On defects in affine Toda field theory

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Abstract

This thesis outlines methods for generating new integrable defects in affine Toda field theory. These methods are grounded in the hypothesis that defects have a particle-like classification with as many species of fundamental defect existing in a particular affine Toda theory as there are species of soliton. The methods employed are:

1. Defect fusing rules, linking different species of defect in the same theory. Defect fusing rules are used in this thesis to find transmission matrices for a new, species 2, defect in $a_3^{(1)}$.

2. Folding, linking defects in simply laced theories to defects in non-simply laced theories. Folding is used in this thesis to find defects in the $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ affine Toda field theories.

Declaration

The work presented in this thesis is based on research carried out in the Department of Mathematical Sciences, Durham University, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapters 2,3 and 4 review the background material that underlies the research presented in this thesis. Chapter 6 is based on my paper *Folding defect affine Toda field theories* [Rob14*a*], published in J. Phys. **A**. Chapters 5 and 7 are based on my paper *Defect fusing rules in affine Toda field theory* [Rob14*b*], also published in J. Phys. **A**.

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1

Overview

This chapter gives an overview of the aims and content of the thesis.

The main aim of this thesis is to further the understanding of defects in affine Toda field theory (ATFT), expanding on the ideas presented in [Rob14*a*] and [Rob14*b*]. Previous consideration of defects in ATFT has been *ad hoc* in nature and what this thesis attempts to do is make steps towards systematising the study of defects in ATFT by means of defect fusing rules and the folding of defect ATFTs. This necessitates the identification of already known defects [BCZ04*b*] as fundamental defects of definite species.

The original research is concentrated in chapters 5, 6 and 7. Chapters 2, 3 and 4 are based on the work of others and are intended to provide the necessary background for chapters 5, 6 and 7. The material in chapters 2, 3 and 4 should therefore not be considered original, although some results such a the folded multisolitons of section 3.1.2.1 and the species r defect transmission matrices (4.41) and (4.42) (for $r > 2$) do not appear in the literature of other authors. Chapter 4 builds on chapter 3, which in turn builds on chapter 2.

Chapter 2 introduces affine Toda field theory (ATFT) at the Lagrangian level and explains how the Lagrangians relate to the Lie algebra root spaces. The folding of the $a_r^{(1)}$ series of ATFTs is then outlined.

Chapter 3 explains solitons in ATFT, particularly in the $a_r^{(1)}$ theories. The chapter begins by introducing solitons in classical ATFT using the Hirota tau function approach originally used by Hollowood [Hol92]. The multisolitons and fusing rules of the $a_r^{(1)}$ solitons are then considered, before the folding of $a_r^{(1)}$ solitons is addressed. Quantum $a_r^{(1)}$ solitons and their scattering is then explained along with the the soliton fusing rules at the quantum level.

Chapter 4 extends the analysis to ATFT with defects. Defects are introduced at the Lagrangian level before the classical transmission of solitons is explained; multiple defect systems are then outlined. Defects are then considered in the quantum theory in terms of the T-matrix, which describes the transmission of solitons through defects.

Chapter 5 explains the fusing rules of defects at the classical level. The $a_2^{(1)}$ ATFT is considered first in some detail followed by more general considerations. The main quantity of interest in this chapter is the classical delay factor a soliton receives when transmitted through a defect.

Chapter 6 is concerned with the construction of defects that do not spoil classical integrability when the theory is folded. Again $a_2^{(1)}$ is used as a motivating case before defects in general $a_r^{(1)}$, that can be folded, are considered. The soliton delay factors are used in tandem with momentum conservation to support the belief that the defects remain classically integrable after folding.

Chapter 7 considers the idea of defect fusing at the quantum level. How defect fusing fits into the Faddeev–Zamolodchikov algebra is explained and $a_2^{(1)}$ is again considered explicitly. The fusing rules are then applied in finding transmission matrices for the species 2 defect of $a_3^{(1)}$ before an attempt is made to apply the defect fusing rules to $a_5^{(1)}$.

Chapter 8 summarises the findings of the thesis and considers various possible extensions of this work.

Appendix A considers a many-defect system in a_1, and is not an integral part of this work.

The ordering of the research chapters 5, 6 and 7 separates the presentation of the classical (chapter 5) and quantum (chapter 7) defect fusing rules, with the folding of defect ATFTs in between. This has been done because the analysis of chapter 5 provides an approach to chapter 6 and because chapters 5 and

6 are both classical and use similar methodology. These chapters are roughly in order of increasing sophistication.

2

Affine Toda field theory

Affine Toda field theory (hereafter often abbreviated to ATFT) is a class of relativistic integrable field theory living in 1+1 dimensions. The study of ATFT has a long history [AFZ79, MOP81, Wil81, OT83a, OT83b, DS84] and has gone through many stages of development such as the construction of S-matrices [BCDS89, BCDS90] and fusing rules [Dor92] in the real coupling theory; the discovery of solitons [Hol92] and soliton S-matrices [Hol93a]; the construction of integrable boundary conditions [GZ94, BCDR95]; and the discovery of integrable defects [KL99, BCZ04a, BCZ04b]. This thesis aims only to cover the background relevant to the research contained within, so little is said about many of these developments. Detailed reviews of many of the pre-defect developments can be found in previous theses such as [Dor90, McG94b, Hal94, Isk95, Har96, Per99].

2.1 Definition

A 1+1 dimensional affine Toda field theory can be associated to each affine Dynkin diagram [OT83b]. ATFT may be thought of as a generalisation of sine-Gordon theory, with sine-Gordon theory associated to the $a_1^{(1)}$ (often referred to as a_1) root data. The Lagrange density (usually referred to as the 'Lagrangian') describing the theory is given by [BCDS89]

$$\mathcal{L} = \tfrac{1}{2}\dot{u} \cdot \dot{u} - \tfrac{1}{2}u' \cdot u' - U(u) \tag{2.1}$$

with potential

$$U(u) = \frac{m^2}{\beta^2} \sum_{j=0}^{r} n_j \left(e^{\beta\alpha_j \cdot u} - 1 \right) \ . \tag{2.2}$$

Equations (2.1) with (2.2) describe an affine Toda field u living in the root space of the underlying Lie algebra (for an introduction to root and weight spaces see [Cah06]), where the root space is of rank r. The *positive simple roots* are $\{\alpha_i\}$ for $i = 1, \dots, r$ and the *marks* $\{n_i\}$ are a characteristic of the underlying algebra. The additional root α_0 is the lowest root in the root space and is given by $\alpha_0 = -\sum_{j=1}^{r} n_j \alpha_j$, with the convention that $n_0 = 1$. The parameter m in (2.2) sets a mass scale; while β is the coupling constant.

The equation of motion comes from applying the Euler–Lagrange equation to the Lagrangian giving

$$\ddot{u} - u'' = -U_u = -\frac{m^2}{\beta} \sum_{j=0}^{r} n_j \alpha_j e^{\beta\alpha_j \cdot u} \ . \tag{2.3}$$

In (2.3), U_u denotes the gradient of the potential U with respect to the argument u - similar notation is used later when considering defects. The equation of motion (2.3) will, in the context of defects, be referred to as the 'bulk' equation of motion.

The energy and momentum associated to the field u is found using the stress tensor $T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu u)} \partial^\nu u - \eta^{\mu\nu}\mathcal{L}$, where $\eta^{00} = 1$, $\eta^{01} = \eta^{10} = 0$ and $\eta^{11} = -1$. The energy is given by the integral of T^{00}, so

$$E = \int_{-\infty}^{\infty} T^{00} \, \mathrm{d}x = \int_{-\infty}^{\infty} \tfrac{1}{2}\dot{u} \cdot \dot{u} + \tfrac{1}{2}u' \cdot u' + U(u) \, \mathrm{d}x \tag{2.4}$$

while the momentum is given by the integral of T^{01}, so

$$P = \int_{-\infty}^{\infty} T^{01} \, \mathrm{d}x = \int_{-\infty}^{\infty} \dot{u} \cdot u' \, \mathrm{d}x \ . \tag{2.5}$$

When the coupling β is real it can be seen in the potential (2.2) that there is only one vacuum, where u is the zero vector in root space, when u is restricted to be real. Real coupling ATFT thus leads to particles which are excitations from the vacuum, referred to here as the fundamental excitations. The masses of the fundamental excitations can be read off from the quadratic part of the expansion of the potential (2.2), and form the components the right Perron–Frobenius eigenvector of the (non-affine) Cartan matrix in question [BCDS90, Fre91]. Since the Lorentz group in 1+1 dimensions consists only of boosts, the particle energy depends only upon the mass and the rapidity of the particle. The real coupling ATFT is well understood, with exact scattering matrices known for all instances of root data [BCDS90, Kha97], but is not the case of relevance to this thesis.

When the coupling is imaginary the potential (2.2) no longer possesses just one vacuum, but has many real vacua when $u = \frac{2\pi}{|\beta|}\lambda$ where λ is any weight of the algebra, satisfying $\lambda \cdot \alpha_i \in \mathbb{Z}$ for all $i = 0, 1, \ldots, r$. There exists in this case the possibility of the field u taking a different vacuum value at $x = -\infty$ to the one at $x = \infty$, suggesting the existence of soliton solutions. There are indeed soliton solutions [Hol92]. Solitons in ATFT are the topic of chapter 3.

2.1.1 $a_r^{(1)}$ affine Toda theory

All of the work in this thesis centres around the $a_r^{(1)}$ series of affine Toda field theories. These form the simplest and best understood series of ATFTs, particularly as they possess the simplest integrable defects, as discussed in chapter 4.

