$$

$$
(4.5.117)
$$

where

$$
\boldsymbol{J}_{2,b\to a}^{(1)}(I,K) = \mathcal{X}_3^{0,idc}(I,K) - \Gamma_{ab}^{(1)}(x). \qquad (4.5.118)
$$

The identity changing dipole functions are free from explicit poles as mentioned in Eq. (3.5.78), while the NLO-like combination of terms ensures that Eq. (4.5.117) is free from implicit divergences and gives a finite contribution to the cross section.

In some special cases where the double real subtraction term does not provide sufficient identity changing contributions, we must add new terms of the type

$$
\mathcal{X}_3^{0,idc}(\widetilde{(IJ)},\widetilde{(JK)})X_3^0(L,M,N)|M_{n+2}^0| \qquad (4.5.119)
$$

as a $\mathrm{d}\hat{\sigma}_{NNLO}^{VS,d}$ contribution. Note that p_L, p_M and p_N are momenta in the $\{p\}_{n+1}$ momentum set while $p_{\widetilde{IJ}}$ and $p_{\widetilde{JK}}$ are mapped momentum in the $\{p\}_n$ momentum set.

Another special case is when, after integration, a single subtraction term from $\mathrm{d}\hat{\sigma}_{NNLO}^{S,c}$ is left that does not fit into any of the terms in Eq. (4.5.117). For example, $\mathcal{X}_3^{0,idc}(I,K)X_3^0(K,M,N)|M_{n+2}^0|$ type of term that would appear on its own where $X_3^0(K,M,N)$ contains only the p_M soft or $p_M//p_N$ limit. The procedure is to add a new term in $\mathrm{d}\hat{\sigma}_{NNLO}^{VS,d}$ to regulate both explicit and implicit divergences, the combination,

$$
\left(\mathcal{X}_3^{0,idc}(I,K) - \mathcal{X}_3^{0,idc}(I,\widetilde{(KM)})\right)X_3^0(K,M,N)|M_{n+2}^0|. \qquad (4.5.120)
$$

The explicit pole cancellation in Eq. (4.5.120) can be achieved if one chooses $\mathcal{X}_3^{0,idc}(I,\widetilde{(KM)})$ to be the same type as $\mathcal{X}_3^{0,idc}(I,K)$. The implicit divergences in $X_3^0(K,M,N)$ are

regulated in the p_M soft and $p_M//p_N$ limits such that,

$$\left(\mathcal{X}_3^{0,idc}(I,K) - \mathcal{X}_3^{0,idc}(I,(\widetilde{KM})) \right) \xrightarrow{p_{\widetilde{KM}} \to p_K} 0. \qquad (4.5.121)$$

The $\mathrm{d}\hat{\sigma}_{NNLO}^{VS,d}$ contribution must be integrated and added back at the double virtual level. From the explicit pole cancellation at the double virtual level, these $\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS,d}$ terms are indispensable and they are constructed on a case by case basis.

4.6 Divergent behaviour of the double virtual contributions at NNLO

4.6.1 Pole structure of two-loop matrix elements

The colour ordered two-loop matrix elements contains explicit divergences coming from the integration of loop momentum in the one and two-loop amplitudes. Following Catani [114, 115], tensorial operators in colour space, $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$, are used to express the infrared singularity structure such that the renormalized two-loop matrix element satisfies,

$$\mathcal{M}^2(\epsilon) = \mathbf{I}^{(1)}(\epsilon)\mathcal{M}^1(\epsilon) + \mathbf{I}^{(2)}(\epsilon)\mathcal{M}^0 + \mathcal{M}^{2,fin}, \qquad (4.6.122)$$

where $\mathcal{M}^1(\epsilon)$ is the renormalized one-loop amplitude defined in Eq. (3.4.64), and $\mathcal{M}^{2,fin}$ is infrared finite function in $\epsilon \to 0$ limit. From the definition of the two-loop matrix element in section 1.3, then for the n-parton scattering process,

$$|M_n^2|^2 = \mathcal{M}_n^0 \mathcal{M}_n^{2\dagger} + \mathcal{M}_n^2 \mathcal{M}_n^{0\dagger} + \mathcal{M}_n^1 \mathcal{M}_n^{1\dagger}. \qquad (4.6.123)$$

For the convenience of matching the integrated double and single radiation from tree and one-loop matrix elements, the pole structure of the colour ordered matrix element $|M_n^2|^2$ can be expressed using the real scalar operators in a dipole formalism such that,

$$|M_n^2(\cdots,i,j,\cdots)|^2 \xrightarrow{pole}$$

$$2\mathbf{I}_n^{(1)}(\epsilon,\cdots,i,j,\cdots)\left(|M_n^1(\cdots,i,j,\cdots)|^2 - \frac{\beta_0}{\epsilon}|M_n^0(\cdots,i,j,\cdots)|^2 \right)$$

$$-2\mathbf{I}_n^{(1)}(\epsilon, \cdots, i, j, \cdots)^2 |M_n^0(\cdots, i, j, \cdots)|^2$$

$$+2e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\mathbf{I}_n^{(1)}(2\epsilon, \cdots, i, j, \cdots)|M_n^0(\cdots, i, j, \cdots)|^2$$

$$+2\mathbf{H}^{(2)}(\epsilon, \cdots, i, j, \cdots)|M_n^0(\cdots, i, j, \cdots)|^2, \qquad (4.6.124)$$

with $\mathbf{I}_n^{(1)}$ defined in Eq. (3.4.65) and (3.4.66), and

$$K = N\left(\frac{67}{18}-\frac{\pi^2}{9}\right) - N_F\frac{10}{9},$$

$$\mathbf{H}^{(2)}(\epsilon, \cdots, i, \cdots) = \sum_i \mathbf{H}_i^{(2)}(\epsilon). \qquad (4.6.125)$$

The $\mathbf{H}_i^{(2)}(\epsilon)$ functions is process- and renormalization scheme-dependent. Using the $\overline{MS}$ scheme, for each gluon involved in the two-loop matrix element [119],

$$\mathbf{H}_g^{(2)}(\epsilon) = \frac{e^{\epsilon\gamma}}{4\,\epsilon\,\Gamma(1-\epsilon)}H_g^{(2)}, \qquad (4.6.126)$$

and for each quark (or anti-quark),

$$\mathbf{H}_q^{(2)}(\epsilon) = \frac{e^{\epsilon\gamma}}{4\,\epsilon\,\Gamma(1-\epsilon)}H_q^{(2)}, \qquad (4.6.127)$$

so that

$$H_q^{(2)} = \left(\frac{7}{4}\zeta_3 + \frac{409}{864} - \frac{11\pi^2}{96}\right)N^2 + \left(-\frac{1}{4}\zeta_3 - \frac{41}{108} - \frac{\pi^2}{96}\right) + \left(-\frac{3}{2}\zeta_3 - \frac{3}{32} + \frac{\pi^2}{8}\right)\frac{1}{N^2}$$

$$+ \left(\frac{\pi^2}{48} - \frac{25}{216}\right)\frac{(N^2-1)N_F}{N}, \qquad (4.6.128)$$

$$H_g^{(2)} = \left(\frac{1}{2}\zeta_3 + \frac{5}{12} + \frac{11\pi^2}{144}\right)N^2 + \frac{5}{27}N_F^2 + \left(-\frac{\pi^2}{72} - \frac{89}{108}\right)NN_F - \frac{N_F}{4N}. \qquad (4.6.129)$$

4.6.2   Mass factorization terms at the double virtual level

Apart from the pole structure of two-loop matrix element, for a proton proton collision process, there is an explicitly divergent contribution coming from the mass factorization of initial state parton radiation in the double virtual level of NNLO calculations. From the discussion in section 1.4, the explicit contribution from the double virtual level mass factorization terms $(\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2}$ in Eq. (1.4.52)) are,

$$\mathrm{d}\hat{\sigma}_{ij,NNLO}^{MF,2}(\xi_1 H_1, \xi_2 H_2) =$$

$$-\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}^2(\epsilon)\left[\delta(1-x_2)\delta_{lj}\left(\mathbf{\Gamma}_{ki}^2(x_1) - [\mathbf{\Gamma}_{ai}^1 \otimes \mathbf{\Gamma}_{ka}^1](x_1)\right)\right.$$

$$+ \delta(1 - x_1)\delta_{ki}\left(\mathbf{\Gamma}^2_{lj}(x_2) - [\mathbf{\Gamma}^1_{aj} \otimes \mathbf{\Gamma}^1_{la}](x_2) \right)$$

$$- \mathbf{\Gamma}^1_{ki}(x_1)\mathbf{\Gamma}^1_{lj}(x_2) \bigg] \mathrm{d}\hat{\sigma}^B_{kl}(x_1\xi_1 H_1, x_2\xi_2 H_2)$$

$$- \int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\bigg[\delta(1 - x_2)\mathbf{\Gamma}^1_{ki}(x_1)\left(\mathrm{d}\hat{\sigma}^V_{kj,NLO} + \int_1 \mathrm{d}\hat{\sigma}^S_{kj,NLO} \right)$$

$$+ \delta(1 - x_1)\mathbf{\Gamma}^1_{lj}(x_2)\left(\mathrm{d}\hat{\sigma}^V_{il,NLO} + \int_1 \mathrm{d}\hat{\sigma}^S_{il,NLO} \right) \bigg] (x_1\xi_1 H_1, x_2\xi_2 H_2).$$

$$(4.6.130)$$

Note that we replaced $\mathrm{d}\hat{\sigma}^V_{NLO} \to \mathrm{d}\hat{\sigma}^V_{NLO} + \int_1 \mathrm{d}\hat{\sigma}^S$ as mentioned in Eq. (4.4.82). By arranging the repeating indices and exploiting the NLO mass factorization term Eq. (3.4.67), Eq. (4.6.130) can be rewritten as,

$$\mathrm{d}\hat{\sigma}^{MF,2}_{ij,NNLO}(\xi_1 H_1, \xi_2 H_2) =$$

$$- \int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}^2(\epsilon)\mathbf{\Gamma}^2_{ij;kl}(x_1, x_2)\mathrm{d}\hat{\sigma}^B_{kl}(x_1\xi_1 H_1, x_2\xi_2 H_2)$$

$$- \int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\mathbf{\Gamma}^1_{ij;kl}(x_1, x_2)\left(\mathrm{d}\hat{\sigma}^V_{kl,NLO} - \mathrm{d}\hat{\sigma}^T_{kl,NLO} \right)(x_1\xi_1 H_1, x_2\xi_2 H_2),$$

$$(4.6.131)$$

where

$$\mathbf{\Gamma}^1_{ij;kl}(x_1, x_2) = \delta(1 - x_2)\delta_{lj}\mathbf{\Gamma}^1_{ki}(x_1) + \delta(1 - x_1)\delta_{ki}\mathbf{\Gamma}^1_{lj}(x_2), \quad (4.6.132)$$

$$\mathbf{\Gamma}^2_{ij;kl}(x_1, x_2) = \delta(1 - x_2)\delta_{lj}\mathbf{\Gamma}^2_{ki}(x_1) + \delta(1 - x_1)\delta_{ki}\mathbf{\Gamma}^2_{lj}(x_2) + \mathbf{\Gamma}^1_{ki}(x_1)\mathbf{\Gamma}^1_{lj}(x_2).$$

$$(4.6.133)$$

It is convenient to introduce $\overline{\mathbf{\Gamma}}^2_{ij;kl}(x_1, x_2)$ such that,

$$\overline{\mathbf{\Gamma}}^2_{ij;kl}(x_1, x_2) = \overline{\mathbf{\Gamma}}^2_{ik}(x_1)\delta_{jl}\delta(1 - x_2) + \overline{\mathbf{\Gamma}}^2_{jl}(x_2)\delta_{ik}\delta(1 - x_1), \quad (4.6.134)$$

where

$$\overline{\mathbf{\Gamma}}^2_{ij}(x) \sim \overline{\mathbf{\Gamma}}^2_{ij}(x) = -\frac{1}{2\epsilon}\left(p^1_{ij}(x) + \frac{\beta_0}{\epsilon}p^0_{ij}(x) \right). \quad (4.6.135)$$

The relation between $\overline{\mathbf{\Gamma}}^2_{ij}(z)$ and $\overline{\mathbf{\Gamma}}^2_{ij}(z)$ is given in appendix of [116], and the details of $p^1_{ij}(x)$ functions can be found in [135]. Now $\mathbf{\Gamma}^2_{ij;kl}(x_1, x_2)$ can be further decomposed as

$$\mathbf{\Gamma}^2_{ij;kl}(x_1, x_2) = \overline{\mathbf{\Gamma}}^2_{ij;kl}(x_1, x_2) - \frac{\beta_0}{\epsilon}\mathbf{\Gamma}^1_{ij;kl}(x_1, x_2) + \frac{1}{2}\left[\mathbf{\Gamma}^1_{ij;ab} \otimes \mathbf{\Gamma}^1_{ab;kl} \right](x_1, x_2).$$

$$(4.6.136)$$

Inserting Eq. (4.6.136) in to (4.6.131), the double virtual mass factorization terms can be arranged into three terms,

$$\mathrm{d}\hat{\sigma}^{MF,2}_{ij,NNLO} = \mathrm{d}\hat{\sigma}^{MF,2,A}_{ij,NNLO} + \mathrm{d}\hat{\sigma}^{MF,2,B}_{ij,NNLO} + \mathrm{d}\hat{\sigma}^{MF,2,C}_{ij,NNLO},$$

$$(4.6.137)$$

where (omitting the dependence of x_1 and x_2)

$$\mathrm{d}\hat{\sigma}^{MF,2,A}_{ij,NNLO} = -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\mathbf{\Gamma}^1_{ij;kl}\left(\mathrm{d}\hat{\sigma}^V_{kl,NLO} - \frac{\beta_0}{\epsilon}\bar{C}(\epsilon)\mathrm{d}\hat{\sigma}^B_{kl}\right)$$

$$\mathrm{d}\hat{\sigma}^{MF,2,B}_{ij,NNLO} = -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\left\{-\mathbf{\Gamma}^1_{ij;kl}\mathrm{d}\hat{\sigma}^T_{kl,NLO} + \frac{1}{2}\left[\mathbf{\Gamma}^1_{ij;ab}\otimes\mathbf{\Gamma}^1_{ab;kl}\right]\bar{C}(\epsilon)\mathrm{d}\hat{\sigma}^B_{kl}\right\}$$

$$\mathrm{d}\hat{\sigma}^{MF,2,C}_{ij,NNLO} = -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}^2(\epsilon)\overline{\mathbf{\Gamma}}^2_{ij;kl}\mathrm{d}\hat{\sigma}^B_{kl}. \qquad (4.6.138)$$

These three terms will be combined with other double virtual subtraction terms and fit into the three blocks as $\mathrm{d}\hat{\sigma}^{U,A}_{ij,NNLO}$, $\mathrm{d}\hat{\sigma}^{U,B}_{ij,NNLO}$ and $\mathrm{d}\hat{\sigma}^{U,C}_{ij,NNLO}$. Details will be discussed in section 4.7.2.

4.7 Antenna subtraction term for the double virtual level at NNLO

4.7.1 Integration of the antenna subtraction terms for VV contribution

All the remaining antenna subtraction terms introduced at the double real level and the new terms introduced at the real-virtual level must be integrated and added back at the double virtual level. From the analysis in section 4.3.3 and 4.5.3, the integrated subtraction terms that contribute at the double virtual level are $\int_2 \mathrm{d}\hat{\sigma}^{S,b_1}_{NNLO}$, $\int_2 \mathrm{d}\hat{\sigma}^{S,d}_{NNLO}$ and $\int_1 \mathrm{d}\hat{\sigma}^{VS}_{NNLO}$. The new types of terms are,

$$\int_2 \mathrm{d}\hat{\sigma}^{S,b_1}_{NNLO} = \mathcal{N}^{VV}_{NNLO}\mathrm{d}\Phi_n \frac{1}{s_{n+2}}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\left\{\sum_{I,L}\mathcal{X}^0_4(I,L)|M^0_{n+2}(\cdots,I,L,\cdots)|^2\right\}$$

$$\times J^{(n)}_n(\{p\}_n), \qquad (4.7.139)$$

$$\int_2 d\hat{\sigma}_{NNLO}^{S,d} = -\mathcal{N}_{NNLO}^{VV} d\Phi_n \frac{1}{s_{n+2}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \left\{ \sum_{I,K} \sum_{L,N} \mathcal{X}_3^0(I,K) \otimes \mathcal{X}_3^0(L,N) \right.$$

$$\left. \times |M_{n+2}^0(\cdots, I, K, \cdots, L, N, \cdots)|^2 \right\} J_n^{(n)}(\{p\}_n), \qquad (4.7.140)$$

$$\int_1 d\hat{\sigma}_{NNLO}^{VS,a} = -\mathcal{N}_{NNLO}^{VV} d\Phi_n \frac{1}{s_{n+2}} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \left\{ \sum_{I,K} \mathcal{X}_3^1(I,K) \right.$$

$$\left. \times |M_{n+2}^0(\cdots, I, K, \cdots)|^2 \right\} J_n^{(n)}(\{p\}_n). \qquad (4.7.141)$$

while the $\int_1 d\hat{\sigma}_{NNLO}^{VS,b,c,d}$ terms have similar structures as in (4.7.140) and therefore are not repeated. The $\mathcal{N}_{NNLO}^{VV}$ factor is a normalisation factor related to strong coupling parameter α_s, momentum flux s and colour factor N. The relation between $\mathcal{N}_{NNLO}^{VV}$ and $\mathcal{N}_{NNLO}^{RR}$ is

$$\frac{\mathcal{N}_{NNLO}^{RR}}{\mathcal{N}_{NNLO}^{VV}} = \frac{1}{C(\epsilon)^2}. \qquad (4.7.142)$$

The integration over the double unresolved sub-phase space in $\mathcal{X}_4^0$ functions depends on the primary mapping $\{p_i, p_j, p_k, p_l\} \to \{p_I, p_L\}$ that,

$$\mathcal{X}_4^0(I,L) \xrightarrow{FF} \mathcal{X}_4^0(s_{IL}, x_1, x_2) = \frac{1}{C^2(\epsilon)} \int \delta(1-x_1)\delta(1-x_2) d\Phi_{X_4} X_4^0(i,j,k,l),$$

$$\mathcal{X}_4^0(I,L) \xrightarrow{IF} \mathcal{X}_4^0(s_{\bar{1}L}, x_1, x_2) = \frac{1}{C^2(\epsilon)} \int \delta(x_1 - \hat{x}_1)\delta(1-x_2) \frac{Q^2}{2\pi} d\Phi_3 X_4^0(1,j,k,l),$$

$$\mathcal{X}_4^0(I,L) \xrightarrow{II} \mathcal{X}_4^0(s_{\bar{1}\bar{2}}, x_1, x_2) = \frac{1}{C^2(\epsilon)} \int \delta(x_1 - \hat{x}_1)\delta(x_2 - \hat{x}_2) x_1 x_2 [dp_j][dp_k] X_4^0(1,j,k,2).$$

$$(4.7.143)$$

The integrated four-parton antennae for various parton types, all $\mathcal{X}_4^0(s_{IL}, x_1, x_2)$ are summarised in tables 4.7 – 4.10. Here the functions with trivial exchange of $x_1 \leftrightarrow x_2$ in $\mathcal{X}_4^0(s_{\bar{1}\bar{2}}, x_1, x_2)$ are not listed. The function $\mathcal{E}_{4,\bar{q}'}^0$ was omitted in the original work [134], and the leading singularities are given by

$$\mathcal{E}_{4,\bar{q}'}^0 = +\frac{1}{\epsilon^3}\left[-1 + \frac{1}{x_1} + \frac{x_1}{2}\right] + \frac{1}{\epsilon^2}\left[-\frac{7}{2} + \frac{55}{12x_1} + \frac{5}{16}x_1 - \frac{1}{3}x_1^2 - \frac{3}{4}H(0,x_1)\right.$$

$$\left. + \frac{2}{x_1}H(0,x_1) + \frac{3}{2}x_1 H(0,x_1) - 2H(1,x_1) + \frac{2}{x_1}H(1,x_1) + x_1 H(1,x_1)\right] + \mathcal{O}(\epsilon^{-1}),$$

$$(4.7.144)$$

where

$$H(0,z) = \ln(z),$$

maple	fortran/form	latex	comment
`calA40FF`	`A40FFint`	$\mathcal{A}_4^0$	Eq. (5.31) of [109].
`calAt40FF`	`At40FFint`	$\tilde{\mathcal{A}}_4^0$	Eq. (5.32) of [109].
`calB40FF`	`B40FFint`	$\mathcal{B}_4^0$	Eq. (5.39) of [109].
`calC40FF`	`C40FFint`	$\mathcal{C}_4^0$	Eq. (5.44) of [109].
`calD40FF`	`D40FFint`	$\mathcal{D}_4^0$	Eq. (6.45) of [109].
`calE40FF`	`E40FFint`	$\mathcal{E}_4^0$	Eq. (6.51) of [109].
`calEt40FF`	`Et40FFint`	$\tilde{\mathcal{E}}_4^0$	Eq. (6.52) of [109].
`calF40FF`	`F40FFint`	$\mathcal{F}_4^0$	Eq. (7.45) of [109].
`calG40FF`	`G40FFint`	$\mathcal{G}_4^0$	Eq. (7.52) of [109].
`calGt40FF`	`Gt40FFint`	$\tilde{\mathcal{G}}_4^0$	Eq. (7.53) of [109].
`calH40FF`	`H40FFint`	$\mathcal{H}_4^0$	Eq. (7.59) of [109].

Table 4.7: $\mathcal{X}_4^0$ antenna functions for final-final state

$$H(1, z) = -\ln(1 - z). \tag{4.7.145}$$

The functions $\mathcal{E}_{4,\bar{q}'g}^0$ and $\mathcal{E}_{4,q\bar{q}'}^0$ were also omitted in the original work [136], and leading singularities are given by,

$$\mathcal{E}_{4,\bar{q}'g}^0 = + \frac{1}{\epsilon^2}\left[-\frac{1}{2} + \frac{1}{2x_1} - \frac{x_2}{x_1} + \frac{x_2^2}{x_1} + x_2 - x_2^2 + \frac{1}{4}x_1 - \frac{1}{2}x_1x_2 + \frac{1}{2}x_1x_2^2 \right]$$

$$+ \mathcal{O}(\epsilon^{-1}), \tag{4.7.146}$$

$$\mathcal{E}_{4,q\bar{q}'}^0 = + \frac{1}{\epsilon^2}\left[-\frac{1}{2} + \frac{1}{x_2(1 + x_1)} - \frac{2}{x_2(x_1 + x_2)} + \frac{1}{2x_2} + \frac{1}{1 + x_1} + \frac{2}{(x_1 + x_2)^2} \right.$$

$$+ \frac{x_2}{2(1 + x_1)} - \frac{x_2}{x_1 + x_2} + \frac{x_2}{4} - \frac{x_2^2}{(x_1 + x_2)^2} + \frac{x_2^3}{x_1 + x_2} - \frac{x_2^4}{(x_1 + x_2)^2}$$

$$\left. + \frac{x_1}{2x_2} - \frac{x_1}{2} + \frac{x_1x_2}{4} \right] + \mathcal{O}(\epsilon^{-1}). \tag{4.7.147}$$

The expressions in Eqs. (4.7.144), (4.7.146) and (4.7.147) reveal that the secondary quark pair in $E_4^0(q, q', \bar{q}', g)$ antenna has no exchange symmetry and the anti-quark $\bar{q}'$ is colour connected to the gluon g. The integrated $\mathcal{E}_4^0$ functions then have four different crossing combinations in the initial-final double unresolved sub-phase space, and have six different crossing combinations in the initial-initial double unresolved

sub-phase space.

Similarly, all $\mathcal{X}_3^1$ functions are summarised in tables 4.11, 4.12 and 4.13.

4.7.2 Antenna subtraction terms $\mathrm{d}\hat{\sigma}_{NNLO}^U$

The double virtual contribution from two-loop matrix can be written as,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{VV} =& \mathcal{N}_{NNLO}^{VV} \mathrm{d}\Phi_n(p_3,\cdots,p_{n+2};p_1,p_2)\frac{1}{s_n}|M_{n+2}^2(\cdots,I,K,\cdots)|^2 \\
& \times J_n^{(n)}(\{p\}_n).
\end{aligned}
\tag{4.7.148}
$$

The jet-function ensures that there is no implicit divergence from unresolved regions of phase space. Nevertheless, the explicit divergences from the pole structure of the two-loop matrix elements and mass factorization terms $(\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2})$ need to be regulated. In the framework of the antenna subtraction method, the integrated double unresolved contributions from the double real subtraction term and the integrated single-unresolved contributions from the real-virtual subtraction term are added back at the double virtual level $(\mathrm{d}\hat{\sigma}_{NNLO}^U)$ such that all counter terms cancel each other at the level of the full NNLO cross section. The integrations of these antenna subtraction terms are carried out in $d-$dimensions and the divergence can be expressed explicitly as a Laurent expansion of the small parameter ϵ. Those explicit divergences cancel with the explicit pole structure from $\mathrm{d}\hat{\sigma}_{NNLO}^{VV}$ such that the double virtual contribution $\mathrm{d}\hat{\sigma}_{NNLO}^{VV} - \mathrm{d}\hat{\sigma}_{NNLO}^U$ is well defined in 4 dimensional phase space integral. The $\mathrm{d}\hat{\sigma}_{NNLO}^U$ term is defined as

$$
\mathrm{d}\hat{\sigma}_{NNLO}^U = -\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS} - \int_2 \mathrm{d}\hat{\sigma}_{NNLO}^{S,2} - \mathrm{d}\hat{\sigma}_{NNLO}^{MF,2},
\tag{4.7.149}
$$

and can be decomposed into three type of contributions:

$$
\mathrm{d}\hat{\sigma}_{NNLO}^U \equiv \mathrm{d}\hat{\sigma}_{NNLO}^{U,A} + \mathrm{d}\hat{\sigma}_{NNLO}^{U,B} + \mathrm{d}\hat{\sigma}_{NNLO}^{U,C}.
\tag{4.7.150}
$$

$\mathrm{d}\hat{\sigma}_{NNLO}^{U,A}$ subtraction term

The $\mathrm{d}\hat{\sigma}_{NNLO}^{U,A}$ term originates from integrated terms from real-virtual subtraction terms ($\mathrm{d}\hat{\sigma}_{NNLO}^{VS;a}$) in Eq. (4.5.108) (the second term), mass factorization terms $(\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2,A})$ in Eq. (4.6.137) and the integrated rescale term proportional to -1 in

maple	fortran/form	latex	comment
calgA40IF	gA40IFint	$\mathcal{A}^0_{4,g}$	gA40. Eq. (6.2) of [134].
calgAt40IF	gAt40IFint	$\tilde{\mathcal{A}}^0_{4,g}$	gAt40. Eq. (6.3) of [134].
calgD402IF	gD402IFint	$\mathcal{D}^0_{4,g'}$	gD402. Eq. (6.9) of [134].
calgD40IF	gD40IFint	$\mathcal{D}^0_{4,g}$	gD40. Eq. (6.8) of [134].
calgE40IF	gE40IFint	$\mathcal{E}^0_{4,g}$	gE40. Eq. (6.10) of [134].
calgEt40IF	gEt40IFint	$\tilde{\mathcal{E}}^0_{4,g}$	gEt40. Eq. (6.11) of [134].
calgF40IF	gF40IFint	$\mathcal{F}^0_{4,g}$	gF40. Eq. (6.16) of [134].
calgG40IF	gG40IFint	$\mathcal{G}^0_{4,g}$	gG40. Eq. (6.17) of [134].
calgGt40IF	gGt40IFint	$\tilde{\mathcal{G}}^0_{4,g}$	gGt40. Eq. (6.18) of [134].
calqA40IF	qA40IFint	$\mathcal{A}^0_{4,q}$	qA40. Eq. (5.2) of [134].
calqAt40IF	qAt40IFint	$\tilde{\mathcal{A}}^0_{4,q}$	qAt40. Eq. (5.3) of [134].
calqB40IF	qB40IFint	$\mathcal{B}^0_{4,q}$	qB40. Eq. (5.4) of [134].
calqC40IF	qC40IFint	$\mathcal{C}^0_{4,q}$	qC40a. Eq. (5.6) of [134].
calqD40IF	qD40IFint	$\mathcal{D}^0_{4,q}$	qD40. Eq. (5.15) of [134].
calqE40IF	qE40IFint	$\mathcal{E}^0_{4,q}$	qE40. Eq. (5.16) of [134].
calqEt40IF	qEt40IFint	$\tilde{\mathcal{E}}^0_{4,q}$	qEt40. Eq. (5.17) of [134].
calqG40IF	qG40IFint	$\mathcal{G}^0_{4,q}$	qG40. Eq. (5.29) of [134].
calqGt40IF	qGt40IFint	$\tilde{\mathcal{G}}^0_{4,q}$	qGt40. Eq. (5.30) of [134].
calqH40IF	qH40IFint	$\mathcal{H}^0_{4,q}$	qH40. Eq. (5.31) of [134].
calqbC40IF	qbC40IFint	$\mathcal{C}^0_{4,\bar{q},\bar{q}q\bar{q}}$	qC40b. Eq. (5.7) of [134].
calqbbC40IF	qbbC40IFint	$\mathcal{C}^0_{4,\bar{q},q\bar{q}\bar{q}}$	qC40c. Eq. (5.8) of [134].
calqbpE40IF	qbpE40IFint	$\mathcal{E}^0_{4,\bar{q}'}$	qbpE40. New in Eq. (4.7.144)
calqpB40IF	qpB40IFint	$\mathcal{B}^0_{4,q'}$	qpB40. Eq. (5.5) of [134].
calqpE40IF	qpE40IFint	$\mathcal{E}^0_{4,q'}$	qpE40. Eq. (5.18) of [134].
calqpEt40IF	qpEt40IFint	$\tilde{\mathcal{E}}^0_{4,q'}$	qpEt40. Eq. (5.19) of [134].

Table 4.8: $\mathcal{X}^0_4$ antenna functions for initial-final state

maple	fortran/form	latex	comment
`calggA40II`	`ggA40IIint`	$\mathcal{A}^0_{4,gg}$	A34. Eq. (5.4) of [136].
`calggAt40II`	`ggAt40IIint`	$\tilde{\mathcal{A}}^0_{4,gg}$	At34. Eq. (5.7) of [136].
`calggD40II`	`ggD40IIint`	$\mathcal{D}^0_{4,gg}$	D23. Eq. (5.17) of [136].
`calggD40nadjII`	`ggD40nadjIIint`	$\mathcal{D}^{0,nadj}_{4,gg}$	D24. Eq. (5.18) of [136].
`calggF40II`	`ggF40IIint`	$\mathcal{F}^0_{4,gg}$	F12. Eq. (5.27) of [136].
`calggF40nadjII`	`ggF40nadjIIint`	$\mathcal{F}^{0,nadj}_{4,gg}$	F13. Eq. (5.28) of [136].
`calggG40II`	`ggG40IIint`	$\mathcal{G}^0_{4,gg}$	G12. Eq. (5.29) of [136].
`calggGt40II`	`ggGt40IIint`	$\tilde{\mathcal{G}}^0_{4,gg}$	Gt12. Eq. (5.33) of [136].
`calgqG40II`	`gqG40IIint`	$\mathcal{G}^0_{4,gq}$	G13. Eq. (5.30) of [136].
`calgqG40nadjII`	`gqG40nadjIIint`	$\mathcal{G}^{0,nadj}_{4,gq}$	G14. Eq. (5.31) of [136].
`calgqGt40II`	`gqGt40IIint`	$\tilde{\mathcal{G}}^0_{4,gq}$	Gt13. Eq. (5.34) of [136].
`calqbpgE40II`	`qbpgE40IIint`	$\mathcal{E}^0_{4,\bar{q}'g}$	E34. New in Eq. (4.7.146)
`calqgA40II`	`qgA40IIint`	$\mathcal{A}^0_{4,qg}$	A13. Eq. (5.2) of [136].
`calqgA40nadjII`	`qgA40nadjIIint`	$\mathcal{A}^{0,nadj}_{4,qg}$	A14. Eq. (5.3) of [136].
`calqgAt40II`	`qgAt40IIint`	$\tilde{\mathcal{A}}^0_{4,qg}$	At13. Eq. (5.6) of [136].
`calqgD40II`	`qgD40IIint`	$\mathcal{D}^0_{4,qg}$	D12. Eq. (5.15) of [136].
`calqgD40nadjII`	`qgD40nadjIIint`	$\mathcal{D}^{0,nadj}_{4,qg}$	D13. Eq. (5.16) of [136].
`calqgE40II`	`qgE40IIint`	$\mathcal{E}^0_{4,qg}$	E14. Eq. (5.20) of [136].
`calqgEt40II`	`qgEt40IIint`	$\tilde{\mathcal{E}}^0_{4,qg}$	Et14. Eq. (6.68) of [137].
`calqpgE40II`	`qpgE40IIint`	$\mathcal{E}^0_{4,q'g}$	E24. Eq. (5.22) of [136].
`calqpgEt40II`	`qpgEt40IIint`	$\tilde{\mathcal{E}}^0_{4,q'g}$	Et24. Eq. (5.26) of [136].

Table 4.9: gluon-gluon, quark-gluon and gluon-quark initiated $\mathcal{X}^0_4$ antenna functions

maple	fortran/form	latex	comment
calqpqpB40II	qpqpB40IIint	$\mathcal{B}^0_{4,q'\bar{q}'}$	B34. Eq. (5.10) of [136].
calqpqpC40II	qpqpC40IIint	$\mathcal{C}^0_{4,q'\bar{q}'}$	C34. Eq. (5.14) of [136].
calqpqpE40II	qpqpE40IIint	$\mathcal{E}^0_{4,q'\bar{q}'}$	E23. Eq. (5.21) of [136].
calqpqpEt40II	qpqpEt40IIint	$\tilde{\mathcal{E}}^0_{4,q'\bar{q}'}$	Et23. Eq. (6.69) of [137].
calqqA40II	qqA40IIint	$\mathcal{A}^0_{4,q\bar{q}}$	A12. Eq. (5.1) of [136].
calqqAt40II	qqAt40IIint	$\tilde{\mathcal{A}}^0_{4,q\bar{q}}$	At12. Eq. (5.5) of [136].
calqqB40II	qqB40IIint	$\mathcal{B}^0_{4,q\bar{q}}$	B12. Eq. (6.71) of [137].
calqqC401II	qqC401IIint	$\mathcal{C}^0_{4,q\dot{q}}$	C13. Eq. (5.12) of [136].
calqqC402II	qqC402IIint	$\mathcal{C}^0_{4,\dot{q}q}$	Eq. (5.12) of [136] with $x_1 \leftrightarrow x_2$.
calqqG40II	qqG40IIint	$\mathcal{G}^0_{4,q\bar{q}}$	G34. Eq. (5.32) of [136].
calqqGt40II	qqGt40IIint	$\tilde{\mathcal{G}}^0_{4,q\bar{q}}$	Gt34. Eq. (5.35) of [136].
calqqH40II	qqH40IIint	$\mathcal{H}^0_{4,q\bar{q}}$	H12. Eq. (6.65) of [137].
calqqbC40II	qqbC40IIint	$\mathcal{C}^0_{4,q\bar{q}}$	C14 = C12. Eq. (5.11) of [136].
calqqbpE40II	qqbpE40IIint	$\mathcal{E}^0_{4,q\bar{q}'}$	E13. New in Eq. (4.7.147).
calqqpB40II	qqpB40IIint	$\mathcal{B}^0_{4,qq'}$	B13. Eq. (6.72) of [137].
calqqpE40II	qqpE40IIint	$\mathcal{E}^0_{4,qq'}$	E12. Eq. (5.19) of [136].
calqqpEt40II	qqpEt40IIint	$\tilde{\mathcal{E}}^0_{4,qq'}$	Et12. Eq. (6.67) of [137].
calqqpH40II	qqpH40IIint	$\mathcal{H}^0_{4,qq'}$	H13. Eq. (6.66) of [137].

Table 4.10: quark-quark initiated $\mathcal{X}^0_4$ antenna functions

maple	fortran/form	latex	comment
calA31FF	A31FFint	$\mathcal{A}_3^1$	Eq. (5.18) of [109].
calAh31FF	Ah31FFint	$\hat{\mathcal{A}}_3^1$	Eq. (5.20) of [109].
calAt31FF	At31FFint	$\tilde{\mathcal{A}}_3^1$	Eq. (5.19) of [109].
calD31FF	D31FFint	$\mathcal{D}_3^1$	Eq. (6.22) of [109].
calDh31FF	Dh31FFint	$\hat{\mathcal{D}}_3^1$	Eq. (6.23) of [109].
calE31FF	E31FFint	$\mathcal{E}_3^1$	Eq. (6.34) of [109].
calEh31FF	Eh31FFint	$\hat{\mathcal{E}}_3^1$	Eq. (6.36) of [109].
calEt31FF	Et31FFint	$\tilde{\mathcal{E}}_3^1$	Eq. (6.35) of [109].
calF31FF	F31FFint	$\mathcal{F}_3^1$	Eq. (7.22) of [109].
calFh31FF	Fh31FFint	$\hat{\mathcal{F}}_3^1$	Eq. (7.23) of [109].
calG31FF	G31FFint	$\mathcal{G}_3^1$	Eq. (7.34) of [109].
calGh31FF	Gh31FFint	$\hat{\mathcal{G}}_3^1$	Eq. (7.36) of [109].
calGt31FF	Gt31FFint	$\tilde{\mathcal{G}}_3^1$	Eq. (7.35) of [109].

Table 4.11: $\mathcal{X}_3^1$ antenna functions for final-final state

maple	fortran/form	latex	comment
`calgA31IF`	`gA31IFint`	$\mathcal{A}^1_{3,g}$	Eq. (6.4) of [134].
`calgAh31IF`	`gAh31IFint`	$\hat{\mathcal{A}}^1_{3,g}$	Eq. (6.6) of [134].
`calgAt31IF`	`gAt31IFint`	$\tilde{\mathcal{A}}^1_{3,g}$	Eq. (6.5) of [134].
`calgD31IF`	`gD31IFint`	$\mathcal{D}^1_{3,g}$	Eq. (6.12) of [134].
`calgDh31IF`	`gDh31IFint`	$\hat{\mathcal{D}}^1_{3,g}$	Eq. (6.13) of [134].
`calgF31IF`	`gF31IFint`	$\mathcal{F}^1_{3,g}$	Eq. (6.19) of [134].
`calgFh31IF`	`gFh31IFint`	$\hat{\mathcal{F}}^1_{3,g}$	Eq. (6.20) of [134].
`calgG31IF`	`gG31IFint`	$\mathcal{G}^1_{3,g}$	Eq. (6.21) of [134].
`calgGh31IF`	`gGh31IFint`	$\hat{\mathcal{G}}^1_{3,g}$	Eq. (6.23) of [134].
`calgGt31IF`	`gGt31IFint`	$\tilde{\mathcal{G}}^1_{3,g}$	Eq. (6.22) of [134].
`calqA31IF`	`qA31IFint`	$\mathcal{A}^1_{3,q}$	Eq. (5.9) of [134].
`calqAh31IF`	`qAh31IFint`	$\hat{\mathcal{A}}^1_{3,q}$	Eq. (5.11) of [134].
`calqAt31IF`	`qAt31IFint`	$\tilde{\mathcal{A}}^1_{3,q}$	Eq. (5.10) of [134].
`calqD31IF`	`qD31IFint`	$\mathcal{D}^1_{3,q}$	Eq. (5.20) of [134].
`calqDh31IF`	`qDh31IFint`	$\hat{\mathcal{D}}^1_{3,q}$	Eq. (5.21) of [134].
`calqE31IF`	`qE31IFint`	$\mathcal{E}^1_{3,q}$	Eq. (5.22) of [134].
`calqEh31IF`	`qEh31IFint`	$\hat{\mathcal{E}}^1_{3,q}$	Eq. (5.24) of [134].
`calqEt31IF`	`qEt31IFint`	$\tilde{\mathcal{E}}^1_{3,q}$	Eq. (5.23) of [134].
`calqpE31IF`	`qpE31IFint`	$\mathcal{E}^1_{3,q'}$	Eq. (5.25) of [134].
`calqpEh31IF`	`qpEh31IFint`	$\hat{\mathcal{E}}^1_{3,q'}$	Eq. (5.27) of [134].
`calqpEt31IF`	`qpEt31IFint`	$\tilde{\mathcal{E}}^1_{3,q'}$	Eq. (5.26) of [134].
`calqpG31IF`	`qpG31IFint`	$\mathcal{G}^1_{3,q'}$	Eq. (5.32) of [134].
`calqpGh31IF`	`qpGh31IFint`	$\hat{\mathcal{G}}^1_{3,q'}$	Eq. (5.34) of [134].
`calqpGt31IF`	`qpGt31IFint`	$\tilde{\mathcal{G}}^1_{3,q'}$	Eq. (5.33) of [134].

Table 4.12: $\mathcal{X}^1_3$ antenna functions for final-initial state

maple	fortran/form	latex	comment
calggD31II	ggD31IIint	$\mathcal{D}^1_{3,gg}$	ggD31
calggDh31II	ggDh31IIint	$\hat{\mathcal{D}}^1_{3,gg}$	ggDh31
calggF31II	ggF31IIint	$\mathcal{F}^1_{3,gg}$	ggF31
calggFh31II	ggFh31IIint	$\hat{\mathcal{F}}^1_{3,gg}$	ggFh31
calgqG31II	gqG31IIint	$\mathcal{G}^1_{3,gq}$	gqG31
calgqGh31II	gqGh31IIint	$\hat{\mathcal{G}}^1_{3,gq}$	gqGh31
calgqGt31II	gqGt31IIint	$\tilde{\mathcal{G}}^1_{3,gq}$	gqGt31
calqgA31II	qgA31IIint	$\mathcal{A}^1_{3,qg}$	qgA31
calqgAh31II	qgAh31IIint	$\hat{\mathcal{A}}^1_{3,qg}$	qgAh31
calqgAt31II	qgAt31IIint	$\tilde{\mathcal{A}}^1_{3,qg}$	qgAt31
calqgD31II	qgD31IIint	$\mathcal{D}^1_{3,qg}$	qgD31. Eq. (B.1) of [138].
calqgDh31II	qgDh31IIint	$\hat{\mathcal{D}}^1_{3,qg}$	qgDh31
calqgG31II	qgG31IIint	$\mathcal{G}^1_{3,qg}$	qgG31
calqgGh31II	qgGh31IIint	$\hat{\mathcal{G}}^1_{3,qg}$	qgGh31
calqgGt31II	qgGt31IIint	$\tilde{\mathcal{G}}^1_{3,qg}$	qgGt31
calqpqE31II	qpqE31IIint	$\mathcal{E}^1_{3,q'q}$	qpqE31
calqpqpE31II	qpqpE31IIint	$\mathcal{E}^1_{3,q'q'}$	qpqpE31
calqpqpEh31II	qpqpEh31IIint	$\hat{\mathcal{E}}^1_{3,q'q'}$	qpqpEh31
calqpqpEt31II	qpqpEt31IIint	$\tilde{\mathcal{E}}^1_{3,q'q'}$	qpqpEt31
calqqA31II	qqA31IIint	$\mathcal{A}^1_{3,qq}$	qqA31
calqqAh31II	qqAh31IIint	$\hat{\mathcal{A}}^1_{3,qq}$	qqAh31
calqqAt31II	qqAt31IIint	$\tilde{\mathcal{A}}^1_{3,qq}$	qqAt31
calqqG31II	qqG31IIint	$\mathcal{G}^1_{3,qq}$	qqG31
calqqGh31II	qqGh31IIint	$\hat{\mathcal{G}}^1_{3,qq}$	qqGh31
calqqGt31II	qqGt31IIint	$\tilde{\mathcal{G}}^1_{3,qq}$	qqGt31
calqqpE31II	qqpE31IIint	$\mathcal{E}^1_{3,qq'}$	qqpE31
calqqpEh31II	qqpEh31IIint	$\hat{\mathcal{E}}^1_{3,qq'}$	qqpEh31
calqqpEt31II	qqpEt31IIint	$\tilde{\mathcal{E}}^1_{3,qq'}$	qqpEt31

Table 4.13: $\mathcal{X}^1_3$ antenna functions for initial-initial state

Eq. (4.5.89). From the definition of $\boldsymbol{J}_n^{(1)}$ in Eq. (3.5.74), the $\mathrm{d}\hat{\sigma}_{NNLO}^{U,A}$ term can be organised as

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{U,A} = & -\mathcal{N}_{NNLO}^{VV} \mathrm{d}\Phi_n \frac{1}{s_n} \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \boldsymbol{J}_{n+2}^{(1)}(\cdots,i,j,\cdots) \\
& \times \left(|M_{n+2}^1(\cdots,i,j,\cdots)|^2 - \frac{\beta_0}{\epsilon} |M_{n+2}^0(\cdots,i,j,\cdots)|^2 \right) J_n^{(n)}(\{p\}_n).
\end{aligned}
$$

(4.7.151)

From the pole structure of dipole functions of $\boldsymbol{J}_2^{(1)}$, the explicit pole structure of Eq. (4.7.151) matches the first line of the pole structure in two-loop matrix elements in Eq. (4.6.124).

$\mathrm{d}\hat{\sigma}_{NNLO}^{U,B}$ subtraction term

The $\mathrm{d}\hat{\sigma}_{NNLO}^{U,B}$ term is the collection of integrated $\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS,b}$ coming from Eq. (4.5.111), $\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS,c_2}$ from (4.5.116), $\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS,d}$, $\int_2 \mathrm{d}\hat{\sigma}_{NNLO}^{S,d}$ from (4.3.77) and the mass factorization term $\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2,B}$ in Eq. (4.6.137). By using the dipole functions $\boldsymbol{J}_n^{(1)}$ and the convolution operation, the $\mathrm{d}\hat{\sigma}_{NNLO}^{U,B}$ term can be organised as

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{NNLO}^{U,B} = & -\mathcal{N}_{NNLO}^{VV} \mathrm{d}\Phi_n \frac{1}{s_n} \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \\
& \times \frac{1}{2} \boldsymbol{J}_{n+2}^{(1)}(\cdots,i,j,\cdots) \otimes \boldsymbol{J}_{n+2}^{(1)}(\cdots,i,j,\cdots) \\
& \times |M_{n+2}^0(\cdots,i,j,\cdots)|^2 J_n^{(n)}(\{p\}_n).
\end{aligned}
$$

(4.7.152)

It can be seen that the explicit pole structure in Eq. (4.7.152) matches the second line of the two-loop pole structure given in Eq. (4.6.124).

$\mathrm{d}\hat{\sigma}_{NNLO}^{U,C}$ subtraction term

The last double virtual antenna subtraction term collects all the remaining contribution from integrated antenna subtraction terms from $\mathrm{d}\hat{\sigma}_{NNLO}^S$, $\mathrm{d}\hat{\sigma}_{NNLO}^T$ and double virtual mass factorization term $\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2}$. More precisely, $\mathrm{d}\hat{\sigma}_{NNLO}^{U,C}$ contains $\int_2 \mathrm{d}\hat{\sigma}_{NNLO}^{S,b_1}$ from (4.3.71), $\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS,a}$ from (4.5.108) (the first term), $\int_1 \mathrm{d}\hat{\sigma}_{NNLO}^{VS,c_1}$ from (4.5.115), the integrated rescaling term proportional to $|s_{ijk}|/\mu_R^2$ in Eq. (4.5.89) and the mass factorization term $\mathrm{d}\hat{\sigma}_{NNLO}^{MF,2,C}$ in Eq. (4.6.137). All terms in $\mathrm{d}\hat{\sigma}_{NNLO}^{U,C}$ are proportional to the tree level matrix elements and can be arranged into dipole functions

so that

$$
\mathrm{d}\hat{\sigma}^{U,C}_{NNLO} = -\mathcal{N}^{VV}_{NNLO}\mathrm{d}\Phi_n \frac{1}{s_n}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}
$$
$$
\times \boldsymbol{J}^{(2)}_{n+2}(\cdots,i,j,\cdots)|M^0_{n+2}(\cdots,i,j,\cdots)|^2 J^{(n)}_n(\{p\}_n), \qquad (4.7.153)
$$

where $\boldsymbol{J}^{(2)}_{n+2}$ is the sum of two-loop dipole functions $\boldsymbol{J}^{(2)}_2$ (analogy to one-loop dipole functions $\boldsymbol{J}^{(1)}_2$),

$$
\boldsymbol{J}^{(2)}_{n+2}(\cdots,i,j,\cdots) = \sum_{i,j}\boldsymbol{J}^{(2)}_2(i,j). \qquad (4.7.154)
$$

Similar to the NLO $\boldsymbol{J}^{(1)}_2$ functions, once determined, the $\boldsymbol{J}^{(2)}_2$ functions with various parton types and colour orderings would match the corresponding explicit poles in the last two lines of dipole structures for two-loop matrix elements in Eq. (4.6.124) such that,

$$
\boldsymbol{J}^{(2)}_2(i,j) \xrightarrow{pole} 2e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\mathbf{I}^{(1)}_{ij}(2\epsilon,s_{ij})+2\left(\mathbf{H}^{(2)}_i(\epsilon)+\mathbf{H}^{(2)}_j(\epsilon)\right) \quad (4.7.155)
$$

The $\boldsymbol{J}^{(2)}_2$ functions are also found to be universal for different scattering processes from the studies of proton proton collisions to Higgs boson plus jet [84], vector boson plus jet and di-jets at NNLO [102, 116, 117]. Details for specific $\boldsymbol{J}^{(2)}_2$ dipole functions will be introduced in chapters 6, 7, 8 and 9.

Now all the three types of double virtual subtraction terms in Eq. (4.7.150) have been introduced. The full $\mathrm{d}\hat{\sigma}^U_{NNLO}$ term has the general structure as the following [116]:

$$
\mathrm{d}\hat{\sigma}^U_{NNLO} = -\mathcal{N}^{VV}_{NNLO}\mathrm{d}\Phi_n \frac{1}{s_n}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\times
$$
$$
\left\{\boldsymbol{J}^{(1)}_{n+2}(\cdots,i,j,\cdots)\left(|M^1_{n+2}(\cdots,i,j,\cdots)|^2 - \frac{\beta_0}{\epsilon}|M^0_{n+2}(\cdots,i,j,\cdots)|^2\right)\right.
$$
$$
+ \frac{1}{2}\boldsymbol{J}^{(1)}_{n+2}(\cdots,i,j,\cdots)\otimes\boldsymbol{J}^{(1)}_{n+2}(\cdots,i,j,\cdots)|M^0_{n+2}(\cdots,i,j,\cdots)|^2
$$
$$
\left.+ \boldsymbol{J}^{(2)}_{n+2}(\cdots,i,j,\cdots)|M^0_{n+2}(\cdots,i,j,\cdots)|^2\right\}J^{(n)}_n(\{p\}_n). \qquad (4.7.156)
$$

The explicit IR divergences in Eq. (4.7.156) is exactly the same as in Eq. (4.7.148). The difference of $\mathrm{d}\hat{\sigma}^{VV}_{NNLO} - \mathrm{d}\hat{\sigma}^U_{NNLO}$ is free from IR divergences. As we simply combine the integrated antenna subtraction terms from $\mathrm{d}\hat{\sigma}^S_{NNLO}$, $\mathrm{d}\hat{\sigma}^T_{NNLO}$ and $\mathrm{d}\hat{\sigma}^{MF,2}_{NNLO}$ to form $\mathrm{d}\hat{\sigma}^U_{NNLO}$, at exclusive cross section level, we introduce no unphysical counter terms.

Chapter 5

Matrix Elements for Higgs Boson Production in Association with a Jet at up to NNLO

The scattering matrix elements relevant for the NNLO corrections to the $pp \rightarrow H+$ jet process are are Higgs boson plus three, four and five partons at tree-level, Higgs boson plus three and four partons at one-loop, and Higgs boson plus three partons at two-loop. In the formalism of colour ordered matrix elements, specific examples have already been introduced in section 3.1.3 and 4.1.2. Helicity amplitudes for these matrix elements have already been calculated by different groups. However, these expressions have not been tested for numerical stability in the unresolved phase regions relevant for NNLO calculations.

In this chapter, we first introduce the modern approaches for calculating tree and one-loop helicity amplitudes. New (and stable) results for tree-level matrix elements for Higgs boson plus five partons are obtained using the BCFW method for the first time. Then, the numerical instability issues for the one-loop Higgs boson plus four parton matrix elements are exposed and subsequently solved by rewriting the analytical expressions to avoid cancellations of divergent terms in the single soft phase space regions.

5.1   Tree-level Higgs boson plus multi-parton matrix elements

In the large top quark mass limit, the tree-level matrix elements for Higgs boson plus multiple light-like partons are constructed using the Feynman rules for gluon and (massless) quark interactions and self-interactions, supplemented by the effective couplings of gluons with the Higgs boson. Traditionally, Feynman rules are used for calculating scattering matrix elements in non-abelian gauge theories. However, with the increasing of number of external particles, the number of Feynman diagrams and kinematic variables for each diagram increase rapidly. Furthermore, when summing all contributions of Feynman diagrams, the results for each diagram also experience large internal cancellations. Modern approaches for calculating tree-level scattering matrix element treat the growth of multiplicities of external particles recursively. Instead of calculating a completely new set of Feynman diagrams for each scattering process, previously constructed (and therefore on-shell) helicity amplitudes with small number of external particles are joined together to calculate amplitudes with higher multiplicities. These modern on-shell approaches include Parke-Taylor helicity amplitudes [139], the CSW method [140], the BCFW method [141] and the CHY method [142]. Here we compute the matrix elements for Higgs plus up to five partons for the first time using the BCFW method. Compared to the previous results calculated from Feynman rules, the modern approaches produce numerically equivalent results while having much simplified expressions and much less internal cancellations.

In this sub-section, we first introduce the general idea of the BCFW method, before introducing the effective field theory used to describe the Higgs coupling with gluons. Finally, we give explicit results for Higgs boson plus up to five parton tree-level helicity amplitudes.

5.1.1 The BCFW method

Spinors and spinor products

Modern approaches for calculating tree and one-loop helicity amplitudes largely rely on the spinor helicity formalism. The spinor products, as a new set of kinematic objects, have nonlinear properties to help simplifying the expressions and also concisely illustrate the divergent behaviour of helicity amplitudes. Following the notation in [105], each massless spinor of quark has two degrees of freedom and can be constructed as

$$
u_+(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p^+} \\ \sqrt{p^-}e^{i\psi_p} \\ \sqrt{p^+} \\ \sqrt{p^-}e^{i\psi_p} \end{pmatrix} \qquad u_-(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p^-}e^{-i\psi_p} \\ -\sqrt{p^+} \\ -\sqrt{p^-}e^{-i\psi_p} \\ \sqrt{p^+} \end{pmatrix}, \tag{5.1.1}
$$

where $p^\mu = (p^0, p^1, p^2, p^3)$ is the four dimensional momentum with energy p^0 and

$$
e^{\pm i\psi_p} \equiv \frac{p^1 \pm ip^2}{\sqrt{(p^1)^2 + (p^2)^2}} = \frac{p^1 \pm ip^2}{\sqrt{p^+p^-}}, \qquad p^\pm = p^0 \pm p^3. \tag{5.1.2}
$$

In a shorthand notation we define

$$
|i^\pm\rangle \equiv |p_i^\pm\rangle \equiv u_\pm(p_i), \qquad \langle i^\pm| \equiv \langle p_i^\pm| \equiv \overline{u}_\pm(p_i), \tag{5.1.3}
$$

where $\overline{u}_\pm(p_i)$ is complex conjugate of $u_\pm(p_i)$. The spinor products can then be defined as

$$
\langle ij\rangle \equiv \langle i^-|j^+\rangle = \overline{u}_-(p_i)u_+(p_j) = \sqrt{|s_{ij}|}e^{i\phi_{ij}}, \tag{5.1.4}
$$

$$
[ij] \equiv \langle i^+|j^-\rangle = \overline{u}_+(p_i)u_-(p_j) = \sqrt{|s_{ij}|}e^{-i(\phi_{ij}+\pi)}, \tag{5.1.5}
$$

where ϕ_{ij} is a phase factor which depends on our choice of phase in Eq. (5.1.1). The physical Lorentz invariants are free from the phase factor choice so that

$$
\langle ij\rangle[ji] = \langle i^-|j^+\rangle\langle j^+|i^-\rangle = 2k_i \cdot k_j = s_{ij}. \tag{5.1.6}
$$

Equation (5.1.6), together with the Gordon identity

$$
\langle i^\pm|\gamma^\mu|i^\pm\rangle = 2k_i^\mu, \tag{5.1.7}
$$

and the Fierz rearrangement

$$\langle i^+|\gamma^\mu|j^+\rangle\langle k^+|\gamma_\mu|l^+\rangle = 2[ik]\langle lj\rangle \tag{5.1.8}$$

are useful in BCFW method.

The polarization vector of a massless gauge boson with momentum p_μ and definite helicity $\lambda = \pm$ can be defined in axial gauge with arbitrary light-like vector n_μ to be,

$$\varepsilon_\mu^\pm(p, n) = \pm\frac{\langle n^\mp|\gamma_\mu|p^\mp\rangle}{\sqrt{2}\langle n^\mp|p^\pm\rangle}. \tag{5.1.9}$$

From the gauge invariant condition, the choice of light-like vector n_μ does not change the form of scattering amplitude. It is also guaranteed by the Dirac equation that the polarization vector in eq. (5.1.9) is orthogonal to the momentum p_μ ($\varepsilon^\pm \cdot p = 0$). The propagator of the gauge boson is defined to be,

$$\Delta_{\mu\nu}(p) = \sum_{\lambda=\pm} \varepsilon_\mu^\lambda(p)\frac{i}{p^2 + i\epsilon}\varepsilon_\nu^{-\lambda}(p). \tag{5.1.10}$$

Using the definition in Eq. (5.1.9), one can check that

$$\sum_{\lambda=\pm}\varepsilon_\mu^\lambda(p, n)\frac{i\delta_{ab}}{p^2 + i\epsilon}(\varepsilon_\nu^\lambda(p, n))^* = -\frac{i\delta_{ab}}{p^2 + i\epsilon}\left[\eta^{\mu\nu} - \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n}\right], \tag{5.1.11}$$

which agrees with the propagator of a massless gauge boson in a light-like axial gauge given in Eq. (1.1.14). The polarization vector is normalised in 4-dimensional space time such that,

$$\sum_{\lambda=\pm}\eta^{\mu\nu}\varepsilon_\mu^\lambda(p, n)(\varepsilon_\nu^\lambda(p, n))^* = -2. \tag{5.1.12}$$

The BCFW method in a nutshell

The BCFW method exploits the fact that for a rational function $\mathcal{M}$ of a complex variable z which vanishes at infinity such that,

$$\lim_{z\to\infty}\frac{\mathcal{M}(z)}{z} = 0, \tag{5.1.13}$$

then,

$$\oint_C \frac{dz}{2\pi i}\frac{\mathcal{M}(z)}{z} = 0, \tag{5.1.14}$$
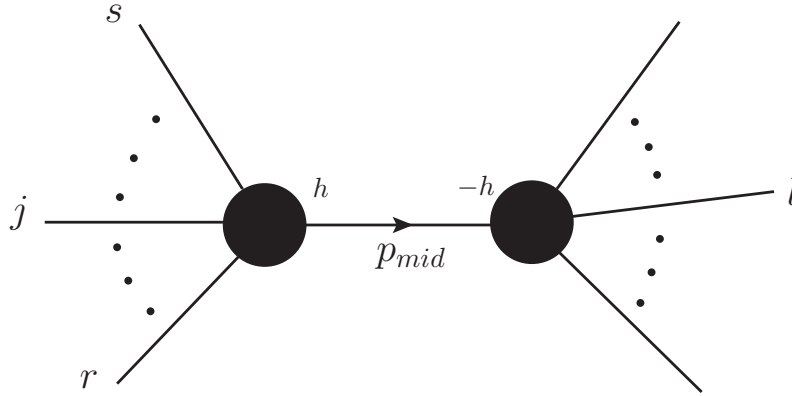
Figure 5.1: Pictorial representation of the separation of the amplitude into left and right partial amplitudes.

where C is the closed contour at infinity. The result of the contour integral is simply the sum of the residues which includes $\mathcal{M}(0)$. The BCFW method [141] is essentially a way of deforming the amplitude into a function of a complex variable z, whose value at $z = 0$ is the desired quantity. In doing so, this enables the recursive use of on-shell partial amplitudes to construct full amplitudes [141].

Let us consider a colour ordered tree-level amplitude $\mathcal{M}^0$ and separate it into left and right partial amplitudes $\mathcal{M}_L^{0,h}$ and $\mathcal{M}_R^{0,-h}$ which are connected by intermediate momentum p_{mid} with helicity h as shown in Fig. 5.1. From momentum conservation, the value of p_{mid} equals to the value of all external momenta in the left (or the right) partial amplitudes (depending on the direction choice of the intermediate momentum). Different values for p_{mid} reflect different separation choices of the full amplitude. Assuming the external momenta in the left partial amplitude are $p_r, p_{r+1}, \cdots, p_s$, and intermediate momentum coming out from the left partial amplitude, one has

$$p_{mid}^\mu = p_r^\mu + p_{r+1}^\mu + \cdots + p_s^\mu \equiv P_{r\cdots s}^\mu. \tag{5.1.15}$$

To deform the amplitude, we introduce the complex variable z by modifying the external states in a way that keeps the momentum conservation and on-shell conditions for the external particles. For example, one can apply the $[j, l\rangle$ shift that

$$\overline{u}(p_j) \to \overline{u}(p_j) - z\overline{u}(p_l), \qquad u(p_l) \to u(p_l) + zu(p_j), \tag{5.1.16}$$

where p_j is one of the momenta from the left partial amplitude ($j \in r, \cdots, s$) and p_l lies in the set of momenta in the right partial amplitude. From Gordon identity (equation (5.1.7)), the $[j, l\rangle$ shift also change the momentum that

$$p_j^\mu \to p_j^\mu(z) = p_j^\mu - \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle,$$

$$p_l^\mu \to p_l^\mu(z) = p_l^\mu + \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle, \tag{5.1.17}$$

while keeping all other external momentum unchanged. The overall momentum conservation is preserved under $[j, l\rangle$ shift, and the shifted momentums $p_j^\mu(z)$ and $p_j^\mu(z)$ are still on-shell (can be proved by using equation (5.1.8)). The most significant change is that p_{mid}^μ now depends on z,

$$p_{mid}^\mu \to p_{mid}^\mu(z) \equiv P_{r\cdots s}^\mu(z) = P_{r\cdots s}^\mu - \frac{z}{2} \langle j^- | \gamma^\mu | l^- \rangle. \tag{5.1.18}$$

One has successfully continued the full amplitude $\mathcal{M}^0$ onto $\mathcal{M}^0(z)$ which is defined in the complex plane with variable z. If $\mathcal{M}^0(z)/z \to 0$ when $z \to \infty$, the integral around the circle C at infinity vanishes as in Eq. (5.1.14). Using Cauchy's theorem, the countour integral can be replaced by the sum of the residues in the complex $z-$plane,

$$0 = \sum_{z = z_i} Res \left\{ \frac{\mathcal{M}^0(z)}{z} \right\}, \tag{5.1.19}$$

where z_i are singular points for function $\mathcal{M}^0(z)/z$ in the $z-$plane. One obvious singular point is at $z = 0$ which yields,

$$Res \left\{ \frac{\mathcal{M}^0(z)}{z} \right\} \bigg|_{z=0} = \mathcal{M}^0(0) \equiv \mathcal{M}^0. \tag{5.1.20}$$

This implies the full amplitude can be calculated by finding all the other residues of $\mathcal{M}^0(z)/z$ on $z-$plane,

$$\mathcal{M}^0 = - \sum_{z = z_i \neq 0} Res \left\{ \frac{\mathcal{M}^0(z)}{z} \right\}. \tag{5.1.21}$$

To find these residues, we can exploit our knowledge of propagators in Eqs. (1.1.14) and (5.1.10) to write the full amplitude in factorized form by joining two off-shell partial amplitudes ($\widetilde{\mathcal{M}}^0$) with a gauge boson propagator,

$$\mathcal{M}^0(z) = \sum_{p_{mid}} \widetilde{\mathcal{M}}_L^{0,h}(z) \frac{i}{p_{mid}^2(z)} \widetilde{\mathcal{M}}_R^{0,-h}(z). \tag{5.1.22}$$

Note that the two polarization vectors ε_μ^λ and $\varepsilon_\nu^{-\lambda}$ in Eq. (5.1.10) have been absorbed into the off-shell partial amplitudes, $\widetilde{\mathcal{M}}_L^{0,h}(z)$ and $\widetilde{\mathcal{M}}_R^{0,-h}(z)$. The sum over all intermediate momentum p_{mid} represents all possible divisions of the full amplitude that respect keeping one shifted momentum in the left hand set and one in the right hand set.

For $p_{mid}^2(z_i) = 0$, one finds a singular point z_i which occurs when the intermediate momentum is on-shell. This means when calculating the residue of $\mathcal{M}^0(z)/z$ at such z_i point, one can replace off-shell amplitudes $\widetilde{\mathcal{M}}_L^{0,h}$ and $\widetilde{\mathcal{M}}_R^{0,-h}$ with on-shell amplitudes. More precisely, by using on-shell momentum p_μ in eq.(5.1.9) for the two polarized vectors ε_μ^λ and $\varepsilon_\nu^{-\lambda}$, the partial amplitudes $\widetilde{\mathcal{M}}_L^{0,h}$ and $\widetilde{\mathcal{M}}_R^{0,-h}$ both become on-shell amplitudes $(\mathcal{M}^0)$,

$$\mathcal{M}^0 = - \sum_{z=z_i \neq 0} \mathcal{M}_L^{0,h}(z_i) Res\left\{ \frac{i}{zp_{mid}^2(z)} \right\}\bigg|_{z=z_i} \mathcal{M}_R^{0,-h}(z_i). \qquad (5.1.23)$$

From Eq. (5.1.18), the on-shell condition $p_{mid}^2(z_i) = 0$ gives,

$$0 = \left(P_{r\cdots s}^\mu - \frac{z}{2}\langle j^-|\gamma^\mu|l^-\rangle \right)^2 = P_{r\cdots s}^2 - z\langle j^-|\slashed{P}_{r\cdots s}|l^-\rangle \qquad (5.1.24)$$

for which the solution is

$$z_i \equiv z_{rs} = \frac{P_{r\cdots s}^2}{\langle j^-|\slashed{P}_{r\cdots s}|l^-\rangle}. \qquad (5.1.25)$$

As z_{rs} depends on the external momentums in the left partial amplitudes. To find all singular points in $z-$plane, one simply find all possible independent combinations of separating left and right partial amplitudes. Evaluating the residues in equation (5.1.23), one have

$$-Res\left\{ \frac{i}{zp_{mid}^2(z)} \right\}\bigg|_{z=z_i} = \frac{i}{P_{r\cdots s}^2}. \qquad (5.1.26)$$

The full amplitude can now be determined by

$$\mathcal{M}^0 = \sum_{r,s,h} \mathcal{M}_L^{0,h}(z_{rs})\frac{i}{P_{r\cdots s}^2}\mathcal{M}_R^{0,-h}(z_{rs}), \qquad (5.1.27)$$

where the sum of helicity h of the intermediate parton is to make sure all the possible combinations of left and right partial amplitudes are considered. From Eq. (5.1.27), we see that we can recursively use on-shell results with lower parton multiplicities in $\mathcal{M}_L^{0,h}(z_{rs})$ and $\mathcal{M}_R^{0,-h}(z_{rs})$ to construct amplitudes with higher parton multiplicities.

The BCFW method is quite general. If the propagating particle is a massless quark, the BCFW method works in the same way because the quark propagator has similar structure to Eq. (5.1.10). To extend this method to Higgs boson plus multi-parton amplitudes, one iteratively builds more complicated helicity amplitudes by linking the vertices of Higgs boson coupling to quarks (standard model) or gluons (effective field theory) and the normal QCD vertices via the appropriate intermediate propagator p^{μ}_{mid}. The $[j, l\rangle$ shift never involves shifting the Higgs boson momentum. Valid choices of the $[j, l\rangle$ shift must always satisfy $\mathcal{M}^0_H(z)/z \to 0$ when $z \to \infty$ and this has to be checked with specific expressions during the calculation.

5.1.2   The effective Higgs boson-gluon interaction

In the Standard Model the Higgs boson couples to gluons through a quark loop. Because the Higgs boson couples to proportional to the quark mass, the dominant contribution is from the top quark. The bottom quark gives a few percent contribution. As introduced in section 2.2, in the large m_t limit, the top quark can be integrated out, leading to the effective interaction,

$$\mathcal{L}^{int}_H = \frac{\mathcal{C}}{2} H tr G_{\mu\nu} G^{\mu\nu}. \tag{5.1.28}$$

To NNLO in α_s, the strength of the interaction in $\overline{MS}$ scheme is given by [31,143,144]

$$\mathcal{C} = \frac{\alpha_s}{6\pi v} \left[1 + \frac{11}{2} \left(\frac{\alpha_s}{2\pi} \right) + \left(\frac{\alpha_s}{2\pi} \right)^2 \left(\frac{2777}{72} + \frac{19}{4} \log \frac{\mu^2_R}{m^2_t} - \frac{67 N_F}{24} + \frac{4 N_F}{3} \log \frac{\mu^2_R}{m^2_t} \right) \right]$$
$$+ \mathcal{O}(\alpha^4_s), \tag{5.1.29}$$

with $v = 246$ GeV.

Analytic expressions for the tree amplitudes for a Higgs boson plus four gluons in the heavy top quark approximation were first computed using traditional Feynman diagram methods by Dawson and Kauffman [27]. Kauffman, Desai and Risal [145, 146] extended these results to the other four-parton processes. Subsequently, analytic formulae for the Higgs boson plus 5 partons were derived by Del Duca, Frizzo and Maltoni [147]. Compact analytic expressions for tree-level amplitudes were obtained for gluonic processes [148] and processes with quarks [149] using MHV rules. In the following sections I will present the compact analytic expressions

for tree-level amplitudes for Higgs boson plus up to five partons in the heavy top quark approximation.

5.1.3   Interpretation in terms of ϕ and $\phi^\dagger$ fields

It is convenient to divide the Higgs boson coupling to gluons into two terms, one for coupling to each of the self dual (SD) and anti-self dual (ASD) gluon field strength combinations, where [148]

$$G_{SD}^{\mu\nu} = \frac{1}{2}(G^{\mu\nu} + {}^*G^{\mu\nu}), \qquad G_{ASD}^{\mu\nu} = \frac{1}{2}(G^{\mu\nu} - {}^*G^{\mu\nu}) \tag{5.1.30}$$

where ${}^*G^{\mu\nu}$ is the dual gluon field strength,

$$ {}^*G^{\mu\nu} \equiv \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}. \tag{5.1.31}$$

At the same time, it is helpful to identify the H field to be the real part of a complex scalar field $\phi = \frac{1}{2}(H + iA)$, such that

$$\mathcal{L}_{H,A}^{int} = \frac{C}{2}(H tr G_{\mu\nu}G^{\mu\nu} + iA tr G_{\mu\nu}^*G^{\mu\nu}) \tag{5.1.32}$$

$$= C(\phi tr G_{SD\mu\nu}G_{SD}^{\mu\nu} + \phi^\dagger tr G_{ASD\mu\nu}^*G_{ASD}^{\mu\nu}). \tag{5.1.33}$$

Due to self duality, the amplitudes for ϕ plus multi-partons, and those for $\phi^\dagger$ plus multi-partons, separately have a simpler structure than the amplitudes for H plus multi-partons. In QCD calculations, amplitudes for $\phi^\dagger$ can simply by obtained by applying charge conjugation to the ϕ amplitudes. In the amplitudes presented by spinor products ($\langle ij \rangle$ and $[ij]$), this simply means to swap $\langle ij \rangle \leftrightarrow [ji]$.

Because $H = \phi + \phi^\dagger$, the Higgs boson amplitudes can be recovered as the sum of the ϕ and $\phi^\dagger$ amplitudes. For tree-level H plus multi-parton amplitudes we have

$$\mathcal{M}_H(P_1^{h_1} \cdots P_n^{h_n}) = \mathcal{M}_\phi(P_1^{h_1} \cdots P_n^{h_n}) + \mathcal{M}_{\phi^\dagger}(P_1^{h_1} \cdots P_n^{h_n}), \tag{5.1.34}$$

where $P_i^{h_i}$ is the label for parton i of type P carrying light-like momentum p_i and helicity h_i.

5.1.4 Identities and properties of ϕ, $\phi^\dagger$ plus parton amplitudes

The properties of the simpler ϕ-($\phi^\dagger$-)amplitudes are summarized below,

$$\mathcal{M}_\phi^0(g_1^-, g_2^-, \cdots, g_n^-) = \frac{(-1)^n m_H^4}{[12][23]\cdots[n1]}, \tag{5.1.35}$$

$$\mathcal{M}_{\phi^\dagger}^0(g_1^+, g_2^+, \cdots, g_n^+) = \frac{m_H^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}, \tag{5.1.36}$$

$$\mathcal{M}_\phi^0(g_1^\pm, g_2^+, g_3^+ \cdots, g_n^+) = 0, \tag{5.1.37}$$

$$\mathcal{M}_{\phi^\dagger}^0(g_1^\pm, g_2^-, g_3^- \cdots, g_n^-) = 0, \tag{5.1.38}$$

$$\mathcal{M}_\phi^0(q_1^-, g_2^+, g_3^+ \cdots, g_{n-1}^+, \bar{q}_n^+) = 0, \tag{5.1.39}$$

$$\mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, g_3^- \cdots, g_{n-1}^-, \bar{q}_n^+) = 0. \tag{5.1.40}$$

Details of the proof of above equations can be found in [148, 149].

5.1.5 Higgs boson plus two parton amplitudes

In the limit where the light fermion mass are ignored and the top quark is treated as much heavier than the Higgs boson mass, the only non vanishing two-parton amplitude is $\mathcal{M}_H^0(g_1, g_2)$. There are four possible helicity combinations. The amplitudes with one positive and one negative helicity are zero while the amplitudes with all plus and all minus helicity are related by parity;

$$\mathcal{M}_H^0(g_1^-, g_2^-) \equiv \mathcal{M}_\phi^0(g_1^-, g_2^-) = -[12]^2, \tag{5.1.41}$$

$$\mathcal{M}_H^0(g_1^+, g_2^-) \equiv \mathcal{M}_\phi^0(g_1^-, g_2^+) = 0. \tag{5.1.42}$$

These amplitudes, together with the three-parton QCD vertex, can be recursively used in BCFW method to extend the number of partons for ϕ (or $\phi^\dagger$) plus multi-parton amplitudes.

5.1.6 Higgs boson plus three parton amplitudes

Tree-level amplitudes: $\mathcal{M}_H^0(g_1, g_2, g_3)$

The three-gluon amplitude has eight possible helicity combinations. By applying parity symmetry, amplitudes where the first gluon has positive helicity can be ob-

tained from those where it has negative helicity,

$$\mathcal{M}_H^0(g_1^+, g_2^{\lambda_2}, g_3^{\lambda_3}) = \mathcal{M}_H^0(g_1^-, g_2^{-\lambda_2}, g_3^{-\lambda_3})^\dagger. \tag{5.1.43}$$

Furthermore, using cyclic symmetry, the three amplitudes with two negative and one positive helicity gluon are related,

$$\mathcal{M}_H^0(g_1^-, g_2^-, g_3^+) = \mathcal{M}_H^0(g_3^+, g_1^-, g_2^-) = \mathcal{M}_H^0(g_2^-, g_3^+, g_1^-). \tag{5.1.44}$$

There are only two independent amplitudes which are,

$$\mathcal{M}_H^0(g_1^-, g_2^-, g_3^+) \equiv \mathcal{M}_\phi^0(g_1^-, g_2^-, g_3^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \tag{5.1.45}$$

$$\mathcal{M}_H^0(g_1^-, g_2^-, g_3^-) \equiv \mathcal{M}_\phi^0(g_1^-, g_2^-, g_3^-) = -\frac{m_H^4}{[12][23][31]}. \tag{5.1.46}$$

Tree-level amplitudes: $\mathcal{M}_H^0(q_1, g_2, \bar{q}_3)$

The two-quark, one-gluon amplitude has four helicity combinations. By applying parity symmetry, one can always reduce the number of independent amplitudes by half, for example, by requiring that the quark always carries negative helicity. Further using line-reverse relation and charge conjugation symmetry

$$\mathcal{M}_H^0(q_1^-, g_2^+, \bar{q}_3^+) = \mathcal{M}_H^0(q_3^-, g_2^-, \bar{q}_1^+)^\dagger \tag{5.1.47}$$

we can always obtain the amplitude for a positive helicity gluon from that with a negative helicity gluon. There is only one independent amplitude.

$$\mathcal{M}_H^0(q_1^-, g_2^-, \bar{q}_3^+) \equiv \mathcal{M}_\phi^0(q_1^-, g_2^-, \bar{q}_3^+) = \frac{\langle 12 \rangle^2}{\langle 13 \rangle}. \tag{5.1.48}$$

5.1.7 Higgs boson plus four parton amplitudes

Tree-level amplitudes: $\mathcal{M}_H^0(g_1, g_2, g_3, g_4)$

The four-gluon amplitude has sixteen possible helicity combinations. Using parity symmetry, amplitudes where the first gluon has positive helicity can be obtained from those where it has negative helicity,

$$\mathcal{M}_H^0(g_1^+, g_2^{\lambda_2}, g_3^{\lambda_3}, g_4^{\lambda_4}) = \mathcal{M}_H^0(g_1^-, g_2^{-\lambda_2}, g_3^{-\lambda_3}, g_4^{-\lambda_4})^\dagger. \tag{5.1.49}$$

Furthermore, by applying cyclic permutation relations,

$$\mathcal{M}_H^0(g_1^-, g_2^+, g_3^-, g_4^-) = \mathcal{M}_H^0(g_2^+, g_3^-, g_4^-, g_1^-),$$

$$\mathcal{M}_H^0(g_1^-, g_2^-, g_3^+, g_4^-) = \mathcal{M}_H^0(g_3^+, g_4^-, g_1^-, g_2^-),$$

$$\mathcal{M}_H^0(g_1^-, g_2^-, g_3^-, g_4^+) = \mathcal{M}_H^0(g_4^+, g_1^-, g_2^-, g_3^-),$$

$$\mathcal{M}_H^0(g_1^-, g_2^+, g_3^+, g_4^-) = \mathcal{M}_H^0(g_2^+, g_3^+, g_4^-, g_1^-), \tag{5.1.50}$$

we are left with only four independent amplitudes,

$$\mathcal{M}_H^0(g_1^-, g_2^-, g_3^-, g_4^-) \equiv \mathcal{M}_\phi^0(g_1^-, g_2^-, g_3^-, g_4^-) = \frac{m_H^4}{[12][23][34][41]}, \tag{5.1.51}$$

$$\mathcal{M}_H^0(g_1^+, g_2^-, g_3^-, g_4^-) \equiv \mathcal{M}_\phi^0(g_1^+, g_2^-, g_3^-, g_4^-)$$

$$= -\frac{m_H^4 \langle 24 \rangle^4}{s_{124} \langle 12 \rangle \langle 14 \rangle \langle 2|k_\phi|3] \langle 4|k_\phi|3]}$$

$$+ \frac{\langle 4|k_H|1]^3}{[12][23]s_{123}\langle 4|k_H|3]} - \frac{\langle 2|k_H|1]^3}{[14][34]s_{134}\langle 2|k_H|3]}, \tag{5.1.52}$$

$$\mathcal{M}_H^0(g_1^+, g_2^+, g_3^-, g_4^-) \equiv \mathcal{M}_\phi^0(g_1^+, g_2^+, g_3^-, g_4^-) + \mathcal{M}_{\phi^\dagger}^0(g_1^+, g_2^+, g_3^-, g_4^-)$$

$$= \frac{\langle 34 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} + \frac{[12]^4}{[12][23][34][41]}, \tag{5.1.53}$$

$$\mathcal{M}_H^0(g_1^-, g_2^+, g_3^-, g_4^+) \equiv \mathcal{M}_\phi^0(g_1^-, g_2^+, g_3^-, g_4^+) + \mathcal{M}_{\phi^\dagger}^0(g_1^-, g_2^+, g_3^-, g_4^+)$$

$$= \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} + \frac{[24]^4}{[12][23][34][41]}. \tag{5.1.54}$$

Here k_H is the momentum of the Higgs boson, $k_H = k_\phi = k_{\phi^\dagger} = -(k_1 + k_2 + k_3 + k_4)$.

Tree-level amplitudes: $\mathcal{M}_H^0(q_1, g_2, g_3, \bar{q}_4)$

The two-quark, two-gluon amplitudes have eight different helicity combinations. By applying parity symmetry, we can again reduce the number of independent amplitudes by requiring that the quark always carries negative helicity. Further using change conjugation symmetry and the line-reversal relation,

$$\mathcal{M}_H^0(q_1^-, g_2^+, g_3^+, \bar{q}_4^+) = -\mathcal{M}_H^0(q_4^-, g_3^-, g_2^-, \bar{q}_1^+)^\dagger. \tag{5.1.55}$$

The three remaining independent amplitudes are given by,

$$\mathcal{M}_H^0(q_1^-, g_2^-, g_3^-, \bar{q}_4^+) \equiv \mathcal{M}_\phi^0(q_1^-, g_2^-, g_3^-, \bar{q}_4^+) =$$

$$\frac{\langle 1|k_H|4]^2}{[23][34]\langle 1|k_H|2]} + \frac{m_H^4 \langle 13 \rangle^3}{\langle 14 \rangle \langle 1|k_H|2]\langle 3|k_H|2]s_{134}} + \frac{\langle 3|k_H|4]^2 \langle 3|k_H|1]}{[12][14]\langle 3|k_H|2]s_{124}}, \tag{5.1.56}$$

$$\mathcal{M}_H^0(q_1^-, g_2^-, g_3^+, \bar{q}_4^+) \equiv \mathcal{M}_\phi^0(q_1^-, g_2^-, g_3^+, \bar{q}_4^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, g_3^+, \bar{q}_4^+)$$

$$= \frac{[13][34]^2}{[12][23][41]} - \frac{\langle 24 \rangle \langle 12 \rangle^2}{\langle 34 \rangle \langle 23 \rangle \langle 41 \rangle}, \tag{5.1.57}$$

$$\mathcal{M}_H^0(q_1^-, g_2^+, g_3^-, \bar{q}_4^+) \equiv \mathcal{M}_\phi^0(q_1^-, g_2^+, g_3^-, \bar{q}_4^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^+, g_3^-, \bar{q}_4^+)$$

$$= \frac{[24]^3}{[23][34][41]} - \frac{\langle 13 \rangle^3}{\langle 12 \rangle \langle 23 \rangle \langle 41 \rangle}. \tag{5.1.58}$$

Tree-level amplitudes: $\mathcal{M}_H^0(q_1, \bar{Q}_2, Q_3, \bar{q}_4)$ and $\mathcal{M}_H^0(q_1, \bar{q}_2, Q_3, \bar{Q}_4)$

There are two separate four-quark amplitudes that are either leading in colour (when the quark of one pair is colour connected to the antiquark of the other pair) $\mathcal{M}_H^0(q_1, \bar{Q}_2, Q_3, \bar{q}_4)$ or colour sub-leading (when the quark and antiquark in the same pair are colour connected) $\mathcal{M}_H^0(q_1, \bar{q}_2, Q_3, \bar{Q}_4)$.

The leading colour amplitude has four helicity combinations. From parity symmetry only two of them are independent,

$$\mathcal{M}_H^0(q_1^-, \bar{Q}_2^+, Q_3^-, \bar{q}_4^+) = \mathcal{M}_\phi^0(q_1^-, \bar{Q}_2^+, Q_3^-, \bar{q}_4^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, \bar{Q}_2^+, Q_3^-, \bar{q}_4^+)$$

$$= -\frac{\langle 13 \rangle^2}{\langle 23 \rangle \langle 41 \rangle} - \frac{[24]^2}{[23][41]}, \tag{5.1.59}$$

$$\mathcal{M}_H^0(q_1^-, \bar{Q}_2^-, Q_3^+, \bar{q}_4^+) = \mathcal{M}_\phi^0(q_1^-, \bar{Q}_2^-, Q_3^+, \bar{q}_4^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, \bar{Q}_2^-, Q_3^+, \bar{q}_4^+)$$

$$= -\frac{\langle 12 \rangle^2}{\langle 23 \rangle \langle 41 \rangle} - \frac{[34]^2}{[23][41]}. \tag{5.1.60}$$

Similarly, the sub-leading colour amplitide has four helicity combinations. In the special case of two quark pairs, they can be obtained directly from the leading colour amplitudes,

$$\mathcal{M}_H^0(q_1^-, \bar{q}_2^+, Q_3^-, \bar{Q}_4^+) = \mathcal{M}_H^0(q_1^-, \bar{Q}_2^+, Q_3^-, \bar{q}_4^+), \tag{5.1.61}$$

$$\mathcal{M}_H^0(q_1^-, \bar{q}_2^+, Q_3^+, \bar{Q}_4^-) = \mathcal{M}_H^0(q_1^-, \bar{Q}_2^-, Q_3^+, \bar{q}_4^+). \tag{5.1.62}$$

5.1.8 Higgs boson plus five parton amplitudes

Tree-level amplitudes: $\mathcal{M}_H^0(g_1, g_2, g_3, g_4, g_5)$

The five-gluon amplitude has thirty-two different helicity combinations. Using parity and cyclic permutation symmetry, it is straightforward to relate each helicity

amplitude to one of four independent helicity amplitudes using equations analogous to (5.1.49) and (5.1.50).

Explicit formulas for the independent amplitudes are,

$$
\begin{aligned}
\mathcal{M}_H^0(g_1^-,g_2^-,g_3^-,g_4^-,g_5^-) &\equiv \mathcal{M}_H^\phi(g_1^-,g_2^-,g_3^-,g_4^-,g_5^-) \\
&= -\frac{m_H^4}{[12][23][34][45][51]}, \tag{5.1.63}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_H^0(g_1^+,g_2^-,g_3^-,g_4^-,g_5^-) &\equiv \mathcal{M}_\phi^0(g_1^+,g_2^-,g_3^-,g_4^-,g_5^-) \\
&= -\frac{m_H^4\langle 23\rangle\langle 5|2+3|1]^4}{s_{23}s_{123}s_{1235}[12]\langle 5|k_\phi|4]\langle 5|1+2|3][4|k_\phi(2+3)|1]} \\
&\quad +\frac{m_H^4\langle 25\rangle^4}{s_{125}[34]\langle 21\rangle\langle 51\rangle\langle 2|1+5|4]\langle 5|1+2|3]} \\
&\quad -\frac{\langle 5|k_\phi|1]^3}{s_{1234}[12][23][34]\langle 5|k_\phi|4]} \\
&\quad -\frac{m_H^4\langle 45\rangle\langle 2|4+5|1]^4}{s_{45}s_{145}s_{1245}[15]\langle 2|k_\phi|3]\langle 2|1+5|4][1|(4+5)k_\phi|3]} \\
&\quad -\frac{\langle 45\rangle[1|(4+5)(2+3)|1]^3}{s_{45}[12][23][15][1|(2+3)k_\phi|4][1|(4+5)k_\phi|3]} \\
&\quad +\frac{\langle 45\rangle\langle 2|k_\phi|1]^3}{s_{45}s_{1345}[15][43]\langle 2|k_\phi|3\rangle}, \tag{5.1.64}
\end{aligned}
$$

$$
\mathcal{M}_H^0(g_1^+,g_2^+,g_3^-,g_4^-,g_5^-) \equiv \mathcal{M}_\phi^0(g_1^+,g_2^+,g_3^-,g_4^-,g_5^-) + \mathcal{M}_{\phi^\dagger}^0(g_1^+,g_2^+,g_3^-,g_4^-,g_5^-), \tag{5.1.65}
$$

with,

$$
\begin{aligned}
\mathcal{M}_\phi^0(g_1^+,g_2^+,g_3^-,g_4^-,g_5^-) &= \frac{m_H^4[12]\langle 35\rangle^4}{s_{12}s_{1235}\langle 23\rangle\langle 15\rangle\langle 3|k_\phi|4]\langle 5|k_\phi|4]} \\
&\quad -\frac{[12]\langle 5|(1+2)(3+4)|5\rangle^3}{s_{12}[34]\langle 51\rangle\langle 5|k_\phi|4]\langle 5|1+2|3]\langle 2|(3+4)k_\phi|5\rangle} \\
&\quad -\frac{[12]\langle 5|(1+2)k_\phi|3\rangle^3}{s_{12}s_{125}s_{1245}\langle 51\rangle\langle 3|k_\phi|4]\langle 2|1+5|4]} \\
&\quad -\frac{\langle 32\rangle\langle 45\rangle^3[21]^3}{s_{23}s_{123}\langle 4|2+3|1]\langle 5|1+2|3]} \\
&\quad +\frac{\langle 45\rangle\langle 3|4+5|1]^3}{s_{45}s_{145}\langle 23\rangle[15]\langle 2|1+5|4]} \\
&\quad +\frac{\langle 34\rangle^3\langle 5|k_\phi|1]^3}{s_{234}s_{1234}\langle 23\rangle\langle 4|2+3|1]\langle 2|(3+4)k_\phi|5\rangle}, \tag{5.1.66}
\end{aligned}
$$

$$
\mathcal{M}_{\phi^\dagger}^0(g_1^+,g_2^+,g_3^-,g_4^-,g_5^-) = -\frac{[12]^4}{[12][23][34][45][51]}, \tag{5.1.67}
$$

and,

$$\mathcal{M}_H^0(g_1^+, g_2^-, g_3^+, g_4^-, g_5^-) \equiv = \mathcal{M}_\phi^0(g_1^+, g_2^-, g_3^+, g_4^-, g_5^-) + \mathcal{M}_{\phi^\dagger}^0(g_1^+, g_2^-, g_3^+, g_4^-, g_5^-),$$

$$(5.1.68)$$

with

$$\mathcal{M}_\phi^0(g_1^+, g_2^-, g_3^+, g_4^-, g_5^-) =$$
$$+\frac{m_H^4[12]\langle 24\rangle^4\langle 25\rangle^4}{s_{12}\langle 15\rangle\langle 34\rangle\langle 32\rangle\langle 2|(3+4)k_\phi|5\rangle\langle 4|k_\phi(1+5)|2\rangle\langle 2|(3+4)(1+5)|2\rangle}$$
$$-\frac{[12]\langle 2|k_\phi|3]^3\langle 25\rangle^4}{s_{12}\langle 15\rangle[34]\langle 2|1+5|4\rangle\langle 2|(3+4)(1+5)|2\rangle\langle 2|k_\phi(3+4)|5\rangle}$$
$$+\frac{[21]\langle 25\rangle^4\langle 4|k_\phi|3]^3}{s_{12}s_{125}s_{1235}\langle 51\rangle\langle 5|1+2|3]\langle 4|k_\phi(1+5)|2\rangle}$$
$$+\frac{\langle 45\rangle^3\langle 2|k_\phi|1]^3}{s_{345}s_{1345}\langle 34\rangle\langle 3|4+5|1]\langle 5|(3+4)k_\phi|2\rangle}$$
$$+\frac{\langle 23\rangle\langle 45\rangle^3[31]^4}{s_{23}s_{123}[21]\langle 4|2+3|1]\langle 5|1+2|3]}$$
$$+\frac{\langle 45\rangle\langle 2|4+5|1]^4}{s_{45}s_{145}\langle 23\rangle[15]\langle 3|4+5|1]\langle 2|1+5|4]}$$
$$-\frac{\langle 24\rangle^4\langle 5|k_\phi|1]^3}{s_{234}s_{1234}\langle 23\rangle\langle 34\rangle\langle 4|2+3|1]\langle 5|k_\phi(3+4)|2\rangle},$$

$$(5.1.69)$$

$$\mathcal{M}_{\phi^\dagger}^0(g_1^+, g_2^-, g_3^+, g_4^-, g_5^-) = -\frac{[13]^4}{[12][23][34][45][51]}.$$

$$(5.1.70)$$

Tree-level amplitudes: $\mathcal{M}_H^0(q_1, g_2, g_3, g_4, \bar{q}_5)$

The two-quark, three-gluon amplitude has sixteen different helicity combinations. By parity symmetry, we can reduce this number to eight. By applying line-reversal and charge conjugation symmetry we can further reduce the number of independent amplitudes to four,

$$\mathcal{M}_H^0(q_1^-, g_2^+, g_3^+, g_4^+, \bar{q}_5^+) = \mathcal{M}_H^0(q_5^-, g_4^-, g_3^-, g_2^-, \bar{q}_1^+)^\dagger, \qquad (5.1.71)$$

$$\mathcal{M}_H^0(q_1^-, g_2^+, g_3^+, g_4^-, \bar{q}_5^+) = \mathcal{M}_H^0(q_5^-, g_4^+, g_3^-, g_2^-, \bar{q}_1^+)^\dagger, \qquad (5.1.72)$$

$$\mathcal{M}_H^0(q_1^-, g_2^+, g_3^-, g_4^+, \bar{q}_5^+) = \mathcal{M}_H^0(q_5^-, g_4^-, g_3^+, g_2^-, \bar{q}_1^+)^\dagger, \qquad (5.1.73)$$

$$\mathcal{M}_H^0(q_1^-, g_2^-, g_3^+, g_4^+, \bar{q}_5^+) = \mathcal{M}_H^0(q_5^-, g_4^-, g_3^-, g_2^+, \bar{q}_1^+)^\dagger. \qquad (5.1.74)$$

Explicit formulas for the four independent amplitudes are:

$$\mathcal{M}_H^0(q_1^-, g_2^-, g_3^-, g_4^-, \bar{q}_5^+) = \mathcal{M}_\phi^0(q_1^-, g_2^-, g_3^-, g_4^-, \bar{q}_5^+)$$

$$
\begin{aligned}
= \quad &+ \frac{m_H^4 \langle 12 \rangle \langle 4|1+2|5]^3 \langle 4|2+5|1]}{s_{12} s_{125} s_{1245} [15] \langle 4|k_\phi|3] \langle 4|1+5|2] [5|(1+2)k_\phi|3]} \\
&- \frac{\langle 34 \rangle [5|(3+4)(1+2)|5]^2 [1|k_\phi(3+4)|5]}{s_{34} [12][15][45][2|k_\phi(3+4)|5][3|k_\phi(1+2)|5]} \\
&+ \frac{m_H^4 \langle 34 \rangle \langle 1|3+4|5]^3}{s_{1345} s_{34} \langle 1|k_\phi|2][45][2|k_\phi(3+4)|5]\langle 1|4+5|3]} \\
&- \frac{\langle 1|k_\phi|5]^2 \langle 34 \rangle}{s_{34}\langle 1|k_\phi|2][32][45]} + \frac{m_H^4 \langle 14 \rangle^3 [15]}{s_{15} s_{145} \langle 23 \rangle \langle 4|1+5|2]\langle 1|4+5|3]} \\
&+ \frac{\langle 4|k_\phi|5]^2 \langle 4|k_\phi|1]}{s_{1235}[12][23][51]\langle 4|k_\phi|3]},
\end{aligned}
\tag{5.1.75}
$$

$$
\mathcal{M}_H^0(q_1^-, g_2^+, g_3^-, g_4^-, \bar{q}_5^+) \equiv \mathcal{M}_\phi^0(q_1^-, g_2^+, g_3^-, g_4^-, \bar{q}_5^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^+, g_3^-, g_4^-, \bar{q}_5^+),
\tag{5.1.76}
$$

with

$$
\begin{aligned}
\mathcal{M}_\phi^0(q_1^-, g_2^+, g_3^-, &g_4^-, \bar{q}_5^+) = \\
&+ \frac{m_H^4 \langle 13 \rangle^4 \langle 14 \rangle^3 [15]}{s_{15} \langle 23 \rangle \langle 21 \rangle \langle 1|(4+5)k_\phi|3]) \langle 1|(2+3)(4+5)|1] \langle 1|(2+3)k_\phi|4 \rangle} \\
&- \frac{[15] \langle 14 \rangle^3 \langle 1|k_\phi|2]^3}{s_{15}[23]\langle 1|(2+3)(4+5)|1]\langle 1|4+5|3]\langle 4|(2+3)k_\phi|1 \rangle} \\
&+ \frac{\langle 3|k_\phi|2]^3 [15] \langle 41 \rangle^3}{s_{1245} s_{145} s_{15} \langle 4|1+5|2]\langle 3|k_\phi(4+5)|1 \rangle} \\
&- \frac{\langle 34 \rangle^3 \langle 1|k_\phi|5]^2}{s_{234} \langle 23 \rangle \langle 2|3+4|5]\langle 4|(2+3)k_\phi|1 \rangle} \\
&- \frac{\langle 13 \rangle^3 \langle 4|k_\phi|5]^2 \langle 3|(1+2)k_\phi|4 \rangle}{s_{123} s_{1235} \langle 12 \rangle \langle 23 \rangle \langle 3|1+2|5] \langle 1|(2+3)k_\phi|4 \rangle} \\
&- \frac{\langle 34 \rangle \langle 1|3+4|5]^3}{s_{34} \langle 12 \rangle [45] \langle 2|3+4|5]\langle 1|4+5|3]} \\
&- \frac{\langle 34 \rangle^3 [25]^3}{s_{125}[15]\langle 3|1+2|5]\langle 4|1+5|2]},
\end{aligned}
\tag{5.1.77}
$$

$$
\mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^+, g_3^-, g_4^-, \bar{q}_5^+) = \frac{[25]^3}{[23][34][45][15]},
\tag{5.1.78}
$$

$$
\mathcal{M}_H^0(q_1^-, g_2^-, g_3^+, g_4^-, \bar{q}_5^+) \equiv \mathcal{M}_\phi^0(q_1^-, g_2^-, g_3^+, g_4^-, \bar{q}_5^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, g_3^+, g_4^-, \bar{q}_5^+),
\tag{5.1.79}
$$

with

$$
\mathcal{M}_\phi^0(q_1^-, g_2^-, g_3^+, g_4^-, \bar{q}_5^+) =
$$

$$-\frac{\langle 12\rangle^2\langle 34\rangle[35]^3\langle 2|4+5|3]}{s_{34}s_{345}[45]\langle 1|4+5|3]\langle 2|3+4|5]}$$

$$-\frac{\langle 12\rangle\langle 4|1+2|5]^3\langle 4|2+5|1]}{s_{12}s_{125}\langle 34\rangle[15]\langle 3|1+2|5]\langle 4|1+5|2]}$$

$$-\frac{\langle 24\rangle^4\langle 1|k_\phi|5]^2}{s_{234}\langle 23\rangle\langle 34\rangle\langle 2|3+4|5]\langle 4|(2+3)k_\phi|1\rangle}$$

$$-\frac{\langle 12\rangle^2\langle 4|k_\phi|5]^2\langle 2|(1+3)k_\phi|4\rangle}{s_{123}s_{1235}\langle 23\rangle\langle 3|1+2|5]\langle 1|(2+3)k_\phi|4\rangle}$$

$$+\frac{m_H^4\langle 24\rangle^4\langle 14\rangle^3[15]}{s_{15}\langle 23\rangle\langle 34\rangle\langle 4|(2+3)(1+5)|4]\langle 4|(2+3)k_\phi|1\rangle\langle 4|(1+5)k_\phi|2\rangle}$$

$$-\frac{\langle 2|k_\phi|3]^3\langle 14\rangle^3[15]}{s_{145}s_{15}s_{1345}\langle 1|4+5|3]\langle 4|(1+5)k_\phi|2\rangle}$$

$$+\frac{\langle 4|k_\phi|3]^3\langle 14\rangle^3[15]}{s_{15}[23]\langle 4|(2+3)(1+5)|4]\langle 4|1+5|2]\langle 1|(2+3)k_\phi|4\rangle)}, \tag{5.1.80}$$

$$\mathcal{M}_{\phi^\dagger}^0(q_1^-,g_2^-,g_3^+,g_4^-,\bar{q}_5^+) = \frac{[31][35]^3}{[12][23][34][45][51]}, \tag{5.1.81}$$

and finally,

$$\mathcal{M}_H^0(q_1^-,g_2^-,g_3^-,g_4^+,\bar{q}_5^+) \equiv \mathcal{M}_\phi^0(q_1^-,g_2^-,g_3^-,g_4^+,\bar{q}_5^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-,g_2^-,g_3^-,g_4^+,\bar{q}_5^+), \tag{5.1.82}$$

with,

$$\mathcal{M}_\phi^0(q_1^-,g_2^-,g_3^-,g_4^+,\bar{q}_5^+) =$$

$$\frac{\langle 15\rangle\langle 23\rangle^3[54]^2[14]}{s_{15}s_{145}\langle 2|1+5|4]\langle 3|4+5|1]} - \frac{\langle 23\rangle\langle 1|2+3|4]^2\langle 5|2+3|4]}{s_{23}s_{234}\langle 51\rangle[34]\langle 5|3+4|2]}$$

$$-\frac{\langle 12\rangle^2\langle 25\rangle\langle 3|k_\phi|4]^3}{s_{1245}s_{125}\langle 51\rangle\langle 2|1+5|4]\langle 5|(1+2)k_\phi|3\rangle}$$

$$+\frac{[45]\langle 35\rangle\langle 3|k_\phi|1]\langle 3|(1+2)(4+5)|3]^2}{s_{45}[12]\langle 34\rangle\langle 3|k_\phi|2]\langle 3|4+5|1]\langle 5|(1+2)k_\phi|3\rangle)}$$

$$-\frac{m_H^4[45]\langle 13\rangle^3\langle 35\rangle}{s_{45}s_{1345}\langle 15\rangle\langle 34\rangle\langle 1|k_\phi|2]\langle 3|k_\phi|2]} + \frac{[45]\langle 35\rangle\langle 1|k_\phi|(4+5)|3]^2}{s_{345}s_{45}\langle 34\rangle\langle 1|k_\phi|2]\langle 5|3+4|2]}, \tag{5.1.83}$$

$$\mathcal{M}_{\phi^\dagger}^0(q_1^-,g_2^-,g_3^-,g_4^+,\bar{q}_5^+) = \frac{[41][45]^2}{[12][23][34][51]}. \tag{5.1.84}$$

Tree-level amplitudes: $\mathcal{M}_H^0(q_1,g_2,\bar{Q}_3,Q_4,\bar{q}_5)$ **and** $\mathcal{M}_H^0(q_1,\bar{Q}_2,Q_3,g_4,\bar{q}_5)$

There are two leading colour four-quark one-gluon amplitudes, $\mathcal{M}_H^0(q_1,g_2,\bar{Q}_3,Q_4,\bar{q}_5)$ and $\mathcal{M}_H^0(q_1,\bar{Q}_2,Q_3,g_4,\bar{q}_5)$. All of the $\mathcal{M}_H^0(q_1,\bar{Q}_2,Q_3,g_4,\bar{q}_5)$ amplitudes are related to the corresponding $\mathcal{M}_H^0(q_1,g_2,\bar{Q}_3,Q_4,\bar{q}_5)$ amplitudes by,

$$\mathcal{M}_H^0(q_1^{\lambda_1},\bar{Q}_2^{\lambda_2},Q_3^{\lambda_3},g_4^{\lambda_4},\bar{q}_5^{\lambda_5}) = \mathcal{M}_H^0(Q_3^{\lambda_3},g_4^{\lambda_4},\bar{q}_5^{\lambda_5},q_1^{\lambda_1},\bar{Q}_2^{\lambda_2}). \tag{5.1.85}$$

With parity symmetry, charge conjugation symmetry and line-reverse relation such as

$$\mathcal{M}_H^0(q_1^-, g_2^-, \bar{Q}_3^+, Q_4^-, \bar{q}_5^+) = \mathcal{M}_H^{0\dagger}(Q_3^-, g_2^+, \bar{q}_1^+, q_5^-, \bar{Q}_4^+), \qquad (5.1.86)$$

only three of the eight helicity combinations are independent.