The $a_r^{(1)}$ affine Toda field theory has all of its marks equal to unity, $n_i = 1$ for $i = 0, 1, \ldots, n$, and all roots of the same length (i.e., it is 'simply laced'), conventionally taken to be $|\alpha_i| = \sqrt{2}$ (conventions often differ for a_1, i.e., sine-Gordon theory). The inner products of the roots are given by the (affine) Cartan matrix, which for $a_r^{(1)}$ is given by

$$C_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} = \alpha_i \cdot \alpha_j = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

where the roots are labelled modulo $h = r+1$, where h is the Coxeter number of the a_r Lie algebra. It is clear that the Cartan matrix for $a_r^{(1)}$ may be re-written as $\alpha_i \cdot \alpha_j = 2\delta_{ij} - \delta_{i(j+1)} - \delta_{i(j-1)}$ with the indices identified modulo the Coxeter

$a_r^{(1)}$

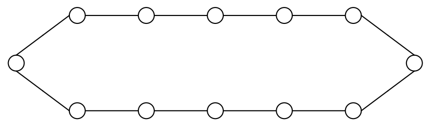

Figure 2.1: The Dynkin diagram of $a_{11}^{(1)}$.

number. The information contained in the Cartan matrix is equivalent to the (Kac–)Dynkin diagram of the theory. Figure 2.1 shows the Dynkin diagram for $a_{11}^{(1)}$. The white nodes denote the roots of length $\sqrt{2}$ while the single lines between nodes denote an inner product of -1 between the corresponding roots. Every $a_r^{(1)}$, apart from a_1, has as its Dynkin diagram a similar picture to figure 2.1 consisting of a loop of $r+1$ nodes.

The fundamental highest weights, $\{\lambda_i\}$, are defined by

$$\lambda_i \cdot \alpha_j = \delta_{ij} , \qquad i, j = 1, \ldots, r$$

with $\lambda_0 = 0$. The fundamental highest weights therefore form a dual basis to $\{\alpha_1, \ldots, \alpha_r\}$. Note then the relation

$$\alpha_i = 2\lambda_i - \lambda_{i-1} - \lambda_{i+1} .$$

The field u in $a_r^{(1)}$ has r components and a convenient basis is needed in later calculations. A convenient non-orthogonal, but useful, basis is given by using the positive simple roots as basis vectors - this basis naturally appears in considering the Hirota tau functions of $a_r^{(1)}$ solitons [Hol92]. So u is given by

$$u = u_1\alpha_1 + u_2\alpha_2 + \ldots + u_r\alpha_r$$

with $u_0 = 0$. Note then that $u_i = \lambda_i \cdot u$.

2.2 Folding $a_r^{(1)}$ affine Toda field theory

Many of the semi-simple Lie algebras have discrete reflection symmetries in their root space. The projection of the the roots onto the invariant subspace under such a symmetry is called *folding* or *reduction*. This can be seen at the level of the Dynkin diagrams, e.g., figure 2.2.

For an ATFT, folding means ensuring that the field u lies in the invariant subspace of the discrete folding symmetry. Folding is a tool that allows the properties of non-simply laced theories (i.e., where not all of the roots have the same length) to be deduced using the properties of a larger ranked (usually simply laced) theory which may be easier to work with. All of the simply laced ATFTs, with the exception of a_1 and e_8, can be folded to get a non-simply laced ATFT [OT83*a*, BCDS90, KS96*b*] and some of the non-simply laced theories can be folded still further. The only ATFTs dealt with in this thesis are the $a_r^{(1)}$ theories and their descendants through folding. All of the foldings of $a_r^{(1)}$ used here are similar in that they all come from making pairwise identifications on the roots of $a_r^{(1)}$, hence may be referred to as 'bivalent' foldings. There exist further foldings in $a_r^{(1)}$ [Sas92, KS96*b*], but these all have redundancies for the purpose of constructing defects.

2.2.1 Canonical folding

The canonical folding of $a_r^{(1)}$ refers to the folding process outlined in [OT83*a*]. When r is odd (or equivalently, h is even) $a_r^{(1)}$ may be folded to a member of the c series of ATFTs. Folding at the level of the algebra is illustrated by figure 2.2. In figure 2.2 the black nodes represent roots of unit length while the white nodes of $c_n^{(1)}$ represent α'_0 and α'_n which have length $\sqrt{2}$. The information in the $c_n^{(1)}$ Dynkin diagram is equivalent to the $c_n^{(1)}$ Cartan matrix. The roots of $c_n^{(1)}$, $\{\alpha'_i\}$, are given by the identification on the roots of $a_{2n-1}^{(1)}$, $\{\alpha_i\}$ via

$$\alpha'_i = \frac{\alpha_i + \alpha_{h-i}}{2} \tag{2.6}$$

where $h = 2n$ is the Coxeter number of a_{2n-1}. Note then that there are two self-identified roots, $\alpha'_0 = \alpha_0$ and $\alpha'_n = \alpha_n$. The marks of $c_n^{(1)}$ are $n_0 = n_n = 1$, $n_i = 2$ for $i = 1, \ldots, n-1$.

The aim is to fold the $a_{2n-1}^{(1)}$ ATFT such that the Lagrangian (2.1) and potential (2.2) describing an $a_{2n-1}^{(1)}$ field $u = \sum_{j=1}^{2n-1} u_j \alpha_j$ reduces to a Lagrangian describing a $c_n^{(1)}$ field $\phi = \sum_{j=1}^{n} \phi_j \alpha'_j$. For the folding to make sense and (2.6) to hold it follows that the component of u proportional to α_i is the same as

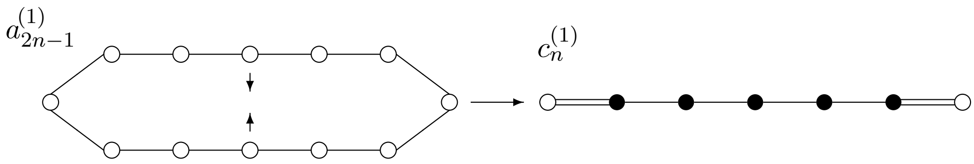

Figure 2.2: $a_{2n-1}^{(1)} \rightarrow c_n^{(1)}$.

the component of u proportional to α_{h-i}, so

$$\begin{aligned} u_i = u_{h-i} &= \frac{\phi_i}{2} \qquad\qquad \text{for } i = 1, \ldots, n{-}1 \\ u_n &= \phi_n \ . \end{aligned}$$

With the above identifications one can easily show that the Lagrangian (2.1) and potential (2.2) describe a $c_n^{(1)}$ field.

2.2.2 Non-canonical folding

The non-canonical folding of $a_r^{(1)}$ and the other affine Dynkin diagrams was described by Khastgir and Sasaki [Sas92, KS96*b*] but they did not develop it further, even where it might have been appropriate [Kha97]. Two series of non-simply laced folded theory arise from these considerations: $d_n^{(2)}$ and $a_{2n}^{(2)}$. These series can be obtained by the canonical foldings $d_{n+1}^{(1)} \rightarrow d_n^{(2)}$ and $d_{2n+2}^{(1)} \rightarrow a_{2n}^{(2)}$, but this is not useful in finding new defects as no $d_s^{(1)}$ defects are known. The non-canonical folding processes of use are

- $a_{2n-1}^{(1)} \rightarrow d_n^{(2)}$: This folding is non-canonical in the sense of [KS96*b*]. It is illustrated by figure 2.3 where the black-in-white nodes represent roots of length $\frac{1}{\sqrt{2}}$ [1]. The root space identification required is

$$\alpha_i' = \frac{\alpha_i + \alpha_{h+1-i}}{2}$$

  where $h = 2n$ and α_0 is chosen to be the lower-left root in figure 2.3.

[1] The conventional normalisation for $d_n^{(2)}$ can be achieved by rescaling the roots $\alpha_i' \rightarrow \sqrt{2}\alpha_i'$. The affine Toda potential obtained from this folding is also non-standard, being twice the conventional potential - this factor of two can be removed in the action by an isotropic space-time rescaling: $x \rightarrow \frac{x}{\sqrt{2}}$, $t \rightarrow \frac{t}{\sqrt{2}}$.

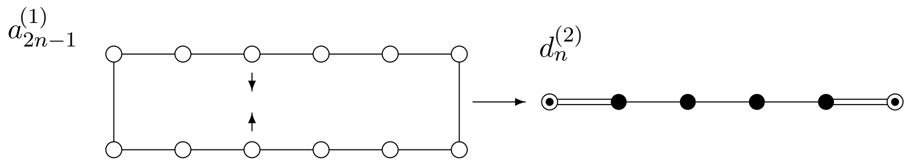

Figure 2.3: $a_{2n-1}^{(1)} \to d_n^{(2)}$.

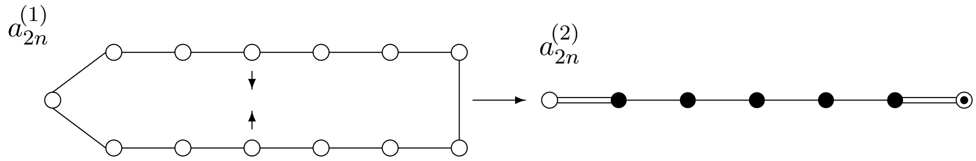

Figure 2.4: $a_{2n}^{(1)} \to a_{2n}^{(2)}$.

The $a_{2n-1}^{(1)}$ affine Toda field is folded to the $d_n^{(2)}$ affine Toda field by setting

$$\begin{aligned} u_{i+1} = u_{h-i} &= \frac{\phi_i}{2} \quad \text{for } i = 1, \dots, n-1 \\ u_1 &= 0\,. \end{aligned}$$

- $a_{2n}^{(1)} \to a_{2n}^{(2)}$: This case is illustrated by figure 2.4, with the root space identification

$$\alpha_i' = \frac{\alpha_i + \alpha_{h-i}}{2}$$

where the Coxeter number is now $h = 2n+1$.

The $a_{2n}^{(1)} \to a_{2n}^{(2)}$ folding is now achieved by setting

$$u_i = u_{h-i} = \frac{\phi_i}{2} \quad \text{for } i = 1, \dots, n\,.$$

Examples of each class of folding are given in chapter 6 where defects are constructed for the folded theories.

2.2.3 Folding *a priori* and *a posteriori*

There is an issue in folding ATFT which has seen little attention, possibly because it presents no problems in the regular ATFT without defects. There are two ways in which the folding can be achieved. The identifications of sections 2.2.1 and 2.2.2 can be applied at the Lagrangian level before taking the equation of motion - what may be referred to as *a priori* (i.e., prior to taking equations of motion) folding. The other way to fold is to take the equation of motion (2.3) before making the identifications - what may be referred to as *a posteriori* (i.e., after taking equation of motion) folding.

The major difference between the two kinds of folding is the number of equations obtained when the equation of motion (2.3) is considered in component form. Consider for concreteness the folding $a_3^{(1)} \to c_2^{(1)}$. *A priori* folding gives two equations for the components of ϕ, the $c_2^{(1)}$ field; while *a posteriori* folding gives three equations for u components, which become three equations for ϕ components. It turns out that the extra equation is a duplicate of one of the other equations, so both methods of folding are equivalent. This extends to every ATFT. However, in the presence of defects the equivalence of *a priori* and *a posteriori* folding is much less clear cut - this issue is addressed briefly in chapter 4.

2.3 Integrability

The first sentence of this chapter states that affine Toda field theory is a class of *integrable* field theory. Integrable field theories are exactly solvable (in principle, at least). At the classical level the integrability of the theory manifests itself in the existence of infinitely many conserved charges in involution. For any given root data these charges can be generated using a Lax pair approach. The Lax pair approach can be extended to include ATFT with boundaries [BCDR95] and ATFT defects [BCZ04*b*]. The existence of solitons which preserve their form after collisions is another indication of classical integrability. At the quantum level the exact solvability of the theory is articulated by the existence of exact scattering matrices (S-matrices) and transmission matrices (T-matrices). These matrices describe the evolution of the system from time $t = -\infty$ through to time $t = \infty$.

3

Solitons in affine Toda field theory

This chapter introduces solitons in ATFT. More general introductions to solitons can be found in the books [DJ89] and [MS04]. As is noted in chapter 2, the potential (2.2) for ATFT has multiple real vacua when the coupling constant β is imaginary[2]. One can assume that β is imaginary in what follows.

This chapter splits broadly into two parts: the classical picture of ATFT solitons; and the quantum picture. The analysis of the classical picture is used in chapters 5 and 6 while the quantum approach is necessary for chapter 7.

3.1 Classical picture

There exist a number of ways to construct the solitons of affine Toda theory. Hollowood was first to construct explicit solutions in $a_r^{(1)}$ and $d_4^{(1)}$ [Hol92], where he used a Hirota tau function [Hir80] approach. This work was extended by MacKay and McGhee to obtain tau functions for the single solitons in all of the other ATFTs [MM93, McG94*b*]. Further developments with the tau function method are found in [McG94*a*, ZC95, HIM95]. General construction of solitons can also be made using the algebraic methods of Olive, Turok and Underwood [OTU93*a*, OTU93*b*]; while for $a_r^{(1)}$ one can construct solitons via Bäcklund transformation [LOT93] or inverse scattering [BJ97, BJ98] methods.

[2] All of the soliton charges are real [Fre95] but the value of the soliton field u is not everywhere real.

The approach taken in this thesis, classically, is the tau function approach. For a general ATFT the soliton solutions may be written in the form [MM93]

$$u = -\frac{1}{\beta} \sum_{j=0}^{r} \eta_j \alpha_j \ln \tau_j \tag{3.1}$$

where $\eta_j = \frac{2}{\alpha_j \cdot \alpha_j}$ with no sum implied and the ATFT corresponds to an affine Dynkin diagram with $r+1$ nodes. The existence of $r+1$ tau functions for an r component field u can be explained using the conformal affine Toda models [CFGZ93]. The components of u in the basis discussed in section 2.1.1 are then

$$u_i = -\frac{1}{\beta} \ln \left(\frac{\tau_i^{\eta_i}}{\tau_0^{\eta_0 \, n_i}} \right) . \tag{3.2}$$

Substituting the ansatz (3.1) into the affine Toda Lagrangian (2.1) and potential (2.2) one can see that the tau functions $\{\tau_i\}$ have no dependence on the coupling β. The tau functions depend upon which soliton or multisoliton solution is being considered which of course depends on the theory under consideration. In order to find the tau functions (3.1) is used in the equation of motion (2.3) along with a decoupling [Hol92]. In general the equation to be solved is

$$\eta_i \left(\ddot{\tau}_i \tau_i - \dot{\tau}_i^2 - \tau_i'' \tau_i + \tau_i'^2 \right) - m^2 n_i \left(\prod_{j=0}^{r} \tau_j^{-\eta_j \alpha_j \cdot \alpha_i} - 1 \right) \tau_i^2 = 0 \, .$$

Note that the kinetic terms are always in Hirota bilinear form $\ddot{\tau}_i \tau_i - \dot{\tau}_i^2 - \tau_i'' \tau_i + \tau_i'^2 = \frac{1}{2} \left(D_t^2 - D_x^2 \right) \tau_i \cdot \tau_i$, where the Hirota derivatives are defined by [MM93] $D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n f(x,t) g(x',t')|_{x=x',t=t'}$; but the potential is only in true bilinear form for $a_r^{(1)}$. This sets $a_r^{(1)}$ apart as it is therefore much easier to write down the multisoliton solutions in that case.

The approach to solving for the tau functions is to take an ansatz such as

$$\tau_j = 1 + \delta_j^{(1)} E + \delta_j^{(2)} E^2 + \ldots + \delta_j^{(N)} E^{N_j}$$

where each $\delta_j^{(k)}$ is taken to be an arbitrary constant while E for a single soliton solution is given by $E = e^{ax-bt+c}$, where a, b and c are arbitrary constants with $\frac{b}{a} = \tanh\theta$, where θ is the soliton rapidity. The tau functions can then be solved order by order in E. The smallest value of N_j for which the series terminates describes the single solitons, and it is only for $a_r^{(1)}$ that solitons can be found that are linear in E for all values of j.

3.1.1 Solitons in $a_r^{(1)}$

For the case of $a_r^{(1)}$ the marks are all equal to unity, $n_i = 1$ for all i, while the theory is simply laced, so $\eta_i = 1$ for all i. Thus, (3.1) reduces to

$$u = -\frac{1}{\beta}\sum_{j=0}^{r} \alpha_j \ln \tau_j$$

while the equation the tau functions must obey simplifies greatly to

$$\ddot{\tau}_i\tau_i - \dot{\tau}_i^2 - \tau_i''\tau_i + \tau_i'^2 = m^2\left(\tau_{i-1}\tau_{i+1} - \tau_i^2\right) \ . \tag{3.3}$$

There are r species of single soliton (fundamental soliton) in $a_r^{(1)}$, and as such the solitons can be associated to the nodes on the $a_r^{(1)}$ Dynkin diagram, with the zero solution (which may be thought of as a species 0 soliton) associated to the α_0 node[3]. The tau functions of the one-soliton solution, of species p, in $a_r^{(1)}$ are of the form

$$\tau_j = 1 + \omega^{pj}E_p \ . \tag{3.4}$$

In (3.4), $\omega = e^{\frac{2\pi i}{h}}$ and $p \in \{1, \dots, r\}$ (note that $p = 0$ denotes the trivial solution), so ω^p encompasses the $(r+1)$-th roots of unity ($h = r + 1$). The spacetime dependence of the soliton is found in

$$E_p = e^{a_p x - b_p t + c_p} \ . \tag{3.5}$$

In (3.5), c_p is constant. The imaginary part of c_p determines the topological

[3] Note that there is just one species of sine-Gordon soliton, the antisoliton is another soliton (of the same species) having opposite topological charge.

charge of the soliton under consideration - there are up to h sectors of differing topological charge [McG94*a*], as is discussed in section 3.1.1.2. The real part of c_p fixes the position of centre of mass of the soliton at time $t = 0$ and may be chosen arbitrarily. The other quantities in (3.5) are given by $a_p = m_p \cosh\theta$ and $b_p = m_p \sinh\theta$ where θ is the soliton rapidity - it is clear then that $\mathrm{Re}(\theta) > 0$ for a right-moving soliton. The quantity m_p is given by

$$m_p = 2m \sin\left(\frac{\pi p}{h}\right) \ . \tag{3.6}$$

The quantity m_p is proportional to the mass of the soliton. It can be seen that m_p^2 is m^2 times the p-th eigenvalue of the $a_r^{(1)}$ Cartan matrix. Most values of m_p appear twice, with only $m_0 = 0$ and, when $r = 2n - 1$, $m_n = 2m$ being non-degenerate[4]. Note that the species n soliton of $a_{2n-1}^{(1)}$ is an embedded sine-Gordon soliton *classically* as can be seen by reducing the system to a_1, but in the quantum theory the species n soliton has many possibilities for its topological charge so does not correspond to the quantum sine-Gordon soliton.