Explicit formulas for the independent amplitudes are:

$$\mathcal{M}_H^0(q_1^-, g_2^+, \bar{Q}_3^+, Q_4^-, \bar{q}_5^+) = \mathcal{M}_\phi^0(q_1^-, g_2^+, \bar{Q}_3^+, Q_4^-, \bar{q}_5^+) + \mathcal{M}_{\phi\dagger}^0(q_1^-, g_2^+, \bar{Q}_3^+, Q_4^-, \bar{q}_5^+), \qquad (5.1.87)$$

$$
\begin{aligned}
\mathcal{M}_{\phi\dagger}^0(q_1^-, g_2^+, \bar{Q}_3^+, Q_4^-, \bar{q}_5^+) = & \\
& - \frac{\langle 4|2+3|5]^2 \langle 3|2+4|5]}{s_{234}\langle 23\rangle\langle 34\rangle[15]\langle 2|3+4|5]} + \frac{[23]^2\langle 14\rangle^2[15]\langle 1|4+5|2]}{s_{145}s_{15}\langle 4|1+5|2]\langle 1|4+5|3]} \\
& + \frac{[25]^3\langle 4|k_\phi|3]^2}{s_{125}[15]\langle 4|1+5|2][3|k_\phi(1+2)|5]} \\
& - \frac{\langle 1|k_\phi|2]^2\langle 34\rangle[35]^2[2|k_\phi(4+5)|3]}{s_{34}s_{1345}s_{345}\langle 1|4+5|3][2|k_\phi(3+4)|5]} \\
& + \frac{m_H^4[25]^3\langle 34\rangle[35]^3}{s_{34}[51][5|(3+4)(1+2)|5][3|k_\phi(1+2)|5][2|k_\phi(3+4)|5]} \\
& - \frac{\langle 1|k_\phi|5]^2\langle 34\rangle[35]^2}{s_{34}\langle 12\rangle\langle 2|3+4|5][5|(3+4)(1+2)|5]}, \qquad (5.1.88)
\end{aligned}
$$

$$\mathcal{M}_\phi^0(q_1^-, g_2^+, \bar{Q}_3^+, Q_4^-, \bar{q}_5^+) = \frac{\langle 31\rangle\langle 14\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 51\rangle}, \qquad (5.1.89)$$

$$\mathcal{M}_H^0(q_1^-, g_2^+, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) = \mathcal{M}_\phi^0(q_1^-, g_2^+, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) + \mathcal{M}_{\phi\dagger}^0(q_1^-, g_2^+, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+), \qquad (5.1.90)$$

$$
\begin{aligned}
\mathcal{M}_{\phi\dagger}^0(q_1^-, g_2^+, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) = & \\
& + \frac{\langle 34\rangle[42]^3\langle 1|k_\phi|5]^2}{s_{34}s_{234}\langle 1|2+3|4][5|k_\phi(3+4)|2]} \\
& + \frac{[45]^2\langle 1|k_\phi|2]^2[2|k_\phi(4+5)|3]}{s_{345}s_{1345}[34]\langle 1|4+5|3][2|k_\phi(3+4)|5]} \\
& - \frac{\langle 1|4+5|2]^3}{s_{145}\langle 15\rangle\langle 1|4+5|3]\langle 4|1+5|2]} + \frac{[45]^2\langle 13\rangle^3[23]}{s_{23}\langle 12\rangle\langle 1|2+3|4]\langle 3|1+2|5]} \\
& - \frac{\langle 3|k_\phi|2]^2\langle 4|k_\phi|2]\langle 15\rangle[25]^3}{s_{15}\langle 34\rangle\langle 4|1+5|2][2|(1+5)(3+4)|2][2|k_\phi(3+4)|5]} \\
& + \frac{m_H^4[24]^3[25]^3\langle 15\rangle}{s_{15}[43][2|(1+5)(3+4)|2][4|k_\phi(1+5)|2][5|k_\phi(3+4)|2]}
\end{aligned}
$$

$$+ \frac{\langle 3|k_\phi|4]^2 \langle 15 \rangle [25]^3}{s_{15} s_{125} \langle 3|1+2|5][4|k_\phi(1+5)|2]}, \tag{5.1.91}$$

$$\mathcal{M}_\phi^0(q_1^-, g_2^+, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) = \frac{\langle 13 \rangle^3}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 15 \rangle}, \tag{5.1.92}$$

$$\mathcal{M}_H^0(q_1^-, g_2^-, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) = \mathcal{M}_\phi^0(q_1^-, g_2^-, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+), \tag{5.1.93}$$

$$\mathcal{M}_\phi^0(q_1^-, g_2^-, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) =$$
$$+ \frac{\langle 13 \rangle^2 [34] \langle 3|k_\phi|5]^2 \langle 3|(2+5)(1+4)|3 \rangle}{s_{34} \langle 3|1+4|2] \langle 3|1+4|5] \langle 3|k_\phi|2] \langle 1|(2+5)k_\phi|3 \rangle}$$
$$+ \frac{m_H^4 \langle 13 \rangle^3 [34]}{s_{34} s_{1345} \langle 51 \rangle \langle 1|k_\phi|2] \langle 3|k_\phi|2]} + \frac{\langle 1|k_\phi|5]^2 \langle 13 \rangle^2 [34]}{s_{34} \langle 1|k_\phi|2] \langle 1|3+4|2] \langle 1|3+4|5]}$$
$$+ \frac{\langle 12 \rangle^2 \langle 25 \rangle \langle 3|k_\phi|4]^2 \langle 1|2+5|4]}{s_{125} \langle 15 \rangle \langle 2|1+5|4] \langle 5|1+2|4] \langle 1|(2+5)k_\phi|3 \rangle}$$
$$- \frac{\langle 23 \rangle \langle 1|2+3|4]^2 \langle 1|2+4|3]}{s_{23} s_{234} \langle 15 \rangle [34] \langle 1|3+4|2]} - \frac{\langle 23 \rangle^2 \langle 15 \rangle [54]^2 \langle 2|1+4|5]}{s_{15} s_{145} \langle 2|1+5|4] \langle 3|1+4|5]}$$
$$+ \frac{\langle 12 \rangle \langle 3|1+2|4]^2}{s_{12} \langle 3|1+4|2] \langle 5|1+2|4]} - \frac{\langle 12 \rangle^2 [45]^2}{s_{345} [34] \langle 1|3+4|5]}, \tag{5.1.94}$$

$$\mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, \bar{Q}_3^-, Q_4^+, \bar{q}_5^+) = \frac{[13][45]^2}{[12][23][34][51]}. \tag{5.1.95}$$

Tree-level amplitudes: $\mathcal{M}_H^0(q_1, g_2, \bar{q}_3, Q_4, \bar{Q}_5)$ **and** $\mathcal{M}_H^0(q_1, \bar{q}_2, Q_3, g_4, \bar{Q}_5)$

The two sub-leading colour four-quark one-gluon amplitudes are $\mathcal{M}_H^0(q_1, g_2, \bar{q}_3, Q_4, \bar{Q}_5)$ and $\mathcal{M}_H^0(q_1, \bar{q}_2, Q_3, g_4, \bar{Q}_5)$. As in the leading colour case, we can obtain the amplitudes for $\mathcal{M}_H(q_1, g_2, \bar{q}_3, Q_4, \bar{Q}_5)$ in a straightforward manner.

$\mathcal{M}_H^0(q_1, g_2, \bar{q}_3, Q_4, \bar{Q}_5)$ has eight possible helicity combinations. Besides the usual charge conjugation symmetry and line-reversal relations, these photon-like colour sub-leading amplitudes have the additional symmetry properties:

$$\mathcal{M}_H^0(q_1, g_2, \bar{q}_3, Q_4, \bar{Q}_5) = \mathcal{M}_H^0(q_1, g_2, \bar{q}_3, Q_5, \bar{Q}_4) = \mathcal{M}_H^0(q_3, g_2, \bar{q}_1, Q_4, \bar{Q}_5), \tag{5.1.96}$$

such that,

$$\mathcal{M}_H^0(q_1^-, g_2^-, \bar{q}_3^+, Q_4^+, \bar{Q}_5^-) = \mathcal{M}_H^0(q_1^-, g_2^-, \bar{q}_3^+, Q_5^-, \bar{Q}_4^+), \tag{5.1.97}$$

$$\mathcal{M}_H^0(q_1^-, g_2^+, \bar{q}_3^+, Q_4^-, \bar{Q}_5^+) = \mathcal{M}_H^{0\dagger}(q_3^-, g_2^-, \bar{q}_1^+, Q_5^-, \bar{Q}_4^+), \tag{5.1.98}$$

$$\mathcal{M}_H^0(q_1^-, g_2^+, \bar{q}_3^+, Q_4^+, \bar{Q}_5^-) = \mathcal{M}_H^{0\dagger}(q_3^-, g_2^-, \bar{q}_1^+, Q_4^-, \bar{Q}_5^+). \tag{5.1.99}$$

There is only one independent amplitude that is given by,

$$\mathcal{M}_H^0(q_1^-, g_2^-, \bar{q}_3^+, Q_4^-, \bar{Q}_5^+) = \mathcal{M}_\phi^0(q_1^-, g_2^-, \bar{q}_3^+, Q_4^-, \bar{Q}_5^+) + \mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, \bar{q}_3^+, Q_4^-, \bar{Q}_5^+), \tag{5.1.100}$$

$$\mathcal{M}_\phi^0(q_1^-, g_2^-, \bar{q}_3^+, Q_4^-, \bar{Q}_5^+) = \frac{\langle 14 \rangle^2 \langle 2|k_\phi|3]^2}{s_{145} s_{1345} \langle 45 \rangle \langle 1|4+5|3]} - \frac{\langle 12 \rangle^2 \langle 45 \rangle [35]^2}{s_{45} s_{345} \langle 1|4+5|3]}$$
$$+ \frac{\langle 12 \rangle \langle 4|1+2|3]^2}{s_{12} s_{123} \langle 45 \rangle [23]}, \tag{5.1.101}$$

$$\mathcal{M}_{\phi^\dagger}^0(q_1^-, g_2^-, \bar{q}_3^+, Q_4^-, \bar{Q}_5^+) = \frac{[35]^2}{[12][23][45]}. \tag{5.1.102}$$

5.2 One-loop Higgs boson plus multi-parton matrix elements

The one-loop amplitudes involving a Higgs boson and up to four partons have been studied analytically by different groups. In the gluonic case, there are several helicity configurations; the all-minus $(-, -, -, -)$ [150], the MHV $(+, +, -, -)$ [151] and $(+, -, +, -)$ [152] and the NMHV $(+, -, -, -)$ [122]. In the two-quark, two-gluon case, there are only two types, the MHV [121] and NMHV [123] amplitudes while expressions for the four-quark amplitudes are also given in Ref. [121]. The method used for calculating those one-loop amplitudes are based on four-dimensional unitarity-based recursion relations. In this section, the unitarity method is briefly introduced to classify different components inside the one-loop amplitude. The numerical stability of the published analytical expressions is studied and improved for the purposes of our NNLO calculation at the real-virtual level in chapters 6, 7, 8 and 9.

5.2.1 Unitarity based recursive relations for one-loop amplitude calculation

The unitarity property of the scattering matrix S is a fundamental requirement from the conservation of probability. Writing $S = 1 + iT$ where T is the transition matrix,

then unitarity implies

$$-i(T - T^\dagger) = T^\dagger T. \qquad (5.2.103)$$

The left-hand side of (5.2.103) refers to the discontinuity in the scattering amplitude while the right-hand side refers to a loop amplitude with a cut on the loop propagators. At the diagrammatic level, this property implies the Cutkosky rules [153] that

$$\frac{i}{p^2 + i\epsilon} \to 2\pi\delta^{(+)}(p^2). \qquad (5.2.104)$$

In the calculation of one-loop amplitudes, one needs to integrate the internal loop momentum over all possible values. From the Cutkosky rules, cutting the loop propagator simplifies the loop integral to a phase-space integral. By evaluating the phase-space integral, one calculates the cut-constructable part of the one-loop amplitude.

In general, $n-$point one-loop integrals would involve many internal propagators which can be simplified to four or fewer-point (four or fewer internal propagators) scalar integrals through tensor reduction procedures [154–158]. Generic one-loop integrals are simplified to a set of well known master integrals after reduction and the key problem is to calculate the coefficient of each master integral. In generalized unitarity [159], quadruple-cuts are applied to calculate the coefficient of the box (four-point) master integrals. For three or fewer-point integrals, triple- and double-cuts can be applied to find the coefficients of the triangle (three-point) and bubble (two-point) master integrals [160–162].

Cutting a propagator means constraining the internal momentum so that the propagating particle is on-shell. The partial amplitudes between two cuts are well defined tree-level components and the full loop amplitudes can be seen as the products of tree-level components with base loop integrals (box-, triangle- and bubble-integrals). Collecting the tree components associated with different base integrals, one can calculate the coefficients of each base integrals. In four dimensional calculations, those tree components can be calculated using the BCFW method in section 5.1.1.

In calculations where the helicities are evaluated in four-dimensions, any rational parts of the loop amplitude cannot be determined directly from loop integrals and

must be constructed using different recursive methods [122,163,164] inspired by the BCFW method discussed in section 5.1.1. However, in D-dimensions, the rational terms are sensitive to the unitarity cuts and can be directly determined [165–169].

5.2.2 Rewriting of the one-loop amplitudes in a numerically stable form

The matrix elements for the one-loop Higgs boson with four partons are analytic and have been numerically implemented in the MCFM code [1] where they have been applied to the study of $H+2$ jets at NLO [170]. However, they have not been studied in the regions of phase space where one of the particles becomes soft - and where we expect the matrix elements to have specific singular behaviour. In this case, there can be large cancellations between different terms in the analytic expressions so that although the behaviour is correct analytically, the numerical evaluation may be unstable.

In this section, we discuss where the numerical stability of these matrix elements is problematic and show how to rewrite the matrix elements in a numerically stable form.

The NMHV amplitude $\mathcal{M}_H^{(1)}(g_1^+, g_2^-, g_3^-, g_4^-)$

The Higgs boson NMHV amplitude $\mathcal{M}_H^{(1)}(g_1^+, g_2^-, g_3^-, g_4^-)$ studied in [122] is the combination of finite cut-constructible contribution and the rational part:

$$\mathcal{M}_H^{(1)}(g_1^+, g_2^-, g_3^-, g_4^-) = -F_4(H, g_1^+, g_2^-, g_3^-, g_4^-) + R_4(H, g_1^+, g_2^-, g_3^-, g_4^-). \quad (5.2.105)$$

The finite cut-constructible contribution given in Eq. (5.12) of [122] is given by

$$\begin{aligned}
&F_4(H, g_1^+, g_2^-, g_3^-, g_4^-) \\
&= \Bigg\{ -\frac{s_{234}^3}{4\langle 1|p_H|2]\langle 1|p_H|4][23][34]} W^{(1)} \\
&\quad - \left(\frac{\langle 2|p_H|1]^3}{2s_{134}\langle 2|p_H|3][34][41]} + \frac{\langle 34\rangle^3 m_H^4}{2s_{134}\langle 1|p_H|2]\langle 3|p_H|2]\langle 41\rangle} \right) W^{(2)}
\end{aligned}$$

---

[1]http://mcfm.fnal.gov/

$$+\frac{1}{4s_{124}}\left(\frac{\langle3|p_H|1]^4}{\langle3|p_H|2]\langle3|p_H|4][21][41]}+\frac{\langle24\rangle^4m_H^4}{\langle12\rangle\langle14\rangle\langle2|p_H|3]\langle4|p_H|3]}\right)W^{(3)}$$

$$+2C_{3;\phi|12|34}(\phi,1_g^+,2_g^-,3_g^-,4_g^-)\,\mathrm{F}_3^{3\mathrm{m}}(m_H^2,s_{12},s_{34})$$

$$+\left(1-\frac{N_f}{4N_c}\right)\left(\frac{\langle3|p_H|1]^2}{s_{124}[24]^2}\,\mathrm{F}_{4\mathrm{F}}^{1\mathrm{m}}(s_{12},s_{14};s_{124})\right.$$

$$\left.-\frac{4\langle24\rangle\langle3|p_H|1]^2}{s_{124}[42]}\hat{L}_1\left(s_{124},s_{12}\right)+\frac{4\langle23\rangle\langle4|p_H|1]^2}{s_{123}[32]}\hat{L}_1\left(s_{123},s_{12}\right)\right)$$

$$-\left(1-\frac{N_f}{N_c}+\frac{N_s}{N_c}\right)\left(\frac{[12][41]\langle3|p_H|2]\langle3|p_H|4]}{2s_{124}[24]^4}\,\mathrm{F}_{4\mathrm{F}}^{1\mathrm{m}}(s_{12},s_{14};s_{124})\right.$$

$$+\frac{2s_{124}\langle24\rangle\langle34\rangle^2[41]^2}{3[42]}\hat{L}_3\left(s_{124},s_{12}\right)$$

$$+\frac{\langle34\rangle[41]\left(3s_{124}\langle34\rangle[41]+\langle24\rangle\langle3|p_H|1][42]\right)}{3[42]^2}\hat{L}_2\left(s_{124},s_{12}\right)$$

$$+\left(\frac{2s_{124}\langle34\rangle^2[41]^2}{\langle24\rangle[42]^3}-\frac{\langle24\rangle\langle3|p_H|1]^2}{3s_{124}[42]}\right)\hat{L}_1\left(s_{124},s_{12}\right)$$

$$+\frac{\langle3|p_H|1](4s_{124}\langle34\rangle[41]+\langle3|p_H|1](2s_{14}+s_{24}))}{s_{124}\langle24\rangle[42]^3}\hat{L}_0\left(s_{124},s_{12}\right)$$

$$-\frac{2s_{123}\langle23\rangle\langle34\rangle^2[31]^2}{3[32]}\hat{L}_3\left(s_{123},s_{12}\right)+\frac{\langle23\rangle\langle34\rangle[31]\langle4|p_H|1]}{3[32]}\hat{L}_2\left(s_{123},s_{12}\right)$$

$$\left.+\frac{\langle23\rangle\langle4|p_H|1]^2}{3s_{123}[32]}\hat{L}_1\left(s_{123},s_{12}\right)\right)\bigg\}+\bigg\{(2\leftrightarrow4)\bigg\},\qquad(5.2.106)$$

while the rational part is given in Eq. (5.13) of [122] (and originally derived in Ref. [120]) is,

$$R_4(H,g_1^+,g_2^-,g_3^-,g_4^-)$$

$$=\quad\bigg\{\left(1-\frac{N_f}{N_c}+\frac{N_s}{N_c}\right)\frac{1}{2}\bigg(\frac{\langle23\rangle\langle34\rangle\langle4|p_H|1][31]}{3s_{123}\langle12\rangle[21][32]}-\frac{\langle3|p_H|1]^2}{s_{124}[42]^2}$$

$$-\frac{\langle24\rangle(s_{23}s_{24}+s_{23}s_{34}+s_{24}s_{34})}{3\langle12\rangle\langle14\rangle[23][34][42]}+\frac{\langle2|p_H|1]\langle4|p_H|1]}{3s_{234}[23][34]}-\frac{2[12]\langle23\rangle[31]^2}{3[23]^2[41][34]}$$

$$+\frac{\langle24\rangle\langle34\rangle\langle3|p_H|1][41]}{3s_{124}s_{12}[42]}-\frac{[14]^2\langle43\rangle^2}{s_{12}[42]^2}\bigg)\bigg\}+\bigg\{(2\leftrightarrow4)\bigg\}.\qquad(5.2.107)$$

Note that in the last term of the first bracket of Eq. (5.2.107), the $2\leftrightarrow4$ swapping is applied compared to the original formula in [120].

$\mathrm{F}_3^{3\mathrm{m}}$ is the three-mass triangle functions while the $W^{(1)}$, $W^{(2)}$ and $W^{(3)}$ functions are combinations of the finite pieces of one-mass ($\mathrm{F}_{4\mathrm{F}}^{1\mathrm{m}}$) and two-mass hard ($\mathrm{F}_{4\mathrm{F}}^{2\mathrm{mh}}$) box functions:

$$W^{(1)}\quad=\quad\mathrm{F}_{4\mathrm{F}}^{1\mathrm{m}}(s_{23},s_{34};s_{234})+\mathrm{F}_{4\mathrm{F}}^{2\mathrm{mh}}(s_{41},s_{234};m_H^2,s_{23})+\mathrm{F}_{4\mathrm{F}}^{2\mathrm{mh}}(s_{12},s_{234};s_{34},m_H^2),$$

$$W^{(2)}\quad=\quad\mathrm{F}_{4\mathrm{F}}^{1\mathrm{m}}(s_{14},s_{34};s_{134})+\mathrm{F}_{4\mathrm{F}}^{2\mathrm{mh}}(s_{12},s_{134};m_H^2,s_{34})+\mathrm{F}_{4\mathrm{F}}^{2\mathrm{mh}}(s_{23},s_{134};s_{14},m_H^2),$$

$$W^{(3)} \;=\; \mathrm{F}^{\mathrm{1m}}_{4\mathrm{F}}(s_{12}, s_{14}; s_{124}) + \mathrm{F}^{\mathrm{2mh}}_{4\mathrm{F}}(s_{23}, s_{124}; m_H^2, s_{14}) + \mathrm{F}^{\mathrm{2mh}}_{4\mathrm{F}}(s_{34}, s_{124}; s_{12}, m_H^2).$$

$$(5.2.108)$$

The coefficient $C_{3;\phi|12|34}$ of the three-mass triangle function $\mathrm{F}_3^{\mathrm{3m}}$ can be written as [122],

$$C_{3;\phi|12|34}(\phi, 1_g^+, 2_g^-, 3_g^-, 4_g^-) = \sum_{\gamma = \gamma_\pm} -\frac{m_\phi^4 \langle K_1^\flat 2 \rangle^3 \langle 34 \rangle^3}{2\gamma(\gamma + m_\phi^2)\langle K_1^\flat 1 \rangle \langle K_1^\flat 3 \rangle \langle K_1^\flat 4 \rangle \langle 12 \rangle} \qquad (5.2.109)$$

where massless vector $K_1^\flat$ is given by,

$$K_1^{\flat\,\mu} = \gamma \frac{\gamma K_1^\mu - K_1^2 K_2^\mu}{\gamma^2 - K_1^2 K_2^2}, \qquad (5.2.110)$$

where K_1, K_2 (and K_3) are the momenta of the three off-shell legs and where γ is determined by the two solutions that ensure that $K_1^\flat$ is light-like, that is,

$$\gamma^2 - 2K_1 \cdot K_2 \gamma + K_1^2 K_2^2 = 0, \qquad (5.2.111)$$

$$\gamma_\pm = K_1 \cdot K_2 \pm \sqrt{(K_1 \cdot K_2)^2 - K_1^2 K_2^2}. \qquad (5.2.112)$$

The momentum conservation condition implies that $K_1 + K_2 + K_3 = 0$ and in this case,

$$K_1^\mu = -p_1^\mu - p_2^\mu - p_3^\mu - p_4^\mu, \qquad K_2^\mu = p_1^\mu + p_2^\mu, \qquad K_3^\mu = p_3^\mu + p_4^\mu. \qquad (5.2.113)$$

The logarithmic cut-completion terms are defined in terms of the following functions,

$$
\begin{aligned}
\hat{L}_3(s,t) &= L_3(s,t) - \frac{1}{2(s-t)^2}\left(\frac{1}{s} + \frac{1}{t}\right), \\
\hat{L}_2(s,t) &= L_2(s,t) - \frac{1}{2(s-t)}\left(\frac{1}{s} + \frac{1}{t}\right), \\
\hat{L}_1(s,t) &= L_1(s,t), \\
\hat{L}_0(s,t) &= L_0(s,t),
\end{aligned}
\qquad (5.2.114)
$$

with

$$L_k(s,t) = \frac{\log(s/t)}{(s-t)^k}. \qquad (5.2.115)$$

There are two separate numerical problems in the evaluation of this amplitude. First, the coefficient of the three-mass triangle $C_{3;\phi|12|34}$ as presented in [122] is

unstable in the limit where the third momentum input of the triangle becomes massless i.e. $(p_3 + p_4)^2 \to 0$. Second, there are large cancellations between the cut-constructable terms $\hat{L}_k(s_{124}, s_{12})$ and the rational contribution in the limit where gluon 2 is soft. In both cases, the expression can be rewritten to give a numerically stable result.

Improving the numerical stability of the coefficients of the three mass triangle integrals

In general the coefficient of the three-mass triangle $C_{3;\phi|12|34}(\phi, 1_g^+, 2_g^-, 3_g^-, 4_g^-)$ is sensitive to the three massive momentum inputs K_1, K_2 and K_3 when one of the legs becomes massless, e.g. $K_3^2 \to 0$. In this limit,

$$-2K_1 \cdot K_2 \to K_1^2 + K_2^2, \tag{5.2.116}$$

and equation (5.2.111) now implies

$$\gamma_+ \to -K_2^2 \qquad \gamma_- \to -K_1^2. \tag{5.2.117}$$

By rearranging Eq. (5.2.110) in terms of K_3,

$$K_1^{\flat-\,\mu} = \gamma_- \frac{\gamma_- K_1^\mu + K_1^2(K_1^\mu + K_3^\mu)}{\gamma_-^2 - K_1^2 K_2^2}, \tag{5.2.118}$$

we immediately see that there is a potentially large cancellation between the first and second terms in the numerator - both K_1^2 and γ_- are large so that the coefficient of K_1 is given by the difference of large quantities. Since $K_1^{\flat\,\mu}$ is repeatedly used in Eq. (5.2.109) there are large numerical instabilities.

Similarly, if we eliminate K_1^μ from the numerator of Eq. (5.2.110),

$$K_1^{\flat+\,\mu} = \gamma_+ \frac{-\gamma_+(K_2^\mu + K_3^\mu) - (K_2 + K_3)^2 K_2^\mu}{\gamma_+^2 - K_1^2 K_2^2}. \tag{5.2.119}$$

When $K_3^2 \to 0$, there are again large numerical cancellations.

The solution is quite straightforward. Quite generally, Eq. (5.2.112) are the solutions of Eq. (5.2.111) and satisfy the following identities:

$$\gamma_+ + \gamma_- = 2K_1 \cdot K_2, \tag{5.2.120}$$

$$\gamma_+ \gamma_- = K_1^2 K_2^2, \tag{5.2.121}$$

$$(\gamma_- + K_1^2)(\gamma_+ + K_1^2) = K_1^2 K_3^2, \tag{5.2.122}$$

$$(\gamma_- + K_2^2)(\gamma_+ + K_2^2) = K_2^2 K_3^2. \tag{5.2.123}$$

In each of the last two equations, the LHS is composed of a "large" bracket and "small" one (obtained by the cancellation of large terms). Therefore, we should systematically make the replacement,

$$(\gamma_- + K_1^2) = \frac{K_1^2 K_3^2}{(\gamma_+ + K_1^2)}, \tag{5.2.124}$$

$$(\gamma_+ + K_2^2) = \frac{K_2^2 K_3^2}{(\gamma_- + K_2^2)}, \tag{5.2.125}$$

ensuring that a large numerical cancellation is replaced by a precise determination of the small remainder proportional to K_3^2.

In other words, rewriting Eq. (5.2.109) as,

$$C_{3;\phi|12|34}(\phi, 1_g^+, 2_g^-, 3_g^-, 4_g^-) = \sum_{\gamma=\gamma_\pm} \frac{m_\phi^4 \langle 34 \rangle^3 \langle 2|K_1^\flat|1] \langle 2|K_1^\flat|3] \langle 2|K_1^\flat|4]}{2\gamma(\gamma + m_\phi^2)\langle 12 \rangle s_{1K_1^\flat} s_{3K_1^\flat} s_{4K_1^\flat}}, \tag{5.2.126}$$

with

$$\langle 2|K_1^\flat|1] = \frac{-\gamma^2(\langle 23 \rangle [31] + \langle 24 \rangle [41])}{\gamma^2 - K_1^2 K_2^2}, \tag{5.2.127}$$

$$\langle 2|K_1^\flat|3] = \frac{-\gamma(K_1^2 + \gamma)\langle 21 \rangle [13] - \gamma^2 \langle 24 \rangle [43]}{\gamma^2 - K_1^2 K_2^2}, \tag{5.2.128}$$

$$\langle 2|K_1^\flat|4] = \frac{-\gamma(K_1^2 + \gamma)\langle 21 \rangle [14] - \gamma^2 \langle 23 \rangle [34]}{\gamma^2 - K_1^2 K_2^2}, \tag{5.2.129}$$

$$s_{1K_1^\flat} = \frac{-\gamma(K_1^2 + \gamma)s_{12} - \gamma^2(s_{13} + s_{14})}{\gamma^2 - K_1^2 K_2^2}, \tag{5.2.130}$$

$$s_{3K_1^\flat} = \frac{-\gamma(K_1^2 + \gamma)(s_{13} + s_{23}) - \gamma^2 s_{34}}{\gamma^2 - K_1^2 K_2^2}, \tag{5.2.131}$$

$$s_{4K_1^\flat} = \frac{-\gamma(K_1^2 + \gamma)(s_{14} + s_{24}) - \gamma^2 s_{34}}{\gamma^2 - K_1^2 K_2^2}, \tag{5.2.132}$$

then $\langle 2|K_1^\flat|3]$, $\langle 2|K_1^\flat|4]$, $s_{1K_1^\flat}$, $s_{3K_1^\flat}$ and $s_{4K_1^\flat}$ can be seen to have large cancellations in the $K_3^2 \to 0$ limits, i.e. p_3 (or p_4)$\to 0$ limit or $p_3 \| p_4$. By repeatedly using the identities in (5.2.124), $s_{3K_1^\flat}^-$ can be rewritten as

$$s_{3K_1^\flat}^- = -\frac{\gamma_- s_{34} m_H^2}{\gamma_+(\gamma_-^2 - m_H^2 s_{12})(m_H^2 + \gamma_+)}\left(\frac{m_H^2 s_{12} s_{34}}{\gamma_- + s_{12}} - (s_{14} + s_{24} + s_{34})\gamma_+\right). \tag{5.2.133}$$

Now $s_{3K_1^\flat}^-$ is explicitly proportional to the small parameter s_{34} and there are no large cancellations. $s_{1K_1^\flat}^-$ and $s_{4K_1^\flat}^-$ can be rewritten in a similar way.

Note that the same numerical instability issue appears in the coefficients of the three mass triangle integrals from the NMHV amplitudes in the two-quark, two-gluon case, namely $C_{3;\phi|12|34}(\phi, 1_{\bar q}^-, 2_q^+, 3_g^-, 4_g^-)$ and $C_{3;\phi|41|23}(\phi, 1_{\bar q}^-, 2_q^+, 3_g^-, 4_g^-)$. The rewriting procedure is exactly the same for the two-quark, two-gluon case as for the four-gluon case.

Improving the numerical stability of the cut-constructable terms and the rational terms

In the single soft gluon limit, $p_2 \to 0$ there are multiple large cancellations between the cut-constructable terms proportional to $\hat{L}_k(s_{124}, s_{12})$ and the rational contribution. In this limit, $\langle 2i \rangle$ and $[2i]$ are proportional to a small quantity Δ while invariants s_{i2} are proportional to Δ^2. The large contributions come from a variety of sources. First there are explicit factors of spinor products and invariants such as the rational contribution,

$$\frac{[14]^2 \langle 43 \rangle^2}{2 s_{12} [42]^2} \propto \mathcal{O}\left(\frac{1}{\Delta^4}\right). \tag{5.2.134}$$

Second, there are hidden divergences within the definition of the rational parts of the $\hat{L}_k(s_{124}, s_{12})$ functions, so that,

$$\frac{s_{124} \langle 34 \rangle^2 [41]^2}{[42]^2} \hat{L}_2(s_{124}, s_{12}) \propto \mathcal{O}\left(\frac{1}{\Delta^4}\right). \tag{5.2.135}$$

These terms need to be rearranged so that the divergence of individual terms is no worse than the overall divergence $\mathcal{O}\left(\frac{1}{\Delta^2}\right)$. Finally, there is a logarithmic divergence present $\ln\left(\frac{s_{124}}{s_{12}}\right)$ which should be explicitly multiplied by spinor factors not more divergent than $\mathcal{O}\left(\frac{1}{\Delta^2}\right)$.

The cut-constructable contribution containing $\hat{L}_i(s_{124}, s_{12})$ $(i = 3, 2, 1, 0)$ in Eq. (5.2.106) is,

$$
\begin{aligned}
F_4(\hat{L}_i(s_{124}, s_{12})) \propto &+ \frac{2 s_{124} \langle 24 \rangle \langle 34 \rangle^2 [41]^2}{3 [42]} \hat{L}_3 \\
&+ \frac{\langle 34 \rangle [41] \left(3 s_{124} \langle 34 \rangle [41] + \langle 24 \rangle \langle 3|p_H|1] [42]\right)}{3 [42]^2} \hat{L}_2 \\
&+ \left(\frac{2 s_{124} \langle 34 \rangle^2 [41]^2}{\langle 24 \rangle [42]^3} - \frac{\langle 24 \rangle \langle 3|p_H|1]^2}{3 s_{124} [42]}\right) \hat{L}_1
\end{aligned}
$$

$$+ \frac{\langle 3|p_H|1](4s_{124}\langle 34\rangle[41] + \langle 3|p_H|1](2s_{14} + s_{24}))}{s_{124}\langle 24\rangle[42]^3} \hat{L}_0. \quad (5.2.136)$$

For the convenience of writing, all $\hat{L}_i(s_{124}, s_{12})$ functions are now in short notation $\hat{L}_i$.

Inserting the definition of p_H and using the property $\langle 3|p_H|1] = -\langle 32\rangle[31] - \langle 34\rangle[41]$, Eq. (5.2.136) can be arranged as,

$$F_4(\hat{L}_i(s_{124}, s_{12})) \propto + \frac{\langle 34\rangle^2[41]^2}{3[42]^2}\left(+ s_{24}(2s_{124}\hat{L}_3 - \hat{L}_2 - \frac{1}{s_{124}}\hat{L}_1)\right.$$

$$+ 3s_{124}\hat{L}_2 - \frac{3}{s_{124}}\hat{L}_0$$

$$\left. + \frac{1}{s_{24}}(6s_{124}\hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}}\hat{L}_0)\right)$$

$$- \frac{\langle 34\rangle[41]\langle 32\rangle[21]}{3[42]^2}\left(+ s_{24}\hat{L}_2 + 2\frac{s_{24}}{s_{124}}\hat{L}_1 + \frac{6}{s_{124}}\hat{L}_0 + \frac{12s_{12}}{s_{124}s_{24}}\hat{L}_0\right)$$

$$+ \frac{\langle 32\rangle^2[21]^2}{3[42]^2}\left(- \frac{s_{24}}{s_{124}}\hat{L}_1 + \frac{3}{s_{124}}\hat{L}_0 + \frac{6s_{14}}{s_{124}s_{24}}\hat{L}_0\right). \quad (5.2.137)$$

The terms subject to large cancellations are concentrated in lines 2 and 3 of Eq. (5.2.137) where the naive divergences of individual terms are $\mathcal{O}(\Delta^{-4})$,

$$\frac{\langle 34\rangle^2[41]^2}{3[42]^2}\left(3s_{124}\hat{L}_2 - \frac{3}{s_{124}}\hat{L}_0\right) \propto \mathcal{O}(\Delta^{-4}), \quad (5.2.138)$$

$$\frac{\langle 34\rangle^2[41]^2}{3[42]^2 s_{24}}\left(6s_{124}\hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}}\hat{L}_0\right) \propto \mathcal{O}(\Delta^{-4}). \quad (5.2.139)$$

According to the definition in Eqs. (5.2.114) and (5.2.115), we have the following identities:

$$s\hat{L}_3(s, t) = t\hat{L}_3(s, t) + \hat{L}_2(s, t),$$

$$s\hat{L}_2(s, t) = t\hat{L}_2(s, t) + \hat{L}_1(s, t) - \frac{1}{2}\left(\frac{1}{s} + \frac{1}{t}\right),$$

$$s\hat{L}_1(s, t) = t\hat{L}_1(s, t) + \hat{L}_0(s, t),$$

$$\frac{1}{s}\hat{L}_1(s, t) = \hat{L}_2(s, t) - \frac{t}{s}\hat{L}_2(s, t) + \frac{1}{2s}\left(\frac{1}{s} + \frac{1}{t}\right),$$

$$\frac{1}{s}\hat{L}_0(s, t) = \hat{L}_1 - \frac{t}{s}\hat{L}_1. \quad (5.2.140)$$

We can use these identities to rewrite Eqs. (5.2.138) and (5.2.139) so that the $\mathcal{O}(\Delta^{-4})$ divergence is made explicit (and can be cancelled directly against the rational contribution (5.2.134)), the $\mathcal{O}(\Delta^{-3})$ is cancelled and all remaining terms are explicitly

$\mathcal{O}(\Delta^{-2})$. Inserting (5.2.140) into (5.2.138) we have,

$$\frac{\langle 34\rangle^2[41]^2}{3[42]^2}\left(3s_{124}\hat{L}_2 - \frac{3}{s_{124}}\hat{L}_0\right) \equiv \frac{\langle 34\rangle^2[41]^2}{[42]^2}\left(s_{12}\hat{L}_2 + \frac{s_{12}}{s_{124}}\hat{L}_1 - \frac{1}{2}\left(\frac{1}{s_{124}} + \frac{1}{s_{12}}\right)\right)$$

$$\sim \mathcal{O}(\Delta^{-2}) - \frac{\langle 34\rangle^2[41]^2}{2s_{12}[42]^2}. \tag{5.2.141}$$

Similarly, (5.2.139) becomes,

$$\frac{\langle 34\rangle^2[41]^2}{3[42]^2 s_{24}}\left(6s_{124}\hat{L}_1 - 6\hat{L}_0 - \frac{6s_{12}}{s_{124}}\hat{L}_0\right) \equiv \frac{\langle 34\rangle^2[41]^2}{[42]^2 s_{24}}\left(\frac{2s_{12}^2}{s_{124}}\hat{L}_1\right) \sim \mathcal{O}(\Delta^{-2}). \tag{5.2.142}$$

With further rewriting of terms in (5.2.137) using identities in (5.2.140) and moving the rational terms from the rewriting into (5.2.107), we have the final numerically stable result:

$$F_4(H, g_1^+, g_2^-, g_3^-, g_4^-)$$
$$= \left\{ -\frac{s_{234}^3}{4\langle 1|p_H|2\rangle\langle 1|p_H|4\rangle[23][34]}W^{(1)}\right.$$
$$-\left(\frac{\langle 2|p_H|1\rangle^3}{2s_{134}\langle 2|p_H|3\rangle[34][41]} + \frac{\langle 34\rangle^3 m_H^4}{2s_{134}\langle 1|p_H|2\rangle\langle 3|p_H|2\rangle\langle 41\rangle}\right)W^{(2)}$$
$$+\frac{1}{4s_{124}}\left(\frac{\langle 3|p_H|1\rangle^4}{\langle 3|p_H|2\rangle\langle 3|p_H|4\rangle[21][41]} + \frac{\langle 24\rangle^4 m_H^4}{\langle 12\rangle\langle 14\rangle\langle 2|p_H|3\rangle\langle 4|p_H|3\rangle}\right)W^{(3)}$$
$$+2C_{3;\phi|12|34}(\phi, 1_g^+, 2_g^-, 3_g^-, 4_g^-)\,\mathrm{F}_3^{3m}(m_H^2, s_{12}, s_{34})$$
$$+\left(1 - \frac{N_f}{4N_c}\right)\left(\frac{\langle 3|p_H|1\rangle^2}{s_{124}[24]^2}\,\mathrm{F}_{4F}^{1m}(s_{12}, s_{14}; s_{124})\right.$$
$$\left.-\frac{4\langle 24\rangle\langle 3|p_H|1\rangle^2}{s_{124}[42]}\hat{L}_1(s_{124}, s_{12}) + \frac{4\langle 23\rangle\langle 4|p_H|1\rangle^2}{s_{123}[32]}\hat{L}_1(s_{123}, s_{12})\right)$$
$$-\left(1 - \frac{N_f}{N_c} + \frac{N_s}{N_c}\right)\left(\frac{[12][41]\langle 3|p_H|2\rangle\langle 3|p_H|4\rangle}{2s_{124}[24]^4}\,\mathrm{F}_{4F}^{1m}(s_{12}, s_{14}; s_{124})\right)$$
$$+\frac{\langle 34\rangle^2[41]^2}{3[42]^2 s_{24}}\left(2s_{24}s_{12}\hat{L}_3(s_{124}, s_{12}) + \frac{s_{24}s_{12}}{s_{124}}\hat{L}_2(s_{124}, s_{12})\right.$$
$$\left.+ 3s_{12}\hat{L}_2(s_{124}, s_{12}) + \frac{3s_{12}}{s_{124}}\hat{L}_1(s_{124}, s_{12}) + \frac{6s_{12}^2}{s_{24}s_{124}}\hat{L}_1(s_{124}, s_{12})\right)$$
$$-\frac{\langle 34\rangle[41]\langle 32\rangle[21]}{3[42]^2}\left(s_{24}\hat{L}_2(s_{124}, s_{12}) + \frac{2s_{24}}{s_{124}}\hat{L}_1(s_{124}, s_{12})\right.$$
$$\left.+ \frac{6}{s_{124}}\hat{L}_0(s_{124}, s_{12}) + \frac{12s_{12}}{s_{124}s_{24}}\hat{L}_0(s_{124}, s_{12})\right)$$
$$+\frac{\langle 32\rangle^2[21]^2}{3[42]^2}\left(-\frac{s_{24}}{s_{124}}\hat{L}_1(s_{124}, s_{12}) + \frac{3}{s_{124}}\hat{L}_0(s_{124}, s_{12}) + \frac{6s_{14}}{s_{124}s_{24}}\hat{L}_0(s_{124}, s_{12})\right)$$
$$-\frac{2s_{123}\langle 23\rangle\langle 34\rangle^2[31]^2}{3[32]}\hat{L}_3(s_{123}, s_{12}) + \frac{\langle 23\rangle\langle 34\rangle[31]\langle 4|p_H|1\rangle}{3[32]}\hat{L}_2(s_{123}, s_{12})$$

$$+\frac{\langle 23\rangle\langle 4|p_H|1]^2}{3s_{123}[32]}\hat{L}_1\left(s_{123},s_{12}\right)\bigg)\bigg\}+\bigg\{(2\leftrightarrow 4)\bigg\}, \tag{5.2.143}$$

$$
\begin{aligned}
&R_4(H,g_1^+,g_2^-,g_3^-,g_4^-)\\
&=\quad\bigg\{\bigg(1-\frac{N_f}{N_c}+\frac{N_s}{N_c}\bigg)\frac{1}{2}\bigg(\frac{\langle 23\rangle\langle 34\rangle\langle 4|p_H|1][31]}{3s_{123}\langle 12\rangle[21][32]}-\frac{\langle 3|p_H|1]^2}{s_{124}[42]^2}\\
&\quad-\frac{\langle 24\rangle(s_{23}s_{24}+s_{23}s_{34}+s_{24}s_{34})}{3\langle 12\rangle\langle 14\rangle[23][34][42]}+\frac{\langle 2|p_H|1]\langle 4|p_H|1]}{3s_{234}[23][34]}-\frac{2[12]\langle 23\rangle[31]^2}{3[23]^2[41][34]}\\
&\quad-\frac{\langle 24\rangle\langle 34\rangle\langle 32\rangle[21][41]}{3s_{124}s_{12}[42]}+\frac{\langle 24\rangle\langle 34\rangle^2[41]^2}{3s_{124}^2[42]}+\frac{[14]^2\langle 43\rangle^2}{s_{124}[42]^2}\bigg)\bigg\}\\
&\quad+\bigg\{(2\leftrightarrow 4)\bigg\}. \tag{5.2.144}
\end{aligned}
$$

With compact (and stable) result for Higgs boson plus up to five parton matrix elements at tree level, stable Higgs boson plus four parton matrix elements at one-loop level and two-loop results for Higgs boson plus three parton matrix elements [119], we have all the matrix elements needed for $pp \to H$+jet process at NNLO. We are going to use those matrix elements repeatedly for different initial state parton channels in chapter 6, 7, 8 and 9.

Chapter 6

Production of a Higgs Boson Plus Jet from Gluon Fusion

The subprocesses contributing to the $pp \rightarrow H$+jet process can be categorized according to the initial state parton identities. In general, for proton proton collision, the initial parton combinations could be of three main types:

1) gg,

2) qg (including $\bar{q}g$, gq $g\bar{q}$),

3) qq (including $q\bar{q}$, $\bar{q}\bar{q}$, $\bar{q}q$, qQ, $q\bar{Q}$ $\bar{Q}q$ and $\bar{q}\bar{Q}$).

In this chapter, I will discuss the $gg \rightarrow H$+jet processes for up to the third order of the perturbative expansions (LO, NLO and NNLO) for the cross section that is fully differential in the Higgs boson plus jet observables. I will take the leading colour contribution in the gluons plus Higgs boson channel as an example of the implementation of the antenna subtraction method as introduced in chapter 3 and 4. The antenna subtraction method is used to construct the explicit subtraction terms that regulate the infrared divergences at NLO and NNLO. Spike plots are introduced to illustrate the numerical quality of the subtraction terms which must mimic the matrix elements in the unresolved regions of phase space. Explicit IR divergences from the matrix elements are analytically cancelled when combined with integrated antenna subtraction terms and mass factorization terms. The numerical results

in terms of differential cross sections for the transverse momentum and rapidity distributions for both the Higgs boson and the associated jet are studied in [84] and are discussed in chapter 10.

6.1 Normalisation factor for $pp \rightarrow H$+jet cross sections

For convenience, we collect all factors involving the strong coupling α_s,

$$\alpha_s = \frac{g^2}{4\pi}, \tag{6.1.1}$$

the Hgg effective coupling in the heavy top mass limit (at LO),

$$\mathcal{C} = \frac{\alpha_s}{6\pi v}, \tag{6.1.2}$$

where v is the energy scale of the electroweak symmetry breaking (the vev of Higgs field introduced in Eq. (2.1.19)), and the leading order colour factor as an overall normalisation factor. At leading order, this overall factor is given by,

$$\mathcal{N}_{LO} = \frac{\mathcal{C}^2}{4} g^2 N(N^2 - 1). \tag{6.1.3}$$

This universal normalisation factor will be used through out all the $pp \rightarrow H$+jet processes at LO, NLO and NNLO in this chapter and chapter 7 and 9 and any leftover dependence on α_s, N and N_F made explicit for each separate contribution.

For example, for the colour ordered tree-level scattering amplitude for a Higgs boson plus three gluons of Eq. (3.1.4), the normalisation factor for A^0_{3gH} is simply,

$$\mathcal{N}^A_{LO} = \mathcal{N}_{LO}, \tag{6.1.4}$$

while for the colour ordered tree-level scattering amplitude for a Higgs boson plus a quark pair and a gluon at tree level given in Eq. (3.1.5), one finds the normalisation factor for B^0_{1gH}:

$$\mathcal{N}^B_{LO} = \frac{1}{N}\mathcal{N}_{LO}. \tag{6.1.5}$$

Each additional radiation (or loop) introduces additional factors of $g^2 N$ so that at NLO we have overall factors for the real rediation and virtual contributions such that (the factor of N is made explicit elsewhere in the cross section),

$$\mathcal{N}_{NLO}^R = \mathcal{N}_{LO} \left(\frac{\alpha_s}{2\pi} \right) \frac{\bar{C}(\epsilon)}{C(\epsilon)}, \tag{6.1.6}$$

$$\mathcal{N}_{NLO}^V = \mathcal{N}_{LO} \left(\frac{\alpha_s}{2\pi} \right) \bar{C}(\epsilon), \tag{6.1.7}$$

where

$$C(\epsilon) = (4\pi)^\epsilon \frac{e^{-\epsilon\gamma}}{8\pi^2}, \tag{6.1.8}$$

$$\bar{C}(\epsilon) = (4\pi)^\epsilon e^{-\epsilon\gamma}. \tag{6.1.9}$$

A factor of $C(\epsilon)^{-1}$ is produced when the parton multiplicity in the phase space is increased by one and is ultimately absorbed into the integration of antenna functions in Eqs. (3.5.72) and (4.7.143). Note that each power of the (bare) coupling is accompanied by a factor of $\bar{C}(\epsilon)$ that is absorbed in the process of renormalization.

At NNLO, we have separate normalisations for the double real, the real-virtual and double virtual contributions,

$$\mathcal{N}_{NNLO}^{RR} = \mathcal{N}_{LO} \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)^2}, \tag{6.1.10}$$

$$\mathcal{N}_{NNLO}^{RV} = \mathcal{N}_{LO} \left(\frac{\alpha_s}{2\pi} \right)^2 \frac{\bar{C}(\epsilon)^2}{C(\epsilon)}, \tag{6.1.11}$$

$$\mathcal{N}_{NNLO}^{VV} = \mathcal{N}_{LO} \left(\frac{\alpha_s}{2\pi} \right)^2 \bar{C}(\epsilon)^2. \tag{6.1.12}$$

6.2 gg **initiated cross sections at LO**

The gluon fusion to Higgs boson plus one jet process at Born level (Leading order) has only one contribution from the $gg \to H + g$ process. The spin and colour averaged differential cross section is

$$d\hat{\sigma}_{gg}^B = \mathcal{N}_{gg} \mathcal{N}_{LO} \, d\Phi_{H+1}(p_3, p_H; p_1, p_2) \left\{ 2A_{3gH}^0(\hat{1}, \hat{2}, 3) \right\} J_1^{(1)}(p_3). \tag{6.2.13}$$

The factor of 2 in above equation comes from the two non-cyclic colour orderings of $A_{3gH}^0(\hat{1}, \hat{2}, 3)$ and $A_{3gH}^0(\hat{1}, 3, \hat{2})$. By using line-reversal symmetry the two colour-ordered matrix elements give equal contributions. The $\mathcal{N}_{ij}$ factor contains overall

multiplicative factors due to spin and colour averaging of the initial state partons,

$$\mathcal{N}_{ij} = \frac{1}{(2s) \cdot 4 \cdot C_i \cdot C_j},$$ (6.2.14)

where $\sqrt{s}$ is the hadron-hadron centre of mass energy and C_i is the number of colour states for initial state parton type i. For a quark or anti-quark $C_q = C_{\bar{q}} = N$ and for a gluon $C_g = N^2 - 1$.

6.3 *gg* initiated cross sections at NLO

6.3.1 Real contribution

The real radiation contribution comes from the $gg \to H + gg$ and $gg \to H + q\bar{q}$ processes,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{gg}^R =& \mathcal{N}_{gg} \mathcal{N}_{NLO}^R \, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2) \Bigg\{ \\
& \frac{2N}{2!} \left[A_{4gH}^0(\hat{1}, \hat{2}, 3, 4) + A_{4gH}^0(\hat{1}, \hat{2}, 4, 3) + A_{4gH}^0(\hat{1}, 3, \hat{2}, 4) \right] \\
& + N_F \left[B_{2gH}^0(3_q, \hat{1}, \hat{2}, 4_{\bar{q}}) + B_{2gH}^0(3_q, \hat{2}, \hat{1}, 4_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}_{2gH}^0(3_q, \widetilde{\hat{1}}, \widetilde{\hat{2}}, 4_{\bar{q}}) \right] \Bigg\} J_1^{(2)}(p_3, p_4).
\end{aligned}
$$ (6.3.15)

The details of the squared matrix elements in Eq.(6.3.15) are introduced in section 3.1.3 and 5.1.7. The $1/2!$ coefficient associated with A_{4gH}^0 matrix elements is the averaging factor for the two identical gluons in the final states. The N_F factor associated with B_{2gH}^0 and $\widetilde{B}_{2gH}^0$ is to sum over all the active quark flavours in the final state.

6.3.2 Virtual contribution

The one-loop contribution is from the $gg \to H + g$ process and the differential cross section is given by,

$$
\mathrm{d}\hat{\sigma}_{gg}^V = \mathcal{N}_{gg} \mathcal{N}_{NLO}^V \, \mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2) \left[2N A_{3gH}^1(\hat{1}, \hat{2}, 3) + 2N_F \widehat{A}_{3gH}^1(\hat{1}, \hat{2}, 3) \right]
$$
$$
\times J_1^{(1)}(p_3)
$$ (6.3.16)

Details of the relevant matrix elements in Eq.(6.3.16) are given in section 3.1.3.

subtraction term	maple	matrix element
$A_{4gH,NLO}^{0,XS}(\hat{1},\hat{2},i,j)$	`A4g0HXSNLO`	$A_{4gH}^{0}(\hat{1},\hat{2},i,j)$
$A_{4gH,NLO}^{0,YS}(\hat{1},i,\hat{2},j)$	`A4g0HYSNLO`	$A_{4gH}^{0}(\hat{1},i,\hat{2},j)$
$B_{2gH,NLO}^{0,XS}(i_q,\hat{1},\hat{2},j_{\bar{q}})$	`ggB2g0HXSNLO`	$B_{2gH}^{0}(i_q,\hat{1},\hat{2},j_{\bar{q}})$
$\widetilde{B}_{2gH,NLO}^{0,XS}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$	`ggBt2g0HXSNLO`	$\widetilde{B}_{2gH}^{0}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$

Table 6.1: NLO antenna subtraction terms for real contributions in $gg \to H+$jet process and their relation to the matrix elements

6.4 gg initiated subtraction terms at NLO

6.4.1 Real subtraction terms

Using the NLO antenna subtraction method introduced in section 3.3.3, one can construct the antenna subtraction terms to mimic the implicit IR divergences in Eq. (6.3.15) such that

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{gg,NLO}^{S} =& \mathcal{N}_{gg}\, \mathcal{N}_{NLO}^{R}\, \mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{ \\
& \frac{2N}{2!}\left[A_{4gH,NLO}^{0,XS}(\hat{1},\hat{2},3,4) + A_{4gH,NLO}^{0,XS}(\hat{1},\hat{2},4,3) + A_{4gH,NLO}^{0,YS}(\hat{1},3,\hat{2},4)\right] \\
& +N_F\left[B_{2gH,NLO}^{0,XS}(3_q,\hat{1},\hat{2},4_{\bar{q}}) + B_{2gH,NLO}^{0,XS}(3_q,\hat{2},\hat{1},4_{\bar{q}})\right. \\
& \left.\left. - \frac{1}{N^2}\widetilde{B}_{2gH,NLO}^{0,XS}(3_q,\widetilde{\hat{1}},\widetilde{\hat{2}},4_{\bar{q}})\right]\right\}.
\end{aligned}
\tag{6.4.17}
$$

Here $A_{4gH,NLO}^{0,XS}(\hat{1},\hat{2},i,j)$, $A_{4gH,NLO}^{0,XS}(\hat{1},i,\hat{2},j)$, $B_{2gH,NLO}^{0,XS}(i_q,\hat{1},\hat{2},j_{\bar{q}})$ and $\widetilde{B}_{2gH,NLO}^{0,XS}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$ are functions designed to remove the IR divergences in $A_{4gH}^{0}(\hat{1},\hat{2},i,j)$, $A_{4gH}^{0}(\hat{1},i,\hat{2},j)$, $B_{2gH}^{0}(i_q,\hat{1},\hat{2},j_{\bar{q}})$ and $\widetilde{B}_{2gH}^{0}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$ with $\{i,j\} \subseteq \{3,4\}$. The corresponding relationships between subtraction terms, file name in the `NNLOJET` maple script and matrix elements are also summarised in Table 6.1.

Explicit formulas for each subtraction term are as follows:

$$
\begin{aligned}
A_{4gH,NLO}^{0,XS}(\hat{1},\hat{2},i,j) =& \\
& +\ f_{3,g}^{0}(2,i,j)\, A_{3gH}^{0}(1,\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)
\end{aligned}
$$

$$+ \quad f_{3,g}^0(1,j,i) \, A_{3gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1), \tag{6.4.18}$$

$$A_{4gH,NLO}^{0,XS}(\hat{1}, i, \hat{2}, j) =$$

$$+ \quad F_{3,gg}^0(1,i,2) \, A_{3gH}^0(\overline{1}, \overline{2}, j) \, J_1^{(1)}(\{p\}_1)$$

$$+ \quad F_{3,gg}^0(1,j,2) \, A_{3gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1), \tag{6.4.19}$$

$$B_{2gH,NLO}^{0,XS}(i_q, \hat{1}, \hat{2}, j_{\bar{q}}) =$$

$$- \quad d_{3,gg \to qg}^0(i,1,2) \, B_{1gH}^0(\overline{1}, \overline{2}, j) \, J_1^{(1)}(\{p\}_1)$$

$$- \quad d_{3,gg \to qg}^0(j,2,1) \, B_{1gH}^0(i, \overline{1}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2} G_{3,g}^0(1,i,j) \, A_{3gH}^0((\widetilde{ij}), \overline{1}, 2) \, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2} G_{3,g}^0(2,j,i) \, A_{3gH}^0((\widetilde{ij}), 1, \overline{2}) \, J_1^{(1)}(\{p\}_1), \tag{6.4.20}$$

$$\widetilde{B}_{2gH,NLO}^{0,XS}(i_q, \widetilde{\hat{1}}, \widetilde{\hat{2}}, j_{\bar{q}}) =$$

$$- \quad A_{3,g \to q}^0(i,1,j) \, B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_{3,g \to q}^0(i,2,j) \, B_{1gH}^0(\overline{2}, 1, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1). \tag{6.4.21}$$

Note that the minus signs multiplying the $d_{3,gg \to qg}^0(i,1,2)$ and $A_{3,g \to q}^0(i,1,j)$ antennae in $B_{2gH,NLO}^{0,XS}(i_q, \hat{1}, \hat{2}, j_{\bar{q}})$ and $\widetilde{B}_{2gH,NLO}^{0,XS}(i_q, \widetilde{\hat{1}}, \widetilde{\hat{2}}, j_{\bar{q}})$ come from the unresolved limit when a final state quark is collinear with initial state gluons. In this limit, crossing a fermion from the final state to the initial state will cause the scattering matrix element to automatically pick up a minus sign. In the subtraction terms that mimic such unresolved behaviour, the minus sign is made explicit.

To numerically test that the antenna subtraction terms given in Eqs. (6.4.18), (6.4.19), (6.4.20) and (6.4.21) remove the implicit IR divergences in Eq. (6.4.17) correctly, we generate a set of *spike plots* to illustrate that the subtraction terms converge to the matrix elements when approaching the unresolved limits. For each unresolved limit, a set of momenta (phase space points) are generated using RAMBO [171] such that the momenta satisfy a set of constraints that allow the unresolved limit to be approached in a controlled manner. For each phase space point in each limit, the ratio of the matrix element to the subtraction term is calculated,

$$R = \frac{d\hat{\sigma}^R}{d\hat{\sigma}_{NLO}^S}. \tag{6.4.22}$$

The calculation is repeated for 1,000 different phase space points in each unresolved limit. The constraints are then tightened to force the generating of phase space

points closer to the unresolved limits and the ratio is calculated for another 1,000 points. The process is repeated for the third time for a even tighter constraint and the histogram of the ratios of the three sets of constraints are plotted.
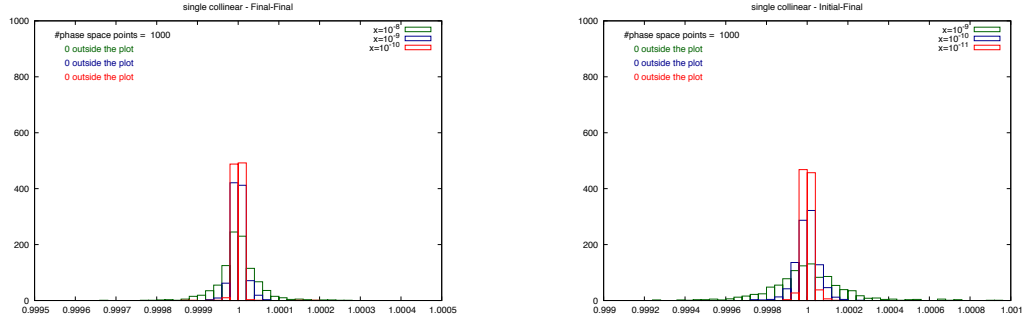
(a) Collinear limit between final-state partons i and j, such that $x = s_{ij}/s_{12}$ approaches zero in the unresolved limit.

(b) Collinear limit between initial-final partons 1 and i, such that $x = s_{1i}/s_{12}$ approaches zero in the unresolved limit.
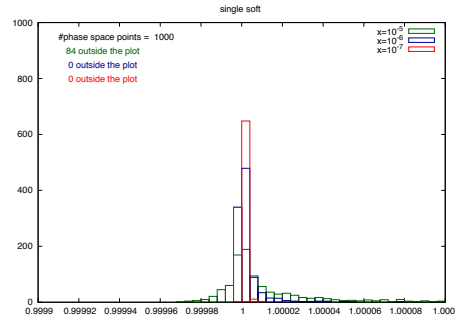
(c) Soft limit for soft parton i, such that $x = (s_{1i}+s_{2i})/s_{12}$ approaches zero in the unresolved limit.

Figure 6.1: Spike plots displaying the convergence of the subtraction terms in $\mathrm{d}\hat{\sigma}^S_{gg,NLO}$ to the matrix elements in $\mathrm{d}\hat{\sigma}^R_{gg}$ in various unresolved limits.

The spike plots for the final-final collinear limit, the initial-final collinear limit and soft limits are displayed in Figure 6.1. As can be seen from each plot, the distribution of ratios typically forms a spike around $R = 1$ indicating that the matrix elements and subtraction terms have similar sizes. Furthermore, as the unresolved limit is approached (green to blue to red), the spike typically becomes sharper.

subtraction term	maple	matrix element
$A_{3gH,NLO}^{1,XT}(\hat{1},\hat{2},i)$	`A3g1HXTNLO`	$A_{3gH}^1(\hat{1},\hat{2},i)$
$\widehat{A}_{3gH,NLO}^{1,XT}(\hat{1},\hat{2},i)$	`Ah3g1HXTNLO`	$\widehat{A}_{3gH}^1(\hat{1},\hat{2},i)$
$\widetilde{B}_{1gH,NLO}^{0,XT}(\hat{1}_q,\hat{2}_g,i_{\bar{q}})$	`ggBt2g0HXTNLO`	IR safe

Table 6.2: NLO antenna subtraction terms for virtual contributions in $gg \to H$+jet process and their relation to the matrix elements.

These plots provide graphical evidence for the convergence of the subtraction term to the matrix element in the unresolved limits.

6.4.2 Virtual subtraction terms

Using the NLO antenna subtraction method introduced in section 3.5.2, one can combine the integrated real subtraction and mass factorization terms to construct the virtual subtraction term, $\mathrm{d}\hat{\sigma}_{NLO}^T$ which removes the explicit IR divergences in Eq. (6.3.16),

$$
\mathrm{d}\hat{\sigma}_{gg,NLO}^T = \mathcal{N}_{gg}\,\mathcal{N}_{NLO}^V\,\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+1}(p_3,p_H;p_1,p_2)\times
$$
$$
\left[2NA_{3gH,NLO}^{1,XT}(\hat{1},\hat{2},3) + 2N_F\widehat{A}_{3gH,NLO}^{1,XT}(\hat{1},\hat{2},3) - \frac{N_F}{N^2}\widetilde{B}_{2gH,NLO}^{0,XT}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})\right].
$$
$$(6.4.23)$$

The corresponding relationships between subtraction terms, file name in the `NNLOJET` maple script and matrix elements are summarised in table 6.2.

The explicit formulae are as follows:

$$
A_{3gH,NLO}^{1,XT}(\hat{1},\hat{2},i) =
$$
$$
- \left[+ J_{2,GG}^{1,II}(s_{12}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,GG}^{1,IF}(s_{1i})\right] A_{3gH}^0(1,2,i)\, J_1^{(1)}(\{p\}_1) \quad (6.4.24)
$$
$$
\widehat{A}_{3gH,NLO}^{1,XT}(\hat{1},\hat{2},i) =
$$
$$
- \left[+ \hat{J}_{2,GG}^{1,FI}(s_{2i}) + \hat{J}_{2,GG}^{1,IF}(s_{1i}) + \hat{J}_{2,GG}^{1,II}(s_{12})\right] A_{3gH}^0(1,2,i)\, J_1^{(1)}(\{p\}_1)
$$
$$
- \frac{1}{2}\, J_{2,QG,gg\to qg}^{1,II}(s_{12})\, B_{1gH}^H(1,2,i)\, J_1^{(1)}(\{p\}_1)
$$
$$
- \frac{1}{2}\, J_{2,GQ,gg\to gq}^{1,II}(s_{12})\, B_{1gH}^H(i,1,2)\, J_1^{(1)}(\{p\}_1)
$$

$$
-\ \frac{1}{2}\ J^{1,II}_{2,QG,gg\to qg}(s_{12})\ B^H_{1gH}(i,2,1)\ J^{(1)}_1(\{p\}_1)
$$

$$
-\ \frac{1}{2}\ J^{1,II}_{2,GQ,gg\to gq}(s_{12})\ B^H_{1gH}(2,1,i)\ J^{(1)}_1(\{p\}_1) \tag{6.4.25}
$$

$$
\widetilde{B}^{0,XT}_{1gH,NLO}(\hat{1}_q,\hat{2}_g,i_{\bar{q}}) =
$$

$$
-\ 2J^{1,IF}_{2,QQ,g\to q}(s_{1i})\ B^H_{1gH}(1,2,i)\ J^{(1)}_1(\{p\}_1)
$$

$$
-\ 2J^{1,FI}_{2,QQ,g\to q}(s_{2i})\ B^H_{1gH}(2,1,i)\ J^{(1)}_1(\{p\}_1) \tag{6.4.26}
$$

Note that the two-quark matrix elements B^0_{1gH} appear in $\widehat{A}^{1,XT}_{3gH,NLO}(\hat{1},\hat{2},i)$ and $\widetilde{B}^{0,XT}_{1gH,NLO}(\hat{1}_q,\hat{2}_g,i_{\bar{q}})$ because of the initial state identity changing collinear limits in $\mathrm{d}\hat{\sigma}^S_{NLO}$ and the NLO mass factorization terms. As discussed in section 3.5.2, all the identity changing dipole functions ($\boldsymbol{J}^{(1)}_{2,a\to b}$) are IR finite. The jet functions in Eqs. (6.4.25) and (6.4.26) guarantee that the reduced tree level matrix elements are also IR safe. Thus the virtual subtraction terms associated with B^0_{1gH} only provide a finite contribution to the differential cross section.

The cancellation of explicit IR divergence between $\mathrm{d}\hat{\sigma}^V_{gg}$ and $\mathrm{d}\hat{\sigma}^T_{gg,NLO}$ can be checked analytically. The dipole functions associated with A^0_{3gH} in Eqs. (6.4.24) and (6.4.25) precisely match the Catani dipole structure [114, 115] in A^1_{3gH} and $\widehat{A}^1_{3gH}$.

6.5 gg initiated contribution at NNLO

6.5.1 Double real contribution

The double real contribution at NNLO for $gg \to H+$jet comes from the $gg \to H + ggg$ and $gg \to H + gq\bar{q}$ processes,

$$
\mathrm{d}\hat{\sigma}^{RR}_{gg} = \mathcal{N}_{gg}\,\mathcal{N}^{RR}_{NNLO}\,\mathrm{d}\Phi_{H+3}(p_3,p_4,p_5,p_H;p_1,p_2)\Bigg\{
$$

$$
+\frac{2N^2}{3!}\sum_{(i,j,k)\in P(3,4,5)}\Bigg[A^0_{5gH}(\hat{1},\hat{2},i,j,k) + A^0_{5gH}(\hat{1},i,\hat{2},j,k)\Bigg]
$$

$$
+NN_F\sum_{P(\hat{1},\hat{2})}\Bigg[B^0_{3gH}(4_q,\hat{1},\hat{2},3,5_{\bar{q}}) + B^0_{3gH}(4_q,3,\hat{1},\hat{2},5_{\bar{q}}) + B^0_{3gH}(4_q,\hat{1},3,\hat{2},5_{\bar{q}})\Bigg]
$$

$$
-\frac{N_F}{N}\sum_{P(\hat{1},\hat{2})}\Bigg[\widetilde{B}^0_{3gH}(4_q,\widetilde{\hat{1}},\hat{2},3,5_{\bar{q}}) + \widetilde{B}^0_{3gH}(4_q,\widetilde{\hat{1}},3,\hat{2},5_{\bar{q}}) + \widetilde{B}^0_{3gH}(4_q,\widetilde{3},\hat{1},\hat{2},5_{\bar{q}})\Bigg]
$$

$$+\frac{N_F(N^2+1)}{N^3}\left[\widetilde{\widetilde{B}}^0_{3gH}(4_q,\widetilde{\widehat{1}},\widetilde{\widehat{2}},\widetilde{3},5_{\bar{q}})\right]\Bigg\}J^{(3)}_1(p_3,p_4,p_5) \tag{6.5.27}$$

The squared matrix elements in Eq. (6.5.27) are discussed in section 4.1.2 and 5.1.8. The 1/3! coefficient associated with A^0_{5gH} matrix elements is the averaging factor for the three identical gluons in the final state.

6.5.2 Real-virtual contribution

The real-virtual contribution at NNLO for $gg \to H$+jet comes from the $gg \to H+gg$ and $gg \to H + q\bar{q}$ processes,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{RV}_{gg} =&\,\mathcal{N}_{gg}\,\mathcal{N}^{RV}_{NNLO}\,\mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{\\
&+\frac{2N^2}{2!}\left[A^1_{4gH}(\hat{1},\hat{2},3,4)+A^1_{4gH}(\hat{1},\hat{2},4,3)+A^1_{4gH}(\hat{1},3,\hat{2},4)\right]\\
&+\frac{2NN_F}{2!}\left[\widehat{A}^1_{4gH}(\hat{1},\hat{2},3,4)+\widehat{A}^1_{4gH}(\hat{1},\hat{2},4,3)+\widehat{A}^1_{4gH}(\hat{1},3,\hat{2},4)\right]\\
&+NN_F\sum_{P(\hat{1},\hat{2})}\left[B^1_{2gH}(3_q,\hat{1},\hat{2},4_{\bar{q}})-\frac{1}{N^2}\widetilde{B}^1_{2gH}(3_q,\hat{1},\hat{2},4_{\bar{q}})+\frac{N_F}{N}\widehat{B}^1_{2gH}(3_q,\hat{1},\hat{2},4_{\bar{q}})\right]\\
&-\frac{N_F}{N}\left[\widetilde{\widetilde{B}}^1_{2gH}(3_q,\widetilde{\widehat{1}},\widetilde{\widehat{2}},4_{\bar{q}})-\frac{1}{N^2}\widetilde{\widetilde{\widetilde{B}}}^1_{2gH}(3_q,\widetilde{\widehat{1}},\widetilde{\widehat{2}},4_{\bar{q}})+\frac{N_F}{N}\widehat{\widetilde{\widetilde{B}}}^1_{2gH}(3_q,\widetilde{\widehat{1}},\widetilde{\widehat{2}},4_{\bar{q}})\right]\\
&+\frac{N_F}{N}\left[\widehat{\widetilde{B}}^1_{2gH}(3_q,\widetilde{\widehat{1}},\widetilde{\widehat{2}},4_{\bar{q}})\right]\Bigg\}J^{(2)}_1(p_3,p_4).
\end{aligned}
\tag{6.5.28}
$$

The squared matrix elements in Eq. (6.5.28) are discussed in section 4.1.2 and 5.2. The 1/2! coefficient associated with A^1_{4gH} and $\widehat{A}^1_{4gH}$ matrix elements is the averaging factor for the two identical gluons in the final state.

6.5.3 Double virtual contribution

The double virtual contribution at NNLO for $gg \to H$+jet comes from the $gg \to H + g$ process,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{VV}_{gg} =&\,\mathcal{N}_{gg}\,\mathcal{N}^{VV}_{NNLO}\,\mathrm{d}\Phi_{H+1}(p_3,p_H;p_1,p_2)\Bigg\{\\
&+2N^2A^2_{3gH}(\hat{1},\hat{2},3)+2NN_F\widehat{A}^2_{3gH}(\hat{1},\hat{2},3)+2N^2_F\widehat{\widehat{A}}^2_{3gH}(\hat{1},\hat{2},3)
\end{aligned}
$$

$$+2\frac{N_F}{N}\widehat{\widetilde{A}}^2_{3gH}(\hat{1},\hat{2},3)+2\widetilde{A}^2_{3gH}(\hat{1},\hat{2},3)+\frac{2}{N^2}\widetilde{\widetilde{A}}^2_{3gH}(\hat{1},\hat{2},3)\Bigg\}J_1^{(1)}(p_3). \quad (6.5.29)$$

The squared matrix elements in Eq. (6.5.29) are discussed in section 4.1.2 and the explicit formulas are in [119].

6.6 gg initiated subtraction terms at NNLO

6.6.1 Double real subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.3.3, one can construct the double real subtraction term that mimics the implicit IR divergences in Eq. (6.5.27),

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{gg}^S =&\mathcal{N}_{gg}\,\mathcal{N}_{NNLO}^{RR}\,\mathrm{d}\Phi_{H+3}(p_3,p_4,p_5,p_H;p_1,p_2)\Bigg\{\\
&+\frac{2N^2}{3!}\Bigg[\sum_{(i,j,k)\in P^c(3,4,5)}A_{5gH}^{0,XS}(\hat{1},\hat{2},i,j,k)+A_{5gH}^{0,FYS}(\hat{1},3,\hat{2},4,5)\Bigg]\\
&+NN_F B_{3gH}^{0,FS}(4_q,\hat{1},\hat{2},3,5_{\bar{q}})\\
&-\frac{N_F}{N}\Bigg[\widetilde{B}_{3gH}^{0,XS}(4_q,\widetilde{3},\hat{1},\hat{2},5_{\bar{q}})+\widetilde{B}_{3gH}^{0,YS}(4_q,\widetilde{\hat{1}},\hat{2},3,5_{\bar{q}})\Bigg]\\
&+\frac{N_F(N^2+1)}{N^3}\Bigg[\widetilde{\widetilde{B}}_{3gH}^{0,XS}(4_q,\widetilde{\hat{1}},\widetilde{\hat{2}},\widetilde{3},5_{\bar{q}})\Bigg]\Bigg\}. \quad (6.6.30)
\end{aligned}
$$

Here $P^c(3,4,5)$ contains the three cyclic permutations ($\{3,4,5\},\{4,5,3\}$ and $\{5,3,4\}$).

The corresponding relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements (or combinations of subtraction terms) in Eq. (6.6.30) are summarised in table 6.3. In general, if the two initial state partons are colour connected or colour almost unconnected, we name the corresponding antenna subtraction terms with X or Y topology. For example, in $A_{5gH}^{0,FYS}(\hat{1},i,\hat{2},j,k)$, the initial state gluons $\hat{1}$, $\hat{2}$ are separated by final state gluon i. We therefore name the subtraction term `A5g0HFYS`.

The $A_{5gH}^{0,FYS}(\hat{1},i,\hat{2},j,k)$ function is the subtraction term for a combination of three $A_{5gH}^{0,YS}(\hat{1},i,\hat{2},j,k)$ functions with cyclic permutations of $P^c(i,j,k)$. Each $A_{5gH}^{0,YS}(\hat{1},i,\hat{2},j,k)$ function does not mimic all the double and single unresolved limits

subtraction term	maple	matrix element
$A_{5gH}^{0,FYS}(\hat{1}, i, \hat{2}, j, k)$	A5g0HFYS	$\displaystyle\sum_{P^C(i,j,k)} A_{5gH}^{0,YS}(\hat{1}, i, \hat{2}, j, k)$
$A_{5gH}^{0,XS}(\hat{1}, \hat{2}, i, j, k)$	A5g0HXS	$A_{5gH}^0(\hat{1}, \hat{2}, i, j, k) + A_{5gH}^0(\hat{1}, \hat{2}, k, j, i)$
$A_{5gH}^{0,YS}(\hat{1}, i, \hat{2}, j, k)$	A5g0HYS	$A_{5gH}^0(\hat{1}, i, \hat{2}, j, k) + A_{5gH}^0(\hat{1}, i, \hat{2}, k, j)$
$B_{3gH}^{0,FS}(j_q, \hat{1}, \hat{2}, i, k_{\bar{q}})$	ggB3g0HFS	$\displaystyle\sum_{P(\hat{1}, i, \hat{2})} B_{3gH}^0(j_q, \hat{1}, \hat{2}, i, k_{\bar{q}})$
$\widetilde{B}_{3gH}^{0,XS}(j_q, \widetilde{i}, \hat{1}, \hat{2}, k_{\bar{q}})$	ggBt3g0HXS	$\displaystyle\sum_{P(\hat{1}, \hat{2})} \widetilde{B}_{3gH}^0(j_q, \widetilde{i}, \hat{1}, \hat{2}, k_{\bar{q}})$
$\widetilde{B}_{3gH}^{0,YS}(j_q, \widetilde{\hat{1}}, \hat{2}, i, k_{\bar{q}})$	ggBt3g0HYS	$\displaystyle\sum_{P(\hat{1}, \hat{2})} [\widetilde{B}_{3gH}^0(j_q, \widetilde{\hat{1}}, \hat{2}, i, k_{\bar{q}}) + \widetilde{B}_{3gH}^0(j_q, \hat{1}, i, \hat{2}, k_{\bar{q}})]$
$\widetilde{\widetilde{B}}_{3gH}^{0,XS}(j_q, \widetilde{\hat{1}}, \widetilde{\hat{2}}, \widetilde{i}, k_{\bar{q}})$	ggBtt3g0HXS	$\widetilde{\widetilde{B}}_{3gH}^0(j_q, \widetilde{\hat{1}}, \widetilde{\hat{2}}, \widetilde{i}, k_{\bar{q}})$

Table 6.3: NNLO antenna subtraction terms for the double real contributions to the $gg \to H$+jet process and their relation to the matrix elements (or combinations of subtraction terms)

of two Y topology matrix elements. Only the three permutations together removes all of the double and single unresolved limits of all six Y topology matrix elements.

The explicit formulae for $A_{5gH}^{0,XS}(\hat{1}, \hat{2}, i, j, k)$ and $A_{5gH}^{0,FYS}(\hat{1}, i, \hat{2}, j, k)$ are,

$$
\begin{aligned}
A_{5gH}^{0,XS}(\hat{1}, \hat{2}, i, j, k) = \\
+ \ & f_{3,g}^0(2, i, j) \, A_{4gH}^0(1, \overline{2}, \widetilde{(ij)}, k) \, J_1^{(2)}(\{p\}_2) \\
+ \ & f_3^0(i, j, k) \, A_{4gH}^0(1, 2, \widetilde{(ij)}, \widetilde{(jk)}) \, J_1^{(2)}(\{p\}_2) \\
+ \ & f_{3,g}^0(1, k, j) \, A_{4gH}^0(\overline{1}, 2, i, \widetilde{(jk)}) \, J_1^{(2)}(\{p\}_2) \\
+ \ & f_{3,g}^0(2, k, j) \, A_{4gH}^0(1, \overline{2}, \widetilde{(kj)}, i) \, J_1^{(2)}(\{p\}_2) \\
+ \ & f_3^0(k, j, i) \, A_{4gH}^0(1, 2, \widetilde{(kj)}, \widetilde{(ji)}) \, J_1^{(2)}(\{p\}_2) \\
+ \ & f_{3,g}^0(1, i, j) \, A_{4gH}^0(\overline{1}, 2, k, \widetilde{(ji)}) \, J_1^{(2)}(\{p\}_2) \\
+ \ & F_4^0(2, i, j, k) \, A_{3gH}^0(1, \overline{2}, \widetilde{(ijk)}) \, J_1^{(1)}(\{p\}_1) \\
- \ & f_{3,g}^0(2, i, j) \, F_{3,g}^0(\overline{2}, \widetilde{(ij)}, k) \, A_{3gH}^0(1, \overline{\overline{2}}, \widetilde{(\widetilde{(ij)}k)}) \, J_1^{(1)}(\{p\}_1) \\
- \ & f_3^0(i, j, k) \, F_{3,g}^0(2, \widetilde{(ij)}, \widetilde{(jk)}) \, A_{3gH}^0(1, \overline{2}, \widetilde{(\widetilde{ij}, \widetilde{jk})}) \, J_1^{(1)}(\{p\}_1) \\
- \ & f_{3,g}^0(2, k, j) \, F_{3,g}^0(\overline{2}, i, \widetilde{(jk)}) \, A_{3gH}^0(1, \overline{\overline{2}}, \widetilde{(i\widetilde{(jk)})}) \, J_1^{(1)}(\{p\}_1)
\end{aligned}
$$

$$+ \ \frac{1}{2} \ f^0_{3,g}(2,i,j) \, f^0_{3,g}(\overline{2},k,(\widetilde{ij})) \, A^0_{3gH}(1,\overline{\overline{2}},(k\widetilde{\widetilde{(ij)}})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ F^0_{3,gg}(1,i,2) \, f^0_{3,g}(\overline{2},k,j) \, A^0_{3gH}(\overline{1},\overline{\overline{2}},(\widetilde{kj})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ f^0_{3,g}(1,i,j) \, f^0_{3,g}(2,k,(\widetilde{ji})) \, A^0_{3gH}(\overline{1},\overline{2},(k\widetilde{\widetilde{(ji)}})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ \left[+ S^{IF}_{2i(\widetilde{ji})} - S^{IF}_{\overline{2}i(\widetilde{\widetilde{(ji)}k})} - S^{IF}_{\overline{1}i2} + S^{IF}_{\overline{1}i\overline{2}} - S^{IF}_{\overline{1}i(\widetilde{ji})} + S^{IF}_{\overline{1}i(\widetilde{\widetilde{(ji)}k})} \right]$$

$$\times f^0_{3,g}(2,k,(\widetilde{ji})) \, A^0_{3gH}(\overline{1},\overline{2},(\widetilde{\widetilde{(ji)}k})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \ \frac{1}{2} \ f^0_{3,g}(2,k,j) \, f^0_{3,g}(\overline{2},i,(\widetilde{kj})) \, A^0_{3gH}(1,\overline{\overline{2}},(i\widetilde{\widetilde{(kj)}})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ F^0_{3,gg}(1,k,2) \, f^0_{3,g}(\overline{2},i,j) \, A^0_{3gH}(\overline{1},\overline{\overline{2}},(\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ f^0_{3,g}(1,k,j) \, f^0_{3,g}(2,i,(\widetilde{jk})) \, A^0_{3gH}(\overline{1},\overline{2},(i\widetilde{\widetilde{(jk)}})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ \left[+ S^{IF}_{2k(\widetilde{jk})} - S^{IF}_{\overline{2}k(\widetilde{\widetilde{(jk)}i})} - S^{IF}_{\overline{1}k2} + S^{IF}_{\overline{1}k\overline{2}} - S^{IF}_{\overline{1}k(\widetilde{jk})} + S^{IF}_{\overline{1}k(\widetilde{\widetilde{(jk)}i})} \right]$$

$$\times f^0_{3,g}(2,i,(\widetilde{jk})) \, A^0_{3gH}(\overline{1},\overline{2},(\widetilde{\widetilde{(jk)}i})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \ F^0_4(1,k,j,i) \, A^0_{3gH}(\overline{1},2,(\widetilde{ijk})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ f^0_{3,g}(1,k,j) \, F^0_{3,g}(\overline{1},(\widetilde{kj}),i) \, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{\widetilde{(kj)}i})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ f^0_3(k,j,i) \, F^0_{3,g}(1,(\widetilde{kj}),(\widetilde{ji})) \, A^0_{3gH}(\overline{1},2,(\widetilde{\widetilde{kj}},\widetilde{\widetilde{ji}})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ f^0_{3,g}(1,i,j) \, F^0_{3,g}(\overline{1},k,(\widetilde{ji})) \, A^0_{3gH}(\overline{\overline{1}},2,(k\widetilde{\widetilde{(ji)}})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \ \frac{1}{2} \ f^0_{3,g}(1,k,j) \, f^0_{3,g}(\overline{1},i,(\widetilde{jk})) \, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{\widetilde{(jk)}i})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ F^0_{3,gg}(1,k,2) \, f^0_{3,g}(\overline{1},i,j) \, A^0_{3gH}(\overline{\overline{1}},\overline{2},(\widetilde{ji})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ f^0_{3,g}(2,k,j) \, f^0_{3,g}(1,i,(\widetilde{kj})) \, A^0_{3gH}(\overline{1},\overline{2},(\widetilde{\widetilde{(kj)}i})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ \left[+ S^{IF}_{1k(\widetilde{jk})} - S^{IF}_{\overline{1}k(\widetilde{\widetilde{(jk)}i})} - S^{IF}_{1k\overline{2}} + S^{IF}_{\overline{1}k\overline{2}} - S^{IF}_{\overline{2}k(\widetilde{jk})} + S^{IF}_{\overline{2}k(\widetilde{\widetilde{(jk)}i})} \right]$$

$$\times f^0_{3,g}(1,i,(\widetilde{kj})) \, A^0_{3gH}(\overline{1},\overline{2},(i\widetilde{\widetilde{(kj)}})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \ \frac{1}{2} \ f^0_{3,g}(1,i,j) \, f^0_{3,g}(\overline{1},k,(\widetilde{ji})) \, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{\widetilde{(ji)}k})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ F^0_{3,gg}(1,i,2) \, f^0_{3,g}(\overline{1},k,j) \, A^0_{3gH}(\overline{\overline{1}},\overline{2},(\widetilde{jk})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ f^0_{3,g}(2,i,j) \, f^0_{3,g}(1,k,(\widetilde{ij})) \, A^0_{3gH}(\overline{1},\overline{2},(\widetilde{\widetilde{(ij)}k})) \, J^{(1)}_1(\{p\}_1)$$

$$- \ \frac{1}{2} \ \left[+ S^{IF}_{1i(\widetilde{ji})} - S^{IF}_{\overline{1}i(\widetilde{\widetilde{(ji)}k})} - S^{IF}_{1i\overline{2}} + S^{IF}_{\overline{1}i\overline{2}} - S^{IF}_{\overline{2}i(\widetilde{ji})} + S^{IF}_{\overline{2}i(\widetilde{\widetilde{(ji)}k})} \right]$$

$$\times f^0_{3,g}(1,k,(\widetilde{ji})) \, A^0_{3gH}(\overline{1},\overline{2},(\widetilde{\widetilde{(ji)}k})) \, J^{(1)}_1(\{p\}_1)$$

$$- \quad F_4^0(1,i,2,k)\, A_{3gH}^0(\overline{1},\overline{2},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad F_{3,gg}^0(1,i,2)\, F_{3,gg}^0(\overline{1},k,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad F_{3,gg}^0(1,k,2)\, F_{3,gg}^0(\overline{1},i,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\, F_{3,gg}^0(1,i,2)\, F_{3,gg}^0(\overline{1},k,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, f_{3,g}^0(1,i,j)\, F_{3,gg}^0(\overline{1},k,2)\, A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ji}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, f_{3,g}^0(2,i,j)\, F_{3,gg}^0(1,k,\overline{2})\, A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\left[-S_{1i\overline{2}}^{IF} + S_{\overline{1}i\overline{2}}^{IF} + S_{1i(\widetilde{ij})}^{IF} - S_{\overline{1}i(\widetilde{ij})}^{IF} + S_{\overline{2}i(\widetilde{ji})}^{IF} - S_{\overline{\overline{2}}i(\widetilde{ji})}^{IF} \right]$$
$$\times F_{3,gg}^0(1,k,\overline{2})\, A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\, F_{3,gg}^0(1,k,2)\, F_{3,gg}^0(\overline{1},i,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, f_{3,g}^0(1,k,j)\, F_{3,gg}^0(\overline{1},i,2)\, A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, f_{3,g}^0(2,k,j)\, F_{3,gg}^0(1,i,\overline{2})\, A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{kj}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\left[-S_{\overline{1}k2}^{IF} + S_{\overline{\overline{1}}k\overline{2}}^{IF} + S_{\overline{1}k(\widetilde{kj})}^{IF} - S_{\overline{\overline{1}}k(\widetilde{kj})}^{IF} + S_{2k(\widetilde{jk})}^{IF} - S_{\overline{2}k(\widetilde{jk})}^{IF} \right]$$
$$\times F_{3,gg}^0(\overline{1},i,2)\, A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1), \tag{6.6.31}$$

$$A_{5gH}^{0,FYS}(\hat{1},i,\hat{2},j,k) =$$

$$\sum_{P^C(i,j,k)} \Bigg\{ \quad + \quad F_{3,gg}^0(1,i,2)\, A_{4gH}^0(\overline{1},\overline{2},j,k)\, J_1^{(2)}(\{p\}_2)$$

$$+ \quad f_{3,g}^0(2,j,k)\, A_{4gH}^0(1,i,\overline{2},(\widetilde{jk}))\, J_1^{(2)}(\{p\}_2)$$

$$+ \quad f_{3,g}^0(1,k,j)\, A_{4gH}^0(\overline{1},i,2,(\widetilde{jk}))\, J_1^{(2)}(\{p\}_2)$$

$$+ \quad F_{3,gg}^0(1,i,2)\, A_{4gH}^0(\overline{1},\overline{2},k,j)\, J_1^{(2)}(\{p\}_2)$$

$$+ \quad f_{3,g}^0(2,k,j)\, A_{4gH}^0(1,i,\overline{2},(\widetilde{kj}))\, J_1^{(2)}(\{p\}_2)$$

$$+ \quad f_{3,g}^0(1,j,k)\, A_{4gH}^0(\overline{1},i,2,(\widetilde{kj}))\, J_1^{(2)}(\{p\}_2)$$

$$+ \quad F_4^0(1,i,2,j)\, A_{3gH}^0(\overline{1},\overline{2},k)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad F_{3,gg}^0(1,i,2)\, F_{3,gg}^0(\overline{1},j,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},k)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad F_{3,gg}^0(2,j,1)\, F_{3,gg}^0(\overline{1},i,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},k)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, F_{3,gg}^0(1,i,2)\, F_{3,gg}^0(\overline{1},j,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},k)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\, f_{3,g}^0(1,i,k)\, F_{3,gg}^0(\overline{1},j,2)\, A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ki}))\, J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(2,i,k) F_{3,gg}^0(1,j,\overline{2}) A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ik})) J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} \left[-S_{1i\overline{2}}^{IF} + S_{\overline{1}i\overline{2}}^{IF} + S_{1i(\widetilde{ik})}^{IF} - S_{\overline{1}i(\widetilde{ik})}^{IF} + S_{\overline{2}i(\widetilde{ki})}^{IF} - S_{\overline{\overline{2}}i(\widetilde{ki})}^{IF} \right]$$
$$\times F_{3,gg}^0(1,j,\overline{2}) A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ki})) J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} F_{3,gg}^0(1,j,2) F_{3,gg}^0(\overline{1},i,\overline{2}) A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},k) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(1,j,k) F_{3,gg}^0(\overline{1},i,2) A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{kj})) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(2,j,k) F_{3,gg}^0(1,i,\overline{2}) A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{jk})) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} \left[+S_{1j\overline{2}}^{IF} - S_{\overline{1}j\overline{\overline{2}}}^{IF} - S_{1j(\widetilde{jk})}^{IF} + S_{\overline{1}j(\widetilde{jk})}^{IF} - S_{\overline{2}j(\widetilde{kj})}^{IF} + S_{\overline{\overline{2}}j(\widetilde{kj})}^{IF} \right]$$
$$\times F_{3,gg}^0(1,i,\overline{2}) A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{kj})) J_1^{(1)}(\{p\}_1)$$

$$+ \quad F_4^0(1,i,2,k) A_{3gH}^0(\overline{1},\overline{2},j) J_1^{(1)}(\{p\}_1)$$

$$- \quad F_{3,gg}^0(1,i,2) F_{3,gg}^0(\overline{1},k,\overline{2}) A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j) J_1^{(1)}(\{p\}_1)$$

$$- \quad F_{3,gg}^0(2,k,1) F_{3,gg}^0(\overline{1},i,\overline{2}) A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j) J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} F_{3,gg}^0(1,i,2) F_{3,gg}^0(\overline{1},k,\overline{2}) A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(1,i,j) F_{3,gg}^0(\overline{1},k,2) A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ji})) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(2,i,j) F_{3,gg}^0(1,k,\overline{2}) A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ij})) J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} \left[-S_{\overline{1}i2}^{IF} + S_{\overline{\overline{1}}i\overline{2}}^{IF} + S_{\overline{1}i(\widetilde{ij})}^{IF} - S_{\overline{\overline{1}}i(\widetilde{ij})}^{IF} + S_{2i(\widetilde{ji})}^{IF} - S_{\overline{2}i(\widetilde{ji})}^{IF} \right]$$
$$\times F_{3,gg}^0(\overline{1},k,2) A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ij})) J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} F_{3,gg}^0(1,k,2) F_{3,gg}^0(\overline{1},i,\overline{2}) A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(1,k,j) F_{3,gg}^0(\overline{1},i,2) A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk})) J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} f_{3,g}^0(2,k,j) F_{3,gg}^0(1,i,\overline{2}) A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{kj})) J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} \left[-S_{\overline{1}k2}^{IF} + S_{\overline{\overline{1}}k\overline{2}}^{IF} + S_{\overline{1}k(\widetilde{kj})}^{IF} - S_{\overline{\overline{1}}k(\widetilde{jk})}^{IF} + S_{2k(\widetilde{jk})}^{IF} - S_{\overline{2}k(\widetilde{jk})}^{IF} \right]$$
$$\times F_{3,gg}^0(\overline{1},i,2) A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk})) J_1^{(1)}(\{p\}_1)$$

$$+ \quad F_4^0(2,j,k,1) A_{3gH}^0(\overline{1},i,\overline{2}) J_1^{(1)}(\{p\}_1)$$

$$- \quad f_{3,g}^0(2,j,k) F_{3,gg}^0(\overline{2},(\widetilde{jk}),1) A_{3gH}^0(\overline{1},i,\overline{\overline{2}}) J_1^{(1)}(\{p\}_1)$$

$$- \quad f_{3,g}^0(1,k,j) F_{3,gg}^0(2,(\widetilde{jk}),\overline{1}) A_{3gH}^0(\overline{\overline{1}},i,\overline{2}) J_1^{(1)}(\{p\}_1)$$

$$+ \quad F_4^0(2,k,j,1) A_{3gH}^0(\overline{1},i,\overline{2}) J_1^{(1)}(\{p\}_1)$$

$$- \ f^0_{3,g}(2,k,j) \, F^0_{3,gg}(\overline{2},(\widetilde{kj}),1) \, A^0_{3gH}(\overline{1},i,\overline{\overline{2}}) \, J^{(1)}_1(\{p\}_1)$$

$$- \ f^0_{3,g}(1,j,k) \, F^0_{3,gg}(2,(\widetilde{kj}),\overline{1}) \, A^0_{3gH}(\overline{\overline{1}},i,\overline{2}) \, J^{(1)}_1(\{p\}_1) \Bigg\}. \qquad (6.6.32)$$

The wide angle soft function, S^{IF}_{ijk}, appearing in Eqs. (6.6.31) and (6.6.32) is the same Eikonal factor introduced in Eq. (3.2.35). The upper index IF here is a reminder that this Eikonal factor needs to be integrated over the Initial-Final three parton phase space at the real-virtual level.

Explicit formulae for the other antenna subtraction terms in table 6.3 can be found in appendix B.1. The double real subtraction terms fit the general structure described in section 4.3.3.