3.1.1.1 Energy and momentum of single solitons

The energy and momentum of the solitons can be found using a construction originating in [OTU93*a*] and applied in a similar way to here in [HIM95], involving the observation that for solitons, in any ATFT, the stress tensor may be written in the form

$$T^{\mu\nu} = \left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) C$$

such that

$$T^{00} = -C'' \qquad\qquad T^{11} = -\ddot{C} \qquad\qquad T^{01} = T^{10} = +\dot{C}' \ .$$

In particular this means that

$$\partial^2 C = T^{00} - T^{11} = 2U = \frac{2m^2}{\beta^2}\sum_{j=0}^{r} n_j \left(e^{\beta\alpha_j\cdot u} - 1\right)$$

[4] It should be noted that there is no zero-mass soliton in the theory: $p = 0$ refers to the trivial solution (hence $m_0 = 0$) and m_0 is included with the soliton masses as the trivial solution is connected to soliton solutions via the soliton fusing rules.

which gives, up to a constant, the solution

$$C = \frac{2}{\beta^2} \sum_{j=0}^{r} \eta_j \ln \tau_j \; .$$

Having C is useful as it allows the energy and momentum to be calculated easily. That is

$$\begin{aligned} E &= \int_{-\infty}^{\infty} T^{00} \mathrm{d}x = -\left[C'\right]_{x=-\infty}^{\infty} \\ P &= \int_{-\infty}^{\infty} T^{01} \mathrm{d}x = \left[\dot{C}\right]_{x=-\infty}^{\infty} \; . \end{aligned}$$

For the species p soliton of $a_r^{(1)}$ the energy and momentum are therefore given by

$$E = -\frac{2h}{\beta^2} m_p \cosh\theta \qquad\qquad P = -\frac{2h}{\beta^2} m_p \sinh\theta$$

where it should be borne in mind that the coupling is imaginary, and as such the energy and momentum are positive for a right-moving soliton. The mass of the species p soliton then follows from the on-shell condition and is given by

$$M_p = \frac{2h}{|\beta|^2} m_p \; . \tag{3.7}$$

Using the semi-classical methods of Dashen, Hasslacher and Neveu [DHN75], Hollowood [Hol93*b*] showed that the classical mass ratios of the $a_r^{(1)}$ solitons are preserved under first-order ($O(\beta^2)$) corrections.

3.1.1.2 Topological charges of single solitons

The topological charge associated to the field u is

$$Q = \frac{\beta}{2\pi i} \int_{-\infty}^{\infty} u' \, \mathrm{d}x = \frac{\beta}{2\pi i} \left(\lim_{x\to\infty} - \lim_{x\to-\infty} \right) u(x,t) \; .$$

Using the form of the soliton solution (3.1), (3.2), the topological charge becomes

$$Q = -\frac{1}{2\pi i} \sum_{j=1}^{r} \left(\lim_{x\to\infty} - \lim_{x\to-\infty} \right) \ln \left(\frac{\tau_j^{\eta_j}}{\tau_0^{\eta_0 n_j}} \right) \alpha_j$$

which, specialising to the species p soliton of $a_r^{(1)}$ reduces to

$$Q = -\frac{1}{2\pi i} \sum_{j=1}^{r} \left(\lim_{x\to\infty} - \lim_{x\to-\infty} \right) \ln \left(\frac{1+\omega^{pj} E_p}{1+E_p} \right) \alpha_j \ .$$

For the soliton solution to make sense the situation of $\tau_j = 0$ must be avoided for all $j = 0, 1, \ldots, r$. This has the effect of splitting the topological charge of the soliton into h sectors, with $\mathrm{Im}(c_p)$ forbidden to take the values $\frac{2\pi k}{h}$ for integer k. The topological charge does not change within a sector and only changes between sectors [McG94*a*]. The 'highest' charge found for the species p soliton, in the first sector, is given by [McG94*a*]

$$Q_p^{(1)} = \sum_{j=0}^{r} \frac{p(h-j) \bmod h}{h} \alpha_j \tag{3.8}$$

while the other classical charges arise from letting c_p go to $c_p - \frac{2\pi i}{h}$, which has the same effect as mapping $\alpha_i \to \alpha_{i+1}$ [McG94*a*], with labels taken modulo h.

As an example, consider the fundamental solitons of $a_3^{(1)}$. For the species 1 soliton the highest charge from (3.8) is

$$Q_1^{(1)} = \tfrac{3}{4}\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{4}\alpha_3$$

which is the highest weight of the **4** representation of a_3, which will be denoted here as l_1. The other charges arise in applying $\alpha_i \to \alpha_{i+1}$, resulting in

$$\begin{aligned} Q_1^{(2)} &= \tfrac{3}{4}\alpha_2 + \tfrac{1}{2}\alpha_3 + \tfrac{1}{4}\alpha_0 = -\tfrac{1}{4}\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{4}\alpha_3 = l_2 \\ Q_1^{(3)} &= -\tfrac{1}{4}\alpha_1 - \tfrac{1}{2}\alpha_2 + \tfrac{1}{4}\alpha_3 = l_3 \\ Q_1^{(4)} &= -\tfrac{1}{4}\alpha_1 - \tfrac{1}{2}\alpha_2 - \tfrac{3}{4}\alpha_3 = l_4 \end{aligned}$$

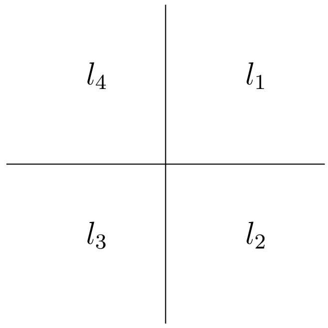

Figure 3.1: Argument of e^{c_p} and soliton charge for species 1 soliton of $a_3^{(1)}$.

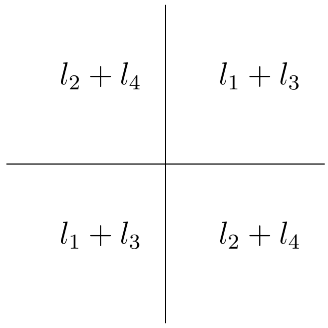

Figure 3.2: Soliton charge sectors for species 2 soliton of $a_3^{(1)}$.

where they have been labelled l_2, l_3 and l_4 as they match the other weights of the **4** representation of a_3. The topological charges of the species 1 soliton thus fill the corresponding representation of a_3. The charge sectors are illustrated by figure 3.1.

For the species 3 soliton similar analysis gives $Q_3^{(1)} = -l_4$, $Q_3^{(2)} = -l_3$, $Q_3^{(3)} = -l_2$ and $Q_3^{(4)} = -l_1$, where l_i is defined as above. The species 3 soliton charges fill the weight space of the $\bar{\mathbf{4}}$ representation of a_3.

The species 2 soliton displays a long-standing problem with the classical ATFT solitons. The charges found by these methods are $Q_2^{(1)} = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_3 = l_1 + l_3 = Q_2^{(3)}$ and $Q_2^{(2)} = -\frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_3 = l_2 + l_4 = Q_2^{(4)}$, as illustrated in figure 3.2. Thus only two of the six weights of the **6** representation appear as topological charges (and the highest charge is not the highest weight). This is symptomatic of a more general trend, that most solitons classically have 'missing charges' [McG94*a*, McG94*b*], although the species 1 and species r solitons of $a_r^{(1)}$ do have charges that fill their corresponding representations.

3.1.1.3 Multisolitons of $a_r^{(1)}$

Constructing the tau functions of the single solitons of the other affine Toda theories is laborious, so few multisoliton solutions have even been constructed [MM93, McG94*b*, Hal94]. In $a_r^{(1)}$, owing to (3.3) being bilinear in form, the general soliton solution can be written as [Hol92]

$$\tau_j = \sum_{\mu_1=0}^{1} \cdots \sum_{\mu_N=0}^{1} \exp\left[\sum_{i=1}^{N} \mu_i \ln\left(\omega^{p_i j} E_{p_i}\right) + \sum_{1\le i<j\le N} \mu_i\mu_j \ln\left(A^{(p_i p_j)}\right)\right] . \quad (3.9)$$

In (3.9), there are thus pairwise interaction terms, which are given by

$$A^{(p_1 p_2)} = -\frac{(a_{p_1} - a_{p_2})^2 - (b_{p_1} - b_{p_2})^2 - m^2_{p_1-p_2}}{(a_{p_1} + a_{p_2})^2 - (b_{p_1} + b_{p_2})^2 - m^2_{p_1+p_2}} \quad (3.10)$$

where $a_{p_1} = m_{p_1}\cosh\theta_1$, $a_{p_2} = m_{p_2}\cosh\theta_2$, etc., and $m_{p_1+p_2}$ is given by substituting $p_1 + p_2$ for p in (3.6). Notice that the interaction parameter vanishes if both p_1 and p_2 are the same and both solitons possess the same rapidity.