As at NLO, we use the spike plots introduced in section 6.4.1 to study the convergence of the double real matrix elements and the subtraction term through the ratio,

$$R = \frac{\mathrm{d}\hat{\sigma}^{RR}}{\mathrm{d}\hat{\sigma}^S_{NNLO}}. \qquad (6.6.33)$$

For Higgs boson plus five parton matrix elements, there are 49 different double or single unresolved limits. The spike plots are calculated and tested for all these unresolved limits. Sample results for four typical double unresolved limits are presented in Figure 6.2 to provide graphical evidence of the convergence of the subtraction terms to the matrix elements in each unresolved limits.
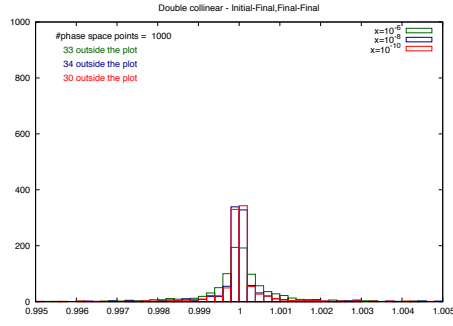
(a) Double Collinear limit between Initial-Final and Final-Final partons (1//i, j//k), where $x = s_{1i}/s_{12} = s_{jk}/s_{12}$ approaches zero in the unresolved limit.
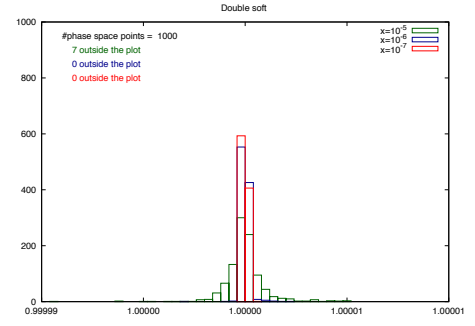
(b) Double soft limit for parton i and j, where $x = (s_{1i} + s_{2i} + s_{2i} + s_{2j})/s_{12}$ approaches zero in the unresolved limit.
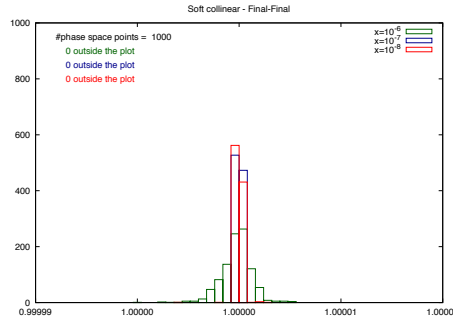
(c) Soft collinear limit for soft partons i and collinear Final-Final partons (j//k), such that $x = s_{jk}/s_{12} = (s_{1i} + s_{2i})/s_{12}$ approaches zero in the unresolved limit.
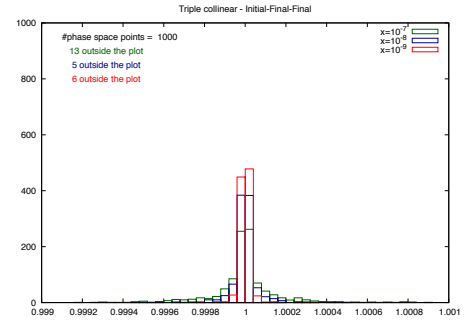
(d) Triple collinear limit for Initial-Final-Final partons $1//i//j$, such that $x = s_{1ij}/s_{12}$ approaches zero in the unresolved limit.

Figure 6.2: Spike plots displaying the convergence of the NNLO subtraction terms in $d\hat{\sigma}_{gg}^{S}$ to the matrix elements in $d\hat{\sigma}_{gg}^{RR}$ in various unresolved limits.

6.6.2 Real-virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.5.3, one can combine the integrated double real level subtraction terms and real-virtual level mass factorization term to construct the real-virtual subtraction term, $d\hat{\sigma}_{gg}^{T}$, which

removes both the explicit and implicit IR divergences from Eq.(6.5.28),

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{gg}^{T} =& \mathcal{N}_{gg}\,\mathcal{N}_{NNLO}^{RV}\,\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{ \\
& \frac{2N^2}{2!}A_{4gH}^{1,FT}(\hat{1},\hat{2},3,4) \\
&+\frac{2NN_F}{2!}\Bigg[\sum_{(i,j)\in P(3,4)}\widehat{A}_{4gH}^{1,XT}(\hat{1},\hat{2},i,j)+\widehat{A}_{4gH}^{1,YT}(\hat{1},3,\hat{2},4)\Bigg] \\
&+NN_F\Bigg[B_{2gH}^{1,XT}(3_q,\hat{1},\hat{2},4_{\bar{q}})-\frac{1}{N^2}\widetilde{B}_{2gH}^{1,XT}(3_q,\hat{1},\hat{2},4_{\bar{q}})+\frac{N_F}{N}\widehat{B}_{2gH}^{1,XT}(3_q,\hat{1},\hat{2},4_{\bar{q}})\Bigg] \\
&-\frac{N_F}{N}\Bigg[\widetilde{\widetilde{B}}_{2gH}^{1,XT}(3_q,\widetilde{\hat{1}},\widetilde{\hat{2}},4_{\bar{q}})-\frac{1}{N^2}\widetilde{\widetilde{\widetilde{B}}}_{2gH}^{1,XT}(3_q,\widetilde{\hat{1}},\widetilde{\hat{2}},4_{\bar{q}})+\frac{N_F}{N}\widehat{\widetilde{\widetilde{B}}}_{2gH}^{1,XT}(3_q,\widetilde{\hat{1}},\widetilde{\hat{2}},4_{\bar{q}})\Bigg]\Bigg\}
\end{aligned}
\tag{6.6.34}
$$

The corresponding relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements (or combinations of subtraction terms) in Eq.(6.5.28) and (6.6.34) are summarised in table 6.4. Note that the $A_{4gH}^{1,XT}(\hat{1},\hat{2},i,j)$ or $A_{4gH}^{1,YT}(\hat{1},i,\hat{2},j)$ subtraction terms do not remove all the single soft IR limits in $A_{4gH}^{1}(\hat{1},\hat{2},i,j)$ or $A_{4gH}^{1}(\hat{1},i,\hat{2},j)$ matrix elements. Only the combination of $A_{4gH}^{1,FT}(\hat{1},\hat{2},i,j)$ removes all the IR divergences of the colour leading one-loop matrix elements with Higgs boson plus four gluons.

Explicit formulae for $A_{4gH}^{1,XT}(\hat{1},\hat{2},i,j)$ and $A_{4gH}^{1,YT}(\hat{1},i,\hat{2},j)$ are,

$$
\begin{aligned}
A_{4gH}^{1,XT}&(\hat{1},\hat{2},i,j)= \\
&-\Bigg[+J_{2,GG}^{1,FI}(s_{i2})+J_{2,GG}^{1,FF}(s_{ij})+J_{2,GG}^{1,IF}(s_{1j})+J_{2,GG}^{1,II}(s_{12})\Bigg]A_{4gH}^{0}(1,2,i,j)\,J_1^{(2)}(\{p\}_2) \\
&+f_{3,g}^{0}(2,i,j)\Bigg[A_{3gH}^{1}(1,\overline{2},(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2) \\
&\quad+\Bigg(+J_{2,GG}^{1,II}(s_{1\overline{2}})+J_{2,GG}^{1,FI}(s_{(\widetilde{ij})\overline{2}})+J_{2,GG}^{1,IF}(s_{1(\widetilde{ij})})\Bigg)A_{3gH}^{0}(1,\overline{2},(\widetilde{ij}))\Bigg]J_1^{(1)}(\{p\}_1) \\
&+f_{3,g}^{0}(1,j,i)\Bigg[A_{3gH}^{1}(\overline{1},2,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2) \\
&\quad+\Bigg(+J_{2,GG}^{1,II}(s_{\overline{1}2})+J_{2,GG}^{1,FI}(s_{(\widetilde{ij})2})+J_{2,GG}^{1,IF}(s_{\overline{1}(\widetilde{ij})})\Bigg)A_{3gH}^{0}(\overline{1},2,(\widetilde{ij}))\Bigg]J_1^{(1)}(\{p\}_1) \\
&+\Bigg[f_{3,g}^{1}(2,i,j)\,\delta(1-x_1)\,\delta(1-x_2) \\
&\quad+\Bigg(+J_{2,GG}^{1,FI}(s_{i2})+J_{2,GG}^{1,FF}(s_{ij})+J_{2,GG}^{1,FI}(s_{j2})-2J_{2,GG}^{1,FI}(s_{(\widetilde{ij})\overline{2}})\Bigg)f_{3,g}^{0}(2,i,j)\Bigg] \\
&\quad\times A_{3gH}^{0}(1,\overline{2},(\widetilde{ij}))\,J_1^{(1)}(\{p\}_1)
\end{aligned}
$$

subtraction term	maple	matrix element
$A_{4gH}^{1,FT}(\hat{1},\hat{2},i,j)$	`A4g1HFT`	$\displaystyle\sum_{P(i,j)} A_{4gH}^{1,XT}(\hat{1},\hat{2},i,j) + A_{4gH}^{1,YT}(\hat{1},i,\hat{2},j)$
$A_{4gH}^{1,XT}(\hat{1},\hat{2},i,j)$	`A4g1HXT`	$A_{4gH}^{1}(\hat{1},\hat{2},i,j)$
$A_{4gH}^{1,YT}(\hat{1},i,\hat{2},j)$	`A4g1HYT`	$A_{4gH}^{1}(\hat{1},i,\hat{2},j)$
$\widehat{A}_{4gH}^{1,XT}(\hat{1},\hat{2},i,j)$	`Ah4g1HXT`	$\widehat{A}_{4gH}^{1}(\hat{1},\hat{2},i,j)$
$\widehat{A}_{4gH}^{1,YT}(\hat{1},i,\hat{2},j)$	`Ah4g1HYT`	$\widehat{A}_{4gH}^{1}(\hat{1},i,\hat{2},j)$
$B_{2gH}^{1,XT}(i_q,\hat{1},\hat{2},j_{\bar{q}})$	`ggB2g1HXT`	$B_{2gH}^{1}(i_q,\hat{1},\hat{2},j_{\bar{q}}) + B_{2gH}^{1}(i_q,\hat{2},\hat{1},j_{\bar{q}})$
$\widehat{B}_{2gH}^{1,XT}(i_q,\hat{1},\hat{2},j_{\bar{q}})$	`ggBh2g1HXT`	$\widehat{B}_{2gH}^{1}(i_q,\hat{1},\hat{2},j_{\bar{q}}) + \widehat{B}_{2gH}^{1}(i_q,\hat{2},\hat{1},j_{\bar{q}})$
$\widetilde{\widetilde{B}}_{2gH}^{1,XT}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$	`ggBttt2g1HXT`	$\widetilde{\widetilde{B}}_{2gH}^{1}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$
$\widehat{\widetilde{\widetilde{B}}}_{2gH}^{1,XT}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$	`ggBtth2g1HXT`	$\widehat{\widetilde{\widetilde{B}}}_{2gH}^{1}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$
$\widetilde{B}_{2gH}^{1,XT}(i_q,\hat{1},\hat{2},j_{\bar{q}})$	`ggBt2g1HXT`	$\widetilde{B}_{2gH}^{1}(i_q,\hat{1},\hat{2},j_{\bar{q}}) + \widetilde{B}_{2gH}^{1}(i_q,\hat{2},\hat{1},j_{\bar{q}})$
$\widetilde{\widetilde{B}}_{2gH}^{1,XT}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$	`ggBtt2g1HXT`	$\widetilde{\widetilde{B}}_{2gH}^{1}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}}) - \widehat{\widetilde{\widetilde{B}}}_{2gH}^{1}(i_q,\widetilde{\hat{1}},\widetilde{\hat{2}},j_{\bar{q}})$

Table 6.4: NNLO antenna subtraction terms for real-virtual contributions in $gg \to H$+jet process and their relation to the matrix elements (or combinations of subtraction terms)

$$
\begin{aligned}
&+ \Bigg[f_{3,g}^1(1,j,i)\,\delta(1-x_1)\,\delta(1-x_2) \\
&\qquad + \bigg(+ J_{2,GG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,FF}(s_{ij}) + J_{2,GG}^{1,IF}(s_{1j}) - 2J_{2,GG}^{1,IF}(s_{\bar{1}(\widetilde{ij})}) \bigg) f_{3,g}^0(1,j,i) \Bigg] \\
&\qquad \times\, A_{3gH}^0(\bar{1},2,(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1) \\
&+ \frac{1}{2}\Bigg[+ J_{2,GG}^{1,FI}(s_{(\widetilde{ij})\bar{2}}) - J_{2,GG}^{1,FI}(s_{j\bar{2}}) - J_{2,GG}^{1,IF}(s_{1(\widetilde{ij})}) \\
&\qquad + J_{2,GG}^{1,IF}(s_{1j}) - J_{2,GG}^{1,II}(s_{1\bar{2}}) + J_{2,GG}^{1,II}(s_{12}) \\
&\qquad - \mathcal{S}^{FI}(s_{\bar{2}(\widetilde{ij})},s_{2j},x_{\bar{2}(\widetilde{ij}),2j}) + \mathcal{S}^{FI}(s_{2j},s_{2j},1) + \mathcal{S}^{FI}(s_{1(\widetilde{ij})},s_{2j},x_{1(\widetilde{ij}),2j}) \\
&\qquad - \mathcal{S}^{FI}(s_{1j},s_{2j},x_{1j,2j}) \Bigg] \times f_{3,g}^0(2,i,j)\, A_{3gH}^0(1,\bar{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1) \\
&+ \frac{1}{2}\Bigg[+ J_{2,GG}^{1,IF}(s_{\bar{1}(\widetilde{ij})}) - J_{2,GG}^{1,IF}(s_{1i}) - J_{2,GG}^{1,FI}(s_{2(\widetilde{ij})})
\end{aligned}
$$

$$+ J_{2,GG}^{1,FI}(s_{2i}) - J_{2,GG}^{1,II}(s_{\overline{1}2}) + J_{2,GG}^{1,II}(s_{12})$$

$$- \mathcal{S}^{IF}(s_{\overline{1}(\widetilde{ij})}, s_{1i}, x_{\overline{1}(\widetilde{ij}),1i}) + \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) + \mathcal{S}^{IF}(s_{2(\widetilde{ij})}, s_{1i}, x_{2(\widetilde{ij}),1i})$$

$$- \mathcal{S}^{IF}(s_{2i}, s_{1i}, x_{2i,1i}) \Bigg] \times f_{3,g}^0(1, j, i)\, A_{3gH}^0(\overline{1}, 2, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1). \qquad (6.6.35)$$

$$A_{4gH}^{1,YT}(\hat{1}, i, \hat{2}, j) =$$

$$- \Bigg[+ J_{2,GG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,FI}(s_{i2}) + J_{2,GG}^{1,FI}(s_{j2}) + J_{2,GG}^{1,IF}(s_{1j}) \Bigg] A_{4gH}^0(\overline{1}, i, \overline{2}, j)\, J_1^{(2)}(\{p\}_2)$$

$$+ F_{3,gg}^0(1, i, 2) \Bigg[A_{3gH}^1(\overline{1}, \overline{2}, j)\, \delta(1 - x_1)\, \delta(1 - x_2)$$

$$+ \left(+ J_{2,GG}^{1,II}(s_{\overline{1}2}) + J_{2,GG}^{1,FI}(s_{j\overline{2}}) + J_{2,GG}^{1,IF}(s_{\overline{1}j}) \right) A_{3gH}^0(\overline{1}, \overline{2}, j) \Bigg] J_1^{(1)}(\{p\}_1)$$

$$+ F_{3,gg}^0(2, j, 1) \Bigg[A_{3gH}^1(\overline{1}, i, \overline{2})\, \delta(1 - x_1)\, \delta(1 - x_2)$$

$$+ \left(+ J_{2,GG}^{1,IF}(s_{\overline{1}i}) + J_{2,GG}^{1,FI}(s_{i\overline{2}}) + J_{2,GG}^{1,II}(s_{\overline{1}2}) \right) A_{3gH}^0(\overline{1}, i, \overline{2}) \Bigg] J_1^{(1)}(\{p\}_1)$$

$$+ \Bigg[F_{3,gg}^1(1, i, 2)\, \delta(1 - x_1)\, \delta(1 - x_2)$$

$$+ \left(+ J_{2,GG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,FI}(s_{i2}) + J_{2,GG}^{1,II}(s_{12}) - 2 J_{2,GG}^{1,II}(s_{\overline{1}2}) \right) F_{3,gg}^0(1, i, 2) \Bigg]$$

$$\times A_{3gH}^0(\overline{1}, \overline{2}, j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \Bigg[F_{3,gg}^1(1, j, 2)\, \delta(1 - x_1)\, \delta(1 - x_2)$$

$$+ \left(+ J_{2,GG}^{1,IF}(s_{1j}) + J_{2,GG}^{1,FI}(s_{j2}) + J_{2,GG}^{1,II}(s_{12}) - 2 J_{2,GG}^{1,II}(s_{\overline{1}2}) \right) F_{3,gg}^0(1, j, 2) \Bigg]$$

$$\times A_{3gH}^0(\overline{1}, i, \overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} \Bigg[+ J_{2,GG}^{1,II}(s_{\overline{1}2}) - J_{2,GG}^{1,II}(s_{12}) - J_{2,GG}^{1,FI}(s_{j\overline{2}})$$

$$+ J_{2,GG}^{1,FI}(s_{j2}) - J_{2,GG}^{1,IF}(s_{\overline{1}j}) + J_{2,GG}^{1,IF}(s_{1j})$$

$$+ \left(- \tilde{\mathcal{S}}^{IF}(s_{\overline{1}2}, s_{\overline{1}j}, x_{\overline{1}2,\overline{1}j}) + \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{2}j}, s_{\overline{1}j}, x_{\overline{2}j,\overline{1}j}) \right.$$

$$\left. - \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}j}, s_{\overline{1}j}, 1) - \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) \right) \Bigg]$$

$$\times F_{3,gg}^0(1, i, 2)\, A_{3gH}^0(\overline{1}, \overline{2}, j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} \Bigg[+ J_{2,GG}^{1,II}(s_{\overline{1}2}) - J_{2,GG}^{1,II}(s_{12}) - J_{2,GG}^{1,FI}(s_{i\overline{2}})$$

$$+ J_{2,GG}^{1,FI}(s_{i2}) - J_{2,GG}^{1,IF}(s_{\bar{1}i}) + J_{2,GG}^{1,IF}(s_{1i})$$

$$+ \Bigg(- \tilde{\mathcal{S}}^{FI}(s_{\overline{12}}, s_{\overline{2}i}, x_{\overline{12},\overline{2}i}) + \mathcal{S}^{FI}(s_{12}, s_{2i}, x_{12,2i}) + \tilde{\mathcal{S}}^{FI}(s_{\overline{2}i}, s_{\overline{2}i}, 1)$$

$$- \mathcal{S}^{FI}(s_{2i}, s_{2i}, 1) + \tilde{\mathcal{S}}^{FI}(s_{\overline{1}i}, s_{\overline{2}i}, x_{\overline{1}i,\overline{2}i}) - \mathcal{S}^{FI}(s_{1i}, s_{2i}, x_{1i,2i}) \Bigg) \Bigg]$$

$$\times F_{3,gg}^0(1, j, 2) \, A_{3gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1). \tag{6.6.36}$$

The integrated wide angle soft functions, $\mathcal{S}^{FI}$ and $\mathcal{S}^{IF}$, appearing in Eqs.(6.6.35) and (6.6.36) are the same integrated Eikonal factors introduced in Eqs.(4.5.102) and (4.5.104). The upper index IF or FI here is a reminder that this integrated Eikonal factor is integrated over the Initial-Final three parton phase space with parton 1 (IF) or parton 2 (FI) in the initial state.

Explicit formulas for the other antenna subtraction terms in table 6.4 can be found in appendix B.2. The real-virtual subtraction terms fit the general structure described in section 4.5.3.

To show that these subtraction terms correctly remove the explicit IR divergences from the matrix elements, we construct spike plots retaining only the terms proportional to ϵ^{-2} and ϵ^{-1} from both the subtraction terms and the matrix elements,

$$R_{\epsilon^{-1}} = \frac{d\hat{\sigma}^{RV}}{d\hat{\sigma}_{NNLO}^T}\bigg|_{\epsilon^{-1}}, \qquad R_{\epsilon^{-2}} = \frac{d\hat{\sigma}^{RV}}{d\hat{\sigma}_{NNLO}^T}\bigg|_{\epsilon^{-2}}. \tag{6.6.37}$$

Since this cancellation takes place everywhere in phase space, Figure 6.3 shows R for general phase space points. Of course, the explicit IR divergences cancellation between $d\hat{\sigma}^{RV}$ and $d\hat{\sigma}_{NNLO}^T$ is analytical and should be achieved when the single unresolved limits are approached. In each single unresolved limit, the spike plots for $R_{\epsilon^{-1}}$ and $R_{\epsilon^{-2}}$ have a similar appearance as Figure 6.3.

The subtraction term should also correctly mimic the matrix elements in the unresolved limits. For Higgs boson plus four partons at RV level, there are 7 different single unresolved limits. We consider the quantity,

$$R_{\epsilon^0} = \frac{d\hat{\sigma}^{RV}}{d\hat{\sigma}_{NNLO}^T}\bigg|_{\epsilon^0}, \tag{6.6.38}$$

where only the finite part of the real-virtual subtraction terms and matrix elements are used. The behaviour of R_{ϵ^0} in various single unresolved limits is illustrated in Figure 6.4.
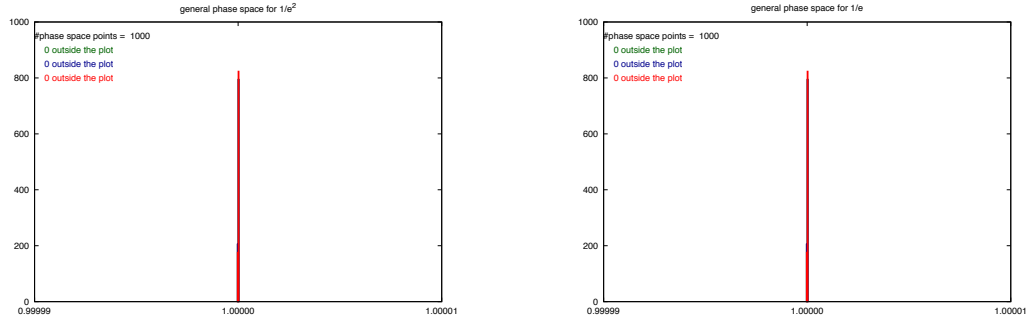
(a) $1/\epsilon^2$ pole cancellation   (b) $1/\epsilon$ pole cancellation

Figure 6.3: Spike plots displaying the explicit IR divergence cancellation between $\mathrm{d}\hat{\sigma}^T_{gg}$ and $\mathrm{d}\hat{\sigma}^{RV}_{gg}$

6.6.3 Double virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.7.2, one can remove the explicit IR divergences in the double virtual matrix elements in Eq.(6.5.29) with the following double virtual subtraction term,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^U_{gg} =&\, \mathcal{N}_{gg}\,\mathcal{N}^{VV}_{NNLO}\,\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+1}(p_3,p_H;p_1,p_2)\Bigg\{ \\
&-N^2 A^{2,XU}_{3gH}(\hat{1},\hat{2},3) - \frac{N_F}{N}\widehat{\widetilde{A}}^{2,XU}_{3gH}(\hat{1},\hat{2},3) - NN_F\widehat{A}^{2,XU}_{3gH}(\hat{1},\hat{2},3) \\
&-N_F^2\widehat{\widehat{A}}^{2,XU}_{3gH}(\hat{1},\hat{2},3) + \frac{N_F^2}{N^2}\widetilde{\widetilde{B}}^{1,XU}_{1gH}(\hat{1}_q,\hat{2},3_{\bar{q}}) - \frac{N_F}{N^3}\widetilde{\widetilde{B}}^{1,XU}_{1gH}(\hat{1}_q,\hat{2},3_{\bar{q}})\Bigg\}J^{(1)}_1(p_3).
\end{aligned}
$$
(6.6.39)

The relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements in Eqs. (6.5.29) and (6.6.39) are summarised in table 6.5.

The explicit formula for $A^{2,XU}_{3gH}(\hat{1},\hat{2},i)$ in terms of integrated antennae is,

$$
\begin{aligned}
A^{2,XU}_{3gH}(\hat{1},\hat{2},i) =&\, \\
-&\Bigg[+2\,\mathcal{F}^0_{3,gg}(s_{12}) - 2\,\Gamma^{(1)}_{gg}(x_1) - 2\,\Gamma^{(1)}_{gg}(x_2) + \mathcal{F}^0_{3,g}(s_{2i}) \\
&\quad + \mathcal{F}^0_{3,g}(s_{1i})\Bigg]A^1_{3gH}(1,2,i) \\
-&\Bigg[+\mathcal{F}^0_{3,gg}(s_{12})\otimes\mathcal{F}^0_{3,gg}(s_{12}) + \frac{1}{4}\,\Gamma^{(1)}_{gg}(x_1)\otimes\Gamma^{(1)}_{gg}(x_1) + \frac{1}{4}\,\Gamma^{(1)}_{gg}(x_2)\otimes\Gamma^{(1)}_{gg}(x_2)
\end{aligned}
$$

subtraction term	maple	matrix element
$A_{3gH}^{2,XU}(\hat{1},\hat{2},i)$	`A3g2HXU`	$A_{3gH}^2(\hat{1},\hat{2},i)$
$\widehat{\widetilde{A}}_{3gH}^{2,XU}(\hat{1},\hat{2},i)$	`Aht3g2HXU`	$\widehat{\widetilde{A}}_{3gH}^2(\hat{1},\hat{2},i)$
$\widehat{A}_{3gH}^{2,XU}(\hat{1},\hat{2},i)$	`Ah3g2HXU`	$\widehat{A}_{3gH}^2(\hat{1},\hat{2},i)$
$\widehat{\widehat{A}}_{3gH}^{2,XU}(\hat{1},\hat{2},i)$	`Ahh3g2HXU`	$\widehat{\widehat{A}}_{3gH}^2(\hat{1},\hat{2},i)$
$\widetilde{\widetilde{\widehat{B}}}_{1gH}^{1,XU}(\hat{1}_q,\hat{2},i_{\bar{q}})$	`ggBtth2g1HXU`	IR safe
$\widetilde{\widetilde{\widehat{B}}}_{1gH}^{1,XU}(\hat{1}_q,\hat{2},i_{\bar{q}})$	`ggBttt2g1HXU`	IR safe

Table 6.5: NNLO antenna subtraction terms for double virtual contributions in $gg \to H+$jet process and their relation to the matrix elements

$$
\begin{aligned}
&\left. - \Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{F}_{3,gg}^0(s_{12}) - \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{F}_{3,gg}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \\
&\times A_{3gH}^0(1,2,i) \\
&- \left[+ \frac{1}{4}\,\mathcal{F}_{3,g}^0(s_{2i}) \otimes \mathcal{F}_{3,g}^0(s_{2i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{F}_{3,g}^0(s_{2i}) + \frac{1}{4}\,\Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_2) \right] \\
&\qquad \times A_{3gH}^0(1,2,i) \\
&- \left[+ \frac{1}{4}\,\mathcal{F}_{3,g}^0(s_{1i}) \otimes \mathcal{F}_{3,g}^0(s_{1i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{F}_{3,g}^0(s_{1i}) + \frac{1}{4}\,\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \\
&\qquad \times A_{3gH}^0(1,2,i) \\
&- \left[+ \mathcal{F}_{3,gg}^0(s_{12}) \otimes \mathcal{F}_{3,g}^0(s_{2i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{F}_{3,g}^0(s_{2i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{F}_{3,g}^0(s_{2i}) \right. \\
&\qquad \left. - \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{F}_{3,gg}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_2) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \\
&\times A_{3gH}^0(1,2,i) \\
&- \left[+ \mathcal{F}_{3,gg}^0(s_{12}) \otimes \mathcal{F}_{3,g}^0(s_{1i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{F}_{3,g}^0(s_{1i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{F}_{3,g}^0(s_{1i}) \right. \\
&\qquad \left. - \Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{F}_{3,gg}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \\
&\times A_{3gH}^0(1,2,i) \\
&- \left[+ \frac{1}{2}\,\mathcal{F}_{3,g}^0(s_{1i}) \otimes \mathcal{F}_{3,g}^0(s_{2i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{F}_{3,g}^0(s_{1i}) - \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{F}_{3,g}^0(s_{2i}) \right. \\
&\qquad \left. + \frac{1}{2}\,\Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_1) \right] A_{3gH}^0(1,2,i)
\end{aligned}
$$

$$
-\left[+ \mathcal{F}_{4,g}^0(s_{2i}) + \mathcal{F}_{3,g}^1(s_{2i}) + \frac{b_0}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{F}_{3,g}^0(s_{2i}) - \frac{b_0}{\epsilon} \mathcal{F}_{3,g}^0(s_{2i}) \right.
$$

$$
\left. - \frac{1}{2} \mathcal{F}_{3,g}^0(s_{2i}) \otimes \mathcal{F}_{3,g}^0(s_{2i}) + \frac{b_0}{\epsilon} \Gamma_{gg}^{(1)}(x_2) - \overline{\Gamma}_{gg}^{(2)}(x_2) \right] A_{3gH}^0(1,2,i)
$$

$$
-\left[+ \mathcal{F}_{4,g}^0(s_{1i}) + \mathcal{F}_{3,g}^1(s_{1i}) + \frac{b_0}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{F}_{3,g}^0(s_{1i}) - \frac{b_0}{\epsilon} \mathcal{F}_{3,g}^0(s_{1i}) \right.
$$

$$
\left. - \frac{1}{2} \mathcal{F}_{3,g}^0(s_{1i}) \otimes \mathcal{F}_{3,g}^0(s_{1i}) + \frac{b_0}{\epsilon} \Gamma_{gg}^{(1)}(x_1) - \overline{\Gamma}_{gg}^{(2)}(x_1) \right] A_{3gH}^0(1,2,i)
$$

$$
-\left[+ 2\mathcal{F}_{4,gg}^0(s_{12}) + \mathcal{F}_{4,gg}^{0,nadj}(s_{12}) - 2\mathcal{F}_{3,gg}^0(s_{12}) \otimes \mathcal{F}_{3,gg}^0(s_{12}) + 2\mathcal{F}_{3,gg}^1(s_{12}) \right.
$$

$$
+ 2\frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} \mathcal{F}_{3,gg}^0(s_{12}) - 2\frac{b_0}{\epsilon} \mathcal{F}_{3,gg}^0(s_{12}) + \frac{b_0}{\epsilon} \Gamma_{gg}^{(1)}(x_1) + \frac{b_0}{\epsilon} \Gamma_{gg}^{(1)}(x_2)
$$

$$
\left. - \overline{\Gamma}_{gg}^{(2)}(x_1) - \overline{\Gamma}_{gg}^{(2)}(x_2) \right] A_{3gH}^0(1,2,i), \tag{6.6.40}
$$

where $\otimes$ is the convolution operation defined in Eq. (1.4.35). The double virtual subtraction term fits the general structure described in section 4.7.2.

Using the dipole functions introduced in section 3.5.2 and 4.7.2, Eq. (6.6.40) can be re-expressed in the same structure as the two-loop poles in Eq. (4.7.156) as,

$$
A_{3gH}^{2,XU}(\hat{1},\hat{2},i) =
$$

$$
-\left[J_{2,GG}^{1,II}(s_{12}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,GG}^{1,IF}(s_{1i}) \right] \times \left[2A_{3gH}^1(1,2,i) - 2\frac{b_0}{\epsilon} A_{3gH}^0(1,2,i) \right]
$$

$$
-\left[J_{2,GG}^{1,II}(s_{12}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,GG}^{1,IF}(s_{1i}) \right] \otimes \left[J_{2,GG}^{1,II}(s_{12}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,GG}^{1,IF}(s_{1i}) \right]
$$

$$
\times A_{3gH}^0(1,2,i)
$$

$$
-\left[J_{2,GG}^{2,II}(s_{12}) + J_{2,GG}^{2,IF}(s_{1i}) + J_{2,GG}^{2,FI}(s_{2i}) \right] \times 2A_{3gH}^0(1,2,i), \tag{6.6.41}
$$

where the NNLO integrated antennae are,

$$
J_{2,GG}^{2,II}(s_{12}) = + \mathcal{F}_{4,gg}^0(s_{12}) + \frac{1}{2}\mathcal{F}_{4,gg}^{0,nadj}(s_{12}) - \mathcal{F}_{3,gg}^0(s_{12}) \otimes \mathcal{F}_{3,gg}^0(s_{12}) + \mathcal{F}_{3,gg}^1(s_{12})
$$

$$
- \frac{b_0}{\epsilon} \mathcal{F}_{3,gg}^0(s_{12}) - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)}(x_1) - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)}(x_2), \tag{6.6.42}
$$

$$
J_{2,GG}^{2,IF}(s_{1i}) = + \frac{1}{2}\mathcal{F}_{4,g}^0(s_{1i}) + \frac{1}{2}\mathcal{F}_{3,g}^1(s_{1i}) + \frac{b_0}{2\epsilon} \left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{F}_{3,g}^0(s_{1i})
$$

$$
- \frac{1}{4} \mathcal{F}_{3,g}^0(s_{1i}) \otimes \mathcal{F}_{3,g}^0(s_{1i}) - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)}(x_1), \tag{6.6.43}
$$

$$
J_{2,GG}^{2,FI}(s_{2i}) = + \frac{1}{2}\mathcal{F}_{4,g}^0(s_{2i}) + \frac{1}{2}\mathcal{F}_{3,g}^1(s_{2i}) + \frac{b_0}{2\epsilon} \left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{F}_{3,g}^0(s_{2i})
$$

$$
- \frac{1}{4} \mathcal{F}_{3,g}^0(s_{2i}) \otimes \mathcal{F}_{3,g}^0(s_{2i}) - \frac{1}{2}\overline{\Gamma}_{gg}^{(2)}(x_2). \tag{6.6.44}
$$

Due to the complexity of the integrated antenna functions, we use a FORM program to analytically check that the explicit IR divergences cancel between $A_{3gH}^{2,XU}(\hat{1}, \hat{2}, i)$ and $A_{3gH}^2(\hat{1}, \hat{2}, i)$. The cancellation of all of the explicit IR divergences in the two-loop matrix elements indicates that all the subtraction counter terms we introduced at the double real and real-virtual levels are correctly compensated at the double virtual level.

Explicit formulas for other antenna subtraction terms in table 6.5 can be found in appendix B.3.

By taking the leading colour contributions to $gg \rightarrow H$+jet at NNLO as an example, we have given explicit examples of how the antenna subtraction terms follows the structure introduced in chapter 4. The leading colour example we have seen in this chapter has no initial state identity changing (idc) behaviour. However the sub-leading colour contributions such as $gg \rightarrow gq\bar{q}H$ at RR level and $gg \rightarrow q\bar{q}H$ at RV level do have such behaviour. More details about how to construct antenna subtraction terms to remove initial state identity changing (idc) limits are introduced in chapter 7 and 9.
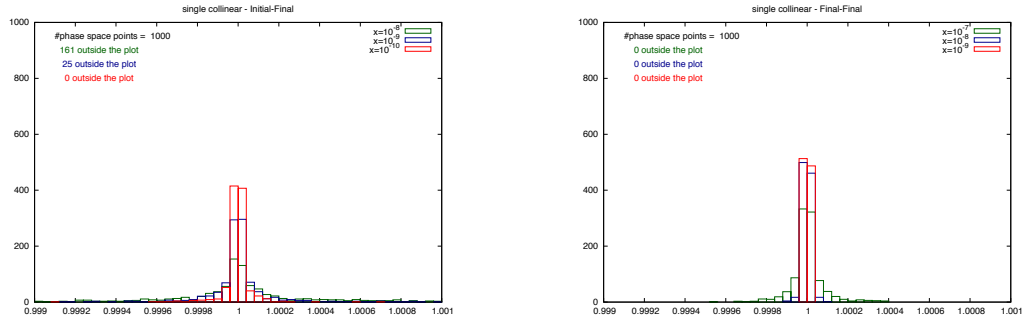
(a) Collinear limit between initial-final partons 1 and i where $x = s_{1i}/s_{12}$ approaches zero in the unresolved limit.

(b) Collinear limit between final-state partons i and j where $x = s_{ij}/s_{12}$ approaches zero in the unresolved limit.
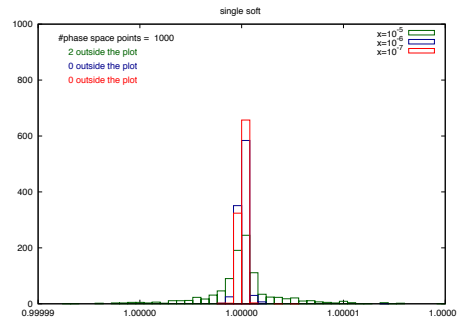
(c) Soft limit for soft partons i, where $x = (s_{1i} + s_{2i})/s_{12}$ approaches zero in the unresolved limit.

Figure 6.4: Spike plots displaying the convergence of the subtraction terms in $\mathrm{d}\hat{\sigma}_{gg}^{T}$ to the matrix elements in $\mathrm{d}\hat{\sigma}_{gg}^{RV}$ in various unresolved limits.

Chapter 7

Production of Higgs Boson Plus Jet from Quark-Gluon Scattering

In this chapter, I will discuss the qg, $\bar{q}g$, gq and $g\bar{q} \to H$+jet contributions to the fully differential cross section for Higgs boson plus jet observables up to NNLO. I will take the leading colour contribution to the $qg \to H + q + X$ channel as an example of the implementation of the antenna subtraction method as introduced in chapters 3 and 4. Just as for the gluon initiated channel discussed in chapter 6, the qg initiated channel contains initial state identity changing (idc) IR divergences. The antenna subtraction terms are more involved to regulate the idc limits in double real, real-virtual and double virtual contributiuons to the cross section and I will illustrate through examples about how these idc limits are treated at NNLO using the techniques introduced in section 4.3.3 and 4.5.3.

The spike plots introduced in chapter 6 are still used as a powerful tool to provide graphical evidence for the convergence of the subtraction term to the matrix element in the unresolved limits. For explicit IR divergences we use FORM programs to analytically check the divergences cancel between $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$ and $\mathrm{d}\hat{\sigma}^{U}_{NNLO}$.

7.1 qg initiated cross sections at LO

The quark-gluon scattering to Higgs boson plus one jet process at Born level has only one contribution from the $qg \to H + q$ process. The spin and colour averaged

differential cross section is given by,

$$\mathrm{d}\hat{\sigma}^B_{qg} = \mathcal{N}_{qg}\,\mathcal{N}_{LO}\,\mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)\left\{\frac{1}{N}B^0_{1gH}(\hat{1}_q, \hat{2}, 3_{\bar{q}})\right\}J^{(1)}_1(p_3). \qquad (7.1.1)$$

Explicit formula for B^0_{1gH} are given in section 3.1.2 and 5.1.6.

Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the other quark gluon initiated channels are related to $\mathrm{d}\hat{\sigma}^B_{qg}$:

$$\mathrm{d}\hat{\sigma}^B_{\bar{q}g} = \mathrm{d}\hat{\sigma}^B_{qg},$$

$$\mathrm{d}\hat{\sigma}^B_{gq} = \mathrm{d}\hat{\sigma}^B_{qg} \qquad (x_1 \leftrightarrow x_2),$$

$$\mathrm{d}\hat{\sigma}^B_{g\bar{q}} = \mathrm{d}\hat{\sigma}^B_{\bar{q}g} \qquad (x_1 \leftrightarrow x_2), \qquad (7.1.2)$$

where x_1, x_2 are the momentum fractions of the initial state partons as introduced in Eq. (1.4.39).

7.2 qg initiated cross sections at NLO

7.2.1 Real cross sections

The real radiation contribution comes from the $qg \to H + qg$ process,

$$\mathrm{d}\hat{\sigma}^R_{qg} = \mathcal{N}_{qg}\,\mathcal{N}^R_{NLO}\,\mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\Bigg\{$$

$$+ B^0_{2gH}(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) + B^0_{2gH}(\hat{1}_q, 3, \hat{2}, 4_{\bar{q}}) - \frac{1}{N^2}\widetilde{B}^0_{2gH}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}})\Bigg\}J^{(2)}_1(p_3, p_4).$$

$$(7.2.3)$$

The squared matrix elements in Eq.(7.2.3) are discussed in section 3.1.3 and 5.1.7.

7.2.2 Virtual cross sections

The one-loop contribution is from the $qg \to H + q$ process and the differential cross section is given by,

$$\mathrm{d}\hat{\sigma}^V_{qg} = \mathcal{N}_{qg}\,\mathcal{N}^V_{NLO}\,\mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)\Bigg\{$$

$$+ B^1_{1gH}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}^1_{1gH}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) + \frac{N_f}{N} \widehat{B}^1_{1gH}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) \Big\} J^{(1)}_1(p_3). \quad (7.2.4)$$

Details of the relevant matrix elements in Eq. (7.2.4) are given in section 3.1.3.

Just as at leading order, the other quark gluon initiated contributions can be obtained from $d\hat{\sigma}_{qg,NLO}$:

$$d\hat{\sigma}_{\bar{q}g,NLO} = d\hat{\sigma}_{qg,NLO},$$

$$d\hat{\sigma}_{gq,NLO} = d\hat{\sigma}_{qg,NLO} \qquad (x_1 \leftrightarrow x_2),$$

$$d\hat{\sigma}_{g\bar{q},NLO} = d\hat{\sigma}^B_{\bar{q}g,NLO} \qquad (x_1 \leftrightarrow x_2). \qquad (7.2.5)$$

7.3 qg initiated subtraction terms at NLO

7.3.1 Real subtraction terms

Using the NLO antenna subtraction method introduced in section 3.3.3, one can construct the antenna subtraction terms to mimic the implicit IR divergences in Eq.(7.2.3) such that

$$d\hat{\sigma}^S_{qg,NLO} = \mathcal{N}_{qg} \mathcal{N}^R_{NLO} d\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\Bigg\{$$

$$B^{0,XS}_{2gH,NLO}(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) + B^{0,YS}_{2gH,NLO}(\hat{1}_q, 3, \hat{2}, 4_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}^{0,XS}_{2gH,NLO}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}})\Bigg\}.$$

$$(7.3.6)$$

The corresponding relationships between subtraction terms, file name in the NNLO-JET maple script and matrix elements are summarised in Table 7.1.

Explicit formulas for each subtraction term are as follows:

$$B^{0,XS}_{2gH,NLO}(\hat{1}_q, \hat{2}, i, j_{\bar{q}}) =$$

$$+ \ d^0_{3,g}(j, i, 2) B^0_{1gH}(1, \overline{2}, (\widetilde{ij})) J^{(1)}_1(\{p\}_1)$$

$$+ \ G^0_{3,qg \to gg}(2, j, 1) A^0_{3gH}(\overline{1}, \overline{2}, i) J^{(1)}_1(\{p\}_1), \qquad (7.3.7)$$

$$B^{0,YS}_{2gH,NLO}(\hat{1}_q, i, \hat{2}, j_{\bar{q}}) =$$

$$+ \ D^0_{3,qg}(1, i, 2) B^0_{1gH}(\overline{1}, \overline{2}, j) J^{(1)}_1(\{p\}_1)$$

$$- \ A^0_{3,qg \to qq}(1, 2, j) B^0_{1gH}(\overline{1}, i, \overline{2}) J^{(1)}_1(\{p\}_1)$$

subtraction term	maple	matrix element
$B^{0,XS}_{2gH,NLO}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$	qgB2g0HXSNLO	$B^0_{2gH}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$
$B^{0,YS}_{2gH,NLO}(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$	qgB2g0HYSNLO	$B^0_{2gH}(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$
$\widetilde{B}^{0,XS}_{2gH,NLO}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$	qgBt2g0HXSNLO	$\widetilde{B}^0_{2gH}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$

Table 7.1: NLO antenna subtraction terms for real contributions in $qg \to H$+jet process and their relation to the matrix elements

$$+ \ G^0_{3,qg\to gg}(2,1,j)\, A^0_{3gH}(\overline{1}, \overline{2}, i)\, J^{(1)}_1(\{p\}_1), \tag{7.3.8}$$

$$\widetilde{B}^{0,XS}_{2gH,NLO}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}}) =$$

$$- \ A^0_{3,qg\to qq}(1,2,j)\, B^0_{1gH}(\overline{1}, i, \overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \ A^0_{3,q}(1,i,j)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ij}))\, J^{(1)}_1(\{p\}_1), \tag{7.3.9}$$

To numerically test that the antenna subtraction terms given in Eqs. (7.3.7), (7.3.8) and (7.3.9) remove the implicit IR divergences in Eq. (7.2.3) correctly, we use the same spike plots defined in section 6.4.1 to test that the subtraction terms converge to the matrix elements when approaching various NLO unresolved limits.

7.3.2   Virtual subtraction terms

Using the NLO antenna subtraction method introduced in section 3.5.2, one can combine the integrated real subtraction and mass factorization terms to construct the virtual subtraction term which removes the explicit IR divergences in Eq.(7.2.4),

$$\mathrm{d}\hat{\sigma}^T_{qg,NLO} = \mathcal{N}_{qg}\, \mathcal{N}^V_{NLO}\, \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)\Bigg\{$$

$$B^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) - \frac{1}{N^2}\widetilde{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) + \frac{N_f}{N}\widehat{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, 3_{\bar{q}})\Bigg\}$$

$$\tag{7.3.10}$$

The corresponding relationships between subtraction terms, file name in the NNLO-JET maple script and matrix elements are summarised in table 7.2.

The explicit formulae are as follows:

$$B^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}}) =$$

subtraction term	maple	matrix element
$B^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}})$	`qgB1g1HXTNLO`	$B^1_{1gH}(\hat{1}_q, \hat{2}, i_{\bar{q}})$
$\widetilde{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}})$	`qgBt1g1HXTNLO`	$\widetilde{B}^1_{1gH}(\hat{1}_q, \hat{2}, i_{\bar{q}})$
$\widehat{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}})$	`qgBh1g1HXTNLO`	$\widehat{B}^1_{1gH}(\hat{1}_q, \hat{2}, i_{\bar{q}})$

Table 7.2: NLO antenna subtraction terms for virtual contributions in $qg \to H$+jet process and their relation to the matrix elements

$$
\begin{aligned}
&- \left[+ J^{1,FI}_{2,QG}(s_{2i}) + J^{1,II}_{2,QG}(s_{12}) \right] B^0_{1gH}(1,2,i)\, J^{(1)}_1(\{p\}_1) \\
&- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, B^0_{1gH}(1,i,2)\, J^{(1)}_1(\{p\}_1) \\
&- 2J^{1,II}_{2,GG,q'g\to gg}(s_{12})\, A^0_{3gH}(1,2,i)\, J^{(1)}_1(\{p\}_1), \qquad\qquad (7.3.11)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}}) = \\
&- J^{1,IF}_{2,QQ}(s_{1i})\, B^0_{1gH}(1,2,i)\, J^{(1)}_1(\{p\}_1) \\
&- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, B^0_{1gH}(1,i,2)\, J^{(1)}_1(\{p\}_1), \qquad\qquad (7.3.12)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}}) = \\
&- \hat{J}^{1,II}_{2,QG}(s_{12})\, B^0_{1gH}(1,2,i)\, J^{(1)}_1(\{p\}_1) \\
&- \hat{J}^{1,FI}_{2,QG}(s_{2i})\, B^0_{1gH}(1,2,i)\, J^{(1)}_1(\{p\}_1), \qquad\qquad (7.3.13)
\end{aligned}
$$

Note the appearance of the three gluon matrix element $A^0_{3gH}(1,2,i)$ and the quark anti-quark initiated matrix element $B^0_{1gH}(1,i,2)$ in the above subtraction terms. This is because of (a) initial state identity changing collinear limits present in the $d\hat{\sigma}^S_{NLO}$ and (b) NLO mass factorization terms. As discussed in section 3.5.2, all the identity changing dipole functions ($\boldsymbol{J}^{(1)}_{2,a\to b}$) are IR finite. The jet functions in Eqs. (7.3.11) and (7.3.12) guarantee that the reduced tree level matrix elements are also IR safe. Thus the virtual subtraction terms associated with $A^0_{3gH}(1,2,i)$ and $B^0_{1gH}(1_q, i, 2_{\bar{q}})$ only provide a finite contribution to the differential cross section.

7.4 qg initiated cross sections at NNLO

7.4.1 Double real cross section

The double real contribution at NNLO for $qg \to H$+jet comes from the $qg \to H + qgg$, $qg \to H + qQ\bar{Q}$ and $qg \to H + qq\bar{q}$ processes,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{qg}^{RR} =\; & \mathcal{N}_{qg} \mathcal{N}_{NNLO}^{RR} \, \mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2) \Bigg\{ \\
& + \frac{N}{2!} \sum_{(i,j)\in P(3,4)} \left[B_{3gH}^0(\hat{1}_q, \hat{2}, i, j, 5_{\bar{q}}) + B_{3gH}^0(\hat{1}_q, i, \hat{2}, j, 5_{\bar{q}}) + B_{3gH}^0(\hat{1}_q, i, j, \hat{2}, 5_{\bar{q}}) \right] \\
& - \frac{1}{2!N} \sum_{(i,j)\in P(3,4)} \left[\widetilde{B}_{3gH}^0(\hat{1}_q, \widetilde{\hat{2}}, i, j, 5_{\bar{q}}) + \widetilde{B}_{3gH}^0(\hat{1}_q, \widetilde{i}, \hat{2}, j, 5_{\bar{q}}) + \widetilde{B}_{3gH}^0(\hat{1}_q, \widetilde{i}, j, \hat{2}, 5_{\bar{q}}) \right] \\
& + \frac{(N^2+1)}{2!N^3} \widetilde{\widetilde{B}}_{3gH}^0(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, \widetilde{4}, 5_{\bar{q}}) \\
& + N_f \left[C_{1gH}^0(\hat{1}_q, \hat{2}, 3_{\bar{Q}}, 4_Q, 5_{\bar{q}}) + C_{1gH}^0(\hat{1}_q, 3_{\bar{Q}}, 4_Q, \hat{2}, 5_{\bar{q}}) \right] \\
& + \frac{N_f}{N^2} \left[\widetilde{C}_{1gH}^0(\hat{1}_q, \hat{2}, 5_{\bar{q}}, 4_Q, 3_{\bar{Q}}) + \widetilde{C}_{1gH}^0(\hat{1}_q, 5_{\bar{q}}, 4_Q, \hat{2}, 3_{\bar{Q}}) - \widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q, 5_{\bar{q}}, 4_Q, 3_{\bar{Q}}, \hat{2}) \right] \\
& - \frac{1}{2!N} \left[D_{1gH}^0(\hat{1}_q, 4_q, 5_{\bar{q}}, 3_{\bar{q}}, \hat{2}) - \widetilde{D}_{1gH}^0(\hat{1}_q, 4_q, 5_{\bar{q}}, 3_{\bar{q}}, \hat{2}) \right] \\
& + \frac{1}{2!N^3} \widetilde{D}_{1gH}^0(\hat{1}_q, 4_q, 5_{\bar{q}}, 3_{\bar{q}}, \hat{2}) \Bigg\} J_1^{(3)}(p_3, p_4, p_5). \qquad (7.4.14)
\end{aligned}
$$

The squared matrix elements in Eq.(7.4.14) are discussed in section 4.1.2 and 5.1.8. The $1/2!$ coefficients in Eq.(7.4.14) are the averaging factors for two identical particles in the final state. The sum over active quark flavours gives the N_F factor to the C_{1gH}^0, $\widetilde{C}_{1gH}^0$ and $\widetilde{\widetilde{C}}_{1gH}^0$ matrix elements. The D_{1gH}^0 and $\widetilde{D}_{1gH}^0$ matrix elements do not have the N_F factor in front as the final state quarks are identical to the initial state quarks where the flavours are fixed.

7.4.2 Real-virtual cross section

The real-virtual contribution at NNLO for $qg \to H$+jet comes from the $qg \to H+qg$ process,

$$
\mathrm{d}\hat{\sigma}_{qg}^{RV} = \mathcal{N}_{qg} \mathcal{N}_{NNLO}^{RV} \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2) \Bigg\{
$$

$$+N \sum_{P(\hat{2},3)} \left[B_{2gH}^1(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}_{2gH}^1(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) + \frac{N_f}{N} \widehat{B}_{2gH}^1(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) \right]$$

$$- \frac{1}{N} \left[\widetilde{\widetilde{B}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) - \frac{1}{N^2} \widetilde{\widetilde{\widetilde{B}}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) + \frac{N_f}{N} \widetilde{\widehat{B}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) \right]$$

$$+ \frac{1}{N} \left[\widehat{\widetilde{B}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) \right] \Bigg\} J_1^{(2)}(p_3, p_4). \tag{7.4.15}$$

The squared matrix elements in Eq.(7.4.15) are discussed in section 4.1.2 and 5.2.

7.4.3 Double virtual cross section

The double virtual contribution at NNLO for $qg \to H$+jet comes from the $qg \to H + q$ process,

$$d\hat{\sigma}_{qg}^{VV} = \mathcal{N}_{qg} \mathcal{N}_{NNLO}^{VV} d\Phi_{H+1}(p_3, p_H; p_1, p_2) \Bigg\{$$

$$+ N B_{1gH}^2(\hat{1}_q, \hat{2}, 3_{\bar{q}}) + N_F \widehat{B}_{1gH}^2(\hat{1}_q, \hat{2}, 3_{\bar{q}}) + \frac{N_F^2}{N} \widehat{\widehat{B}}_{1gH}^2(\hat{1}_q, \hat{2}, 3_{\bar{q}})$$

$$+ \frac{N_F}{N^2} \widehat{\widetilde{B}}_{3gH}^2(\hat{1}_q, \hat{2}, 3_{\bar{q}}) + \frac{1}{N} \widetilde{B}_{1gH}^2(\hat{1}_q, \hat{2}, 3_{\bar{q}}) + \frac{1}{N^3} \widetilde{\widetilde{B}}_{1gH}^2(\hat{1}_q, \hat{2}, 3_{\bar{q}}) \Bigg\} J_1^{(1)}(p_3)$$

$$\tag{7.4.16}$$

The squared matrix elements in Eq. (7.4.16) are discussed in section 4.1.2 and the explicit formulas are given in [119].

Similar as at LO and NLO, the other quark gluon initiated cross section at NNLO are related to $d\hat{\sigma}_{qg,NNLO}$:

$$d\hat{\sigma}_{\bar{q}g,NNLO} = d\hat{\sigma}_{qg,NNLO},$$

$$d\hat{\sigma}_{gq,NNLO} = d\hat{\sigma}_{qg,NNLO} \qquad (x_1 \leftrightarrow x_2),$$

$$d\hat{\sigma}_{g\bar{q},NNLO} = d\hat{\sigma}_{\bar{q}g,NNLO}^B \qquad (x_1 \leftrightarrow x_2). \tag{7.4.17}$$

7.5 *qg* initiated subtraction terms at NNLO

7.5.1 Double real subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.3.3, one can construct the double real subtraction term that mimics the implicit IR divergences

in Eq. (7.4.14),

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{qg}^{RR} = \mathcal{N}_{qg}\mathcal{N}_{NNLO}^{RR}\mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2)\Bigg\{ & \\
\frac{N}{2!}\Bigg[B_{3gH}^{0,FXS}(\hat{1}_q, \hat{2}, 3, 4, 5_{\bar{q}}) + B_{3gH}^{0,YS}(\hat{1}_q, 3, \hat{2}, 4, 5_{\bar{q}}) \Bigg] & \\
-\frac{1}{2!N}\sum_{(i,j)\in P(3,4)}\Bigg[\widetilde{B}_{3gH}^{0,XS}(\hat{1}_q, \widetilde{\hat{2}}, i, j, 5_{\bar{q}}) + \widetilde{B}_{3gH}^{0,YS}(\hat{1}_q, \widetilde{i}, \hat{2}, j, 5_{\bar{q}}) \Bigg] & \\
+\frac{(N^2+1)}{2!N^3}\widetilde{\widetilde{B}}_{3gH}^{0,XS}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, \widetilde{4}, 5_{\bar{q}}) + N_f C_{1gH}^{0,XS}(\hat{1}_q, \hat{2}, 3_{\bar{Q}}, 4_Q, 5_{\bar{q}}) & \\
+\frac{N_f}{N^2}\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q, \hat{2}, 5_{\bar{q}}, 4_Q, 3_{\bar{Q}}) & \\
-\frac{1}{2N}\Bigg[D_{1gH}^{0,XS}(\hat{1}_q, 4_q, 5_{\bar{q}}, 3_{\bar{q}}, \hat{2}) - \widetilde{D}_{1gH}^{0,XS}(\hat{1}_q, 4_q, 5_{\bar{q}}, 3_{\bar{q}}, \hat{2}) \Bigg] & \\
+\frac{1}{2N^3}\widetilde{D}_{1gH}^{0,XS}(\hat{1}_q, 4_q, 5_{\bar{q}}, 3_{\bar{q}}, \hat{2}) \Bigg\}. &
\end{aligned}
\tag{7.5.18}
$$

The corresponding relationships between the subtraction terms, file name in the NNLOJET maple script and matrix elements (or combinations of subtraction terms) in Eq. (7.5.18) are summarised in table 7.3.

The $B_{3gH}^{0,FXS}(\hat{1}_q, \hat{2}, i, j, k_{\bar{q}})$ function is the subtraction term for a combination of two $B_{3gH}^{0,XS}(\hat{1}_q, \hat{2}, i, j, k_{\bar{q}})$ functions with permutations of the two gluons. Each $B_{3gH}^{0,XS}(\hat{1}_q, \hat{2}, i, j, k_{\bar{q}})$ function does not mimic all the double and single unresolved limits of two X topology matrix elements. Only the two permutations together removes all of the double and single unresolved limits of all four X topology matrix elements.

In the $\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q, \hat{2}, k_{\bar{q}}, i_Q, j_{\bar{Q}})$ subtraction term, the momenta of the secondary quark-antiquark pair $(Q\bar{Q})$ can be symmetrized as the jet function does not distinguish quarks and antiquarks. This symmetrization is designed to remove the implicit IR divergences from the interference terms present in $\widetilde{\widetilde{C}}_{1gH}^{0}$ (see Eq.(4.1.22)), since those interference terms are odd under the $Q \leftrightarrow \bar{Q}$ interchange. The symmetrized subtraction term $\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q, \hat{2}, k_{\bar{q}}, i_Q, j_{\bar{Q}})$ mimics the implicit IR divergences in the combination,

$$
\frac{1}{2}\sum_{P(i,j)}\Bigg[\widetilde{C}_{1gH}^{0}(\hat{1}_q, \hat{2}, k_{\bar{q}}, i_Q, j_{\bar{Q}}) + \widetilde{C}_{1gH}^{0}(\hat{1}_q, k_{\bar{q}}, i_Q, \hat{2}, j_{\bar{Q}}) - \widetilde{\widetilde{C}}_{1gH}^{0}(\hat{1}_q, k_{\bar{q}}, i_Q, j_{\bar{Q}}, \hat{2}) \Bigg].
$$

subtraction term	maple	matrix element
$B_{3gH}^{0,FXS}(\hat{1}_q,\hat{2},i,j,k_{\bar{q}})$	qgB3g0HFXS	$\sum\limits_{P(i,j)} B_{3gH}^{0,XS}(\hat{1}_q,\hat{2},i,j,k_{\bar{q}})$
$B_{3gH}^{0,XS}(\hat{1}_q,\hat{2},i,j,k_{\bar{q}})$	qgB3g0HXS	$B_{3gH}^{0}(\hat{1}_q,\hat{2},i,j,k_{\bar{q}}) + B_{3gH}^{0}(\hat{1}_q,j,i,\hat{2},k_{\bar{q}})$
$B_{3gH}^{0,YS}(\hat{1}_q,i,\hat{2},j,k_{\bar{q}})$	qgB3g0HYS	$B_{3gH}^{0}(\hat{1}_q,i,\hat{2},j,k_{\bar{q}}) + B_{3gH}^{0}(\hat{1}_q,j,\hat{2},i,k_{\bar{q}})$
$\widetilde{B}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{\hat{2}},i,j,k_{\bar{q}})$	qgBt3g0HXS	$\widetilde{B}_{3gH}^{0}(\hat{1}_q,\widetilde{\hat{2}},i,j,k_{\bar{q}})$
$\widetilde{B}_{3gH}^{0,YS}(\hat{1}_q,\widetilde{i},\hat{2},j,k_{\bar{q}})$	qgBt3g0HYS	$\widetilde{B}_{3gH}^{0}(\hat{1}_q,\widetilde{i},\hat{2},j,k_{\bar{q}}) + \widetilde{B}_{3gH}^{0}(\hat{1}_q,\widetilde{i},j,\hat{2},k_{\bar{q}})$
$\widetilde{\widetilde{B}}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{\hat{2}},\widetilde{i},\widetilde{j},k_{\bar{q}})$	qgBtt3g0HXS	$\widetilde{\widetilde{B}}_{3gH}^{0}(\hat{1}_q,\widetilde{\hat{2}},\widetilde{i},\widetilde{j},k_{\bar{q}})$
$C_{1gH}^{0,XS}(\hat{1}_q,\hat{2},i_{\bar{Q}},j_Q,k_{\bar{q}})$	qgC1g0HXS	$C_{1gH}^{0}(\hat{1}_q,\hat{2},i_{\bar{Q}},j_Q,k_{\bar{q}}) + C_{1gH}^{0}(\hat{1}_q,i_{\bar{Q}},j_Q,\hat{2},k_{\bar{q}})$
$\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q,\hat{2},k_{\bar{q}},i_Q,j_{\bar{Q}})$	qgCt1g0HFXS	$\frac{1}{2}\sum\limits_{P(i,j)} \widetilde{C}_{1gH}^{0,XS}(\hat{1}_q,\hat{2},k_{\bar{q}},i_Q,j_{\bar{Q}})$
$\widetilde{C}_{1gH}^{0,XS}(\hat{1}_q,\hat{2},k_{\bar{q}},i_Q,j_{\bar{Q}})$	qgCt1g0HXS	$\widetilde{C}_{1gH}^{0}(\hat{1}_q,\hat{2},k_{\bar{q}},i_Q,j_{\bar{Q}}) + \widetilde{C}_{1gH}^{0}(\hat{1}_q,k_{\bar{q}},i_Q,\hat{2},j_{\bar{Q}})$ $-\widetilde{\widetilde{C}}_{1gH}^{0}(\hat{1}_q,k_{\bar{q}},i_Q,j_{\bar{Q}},\hat{2})$
$D_{1gH}^{0,XS}(\hat{1}_q,i_q,j_{\bar{q}},k_{\bar{q}},\hat{2})$	qgD1g0HXS	$D_{1gH}^{0}(\hat{1}_q,i_q,j_{\bar{q}},k_{\bar{q}},\hat{2})$
$\widetilde{D}_{1gH}^{0,XS}(\hat{1}_q,i_q,j_{\bar{q}},k_{\bar{q}},\hat{2})$	qgDt1g0HXS	$\widetilde{D}_{1gH}^{0}(\hat{1}_q,i_q,j_{\bar{q}},k_{\bar{q}},\hat{2})$

Table 7.3: NNLO antenna subtraction terms for double real contributions in $qg \rightarrow H$+jet process and their relation to the matrix elements (or combinations of subtraction terms)

Taking only the colour leading contribution as an example, the explicit formulae for $B_{3gH}^{0,FXS}(\hat{1}_q,\hat{2},i,j,k_{\bar{q}})$ and $B_{3gH}^{0,YS}(\hat{1}_q,i,\hat{2},j,k_{\bar{q}})$ are,

$$B_{3gH}^{0,FXS}(\hat{1}_q,\hat{2},i,j,k_{\bar{q}}) =$$
$$\sum_{P(i,j)} \Big\{ \quad + \quad f_{3,g}^{0}(2,i,j)\, B_{2gH}^{0}(1,\bar{2},(\widetilde{ij}),k)\, J_1^{(2)}(\{p\}_2)$$
$$+ \quad d_3^{0}(k,j,i)\, B_{2gH}^{0}(1,2,(\widetilde{ij}),(\widetilde{jk}))\, J_1^{(2)}(\{p\}_2)$$
$$+ \quad d_{3,q}^{0}(1,j,i)\, B_{2gH}^{0}(\bar{1},(\widetilde{ji}),2,k)\, J_1^{(2)}(\{p\}_2)$$
$$+ \quad f_{3,g}^{0}(2,i,j)\, B_{2gH}^{0}(1,(\widetilde{ji}),\bar{2},k)\, J_1^{(2)}(\{p\}_2)$$
$$- \quad A_{3,qg\rightarrow qq}^{0}(1,2,k)\, B_{2gH}^{0}(\bar{1},j,i,\bar{2})\, J_1^{(2)}(\{p\}_2)$$
$$+ \quad D_4^{0}(1,2,i,j)\, B_{1gH}^{0}(\bar{1},\bar{2},k)\, J_1^{(1)}(\{p\}_1)$$
$$- \quad f_{3,g}^{0}(2,i,j)\, D_{3,qg}^{0}(1,\bar{2},(\widetilde{ij}))\, B_{1gH}^{0}(\bar{1},\bar{\bar{2}},k)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad d_{3,q}^0(1,j,i)\, D_{3,qg}^0(\overline{1},2,(\widetilde{ji}))\, B_{1gH}^0(\overline{\overline{1}},\overline{2},k)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad D_4^0(k,j,i,2)\, B_{1gH}^0(1,\overline{2},(\widetilde{kji}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad d_{3,g}^0(k,j,i)\, D_{3,g}^0((\widetilde{kj}),(\widetilde{ji}),2)\, B_{1gH}^0(1,\overline{2},(\widetilde{\widetilde{kj}},\widetilde{\widetilde{ji}}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad f_{3,g}^0(2,i,j)\, D_{3,g}^0(k,(\widetilde{ij}),\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},(\widetilde{k(\widetilde{ij})}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \tilde{A}_4^0(1,2,j,k)\, B_{1gH}^0(\overline{1},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad A_{3,q}^0(1,j,k)\, A_{3,qg\to qq}^0(\overline{1},2,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_4^0(1,i,2,k)\, B_{1gH}^0(\overline{1},\overline{2},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad D_{3,qg}^0(1,i,2)\, A_{3,qg\to qq}^0(\overline{1},\overline{2},k)\, B_{1gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad f_{3,g}^0(2,i,j)\, A_{3,qg\to qq}^0(1,\overline{2},k)\, B_{1gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad D_{3,qg}^0(1,i,2)\, A_{3,qg\to qq}^0(\overline{1},\overline{2},k)\, B_{1gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad d_3^0(k,i,j)\, A_{3,qg\to qq}^0(1,2,(\widetilde{ki}))\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_{3,q}^0(1,i,k)\, A_{3,qg\to qq}^0(\overline{1},2,(\widetilde{ik}))\, B_{1gH}^0(\overline{\overline{1}},\overline{2},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad 2\, d_{3,q}^0(1,i,j)\, A_{3,qg\to qq}^0(\overline{1},2,k)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \Big[+ S_{\overline{1}i2}^{IF} + S_{\overline{1}ik}^{IF} - 2S_{\overline{\overline{1}}i\overline{2}}^{IF} - S_{2i(\widetilde{ij})}^{IF} - S_{ki(\widetilde{ij})}^{IF} + 2S_{\overline{2}i(\widetilde{ij})}^{IF}$$

$$\qquad - 2S_{\overline{1}i(\widetilde{ij})}^{IF} + 2S_{\overline{\overline{1}}i(\widetilde{ij})}^{IF} \Big]$$

$$\times A_{3,qg\to qq}^0(\overline{1},2,k)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad 2\, A_{3,qg\to qq}^0(1,2,k)\, A_{3,q\bar{q}}^0(\overline{1},i,\overline{2})\, B_{1gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad 2\, A_{3,qg\to qq}^0(1,2,k)\, d_{3,q}^0(\overline{2},i,j)\, B_{1gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad 2\, A_{3,qg\to qq}^0(1,2,k)\, d_{3,q}^0(\overline{1},i,j)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_4^0(1,i,2,k)\, B_{1gH}^0(\overline{1},j,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad D_{3,qg}^0(1,i,2)\, A_{3,qg\to qq}^0(\overline{1},\overline{2},k)\, B_{1gH}^0(\overline{\overline{1}},j,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad A_{3,qg\to qq}^0(1,2,k)\, A_{3,q\bar{q}}^0(\overline{1},i,\overline{2})\, B_{1gH}^0(\overline{\overline{1}},j,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad f_{3,g}^0(2,i,j)\, A_{3,qg\to qq}^0(1,\overline{2},k)\, B_{1gH}^0(\overline{1},(\widetilde{ij}),\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad d_{3,q}^0(1,j,i)\, A_{3,qg\to qq}^0(\overline{1},2,k)\, B_{1gH}^0(\overline{\overline{1}},(\widetilde{ji}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$- \quad D_{3,qg}^0(1,i,2)\, A_{3,qg\to qq}^0(\overline{1},\overline{2},k)\, B_{1gH}^0(\overline{\overline{1}},j,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \Big[+ S_{\overline{1}i2}^{IF} - S_{\overline{\overline{1}}i\overline{2}}^{IF} - S_{2i(\widetilde{ij})}^{IF} + S_{\overline{2}i(\widetilde{ij})}^{IF} - S_{\overline{1}i(\widetilde{ij})}^{IF} + S_{\overline{\overline{1}}i(\widetilde{ij})}^{IF} \Big]$$

$$\times A_{3,qg\to qq}^0(\overline{1},2,k)\, B_{1gH}^0(\overline{\overline{1}},(\widetilde{ij}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$
\begin{aligned}
&- \quad G^0_{3,qg\to gg}(2,k,1)\, A^0_{4gH}(\overline{1},\overline{2},i,j)\, J^{(2)}_1(\{p\}_2) \\
&- \quad G^0_{3,qg\to gg}(2,k,1)\, A^0_{4gH}(\overline{1},j,i,\overline{2})\, J^{(2)}_1(\{p\}_2) \\
&- \quad G^0_4(2,k,1,j)\, A^0_{3gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1) \\
&+ \quad G^0_{3,qg\to gg}(2,k,1)\, F^0_{3,gg}(\overline{1},j,\overline{2})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1) \\
&+ \quad D^0_{3,qg}(1,j,2)\, G^0_{3,qg\to gg}(\overline{2},k,\overline{1})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1) \\
&+ \quad A^0_{3,qg\to qq}(1,2,k)\, G^0_{3,qg\to gg}(j,\overline{2},\overline{1})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1) \\
&- \quad G^0_4(2,1,k,j)\, A^0_{3gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1) \\
&+ \quad G^0_{3,qg\to gg}(2,1,k)\, F^0_{3,gg}(\overline{1},j,\overline{2})\, A^0_{3gH}(\overline{\overline{1}},i,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1) \\
&+ \quad d^0_{3,g}(k,j,2)\, G^0_{3,qg\to gg}(\overline{2},(\widetilde{kj}),1)\, A^0_{3gH}(\overline{1},i,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1) \\
&- \quad D^0_{3,qg}(1,j,2)\, G^0_{3,qg\to gg}(\overline{2},\overline{1},k)\, A^0_{3gH}(\overline{\overline{1}},\overline{2},i)\, J^{(1)}_1(\{p\}_1) \\
&+ \quad d^0_{3,q}(1,j,i)\, G^0_{3,qg\to gg}(2,\overline{1},k)\, A^0_{3gH}(\overline{\overline{1}},(\widetilde{ji}),\overline{2})\, J^{(1)}_1(\{p\}_1) \\
&+ \quad f^0_{3,g}(2,j,i)\, G^0_{3,qg\to gg}(\overline{2},1,k)\, A^0_{3gH}(\overline{1},\overline{\overline{2}},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1) \\
&+ \quad \left[+ S^{IF}_{\overline{1}j2} - S^{IF}_{\overline{1}j\overline{2}} - S^{IF}_{2j(\widetilde{ij})} + S^{IF}_{\overline{2}j(\widetilde{ij})} - S^{IF}_{\overline{1}j(\widetilde{ij})} + S^{IF}_{\overline{\overline{1}}j(\widetilde{ij})} \right] \\
&\qquad \times G^0_{3,qg\to gg}(2,\overline{1},k)\, A^0_{3gH}(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1) \\
&- \quad d^0_{3,g}(k,j,2)\, G^0_{3,qg\to gg}(\overline{2},1,(\widetilde{kj}))\, A^0_{3gH}(\overline{1},i,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1) \\
&+ \quad d^0_3(k,j,i)\, G^0_{3,qg\to gg}(2,1,(\widetilde{jk}))\, A^0_{3gH}(\overline{1},\overline{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1) \\
&+ \quad f^0_{3,g}(2,j,i)\, G^0_{3,qg\to gg}(\overline{2},1,k)\, A^0_{3gH}(\overline{1},\overline{\overline{2}},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1) \\
&+ \quad \left[+ S^{IF}_{kj2} - S^{IF}_{\overline{\overline{1}}j\overline{2}} - S^{IF}_{2j(\widetilde{ij})} + S^{IF}_{\overline{2}j(\widetilde{ij})} - S^{IF}_{kj(\widetilde{ij})} + S^{IF}_{\overline{\overline{1}}j(\widetilde{ij})} \right] \\
&\qquad \times G^0_{3,qg\to gg}(2,\overline{1},k)\, A^0_{3gH}(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1) \Bigg\} \qquad (7.5.19)
\end{aligned}
$$

$$
\begin{aligned}
B^{0,YS}_{3gH}&(\hat{1}_q,i,\hat{2},j,k_{\bar{q}}) = \\
&+ \quad D^0_{3,qg}(1,i,2)\, B^0_{2gH}(\overline{1},\overline{2},j,k)\, J^{(2)}_1(\{p\}_2) \\
&+ \quad d^0_{3,g}(k,j,2)\, B^0_{2gH}(1,i,\overline{2},(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2) \\
&+ \quad D^0_{3,qg}(1,j,2)\, B^0_{2gH}(\overline{1},\overline{2},i,k)\, J^{(2)}_1(\{p\}_2) \\
&+ \quad d^0_{3,g}(k,i,2)\, B^0_{2gH}(1,j,\overline{2},(\widetilde{ik}))\, J^{(2)}_1(\{p\}_2) \\
&+ \quad D^0_4(1,i,2,j)\, B^0_{1gH}(\overline{1},\overline{2},k)\, J^{(1)}_1(\{p\}_1) \\
&- \quad D^0_{3,qg}(1,i,2)\, D^0_{3,qg}(\overline{1},\overline{2},j)\, B^0_{1gH}(\overline{\overline{1}},\overline{\overline{2}},k)\, J^{(1)}_1(\{p\}_1)
\end{aligned}
$$

$$- \quad D^0_{3,qg}(1,j,2)\, D^0_{3,qg}(\overline{1},\overline{2},i)\, B^0_{1gH}(\overline{\overline{1}},\overline{\overline{2}},k)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, D^0_{3,qg}(1,j,2)\, D^0_{3,qg}(\overline{1},i,\overline{2})\, B^0_{1gH}(\overline{\overline{1}},\overline{\overline{2}},k)\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, d^0_{3,g}(k,j,2)\, D^0_{3,qg}(1,i,\overline{2})\, B^0_{1gH}(\overline{1},\overline{\overline{2}},\widetilde{(jk)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, A^0_{3,q}(1,j,k)\, D^0_{3,qg}(\overline{1},i,2)\, B^0_{1gH}(\overline{\overline{1}},\overline{2},\widetilde{(kj)})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\left[- S^{IF}_{\overline{1}j2} + S^{IF}_{\overline{\overline{1}}j\overline{2}} + S^{IF}_{2j\widetilde{(jk)}} - S^{IF}_{\overline{2}j\widetilde{(jk)}} + S^{IF}_{\overline{1}j\widetilde{(jk)}} - S^{IF}_{\overline{\overline{1}}j\widetilde{(jk)}} \right]$$
$$\times D^0_{3,qg}(\overline{1},i,2)\, B^0_{1gH}(\overline{\overline{1}},\overline{2},\widetilde{(jk)})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, D^0_{3,qg}(1,i,2)\, D^0_{3,qg}(\overline{1},j,\overline{2})\, B^0_{1gH}(\overline{\overline{1}},\overline{\overline{2}},k)\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, d^0_{3,g}(k,i,2)\, D^0_{3,qg}(1,j,\overline{2})\, B^0_{1gH}(\overline{1},\overline{\overline{2}},\widetilde{(ik)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, A^0_{3,q}(1,i,k)\, D^0_{3,qg}(\overline{1},j,2)\, B^0_{1gH}(\overline{\overline{1}},\overline{2},\widetilde{(ki)})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\left[- S^{IF}_{\overline{1}i2} + S^{IF}_{\overline{\overline{1}}i\overline{2}} + S^{IF}_{2i\widetilde{(ik)}} - S^{IF}_{\overline{2}i\widetilde{(ik)}} + S^{IF}_{\overline{1}i\widetilde{(ik)}} - S^{IF}_{\overline{\overline{1}}i\widetilde{(ik)}} \right]$$
$$\times D^0_{3,qg}(\overline{1},j,2)\, B^0_{1gH}(\overline{\overline{1}},\overline{2},\widetilde{(ik)})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad D^0_4(k,j,2,i)\, B^0_{1gH}(1,\overline{2},\widetilde{(ijk)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad d^0_{3,g}(k,j,2)\, D^0_{3,g}(\widetilde{(kj)},i,\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},\widetilde{(i\widetilde{(jk)})})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad d^0_{3,g}(k,i,2)\, D^0_{3,g}(\widetilde{(ki)},\overline{2},j)\, B^0_{1gH}(1,\overline{\overline{2}},\widetilde{(j\widetilde{(ik)})})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, d^0_{3,g}(k,i,2)\, d^0_{3,g}(\widetilde{(ki)},j,\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},\widetilde{(\widetilde{(ki)}j)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, D^0_{3,qg}(1,i,2)\, d^0_{3,g}(k,j,\overline{2})\, B^0_{1gH}(\overline{1},\overline{\overline{2}},\widetilde{(kj)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, A^0_{3,q}(1,i,k)\, d^0_{3,g}(\widetilde{(ki)},j,2)\, B^0_{1gH}(\overline{1},\overline{2},\widetilde{(\widetilde{(ki)}j)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\left[+ S^{IF}_{2i\widetilde{(ki)}} - S^{IF}_{\overline{2}i(\widetilde{(ki)},j)} - S^{IF}_{2i\overline{1}} + S^{IF}_{\overline{2}i\overline{1}} - S^{IF}_{\overline{1}i\widetilde{(ki)}} + S^{IF}_{\overline{1}i(\widetilde{(ki)},j)} \right]$$
$$\times d^0_{3,g}(\widetilde{(ki)},j,2)\, B^0_{1gH}(\overline{1},\overline{2},\widetilde{(ki)},j)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, d^0_{3,g}(k,j,2)\, d^0_{3,g}(\widetilde{(kj)},i,\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},\widetilde{(\widetilde{(kj)},i)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, D^0_{3,qg}(1,j,2)\, d^0_{3,g}(k,i,\overline{2})\, B^0_{1gH}(\overline{1},\overline{\overline{2}},\widetilde{(ki)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, A^0_{3,q}(1,j,k)\, d^0_{3,g}(\widetilde{(kj)},i,2)\, B^0_{1gH}(\overline{1},\overline{2},\widetilde{(\widetilde{(kj)},i)})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\left[+ S^{IF}_{2j\widetilde{(kj)}} - S^{IF}_{\overline{2}j(\widetilde{(kj)},i)} - S^{IF}_{2j\overline{1}} + S^{IF}_{\overline{2}j\overline{1}} - S^{IF}_{\overline{1}j\widetilde{(kj)}} + S^{IF}_{\overline{1}j(\widetilde{(kj)},i)} \right]$$
$$\times d^0_{3,g}(\widetilde{(kj)},i,2)\, B^0_{1gH}(\overline{1},\overline{2},\widetilde{(kj)},i)\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \tilde{A}_4^0(1,i,j,k)\, B_{1gH}^0(\overline{1},2,(\widetilde{ijk}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad A_{3,q}^0(1,i,k)\, A_{3,q}^0(\overline{1},j,(\widetilde{ik}))\, B_{1gH}^0(\overline{\overline{1}},2,\widetilde{(j,(\widetilde{ik}))})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad A_{3,q}^0(1,j,k)\, A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},2,\widetilde{(i,(\widetilde{jk}))})\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\, A_{3,q}^0(1,j,k)\, A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},2,\widetilde{(i,(\widetilde{jk}))})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, d_{3,g}^0(k,j,2)\, A_{3,q}^0(1,i,(\widetilde{kj}))\, B_{1gH}^0(\overline{1},\overline{2},\widetilde{(i,(\widetilde{kj}))})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, D_{3,qg}^0(1,j,2)\, A_{3,q}^0(\overline{1},i,k)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ik}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\left[-S_{\overline{1}j(\widetilde{jk})}^{IF} + S_{\overline{\overline{1}}j(i,(\widetilde{jk}))}^{IF} + S_{2j(\widetilde{jk})}^{IF} - S_{2j(i,(\widetilde{jk}))}^{IF} + S_{2j\overline{1}}^{IF} - S_{2j\overline{\overline{1}}}^{IF} \right]$$

$$\times A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},2,\widetilde{(i,(\widetilde{jk}))})\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\, A_{3,q}^0(1,i,k)\, A_{3,q}^0(\overline{1},j,(\widetilde{ik}))\, B_{1gH}^0(\overline{\overline{1}},2,\widetilde{(j,(\widetilde{ik}))})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, d_{3,g}^0(k,i,2)\, A_{3,q}^0(1,j,(\widetilde{ki}))\, B_{1gH}^0(\overline{1},\overline{2},\widetilde{(j,(\widetilde{ki}))})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, D_{3,qg}^0(1,i,2)\, A_{3,q}^0(\overline{1},j,k)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$- \quad \frac{1}{2}\left[-S_{\overline{1}i(\widetilde{ik})}^{IF} + S_{\overline{\overline{1}}i(j,(\widetilde{ik}))}^{IF} + S_{2i(\widetilde{ik})}^{IF} - S_{2i(j,(\widetilde{ik}))}^{IF} + S_{2i\overline{1}}^{IF} - S_{2i\overline{\overline{1}}}^{IF} \right]$$

$$\times A_{3,q}^0(\overline{1},j,(\widetilde{ik}))\, B_{1gH}^0(\overline{\overline{1}},2,\widetilde{(j,(\widetilde{ik}))})\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_4^0(1,2,j,k)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad d_{3,g}^0(k,j,2)\, A_{3,qg\to qq}^0(1,\overline{2},(\widetilde{jk}))\, B_{1gH}^0(\overline{1},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_4^0(1,2,j,k)\, B_{1gH}^0(\overline{1},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad d_{3,g}^0(k,j,2)\, A_{3,qg\to qq}^0(1,\overline{2},(\widetilde{jk}))\, B_{1gH}^0(\overline{1},\overline{\overline{2}},i)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_4^0(1,2,i,k)\, B_{1gH}^0(\overline{1},j,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad d_{3,g}^0(k,i,2)\, A_{3,qg\to qq}^0(1,\overline{2},(\widetilde{ik}))\, B_{1gH}^0(\overline{1},j,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- \quad A_4^0(1,2,i,k)\, B_{1gH}^0(\overline{1},\overline{2},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad d_{3,g}^0(k,i,2)\, A_{3,qg\to qq}^0(1,\overline{2},(\widetilde{ik}))\, B_{1gH}^0(\overline{1},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$- \quad G_{3,qg\to gg}^0(2,k,1)\, A_{4gH}^0(\overline{1},i,\overline{2},j)\, J_1^{(2)}(\{p\}_2)$$

$$- \quad G_{3,qg\to gg}^0(2,k,1)\, A_{4gH}^0(\overline{1},j,\overline{2},i)\, J_1^{(2)}(\{p\}_2)$$

$$- \quad G_4^0(2,k,1,i)\, A_{3gH}^0(\overline{1},\overline{2},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad G_{3,qg\to gg}^0(2,k,1)\, F_{3,gg}^0(\overline{1},i,\overline{2})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad D_{3,qg}^0(1,i,2)\, G_{3,qg\to gg}^0(\overline{2},k,\overline{1})\, A_{3gH}^0(\overline{\overline{1}},\overline{\overline{2}},j)\, J_1^{(1)}(\{p\}_1)$$

$$+ \quad A^0_{3,qg\to qq}(1,2,k)\, G^0_{3,qg\to gg}(i,\overline{2},\overline{1})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},j)\, J^{(1)}_1(\{p\}_1)$$

$$- \quad G^0_4(2,1,k,i)\, A^0_{3gH}(\overline{1},\overline{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad G^0_{3,qg\to gg}(2,1,k)\, F^0_{3,gg}(\overline{1},\overline{2},i)\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},j)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad d^0_{3,g}(k,i,2)\, G^0_{3,qg\to gg}(\overline{2},1,(\widetilde{ki}))\, A^0_{3gH}(\overline{1},\overline{\overline{2}},j)\, J^{(1)}_1(\{p\}_1)$$

$$- \quad G^0_4(2,k,1,j)\, A^0_{3gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad G^0_{3,qg\to gg}(2,k,1)\, F^0_{3,gg}(\overline{1},j,\overline{2})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad D^0_{3,qg}(1,j,2)\, G^0_{3,qg\to gg}(\overline{2},k,\overline{1})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad A^0_{3,qg\to qq}(1,2,k)\, G^0_{3,qg\to gg}(j,\overline{2},\overline{1})\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1)$$

$$- \quad G^0_4(2,1,k,j)\, A^0_{3gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad G^0_{3,qg\to gg}(2,1,k)\, F^0_{3,gg}(\overline{1},\overline{2},j)\, A^0_{3gH}(\overline{\overline{1}},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad d^0_{3,g}(k,j,2)\, G^0_{3,qg\to gg}(\overline{2},1,(\widetilde{kj}))\, A^0_{3gH}(\overline{1},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1). \tag{7.5.20}$$

In Eq.(7.5.19) and (7.5.20), the initial state identity changing limits could change an initial state quark into an initial state gluon, $1_q \to \overline{1}_g$. We use $G^0_{3,q\to g}$ and G^0_4 antenna functions to mimic the single and double unresolved limits related to these idc behaviour.

Similarly, the initial state identity changing limits could change an initial state gluon into an initial state quark (anti-quark), $2_g \to \overline{2}_q$. In this case, we use $A^0_{3,qg\to qq}$ and A^0_4 antenna functions to mimic the corresponding single and double implicit IR divergences involving the idc behaviour.

In the initial state identity preserving (idp) limits, we use antenna functions like $D^0_4(k_q, j_g, i_g, 2_g)$ in association with idp reduced matrix elements to remove the triple collinear ($j//i//2$), double collinear ($k//j; i//2$), double soft ($i, j \to 0$) and soft collinear limits ($j \to 0; i//2$) with a term like

$$+ D^0_4(k,j,i,2)\, B^0_{1gH}(1,\overline{2},(\widetilde{kji}))\, J^{(1)}_1(\{p\}_1). \tag{7.5.21}$$

However, the $D^0_4(k_q, j_g, i_g, 2_g)$ antenna function also contains implicit IR divergent in idc limits like $k_q//2_g$, $k_q//j_g//2_g$ and $k_q//2_g//i_g$ because the quark k is also colour connected to the gluon 2. These limits are idc because the initial state particle will be a quark, $k_q//2_g \to \overline{2}_q$. However, these idc limits are unphysical in Eq.(7.5.21) because the reduced matrix element requires parton 2 to be a gluon.

The solution to remove these unphysical idc limits requires two steps. The first step is to remove the idc double unresolved limits and idp single unresolved limits using the following terms:

$$
\begin{aligned}
&+ D_4^0(k, j, i, 2)\, B_{1gH}^0(1, \overline{2}, (\widetilde{kji}))\, J_1^{(1)}(\{p\}_1) \\
&- d_3^0(k, j, i)\, D_{3,g}^0((\widetilde{kj}), (\widetilde{ji}), 2)\, B_{1gH}^0(1, \overline{2}, (\widetilde{\widetilde{kj}}, \widetilde{ji})\, J_1^{(1)}(\{p\}_1) \\
&- f_{3,g}^0(2, i, j)\, D_{3,g}^0(k, (\widetilde{ij}), \overline{2})\, B_{1gH}^0(1, \overline{\overline{2}}, (\widetilde{k(\widetilde{ij})}))\, J_1^{(1)}(\{p\}_1) \\
&- \tilde{A}_4^0(1, 2, j, k)\, B_{1gH}^0(\overline{1}, \overline{2}, i)\, J_1^{(1)}(\{p\}_1) \\
&+ A_{3,q}^0(1, j, k)\, A_{3,qg \to qq}^0(\overline{1}, 2, (\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}}, \overline{2}, i)\, J_1^{(1)}(\{p\}_1) \\
&- A_4^0(1, i, 2, k)\, B_{1gH}^0(\overline{1}, \overline{2}, j)\, J_1^{(1)}(\{p\}_1) \\
&+ D_{3,qg}^0(1, i, 2)\, A_{3,qg \to qq}^0(\overline{1}, \overline{2}, k)\, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, j)\, J_1^{(1)}(\{p\}_1). \quad (7.5.22)
\end{aligned}
$$

We use $\tilde{A}_4^0(1_q, 2_g, j_g, k_{\bar{q}})$ and $A_4^0(1_q, i_g, 2_g, k_{\bar{q}})$ to remove the $k_q // j_g // 2_g$ and $k_q // 2_g // i_g$ triple collinear idc limits in $D_4^0(k_q, j_g, i_g, 2_g)$. The $d\hat{\sigma}_{NNLO}^{S,b_2}$ terms in Eq.(7.5.22) are designed to remove only the single unresolved idp limits.

The second step is to remove the single unresolved idc limits from $D_4^0(k_q, j_g, i_g, 2_g)$, $\tilde{A}_4^0(1_q, 2_g, j_g, k_{\bar{q}})$ and $A_4^0(1_q, i_g, 2_g, k_{\bar{q}})$. By using the following terms together with the colour permutation $(i \leftrightarrow j)$ in Eq.(7.5.19) we can remove the $k_q // 2_g$ limit,

$$
\begin{aligned}
&+2\, A_{3,qg \to qq}^0(1, 2, k)\, A_{3,q\bar{q}}^0(\overline{1}, i, \overline{2})\, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, j)\, J_1^{(1)}(\{p\}_1) \\
&-2\, A_{3,qg \to qq}^0(1, 2, k)\, d_{3,q}^0(\overline{2}, i, j)\, B_{1gH}^0(\overline{1}, \overline{\overline{2}}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1). \quad (7.5.23)
\end{aligned}
$$

The boosted momenta p_i, p_j after the initial-initial mapping in Eq.(7.5.23) preserve the Lorentz invariant s_{ij}. This means that the $d_{3,q}^0(\overline{2}, i, j)$ antenna function in Eq.(7.5.23) will have an implicit IR divergence in the $i // j$ collinear limit. To remove this implicit IR divergence, we introduce two new terms

$$
\begin{aligned}
&+ f_{3,g}^0(2, i, j)\, A_{3,qg \to qq}^0(1, \overline{2}, k)\, B_{1gH}^0(\overline{1}, \overline{\overline{2}}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1) \\
&+ d_3^0(k, i, j)\, A_{3,qg \to qq}^0(1, 2, (\widetilde{ki}))\, B_{1gH}^0(\overline{\overline{1}}, \overline{2}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1). \quad (7.5.24)
\end{aligned}
$$

Note that the $A_{3,qg \to qq}^0(1, \overline{2}, k)$ function in Eq.(7.5.24) has a $\overline{2} // k$ limit which is designed to remove the corresponding limit in $d\hat{\sigma}_{NNLO}^{S,a}$. However, we have now over subtracted implicit IR divergences in the $2 // i$ and $k // i$ limits present in $f_{3,g}^0(2, i, j)$ and $d_3^0(k, i, j)$ in Eq.(7.5.24).

In order to close the loop of removing over subtracted limits by introducing new over subtracted limits, we note that the combination

$$+2\, d^0_{3,q}(1, i, j)\, A^0_{3,qg \to qq}(\overline{1}, 2, k)\, B^0_{1gH}(\overline{\overline{1}}, \overline{2}, (\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$
$$-2\, A^0_{3,qg \to qq}(1, 2, k)\, d^0_{3,q}(\overline{1}, i, j)\, B^0_{1gH}(\overline{\overline{1}}, \overline{2}, (\widetilde{ij}))\, J^{(1)}_1(\{p\}_1). \tag{7.5.25}$$

only gives a contribution in the $1//i$ limit. In the $i//j$ and $2//k$ limits, the two lines in Eq.(7.5.25) would cancel each other. The only remaining implicit IR divergence is from $1//i$ limit in $d^0_{3,q}(1, i, j)$.

Therefore, we can use the $D^0_3(1_q, i_g, 2_g)$ and $A^0_3(1_q, i_g, k_{\bar{q}})$ antennae to remove all four over subtracted limits,

$$-\, D^0_{3,qg}(1, i, 2)\, A^0_{3,qg \to qq}(\overline{1}, \overline{2}, k)\, B^0_{1gH}(\overline{\overline{1}}, \overline{\overline{2}}, j)\, J^{(1)}_1(\{p\}_1)$$
$$-\, A^0_{3,q}(1, i, k)\, A^0_{3,qg \to qq}(\overline{1}, 2, (\widetilde{ik}))\, B^0_{1gH}(\overline{\overline{1}}, \overline{2}, j)\, J^{(1)}_1(\{p\}_1). \tag{7.5.26}$$

The eight terms in Eqs.(7.5.23), (7.5.24), (7.5.25) and (7.5.26) remove the single unresolved idc limits in Eq.(7.5.22) under colour permutation $i \leftrightarrow j$. In addition to the collinear limits we just analyzed, the single soft limits need to be removed by eight corresponding large angle soft terms.

The usage of $D^0_4(k_q, j_g, 2_g, i_g)$ in Eq.7.5.20 contains double unresolved idc limits ($k//j//2$ and $k//i//2$). Just as in Eq.(7.5.22), we use $A^0_4(1_q, 2_g, i_g, k_g)$ and $A^0_4(1_q, 2_g, j_g, k_g)$ to remove the idc double unresolved limits. There are no single unresolved idc limits in $D^0_4(k_q, j_g, 2_g, i_g)$.

Explicit formulae for the other antenna subtraction terms in table 7.3 can be found in appendix C.1. The double real subtraction terms fit the general structure described in section 4.3.3.

To numerically test that the antenna subtraction terms given in Eq.(7.5.18) remove the implicit IR divergences in Eq.(7.4.14) correctly, we use the same spike plots for double real contribution defined in section 6.6.1 to test that the subtraction terms converge to the matrix elements when approaching various double and single unresolved limits at NNLO.

7.5.2 Real-virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.5.3, one can combine the integrated double real subtraction terms and real-virtual mass factorization term to construct the real-virtual subtraction term, $d\hat{\sigma}_{qg}^T$, which removes both the explicit and implicit IR divergences from Eq.(7.4.15),

$$
\begin{aligned}
d\hat{\sigma}_{qg}^T = {} & \mathcal{N}_{qg}\, \mathcal{N}_{NNLO}^{RV}\, \frac{dx_1}{x_1}\frac{dx_2}{x_2} d\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\Bigg\{ \\
& N\left[B_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) - \frac{1}{N^2}\widetilde{B}_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) + \frac{N_f}{N}\widehat{B}_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, 3, 4_{\bar{q}}) \right] \\
& -\frac{1}{N}\left[\widetilde{B}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) - \frac{1}{N^2}\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) + \frac{N_f}{N}\widehat{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{3}, 4_{\bar{q}}) \right]\Bigg\}.
\end{aligned}
$$

$$(7.5.27)$$

The corresponding relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements (or combinations of subtraction terms) in Eq.(7.4.15) and (7.5.27) are summarised in table 7.4. Note that the $\widehat{B}_{2gH}^{1,XT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$ or $\widehat{B}_{2gH}^{1,YT}(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$ subtraction terms do not remove all the single soft IR limits in $\widehat{B}_{2gH}^{1}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$ or $\widehat{B}_{2gH}^{1}(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$ matrix elements. Only the combination of $\widehat{B}_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$ removes all the IR divergence of the N_F contributions of the one-loop matrix elements with Higgs boson plus two gluons and one quark pair.