Consider, in particular, a two-soliton solution consisting of a soliton of species p_1 with rapidity θ_1 and a soliton of species p_2 with rapidity θ_2. The tau functions are

$$\tau_j = 1 + \omega^{p_1 j} E_{p_1} + \omega^{p_2 j} E_{p_2} + A^{(p_1 p_2)} \omega^{(p_1+p_2)j} E_{p_1} E_{p_2} . \quad (3.11)$$

In equation (3.11), there is a single interaction parameter $A^{(p_1p_2)}$, defined as in (3.10). The tau functions (3.11) can be analysed to reveal how the solitons interact. If $\theta_1 \neq \theta_2$ the solitons will be far apart as $t \to \pm\infty$. Consider the case of $\theta_1 > \theta_2$ in the frame of the species p_1 soliton (so $\theta_1 = 0$). As $t \to -\infty$, the tau function resembles the one soliton solution $\tau_j \sim 1 + \omega^{p_1 j} E_{p_1}$. As $t \to \infty$, such that the solitons have interacted, $\tau_j \sim 1 + A^{(p_1p_2)}\omega^{p_1 j} E_{p_1}$. The interaction parameter, $A^{(p_1p_2)}$, can thus be incorporated into E_{p_1} as the time delay/advance caused by the soliton interaction.

Note than in $a_3^{(1)}$, a species 2 soliton gives the same interaction parameter (and hence delay) to a species 1 soliton as it does to a species 3 soliton of the same rapidity. Something similar is seen with defects in chapter 5.

3.1.1.4 Fusing rules

An interesting property of the solitons of ATFT is that they possess fusing rules akin to the fusing rules of the fundamental excitations [Dor92]. These fusing rules were examined algebraically in [OTU93*b*] and explicitly in the tau function approach in [Hal94]. The fusing rules allow the properties of all of the fundamental solitons to be determined in terms of a minimal set of 'basic' solitons and in the case of $a_r^{(1)}$ only one basic soliton is needed, the species 1 soliton (one could equally take the species r soliton as the basic soliton). At the level of the tau functions, the fusing process reduces a two-soliton solution (in the form of (3.11) for $a_r^{(1)}$) to a one-soliton solution (in the form of (3.4) for $a_r^{(1)}$).

The fusing process is now demonstrated in $a_r^{(1)}$. By redefining $E_{p_1} \to \left(A^{(p_1p_2)}\right)^{-\frac{1}{2}} E_{p_1}$ and $E_{p_2} \to \left(A^{(p_1p_2)}\right)^{-\frac{1}{2}} E_{p_2}$ the tau functions (3.11) become

$$\tau_j = 1 + \omega^{p_1 j} \left(A^{(p_1p_2)}\right)^{-\frac{1}{2}} E_{p_1} + \omega^{p_2 j} \left(A^{(p_1p_2)}\right)^{-\frac{1}{2}} E_{p_2} + \omega^{(p_1+p_2)j} E_{p_1} E_{p_2} \,. \tag{3.12}$$

The aim is to reduce this to a one-soliton solution and the first step is to remove the linear terms, which occurs when $A^{(p_1p_2)}$ has a pole. By examining (3.10) and making use of (3.6) this is seen to happen when $\theta_1 - \theta_2 = \pm \frac{i\pi(p_1+p_2)}{h}$, i.e., the fusing angle is $\frac{\pi(p_1+p_2)}{h}$, as it is for fusing species p_1 and species p_2 fundamental excitations.

When the rapidities of the solitons are judiciously chosen to equal

$$\theta_1 = \theta + \frac{i\pi p_2}{h} \qquad\qquad \theta_2 = \theta - \frac{i\pi p_1}{h}$$

the tau functions (3.12) reduce to

$$\tau_j = 1 + \omega^{(p_1+p_2)j} E_{p_1+p_2}(\theta)$$

where $E_{p_1+p_2}(\theta) = E_{p_1}(\theta + \frac{i\pi p_2}{h}) E_{p_2}(\theta - \frac{i\pi p_1}{h}) = e^{a_{p_1+p_2}x - b_{p_1+p_2}t + c_{p_1} + c_{p_2}}$. This is clearly in the form of a one-soliton solution, (3.4), of species $p = p_1 + p_2$. Note that if $p_1 + p_2 = 0 \bmod h$, then what happens is soliton-antisoliton annihilation.

One thing to note about the fusing process is that it does not generally conserve topological charge. For example, in $a_3^{(1)}$ one can take two species 1 solitons with topological charges l_1 and l_2, and the resulting species 2 soliton has a topological charge of either $l_1 + l_3$ or $l_2 + l_4$, see section 3.1.1.2.

3.1.2 Folding $a_r^{(1)}$ solitons

Folding is the tool that was used to get explicit solutions for the tau functions of the non-simply laced theories [MM93, McG94*b*]. One merely has to construct a solution in the appropriate simply laced theory which has the symmetry of the reduced theory.

In $a_{2n-1}^{(1)}$ the requirement that the field has the folded symmetry, as explained in section 2.2, is that $u_i = u_{h-i}$ (for folding to $c_n^{(1)}$) or $u_i = u_{h+1-i}$ (folding to $d_n^{(2)}$). In folding $a_{2n}^{(1)}$ to $a_{2n}^{(2)}$ the requirement is[5] $u_i = u_{h-i}$. In light of (3.2), the condition $u_i = u_{h-i}$ is equivalent to $\tau_i = \tau_{h-i}$. This cannot be realised by any one-soliton solution of $a_r^{(1)}$(except for when $p = n$ in $a_{2n-1}^{(1)}$), (3.4), but can be done with a two soliton solution.

To find the tau functions for single solitons of species p in the folded theories take a two soliton $a_r^{(1)}$ solution with $p_1 = p$ and $p_2 = h - p$ in (3.11). In order for the two solitons to be thought of as a single soliton after folding it is a physical requirement that they must already constitute one entity. The requirement is that the solitons are 'combined'. What is meant by 'combined' is that the two $a_r^{(1)}$ solitons are given the same centre of mass at time $t = 0$, meaning that $\mathrm{Re}(c_{p_1}) = \mathrm{Re}(c_{p_2})$; and that the solitons retain the same centre of mass as each other at other times (the single folded soliton does not dissociate into two separate solitons), so must possess the same rapidity, $\theta_1 = \theta_2$. This means that E_{p_1} and E_{p_2} may differ only in $\mathrm{Im}(c)$.

When $\theta_1 = \theta_2$, the interaction parameter (3.10) becomes

$$A^{(p(h-p))} = \cos^2\left(\frac{\pi p}{h}\right) \equiv A$$

[5] In what follows there is an assumption that the folding is to $c_n^{(1)}$ or $a_{2n}^{(2)}$. If folding to $d_n^{(2)}$ replace h with $h+1$ in the powers of ω and let $A \to A\omega^p$.

	$c_n^{(1)}$	$d_n^{(2)}$	$a_{2n}^{(2)}$
τ_0'	τ_0	$(\tau_1)^{\frac{1}{2}}$	τ_0
τ_i' $i \neq 0, n-1, n$	τ_i	τ_{1+i}	τ_i
τ_{n-1}'	τ_{n-1}	$(\tau_n)^{\frac{1}{2}}$	τ_{n-1}
τ_n'	τ_n	$-$	$(\tau_n)^{\frac{1}{2}}$

Table 3.1: Tau function identifications in the folded theories.

and so the tau functions compatible with folding (to a one soliton folded solution) possess the form[6]

$$\tau_j = 1 + \left(\omega^{pj} + \omega^{p(h-j)}\right) E_p + AE_p^2 \ . \tag{3.13}$$

It can be shown that these folded solitons are the same as those found in [McG94*b*] with the identifications in table 3.1. To match the tau functions for $d_n^{(2)}$ and $a_{2n}^{(2)}$ to those in [McG94*b*] one must first shift E_p such that the quadratic term in the tau functions becomes just E_p^2.

3.1.2.1 Folded multisolitons

It is only for $a_r^{(1)}$ that a formula like (3.9) is known. Once the basic tau functions for one soliton solutions in the folded theories are known, multisoliton solutions in these models may be constructed directly; however, this requires knowledge of the generally complicated interaction parameters of the folded model. This problem may be obviated by instead constructing these multisolitons in the $a_r^{(1)}$ model. This was noted in [ZC95], but was only applied to $c_n^{(1)}$. This means that the analysis here is original for $d_n^{(2)}$ and $a_{2n}^{(2)}$. In particular, the two soliton

[6] For the case of $d_n^{(2)}$ this becomes $\tau_j = 1 + \left(\omega^{pj} + \omega^{p(h+1-j)}\right) E_p + A\omega^p E_p^2$.

solution in the folded theory takes the form

$$\begin{aligned}
\tau_j = 1 &+ \left(\omega^{p_1 j} + \omega^{p_1(h-j)}\right) E_{p_1} + \left(\omega^{p_2 j} + \omega^{p_2(h-j)}\right) E_{p_2} + A^{(12)} E_{p_1}^2 + A^{(34)} E_{p_2}^2 \\
&+ A^{(13)} \left(\omega^{p_1 j + p_2 j} + \omega^{p_1(h-j)+p_2(h-j)}\right) E_{p_1} E_{p_2} \\
&+ A^{(14)} \left(\omega^{p_1 j + p_2(h-j)} + \omega^{p_1(h-j)+p_2 j}\right) E_{p_1} E_{p_2} \\
&+ A^{(12)} A^{(13)} A^{(14)} \left(\omega^{p_2 j} + \omega^{p_2(h-j)}\right) E_{p_1}^2 E_{p_2} \\
&+ A^{(34)} A^{(13)} A^{(14)} \left(\omega^{p_1 j} + \omega^{p_1(h-j)}\right) E_{p_1} E_{p_2}^2 \\
&+ A^{(12)} A^{(34)} \left(A^{(13)}\right)^2 \left(A^{(14)}\right)^2 E_{p_1}^2 E_{p_2}^2 \, . \qquad (3.14)
\end{aligned}$$

Note that (3.14) contains four interaction parameters - a fact that is not obvious, should one wish to construct folded solitons using the folded theory as a starting point.