The explicit formula for $B_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$ is,

$$
\begin{aligned}
B_{2gH}^{1,FT}&(\hat{1}_q, \hat{2}, i, j_{\bar{q}}) = \\
- &\left[+ J_{2,QG}^{1,II}(s_{12}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,GQ}^{1,FF}(s_{ij}) \right] B_{2gH}^0(1, 2, i, j)\, J_1^{(2)}(\{p\}_2) \\
- &\left[+ J_{2,QG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,QG}^{1,FI}(s_{2j}) \right] B_{2gH}^0(1, i, 2, j)\, J_1^{(2)}(\{p\}_2) \\
+ &\quad d_{3,g}^0(j, i, 2)\left[B_{1gH}^1(1, \bar{2}, (\widetilde{ij}))\, \delta(1-x_1)\, \delta(1-x_2) \right. \\
& \left. + \left(+ J_{2,QG}^{1,II}(s_{1\bar{2}}) + J_{2,QG}^{1,FI}(s_{\bar{2}(\widetilde{ij})}) \right) d_{3,g}^0(j, i, 2)\, B_{1gH}^0(1, \bar{2}, (\widetilde{ij})) \right] J_1^{(1)}(\{p\}_1) \\
+ &\quad\left[D_{3,g}^1(j, i, 2)\, \delta(1-x_1)\, \delta(1-x_2) \right. \\
& \left. + \left(+ J_{2,QG}^{1,FF}(s_{ji}) + J_{2,GG}^{1,FI}(s_{i2}) + J_{2,QG}^{1,FI}(s_{j2}) - 2J_{2,QG}^{1,FI}(s_{\bar{2}(\widetilde{ij})}) \right) D_{3,g}^0(j, i, 2) \right] \\
& \times B_{1gH}^0(1, \bar{2}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)
\end{aligned}
$$

subtraction term	maple	matrix element
$B_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$	`qgB2g1HFT`	$B_{2gH}^1(\hat{1}_q, \hat{2}, i, j_{\bar{q}}) + B_{2gH}^1(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$
$\widehat{B}_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$	`qgBh2g1HFT`	$\widehat{B}_{2gH}^{1,XT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}}) + \widehat{B}_{2gH}^{1,YT}(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$
$\widehat{B}_{2gH}^{1,XT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$	`qgBh2g1HXT`	$\widehat{B}_{2gH}^1(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$
$\widehat{B}_{2gH}^{1,YT}(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$	`qgBh2g1HYT`	$\widehat{B}_{2gH}^1(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$
$\widehat{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$	`qgBtth2g1HXT`	$\widehat{\widetilde{B}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$
$\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$	`qgBttt2g1HXT`	$\widetilde{\widetilde{B}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$
$\widetilde{B}_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$	`qgBt2g1HFT`	$\widetilde{B}_{2gH}^1(\hat{1}_q, \hat{2}, i, j_{\bar{q}}) + \widetilde{B}_{2gH}^1(\hat{1}_q, i, \hat{2}, j_{\bar{q}})$
$\widetilde{B}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$	`qgBtt2g1HXT`	$\widetilde{B}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}}) - \widehat{\widetilde{B}}_{2gH}^1(\hat{1}_q, \widetilde{\hat{2}}, \widetilde{i}, j_{\bar{q}})$

Table 7.4: NNLO antenna subtraction terms for real-virtual contributions in $qg \to H$+jet process and their relation to the matrix elements (or combinations of subtraction terms)

$$
- \frac{1}{2}\Big[+ J_{2,QG}^{1,FI}(s_{j2}) - J_{2,QG}^{1,FI}(s_{(\widetilde{ij})\bar{2}}) - J_{2,QG}^{1,II}(s_{12})
$$

$$
+ J_{2,QG}^{1,II}(s_{1\bar{2}}) - J_{2,QQ}^{1,IF}(s_{1j}) + J_{2,QQ}^{1,IF}(s_{1(\widetilde{ij})})
$$

$$
+ \mathcal{S}^{IF}(s_{\bar{2}(\widetilde{ij})}, s_{1j}, x_{\bar{2}(\widetilde{ij}),1j}) - \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) - \mathcal{S}^{IF}(s_{1\bar{2}}, s_{1j}, x_{1\bar{2},1j})
$$

$$
+ \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) - \mathcal{S}^{IF}(s_{1(\widetilde{ij})}, s_{1j}, x_{1(\widetilde{ij}),1j}) + \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1)\Big]
$$

$$
\times d_{3,g}^0(j, i, 2)\, B_{1gH}^0(1, \bar{2}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)
$$

$$
+ \ D_{3,qg}^0(1, i, 2)\Bigg[B_{1gH}^1(\bar{1}, \bar{2}, j)\, \delta(1 - x_1)\, \delta(1 - x_2)
$$

$$
+ \ \Big(+ J_{2,QG}^{1,II}(s_{1\bar{2}}) + J_{2,QG}^{1,FI}(s_{\bar{2}j}) \Big)\, D_{3,qg}^0(1, i, 2)\, B_{1gH}^0(\bar{1}, \bar{2}, j) \Bigg] J_1^{(1)}(\{p\}_1)
$$

$$
+ \ \Bigg[D_{3,qg}^1(1, i, 2)\, \delta(1 - x_1)\, \delta(1 - x_2)
$$

$$
+ \Big(+ J_{2,QG}^{1,II}(s_{12}) + J_{2,GG}^{1,FI}(s_{2i}) + J_{2,QG}^{1,IF}(s_{1i}) - 2 J_{2,QG}^{1,II}(s_{1\bar{2}}) \Big)\, D_{3,qg}^0(1, i, 2) \Bigg]
$$

$$
\times B_{1gH}^0(\bar{1}, \bar{2}, j)\, J_1^{(1)}(\{p\}_1)
$$

$$
- \frac{1}{2}\Big[+ J_{2,QG}^{1,II}(s_{12}) - J_{2,QG}^{1,II}(s_{1\bar{2}}) - J_{2,QG}^{1,FI}(s_{j2})
$$

$$
+ J_{2,QG}^{1,FI}(s_{j\bar{2}}) - J_{2,QQ}^{1,IF}(s_{1j}) + J_{2,QQ}^{1,IF}(s_{\bar{1}j})
$$

$$+ \tilde{\mathcal{S}}^{IF}(s_{\overline{12}}, s_{\overline{1}j}, x_{\overline{12},\overline{1}j}) - \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) - \tilde{\mathcal{S}}^{IF}(s_{\overline{2}j}, s_{\overline{1}j}, x_{\overline{2}j,\overline{1}j})$$

$$+ \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) - \tilde{\mathcal{S}}^{IF}(s_{\overline{1}j}, s_{\overline{1}j}, 1) + \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) \bigg]$$

$$\times D^0_{3,qg}(1, i, 2) \, B^0_{1gH}(\overline{1}, \overline{2}, j) \, J^{(1)}_1(\{p\}_1)$$

$$- \bigg[\tilde{A}^1_{3,q}(1, i, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J^{1,IF}_{2,QQ}(s_{j1}) - J^{1,IF}_{2,QQ}(s_{(\widetilde{ij})\overline{1}}) \bigg) A^0_{3,q}(1, i, j) \bigg] B^0_{1gH}(\overline{1}, 2, (\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \frac{1}{2} \bigg[+ J^{1,IF}_{2,QQ}(s_{1j}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) - J^{1,FI}_{2,QG}(s_{j2})$$

$$+ J^{1,FI}_{2,QG}(s_{(\widetilde{ij})2}) - J^{1,II}_{2,QG}(s_{12}) + J^{1,II}_{2,QG}(s_{\overline{12}})$$

$$- \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) + \mathcal{S}^{IF}(s_{\overline{1}(\widetilde{ij})}, s_{1j}, x_{\overline{1}(\widetilde{ij}),1j}) + \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j})$$

$$- \mathcal{S}^{IF}(s_{\overline{12}}, s_{1j}, x_{\overline{12},1j}) + \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) - \mathcal{S}^{IF}(s_{2(\widetilde{ij})}, s_{1j}, x_{2(\widetilde{ij}),1j}) \bigg]$$

$$\times A^0_{3,q}(1, i, j) \, B^0_{1gH}(\overline{1}, 2, (\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$- \bigg[\tilde{A}^1_{3,qg \to qq}(1, 2, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J^{1,IF}_{2,QQ}(s_{1j}) - J^{1,II}_{2,QQ}(s_{\overline{12}}) \bigg) A^0_{3,qg \to qq}(1, 2, j) \bigg] B^0_{1gH}(\overline{1}, \overline{2}, i) \, J^{(1)}_1(\{p\}_1)$$

$$- \bigg[A^1_{3,qg \to qq}(1, 2, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J^{1,II}_{2,QG}(s_{12}) + J^{1,FI}_{2,QG}(s_{2j}) - J^{1,II}_{2,QQ}(s_{\overline{12}}) \bigg) A^0_{3,qg \to qq}(1, 2, j) \bigg]$$

$$\times B^0_{1gH}(\overline{1}, \overline{2}, i) \, J^{(1)}_1(\{p\}_1)$$

$$- \bigg[+ J^{1,FI}_{2,GG}(s_{2i}) + J^{1,FF}_{2,QG}(s_{ij}) - J^{1,IF}_{2,QQ}(s_{1j}) - J^{1,II}_{2,QG}(s_{12})$$

$$+ 2 J^{1,IF}_{2,QG}(s_{1i}) - 2 J^{1,IF}_{2,QG}(s_{\overline{1}i}) + 2 J^{1,II}_{2,QQ}(s_{\overline{12}}) - 2 J^{1,FI}_{2,GQ}(s_{\overline{2}i})$$

$$- \mathcal{S}^{IF}(s_{2i}, s_{1i}, x_{2i,1i}) - \mathcal{S}^{IF}(s_{ji}, s_{1i}, x_{ji,1i}) + 2 \tilde{\mathcal{S}}^{IF}(s_{\overline{2}i}, s_{\overline{1}i}, x_{\overline{2}i,\overline{1}i})$$

$$+ \mathcal{S}^{IF}(s_{1j}, s_{1i}, x_{1j,1i}) + \mathcal{S}^{IF}(s_{12}, s_{1i}, x_{12,1i}) - 2 \tilde{\mathcal{S}}^{IF}(s_{\overline{12}}, s_{\overline{1}i}, x_{\overline{12},\overline{1}i})$$

$$- 2 \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) + 2 \tilde{\mathcal{S}}^{IF}(s_{\overline{1}i}, s_{\overline{1}i}, 1)$$

$$\bigg] \times A^0_{3,qg \to qq}(1, 2, j) \, B^0_{1gH}(\overline{1}, \overline{2}, i) \, J^{(1)}_1(\{p\}_1)$$

$$+ \bigg[- 2 J^{1,FI}_{2,GQ}(s_{\overline{2}(\widetilde{ij})}) + 2 J^{1,FI}_{2,QG}(s_{\overline{2}(\widetilde{ij})}) \bigg] d^0_{3,g \to q}(j, 2, i) \, B^0_{1gH}(1, \overline{2}, (\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[+ J^{1,II}_{2,QQ,qg\to qq}(s_{12}) - J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2}) \right] A^0_{3,q\bar q}(1,i,2)\, B^0_{1gH}(\overline{1},\overline{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[- J^{1,II}_{2,QQ,qg\to qq}(s_{12}) + J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2}) \right] d^0_{3,q}(1,i,j)\, B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[- J^{1,II}_{2,QQ,qg\to qq}(s_{12}) + J^{1,II}_{2,QQ,qg\to qq}(s_{1\overline{2}}) \right] d^0_{3,q}(2,i,j)\, B^0_{1gH}(1,\overline{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[+ J^{1,II}_{2,QQ,qg\to qq}(s_{12}) - J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2}) \right] A^0_{3,q\bar q}(1,j,2)\, B^0_{1gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[- J^{1,II}_{2,QQ,qg\to qq}(s_{12}) + J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2}) \right] d^0_{3,q}(1,j,i)\, B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[- J^{1,II}_{2,QQ,qg\to qq}(s_{12}) + J^{1,II}_{2,QQ,qg\to qq}(s_{1\overline{2}}) \right] d^0_{3,q}(2,j,i)\, B^0_{1gH}(1,\overline{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, B^0_{2gH}(1,i,j,2)\, J^{(2)}_1(\{p\}_2)$$

$$+ \quad \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2})\, d^0_{3,q}(1,i,j)\, B^0_{1gH}(\overline{1},(\widetilde{ij}),2)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{1\overline{2}})\, d^0_{3,q}(2,j,i)\, B^0_{1gH}(1,(\widetilde{ji}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[+ \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12}) - \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2}) \right] A^0_{3,q\bar q}(1,i,2)\, B^0_{1gH}(\overline{1},j,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, B^0_{2gH}(1,j,i,2)\, J^{(2)}_1(\{p\}_2)$$

$$+ \quad \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2})\, d^0_{3,q}(1,j,i)\, B^0_{1gH}(\overline{1},(\widetilde{ij}),2)\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{1\overline{2}})\, d^0_{3,q}(2,i,j)\, B^0_{1gH}(1,(\widetilde{ji}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[+ \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12}) - \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{1}2}) \right] A^0_{3,q\bar q}(1,j,2)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- \quad A^0_{3,qg\to qq}(1,2,j)\left[B^1_{1gH}(\overline{1},i,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,IF}_{2,QG}(s_{\overline{1}i}) + J^{1,FI}_{2,GQ}(s_{i\overline{2}}) \right) A^0_{3,qg\to qq}(1,2,j)\, B^0_{1gH}(\overline{1},i,\overline{2}) \right] J^{(1)}_1(\{p\}_1)$$

$$- \quad \left[A^1_{3,qg\to qq}(1,2,j)\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,II}_{2,QG}(s_{12}) + J^{1,FI}_{2,QG}(s_{2j}) - J^{1,II}_{2,QQ}(s_{\overline{1}\overline{2}}) \right) A^0_{3,qg\to qq}(1,2,j) \right]$$

$$\times B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \quad \left[+ J^{1,II}_{2,QG}(s_{12}) - J^{1,II}_{2,QQ}(s_{\overline{1}\overline{2}}) - J^{1,IF}_{2,QG}(s_{1i}) + J^{1,IF}_{2,QG}(s_{\overline{1}i}) - J^{1,FI}_{2,GG}(s_{2i}) + J^{1,FI}_{2,GQ}(s_{i\overline{2}}) \right.$$

$$+ \left(- \mathcal{S}^{IF}(s_{12},s_{1i},x_{12,1i}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}\overline{2}},s_{\overline{1}i},x_{\overline{1}\overline{2},\overline{1}i}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}i},s_{\overline{1}i},1) \right.$$

$$\left.\left. - \mathcal{S}^{IF}(s_{1i},s_{1i},1) + \tilde{\mathcal{S}}^{IF}(s_{\overline{2}i},s_{\overline{1}i},x_{\overline{2}i,\overline{1}i}) - \mathcal{S}^{IF}(s_{2i},s_{1i},x_{2i,1i}) \right) \right]$$

$$\times A^0_{3,qg\to qq}(1,2,j)\,B^0_{1gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\ 2\,G^0_{3,gq\to gg}(2,1,j)\left[A^1_{3gH}(\overline{1},\overline{2},i)\,\delta(1-x_1)\,\delta(1-x_2)+\left(J^{1,II}_{2,GG}(s_{\overline{12}})\right.\right.$$

$$\left.\left.+\ J^{1,FI}_{2,GG}(s_{\overline{2}i})+J^{1,IF}_{2,GG}(s_{\overline{1}i})\right)G^0_{3,gq\to gg}(2,1,j)\,A^0_{3gH}(\overline{1},\overline{2},i)\right]J^{(1)}_1(\{p\}_1)$$

$$-\ 2\left[G^1_{3,gq\to gg}(2,1,j)\,\delta(1-x_1)\,\delta(1-x_2)+\left(J^{1,II}_{2,QG}(s_{12})\right.\right.$$

$$\left.\left.+\ J^{1,FI}_{2,QG}(s_{j2})-2J^{1,II}_{2,GG}(s_{\overline{12}})\right)G^0_{3,gq\to gg}(2,1,j)\right]A^0_{3gH}(\overline{1},\overline{2},i)\,J^{(1)}_1(\{p\}_1)$$

$$+\ \left[+\,J^{1,II}_{2,QG}(s_{12})-2J^{1,II}_{2,GG}(s_{\overline{12}})-2J^{1,FI}_{2,GG}(s_{2i})+2J^{1,FI}_{2,GG}(s_{\overline{2}i})\right.$$

$$-\ J^{1,IF}_{2,QG}(s_{1i})+2J^{1,IF}_{2,GG}(s_{\overline{1}i})+J^{1,FI}_{2,QG}(s_{2j})-J^{1,FF}_{2,QG}(s_{ij})$$

$$-\ \mathcal{S}^{IF}(s_{12},s_{1i},x_{12,1i})+2\tilde{\mathcal{S}}^{IF}(s_{\overline{12}},s_{\overline{1}i},x_{\overline{12},\overline{1}i})+2\mathcal{S}^{IF}(s_{2i},s_{1i},x_{2i,1i})$$

$$-\ 2\tilde{\mathcal{S}}^{IF}(s_{\overline{2}i},s_{\overline{1}i},x_{\overline{2}i,\overline{1}i})+\mathcal{S}^{IF}(s_{1i},s_{1i},1)-2\tilde{\mathcal{S}}^{IF}(s_{\overline{1}i},s_{\overline{1}i},1)$$

$$\left.-\ \mathcal{S}^{IF}(s_{2j},s_{1i},x_{2j,1i})+\mathcal{S}^{IF}(s_{ji},s_{1i},x_{ji,1i})\right.$$

$$\left.\vphantom{\Big]}\right]\times G^0_{3,gq\to gg}(2,1,j)\,A^0_{3gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\ J^{1,II}_{2,GG,q'g\to gg}(s_{12})\,A^{H,0}_4(1,2,i,j)\,J^{(2)}_1(\{p\}_2)$$

$$+\ J^{1,II}_{2,GG,q'g\to gg}(s_{1\overline{2}})\,f^0_{3,g}(2,i,j)\,A^0_{3gH}(1,\overline{2},(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+\ J^{1,II}_{2,GG,q'g\to gg}(s_{\overline{1}2})\,f^0_{3,g}(1,j,i)\,A^0_{3gH}(\overline{1},2,(\widetilde{ji}))\,J^{(1)}_1(\{p\}_1)$$

$$+\ \left[-\,J^{1,II}_{2,GG,q'g\to gg}(s_{\overline{12}})+J^{1,II}_{2,GG,q'g\to gg}(s_{12})\right]F^0_{3,gg}(1,i,2)\,A^0_{3gH}(\overline{1},\overline{2},j)\,J^{(1)}_1(\{p\}_1)$$

$$-\ J^{1,II}_{2,GG,q'g\to gg}(s_{12})\,A^{H,0}_4(1,2,j,i)\,J^{(2)}_1(\{p\}_2)$$

$$+\ J^{1,II}_{2,GG,q'g\to gg}(s_{1\overline{2}})\,f^0_{3,g}(2,j,i)\,A^0_{3gH}(1,\overline{2},(\widetilde{ji}))\,J^{(1)}_1(\{p\}_1)$$

$$+\ J^{1,II}_{2,GG,q'g\to gg}(s_{\overline{1}2})\,f^0_{3,g}(1,i,j)\,A^0_{3gH}(\overline{1},2,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+\ \left[-\,J^{1,II}_{2,GG,q'g\to gg}(s_{\overline{12}})+J^{1,II}_{2,GG,q'g\to gg}(s_{12})\right]F^0_{3,gg}(1,j,2)\,A^0_{3gH}(\overline{1},\overline{2},i)\,J^{(1)}_1(\{p\}_1)$$

$$-\ J^{1,II}_{2,GG,q'g\to gg}(s_{12})\,A^{H,0}_4(1,i,2,j)\,J^{(2)}_1(\{p\}_2)$$

$$+\ J^{1,II}_{2,GG,q'g\to gg}(s_{12})\,F^0_{3,gg}(1,i,2)\,A^0_{3gH}(\overline{1},\overline{2},j)\,J^{(1)}_1(\{p\}_1)$$

$$+\ J^{1,II}_{2,GG,q'g\to gg}(s_{12})\,F^0_{3,gg}(1,j,2)\,A^0_{3gH}(\overline{1},\overline{2},i)\,J^{(1)}_1(\{p\}_1)$$

$$+\ \left[+\,J^{1,II}_{2,QQ,qg\to qq}(s_{12})-J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{12}})\right]G^0_{3,q\bar{q}}(i,2,1)\,A^0_{3gH}(\overline{1},j,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+\ \left[+\,J^{1,II}_{2,QQ,qg\to qq}(s_{12})-J^{1,II}_{2,QQ,qg\to qq}(s_{\overline{12}})\right]G^0_{3,q\bar{q}}(j,2,1)\,A^0_{3gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$(7.5.28)$$

Eq.(7.5.28) shares some unconventional structures designed to remove initial state identity changing IR divergences. Structures like

$$\left[-2J_{2,GQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) + 2J_{2,QG}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) \right] d_{3,g\to q}^0(j,2,i) \, B_{1gH}^0(1,\overline{2},(\widetilde{ij})) \, J_1^{(1)}(\{p\}_1) \quad (7.5.29)$$

are designed to remove the leftover contributions in the $j//2$ collinear limit. The terms in the square bracket are free from explicit IR divergences and the implicit IR divergences appear only when the $j//2$ limit is approached. The terms in Eq.(7.5.29) are newly introduced in $d\hat{\sigma}_{NNLO}^T$ and need to be integrated and compensated at the double virtual level.

The second unconventional structure is

$$+ \left[+J_{2,QQ,qg\to qq}^{1,II}(s_{12}) - J_{2,QQ,qg\to qq}^{1,II}(s_{\overline{12}}) \right] A_{3,q\bar{q}}^0(1,i,2) \, B_{1gH}^0(\overline{1},\overline{2},j) \, J_1^{(1)}(\{p\}_1)$$

$$+ \left[-J_{2,QQ,qg\to qq}^{1,II}(s_{12}) + J_{2,QQ,qg\to qq}^{1,II}(s_{\overline{12}}) \right] d_{3,q}^0(1,i,j) \, B_{1gH}^0(\overline{1},2,(\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$+ \left[-J_{2,QQ,qg\to qq}^{1,II}(s_{12}) + J_{2,QQ,qg\to qq}^{1,II}(s_{1\overline{2}}) \right] d_{3,q}^0(2,i,j) \, B_{1gH}^0(1,\overline{2},(\widetilde{ij})) \, J_1^{(1)}(\{p\}_1).$$

$$(7.5.30)$$

Eq.(7.5.30) contains integrated functions involving idc limits, however the full structure only give finite contributions in various single unresolved limits and the idc divergences are well regulated. The terms in the square brackets are free from explicit IR divergences. As we approach the $p_i \to 0$ single soft limit, the finite terms in the square bracket would cancel each other as discussed in the special case in Eq.(4.5.120). The implicit divergence from the single soft behaviour is then regulated by the term in each square bracket. Similarly, the terms in the square brackets in the last two lines of Eq.(7.5.30) regulate the $i//j$ collinear limit. In the $1//i$ or $2//i$ collinear limits, the $J_{2,QQ,qg\to qq}^{1,II}(s_{\overline{12}})$ term tends to $J_{2,QQ,qg\to qq}^{1,II}(s_{\overline{1}2})$ or $J_{2,QQ,qg\to qq}^{1,II}(s_{1\overline{2}})$ and cancels against the second or third line in Eq.(7.5.30). The implicit divergences from $1//i$ or $2//i$ collinear limits are then regulated by the net effects of the cancellation.

The third unconventional structure is

$$-\frac{1}{2} J_{2,QQ,qg\to qq}^{1,II}(s_{12}) \, B_{2gH}^0(1,i,j,2) \, J_1^{(2)}(\{p\}_2)$$

$$+\frac{1}{2} J_{2,QQ,qg\to qq}^{1,II}(s_{\overline{12}}) \, d_{3,q}^0(1,i,j) \, B_{1gH}^0(\overline{1},(\widetilde{ij}),2) \, J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2}\,J_{2,QQ,qg\rightarrow qq}^{1,II}(s_{1\overline{2}})\,d_{3,q}^{0}(2,j,i)\,B_{1gH}^{0}(1,(\widetilde{ji}),\overline{2})\,J_{1}^{(1)}(\{p\}_{1})$$

$$+\left[+\frac{1}{2}\,J_{2,QQ,qg\rightarrow qq}^{1,II}(s_{12})-\frac{1}{2}\,J_{2,QQ,qg\rightarrow qq}^{1,II}(s_{\overline{12}})\right]A_{3,q\bar{q}}^{0}(1,i,2)\,B_{1gH}^{0}(\overline{1},j,\overline{2})\,J_{1}^{(1)}(\{p\}_{1}).$$

$$(7.5.31)$$

As introduced in Eq.(4.5.117), the NLO-like combination of terms ensures that Eq.(7.5.31) is free from implicit divergences in the p_i soft and $i//j$ collinear limits. In the $1//i$ collinear limit, the last line in Eq.(7.5.31) cancels against the first two lines. In the $2//j$ collinear limit, a similar NLO-like combination of terms with $i \leftrightarrow j$ exchange would regulate the implicit IR divergences in the same fashion. Eq.(7.5.31) together with the $i \leftrightarrow j$ exchange does not introduce explicit or implicit IR divergences to the real-virtual subtraction term $B_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}})$.

Explicit formulas for the other antenna subtraction terms in table 7.4 can be found in appendix C.2. The real-virtual subtraction terms fit the general structure described in section 4.5.3.

To numerically test that the antenna subtraction terms given in Eq.(7.5.27) remove the implicit IR divergences in Eq.(7.4.15) correctly, we use the same spike plots for defined in section 6.6.2 to test that the subtraction terms remove the explicit IR divergence of the matrix elements in any phase space point and converge to the matrix elements when approaching various single unresolved limits at the real-virtual level.

7.5.3 Double virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.7.2, one can remove the explicit IR divergences in the double virtual matrix elements in Eq.(7.4.16) with the following double virtual subtraction term,

$$d\hat{\sigma}_{qg}^{U} = \mathcal{N}_{qg}\mathcal{N}_{NNLO}^{VV}\frac{dx_1}{x_1}\frac{dx_2}{x_2}d\Phi_{H+1}(p_3,p_H;p_1,p_2)\Bigg\{$$

$$-NB_{1gH}^{2,XU}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})-\frac{1}{N}\widetilde{B}_{1gH}^{2,XU}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})-\frac{1}{N^3}\widetilde{\widetilde{B}}_{1gH}^{2,XU}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})$$

$$-\frac{N_f}{N^2}\widehat{\widetilde{B}}_{1gH}^{2,XU}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})-N_f\widehat{B}_{1gH}^{2,XU}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})-\frac{N_f^2}{N}\widehat{B}_{1gH}^{2,XU}(\hat{1}_q,\hat{2}_g,3_{\bar{q}})\Bigg\}J_1^{(1)}(p_3).$$

$$(7.5.32)$$

subtraction term	maple	matrix element
$B_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$	qgB1g2HXU	$B_{1gH}^{2}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$
$\widetilde{B}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$	qgBt1g2HXU	$\widetilde{B}_{1gH}^{2}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$
$\widetilde{\widetilde{B}}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$	qgBtt1g2HXU	$\widetilde{\widetilde{B}}_{1gH}^{2}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$
$\widehat{\widetilde{B}}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$	qgBth1g2HXU	$\widehat{\widetilde{B}}_{1gH}^{2}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$
$\widehat{B}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$	qgBh1g2HXU	$\widehat{B}_{1gH}^{2}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$
$\widehat{\widehat{B}}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$	qgBhh1g2HXU	$\widehat{\widehat{B}}_{1gH}^{2}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$

Table 7.5: NNLO antenna subtraction terms for double virtual contributions in $qg \rightarrow H+$jet process and their relation to the matrix elements

The relationships between the subtraction terms, file name in the NNLOJET maple script and matrix elements in Eqs.(7.4.16) and (7.5.32) are summarised in table 7.5.

The explicit formula for $B_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}})$ in terms of integrated antennae is,

$$
\begin{aligned}
B_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}}) = \\
&-\Bigg[\mathcal{D}_{3,qg}^0(s_{12}) - \Gamma_{qq}^{(1)}(x_1) \\
&\quad - \Gamma_{gg}^{(1)}(x_2) + \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \Bigg] \left(B_{1gH}^1(1,2,i) - \frac{b_0}{\epsilon} B_{1gH}^0(1,2,i) \right) \\
&-\Bigg[+ \frac{1}{2} \mathcal{D}_{3,qg}^0(s_{12}) \otimes \mathcal{D}_{3,qg}^0(s_{12}) - \Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{D}_{3,qg}^0(s_{12}) \\
&\quad - \frac{1}{2} \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{D}_{3,qg}^0(s_{12}) + \frac{1}{2} \Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{qq}^{(1)}(x_1) \\
&\quad + \frac{1}{2} \Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_2) + \frac{1}{8} \Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_2) \Bigg] B_{1gH}^0(1,2,i) \\
&-\Bigg[- \frac{1}{2} \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{D}_{3,qg}^0(s_{12}) + \mathcal{D}_{3,qg}^0(s_{12}) \otimes \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \\
&\quad + \frac{1}{2} \Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_2) - \Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \\
&\quad + \frac{1}{4} \Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_2) - \frac{1}{2} \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \Bigg] B_{1gH}^0(1,2,i) \\
&-\Bigg[+ \frac{1}{8} \Gamma_{gg}^{(1)}(x_2) \otimes \Gamma_{gg}^{(1)}(x_2) - \frac{1}{2} \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \\
&\quad + \frac{1}{2} \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \otimes \mathcal{D}_{3,g \rightarrow g}^0(s_{2i}) \Bigg] B_{1gH}^0(1,2,i)
\end{aligned}
$$

$$
-\bigg[+ \mathcal{D}^0_{4,g}(s_{2i}) + \frac{1}{2}\,\mathcal{D}^0_{4,g'}(s_{2i}) + \mathcal{D}^1_{3,g}(s_{2i}) - \mathcal{D}^0_{3,g\to g}(s_{2i}) \otimes \mathcal{D}^0_{3,g\to g}(s_{2i})
$$

$$
- \mathcal{D}^0_{3,g\to q}(s_{2i}) \otimes \mathcal{D}^0_{3,g\to q}(s_{2i}) + 2\,\Gamma^{(1)}_{qq}(x_2) \otimes \mathcal{D}^0_{3,g\to q}(s_{2i}) - \Gamma^{(1)}_{gg}(x_2) \otimes \mathcal{D}^0_{3,g\to q}(s_{2i})
$$

$$
+ \frac{b_0}{\epsilon}\,\mathcal{D}^0_{3,g\to g}(s_{2i}) \left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} + \frac{b_0}{\epsilon}\,\mathcal{D}^0_{3,g\to q}(s_{2i}) \left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon}
$$

$$
- \frac{b_0}{\epsilon}\,\mathcal{D}^0_{3,g\to q}(s_{2i}) - \frac{1}{2}\,\bar{\Gamma}^{(2)}_{gg}(x_2) \bigg] B^0_{1gH}(1,2,i)
$$

$$
-\bigg[+ \mathcal{D}^0_{4,qg}(s_{12}) + \frac{1}{2}\,\mathcal{D}^{0,nadj}_{4,qg}(s_{12}) + \mathcal{D}^1_{3,qg}(s_{12}) - \mathcal{D}^0_{3,qg}(s_{12}) \otimes \mathcal{D}^0_{3,qg}(s_{12})
$$

$$
+ \frac{b_0}{\epsilon}\,\mathcal{D}^0_{3,qg}(s_{12}) \left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} - \bar{\Gamma}^{(2)}_{qq}(x_1) - \frac{1}{2}\,\bar{\Gamma}^{(2)}_{gg}(x_2) \bigg] B^0_{1gH}(1,2,i)
$$

$$
-\bigg[- \tilde{\mathcal{A}}^0_{4,qg}(s_{12}) - \mathcal{A}^{0,nadj}_{4,qg}(s_{12}) - \mathcal{A}^0_{4,qg}(s_{12}) - \mathcal{A}^1_{3,qg}(s_{12}) - \tilde{\mathcal{A}}^1_{3,qg}(s_{12})
$$

$$
+ \Gamma^{(1)}_{gg}(x_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) - 2\,\Gamma^{(1)}_{qq}(x_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12})
$$

$$
+ 2\,\mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{A}^0_{3,qq}(s_{12}) - \frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} \mathcal{A}^0_{3,qg\to qq}(s_{12})
$$

$$
+ \frac{b_0}{\epsilon}\,\mathcal{A}^0_{3,qg\to qq}(s_{12}) \bigg] B^0_{1gH}(1,2,i)
$$

$$
-\bigg[- \frac{1}{2}\,\tilde{\mathcal{A}}^0_{4,q}(s_{1i}) - \tilde{\mathcal{A}}^1_{3,q}(s_{1i}) + \frac{1}{2}\,\mathcal{A}^0_{3,q}(s_{1i}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) \bigg] B^0_{1gH}(1,2,i)
$$

$$
-\bigg[- 2\,\mathcal{G}^0_{3,qg\to gg}(s_{12}) - 2\,S_{q\to g}\Gamma^{(1)}_{gq}(x_1) \bigg] A^1_{3gH}(1,2,i)
$$

$$
-\bigg[- \mathcal{G}^0_{3,qg\to gg}(s_{12}) \otimes \mathcal{F}^0_{3,g}(s_{2i}) - S_{q\to g}\Gamma^{(1)}_{gq}(x_1) \otimes \mathcal{F}^0_{3,g}(s_{2i})
$$

$$
+ \Gamma^{(1)}_{gg}(x_2) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12}) + S_{q\to g}\Gamma^{(1)}_{gg}(x_2) \otimes \Gamma^{(1)}_{gq}(x_1) \bigg] A^0_{3gH}(1,2,i)
$$

$$
-\bigg[- \mathcal{G}^0_{3,qg\to gg}(s_{12}) \otimes \mathcal{F}^0_{3,g}(s_{1i}) - S_{q\to g}\Gamma^{(1)}_{gq}(x_1) \otimes \mathcal{F}^0_{3,g}(s_{1i})
$$

$$
+ \Gamma^{(1)}_{gg}(x_1) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12}) + S_{q\to g}\Gamma^{(1)}_{gg}(x_1) \otimes \Gamma^{(1)}_{gq}(x_1) \bigg] A^0_{3gH}(1,2,i)
$$

$$
-\bigg[- 2\,\mathcal{G}^0_{3,qg\to gg}(s_{12}) \otimes \mathcal{F}^0_{3,gg}(s_{12}) - 2\,S_{q\to g}\Gamma^{(1)}_{gq}(x_1) \otimes \mathcal{F}^0_{3,gg}(s_{12})
$$

$$
+ \Gamma^{(1)}_{gg}(x_1) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12}) + S_{q\to g}\Gamma^{(1)}_{gg}(x_1) \otimes \Gamma^{(1)}_{gq}(x_1)
$$

$$
+ \Gamma^{(1)}_{gg}(x_2) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12}) + S_{q\to g}\Gamma^{(1)}_{gg}(x_2) \otimes \Gamma^{(1)}_{gq}(x_1) \bigg] A^0_{3gH}(1,2,i)
$$

$$
-\bigg[- 2\,\mathcal{G}^0_{4,qg}(s_{12}) - 2\,\mathcal{G}^{0,nadj}_{4,qg}(s_{12}) - 2\,\mathcal{G}^1_{3,qg}(s_{12}) - 2\frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} \mathcal{G}^0_{3,qg\to gg}(s_{12})
$$

$$+ 2\frac{b_0}{\epsilon}\, \mathcal{G}^0_{3,qg\to gg}(s_{12}) + 4\, \mathcal{G}^0_{3,qg\to gg}(s_{12}) \otimes \mathcal{F}^0_{3,gg}(s_{12})$$

$$+ 2\, \mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{G}^0_{3,qq}(s_{12}) - 2\, \Gamma^{(1)}_{gg}(x_1) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12})$$

$$+ 2\, \Gamma^{(1)}_{qq}(x_1) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12}) + 2\frac{b_0}{\epsilon}\, S_{q\to g}\Gamma^{(1)}_{gq}(x_1)$$

$$- S_{q\to g}\Gamma^{(1)}_{gg}(x_1) \otimes \Gamma^{(1)}_{gq}(x_1) + S_{q\to g}\Gamma^{(1)}_{qq}(x_1) \otimes \Gamma^{(1)}_{gq}(x_1)$$

$$- 2\, S_{q\to g}\bar{\Gamma}^{(2)}_{gq}(x_1)\Bigg] A^0_{3gH}(1,2,i)$$

$$-\Bigg[- \mathcal{A}^0_{3,qg\to qq}(s_{12}) - S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \Bigg]\left(+B^1_{1gH}(1,i,2) - \frac{b_0}{\epsilon}B^0_{1gH}(1,i,2) \right)$$

$$-\Bigg[+ \Gamma^{(1)}_{qq}(x_1) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) + S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \otimes \Gamma^{(1)}_{qq}(x_1)$$

$$- \frac{1}{2}\, \mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{D}^0_{3,q}(s_{1i}) - \frac{1}{2}\, S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \otimes \mathcal{D}^0_{3,q}(s_{1i}) \Bigg] B^0_{1gH}(1,i,2)$$

$$-\Bigg[+ \Gamma^{(1)}_{qq}(x_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) + S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \otimes \Gamma^{(1)}_{qq}(x_2)$$

$$- \frac{1}{2}\, \mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{D}^0_{3,q}(s_{2i}) - \frac{1}{2}\, S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \otimes \mathcal{D}^0_{3,q}(s_{2i}) \Bigg] B^0_{1gH}(1,i,2)$$

$$-\Bigg[- \mathcal{A}^{0,nadj}_{4,qg}(s_{12}) - \mathcal{A}^0_{4,qg}(s_{12}) - \mathcal{A}^1_{3,qg}(s_{12}) - \frac{b_0}{\epsilon}\left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}^0_{3,qg\to qq}(s_{12})$$

$$+ \mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{A}^0_{3,qq}(s_{12}) - \Gamma^{(1)}_{qq}(x_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12})$$

$$+ \Gamma^{(1)}_{gg}(x_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) + \frac{1}{2}\, S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \otimes \Gamma^{(1)}_{gg}(x_2)$$

$$- \frac{1}{2}\, S_{g\to q}\Gamma^{(1)}_{qg}(x_2) \otimes \Gamma^{(1)}_{qq}(x_2) - S_{g\to q}\bar{\Gamma}^{(2)}_{qg}(x_2) \Bigg] B^0_{1gH}(1,i,2) \tag{7.5.33}$$

The integrated antenna subtraction terms in Eq. (7.5.33) compensate the double real and real-virtual sutraction terms we introduced in Eq.(7.5.19), (7.5.20) and (7.5.28). These terms are arranged by the Lorentz invariant products s_{ij} to further illustrate the dipole structure of the antenna subtraction method.

Using the dipole functions introduced in section 3.5.2 and 4.7.2, Eq. (7.5.33) can be re-expressed in a similar structure as the two-loop Catani pole structure of Eq. (4.7.156) as,

$$B^{2,XU}_{1gH}(\hat{1}_q, \hat{2}_g, i_{\bar{q}}) =$$
$$-\Bigg[- J^{1,II}_{2,QG}(s_{12}) - J^{1,FI}_{2,QG}(s_{2i}) \Bigg]\left(-B^1_{1gH}(1,2,i) + \frac{b_0}{\epsilon}B^0_{1gH}(1,2,i) \right)$$

$$-\frac{1}{2}\left[J_{2,QG}^{1,II}(s_{12}) + J_{2,QG}^{1,FI}(s_{2i}) \right] \otimes \left[J_{2,QG}^{1,II}(s_{12}) + J_{2,QG}^{1,FI}(s_{2i}) \right] B_{1gH}^0(1,2,i)$$

$$-\left[J_{2,QG}^{2,FI}(s_{2i}) + J_{2,QG}^{2,II}(s_{12}) + \bar{J}_{2,QQ,qg\rightarrow qq}^{2,II}(s_{12}) + \bar{J}_{2,QQ}^{2,IF}(s_{1i}) \right] B_{1gH}^0(1,2,i)$$

$$-2\, J_{2,GG,q'g\rightarrow gg}^{1,II}(s_{12}) \times A_{3gH}^1(1,2,i)$$

$$-2\, J_{2,GG,q'g\rightarrow gg}^{1,II}(s_{12}) \otimes \left[J_{2,GG}^{1,FI}(s_{2i}) + J_{2,GG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,II}(s_{12}) \right] A_{3gH}^0(1,2,i)$$

$$-2 J_{2,GG,q'g\rightarrow gg}^{2,II}(s_{12}) \times A_{3gH}^0(1,2,i)$$

$$-J_{2,QQ,qg\rightarrow qq}^{1,II}(s_{12}) \times \left(+B_{1gH}^1(1,i,2) - \frac{b_0}{\epsilon} B_{1gH}^0(1,i,2) \right)$$

$$-J_{2,QQ,qg\rightarrow qq}^{1,II}(s_{12}) \otimes \left[J_{2,GQ}^{1,IF}(s_{1i}) + J_{2,QG}^{1,FI}(s_{2i}) \right] B_{1gH}^0(1,i,2)$$

$$-J_{2,QQ,qg\rightarrow qq}^{2,II}(s_{12}) \times B_{1gH}^0(1,i,2) \tag{7.5.34}$$

where the NNLO integrated antennae are,

$$J_{2,QG}^{2,FI}(s_{2i}) = + \mathcal{D}_{4,g}^0(s_{2i}) + \frac{1}{2}\,\mathcal{D}_{4,g'}^0(s_{2i}) + \mathcal{D}_{3,g}^1(s_{2i}) - \mathcal{D}_{3,g\rightarrow g}^0(s_{2i}) \otimes \mathcal{D}_{3,g\rightarrow g}^0(s_{2i})$$

$$- \mathcal{D}_{3,g\rightarrow q}^0(s_{2i}) \otimes \mathcal{D}_{3,q}^0(s_{2i}) + 2\,\Gamma_{qq}^{(1)}(x_2) \otimes \mathcal{D}_{3,g\rightarrow q}^0(s_{2i})$$

$$- \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{D}_{3,g\rightarrow q}^0(s_{2i}) + \frac{b_0}{\epsilon}\, \mathcal{D}_{3,g\rightarrow g}^0(s_{2i}) \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon}$$

$$+ \frac{b_0}{\epsilon}\, \mathcal{D}_{3,g\rightarrow q}^0(s_{2i}) \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} - \frac{b_0}{\epsilon}\, \mathcal{D}_{3,g\rightarrow q}^0(s_{2i}) - \frac{1}{2}\, \bar{\Gamma}_{gg}^{(2)}(x_2), \tag{7.5.35}$$

$$J_{2,QG}^{2,II}(s_{12}) = + \mathcal{D}_{4,qg}^0(s_{12}) + \frac{1}{2}\,\mathcal{D}_{4,qg}^{0,nadj}(s_{12}) + \mathcal{D}_{3,qg}^1(s_{12}) - \mathcal{D}_{3,qg}^0(s_{12}) \otimes \mathcal{D}_{3,qg}^0(s_{12})$$

$$+ \frac{b_0}{\epsilon}\, \mathcal{D}_{3,qg}^0(s_{12}) \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} - \bar{\Gamma}_{qq}^{(2)}(x_1) - \frac{1}{2}\, \bar{\Gamma}_{gg}^{(2)}(x_2), \tag{7.5.36}$$

$$\bar{J}_{2,QQ,qg\rightarrow qq}^{2,II}(s_{12}) = - \tilde{\mathcal{A}}_{4,qg}^0(s_{12}) - \mathcal{A}_{4,qg}^{0,nadj}(s_{12}) - \mathcal{A}_{4,qg}^0(s_{12}) - \mathcal{A}_{3,qg}^1(s_{12}) - \tilde{\mathcal{A}}_{3,qg}^1(s_{12})$$

$$+ \Gamma_{gg}^{(1)}(x_2) \otimes \mathcal{A}_{3,qg\rightarrow qq}^0(s_{12}) - 2\,\Gamma_{qq}^{(1)}(x_2) \otimes \mathcal{A}_{3,qg\rightarrow qq}^0(s_{12})$$

$$+ 2\,\mathcal{A}_{3,qg\rightarrow qq}^0(s_{12}) \otimes \mathcal{A}_{3,qq}^0(s_{12}) - \frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}_{3,qg\rightarrow qq}^0(s_{12})$$

$$+ \frac{b_0}{\epsilon}\, \mathcal{A}_{3,qg\rightarrow qq}^0(s_{12}), \tag{7.5.37}$$

$$\bar{J}_{2,QQ}^{2,IF}(s_{1i}) = - \frac{1}{2}\, \tilde{\mathcal{A}}_{4,q}^0(s_{1i}) - \tilde{\mathcal{A}}_{3,q}^1(s_{1i}) + \frac{1}{2}\, \mathcal{A}_{3,q}^0(s_{1i}) \otimes \mathcal{A}_{3,q}^0(s_{1i}), \tag{7.5.38}$$

$$J_{2,GG,q'g\rightarrow gg}^{2,II}(s_{12}) = - \mathcal{G}_{4,qg}^0(s_{12}) - \mathcal{G}_{4,qg}^{0,nadj}(s_{12}) - \mathcal{G}_{3,qg}^1(s_{12}) - \frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{G}_{3,qg\rightarrow gg}^0(s_{12})$$

$$+ \frac{b_0}{\epsilon}\, \mathcal{G}_{3,qg\rightarrow gg}^0(s_{12}) + 2\,\mathcal{G}_{3,qg\rightarrow gg}^0(s_{12}) \otimes \mathcal{F}_{3,gg}^0(s_{12})$$

$$+ \ \mathcal{A}^0_{3,qg \to qq}(s_{12}) \otimes \mathcal{G}^0_{3,qq}(s_{12}) - \Gamma^{(1)}_{gg}(x_1) \otimes \mathcal{G}^0_{3,qg \to gg}(s_{12})$$

$$+ \ \Gamma^{(1)}_{qq}(x_1) \otimes \mathcal{G}^0_{3,qg \to gg}(s_{12}) + \frac{b_0}{\epsilon} S_{q \to g} \Gamma^{(1)}_{gq}(x_1) - S_{q \to g} \bar{\Gamma}^{(2)}_{gq}(x_1)$$

$$- \ \frac{1}{2} S_{q \to g} \Gamma^{(1)}_{gg}(x_1) \otimes \Gamma^{(1)}_{gq}(x_1) + \frac{1}{2} S_{q \to g} \Gamma^{(1)}_{qq}(x_1) \otimes \Gamma^{(1)}_{gq}(x_1),$$

$$(7.5.39)$$

$$J^{2,II}_{2,QQ,qg \to qq}(s_{12}) = - \ \mathcal{A}^{0,nadj}_{4,qg}(s_{12}) - \mathcal{A}^0_{4,qg}(s_{12}) - \mathcal{A}^1_{3,qg}(s_{12}) - \frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}^0_{3,qg \to qq}(s_{12})$$

$$+ \ \mathcal{A}^0_{3,qg \to qq}(s_{12}) \otimes \mathcal{A}^0_{3,qq}(s_{12}) - \Gamma^{(1)}_{qq}(x_2) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12})$$

$$+ \ \Gamma^{(1)}_{gg}(x_2) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12}) + \frac{1}{2} S_{g \to q} \Gamma^{(1)}_{qg}(x_2) \otimes \Gamma^{(1)}_{gg}(x_2)$$

$$- \ \frac{1}{2} S_{g \to q} \Gamma^{(1)}_{qg}(x_2) \otimes \Gamma^{(1)}_{qq}(x_2) - S_{g \to q} \bar{\Gamma}^{(2)}_{qg}(x_2). \qquad (7.5.40)$$

In Eq.(7.5.33) and (7.5.34), the terms proportional to $B^1_{1gH}(1,2,i)$ and $B^0_{1gH}(1,2,i)$ cancel directly against the explicit IR divergence in the two-loop matrix elements in Eq.(7.4.16). Other terms proportional to $B^1_{1gH}(1,i,2)$, $B^0_{1gH}(1,i,2)$, $A^1_{3gH}(1,2,i)$ and $A^0_{3gH}(1,2,i)$ contain the initial state identity changing (idc) behaviours and thus have no corresponding double virtual contributions from the matrix elements. We use a FORM program to analytically check that these idc terms are explicitly finite.

By taking the leading colour contributions to $qg \to H+$jet at NNLO as an example, we have given an explicit example of how the antenna subtraction terms removes the IR divergences from the quark-gluon initial state.

Explicit formulas for other antenna subtraction terms in table 7.5 can be found in appendix C.3. For gq, $\bar{q}g$ and $g\bar{q}$ initiated processes, the NNLO subtraction terms are related as in Eq. (7.4.17).

Chapter 8

Production of Higgs Boson Plus Jet from Quark-anti-Quark Scattering

In this chapter, I will focus the discussion on the $q\bar{q} \to H$+jet contribution to the fully differential cross section for Higgs plus jet observables up to NNLO. The leading colour contribution to the $q\bar{q} \to H$+jet channel will be taken as an example of the implementation of the antenna subtraction method as introduced in chapters 3 and 4.

8.1 $q\bar{q}$ initiated cross section at LO

At Born level, only the $q\bar{q} \to H + g$ process contributes. The spin and colour averaged differential cross section is

$$d\hat{\sigma}_{q\bar{q}}^B = \mathcal{N}_{q\bar{q}} \mathcal{N}_{LO} \, d\Phi_{H+1}(p_3, p_H; p_1, p_2) \left\{ \frac{1}{N} B_{1gH}^0(\hat{1}_q, 3, \hat{2}_{\bar{q}}) \right\} J_1^{(1)}(p_3). \qquad (8.1.1)$$

An explicit formula for B_{1gH}^0 is given in section 3.1.2 and 5.1.6.

Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the $\bar{q}q$ initiated channels are related to $d\hat{\sigma}_{q\bar{q}}^B$:

$$d\hat{\sigma}_{q\bar{q}}^B = d\hat{\sigma}_{\bar{q}q}^B \qquad (x_1 \leftrightarrow x_2), \qquad (8.1.2)$$

where x_1, x_2 are the momentum fractions of the initial state partons as introduced in Eq. (1.4.39).

8.2 $q\bar{q}$ initiated cross section at NLO

8.2.1 Real cross section

The real radiation contribution comes from the $q\bar{q} \to H + gg$ and $q\bar{q} \to H + Q\bar{Q}$ processes,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{q\bar{q}}^R = &\mathcal{N}_{q\bar{q}} \mathcal{N}_{NLO}^R \, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\Bigg\{ \\
&+ \frac{1}{2!}\left[B_{2gH}^0(\hat{1}_q, 3, 4, \hat{2}_{\bar{q}}) + B_{2gH}^0(\hat{1}_q, 4, 3, \hat{2}_{\bar{q}}) - \frac{1}{N^2}\widetilde{B}_{2gH}^0(\hat{1}_q, \widetilde{3}, \widetilde{4}, \hat{2}_{\bar{q}}) \right] \\
&+ \frac{N_f}{N} C_{0gH}^0(\hat{1}_q, 3_{\bar{Q}}, 4_Q, \hat{2}_{\bar{q}}) + \frac{1}{N} C_{0gH}^0(\hat{1}_q, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) - \frac{1}{N^2} D_{0gH}^0(\hat{1}_q, 3_{\bar{q}}, 4_q, \hat{2}_{\bar{q}}) \Bigg\} \\
&\times J_1^{(2)}(p_3, p_4) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (8.2.3)
\end{aligned}
$$

The squared matrix elements in Eq.(8.2.3) are discussed in section 3.1.3 and 5.1.7.

8.2.2 Virtual cross section

The one-loop contribution is from the $q\bar{q} \to H + g$ process and the differential cross section is given by,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{q\bar{q}}^V = &\mathcal{N}_{q\bar{q}} \mathcal{N}_{NLO}^V \, \mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)\Bigg\{ \\
&+ B_{1gH}^1(\hat{1}_q, 3, \hat{2}_{\bar{q}}) - \frac{1}{N^2}\widetilde{B}_{1gH}^1(\hat{1}_q, 3, \hat{2}_{\bar{q}}) + \frac{N_f}{N}\widehat{B}_{1gH}^1(\hat{1}_q, 3, \hat{2}_{\bar{q}}) \Bigg\} J_1^{(1)}(p_3) \quad (8.2.4)
\end{aligned}
$$

Details of the relevant matrix elements in Eq. (8.2.4) are given in section 3.1.3.

Just as at leading order, the NLO $\bar{q}q$ initiated channels are related to the $q\bar{q}$ contributions:

$$
\mathrm{d}\hat{\sigma}_{q\bar{q},NLO} = \mathrm{d}\hat{\sigma}_{\bar{q}q,NLO} \qquad (x_1 \leftrightarrow x_2). \qquad\qquad (8.2.5)
$$

subtraction term	maple	matrix element
$B_{2gH,NLO}^{0,XS}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$	`B2g0HXSNLO`	$B_{2gH}^{0}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$
$\widetilde{B}_{2gH,NLO}^{0,XS}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$	`Bt2g0HXSNLO`	$\widetilde{B}_{2gH}^{0}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$
$C_{0gH,NLO}^{0,XS}(\hat{1}_q, i_{\bar{Q}}, j_Q, \hat{2}_{\bar{q}})$	`qqbC0g0HXSNLO`	$C_{0gH}^{0}(\hat{1}_q, i_{\bar{Q}}, j_Q, \hat{2}_{\bar{q}})$
$C_{0gH,NLO}^{0,XS}(\hat{1}_q, \hat{2}_{\bar{Q}}, j_Q, i_{\bar{q}})$	`qqpbC0g0HXSNLO`	$C_{0gH}^{0}(\hat{1}_q, \hat{2}_{\bar{Q}}, j_Q, i_{\bar{q}})$

Table 8.1: NLO antenna subtraction terms for real contributions in $q\bar{q} \to H+$jet process and their relation to the matrix elements

8.3 $q\bar{q}$ initiated subtraction terms at NLO

8.3.1 Real subtraction terms

Using the NLO antenna subtraction method introduced in section 3.3.3, one can construct the antenna subtraction terms to mimic the implicit IR divergences in Eq.(7.2.3) such that

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{q\bar{q},NLO}^{S} = & \mathcal{N}_{q\bar{q}} \mathcal{N}_{NLO}^{R} \, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2) \Bigg\{ \\
& + \frac{1}{2!} \left[B_{2gH,NLO}^{0,XS}(\hat{1}_q, 3, 4, \hat{2}_{\bar{q}}) + B_{2gH,NLO}^{0,XS}(\hat{1}_q, 4, 3, \hat{2}_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}_{2gH,NLO}^{0,XS}(\hat{1}_q, \widetilde{3}, \widetilde{4}, \hat{2}_{\bar{q}}) \right] \\
& + \frac{N_f}{N} C_{0gH,NLO}^{0,XS}(\hat{1}_q, 3_{\bar{Q}}, 4_Q, \hat{2}_{\bar{q}}) + \frac{1}{N} C_{0gH,NLO}^{0,XS}(\hat{1}_q, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) \Bigg\}
\end{aligned}
\tag{8.3.6}
$$

The corresponding relationships between subtraction terms, file name in the NNLO-JET maple script and matrix elements are summarised in Table 8.1.

Explicit formulas for each subtraction term in table 8.1 are as follows:

$$
\begin{aligned}
B_{2gH,NLO}^{0,XS}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) = & \\
& + d_{3,q}^{0}(1, i, j) \, B_{1gH}^{0}(\overline{1}, (\widetilde{ij}), 2) \, J_1^{(1)}(\{p\}_1) \\
& + d_{3,q}^{0}(2, j, i) \, B_{1gH}^{0}(1, (\widetilde{ij}), \overline{2}) \, J_1^{(1)}(\{p\}_1)
\end{aligned}
\tag{8.3.7}
$$

$$
\begin{aligned}
\widetilde{B}_{2gH,NLO}^{0,XS}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}}) = & \\
& + A_{3,q\bar{q}}^{0}(1, i, 2) \, B_{1gH}^{0}(\overline{1}, j, \overline{2}) \, J_1^{(1)}(\{p\}_1) \\
& + A_{3,q\bar{q}}^{0}(1, j, 2) \, B_{1gH}^{0}(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)
\end{aligned}
\tag{8.3.8}
$$

$$C_{0gH,NLO}^{0,XS}(\hat{1}_q, i_{\bar{Q}}, j_Q, \hat{2}_{\bar{q}}) =$$
$$+\frac{1}{2} E_{3,q}^0(1, i, j) B_{1gH}^0(\overline{1}, (\widetilde{ij}), 2) J_1^{(1)}(\{p\}_1)$$
$$+\frac{1}{2} E_{3,q}^0(2, j, i) B_{1gH}^0(1, (\widetilde{ij}), \overline{2}) J_1^{(1)}(\{p\}_1) \tag{8.3.9}$$
$$C_{0gH,NLO}^{0,XS}(\hat{1}_q, \hat{2}_{\bar{Q}}, j_Q, i_{\bar{q}}) =$$
$$-E_{3,q'\to g}^0(i, j, 2) B_{1gH}^0(1, \overline{2}, (\widetilde{ij})) J_1^{(1)}(\{p\}_1)$$
$$-E_{3,q'\to g}^0(j, i, 1) B_{1gH}^0((\widetilde{ij}), \overline{1}, 2) J_1^{(1)}(\{p\}_1) \tag{8.3.10}$$

To numerically test that the antenna subtraction terms given in Eqs. (8.3.7), (8.3.8), (8.3.9) and (8.3.10) remove the implicit IR divergences in Eq. (8.2.3) correctly, we use the same spike plots defined in section 6.4.1 to test that the subtraction terms converge to the matrix elements when approaching various NLO unresolved limits.

8.3.2 Virtual subtraction terms

Using the NLO antenna subtraction method introduced in section 3.5.2, one can combine the integrated real subtraction and mass factorization terms to construct the virtual subtraction term which removes the explicit IR divergences in Eq.(8.2.4),

$$\mathrm{d}\hat{\sigma}_{q\bar{q},NLO}^T = \mathcal{N}_{q\bar{q}} \mathcal{N}_{NLO}^V \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2) \Bigg\{$$
$$+B_{1gH,NLO}^{1,XT}(\hat{1}_q, 3, \hat{2}_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}_{1gH,NLO}^{1,XT}(\hat{1}_q, 3, \hat{2}_{\bar{q}}) + \frac{N_f}{N} \widehat{B}_{1gH,NLO}^{1,XT}(\hat{1}_q, 3, \hat{2}_{\bar{q}})$$
$$+\frac{1}{N} C_{0gH,NLO}^{0,XT}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) \Bigg\} \tag{8.3.11}$$

The corresponding relationships between subtraction terms, file name in the NNLO-JET maple script and matrix elements are summarised in table 8.2.

The explicit formulae are as follows:

$$B_{1gH,NLO}^{1,XT}(\hat{1}_q, i, \hat{2}_{\bar{q}}) =$$
$$-\Big[+ J_{2,QG}^{1,IF}(s_{1i}) + J_{2,GQ}^{1,FI}(s_{2i}) \Big] B_{1gH}^0(1, i, 2) J_1^{(1)}(\{p\}_1) \tag{8.3.12}$$
$$\widetilde{B}_{1gH,NLO}^{1,XT}(\hat{1}_q, i, \hat{2}_{\bar{q}}) =$$
$$-\Big[+ \frac{1}{2} J_{2,QQ}^{1,II}(s_{12}) + \frac{1}{2} J_{2,QQ}^{1,II}(s_{21}) \Big] B_{1gH}^0(1, i, 2) J_1^{(1)}(\{p\}_1) \tag{8.3.13}$$

subtraction term	maple	matrix element
$B^{1,XT}_{1gH,NLO}(\hat{1}_q, i, \hat{2}_{\bar{q}})$	B1g1HXTNLO	$B^1_{1gH}(\hat{1}_q, i, \hat{2}_{\bar{q}})$
$\widetilde{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, i, \hat{2}_{\bar{q}})$	Bt1g1HXTNLO	$\widetilde{B}^1_{1gH}(\hat{1}_q, i, \hat{2}_{\bar{q}})$
$\widehat{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, i, \hat{2}_{\bar{q}})$	Bh1g1HXTNLO	$\widehat{B}^1_{1gH}(\hat{1}_q, i, \hat{2}_{\bar{q}})$
$C^{0,XT}_{0gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}})$	qqpbC0g0HXTNLO	IR safe

Table 8.2: NLO antenna subtraction terms for virtual contributions in $q\bar{q} \to H$+jet process and their relation to the matrix elements

$$
\begin{aligned}
\widehat{B}^{1,XT}_{1gH,NLO}(\hat{1}_q, i, \hat{2}_{\bar{q}}) = \\
-\left[+ \hat{J}^{1,IF}_{2,QG}(s_{1i}) + \hat{J}^{1,FI}_{2,GQ}(s_{2i}) \right] B^0_{1gH}(1, i, 2)\, J^{(1)}_1(\{p\}_1)
\end{aligned}
\tag{8.3.14}
$$

$$
\begin{aligned}
C^{0,XT}_{0gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}}) = \\
-J^{1,IF}_{2,GQ,q'\to g}(s_{1i})\, B^0_{1gH}(2, 1, i)\, J^{(1)}_1(\{p\}_1) \\
-J^{1,FI}_{2,QG,q'\to g}(s_{2i})\, B^0_{1gH}(1, 2, i)\, J^{(1)}_1(\{p\}_1)
\end{aligned}
\tag{8.3.15}
$$

$C^{0,XT}_{0gH,NLO}(\hat{1}_q, \hat{2}, i_{\bar{q}})$ comes from the integrated idc term in Eq.(8.3.10) and NLO mass factorization terms and only provides a finite contribution to the cross section.

8.4 $q\bar{q}$ initiated cross section at NNLO

8.4.1 Double real cross section

The double real contribution at NNLO for $q\bar{q} \to H$+jet comes from the $q\bar{q} \to H + ggg$, $q\bar{q} \to H + gQ\bar{Q}$ and $q\bar{q} \to H + gq\bar{q}$ processes,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{RR}_{q\bar{q}} = &\, \mathcal{N}_{q\bar{q}} \mathcal{N}^{RR}_{NNLO}\, \mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2) \Bigg\{ \\
& + \frac{N}{3!} \sum_{(i,j,k)\in P(3,4,5)} \left[B^0_{3gH}(\hat{1}_q, i, j, k, \hat{2}_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}^0_{3gH}(\hat{1}_q, i, j, \widetilde{k}, \hat{2}_{\bar{q}}) \right] \\
& + \frac{(N^2 + 1)}{3! N^3} \left[\widetilde{\widetilde{B}}^0_{3gH}(\hat{1}_q, \widetilde{3}, \widetilde{4}, \widetilde{5}, \hat{2}_{\bar{q}}) \right] \\
& + N_f \left[C^0_{1gH}(\hat{1}_q, 5, 3_{\bar{Q}}, 4_Q, \hat{2}_{\bar{q}}) + C^0_{1gH}(\hat{1}_q, 3_{\bar{Q}}, 4_Q, 5, \hat{2}_{\bar{q}}) \right] \\
& + \frac{N_f}{N^2} \left[\widetilde{C}^0_{1gH}(\hat{1}_q, 5, \hat{2}_{\bar{q}}, 4_Q, 3_{\bar{Q}}) + \widetilde{C}^0_{1gH}(\hat{1}_q, \hat{2}_{\bar{q}}, 4_Q, 5, 3_{\bar{Q}}) - \widetilde{\widetilde{C}}^0_{1gH}(\hat{1}_q, \hat{2}_{\bar{q}}, 4_Q, 3_{\bar{Q}}, 5) \right]
\end{aligned}
$$

$$+\left[C^0_{1gH}(\hat{1}_q, 5, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) + C^0_{1gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 4_Q, 5, 3_{\bar{q}})\right]$$

$$+\frac{1}{N^2}\left[\widetilde{C}^0_{1gH}(\hat{1}_q, 5, 3_{\bar{q}}, 4_Q, \hat{2}_{\bar{Q}}) + \widetilde{C}^0_{1gH}(\hat{1}_q, 3_{\bar{q}}, 4_Q, 5, \hat{2}_{\bar{Q}}) - \widetilde{\widetilde{C}}^0_{1gH}(\hat{1}_q, 3_{\bar{q}}, 4_Q, \hat{2}_{\bar{Q}}, 5)\right]$$

$$-\frac{1}{N}\left[D^0_{1gH}(\hat{1}_q, 4_q, \hat{2}_{\bar{q}}, 3_{\bar{q}}, 5) - \widetilde{D}^0_{1gH}(\hat{1}_q, 4_q, \hat{2}_{\bar{q}}, 3_{\bar{q}}, 5)\right]$$

$$\left.+\frac{1}{N^3}\widetilde{D}^0_{1gH}(\hat{1}_q, 4_q, \hat{2}_{\bar{q}}, 3_{\bar{q}}, 5)\right\} J^{(3)}_1(p_3, p_4, p_5) \tag{8.4.16}$$

The squared matrix elements in Eq.(8.4.16) are discussed in section 4.1.2 and 5.1.8. The $1/3!$ coefficients in Eq.(8.4.16) are the averaging factors for three identical gluons in the final state. The sum over active quark flavours gives the N_F factor to the C^0_{1gH}, $\widetilde{C}^0_{1gH}$ and $\widetilde{\widetilde{C}}^0_{1gH}$ matrix elements. The D^0_{1gH} and $\widetilde{D}^0_{1gH}$ matrix elements don not have the N_F factor in front as the final state quarks are identical to the initial state quarks where the flavours are fixed.

8.4.2 Real-virtual cross section

The real-virtual contribution at NNLO for $q\bar{q} \to H+$jet comes from the $q\bar{q} \to H+gg$, $q\bar{q} \to H + Q\bar{Q}$ and $q\bar{q} \to H + q\bar{q}$ process,

$$\mathrm{d}\hat{\sigma}^{RV}_{q\bar{q}} = \mathcal{N}_{q\bar{q}} \mathcal{N}^{RV}_{NNLO}\, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\left\{\vphantom{\sum}\right.$$

$$+\frac{N}{2!}\sum_{(i,j)\in P(3,4)}\left[B^1_{2gH}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) - \frac{1}{N^2}\widetilde{B}^1_{2gH}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) + \frac{N_f}{N}\widehat{B}^1_{2gH}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})\right]$$

$$-\frac{1}{2!N}\left[\widetilde{\widetilde{B}}^1_{2gH}(\hat{1}_q, \widetilde{3}, \widetilde{4}, \hat{2}_{\bar{q}}) - \frac{1}{N^2}\widetilde{\widetilde{\widetilde{B}}}^1_{2gH}(\hat{1}_q, \widetilde{3}, \widetilde{4}, \hat{2}_{\bar{q}}) + \frac{N_f}{N}\widehat{\widetilde{\widetilde{B}}}^1_{2gH}(\hat{1}_q, \widetilde{3}, \widetilde{4}, \hat{2}_{\bar{q}})\right]$$

$$+\frac{1}{2!N}\left[\widehat{\widetilde{B}}^1_{2gH}(\hat{1}_q, \widetilde{3}, \widetilde{4}, \hat{2}_{\bar{q}})\right]$$

$$+N_f\left[C^1_{0gH}(\hat{1}_q, 4_{\bar{Q}}, 3_Q, \hat{2}_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}^1_{0gH}(\hat{1}_q, 4_{\bar{Q}}, 3_Q, \hat{2}_{\bar{q}}) + \frac{N_f}{N}\widehat{C}^1_{0gH}(\hat{1}_q, 4_{\bar{Q}}, 3_Q, \hat{2}_{\bar{q}})\right]$$

$$+\left[C^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) + \frac{N_f}{N}\widehat{C}^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}})\right]$$

$$\left.-\frac{1}{N}\left[D^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{q}}, 3_q, 4_{\bar{q}}) + \frac{1}{N^2}\widetilde{D}^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{q}}, 3_q, 4_{\bar{q}}) - \frac{N_f}{N}\widehat{D}^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{q}}, 3_q, 4_{\bar{q}})\right]\right\}$$

$$\times J^{(2)}_1(p_3, p_4) \tag{8.4.17}$$

The squared matrix elements in Eq.(8.4.17) are discussed in section 4.1.2 and 5.2. The 1/2! coefficients in Eq.(8.4.17) are the averaging factors for two identical gluons in the final state.

8.4.3 Double virtual cross section

The double virtual contribution at NNLO for $q\bar{q} \to H$+jet comes from the $q\bar{q} \to H + g$ process,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{q\bar{q}}^{VV} =& \mathcal{N}_{q\bar{q}} \mathcal{N}_{NNLO}^{VV} \, \mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2) \Bigg\{ \\
& + N B_{1gH}^2(\hat{1}_q, 3, \hat{2}_{\bar{q}}) + N_F \widehat{B}_{1gH}^2(\hat{1}_q, 3, \hat{2}_{\bar{q}}) + \frac{N_F^2}{N} \widehat{\widehat{B}}_{1gH}^2(\hat{1}_q, 3, \hat{2}_{\bar{q}}) \\
& + \frac{N_F}{N^2} \widetilde{\widehat{B}}_{3gH}^2(\hat{1}_q, 3, \hat{2}_{\bar{q}}) + \frac{1}{N} \widetilde{B}_{1gH}^2(\hat{1}_q, 3, \hat{2}_{\bar{q}}) + \frac{1}{N^3} \widetilde{\widetilde{B}}_{1gH}^2(\hat{1}_q, 3, \hat{2}_{\bar{q}}) \Bigg\} J_1^{(1)}(p_3)
\end{aligned}
$$

(8.4.18)

The squared matrix elements in Eq. (8.4.18) are discussed in section 4.1.2 and the explicit formulas are given in [119].

Similar as at LO and NLO, the other quark anti-quark initiated cross section at NNLO are related to $\mathrm{d}\hat{\sigma}_{q\bar{q},NNLO}$:

$$
\mathrm{d}\hat{\sigma}_{\bar{q}q,NNLO} = \mathrm{d}\hat{\sigma}_{q\bar{q},NNLO} \qquad (x_1 \leftrightarrow x_2).
$$

(8.4.19)

8.5 $q\bar{q}$ initiated subtraction terms at NNLO

8.5.1 Double real subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.3.3, one can construct the double real subtraction term that mimics the implicit IR divergences in Eq. (8.4.16),

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{q\bar{q}}^{S} =& \mathcal{N}_{q\bar{q}} \mathcal{N}_{NNLO}^{RR} \, \mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2) \Bigg\{ \\
& + \frac{N}{3!} \Bigg[\sum_{(i,j,k) \in P^c(3,4,5)} B_{3gH}^{0,XS}(\hat{1}_q, i, j, k, \hat{2}_{\bar{q}}) - \frac{1}{N^2} \widetilde{B}_{3gH}^{0,FXS}(\hat{1}_q, \widetilde{3}, 4, 5, \hat{2}_{\bar{q}}) \Bigg]
\end{aligned}
$$

$$+\frac{(N^2+1)}{3!N^3}\left[\widetilde{\widetilde{B}}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{3},\widetilde{4},\widetilde{5},\hat{2}_{\bar{q}})\right]$$

$$+N_f C_{1gH}^{0,XS}(\hat{1}_q,5,3_{\bar{Q}},4_Q,\hat{2}_{\bar{q}})+\frac{N_f}{N^2}\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q,5,\hat{2}_{\bar{q}},4_Q,3_{\bar{Q}})$$

$$+C_{1gH}^{0,YS}(\hat{1}_q,5,\hat{2}_{\bar{Q}},4_Q,3_{\bar{q}})+\frac{1}{N^2}\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q,5,3_{\bar{q}},4_Q,\hat{2}_{\bar{Q}})$$

$$-\frac{1}{N}\left[D_{1gH}^{0,XS}(\hat{1}_q,4_q,\hat{2}_{\bar{q}},3_{\bar{q}},5)-\widetilde{D}_{1gH}^{0,XS}(\hat{1}_q,4_q,\hat{2}_{\bar{q}},3_{\bar{q}},5)\right]$$

$$+\frac{1}{N^3}\widetilde{D}_{1gH}^{0,XS}(\hat{1}_q,4_q,\hat{2}_{\bar{q}},3_{\bar{q}},5)\Bigg\} \tag{8.5.20}$$

The corresponding relationships between the subtraction terms, file name in the NNLOJET maple script and matrix elements (or combinations of subtraction terms) in Eq.(8.5.20) are summarised in table 8.3.

The $\widetilde{B}_{3gH}^{0,FXS}(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$ function is the subtraction term for a combination of six $\widetilde{B}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$ functions with full permutations of the three gluons (one of which is a photo-like gluon). Each $\widetilde{B}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$ function does not mimic all the double and single unresolved limits of one X topology matrix element. Only the six permutations together removes all of the double and single unresolved limits of all six X topology matrix elements.

In the $\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}})$ subtraction term, the momenta of the secondary quark-antiquark pair $(Q\bar{Q})$ can be symmetrized as the jet function does not distinguish quarks and antiquarks. Similar to the structure introduced in `qgCt1g0HFXS` in section 7.5.1, the symmetrized subtraction term $\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}})$ mimics the implicit IR divergences in the combination,

$$\frac{1}{2}\sum_{P(i,j)}\left[\widetilde{C}_{1gH}^0(\hat{1}_q,k,\hat{2}_{\bar{q}},j_Q,i_{\bar{Q}})+\widetilde{C}_{1gH}^0(\hat{1}_q,\hat{2}_{\bar{q}},j_Q,k,i_{\bar{Q}})-\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,\hat{2}_{\bar{q}},j_Q,i_{\bar{Q}},k)\right].$$

In the $\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q,i,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}})$ subtraction term, the quark labeled with $\hat{2}_{\bar{Q}}$ is in the initial state. This means that for matrix elements,

$$\widetilde{C}_{1gH}^0(\hat{1}_q,i,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}})+\widetilde{C}_{1gH}^0(\hat{1}_q,j_{\bar{q}},k_Q,i,\hat{2}_{\bar{Q}})-\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}},i),$$

we can not use the $Q\leftrightarrow\bar{Q}$ interchange technique to avoid constructing subtraction terms for $\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,k_{\bar{q}},\hat{2}_Q,j_{\bar{Q}},i)$.

Taking only the colour leading contribution as an example, the explicit formula

subtraction term	maple	matrix element
$B_{3gH}^{0,XS}(\hat{1}_q,i,j,k,\hat{2}_{\bar{q}})$	B3g0HXS	$B_{3gH}^0(\hat{1}_q,i,j,k,\hat{2}_{\bar{q}}) + B_{3gH}^0(\hat{1}_q,k,j,i,\hat{2}_{\bar{q}})$
$\widetilde{B}_{3gH}^{0,FXS}(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$	Bt3g0HFXS	$\sum_{P(i,j,k)} \widetilde{B}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$
$\widetilde{B}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$	Bt3g0HXS	$\widetilde{B}_{3gH}^0(\hat{1}_q,\widetilde{i},j,k,\hat{2}_{\bar{q}})$
$\widetilde{\widetilde{B}}_{3gH}^{0,XS}(\hat{1}_q,\widetilde{i},\widetilde{j},\widetilde{k},\hat{2}_{\bar{q}})$	Btt3g0HXS	$\widetilde{\widetilde{B}}_{3gH}^0(\hat{1}_q,\widetilde{i},\widetilde{j},\widetilde{k},\hat{2}_{\bar{q}})$
$C_{1gH}^{0,XS}(\hat{1}_q,i,j_{\bar{Q}},k_Q,\hat{2}_{\bar{q}})$	qqbC1g0HXS	$C_{1gH}^0(\hat{1}_q,i,j_{\bar{Q}},k_Q,\hat{2}_{\bar{q}})$ $+C_{1gH}^0(\hat{1}_q,j_{\bar{Q}},k_Q,i,\hat{2}_{\bar{q}})$
$C_{1gH}^{0,YS}(\hat{1}_q,i,\hat{2}_{\bar{Q}},k_Q,j_{\bar{q}})$	qqpbC1g0HYS	$C_{1gH}^0(\hat{1}_q,i,\hat{2}_{\bar{Q}},k_Q,j_{\bar{q}})$ $+C_{1gH}^0(\hat{1}_q,\hat{2}_{\bar{Q}},k_Q,i,j_{\bar{q}})$
$\widetilde{C}_{1gH}^{0,FXS}(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}})$	qqbCt1g0HFXS	$\frac{1}{2}\sum_{P(j,k)} \widetilde{C}_{1gH}^{0,XS}(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}})$
$\widetilde{C}_{1gH}^{0,XS}(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}})$	qqbCt1g0HXS	$\widetilde{C}_{1gH}^0(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}})+\widetilde{C}_{1gH}^0(\hat{1}_q,\hat{2}_{\bar{q}},k_Q,i,j_{\bar{Q}})$ $-\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}},i)$
$\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q,i,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}})$	qqpbCt1g0HYS	$\widetilde{C}_{1gH}^0(\hat{1}_q,i,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}})+\widetilde{C}_{1gH}^0(\hat{1}_q,j_{\bar{q}},k_Q,i,\hat{2}_{\bar{Q}})$ $-\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}},i)$
$D_{1gH}^{0,XS}(\hat{1}_q,k_q,\hat{2}_{\bar{q}},j_{\bar{q}},i)$	qqbD1g0HXS	$D_{1gH}^0(\hat{1}_q,k_q,\hat{2}_{\bar{q}},j_{\bar{q}},i)$
$\widetilde{D}_{1gH}^{0,XS}(\hat{1}_q,k_q,\hat{2}_{\bar{q}},j_{\bar{q}},i)$	qqbDt1g0HXS	$\widetilde{D}_{1gH}^0(\hat{1}_q,k_q,\hat{2}_{\bar{q}},j_{\bar{q}},i)$

Table 8.3: NNLO antenna subtraction terms for double real contributions in $q\bar{q} \to H+$jet process and their relation to the matrix elements (or combinations of subtraction terms)

for $B_{3gH}^{0,XS}(\hat{1}_q, i, j, k, \hat{2}_{\bar{q}})$ is,

$$
\begin{aligned}
B_{3gH}^{0,XS}(\hat{1}_q, i, j, k, \hat{2}_{\bar{q}}) = \\
&+ d_{3,q}^0(1,i,j)\, B_{2gH}^0(\bar{1}, (\widetilde{ij}), k, 2)\, J_1^{(2)}(\{p\}_2) \\
&+ f_3^0(i,j,k)\, B_{2gH}^0(1, (\widetilde{ij}), (\widetilde{jk}), 2)\, J_1^{(2)}(\{p\}_2) \\
&+ d_{3,q}^0(2,k,j)\, B_{2gH}^0(1, i, (\widetilde{jk}), \bar{2})\, J_1^{(2)}(\{p\}_2) \\
&+ d_{3,q}^0(1,k,j)\, B_{2gH}^0(\bar{1}, (\widetilde{kj}), i, 2)\, J_1^{(2)}(\{p\}_2) \\
&+ f_3^0(k,j,i)\, B_{2gH}^0(1, (\widetilde{kj}), (\widetilde{ji}), 2)\, J_1^{(2)}(\{p\}_2) \\
&+ d_{3,q}^0(2,i,j)\, B_{2gH}^0(1, k, (\widetilde{ji}), \bar{2})\, J_1^{(2)}(\{p\}_2) \\
&+ D_4^0(1,i,j,k)\, B_{1gH}^0(\bar{1}, (\widetilde{ijk}), 2)\, J_1^{(1)}(\{p\}_1) \\
&- d_{3,q}^0(1,i,j)\, D_{3,q}^0(\bar{1}, (\widetilde{ij}), k)\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{(\widetilde{ij})k}), 2)\, J_1^{(1)}(\{p\}_1) \\
&- f_3^0(i,j,k)\, D_{3,q}^0(1, (\widetilde{ij}), (\widetilde{jk}))\, B_{1gH}^0(\bar{1}, (\widetilde{\widetilde{ij},\widetilde{jk}}), 2)\, J_1^{(1)}(\{p\}_1) \\
&- d_{3,q}^0(1,k,j)\, D_{3,q}^0(\bar{1}, i, (\widetilde{kj}))\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{i(\widetilde{kj})}), 2)\, J_1^{(1)}(\{p\}_1) \\
&+ \frac{1}{2}\, d_{3,q}^0(1,i,j)\, d_{3,q}^0(\bar{1}, k, (\widetilde{ij}))\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{(\widetilde{ij})k}), 2)\, J_1^{(1)}(\{p\}_1) \\
&- \frac{1}{2}\, d_{3,q}^0(2,i,j)\, d_{3,q}^0(1, k, (\widetilde{ij}))\, B_{1gH}^0(\bar{1}, (\widetilde{k(\widetilde{ij})}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&- \frac{1}{2}\, A_{3,q\bar{q}}^0(1,i,2)\, d_{3,q}^0(\bar{1}, k, j)\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{kj}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&- \frac{1}{2}\left[+ S_{1i(\widetilde{ij})}^{IF} - S_{\bar{1}i(\widetilde{k(\widetilde{ij})})}^{IF} - S_{\bar{2}i(\widetilde{ij})}^{IF} + S_{\bar{2}i(\widetilde{k(\widetilde{ij})})}^{IF} - S_{1i\bar{2}}^{IF} + S_{\bar{1}i\bar{2}}^{IF} \right] \\
&\times d_{3,q}^0(1, k, (\widetilde{ij}))\, B_{1gH}^0(\bar{1}, (\widetilde{k(\widetilde{ij})}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&+ \frac{1}{2}\, d_{3,q}^0(1,k,j)\, d_{3,q}^0(\bar{1}, i, (\widetilde{kj}))\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{(\widetilde{kj})i}), 2)\, J_1^{(1)}(\{p\}_1) \\
&- \frac{1}{2}\, d_{3,q}^0(2,k,j)\, d_{3,q}^0(1, i, (\widetilde{kj}))\, B_{1gH}^0(\bar{1}, (\widetilde{i(\widetilde{kj})}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&- \frac{1}{2}\, A_{3,q\bar{q}}^0(1,k,2)\, d_{3,q}^0(\bar{1}, i, j)\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{ij}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&- \frac{1}{2}\left[+ S_{1k(\widetilde{kj})}^{IF} - S_{\bar{1}k(\widetilde{(\widetilde{kj})i})}^{IF} - S_{\bar{2}k(\widetilde{kj})}^{IF} + S_{\bar{2}k(\widetilde{i(\widetilde{kj})})}^{IF} - S_{1k\bar{2}}^{IF} + S_{\bar{1}k\bar{2}}^{IF} \right] \\
&\times d_{3,q}^0(1, i, (\widetilde{kj}))\, B_{1gH}^0(\bar{1}, (\widetilde{i(\widetilde{kj})}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&+ D_4^0(2,k,j,i)\, B_{1gH}^0(1, (\widetilde{ijk}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&- d_{3,q}^0(2,k,j)\, D_{3,q}^0(\bar{2}, (\widetilde{kj}), i)\, B_{1gH}^0(1, (\widetilde{i(\widetilde{jk})}), \bar{\bar{2}})\, J_1^{(1)}(\{p\}_1) \\
&- f_3^0(k,j,i)\, D_{3,q}^0(2, (\widetilde{kj}), (\widetilde{ji}))\, B_{1gH}^0(1, (\widetilde{\widetilde{kj},\widetilde{ji}}), \bar{2})\, J_1^{(1)}(\{p\}_1) \\
&- d_{3,q}^0(2,i,j)\, D_{3,q}^0(\bar{2}, k, (\widetilde{ij}))\, B_{1gH}^0(1, (\widetilde{k(\widetilde{ij})}), \bar{\bar{2}})\, J_1^{(1)}(\{p\}_1)
\end{aligned}
$$

$$+\frac{1}{2}\,d^0_{3,q}(2,k,j)\,d^0_{3,q}(\overline{2},i,(\widetilde{kj}))\,B^0_{1gH}(1,(\widetilde{(\widetilde{kj})i}),\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\,d^0_{3,q}(1,k,j)\,d^0_{3,q}(2,i,(\widetilde{kj}))\,B^0_{1gH}(\overline{1},(\widetilde{i(\widetilde{kj})}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\,A^0_{3,q\bar{q}}(1,k,2)\,d^0_{3,q}(\overline{2},i,j)\,B^0_{1gH}(\overline{1},(\widetilde{ij}),\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\left[\,+\,S^{IF}_{2k(\widetilde{kj})}-S^{IF}_{\overline{2}k(i(\widetilde{kj}))}-S^{IF}_{\overline{1}k(\widetilde{kj})}+S^{IF}_{\overline{1}k(i(\widetilde{kj}))}-S^{IF}_{\overline{1}k2}+S^{IF}_{\overline{1}k\overline{2}}\right]$$

$$\times d^0_{3,q}(2,i,(\widetilde{kj}))\,B^0_{1gH}(\overline{1},(\widetilde{i(\widetilde{kj})}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+\frac{1}{2}\,d^0_{3,q}(2,i,j)\,d^0_{3,q}(\overline{2},k,(\widetilde{ij}))\,B^0_{1gH}(1,(\widetilde{(\widetilde{ij})k}),\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\,d^0_{3,q}(1,i,j)\,d^0_{3,q}(2,k,(\widetilde{ij}))\,B^0_{1gH}(\overline{1},(\widetilde{k(\widetilde{ij})}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\,A^0_{3,q\bar{q}}(1,i,2)\,d^0_{3,q}(\overline{2},k,j)\,B^0_{1gH}(\overline{1},(\widetilde{kj}),\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\left[\,+\,S^{IF}_{2i(\widetilde{ij})}-S^{IF}_{\overline{2}i(k(\widetilde{ij}))}-S^{IF}_{\overline{1}i(\widetilde{ij})}+S^{IF}_{\overline{1}i(k(\widetilde{ij}))}-S^{IF}_{\overline{1}i2}+S^{IF}_{\overline{1}i\overline{2}}\right]$$

$$\times d^0_{3,q}(2,k,(\widetilde{ij}))\,B^0_{1gH}(\overline{1},(\widetilde{k(\widetilde{ij})}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\,\tilde{A}^0_4(1,i,k,2)\,B^0_{1gH}(\overline{1},j,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+\,A^0_{3,q\bar{q}}(1,k,2)\,A^0_{3,q\bar{q}}(\overline{1},i,\overline{2})\,B^0_{1gH}(\overline{\overline{1}},j,\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$+\,A^0_{3,q\bar{q}}(1,i,2)\,A^0_{3,q\bar{q}}(\overline{1},k,\overline{2})\,B^0_{1gH}(\overline{\overline{1}},j,\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\,A^0_{3,q\bar{q}}(1,k,2)\,A^0_{3,q\bar{q}}(\overline{1},i,\overline{2})\,B^0_{1gH}(\overline{\overline{1}},j,\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$+\frac{1}{2}\,d^0_{3,q}(2,k,j)\,A^0_{3,q\bar{q}}(1,i,\overline{2})\,B^0_{1gH}(\overline{1},(\widetilde{kj}),\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$+\frac{1}{2}\,d^0_{3,q}(1,k,j)\,A^0_{3,q\bar{q}}(\overline{1},i,2)\,B^0_{1gH}(\overline{\overline{1}},(\widetilde{kj}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\left[\,-\,S^{IF}_{\overline{1}k2}+S^{IF}_{\overline{\overline{1}}k\overline{2}}+S^{IF}_{2k(\widetilde{kj})}-S^{IF}_{\overline{2}k(\widetilde{kj})}+S^{IF}_{\overline{1}k(\widetilde{kj})}-S^{IF}_{\overline{\overline{1}}k(\widetilde{kj})}\right]$$

$$\times A^0_{3,q\bar{q}}(\overline{1},i,2)\,B^0_{1gH}(\overline{\overline{1}},(\widetilde{jk}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\,A^0_{3,q\bar{q}}(1,i,2)\,A^0_{3,q\bar{q}}(\overline{1},k,\overline{2})\,B^0_{1gH}(\overline{\overline{1}},j,\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$+\frac{1}{2}\,d^0_{3,q}(2,i,j)\,A^0_{3,q\bar{q}}(1,k,\overline{2})\,B^0_{1gH}(\overline{1},(\widetilde{ij}),\overline{\overline{2}})\,J^{(1)}_1(\{p\}_1)$$

$$+\frac{1}{2}\,d^0_{3,q}(1,i,j)\,A^0_{3,q\bar{q}}(\overline{1},k,2)\,B^0_{1gH}(\overline{\overline{1}},(\widetilde{ij}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\frac{1}{2}\left[\,-\,S^{IF}_{\overline{1}i2}+S^{IF}_{\overline{\overline{1}}i\overline{2}}+S^{IF}_{2i(\widetilde{ij})}-S^{IF}_{\overline{2}i(\widetilde{ij})}+S^{IF}_{\overline{1}i(\widetilde{ij})}-S^{IF}_{\overline{\overline{1}}i(\widetilde{ij})}\right]$$

$$\times A^0_{3,q\bar{q}}(\overline{1},k,2)\,B^0_{1gH}(\overline{\overline{1}},(\widetilde{ij}),\overline{2})\,J^{(1)}_1(\{p\}_1).\qquad(8.5.21)$$

Explicit formulae for the other antenna subtraction terms in table 8.3 can be

found in appendix D.1. The double real subtraction terms fit the general structure described in section 4.3.3.

To numerically test that the antenna subtraction terms given in Eq.(8.5.20) remove the implicit IR divergences in Eq.(8.4.16) correctly, we use the same spike plots for double real contribution defined in section 6.6.1 to test that the subtraction terms converge to the matrix elements when approaching various double and single unresolved limits at NNLO.

8.5.2   Real-virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.5.3, one can combine the integrated double real subtraction terms and real-virtual mass factorization term to construct the real-virtual subtraction term, $\mathrm{d}\hat{\sigma}_{q\bar{q}}^{T}$, which removes both the explicit and implicit IR divergences from Eq.(8.4.17),

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{q\bar{q}}^{T} =&\, \mathcal{N}_{q\bar{q}}\mathcal{N}_{NNLO}^{RV}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{ \\
&+\frac{N}{2!}\Bigg[B_{2gH}^{1,FXT}(\hat{1}_q,3,4,\hat{2}_{\bar{q}})-\frac{1}{N^2}\widetilde{B}_{2gH}^{1,XT}(\hat{1}_q,3,4,\hat{2}_{\bar{q}})+\sum_{(i,j)\in P(3,4)}\frac{N_f}{N}\widehat{B}_{2gH}^{1,XT}(\hat{1}_q,i,j,\hat{2}_{\bar{q}})\Bigg] \\
&-\frac{1}{2!N}\Bigg[\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q,\widetilde{3},\widetilde{4},\hat{2}_{\bar{q}})-\frac{1}{N^2}\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q,\widetilde{3},\widetilde{4},\hat{2}_{\bar{q}})+\frac{N_f}{N}\widehat{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q,\widetilde{3},\widetilde{4},\hat{2}_{\bar{q}})\Bigg] \\
&+N_f\Bigg[C_{0gH}^{1,FXT}(\hat{1}_q,4_{\bar{Q}},3_Q,\hat{2}_{\bar{q}})-\frac{1}{N^2}\widetilde{C}_{0gH}^{1,FT}(\hat{1}_q,4_{\bar{Q}},3_Q,\hat{2}_{\bar{q}})+\frac{N_f}{N}\widehat{C}_{0gH}^{1,XT}(\hat{1}_q,4_{\bar{Q}},3_Q,\hat{2}_{\bar{q}})\Bigg] \\
&+\Bigg[C_{0gH}^{1,YT}(\hat{1}_q,\hat{2}_{\bar{Q}},3_Q,4_{\bar{q}})-\frac{1}{N^2}\widetilde{C}_{0gH}^{1,YT}(\hat{1}_q,\hat{2}_{\bar{Q}},3_Q,4_{\bar{q}})+\frac{N_f}{N}\widehat{C}_{0gH}^{1,YT}(\hat{1}_q,\hat{2}_{\bar{Q}},3_Q,4_{\bar{q}})\Bigg] \\
&-\frac{1}{N}\Bigg[D_{0gH}^{1,XT}(\hat{1}_q,\hat{2}_{\bar{q}},3_q,4_{\bar{q}})+\frac{1}{N^2}\widetilde{D}_{0gH}^{1,XT}(\hat{1}_q,4_{\bar{q}},3_q,\hat{2}_{\bar{q}})\Bigg]\Bigg\} \qquad (8.5.22)
\end{aligned}
$$

The corresponding relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements (or combinations of subtraction terms) in Eq.(8.4.17) and (8.5.22) are summarised in table 8.4. Note that the antenna subtraction terms $B_{2gH}^{1,FXT}(\hat{1}_q,i,j,\hat{2}_{\bar{q}})$, $C_{0gH}^{1,FXT}(\hat{1}_q,j_{\bar{Q}},i_Q,\hat{2}_{\bar{q}})$ and $\widetilde{C}_{0gH}^{1,FT}(\hat{1}_q,j_{\bar{Q}},i_Q,\hat{2}_{\bar{q}})$ are combinations of sub-layer subtraction terms. Only the combination of the full terms remove all the IR divergences from the corresponding matrix elements at RV level.

subtraction term	maple	matrix element
$B_{2gH}^{1,FXT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$	`B2g1HFXT`	$B_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) + B_{2gH}^{1,XT}(\hat{1}_q, j, i, \hat{2}_{\bar{q}})$
$B_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$	`B2g1HXT`	$B_{2gH}^{1}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$
$\widetilde{B}_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$	`Bt2g1HXT`	$\displaystyle\sum_{P(i,j)} \widetilde{B}_{2gH}^{1}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$
$\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$	`Btt2g1HXT`	$\widetilde{\widetilde{B}}_{2gH}^{1}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}}) - \widehat{\widetilde{B}}_{2gH}^{1}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$
$\widehat{B}_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$	`Bh2g1HXT`	$\widehat{B}_{2gH}^{1}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$
$\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$	`Bttt2g1HXT`	$\widetilde{\widetilde{\widetilde{B}}}_{2gH}^{1}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$
$\widehat{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$	`Btth2g1HXT`	$\widehat{\widetilde{\widetilde{B}}}_{2gH}^{1}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}})$
$C_{0gH}^{1,FXT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$	`qqbC0g1HFXT`	$\frac{1}{2} \displaystyle\sum_{P(i,j)} C_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$
$C_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$	`qqbC0g1HXT`	$C_{0gH}^{1}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$
$\widetilde{C}_{0gH}^{1,FT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$	`qqbCt0g1HFXT`	$\frac{1}{2} \displaystyle\sum_{P(i,j)} \widetilde{C}_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$
$\widetilde{C}_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$	`qqbCt0g1HXT`	$\widetilde{C}_{0gH}^{1}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$
$\widehat{C}_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$	`qqbCh0g1HXT`	$\widehat{C}_{0gH}^{1}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}})$
$C_{0gH}^{1,YT}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}})$	`qqpbC0g1HYT`	$C_{0gH}^{1}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}})$
$\widetilde{C}_{0gH}^{1,YT}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}})$	`qqpbCt0g1HYT`	$\widetilde{C}_{0gH}^{1}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}})$
$\widehat{C}_{0gH}^{1,YT}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}})$	`qqpbCh0g1HYT`	$\widehat{C}_{0gH}^{1}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}})$
$D_{0gH}^{1,XT}(\hat{1}_q, \hat{2}_{\bar{q}}, i_q, j_{\bar{q}})$	`qqbD0g1HXT`	$D_{0gH}^{1}(\hat{1}_q, \hat{2}_{\bar{q}}, i_q, j_{\bar{q}})$
$\widetilde{D}_{0gH}^{1,XT}(\hat{1}_q, i_{\bar{q}}, j_q, \hat{2}_{\bar{q}})$	`qqbDt0g1HXT`	$\widetilde{D}_{0gH}^{1}(\hat{1}_q, \hat{2}_{\bar{q}}, j_q, i_{\bar{q}})$

Table 8.4: NNLO antenna subtraction terms for real-virtual contributions in $q\bar{q} \to H+$jet process and their relation to the matrix elements (or combinations of subtraction terms)

The explicit formula for $B_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}})$ is,

$$
\begin{aligned}
B_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) = & \\
& - \Big[+ J_{2,QG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,FF}(s_{ij}) + J_{2,GQ}^{1,FI}(s_{2j}) \Big] B_{2gH}^0(1, i, j, 2)\, J_1^{(2)}(\{p\}_2) \\
& + d_{3,q}^0(1, i, j) \bigg[B_{1gH}^1(\overline{1}, (\widetilde{ij}), 2)\, \delta(1 - x_1)\, \delta(1 - x_2) \\
& \quad + \Big(+ J_{2,QG}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) + J_{2,GQ}^{1,FI}(s_{2(\widetilde{ij})}) \Big) d_{3,q}^0(1, i, j)\, B_{1gH}^0(\overline{1}, (\widetilde{ij}), 2) \bigg] J_1^{(1)}(\{p\}_1) \\
& + \bigg[d_{3,q}^1(1, i, j)\, \delta(1 - x_1)\, \delta(1 - x_2) + \Big(J_{2,QG}^{1,IF}(s_{1i}) + J_{2,GG}^{1,FF}(s_{ij}) \\
& \quad + J_{2,QG}^{1,IF}(s_{1j}) - 2 J_{2,QG}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \Big) d_{3,q}^0(1, i, j) \bigg] B_{1gH}^0(\overline{1}, (\widetilde{ij}), 2)\, J_1^{(1)}(\{p\}_1) \\
& + d_{3,q}^0(2, j, i) \bigg[B_{1gH}^1(1, (\widetilde{ij}), \overline{2})\, \delta(1 - x_1)\, \delta(1 - x_2) \\
& \quad + \Big(+ J_{2,QG}^{1,IF}(s_{1(\widetilde{ij})}) + J_{2,GQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) \Big) d_{3,q}^0(2, j, i)\, B_{1gH}^0(1, (\widetilde{ij}), \overline{2}) \bigg] J_1^{(1)}(\{p\}_1) \\
& + \bigg[d_{3,q}^1(2, j, i)\, \delta(1 - x_1)\, \delta(1 - x_2) + \Big(J_{2,GQ}^{1,FI}(s_{2j}) + J_{2,GG}^{1,FF}(s_{ij}) \\
& \quad + J_{2,GQ}^{1,FI}(s_{2i}) - 2 J_{2,GQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) \Big) d_{3,q}^0(2, j, i) \bigg] B_{1gH}^0(1, (\widetilde{ij}), \overline{2})\, J_1^{(1)}(\{p\}_1) \\
& - \bigg[\tilde{A}_{3,q\bar{q}}^1(1, i, 2)\, \delta(1 - x_1)\, \delta(1 - x_2) \\
& \quad + \Big(+ J_{2,QQ}^{1,II}(s_{12}) - J_{2,QQ}^{1,II}(s_{\overline{12}}) \Big) A_{3,q\bar{q}}^0(1, i, 2) \bigg] B_{1gH}^0(\overline{1}, j, \overline{2})\, J_1^{(1)}(\{p\}_1) \\
& + \frac{1}{2} \bigg[+ J_{2,QG}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) - J_{2,QG}^{1,IF}(s_{1j}) - J_{2,GQ}^{1,FI}(s_{2(\widetilde{ij})}) + J_{2,GQ}^{1,FI}(s_{2j}) - J_{2,QQ}^{1,II}(s_{\overline{12}}) \\
& \quad + J_{2,QQ}^{1,II}(s_{12}) + \Big(- \mathcal{S}^{FI}(s_{\overline{1}(\widetilde{ij})}, s_{2j}, x_{\overline{1}(\widetilde{ij}),2j}) + \mathcal{S}^{FI}(s_{1j}, s_{2j}, x_{1j,2j}) \\
& \quad + \mathcal{S}^{FI}(s_{2(\widetilde{ij})}, s_{2j}, x_{2(\widetilde{ij}),2j}) - \mathcal{S}^{FI}(s_{2j}, s_{2j}, 1) \Big) \bigg] \\
& \times d_{3,q}^0(1, i, j)\, B_{1gH}^0(\overline{1}, (\widetilde{ij}), 2)\, J_1^{(1)}(\{p\}_1) \\
& + \frac{1}{2} \bigg[+ J_{2,GQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) - J_{2,GQ}^{1,FI}(s_{2i}) - J_{2,QG}^{1,IF}(s_{1(\widetilde{ij})}) + J_{2,QG}^{1,IF}(s_{1i}) - J_{2,QQ}^{1,II}(s_{1\overline{2}}) \\
& \quad + J_{2,QQ}^{1,II}(s_{12}) + \Big(- \mathcal{S}^{IF}(s_{\overline{2}(\widetilde{ij})}, s_{1i}, x_{\overline{2}(\widetilde{ij}),1i}) + \mathcal{S}^{IF}(s_{2i}, s_{1i}, x_{2i,1i}) \\
& \quad + \mathcal{S}^{IF}(s_{1(\widetilde{ij})}, s_{1i}, x_{1(\widetilde{ij}),1i}) - \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) \Big) \bigg] \\
& \times d_{3,q}^0(2, j, i)\, B_{1gH}^0(1, (\widetilde{ij}), \overline{2})\, J_1^{(1)}(\{p\}_1) \\
& - \frac{1}{2} \bigg[+ J_{2,QQ}^{1,II}(s_{\overline{12}}) - J_{2,QQ}^{1,II}(s_{12}) - J_{2,QG}^{1,IF}(s_{\overline{1}j}) + J_{2,QG}^{1,IF}(s_{1j}) - J_{2,GQ}^{1,FI}(s_{\overline{2}j})
\end{aligned}
$$

$$+ J_{2,GQ}^{1,FI}(s_{2j}) + \left(-\tilde{\mathcal{S}}^{IF}(s_{\overline{12}}, s_{\overline{1}j}, x_{\overline{12},\overline{1}j}) + \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) \right.$$

$$+ \tilde{\mathcal{S}}^{IF}(s_{\overline{1}j}, s_{\overline{1}j}, 1) - \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) + \tilde{\mathcal{S}}^{IF}(s_{\overline{2}j}, s_{\overline{1}j}, x_{\overline{2}j,\overline{1}j})$$

$$\left. - \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) \right) \Bigg] A_{3,q\bar{q}}^0(1, i, 2) \, B_{1gH}^0(\overline{1}, j, \overline{2}) \, J_1^{(1)}(\{p\}_1). \qquad (8.5.23)$$

Explicit formulas for the other antenna subtraction terms in table 8.4 can be found in appendix D.2. The real-virtual subtraction terms fit the general structure described in section 4.5.3.

To numerically test that the antenna subtraction terms given in Eq.(8.5.22) remove the implicit IR divergens in Eq.(8.4.17) correctly, we use the same spike plots for defined in section 6.6.2 to test that the subtraction terms remove the explicit IR divergence of the matrix elements in any phase space point and converge to the matrix elements when approaching various single unresolved limits at the real-virtual level.

8.5.3   Double virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.7.2, one can remove the explicit IR divergences in the double virtual matrix elements in Eq.(8.4.18) with the following double virtual subtraction term,

$$d\hat{\sigma}_{q\bar{q}}^U = \mathcal{N}_{q\bar{q}} \mathcal{N}_{NNLO}^{VV} \frac{dx_1}{x_1} \frac{dx_2}{x_2} d\Phi_{H+1}(p_3, p_H; p_1, p_2) \Bigg\{$$

$$- N B_{1gH}^{2,XU}(1_q, 3, 2_{\bar{q}}) - \frac{1}{N} \widetilde{B}_{1gH}^{2,XU}(1_q, 3, 2_{\bar{q}}) - \frac{1}{N^3} \widetilde{\widetilde{B}}_{1gH}^{2,XU}(1_q, 3, 2_{\bar{q}})$$

$$- \frac{N_f}{N^2} \widehat{\widetilde{B}}_{1gH}^{2,XU}(1_q, 3, 2_{\bar{q}}) - N_f \widehat{B}_{1gH}^{2,XU}(1_q, 3, 2_{\bar{q}}) - \frac{N_f^2}{N} \widehat{\widehat{B}}_{1gH}^{2,XU}(1_q, 3, 2_{\bar{q}})$$

$$- C_{1gH}^{1,XU}(1_q, 3, 2_{\bar{q}}) + \frac{1}{N^2} \widetilde{C}_{1gH}^{1,XU}(1_q, 3, 2_{\bar{q}}) - \frac{N_F}{N} \widehat{C}_{1gH}^{1,XU}(1_q, 3, 2_{\bar{q}}) \Bigg\} J_1^{(1)}(p_3)$$

$$(8.5.24)$$

The relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements in Eqs.(8.4.18) and (8.5.24) are summarised in table 8.5.

The explicit formula for $B_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$ in terms of integrated antennae is,

$$B_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}}) =$$

subtraction term	maple	matrix element
$B_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$	`B1g2HXU`	$B_{1gH}^{2}(1_q, i, 2_{\bar{q}})$
$\widetilde{B}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$	`Bt1g2HXU`	$\widetilde{B}_{1gH}^{2}(1_q, i, 2_{\bar{q}})$
$\widetilde{\widetilde{B}}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$	`Btt1g2HXU`	$\widetilde{\widetilde{B}}_{1gH}^{2}(1_q, i, 2_{\bar{q}})$
$\widehat{\widetilde{B}}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$	`Bth1g2HXU`	$\widehat{\widetilde{B}}_{1gH}^{2}(1_q, i, 2_{\bar{q}})$
$\widehat{B}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$	`Bh1g2HXU`	$\widehat{B}_{1gH}^{2}(1_q, i, 2_{\bar{q}})$
$\widehat{\widehat{B}}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}})$	`Bhh1g2HXU`	$\widehat{\widehat{B}}_{1gH}^{2}(1_q, i, 2_{\bar{q}})$
$C_{1gH}^{1,XU}(1_q, i, 2_{\bar{q}})$	`qqpbC0g1HXU`	IR safe
$\widetilde{C}_{1gH}^{1,XU}(1_q, i, 2_{\bar{q}})$	`qqpbCt0g1HXU`	IR safe
$\widehat{C}_{1gH}^{1,XU}(1_q, i, 2_{\bar{q}})$	`qqpbCh0g1HXU`	IR safe

Table 8.5: NNLO antenna subtraction terms for double virtual contributions in $q\bar{q} \to H$+jet process and their relation to the matrix elements (or combinations of subtraction terms)

$$
-\left[+\frac{1}{2}\,\mathcal{D}_{3,q}^{0}(s_{23}) - \Gamma_{qq}^{(1)}(z_2) + \frac{1}{2}\,\mathcal{D}_{3,q}^{0}(s_{13}) - \Gamma_{qq}^{(1)}(z_1) \right]
$$
$$
\times \left(B_{1gH}^{1}(1,3,2) - \frac{b_0}{\epsilon} B_{1gH}^{0}(1,3,2) \right)
$$
$$
-\left[+\frac{1}{8}\,\mathcal{D}_{3,q}^{0}(s_{23}) \otimes \mathcal{D}_{3,q}^{0}(s_{23}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{D}_{3,q}^{0}(s_{23}) + \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_2) \right]
$$
$$
\times B_{1gH}^{0}(1,3,2)
$$
$$
-\left[\frac{1}{8}\,\mathcal{D}_{3,q}^{0}(s_{13}) \otimes \mathcal{D}_{3,q}^{0}(s_{13}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{D}_{3,q}^{0}(s_{13}) + \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1) \right]
$$
$$
\times B_{1gH}^{0}(1,3,2)
$$
$$
-\left[+\frac{1}{4}\,\mathcal{D}_{3,q}^{0}(s_{13}) \otimes \mathcal{D}_{3,q}^{0}(s_{23}) + \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_2) \right.
$$
$$
\left. -\frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{D}_{3,q}^{0}(s_{13}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{D}_{3,q}^{0}(s_{23}) \right] B_{1gH}^{0}(1,3,2)
$$
$$
-\left[+\frac{1}{2}\,\mathcal{D}_{4,q}^{0}(s_{13}) + \frac{1}{2}\,\mathcal{D}_{3,q}^{1}(s_{13}) + \frac{b_0}{2\epsilon}\left(\frac{s_{13}}{\mu_R^2}\right)^{-\epsilon} \mathcal{D}_{3,q}^{0}(s_{13}) - \frac{1}{4}\,\mathcal{D}_{3,q}^{0}(s_{13}) \otimes \mathcal{D}_{3,q}^{0}(s_{13}) \right.
$$
$$
\left. -\bar{\Gamma}_{qq}^{(2)}(z_1) \right] B_{1gH}^{0}(1,3,2)
$$

$$-\left[+\frac{1}{2}\,\mathcal{D}_{4,q}^0(s_{23})+\frac{1}{2}\,\mathcal{D}_{3,q}^1(s_{23})+\frac{b_0}{2\epsilon}\left(\frac{s_{23}}{\mu_R^2}\right)^{-\epsilon}\mathcal{D}_{3,q}^0(s_{23})-\frac{1}{4}\,\mathcal{D}_{3,q}^0(s_{23})\otimes\mathcal{D}_{3,q}^0(s_{23})\right.$$

$$\left.-\,\bar{\Gamma}_{qq}^{(2)}(z_2)\right]B_{1gH}^0(1,3,2)$$

$$-\left[-\frac{1}{2}\,\tilde{\mathcal{A}}_{4,q\bar{q}}^0(s_{12})-\tilde{\mathcal{A}}_{3,qq}^1(s_{12})+\frac{1}{2}\,\mathcal{A}_{3,qq}^0(s_{12})\otimes\mathcal{A}_{3,qq}^0(s_{12})\right]B_{1gH}^0(1,3,2). \quad (8.5.25)$$

Using the dipole functions introduced in section 3.5.2 and 4.7.2, Eq. (8.5.25) can be re-expressed in a similar structure as the two-loop Catani pole structure of Eq. (4.7.156) as,

$$B_{1gH}^{2,XU}(1_q,i,2_{\bar{q}})=$$

$$-\left[+J_{2,QG}^{1,IF}(s_{13})+J_{2,QG}^{1,FI}(s_{23})\right]\left(B_{1gH}^1(1,3,2)-\frac{b_0}{\epsilon}B_{1gH}^0(1,3,2)\right)$$

$$-\frac{1}{2}\left[+J_{2,QG}^{1,IF}(s_{13})+J_{2,QG}^{1,FI}(s_{23})\right]\otimes\left[+J_{2,QG}^{1,IF}(s_{13})+J_{2,QG}^{1,FI}(s_{23})\right]B_{1gH}^0(1,3,2)$$

$$-\left[+J_{2,QG}^{2,IF}(s_{13})+J_{2,QG}^{2,FI}(s_{23})+\bar{J}_{2,QQ}^{2,II}(s_{12})\right]B_{1gH}^0(1,3,2), \quad (8.5.26)$$

where the NNLO integrated antennae are,

$$J_{2,QG}^{2,IF}(s_{1i})=+\frac{1}{2}\,\mathcal{D}_{4,q}^0(s_{1i})+\frac{1}{2}\,\mathcal{D}_{3,q}^1(s_{1i})+\frac{b_0}{2\epsilon}\left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon}\mathcal{D}_{3,q}^0(s_{1i})$$

$$-\frac{1}{4}\,\mathcal{D}_{3,q}^0(s_{1i})\otimes\mathcal{D}_{3,q}^0(s_{1i})-\bar{\Gamma}_{qq}^{(2)}(z_1), \quad (8.5.27)$$

$$J_{2,QG}^{2,FI}(s_{2i})=+\frac{1}{2}\,\mathcal{D}_{4,q}^0(s_{2i})+\frac{1}{2}\,\mathcal{D}_{3,q}^1(s_{2i})+\frac{b_0}{2\epsilon}\left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon}\mathcal{D}_{3,q}^0(s_{2i})$$

$$-\frac{1}{4}\,\mathcal{D}_{3,q}^0(s_{2i})\otimes\mathcal{D}_{3,q}^0(s_{2i})-\bar{\Gamma}_{qq}^{(2)}(z_2), \quad (8.5.28)$$

$$\bar{J}_{2,QQ}^{2,II}(s_{12})=-\frac{1}{2}\,\tilde{\mathcal{A}}_{4,q\bar{q}}^0(s_{12})-\tilde{\mathcal{A}}_{3,qq}^1(s_{12})+\frac{1}{2}\,\mathcal{A}_{3,qq}^0(s_{12})\otimes\mathcal{A}_{3,qq}^0(s_{12}). \quad (8.5.29)$$

Explicit formulas for other antenna subtraction terms in table 8.5 can be found in appendix D.3. For $\bar{q}q$ initiated processes, the NNLO subtraction terms are related as in Eq. (8.4.19).

Chapter 9

Production of Higgs Boson Plus Jet from Quark-Quark Scattering

In this chapter, I will focus the discussion on the $qq \to H+$jet contributions to the fully differential cross section for Higgs plus jet observables up to NNLO. The leading colour contribution to the $qq \to H+$jet channel will be taken as an example of the implementation of the antenna subtraction method as introduced in chapters 3 and 4. The quark (anti-)quark initiated channels contain identity changing initial state divergences. However, as there is no soft-quark singularity, these processes have a less complicated IR divergence structure than the processes considered in chapter 7.

The other quark (anti-)quark initiated channels are qQ, $\bar{q}\bar{q}$, $\bar{q}\bar{Q}$, $\bar{q}Q$ and $q\bar{Q} \to H+$jet (q and Q are used to denote different quarks). These channels give distinct physical contributions to the total cross section. However, both the matrix elements and antenna subtraction terms are closely related to those $qq \to H+$jet channel. Details of the relationships between the different channels are discussed in appendix A.

9.1 qq initiated cross sections at LO

As at least two quark pairs with the same flavours are involved in the qq initiated cross sections and there is no Born-level qq initiated process. Similarly, there is no

virtual NLO contribution, and no double virtual NNLO contribution.

9.2 *qq* initiated cross sections at NLO

9.2.1 Real cross sections

The real radiation contribution for the identical quark pair initiated contribution comes from the $qq \to H + qq$ process,

$$
\mathrm{d}\hat{\sigma}_{qq}^R = \mathcal{N}_{qq} \mathcal{N}_{NLO}^R \, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2) \Bigg\{
$$
$$
+ \frac{1}{N} C_{0gH}^0(\hat{1}_q, 3_{\bar{Q}}, \hat{2}_Q, 4_{\bar{q}}) - \frac{1}{2N^2} D_{0gH}^0(\hat{1}_q, 3_{\bar{q}}, \hat{2}_q, 4_{\bar{q}}) \Bigg\} J_1^{(2)}(p_3, p_4), \qquad (9.2.1)
$$

The squared matrix elements in Eq.(9.2.1) are discussed in section 3.1.3 and 5.1.7.

There is no virtual contribution from the qq initiated channel. Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the $\bar{q}\bar{q}$ initiated channels are related to $\mathrm{d}\hat{\sigma}_{qq,NLO}$:

$$
\mathrm{d}\hat{\sigma}_{qq}^R = \mathrm{d}\hat{\sigma}_{\bar{q}\bar{q}}^R. \qquad (9.2.2)
$$

9.3 *qq* initiated subtraction terms at NLO

9.3.1 Real subtraction terms

Using the NLO antenna subtraction method introduced in section 3.3.3, one can construct the antenna subtraction terms to mimic the implicit IR divergences in Eq.(9.2.1) such that

$$
\mathrm{d}\hat{\sigma}_{qq,NLO}^S = \mathcal{N}_{qq} \mathcal{N}_{NLO}^R \, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2) \Bigg\{
$$
$$
+ \frac{1}{N} C_{0gH,NLO}^{0,YS}(\hat{1}_q, 3_{\bar{Q}}, \hat{2}_Q, 4_{\bar{q}}) \Bigg\}. \qquad (9.3.3)
$$

Note that the $D_{0gH}^0(\hat{1}_q, 3_{\bar{q}}, \hat{2}_q, 4_{\bar{q}})$ matrix element in Eq.(9.2.1) has no implicit IR divergence. We only need to construct $C_{0gH,NLO}^{0,YS}$ subtraction term to remove the implicit IR divergences from C_{0gH}^0.

subtraction term	maple	matrix element
$C_{0gH,NLO}^{0,YS}(\hat{1}_q, i_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}})$	`qqpC0g0HYSNLO`	$C_{0gH}^0(\hat{1}_q, i_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}})$

Table 9.1: NLO antenna subtraction terms for real contributions in $qq \to H$+jet process and their relation to the matrix elements

subtraction term	maple	matrix element
$C_{0gH,NLO}^{0,YT}(\hat{1}, \hat{2}, i)$	`qqpC0g0HYTNLO`	IR safe

Table 9.2: NLO antenna subtraction terms for virtual contributions in $qq \to H$+jet process and their relation to the matrix elements

The corresponding relationships between subtraction terms, file name in the NNLOJET maple script and matrix elements are summarised in Table 9.1.

Explicit formula for $C_{0gH,NLO}^{0,YS}(\hat{1}_q, i_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}})$ is

$$
\begin{aligned}
C_{0gH,NLO}^{0,YS}(\hat{1}_q, i_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}}) = \\
-E_{3,q'\to g}^0(j, 2, i)\, B_{1gH}^0(1, \bar{2}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1) \\
-E_{3,q'\to g}^0(i, 1, j)\, B_{1gH}^0((\widetilde{ij}), \bar{1}, 2)\, J_1^{(1)}(\{p\}_1).
\end{aligned}
\tag{9.3.4}
$$

9.3.2 Virtual subtraction terms

The virtual subtraction terms is a collection of integrated real subtraction and mass factorization terms. As there is no virtual contribution from the matrix elements. The following virtual subtraction terms must have no IR divergence,

$$
\begin{aligned}
d\hat{\sigma}_{qq,NLO}^T = \mathcal{N}_{qq}\,\mathcal{N}_{NLO}^V\,\frac{dx_1}{x_1}\frac{dx_2}{x_2}d\Phi_{H+1}(p_3, p_H; p_1, p_2)\Bigg\{ \\
+\frac{1}{N}C_{0gH,NLO}^{0,YT}(\hat{1}, \hat{2}, 3)\Bigg\},
\end{aligned}
\tag{9.3.5}
$$

The corresponding relationships between subtraction terms, file name in the NNLOJET maple script and matrix elements are summarised in Table 9.2.

Explicit formula for $C_{0gH,NLO}^{0,YT}(\hat{1}, \hat{2}, i)$ is

$$
C_{0gH,NLO}^{0,YT}(\hat{1}, \hat{2}, i) =
$$

$$-J_{2,GQ,q'\to g}^{1,IF}(s_{1i})\,B_{1gH}^0(2,1,i)\,J_1^{(1)}(\{p\}_1)$$

$$-J_{2,QG,q'\to g}^{1,FI}(s_{2i})\,B_{1gH}^0(1,2,i)\,J_1^{(1)}(\{p\}_1). \tag{9.3.6}$$

As usual, spike plots are used at real to provide graphic evidence of the convergence between matrix element and antenna subtraction terms. The virtual subtraction terms only have idc dipole functions which have no explicit IR divergence.

The other NLO quark (anti-)quark initiated channels ($\bar{q}\bar{q}$, qQ, $q\bar{Q}$, $\bar{q}\bar{Q}$ and $\bar{Q}q$) are not independent from the qq channel. The details for the matrix elements and antenna subtraction term for those channels are introduced in appendix A.

9.4 qq initiated cross sections at NNLO

9.4.1 Double real cross sections

The double real contribution at NNLO for $qq \to H+$jet comes from the $qq \to H+qqg$ process,

$$\begin{aligned}
\mathrm{d}\hat{\sigma}_{qq}^{RR} = {}&\mathcal{N}_{qq}\,\mathcal{N}_{NNLO}^{RR}\,\mathrm{d}\Phi_{H+3}(p_3,p_4,p_5,p_H;p_1,p_2)\Bigg\{ \\
&+\left[C_{1gH}^0(\hat{1}_q,5,3_{\bar{Q}},\hat{2}_Q,4_{\bar{q}}) + C_{1gH}^0(\hat{1}_q,3_{\bar{Q}},\hat{2}_Q,5,4_{\bar{q}})\right] \\
&+\frac{1}{N^2}\left[\widetilde{C}_{1gH}^0(\hat{1}_q,5,4_{\bar{q}},\hat{2}_Q,3_{\bar{Q}}) + \widetilde{C}_{1gH}^0(\hat{1}_q,4_{\bar{q}},\hat{2}_Q,5,3_{\bar{Q}}) - \widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,4_{\bar{q}},\hat{2}_Q,3_{\bar{Q}},5)\right] \\
&-\frac{1}{2!N}\left[D_{1gH}^0(\hat{1}_q,\hat{2}_q,4_{\bar{q}},3_{\bar{q}},5) - \widetilde{D}_{1gH}^0(\hat{1}_q,\hat{2}_q,4_{\bar{q}},3_{\bar{q}},5)\right] \\
&+\frac{1}{2!N^3}\widetilde{D}_{1gH}^0(\hat{1}_q,\hat{2}_q,4_{\bar{q}},3_{\bar{q}},5)\Bigg\}J_1^{(3)}(p_3,p_4,p_5)
\end{aligned} \tag{9.4.7}$$

The squared matrix elements in Eq.(9.4.7) are discussed in section 4.1.2 and 5.1.8. The 1/2! coefficients in Eq.(9.4.7) are the averaging factors for two identical anti-quarks in the final state. For identical quark pairs, we should have matrix elements C_{1gH}^0, $\widetilde{C}_{1gH}^0$ and $\widetilde{\widetilde{C}}_{1gH}^0$ together with the $Q \leftrightarrow q$ permutation as introduced in section 3.1.1 (also with the 1/2! averaging factor). In the qq initiated channel with massless active quark assumption, the final state Q and q quarks can not be distinguished by the jet function. This means that after phase space integration, the result for differential cross sections involving C_{1gH}^0, $\widetilde{C}_{1gH}^0$ and $\widetilde{\widetilde{C}}_{1gH}^0$ will be the same before or

after the $Q \leftrightarrow q$ permutation. Here we combine the $Q \leftrightarrow q$ permutation and the extra factor of 2 cancels with the $1/2!$ averaging factor.

9.4.2 Real-virtual cross sections

The real-virtual contribution at NNLO for $qq \to H+$jet comes from the $qq \to H+qq$ process,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{qq}^{RV} =& \mathcal{N}_{qq} \mathcal{N}_{NNLO}^{RV} \, \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2) \Bigg\{ \\
&+ \left[C_{0gH}^1(\hat{1}_q, 4_{\bar{Q}}, \hat{2}_Q, 3_{\bar{q}}) - \frac{1}{N^2} \widetilde{C}_{0gH}^1(\hat{1}_q, 4_{\bar{Q}}, \hat{2}_Q, 3_{\bar{q}}) + \frac{N_f}{N} \widehat{C}_{0gH}^1(\hat{1}_q, 4_{\bar{Q}}, \hat{2}_Q, 3_{\bar{q}}) \right] \\
&- \frac{1}{2N} \left[D_{0gH}^1(\hat{1}_q, 3_{\bar{q}}, \hat{2}_q, 4_{\bar{q}}) + \frac{1}{N^2} \widetilde{D}_{0gH}^1(\hat{1}_q, 3_{\bar{q}}, \hat{2}_q, 4_{\bar{q}}) - \frac{N_f}{N} \widehat{D}_{0gH}^1(\hat{1}_q, 3_{\bar{q}}, \hat{2}_q, 4_{\bar{q}}) \right] \Bigg\} \\
&\times J_1^{(2)}(p_3, p_4)
\end{aligned}
\tag{9.4.8}
$$

The squared matrix elements in Eq.(9.4.8) are discussed in section 4.1.2 and 5.2. Here we sum the $Q \leftrightarrow q$ permutation in the same fashion as in section 9.4.1.

There is no contribution from the two-loop matrix element for qq initiated channels at double virtual level. However, we will have subtraction terms at the double virtual level for this channel which contain no IR divergence.

Just as at NLO, the anti-quark anti-quark initiated cross section at NNLO are related to $\mathrm{d}\hat{\sigma}_{qq,NNLO}$:

$$
\mathrm{d}\hat{\sigma}_{\bar{q}\bar{q},NNLO} = \mathrm{d}\hat{\sigma}_{qq,NNLO}.
\tag{9.4.9}
$$

9.5 qq initiated subtraction terms at NNLO

9.5.1 Double real subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.3.3, one can construct the double real subtraction term that mimics the implicit IR divergences in Eq. (9.4.7),

$$
\mathrm{d}\hat{\sigma}_{qq}^S = \mathcal{N}_{qq} \mathcal{N}_{NNLO}^{RR} \, \mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2) \Bigg\{
$$

subtraction term	maple	matrix element
$C_{1gH}^{0,YS}(\hat{1}_q, i, k_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}})$	qqpC1g0HYS	$C_{1gH}^0(\hat{1}_q, i, k_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}}) + C_{1gH}^0(\hat{1}_q, k_{\bar{Q}}, \hat{2}_Q, i, j_{\bar{q}})$
$\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q, i, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}})$	qqpCt1g0HYS	$\widetilde{C}_{1gH}^0(\hat{1}_q, i, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}}) + \widetilde{C}_{1gH}^0(\hat{1}_q, j_{\bar{q}}, \hat{2}_Q, i, k_{\bar{Q}})$ $-\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}}, i)$
$D_{1gH}^{0,YS}(\hat{1}_q, \hat{2}_q, k_{\bar{q}}, j_{\bar{q}}, i)$	qqD1g0HYS	$D_{1gH}^0(\hat{1}_q, \hat{2}_q, k_{\bar{q}}, j_{\bar{q}}, i)$
$\widetilde{D}_{1gH}^{0,YS}(\hat{1}_q, \hat{2}_q, k_{\bar{q}}, j_{\bar{q}}, i)$	qqDt1g0HYS	$\widetilde{D}_{1gH}^0(\hat{1}_q, \hat{2}_q, k_{\bar{q}}, j_{\bar{q}}, i)$

Table 9.3: NNLO antenna subtraction terms for double real contributions in $qq \to$ H+jet process and their relation to the matrix elements

$$
\begin{aligned}
&+ C_{1gH}^{0,YS}(\hat{1}_q, 5, 3_{\bar{Q}}, \hat{2}_Q, 4_{\bar{q}}) + \frac{1}{N^2}\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q, 5, 4_{\bar{q}}, \hat{2}_Q, 3_{\bar{Q}}) \\
&- \frac{1}{2N}\left[D_{1gH}^{0,YS}(\hat{1}_q, \hat{2}_q, 4_{\bar{q}}, 3_{\bar{q}}, 5) - \widetilde{D}_{1gH}^{0,YS}(\hat{1}_q, \hat{2}_q, 4_{\bar{q}}, 3_{\bar{q}}, 5) \right] \\
&+ \frac{1}{2N^3}\widetilde{D}_{1gH}^{0,YS}(\hat{1}_q, \hat{2}_q, 4_{\bar{q}}, 3_{\bar{q}}, 5) \Bigg\}
\end{aligned}
\tag{9.5.10}
$$

The corresponding relationships between the subtraction terms, file name in the NNLOJET maple script and matrix elements (or combinations of subtraction terms) in Eq. (9.5.10) are summarised in table 9.3.

Note that the quark labeled with $\hat{2}_Q$ is in the initial state in the $qQ(qq)$ initiated processes. This means that for matrix elements,

$$
\widetilde{C}_{1gH}^0(\hat{1}_q, i, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}}) + \widetilde{C}_{1gH}^0(\hat{1}_q, j_{\bar{q}}, \hat{2}_Q, i, k_{\bar{Q}}) - \widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}}, i),
$$

we can not use the $Q \leftrightarrow \bar{Q}$ interchange technique as introduced in qgCt1g0HFXS (section 7.5.1) to avoid constructing subtraction terms for $\widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}}, i)$.

Taking only the colour leading contribution as an example, the explicit formula for $C_{1gH}^{0,YS}(\hat{1}_q, i, k_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}})$ is,

$$
\begin{aligned}
C_{1gH}^{0,YS}&(\hat{1}_q, i, k_{\bar{Q}}, \hat{2}_Q, j_{\bar{q}}) = \\
&+ A_{3,q}^0(1, i, k)\, C_{0gH}^0(\bar{1}, (\widetilde{ik}), 2, j)\, J_1^{(2)}(\{p\}_2) \\
&- E_{3,qq' \to qg}^0(1, k, 2)\, B_{2gH}^0(\bar{1}, i, \bar{2}, j)\, J_1^{(2)}(\{p\}_2) \\
&- E_{3,qq' \to qg}^0(2, j, 1)\, B_{2gH}^0(\bar{2}, \bar{1}, i, k)\, J_1^{(2)}(\{p\}_2) \\
&- E_4^0(1, 2, k, i)\, B_{1gH}^0(\bar{1}, \bar{2}, j)\, J_1^{(1)}(\{p\}_1)
\end{aligned}
$$

$$+ E_{3,qq' \to qg}^0(1, k, 2) \, D_{3,qg}^0(\overline{1}, \overline{2}, i) \, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, j) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(1, i, k) \, E_{3,qq' \to qg}^0(\overline{1}, 2, (\widetilde{ik})) \, B_{1gH}^0(\overline{\overline{1}}, \overline{2}, j) \, J_1^{(1)}(\{p\}_1)$$

$$- E_4^0(2, j, 1, i) \, B_{1gH}^0(\overline{2}, \overline{1}, k) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_{3,qq' \to qg}^0(2, j, 1) \, D_{3,gq}^0(\overline{2}, i, \overline{1}) \, B_{1gH}^0(\overline{\overline{2}}, \overline{\overline{1}}, k) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q\bar{q}}^0(1, i, 2) \, E_{3,qq' \to qg}^0(\overline{2}, j, \overline{1}) \, B_{1gH}^0(\overline{\overline{2}}, \overline{\overline{1}}, k) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q\bar{q}}^0(1, i, 2) \, E_{3,qq' \to qg}^0(\overline{2}, j, \overline{1}) \, B_{1gH}^0(\overline{\overline{2}}, \overline{\overline{1}}, k) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(1, i, k) \, E_{3,qq' \to qg}^0(2, j, \overline{1}) \, B_{1gH}^0(\overline{2}, \overline{\overline{1}}, (\widetilde{ik})) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(2, i, k) \, E_{3,qq' \to qg}^0(\overline{2}, j, 1) \, B_{1gH}^0(\overline{\overline{2}}, \overline{1}, (\widetilde{ik})) \, J_1^{(1)}(\{p\}_1)$$

$$+ \Bigg[- S_{1i(\widetilde{ik})}^{IF} + S_{\overline{1}i(\widetilde{ik})}^{IF} - S_{2i(\widetilde{ik})}^{IF} + S_{\overline{2}i(\widetilde{ik})}^{IF} + S_{1i\overline{2}}^{IF} - S_{\overline{1}i\overline{\overline{2}}}^{IF} \Bigg]$$

$$\times E_{3,qq' \to qg}^0(\overline{2}, j, 1) \, B_{1gH}^0(\overline{\overline{2}}, \overline{1}, (\widetilde{ik})) \, J_1^{(1)}(\{p\}_1)$$

$$+ B_4^0(1, k, 2, j) \, B_{1gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,qq' \to qg}^0(1, k, 2) \, A_{3,qg \to qq}^0(\overline{1}, \overline{2}, j) \, B_{1gH}^0(\overline{\overline{1}}, i, \overline{\overline{2}}) \, J_1^{(1)}(\{p\}_1)$$

$$+ H_4^0(1, j, 2, k) \, A_{3gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,qq' \to qg}^0(2, j, 1) \, G_{3,gq \to gg}^0(\overline{1}, \overline{2}, k) \, A_{3gH}^0(\overline{\overline{1}}, i, \overline{\overline{2}}) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,qq' \to qg}^0(1, k, 2) \, G_{3,qg \to gg}^0(\overline{2}, \overline{1}, j) \, A_{3gH}^0(\overline{\overline{2}}, i, \overline{\overline{1}}) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(2, i, j) \, C_{0gH}^0(1, k, \overline{2}, (\widetilde{ij})) \, J_1^{(2)}(\{p\}_2)$$

$$- E_{3,qq' \to qg}^0(1, k, 2) \, B_{2gH}^0(\overline{1}, \overline{2}, i, j) \, J_1^{(2)}(\{p\}_2)$$

$$- E_{3,qq' \to qg}^0(2, j, 1) \, B_{2gH}^0(\overline{2}, i, \overline{1}, k) \, J_1^{(2)}(\{p\}_2)$$

$$- E_4^0(1, k, 2, i) \, B_{1gH}^0(\overline{1}, \overline{2}, j) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_{3,qq' \to qg}^0(1, k, 2) \, D_{3,qg}^0(\overline{1}, i, \overline{2}) \, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, j) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q\bar{q}}^0(2, i, 1) \, E_{3,qq' \to qg}^0(\overline{1}, k, \overline{2}) \, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, j) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q\bar{q}}^0(2, i, 1) \, E_{3,qq' \to qg}^0(\overline{1}, k, \overline{2}) \, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, j) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(2, i, j) \, E_{3,qq' \to qg}^0(1, k, \overline{2}) \, B_{1gH}^0(\overline{1}, \overline{\overline{2}}, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(1, i, j) \, E_{3,qq' \to qg}^0(\overline{1}, k, 2) \, B_{1gH}^0(\overline{\overline{1}}, \overline{2}, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- \Bigg[+ S_{\overline{1}i(\widetilde{ij})}^{IF} - S_{\overline{\overline{1}}i(\widetilde{ij})}^{IF} + S_{2i(\widetilde{ij})}^{IF} - S_{\overline{2}i(\widetilde{ij})}^{IF} - S_{2i\overline{1}}^{IF} + S_{\overline{2}i\overline{\overline{1}}}^{IF} \Bigg]$$

$$\times E_{3,qq' \to qg}^0(\overline{1}, k, 2) \, B_{1gH}^0(\overline{\overline{1}}, \overline{2}, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- E_4^0(2, 1, j, i) \, B_{1gH}^0(\overline{2}, \overline{1}, k) \, J_1^{(1)}(\{p\}_1)$$

$$+ E^0_{3,qq'\to qg}(2,j,1)\, D^0_{3,gq}(\overline{2},\overline{1},i)\, B^0_{1gH}(\overline{\overline{2}},\overline{\overline{1}},k)\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,q}(2,i,j)\, E^0_{3,qq'\to qg}(\overline{2},1,(\widetilde{ij}))\, B^0_{1gH}(\overline{\overline{2}},\overline{1},k)\, J^{(1)}_1(\{p\}_1)$$

$$+ B^0_4(2,j,1,k)\, B^0_{1gH}(\overline{2},i,\overline{1})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,qq'\to qg}(2,j,1)\, A^0_{3,qg\to qq}(\overline{2},\overline{1},k)\, B^0_{1gH}(\overline{\overline{2}},i,\overline{\overline{1}})\, J^{(1)}_1(\{p\}_1)$$

$$+ H^0_4(1,j,2,k)\, A^0_{3gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,qq'\to qg}(2,j,1)\, G^0_{3,gq\to gg}(\overline{1},\overline{2},k)\, A^0_{3gH}(\overline{\overline{1}},i,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,qq'\to qg}(1,k,2)\, G^0_{3,qg\to gg}(\overline{2},\overline{1},j)\, A^0_{3gH}(\overline{\overline{2}},i,\overline{\overline{1}})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,qq'\to qg}(2,j,1)\, A^0_{3,q}(\overline{2},i,k)\, B^0_{1gH}(\overline{\overline{2}},\overline{1},(\widetilde{ik}))\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,qq'\to qg}(1,k,2)\, A^0_{3,q}(\overline{1},i,j)\, B^0_{1gH}(\overline{\overline{1}},\overline{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1). \qquad (9.5.11)$$

Explicit formulae for the other antenna subtraction terms in table 9.3 can be found in appendix E.1. The double real subtraction terms fit the general structure described in section 4.3.3.

To numerically test that the antenna subtraction terms given in Eq.(9.5.10) remove the implicit IR divergences in Eq.(9.4.7) correctly, we use the same spike plots for double real contribution defined in section 6.6.1 to test that the subtraction terms converge to the matrix elements when approaching various double and single unresolved limits at NNLO.

9.5.2 Real-virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.5.3, one can construct the real-virtual subtraction term, $\mathrm{d}\hat{\sigma}^T_{qq}$, which removes both the explicit and implicit IR divergences from Eq.(9.4.8),

$$\mathrm{d}\hat{\sigma}^T_{qq} = \mathcal{N}_{qq}\,\mathcal{N}^{RV}_{NNLO}\,\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{$$

$$+ \left[C^{1,YT}_{0gH}(\hat{1}_q,4_{\bar{Q}},\hat{2}_Q,3_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}^{1,YT}_{0gH}(\hat{1}_q,4_{\bar{Q}},\hat{2}_Q,3_{\bar{q}}) + \frac{N_f}{N}\widehat{C}^{1,YT}_{0gH}(\hat{1}_q,4_{\bar{Q}},\hat{2}_Q,3_{\bar{q}}) \right]$$

$$- \frac{1}{2N}\left[D^{1,YT}_{0gH}(\hat{1}_q,3_{\bar{q}},\hat{2}_q,4_{\bar{q}}) + \frac{1}{N^2}\widetilde{D}^{1,YT}_{0gH}(\hat{1}_q,3_{\bar{q}},\hat{2}_q,4_{\bar{q}}) \right]\Bigg\} \qquad (9.5.12)$$

The corresponding relationships between the subtraction terms, file name in the `NNLOJET` maple script and matrix elements (or combinations of subtraction terms)

subtraction term	maple	matrix element
$C_{0gH}^{1,YT}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$	qqpC0g1HYT	$C_{0gH}^{1}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$
$\widetilde{C}_{0gH}^{1,YT}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$	qqpCt0g1HYT	$\widetilde{C}_{0gH}^{1}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$
$\widehat{C}_{0gH}^{1,YT}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$	qqpCh0g1HYT	$\widehat{C}_{0gH}^{1}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$
$D_{0gH}^{1,YT}(\hat{1}_q, i_{\bar{q}}, \hat{2}_q, j_{\bar{q}})$	qqD0g1HYT	$D_{0gH}^{1}(\hat{1}_q, i_{\bar{q}}, \hat{2}_q, j_{\bar{q}})$
$\widetilde{D}_{0gH}^{1,YT}(\hat{1}_q, i_{\bar{q}}, \hat{2}_q, j_{\bar{q}})$	qqDt0g1HYT	$\widetilde{D}_{0gH}^{1}(\hat{1}_q, i_{\bar{q}}, \hat{2}_q, j_{\bar{q}})$

Table 9.4: NNLO antenna subtraction terms for real-virtual contributions in $qq \rightarrow H$+jet process and their relation to the matrix elements

in Eq.(9.4.8) and (9.5.12) are summarised in table 9.4.

The explicit formula for $C_{0gH}^{1,YT}(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}})$ is,

$$
\begin{aligned}
C_{0gH}^{1,YT}&(\hat{1}_q, j_{\bar{Q}}, \hat{2}_Q, i_{\bar{q}}) = \\
&- \left[+ J_{2,QQ}^{1,IF}(s_{1j}) + J_{2,QQ}^{1,FI}(s_{2i}) \right] C_{0gH}^0(1,j,2,i) \, J_1^{(2)}(\{p\}_2) \\
&- E_{3,qq' \rightarrow qg}^0(1,j,2) \left[B_{1gH}^1(\overline{1},\overline{2},i) \, \delta(1-x_1) \, \delta(1-x_2) \right. \\
&+ \left(+ J_{2,QG}^{1,II}(s_{\overline{12}}) + J_{2,QG}^{1,FI}(s_{i\overline{2}}) \right) E_{3,qq' \rightarrow qg}^0(1,j,2) \, B_{1gH}^0(\overline{1},\overline{2},i) \bigg] J_1^{(1)}(\{p\}_1) \\
&- \left[E_{3,qq' \rightarrow qg}^1(1,j,2) \, \delta(1-x_1) \, \delta(1-x_2) + \left(+ J_{2,QQ}^{1,II}(s_{12}) \right.\right. \\
&\left. + J_{2,QQ}^{1,IF}(s_{1j}) - 2J_{2,QG}^{1,II}(s_{\overline{12}}) \right) E_{3,qq' \rightarrow qg}^0(1,j,2) \bigg] B_{1gH}^0(\overline{1},\overline{2},i) \, J_1^{(1)}(\{p\}_1) \\
&- E_{3,qq' \rightarrow qg}^0(2,i,1) \left[B_{1gH}^1(\overline{2},\overline{1},j) \, \delta(1-x_1) \, \delta(1-x_2) \right. \\
&+ \left(+ J_{2,GQ}^{1,II}(s_{\overline{12}}) + J_{2,GQ}^{1,IF}(s_{j\overline{1}}) \right) E_{3,qq' \rightarrow qg}^0(2,i,1) \, B_{1gH}^0(\overline{2},\overline{1},j) \bigg] J_1^{(1)}(\{p\}_1) \\
&- \left[E_{3,qq' \rightarrow qg}^1(2,i,1) \, \delta(1-x_1) \, \delta(1-x_2) + \left(+ J_{2,QQ}^{1,II}(s_{12}) \right.\right. \\
&\left. + J_{2,QQ}^{1,FI}(s_{2i}) - 2J_{2,GQ}^{1,II}(s_{\overline{12}}) \right) E_{3,qq' \rightarrow qg}^0(2,i,1) \bigg] B_{1gH}^0(\overline{2},\overline{1},j) \, J_1^{(1)}(\{p\}_1) \\
&- \left[+ J_{2,QG}^{1,II}(s_{\overline{12}}) - J_{2,QQ}^{1,II}(s_{12}) - J_{2,QG}^{1,FI}(s_{i\overline{2}}) + J_{2,QQ}^{1,FI}(s_{2i}) - J_{2,QQ}^{1,IF}(s_{\overline{1}i}) + J_{2,QQ}^{1,IF}(s_{1i}) \right. \\
&+ \left(- \tilde{\mathcal{S}}^{IF}(s_{\overline{12}}, s_{\overline{1}i}, x_{\overline{12},\overline{1}i}) + \mathcal{S}^{IF}(s_{12}, s_{1i}, x_{12,1i}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{2}i}, s_{\overline{1}i}, x_{\overline{2}i,\overline{1}i}) \right. \\
&\left.\left. - \mathcal{S}^{IF}(s_{2i}, s_{1i}, x_{2i,1i}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}i}, s_{\overline{1}i}, 1) - \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) \right) \right]
\end{aligned}
$$

$$\times E^0_{3,qq'\to qg}(1,j,2)\, B^0_{1gH}(\bar{1},\bar{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$-\Bigg[+ J^{1,II}_{2,GQ}(s_{\overline{12}}) - J^{1,II}_{2,QQ}(s_{12}) - J^{1,IF}_{2,GQ}(s_{j\bar{1}}) + J^{1,IF}_{2,QQ}(s_{1j}) - J^{1,FI}_{2,QQ}(s_{\bar{2}j}) + J^{1,FI}_{2,QQ}(s_{2j})$$

$$+\Bigg(-\tilde{\mathcal{S}}^{FI}(s_{\overline{12}},s_{\bar{2}j},x_{\overline{12},\bar{2}j}) + \mathcal{S}^{FI}(s_{12},s_{2j},x_{12,2j}) + \tilde{\mathcal{S}}^{FI}(s_{\bar{1}j},s_{\bar{2}j},x_{\bar{1}j,\bar{2}j})$$

$$-\mathcal{S}^{FI}(s_{1j},s_{2j},x_{1j,2j}) + \tilde{\mathcal{S}}^{FI}(s_{\bar{2}j},s_{\bar{2}j},1) - \mathcal{S}^{FI}(s_{2j},s_{2j},1)\Bigg)\Bigg]$$

$$\times E^0_{3,qq'\to qg}(2,i,1)\, B^0_{1gH}(\bar{2},\bar{1},j)\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\, B^0_{2gH}(2,1,j,i)\, J^{(2)}_1(\{p\}_2)$$

$$+J^{1,II}_{2,GQ,q'q\to gq}(s_{\overline{12}})\, d^0_{3,g}(i,j,1)\, B^0_{1gH}(2,\bar{1},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\, G^0_{3,gq\to gg}(1,i,2)\, A^0_{3gH}(\bar{1},\bar{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\, B^0_{2gH}(2,j,1,i)\, J^{(2)}_1(\{p\}_2)$$

$$+J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\, D^0_{3,gq}(2,j,1)\, B^0_{1gH}(\bar{2},\bar{1},i)\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\, A^0_{3,gq\to qq}(2,1,i)\, B^0_{1gH}(\bar{2},j,\bar{1})\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\, G^0_{3,gq\to gg}(1,i,2)\, A^0_{3gH}(\bar{1},\bar{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$+\Bigg[-J^{1,II}_{2,GQ,q'q\to gq}(s_{\overline{12}}) + J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\Bigg] D^0_{3,gq}(2,j,1)\, B^0_{1gH}(\bar{2},\bar{1},i)\, J^{(1)}_1(\{p\}_1)$$

$$+\Bigg[+J^{1,II}_{2,GQ,q'q\to gq}(s_{1\bar{2}}) - J^{1,II}_{2,GQ,q'q\to gq}(s_{12})\Bigg] A^0_{3,q}(2,j,i)\, B^0_{1gH}(\bar{2},1,(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,QG,qq'\to qg}(s_{12})\, B^0_{2gH}(1,2,i,j)\, J^{(2)}_1(\{p\}_2)$$

$$+J^{1,II}_{2,QG,qq'\to qg}(s_{1\bar{2}})\, d^0_{3,g}(j,i,2)\, B^0_{1gH}(1,\bar{2},(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,QG,qq'\to qg}(s_{12})\, G^0_{3,qg\to gg}(2,j,1)\, A^0_{3gH}(\bar{1},\bar{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,QG,qq'\to qg}(s_{12})\, B^0_{2gH}(1,i,2,j)\, J^{(2)}_1(\{p\}_2)$$

$$+J^{1,II}_{2,QG,qq'\to qg}(s_{12})\, D^0_{3,qg}(1,i,2)\, B^0_{1gH}(\bar{1},\bar{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,QG,qq'\to qg}(s_{12})\, A^0_{3,qg\to qq}(1,2,j)\, B^0_{1gH}(\bar{1},i,\bar{2})\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,II}_{2,QG,qq'\to qg}(s_{12})\, G^0_{3,qg\to gg}(2,j,1)\, A^0_{3gH}(\bar{1},\bar{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+\Bigg[-J^{1,II}_{2,QG,qq'\to qg}(s_{\overline{12}}) + J^{1,II}_{2,QG,qq'\to qg}(s_{12})\Bigg] D^0_{3,qg}(1,i,2)\, B^0_{1gH}(\bar{1},\bar{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$+\Bigg[+J^{1,II}_{2,QG,qq'\to qg}(s_{\bar{1}2}) - J^{1,II}_{2,QG,qq'\to qg}(s_{12})\Bigg] A^0_{3,q}(1,i,j)\, B^0_{1gH}(\bar{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1).$$

$$(9.5.13)$$

Explicit formulas for the other antenna subtraction terms in table 9.4 can be found in appendix E.2. The real-virtual subtraction terms fit the general structure

subtraction term	maple	matrix element
$C_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}})$	`qqpC0g1HXU`	IR safe
$\widetilde{C}_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}})$	`qqpCt0g1HXU`	IR safe
$\widehat{C}_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}})$	`qqpCh0g1HXU`	IR safe
$D_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}})$	`qqD0g1HXU`	IR safe
$\widetilde{D}_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}})$	`qqDt0g1HXU`	IR safe

Table 9.5: NNLO antenna subtraction terms for double virtual contributions in $qq \to H$+jet process and their relation to the matrix elements

described in section 4.5.3.

To numerically test that the antenna subtraction terms given in Eq.(9.5.12) remove the implicit IR divergences in Eq.(9.4.8) correctly, we use the same spike plots for defined in section 6.6.2 to test that the subtraction terms remove the explicit IR divergence of the matrix elements in any phase space point and converge to the matrix elements when approaching various single unresolved limits at the real-virtual level.

9.5.3   Double virtual subtraction terms

Using the NNLO antenna subtraction method introduced in section 4.7.2, one can combine the integrated antenna subtraction terms from $d\hat{\sigma}_{NNLO}^S$ and $d\hat{\sigma}_{NNLO}^T$ with double virtual mass factorization terms to the following double virtual subtraction term,

$$
\begin{aligned}
d\hat{\sigma}_{qq}^U = \mathcal{N}_{qq}\,\mathcal{N}_{NNLO}^{VV}\, \frac{dx_1}{x_1}\frac{dx_2}{x_2} d\Phi_{H+1}(p_3, p_H; p_1, p_2) \Bigg\{ \\
- C_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) + \frac{1}{N^2}\widetilde{C}_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) - \frac{N_F}{N}\widehat{C}_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) \\
+ \frac{1}{N}D_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) - \frac{1}{N^3}\widetilde{D}_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) \Bigg\} J_1^{(1)}(p_3)
\end{aligned}
\tag{9.5.14}
$$

The relationships between the subtraction terms and file names in the `NNLOJET` maple script in (9.5.14) are summarised in table 7.5.

The explicit formula for $C_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}})$ in terms of integrated antennae is,

$$
\begin{aligned}
C_{1gH}^{1,XU}(1_q, 2, i_{\bar{q}}) = \\
& -\left[- \mathcal{E}_{3,qq'\to qg}^0(s_{12}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \right] B_{1gH}^1(1,2,3) \\
& -\left[- \mathcal{D}_{3,g\to g}^0(s_{23}) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(z_2) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) \right. \\
& \quad \left. - S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \mathcal{D}_{3,g\to g}^0(s_{23}) + \frac{1}{2}\,S_{q\to g}\,\Gamma_{gg}^{(1)}(z_2) \otimes \Gamma_{gq}^{(1)}(z_2) \right] B_{1gH}^0(1,2,3) \\
& -\left[- \mathcal{D}_{3,qg}^0(s_{12}) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(z_2) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) \right. \\
& \quad + \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \mathcal{D}_{3,qg}^0(s_{12}) \\
& \quad \left. + S_{q\to g}\Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{gq}^{(1)}(z_2) + \frac{1}{2}\,S_{q\to g}\,\Gamma_{gg}^{(1)}(z_2) \otimes \Gamma_{gq}^{(1)}(z_2) \right] B_{1gH}^0(1,2,3) \\
& -\left[- \mathcal{E}_{4,qq'}^0(s_{12}) - \mathcal{E}_{4,q\bar{q}'}^0(s_{12}) - \mathcal{E}_{3,qq'}^1(s_{12}) - \frac{b_0}{\epsilon}\left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} \mathcal{E}_{3,qq'\to qg}^0(s_{12}) \right. \\
& \quad + \frac{b_0}{\epsilon}\,\mathcal{E}_{3,qq'\to qg}^0(s_{12}) + 2\,\mathcal{D}_{3,qg}^0(s_{12}) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) + \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) \\
& \quad - \Gamma_{gg}^{(1)}(z_2) \otimes \mathcal{E}_{3,qq'\to qg}^0(s_{12}) - \frac{1}{2}\,S_{q\to g}\,\Gamma_{gg}^{(1)}(z_2) \otimes \Gamma_{gq}^{(1)}(z_2) \\
& \quad \left. + \frac{1}{2}\,S_{q\to g}\,\Gamma_{qq}^{(1)}(z_2) \otimes \Gamma_{gq}^{(1)}(z_2) + S_{q\to g}\frac{b_0}{\epsilon}\Gamma_{gq}^{(1)}(z_2) - S_{q\to g}\bar{\Gamma}_{gq}^{(2)}(z_2) \right] B_{1gH}^0(1,2,3) \\
& -\left[- \mathcal{E}_{3,q'q\to gq}^0(s_{12}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \right] B_{1gH}^1(2,1,3) \\
& -\left[- \mathcal{D}_{3,g\to g}^0(s_{13}) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) \right. \\
& \quad \left. - S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{D}_{3,g\to g}^0(s_{13}) + \frac{1}{2}\,S_{q\to g}\,\Gamma_{gg}^{(1)}(z_1) \otimes \Gamma_{gq}^{(1)}(z_1) \right] B_{1gH}^0(2,1,3) \\
& -\left[- \mathcal{D}_{3,gq}^0(s_{12}) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) + \frac{1}{2}\,\Gamma_{gg}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) \right. \\
& \quad + \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{D}_{3,gq}^0(s_{12}) \\
& \quad \left. + S_{q\to g}\Gamma_{qq}^{(1)}(z_2) \otimes \Gamma_{gq}^{(1)}(z_1) + \frac{1}{2}\,S_{q\to g}\,\Gamma_{gg}^{(1)}(z_1) \otimes \Gamma_{gq}^{(1)}(z_1) \right] B_{1gH}^0(2,1,3) \\
& -\left[- \mathcal{E}_{4,q'q}^0(s_{12}) - \mathcal{E}_{4,\bar{q}'q}^0(s_{12}) - \mathcal{E}_{3,q'q}^1(s_{12}) - \frac{b_0}{\epsilon}\,\mathcal{E}_{3,q'q\to gq}^0(s_{12})\left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} \right. \\
& \quad + \frac{b_0}{\epsilon}\,\mathcal{E}_{3,q'q\to gq}^0(s_{12}) + 2\,\mathcal{D}_{3,gq}^0(s_{12}) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) + \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) \\
& \quad \left. - \Gamma_{gg}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'q\to gq}^0(s_{12}) - \frac{1}{2}\,S_{q\to g}\,\Gamma_{gg}^{(1)}(z_1) \otimes \Gamma_{gq}^{(1)}(z_1) \right.
\end{aligned}
$$

$$+ \frac{1}{2} S_{q \to g} \Gamma^{(1)}_{qq}(z_1) \otimes \Gamma^{(1)}_{gq}(z_1) + \frac{b_0}{\epsilon} S_{q \to g} \Gamma^{(1)}_{gq}(z_1) - S_{q \to g} \bar{\Gamma}^{(2)}_{gq}(z_1) \Bigg] B^0_{1gH}(2,1,3)$$

$$- \Bigg[+ \mathcal{B}^0_{4,qq'}(s_{12}) + \mathcal{B}^0_{4,q'q}(s_{12}) + S_{q \to g} \Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12})$$

$$+ S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{A}^0_{3,gq \to qq}(s_{12}) + \frac{1}{2} \Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{qg}(z_1) + \frac{1}{2} \Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{qg}(z_2)$$

$$- \frac{1}{4} \bar{\Gamma}^{(2)}_{q\bar{q}}(z_1) - \frac{1}{4} \bar{\Gamma}^{(2)}_{q\bar{q}}(z_2) \Bigg] B^0_{1gH}(1,3,2)$$

$$- \Bigg[+ 2 \mathcal{H}^0_{4,qq'}(s_{12}) + 2 S_{q \to g} \Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{G}^0_{3,qg \to gg}(s_{12}) + 2 \Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{G}^0_{3,gq \to gg}(s_{12})$$

$$+ 2 S^2_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gq}(z_2) \Bigg] A^0_{3gH}(1,2,3). \tag{9.5.15}$$

Using the dipole functions introduced in section 3.5.2 and 4.7.2, Eq. (9.5.15) can be re-expressed as,

$$C^{1,XU}_{1gH}(1_q, 2, i_{\bar{q}}) =$$

$$- J^{1,II}_{2,QG,qq' \to qg}(s_{12}) B^1_{1gH}(1,2,3)$$

$$- \Big[J^{1,FI}_{2,GQ}(s_{23}) + J^{1,II}_{2,QG}(s_{12}) \Big] \otimes J^{1,II}_{2,QG,qq' \to qg}(s_{12}) B^0_{1gH}(1,2,3)$$

$$- J^{2,II}_{2,QG,q \to g}(s_{12}) B^0_{1gH}(1,2,3)$$

$$- J^{1,II}_{2,GQ,q'q \to gq}(s_{12}) B^1_{1gH}(2,1,3)$$

$$- \Big[J^{1,IF}_{2,GQ}(s_{13}) + J^{1,II}_{2,GQ}(s_{12}) \Big] \otimes J^{1,II}_{2,GQ,q'q \to gq}(s_{12}) B^0_{1gH}(2,1,3)$$

$$- J^{2,II}_{2,GQ,q \to g}(s_{12}) B^0_{1gH}(2,1,3)$$

$$- J^{2,II}_{2,QQ,q \to g \to q}(s_{12}) B^0_{1gH}(1,3,2)$$

$$- J^{2,II}_{2,GG,q \to g,q \to g}(s_{12}) A^0_{3gH}(1,2,3), \tag{9.5.16}$$

where the NNLO integrated antenna functions are,

$$J^{2,II}_{2,QG,q \to g}(s_{12}) = - \mathcal{E}^0_{4,qq'}(s_{12}) - \mathcal{E}^0_{4,q\bar{q}'}(s_{12}) - \mathcal{E}^1_{3,qq'}(s_{12}) - \frac{b_0}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{E}^0_{3,qq' \to qg}(s_{12})$$

$$+ \frac{b_0}{\epsilon} \mathcal{E}^0_{3,qq' \to qg}(s_{12}) + 2 \mathcal{D}^0_{3,qg}(s_{12}) \otimes \mathcal{E}^0_{3,qq' \to qg}(s_{12})$$

$$+ \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{E}^0_{3,qq' \to qg}(s_{12}) - \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{E}^0_{3,qq' \to qg}(s_{12})$$

$$- \frac{1}{2} S_{q \to g} \Gamma^{(1)}_{gg}(z_2) \otimes \Gamma^{(1)}_{gq}(z_2) + \frac{1}{2} S_{q \to g} \Gamma^{(1)}_{qq}(z_2) \otimes \Gamma^{(1)}_{gq}(z_2)$$

$$+ S_{q \to g} \frac{b_0}{\epsilon} \Gamma^{(1)}_{gq}(z_2) - S_{q \to g} \bar{\Gamma}^{(2)}_{gq}(z_2) \tag{9.5.17}$$

$$J^{2,II}_{2,QQ,q \to g \to q}(s_{12}) = + \mathcal{B}^0_{4,qq'}(s_{12}) + \mathcal{B}^0_{4,q'q}(s_{12}) + S_{q \to g} \Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12})$$

$$+ \; S_{q \to g} \Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{A}_{3,gq \to qq}^0(s_{12}) + \frac{1}{2} \; \Gamma_{gq}^{(1)}(z_1) \otimes \Gamma_{qg}^{(1)}(z_1)$$

$$+ \frac{1}{2} \; \Gamma_{gq}^{(1)}(z_2) \otimes \Gamma_{qg}^{(1)}(z_2) - \frac{1}{4} \; \bar{\Gamma}_{q\bar{q}}^{(2)}(z_1) - \frac{1}{4} \; \bar{\Gamma}_{q\bar{q}}^{(2)}(z_2) \qquad (9.5.18)$$

$$J_{2,GG,q \to g,q \to g}^{2,II}(s_{12}) = + \, 2 \, \mathcal{H}_{4,qq'}^0(s_{12}) + 2 \, S_{q \to g} \Gamma_{gq}^{(1)}(z_2) \otimes \mathcal{G}_{3,qg \to gg}^0(s_{12})$$

$$+ \, 2 \, \Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{G}_{3,gq \to gg}^0(s_{12}) + 2 \, S_{q \to g}^2 \Gamma_{gq}^{(1)}(z_1) \otimes \Gamma_{gq}^{(1)}(z_2).$$

$$(9.5.19)$$

Explicit formulas for other antenna subtraction terms in table 9.5 can be found in appendix E.3. All the antenna subtraction terms in table 9.5 contain the initial state identity changing (idc) behaviour and thus have no corresponding double virtual contributions from the matrix elements. We use the same FORM program to analytically check that these idc terms are explicitly IR finite.

For $\bar{q}\bar{q}$ initiated process, the NNLO subtraction terms are related as in Eq. (9.4.9). The other NNLO quark (anti-)quark initiated channels ($qQ, q\bar{Q}, \bar{q}\bar{Q}$ and $\bar{Q}q$) are not independent from the $q\bar{q}$ and qq channels. The details for the matrix elements and antenna subtraction term for those channels are introduced in appendix A.

Chapter 10

Numerical Results

In this chapter I will discuss the numerical implementation of the parton level $pp \rightarrow H+$ jet processes up to NNLO. Using the numerically stable matrix elements introduced in chapter 5 and the antenna subtraction terms from chapter 6, 7, 8 and 9, we can compute the exclusive $pp \rightarrow H +$ jet cross section at NNLO

$$
\begin{aligned}
\hat{\sigma}_{NNLO} &= \int_{\mathrm{d}\Phi_{H+3}} \left[\mathrm{d}\hat{\sigma}_{NNLO}^{RR} - \mathrm{d}\hat{\sigma}_{NNLO}^{S} \right] \\
&+ \int_{\mathrm{d}\Phi_{H+2}} \left[\mathrm{d}\hat{\sigma}_{NNLO}^{RV} - \mathrm{d}\hat{\sigma}_{NNLO}^{T} \right] \\
&+ \int_{\mathrm{d}\Phi_{H+1}} \left[\mathrm{d}\hat{\sigma}_{NNLO}^{VV} - \mathrm{d}\hat{\sigma}_{NNLO}^{U} \right] , \qquad (10.0.1)
\end{aligned}
$$

where each of the square brackets is finite and well behaved in the infrared singular regions. The convolution of the partonic cross section and the parton distribution functions as in Eq.(1.4.43) yields partonic predictions for final state observables at a hadron collider like the LHC.

Analytic results for the matrix elements and the corresponding antenna subtraction terms have been discussed in the earlier chapters (and appendices) for $pp \rightarrow H$ + jet processes at LO, NLO and NNLO and we now have in principle all of the ingredients necessary to develop a parton-level event generator for Higgs boson-plus-jet production through to NNLO. Currently, we have implemented all purely gluonic subprocesses [84]. With this numerical implementation, we can compute any infrared safe observable related to $H+$ jet final states to NNLO accuracy. The Higgs boson decay to two photons is included, such that realistic event selection cuts on

the photons can be applied. Renormalization and factorization scales can be chosen on an event-by-event basis.

For our numerical computations, we use the NNPDF2.3 parton distribution functions [172] with the corresponding value of $\alpha_s(M_Z) = 0.118$ at NNLO, and $m_H = 125$ GeV. Default values for the factorization and renormalization scales are $\mu_F = \mu_R = m_H$, with theory errors estimated from the envelope of the scale variation between $m_H/2$ and $2m_H$. To compare with previously obtained results for the total cross section for purely gluonic $H+$ jet production at $\sqrt{s} = 8$ TeV, we use the same cuts as in [83]: jets are reconstructed in the k_T algorithm with $R = 0.5$, and accepted if $p_T > 30$ GeV. With this, we obtain the total cross section at different perturbative orders as

$$
\begin{aligned}
\sigma_{LO} &= 2.72^{+1.22}_{-0.78} \text{ pb}, \\
\sigma_{NLO} &= 4.38^{+0.76}_{-0.74} \text{ pb}, \\
\sigma_{NNLO} &= 6.34^{+0.28}_{-0.49} \text{ pb},
\end{aligned}
\tag{10.0.2}
$$

in very good agreement with [83].

In our kinematical distributions and ratio plots, the error band describes the scale variation envelope as described above, where the denominator in the ratio plots is evaluated at fixed central scale, such that the band only reflects the variation of the numerator. Error bars on the distributions indicate the numerical integration errors on individual bins.

The transverse momentum distribution of the Higgs boson is particularly important for discriminating between different Higgs production modes. A first measurement has been studied by ATLAS [92], demonstrating the feasibility and future experimental prospects for this observable. It has been computed previously to NLO [173], combined with resummation to third logarithmic order (NNLL) [174–178]. In Figure 10.1, we observe that the Higgs boson transverse momentum distribution receives NLO corrections which change both normalisation and shape compare to LO contributions. The NLO and LO corrections are validated against the MCFM program. We also observe that the Higgs boson transverse momentum distribution receives sizable NNLO corrections throughout the whole range in p_T, which enhance

the NLO cross section by a quasi constant factor of about 1.4, slightly decreasing towards higher p_T. Using the same scale variations pattern as for the inclusive cross section above, we observe that the p_T distribution of the Higgs boson has a residual NNLO theory uncertainty ranging between 5% and 16%. At high values of p_T, the effective theory approximation used for the coupling of the Higgs boson to gluons breaks down, since the large momentum transfer in the process starts resolving the top quark loop. Consequently, one expects top quark mass effects for $p_T > m_t$ to be more important than the higher order corrections in the effective theory [32, 33]. Nevertheless, within the effective field theory in high p_T region, we observe large and negative NLO effects such that NLO/LO ratio is much less than one. The NNLO effects are small compare to LO but produces a large NNLO/NLO ratio with large numerical uncertainty.

We note that at leading order $p_{T,H}$ is kinematically forced to be equal to the transverse momentum of the jet, and is consequently larger than the transverse momentum cut on the jet. At higher orders, higher multiplicity final states are allowed and this kinematical restriction no longer applies. A similar pattern to the Higgs boson p_T distribution, is also observed for the leading jet, Figure 10.2, which displays a slightly smaller scale uncertainty amounting up to 12%, and displays rising NNLO corrections for very large values of p_T, again likely beyond the applicability of the effective theory approximation.

The rapidity distribution of the Higgs boson and the leading jet are displayed in Figures 10.3 and 10.4 respectively. In both cases, we observe that the NLO corrections are largest at central rapidity, while becoming moderate at larger rapidities, while the ratio NNLO/NLO remains rather constant throughout the rapidity range. The jet cut at high rapidity implies large momentum flow into the effective Hgg coupling where our prediction ability is limited. The residual theory uncertainty at NNLO is quasi constant for both distributions, and amounts to 9%. Both the transverse momentum and rapidity distributions highlight the fact that the NNLO corrections to $H + jet$ production in the gluon-only channel substantially enhance the normalization of NLO predictions, while not modifying the NLO shape, except around the production threshold.

Our numerical implementation in purely gluonic subprocesses for $pp \to H + jet$ through to NNLO reveals substantial NNLO corrections in the transverse momentum and rapidity distributions of the Higgs boson and the leading jet. However, the shapes of the distributions do not change dramatically from NLO to NNLO, except around the production threshold in p_T. For all of the observables considered here, we observed a reduction of the respective uncertainties in the theory prediction due to variations of the factorization and renormalization scales, resulting in a residual uncertainty of around 9% on the normalization of the distributions. With this program, we could expect to implement processes involving quarks in a similar fashion using the available matrix elements from chapter 5 and antenna subtraction terms from chapter 6, 7, 8 and 9.

(a)

(b)

Figure 10.1: (a) Transverse momentum distribution of the Higgs boson in inclusive $H + jet$ production in pp collisions with $\sqrt{s} = 8$ TeV at LO, NLO, NNLO and (b) Ratios of different perturbative orders, NLO/LO, NNLO/LO and NNLO/NLO.

(a)

(b)
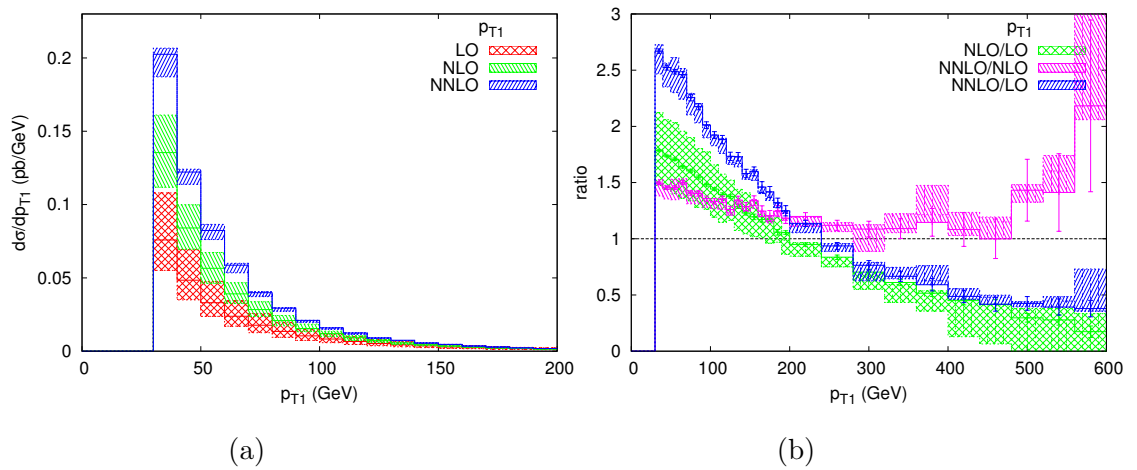
Figure 10.2: (a) Transverse momentum distribution of the leading jet in inclusive $H + jet$ production in pp collisions with $\sqrt{s} = 8$ TeV at LO, NLO, NNLO and (b) Ratios of different perturbative orders, NLO/LO, NNLO/LO and NNLO/NLO.

(a)   (b)
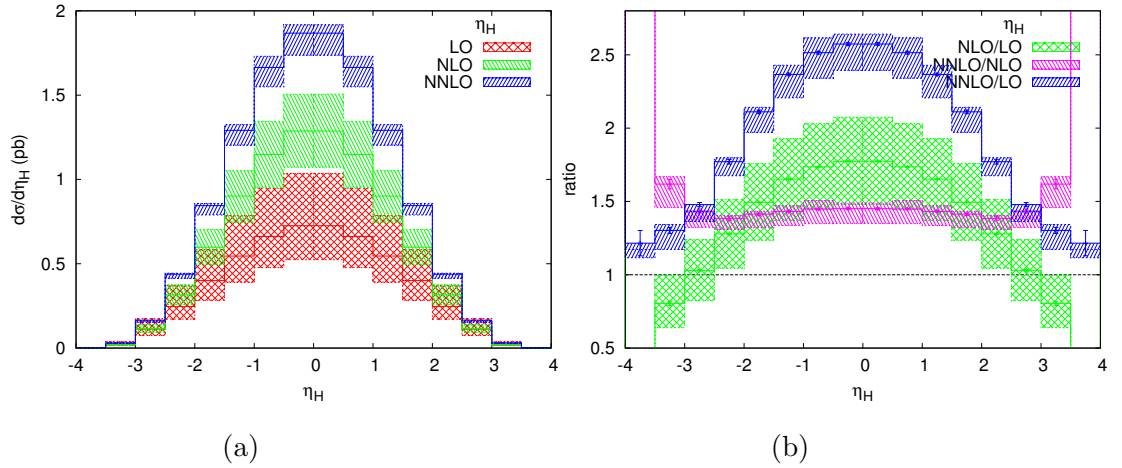
Figure 10.3: (a) Rapidity distribution of the Higgs boson in inclusive $H + jet$ production in pp collisions with $\sqrt{s} = 8$ TeV at LO, NLO, NNLO and (b) Ratios of different perturbative orders, NLO/LO, NNLO/LO and NNLO/NLO.
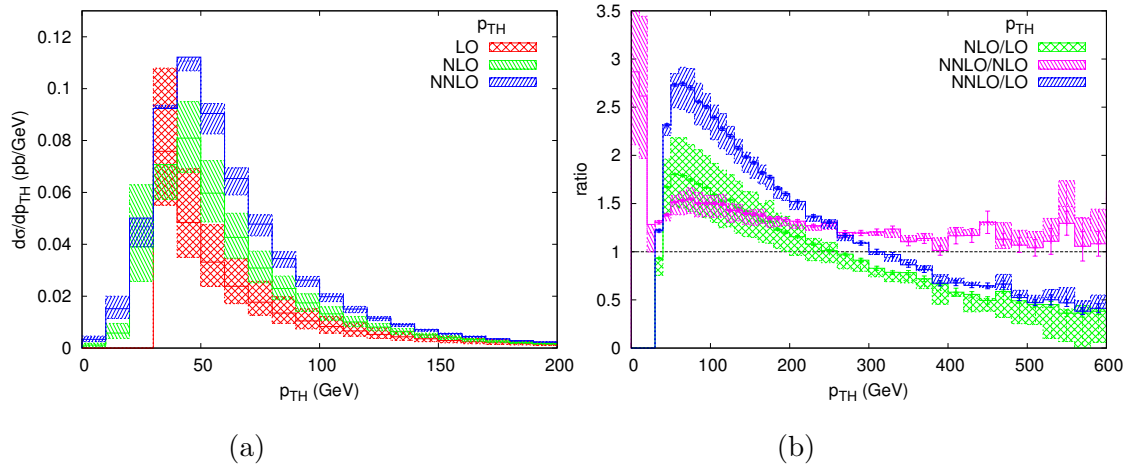

(a)   (b)
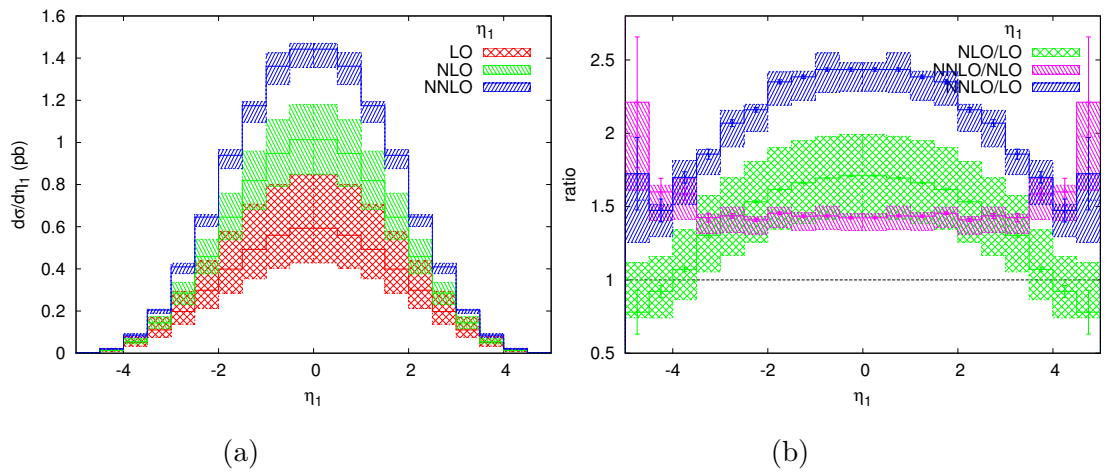
Figure 10.4: (a) Rapidity distribution of the leading jet in inclusive $H + jet$ production in pp collisions with $\sqrt{s} = 8$ TeV at LO, NLO, NNLO and (b) ratios of different perturbative orders, NLO/LO, NNLO/LO and NNLO/NLO.

Chapter 11

Conclusions

At the Large Hadron Collider, a new boson signal has been discovered and current studies provide strong evidence that this new boson looks very much like the Standard Model Higgs boson. Contemporary Higgs boson related research now focuses on the detailed measurement of the couplings and dynamic properties of this new particle. Boosted Higgs boson dynamics, differential cross sections and jet-bin analysis at high precision are desired to improve signal to background ratio, test $Hb\bar{b}$ and $Ht\bar{t}$ coupling, analysis jet-veto efficiency as well as many other phenomenology studies. In this thesis we have studied the hadronic production of a Higgs boson in association with one jet at NNLO accuracy. We perform the first fully differential study of $pp \to H$+jet at NNLO accuracy which hopefully will help improve our understanding of the new particle and electroweak symmetry breaking.

In this thesis, we concentrated on three main aspects relevant for $pp \to H$+jet at NNLO. First, numerically stable matrix elements for Higgs boson plus five partons at tree level and Higgs boson plus four partons at one-loop level were calculated and tested in the various single and double unresolved phase space regions. Second, we constructed and tested the subtraction terms in the antenna subtraction formalism for $pp \to H$+jet processes through to NNLO in all parton channels. Finally, with all of the ingredients necessary to develop a parton-level event generator through to NNLO in place, we implemented the purely gluonic subprocesses (the dominant channel) as an example for phenomenological studies.

In chapter 5, the scattering matrix elements relevant for the NNLO corrections

to the $pp \to H+$ jet processes are discussed. Although the tree and one-loop level matrix elements involved have been calculated by different groups before, these expressions had not been tested for numerical stability in the unresolved phase regions relevant for NNLO calculations. We used the BCFW method to calculated tree-level matrix elements for Higgs boson plus five partons where the expressions are more compact and stable. For one-loop matrix elements of Higgs boson plus four partons, numerical instability issues are solved by rewriting the analytical expressions to avoid cancellations of divergent terms in the single soft phase space regions. With these numerically stable matrix elements, we can perform meaningful testings for the convergence between matrix elements and subtraction terms in unresolved phase space regions.

In chapter 6, 7, 8, 9 together with appendix A, the differential cross sections and antenna subtraction terms related to $pp \to H+$ jet process for all parton channels are studied. A maple script is developed to convert antenna subtraction terms written in a ".map" format to Fortran and FORM programs for testing IR divergence cancellations and for Monte Carlo simulations. The main challenge for constructing NNLO antenna subtraction terms for $pp \to H+$ jet processes is to regulate the IR divergences involving initial state identity changing (idc) behaviour. New structures in the antenna subtraction framework involving idc limits are introduced. The methodology is summarised in chapter 4 while explicit examples are given in chapters 7 and 9. The new structures for regulating idc divergences fit precisely into the dipole prescription of the antenna subtraction method.

In chapter 10, we implement our analytical formulae of all purely gluonic sub-processes as an example in a parton-level event generator through to NNLO [84]. With all parameter settings the same as used in previously obtained results for the total cross section, our results are in very good agreement with [83]. Our event generator further produce the first fully differential cross sections. We observe substantial NNLO corrections in the transverse momentum and rapidity distributions of the Higgs boson and the leading jet. The shapes of the distributions do not change dramatically from NLO to NNLO, except around the production threshold in p_T. For all of the observables considered here, we observed a reduction of the re-

spective uncertainties in the theory prediction due to variations of the factorization and renormalization scales, resulting in a residual uncertainty of around 9% on the normalization of the distributions.

The next step is to implement processes involving quarks in our event generator. Sizable NNLO corrections from qg initiated channels are expected in the large p_T region. Fully exclusive results for the physical cross section will enable a more detailed comparison with LHC data from Run 1.

It is anticipated that beyond the value of the explicit examples presented here, the techniques and structures derived during the research for this thesis will accelerate the calculations of other processes and provide a greater understanding of precision QCD and Higgs boson phenomenology. For example, the development of the antenna subtraction to more complicated processes in which the identity of the initial state parton changes can be immediately applied to $pp \rightarrow X$+jet processes where X could be one of the Standard Model vector bosons or even Beyond the Standard Model particles which couple to quarks and/or gluons. Because of the universality of the antenna subtraction method and in particular the two-loop integrated antenna functions $\boldsymbol{J}_2^{(2)}$, the various identity preserving and identity changing $\boldsymbol{J}_2^{(2)}$ functions derived in this thesis will also appear in processes such as $pp \rightarrow 2$ jets or $pp \rightarrow X$+2 jets.

Appendix A

Cross section and antenna subtraction terms for $q\bar{Q}$, qQ, $\bar{Q}q$ and $\bar{q}\bar{Q} \to H$+jet processes at NLO and NNLO

A.1 Contributions at NLO

A.1.1 $q\bar{Q}$ initiated cross sections at NLO

The real radiation contribution for the $q\bar{Q} \to H + Q\bar{q}$ process is,

$$
\mathrm{d}\hat{\sigma}^R_{q\bar{Q}} = \mathcal{N}_{q\bar{Q}} \mathcal{N}^R_{NLO} \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)
$$
$$
\times \left\{ \frac{1}{N} C^0_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) \right\} J^{(2)}_1(p_3, p_4). \tag{A.1.1}
$$

There is no virtual contribution from the $q\bar{Q}$ initiated channel.

$$
\mathrm{d}\hat{\sigma}^V_{q\bar{Q}} = 0. \tag{A.1.2}
$$

The NLO antenna subtraction terms for $q\bar{Q}$ initiated channel have been introduced as part of the subtraction terms for the $q\bar{q}$ initiated channels in section 8.3. Here I only list the results for the differential cross section,

$$
\mathrm{d}\hat{\sigma}^S_{q\bar{Q},NLO} = \mathcal{N}_{q\bar{Q}} \mathcal{N}^R_{NLO} \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)
$$

$$\times \left\{ \frac{1}{N} C_{0gH,NLO}^{0,XS}(\hat{1}_q, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) \right\} J_1^{(2)}(p_3, p_4) \tag{A.1.3}$$

$$\mathrm{d}\hat{\sigma}_{q\bar{Q},NLO}^T = \mathcal{N}_{q\bar{Q}} \mathcal{N}_{NLO}^V \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)$$

$$\times \left\{ \frac{1}{N} C_{0gH,NLO}^{0,XT}(\hat{1}_q, \hat{2}, 3_{\bar{q}}) \right\} J_1^{(2)}(p_3, p_4) \tag{A.1.4}$$

A.1.2 $\bar{Q}q$ initiated cross sections at NLO

Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the contributions to the $\bar{Q}q$ initiated channels are related to those from the $q\bar{Q}$ channels,

$$\mathrm{d}\hat{\sigma}_{\bar{Q}q,NLO} = \mathrm{d}\hat{\sigma}_{q\bar{Q},NLO} \qquad (x_1 \leftrightarrow x_2), \tag{A.1.5}$$

where x_1, x_2 are the momentum fractions of the initial state partons as introduced in Eq. (1.4.39).

A.1.3 qQ initiated cross sections at NLO

The real radiation contribution for the $qQ \to H + \bar{Q}\bar{q}$ process is,

$$\mathrm{d}\hat{\sigma}_{qQ}^R = \mathcal{N}_{qQ} \mathcal{N}_{NLO}^R \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)$$

$$\times \left\{ \frac{1}{N} C_{0gH}^0(\hat{1}_q, 3_{\bar{Q}}, \hat{2}_Q, 4_{\bar{q}}) \right\} J_1^{(2)}(p_3, p_4). \tag{A.1.6}$$

There is no virtual contribution from the qQ initiated channel.

$$\mathrm{d}\hat{\sigma}_{qQ}^V = 0. \tag{A.1.7}$$

The NLO antenna subtraction terms for qQ initiated channel have been introduced as part of the subtraction terms for qq initiated channels in section 9.2. Here I only list the results for the differential cross section,

$$\mathrm{d}\hat{\sigma}_{qQ,NLO}^S = \mathcal{N}_{qQ} \mathcal{N}_{NLO}^R \mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)$$

$$\times \left\{ \frac{1}{N} C_{0gH,NLO}^{0,YS}(\hat{1}_q, 3_{\bar{Q}}, \hat{2}_Q, 4_{\bar{q}}) \right\} J_1^{(2)}(p_3, p_4), \tag{A.1.8}$$

$$\mathrm{d}\hat{\sigma}^T_{qQ,NLO} = \mathcal{N}_{qQ}\mathcal{N}^V_{NLO}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)$$

$$\times \left\{ \frac{1}{N}C^{0,YT}_{0gH,NLO}(\hat{1}_q, 3_{\bar{Q}}, \hat{2}_Q, 4_{\bar{q}}) \right\} J^{(2)}_1(p_3, p_4). \tag{A.1.9}$$

A.1.4 $\bar{Q}\bar{q}$ initiated cross sections at NLO

Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the contributions to the $\bar{q}\bar{Q}$ initiated channels are related to those from the qQ channels,

$$\mathrm{d}\hat{\sigma}_{\bar{q}\bar{Q},NLO} = \mathrm{d}\hat{\sigma}_{qQ,NLO}. \tag{A.1.10}$$

A.2 $q\bar{Q}$ initiated cross sections at NNLO

A.2.1 Double real contribution

The double real contribution at NNLO for $q\bar{Q} \to H+$jet comes from the $q\bar{Q} \to H + gQ\bar{q}$ process,

$$\mathrm{d}\hat{\sigma}^{RR}_{q\bar{Q}} = \mathcal{N}_{q\bar{Q}}\mathcal{N}^{RR}_{NNLO}\mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2)\Bigg\{$$

$$+ \left[C^0_{1gH}(\hat{1}_q, 5, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) + C^0_{1gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 4_Q, 5, 3_{\bar{q}}) \right]$$

$$+ \frac{1}{N^2}\left[\widetilde{C}^0_{1gH}(\hat{1}_q, 5, 3_{\bar{q}}, 4_Q, \hat{2}_{\bar{Q}}) + \widetilde{C}^0_{1gH}(\hat{1}_q, 3_{\bar{q}}, 4_Q, 5, \hat{2}_{\bar{Q}}) - \widetilde{\widetilde{C}}^0_{1gH}(\hat{1}_q, 3_{\bar{q}}, 4_Q, \hat{2}_{\bar{Q}}, 5) \right]\Bigg\}$$

$$\times J^{(3)}_1(p_3, p_4, p_5). \tag{A.2.11}$$

A.2.2 Real-virtual contribution

The real-virtual contribution at NNLO for $q\bar{Q} \to H+$jet comes from the $q\bar{Q} \to H + Q\bar{q}$ process,

$$\mathrm{d}\hat{\sigma}^{RV}_{q\bar{Q}} = \mathcal{N}_{q\bar{Q}}\mathcal{N}^{RV}_{NNLO}\mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\Bigg\{$$

$$+ \left[C^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) + \frac{N_f}{N}\widehat{C}^1_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) \right]\Bigg\}$$

$$\times J^{(2)}_1(p_3, p_4). \tag{A.2.12}$$

A.2.3 $q\bar{Q}$ initiated subtraction terms at NNLO

The NNLO antenna subtraction terms for $q\bar{Q}$ initiated channel have been introduced as part of the subtraction terms for $q\bar{q}$ initiated channels in section 8.4.

$$
\mathrm{d}\hat{\sigma}^S_{q\bar{Q}} = \mathcal{N}_{q\bar{Q}}\mathcal{N}^{RR}_{NNLO}\mathrm{d}\Phi_{H+3}(p_3, p_4, p_5, p_H; p_1, p_2)\bigg\{
$$
$$
+ C^{0,YS}_{1gH}(\hat{1}_q, 5, \hat{2}_{\bar{Q}}, 4_Q, 3_{\bar{q}}) + \frac{1}{N^2}\widetilde{C}^{0,YS}_{1gH}(\hat{1}_q, 5, 3_{\bar{q}}, 4_Q, \hat{2}_{\bar{Q}})\bigg\}, \tag{A.2.13}
$$

$$
\mathrm{d}\hat{\sigma}^T_{q\bar{Q}} = \mathcal{N}_{q\bar{Q}}\mathcal{N}^{RV}_{NNLO}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+2}(p_3, p_4, p_H; p_1, p_2)\bigg\{
$$
$$
+ C^{1,YT}_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}^{1,YT}_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}}) + \frac{N_f}{N}\widehat{C}^{1,YT}_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, 3_Q, 4_{\bar{q}})\bigg\}, \tag{A.2.14}
$$

$$
\mathrm{d}\hat{\sigma}^U_{q\bar{Q}} = \mathcal{N}_{q\bar{Q}}\mathcal{N}^{VV}_{NNLO}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)\bigg\{
$$
$$
- C^{1,XU}_{1gH}(1_q, 3, 2_{\bar{q}}) + \frac{1}{N^2}\widetilde{C}^{1,XU}_{1gH}(1_q, 3, 2_{\bar{q}}) - \frac{N_F}{N}\widehat{C}^{1,XU}_{1gH}(1_q, 3, 2_{\bar{q}})\bigg\}J^{(1)}_1(p_3). \tag{A.2.15}
$$

A.3 $\bar{Q}q$ initiated cross sections at NNLO

Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the contributions to the $\bar{Q}q$ initiated channels are related to the $q\bar{Q}$ subtraction terms:

$$
\mathrm{d}\hat{\sigma}_{\bar{Q}q,NNLO} = \mathrm{d}\hat{\sigma}_{q\bar{Q},NNLO} \qquad (x_1 \leftrightarrow x_2), \tag{A.3.16}
$$

where x_1, x_2 are the momentum fractions of the initial state partons as introduced in Eq. (1.4.39).

A.4 qQ initiated cross sections at NNLO

A.4.1   Double real contribution

The double real contribution at NNLO for $qQ \to H$+jet comes from the $qQ \to H + g\bar{Q}\bar{q}$ process,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{qQ}^{RR} =& \mathcal{N}_{qQ}\mathcal{N}_{NNLO}^{RR}\mathrm{d}\Phi_{H+3}(p_3,p_4,p_5,p_H;p_1,p_2)\Bigg\{ \\
&+ \left[C_{1gH}^0(\hat{1}_q,5,3_{\bar{Q}},\hat{2}_Q,4_{\bar{q}}) + C_{1gH}^0(\hat{1}_q,3_{\bar{Q}},\hat{2}_Q,5,4_{\bar{q}}) \right] \\
&+ \frac{1}{N^2} \left[\widetilde{C}_{1gH}^0(\hat{1}_q,5,4_{\bar{q}},\hat{2}_Q,3_{\bar{Q}}) + \widetilde{C}_{1gH}^0(\hat{1}_q,4_{\bar{q}},\hat{2}_Q,5,3_{\bar{Q}}) - \widetilde{\widetilde{C}}_{1gH}^0(\hat{1}_q,4_{\bar{q}},\hat{2}_Q,3_{\bar{Q}},5) \right] \Bigg\} \\
&\times J_1^{(3)}(p_3,p_4,p_5)
\end{aligned}
\tag{A.4.17}
$$

A.4.2   Real-virtual contribution

The real-virtual contribution at NNLO for $qQ \to H$+jet comes from the $qQ \to H + \bar{Q}\bar{q}$ process,

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{qQ}^{RV} =& \mathcal{N}_{qQ}\mathcal{N}_{NNLO}^{RV}\mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{ \\
&+ C_{0gH}^1(\hat{1}_q,4_{\bar{Q}},\hat{2}_Q,3_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}_{0gH}^1(\hat{1}_q,4_{\bar{Q}},\hat{2}_Q,3_{\bar{q}}) + \frac{N_f}{N}\widehat{C}_{0gH}^1(\hat{1}_q,4_{\bar{Q}},\hat{2}_Q,3_{\bar{q}}) \Bigg\} \\
&\times J_1^{(2)}(p_3,p_4)
\end{aligned}
\tag{A.4.18}
$$

A.4.3 qQ initiated subtraction terms at NNLO

The NNLO antenna subtraction terms for qQ initiated channel have been introduced as part of the subtraction terms for qq initiated channels in section 9.4.

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}_{qQ}^{S} =& \mathcal{N}_{qQ}\mathcal{N}_{NNLO}^{RR}\mathrm{d}\Phi_{H+3}(p_3,p_4,p_5,p_H;p_1,p_2)\Bigg\{ \\
&+ C_{1gH}^{0,YS}(\hat{1}_q,5,3_{\bar{Q}},\hat{2}_Q,4_{\bar{q}}) + \frac{1}{N^2}\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q,5,4_{\bar{q}},\hat{2}_Q,3_{\bar{Q}}) \Bigg\},
\end{aligned}
\tag{A.4.19}
$$

$$
\mathrm{d}\hat{\sigma}_{qQ}^{T} = \mathcal{N}_{qQ}\mathcal{N}_{NNLO}^{RV}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+2}(p_3,p_4,p_H;p_1,p_2)\Bigg\{
$$

$$
+ C_{0gH}^{1,YT}(\hat{1}_q, 4_{\bar{Q}}, \hat{2}_Q, 3_{\bar{q}}) - \frac{1}{N^2}\widetilde{C}_{0gH}^{1,YT}(\hat{1}_q, 4_{\bar{Q}}, \hat{2}_Q, 3_{\bar{q}}) + \frac{N_f}{N}\widehat{C}_{0gH}^{1,YT}(\hat{1}_q, 4_{\bar{Q}}, \hat{2}_Q, 3_{\bar{q}}) \Bigg\},
$$

$$(A.4.20)$$

$$
\mathrm{d}\hat{\sigma}_{qQ}^U = \mathcal{N}_{qQ}\mathcal{N}_{NNLO}^{VV}\frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\mathrm{d}\Phi_{H+1}(p_3, p_H; p_1, p_2)\Bigg\{
$$

$$
- C_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) + \frac{1}{N^2}\widetilde{C}_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}}) - \frac{N_F}{N}\widehat{C}_{1gH}^{1,XU}(1_q, 2, 3_{\bar{q}})\Bigg\}J_1^{(1)}(p_3).
$$

$$(A.4.21)$$

A.5 $\quad \bar{q}\bar{Q}$ initiated cross sections at NNLO

Using the line-reversal relation and charge conjugation symmetry introduced in chapter 5, the contributions to the $\bar{q}\bar{Q}$ initiated channels are related to the qQ subtraction terms:

$$
\mathrm{d}\hat{\sigma}_{\bar{q}\bar{Q},NNLO} = \mathrm{d}\hat{\sigma}_{qQ,NNLO}. \tag{A.5.22}
$$

Appendix B

Explicit results of antenna subtraction terms for $gg \to H+$jet processes at NNLO