Using that $a_{p_1} = m_{p_1} \cosh\theta_1, a_{p_2} = m_{p_2} \cosh\theta_2, b_{p_1} = m_{p_1} \sinh\theta_1, b_{p_2} = m_{p_2} \sinh\theta_2$ and denoting the two rapidities by $\theta_1 = \theta + \psi$ and $\theta_2 = \theta - \psi$ gives, in particular, the interaction parameter

$$A^{(13)} = \frac{m_{p_1 - p_2}^2 + (m_{p_1} + m_{p_2})^2 \sinh^2\psi - (m_{p_1} - m_{p_2})^2 \cosh^2\psi}{(m_{p_1} + m_{p_2})^2 \cosh^2\psi - (m_{p_1} - m_{p_2})^2 \sinh^2\psi - m_{p_1 + p_2}^2} \, . \qquad (3.15)$$

Among the two soliton solutions there are two interesting cases that can occur when the the relative rapidity $\theta_1 - \theta_2 = 2\psi$ between the solitons is imaginary:

- The solitons possess fusing rules, which are just $a_r^{(1)}$ fusing rules. Fusion of the solitons occurs when the denominator of $A^{(13)}$ in equation (3.15) vanishes (one should first make the redefinitions $E_{p_1} \to \left(A^{(13)}\right)^{-\frac{1}{2}} E_{p_1}$ and $E_{p_2} \to \left(A^{(13)}\right)^{-\frac{1}{2}} E_{p_2}$ in equation (3.14)). This occurs when $\psi = \pm i \frac{\pi(p_1+p_2)}{2(r+1)} \equiv \pm i\tilde{\psi}$. The resulting tau functions describe a species $p = p_1 + p_2$ single folded soliton with rapidity $\tilde{\theta} = \theta + i\frac{\pi(p_1-p_2)}{2(r+1)}$.
- The existence of breather solutions in ATFT has been known for some time [OTU93*b*] and solutions have been considered in Hirota form for $a_r^{(1)}$ [HIM95] and $d_4^{(1)}$ [Isk95]. For equation (3.14) to describe a folded breather the constituent solitons must be of the same species, $p_1 = p_2$, with the same centre of mass $\mathrm{Re}(c_{p_1}) = \mathrm{Re}(c_{p_2})$ and with an imaginary rapidity difference which must be less than the fusing angle. Note that

in the cases of $d_n^{(2)}$ and $a_{2n}^{(2)}$ these breather tau functions are also the tau functions of particular breathers in $d_s^{(1)}$.

3.2 Quantum picture

There are a number of differences between the solitons in the classical and the quantum picture. One difference is that the quantum solitons do not have missing charges. A consistency requirement of the proposed soliton scattering matrices is that the species p soliton of $a_r^{(1)}$ can take any of the weights of the p-th fundamental representation as its topological charge [Hol93a], which is not usually the case in the classical theory [McG94a]. The focus here is on the scattering of $a_r^{(1)}$ fundamental solitons, for a more broad review of quantum affine Toda solitons, see [Mac].

3.2.1 Faddeev–Zamolodchikov algebra

Quantum integrable field theories are specified by their exact scattering matrices (S-matrices). In two dimensions these theories have the special property of factorised scattering [ZZ79], meaning that all scattering processes can be broken down into two-particle processes.

The quantum scattering of $a_r^{(1)}$ affine Toda solitons is well described by the Faddeev–Zamolodchikov (FZ) algebra [ZZ79, Fad80]. The FZ algebra is an associative algebra which describes S-matrices through the use of non-commuting creation operators. Let the species p soliton of $a_r^{(1)}$, carrying a topological charge labelled by i, moving at rapidity θ be denoted by

$$ {}^{p}A_i(\theta) \ . \tag{3.16} $$

The quantity i in (3.16) ranges through the number of weights in the p-th fundamental representation of a_r, as it is a label for one of those weights. For the species 1 soliton the label i corresponds to a topological charge equal to the weight $l_i = -\frac{1}{h}\sum_{j=i}^{i+r} j\alpha_j$, where the roots are labelled modulo h. For the species r soliton the label i is taken to mean a charge of $-l_i$ [CZ07, CZ09a].

In the limit $t \to -\infty$ in a two-soliton solution, the overlap of the soliton wavefunctions will become negligible provided that the solitons are moving with different rapidities. If $\theta_1 > \theta_2$ (or at least $\mathrm{Re}(\theta_1) > \mathrm{Re}(\theta_2)$) then the soliton possessing the rapidity θ_1 will be far to the left of the soliton with rapidity θ_2 as $t \to -\infty$. The two-soliton state in the far past can thus be represented by

$$ {}^{p_1}A_j(\theta_1)\, {}^{p_2}A_k(\theta_2) \ . \tag{3.17}$$

Equation (3.17) represents a species p_1 soliton with charge label j and a species p_2 soliton with charge label k, with $\theta_1 > \theta_2$. In the limit $t \to \infty$, with $\theta_1 > \theta_2$, the solitons will be in the opposite order with negligible overlap, so the two-soliton state will be represented by

$$ {}^{p_2}A_m(\theta_2)\, {}^{p_1}A_n(\theta_1) $$

where the soliton topological charges may have changed from what they were at $t = -\infty$.

The S-matrix is the quantity that relates the states in the far past to the states in the far future. Some of the properties of the S-matrix are detailed in section 3.2.2. The S-matrix in a two-particle scattering process in an relativistic integrable field theory depends upon the difference in the rapidities of the particles, so the scattering process is given in the FZ algebra by

$$ {}^{p_1}A_j(\theta_1)\, {}^{p_2}A_k(\theta_2) = {}^{p_1p_2}S_{jk}^{mn}(\theta_1 - \theta_2)\, {}^{p_2}A_m(\theta_2)\, {}^{p_1}A_n(\theta_1) \ . \tag{3.18}$$

The S-matrix in (3.18), ${}^{p_1p_2}S_{jk}^{mn}(\theta_1 - \theta_2)$ depends on the species of solitons involved (p_1 and p_2) and the soliton topological charges. Overall topological charge is conserved, so if both solitons were of species 1 that would mean $l_j + l_k = l_m + l_n$. The fact that two particles always scatter to two particles is a consequence of the integrability of the theory [Dor].

3.2.2 The S-matrix

The scattering of affine Toda solitons, by factorisation, depends only upon the two-particle S-matrices. These S-matrices can be written down exactly due to their integrability. Introductions to exact S-matrices can be found in [Dor] and [Mus10].

The S-matrices for $a_r^{(1)}$ solitons were originally postulated by Hollowood [Hol93a]. One of the key properties of the S-matrices is that they satisfy the Yang–Baxter equation. The Yang–Baxter equation is a condition of factorisability and for the case of three solitons, as illustrated in figure 3.3 with $\theta_1 > \theta_2 > \theta_3$, the equation is

$$S_{ij}^{mn}(\theta_1-\theta_2)S_{nk}^{lt}(\theta_1-\theta_3)S_{ml}^{rs}(\theta_2-\theta_3) = S_{jk}^{lm}(\theta_2-\theta_3)S_{il}^{rn}(\theta_1-\theta_3)S_{nm}^{st}(\theta_1-\theta_2) \quad (3.19)$$

where l, m and n are summed over the appropriate possibilities. The species labels on the S-matrices in (3.19) have been suppressed, as the Yang–Baxter equation must hold for any set of soliton species. Overall topological charge is conserved though care should be taken over the meaning of the indices when different soliton species are involved: when all solitons are of species 1 the topological charge conservation means that $l_i + l_j + l_k = l_r + l_s + l_t$.

While the Yang–Baxter equation puts constraints on the S-matrix (or the R-matrix), it does not have any power to constraint the scalar prefactor, $\rho(\theta)$ of the S-matrix, as the same prefactors appear on both sides of the equation (3.19). Two constraints that do constrain the prefactor are:

- Unitarity, giving

$$S_{jk}^{mn}(\theta_1-\theta_2)S_{mn}^{ts}(\theta_2-\theta_1) = \delta_j^t\delta_k^s\ . \quad (3.20)$$

- Crossing symmetry, as illustrated in figure 3.4, where viewing the process with time running upwards and viewing it with time running left-to-right are equivalent.

$${}^{p_1p_2}S_{jk}^{st}(\theta_1-\theta_2) = {}^{p_1(h-p_2)}S_{j\bar{s}}^{\bar{k}t}(i\pi-\theta_1+\theta_2)\ . \quad (3.21)$$

  In (3.21) it is important to note the species of the solitons, as the alternative viewpoint sees one of them as an antisoliton. The label $\bar{k}$ indicates the opposite topological charge to the label k.