B.1 $\quad gg \to H+$jet at RR

The double real subtraction terms $\mathrm{d}\hat{\sigma}^S_{NNLO}$ mentioned in section 6.6.1 are:

$$
B^{0,FS}_{3gH}(j_q, \hat{1}, \hat{2}, i, k_{\bar{q}}) =
$$

$$
+F^0_{3,gg}(1, i, 2)\, B^0_{2gH}(j, \bar{1}, \bar{2}, k)\, J^{(2)}_1(\{p\}_2)
$$

$$
+F^0_{3,gg}(2, i, 1)\, B^0_{2gH}(j, \bar{2}, \bar{1}, k)\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,g\to q}(j, 1, i)\, B^0_{2gH}(\bar{1}, (\widetilde{ji}), 2, k)\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,g\to q}(k, 2, i)\, B^0_{2gH}(j, 1, (\widetilde{ki}), \bar{2})\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,g\to q}(j, 2, i)\, B^0_{2gH}(\bar{2}, (\widetilde{ji}), 1, k)\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,g\to q}(k, 1, i)\, B^0_{2gH}(j, 2, (\widetilde{ki}), \bar{1})\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,gg\to qg}(j, 1, 2)\, B^0_{2gH}(\bar{1}, \bar{2}, i, k)\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,gg\to qg}(j, 2, 1)\, B^0_{2gH}(\bar{2}, \bar{1}, i, k)\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,gg\to qg}(k, 1, 2)\, B^0_{2gH}(j, i, \bar{2}, \bar{1})\, J^{(2)}_1(\{p\}_2)
$$

$$
-d^0_{3,gg\to qg}(k, 2, 1)\, B^0_{2gH}(j, i, \bar{1}, \bar{2})\, J^{(2)}_1(\{p\}_2)
$$

$$
+d^0_{3,g}(j, i, 1)\, B^0_{2gH}((\widetilde{ji}), \bar{1}, 2, k)\, J^{(2)}_1(\{p\}_2)
$$

$$+ d^0_{3,g}(j,i,2)\, B^0_{2gH}((\widetilde{ji}),\overline{2},1,k)\, J^{(2)}_1(\{p\}_2)$$

$$+ d^0_{3,g}(k,i,1)\, B^0_{2gH}(j,2,\overline{1},(\widetilde{ki}))\, J^{(2)}_1(\{p\}_2)$$

$$+ d^0_{3,g}(k,i,2)\, B^0_{2gH}(j,1,\overline{2},(\widetilde{ki}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(2,j,k)\, A^0_{4gH}(1,i,\overline{2},(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(1,k,j)\, A^0_{4gH}(\overline{1},(\widetilde{kj}),2,i)\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(1,j,k)\, A^0_{4gH}(\overline{1},i,2,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(2,k,j)\, A^0_{4gH}(1,(\widetilde{kj}),\overline{2},i)\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_3(i,j,k)\, A^0_{4gH}(1,2,(\widetilde{ij}),(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_3(i,k,j)\, A^0_{4gH}(1,2,(\widetilde{ik}),(\widetilde{kj}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_3(i,k,j)\, A^0_{4gH}(1,2,(\widetilde{kj}),(\widetilde{ik}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_3(i,j,k)\, A^0_{4gH}(1,2,(\widetilde{kj}),(\widetilde{ji}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(1,j,k)\, A^0_{4gH}(\overline{1},2,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(1,k,j)\, A^0_{4gH}(\overline{1},2,i,(\widetilde{kj}))\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(2,k,j)\, A^0_{4gH}(1,\overline{2},(\widetilde{kj}),i)\, J^{(2)}_1(\{p\}_2)$$

$$+ \frac{1}{2}\, G^0_{3,g}(2,j,k)\, A^0_{4gH}(1,\overline{2},(\widetilde{jk}),i)\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_4(k,1,i,j)\, B^0_{1gH}(\overline{1},2,(\widetilde{kij}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(j,i,1)\, A^0_{3,g\to q}(k,\overline{1},(\widetilde{ji}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{k(\widetilde{ji})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g\to q}(k,1,i)\, A^0_{3,q}(\overline{1},(\widetilde{ki}),j)\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{(\widetilde{ki})j}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_4(k,i,1,j)\, B^0_{1gH}(\overline{1},2,(\widetilde{kij}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(k,i,1)\, A^0_{3,g\to q}(j,\overline{1},(\widetilde{ki}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ki})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g\to q}(j,1,i)\, A^0_{3,q}(\overline{1},(\widetilde{ji}),k)\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{(\widetilde{ji})k}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_4(k,2,i,j)\, B^0_{1gH}((\widetilde{kij}),1,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(j,i,2)\, A^0_{3,g\to q}(k,\overline{2},(\widetilde{ji}))\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{k(\widetilde{ji})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g\to q}(k,2,i)\, A^0_{3,q}(\overline{2},(\widetilde{ki}),j)\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{(\widetilde{ki})j}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_4(k,i,2,j)\, B^0_{1gH}((\widetilde{kij}),1,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(k,i,2)\, A^0_{3,g\to q}(j,\overline{2},(\widetilde{ki}))\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{j(\widetilde{ki})}))\, J^{(1)}_1(\{p\}_1)$$

$$+\, d_{3,g\to q}^0(j,2,i)\, A_{3,q}^0(\overline{2},(\widetilde{ji}),k)\, B_{1gH}^0(\overline{\overline{2}},1,(\widetilde{(\widetilde{ji})k}))\, J_1^{(1)}(\{p\}_1)$$

$$+\, G_4^0(1,j,k,i)\, A_{3gH}^0(\overline{1},2,(\widetilde{jki}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g\to q}^0(j,1,i)\, G_{3,q'\to g}^0((\widetilde{ji}),k,\overline{1})\, A_{3gH}^0(2,\overline{\overline{1}},(\widetilde{(\widetilde{ji})k}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g}^0(k,i,1)\, G_{3,g}^0(\overline{1},(\widetilde{ki}),j)\, A_{3gH}^0(2,\overline{\overline{1}},(\widetilde{(\widetilde{ki})j}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_3^0(i,k,j)\, F_{3,g}^0(1,(\widetilde{kj}),(\widetilde{ik}))\, A_{3gH}^0(2,\overline{1},(\widetilde{\widetilde{kj},\widetilde{ik}}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_{3,g}^0(1,j,k)\, F_{3,g}^0(\overline{1},(\widetilde{jk}),i)\, A_{3gH}^0(\overline{\overline{1}},(\widetilde{(\widetilde{jk})i}),2)\, J_1^{(1)}(\{p\}_1)$$

$$+\, G_4^0(2,k,j,i)\, A_{3gH}^0(1,\overline{2},(\widetilde{kji}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g\to q}^0(k,2,i)\, G_{3,q'\to g}^0((\widetilde{ki}),j,\overline{2})\, A_{3gH}^0(1,\overline{\overline{2}},(\widetilde{(\widetilde{ki})j}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g}^0(j,i,2)\, G_{3,g}^0(\overline{2},(\widetilde{ji}),k)\, A_{3gH}^0(1,\overline{\overline{2}},(\widetilde{(\widetilde{ji})k}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_{3,g}^0(2,k,j)\, F_{3,g}^0(\overline{2},(\widetilde{kj}),i)\, A_{3gH}^0(\overline{\overline{2}},(\widetilde{(\widetilde{kj})i}),1)\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_3^0(i,j,k)\, F_{3,g}^0(2,(\widetilde{jk}),(\widetilde{ij}))\, A_{3gH}^0(1,\overline{2},(\widetilde{\widetilde{jk},\widetilde{ij}}))\, J_1^{(1)}(\{p\}_1)$$

$$+\, G_4^0(2,j,k,i)\, A_{3gH}^0(\overline{2},1,(\widetilde{jki}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g\to q}^0(j,2,i)\, G_{3,q'\to g}^0((\widetilde{ji}),k,\overline{2})\, A_{3gH}^0(1,\overline{\overline{2}},(\widetilde{(\widetilde{ji})k}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g}^0(k,i,2)\, G_{3,g}^0(\overline{2},(\widetilde{ki}),j)\, A_{3gH}^0(1,\overline{\overline{2}},(\widetilde{(\widetilde{ki})j}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_3^0(i,k,j)\, F_{3,g}^0(2,(\widetilde{kj}),(\widetilde{ik}))\, A_{3gH}^0(1,\overline{2},(\widetilde{\widetilde{kj},\widetilde{ik}}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_{3,g}^0(2,j,k)\, F_{3,g}^0(\overline{2},(\widetilde{jk}),i)\, A_{3gH}^0(\overline{\overline{2}},(\widetilde{(\widetilde{jk})i}),1)\, J_1^{(1)}(\{p\}_1)$$

$$+\, G_4^0(1,k,j,i)\, A_{3gH}^0(2,\overline{1},(\widetilde{kji}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g\to q}^0(k,1,i)\, G_{3,q'\to g}^0((\widetilde{ki}),j,\overline{1})\, A_{3gH}^0(2,\overline{\overline{1}},(\widetilde{(\widetilde{ki})j}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,g}^0(j,i,1)\, G_{3,g}^0(\overline{1},(\widetilde{ji}),k)\, A_{3gH}^0(2,\overline{\overline{1}},(\widetilde{(\widetilde{ji})k}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_3^0(i,j,k)\, F_{3,g}^0(1,(\widetilde{jk}),(\widetilde{ij}))\, A_{3gH}^0(2,\overline{1},(\widetilde{\widetilde{jk},\widetilde{ij}}))\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_{3,g}^0(1,k,j)\, F_{3,g}^0(\overline{1},(\widetilde{kj}),i)\, A_{3gH}^0(\overline{\overline{1}},(\widetilde{(\widetilde{kj})i}),2)\, J_1^{(1)}(\{p\}_1)$$

$$+\, G_4^0(1,j,k,2)\, A_{3gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,gg\to qg}^0(j,1,2)\, G_{3,qg\to gg}^0(\overline{2},\overline{1},k)\, A_{3gH}^0(\overline{\overline{1}},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$-\, d_{3,gg\to qg}^0(k,2,1)\, G_{3,gq\to gg}^0(\overline{1},\overline{2},j)\, A_{3gH}^0(\overline{\overline{1}},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_{3,g}^0(1,j,k)\, F_{3,gg}^0(\overline{1},(\widetilde{jk}),2)\, A_{3gH}^0(\overline{\overline{1}},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$-\, \frac{1}{2}\, G_{3,g}^0(2,j,k)\, F_{3,gg}^0(1,(\widetilde{jk}),\overline{2})\, A_{3gH}^0(\overline{\overline{2}},i,\overline{1})\, J_1^{(1)}(\{p\}_1)$$

$$+ G_4^0(2,j,k,1)\, A_{3gH}^0(\overline{2},i,\overline{1})\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,gg\to qg}^0(j,2,1)\, G_{3,gq\to gg}^0(\overline{1},\overline{2},k)\, A_{3gH}^0(\overline{\overline{1}},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,gg\to qg}^0(k,1,2)\, G_{3,qg\to gg}^0(\overline{2},\overline{1},j)\, A_{3gH}^0(\overline{\overline{1}},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, G_{3,g}^0(2,j,k)\, F_{3,gg}^0(\overline{2},(\widetilde{jk}),1)\, A_{3gH}^0(\overline{\overline{2}},i,\overline{1})\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, G_{3,g}^0(1,j,k)\, F_{3,gg}^0(\overline{1},(\widetilde{jk}),2)\, A_{3gH}^0(\overline{\overline{1}},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$- G_{3,g}^0(2,j,k)\, f_{3,g}^0(1,i,(\widetilde{jk}))\, A_{3gH}^0(\overline{1},\overline{2},(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ G_{3,g}^0(1,j,k)\, f_{3,g}^0(\overline{1},i,(\widetilde{jk}))\, A_{3gH}^0(\overline{\overline{1}},(\widetilde{i(\widetilde{jk})}),2)\, J_1^{(1)}(\{p\}_1)$$

$$- G_{3,g}^0(1,j,k)\, f_{3,g}^0(2,i,(\widetilde{jk}))\, A_{3gH}^0(\overline{1},\overline{2},(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ G_{3,g}^0(2,j,k)\, f_{3,g}^0(\overline{2},i,(\widetilde{jk}))\, A_{3gH}^0(1,(\widetilde{i(\widetilde{jk})}),\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, G_{3,g}^0(2,j,k)\, F_{3,gg}^0(1,i,\overline{2})\, A_{3gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, G_{3,g}^0(1,k,j)\, F_{3,gg}^0(\overline{1},i,2)\, A_{3gH}^0(\overline{\overline{1}},(\widetilde{kj}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, G_{3,g}^0(1,j,k)\, F_{3,gg}^0(\overline{1},i,2)\, A_{3gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, G_{3,g}^0(2,k,j)\, F_{3,gg}^0(1,i,\overline{2})\, A_{3gH}^0(\overline{1},(\widetilde{kj}),\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,g}^0(j,i,1)\, a_{3,g\to q}^0(k,\overline{1},(\widetilde{ji}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{k(\widetilde{ji})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ F_{3,gg}^0(1,i,2)\, a_{3,g\to q}^0(k,\overline{1},j)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{kj}))\, J_1^{(1)}(\{p\}_1)$$

$$+ d_{3,g}^0(j,i,2)\, a_{3,g\to q}^0(k,1,(\widetilde{ji}))\, B_{1gH}^0(\overline{1},\overline{2},(\widetilde{k(\widetilde{ji})}))\, J_1^{(1)}(\{p\}_1)$$

$$+\left[+ S_{(\widetilde{ji})i\overline{1}}^{IF} - S_{(k\widetilde{(ji)})i\overline{\overline{1}}}^{IF} - S_{\overline{1}i2}^{IF} + S_{\overline{\overline{1}}i2}^{IF} - S_{(\widetilde{ji})i2}^{IF} + S_{(k\widetilde{(ji)})i2}^{IF} \right]$$

$$\times a_{3,g\to q}^0(k,\overline{1},(\widetilde{ji}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{k(\widetilde{ji})}))\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,g}^0(k,i,1)\, a_{3,g\to q}^0(j,\overline{1},(\widetilde{ki}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ki})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ F_{3,gg}^0(1,i,2)\, a_{3,g\to q}^0(j,\overline{1},k)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$+ d_{3,g}^0(k,i,2)\, a_{3,g\to q}^0(j,1,(\widetilde{ki}))\, B_{1gH}^0(\overline{1},\overline{2},(\widetilde{j(\widetilde{ki})}))\, J_1^{(1)}(\{p\}_1)$$

$$+\left[+ S_{(\widetilde{ki})i\overline{1}}^{IF} - S_{(j\widetilde{(ki)})i\overline{\overline{1}}}^{IF} - S_{\overline{1}i2}^{IF} + S_{\overline{\overline{1}}i2}^{IF} - S_{(\widetilde{ki})i2}^{IF} + S_{(j\widetilde{(ki)})i2}^{IF} \right]$$

$$\times a_{3,g\to q}^0(j,\overline{1},(\widetilde{ki}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ki})}))\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,g}^0(j,i,2)\, a_{3,g\to q}^0(k,\overline{2},(\widetilde{ji}))\, B_{1gH}^0(\overline{\overline{2}},1,(\widetilde{k(\widetilde{ji})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ F_{3,gg}^0(1,i,2)\, a_{3,g\to q}^0(k,\overline{2},j)\, B_{1gH}^0(\overline{\overline{2}},\overline{1},(\widetilde{kj}))\, J_1^{(1)}(\{p\}_1)$$

$$+ d_{3,g}^0(j,i,1)\, a_{3,g\to q}^0(k,2,(\widetilde{ji}))\, B_{1gH}^0(\overline{2},\overline{1},(\widetilde{k(\widetilde{ji})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \left[+ S^{IF}_{(\widetilde{ji})i\bar{2}} - S^{IF}_{(k(\widetilde{ji}))i\bar{\bar{2}}} - S^{IF}_{1i\bar{2}} + S^{IF}_{1i\bar{\bar{2}}} - S^{IF}_{(\widetilde{ji})i1} + S^{IF}_{(k(\widetilde{ji}))i1} \right]$$

$$\times a^0_{3,g \to q}(k, \bar{2}, (\widetilde{ji})) \, B^0_{1gH}(\bar{\bar{2}}, 1, (k\widetilde{(ji)})) \, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g}(k, i, 2) \, a^0_{3,g \to q}(j, \bar{2}, (\widetilde{ki})) \, B^0_{1gH}(\bar{\bar{2}}, 1, (j\widetilde{(ki)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ F^0_{3,gg}(1, i, 2) \, a^0_{3,g \to q}(j, \bar{2}, k) \, B^0_{1gH}(\bar{\bar{2}}, \bar{1}, (\widetilde{jk})) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(k, i, 1) \, a^0_{3,g \to q}(j, 2, (\widetilde{ki})) \, B^0_{1gH}(\bar{2}, \bar{1}, (j\widetilde{(ki)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \left[+ S^{IF}_{(\widetilde{ki})i\bar{2}} - S^{IF}_{(j(\widetilde{ki}))i\bar{\bar{2}}} - S^{IF}_{1i\bar{2}} + S^{IF}_{1i\bar{\bar{2}}} - S^{IF}_{(\widetilde{ki})i1} + S^{IF}_{(j(\widetilde{ki}))i1} \right]$$

$$\times a^0_{3,g \to q}(j, \bar{2}, (\widetilde{ki})) \, B^0_{1gH}(\bar{\bar{2}}, 1, (j\widetilde{(ki)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ a^0_{3,g \to q}(k, 1, j) \, A^0_{3,q}((\widetilde{kj}), i, \bar{1}) \, B^0_{1gH}(\bar{\bar{1}}, 2, ((\widetilde{kj})i)) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(k, 1, j) \, D^0_{3,qg}(\bar{1}, i, 2) \, B^0_{1gH}(\bar{\bar{1}}, \bar{2}, (\widetilde{kj})) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(k, 1, j) \, d^0_{3,g}((\widetilde{kj}), i, 2) \, B^0_{1gH}(\bar{1}, \bar{2}, ((\widetilde{kj})i)) \, J^{(1)}_1(\{p\}_1)$$

$$+ a^0_{3,g \to q}(j, 1, k) \, A^0_{3,q}((\widetilde{jk}), i, \bar{1}) \, B^0_{1gH}(\bar{\bar{1}}, 2, ((\widetilde{jk})i)) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(j, 1, k) \, D^0_{3,qg}(\bar{1}, i, 2) \, B^0_{1gH}(\bar{\bar{1}}, \bar{2}, (\widetilde{jk})) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(j, 1, k) \, d^0_{3,g}((\widetilde{jk}), i, 2) \, B^0_{1gH}(\bar{1}, \bar{2}, ((\widetilde{jk})i)) \, J^{(1)}_1(\{p\}_1)$$

$$+ a^0_{3,g \to q}(k, 2, j) \, A^0_{3,q}((\widetilde{kj}), i, \bar{2}) \, B^0_{1gH}(\bar{\bar{2}}, 1, ((\widetilde{kj})i)) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(k, 2, j) \, D^0_{3,gq}(\bar{2}, i, 1) \, B^0_{1gH}(\bar{\bar{2}}, \bar{1}, (\widetilde{kj})) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(k, 2, j) \, d^0_{3,g}((\widetilde{kj}), i, 1) \, B^0_{1gH}(\bar{2}, \bar{1}, ((\widetilde{kj})i)) \, J^{(1)}_1(\{p\}_1)$$

$$+ a^0_{3,g \to q}(j, 2, k) \, A^0_{3,q}((\widetilde{jk}), i, \bar{2}) \, B^0_{1gH}(\bar{\bar{2}}, 1, ((\widetilde{jk})i)) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(j, 2, k) \, D^0_{3,gq}(\bar{2}, i, 1) \, B^0_{1gH}(\bar{\bar{2}}, \bar{1}, (\widetilde{jk})) \, J^{(1)}_1(\{p\}_1)$$

$$- a^0_{3,g \to q}(j, 2, k) \, d^0_{3,g}((\widetilde{jk}), i, 1) \, B^0_{1gH}(\bar{2}, \bar{1}, ((\widetilde{jk})i)) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,gg \to qg}(j, 1, 2) \, d^0_{3,g}(k, i, \bar{2}) \, B^0_{1gH}(\bar{1}, \bar{\bar{2}}, (\widetilde{ki})) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,gg \to qg}(j, 2, 1) \, d^0_{3,g}(k, i, \bar{1}) \, B^0_{1gH}(\bar{2}, \bar{\bar{1}}, (\widetilde{ki})) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,gg \to qg}(k, 1, 2) \, d^0_{3,g}(j, i, \bar{2}) \, B^0_{1gH}((\widetilde{ji}), \bar{\bar{2}}, \bar{1}) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,gg \to qg}(k, 2, 1) \, d^0_{3,g}(j, i, \bar{1}) \, B^0_{1gH}((\widetilde{ji}), \bar{\bar{1}}, \bar{2}) \, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g \to q}(k, 1, i) \, A^0_{3,q}(j, (\widetilde{ki}), \bar{1}) \, B^0_{1gH}(\bar{\bar{1}}, 2, (j\widetilde{(ki)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g \to q}(k, 1, i) \, D^0_{3,qg}(\bar{1}, (\widetilde{ki}), 2) \, B^0_{1gH}(\bar{\bar{1}}, \bar{2}, j) \, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g \to q}(j, 1, i) \, A^0_{3,q}(k, (\widetilde{ji}), \bar{1}) \, B^0_{1gH}(\bar{\bar{1}}, 2, (k\widetilde{(ji)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g\to q}(j,1,i)\, D^0_{3,qg}(\overline{1},(\widetilde{ji}),2)\, B^0_{1gH}(\overline{\overline{1}},\overline{2},k)\, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g\to q}(k,2,i)\, A^0_{3,q}(j,(\widetilde{ki}),\overline{2})\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{j(\widetilde{ki})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g\to q}(k,2,i)\, D^0_{3,qg}(\overline{2},(\widetilde{ki}),1)\, B^0_{1gH}(\overline{\overline{2}},\overline{1},j)\, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g\to q}(j,2,i)\, A^0_{3,q}(k,(\widetilde{ji}),\overline{2})\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{k(\widetilde{ji})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g\to q}(j,2,i)\, D^0_{3,qg}(\overline{2},(\widetilde{ji}),1)\, B^0_{1gH}(\overline{\overline{2}},\overline{1},k)\, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g\to q}(k,2,i)\, d^0_{3,g\to q}(j,1,(\widetilde{ki}))\, B^0_{1gH}(1,(\widetilde{j(\widetilde{ki})}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- d^0_{3,g\to q}(k,1,i)\, d^0_{3,g\to q}(j,2,(\widetilde{ki}))\, B^0_{1gH}(2,(\widetilde{j(\widetilde{ki})}),\overline{1})\, J^{(1)}_1(\{p\}_1) \qquad \text{(B.1.1)}$$

$$\widetilde{B}^{0,XS}_{3gH}(j_q,\widetilde{i},\hat{1},\hat{2},k_{\bar{q}}) =$$

$$+ A^0_3(j,i,k)\, B^0_{2gH}((\widetilde{ji}),1,2,(\widetilde{ik}))\, J^{(2)}_1(\{p\}_2)$$

$$+ A^0_3(j,i,k)\, B^0_{2gH}((\widetilde{ji}),2,1,(\widetilde{ik}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g\to q}(j,1,k)\, \tilde{B}^0_{2gH}(\overline{1},2,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g\to q}(j,2,k)\, \tilde{B}^0_{2gH}(\overline{2},1,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$+ A^0_4(j,1,2,k)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_4(j,2,1,k)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g\to q}(j,1,k)\, A^0_{3,qg\to qq}(\overline{1},2,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{1}},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g\to q}(j,2,k)\, A^0_{3,qg\to qq}(\overline{2},1,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{2}},i,\overline{1})\, J^{(1)}_1(\{p\}_1)$$

$$- \tilde{A}^0_4(j,1,i,k)\, B^0_{1gH}(\overline{1},2,(\widetilde{jik}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,g\to q}(j,1,k)\, A^0_{3,q}(\overline{1},i,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(j,i,k)\, A^0_{3,g\to q}((\widetilde{ji}),1,(\widetilde{ik}))\, B^0_{1gH}(\overline{1},2,(\widetilde{\widetilde{ji},\widetilde{ik}}))\, J^{(1)}_1(\{p\}_1)$$

$$- \tilde{A}^0_4(j,2,i,k)\, B^0_{1gH}(\overline{2},1,(\widetilde{jik}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,g\to q}(j,2,k)\, A^0_{3,q}(\overline{2},i,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(j,i,k)\, A^0_{3,g\to q}((\widetilde{ji}),2,(\widetilde{ik}))\, B^0_{1gH}(\overline{2},1,(\widetilde{\widetilde{ji},\widetilde{ik}}))\, J^{(1)}_1(\{p\}_1)$$

$$+ \tilde{G}^0_4(1,j,k,2)\, A^0_{3gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ \tilde{G}^0_4(1,j,k,i)\, A^0_{3gH}(\overline{1},2,(\widetilde{jki}))\, J^{(1)}_1(\{p\}_1)$$

$$+ \tilde{G}^0_4(2,j,k,i)\, A^0_{3gH}(\overline{2},1,(\widetilde{jki}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g\to q}(j,1,k)\, G^0_{3,gq\to gg}(2,\overline{1},(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{1}},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g \to q}(j,2,k)\, G^0_{3,gq \to gg}(1,\overline{2},(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{2}},\overline{1},i)\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g \to q}(j,1,k)\, G^0_{3,q' \to g}(i,\overline{1},(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{i(jk)}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_3(j,i,k)\, G^0_{3,g}(1,(\widetilde{ji}),(\widetilde{ik}))\, A^0_{3gH}(\overline{1},2,(\widetilde{\widetilde{ji},\widetilde{ik}}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g \to q}(j,2,k)\, G^0_{3,q' \to g}(i,\overline{2},(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{2}},1,(\widetilde{i(jk)}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_3(j,i,k)\, G^0_{3,g}(2,(\widetilde{ji}),(\widetilde{ik}))\, A^0_{3gH}(\overline{2},1,(\widetilde{\widetilde{ji},\widetilde{ik}}))\, J^{(1)}_1(\{p\}_1) \tag{B.1.2}$$

$$\widetilde{B}^{0,YS}_{3gH}(j_q,\widetilde{\hat{1}},\hat{2},i,k_{\bar{q}}) =$$

$$- A^0_{3,g \to q}(j,1,k)\, B^0_{2gH}(\overline{1},2,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g \to q}(j,1,k)\, B^0_{2gH}(\overline{1},i,2,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g \to q}(j,2,k)\, B^0_{2gH}(\overline{2},1,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g \to q}(j,2,k)\, B^0_{2gH}(\overline{2},i,1,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$+ d^0_{3,g}(j,i,1)\, \tilde{B}^0_{2gH}((\widetilde{ji}),\overline{1},2,k)\, J^{(2)}_1(\{p\}_2)$$

$$+ d^0_{3,g}(k,i,1)\, \tilde{B}^0_{2gH}(j,\overline{1},2,(\widetilde{ki}))\, J^{(2)}_1(\{p\}_2)$$

$$+ d^0_{3,g}(j,i,2)\, \tilde{B}^0_{2gH}((\widetilde{ji}),\overline{2},1,k)\, J^{(2)}_1(\{p\}_2)$$

$$+ d^0_{3,g}(k,i,2)\, \tilde{B}^0_{2gH}(j,\overline{2},1,(\widetilde{ki}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g \to q}(j,1,k)\, \tilde{B}^0_{2gH}(\overline{1},2,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g \to q}(j,2,k)\, \tilde{B}^0_{2gH}(\overline{2},1,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_4(j,1,i,k)\, B^0_{1gH}(\overline{1},2,(\widetilde{jik}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_4(j,i,1,k)\, B^0_{1gH}(\overline{1},2,(\widetilde{jik}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_4(j,2,i,k)\, B^0_{1gH}(\overline{2},1,(\widetilde{jik}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_4(j,i,2,k)\, B^0_{1gH}(\overline{2},1,(\widetilde{jik}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(k,i,1)\, A^0_{3,g \to q}(j,\overline{1},(\widetilde{ik}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ik})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(j,i,1)\, A^0_{3,g \to q}((\widetilde{ji}),\overline{1},k)\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{k(\widetilde{ji})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(k,i,2)\, A^0_{3,g \to q}(j,\overline{2},(\widetilde{ik}))\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{j(\widetilde{ik})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ d^0_{3,g}(j,i,2)\, A^0_{3,g \to q}((\widetilde{ji}),\overline{2},k)\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{k(\widetilde{ji})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,g \to q}(j,1,k)\, A^0_{3,q}(\overline{1},i,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,g \to q}(j,2,k)\, A^0_{3,q}(\overline{2},i,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{2}},1,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+2\,\tilde{A}_4^0(j,1,2,k)\,B_{1gH}^0(\overline{1},i,\overline{2})\,J_1^{(1)}(\{p\}_1)$$

$$-2\,A_{3,g\to q}^0(j,1,k)\,A_{3,qg\to qq}^0(\overline{1},2,(\widetilde{jk}))\,B_{1gH}^0(\overline{\overline{1}},i,\overline{2})\,J_1^{(1)}(\{p\}_1)$$

$$-2\,A_{3,g\to q}^0(j,2,k)\,A_{3,qg\to qq}^0(\overline{2},1,(\widetilde{jk}))\,B_{1gH}^0(\overline{\overline{2}},i,\overline{1})\,J_1^{(1)}(\{p\}_1)$$

$$-\,\tilde{A}_4^0(j,1,i,k)\,B_{1gH}^0(\overline{1},2,(\widetilde{jik}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,A_{3,g\to q}^0(j,1,k)\,A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\,B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,A_3^0(j,i,k)\,A_{3,g\to q}^0((\widetilde{ji}),1,(\widetilde{ik}))\,B_{1gH}^0(\overline{1},2,(\widetilde{\widetilde{ji},\widetilde{ik}}))\,J_1^{(1)}(\{p\}_1)$$

$$-\,\tilde{A}_4^0(j,2,i,k)\,B_{1gH}^0(\overline{2},1,(\widetilde{jik}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,A_{3,g\to q}^0(j,2,k)\,A_{3,q}^0(\overline{2},i,(\widetilde{jk}))\,B_{1gH}^0(\overline{\overline{2}},1,(\widetilde{i(\widetilde{jk})}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,A_3^0(j,i,k)\,A_{3,g\to q}^0((\widetilde{ji}),2,(\widetilde{ik}))\,B_{1gH}^0(\overline{2},1,(\widetilde{\widetilde{ji},\widetilde{ik}}))\,J_1^{(1)}(\{p\}_1)$$

$$-\,A_3^0(j,i,k)\,A_{3,g\to q}^0((\widetilde{ji}),1,(\widetilde{ik}))\,B_{1gH}^0(\overline{1},2,(\widetilde{\widetilde{ji},\widetilde{ik}}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,d_{3,g}^0(j,i,2)\,A_{3,g\to q}^0(k,1,(\widetilde{ji}))\,B_{1gH}^0(\overline{1},\overline{2},(\widetilde{k(\widetilde{ji})}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,d_{3,g}^0(k,i,2)\,A_{3,g\to q}^0(j,1,(\widetilde{ki}))\,B_{1gH}^0(\overline{1},\overline{2},(\widetilde{j(\widetilde{ki})}))\,J_1^{(1)}(\{p\}_1)$$

$$+\left[+S_{(\widetilde{ij})ik}^{IF}-S_{\overline{1}i((\widetilde{ij})k)}^{IF}-S_{(\widetilde{ij})i\overline{2}}^{IF}+S_{((\widetilde{ij})k)i\overline{2}}^{IF}-S_{ki\overline{2}}^{IF}+S_{\overline{1}i\overline{2}}^{IF}\right]$$

$$\times A_{3,g\to q}^0((\widetilde{ij}),1,k)\,B_{1gH}^0(\overline{1},\overline{2},((\widetilde{\widetilde{ij})k}))\,J_1^{(1)}(\{p\}_1)$$

$$-\,A_3^0(j,i,k)\,A_{3,g\to q}^0((\widetilde{ji}),2,(\widetilde{ik}))\,B_{1gH}^0(\overline{2},1,(\widetilde{\widetilde{ji},\widetilde{ik}}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,d_{3,g}^0(j,i,1)\,A_{3,g\to q}^0(k,2,(\widetilde{ji}))\,B_{1gH}^0(\overline{2},\overline{1},(\widetilde{k(\widetilde{ji})}))\,J_1^{(1)}(\{p\}_1)$$

$$+\,d_{3,g}^0(k,i,1)\,A_{3,g\to q}^0(j,2,(\widetilde{ki}))\,B_{1gH}^0(\overline{2},\overline{1},(\widetilde{j(\widetilde{ki})}))\,J_1^{(1)}(\{p\}_1)$$

$$+\left[+S_{(\widetilde{ik})ij}^{IF}-S_{\overline{2}i((\widetilde{ik})j)}^{IF}-S_{(\widetilde{ik})i\overline{1}}^{IF}+S_{((\widetilde{ik})j)i\overline{1}}^{IF}-S_{ji\overline{1}}^{IF}+S_{\overline{2}i\overline{1}}^{IF}\right]$$

$$\times A_{3,g\to q}^0((\widetilde{ik}),2,j)\,B_{1gH}^0(\overline{2},\overline{1},((\widetilde{\widetilde{ik})j}))\,J_1^{(1)}(\{p\}_1) \tag{B.1.3}$$

$$\widetilde{\widetilde{B}}_{3gH}^{0,XS}\left(j_q,\widetilde{\widetilde{\overline{1}}},\widetilde{\widetilde{\overline{2}}},\widetilde{i},k_{\bar{q}}\right)=$$

$$-A_{3,g\to q}^0(j,1,k)\,\tilde{B}_{2gH}^0(\overline{1},2,i,(\widetilde{jk}))\,J_1^{(2)}(\{p\}_2)$$

$$-A_{3,g\to q}^0(j,2,k)\,\tilde{B}_{2gH}^0(\overline{2},1,i,(\widetilde{jk}))\,J_1^{(2)}(\{p\}_2)$$

$$+A_3^0(j,i,k)\,\tilde{B}_{2gH}^0((\widetilde{ji}),1,2,(\widetilde{ik}))\,J_1^{(2)}(\{p\}_2)$$

$$-\tilde{A}_4^0(j,1,i,k)\,B_{1gH}^0(\overline{1},2,(\widetilde{ijk}))\,J_1^{(1)}(\{p\}_1)$$

$$-\tilde{A}_4^0(j,2,i,k)\,B_{1gH}^0(\overline{2},1,(\widetilde{ijk}))\,J_1^{(1)}(\{p\}_1)$$

$$+\tilde{A}_4^0(j,1,2,k)\,B_{1gH}^0(\overline{1},i,\overline{2})\,J_1^{(1)}(\{p\}_1)$$

$$+ A^0_{3,g \rightarrow q}(j,1,k) \, A^0_{3,q}(\overline{1},i,(\widetilde{jk})) \, B^0_{1gH}(\overline{\overline{1}},2,(i\widetilde{(jk)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(j,i,k) \, A^0_{3,g \rightarrow q}((\widetilde{ji}),1,(i\widetilde{k})) \, B^0_{1gH}(\overline{1},2,(\widetilde{\widetilde{ji},i\widetilde{k}})) \, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,g \rightarrow q}(j,2,k) \, A^0_{3,q}(\overline{2},i,(\widetilde{jk})) \, B^0_{1gH}(\overline{\overline{2}},1,(i\widetilde{(jk)})) \, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(j,i,k) \, A^0_{3,g \rightarrow q}((\widetilde{ji}),2,(i\widetilde{k})) \, B^0_{1gH}(\overline{2},1,(\widetilde{\widetilde{ji},i\widetilde{k}})) \, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g \rightarrow q}(j,1,k) \, A^0_{3,qg \rightarrow qq}(\overline{1},2,(\widetilde{jk})) \, B^0_{1gH}(\overline{\overline{1}},i,\overline{2}) \, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g \rightarrow q}(j,2,k) \, A^0_{3,qg \rightarrow qq}(\overline{2},1,(\widetilde{jk})) \, B^0_{1gH}(\overline{\overline{2}},i,\overline{1}) \, J^{(1)}_1(\{p\}_1) \qquad \text{(B.1.4)}$$

B.2 $gg \rightarrow H$+jet at RV

The real-virtual subtraction terms $\mathrm{d}\hat{\sigma}^T_{NNLO}$ mentioned in section 6.6.2 are:

$$\widehat{A}^{1,XT}_{4gH}(\hat{1},\hat{2},i,j) =$$

$$- \left[+ \hat{J}^{1,FI}_{2,GG}(s_{2i}) + \hat{J}^{1,FF}_{2,GG}(s_{ij}) + \hat{J}^{1,IF}_{2,GG}(s_{1j}) + \hat{J}^{1,II}_{2,GG}(s_{12}) \right] A^0_{4gH}(1,2,i,j) \, J^{(2)}_1(\{p\}_2)$$

$$+ f^0_{3,g}(2,i,j) \left[\hat{A}^1_{3gH}(1,\overline{2},(\widetilde{ij})) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,II}_{2,GG}(s_{1\overline{2}}) + \hat{J}^{1,FI}_{2,GG}(s_{\overline{2}(\widetilde{ij})}) + \hat{J}^{1,IF}_{2,GG}(s_{1(\widetilde{ij})}) \right) A^0_{3gH}(1,\overline{2},(\widetilde{ij})) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[\hat{f}^1_{3,g}(2,i,j) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,FI}_{2,GG}(s_{2i}) + \hat{J}^{1,FF}_{2,GG}(s_{ij}) + \hat{J}^{1,FI}_{2,GG}(s_{2j}) - 2\hat{J}^{1,FI}_{2,GG}(s_{\overline{2}(\widetilde{ij})}) \right) f^0_{3,g}(2,i,j) \right]$$

$$\times A^0_{3gH}(1,\overline{2},(\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$+ f^0_{3,g}(1,j,i) \left[\hat{A}^1_{3gH}(\overline{1},2,(\widetilde{ij})) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) + \hat{J}^{1,FI}_{2,GG}(s_{2(\widetilde{ij})}) + \hat{J}^{1,IF}_{2,GG}(s_{\overline{1}(\widetilde{ij})}) \right) A^0_{3gH}(\overline{1},2,(\widetilde{ij})) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[\hat{f}^1_{3,g}(1,j,i) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,IF}_{2,GG}(s_{1i}) + \hat{J}^{1,FF}_{2,GG}(s_{ij}) + \hat{J}^{1,IF}_{2,GG}(s_{1j}) - 2\hat{J}^{1,IF}_{2,GG}(s_{\overline{1}(\widetilde{ij})}) \right) f^0_{3,g}(1,j,i) \right]$$

$$\times A^0_{3gH}(\overline{1},2,(\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \left[+ \hat{J}^{1,II}_{2,GG}(s_{12}) - \hat{J}^{1,II}_{2,GG}(s_{1\overline{2}}) + \hat{J}^{1,IF}_{2,GG}(s_{1j}) - \hat{J}^{1,IF}_{2,GG}(s_{1(\widetilde{ij})}) \right.$$

$$\left. - \hat{J}^{1,FI}_{2,GG}(s_{2j}) + \hat{J}^{1,FI}_{2,GG}(s_{\overline{2}(\widetilde{ij})}) \right] f^0_{3,g}(2,i,j) \, A^0_{3gH}(1,\overline{2},(\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$+ \left[+ \hat{J}^{1,II}_{2,GG}(s_{12}) - \hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) + \hat{J}^{1,FI}_{2,GG}(s_{2i}) - \hat{J}^{1,FI}_{2,GG}(s_{2(\widetilde{ji})}) \right.$$

$$\left. - \hat{J}^{1,IF}_{2,GG}(s_{1i}) + \hat{J}^{1,IF}_{2,GG}(s_{\overline{1}(\widetilde{ji})}) \right] f^0_{3,g}(1,j,i) \, A^0_{3gH}(\overline{1},2,(\widetilde{ji})) \, J^{(1)}_1(\{p\}_1) \qquad \text{(B.2.5)}$$

$$\widehat{A}^{1,YT}_{4gH}(\hat{1},i,\hat{2},j) =$$

$$- \left[+ \hat{J}^{1,IF}_{2,GG}(s_{1i}) + \hat{J}^{1,FI}_{2,GG}(s_{2i}) + \hat{J}^{1,FI}_{2,GG}(s_{2j}) + \hat{J}^{1,IF}_{2,GG}(s_{1j}) \right] A^0_{4gH}(1,i,2,j) \, J^{(2)}_1(\{p\}_2)$$

$$+ F^0_{3,gg}(1,i,2) \left[\hat{A}^1_{3gH}(\overline{1},\overline{2},j) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) + \hat{J}^{1,FI}_{2,GG}(s_{\overline{2}j}) + \hat{J}^{1,IF}_{2,GG}(s_{\overline{1}j}) \right) A^0_{3gH}(\overline{1},\overline{2},j) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[\hat{F}^1_{3,gg}(1,i,2) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,IF}_{2,GG}(s_{1i}) + \hat{J}^{1,FI}_{2,GG}(s_{2i}) + \hat{J}^{1,II}_{2,GG}(s_{12}) - 2\hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) \right) F^0_{3,gg}(1,i,2) \right]$$

$$\times A^0_{3gH}(\overline{1},\overline{2},j) \, J^{(1)}_1(\{p\}_1)$$

$$+ F^0_{3,gg}(1,j,2) \left[\hat{A}^1_{3gH}(\overline{1},\overline{2},i) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) + \hat{J}^{1,FI}_{2,GG}(s_{\overline{2}i}) + \hat{J}^{1,IF}_{2,GG}(s_{\overline{1}i}) \right) A^0_{3gH}(\overline{1},\overline{2},i) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[\hat{F}^1_{3,gg}(1,j,2) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ \hat{J}^{1,IF}_{2,GG}(s_{1j}) + \hat{J}^{1,FI}_{2,GG}(s_{2j}) + \hat{J}^{1,II}_{2,GG}(s_{12}) - 2\hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) \right) F^0_{3,gg}(1,j,2) \right]$$

$$\times A^0_{3gH}(\overline{1},\overline{2},i) \, J^{(1)}_1(\{p\}_1)$$

$$+ \left[+ \hat{J}^{1,FI}_{2,GG}(s_{2j}) - \hat{J}^{1,FI}_{2,GG}(s_{\overline{2}j}) + \hat{J}^{1,IF}_{2,GG}(s_{1j}) - \hat{J}^{1,IF}_{2,GG}(s_{\overline{1}j}) \right.$$

$$\left. - \hat{J}^{1,II}_{2,GG}(s_{12}) + \hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) \right] F^0_{3,gg}(1,i,2) \, A^0_{3gH}(\overline{1},\overline{2},j) \, J^{(1)}_1(\{p\}_1)$$

$$+ \left[+ \hat{J}^{1,FI}_{2,GG}(s_{2i}) - \hat{J}^{1,FI}_{2,GG}(s_{\overline{2}i}) + \hat{J}^{1,IF}_{2,GG}(s_{1i}) - \hat{J}^{1,IF}_{2,GG}(s_{\overline{1}i}) \right.$$

$$\left. - \hat{J}^{1,II}_{2,GG}(s_{12}) + \hat{J}^{1,II}_{2,GG}(s_{\overline{1}2}) \right] F^0_{3,gg}(1,j,2) \, A^0_{3gH}(\overline{1},\overline{2},i) \, J^{(1)}_1(\{p\}_1) \qquad \text{(B.2.6)}$$

$$B^{1,XT}_{2gH}(i_q,\hat{1},\hat{2},j_{\bar{q}}) =$$

$$- \left[+ J^{1,IF}_{2,GQ}(s_{i1}) + J^{1,II}_{2,GG}(s_{12}) + J^{1,FI}_{2,QG}(s_{2j}) \right] B^0_{2gH}(i,1,2,j) \, J^{(2)}_1(\{p\}_2)$$

$$- \left[+ J^{1,FI}_{2,QG}(s_{i2}) + J^{1,II}_{2,GG}(s_{21}) + J^{1,IF}_{2,GQ}(s_{1j}) \right] B^0_{2gH}(i,2,1,j) \, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,g\to q}(i,1,j) \Bigg[B^1_{1gH}(\overline{1},2,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,II}_{2,QG}(s_{\overline{1}2}) + J^{1,FI}_{2,QG}(s_{(\widetilde{ij})2}) \Bigg) B^0_{1gH}(\overline{1},2,(\widetilde{ij})) \Bigg] J^{(1)}_1(\{p\}_1)$$

$$- \Bigg[A^1_{3,g}(i,1,j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,IF}_{2,GQ}(s_{i1}) + J^{1,IF}_{2,GQ}(s_{j1}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) \Bigg) A^0_{3,g\to q}(i,1,j) \Bigg]$$

$$\times B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,g\to q}(i,2,j) \Bigg[B^1_{1gH}(\overline{2},1,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,II}_{2,GQ}(s_{1\overline{2}}) + J^{1,IF}_{2,GQ}(s_{(\widetilde{ij})1}) \Bigg) B^0_{1gH}(\overline{2},1,(\widetilde{ij})) \Bigg] J^{(1)}_1(\{p\}_1)$$

$$- \Bigg[A^1_{3,g}(i,2,j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,FI}_{2,QG}(s_{i2}) + J^{1,FI}_{2,QG}(s_{j2}) - J^{1,FI}_{2,QQ}(s_{\overline{2}(\widetilde{ij})}) \Bigg) A^0_{3,g\to q}(i,2,j) \Bigg]$$

$$\times B^0_{1gH}(\overline{2},1,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+ G^0_{3,g}(1,i,j) \Bigg[A^1_{3gH}(\overline{1},2,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,IF}_{2,GG}(s_{\overline{1}(\widetilde{ij})}) + J^{1,FI}_{2,GG}(s_{2(\widetilde{ij})}) + J^{1,II}_{2,GG}(s_{\overline{1}2}) \Bigg) A^0_{3gH}(\overline{1},2,(\widetilde{ij})) \Bigg] J^{(1)}_1(\{p\}_1)$$

$$+ \Bigg[G^1_{3,g}(1,i,j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,IF}_{2,GQ}(s_{1i}) + J^{1,IF}_{2,GQ}(s_{1j}) - 2J^{1,IF}_{2,GG}(s_{\overline{1}(\widetilde{ij})}) \Bigg) G^0_{3,g}(1,i,j) \Bigg]$$

$$\times A^0_{3gH}(\overline{1},2,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+ G^0_{3,g}(2,i,j) \Bigg[A^1_{3gH}(\overline{2},1,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,IF}_{2,GG}(s_{1(\widetilde{ij})}) + J^{1,FI}_{2,GG}(s_{\overline{2}(\widetilde{ij})}) + J^{1,II}_{2,GG}(s_{\overline{2}1}) \Bigg) A^0_{3gH}(\overline{2},1,(\widetilde{ij})) \Bigg] J^{(1)}_1(\{p\}_1)$$

$$+ \Bigg[G^1_{3,g}(2,i,j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(+ J^{1,FI}_{2,QG}(s_{2i}) + J^{1,FI}_{2,QG}(s_{2j}) - 2J^{1,FI}_{2,GG}(s_{\overline{2}(\widetilde{ij})}) \Bigg) G^0_{3,g}(2,i,j) \Bigg]$$

$$\times A^0_{3gH}(\overline{2},1,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+ \Bigg[+ J^{1,IF}_{2,GQ}(s_{j1}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) - J^{1,II}_{2,GG}(s_{12})$$

$$+ J^{1,II}_{2,QG}(s_{\overline{1}2}) - J^{1,FI}_{2,QG}(s_{2j}) + J^{1,FI}_{2,QG}(s_{(\widetilde{ij})2})$$

$$+ \Big(- \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) + \mathcal{S}^{IF}(s_{\overline{1}(\widetilde{ij})}, s_{1j}, x_{\overline{1}(\widetilde{ij}),1j}) + \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j})$$

$$- \mathcal{S}^{IF}(s_{\overline{1}2}, s_{1j}, x_{\overline{1}2,1j}) + \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) - \mathcal{S}^{IF}(s_{2(\widetilde{ij})}, s_{1j}, x_{2(\widetilde{ij}),1j}) \Big) \Big]$$

$$\times a^0_{3,g\to q}(i, 1, j)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \Big[+ J^{1,IF}_{2,GQ}(s_{i1}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) - J^{1,II}_{2,GG}(s_{12})$$

$$+ J^{1,II}_{2,QG}(s_{\overline{1}2}) - J^{1,FI}_{2,QG}(s_{2i}) + J^{1,FI}_{2,QG}(s_{(\widetilde{ij})2})$$

$$+ \Big(- \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) + \mathcal{S}^{IF}(s_{\overline{1}(\widetilde{ij})}, s_{1i}, x_{\overline{1}(\widetilde{ij}),1i}) + \mathcal{S}^{IF}(s_{12}, s_{1i}, x_{12,1i})$$

$$- \mathcal{S}^{IF}(s_{\overline{1}2}, s_{1i}, x_{\overline{1}2,1i}) + \mathcal{S}^{IF}(s_{2i}, s_{1i}, x_{2i,1i}) - \mathcal{S}^{IF}(s_{2(\widetilde{ij})}, s_{1i}, x_{2(\widetilde{ij}),1i}) \Big) \Big]$$

$$\times a^0_{3,g\to q}(j, 1, i)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \Big[+ J^{1,FI}_{2,QG}(s_{j2}) - J^{1,FI}_{2,QQ}(s_{\overline{2}(\widetilde{ij})}) - J^{1,II}_{2,GG}(s_{12})$$

$$+ J^{1,II}_{2,GQ}(s_{1\overline{2}}) - J^{1,IF}_{2,GQ}(s_{1j}) + J^{1,IF}_{2,GQ}(s_{(\widetilde{ij})1})$$

$$+ \Big(- \mathcal{S}^{FI}(s_{2j}, s_{2j}, 1) + \mathcal{S}^{FI}(s_{\overline{2}(\widetilde{ij})}, s_{2j}, x_{\overline{2}(\widetilde{ij}),2j}) + \mathcal{S}^{FI}(s_{12}, s_{2j}, x_{12,2j})$$

$$- \mathcal{S}^{FI}(s_{\overline{2}1}, s_{2j}, x_{\overline{2}1,2j}) + \mathcal{S}^{FI}(s_{1j}, s_{2j}, x_{1j,2j}) - \mathcal{S}^{FI}(s_{1(\widetilde{ij})}, s_{2j}, x_{1(\widetilde{ij}),2j}) \Big) \Big]$$

$$\times a^0_{3,g\to q}(i, 2, j)\, B^0_{1gH}(\overline{2}, 1, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+ \Big[+ J^{1,FI}_{2,QG}(s_{i2}) - J^{1,FI}_{2,QQ}(s_{\overline{2}(\widetilde{ij})}) - J^{1,II}_{2,GG}(s_{12})$$

$$+ J^{1,II}_{2,GQ}(s_{1\overline{2}}) - J^{1,IF}_{2,GQ}(s_{1i}) + J^{1,IF}_{2,GQ}(s_{(\widetilde{ij})1})$$

$$+ \Big(- \mathcal{S}^{FI}(s_{2i}, s_{2i}, 1) + \mathcal{S}^{FI}(s_{\overline{2}(\widetilde{ij})}, s_{2i}, x_{\overline{2}(\widetilde{ij}),2i}) + \mathcal{S}^{FI}(s_{12}, s_{2i}, x_{12,2i})$$

$$- \mathcal{S}^{FI}(s_{\overline{2}1}, s_{2i}, x_{\overline{2}1,2i}) + \mathcal{S}^{FI}(s_{1i}, s_{2i}, x_{1i,2i}) - \mathcal{S}^{FI}(s_{1(\widetilde{ij})}, s_{2i}, x_{1(\widetilde{ij}),2i}) \Big) \Big]$$

$$\times a^0_{3,g\to q}(j, 2, i)\, B^0_{1gH}(\overline{2}, 1, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- 2 J^{1,IF}_{2,QG,g\to q}(s_{1i})\, B^0_{2gH}(1, i, 2, j)\, J_1^{(2)}(\{p\}_2)$$

$$- 2 J^{1,IF}_{2,QG,g\to q}(s_{1i})\, G^0_{3,q'\to g}(i, j, 1)\, A^0_{3gH}(\overline{1}, 2, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+ 2 J^{1,IF}_{2,QG,g\to q}(s_{1i})\, D^0_{3,qg}(1, i, 2)\, B^0_{1gH}(\overline{1}, \overline{2}, j)\, J_1^{(1)}(\{p\}_1)$$

$$- J^{1,IF}_{2,QG,g\to q}(s_{1i})\, d^0_{3,g\to q}(j, 2, i)\, B^0_{1gH}(1, (\widetilde{ji}), \overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$- J^{1,IF}_{2,QG,g\to q}(s_{1(\widetilde{ji})})\, d^0_{3,g\to q}(j, 2, i)\, B^0_{1gH}(1, (\widetilde{ji}), \overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$- 2 J^{1,FI}_{2,GQ,g\to q}(s_{2i})\, B^0_{2gH}(2, i, 1, j)\, J_1^{(2)}(\{p\}_2)$$

$$- 2 J^{1,FI}_{2,GQ,g\to q}(s_{2i})\, G^0_{3,q'\to g}(i, j, 2)\, A^0_{3gH}(1, \overline{2}, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+2J^{1,FI}_{2,GQ,g\to q}(s_{2i})\, D^0_{3,qg}(2,i,1)\, B^0_{1gH}(\overline{2},\overline{1},j)\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,FI}_{2,GQ,g\to q}(s_{2i})\, d^0_{3,g\to q}(j,1,i)\, B^0_{1gH}(2,(\widetilde{ji}),\overline{1})\, J^{(1)}_1(\{p\}_1)$$

$$-J^{1,FI}_{2,GQ,g\to q}(s_{2(\widetilde{ji})})\, d^0_{3,g\to q}(j,1,i)\, B^0_{1gH}(2,(\widetilde{ji}),\overline{1})\, J^{(1)}_1(\{p\}_1)$$

$$-2J^{1,II}_{2,QG,gg\to qg}(s_{12})\, B^0_{2gH}(1,2,i,j)\, J^{(2)}_1(\{p\}_2)$$

$$-2J^{1,II}_{2,QG,gg\to qg}(s_{12})\, G^0_{3,qg\to gg}(2,1,j)\, A^0_{3gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+2J^{1,II}_{2,QG,gg\to qg}(s_{12})\, d^0_{3,g}(j,i,2)\, B^0_{1gH}(1,\overline{2},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$-2J^{1,II}_{2,GQ,gg\to gq}(s_{12})\, B^0_{2gH}(2,1,i,j)\, J^{(2)}_1(\{p\}_2)$$

$$-2J^{1,II}_{2,GQ,gg\to gq}(s_{12})\, G^0_{3,qg\to gg}(1,2,j)\, A^0_{3gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$+2J^{1,II}_{2,GQ,gg\to gq}(s_{12})\, d^0_{3,g}(j,i,1)\, B^0_{1gH}(2,\overline{1},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$+\left[+2J^{1,IF}_{2,QQ,g\to q}(s_{1j})-2J^{1,IF}_{2,QQ,g\to q}(s_{\overline{1}(\widetilde{ji})})\right]A^0_{3,q}(j,i,1)\, B^0_{1gH}(\overline{1},2,(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$+\left[-2J^{1,IF}_{2,QQ,g\to q}(s_{1j})+2J^{1,IF}_{2,QQ,g\to q}(s_{\overline{1}j})\right]D^0_{3,qg}(1,i,2)\, B^0_{1gH}(\overline{1},\overline{2},j)\, J^{(1)}_1(\{p\}_1)$$

$$+\left[-2J^{1,IF}_{2,QQ,g\to q}(s_{1j})+2J^{1,IF}_{2,QQ,g\to q}(s_{1(\widetilde{ji})})\right]d^0_{3,g}(j,i,2)\, B^0_{1gH}(1,\overline{2},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$+\left[+2J^{1,FI}_{2,QQ,g\to q}(s_{2j})-2J^{1,FI}_{2,QQ,g\to q}(s_{\overline{2}(\widetilde{ji})})\right]A^0_{3,q}(j,i,2)\, B^0_{1gH}(\overline{2},1,(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$+\left[-2J^{1,FI}_{2,QQ,g\to q}(s_{2j})+2J^{1,FI}_{2,QQ,g\to q}(s_{\overline{2}j})\right]D^0_{3,qg}(2,i,1)\, B^0_{1gH}(\overline{2},\overline{1},j)\, J^{(1)}_1(\{p\}_1)$$

$$+\left[-2J^{1,FI}_{2,QQ,g\to q}(s_{2j})+2J^{1,FI}_{2,QQ,g\to q}(s_{2(\widetilde{ji})})\right]d^0_{3,g}(j,i,1)\, B^0_{1gH}(2,\overline{1},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$\text{(B.2.7)}$$

$$\widehat{B}^{1,XT}_{2gH}(i_q,\hat{1},\hat{2},j_{\bar{q}})=$$

$$-\left[+\hat{J}^{1,IF}_{2,GQ}(s_{i1})+\hat{J}^{1,II}_{2,GG}(s_{12})+\hat{J}^{1,FI}_{2,QG}(s_{j2})\right]B^0_{2gH}(i,1,2,j)\, J^{(2)}_1(\{p\}_2)$$

$$-\left[+\hat{J}^{1,IF}_{2,GQ}(s_{i1})+\hat{J}^{1,II}_{2,GG}(s_{12})+\hat{J}^{1,FI}_{2,QG}(s_{j2})\right]B^0_{2gH}(i,2,1,j)\, J^{(2)}_1(\{p\}_2)$$

$$-A^0_{3,g\to q}(i,1,j)\, \widehat{B}^1_{1gH}(\overline{1},2,(\widetilde{ij}))\, \delta(1-x_1)\, \delta(1-x_2)$$

$$-\left[\hat{A}^1_{3,g}(i,1,j)\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$+\left(+\hat{J}^{1,IF}_{2,GQ}(s_{i1})+\hat{J}^{1,II}_{2,GG}(s_{12})+\hat{J}^{1,FI}_{2,QG}(s_{j2})\right)A^0_{3,g\to q}(i,1,j)\bigg]$$

$$\times B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)$$

$$-A^0_{3,g\to q}(i,2,j)\, \widehat{B}^1_{1gH}((\widetilde{ij}),1,\overline{2})\, \delta(1-x_1)\, \delta(1-x_2)$$

$$
-\Bigg[\hat{A}^1_{3,g}(i,2,j)\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\left(+\,\hat{J}^{1,IF}_{2,GQ}(s_{i1}) + \hat{J}^{1,II}_{2,GG}(s_{12}) + \hat{J}^{1,FI}_{2,QG}(s_{j2}) \right) A^0_{3,g\rightarrow q}(i,2,j) \Bigg]
$$

$$
\times\, B^0_{1gH}((\widetilde{ij}),1,\overline{2})\, J^{(1)}_1(\{p\}_1)
$$

$$
+\, G^0_{3,g}(1,i,j)\, \hat{A}^1_{3gH}((\widetilde{ij}),\overline{1},2)\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\Bigg[\hat{G}^1_{3,g}(1,i,j)\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\left(+\,\hat{J}^{1,IF}_{2,GQ}(s_{i1}) + \hat{J}^{1,II}_{2,GG}(s_{12}) + \hat{J}^{1,FI}_{2,QG}(s_{j2}) \right) G^0_{3,g}(1,i,j) \Bigg]
$$

$$
\times\, A^0_{3gH}((\widetilde{ij}),\overline{1},2)\, J^{(1)}_1(\{p\}_1)
$$

$$
+\, G^0_{3,g}(2,j,i)\, \hat{A}^1_{3gH}((\widetilde{ji}),1,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\Bigg[\hat{G}^1_{3,g}(2,j,i)\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\left(+\,\hat{J}^{1,IF}_{2,GQ}(s_{i1}) + \hat{J}^{1,II}_{2,GG}(s_{12}) + \hat{J}^{1,FI}_{2,QG}(s_{j2}) \right) G^0_{3,g}(2,j,i) \Bigg]
$$

$$
\times\, A^0_{3gH}((\widetilde{ji}),1,\overline{2})\, J^{(1)}_1(\{p\}_1) \tag{B.2.8}
$$

$$
\widetilde{\widetilde{\widetilde{B}}}^{1,XT}_{2gH}(i_q,\widetilde{\overline{1}},\widetilde{\overline{2}},j_{\bar{q}}) =
$$

$$
-\, J^{1,FF}_{2,QQ}(s_{ij})\, \tilde{B}^0_{2gH}(i,1,2,j)\, J^{(2)}_1(\{p\}_2)
$$

$$
-\, A^0_{3,g\rightarrow q}(i,1,j)\Bigg[\tilde{B}^1_{1gH}(\overline{1},2,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\, J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})})\, B^0_{1gH}(\overline{1},2,(\widetilde{ij})) \Bigg] J^{(1)}_1(\{p\}_1)
$$

$$
-\Bigg[\tilde{A}^1_{3,g}(i,1,j)\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\left(+\, J^{1,FF}_{2,QQ}(s_{ij}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) \right) A^0_{3,g\rightarrow q}(i,1,j) \Bigg] B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)
$$

$$
-\, A^0_{3,g\rightarrow q}(i,2,j)\Bigg[\tilde{B}^1_{1gH}(\overline{2},1,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\, J^{1,FI}_{2,QQ}(s_{\overline{2}(\widetilde{ij})})\, B^0_{1gH}(\overline{2},1,(\widetilde{ij})) \Bigg] J^{(1)}_1(\{p\}_1)
$$

$$
-\Bigg[\tilde{A}^1_{3,g}(i,2,j)\,\delta(1-x_1)\,\delta(1-x_2)
$$

$$
+\left(+\, J^{1,FF}_{2,QQ}(s_{ij}) - J^{1,FI}_{2,QQ}(s_{\overline{2}(\widetilde{ij})}) \right) A^0_{3,g\rightarrow q}(i,2,j) \Bigg] B^0_{1gH}(\overline{2},1,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1) \tag{B.2.9}
$$

$$\widetilde{\widehat{\widetilde{B}}}_{2gH}^{1,XT}(i_q, \widetilde{\widehat{1}}, \widetilde{\widehat{2}}, j_{\bar{q}}) =$$

$$-\left[+ \hat{J}_{2,GQ}^{1,IF}(s_{1i}) + \hat{J}_{2,GG}^{1,II}(s_{12}) + \hat{J}_{2,QG}^{1,FI}(s_{2j}) \right] \tilde{B}_{2gH}^0(i, 1, 2, j) \, J_1^{(2)}(\{p\}_2)$$

$$- A_{3,g\to q}^0(i, 2, j) \, \hat{B}_{1gH}^1(\overline{2}, 1, (\widetilde{ij})) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$-\left[\hat{A}_{3,g}^1(i, 2, j) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$\left. + \left(+ \hat{J}_{2,GQ}^{1,IF}(s_{1i}) + \hat{J}_{2,GG}^{1,II}(s_{12}) + \hat{J}_{2,QG}^{1,FI}(s_{2j}) \right) A_{3,g\to q}^0(i, 2, j) \right]$$

$$\times B_{1gH}^0(\overline{2}, 1, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,g\to q}^0(i, 1, j) \, \hat{B}_{1gH}^1(\overline{1}, 2, (\widetilde{ij})) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$-\left[\hat{A}_{3,g}^1(i, 1, j) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$\left. + \left(+ \hat{J}_{2,GQ}^{1,IF}(s_{1i}) + \hat{J}_{2,GG}^{1,II}(s_{12}) + \hat{J}_{2,QG}^{1,FI}(s_{2j}) \right) A_{3,g\to q}^0(i, 1, j) \right]$$

$$\times B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1) \tag{B.2.10}$$

$$\widetilde{B}_{2gH}^{1,XT}(i_q, \hat{1}, \hat{2}, j_{\bar{q}}) =$$

$$- J_{2,QQ}^{1,FF}(s_{ij}) \, B_{2gH}^0(i, 1, 2, j) \, J_1^{(2)}(\{p\}_2)$$

$$- J_{2,QQ}^{1,FF}(s_{ij}) \, B_{2gH}^0(i, 2, 1, j) \, J_1^{(2)}(\{p\}_2)$$

$$- A_{3,g\to q}^0(i, 1, j) \left[\tilde{B}_{1gH}^1(\overline{1}, 2, (\widetilde{ij})) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$\left. + J_{2,QQ}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \, B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \right] J_1^{(1)}(\{p\}_1)$$

$$- A_{3,g\to q}^0(i, 2, j) \left[\tilde{B}_{1gH}^1(\overline{2}, 1, (\widetilde{ij})) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$\left. + J_{2,QQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) \, B_{1gH}^0(\overline{2}, 1, (\widetilde{ij})) \right] J_1^{(1)}(\{p\}_1)$$

$$-\left[\tilde{A}_{3,g}^1(i, 1, j) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$\left. + \left(+ J_{2,QQ}^{1,FF}(s_{ij}) - J_{2,QQ}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \right) A_{3,g\to q}^0(i, 1, j) \right] B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$-\left[\tilde{A}_{3,g}^1(i, 2, j) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$\left. + \left(+ J_{2,QQ}^{1,FF}(s_{ij}) - J_{2,QQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) \right) A_{3,g\to q}^0(i, 2, j) \right] B_{1gH}^0(\overline{2}, 1, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$+\left[\tilde{G}_{3,g}^1(1, i, j) \, \delta(1 - x_1) \, \delta(1 - x_2) \right.$$

$$+ J_{2,QQ}^{1,FF}(s_{ij}) \, G_{3,g}^0(1,i,j) \Big] A_{3gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$+ \Big[\tilde{G}_{3,g}^1(2,i,j) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$\qquad + J_{2,QQ}^{1,FF}(s_{ij}) \, G_{3,g}^0(2,i,j) \Big] A_{3gH}^0(1, \overline{2}, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, \tilde{B}_{2gH}^0(1,2,i,j) \, J_1^{(2)}(\{p\}_2)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, A_{3,qg\to qq}^0(1,2,j) \, B_{1gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$+ 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, A_{3,q}^0(1,i,j) \, B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, \tilde{B}_{2gH}^0(2,1,i,j) \, J_1^{(2)}(\{p\}_2)$$

$$- 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, A_{3,gq\to qq}^0(2,1,j) \, B_{1gH}^0(\overline{2}, i, \overline{1}) \, J_1^{(1)}(\{p\}_1)$$

$$+ 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, A_{3,q}^0(2,i,j) \, B_{1gH}^0(\overline{2}, 1, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, B_{2gH}^0(1,2,i,j) \, J_1^{(2)}(\{p\}_2)$$

$$+ 2 J_{2,QQ,g\to q}^{1,IF}(s_{1(\widetilde{ij})}) \, d_{3,g}^0(j,i,2) \, B_{1gH}^0(1, \overline{2}, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, G_{3,qg\to gg}^0(2,1,j) \, A_{3gH}^0(\overline{1}, \overline{2}, i) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, B_{2gH}^0(1,i,2,j) \, J_1^{(2)}(\{p\}_2)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, A_{3,qg\to qq}^0(1,2,j) \, B_{1gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$+ 2 J_{2,QQ,g\to q}^{1,IF}(s_{\overline{1}j}) \, D_{3,qg}^0(1,i,2) \, B_{1gH}^0(\overline{1}, \overline{2}, j) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \, G_{3,q'\to g}^0(i,1,j) \, A_{3gH}^0(\overline{1}, (\widetilde{ij}), 2) \, J_1^{(1)}(\{p\}_1)$$

$$+ \Big[- 2 J_{2,QQ,g\to q}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) + 2 J_{2,QQ,g\to q}^{1,IF}(s_{1j}) \Big] A_{3,q}^0(1,i,j) \, B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, B_{2gH}^0(2,1,i,j) \, J_1^{(2)}(\{p\}_2)$$

$$+ 2 J_{2,QQ,g\to q}^{1,FI}(s_{2(\widetilde{ij})}) \, d_{3,g}^0(j,i,1) \, B_{1gH}^0(2, \overline{1}, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, G_{3,gq\to gg}^0(1,2,j) \, A_{3gH}^0(\overline{1}, \overline{2}, i) \, J_1^{(1)}(\{p\}_1)$$

$$- 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, B_{2gH}^0(2,i,1,j) \, J_1^{(2)}(\{p\}_2)$$

$$- 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, A_{3,gq\to qq}^0(2,1,j) \, B_{1gH}^0(\overline{2}, i, \overline{1}) \, J_1^{(1)}(\{p\}_1)$$

$$+ 2 J_{2,QQ,g\to q}^{1,FI}(s_{\overline{2}j}) \, D_{3,gq}^0(2,i,1) \, B_{1gH}^0(\overline{2}, \overline{1}, j) \, J_1^{(1)}(\{p\}_1)$$

$$+ 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \, G_{3,q'\to g}^0(i,2,j) \, A_{3gH}^0(\overline{2}, (\widetilde{ij}), 1) \, J_1^{(1)}(\{p\}_1)$$

$$+ \Big[- 2 J_{2,QQ,g\to q}^{1,FI}(s_{\overline{2}(\widetilde{ij})}) + 2 J_{2,QQ,g\to q}^{1,FI}(s_{2j}) \Big] A_{3,q}^0(2,i,j) \, B_{1gH}^0(\overline{2}, 1, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$\text{(B.2.11)}$$

$$\widetilde{\widetilde{B}}_{2gH}^{1,XT}\left(i_q, \widetilde{\widetilde{1}}, \widetilde{\widetilde{2}}, j_{\bar{q}}\right) =$$

$$-\left[+ J_{2,QG}^{1,FI}(s_{2j}) + J_{2,QG}^{1,FI}(s_{2i}) + J_{2,GQ}^{1,IF}(s_{1j})\right.$$

$$\left.+ J_{2,GQ}^{1,IF}(s_{1i}) - J_{2,QQ}^{1,FF}(s_{ij})\right] \tilde{B}_{2gH}^0(i,1,2,j) J_1^{(2)}(\{p\}_2)$$

$$- A_{3,g\to q}^0(i,1,j) \left[B_{1gH}^1(\bar{1}, 2, (\widetilde{ij})) \, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+ \left(+ J_{2,QG}^{1,II}(s_{\bar{1}2}) + J_{2,QG}^{1,FI}(s_{2(\widetilde{ij})}) \right) B_{1gH}^0(\bar{1}, 2, (\widetilde{ij})) \right] J_1^{(1)}(\{p\}_1)$$

$$- \left[A_{3,g}^1(i,1,j)\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+ \left(+ J_{2,GQ}^{1,IF}(s_{1i}) + J_{2,GQ}^{1,IF}(s_{1j}) - J_{2,QQ}^{1,IF}(s_{\bar{1}(\widetilde{ij})}) \right) A_{3,g\to q}^0(i,1,j) \right]$$

$$\times B_{1gH}^0(\bar{1}, 2, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,g\to q}^0(i,2,j) \left[B_{1gH}^1((\widetilde{ij}), 1, \bar{2})\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+ \left(+ J_{2,GQ}^{1,II}(s_{1\bar{2}}) + J_{2,GQ}^{1,IF}(s_{1(\widetilde{ij})}) \right) B_{1gH}^0((\widetilde{ij}), 1, \bar{2}) \right] J_1^{(1)}(\{p\}_1)$$

$$- \left[A_{3,g}^1(i,2,j)\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+ \left(+ J_{2,QG}^{1,FI}(s_{2i}) + J_{2,QG}^{1,FI}(s_{2j}) - J_{2,QQ}^{1,FI}(s_{\bar{2}(\widetilde{ij})}) \right) A_{3,g\to q}^0(i,2,j) \right]$$

$$\times B_{1gH}^0((\widetilde{ij}), 1, \bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$- \left[+ J_{2,QG}^{1,FI}(s_{2j}) - J_{2,QG}^{1,II}(s_{\bar{1}2}) + J_{2,QG}^{1,FI}(s_{2i})\right.$$

$$- J_{2,QG}^{1,FI}(s_{2(\widetilde{ij})}) - J_{2,QQ}^{1,FF}(s_{ij}) + J_{2,QQ}^{1,IF}(s_{\bar{1}(\widetilde{ij})})$$

$$+ \left(- \mathcal{S}^{FI}(s_{2j}, s_{2j}, 1) + \mathcal{S}^{FI}(s_{\bar{1}2}, s_{2(\widetilde{ij})}, x_{\bar{1}2,2(\widetilde{ij})}) - \mathcal{S}^{FI}(s_{2i}, s_{2j}, x_{2i,2j}) \right.$$

$$\left.\left.+ \mathcal{S}^{FI}(s_{2(\widetilde{ij})}, s_{2(\widetilde{ij})}, 1) + \mathcal{S}^{FI}(s_{ij}, s_{2j}, x_{ij,2j}) - \mathcal{S}^{FI}(s_{\bar{1}(\widetilde{ij})}, s_{2(\widetilde{ij})}, x_{\bar{1}(\widetilde{ij}),2(\widetilde{ij})}) \right) \right]$$

$$\times A_{3,g\to q}^0(i,1,j)\, B_{1gH}^0(\bar{1}, 2, (\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$- \left[+ J_{2,GQ}^{1,IF}(s_{1j}) - J_{2,GQ}^{1,II}(s_{1\bar{2}}) + J_{2,GQ}^{1,IF}(s_{1i})\right.$$

$$- J_{2,GQ}^{1,IF}(s_{1(\widetilde{ij})}) - J_{2,QQ}^{1,FF}(s_{ij}) + J_{2,QQ}^{1,FI}(s_{\bar{2}(\widetilde{ij})})$$

$$+ \left(- \mathcal{S}^{IF}(s_{1j}, s_{1i}, x_{1j,1i}) + \mathcal{S}^{IF}(s_{\bar{2}1}, s_{1(\widetilde{ij})}, x_{\bar{2}1,1(\widetilde{ij})}) - \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) \right.$$

$$\left.\left.+ \mathcal{S}^{IF}(s_{1(\widetilde{ij})}, s_{1(\widetilde{ij})}, 1) + \mathcal{S}^{IF}(s_{ji}, s_{1i}, x_{ji,1i}) - \mathcal{S}^{IF}(s_{\bar{2}(\widetilde{ij})}, s_{1(\widetilde{ij})}, x_{\bar{2}(\widetilde{ij}),1(\widetilde{ij})}) \right) \right]$$

$$\times A_{3,g\to q}^0(i,2,j)\, B_{1gH}^0((\widetilde{ij}), 1, \bar{2})\, J_1^{(1)}(\{p\}_1) \tag{B.2.12}$$

B.3 $gg \to H$+jet at VV

The double virtual subtraction terms $\mathrm{d}\hat{\sigma}^U_{NNLO}$ mentioned in section 6.6.3 are:

$$
\widehat{\widetilde{A}}^{2,XU}_{3gH}(\hat{1},\hat{2},i) =
$$

$$
-\Bigg[+ 2\,\mathcal{A}^0_{4,gg}(s_{12}) + \tilde{\mathcal{A}}^0_{4,gg}(s_{12}) + 4\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12})
$$

$$
+ 4\,S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{A}^0_{3,gq\to qq}(s_{12}) + 4\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \Bigg] B^0_{1gH}(1,i,2)
$$

$$
-\Bigg[- \tilde{\mathcal{A}}^0_{4,g}(s_{1i}) - 2\,\mathcal{A}^0_{4,g}(s_{1i}) - \mathcal{A}^1_{3,g}(s_{1i})
$$

$$
- \frac{b_0}{\epsilon}\left(\frac{s_{1i}}{\mu^2_R}\right)^{-\epsilon} \mathcal{A}^0_{3,g\to q}(s_{1i}) + \frac{b_0}{\epsilon}\,\mathcal{A}^0_{3,g\to q}(s_{1i}) - \tilde{\mathcal{A}}^1_{3,g}(s_{1i})
$$

$$
+ 2\,\mathcal{A}^0_{3,q}(s_{1i}) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + \Gamma^{(1)}_{gg}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) - 2\,\Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i})
$$

$$
- 2\,S_{g\to q}\overline{\Gamma}^{(2)}_{qg}(z_1) + 2\,S_{q\to g}\widetilde{\overline{\Gamma}}^{(2)}_{qg}(z_1) + \ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{gg}(z_1)
$$

$$
- 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1) + 2\frac{b_0}{\epsilon}\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \Bigg] B^0_{1gH}(1,2,i)
$$

$$
-\Bigg[- \mathcal{A}^0_{3,g\to q}(s_{1i}) - 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \Bigg] B^1_{1gH}(1,2,i)
$$

$$
-\Bigg[- \mathcal{D}^0_{3,qg}(s_{12}) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) - \mathcal{D}^0_{3,g\to g}(s_{2i}) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i})
$$

$$
- 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{D}^0_{3,qg}(s_{12}) - 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{D}^0_{3,g\to g}(s_{2i})
$$

$$
+ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1)
$$

$$
+ \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{gg}(z_2) \Bigg] B^0_{1gH}(1,2,i)
$$

$$
-\Bigg[- \mathcal{A}^0_{3,g\to q}(s_{1i}) - 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \Bigg] \tilde{B}^1_{1gH}(1,2,i)
$$

$$
-\Bigg[- \mathcal{A}^0_{3,q}(s_{1i}) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) - 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{A}^0_{3,q}(s_{1i})
$$

$$
+ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1) \Bigg] B^0_{1gH}(1,2,i)
$$

$$
-\Bigg[- \tilde{\mathcal{A}}^0_{4,g}(s_{2i}) - 2\,\mathcal{A}^0_{4,g}(s_{2i}) - \mathcal{A}^1_{3,g}(s_{2i})
$$

$$
- \frac{b_0}{\epsilon}\left(\frac{s_{2i}}{\mu^2_R}\right)^{-\epsilon} \mathcal{A}^0_{3,g\to q}(s_{2i}) + \frac{b_0}{\epsilon}\,\mathcal{A}^0_{3,g\to q}(s_{2i}) - \tilde{\mathcal{A}}^1_{3,g}(s_{2i})
$$

$$
+ 2\,\mathcal{A}^0_{3,q}(s_{2i}) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i}) + \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i}) - 2\,\Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i})
$$

$$
\begin{aligned}
&- 2\ S_{g\to q}\overline{\Gamma}_{qg}^{(2)}(z_2) + 2\ S_{q\to g}\widetilde{\overline{\Gamma}}_{qg}^{(2)}(z_2) + \ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\Gamma_{gg}^{(1)}(z_2)
\end{aligned}
$$

$$
\left. - 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\Gamma_{qq}^{(1)}(z_2) + 2\frac{b_0}{\epsilon}\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\right]B_{1gH}^0(2,1,i)
$$

$$
-\left[-\ \mathcal{A}_{3,g\to q}^0(s_{2i}) - 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\right]B_{1gH}^1(2,1,i)
$$

$$
-\left[-\ \mathcal{D}_{3,gq}^0(s_{12})\otimes\mathcal{A}_{3,g\to q}^0(s_{2i}) -\ \mathcal{D}_{3,g\to g}^0(s_{1i})\otimes\mathcal{A}_{3,g\to q}^0(s_{2i})\right.
$$

$$
- 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\mathcal{D}_{3,gq}^0(s_{12}) - 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\mathcal{D}_{3,g\to g}^0(s_{1i})
$$

$$
+ \Gamma_{qq}^{(1)}(z_2)\otimes\mathcal{A}_{3,g\to q}^0(s_{2i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\Gamma_{qq}^{(1)}(z_2)
$$

$$
\left. + \Gamma_{gg}^{(1)}(z_1)\otimes\mathcal{A}_{3,g\to q}^0(s_{2i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\Gamma_{gg}^{(1)}(z_1)\right]B_{1gH}^0(2,1,i)
$$

$$
-\left[-\ \mathcal{A}_{3,g\to q}^0(s_{2i}) - 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\right]\tilde{B}_{1gH}^1(2,1,i)
$$

$$
-\left[-\ \mathcal{A}_{3,q}^0(s_{2i})\otimes\mathcal{A}_{3,g\to q}^0(s_{2i}) - 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\mathcal{A}_{3,q}^0(s_{2i})\right.
$$

$$
\left. + \Gamma_{qq}^{(1)}(z_2)\otimes\mathcal{A}_{3,g\to q}^0(s_{2i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\Gamma_{qq}^{(1)}(z_2)\right]B_{1gH}^0(2,1,i)
$$

$$
-\left[+\ \tilde{\mathcal{G}}_{4,gg}^0(s_{12}) +\ \tilde{\mathcal{G}}_{4,g}^0(s_{1i}) +\ \tilde{\mathcal{G}}_{4,g}^0(s_{2i})\right.
$$

$$
+ \tilde{\mathcal{G}}_{3,g}^1(s_{1i}) + \tilde{\mathcal{G}}_{3,g}^1(s_{2i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\otimes\mathcal{G}_{3,qg\to gg}^0(s_{12})
$$

$$
+ 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\otimes\mathcal{G}_{3,q'\to g}^0(s_{1i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\mathcal{G}_{3,gq\to gg}^0(s_{12})
$$

$$
+ 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\mathcal{G}_{3,q'\to g}^0(s_{2i}) + 2\widetilde{\overline{\Gamma}}_{gg,F}^{(2)}(z_1) + 2\widetilde{\overline{\Gamma}}_{gg,F}^{(2)}(z_2)
$$

$$
\left. + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\otimes S_{q\to g}\Gamma_{gq}^{(1)}(z_1) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes S_{q\to g}\Gamma_{gq}^{(1)}(z_2)\right]A_{3gH}^0(1,2,i)
$$

$$
(\text{B.3.13})
$$

$$
\widehat{A}_{3gH}^{2,XU}(\hat{1},\hat{2},i) =
$$

$$
-\left[+ 2\,\mathcal{D}_{3,g\to q}^0(s_{1i})\otimes\mathcal{D}_{3,g\to q}^0(s_{2i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\otimes\mathcal{D}_{3,g\to q}^0(s_{2i})\right.
$$

$$
\left. + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\otimes\mathcal{D}_{3,g\to q}^0(s_{1i}) + 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\otimes S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\right]B_{1gH}^0(1,i,2)
$$

$$
-\left[-\ \mathcal{A}_{3,g\to q}^0(s_{1i}) - 2\ S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\right]B_{1gH}^1(1,2,i)
$$

$$
-\left[-\ \mathcal{D}_{3,qg}^0(s_{12})\otimes\mathcal{A}_{3,g\to q}^0(s_{1i}) + \Gamma_{qq}^{(1)}(z_1)\otimes\mathcal{A}_{3,g\to q}^0(s_{1i})\right.
$$

$$+ \frac{1}{2} \, \Gamma_{gg}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i}) - 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \mathcal{D}_{3,qg}^0(s_{12})$$

$$+ \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{gg}^{(1)}(z_2) + 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1) \Big] B_{1gH}^0(1, 2, i)$$

$$- \Big[- \mathcal{D}_{3,g\to g}^0(s_{2i}) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i}) + \frac{1}{2} \, \Gamma_{gg}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i})$$

$$- 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \mathcal{D}_{3,g\to g}^0(s_{2i}) + \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{gg}^{(1)}(z_2) \Big] B_{1gH}^0(1, 2, i)$$

$$- \Big[- 2 \, \mathcal{A}_{4,g}^0(s_{1i}) - \mathcal{A}_{3,g}^1(s_{1i}) - \frac{b_0}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}_{3,g\to q}^0(s_{1i})$$

$$+ \frac{b_0}{\epsilon} \, \mathcal{A}_{3,g\to q}^0(s_{1i}) + \, \mathcal{A}_{3,q}^0(s_{1i}) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i}) + \Gamma_{gg}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i})$$

$$- \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i}) - 2 \, S_{g\to q} \overline{\Gamma}_{qg}^{(2)}(z_1) + 2 \frac{b_0}{\epsilon} \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1)$$

$$+ \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{gg}^{(1)}(z_1) - \, S_{g\to q} \Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1) \Big] B_{1gH}^0(1, 2, i)$$

$$- \Big[- \mathcal{A}_{3,g\to q}^0(s_{2i}) - 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \Big] B_{1gH}^1(2, 1, i)$$

$$- \Big[- \mathcal{D}_{3,gq}^0(s_{12}) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) + \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$+ \frac{1}{2} \, \Gamma_{gg}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) - 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \mathcal{D}_{3,gq}^0(s_{12})$$

$$+ \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg}^{(1)}(z_1) + 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_2) \Big] B_{1gH}^0(2, 1, i)$$

$$- \Big[- \mathcal{D}_{3,g\to g}^0(s_{1i}) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) + \frac{1}{2} \, \Gamma_{gg}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$- 2 \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \mathcal{D}_{3,g\to g}^0(s_{1i}) + \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg}^{(1)}(z_1) \Big] B_{1gH}^0(2, 1, i)$$

$$- \Big[- 2 \, \mathcal{A}_{4,g}^0(s_{2i}) - \mathcal{A}_{3,g}^1(s_{2i}) - \frac{b_0}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$+ \frac{b_0}{\epsilon} \, \mathcal{A}_{3,g\to q}^0(s_{2i}) + \, \mathcal{A}_{3,q}^0(s_{2i}) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) + \Gamma_{gg}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$- \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) - 2 \, S_{g\to q} \overline{\Gamma}_{qg}^{(2)}(z_2) + 2 \frac{b_0}{\epsilon} \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2)$$

$$+ \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg}^{(1)}(z_2) - \, S_{g\to q} \Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_2) \Big] B_{1gH}^0(2, 1, i)$$

$$- \Big[+ \mathcal{G}_{3,g}^0(s_{1i}) + \mathcal{G}_{3,g}^0(s_{2i}) - 2 \, \Gamma_{gg,F}^{(1)}(z_1)$$

$$- 2 \, \Gamma_{gg,F}^{(1)}(z_2) \Big] \left(+ A_{3gH}^1(1, 2, i) - \frac{b_0}{\epsilon} A_{3gH}^0(1, 2, i) \right)$$

$$-\Big[+ \mathcal{F}^0_{3,g}(s_{2i}) + \mathcal{F}^0_{3,g}(s_{1i}) + 2\,\mathcal{F}^0_{3,gg}(s_{12})$$

$$- 2\,\Gamma^{(1)}_{gg}(z_1) - 2\,\Gamma^{(1)}_{gg}(z_2) \Big] \left(+\hat{A}^1_{3gH}(1,2,i) - \frac{b_F}{\epsilon} A^0_{3gH}(1,2,i) \right)$$

$$-\Big[+ \frac{1}{2}\,\mathcal{G}^0_{3,g}(s_{1i}) \otimes \mathcal{F}^0_{3,g}(s_{1i}) + \frac{1}{2}\,\mathcal{F}^0_{3,g}(s_{2i}) \otimes \mathcal{G}^0_{3,g}(s_{1i}) + \mathcal{G}^0_{3,g}(s_{1i}) \otimes \mathcal{F}^0_{3,gg}(s_{12})$$

$$- \Gamma^{(1)}_{gg}(z_1) \otimes \mathcal{G}^0_{3,g}(s_{1i}) - \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{G}^0_{3,g}(s_{1i}) + \frac{1}{2}\,\mathcal{G}^0_{3,g}(s_{2i}) \otimes \mathcal{F}^0_{3,g}(s_{2i})$$

$$+ \frac{1}{2}\,\mathcal{F}^0_{3,g}(s_{1i}) \otimes \mathcal{G}^0_{3,g}(s_{2i}) + \mathcal{G}^0_{3,g}(s_{2i}) \otimes \mathcal{F}^0_{3,gg}(s_{12}) - \Gamma^{(1)}_{gg}(z_1) \otimes \mathcal{G}^0_{3,g}(s_{2i})$$

$$- \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{G}^0_{3,g}(s_{2i}) - 2\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{F}^0_{3,gg}(s_{12}) - 2\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{F}^0_{3,gg}(s_{12})$$

$$- \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{F}^0_{3,g}(s_{1i}) - \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{F}^0_{3,g}(s_{1i}) - \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{F}^0_{3,g}(s_{2i})$$

$$- \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{F}^0_{3,g}(s_{2i}) + \Gamma^{(1)}_{gg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) + \Gamma^{(1)}_{gg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_1)$$

$$+ \Gamma^{(1)}_{gg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1) + \Gamma^{(1)}_{gg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_2) \Big] A^0_{3gH}(1,2,i)$$

$$-\Big[+ \mathcal{G}^1_{3,g}(s_{1i}) + \frac{b_0}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{G}^0_{3,g}(s_{1i}) + \hat{\mathcal{F}}^1_{3,g}(s_{1i})$$

$$+ \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{F}^0_{3,g}(s_{1i}) + 2\,\mathcal{G}^0_{4,g}(s_{1i}) - \mathcal{G}^0_{3,g}(s_{1i}) \otimes \mathcal{F}^0_{3,g}(s_{1i})$$

$$+ 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{G}^0_{3,q'\to g}(s_{1i}) - \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{F}^0_{3,g}(s_{1i})$$

$$- \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{F}^0_{3,g}(s_{1i}) - 2\,\overline{\Gamma}^{(2)}_{gg,F}(z_1)$$

$$+ 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes S_{q\to g}\Gamma^{(1)}_{gq}(z_1) + \Gamma^{(1)}_{gg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1) \Big] A^0_{3gH}(1,2,i)$$

$$-\Big[+ \mathcal{G}^1_{3,g}(s_{2i}) + \frac{b_0}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{G}^0_{3,g}(s_{2i}) + \hat{\mathcal{F}}^1_{3,g}(s_{2i})$$

$$+ \frac{b_F}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{F}^0_{3,g}(s_{2i}) + 2\,\mathcal{G}^0_{4,g}(s_{2i}) - \mathcal{G}^0_{3,g}(s_{2i}) \otimes \mathcal{F}^0_{3,g}(s_{2i})$$

$$+ 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{G}^0_{3,q'\to g}(s_{2i}) - \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{F}^0_{3,g}(s_{2i})$$

$$- \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{F}^0_{3,g}(s_{2i}) - 2\,\overline{\Gamma}^{(2)}_{gg,F}(z_2)$$

$$+ 2\,S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes S_{q\to g}\Gamma^{(1)}_{gq}(z_2) + \Gamma^{(1)}_{gg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_2) \Big] A^0_{3gH}(1,2,i)$$

$$-\Big[+ 2\,\hat{\mathcal{F}}^1_{3,gg}(s_{12}) + 2\frac{b_F}{\epsilon}\,\mathcal{F}^0_{3,gg}(s_{12}) \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} + 2\,\mathcal{G}^0_{4,gg}(s_{12})$$

$$+ 2\, S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{G}^0_{3,gq\to gg}(s_{12}) + 2\, S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{G}^0_{3,qg\to gg}(s_{12})$$

$$+ \Gamma^{(1)}_{gg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) + \Gamma^{(1)}_{gg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_1) \Bigg] A^0_{3gH}(1,2,i) \tag{B.3.14}$$

$$\widehat{\widetilde{A}}^{2,XU}_{3gH}(\hat{1},\hat{2},i) =$$

$$-\Bigg[-\mathcal{G}^0_{3,g}(s_{1i}) - \mathcal{G}^0_{3,g}(s_{2i}) \Bigg] \left(-\hat{A}^1_{3gH}(1,2,i) + \frac{b_F}{\epsilon} A^0_{3gH}(1,2,i) \right)$$

$$-\Bigg[-2\,\Gamma^{(1)}_{gg,F}(z_1) - 2\,\Gamma^{(1)}_{gg,F}(z_2) \Bigg] \hat{A}^1_{3gH}(1,2,i)$$

$$-\Bigg[-\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{G}^0_{3,g}(s_{1i}) - \Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{G}^0_{3,g}(s_{2i}) - \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{G}^0_{3,g}(s_{1i})$$

$$- \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{G}^0_{3,g}(s_{2i}) \Bigg] A^0_{3gH}(1,2,i)$$

$$-\Bigg[+\hat{\mathcal{G}}^1_{3,g}(s_{1i}) + \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{G}^0_{3,g}(s_{1i}) + \hat{\mathcal{G}}^1_{3,g}(s_{2i})$$

$$+ \frac{b_F}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{G}^0_{3,g}(s_{2i}) + 2\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) \Bigg] A^0_{3gH}(1,2,i)$$

$$-\Bigg[-\mathcal{A}^0_{3,g\to q}(s_{1i}) - 2\, S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \Bigg] \hat{B}^1_{1gH}(1,2,i)$$

$$-\Bigg[+\frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i})$$

$$+ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1) + S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) \Bigg] B^0_{1gH}(1,2,i)$$

$$-\Bigg[-\hat{\mathcal{A}}^1_{3,g}(s_{1i}) - \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}^0_{3,g\to q}(s_{1i}) + \frac{b_F}{\epsilon} \mathcal{A}^0_{3,g\to q}(s_{1i})$$

$$+ \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i})$$

$$+ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) + S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1)$$

$$+ 4\, S_{g\to q}\overline{\Gamma}^{(2)}_{qg,F}(z_1) \Bigg] B^0_{1gH}(1,2,i)$$

$$-\Bigg[-\mathcal{A}^0_{3,g\to q}(s_{2i}) - 2\, S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \Bigg] \hat{B}^1_{1gH}(2,1,i)$$

$$-\Bigg[+\frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i})$$

$$+ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_2) + S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_1) \Bigg] B^0_{1gH}(2,1,i)$$

$$-\Bigg[-\hat{\mathcal{A}}^1_{3,g}(s_{2i}) - \frac{b_F}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}^0_{3,g\to q}(s_{2i}) + \frac{b_F}{\epsilon} \mathcal{A}^0_{3,g\to q}(s_{2i})$$

$$+ \frac{1}{2} \; \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) + \frac{1}{2} \; \Gamma_{gg,F}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$+ \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_1) + \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_2)$$

$$+ 4 \; S_{g\to q}\overline{\Gamma}_{qg,F}^{(2)}(z_2)\bigg] B_{1gH}^0(2,1,i) \tag{B.3.15}$$

$$\widehat{\widetilde{\widehat{B}}}_{1gH}^{1,XU}(\hat{1}_q, \hat{2}, i_{\bar{q}}) =$$

$$-\bigg[- \mathcal{A}_{3,g\to q}^0(s_{2i}) - 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\bigg] \hat{B}_{1gH}^1(2,1,i)$$

$$-\bigg[- 2\frac{b_F}{\epsilon} \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2)\bigg] B_{1gH}^0(2,1,i)$$

$$-\bigg[+ \Gamma_{gg,F}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i}) + \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$+ 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_1)\bigg] B_{1gH}^0(2,1,i)$$

$$-\bigg[- \hat{\mathcal{A}}_{3,g}^1(s_{2i}) - \frac{b_F}{\epsilon}\left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{A}_{3,g\to q}^0(s_{2i}) + \frac{b_F}{\epsilon} \mathcal{A}_{3,g\to q}^0(s_{2i})$$

$$+ 2\frac{b_F}{\epsilon} \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2) + 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_2) + 4 \; S_{g\to q}\overline{\Gamma}_{qg,F}^{(2)}(z_2)$$

$$\bigg] B_{1gH}^0(2,1,i)$$

$$-\bigg[- \mathcal{A}_{3,g\to q}^0(s_{1i}) - 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\bigg] \hat{B}_{1gH}^1(1,2,i)$$

$$-\bigg[- 2\frac{b_F}{\epsilon} \; S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\bigg] B_{1gH}^0(1,2,i)$$

$$-\bigg[+ \Gamma_{gg,F}^{(1)}(z_1) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i}) + \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{A}_{3,g\to q}^0(s_{1i})$$

$$+ 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{gg,F}^{(1)}(z_1)\bigg] B_{1gH}^0(1,2,i)$$

$$-\bigg[- \hat{\mathcal{A}}_{3,g}^1(s_{1i}) - \frac{b_F}{\epsilon}\left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{A}_{3,g\to q}^0(s_{1i}) + \frac{b_F}{\epsilon} \mathcal{A}_{3,g\to q}^0(s_{1i})$$

$$+ 2\frac{b_F}{\epsilon} \; S_{g\to q}\Gamma_{qg}^{(1)}(z_1) + 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_1) \otimes \Gamma_{gg,F}^{(1)}(z_2) + 4 \; S_{g\to q}\overline{\Gamma}_{qg,F}^{(2)}(z_1)$$

$$\bigg] B_{1gH}^0(1,2,i) \tag{B.3.16}$$

$$\widetilde{\widetilde{\widetilde{B}}}_{1gH}^{1,XU}(\hat{1}_q, \hat{2}, i_{\bar{q}}) =$$

$$-\bigg[- \mathcal{A}_{3,g\to q}^0(s_{1i}) - 2 \; S_{g\to q}\Gamma_{qg}^{(1)}(z_1)\bigg] \tilde{B}_{1gH}^1(1,2,i)$$

$$
-\Bigg[-\mathcal{A}^0_{3,g\to q}(s_{1i}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) - 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{A}^0_{3,q}(s_{1i})
$$

$$
+ \ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) + 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1) \Bigg] B^0_{1gH}(1,2,i)
$$

$$
-\Bigg[-\tilde{\mathcal{A}}^0_{4,g}(s_{1i}) - \tilde{\mathcal{A}}^1_{3,g}(s_{1i}) + \mathcal{A}^0_{3,g\to q}(s_{1i}) \otimes \mathcal{A}^0_{3,q}(s_{1i})
$$

$$
- \ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i}) - \ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1) + 2 \ S_{q\to g}\widetilde{\overline{\Gamma}}^{(2)}_{qg}(z_1)
$$

$$
\Bigg] B^0_{1gH}(1,2,i)
$$

$$
-\Bigg[-\mathcal{A}^0_{3,g\to q}(s_{2i}) - 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \Bigg] \tilde{B}^1_{1gH}(2,1,i)
$$

$$
-\Bigg[-\mathcal{A}^0_{3,g\to q}(s_{2i}) \otimes \mathcal{A}^0_{3,q}(s_{2i}) - 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{A}^0_{3,q}(s_{2i})
$$

$$
+ \ \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i}) + 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{qq}(z_2) \Bigg] B^0_{1gH}(2,1,i)
$$

$$
-\Bigg[-\tilde{\mathcal{A}}^0_{4,g}(s_{2i}) - \tilde{\mathcal{A}}^1_{3,g}(s_{2i}) + \mathcal{A}^0_{3,g\to q}(s_{2i}) \otimes \mathcal{A}^0_{3,q}(s_{2i})
$$

$$
- \ \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{A}^0_{3,g\to q}(s_{2i}) - \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{qq}(z_2) + 2 \ S_{q\to g}\widetilde{\overline{\Gamma}}^{(2)}_{qg}(z_2)
$$

$$
\Bigg] B^0_{1gH}(2,1,i)
$$

$$
-\Bigg[+\tilde{\mathcal{A}}^0_{4,gg}(s_{12}) + 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) + 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{A}^0_{3,gq\to qq}(s_{12})
$$

$$
+ 2 \ S_{g\to q}\Gamma^{(1)}_{qg}(z_1) \otimes \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \Bigg] B^0_{1gH}(1,i,2) \tag{B.3.17}
$$

Appendix C

Explicit results of antenna subtraction terms for $qg \to H{+}$jet processes at NNLO

C.1 $qg \to H{+}$jet at RR

The double real subtraction terms $\mathrm{d}\hat{\sigma}^S_{NNLO}$ mentioned in section 7.5.1 are:

$$\widetilde{B}^{0,XS}_{3gH}(\hat{1}_q, \widetilde{\hat{2}}, i, j, k_{\bar{q}}) =$$

$$-A^0_{3,qg \to qq}(1,2,k)\, B^0_{2gH}(\overline{1}, i, j, \overline{2})\, J^{(2)}_1(\{p\}_2)$$

$$+d^0_{3,q}(1,i,j)\, \tilde{B}^0_{2gH}(\overline{1}, 2, (\widetilde{ij}), k)\, J^{(2)}_1(\{p\}_2)$$

$$+d^0_3(k,j,i)\, \tilde{B}^0_{2gH}(1, 2, (\widetilde{ji}), (\widetilde{kj}))\, J^{(2)}_1(\{p\}_2)$$

$$-\tilde{A}^0_4(1,2,j,k)\, B^0_{1gH}(\overline{1}, i, \overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+A^0_{3,qg \to qq}(1,2,k)\, A^0_{3,q\bar{q}}(\overline{1}, j, \overline{2})\, B^0_{1gH}(\overline{\overline{1}}, i, \overline{\overline{2}})\, J^{(1)}_1(\{p\}_1)$$

$$+A^0_{3,q}(1,j,k)\, A^0_{3,qg \to qq}(\overline{1}, 2, (\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{1}}, i, \overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+A^0_4(1,i,j,k)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1)$$

$$-d^0_{3,q}(1,i,j)\, A^0_{3,q}(\overline{1}, (\widetilde{ij}), k)\, B^0_{1gH}(\overline{\overline{1}}, 2, ((\widetilde{k(\widetilde{ij})})))\, J^{(1)}_1(\{p\}_1)$$

$$-d^0_3(k,j,i)\, A^0_{3,q}(1, (\widetilde{ij}), (\widetilde{kj}))\, B^0_{1gH}(\overline{1}, 2, (\widetilde{\widetilde{ij}, \widetilde{kj}}))\, J^{(1)}_1(\{p\}_1)$$

$$-A^0_{3,q}(1,i,k)\, A^0_{3,qg \to qq}(\overline{1}, 2, (\widetilde{ik}))\, B^0_{1gH}(\overline{1}, j, \overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+d^0_{3,q}(1,i,j)\, A^0_{3,qg \to qq}(\overline{1}, 2, k)\, B^0_{1gH}(\overline{\overline{1}}, (\widetilde{ij}), \overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ d_3^0(k,i,j)\, A_{3,qg\to qq}^0(1,2,(\widetilde{ik}))\, B_{1gH}^0(\overline{1},(\widetilde{ij}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ \left[+ S_{\overline{1}ik}^{IF} - S_{\overline{\overline{1}}i\overline{2}}^{IF} - S_{ki(\widetilde{ij})}^{IF} + S_{\overline{2}i(\widetilde{ij})}^{IF} - S_{\overline{1}i(\widetilde{ij})}^{IF} + S_{\overline{\overline{1}}i(\widetilde{ij})}^{IF} \right]$$

$$\times A_{3,qg\to qq}^0(\overline{1},2,k)\, B_{1gH}^0(\overline{\overline{1}},(\widetilde{ij}),\overline{2})\, J_1^{(1)}(\{p\}_1) \tag{C.1.1}$$

$\widetilde{B}_{3gH}^{0,YS}(\hat{1}_q, \widetilde{i}, \hat{2}, j, k_{\bar{q}}) =$

$$+ A_{3,q}^0(1,i,k)\, B_{2gH}^0(\overline{1},j,2,(\widetilde{ik}))\, J_1^{(2)}(\{p\}_2)$$

$$+ D_{3,qg}^0(1,j,2)\, \tilde{B}_{2gH}^0(\overline{1},i,\overline{2},k)\, J_1^{(2)}(\{p\}_2)$$

$$- A_{3,qg\to qq}^0(1,2,k)\, \tilde{B}_{2gH}^0(\overline{1},i,j,\overline{2})\, J_1^{(2)}(\{p\}_2)$$

$$+ A_{3,q}^0(1,i,k)\, B_{2gH}^0(\overline{1},2,j,(\widetilde{ik}))\, J_1^{(2)}(\{p\}_2)$$

$$+ d_{3,g}^0(k,j,2)\, \tilde{B}_{2gH}^0(1,i,\overline{2},(\widetilde{kj}))\, J_1^{(2)}(\{p\}_2)$$

$$+ \tilde{A}_4^0(1,i,j,k)\, B_{1gH}^0(\overline{1},2,(\widetilde{ijk}))\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q}^0(1,i,k)\, A_{3,q}^0(\overline{1},j,(\widetilde{ik}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ik})}))\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q}^0(1,j,k)\, A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$- \tilde{A}_4^0(1,i,2,k)\, B_{1gH}^0(\overline{1},j,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(1,i,k)\, A_{3,qg\to qq}^0(\overline{1},2,(\widetilde{ki}))\, B_{1gH}^0(\overline{\overline{1}},j,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,qg\to qq}^0(1,2,k)\, A_{3,q\bar{q}}^0(\overline{1},i,\overline{2})\, B_{1gH}^0(\overline{\overline{1}},j,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- A_4^0(1,j,2,k)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ D_{3,qg}^0(1,j,2)\, A_{3,qg\to qq}^0(\overline{1},\overline{2},k)\, B_{1gH}^0(\overline{\overline{1}},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,qg\to qq}^0(1,2,k)\, A_{3,q\bar{q}}^0(\overline{1},j,\overline{2})\, B_{1gH}^0(\overline{\overline{1}},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- A_4^0(1,2,j,k)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ d_{3,g}^0(k,j,2)\, A_{3,qg\to qq}^0(1,\overline{2},(\widetilde{kj}))\, B_{1gH}^0(\overline{1},i,\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(1,j,k)\, A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$- D_{3,qg}^0(1,j,2)\, A_{3,q}^0(\overline{1},i,k)\, B_{1gH}^0(\overline{\overline{1}},\overline{2},(\widetilde{ik}))\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,g}^0(k,j,2)\, A_{3,q}^0(1,i,(\widetilde{jk}))\, B_{1gH}^0(\overline{1},\overline{2},(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$- \left[+ S_{\overline{1}j2}^{IF} - S_{\overline{\overline{1}}j2}^{IF} - S_{2j(\widetilde{jk})}^{IF} + S_{2j(\widetilde{i(\widetilde{jk})})}^{IF} + S_{\overline{1}j(\widetilde{jk})}^{IF} - S_{\overline{\overline{1}}j(\widetilde{i(\widetilde{jk})})}^{IF} \right]$$

$$\times A_{3,q}^0(\overline{1},i,(\widetilde{jk}))\, B_{1gH}^0(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$- \tilde{G}_4^0(j,1,k,i)\, A_{3gH}^0(\overline{1},2,(\widetilde{jki}))\, J_1^{(1)}(\{p\}_1)$$

$$+ A^0_{3,q}(1,j,k)\, G^0_{3,q'\to g}(i,\overline{1},(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,q}(1,i,k)\, G^0_{3,q'\to g}(j,\overline{1},(\widetilde{ik}))\, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ik})}))\, J^{(1)}_1(\{p\}_1) \qquad \text{(C.1.2)}$$

$$\widetilde{\widetilde{B}}^{0,XS}_{3gH}(\hat{1}_q,\widetilde{\hat{2}},\widetilde{i},\widetilde{j},k_{\bar{q}}) =$$

$$-A^0_{3,qg\to qq}(1,2,k)\, \tilde{B}^0_{2gH}(\overline{1},i,j,\overline{2})\, J^{(2)}_1(\{p\}_2)$$

$$+A^0_{3,q}(1,i,k)\, \tilde{B}^0_{2gH}(\overline{1},2,j,(\widetilde{ik}))\, J^{(2)}_1(\{p\}_2)$$

$$+A^0_{3,q}(1,j,k)\, \tilde{B}^0_{2gH}(\overline{1},2,i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- \tilde{A}^0_4(1,2,i,k)\, B^0_{1gH}(\overline{1},j,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,qg\to qq}(1,2,k)\, A^0_{3,q\bar{q}}(\overline{1},i,\overline{2})\, B^0_{1gH}(\overline{\overline{1}},j,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,q}(1,i,k)\, A^0_{3,qg\to qq}(\overline{1},2,(\widetilde{ik}))\, B^0_{1gH}(\overline{\overline{1}},j,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \tilde{A}^0_4(1,i,j,k)\, B^0_{1gH}(\overline{1},2,(\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,q}(1,i,k)\, A^0_{3,q}(\overline{1},j,(\widetilde{ik}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{j(\widetilde{ik})}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,q}(1,j,k)\, A^0_{3,q}(\overline{1},i,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$- \tilde{A}^0_4(1,2,j,k)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,qg\to qq}(1,2,k)\, A^0_{3,q\bar{q}}(\overline{1},j,\overline{2})\, B^0_{1gH}(\overline{\overline{1}},i,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,q}(1,j,k)\, A^0_{3,qg\to qq}(\overline{1},2,(\widetilde{jk}))\, B^0_{1gH}(\overline{\overline{1}},i,\overline{2})\, J^{(1)}_1(\{p\}_1) \qquad \text{(C.1.3)}$$

$$C^{0,XS}_{1gH}(\hat{1}_q,\hat{2},i_{\bar{Q}},j_Q,k_{\bar{q}}) =$$

$$-A^0_{3,qg\to qq}(i,2,1)\, C^0_{0gH}(\overline{1},\overline{2},j,k)\, J^{(2)}_1(\{p\}_2)$$

$$+E^0_3(k,j,i)\, B^0_{2gH}(1,2,(\widetilde{ji}),(\widetilde{kj}))\, J^{(2)}_1(\{p\}_2)$$

$$-E^0_{3,q'\to g}(j,k,1)\, B^0_{2gH}((\widetilde{jk}),\overline{1},2,i)\, J^{(2)}_1(\{p\}_2)$$

$$+ E^0_4(k,j,i,2)\, B^0_{1gH}(1,\overline{2},(\widetilde{kji}))\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_3(k,j,i)\, D^0_{3,g}((\widetilde{kj}),(\widetilde{ji}),2)\, B^0_{1gH}(1,\overline{2},(\widetilde{\widetilde{kj},\widetilde{ji}}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,qg\to qq}(k,2,1)\, E^0_{3,q}(\overline{2},j,i)\, B^0_{1gH}(1,\overline{\overline{2}},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_3(k,j,i)\, A^0_{3,qg\to qq}((\widetilde{kj}),2,1)\, B^0_{1gH}(\overline{1},\overline{\overline{2}},(\widetilde{ji}))\, J^{(1)}_1(\{p\}_1)$$

$$- {}_{-}4^0(i,j,1,k)\, A^0_{3gH}(\overline{1},2,(\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_{3,q'\to g}(j,k,1)\, G^0_{3,g}(\overline{1},i,(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{1}},2,(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E_3^0(k,j,i) \, G_{3,q' \to g}^0((\widetilde{ji}), 1, (\widetilde{kj})) \, A_{3gH}^0(\overline{1}, 2, (\widetilde{\widetilde{ji}, \widetilde{kj}})) \, J_1^{(1)}(\{p\}_1)$$

$$+ B_4^0(j,k,1,i) \, B_{1gH}^0(\overline{1}, 2, (\widetilde{jki})) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,q' \to g}^0(j,k,1) \, A_{3,g \to q}^0((\widetilde{jk}), \overline{1}, i) \, B_{1gH}^0(\overline{\overline{1}}, 2, ((\widetilde{\widetilde{jk}})i)) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,qg \to qq}^0(j,2,1) \, C_{0gH}^0(\overline{1}, i, \overline{2}, k) \, J_1^{(2)}(\{p\}_2)$$

$$- A_{3,qg \to qq}^0(k,2,1) \, C_{0gH}^0(\overline{1}, i, j, \overline{2}) \, J_1^{(2)}(\{p\}_2)$$

$$+ E_3^0(k,i,j) \, B_{2gH}^0(1, (\widetilde{ij}), 2, (\widetilde{ki})) \, J_1^{(2)}(\{p\}_2)$$

$$- E_{3,q' \to g}^0(j,k,1) \, B_{2gH}^0(i, \overline{1}, 2, (\widetilde{jk})) \, J_1^{(2)}(\{p\}_2)$$

$$+ E_4^0(1,i,j,2) \, B_{1gH}^0(\overline{1}, \overline{2}, k) \, J_1^{(1)}(\{p\}_1)$$

$$- E_3^0(k,i,j) \, D_{3,qg}^0(1, (\widetilde{ij}), 2) \, B_{1gH}^0(\overline{1}, \overline{2}, (\widetilde{ki})) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,qg \to qq}^0(j,2,1) \, E_{3,qq' \to qg}^0(\overline{1}, i, \overline{2}) \, B_{1gH}^0(\overline{\overline{1}}, \overline{\overline{2}}, k) \, J_1^{(1)}(\{p\}_1)$$

$$- 4^0(1,k,i,j) \, A_{3gH}^0(\overline{1}, 2, (\widetilde{kij})) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_{3,q' \to g}^0(j,k,1) \, G_{3,g}^0(\overline{1}, i, (\widetilde{jk})) \, A_{3gH}^0(\overline{\overline{1}}, 2, (i(\widetilde{\widetilde{jk}}))) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_3^0(k,i,j) \, G_{3,q' \to g}^0((\widetilde{ij}), 1, (\widetilde{ki})) \, A_{3gH}^0(\overline{1}, 2, (\widetilde{\widetilde{ij}, \widetilde{ki}})) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_4^0(j,k,1,2) \, B_{1gH}^0(i, \overline{1}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,q' \to g}^0(j,k,1) \, D_{3,gg}^0((\widetilde{jk}), \overline{1}, 2) \, B_{1gH}^0(i, \overline{\overline{1}}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,qg \to qq}^0(j,2,1) \, E_{3,qq' \to qg}^0(\overline{2}, k, \overline{1}) \, B_{1gH}^0(i, \overline{\overline{1}}, \overline{\overline{2}}) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_4^0(i,k,1,2) \, B_{1gH}^0(j, \overline{1}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,q' \to g}^0(i,k,1) \, D_{3,gg}^0((\widetilde{ik}), \overline{1}, 2) \, B_{1gH}^0(j, \overline{\overline{1}}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,qg \to qq}^0(i,2,1) \, E_{3,qq' \to qg}^0(\overline{2}, k, \overline{1}) \, B_{1gH}^0(j, \overline{\overline{1}}, \overline{\overline{2}}) \, J_1^{(1)}(\{p\}_1)$$

$$- B_4^0(j,k,1,i) \, B_{1gH}^0((\widetilde{jki}), \overline{1}, 2) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_{3,q' \to g}^0(i,k,1) \, A_{3,g \to q}^0(j, \overline{1}, (\widetilde{ik})) \, B_{1gH}^0((j(\widetilde{\widetilde{ik}})), \overline{\overline{1}}, 2) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_3^0(k,i,j) \, A_{3,qg \to qq}^0((\widetilde{ki}), 2, 1) \, B_{1gH}^0(\overline{1}, (\widetilde{ij}), \overline{\overline{2}}) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,qg \to qq}^0(i,2,1) \, E_{3,q' \to g}^0(k,j,\overline{2}) \, B_{1gH}^0(\overline{1}, \overline{\overline{2}}, (\widetilde{kj})) \, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,g \to q}^0(i,2,j) \, E_{3,q' \to g}^0((\widetilde{ij}), k, 1) \, B_{1gH}^0(\overline{2}, \overline{1}, ((\widetilde{\widetilde{ij}})k)) \, J_1^{(2)}(\{p\}_1)$$

$$+ E_{3,q' \to g}^0(j,k,1) \, d_{3,gg \to qg}^0((\widetilde{jk}), \overline{1}, 2) \, B_{1gH}^0(i, \overline{\overline{1}}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- E_{3,q' \to g}^0(i,k,1) \, d_{3,gg \to qg}^0(j, \overline{1}, 2) \, B_{1gH}^0((\widetilde{ik}), \overline{\overline{1}}, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$+ E_{3,q' \to g}^0(i,k,1) \, A_{3,g \to q}^0((\widetilde{ik}), 2, j) \, B_{1gH}^0(\overline{2}, \overline{1}, ((\widetilde{\widetilde{ik}})j)) \, J_1^{(2)}(\{p\}_1) \quad \text{(C.1.4)}$$

$$\widetilde{C}^{0,XS}_{1gH}(\hat{1}_q, \hat{2}, k_{\bar{q}}, i_Q, j_{\bar{Q}}) =$$

$$+ A^0_{3,qg \to qq}(1,2,k)\, C^0_{0gH}(\overline{1}, i, j, \overline{2})\, J^{(2)}_1(\{p\}_2)$$

$$+ A^0_{3,g \to q}(i,2,j)\, C^0_{0gH}(1, \overline{2}, (\widetilde{ij}), k)\, J^{(2)}_1(\{p\}_2)$$

$$- E^0_3(k,i,j)\, \tilde{B}^0_{2gH}(1, 2, (\widetilde{ij}), (\widetilde{ki}))\, J^{(2)}_1(\{p\}_2)$$

$$+ E^0_{3,q}(i,k,1)\, \tilde{B}^0_{2gH}((\widetilde{ik}), \overline{1}, 2, j)\, J^{(2)}_1(\{p\}_2)$$

$$- B^0_4(1,i,j,k)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_3(k,j,i)\, A^0_{3,q}(1, (\widetilde{ji}), (\widetilde{kj}))\, B^0_{1gH}(\overline{1}, 2, (\widetilde{\widetilde{ji}, \widetilde{kj}}))\, J^{(1)}_1(\{p\}_1)$$

$$- B^0_4(i,1,k,j)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ikj}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_{3,q' \to g}(i,1,k)\, A^0_{3,g \to q}((\widetilde{ik}), \overline{1}, j)\, B^0_{1gH}(\overline{\overline{1}}, 2, (\widetilde{j(\widetilde{ik})}))\, J^{(1)}_1(\{p\}_1)$$

$$- \tilde{E}^0_4(k,i,2,j)\, B^0_{1gH}(1, \overline{2}, (\widetilde{kij}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,g \to q}(i,2,j)\, E^0_{3,q' \to g}(k, \overline{2}, (\widetilde{ij}))\, B^0_{1gH}(1, \overline{\overline{2}}, (\widetilde{k(\widetilde{ij})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_{3,q}(i,k,1)\, A^0_{3,g \to q}((\widetilde{ik}), 2, j)\, B^0_{1gH}(((\widetilde{ik})j), \overline{1}, \overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_3(k,i,j)\, A^0_{3,qg \to qq}(1, 2, (\widetilde{ki}))\, B^0_{1gH}(\overline{1}, (\widetilde{ij}), \overline{2})\, J^{(1)}_1(\{p\}_1) \qquad \text{(C.1.5)}$$

$$D^{0,XS}_{1gH}(\hat{1}_q, i_q, j_{\bar{q}}, k_{\bar{q}}, \hat{2}) =$$

$$- 2 A^0_{3,qg \to qq}(1,2,j)\, D^0_{0gH}(\overline{1}, \overline{2}, i, k)\, J^{(2)}_1(\{p\}_2)$$

$$- 2 A^0_{3,qg \to qq}(1,2,k)\, D^0_{0gH}(\overline{1}, j, i, \overline{2})\, J^{(2)}_1(\{p\}_2)$$

$$- 2 A^0_{3,qg \to qq}(1,2,i)\, D^0_{0gH}(\overline{1}, j, \overline{2}, k)\, J^{(2)}_1(\{p\}_2)$$

$$+ 4\, C^0_4(1,i,j,k)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1)$$

$$+ 4\, C^0_4(j,k,i,1)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{jki}))\, J^{(1)}_1(\{p\}_1)$$

$$+ 4\, C^0_4(k,j,i,1)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{kji}))\, J^{(1)}_1(\{p\}_1)$$

$$+ 4\, C^0_4(i,1,j,k)\, B^0_{1gH}(\overline{1}, 2, (\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1) \qquad \text{(C.1.6)}$$

$$\widetilde{D}^{0,XS}_{1gH}(\hat{1}_q, i_q, j_{\bar{q}}, k_{\bar{q}}, \hat{2}) =$$

$$- A^0_{3,qg \to qq}(1,2,j)\, D^0_{0gH}(\overline{1}, \overline{2}, i, k)\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,qg \to qq}(1,2,i)\, D^0_{0gH}(\overline{1}, j, \overline{2}, k)\, J^{(2)}_1(\{p\}_2)$$

$$-A^0_{3,qg\to qq}(1,2,k)\, D^0_{0gH}(\overline{1},j,i,\overline{2})\, J^{(2)}_1(\{p\}_2)$$

$$+2\, C^0_4(1,i,j,k)\, B^0_{1gH}(\overline{1},2,(\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1)$$

$$+2\, C^0_4(k,j,i,1)\, B^0_{1gH}(\overline{1},2,(\widetilde{kji}))\, J^{(1)}_1(\{p\}_1)$$

$$+2\, C^0_4(j,k,i,1)\, B^0_{1gH}(\overline{1},2,(\widetilde{jki}))\, J^{(1)}_1(\{p\}_1)$$

$$+2\, C^0_4(i,1,j,k)\, B^0_{1gH}(\overline{1},2,(\widetilde{ijk}))\, J^{(1)}_1(\{p\}_1) \tag{C.1.7}$$

C.2 $qg \to H$+**jet at RV**

The real-virtual subtraction terms $\mathrm{d}\hat{\sigma}^T_{NNLO}$ mentioned in section 7.5.2 are:

$$\widehat{B}^{1,XT}_{2gH}(\hat{1}_q,\hat{2},i,j_{\bar{q}}) =$$

$$-\left[+2\hat{J}^{1,FF}_{2,QG}(s_{ij})+2\hat{J}^{1,II}_{2,QG}(s_{12})\right]B^0_{2gH}(1,2,i,j)\,J^{(2)}_1(\{p\}_2)$$

$$+D^0_{3,g}(j,i,2)\,\hat{B}^1_{1gH}(1,\overline{2},(\widetilde{ji}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+\left[\hat{D}^1_{3,g}(j,i,2)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$\left.+\left(+2\hat{J}^{1,FF}_{2,QG}(s_{ij})+2\hat{J}^{1,II}_{2,QG}(s_{12})\right)D^0_{3,g}(j,i,2)\right]B^0_{1gH}(1,\overline{2},(\widetilde{ji}))\,J^{(1)}_1(\{p\}_1)$$

$$-A^0_{3,qg\to qq}(1,2,j)\,\hat{B}^1_{1gH}(\overline{1},\overline{2},i)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$-\left[\hat{A}^1_{3,qg\to qq}(1,2,j)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$\left.+\left(+2\hat{J}^{1,FF}_{2,QG}(s_{ij})+2\hat{J}^{1,II}_{2,QG}(s_{12})\right)A^0_{3,qg\to qq}(1,2,j)\right]B^0_{1gH}(\overline{1},\overline{2},i)\,J^{(1)}_1(\{p\}_1)$$

$$-G^0_{3,q'\to g}(i,j,1)\,\hat{A}^1_{3gH}(\overline{1},2,(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$-\left[\hat{G}^1_{3,q'}(i,j,1)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$\left.-\left(-2\hat{J}^{1,FF}_{2,QG}(s_{ij})-2\hat{J}^{1,II}_{2,QG}(s_{12})\right)G^0_{3,q'\to g}(i,j,1)\right]A^0_{3gH}(\overline{1},2,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$\tag{C.2.8}$$

$$\widehat{B}^{1,YT}_{2gH}(\hat{1}_q,i,\hat{2},j_{\bar{q}}) =$$

$$-\left[+2\hat{J}^{1,FF}_{2,QG}(s_{ij})+2\hat{J}^{1,II}_{2,QG}(s_{12})\right]B^0_{2gH}(1,i,2,j)\,J^{(2)}_1(\{p\}_2)$$

$$+D^0_{3,qg}(1,i,2)\,\hat{B}^1_{1gH}(\overline{1},\overline{2},j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+\left[\hat{D}^1_{3,qg}(1,i,2)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$\left.+\left(+2\hat{J}^{1,FF}_{2,QG}(s_{ij})+2\hat{J}^{1,II}_{2,QG}(s_{12})\right)D^0_{3,qg}(1,i,2)\right]B^0_{1gH}(\overline{1},\overline{2},j)\,J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,qg\to qq}(1,2,j)\,\hat{B}^1_{1gH}(\overline{1},i,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2)$$

$$- \Bigg[\, \hat{A}^1_{3,qg\to qq}(1,2,j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \Bigg(\, + 2\hat{J}^{1,FF}_{2,QG}(s_{ij}) + 2\hat{J}^{1,II}_{2,QG}(s_{12})\Bigg)\,A^0_{3,qg\to qq}(1,2,j)\Bigg]\,B^0_{1gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$- G^0_{3,q'\to g}(i,1,j)\,\hat{A}^1_{3gH}(\overline{1},(\widetilde{ij}),2)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$- \Bigg[\, \hat{G}^1_{3,q'}(i,1,j)\,\delta(1-x_1)\,\delta(1-x_2)$$

$$- \Bigg(\, - 2\hat{J}^{1,FF}_{2,QG}(s_{ij}) - 2\hat{J}^{1,II}_{2,QG}(s_{12})\Bigg)\,G^0_{3,q'\to g}(i,1,j)\Bigg]\,A^0_{3gH}(\overline{1},(\widetilde{ij}),2)\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,B^0_{2gH}(i,1,2,j)\,J^{(2)}_1(\{p\}_2)$$

$$- J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,B^0_{2gH}(i,2,1,j)\,J^{(2)}_1(\{p\}_2)$$

$$+ J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,G^0_{3,g}(1,i,j)\,A^0_{3gH}((\widetilde{ij}),\overline{1},2)\,J^{(1)}_1(\{p\}_1)$$

$$+ J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,G^0_{3,g}(1,j,i)\,A^0_{3gH}((\widetilde{ji}),\overline{1},2)\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,A^0_{3,g\to q}(i,1,j)\,B^0_{1gH}((\widetilde{ij}),2,\overline{1})\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,D^0_{3,gg}(i,1,2)\,B^0_{1gH}(j,\overline{1},\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+ J^{1,IF}_{2,GQ,q'\to g}(s_{i1})\,d^0_{3,gg\to qg}(i,1,2)\,B^0_{1gH}(j,\overline{1},\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,IF}_{2,GQ,q'\to g}(s_{i\overline{1}})\,d^0_{3,gg\to qg}(j,2,1)\,B^0_{1gH}(i,\overline{1},\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,C^0_{0gH}(1,i,j,2)\,J^{(2)}_1(\{p\}_2)$$

$$+ J^{1,II}_{2,QQ,qg\to qq}(s_{1\overline{2}})\,E^0_{3,q}(2,j,i)\,B^0_{1gH}(1,(\widetilde{ij}),\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,C^0_{0gH}(1,2,j,i)\,J^{(2)}_1(\{p\}_2)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,E^0_{3,qq'\to qg}(2,1,i)\,B^0_{1gH}(\overline{2},\overline{1},j)\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,E^0_{3,q'\to g}(i,j,2)\,B^0_{1gH}(1,\overline{2},(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,C^0_{0gH}(1,i,2,j)\,J^{(2)}_1(\{p\}_2)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,E^0_{3,qq'\to qg}(1,i,2)\,B^0_{1gH}(\overline{1},\overline{2},j)\,J^{(1)}_1(\{p\}_1)$$

$$- J^{1,II}_{2,QQ,qg\to qq}(s_{12})\,E^0_{3,qq'\to qg}(2,j,1)\,B^0_{1gH}(\overline{2},\overline{1},i)\,J^{(1)}_1(\{p\}_1)$$

$$+ \Bigg[\, - J^{1,II}_{2,QQ,qg\to qq}(s_{12}) + J^{1,II}_{2,QQ,qg\to qq}(s_{1\overline{2}})\Bigg]\,E^0_{3,q}(2,j,i)\,B^0_{1gH}(1,\overline{2},(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+ \Bigg[\, - 2J^{1,FI}_{2,QQ,g\to q}(s_{2j}) + 2J^{1,FI}_{2,QQ,g\to q}(s_{2(\widetilde{ji})})\Bigg]\,E^0_{3,q}(j,i,1)\,B^0_{1gH}(2,\overline{1},(\widetilde{ji}))\,J^{(1)}_1(\{p\}_1)$$

$$+ \Bigg[\, - J^{1,IF}_{2,GQ,q'\to g}(s_{j1}) + J^{1,IF}_{2,GQ,q'\to g}(s_{i\overline{1}})\Bigg]\,D^0_{3,gg}(j,1,2)\,B^0_{1gH}(\overline{2},\overline{1},i)\,J^{(1)}_1(\{p\}_1)$$

$$+\left[+J_{2,GQ,q'\to g}^{1,IF}(s_{j1}) - J_{2,GQ,q'\to g}^{1,IF}(s_{(\widetilde{ij})\overline{1}})\right] A_{3,g\to q}^0(i,1,j)\, B_{1gH}^0((\widetilde{ij}),\overline{1},2)\, J_1^{(1)}(\{p\}_1)$$

$$+\left[-J_{2,GQ,q'\to g}^{1,IF}(s_{j1}) + J_{2,GQ,q'\to g}^{1,IF}(s_{j\overline{1}})\right] d_{3,gg\to qg}^0(i,1,2)\, B_{1gH}^0(j,\overline{1},\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+\left[+J_{2,GQ,q'\to g}^{1,IF}(s_{j1}) - J_{2,GQ,q'\to g}^{1,IF}(s_{(\widetilde{ji})1})\right] A_{3,g\to q}^0(j,2,i)\, B_{1gH}^0(\overline{2},1,(\widetilde{ji}))\, J_1^{(1)}(\{p\}_1)$$

$$\tag{C.2.9}$$

$$\widehat{\widetilde{\widetilde{B}}}_{2gH}^{1,XT}(\hat{1}_q, \hat{\widetilde{2}}, \widetilde{i}, j_{\bar{q}}) =$$

$$-\left[+2\hat{J}_{2,QG}^{1,FF}(s_{ji}) + 2\hat{J}_{2,QG}^{1,II}(s_{12})\right] \tilde{B}_{2gH}^0(1,2,i,j)\, J_1^{(2)}(\{p\}_2)$$

$$-A_{3,qg\to qq}^0(1,2,j)\, \hat{B}_{1gH}^1(\overline{1},i,\overline{2})\, \delta(1-x_1)\,\delta(1-x_2)$$

$$-\left[\hat{A}_{3,qg\to qq}^1(1,2,j)\, \delta(1-x_1)\,\delta(1-x_2)\right.$$

$$\left.+\left(+2\hat{J}_{2,QG}^{1,FF}(s_{ji}) + 2\hat{J}_{2,QG}^{1,II}(s_{12})\right) A_{3,qg\to qq}^0(1,2,j)\right] B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+A_{3,q}^0(1,i,j)\, \hat{B}_{1gH}^1(\overline{1},2,(\widetilde{ij}))\, \delta(1-x_1)\,\delta(1-x_2)$$

$$+\left[\hat{A}_{3,q}^1(1,i,j)\, \delta(1-x_1)\,\delta(1-x_2)\right.$$

$$\left.+\left(+2\hat{J}_{2,QG}^{1,FF}(s_{ji}) + 2\hat{J}_{2,QG}^{1,II}(s_{12})\right) A_{3,q}^0(1,i,j)\right] B_{1gH}^0(\overline{1},2,(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+J_{2,GQ,q'\to g}^{1,IF}(s_{1i})\, \tilde{B}_{2gH}^0(i,1,2,j)\, J_1^{(2)}(\{p\}_2)$$

$$+J_{2,GQ,q'\to g}^{1,IF}(s_{1i})\, A_{3,g\to q}^0(i,1,j)\, B_{1gH}^0(\overline{1},2,(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+J_{2,GQ,q'\to g}^{1,IF}(s_{1i})\, A_{3,g\to q}^0(i,2,j)\, B_{1gH}^0(\overline{2},1,(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+J_{2,QQ,qg\to qq}^{1,II}(s_{12})\, C_{0gH}^0(1,i,j,2)\, J_1^{(2)}(\{p\}_2)$$

$$-\frac{1}{2}\, J_{2,QQ,qg\to qq}^{1,II}(s_{\overline{1}2})\, E_{3,q}^0(1,i,j)\, B_{1gH}^0(\overline{1},(\widetilde{ij}),2)\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, J_{2,QQ,qg\to qq}^{1,II}(s_{1\overline{2}})\, E_{3,q}^0(2,i,j)\, B_{1gH}^0(1,(\widetilde{ij}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+2J_{2,QQ,g\to q}^{1,FI}(s_{2i})\, C_{0gH}^0(1,2,i,j)\, J_1^{(2)}(\{p\}_2)$$

$$+2J_{2,QQ,g\to q}^{1,FI}(s_{2i})\, E_{3,q'\to g}^0(j,2,i)\, B_{1gH}^0(1,\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+2J_{2,QQ,g\to q}^{1,FI}(s_{2(\widetilde{ij})})\, E_{3,q'\to g}^0(i,1,j)\, B_{1gH}^0(2,\overline{1},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1) \tag{C.2.10}$$

$$\widetilde{\widetilde{\widetilde{B}}}_{2gH}^{1,XT}(\hat{1}_q, \hat{\widetilde{2}}, \widetilde{i}, j_{\bar{q}}) =$$

$$-J_{2,QQ}^{1,IF}(s_{1j})\, \tilde{B}_{2gH}^0(1,2,i,j)\, J_1^{(2)}(\{p\}_2)$$

$$-A_{3,qg\to qq}^0(1,2,j)\left[\tilde{B}_{1gH}^1(\overline{1},i,\overline{2})\, \delta(1-x_1)\,\delta(1-x_2)\right.$$

$$+ J_{2,QQ}^{1,II}(s_{\overline{12}}) \, B_{1gH}^0(\overline{1}, i, \overline{2}) \bigg] J_1^{(1)}(\{p\}_1)$$

$$- \bigg[\tilde{A}_{3,qg \to qq}^1(1, 2, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J_{2,QQ}^{1,IF}(s_{1j}) - J_{2,QQ}^{1,II}(s_{\overline{12}}) \bigg) A_{3,qg \to qq}^0(1, 2, j) \bigg] B_{1gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q}^0(1, i, j) \bigg[\tilde{B}_{1gH}^1(\overline{1}, 2, (\widetilde{ij})) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ J_{2,QQ}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \, B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \bigg] J_1^{(1)}(\{p\}_1)$$

$$+ \bigg[\tilde{A}_{3,q}^1(1, i, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J_{2,QQ}^{1,IF}(s_{1j}) - J_{2,QQ}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \bigg) A_{3,g \to q}^0(1, i, j) \bigg] B_{1gH}^0(\overline{1}, 2, (\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, \tilde{B}_{2gH}^0(1, i, j, 2) \, J_1^{(2)}(\{p\}_2)$$

$$+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, A_{3,q\bar{q}}^0(1, i, 2) \, B_{1gH}^0(\overline{1}, j, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, A_{3,q\bar{q}}^0(1, j, 2) \, B_{1gH}^0(\overline{1}, i, \overline{2}) \, J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, D_{0gH}^0(1, i, 2, j) \, J_1^{(2)}(\{p\}_2)$$

$$- \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, D_{0gH}^0(1, i, j, 2) \, J_1^{(2)}(\{p\}_2)$$

$$- \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, D_{0gH}^0(1, 2, i, j) \, J_1^{(2)}(\{p\}_2) \tag{C.2.11}$$

$$\widetilde{B}_{2gH}^{1,FT}(\hat{1}_q, \hat{2}, i, j_{\bar{q}}) =$$

$$- J_{2,QQ}^{1,IF}(s_{1j}) \, B_{2gH}^0(1, i, 2, j) \, J_1^{(2)}(\{p\}_2)$$

$$- J_{2,QQ}^{1,IF}(s_{1j}) \, B_{2gH}^0(1, 2, i, j) \, J_1^{(2)}(\{p\}_2)$$

$$+ D_{3,qg}^0(1, i, 2) \bigg[\tilde{B}_{1gH}^1(\overline{1}, \overline{2}, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ J_{2,QQ}^{1,IF}(s_{\overline{1}j}) \, B_{1gH}^0(\overline{1}, \overline{2}, j) \bigg] J_1^{(1)}(\{p\}_1)$$

$$+ d_{3,g}^0(j, i, 2) \bigg[\tilde{B}_{1gH}^1(1, \overline{2}, (\widetilde{ij})) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ J_{2,QQ}^{1,IF}(s_{1(\widetilde{ij})}) \, B_{1gH}^0(1, \overline{2}, (\widetilde{ij})) \bigg] J_1^{(1)}(\{p\}_1)$$

$$+ \bigg[\tilde{A}_{3,q}^1(1, i, j) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$
+ \left(+ J_{2,QQ}^{1,IF}(s_{1j}) - J_{2,QQ}^{1,IF}(s_{\bar{1}(\widetilde{ij})}) \right) A_{3,q}^0(1,i,j) \Bigg] B_{1gH}^0(\bar{1},2,(\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)
$$

$$
- A_{3,qg \to qq}^0(1,2,j) \Bigg[\tilde{B}_{1gH}^1(\bar{1},i,\bar{2}) \, \delta(1-x_1) \, \delta(1-x_2)
$$

$$
+ J_{2,QQ}^{1,II}(s_{\overline{12}}) \, B_{1gH}^0(\bar{1},i,\bar{2}) \Bigg] J_1^{(1)}(\{p\}_1)
$$

$$
- \Bigg[\tilde{A}_{3,qg \to qq}^1(1,2,j) \, \delta(1-x_1) \, \delta(1-x_2)
$$

$$
+ \left(+ J_{2,QQ}^{1,IF}(s_{1j}) - J_{2,QQ}^{1,II}(s_{\overline{12}}) \right) A_{3,qg \to qq}^0(1,2,j) \Bigg] B_{1gH}^0(\bar{1},i,\bar{2}) \, J_1^{(1)}(\{p\}_1)
$$

$$
- 2 \Bigg[\tilde{G}_{3,q'}^1(i,1,j) \, \delta(1-x_1) \, \delta(1-x_2) + J_{2,QQ}^{1,IF}(s_{1j}) \, G_{3,q' \to g}^0(i,1,j) \Bigg]
$$

$$
\times A_{3gH}^0(\bar{1},2,(\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)
$$

$$
- \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, B_{2gH}^0(1,i,j,2) \, J_1^{(2)}(\{p\}_2)
$$

$$
+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{\overline{1}2}) \, d_{3,q}^0(1,i,j) \, B_{1gH}^0(\bar{1},(\widetilde{ij}),2) \, J_1^{(1)}(\{p\}_1)
$$

$$
+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{1\bar{2}}) \, d_{3,q}^0(2,j,i) \, B_{1gH}^0(1,(\widetilde{ji}),\bar{2}) \, J_1^{(1)}(\{p\}_1)
$$

$$
+ \Bigg[+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) - \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{\overline{12}}) \Bigg]
$$

$$
\times A_{3,q\bar{q}}^0(1,i,2) \, B_{1gH}^0(\bar{1},j,\bar{2}) \, J_1^{(1)}(\{p\}_1)
$$

$$
- \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) \, B_{2gH}^0(1,j,i,2) \, J_1^{(2)}(\{p\}_2)
$$

$$
+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{\overline{1}2}) \, d_{3,q}^0(1,j,i) \, B_{1gH}^0(\bar{1},(\widetilde{ji}),2) \, J_1^{(1)}(\{p\}_1)
$$

$$
+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{1\bar{2}}) \, d_{3,q}^0(2,i,j) \, B_{1gH}^0(1,(\widetilde{ji}),\bar{2}) \, J_1^{(1)}(\{p\}_1)
$$

$$
+ \Bigg[+ \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{12}) - \frac{1}{2} \, J_{2,QQ,qg \to qq}^{1,II}(s_{\overline{12}}) \Bigg]
$$

$$
\times A_{3,q\bar{q}}^0(1,j,2) \, B_{1gH}^0(\bar{1},i,\bar{2}) \, J_1^{(1)}(\{p\}_1) \tag{C.2.12}
$$

$$
\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q,\widetilde{\hat{2}},\widetilde{i},j_{\bar{q}}) =
$$

$$
- \Bigg[+ J_{2,QG}^{1,II}(s_{12}) + J_{2,QG}^{1,FI}(s_{2j}) + J_{2,QG}^{1,IF}(s_{1i}) + J_{2,QG}^{1,FF}(s_{ij}) - J_{2,QQ}^{1,IF}(s_{1j}) \Bigg]
$$

$$
\times \tilde{B}_{2gH}^0(1,2,i,j) \, J_1^{(2)}(\{p\}_2)
$$

$$
+ A_{3,q}^0(1,i,j) \Bigg[B_{1gH}^1(\bar{1},2,(\widetilde{ij})) \, \delta(1-x_1) \, \delta(1-x_2)
$$

$$
+ \left(+ J_{2,QG}^{1,II}(s_{\overline{1}2}) + J_{2,QG}^{1,FI}(s_{2(\widetilde{ij})}) \right) B_{1gH}^0(\bar{1},2,(\widetilde{ij})) \Bigg] J_1^{(1)}(\{p\}_1)
$$

$$
+ \bigg[A^1_{3,q}(1,i,j)\, \delta(1-x_1)\, \delta(1-x_2)
$$

$$
+ \bigg(+ J^{1,IF}_{2,QG}(s_{1i}) + J^{1,FF}_{2,QG}(s_{ij}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) \bigg) A^0_{3,q}(1,i,j) \bigg]
$$

$$
\times B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)
$$

$$
- \bigg[+ J^{1,IF}_{2,QQ}(s_{1j}) - J^{1,IF}_{2,QQ}(s_{\overline{1}(\widetilde{ij})}) - J^{1,FI}_{2,QG}(s_{2j})
$$

$$
+ J^{1,FI}_{2,QG}(s_{2(\widetilde{ij})}) - J^{1,II}_{2,QG}(s_{12}) + J^{1,II}_{2,QG}(s_{\overline{1}2})
$$

$$
+ \bigg(- \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) + \mathcal{S}^{IF}(s_{\overline{1}(\widetilde{ij})}, s_{1j}, x_{\overline{1}(\widetilde{ij}),1j}) + \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j})
$$

$$
- \mathcal{S}^{IF}(s_{2(\widetilde{ij})}, s_{1j}, x_{2(\widetilde{ij}),1j}) + \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) - \mathcal{S}^{IF}(s_{\overline{1}2}, s_{1j}, x_{\overline{1}2,1j}) \bigg) \bigg]
$$

$$
\times A^0_{3,q}(1,i,j)\, B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\, J^{(1)}_1(\{p\}_1)
$$

$$
- A^0_{3,qg\to qq}(1,2,j) \bigg[B^1_{1gH}(\overline{1},i,\overline{2})\, \delta(1-x_1)\, \delta(1-x_2)
$$

$$
+ \bigg(+ J^{1,IF}_{2,QG}(s_{\overline{1}i}) + J^{1,FI}_{2,GQ}(s_{i\overline{2}}) \bigg) B^0_{1gH}(\overline{1},i,\overline{2}) \bigg]\, J^{(1)}_1(\{p\}_1)
$$

$$
- \bigg[A^1_{3,qg\to qq}(1,2,j)\, \delta(1-x_1)\, \delta(1-x_2)
$$

$$
+ \bigg(+ J^{1,II}_{2,QG}(s_{12}) + J^{1,FI}_{2,QG}(s_{2j}) - J^{1,II}_{2,QQ}(s_{\overline{1}2}) \bigg) A^0_{3,qg\to qq}(1,2,j) \bigg]
$$

$$
\times B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)
$$

$$
- \bigg[+ J^{1,FF}_{2,GQ}(s_{ij}) - J^{1,FI}_{2,GQ}(s_{i\overline{2}}) - J^{1,IF}_{2,QQ}(s_{1j})
$$

$$
+ J^{1,II}_{2,QQ}(s_{\overline{1}2}) + J^{1,IF}_{2,QG}(s_{1i}) - J^{1,IF}_{2,QG}(s_{\overline{1}i})
$$

$$
+ \bigg(- \mathcal{S}^{IF}(s_{ji}, s_{1i}, x_{ji,1i}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{2}i}, s_{\overline{1}i}, x_{\overline{2}i,\overline{1}i}) + \mathcal{S}^{IF}(s_{1j}, s_{1i}, x_{1j,1i})
$$

$$
- \tilde{\mathcal{S}}^{IF}(s_{\overline{1}2}, s_{\overline{1}i}, x_{\overline{1}2,\overline{1}i}) - \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}i}, s_{\overline{1}i}, 1) \bigg) \bigg]
$$

$$
\times A^0_{3,qg\to qq}(1,2,j)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)
$$

$$
- \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, \tilde{B}^0_{2gH}(1,i,j,2)\, J^{(2)}_1(\{p\}_2)
$$

$$
+ \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, A^0_{3,q\bar{q}}(1,i,2)\, B^0_{1gH}(\overline{1},j,\overline{2})\, J^{(1)}_1(\{p\}_1)
$$

$$
+ \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, A^0_{3,q\bar{q}}(1,j,2)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)
$$

$$
- \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, D^0_{0gH}(1,i,2,j)\, J^{(2)}_1(\{p\}_2)
$$

$$
- \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, D^0_{0gH}(1,i,j,2)\, J^{(2)}_1(\{p\}_2)
$$

$$
- \frac{1}{2}\, J^{1,II}_{2,QQ,qg\to qq}(s_{12})\, D^0_{0gH}(1,2,i,j)\, J^{(2)}_1(\{p\}_2) \tag{C.2.13}
$$

C.3 $qg \to H$+jet at VV

The double virtual subtraction terms $\mathrm{d}\hat{\sigma}^U_{NNLO}$ mentioned in section 7.5.3 are:

$$
\widetilde{B}^{2,XU}_{1gH}(\hat{1}_q, \hat{2}_g, i_{\bar{q}}) =
$$

$$
-\left[+ \mathcal{D}^0_{3,qg}(s_{12}) - \Gamma^{(1)}_{qq}(z_1) - \Gamma^{(1)}_{gg}(z_2) + \mathcal{D}^0_{3,g\to g}(s_{2i}) \right] \tilde{B}^1_{1gH}(1,2,i)
$$

$$
-\left[+ \mathcal{A}^0_{3,q}(s_{1i}) - \Gamma^{(1)}_{qq}(z_1) \right] \left(+B^1_{1gH}(1,2,i) - \frac{b_0}{\epsilon} B^0_{1gH}(1,2,i) \right)
$$

$$
-\Big[+ \mathcal{D}^0_{3,qg}(s_{12}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) - \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,q}(s_{1i}) - \frac{1}{2} \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{A}^0_{3,q}(s_{1i})
$$

$$
- \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{D}^0_{3,qg}(s_{12}) + \Gamma^{(1)}_{qq}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1) + \frac{1}{2} \Gamma^{(1)}_{qq}(z_1) \otimes \Gamma^{(1)}_{gg}(z_2)
$$

$$
\Big] B^0_{1gH}(1,2,i)
$$

$$
-\Big[+ \mathcal{D}^0_{3,g\to g}(s_{2i}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) - \frac{1}{2} \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{A}^0_{3,q}(s_{1i})
$$

$$
- \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{D}^0_{3,g\to g}(s_{2i}) + \frac{1}{2} \Gamma^{(1)}_{qq}(z_1) \otimes \Gamma^{(1)}_{gg}(z_2) \Big] B^0_{1gH}(1,2,i)
$$

$$
-\Big[+ \mathcal{A}^0_{4,q}(s_{1i}) + \mathcal{A}^1_{3,q}(s_{1i}) + \frac{b_0}{\epsilon} \mathcal{A}^0_{3,q}(s_{1i}) \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon}
$$

$$
- \mathcal{A}^0_{3,q}(s_{1i}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) - \overline{\Gamma}^{(2)}_{qq}(z_1) \Big] B^0_{1gH}(1,2,i)
$$

$$
-\Big[+ \frac{1}{2} \tilde{\mathcal{A}}^0_{4,q}(s_{1i}) + \tilde{\mathcal{A}}^1_{3,q}(s_{1i}) + \mathcal{C}^0_{4,\bar{q},\bar{q}q\bar{q}}(s_{1i}) + 2\,\mathcal{C}^0_{4,q}(s_{1i}) + \mathcal{C}^0_{4,\bar{q},q\bar{q}\bar{q}}(s_{1i})
$$

$$
+ \overline{\Gamma}^{(2)}_{q\bar{q}} t(z_1) + \widetilde{\overline{\widetilde{\Gamma}}}^{(2)}_{qq}(z_1) \Big] B^0_{1gH}(1,2,i)
$$

$$
-\Big[- \mathcal{A}^0_{3,qg\to qq}(s_{12}) - S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \Big] \tilde{B}^1_{1gH}(1,i,2)
$$

$$
-\Big[+ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) + \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12})
$$

$$
- \mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{A}^0_{3,qq}(s_{12}) + S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{qq}(z_1)
$$

$$
+ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{qq}(z_2) - S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{A}^0_{3,qq}(s_{12}) \Big] B^0_{1gH}(1,i,2)
$$

$$
-\Big[- \tilde{\mathcal{A}}^0_{4,qg}(s_{12}) - \tilde{\mathcal{A}}^1_{3,qg}(s_{12}) + \mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{A}^0_{3,qq}(s_{12})
$$

$$
- \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) - \frac{1}{2} S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{qq}(z_2) + S_{q\to g}\widetilde{\overline{\Gamma}}^{(2)}_{qg}(z_2)
$$

$$
\Big] B^0_{1gH}(1,i,2)
$$

$$
-\left[-\ \mathcal{A}^0_{3,qg\to qq}(s_{12}) -\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \right] \left(+B^1_{1gH}(1,i,2) - \frac{b_0}{\epsilon}B^0_{1gH}(1,i,2) \right)
$$

$$
-\left[+\ \Gamma^{(1)}_{qq}(z_1)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12}) - \frac{1}{2}\ \mathcal{A}^0_{3,qg\to qq}(s_{12})\otimes\mathcal{D}^0_{3,q}(s_{1i}) \right.
$$

$$
\left. +\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{qq}(z_1) - \frac{1}{2}\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\mathcal{D}^0_{3,q}(s_{1i}) \right] B^0_{1gH}(1,i,2)
$$

$$
-\left[-\frac{1}{2}\ \mathcal{A}^0_{3,qg\to qq}(s_{12})\otimes\mathcal{D}^0_{3,q}(s_{2i}) - \frac{1}{2}\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\mathcal{D}^0_{3,q}(s_{2i}) \right.
$$

$$
\left. +\ \Gamma^{(1)}_{qq}(z_2)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12}) +\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{qq}(z_2) \right] B^0_{1gH}(1,i,2)
$$

$$
-\left[-\ \mathcal{A}^{0,nadj}_{4,qg}(s_{12}) -\ \mathcal{A}^0_{4,qg}(s_{12}) -\ \mathcal{A}^1_{3,qg}(s_{12}) - \frac{b_0}{\epsilon}\ \mathcal{A}^0_{3,qg\to qq}(s_{12})\left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon} \right.
$$

$$
+\ \mathcal{A}^0_{3,qg\to qq}(s_{12})\otimes\mathcal{A}^0_{3,qq}(s_{12}) -\ \Gamma^{(1)}_{qq}(z_2)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12})
$$

$$
+\ \Gamma^{(1)}_{gg}(z_2)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12}) + \frac{1}{2}\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{gg}(z_2)
$$

$$
\left. -\frac{1}{2}\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{qq}(z_2) -\ S_{g\to q}\overline{\Gamma}^{(2)}_{qg}(z_2) \right] B^0_{1gH}(1,i,2)
$$

$$
-\left[-\ \tilde{\mathcal{G}}^0_{4,q}(s_{1i}) - 2\,\tilde{\mathcal{G}}^1_{3,q'}(s_{1i}) + 2\,\Gamma^{(1)}_{qq}(z_1)\otimes\mathcal{G}^0_{3,q'\to g}(s_{1i}) \right.
$$

$$
\left. + 2\,S_{q\to g}\widetilde{\overline{\tilde{\Gamma}}}^{(2)}_{gq}(z_1) +\ S_{q\to g}\Gamma^{(1)}_{gq}(z_1)\otimes\Gamma^{(1)}_{qq}(z_1) \right] A^0_{3gH}(1,2,i) \tag{C.3.14}
$$

$$
\widetilde{\widetilde{B}}^{2,XU}_{1gH}(\hat{1}_q,\hat{2}_g,i_{\bar{q}}) =
$$

$$
-\left[-\ \tilde{\mathcal{A}}^0_{4,qg}(s_{12}) -\ \tilde{\mathcal{A}}^1_{3,qg}(s_{12}) +\ \mathcal{A}^0_{3,qg\to qq}(s_{12})\otimes\mathcal{A}^0_{3,qq}(s_{12}) \right.
$$

$$
-\ \Gamma^{(1)}_{qq}(z_2)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12}) +\ S_{q\to g}\widetilde{\overline{\tilde{\Gamma}}}^{(2)}_{qg}(z_2) - \frac{1}{2}\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{qq}(z_2)
$$

$$
\left. \vphantom{\frac{1}{2}} \right] B^0_{1gH}(1,i,2)
$$

$$
-\left[-\ \mathcal{A}^0_{3,qg\to qq}(s_{12}) -\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \right] \tilde{B}^1_{1gH}(1,i,2)
$$

$$
-\left[-\ \mathcal{A}^0_{3,qg\to qq}(s_{12})\otimes\mathcal{A}^0_{3,qq}(s_{12}) -\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\mathcal{A}^0_{3,qq}(s_{12}) \right.
$$

$$
+\ \Gamma^{(1)}_{qq}(z_1)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12}) +\ \Gamma^{(1)}_{qq}(z_2)\otimes\mathcal{A}^0_{3,qg\to qq}(s_{12})
$$

$$
\left. +\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{qq}(z_1) +\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2)\otimes\Gamma^{(1)}_{qq}(z_2) \right] B^0_{1gH}(1,i,2)
$$

$$
-\left[+\frac{1}{2}\ \tilde{\mathcal{A}}^0_{4,q}(s_{1i}) +\ \mathcal{C}^0_{4,\bar{q},\bar{q}q\bar{q}}(s_{1i}) + 2\,\mathcal{C}^0_{4,q}(s_{1i}) +\ \mathcal{C}^0_{4,\bar{q},q q\bar{q}}(s_{1i}) +\ \tilde{\mathcal{A}}^1_{3,q}(s_{1i}) \right.
$$

$$- \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{A}_{3,q}^0(s_{1i}) + \overline{\Gamma}_{q\bar{q}}^{(2)} t(z_1) + \widetilde{\widetilde{\overline{\Gamma}}}_{qq}^{(2)}(z_1) + \frac{1}{2} \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1)$$

$$\Bigg] B_{1gH}^0(1,2,i)$$

$$- \Bigg[+ \mathcal{A}_{3,q}^0(s_{1i}) - \Gamma_{qq}^{(1)}(z_1) \Bigg] \tilde{B}_{1gH}^1(1,2,i) \tag{C.3.15}$$

$$\widehat{\widetilde{\widetilde{B}}}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}}) =$$

$$- \Bigg[- \mathcal{A}_{3,g \to q}^0(s_{2i}) \otimes \mathcal{E}_{3,q' \to g}^0(s_{1i}) - S_{q \to g} \Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{A}_{3,g \to q}^0(s_{2i})$$

$$- 2 S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \otimes \mathcal{E}_{3,q' \to g}^0(s_{1i}) - 2 S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \otimes S_{q \to g} \Gamma_{gq}^{(1)}(z_1) \Bigg] B_{1gH}^0(2,1,i)$$

$$- \Bigg[+ \mathcal{A}_{3,qg \to qq}^0(s_{12}) + S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \Bigg] \hat{B}_{1gH}^1(1,i,2)$$

$$- \Bigg[+ \frac{1}{2} \mathcal{A}_{3,qg \to qq}^0(s_{12}) \otimes \mathcal{E}_{3,q}^0(s_{1i}) + \frac{1}{2} \mathcal{A}_{3,qg \to qq}^0(s_{12}) \otimes \mathcal{E}_{3,q}^0(s_{2i})$$

$$+ \frac{1}{2} S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \otimes \mathcal{E}_{3,q}^0(s_{1i}) + \frac{1}{2} S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \otimes \mathcal{E}_{3,q}^0(s_{2i}) \Bigg] B_{1gH}^0(1,i,2)$$

$$- \Bigg[+ \hat{\mathcal{A}}_{3,qg}^1(s_{12}) + \frac{b_F}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}_{3,qg \to qq}^0(s_{12}) - \frac{b_F}{\epsilon} \mathcal{A}_{3,qg \to qq}^0(s_{12})$$

$$- \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{A}_{3,qg \to qq}^0(s_{12}) + \frac{1}{2} S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_2) + S_{g \to q} \overline{\Gamma}_{qg,F}^{(2)}(z_2)$$

$$\Bigg] B_{1gH}^0(1,i,2)$$

$$- \Bigg[- \mathcal{A}_{3,q}^0(s_{1i}) + \Gamma_{qq}^{(1)}(z_1) \Bigg] \hat{B}_{1gH}^1(1,2,i)$$

$$- \Bigg[+ \Gamma_{gg,F}^{(1)}(z_2) \Bigg] \tilde{B}_{1gH}^1(1,2,i)$$

$$- \Bigg[- \mathcal{B}_{4,q}^0(s_{1i}) - \mathcal{B}_{4,q'}^0(s_{1i}) - \tilde{\mathcal{E}}_{4,g}^0(s_{2i}) - \hat{\mathcal{A}}_{3,q}^1(s_{1i}) - \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}_{3,q}^0(s_{1i})$$

$$+ \frac{b_F}{\epsilon} \mathcal{A}_{3,q}^0(s_{1i}) - S_{q \to g} \Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{A}_{3,g \to q}^0(s_{1i}) - 2 S_{g \to q} \Gamma_{qg}^{(1)}(z_2) \otimes \mathcal{E}_{3,q' \to g}^0(s_{2i})$$

$$+ \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{A}_{3,q}^0(s_{1i}) - S_{q \to g} \Gamma_{gq}^{(1)}(z_1) \otimes S_{g \to q} \Gamma_{qg}^{(1)}(z_1)$$

$$- S_{q \to g} \Gamma_{gq}^{(1)}(z_2) \otimes S_{g \to q} \Gamma_{qg}^{(1)}(z_2) - \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{gg,F}^{(1)}(z_2)$$

$$- \frac{b_F}{\epsilon} \Gamma_{qq}^{(1)}(z_1) + \overline{\Gamma}_{qq,F}^{(2)}(z_1) + \overline{\Gamma}_{qQ}^{(2)}(z_1) + \overline{\Gamma}_{q\bar{Q}}^{(2)}(z_1) - \widetilde{\widetilde{\Gamma}}_{gg,F}^{(2)}(z_2) \Bigg] B_{1gH}^0(1,2,i) \tag{C.3.16}$$

$$\widehat{B}_{1gH}^{2,XU}(\hat{1}_q, \hat{2}_g, i_{\bar{q}}) =$$

$$- \Bigg[+ \mathcal{D}^0_{3,g}(s_{2i}) + \mathcal{D}^0_{3,qg}(s_{12}) - \mathcal{A}^0_{3,qg \to qq}(s_{12}) - \Gamma^{(1)}_{qq}(z_1) - \Gamma^{(1)}_{gg}(z_2) \Bigg] \hat{B}^1_{1gH}(1,2,i)$$

$$- \Bigg[- \Gamma^{(1)}_{gg,F}(z_2) \Bigg] B^1_{1gH}(1,2,i)$$

$$- \Bigg[+ S_{g \to q} \Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{E}^0_{3,qq' \to qg}(s_{12}) - \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{D}^0_{3,g}(s_{2i})$$

$$- \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{D}^0_{3,qg}(s_{12}) + \Gamma^{(1)}_{gg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_2) + \frac{b_F}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{D}^0_{3,qg}(s_{12})$$

$$+ \frac{b_F}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{D}^0_{3,g}(s_{2i}) + \frac{b_0}{\epsilon} \Gamma^{(1)}_{gg,F}(z_2) + \frac{b_F}{\epsilon} \Gamma^{(1)}_{gg}(z_2) - \frac{b_F}{\epsilon} \mathcal{D}^0_{3,qg}(s_{12})$$

$$- \frac{b_F}{\epsilon} \mathcal{D}^0_{3,g}(s_{2i}) + \frac{b_F}{\epsilon} \mathcal{A}^0_{3,qg \to qq}(s_{12}) - \hat{\mathcal{A}}^1_{3,qg}(s_{12}) + \mathcal{B}^0_{4,q'}(s_{1i})$$

$$- \overline{\Gamma}^{(2)}_{qq,F}(z_1) - 2 \overline{\Gamma}^{(2)}_{qQ}(z_1)$$

$$+ \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12}) + S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{A}^0_{3,g \to q}(s_{1i})$$

$$+ S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes S_{g \to q} \Gamma^{(1)}_{qg}(z_1) + S_{q \to g} \Gamma^{(1)}_{gq}(z_2) \otimes S_{g \to q} \Gamma^{(1)}_{qg}(z_2)$$

$$+ \Gamma^{(1)}_{qq}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) - \mathcal{A}^0_{3,qg \to qq}(s_{12}) \otimes \mathcal{E}^0_{3,q}(s_{2i})$$

$$+ S_{g \to q} \Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{E}^0_{3,q' \to g}(s_{2i}) + \hat{\mathcal{D}}^1_{3,g}(s_{2i}) - \overline{\Gamma}^{(2)}_{gg,F}(z_2)$$

$$+ \mathcal{E}^0_{4,g}(s_{2i}) + \mathcal{E}^0_{4,qg}(s_{12}) + \hat{\mathcal{D}}^1_{3,qg}(s_{12})$$

$$- \frac{b_F}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}^0_{3,qg \to qq}(s_{12}) + \frac{b_F}{\epsilon} \Gamma^{(1)}_{qq}(z_1) \Bigg] B^0_{1gH}(1,2,i)$$

$$- \Bigg[- 2 \mathcal{G}^0_{3,q' \to g}(s_{1i}) - 2 S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \Bigg] \hat{A}^1_{3gH}(1,2,i)$$

$$- \Bigg[- 2 \mathcal{H}^0_{4,q}(s_{1i}) - 2 \hat{\mathcal{G}}^1_{3,q'}(s_{1i}) - 2 \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{G}^0_{3,q' \to g}(s_{1i})$$

$$+ 2 \frac{b_F}{\epsilon} \mathcal{G}^0_{3,q' \to g}(s_{1i}) - 2 S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{G}^0_{3,g}(s_{1i}) + 2 \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{G}^0_{3,q' \to g}(s_{1i})$$

$$+ 2 S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1) - S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_2) - 2 S_{q \to g} \overline{\Gamma}^{(2)}_{gq,F}(z_1)$$

$$\Bigg] A^0_{3gH}(1,2,i)$$

$$- \Bigg[- \mathcal{A}^0_{3,qg \to qq}(s_{12}) - S_{g \to q} \Gamma^{(1)}_{qg}(z_2) \Bigg] \hat{B}^1_{1gH}(1,i,2)$$

$$- \Bigg[- \mathcal{A}^0_{3,qg \to qq}(s_{12}) \otimes \mathcal{E}^0_{3,q}(s_{2i}) - S_{g \to q} \Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{E}^0_{3,q}(s_{2i}) \Bigg] B^0_{1gH}(1,i,2)$$

$$- \Bigg[- \hat{\mathcal{A}}^1_{3,qg}(s_{12}) - \frac{b_F}{\epsilon} \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} \mathcal{A}^0_{3,qg\to qq}(s_{12}) + \frac{b_F}{\epsilon} \mathcal{A}^0_{3,qg\to qq}(s_{12})$$

$$+ \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) - \frac{1}{2} \ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \Gamma^{(1)}_{gg,F}(z_2) - \ S_{g\to q}\overline{\Gamma}^{(2)}_{qg,F}(z_2)$$

$$\Bigg] B^0_{1gH}(1,i,2)$$

$$- \Bigg[+ 2\, \mathcal{E}^0_{4,\bar{q}'g}(s_{12}) - \mathcal{B}^0_{4,q'}(s_{1i}) + \mathcal{E}^0_{3,q'\to g}(s_{1i}) \otimes \mathcal{D}^0_{3,gg\to gq}(s_{12})$$

$$+ \ S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{D}^0_{3,gg\to gq}(s_{12}) - \mathcal{E}^0_{3,q'\to g}(s_{1i}) \otimes \mathcal{D}^0_{3,gg\to qg}(s_{12})$$

$$- \ S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{D}^0_{3,gg\to qg}(s_{12}) - \mathcal{E}^0_{3,q'\to g}(s_{1i}) \otimes \mathcal{D}^0_{3,really}(s_{12})$$

$$+ \ S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{D}^0_{3,really}(s_{12}) + \mathcal{E}^0_{3,q'\to g}(s_{1i}) \otimes \mathcal{A}^0_{3,g\to q}(s_{1i})$$

$$+ 2\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \mathcal{E}^0_{3,q'q\to gq}(s_{12}) + 2\ S_{g\to q}\Gamma^{(1)}_{qg}(z_2) \otimes \ S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \Bigg] B^0_{1gH}(2,1,i)$$

$$\tag{C.3.17}$$

$$\widehat{\widehat{B}}^{2,XU}_{1gH}(\hat{1}_q, \hat{2}_g, i_{\bar{q}}) = \Gamma^{(1)}_{gg,F}(z_2)\hat{B}^1_{1gH}(1,2,i) \tag{C.3.18}$$

Appendix D

Explicit results of antenna subtraction terms for $q\bar{q} \to H+$jet processes at NNLO

D.1 $q\bar{q} \to H+$jet at RR

The double real subtraction terms $\mathrm{d}\hat{\sigma}_{NNLO}^{S}$ mentioned in section 8.5.1 are:

$$\widetilde{B}_{3gH}^{0,XS}(\hat{1}_q, \widetilde{i}, j, k, \hat{2}_{\bar{q}}) =$$

$$+ A_{3,q\bar{q}}^0(1,i,2)\, B_{2gH}^0(\bar{1},j,k,\bar{2})\, J_1^{(2)}(\{p\}_2)$$

$$+ d_{3,q}^0(1,j,k)\, \tilde{B}_{2gH}^0(\bar{1},i,\widetilde{(jk)},2)\, J_1^{(2)}(\{p\}_2)$$

$$+ d_{3,q}^0(2,k,j)\, \tilde{B}_{2gH}^0(1,i,\widetilde{(jk)},\bar{2})\, J_1^{(2)}(\{p\}_2)$$

$$+ A_4^0(1,j,k,2)\, B_{1gH}^0(\bar{1},i,\bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,q}^0(1,j,k)\, A_{3,q\bar{q}}^0(\bar{1},\widetilde{(jk)},2)\, B_{1gH}^0(\bar{\bar{1}},i,\bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$- d_{3,q}^0(2,k,j)\, A_{3,q\bar{q}}^0(1,\widetilde{(kj)},\bar{2})\, B_{1gH}^0(\bar{1},i,\bar{\bar{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ \tilde{A}_4^0(1,i,j,2)\, B_{1gH}^0(\bar{1},k,\bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q\bar{q}}^0(1,i,2)\, A_{3,q\bar{q}}^0(\bar{1},j,\bar{2})\, B_{1gH}^0(\bar{\bar{1}},k,\bar{\bar{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q\bar{q}}^0(1,j,2)\, A_{3,q\bar{q}}^0(\bar{1},i,\bar{2})\, B_{1gH}^0(\bar{\bar{1}},k,\bar{\bar{2}})\, J_1^{(1)}(\{p\}_1)$$

$$+ \frac{1}{2}\, A_{3,q\bar{q}}^0(1,i,2)\, A_{3,q\bar{q}}^0(\bar{1},j,\bar{2})\, B_{1gH}^0(\bar{\bar{1}},k,\bar{\bar{2}})\, J_1^{(1)}(\{p\}_1)$$

$$- \frac{1}{2}\, d_{3,q}^0(1,i,k)\, A_{3,q\bar{q}}^0(\bar{1},j,2)\, B_{1gH}^0(\bar{\bar{1}},\widetilde{(ik)},\bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$-\frac{1}{2}\, d_{3,q}^{0}(2,i,k)\, A_{3,q\bar{q}}^{0}(1,j,\overline{2})\, B_{1gH}^{0}(\overline{1},(\widetilde{ik}),\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$+\frac{1}{2}\left[-S_{\overline{1}i2}^{IF}+S_{\overline{\overline{1}}i\overline{2}}^{IF}+S_{\overline{1}i(\widetilde{ik})}^{IF}-S_{\overline{\overline{1}}i(\widetilde{ik})}^{IF}+S_{2i(\widetilde{ik})}^{IF}-S_{\overline{2}i(\widetilde{ik})}^{IF}\right]$$

$$\times A_{3,q\bar{q}}^{0}(\overline{1},j,2)\, B_{1gH}^{0}(\overline{\overline{1}},(\widetilde{ik}),\overline{2})\, J_{1}^{(1)}(\{p\}_{1})$$

$$+\frac{1}{2}\, A_{3,q\bar{q}}^{0}(1,j,2)\, A_{3,q\bar{q}}^{0}(\overline{1},i,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},k,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-\frac{1}{2}\, d_{3,q}^{0}(1,j,k)\, A_{3,q\bar{q}}^{0}(\overline{1},i,2)\, B_{1gH}^{0}(\overline{\overline{1}},(\widetilde{jk}),\overline{2})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-\frac{1}{2}\, d_{3,q}^{0}(2,j,k)\, A_{3,q\bar{q}}^{0}(1,i,\overline{2})\, B_{1gH}^{0}(\overline{1},(\widetilde{jk}),\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$+\frac{1}{2}\left[-S_{\overline{1}j2}^{IF}+S_{\overline{\overline{1}}j\overline{2}}^{IF}+S_{\overline{1}j(\widetilde{jk})}^{IF}-S_{\overline{\overline{1}}j(\widetilde{jk})}^{IF}+S_{2j(\widetilde{jk})}^{IF}-S_{\overline{2}j(\widetilde{jk})}^{IF}\right]$$

$$\times A_{3,q\bar{q}}^{0}(\overline{1},i,2)\, B_{1gH}^{0}(\overline{\overline{1}},(\widetilde{jk}),\overline{2})\, J_{1}^{(1)}(\{p\}_{1}) \tag{D.1.1}$$

$$\widetilde{\widetilde{B}}_{3gH}^{0,XS}\,(\hat{1}_{q},\widetilde{i},\widetilde{j},\widetilde{k},\hat{2}_{\bar{q}}) =$$

$$+A_{3,q\bar{q}}^{0}(1,i,2)\, \tilde{B}_{2gH}^{0}(\overline{1},j,k,\overline{2})\, J_{1}^{(2)}(\{p\}_{2})$$

$$+A_{3,q\bar{q}}^{0}(1,j,2)\, \tilde{B}_{2gH}^{0}(\overline{1},i,k,\overline{2})\, J_{1}^{(2)}(\{p\}_{2})$$

$$+A_{3,q\bar{q}}^{0}(1,k,2)\, \tilde{B}_{2gH}^{0}(\overline{1},i,j,\overline{2})\, J_{1}^{(2)}(\{p\}_{2})$$

$$+\tilde{A}_{4}^{0}(1,i,j,2)\, B_{1gH}^{0}(\overline{1},k,\overline{2})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-A_{3,q\bar{q}}^{0}(1,i,2)\, A_{3,q\bar{q}}^{0}(\overline{1},j,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},k,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-A_{3,q\bar{q}}^{0}(1,j,2)\, A_{3,q\bar{q}}^{0}(\overline{1},i,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},k,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$+\tilde{A}_{4}^{0}(1,j,k,2)\, B_{1gH}^{0}(\overline{1},i,\overline{2})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-A_{3,q\bar{q}}^{0}(1,j,2)\, A_{3,q\bar{q}}^{0}(\overline{1},k,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},i,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-A_{3,q\bar{q}}^{0}(1,k,2)\, A_{3,q\bar{q}}^{0}(\overline{1},j,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},i,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$+\tilde{A}_{4}^{0}(1,i,k,2)\, B_{1gH}^{0}(\overline{1},j,\overline{2})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-A_{3,q\bar{q}}^{0}(1,i,2)\, A_{3,q\bar{q}}^{0}(\overline{1},k,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},j,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1})$$

$$-A_{3,q\bar{q}}^{0}(1,k,2)\, A_{3,q\bar{q}}^{0}(\overline{1},i,\overline{2})\, B_{1gH}^{0}(\overline{\overline{1}},j,\overline{\overline{2}})\, J_{1}^{(1)}(\{p\}_{1}) \tag{D.1.2}$$

$$C_{1gH}^{0,XS}(\hat{1}_{q},i,j_{\bar{Q}},k_{Q},\hat{2}_{\bar{q}}) =$$

$$+A_{3,q}^{0}(1,i,j)\, C_{0gH}^{0}(\overline{1},(\widetilde{ij}),k,2)\, J_{1}^{(2)}(\{p\}_{2})$$

$$+A_{3,q}^{0}(2,i,k)\, C_{0gH}^{0}(1,j,(\widetilde{ki}),\overline{2})\, J_{1}^{(2)}(\{p\}_{2})$$

$$+G_3^0(i,j,k)\, B_{2gH}^0(1,(\widetilde{ij}),(\widetilde{jk}),2)\, J_1^{(2)}(\{p\}_2)$$

$$+G_3^0(i,k,j)\, B_{2gH}^0(1,(\widetilde{jk}),(\widetilde{ki}),2)\, J_1^{(2)}(\{p\}_2)$$

$$+E_4^0(1,k,j,i)\, B_{1gH}^0(\overline{1},(\widetilde{ijk}),2)\, J_1^{(1)}(\{p\}_1)$$

$$-G_3^0(i,j,k)\, D_{3,q}^0(1,(\widetilde{jk}),(\widetilde{ij}))\, B_{1gH}^0(\overline{1},(\widetilde{\widetilde{jk},\widetilde{ij}}),2)\, J_1^{(1)}(\{p\}_1)$$

$$-A_{3,q}^0(1,i,j)\, E_{3,q}^0(\overline{1},k,(\widetilde{ji}))\, B_{1gH}^0(\overline{\overline{1}},(\widetilde{k(\widetilde{ji})}),2)\, J_1^{(1)}(\{p\}_1)$$

$$+E_4^0(2,j,k,i)\, B_{1gH}^0(1,(\widetilde{jki}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$-G_3^0(i,k,j)\, D_{3,q}^0(2,(\widetilde{jk}),(\widetilde{ki}))\, B_{1gH}^0(1,(\widetilde{\widetilde{jk},\widetilde{ki}}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$-A_{3,q}^0(2,i,k)\, E_{3,q}^0(\overline{2},j,(\widetilde{ki}))\, B_{1gH}^0(1,(\widetilde{j(\widetilde{ki})}),\overline{\overline{2}})\, J_1^{(1)}(\{p\}_1) \tag{D.1.3}$$

$$C_{1gH}^{0,YS}(\hat{1}_q,i,\hat{2}_{\bar{Q}},k_Q,j_{\bar{q}}) =$$

$$+A_{3,q\bar{q}}^0(1,i,2)\, C_{0gH}^0(\overline{1},\overline{2},k,j)\, J_1^{(2)}(\{p\}_2)$$

$$-E_{3,q'\to g}^0(j,k,2)\, B_{2gH}^0(1,i,\overline{2},(\widetilde{jk}))\, J_1^{(2)}(\{p\}_2)$$

$$-E_{3,q'\to g}^0(k,j,1)\, B_{2gH}^0((\widetilde{kj}),\overline{1},i,2)\, J_1^{(2)}(\{p\}_2)$$

$$-E_4^0(j,k,2,i)\, B_{1gH}^0(1,\overline{2},(\widetilde{jki}))\, J_1^{(1)}(\{p\}_1)$$

$$+E_{3,q'\to g}^0(j,k,2)\, D_{3,g}^0((\widetilde{jk}),\overline{2},i)\, B_{1gH}^0(1,\overline{\overline{2}},(\widetilde{i(\widetilde{jk})}))\, J_1^{(1)}(\{p\}_1)$$

$$+A_{3,q}^0(2,i,j)\, E_{3,q'\to g}^0((\widetilde{ij}),k,\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},((\widetilde{\widetilde{ij})k}))\, J_1^{(1)}(\{p\}_1)$$

$$+A_{3,q}^0(1,i,j)\, E_{3,q'\to g}^0((\widetilde{ij}),k,2)\, B_{1gH}^0(\overline{1},\overline{2},((\widetilde{\widetilde{ij})k}))\, J_1^{(1)}(\{p\}_1)$$

$$+A_{3,q\bar{q}}^0(1,i,2)\, E_{3,q'\to g}^0(j,k,\overline{2})\, B_{1gH}^0(\overline{1},\overline{\overline{2}},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$-A_{3,q}^0(2,i,j)\, E_{3,q'\to g}^0((\widetilde{ij}),k,\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},((\widetilde{\widetilde{ij})k}))\, J_1^{(1)}(\{p\}_1)$$

$$+\left[-S_{\overline{1}i(\widetilde{ij})}^{IF} + S_{\overline{1}i((\widetilde{\widetilde{ij})k})}^{IF} - S_{\overline{1}i2}^{IF} + S_{\overline{1}i\overline{2}}^{IF} + S_{2i(\widetilde{ij})}^{IF} - S_{\overline{2}i((\widetilde{\widetilde{ij})k})}^{IF}\right]$$

$$\times E_{3,q'\to g}^0((\widetilde{ij}),k,2)\, B_{1gH}^0(\overline{1},\overline{2},((\widetilde{\widetilde{ij})k}))\, J_1^{(1)}(\{p\}_1)$$

$$+B_4^0(1,2,k,j)\, B_{1gH}^0(\overline{1},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$-E_{3,q'\to g}^0(j,k,2)\, A_{3,qg\to qq}^0(1,\overline{2},(\widetilde{jk}))\, B_{1gH}^0(\overline{1},\overline{\overline{2}},i)\, J_1^{(1)}(\{p\}_1)$$

$$-E_4^0(k,j,1,i)\, B_{1gH}^0((\widetilde{kji}),\overline{1},2)\, J_1^{(1)}(\{p\}_1)$$

$$+E_{3,q'\to g}^0(k,j,1)\, D_{3,g}^0((\widetilde{kj}),\overline{1},i)\, B_{1gH}^0(2,\overline{\overline{1}},(\widetilde{i(\widetilde{kj})}))\, J_1^{(1)}(\{p\}_1)$$

$$+A_{3,q}^0(1,i,k)\, E_{3,q'\to g}^0((\widetilde{ik}),j,\overline{1})\, B_{1gH}^0(2,\overline{\overline{1}},((\widetilde{\widetilde{ik})j}))\, J_1^{(1)}(\{p\}_1)$$

$$+A_{3,q}^0(2,i,k)\, E_{3,q'\to g}^0((\widetilde{ik}),j,1)\, B_{1gH}^0(\overline{2},\overline{1},((\widetilde{\widetilde{ik})j}))\, J_1^{(1)}(\{p\}_1)$$

$$+ A^0_{3,q\bar{q}}(1,i,2)\, E^0_{3,q'\to g}(k,j,\overline{1})\, B^0_{1gH}(\overline{2},\overline{1},(\widetilde{kj}))\, J^{(1)}_1(\{p\}_1)$$

$$- A^0_{3,q}(1,i,k)\, E^0_{3,q'\to g}((\widetilde{ik}),j,\overline{1})\, B^0_{1gH}(2,\overline{\overline{1}},(\widetilde{(ik)j}))\, J^{(1)}_1(\{p\}_1)$$

$$+ \left[- S^{IF}_{\overline{2}i(\widetilde{ik})} + S^{IF}_{\overline{2}i(\widetilde{(ik)j})} - S^{IF}_{1i\overline{2}} + S^{IF}_{\overline{1}i\overline{2}} + S^{IF}_{1i(\widetilde{ik})} - S^{IF}_{\overline{1}i(\widetilde{(ik)j})} \right]$$

$$\times E^0_{3,q'\to g}((\widetilde{ik}),j,1)\, B^0_{1gH}(\overline{2},\overline{1},(\widetilde{(ik)j}))\, J^{(1)}_1(\{p\}_1)$$

$$+ B^0_4(k,j,1,2)\, B^0_{1gH}(i,\overline{1},\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(k,j,1)\, A^0_{3,qg\to qq}(2,\overline{1},(\widetilde{kj}))\, B^0_{1gH}(i,\overline{\overline{1}},\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ B^0_4(1,k,2,j)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(j,k,2)\, A^0_{3,qg\to qq}(1,\overline{2},(\widetilde{jk}))\, B^0_{1gH}(\overline{1},i,\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1)$$

$$+ B^0_4(k,1,j,2)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(k,j,1)\, A^0_{3,qg\to qq}(2,\overline{1},(\widetilde{kj}))\, B^0_{1gH}(\overline{2},i,\overline{\overline{1}})\, J^{(1)}_1(\{p\}_1)$$

$$+ {}_-A^0(1,j,k,2)\, A^0_{3gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(k,j,1)\, G^0_{3,gq\to gg}(\overline{1},2,(\widetilde{kj}))\, A^0_{3gH}(\overline{\overline{1}},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(j,k,2)\, G^0_{3,gq\to gg}(\overline{2},1,(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{2}},i,\overline{1})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(k,j,1)\, A^0_{3,q}((\widetilde{kj}),i,2)\, B^0_{1gH}(\overline{2},\overline{1},(\widetilde{i(\widetilde{kj})}))\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(j,k,2)\, A^0_{3,q}((\widetilde{jk}),i,1)\, B^0_{1gH}(\overline{1},\overline{2},(\widetilde{i(\widetilde{jk})}))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(k,i,j)\, C^0_{0gH}(1,2,(\widetilde{ki}),(\widetilde{ij}))\, J^{(2)}_1(\{p\}_2)$$

$$- E^0_{3,q'\to g}(j,k,2)\, B^0_{2gH}(1,\overline{2},i,(\widetilde{jk}))\, J^{(2)}_1(\{p\}_2)$$

$$- E^0_{3,q'\to g}(k,j,1)\, B^0_{2gH}((\widetilde{kj}),i,\overline{1},2)\, J^{(2)}_1(\{p\}_2)$$

$$- E^0_4(j,2,k,i)\, B^0_{1gH}(1,\overline{2},(\widetilde{jki}))\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_{3,q'\to g}(j,k,2)\, D^0_{3,g}((\widetilde{jk}),i,\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},((\widetilde{(jk)i})))\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(j,i,k)\, E^0_{3,q'\to g}((\widetilde{ji}),(\widetilde{ik}),2)\, B^0_{1gH}(1,\overline{2},(\widetilde{ji,ik}))\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_4(k,1,j,i)\, B^0_{1gH}((\widetilde{ijk}),\overline{1},2)\, J^{(1)}_1(\{p\}_1)$$

$$+ E^0_{3,q'\to g}(k,j,1)\, D^0_{3,g}((\widetilde{kj}),i,\overline{1})\, B^0_{1gH}(((\widetilde{(kj)i})),\overline{\overline{1}},2)\, J^{(1)}_1(\{p\}_1)$$

$$+ A^0_3(k,i,j)\, E^0_{3,q'\to g}((\widetilde{ki}),(\widetilde{ij}),1)\, B^0_{1gH}((\widetilde{ki,ij}),\overline{1},2)\, J^{(1)}_1(\{p\}_1)$$

$$+ B^0_4(1,k,2,j)\, B^0_{1gH}(\overline{1},\overline{2},i)\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(j,k,2)\, A^0_{3,qg\to qq}(1,\overline{2},(\widetilde{jk}))\, B^0_{1gH}(\overline{1},\overline{\overline{2}},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ B^0_4(2,j,1,k)\, B^0_{1gH}(i,\overline{1},\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(k,j,1)\, A^0_{3,qg\to qq}(2,\overline{1},(\widetilde{kj}))\, B^0_{1gH}(\overline{2},\overline{\overline{1}},i)\, J^{(1)}_1(\{p\}_1)$$

$$+ {}_-A^0(1,j,k,2)\, A^0_{3gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(k,j,1)\, G^0_{3,gq\to gg}(\overline{1},2,(\widetilde{kj}))\, A^0_{3gH}(\overline{\overline{1}},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- E^0_{3,q'\to g}(j,k,2)\, G^0_{3,gq\to gg}(\overline{2},1,(\widetilde{jk}))\, A^0_{3gH}(\overline{\overline{2}},i,\overline{1})\, J^{(1)}_1(\{p\}_1) \qquad \text{(D.1.4)}$$

$$\widetilde{C}^{0,XS}_{1gH}(\hat{1}_q,i,\hat{2}_{\bar{q}},k_Q,j_{\bar{Q}}) =$$

$$- A^0_{3,q\bar{q}}(1,i,2)\, C^0_{0gH}(\overline{1},j,k,\overline{2})\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_3(k,i,j)\, C^0_{0gH}(1,(\widetilde{ij}),(\widetilde{ik}),2)\, J^{(2)}_1(\{p\}_2)$$

$$- G^0_3(i,k,j)\, \tilde{B}^0_{2gH}(1,(\widetilde{ik}),(\widetilde{kj}),2)\, J^{(2)}_1(\{p\}_2)$$

$$- B^0_4(1,j,k,2)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ G^0_3(i,k,j)\, A^0_{3,q\bar{q}}(1,(\widetilde{ik}),2)\, B^0_{1gH}(\overline{1},(\widetilde{kj}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- \frac{1}{2}\, \tilde{E}^0_4(1,k,i,j)\, B^0_{1gH}(\overline{1},(\widetilde{kij}),2)\, J^{(1)}_1(\{p\}_1)$$

$$+ \frac{1}{2}\, G^0_3(i,k,j)\, A^0_{3,q\bar{q}}(1,(\widetilde{kj}),2)\, B^0_{1gH}(\overline{1},(\widetilde{ik}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \frac{1}{2}\, A^0_3(k,i,j)\, E^0_{3,q}(1,(\widetilde{ki}),(\widetilde{ij}))\, B^0_{1gH}(\overline{1},(\widetilde{\widetilde{ki},\widetilde{ij}}),2)\, J^{(1)}_1(\{p\}_1)$$

$$- \frac{1}{2}\, \tilde{E}^0_4(2,k,i,j)\, B^0_{1gH}(1,(\widetilde{kij}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \frac{1}{2}\, G^0_3(i,k,j)\, A^0_{3,q\bar{q}}(1,(\widetilde{kj}),2)\, B^0_{1gH}(\overline{1},(\widetilde{ik}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \frac{1}{2}\, A^0_3(k,i,j)\, E^0_{3,q}(2,(\widetilde{ki}),(\widetilde{ij}))\, B^0_{1gH}(1,(\widetilde{\widetilde{ki},\widetilde{ij}}),\overline{2})\, J^{(1)}_1(\{p\}_1) \qquad \text{(D.1.5)}$$

$$\widetilde{C}^{0,YS}_{1gH}(\hat{1}_q,i,j_{\bar{q}},k_Q,\hat{2}_{\bar{Q}}) =$$

$$- A^0_{3,q}(1,i,j)\, C^0_{0gH}(\overline{1},2,k,(\widetilde{ij}))\, J^{(2)}_1(\{p\}_2)$$

$$- A^0_{3,q}(2,i,k)\, C^0_{0gH}(1,\overline{2},(\widetilde{ik}),j)\, J^{(2)}_1(\{p\}_2)$$

$$+ G^0_{3,q'\to g}(i,k,2)\, \tilde{B}^0_{2gH}(1,(\widetilde{ik}),\overline{2},j)\, J^{(2)}_1(\{p\}_2)$$

$$+ G^0_{3,q'\to g}(i,j,1)\, \tilde{B}^0_{2gH}(k,(\widetilde{ij}),\overline{1},2)\, J^{(2)}_1(\{p\}_2)$$

$$- B^0_4(1,k,2,j)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ G^0_{3,q'\to g}(i,k,2)\, A^0_{3,qg\to qq}(1,\overline{2},j)\, B^0_{1gH}(\overline{1},(\widetilde{ik}),\overline{\overline{2}})\, J^{(1)}_1(\{p\}_1)$$

$$- B^0_4(k,1,j,2)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ G^0_{3,q'\to g}(i,j,1)\, A^0_{3,gq\to qq}(k,\overline{1},2)\, B^0_{1gH}(\overline{\overline{1}},(\widetilde{ij}),\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+ \tilde{E}_4^0(k,j,i,1)\, B_{1gH}^0((\widetilde{kji}),\overline{1},2)\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q}^0(1,i,j)\, E_{3,q'\to g}^0(k,(\widetilde{ij}),\overline{1})\, B_{1gH}^0((\widetilde{k(\widetilde{ij})}),\overline{\overline{1}},2)\, J_1^{(1)}(\{p\}_1)$$

$$+ \tilde{E}_4^0(j,k,i,2)\, B_{1gH}^0(1,\overline{2},(\widetilde{jki}))\, J_1^{(1)}(\{p\}_1)$$

$$- A_{3,q}^0(k,i,2)\, E_{3,q'\to g}^0(j,(\widetilde{ki}),\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},(\widetilde{j(\widetilde{ki})}))\, J_1^{(1)}(\{p\}_1)$$

$$- G_{3,q'\to g}^0(i,j,1)\, A_{3,q}^0(2,(\widetilde{ij}),k)\, B_{1gH}^0(((\widetilde{\widetilde{ij})k}),\overline{1},\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$- G_{3,q'\to g}^0(i,k,2)\, A_{3,q}^0(1,(\widetilde{ik}),j)\, B_{1gH}^0(\overline{1},\overline{2},((\widetilde{\widetilde{ik})j}))\, J_1^{(1)}(\{p\}_1)$$

$$+ 2 A_{3,q}^0(1,i,k)\, C_{0gH}^0(\overline{1},2,(\widetilde{ik}),j)\, J_1^{(2)}(\{p\}_2)$$

$$+ 2 A_{3,q}^0(2,i,j)\, C_{0gH}^0(1,\overline{2},k,(\widetilde{ij}))\, J_1^{(2)}(\{p\}_2)$$

$$- 2 A_{3,q\bar{q}}^0(1,i,2)\, C_{0gH}^0(\overline{1},\overline{2},k,j)\, J_1^{(2)}(\{p\}_2)$$

$$- 2 A_3^0(k,i,j)\, C_{0gH}^0(1,2,(\widetilde{ki}),(\widetilde{ij}))\, J_1^{(2)}(\{p\}_2)$$

$$+ 2 A_{3,q}^0(1,i,k)\, E_{3,q'\to g}^0((\widetilde{ik}),j,\overline{1})\, B_{1gH}^0(((\widetilde{\widetilde{ik})j}),\overline{\overline{1}},2)\, J_1^{(1)}(\{p\}_1)$$

$$+ 2 A_{3,q}^0(1,i,k)\, E_{3,q'\to g}^0(j,(\widetilde{ik}),2)\, B_{1gH}^0(\overline{1},\overline{2},(\widetilde{j(\widetilde{ik})}))\, J_1^{(1)}(\{p\}_1)$$

$$+ 2 A_{3,q}^0(2,i,j)\, E_{3,q'\to g}^0((\widetilde{ij}),k,\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},((\widetilde{\widetilde{ij})k}))\, J_1^{(1)}(\{p\}_1)$$

$$+ 2 A_{3,q}^0(2,i,j)\, E_{3,q'\to g}^0(k,(\widetilde{ij}),1)\, B_{1gH}^0((\widetilde{k(\widetilde{ij})}),\overline{1},\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$- 2 A_{3,q\bar{q}}^0(1,i,2)\, E_{3,q'\to g}^0(j,k,\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},(\widetilde{jk}))\, J_1^{(1)}(\{p\}_1)$$

$$- 2 A_{3,q\bar{q}}^0(1,i,2)\, E_{3,q'\to g}^0(k,j,\overline{1})\, B_{1gH}^0((\widetilde{kj}),\overline{\overline{1}},2)\, J_1^{(1)}(\{p\}_1)$$

$$- 2 A_3^0(k,i,j)\, E_{3,q'\to g}^0((\widetilde{ki}),(\widetilde{ij}),1)\, B_{1gH}^0((\widetilde{\widetilde{ki},\widetilde{ij}}),\overline{1},2)\, J_1^{(1)}(\{p\}_1)$$

$$- 2 A_3^0(k,i,j)\, E_{3,q'\to g}^0((\widetilde{ij}),(\widetilde{ki}),2)\, B_{1gH}^0(1,\overline{2},(\widetilde{\widetilde{ij},\widetilde{ki}}))\, J_1^{(1)}(\{p\}_1)$$

$$- 2\left[+ S_{\overline{1}ik}^{IF} - S_{\overline{1}i2}^{IF} + S_{2i(\widetilde{ij})}^{IF} - S_{ki(\widetilde{ij})}^{IF} \right]$$

$$\times E_{3,q'\to g}^0(k,(\widetilde{ij}),\overline{1})\, B_{1gH}^0((\widetilde{k(\widetilde{ij})}),\overline{\overline{1}},2,6)\, J_1^{(1)}(\{p\}_1)$$

$$- 2\left[+ S_{1i(\widetilde{ik})}^{IF} - S_{1i\overline{2}}^{IF} + S_{\overline{2}ij}^{IF} - S_{(\widetilde{ik})ij}^{IF} \right]$$

$$\times E_{3,q'\to g}^0(j,(\widetilde{ik}),\overline{2})\, B_{1gH}^0(1,\overline{\overline{2}},(\widetilde{j(\widetilde{ik})}),6)\, J_1^{(1)}(\{p\}_1) \tag{D.1.6}$$

$$D_{1gH}^{0,XS}(\hat{1}_q,k_q,\hat{2}_{\bar{q}},j_{\bar{q}},i) =$$

$$+ A_{3,q}^0(1,i,j)\, D_{0gH}^0(\overline{1},(\widetilde{ji}),k,2)\, J_1^{(2)}(\{p\}_2)$$

$$+ A_{3,q}^0(2,i,k)\, D_{0gH}^0(1,j,(\widetilde{ki}),\overline{2})\, J_1^{(2)}(\{p\}_2)$$

$$+ A_{3,q\bar{q}}^0(1,i,2)\, D_{0gH}^0(\overline{1},j,k,\overline{2})\, J_1^{(2)}(\{p\}_2)$$

$$+A_3^0(k,i,j)\, D_{0gH}^0(1,(\widetilde{ji}),(\widetilde{ki}),2)\, J_1^{(2)}(\{p\}_2)$$

$$+4\, C_4^0(1,k,j,2)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+4\, C_4^0(2,j,k,1)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1) \qquad\text{(D.1.7)}$$

$$\widetilde{D}_{1gH}^{0,XS}(\hat{1}_q, k_q, \hat{2}_{\bar{q}}, j_{\bar{q}}, i) =$$

$$+A_{3,q}^0(1,i,j)\, D_{0gH}^0(\overline{1},(\widetilde{ji}),k,2)\, J_1^{(2)}(\{p\}_2)$$

$$+A_{3,q}^0(2,i,k)\, D_{0gH}^0(1,j,(\widetilde{ki}),\overline{2})\, J_1^{(2)}(\{p\}_2)$$

$$+A_{3,q\bar{q}}^0(1,i,2)\, D_{0gH}^0(\overline{1},j,k,\overline{2})\, J_1^{(2)}(\{p\}_2)$$

$$+A_3^0(k,i,j)\, D_{0gH}^0(1,(\widetilde{ji}),(\widetilde{ki}),2)\, J_1^{(2)}(\{p\}_2)$$

$$-A_{3,q}^0(1,i,k)\, D_{0gH}^0(\overline{1},j,(\widetilde{ki}),2)\, J_1^{(2)}(\{p\}_2)$$

$$-A_{3,q}^0(2,i,j)\, D_{0gH}^0(1,(\widetilde{ji}),k,\overline{2})\, J_1^{(2)}(\{p\}_2)$$

$$+2\, C_4^0(1,k,j,2)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+2\, C_4^0(2,j,k,1)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1) \qquad\text{(D.1.8)}$$

D.2 $q\bar{q} \to H+$jet at RV

The real-virtual subtraction terms $\mathrm{d}\hat{\sigma}_{NNLO}^T$ mentioned in section 8.5.2 are:

$$\widetilde{B}_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) =$$

$$-J_{2,QQ}^{1,II}(s_{12})\, B_{2gH}^0(1,i,j,2)\, J_1^{(2)}(\{p\}_2)$$

$$-J_{2,QQ}^{1,II}(s_{12})\, B_{2gH}^0(1,j,i,2)\, J_1^{(2)}(\{p\}_2)$$

$$+d_{3,q}^0(1,i,j)\left[\, \tilde{B}_{1gH}^1(\overline{1},(\widetilde{ij}),2)\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+\, J_{2,QQ}^{1,II}(s_{\overline{1}2})\, B_{1gH}^0(\overline{1},(\widetilde{ij}),2)\right]\, J_1^{(1)}(\{p\}_1)$$

$$+d_{3,q}^0(2,j,i)\left[\, \tilde{B}_{1gH}^1(1,(\widetilde{ij}),\overline{2})\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+\, J_{2,QQ}^{1,II}(s_{1\overline{2}})\, B_{1gH}^0(1,(\widetilde{ij}),\overline{2})\right]\, J_1^{(1)}(\{p\}_1)$$

$$+d_{3,q}^0(1,j,i)\left[\, \tilde{B}_{1gH}^1(\overline{1},(\widetilde{ji}),2)\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left.+\, J_{2,QQ}^{1,II}(s_{\overline{1}2})\, B_{1gH}^0(\overline{1},(\widetilde{ji}),2)\right]\, J_1^{(1)}(\{p\}_1)$$

$$+ d^0_{3,q}(2,i,j) \left[\tilde{B}^1_{1gH}(1,(\widetilde{ji}),\overline{2})\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + J^{1,II}_{2,QQ}(s_{1\overline{2}})\,B^0_{1gH}(1,(\widetilde{ji}),\overline{2}) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[\tilde{A}^1_{3,q\bar{q}}(1,i,2)\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,II}_{2,QQ}(s_{12}) - J^{1,II}_{2,QQ}(s_{\overline{12}}) \right) A^0_{3,q\bar{q}}(1,i,2) \right] B^0_{1gH}(\overline{1},j,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+ \left[\tilde{A}^1_{3,q\bar{q}}(1,j,2)\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,II}_{2,QQ}(s_{12}) - J^{1,II}_{2,QQ}(s_{\overline{12}}) \right) A^0_{3,q\bar{q}}(1,j,2) \right] B^0_{1gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1) \qquad \text{(D.2.9)}$$

$$\widetilde{\widetilde{B}}^{1,XT}_{2gH}(\hat{1}_q,\tilde{i},\tilde{j},\hat{2}_{\bar{q}}) =$$

$$- \left[+ J^{1,FI}_{2,GQ}(s_{2j}) + J^{1,FI}_{2,GQ}(s_{2i}) + J^{1,IF}_{2,QG}(s_{1j}) + J^{1,IF}_{2,QG}(s_{1i}) - J^{1,II}_{2,QQ}(s_{12}) \right]$$

$$\times \tilde{B}^0_{2gH}(1,i,j,2)\,J^{(2)}_1(\{p\}_2)$$

$$+ A^0_{3,q\bar{q}}(1,i,2) \left[B^1_{1gH}(\overline{1},j,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,IF}_{2,QG}(s_{\overline{1}j}) + J^{1,FI}_{2,GQ}(s_{\overline{2}j}) \right) B^0_{1gH}(\overline{1},j,\overline{2}) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[A^1_{3,q\bar{q}}(1,i,2)\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,IF}_{2,QG}(s_{1i}) + J^{1,FI}_{2,GQ}(s_{2i}) - J^{1,II}_{2,QQ}(s_{\overline{12}}) \right) A^0_{3,q\bar{q}}(1,i,2) \right] B^0_{1gH}(\overline{1},j,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+ A^0_{3,q\bar{q}}(1,j,2) \left[B^1_{1gH}(\overline{1},i,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,IF}_{2,QG}(s_{\overline{1}i}) + J^{1,FI}_{2,GQ}(s_{\overline{2}i}) \right) B^0_{1gH}(\overline{1},i,\overline{2}) \right] J^{(1)}_1(\{p\}_1)$$

$$+ \left[A^1_{3,q\bar{q}}(1,j,2)\,\delta(1-x_1)\,\delta(1-x_2) \right.$$

$$\left. + \left(+ J^{1,IF}_{2,QG}(s_{1j}) + J^{1,FI}_{2,GQ}(s_{2j}) - J^{1,II}_{2,QQ}(s_{\overline{12}}) \right) A^0_{3,q\bar{q}}(1,j,2) \right] B^0_{1gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+ \left[+ J^{1,II}_{2,QQ}(s_{\overline{12}}) - J^{1,II}_{2,QQ}(s_{12}) - J^{1,IF}_{2,QG}(s_{\overline{1}j}) + J^{1,IF}_{2,QG}(s_{1j}) - J^{1,FI}_{2,GQ}(s_{\overline{2}j}) + J^{1,FI}_{2,GQ}(s_{2j}) \right.$$

$$+ \left(- \tilde{\mathcal{S}}^{IF}(s_{\overline{12}}, s_{\overline{1}j}, x_{\overline{12},\overline{1}j}) + \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}j}, s_{\overline{1}j}, 1) - \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) \right.$$

$$\left. \left. + \tilde{\mathcal{S}}^{IF}(s_{\overline{2}j}, s_{\overline{1}j}, x_{\overline{2}j,\overline{1}j}) - \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) \right) \right]$$

$$\times A^0_{3,q\bar{q}}(1,i,2)\,B^0_{1gH}(\overline{1},j,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$+ \bigg[+ J_{2,QQ}^{1,II}(s_{\overline{12}}) - J_{2,QQ}^{1,II}(s_{12}) - J_{2,QG}^{1,IF}(s_{\overline{1}i}) + J_{2,QG}^{1,IF}(s_{1i}) - J_{2,GQ}^{1,FI}(s_{\overline{2}i}) + J_{2,GQ}^{1,FI}(s_{2i})$$

$$+ \bigg(- \tilde{\mathcal{S}}^{IF}(s_{\overline{12}}, s_{\overline{1}i}, x_{\overline{12},\overline{1}i}) + \mathcal{S}^{IF}(s_{12}, s_{1i}, x_{12,1i}) + \tilde{\mathcal{S}}^{IF}(s_{\overline{1}i}, s_{\overline{1}i}, 1) - \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1)$$

$$+ \tilde{\mathcal{S}}^{IF}(s_{\overline{2}i}, s_{\overline{1}i}, x_{\overline{2}i,\overline{1}i}) - \mathcal{S}^{IF}(s_{2i}, s_{1i}, x_{2i,1i}) \bigg) \bigg]$$

$$\times A_{3,q\bar{q}}^0(1,j,2)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1) \tag{D.2.10}$$

$$\widehat{B}_{2gH}^{1,XT}(\hat{1}_q, i, j, \hat{2}_{\bar{q}}) =$$

$$-2\hat{J}_{2,GG}^{1,FF}(s_{ij})\, B_{2gH}^0(1,i,j,2)\, J_1^{(2)}(\{p\}_2)$$

$$+ d_{3,q}^0(1,i,j)\, \hat{B}_{1gH}^1(\overline{1}, (\widetilde{ij}), 2)\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+ \bigg[\hat{d}_{3,q}^1(1,i,j)\, \delta(1-x_1)\, \delta(1-x_2) + 2\hat{J}_{2,GG}^{1,FF}(s_{ij})\, d_{3,q}^0(1,i,j) \bigg]$$

$$\times B_{1gH}^0(\overline{1}, (\widetilde{ij}), 2)\, J_1^{(1)}(\{p\}_1)$$

$$+ d_{3,q}^0(2,j,i)\, \hat{B}_{1gH}^1(1, (\widetilde{ij}), \overline{2})\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+ \bigg[\hat{d}_{3,q}^1(2,j,i)\, \delta(1-x_1)\, \delta(1-x_2) + 2\hat{J}_{2,GG}^{1,FF}(s_{ij})\, d_{3,q}^0(2,j,i) \bigg]$$

$$\times B_{1gH}^0(1, (\widetilde{ij}), \overline{2})\, J_1^{(1)}(\{p\}_1) \tag{D.2.11}$$

$$\widetilde{\widetilde{B}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}}) =$$

$$- J_{2,QQ}^{1,II}(s_{12})\, \tilde{B}_{2gH}^0(1,i,j,2)\, J_1^{(2)}(\{p\}_2)$$

$$+ A_{3,q\bar{q}}^0(1,j,2) \bigg[\tilde{B}_{1gH}^1(\overline{1},i,\overline{2})\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+ J_{2,QQ}^{1,II}(s_{\overline{12}})\, B_{1gH}^0(\overline{1},i,\overline{2}) \bigg] J_1^{(1)}(\{p\}_1)$$

$$+ \bigg[\tilde{A}_{3,q\bar{q}}^1(1,j,2)\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+ \bigg(+ J_{2,QQ}^{1,II}(s_{12}) - J_{2,QQ}^{1,II}(s_{\overline{12}}) \bigg) A_{3,q\bar{q}}^0(1,j,2) \bigg] B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q\bar{q}}^0(1,i,2) \bigg[\tilde{B}_{1gH}^1(\overline{1},j,\overline{2})\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+ J_{2,QQ}^{1,II}(s_{\overline{12}})\, B_{1gH}^0(\overline{1},j,\overline{2}) \bigg] J_1^{(1)}(\{p\}_1)$$

$$+ \bigg[\tilde{A}_{3,q\bar{q}}^1(1,i,2)\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+ \bigg(+ J_{2,QQ}^{1,II}(s_{12}) - J_{2,QQ}^{1,II}(s_{\overline{12}}) \bigg) A_{3,q\bar{q}}^0(1,i,2) \bigg] B_{1gH}^0(\overline{1},j,\overline{2})\, J_1^{(1)}(\{p\}_1) \tag{D.2.12}$$

$$\widehat{\widetilde{\widetilde{B}}}_{2gH}^{1,XT}(\hat{1}_q, \widetilde{i}, \widetilde{j}, \hat{2}_{\bar{q}}) =$$

$$-2\hat{J}_{2,GG}^{1,FF}(s_{ij})\,\tilde{B}_{2gH}^0(1,i,j,2)\,J_1^{(2)}(\{p\}_2)$$

$$+ A_{3,q\bar{q}}^0(1,i,2)\,\hat{B}_{1gH}^1(\overline{1},j,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \left[\, \hat{A}_{3,q\bar{q}}^1(1,i,2)\,\delta(1-x_1)\,\delta(1-x_2) + 2\hat{J}_{2,GG}^{1,FF}(s_{ij})\,A_{3,q\bar{q}}^0(1,i,2)\right]$$

$$\times B_{1gH}^0(\overline{1},j,\overline{2})\,J_1^{(1)}(\{p\}_1)$$

$$+ A_{3,q\bar{q}}^0(1,j,2)\,\hat{B}_{1gH}^1(\overline{1},i,\overline{2})\,\delta(1-x_1)\,\delta(1-x_2)$$

$$+ \left[\, \hat{A}_{3,q\bar{q}}^1(1,j,2)\,\delta(1-x_1)\,\delta(1-x_2) + 2\hat{J}_{2,GG}^{1,FF}(s_{ij})\,A_{3,q\bar{q}}^0(1,j,2)\right]$$

$$\times B_{1gH}^0(\overline{1},i,\overline{2})\,J_1^{(1)}(\{p\}_1) \tag{D.2.13}$$

$$C_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}}) =$$

$$-\left[\, + J_{2,QQ}^{1,IF}(s_{1j}) + J_{2,QQ}^{1,FI}(s_{2i})\right] C_{0gH}^0(1,j,i,2)\,J_1^{(2)}(\{p\}_2)$$

$$+\frac{1}{2}\,E_{3,q}^0(2,i,j)\left[\,B_{1gH}^1(1,(\widetilde{ij}),\overline{2})\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$+\left(\,+J_{2,QG}^{1,IF}(s_{1(\widetilde{ij})}) + J_{2,GQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})})\right)B_{1gH}^0(1,(\widetilde{ij}),\overline{2})\bigg]\,J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2}\left[\,E_{3,q}^1(2,i,j)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$+\left(\,+J_{2,QQ}^{1,FI}(s_{2i}) + J_{2,QQ}^{1,FI}(s_{2j}) - 2J_{2,GQ}^{1,FI}(s_{\overline{2}(\widetilde{ij})})\right)E_{3,q}^0(2,i,j)\bigg]$$

$$\times B_{1gH}^0(1,(\widetilde{ij}),\overline{2})\,J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2}\,E_{3,q}^0(1,j,i)\left[\,B_{1gH}^1(\overline{1},(\widetilde{ij}),2)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$+\left(\,+J_{2,QG}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) + J_{2,GQ}^{1,FI}(s_{2(\widetilde{ij})})\right)B_{1gH}^0(\overline{1},(\widetilde{ij}),2)\bigg]\,J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2}\left[\,E_{3,q}^1(1,j,i)\,\delta(1-x_1)\,\delta(1-x_2)\right.$$

$$+\left(\,+J_{2,QQ}^{1,IF}(s_{1i}) + J_{2,QQ}^{1,IF}(s_{1j}) - 2J_{2,QG}^{1,IF}(s_{\overline{1}(\widetilde{ij})})\right)E_{3,q}^0(1,j,i)\bigg]$$

$$\times B_{1gH}^0(\overline{1},(\widetilde{ij}),2)\,J_1^{(1)}(\{p\}_1) \tag{D.2.14}$$

$$\widetilde{C}_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}}) =$$

$$-\left[+ J_{2,QQ}^{1,II}(s_{12}) + J_{2,QQ}^{1,FF}(s_{ij})\right] C_{0gH}^0(1,i,j,2) \, J_1^{(2)}(\{p\}_2)$$

$$+\frac{1}{2} \, E_{3,q}^0(1,i,j) \left[\tilde{B}_{1gH}^1(\overline{1},(\widetilde{ij}),2) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + J_{2,QQ}^{1,II}(s_{\overline{1}2}) \, B_{1gH}^0(\overline{1},(\widetilde{ij}),2)\right] J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2} \left[\tilde{E}_{3,q}^1(1,i,j) \, \delta(1-x_1) \, \delta(1-x_2) + J_{2,QQ}^{1,FF}(s_{ij}) \, E_{3,q}^0(1,i,j)\right]$$

$$\times B_{1gH}^0(\overline{1},(\widetilde{ij}),2) \, J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2} \, E_{3,q}^0(2,i,j) \left[\tilde{B}_{1gH}^1(1,(\widetilde{ij}),\overline{2}) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + J_{2,QQ}^{1,II}(s_{1\overline{2}}) \, B_{1gH}^0(1,(\widetilde{ij}),\overline{2})\right] J_1^{(1)}(\{p\}_1)$$

$$+\frac{1}{2} \left[\tilde{E}_{3,q}^1(2,i,j) \, \delta(1-x_1) \, \delta(1-x_2) + J_{2,QQ}^{1,FF}(s_{ij}) \, E_{3,q}^0(2,i,j)\right]$$

$$\times B_{1gH}^0(1,(\widetilde{ij}),\overline{2}) \, J_1^{(1)}(\{p\}_1) \tag{D.2.15}$$

$$\widehat{C}_{0gH}^{1,XT}(\hat{1}_q, j_{\bar{Q}}, i_Q, \hat{2}_{\bar{q}}) =$$

$$+\frac{1}{2} \, E_{3,q}^0(1,i,j) \, \hat{B}_{1gH}^1(\overline{1},(\widetilde{ij}),2) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$+\frac{1}{2} \, \hat{E}_{3,q}^1(1,i,j) \, B_{1gH}^0(\overline{1},(\widetilde{ij}),2) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$+\frac{1}{2} \, E_{3,q}^0(2,j,i) \, \hat{B}_{1gH}^1(1,(\widetilde{ji}),\overline{2}) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$+\frac{1}{2} \, \hat{E}_{3,q}^1(2,j,i) \, B_{1gH}^0(1,(\widetilde{ji}),\overline{2}) \, \delta(1-x_1) \, \delta(1-x_2) \tag{D.2.16}$$

$$C_{0gH}^{1,YT}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}}) =$$

$$-\left[+ J_{2,QQ}^{1,II}(s_{12}) + J_{2,QQ}^{1,FF}(s_{ji})\right] C_{0gH}^0(1,2,i,j) \, J_1^{(2)}(\{p\}_2)$$

$$- E_{3,q'\to g}^0(j,i,2) \left[B_{1gH}^1(1,\overline{2},(\widetilde{ij})) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ J_{2,QG}^{1,II}(s_{1\overline{2}}) + J_{2,QG}^{1,FI}(s_{(\widetilde{ij})\overline{2}})\right) B_{1gH}^0(1,\overline{2},(\widetilde{ij}))\right] J_1^{(1)}(\{p\}_1)$$

$$-\left[E_{3,q'}^1(j,i,2) \, \delta(1-x_1) \, \delta(1-x_2) \right.$$

$$\left. + \left(+ J_{2,QQ}^{1,FF}(s_{ji}) + J_{2,QQ}^{1,FI}(s_{2j}) - 2 J_{2,QG}^{1,FI}(s_{(\widetilde{ij})\overline{2}})\right) E_{3,q'\to g}^0(j,i,2)\right]$$

$$\times B_{1gH}^0(1,\overline{2},(\widetilde{ij})) \, J_1^{(1)}(\{p\}_1)$$

$$-\bigg[+ J_{2,QG}^{1,FI}(s_{(\widetilde{ij})\overline{2}}) - J_{2,QQ}^{1,FI}(s_{2j}) - J_{2,QG}^{1,II}(s_{1\overline{2}})$$

$$+ J_{2,QQ}^{1,II}(s_{12}) - J_{2,QQ}^{1,IF}(s_{1(\widetilde{ij})}) + J_{2,QQ}^{1,IF}(s_{1j})$$

$$+ \bigg(- \mathcal{S}^{IF}(s_{\overline{2}(\widetilde{ij})}, s_{1j}, x_{\overline{2}(\widetilde{ij}),1j}) + \mathcal{S}^{IF}(s_{2j}, s_{1j}, x_{2j,1j}) + \mathcal{S}^{IF}(s_{1\overline{2}}, s_{1j}, x_{1\overline{2},1j})$$

$$- \mathcal{S}^{IF}(s_{12}, s_{1j}, x_{12,1j}) + \mathcal{S}^{IF}(s_{1(\widetilde{ij})}, s_{1j}, x_{1(\widetilde{ij}),1j}) - \mathcal{S}^{IF}(s_{1j}, s_{1j}, 1) \bigg) \bigg]$$

$$\times E_{3,q'\to g}^{0}(j, i, 2) \, B_{1gH}^{0}(1, \overline{2}, (\widetilde{ij})) \, J_{1}^{(1)}(\{p\}_1)$$

$$- E_{3,q'\to g}^{0}(i, j, 1) \bigg[B_{1gH}^{1}((\widetilde{ij}), \overline{1}, 2) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J_{2,GQ}^{1,II}(s_{\overline{1}2}) + J_{2,GQ}^{1,IF}(s_{(\widetilde{ij})\overline{1}}) \bigg) B_{1gH}^{0}((\widetilde{ij}), \overline{1}, 2) \bigg] J_{1}^{(1)}(\{p\}_1)$$