Further conditions may be found by requiring the S-matrix to be consistent with the bootstrap principle. This alludes to the soliton fusing rules of section

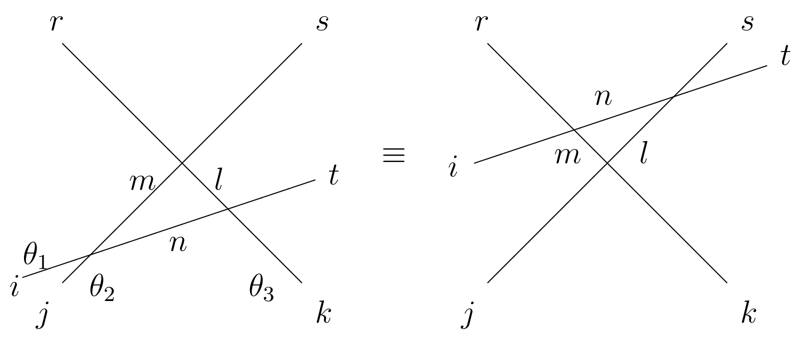

Figure 3.3: Illustration of Yang–Baxter relations for three solitons, given by equation (3.19).

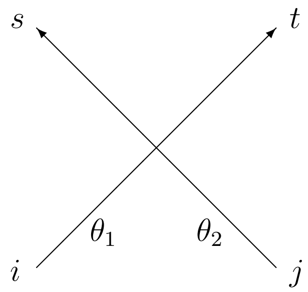

Figure 3.4: Crossing relation for solitons, given in equation (3.21).

3.2.3.

Applying the Yang–Baxter relation, crossing and unitarity to the species 1 soliton S-matrix gives [CZ07]

$$\begin{aligned}
{}^{11}S^{jj}_{jj}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2)\left(Q^{-1}X - QX^{-1}\right) \\
{}^{11}S^{kj}_{jk}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2)\left(X - X^{-1}\right) \qquad j \neq k \\
{}^{11}S^{jk}_{jk}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2)\left(Q^{-1} - Q\right)\begin{cases} X^{(1-\frac{2|l|}{h})}\big|_{l=j-k<0} \\ X^{-(1-\frac{2|l|}{h})}\big|_{l=j-k>0} \end{cases}
\end{aligned} \tag{3.22}$$

where $X = e^{\frac{h\gamma(\theta_1-\theta_2)}{2}}$ and $Q = -e^{i\pi\gamma}$, where the coupling dependence comes through $\gamma = \frac{4\pi}{\beta^2} - 1$, with β the (bulk) coupling of the theory. The prefactor ρ

is given by [Hol93a]

$$\rho(\Theta) = \frac{\Gamma(1+\frac{h\gamma i\Theta}{2\pi})\Gamma(1-\frac{h\gamma i\Theta}{2\pi}-\gamma)}{2\pi i}\frac{\sinh(\frac{\Theta}{2}+\frac{i\pi}{h})}{\sinh(\frac{\Theta}{2}-\frac{i\pi}{h})}$$
$$\times\prod_{k=1}^{\infty}\frac{F_k(\Theta)F_k(\frac{2\pi i}{h}-\Theta)}{F_k(\frac{2\pi i}{h}+\Theta)F_k(2\pi i-\Theta)}$$

where

$$F_k(\Theta) = \frac{\Gamma(1+\frac{h\gamma i\Theta}{2\pi}+hk\gamma)}{\Gamma(\frac{h\gamma i\Theta}{2\pi}+(hk+1)\gamma)} .$$

One can use the same conditions; the Yang–Baxter relations (3.19), unitarity (3.20) and crossing (3.21); to obtain the S-matrices for any species of solitons, but the full expression is generally much more unwieldy than (3.22). An exception to this is the S-matrix for scattering two species r solitons. If the labels on ${}^{11}S$ are that k denotes the charge l_k, and the labels on ${}^{rr}S$ are that k denotes the charge $-l_k$, then it is the case that ${}^{rr}S^{mn}_{jk}(\theta_1-\theta_2) = {}^{11}S^{kj}_{nm}(\theta_1-\theta_2)$. The other S-matrices are best obtained using the fusing rules.

3.2.3 S-matrices and fusing rules

The fusing rules of an ATFT allow all of the soliton S-matrices of the theory to be written in terms of a few basic S-matrices. For the case of $a_r^{(1)}$ there is just one basic S-matrix needed with the usual choice being that of scattering two species 1 solitons, which is given by (3.22).

There is an important caveat though when considering the fusing rules of $a_r^{(1)}$. It was shown by Saleur and Wehefritz-Kaufmann [SW00] that when bound states of solitons (breathers) are taken into account the bootstrap does not close in $a_2^{(1)}$ with the proposed S-matrices (3.22). This should not invalidate the use of the fusing rules in this thesis, where little mention is made of breathers.

The soliton fusing rules in the quantum picture are broadly similar to the fusing rules of the classical picture. The fused soliton is again formed by combining two solitons which have a rapidity difference of i times the fusing angle. There is however one major difference between fusing in the classical and quantum theories and that is that the topological charge is conserved in the quantum fusing process. This means that if a species p_1 and a species p_2 soliton fuse to form a species p soliton, the topological charges of the species p_1 and p_2 solitons must sum up to one of the weights of the p-th fundamental representation. In particular, the charges of the species p_1 and p_2 solitons cannot be the same if fusing is to occur.

The fusing rules can thus be seen at the level of the FZ algebra. for a species 2 soliton of $a_r^{(1)}$ the operator may be written in terms of species 1 operators as

$$ {}^2A_{(jk)}(\theta) = c^{(jk)}\ {}^1A_j(\theta - \tfrac{i\pi}{h})\ {}^1A_k(\theta + \tfrac{i\pi}{h}) + c^{(kj)}\ {}^1A_k(\theta - \tfrac{i\pi}{h})\ {}^1A_j(\theta + \tfrac{i\pi}{h})\ . \tag{3.23} $$

The $\frac{r(r+1)}{2}$ weights, $\{(jk)\}$, of the second fundamental representation of a_r are constructed by taking pairs of weights $\{j,k\}$, such that $j \neq k$, from the r-dimensional first fundamental representation. The charge of the species 2 soliton is then uniquely labelled by $(jk) = (kj)$ and represents a charge of $l_j + l_k$. The equation (3.23) demonstrates that there are only two ways in which the charge (jk) can be obtained by fusing: one species 1 constituent soliton has charge j and the other has charge k. The quantities $\{c^{(jk)}\}$, with $j,k = 1,\ldots,h$, in (3.23) are the soliton fusing couplings for this process, which possess the following properties:

- The couplings only depend on the differences between the arguments, so

$$ c^{(i(i+k))} = c^{(j(j+k))} \tag{3.24} $$

  where i does not have to equal j and $k = 1,\ldots,r$. The labels $i, i+k, j, j+k$ are all to be taken modulo $h = r+1$ but with any label equalling zero written instead as $h = r+1$.

- The coupling ratios are determined by

$$ \frac{c^{(i(j+l))}}{c^{(ij)}} = (-Q)^{-\frac{l}{h}} \tag{3.25} $$

  for $j + l = 1, 2, \ldots, i-1, i+1, \ldots, h$ and assuming that $j \neq i$. One can define $c^{(ii)} = 0$ but the relations (3.25) assume that such a quantity is avoided. The coupling ratios for $a_3^{(1)}$ and $a_5^{(1)}$ are given in chapter 7.

By using the S-matrix for scattering two species 1 solitons, (3.22), and the properties of the soliton fusing couplings, (3.24) and (3.25), the operator (3.23) can be used in the FZ algebra (3.18) to find, with some effort, the S-matrix between a species 1 and a species 2 soliton, given by

$$
\begin{aligned}
{}^{12}S^{(jk)i}_{i(jk)}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2+\tfrac{i\pi}{h})\rho(\theta_1-\theta_2-\tfrac{i\pi}{h})\left(X^2+X^{-2}+Q^{-1}+Q\right)\\
{}^{12}S^{(ik)i}_{i(ik)}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2+\tfrac{i\pi}{h})\rho(\theta_1-\theta_2-\tfrac{i\pi}{h})\left(Q^{-1}X^2+QX^{-2}+Q^{-2}+Q^2\right)\\
{}^{12}S^{(ij)k}_{i(jk)}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2+\tfrac{i\pi}{h})\rho(\theta_1-\theta_2-\tfrac{i\pi}{h})\left(Q^{-1}-Q\right)\begin{cases} X^{-\frac{2|l|}{h}}\left(X^2+Q^{-1}\right)|_{l=i-k<0}\\ X^{\frac{2|l|}{h}}\left(-X^{-2}-Q\right)|_{l=i-k>0}\end{cases}
\end{aligned}
$$

where i, j and k all have different values. Only the last process ${}^{12}S^{(ij)k}_{i(jk)}$ will involve the solitons exchanging topological charge. Compare to ${}^{12}S$ for $a_2^{(1)}$ found in [CZ07].

The fusing process should generalise to involve any soliton species p_1, p_2 and $p = p_1 + p_2$, with p taken modulo h. Similar to (3.23), the species p operator will be

$$
{}^{p}A_i(\theta) = \sum_{j,k} c^{(jk)}_{p_1p_2}\ {}^{p_1}A_j(\theta - \tfrac{i\pi p_2}{h})\ {}^{p_2}A_k(\theta + \tfrac{i\pi p_1}{h})
$$

where the sum is over all possibilities where the topological charges labelled by j and k sum up to the charge labelled by i. The fusing couplings $c^{(jk)}_{p_1p_2}$ may generally have a complicated form.

4

Defects in affine Toda field theory

Over twenty years ago the framework for integrable defects was laid down by Delfino, Mussardo and Simonetti [DMS94*a*, DMS94*b*]. They showed that a non-trivial local integrable quantum field theory in two dimensions can only encompass an impurity and remain integrable if the impurity is either purely reflecting or purely transmitting. This was confirmed in [CFG], but other possibilities exist in a different framework [MRS02].