$$- \bigg[E_{3,q'}^{1}(i, j, 1) \, \delta(1 - x_1) \, \delta(1 - x_2)$$

$$+ \bigg(+ J_{2,QQ}^{1,FF}(s_{ji}) + J_{2,QQ}^{1,IF}(s_{1i}) - 2 J_{2,GQ}^{1,IF}(s_{(\widetilde{ij})\overline{1}}) \bigg) E_{3,q'\to g}^{0}(i, j, 1) \bigg]$$

$$\times B_{1gH}^{0}((\widetilde{ij}), \overline{1}, 2) \, J_{1}^{(1)}(\{p\}_1)$$

$$- \bigg[+ J_{2,GQ}^{1,IF}(s_{(\widetilde{ij})\overline{1}}) - J_{2,QQ}^{1,IF}(s_{1i}) - J_{2,GQ}^{1,II}(s_{\overline{1}2})$$

$$+ J_{2,QQ}^{1,II}(s_{12}) - J_{2,QQ}^{1,FI}(s_{2(\widetilde{ij})}) + J_{2,QQ}^{1,FI}(s_{2i})$$

$$+ \bigg(- \mathcal{S}^{FI}(s_{\overline{1}(\widetilde{ij})}, s_{2i}, x_{\overline{1}(\widetilde{ij}),2i}) + \mathcal{S}^{FI}(s_{1i}, s_{2i}, x_{1i,2i}) + \mathcal{S}^{FI}(s_{\overline{1}2}, s_{2i}, x_{\overline{1}2,2i})$$

$$- \mathcal{S}^{FI}(s_{12}, s_{2i}, x_{12,2i}) + \mathcal{S}^{FI}(s_{2(\widetilde{ij})}, s_{2i}, x_{2(\widetilde{ij}),2i}) - \mathcal{S}^{FI}(s_{2i}, s_{2i}, 1) \bigg) \bigg]$$

$$\times E_{3,q'\to g}^{0}(i, j, 1) \, B_{1gH}^{0}((\widetilde{ij}), \overline{1}, 2) \, J_{1}^{(1)}(\{p\}_1)$$

$$- J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) \, B_{2gH}^{0}(2, 1, i, j) \, J_{1}^{(2)}(\{p\}_2)$$

$$+ J_{2,GQ,q'\to g}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \, d_{3,g}^{0}(j, i, 1) \, B_{1gH}^{0}(2, \overline{1}, (\widetilde{ij})) \, J_{1}^{(1)}(\{p\}_1)$$

$$- J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) \, G_{3,gq\to gg}^{0}(1, 2, j) \, A_{3gH}^{0}(\overline{1}, \overline{2}, i) \, J_{1}^{(1)}(\{p\}_1)$$

$$- J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) \, B_{2gH}^{0}(2, i, 1, j) \, J_{1}^{(2)}(\{p\}_2)$$

$$+ J_{2,GQ,q'\to g}^{1,IF}(s_{\overline{1}j}) \, D_{3,gq}^{0}(2, i, 1) \, B_{1gH}^{0}(\overline{2}, \overline{1}, j) \, J_{1}^{(1)}(\{p\}_1)$$

$$- J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) \, A_{3,gq\to qq}^{0}(2, 1, j) \, B_{1gH}^{0}(\overline{1}, i, \overline{2}) \, J_{1}^{(1)}(\{p\}_1)$$

$$- J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) \, G_{3,gq\to gg}^{0}(1, 2, j) \, A_{3gH}^{0}(\overline{1}, \overline{2}, i) \, J_{1}^{(1)}(\{p\}_1)$$

$$+ \bigg[+ 2 J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) - 2 J_{2,GQ,q'\to g}^{1,IF}(s_{\overline{1}(\widetilde{ij})}) \bigg] D_{3,g}^{0}(j, i, 1) \, B_{1gH}^{0}(2, \overline{1}, (\widetilde{ij})) \, J_{1}^{(1)}(\{p\}_1)$$

$$+ \bigg[+ 2 J_{2,GQ,q'\to g}^{1,IF}(s_{\overline{1}i}) - 2 J_{2,GQ,q'\to g}^{1,IF}(s_{1j}) \bigg] A_{3,gq\to qq}^{0}(2, 1, j) \, B_{1gH}^{0}(\overline{2}, \overline{1}, i) \, J_{1}^{(1)}(\{p\}_1)$$

$$+\left[\, + J_{2,GQ,q'\to g}^{1,IF}(s_{1(\widetilde{ij})}) - J_{2,GQ,q'\to g}^{1,IF}(s_{1j})\right] A_{3,q}^0(2,i,j)\, B_{1gH}^0(\overline{2},1,(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$-J_{2,QG,q'\to g}^{1,FI}(s_{2j})\, B_{2gH}^0(1,2,i,j)\, J_1^{(2)}(\{p\}_2)$$

$$+J_{2,QG,q'\to g}^{1,FI}(s_{\overline{2}(\widetilde{ij})})\, d_{3,g}^0(j,i,2)\, B_{1gH}^0(1,\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$-J_{2,QG,q'\to g}^{1,FI}(s_{2j})\, G_{3,qg\to gg}^0(2,1,j)\, A_{3gH}^0(\overline{1},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$-J_{2,QG,q'\to g}^{1,FI}(s_{2j})\, B_{2gH}^0(1,i,2,j)\, J_1^{(2)}(\{p\}_2)$$

$$+J_{2,QG,q'\to g}^{1,FI}(s_{\overline{2}j})\, D_{3,qg}^0(1,i,2)\, B_{1gH}^0(\overline{1},\overline{2},j)\, J_1^{(1)}(\{p\}_1)$$

$$-J_{2,QG,q'\to g}^{1,FI}(s_{2j})\, A_{3,qg\to qq}^0(1,2,j)\, B_{1gH}^0(\overline{1},i,\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$-J_{2,QG,q'\to g}^{1,FI}(s_{2j})\, G_{3,qg\to gg}^0(2,1,j)\, A_{3gH}^0(\overline{1},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$+\left[\, + 2J_{2,QG,q'\to g}^{1,FI}(s_{2j}) - 2J_{2,QG,q'\to g}^{1,FI}(s_{\overline{2}(\widetilde{ij})})\right] D_{3,g}^0(j,i,2)\, B_{1gH}^0(1,\overline{2},(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$+\left[\, + 2J_{2,QG,q'\to g}^{1,FI}(s_{\overline{2}i}) - 2J_{2,QG,q'\to g}^{1,FI}(s_{2j})\right] A_{3,qg\to qq}^0(1,2,j)\, B_{1gH}^0(\overline{1},\overline{2},i)\, J_1^{(1)}(\{p\}_1)$$

$$+\left[\, + J_{2,QG,q'\to g}^{1,FI}(s_{2(\widetilde{ij})}) - J_{2,QG,q'\to g}^{1,FI}(s_{2j})\right] A_{3,q}^0(1,i,j)\, B_{1gH}^0(\overline{1},2,(\widetilde{ij}))\, J_1^{(1)}(\{p\}_1)$$

$$\tag{D.2.17}$$

$$\widetilde{C}_{0gH}^{1,YT}(\hat{1}_q,\hat{2}_{\bar{Q}},i_Q,j_{\bar{q}}) =$$

$$-\left[\, + \mathcal{A}_{3,q}^0(s_{1j}) + \mathcal{A}_{3,q}^0(s_{2i}) - 2\mathcal{A}_{3,q}^0(s_{1i})\right.$$

$$\left. - 2\mathcal{A}_{3,q}^0(s_{2j}) + 2\mathcal{A}_3^0(s_{ij}) + 2\mathcal{A}_{3,qq}^0(s_{12})\right] C_{0gH}^0(1,2,i,j)\, J_1^{(2)}(\{p\}_2)$$

$$-E_{3,q'\to g}^0(j,i,2)\left[\, \tilde{B}_{1gH}^1(1,\overline{2},(\widetilde{ji}))\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left. + \mathcal{A}_{3,q}^0(s_{1(\widetilde{ji})})\, B_{1gH}^0(1,\overline{2},(\widetilde{ji}))\right] J_1^{(1)}(\{p\}_1)$$

$$-\left[\, \tilde{E}_{3,q'}^1(j,i,2)\, \delta(1-x_1)\, \delta(1-x_2) + \mathcal{A}_{3,q}^0(s_{i2})\, E_{3,q'\to g}^0(j,i,2)\right]$$

$$\times B_{1gH}^0(1,\overline{2},(\widetilde{ji}))\, J_1^{(1)}(\{p\}_1)$$

$$-E_{3,q'\to g}^0(i,j,1)\left[\, \tilde{B}_{1gH}^1((\widetilde{ij}),\overline{1},2)\, \delta(1-x_1)\, \delta(1-x_2)\right.$$

$$\left. + \mathcal{A}_{3,q}^0(s_{2(\widetilde{ij})})\, B_{1gH}^0((\widetilde{ij}),\overline{1},2)\right] J_1^{(1)}(\{p\}_1)$$

$$-\left[\, \tilde{E}_{3,q'}^1(i,j,1)\, \delta(1-x_1)\, \delta(1-x_2) + \mathcal{A}_{3,q}^0(s_{j1})\, E_{3,q'\to g}^0(i,j,1)\right]$$

$$\times B_{1gH}^0((\widetilde{ij}),\overline{1},2)\, J_1^{(1)}(\{p\}_1)$$

$$+2\left[\, + \mathcal{A}_{3,q}^0(s_{1i}) + \mathcal{A}_{3,q}^0(s_{2j}) - \mathcal{A}_{3,qq}^0(s_{12}) - \mathcal{A}_3^0(s_{ij})\right.$$

$$+\bigg(- \mathcal{S}^{IF}(s_{1i}, s_{1i}, 1) + \mathcal{S}^{IF}(s_{12}, s_{1i}, x_{12,1i}) - \mathcal{S}^{IF}(s_{2j}, s_{1i}, x_{2j,1i})$$

$$+ \mathcal{S}^{IF}(s_{ji}, s_{1i}, x_{ji,1i}) \bigg) \bigg] E^0_{3,q'\to g}(i,j,1) \, B^0_{1gH}((\widetilde{ij}),\overline{1},2) \, J^{(1)}_1(\{p\}_1)$$

$$+2 \bigg[+ \mathcal{A}^0_{3,q}(s_{1i}) + \mathcal{A}^0_{3,q}(s_{2j}) - \mathcal{A}^0_{3,qq}(s_{12}) - \mathcal{A}^0_3(s_{ij})$$

$$+\bigg(- \mathcal{S}^{FI}(s_{1i}, s_{2j}, x_{1i,2j}) + \mathcal{S}^{FI}(s_{12}, s_{2j}, x_{12,2j}) - \mathcal{S}^{FI}(s_{2j}, s_{2j}, 1)$$

$$+ \mathcal{S}^{FI}(s_{ij}, s_{2j}, x_{ij,2j}) \bigg) \bigg] E^0_{3,q'\to g}(j,i,2) \, B^0_{1gH}(1,\overline{2},(\widetilde{ji})) \, J^{(1)}_1(\{p\}_1)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i2}) \, \tilde{B}^0_{2gH}(1,i,2,j) \, J^{(2)}_1(\{p\}_2)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i2}) \, A^0_{3,qg\to qq}(1,2,j) \, B^0_{1gH}(\overline{1},i,\overline{2}) \, J^{(1)}_1(\{p\}_1)$$

$$-\mathcal{G}^0_{3,q'\to g}(s_{i2}) \, A^0_{3,q}(1,i,j) \, B^0_{1gH}(\overline{1},2,(\widetilde{ij})) \, J^{(1)}_1(\{p\}_1)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i1}) \, \tilde{B}^0_{2gH}(2,1,i,j) \, J^{(2)}_1(\{p\}_2)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i1}) \, A^0_{3,gq\to qq}(j,1,2) \, B^0_{1gH}(\overline{1},i,\overline{2}) \, J^{(1)}_1(\{p\}_1)$$

$$-\mathcal{G}^0_{3,q'\to g}(s_{i1}) \, A^0_{3,q}(2,i,j) \, B^0_{1gH}(\overline{2},1,(\widetilde{ij})) \, J^{(1)}_1(\{p\}_1) \hfill \text{(D.2.18)}$$

$$\widehat{C}^{1,YT}_{0gH}(\hat{1}_q, \hat{2}_{\bar{Q}}, i_Q, j_{\bar{q}}) =$$

$$- E^0_{3,q'\to g}(j,1,i) \, \hat{B}^1_{1gH}((\widetilde{ji}),\overline{1},2) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$- \hat{E}^1_{3,q'}(j,1,i) \, B^0_{1gH}((\widetilde{ji}),\overline{1},2) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$- E^0_{3,q'\to g}(i,2,j) \, \hat{B}^1_{1gH}(1,\overline{2},(\widetilde{ij})) \, \delta(1-x_1) \, \delta(1-x_2)$$

$$- \hat{E}^1_{3,q'}(i,2,j) \, B^0_{1gH}(1,\overline{2},(\widetilde{ij})) \, \delta(1-x_1) \, \delta(1-x_2) \hfill \text{(D.2.19)}$$

$$D^{1,XT}_{0gH}(\hat{1}_q, \hat{2}_{\bar{q}}, i_q, j_{\bar{q}}) =$$

$$- \bigg[+ J^{1,IF}_{2,QQ}(s_{1i}) + J^{1,FI}_{2,QQ}(s_{2j}) \bigg] D^0_{0gH}(1,2,i,j) \, J^{(2)}_1(\{p\}_2)$$

$$\hfill \text{(D.2.20)}$$

$$\widetilde{D}^{1,XT}_{0gH}(\hat{1}_q, i_{\bar{q}}, j_q, \hat{2}_{\bar{q}}) =$$

$$+ \bigg[+ J^{1,IF}_{2,QQ}(s_{1i}) + J^{1,FI}_{2,QQ}(s_{2j}) + J^{1,II}_{2,QQ}(s_{12}) + J^{1,FF}_{2,QQ}(s_{ij}) - J^{1,IF}_{2,QQ}(s_{1j}) - J^{1,FI}_{2,QQ}(s_{2i}) \bigg]$$

$$\times D^0_{0gH}(1,i,j,2) \, J^{(2)}_1(\{p\}_2) \hfill \text{(D.2.21)}$$

D.3 $q\bar{q} \to H+$jet at VV

The double virtual subtraction terms $\mathrm{d}\hat{\sigma}_{NNLO}^{U}$ mentioned in section 8.5.3 are:

$$
\widetilde{B}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}}) =
$$

$$
-\left[-\frac{1}{2}\, \mathcal{D}_{3,q}^0(s_{1i}) + \Gamma_{qq}^{(1)}(z_1) - \frac{1}{2}\, \mathcal{D}_{3,q}^0(s_{2i}) + \Gamma_{qq}^{(1)}(z_2) \right] \tilde{B}_{1gH}^1(1, i, 2)
$$

$$
-\left[-\mathcal{A}_{3,qq}^0(s_{12}) + \Gamma_{qq}^{(1)}(z_1) + \Gamma_{qq}^{(1)}(z_2) \right] B_{1gH}^1(1, i, 2)
$$

$$
-\left[-\frac{1}{2}\, \mathcal{D}_{3,q}^0(s_{1i}) \otimes \mathcal{A}_{3,qq}^0(s_{12}) + \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{A}_{3,qq}^0(s_{12}) + \frac{1}{2}\, \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{D}_{3,q}^0(s_{1i}) \right.
$$

$$
\left. + \frac{1}{2}\, \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{D}_{3,q}^0(s_{1i}) - \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1) - \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_2) \right.
$$

$$
\left. \vphantom{\frac{1}{2}} \right] B_{1gH}^0(1, i, 2)
$$

$$
-\left[-\frac{1}{2}\, \mathcal{D}_{3,q}^0(s_{2i}) \otimes \mathcal{A}_{3,qq}^0(s_{12}) + \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{A}_{3,qq}^0(s_{12}) + \frac{1}{2}\, \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{D}_{3,q}^0(s_{2i}) \right.
$$

$$
\left. + \frac{1}{2}\, \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{D}_{3,q}^0(s_{2i}) - \Gamma_{qq}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_2) - \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_2) \right.
$$

$$
\left. \vphantom{\frac{1}{2}} \right] B_{1gH}^0(1, i, 2)
$$

$$
-\left[-\mathcal{A}_{4,q\bar{q}}^0(s_{12}) - \frac{1}{2}\, \tilde{\mathcal{A}}_{4,q\bar{q}}^0(s_{12}) - 2\, \mathcal{C}_{4,q\bar{q}}^0(s_{12}) - 2\, \mathcal{C}_{4,\bar{q}q}^0(s_{12}) - \mathcal{A}_{3,qq}^1(s_{12}) \right.
$$

$$
\left. - \frac{b_0}{\epsilon}\, \mathcal{A}_{3,qq}^0(s_{12}) \left(\frac{s_{12}}{\mu_R^2} \right)^{-\epsilon} + \frac{b_0}{\epsilon}\, \mathcal{A}_{3,qq}^0(s_{12}) - \tilde{\mathcal{A}}_{3,qq}^1(s_{12}) + \mathcal{A}_{3,qq}^0(s_{12}) \otimes \mathcal{A}_{3,qq}^0(s_{12}) \right.
$$

$$
\left. + \overline{\Gamma}_{qq}^{(2)}(z_2) + \overline{\Gamma}_{qq}^{(2)}(z_1) - \frac{b_0}{\epsilon}\, \Gamma_{qq}^{(1)}(z_1) - \frac{b_0}{\epsilon}\, \Gamma_{qq}^{(1)}(z_2) \right.
$$

$$
\left. - \widetilde{\overline{\Gamma}}_{qq}^{(2)}(z_2) - \widetilde{\overline{\Gamma}}_{qq}^{(2)}(z_1) \right] B_{1gH}^0(1, i, 2) \tag{D.3.22}
$$

$$
\widetilde{\widetilde{B}}_{1gH}^{2,XU}(1_q, i, 2_{\bar{q}}) =
$$

$$
-\left[+\mathcal{A}_{3,qq}^0(s_{12}) - \Gamma_{qq}^{(1)}(z_1) - \Gamma_{qq}^{(1)}(z_2) \right] \tilde{B}_{1gH}^1(1, i, 2)
$$

$$
-\left[+\frac{1}{2}\, \tilde{\mathcal{A}}_{4,q\bar{q}}^0(s_{12}) + 2\, \mathcal{C}_{4,q\bar{q}}^0(s_{12}) + 2\, \mathcal{C}_{4,\bar{q}q}^0(s_{12}) \right.
$$

$$
\left. + \tilde{\mathcal{A}}_{3,qq}^1(s_{12}) - \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{A}_{3,qq}^0(s_{12}) - \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{A}_{3,qq}^0(s_{12}) \right.
$$

$$
\left. + \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_2) + \frac{1}{2}\, \Gamma_{qq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1) + \frac{1}{2}\, \Gamma_{qq}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_2) \right.
$$

$$+ \widetilde{\widetilde{\Gamma}}_{qq}^{(2)}(z_1) + \widetilde{\widetilde{\Gamma}}_{qq}^{(2)}(z_2)\Bigg] B_{1gH}^0(1,i,2) \tag{D.3.23}$$

$$\widehat{\widetilde{\widetilde{B}}}_{1gH}^{2,XU}(1_q,i,2_{\bar{q}}) =$$

$$-\Bigg[-\frac{1}{2}\,\mathcal{E}_{3,q}^0(s_{1i}) - \frac{1}{2}\,\mathcal{E}_{3,q}^0(s_{2i})\Bigg]\tilde{B}_{1gH}^1(1,i,2)$$

$$-\Bigg[-\mathcal{A}_{3,qq}^0(s_{12}) + \Gamma_{qq}^{(1)}(z_1) + \Gamma_{qq}^{(1)}(z_2)\Bigg]\left(+\hat{B}_{1gH}^1(1,i,2) - \frac{b_F}{\epsilon}B_{1gH}^0(1,i,2)\right)$$

$$-\Bigg[-\frac{1}{2}\,\mathcal{E}_{3,q}^0(s_{1i}) \otimes \mathcal{A}_{3,qq}^0(s_{12}) - \frac{1}{2}\,\mathcal{E}_{3,q}^0(s_{2i}) \otimes \mathcal{A}_{3,qq}^0(s_{12}) + \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q}^0(s_{1i})$$

$$+ \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q}^0(s_{1i}) + \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q}^0(s_{2i}) + \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q}^0(s_{2i})$$

$$\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[-\mathcal{B}_{4,q\bar{q}}^0(s_{12}) - \hat{\mathcal{A}}_{3,qq}^1(s_{12}) - \frac{b_F}{\epsilon}\,\mathcal{A}_{3,qq}^0(s_{12})\left(\frac{s_{12}}{\mu_R^2}\right)^{-\epsilon}\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[-\frac{1}{2}\,\tilde{\mathcal{E}}_{4,q}^0(s_{1i}) - \frac{1}{2}\,\tilde{\mathcal{E}}_{3,q}^1(s_{1i}) + \overline{\Gamma}_{qq,F}^{(2)}(z_1)\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[-\frac{1}{2}\,\tilde{\mathcal{E}}_{4,q}^0(s_{2i}) - \frac{1}{2}\,\tilde{\mathcal{E}}_{3,q}^1(s_{2i}) + \overline{\Gamma}_{qq,F}^{(2)}(z_2)\Bigg] B_{1gH}^0(1,i,2) \tag{D.3.24}$$

$$\widehat{B}_{1gH}^{2,XU}(1_q,i,2_{\bar{q}}) =$$

$$-\Bigg[+\frac{1}{2}\,\mathcal{E}_{3,q}^0(s_{1i}) + \frac{1}{2}\,\mathcal{E}_{3,q}^0(s_{2i})\Bigg] B_{1gH}^1(1,i,2)$$

$$-\Bigg[+\frac{1}{2}\,\mathcal{D}_{3,q}^0(s_{1i}) - \Gamma_{qq}^{(1)}(z_1) + \frac{1}{2}\,\mathcal{D}_{3,q}^0(s_{2i}) - \Gamma_{qq}^{(1)}(z_2)\Bigg]\hat{B}_{1gH}^1(1,i,2)$$

$$-\Bigg[-\frac{1}{4}\,\mathcal{E}_{3,q}^0(s_{1i}) \otimes \mathcal{D}_{3,q}^0(s_{1i}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q}^0(s_{1i})\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[+\frac{1}{4}\,\mathcal{E}_{3,q}^0(s_{1i}) \otimes \mathcal{D}_{3,q}^0(s_{2i}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q}^0(s_{1i})\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[+\frac{1}{4}\,\mathcal{E}_{3,q}^0(s_{2i}) \otimes \mathcal{D}_{3,q}^0(s_{1i}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q}^0(s_{2i})\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[-\frac{1}{4}\,\mathcal{E}_{3,q}^0(s_{2i}) \otimes \mathcal{D}_{3,q}^0(s_{2i}) - \frac{1}{2}\,\Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q}^0(s_{2i})\Bigg] B_{1gH}^0(1,i,2)$$

$$-\Bigg[+\mathcal{E}_{4,q}^0(s_{1i}) + \frac{1}{2}\,\hat{\mathcal{D}}_{3,q}^1(s_{1i}) + \frac{b_F}{2\epsilon}\,\mathcal{D}_{3,q}^0(s_{1i})\left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} - \frac{b_F}{2\epsilon}\,\mathcal{D}_{3,q}^0(s_{1i})$$

$$+ \frac{1}{2}\,\mathcal{E}_{3,q}^1(s_{1i}) + \frac{b_0}{2\epsilon}\,\mathcal{E}_{3,q}^0(s_{1i})\left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} - \frac{b_0}{2\epsilon}\,\mathcal{E}_{3,q}^0(s_{1i})$$

$$+ \frac{b_F}{\epsilon}\,\Gamma_{qq}^{(1)}(z_1) - \overline{\Gamma}_{qq,F}^{(2)}(z_1)\Bigg] B_{1gH}^0(1,i,2)$$

$$-\left[+ \mathcal{E}^0_{4,q}(s_{2i}) + \frac{1}{2}\,\hat{\mathcal{D}}^1_{3,q}(s_{2i}) + \frac{b_F}{2\epsilon}\,\mathcal{D}^0_{3,q}(s_{2i})\left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} - \frac{b_F}{2\epsilon}\,\mathcal{D}^0_{3,q}(s_{2i})\right.$$

$$+ \frac{1}{2}\,\mathcal{E}^1_{3,q}(s_{2i}) + \frac{b_0}{2\epsilon}\,\mathcal{E}^0_{3,q}(s_{2i})\left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} - \frac{b_0}{2\epsilon}\,\mathcal{E}^0_{3,q}(s_{2i})$$

$$\left. + \frac{b_F}{\epsilon}\,\Gamma^{(1)}_{qq}(z_2) - \overline{\Gamma}^{(2)}_{qq,F}(z_2)\right] B^0_{1gH}(1,i,2) \tag{D.3.25}$$

$$\widehat{\widehat{B}}^{2,XU}_{1gH}(1_q,i,2_{\bar{q}}) =$$

$$-\left[+ \frac{1}{2}\,\mathcal{E}^0_{3,q}(s_{1i}) + \frac{1}{2}\,\mathcal{E}^0_{3,q}(s_{2i})\right]\left(\hat{B}^1_{1gH}(1,i,2) - \frac{b_F}{\epsilon} B^0_{1gH}(1,i,2)\right)$$

$$-\left[+ \frac{1}{2}\,\hat{\mathcal{E}}^1_{3,q}(s_{1i}) + \frac{b_F}{2\epsilon}\left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon}\mathcal{E}^0_{3,q}(s_{1i})\right] B^0_{1gH}(1,i,2)$$

$$-\left[+ \frac{1}{2}\,\hat{\mathcal{E}}^1_{3,q}(s_{2i}) + \frac{b_F}{2\epsilon}\,\mathcal{E}^0_{3,q}(s_{2i})\left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon}\right] B^0_{1gH}(1,i,2) \tag{D.3.26}$$

$$C^{1,XU}_{1gH}(1_q,i,2_{\bar{q}}) =$$

$$-\left[- \mathcal{E}^0_{3,q'\to g}(s_{2i}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_2)\right] B^1_{1gH}(1,2,i)$$

$$-\left[- \mathcal{D}^0_{3,g\to g}(s_{2i}) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i})\right.$$

$$\left. - S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{D}^0_{3,g\to g}(s_{2i}) + \frac{1}{2}\,S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{gg}(z_2)\right] B^0_{1gH}(1,2,i)$$

$$-\left[- \mathcal{D}^0_{3,qg}(s_{12}) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i})\right.$$

$$+ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{D}^0_{3,qg}(s_{12})$$

$$\left. + S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{qq}(z_1) + \frac{1}{2}\,S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{gg}(z_2)\right] B^0_{1gH}(1,2,i)$$

$$-\left[- \mathcal{E}^0_{4,\bar{q}'}(s_{2i}) - \mathcal{E}^0_{4,q'}(s_{2i}) + 2\,\mathcal{B}^0_{4,qq'}(s_{12})\right.$$

$$- \mathcal{E}^1_{3,q'}(s_{2i}) - \frac{b_0}{\epsilon}\left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon}\mathcal{E}^0_{3,q'\to g}(s_{2i}) + \frac{b_0}{\epsilon}\,\mathcal{E}^0_{3,q'\to g}(s_{2i})$$

$$+ 2\,\mathcal{D}^0_{3,g}(s_{2i}) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) - 2\,\mathcal{A}^0_{3,qg\to qq}(s_{12}) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i})$$

$$+ \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) - \Gamma^{(1)}_{gg}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i})$$

$$+ \frac{b_0}{\epsilon}\,S_{q\to g}\Gamma^{(1)}_{gq}(z_2) - \frac{1}{2}\,S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{gg}(z_2)$$

$$\left. + \frac{1}{2}\,S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{qq}(z_2) - S_{q\to g}\overline{\Gamma}^{(2)}_{gq}(z_2)\right] B^0_{1gH}(1,2,i)$$

$$- \left[- \mathcal{E}^0_{3,q' \to g}(s_{1i}) - S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \right] B^1_{1gH}(2,1,i)$$

$$- \left[- \mathcal{D}^0_{3,g \to g}(s_{1i}) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg}(z_1) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) \right.$$

$$\left. - S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{D}^0_{3,g \to g}(s_{1i}) + \frac{1}{2}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg}(z_1) \right] B^0_{1gH}(2,1,i)$$

$$- \left[- \mathcal{D}^0_{3,gq}(s_{12}) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) + \frac{1}{2}\,\Gamma^{(1)}_{gg}(z_1) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) \right.$$

$$+ \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) - S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{D}^0_{3,gq}(s_{12})$$

$$\left. + S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{qq}(z_2) + \frac{1}{2}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg}(z_1) \right] B^0_{1gH}(2,1,i)$$

$$- \left[- \mathcal{E}^0_{4,\bar{q}'}(s_{1i}) - \mathcal{E}^0_{4,q'}(s_{1i}) + 2\,\mathcal{B}^0_{4,q'q}(s_{12}) \right.$$

$$- \mathcal{E}^1_{3,q'}(s_{1i}) - \frac{b_0}{\epsilon}\left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{E}^0_{3,q' \to g}(s_{1i}) + \frac{b_0}{\epsilon}\,\mathcal{E}^0_{3,q' \to g}(s_{1i})$$

$$+ 2\,\mathcal{D}^0_{3,g}(s_{1i}) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) - 2\,\mathcal{A}^0_{3,gq \to qq}(s_{12}) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i})$$

$$+ \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) - \Gamma^{(1)}_{gg}(z_1) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) + \frac{b_0}{\epsilon}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1)$$

$$- \frac{1}{2}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg}(z_1) + \frac{1}{2}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1)$$

$$\left. - S_{q \to g}\overline{\Gamma}^{(2)}_{gq}(z_1) \right] B^0_{1gH}(2,1,i)$$

$$- \left[+ \mathcal{B}^0_{4,qq'}(s_{12}) + \mathcal{B}^0_{4,q'q}(s_{12}) + S_{q \to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12}) \right.$$

$$+ S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{A}^0_{3,qg \to qq}(s_{12}) + \frac{1}{2}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes S_{g \to q}\Gamma^{(1)}_{qg}(z_1)$$

$$\left. + \frac{1}{2}\,S_{q \to g}\Gamma^{(1)}_{gq}(z_2) \otimes S_{g \to q}\Gamma^{(1)}_{qg}(z_2) - \overline{\Gamma}^{(2)}_{qQ}(z_1) - \overline{\Gamma}^{(2)}_{qQ}(z_2) \right] B^0_{1gH}(1,i,2)$$

$$- \left[+ 2\,\mathcal{H}^0_{4,qq'}(s_{12}) + 2\,S_{q \to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{G}^0_{3,qg \to gg}(s_{12}) \right.$$

$$\left. + 2\,S_{q \to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{G}^0_{3,qg \to gg}(s_{12}) + 2\,S^2_{q \to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{gq}(z_1) \right] A^0_{3gH}(1,2,i)$$

$$\text{(D.3.27)}$$

$$\widetilde{C}^{1,XU}_{1gH}(1_q, i, 2_{\bar{q}}) =$$

$$- \left[+ \mathcal{E}^0_{3,q' \to g}(s_{2i}) + S_{q \to g}\Gamma^{(1)}_{gq}(z_2) \right] \tilde{B}^1_{1gH}(1,2,i)$$

$$- \left[+ \mathcal{E}^0_{3,q' \to g}(s_{2i}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) + S_{q \to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{A}^0_{3,q}(s_{1i}) \right.$$

$$- \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'\to g}^0(s_{2i}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_1)\Big] B_{1gH}^0(1,2,i)$$

$$- \Big[+ \tilde{\mathcal{E}}_{4,q'}^0(s_{2i}) + \tilde{\mathcal{E}}_{3,q'}^1(s_{2i}) - \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q'\to g}^0(s_{2i})$$

$$- \frac{1}{2} S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \Gamma_{qq}^{(1)}(z_2) - \Gamma_{gq}^{(2)}t(z_2)\Big] B_{1gH}^0(1,2,i)$$

$$- \Big[+ \mathcal{E}_{3,q'\to g}^0(s_{1i}) + S_{q\to g}\Gamma_{gq}^{(1)}(z_1)\Big] \tilde{B}_{1gH}^1(2,1,i)$$

$$- \Big[+ \mathcal{E}_{3,q'\to g}^0(s_{1i}) \otimes \mathcal{A}_{3,q}^0(s_{2i}) + S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{A}_{3,q}^0(s_{2i})$$

$$- \Gamma_{qq}^{(1)}(z_2) \otimes \mathcal{E}_{3,q'\to g}^0(s_{1i}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_2)\Big] B_{1gH}^0(2,1,i)$$

$$- \Big[+ \tilde{\mathcal{E}}_{4,q'}^0(s_{1i}) + \tilde{\mathcal{E}}_{3,q'}^1(s_{1i}) - \Gamma_{qq}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'\to g}^0(s_{1i})$$

$$- \frac{1}{2} S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \Gamma_{qq}^{(1)}(z_1) - \Gamma_{gq}^{(2)}t(z_1)\Big] B_{1gH}^0(2,1,i)$$

$$- \Big[- S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \mathcal{A}_{3,qg\to qq}^0(s_{12}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \mathcal{A}_{3,gq\to qq}^0(s_{12})$$

$$- \mathcal{B}_{4,qq'}^0(s_{12}) - \mathcal{B}_{4,q'q}^0(s_{12}) + \Gamma_{qQ}^{(2)}(z_1) + \Gamma_{qQ}^{(2)}(z_2)$$

$$- \frac{1}{2} \Gamma_{gq}^{(1)}(z_1) \otimes \Gamma_{qg}^{(1)}(z_1) - \frac{1}{2} \Gamma_{gq}^{(1)}(z_2) \otimes \Gamma_{qg}^{(1)}(z_2)\Big] B_{1gH}^0(1,i,2) \hspace{2cm} \text{(D.3.28)}$$

$$\widehat{C}_{1gH}^{1,XU}(1_q, i, 2_{\bar{q}}) =$$

$$- \Big[- \mathcal{E}_{3,q'\to g}^0(s_{2i}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_2)\Big] \hat{B}_{1gH}^1(1,2,i)$$

$$- \Big[+ \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{E}_{3,q'\to g}^0(s_{2i}) + S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_2)\Big] B_{1gH}^0(1,2,i)$$

$$- \Big[- \mathcal{E}_{3,q'\to g}^0(s_{1i}) - S_{q\to g}\Gamma_{gq}^{(1)}(z_1)\Big] \hat{B}_{1gH}^1(2,1,i)$$

$$- \Big[+ \Gamma_{gg,F}^{(1)}(z_1) \otimes \mathcal{E}_{3,q'\to g}^0(s_{1i}) + S_{q\to g}\Gamma_{gq}^{(1)}(z_1) \otimes \Gamma_{gg,F}^{(1)}(z_1)\Big] B_{1gH}^0(2,1,i)$$

$$- \Big[- \hat{\mathcal{E}}_{3,q'}^1(s_{2i}) - \frac{b_F}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{E}_{3,q'\to g}^0(s_{2i}) + \frac{b_F}{\epsilon} \mathcal{E}_{3,q'\to g}^0(s_{2i})$$

$$- \Gamma_{gg,F}^{(1)}(z_2) \otimes \mathcal{E}_{3,q'\to g}^0(s_{2i}) - \frac{3}{2} S_{q\to g}\Gamma_{gq}^{(1)}(z_2) \otimes \Gamma_{gg,F}^{(1)}(z_2)$$

$$- S_{q\to g}\overline{\Gamma}_{gq,F}^{(2)}(z_2)\Big] B_{1gH}^0(1,2,i)$$

$$- \Big[- \hat{\mathcal{E}}_{3,q'}^1(s_{1i}) - \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2}\right)^{-\epsilon} \mathcal{E}_{3,q'\to g}^0(s_{1i}) + \frac{b_F}{\epsilon} \mathcal{E}_{3,q'\to g}^0(s_{1i})$$

$$- \Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{E}^0_{3,q' \to g}(s_{1i}) - \frac{3}{2} \, S_{q \to g} \Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1)$$

$$- \, S_{q \to g} \overline{\Gamma}^{(2)}_{gq,F}(z_1) \bigg] B^0_{1gH}(2, 1, i) \tag{D.3.29}$$

Appendix E

Explicit results of antenna subtraction terms for $qq \to H+$jet processes at NNLO

E.1 $\quad qq \to H+$jet at RR

The double real subtraction terms $\mathrm{d}\hat{\sigma}_{NNLO}^S$ mentioned in section 9.5.1 are:

$$\widetilde{C}_{1gH}^{0,YS}(\hat{1}_q, i, j_{\bar{q}}, \hat{2}_Q, k_{\bar{Q}}) =$$

$$-A_{3,q}^0(1, i, j)\, C_{0gH}^0(\bar{1}, k, 2, (\widetilde{ij}))\, J_1^{(2)}(\{p\}_2)$$

$$-A_{3,q}^0(2, i, k)\, C_{0gH}^0(1, (\widetilde{ik}), \bar{2}, j)\, J_1^{(2)}(\{p\}_2)$$

$$+G_{3,q'\to g}^0(i, k, 2)\, \tilde{B}_{2gH}^0(1, (\widetilde{ik}), \bar{2}, j)\, J_1^{(2)}(\{p\}_2)$$

$$+G_{3,q'\to g}^0(i, j, 1)\, \tilde{B}_{2gH}^0(2, (\widetilde{ij}), \bar{1}, k)\, J_1^{(2)}(\{p\}_2)$$

$$-B_4^0(1, k, 2, j)\, B_{1gH}^0(\bar{1}, i, \bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$+G_{3,q'\to g}^0(i, k, 2)\, A_{3,qg\to qq}^0(1, \bar{2}, j)\, B_{1gH}^0(\bar{1}, (\widetilde{ik}), \bar{\bar{2}})\, J_1^{(1)}(\{p\}_1)$$

$$-B_4^0(k, 1, j, 2)\, B_{1gH}^0(\bar{1}, i, \bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$+G_{3,q'\to g}^0(i, j, 1)\, A_{3,gq\to qq}^0(k, \bar{1}, 2)\, B_{1gH}^0(\bar{\bar{1}}, (\widetilde{ij}), \bar{2})\, J_1^{(1)}(\{p\}_1)$$

$$+\tilde{E}_4^0(k, j, i, 1)\, B_{1gH}^0((\widetilde{kji}), \bar{1}, 2)\, J_1^{(1)}(\{p\}_1)$$

$$-A_{3,q}^0(1, i, j)\, E_{3,q'\to g}^0(k, (\widetilde{ij}), \bar{1})\, B_{1gH}^0((\widetilde{k(ij)}), \bar{\bar{1}}, 2)\, J_1^{(1)}(\{p\}_1)$$

$$+\tilde{E}_4^0(j, k, i, 2)\, B_{1gH}^0(1, \bar{2}, (\widetilde{jki}))\, J_1^{(1)}(\{p\}_1)$$

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$$- A^0_{3,q}(k,i,2)\, E^0_{3,q'\to g}(j,(\widetilde{ki}),\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},(\widetilde{j(\widetilde{ki})}))\, J^{(1)}_1(\{p\}_1)$$

$$- G^0_{3,q'\to g}(i,j,1)\, A^0_{3,q}(2,(\widetilde{ij}),k)\, B^0_{1gH}(((\widetilde{(ij)k})),\overline{1},\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$- G^0_{3,q'\to g}(i,k,2)\, A^0_{3,q}(1,(\widetilde{ik}),j)\, B^0_{1gH}(\overline{1},\overline{2},((\widetilde{(ik)j}))\, J^{(1)}_1(\{p\}_1)$$

$$+2A^0_{3,q\bar{q}}(1,i,2)\, C^0_{0gH}(\overline{1},k,\overline{2},j)\, J^{(2)}_1(\{p\}_2)$$

$$+2A^0_3(k,i,j)\, C^0_{0gH}(1,(\widetilde{ki}),2,(\widetilde{ij}))\, J^{(2)}_1(\{p\}_2)$$

$$-2A^0_{3,q}(1,i,k)\, C^0_{0gH}(\overline{1},(\widetilde{ik}),2,j)\, J^{(2)}_1(\{p\}_2)$$

$$-2A^0_{3,q}(2,i,j)\, C^0_{0gH}(1,k,\overline{2},(\widetilde{ij}))\, J^{(2)}_1(\{p\}_2)$$

$$+2\, A^0_{3,q\bar{q}}(1,i,2)\, E^0_{3,q'\to g}(k,j,\overline{1})\, B^0_{1gH}((\widetilde{kj}),\overline{\overline{1}},\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+2\, A^0_{3,q\bar{q}}(1,i,2)\, E^0_{3,q'\to g}(j,k,\overline{2})\, B^0_{1gH}(\overline{1},\overline{\overline{2}},(\widetilde{jk}))\, J^{(1)}_1(\{p\}_1)$$

$$+2\, A^0_3(k,i,j)\, E^0_{3,q'\to g}((\widetilde{ij}),(\widetilde{ki}),2)\, B^0_{1gH}(1,\overline{2},(\widetilde{\widetilde{ij},\widetilde{ki}})\, J^{(1)}_1(\{p\}_1)$$

$$+2\, A^0_3(k,i,j)\, E^0_{3,q'\to g}((\widetilde{ki}),(\widetilde{ij}),1)\, B^0_{1gH}((\widetilde{\widetilde{ki},\widetilde{ij}}),\overline{1},2)\, J^{(1)}_1(\{p\}_1)$$

$$-2\, A^0_{3,q}(1,i,k)\, E^0_{3,q'\to g}(j,(\widetilde{ik}),2)\, B^0_{1gH}(\overline{1},\overline{2},(\widetilde{j(\widetilde{ik})}))\, J^{(1)}_1(\{p\}_1)$$

$$-2\, A^0_{3,q}(1,i,k)\, E^0_{3,q'\to g}((\widetilde{ik}),j,\overline{1})\, B^0_{1gH}(((\widetilde{(ik)j}),\overline{\overline{1}},2)\, J^{(1)}_1(\{p\}_1)$$

$$-2\, A^0_{3,q}(2,i,j)\, E^0_{3,q'\to g}(k,(\widetilde{ij}),1)\, B^0_{1gH}((\widetilde{k(\widetilde{ij})}),\overline{1},\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$-2\, A^0_{3,q}(2,i,j)\, E^0_{3,q'\to g}((\widetilde{ij}),k,\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},((\widetilde{(ij)k}))\, J^{(1)}_1(\{p\}_1)$$

$$+2\left[\, + S^{IF}_{\overline{1}ik} - S^{IF}_{\overline{1}i2} + S^{IF}_{2i(\widetilde{ij})} - S^{IF}_{ki(\widetilde{ij})}\right]$$

$$\times E^0_{3,q'\to g}(k,(\widetilde{ij}),\overline{1})\, B^0_{1gH}((\widetilde{k(\widetilde{ij})}),\overline{\overline{1}},2,6)\, J^{(1)}_1(\{p\}_1)$$

$$+2\left[\, + S^{IF}_{1i(\widetilde{ik})} - S^{IF}_{1i\overline{2}} + S^{IF}_{\overline{2}ij} - S^{IF}_{(\widetilde{ik})ij}\right]$$

$$\times E^0_{3,q'\to g}(j,(\widetilde{ik}),\overline{2})\, B^0_{1gH}(1,\overline{\overline{2}},(\widetilde{j(\widetilde{ik})}),6)\, J^{(1)}_1(\{p\}_1) \tag{E.1.1}$$

$$D^{0,YS}_{1gH}(\hat{1}_q,\hat{2}_q,k_{\bar{q}},j_{\bar{q}},i) =$$

$$+A^0_{3,q}(1,i,j)\, D^0_{0gH}(\overline{1},(\widetilde{ji}),2,k)\, J^{(2)}_1(\{p\}_2)$$

$$+A^0_{3,q}(2,i,k)\, D^0_{0gH}(1,j,\overline{2},(\widetilde{ik}))\, J^{(2)}_1(\{p\}_2)$$

$$+A^0_{3,q}(1,i,k)\, D^0_{0gH}(\overline{1},j,\overline{2},(\widetilde{ki}))\, J^{(2)}_1(\{p\}_2)$$

$$+A^0_{3,q}(2,i,j)\, D^0_{0gH}(1,(\widetilde{ji}),\overline{2},k)\, J^{(2)}_1(\{p\}_2)$$

$$+4\, C^0_4(1,2,j,k)\, B^0_{1gH}(\overline{1},i,\overline{2})\, J^{(1)}_1(\{p\}_1)$$

$$+4\, C_4^0(2,1,k,j)\, B_{1gH}^0(\overline{2},i,\overline{1})\, J_1^{(1)}(\{p\}_1) \tag{E.1.2}$$

$$\widetilde{D}_{1gH}^{0,YS}(\hat{1}_q,\hat{2}_q,k_{\bar{q}},j_{\bar{q}},i) =$$

$$+A_{3,q}^0(1,i,j)\, D_{0gH}^0(\overline{1},(\widetilde{ji}),2,k)\, J_1^{(2)}(\{p\}_2)$$

$$+A_{3,q}^0(2,i,k)\, D_{0gH}^0(1,j,\overline{2},(\widetilde{ik}))\, J_1^{(2)}(\{p\}_2)$$

$$+A_{3,q}^0(1,i,k)\, D_{0gH}^0(\overline{1},j,\overline{2},(\widetilde{ki}))\, J_1^{(2)}(\{p\}_2)$$

$$+A_{3,q}^0(2,i,j)\, D_{0gH}^0(1,(\widetilde{ji}),\overline{2},k)\, J_1^{(2)}(\{p\}_2)$$

$$-A_{3,q\bar{q}}^0(1,i,2)\, D_{0gH}^0(\overline{1},j,\overline{2},k)\, J_1^{(2)}(\{p\}_2)$$

$$-A_3^0(k,i,j)\, D_{0gH}^0(1,(\widetilde{ji}),\overline{2},(\widetilde{ki}))\, J_1^{(2)}(\{p\}_2)$$

$$+2\, C_4^0(i,2,j,k)\, B_{1gH}^0(1,(\widetilde{ijk}),\overline{2})\, J_1^{(1)}(\{p\}_1)$$

$$+2\, C_4^0(i,1,k,j)\, B_{1gH}^0(2,(\widetilde{ijk}),\overline{1})\, J_1^{(1)}(\{p\}_1) \tag{E.1.3}$$

E.2 $qq \to H+$jet at RV

The real-virtual subtraction terms $d\hat{\sigma}_{NNLO}^T$ mentioned in section 9.5.2 are:

$$\widetilde{C}_{0gH}^{1,YT}(\hat{1}_q,i_{\bar{Q}},\hat{2}_Q,j_{\bar{q}}) =$$

$$-\Bigg[+\mathcal{A}_{3,q}^0(s_{1j}) + \mathcal{A}_{3,q}^0(s_{2i}) - 2\mathcal{A}_{3,qq}^0(s_{12})$$

$$-2\mathcal{A}_3^0(s_{ij}) + 2\mathcal{A}_{3,q}^0(s_{2j}) + 2\mathcal{A}_{3,q}^0(s_{1i}) \Bigg] C_{0gH}^0(1,i,2,j)\, J_1^{(2)}(\{p\}_2)$$

$$-E_{3,q'\to g}^0(j,i,2) \Bigg[\tilde{B}_{1gH}^1(1,\overline{2},(\widetilde{ji}))\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+\mathcal{A}_{3,q}^0(s_{1(\widetilde{ji})})\, B_{1gH}^0(1,\overline{2},(\widetilde{ji})) \Bigg] J_1^{(1)}(\{p\}_1)$$

$$-\Bigg[\tilde{E}_{3,q'}^1(j,i,2)\, \delta(1-x_1)\, \delta(1-x_2) + \mathcal{A}_{3,q}^0(s_{i2})\, E_{3,q'\to g}^0(j,i,2) \Bigg]$$

$$\times B_{1gH}^0(1,\overline{2},(\widetilde{ji}))\, J_1^{(1)}(\{p\}_1)$$

$$-E_{3,q'\to g}^0(i,j,1) \Bigg[\tilde{B}_{1gH}^1((\widetilde{ij}),\overline{1},2)\, \delta(1-x_1)\, \delta(1-x_2)$$

$$+\mathcal{A}_{3,q}^0(s_{2(\widetilde{ij})})\, B_{1gH}^0((\widetilde{ij}),\overline{1},2) \Bigg] J_1^{(1)}(\{p\}_1)$$

$$-\Bigg[\tilde{E}_{3,q'}^1(i,j,1)\, \delta(1-x_1)\, \delta(1-x_2) + \mathcal{A}_{3,q}^0(s_{j1})\, E_{3,q'\to g}^0(i,j,1) \Bigg]$$

$$\times B_{1gH}^0((\widetilde{ij}),\overline{1},2)\, J_1^{(1)}(\{p\}_1)$$

$$+2\Bigg[+\mathcal{A}^0_{3,qq}(s_{12})+\mathcal{A}^0_3(s_{ij})-\mathcal{A}^0_{3,q}(s_{1i})-\mathcal{A}^0_{3,q}(s_{2j})$$

$$+\Bigg(+\mathcal{S}^{IF}(s_{1i},s_{1i},1)-\mathcal{S}^{IF}(s_{12},s_{1i},x_{12,1i})+\mathcal{S}^{IF}(s_{2j},s_{1i},x_{2j,1i})$$

$$-\mathcal{S}^{IF}(s_{ji},s_{1i},x_{ji,1i})\Bigg)\Bigg]E^0_{3,q'\to g}(i,j,1)\,B^0_{1gH}((\widetilde{ij}),\overline{1},2)\,J^{(1)}_1(\{p\}_1)$$

$$+2\Bigg[+\mathcal{A}^0_{3,qq}(s_{12})+\mathcal{A}^0_3(s_{ij})-\mathcal{A}^0_{3,q}(s_{1i})-\mathcal{A}^0_{3,q}(s_{2j})$$

$$+\Bigg(+\mathcal{S}^{FI}(s_{1i},s_{2j},x_{1i,2j})-\mathcal{S}^{FI}(s_{12},s_{2j},x_{12,2j})+\mathcal{S}^{FI}(s_{2j},s_{2j},1)$$

$$-\mathcal{S}^{FI}(s_{ij},s_{2j},x_{ij,2j})\Bigg)\Bigg]E^0_{3,q'\to g}(j,i,2)\,B^0_{1gH}(1,\overline{2},(\widetilde{ji}))\,J^{(1)}_1(\{p\}_1)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i2})\,\tilde{B}^0_{2gH}(1,i,2,j)\,J^{(2)}_1(\{p\}_2)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i2})\,A^0_{3,qg\to qq}(1,2,j)\,B^0_{1gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\mathcal{G}^0_{3,q'\to g}(s_{i2})\,A^0_{3,q}(1,i,j)\,B^0_{1gH}(\overline{1},2,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i1})\,\tilde{B}^0_{2gH}(2,1,i,j)\,J^{(2)}_1(\{p\}_2)$$

$$+\mathcal{G}^0_{3,q'\to g}(s_{i1})\,A^0_{3,gq\to qq}(j,1,2)\,B^0_{1gH}(\overline{1},i,\overline{2})\,J^{(1)}_1(\{p\}_1)$$

$$-\mathcal{G}^0_{3,q'\to g}(s_{i1})\,A^0_{3,q}(2,i,j)\,B^0_{1gH}(\overline{2},1,(\widetilde{ij}))\,J^{(1)}_1(\{p\}_1) \tag{E.2.4}$$

$$\widehat{C}^{1,YT}_{0gH}(\hat{1}_q,j_{\bar{Q}},\hat{2}_Q,i_{\bar{q}})=$$

$$-E^0_{3,q'\to g}(j,1,i)\,\hat{B}^1_{1gH}(2,\overline{1},(\widetilde{ji}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$-\hat{E}^1_{3,q'}(j,1,i)\,B^0_{1gH}(2,\overline{1},(\widetilde{ji}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$-E^0_{3,q'\to g}(i,2,j)\,\hat{B}^1_{1gH}(1,\overline{2},(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2)$$

$$-\hat{E}^1_{3,q'}(i,2,j)\,B^0_{1gH}(1,\overline{2},(\widetilde{ij}))\,\delta(1-x_1)\,\delta(1-x_2) \tag{E.2.5}$$

$$D^{1,YT}_{0gH}(\hat{1}_q,i_{\bar{q}},\hat{2}_q,j_{\bar{q}})=$$

$$-\Bigg[+J^{1,II}_{2,QQ}(s_{12})+J^{1,FF}_{2,QQ}(s_{ji})\Bigg]D^0_{0gH}(1,i,2,j)\,J^{(2)}_1(\{p\}_2) \tag{E.2.6}$$

$$\widetilde{D}^{1,YT}_{0gH}(\hat{1}_q,i_{\bar{q}},\hat{2}_q,j_{\bar{q}})=$$

$$+\Bigg[+J^{1,IF}_{2,QQ}(s_{1i})+J^{1,IF}_{2,QQ}(s_{1j})+J^{1,FI}_{2,QQ}(s_{2i})+J^{1,FI}_{2,QQ}(s_{2j})-J^{1,II}_{2,QQ}(s_{12})-J^{1,FF}_{2,QQ}(s_{ij})\Bigg]$$

$$\times D^0_{0gH}(1,i,2,j)\,J^{(2)}_1(\{p\}_2) \tag{E.2.7}$$

E.3 $qq \to H+$jet at VV

The double virtual subtraction terms $\mathrm{d}\hat{\sigma}^U_{NNLO}$ mentioned in section 9.5.3 are:

$$\widetilde{C}^{1,XU}_{1gH}(1_q, 2, i_{\bar{q}}) =$$

$$- \left[+ \mathcal{E}^0_{3,q'\to g}(s_{2i}) + S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \right] \tilde{B}^1_{1gH}(1, 2, i)$$

$$- \left[+ \mathcal{E}^0_{3,q'\to g}(s_{2i}) \otimes \mathcal{A}^0_{3,q}(s_{1i}) + S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{A}^0_{3,q}(s_{1i}) \right.$$

$$\left. - \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{qq}(z_1) \right] B^0_{1gH}(1, 2, i)$$

$$- \left[+ \tilde{\mathcal{E}}^0_{4,q'}(s_{2i}) + \tilde{\mathcal{E}}^1_{3,q'}(s_{2i}) - \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) \right.$$

$$\left. - \frac{1}{2} S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{qq}(z_2) - \Gamma^{(2)}_{gq}t(z_2) \right] B^0_{1gH}(1, 2, i)$$

$$- \left[+ \mathcal{E}^0_{3,q'\to g}(s_{1i}) + S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \right] \tilde{B}^1_{1gH}(2, 1, i)$$

$$- \left[+ \mathcal{E}^0_{3,q'\to g}(s_{1i}) \otimes \mathcal{A}^0_{3,q}(s_{2i}) + S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{A}^0_{3,q}(s_{2i}) \right.$$

$$\left. - \Gamma^{(1)}_{qq}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{1i}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{qq}(z_2) \right] B^0_{1gH}(2, 1, i)$$

$$- \left[+ \tilde{\mathcal{E}}^0_{4,q'}(s_{1i}) + \tilde{\mathcal{E}}^1_{3,q'}(s_{1i}) - \Gamma^{(1)}_{qq}(z_1) \otimes \mathcal{E}^0_{3,q'\to g}(s_{1i}) \right.$$

$$\left. - \frac{1}{2} S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{qq}(z_1) - \Gamma^{(2)}_{gq}t(z_1) \right] B^0_{1gH}(2, 1, i)$$

$$- \left[- S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \otimes \mathcal{A}^0_{3,qg\to qq}(s_{12}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \mathcal{A}^0_{3,gq\to qq}(s_{12}) \right.$$

$$- \mathcal{B}^0_{4,qq'}(s_{12}) - \mathcal{B}^0_{4,q'q}(s_{12}) + \Gamma^{(2)}_{qQ}B(z_1) + \Gamma^{(2)}_{qQ}B(z_2)$$

$$\left. - \frac{1}{2} \Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{qg}(z_1) - \frac{1}{2} \Gamma^{(1)}_{gq}(z_2) \otimes \Gamma^{(1)}_{qg}(z_2) \right] B^0_{1gH}(1, i, 2) \qquad \text{(E.3.8)}$$

$$\widehat{C}^{1,XU}_{1gH}(1_q, 2, i_{\bar{q}}) =$$

$$- \left[- \mathcal{E}^0_{3,q'\to g}(s_{2i}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \right] \hat{B}^1_{1gH}(1, 2, i)$$

$$- \left[+ \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) + \Gamma^{(1)}_{gg,F}(z_2) \otimes S_{q\to g}\Gamma^{(1)}_{gq}(z_2) \right] B^0_{1gH}(1, 2, i)$$

$$- \left[- \mathcal{E}^0_{3,q'\to g}(s_{1i}) - S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \right] \hat{B}^1_{1gH}(2, 1, i)$$

$$- \left[+ \Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{E}^0_{3,q'\to g}(s_{1i}) + S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1) \right] B^0_{1gH}(2, 1, i)$$

$$-\left[-\hat{\mathcal{E}}^1_{3,q'}(s_{2i}) - \frac{b_F}{\epsilon} \left(\frac{s_{2i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{E}^0_{3,q'\to g}(s_{2i}) + \frac{b_F}{\epsilon} \mathcal{E}^0_{3,q'\to g}(s_{2i}) \right.$$

$$- \Gamma^{(1)}_{gg,F}(z_2) \otimes \mathcal{E}^0_{3,q'\to g}(s_{2i}) - \frac{3}{2} \Gamma^{(1)}_{gg,F}(z_2) \otimes S_{q\to g}\Gamma^{(1)}_{gq}(z_2)$$

$$\left. - S_{q\to g}\overline{\Gamma}^{(2)}_{gq,F}(z_2) \right] B^0_{1gH}(1,2,i)$$

$$-\left[-\hat{\mathcal{E}}^1_{3,q'}(s_{1i}) - \frac{b_F}{\epsilon} \left(\frac{s_{1i}}{\mu_R^2} \right)^{-\epsilon} \mathcal{E}^0_{3,q'\to g}(s_{1i}) + \frac{b_F}{\epsilon} \mathcal{E}^0_{3,q'\to g}(s_{1i}) \right.$$

$$- \Gamma^{(1)}_{gg,F}(z_1) \otimes \mathcal{E}^0_{3,q'\to g}(s_{1i}) - \frac{3}{2} S_{q\to g}\Gamma^{(1)}_{gq}(z_1) \otimes \Gamma^{(1)}_{gg,F}(z_1)$$

$$\left. - S_{q\to g}\overline{\Gamma}^{(2)}_{gq,F}(z_1) \right] B^0_{1gH}(2,1,i) \tag{E.3.9}$$

$$D^{1,XU}_{1gH}(1_q,2,i_{\bar{q}}) =$$
$$-\left[+ \mathcal{C}^0_{4,q\dot{q}}(s_{12}) + \mathcal{C}^0_{4,\dot{q}q}(s_{12}) + \widetilde{\overline{\Gamma}}^{(2)}_{q\bar{q}}(z_1) + \widetilde{\overline{\Gamma}}^{(2)}_{q\bar{q}}(z_2) \right] B^0_{1gH}(1,i,2) \tag{E.3.10}$$

$$\widetilde{D}^{1,XU}_{1gH}(1_q,2,i_{\bar{q}}) =$$
$$-\left[- \mathcal{C}^0_{4,q\dot{q}}(s_{12}) - \mathcal{C}^0_{4,\dot{q}q}(s_{12}) - \widetilde{\overline{\Gamma}}^{(2)}_{q\bar{q}}(z_1) - \widetilde{\overline{\Gamma}}^{(2)}_{q\bar{q}}(z_2) \right] B^0_{1gH}(1,i,2) \tag{E.3.11}$$

Appendix F

Mass Factorisation terms at NLO

In this appendix, we review the conventions and notation for the mass factorisation terms that contribute at NLO. In particular, we define the one-loop anomalous dimensions $\mathbf{\Gamma}^1_{ij}$ that appear in the NLO mass factorisation counter term,

$$\mathrm{d}\hat{\sigma}^{MF}_{ij,NLO}(\xi_1 H_1, \xi_2 H_2) = -\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\left[\delta(1-x_2)\mathbf{\Gamma}^1_{ki}(x_1)\mathrm{d}\hat{\sigma}^B_{kj}\right.$$
$$\left. +\delta(1-x_1)\mathbf{\Gamma}^1_{lj}(x_2)\mathrm{d}\hat{\sigma}^B_{il}\right](x_1\xi_1 H_1, x_2\xi_2 H_2),$$

$$(\text{F.0.1})$$

and explictly write out the NLO contributions to the various initial states.

F.1   Conventions

As usual, we consider the parton types to be the gluons, g, and N_f different light quarks q and their antiquark partners $\bar{q}$. Altogether there are $2N_f + 1$ different types of parton. When we study processes involving quarks of different flavours we discriminate from q using the label Q (which carries $(N_f - 1)$ different flavours).

The link between the colourful anomalous dimension $\mathbf{\Gamma}^1$ and the colour stripped anomalous dimension Γ^1 is given by,

$$\mathbf{\Gamma}^1_{qq}(x) = \left(\frac{N^2 - 1}{N}\right)\Gamma^1_{qq}(x), \qquad (\text{F.1.2})$$

$$\mathbf{\Gamma}^1_{qg}(x) = \Gamma^1_{qg}(x), \qquad (\text{F.1.3})$$

$$\mathbf{\Gamma}^1_{gq}(x) = \left(\frac{N^2 - 1}{N}\right)\Gamma^1_{gq}(x), \qquad (\text{F.1.4})$$

$$\mathbf{\Gamma}^1_{gg}(x) = N\Gamma^1_{gg}(x) + N_f\hat{\Gamma}^1_{gg,F}(x). \tag{F.1.5}$$

The colour stripped anomalous dimensions are related to the LO splitting functions through,

$$\Gamma^1_{ij}(x) = -\frac{1}{\epsilon}p^{(0)}_{ij}(x), \qquad \hat{\Gamma}^1_{ij}(x) = -\frac{1}{\epsilon}p^{(0)}_{ij,F}(x), \tag{F.1.6}$$

while the splitting functions are given by,

$$p^{(0)}_{gg}(x) = 2\mathcal{D}_0(x) + \frac{2}{x} - 4 + 2x - 2x^2 + b_0\delta(1-x),$$

$$p^{(0)}_{gg,F}(x) = b_{0,F}\delta(1-x),$$

$$p^{(0)}_{qq}(x) = \mathcal{D}_0(x) - \frac{1+x}{2} + \frac{3}{4}\delta(1-x),$$

$$p^{(0)}_{gq}(x) = \frac{1}{x} - 1 + \frac{x}{2},$$

$$p^{(0)}_{qg}(x) = \frac{1}{2} - x + x^2,$$

$$p^{(0)}_{q\bar{q}}(x) = p^{(0)}_{qQ}(x) = p^{(0)}_{q\bar{Q}}(x) = 0,$$

$$p^{(0)}_{qg}(x) = p^{(0)}_{\bar{q}g}(x) = p^{(0)}_{Qg}(x),$$

$$p^{(0)}_{gq}(x) = p^{(0)}_{g\bar{q}}(x) = p^{(0)}_{gQ}(x), \tag{F.1.7}$$

where

$$b_0 = \frac{11}{6}, \qquad b_{0,F} = -\frac{1}{3}. \tag{F.1.8}$$

F.2 NLO MF terms for various initial states

It is useful to expand Eq. (F.0.1) according to the initial parton type and colour factor.

gg

$$\mathrm{d}\hat{\sigma}^{MF}_{gg,NLO}(\xi_1 H_1, \xi_2 H_2) =$$
$$-\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\left[N\left(\Gamma^1_{gg}(x_1)\delta(1-x_2) + \Gamma^1_{gg}(x_2)\delta(1-x_1)\right)\right.$$
$$\left. + N_f\left(\Gamma^1_{gg,F}(x_1)\delta(1-x_2) + \Gamma^1_{gg,F}(x_2)\delta(1-x_1)\right)\right]\mathrm{d}\hat{\sigma}^B_{gg}$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[N_f S_{g \to q} \Gamma^1_{qg}(x_1) \delta(1 - x_2) (d\hat{\sigma}^B_{qg} + d\hat{\sigma}^B_{\bar{q}g})
$$

$$
+ N_f S_{g \to q} \Gamma^1_{qg}(x_2) \delta(1 - x_1) (d\hat{\sigma}^B_{gq} + d\hat{\sigma}^B_{g\bar{q}}) \Bigg]. \tag{F.2.9}
$$

qg

$$
d\hat{\sigma}^{MF}_{qg,NLO}(\xi_1 H_1, \xi_2 H_2) =
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} S_{q \to g} \Gamma^1_{gq}(x_1) \delta(1 - x_2) \Bigg] d\hat{\sigma}^B_{gg}
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} \Gamma^1_{qq}(x_1) \delta(1 - x_2)
$$

$$
+ N \Gamma^1_{gg}(x_2) \delta(1 - x_1) + N_f \Gamma^1_{gg,F}(x_2) \delta(1 - x_1) \Bigg] d\hat{\sigma}^B_{qg}
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[S_{g \to q} \Gamma^1_{qg}(x_2) \delta(1 - x_1) \Big(d\hat{\sigma}^B_{qq} + d\hat{\sigma}^B_{q\bar{q}} + (N_f - 1)(d\hat{\sigma}^B_{qQ} + d\hat{\sigma}^B_{q\bar{Q}}) \Big) \Bigg]. \tag{F.2.10}
$$

q$\bar{\text{q}}$

$$
d\hat{\sigma}^{MF}_{q\bar{q},NLO}(\xi_1 H_1, \xi_2 H_2) =
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} \Big(S_{q \to g} \Gamma^1_{gq}(x_1) \delta(1 - x_2) d\hat{\sigma}^B_{gq} + S_{q \to g} \Gamma^1_{gq}(x_2) \delta(1 - x_1) d\hat{\sigma}^B_{qg} \Big) \Bigg]
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} \Big(\Gamma^1_{qq}(x_1) \delta(1 - x_2) + \Gamma^1_{qq}(x_2) \delta(1 - x_1) \Big) \Bigg] d\hat{\sigma}^B_{q\bar{q}}. \tag{F.2.11}
$$

qq

$$
d\hat{\sigma}^{MF}_{qq,NLO}(\xi_1 H_1, \xi_2 H_2) =
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} \Big(S_{q \to g} \Gamma^1_{gq}(x_1) \delta(1 - x_2) d\hat{\sigma}^B_{gq} + S_{q \to g} \Gamma^1_{gq}(x_2) \delta(1 - x_1) d\hat{\sigma}^B_{qg} \Big) \Bigg]
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} \Big(\Gamma^1_{qq}(x_1) \delta(1 - x_2) + \Gamma^1_{qq}(x_2) \delta(1 - x_1) \Big) \Bigg] d\hat{\sigma}^B_{qq}. \tag{F.2.12}
$$

qQ

$$
d\hat{\sigma}^{MF}_{qQ,NLO}(\xi_1 H_1, \xi_2 H_2) =
$$

$$
- \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \bar{C}(\epsilon) \Bigg[\frac{N^2 - 1}{N} \Big(S_{q \to g} \Gamma^1_{gq}(x_1) \delta(1 - x_2) d\hat{\sigma}^B_{gq} + S_{q \to g} \Gamma^1_{gq}(x_2) \delta(1 - x_1) d\hat{\sigma}^B_{qg} \Big) \Bigg]
$$

$$-\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\left[\frac{N^2-1}{N}\left(\Gamma^1_{qq}(x_1)\delta(1-x_2)\mathrm{d}\hat{\sigma}^B_{qQ} + \Gamma^1_{qq}(x_2)\delta(1-x_1)\mathrm{d}\hat{\sigma}^B_{Qq}\right)\right].$$

$$(\text{F.2.13})$$

q$\bar{\text{Q}}$

$$\mathrm{d}\hat{\sigma}^{MF}_{q\bar{Q},NLO}(\xi_1 H_1, \xi_2 H_2) =$$
$$-\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\left[\frac{N^2-1}{N}\left(S_{q\rightarrow g}\Gamma^1_{gq}(x_1)\delta(1-x_2)\mathrm{d}\hat{\sigma}^B_{gq} + S_{q\rightarrow g}\Gamma^1_{gq}(x_2)\delta(1-x_1)\mathrm{d}\hat{\sigma}^B_{qg}\right)\right]$$
$$-\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{C}(\epsilon)\left[\frac{N^2-1}{N}\left(\Gamma^1_{qq}(x_1)\delta(1-x_2)\mathrm{d}\hat{\sigma}^B_{q\bar{Q}} + \Gamma^1_{qq}(x_2)\delta(1-x_1)\mathrm{d}\hat{\sigma}^B_{\bar{Q}q}\right)\right].$$

$$(\text{F.2.14})$$

Bibliography

[1] G. Passarino and M.J.G. Veltman. One Loop Corrections for e^+e^- Annihilation Into $\mu^+\mu^-$ in the Weinberg Model. *Nucl.Phys.*, B160:151, 1979.

[2] J.C. Collins. Renormalization. An Introduction To Renormalization, The Renormalization Group, And The Operator Product Expansion. 1984.

[3] P. Pascual and R. Tarrach. QCD: Renormalization For The Practitioner. *Lect.Notes Phys.*, 194:1–277, 1984.

[4] G. 't Hooft and M.J.G. Veltman. Regularization and Renormalization of Gauge Fields. *Nucl.Phys.*, B44:189–213, 1972.

[5] S. Catani, M.H. Seymour, and Z. Trocsanyi. Regularization scheme independence and unitarity in QCD cross-sections. *Phys.Rev.*, D55:6819–6829, 1997.

[6] C. Anastasiou and K. Melnikov. Higgs boson production at hadron colliders in NNLO QCD. *Nucl.Phys.*, B646:220–256, 2002.

[7] A. Gehrmann-De Ridder, E.W.N. Glover, and J. Pires. Real-Virtual corrections for gluon scattering at NNLO. *JHEP*, 1202:141, 2012.

[8] F. Bloch and A. Nordsieck. Note on the Radiation Field of the electron. *Phys.Rev.*, 52:54–59, 1937.

[9] T. Kinoshita. Mass singularities of Feynman amplitudes. *J.Math.Phys.*, 3:650–677, 1962.

[10] T.D. Lee and M. Nauenberg. Degenerate Systems and Mass Singularities. *Phys.Rev.*, 133:B1549–B1562, 1964.

[11] Search for Dijet Resonances in the Dijet Mass Distribution in pp Collisions at $\sqrt{s} = 7$ TeV. 2010.

[12] Stephen D. Ellis and Davison E. Soper. Successive combination jet algorithm for hadron collisions. *Phys.Rev.*, D48:3160–3166, 1993.

[13] G.P. Salam. Towards Jetography. *Eur.Phys.J.*, C67:637–686, 2010.

[14] Peter W. Higgs. Broken symmetries, massless particles and gauge fields. *Phys.Lett.*, 12:132–133, 1964.

[15] F. Englert and R. Brout. Broken Symmetry and the Mass of Gauge Vector Mesons. *Phys.Rev.Lett.*, 13:321–323, 1964.

[16] Peter W. Higgs. Broken Symmetries and the Masses of Gauge Bosons. *Phys.Rev.Lett.*, 13:508–509, 1964.

[17] G.S. Guralnik, C.R. Hagen, and T.W.B. Kibble. Global Conservation Laws and Massless Particles. *Phys.Rev.Lett.*, 13:585–587, 1964.

[18] Peter W. Higgs. Spontaneous Symmetry Breakdown without Massless Bosons. *Phys.Rev.*, 145:1156–1163, 1966.

[19] G. Aad et al. A particle consistent with the Higgs Boson observed with the ATLAS Detector at the Large Hadron Collider. *Science*, 338:1576–1582, 2012.

[20] G. Aad et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Phys.Lett.*, B716:1–29, 2012.

[21] S. Chatrchyan et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Phys.Lett.*, B716:30–61, 2012.

[22] S. Chatrchyan et al. Observation of a new boson with mass near 125 GeV in pp collisions at $\sqrt{s} = 7$ and 8 TeV. *JHEP*, 1306:081, 2013.

[23] T.P. Cheng and L.F. Li. Gauge theory of elementary particle physics: Problems and solutions. 2000.

[24] Y. Nambu. Quasiparticles and Gauge Invariance in the Theory of Superconductivity. *Phys.Rev.*, 117:648–663, 1960.

[25] J. Goldstone. Field Theories with Superconductor Solutions. *Nuovo Cim.*, 19:154–164, 1961.

[26] M. A. Shifman, A.I. Vainshtein, and V. I. Zakharov. Remarks on Higgs Boson Interactions with Nucleons. *Phys.Lett.*, B78:443, 1978.

[27] S. Dawson and R.P. Kauffman. Higgs boson plus multi - jet rates at the SSC. *Phys.Rev.Lett.*, 68:2273–2276, 1992.

[28] F. Wilczek. Decays of Heavy Vector Mesons Into Higgs Particles. *Phys.Rev.Lett.*, 39:1304, 1977.

[29] T. Inami, T. Kubota, and Y. Okada. Effective Gauge Theory and the Effect of Heavy Quarks in Higgs Boson Decays. *Z.Phys.*, C18:69, 1983.

[30] K.G. Chetyrkin, B. A. Kniehl, and M. Steinhauser. Decoupling relations to $\mathcal{O}(\alpha_s^3)$ and their connection to low-energy theorems. *Nucl.Phys.*, B510:61–87, 1998.

[31] K.G. Chetyrkin, Bernd A. Kniehl, and M. Steinhauser. Hadronic Higgs decay to order α_s^4. *Phys.Rev.Lett.*, 79:353–356, 1997.

[32] Robert V. Harlander, T. Neumann, Kemal J. Ozeren, and M. Wiesemann. Top-mass effects in differential Higgs production through gluon fusion at order α_s^4. *JHEP*, 1208:139, 2012.

[33] T. Neumann and M. Wiesemann. Finite top-mass effects in gluon-induced Higgs production with a jet-veto at NNLO. *JHEP*, 1411:150, 2014.

[34] LHC Higgs Cross Section Working Group, S. Dittmaier, C. Mariotti, G. Passarino, and R. Tanaka (Eds.). Handbook of LHC Higgs Cross Sections: 2. Differential Distributions. *CERN-2012-002*, CERN, Geneva, 2012.

[35] LHC Higgs Cross Section Working Group, S. Heinemeyer, C. Mariotti, G. Passarino, and R. Tanaka (Eds.). Handbook of LHC Higgs Cross Sections: 3. Higgs Properties. *CERN-2013-004*, CERN, Geneva, 2013.

[36] LHC Higgs Cross Section Working Group, S. Dittmaier, C. Mariotti, G. Passarino, and R. Tanaka (Eds.). Handbook of LHC Higgs Cross Sections: 1. Inclusive Observables. *CERN-2011-002*, CERN, Geneva, 2011.

[37] A. Djouadi, M. Spira, and P.M. Zerwas. Production of Higgs bosons in proton colliders: QCD corrections. *Phys.Lett.*, B264:440–446, 1991.

[38] S. Dawson. Radiative corrections to Higgs boson production. *Nucl.Phys.*, B359:283–300, 1991.

[39] M. Spira, A. Djouadi, D. Graudenz, and P.M. Zerwas. Higgs boson production at the LHC. *Nucl.Phys.*, B453:17–82, 1995.

[40] V. Ravindran, J. Smith, and W. L. van Neerven. NNLO corrections to the total cross-section for Higgs boson production in hadron hadron collisions. *Nucl.Phys.*, B665:325–366, 2003.

[41] C. Anastasiou, K. Melnikov, and F. Petriello. Higgs boson production at hadron colliders: Differential cross sections through next-to-next-to-leading order. *Phys.Rev.Lett.*, 93:262002, 2004.

[42] C. Anastasiou, K. Melnikov, and F. Petriello. Fully differential Higgs boson production and the di-photon signal through next-to-next-to-leading order. *Nucl.Phys.*, B724:197–246, 2005.

[43] C. Anastasiou, G. Dissertori, and F. Stckli. NNLO QCD predictions for the $H \to WW \to \ell\nu\ell\nu$ signal at the LHC. *JHEP*, 0709:018, 2007.

[44] S. Catani and M. Grazzini. An NNLO subtraction formalism in hadron collisions and its application to Higgs boson production at the LHC. *Phys.Rev.Lett.*, 98:222002, 2007.

[45] M. Grazzini. NNLO predictions for the Higgs boson signal in the $H \to WW \to \ell\nu\ell\nu$ and $H \to ZZ \to 4l$ decay channels. *JHEP*, 0802:043, 2008.

[46] C. Anastasiou, S. Bucherer, and Z. Kunszt. HPro: A NLO Monte-Carlo for Higgs production via gluon fusion with finite heavy quark masses. *JHEP*, 0910:068, 2009.

[47] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, et al. Higgs boson gluonfusion production at threshold in N^{3}LO QCD. *Phys.Lett.*, B737:325–328, 2014.

[48] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, T. Gehrmann, et al. Higgs boson gluon-fusion production beyond threshold in N3LO QCD. 2014.

[49] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog, and B. Mistlberger. Higgs boson gluon-fusion production in N3LO QCD. 2015.

[50] S. Marzani, Richard D. Ball, V. Del Duca, S. Forte, and A. Vicini. Higgs production via gluon-gluon fusion with finite top mass beyond next-to-leading order. *Nucl.Phys.*, B800:127–145, 2008.

[51] A. Pak, M. Rogal, and M. Steinhauser. Finite top quark mass effects in NNLO Higgs boson production at LHC. *JHEP*, 1002:025, 2010.

[52] Robert V. Harlander and Kemal J. Ozeren. Finite top mass effects for hadronic Higgs production at next-to-next-to-leading order. *JHEP*, 0911:088, 2009.

[53] Robert V. Harlander and Kemal J. Ozeren. Top mass effects in Higgs production at next-to-next-to-leading order QCD: Virtual corrections. *Phys.Lett.*, B679:467–472, 2009.

[54] A. Pak, M. Rogal, and M. Steinhauser. Virtual three-loop corrections to Higgs boson production in gluon fusion for finite top quark mass. *Phys.Lett.*, B679:473–477, 2009.

[55] Robert V. Harlander, H. Mantler, S. Marzani, and Kemal J. Ozeren. Higgs production in gluon fusion at next-to-next-to-leading order QCD for finite top mass. *Eur.Phys.J.*, C66:359–372, 2010.

[56] C. Adrian. Higgs Boson Makes Its Debut After Decades-Long Search. *Science*, 337:141–143, 2012.

[57] V. Khachatryan et al. Precise determination of the mass of the Higgs boson and tests of compatibility of its couplings with the standard model predictions using proton collisions at 7 and 8 TeV. 2014.

[58] G. Aad et al. Measurement of the Higgs boson mass from the $H \to \gamma\gamma$ and $H \to ZZ^* \to 4\ell$ channels with the ATLAS detector using 25 fb^{-1} of pp collision data. *Phys.Rev.*, D90(5):052004, 2014.

[59] S. Chatrchyan et al. Measurement of Higgs boson production and properties in the WW decay channel with leptonic final states. *JHEP*, 1401:096, 2014.

[60] S. Chatrchyan et al. Evidence for the direct decay of the 125 GeV Higgs boson to fermions. *Nature Phys.*, 10:557–560, 2014.

[61] G. Aad et al. Search for the $b\bar{b}$ decay of the Standard Model Higgs boson in associated $(W/Z)H$ production with the ATLAS detector. *JHEP*, 1501:069, 2015.

[62] G. Aad et al. Evidence for the Higgs-boson Yukawa coupling to tau leptons with the ATLAS detector. 2015.

[63] S. Chatrchyan et al. Study of the Mass and Spin-Parity of the Higgs Boson Candidate Via Its Decays to Z Boson Pairs. *Phys.Rev.Lett.*, 110(8):081803, 2013.

[64] S. Chatrchyan et al. Measurement of the properties of a Higgs boson in the four-lepton final state. *Phys.Rev.*, D89(9):092007, 2014.

[65] G. Aad et al. Evidence for the spin-0 nature of the Higgs boson using ATLAS data. *Phys.Lett.*, B726:120–144, 2013.

[66] G. Aad et al. Determination of spin and parity of the Higgs boson in the $WW^* \to e\nu\mu\nu$ decay channel with the ATLAS detector. 2015.

[67] J. F. Gunion, H. E. Haber, G. L. Kane, and S. Dawson. The Higgs Hunter's Guide. *Front.Phys.*, 80:1–448, 2000.

[68] CMS Collaboration. Updated results on the new boson discovered in the search for the standard model Higgs boson in the ZZ to 4 leptons channel in pp collisions at $\sqrt{s} = 7$ and 8 TeV. 2012.

[69] G. Aad et al. Search for neutral Higgs bosons of the minimal supersymmetric standard model in pp collisions at $\sqrt{s} = 8$ TeV with the ATLAS detector. *JHEP*, 1411:056, 2014.

[70] V. Khachatryan et al. Search for neutral MSSM Higgs bosons decaying to a pair of tau leptons in pp collisions. *JHEP*, 1410:160, 2014.

[71] CMS Collaboration. Performance of b tagging at $\sqrt{s} = 8$ TeV in multijet, ttbar and boosted topology events. 2013.

[72] J. Erdmann. Experimental results with boosted top quarks in the final state. 2014.

[73] T. Plehn, Gavin P. Salam, and M. Spannowsky. Fat Jets for a Light Higgs. *Phys.Rev.Lett.*, 104:111801, 2010.

[74] Jonathan M. Butterworth, Adam R. Davison, M. Rubin, and Gavin P. Salam. Jet substructure as a new Higgs search channel at the LHC. *Phys.Rev.Lett.*, 100:242001, 2008.

[75] Search for the bb decay of the Standard Model Higgs boson in associated W/ZH production with the ATLAS detector. Technical Report ATLAS-CONF-2013-079, CERN, Geneva, Jul 2013.

[76] Search for associated production of the Higgs boson in the $WH \to WWW^{(*)} \to \ell\nu\ell\nu\ell\nu$ and $ZH \to ZWW^{(*)} \to \ell\ell\ell\nu\ell\nu$ channels with the AT-

LAS detector at the LHC. Technical Report ATLAS-CONF-2013-075, CERN, Geneva, Jul 2013.

[77] R. Godbole, D. J. Miller, K. Mohan, and C. D. White. Boosting Higgs CP properties via VH Production at the Large Hadron Collider. *Phys.Lett.*, B730:275–279, 2014.

[78] M. Schlaffer, M. Spannowsky, M. Takeuchi, A. Weiler, and C. Wymant. Boosted Higgs Shapes. *Eur.Phys.J.*, C74(10):3120, 2014.

[79] R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, R. Pittau, et al. Scalar and pseudoscalar Higgs production in association with a top-antitop pair. *Phys.Lett.*, B701:427–433, 2011.

[80] John M. Campbell, R. K. Ellis, R. Frederix, P. Nason, C. Oleari, et al. NLO Higgs Boson Production Plus One and Two Jets Using the POWHEG BOX, MadGraph4 and MCFM. *JHEP*, 1207:092, 2012.

[81] F. Cascioli, S. Hche, F. Krauss, P. Maierhfer, S. Pozzorini, et al. Precise Higgs-background predictions: merging NLO QCD and squared quark-loop corrections to four-lepton + 0,1 jet production. *JHEP*, 1401:046, 2014.

[82] G. Cullen, H. van Deurzen, N. Greiner, G. Heinrich, G. Luisoni, et al. NLO QCD Production of Higgs plus jets with GoSam. *PoS*, EPS-HEP2013:446, 2013.

[83] R. Boughezal, F. Caola, K. Melnikov, F. Petriello, and M. Schulze. Higgs boson production in association with a jet at next-to-next-to-leading order in perturbative QCD. *JHEP*, 1306:072, 2013.

[84] X. Chen, T. Gehrmann, E.W.N. Glover, and M. Jaquier. Precise QCD predictions for the production of Higgs + jet final states. *Phys.Lett.*, B740:147–150, 2015.

[85] G. Ferrera, M. Grazzini, and F. Tramontano. Associated WH production at hadron colliders: a fully exclusive QCD calculation at NNLO. *Phys.Rev.Lett.*, 107:152003, 2011.

[86] G. Ferrera, M. Grazzini, and F. Tramontano. Associated ZH production at hadron colliders: the fully differential NNLO QCD calculation. *Phys.Lett.*, B740:51–55, 2015.

[87] S. Dittmaier. Associated Higgs production with heavy quark pairs at hadron colliders. pages 732–739, 2002.

[88] S. Dawson, C.B. Jackson, L.H. Orr, L. Reina, and D. Wackeroth. NLO QCD corrections to hadronic Higgs production with heavy quarks. *Nucl.Phys.Proc.Suppl.*, 133:111–116, 2004.

[89] M. Klasen, K. Kovarik, P. Nason, and C. Weydert. Associated production of charged Higgs bosons and top quarks with POWHEG. *Eur.Phys.J.*, C72:2088, 2012.

[90] G. Cullen, H. van Deurzen, N. Greiner, G. Heinrich, G. Luisoni, et al. GoSam @ LHC: algorithms and applications to Higgs production. *PoS*, RAD-COR2013:029, 2013.

[91] Heribertus B. Hartanto, B. Jager, L. Reina, and D. Wackeroth. Higgs boson production in association with top quarks in the POWHEG BOX. 2015.

[92] G. Aad et al. Measurements of fiducial and differential cross sections for Higgs boson production in the diphoton decay channel at $\sqrt{s} = 8$ TeV with ATLAS. *JHEP*, 1409:112, 2014.

[93] G. Aad et al. Fiducial and differential cross sections of Higgs boson production measured in the four-lepton decay channel in pp collisions at $\sqrt{s}=8$ TeV with the ATLAS detector. *Phys.Lett.*, B738:234–253, 2014.

[94] A. Banfi, P. F. Monni, G. P. Salam, and G. Zanderighi. Higgs and Z-boson production with a jet veto. *Phys.Rev.Lett.*, 109:202001, 2012.

[95] A. Banfi, P. F. Monni, and G. Zanderighi. Quark masses in Higgs production with a jet veto. *JHEP*, 1401:097, 2014.

[96] F. J. Tackmann, J. R. Walsh, and S. Zuberi. Resummation Properties of Jet Vetoes at the LHC. *Phys.Rev.*, D86:053011, 2012.

[97] I. W. Stewart, F. J. Tackmann, J. R. Walsh, and S. Zuberi. Jet p_T resummation in Higgs production at $NNLL' + NNLO$. *Phys.Rev.*, D89(5):054001, 2014.

[98] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, and G. Heinrich. Infrared structure of $e^+e^- \rightarrow 3$ jets at NNLO. *JHEP*, 0711:058, 2007.

[99] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, and G. Heinrich. EERAD3: Event shapes and jet rates in electron-positron annihilation at order α_s^3. *Comput.Phys.Commun.*, 185:3331, 2014.

[100] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, and J. Pires. Second order QCD corrections to jet production at hadron colliders: the all-gluon contribution. *Phys.Rev.Lett.*, 110(16):162003, 2013.

[101] J. Currie, A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, and J. Pires. NNLO QCD corrections to dijet production at hadron colliders. *PoS*, RADCOR2013:004, 2013.

[102] J. Currie, A. Gehrmann-De Ridder, E.W.N. Glover, and J. Pires. NNLO QCD corrections to jet production at hadron colliders from gluon scattering. *JHEP*, 1401:110, 2014.

[103] R.K. Ellis, G. Marchesini, and B.R. Webber. Soft Radiation in Parton Parton Scattering. *Nucl.Phys.*, B286:643, 1987.

[104] G. Altarelli and G. Parisi. Asymptotic Freedom in Parton Language. *Nucl.Phys.*, B126:298, 1977.

[105] L.J. Dixon. Calculating scattering amplitudes efficiently. 1996.

[106] D.A. Kosower. Antenna factorization of gauge theory amplitudes. *Phys.Rev.*, D57:5410–5416, 1998.

[107] D.A. Kosower. Antenna factorization in strongly ordered limits. *Phys.Rev.*, D71:045016, 2005.

[108] A. Daleo, T. Gehrmann, and D. Maitre. Antenna subtraction with hadronic initial states. *JHEP*, 0704:016, 2007.

[109] A. Gehrmann-De Ridder, T. Gehrmann, and E.W.N. Glover. Antenna subtraction at NNLO. *JHEP*, 0509:056, 2005.

[110] A. Gehrmann-De Ridder, T. Gehrmann, and E.W.N. Glover. Infrared structure of $e^+e^- \to 2$ jets at NNLO. *Nucl.Phys.*, B691:195–222, 2004.

[111] A. Gehrmann-De Ridder, T. Gehrmann, and E.W.N. Glover. Quark-gluon antenna functions from neutralino decay. *Phys.Lett.*, B612:36–48, 2005.

[112] A. Gehrmann-De Ridder, T. Gehrmann, and E.W.N. Glover. Gluon-gluon antenna functions from Higgs boson decay. *Phys.Lett.*, B612:49–60, 2005.

[113] E.W.N. Glover and J. Pires. Antenna subtraction for gluon scattering at NNLO. *JHEP*, 1006:096, 2010.

[114] S. Catani. The Singular behavior of QCD amplitudes at two loop order. *Phys.Lett.*, B427:161–171, 1998.

[115] G.F. Sterman and M.E. Tejeda-Yeomans. Multiloop amplitudes and resummation. *Phys.Lett.*, B552:48–56, 2003.

[116] J. Currie, E.W.N. Glover, and S. Wells. Infrared Structure at NNLO Using Antenna Subtraction. *JHEP*, 1304:066, 2013.

[117] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, and J. Pires. Double Virtual corrections for gluon scattering at NNLO. *JHEP*, 1302:026, 2013.

[118] E.W.N. Glover, C. Oleari, and M.E. Tejeda-Yeomans. Two loop QCD corrections to gluon-gluon scattering. *Nucl.Phys.*, B605:467–485, 2001.

[119] T. Gehrmann, M. Jaquier, E.W.N. Glover, and A. Koukoutsakis. Two-Loop QCD Corrections to the Helicity Amplitudes for $H \to 3$ partons. *JHEP*, 1202:056, 2012.

[120] C.F. Berger, V. Del Duca, and L.J. Dixon. Recursive Construction of Higgs-Plus-Multiparton Loop Amplitudes: The Last of the Phi-nite Loop Amplitudes. *Phys.Rev.*, D74:094021, 2006.

[121] L.J. Dixon and Y. Sofianatos. Analytic one-loop amplitudes for a Higgs boson plus four partons. *JHEP*, 0908:058, 2009.

[122] S. Badger, E.W.N. Glover, P. Mastrolia, and C. Williams. One-loop Higgs plus four gluon amplitudes: Full analytic results. *JHEP*, 1001:036, 2010.

[123] S. Badger, J.M. Campbell, R.K. Ellis, and C. Williams. Analytic results for the one-loop NMHV Hqqgg amplitude. *JHEP*, 0912:035, 2009.

[124] F.A. Berends and W.T. Giele. Multiple Soft Gluon Radiation in Parton Processes. *Nucl.Phys.*, B313:595, 1989.

[125] J.M. Campbell and E.W.N. Glover. Double unresolved approximations to multiparton scattering amplitudes. *Nucl.Phys.*, B527:264–288, 1998.

[126] A. Gehrmann-De Ridder and E.W.N. Glover. A Complete $\mathcal{O}(\alpha\alpha_s)$ calculation of the photon + 1 jet rate in e^+e^- annihilation. *Nucl.Phys.*, B517:269–323, 1998.

[127] S. Catani and M. Grazzini. Collinear factorization and splitting functions for next-to-next-to-leading order QCD calculations. *Phys.Lett.*, B446:143–152, 1999.

[128] S. Catani and M. Grazzini. Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond. *Nucl.Phys.*, B570:287–325, 2000.

[129] D.A. Kosower. Multiple singular emission in gauge theories. *Phys.Rev.*, D67:116003, 2003.

[130] S. Weinzierl. The infrared structure of $e^+e^- \rightarrow 3$ jets at NNLO reloaded. *JHEP*, 0907:009, 2009.

[131] Z. Bern, V. Del Duca, and C.R. Schmidt. The Infrared behavior of one loop gluon amplitudes at next-to-next-to-leading order. *Phys.Lett.*, B445:168–177, 1998.

[132] Z. Bern, V. Del Duca, W.B. Kilgore, and C.R. Schmidt. The infrared behavior of one loop QCD amplitudes at next-to-next-to leading order. *Phys.Rev.*, D60:116001, 1999.

[133] Z. Bern, L.J. Dixon, D.C. Dunbar, and D.A. Kosower. One loop n point gauge theory amplitudes, unitarity and collinear limits. *Nucl.Phys.*, B425:217–260, 1994.

[134] A. Daleo, A.and Gehrmann-De Ridder, T. Gehrmann, and G. Luisoni. Antenna subtraction at NNLO with hadronic initial states: initial-final configurations. *JHEP*, 1001:118, 2010.

[135] R.K. Ellis, W.J. Stirling, and B.R. Webber. QCD and collider physics. *Camb.Monogr.Part.Phys.Nucl.Phys.Cosmol.*, 8:1–435, 1996.

[136] A. Gehrmann-De Ridder, T. Gehrmann, and M. Ritzmann. Antenna subtraction at NNLO with hadronic initial states: double real initial-initial configurations. *JHEP*, 1210:047, 2012.

[137] R. Boughezal, A. Gehrmann-De Ridder, and M. Ritzmann. Antenna subtraction at NNLO with hadronic initial states: double real radiation for initial-initial configurations with two quark flavours. *JHEP*, 1102:098, 2011.

[138] T. Gehrmann and P.F. Monni. Antenna subtraction at NNLO with hadronic initial states: real-virtual initial-initial configurations. *JHEP*, 1112:049, 2011.

[139] S.J. Parke and T.R. Taylor. An Amplitude for n Gluon Scattering. *Phys.Rev.Lett.*, 56:2459, 1986.

[140] F. Cachazo, P. Svrcek, and E. Witten. MHV vertices and tree amplitudes in gauge theory. *JHEP*, 0409:006, 2004.

[141] R. Britto, F. Cachazo, B. Feng, and E. Witten. Direct proof of tree-level recursion relation in Yang-Mills theory. *Phys.Rev.Lett.*, 94:181602, 2005.

[142] F. Cachazo, S. He, and E.Y. Yuan. Scattering of Massless Particles in Arbitrary Dimensions. *Phys.Rev.Lett.*, 113(17):171601, 2014.

[143] K.G. Chetyrkin, Johann H. Kuhn, and Christian Sturm. QCD decoupling at four loops. *Nucl.Phys.*, B744:121–135, 2006.

[144] Y. Schroder and M. Steinhauser. Four-loop decoupling relations for the strong coupling. *JHEP*, 0601:051, 2006.

[145] R.P. Kauffman, S.V. Desai, and D. Risal. Production of a Higgs boson plus two jets in hadronic collisions. *Phys.Rev.*, D55:4005–4015, 1997.

[146] R.P. Kauffman, S.V. Desai, and D. Risal. Amplitudes for Higgs bosons plus four partons. 1999.

[147] V. Del Duca, A. Frizzo, and F. Maltoni. Higgs boson production in association with three jets. *JHEP*, 0405:064, 2004.

[148] L.J. Dixon, E.W.N. Glover, and V.V. Khoze. MHV rules for Higgs plus multi-gluon amplitudes. *JHEP*, 0412:015, 2004.

[149] S.D. Badger, E.W.N. Glover, and V.V. Khoze. MHV rules for Higgs plus multi-parton amplitudes. *JHEP*, 0503:023, 2005.

[150] S.D. Badger and E.W.N. Glover. One-loop helicity amplitudes for H → gluons: The All-minus configuration. *Nucl.Phys.Proc.Suppl.*, 160:71–75, 2006.

[151] S.D. Badger, E.W.N. Glover, and K. Risager. One-loop phi-MHV amplitudes using the unitarity bootstrap. *JHEP*, 0707:066, 2007.

[152] E.W.N. Glover, P. Mastrolia, and C. Williams. One-loop phi-MHV amplitudes using the unitarity bootstrap: The General helicity case. *JHEP*, 0808:017, 2008.

[153] R. E. Cutkosky. Singularities and Discontinuities of Feynman Amplitudes. *J. Math. Phys.*, 1:429, 1960.

[154] L. M. Brown and R. P. Feynman. Radiative Corrections to Compton Scattering. *Phys. Rev.*, 85:231, 1952.

[155] G. Passarino and M. J. G. Veltman. One-loop corrections for e^+e annihilation into $\mu^+\mu$ in the Weinberg model. *Nucl. Phys. B*, 160:151, 1979.

[156] J. A. M. Vermaseren W. L. van Neerven. Large loop integrals. *Phys. Lett. B*, 137:241, 1984.

[157] R. Pittau. A Simple method for multileg loop calculations. 2. A General algorithm. *Comput.Phys.Commun.*, 111:48–52, 1998.

[158] S. Weinzierl. Reduction of multileg loop integrals. *Phys.Lett.*, B450:234–240, 1999.

[159] R. Britto, F. Cachazo, and B. Feng. Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills. *Nucl.Phys.*, B725:275–305, 2005.

[160] D. Forde. Direct extraction of one-loop integral coefficients. *Phys.Rev.*, D75:125019, 2007.

[161] P. Mastrolia. On Triple-cut of scattering amplitudes. *Phys.Lett.*, B644:272–283, 2007.

[162] P. Mastrolia. Double-Cut of Scattering Amplitudes and Stokes' Theorem. *Phys.Lett.*, B678:246–249, 2009.

[163] Z. Bern, L.J. Dixon, and D.A. Kosower. On-shell recurrence relations for one-loop QCD amplitudes. *Phys.Rev.*, D71:105013, 2005.

[164] C.F. Berger, Z. Bern, L.J. Dixon, D. Forde, and D.A. Kosower. Bootstrapping One-Loop QCD Amplitudes with General Helicities. *Phys.Rev.*, D74:036009, 2006.

[165] Z. Bern and A.G. Morgan. Massive loop amplitudes from unitarity. *Nucl.Phys.*, B467:479–509, 1996.

[166] Z. Bern, L.J. Dixon, D.C. Dunbar, and D.A. Kosower. One loop selfdual and N=4 superYang-Mills. *Phys.Lett.*, B394:105–115, 1997.

[167] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia. Unitarity cuts and Reduction to master integrals in d dimensions for one-loop amplitudes. *JHEP*, 0703:111, 2007.

[168] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia. D-dimensional unitarity cut method. *Phys.Lett.*, B645:213–216, 2007.

[169] S.D. Badger. Direct Extraction Of One Loop Rational Terms. *JHEP*, 0901:049, 2009.

[170] J.M. Campbell, R.K. Ellis, and C. Williams. Hadronic production of a Higgs boson and two jets at next-to-leading order. *Phys.Rev.*, D81:074023, 2010.

[171] R. Kleiss, W.J. Stirling, and S.D. Ellis. A New Monte Carlo Treatment of Multiparticle Phase Space at High-energies. *Comput.Phys.Commun.*, 40:359, 1986.

[172] R. D. Ball, V. Bertone, S. Carrazza, C. S. Deans, L. Del Debbio, et al. Parton distributions with LHC data. *Nucl.Phys.*, B867:244–289, 2013.

[173] V. Ravindran, J. Smith, and W.L. Van Neerven. Next-to-leading order QCD corrections to differential distributions of Higgs boson production in hadron hadron collisions. *Nucl.Phys.*, B634:247–290, 2002.

[174] C. Balazs and C.P. Yuan. Higgs boson production at the LHC with soft gluon effects. *Phys.Lett.*, B478:192–198, 2000.

[175] G. Bozzi, S. Catani, D. de Florian, and M. Grazzini. Transverse-momentum resummation and the spectrum of the Higgs boson at the LHC. *Nucl.Phys.*, B737:73–120, 2006.

[176] V. Ahrens, T. Becher, M. Neubert, and L. L. Yang. Renormalization-Group Improved Prediction for Higgs Production at Hadron Colliders. *Eur.Phys.J.*, C62:333–353, 2009.

[177] D. de Florian, G. Ferrera, M. Grazzini, and D. Tommasini. Transverse-momentum resummation: Higgs boson production at the Tevatron and the LHC. *JHEP*, 1111:064, 2011.

[178] D. de Florian, G. Ferrera, M. Grazzini, and D. Tommasini. Higgs boson production at the LHC: transverse momentum resummation effects in the $H \rightarrow 2\gamma$, $H \rightarrow WW \rightarrow \ell\nu\ell\nu$ and $H \rightarrow ZZ \rightarrow 4\ell$ decay modes. *JHEP*, 1206:132, 2012.