The case of the purely reflecting impurity - in other words, the integrable boundary - had already been considered in ATFT [GZ94, FK94]. Further developments in affine Toda field theory on the half-line are found in e.g., [FK95, BCDR95, Kim96, DG99, Per99, Doi08].

The case of the purely transmitting impurity, the integrable defect, can be thought of as having an internal boundary, or interface, between two 'bulk' regions. The quantum sine-Gordon defect was first considered by Konik and LeClair [KL99] before the Lagrangian framework was discovered by Bowcock, Corrigan and Zambon [BCZ04*a*], who extended the framework to include $a_r^{(1)}$ defects [BCZ04*b*]. Most of the literature on affine Toda defects has focussed on the sine-Gordon case [KL99, BCZ04*a*, BCZ05, HK08, BS08, Nem10, CZ10, AAGZ11, AD12] but it is the $a_r^{(1)}$ defects for $r \geq 2$ that are of most interest here. See [Cor] for a review of integrable defects in ATFT. Integrable defects have also been studied in other theories such as nonlinear Schrödinger [CMR04, CZ06, AD11], supersymmetric sinh-Gordon theories

[GYZ06, GYZ08] and sigma models [DK13].

This chapter, like chapter 3, is split into two parts - the classical and quantum viewpoints of defects in ATFT. The classical analysis is used mainly in chapters 5 and 6. The quantum properties of the defect are required for chapter 7.

4.1 Classical picture

4.1.1 The type I defects of $a_r^{(1)}$

Bowcock, Corrigan and Zambon attempted to generalise their result for the sine-Gordon defect with the type I ansatz. Placing a defect at $x = 0$, with fields u to the left and v to the right gives

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(\frac{1}{2}uA\dot{u} + uB\dot{v} + vC\dot{v} - D(u,v)\right) \tag{4.1}$$

where $\theta(x)$ is the Heaviside step function; A, B and C are constant matrices which are $r \times r$ if the algebra is of rank r, assuming that the fields u and v are of the same theory[7]; the defect potential is given by D. It is clear that any symmetric parts of A and C can be integrated out of the action, so A and C are antisymmetric. The Lagrangians $\mathcal{L}_u$ and $\mathcal{L}_v$ are both bulk Lagrangians of the form (2.1).

By considering a Lax pair approach taking into account the defect, it was shown that the ansatz (4.1) could only work in the $a_r^{(1)}$ theories, with

$$A = C = 1 - B$$

and

$$D(u,v) = \frac{m}{\beta^2}e^{-\eta}\sum_{j=0}^{r} e^{\frac{\beta}{2}\alpha_j\cdot(B^T u + Bv)} + \frac{m}{\beta^2}e^{\eta}\sum_{j=0}^{r} e^{\frac{\beta}{2}\alpha_j\cdot B(u-v)}\,. \tag{4.2}$$

The parameter η in the defect potential (4.2) is the defect 'rapidity'. Under the action of Lorentz boosts η transforms as a rapidity does [BCZ05] but it should

[7] It is very unlikely that integrability, conserving infinitely many higher-spin charges, could be maintained if u and v belong to different root data.

be noted that a defect may have a non-zero rapidity and still be stationary. The quantity m is the same mass parameter as appears in the bulk affine Toda potential (2.2), while β is the coupling.

In addition to the bulk equations of motion, the Euler–Lagrange equations for (4.1) give conditions at $x = 0$. The Euler–Lagrange equations are

$$\ddot{u} - u'' = -U_u \mid_{x<0}$$
$$\ddot{v} - v'' = -V_v \mid_{x>0}$$
$$u' = A\dot{u} + B\dot{v} - D_u \mid_{x=0} \tag{4.3}$$
$$v' = -A\dot{v} + B^T\dot{u} + D_v \mid_{x=0} \tag{4.4}$$

where U and V are bulk potentials of the form (2.2) for the fields u and v. A subscript u (or v) denotes the gradient of the potential with respect to u (or v) - for the defect potential D_u is the gradient with respect to u whilst keeping v fixed. Something to note about the defect conditions (4.3) and (4.4) is that together they give a Bäcklund transformation (with appropriate identification of B), though fixed at $x = 0$ [BCZ04b].

It was later shown [CZ09a] that demanding that the defect conserves a modified momentum gives the same conditions as the Lax pair approach. The type I defect is then specified by the matrix B, and there are two solutions for B, when $r \geq 2$, which have been developed and are of use here. They are

$$B_1 = 2\sum_{j=1}^{r} (\lambda_j - \lambda_{j+1})\,\lambda_j^T \tag{4.5}$$

and

$$B_r = B_1^T = 2\sum_{j=1}^{r} (\lambda_j - \lambda_{j-1})\,\lambda_j^T \tag{4.6}$$

where $\{\lambda_i\}$ are the fundamental highest weights of $a_r^{(1)}$ which satisfy $\lambda_i \cdot \alpha_j = \delta_{ij}$ for $i, j = 1, \ldots, r$ and $\lambda_0 = 0$. These matrices obey

$$\alpha_i B_1 \alpha_j = \begin{cases} 2 & \text{if } i = j \\ -2 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases} \qquad \alpha_i B_r \alpha_j = \begin{cases} 2 & \text{if } i = j \\ -2 & \text{if } i = j-1 \\ 0 & \text{otherwise} \end{cases} \tag{4.7}$$

where the roots are labelled modulo $h = r + 1$. Note that for either choice $B + B^T = 2$. The labels B_1 and B_r are deliberate, alluding to the existence of a defect 'hierarchy' where the type I defect specified by B_1 is a species 1 defect; the defect specified by B_r is a species r defect. The opposite identification is also possible, as long as the type I defects specified by (4.5) and by (4.6) are associated to the end nodes of the a_r Dynkin diagram. The identification of the type I defects as species 1 and species r is based on two things. One is the fact that they possess the simplest form of Lagrangian known for an affine Toda defect (type I form), somewhat analogous to how the first and final fundamental representations of a_r have the simplest weight spaces. The other reason for this identification is that the species 1 and species r defects are conjugate to each other with B_r being the transpose of B_1; additionally the soliton delay factors of the defects (4.19) and (4.20) are complex conjugates of one another. The existence of different species of defects demands that defect fusing rules be considered - that is the topic of chapters 5 and 7.

4.1.1.1 Topological charge and energy

An important property of defects is that they possess topological charge. Consider the static solution where $u = \frac{2\pi i}{\beta}\lambda$ and $v = \frac{2\pi i}{\beta}\tilde{\lambda}$ with λ and $\tilde{\lambda}$ both weights of a_r, hence minimising the bulk potentials $U(u)$ and $V(v)$ of the form (2.2). The topological charge possessed by the defect is then proportional to the difference of the two vacua, with the charge being $\tilde{\lambda} - \lambda$, which lies in the weight space of the theory. With these solutions for u and v, and using the property that $B = 2 - B^T$, the defect potential becomes

$$D(\tfrac{2\pi i}{\beta}\lambda, \tfrac{2\pi i}{\beta}\tilde{\lambda}) = \frac{m}{\beta^2} e^{-\eta} \sum_{j=0}^{r} e^{2\pi i\left(\frac{1}{2}\alpha_j B^T(\lambda-\tilde{\lambda}) + \alpha_j\cdot\tilde{\lambda}\right)} + \frac{m}{\beta^2} e^{\eta} \sum_{j=0}^{r} e^{2\pi i\left(\frac{1}{2}\alpha_j\cdot B(\lambda-\tilde{\lambda})\right)} .$$

The properties of the weights are such that $\alpha_j\cdot\tilde{\lambda} \in \mathbb{Z}$ for all j; while (4.7) shows, for either choice of B, that if $\lambda - \tilde{\lambda}$ is in the root space then $\frac{1}{2}\alpha_j B^T(\lambda - \tilde{\lambda}) \in \mathbb{Z}$ and $\frac{1}{2}\alpha_j B(\lambda - \tilde{\lambda}) \in \mathbb{Z}$ for all j. The situation of $\lambda - \tilde{\lambda}$ lying in the root space occurs if and only if λ and $\tilde{\lambda}$ are weights of the same representation, so

$$D(\tfrac{2\pi i}{\beta}\lambda, \tfrac{2\pi i}{\beta}\tilde{\lambda}) = \frac{2mh}{\beta^2}\cosh\eta \quad \text{if } \lambda \text{ and } \tilde{\lambda} \text{ are weights of the same representation.}$$

When u and v are proportional to weights in different representations the result is different. Say $\lambda = \lambda_{p_1}$ and $\tilde{\lambda} = \lambda_{p_2}$, where λ_{p_1} and λ_{p_2} are the highest weights of the p_1 and p_2 fundamental representations, in that case the defect potential becomes [CZ07]

$$D(\tfrac{2\pi i}{\beta}\lambda_{p_1}, \tfrac{2\pi i}{\beta}\lambda_{p_2}) = \frac{2mh}{\beta^2}\cosh\left(\eta + \frac{2\pi i(p_1 - p_2)}{h}\right) . \tag{4.8}$$

The energy associated to the defect Lagrangian (4.1) is a modification of (2.4), to account for the defect. The contribution from the bulk is

$$E = \int_{-\infty}^{0} \tfrac{1}{2}\dot{u}\cdot\dot{u} + \tfrac{1}{2}u'\cdot u' + U(u)\,\mathrm{d}x + \int_{0}^{\infty} \tfrac{1}{2}\dot{v}\cdot\dot{v} + \tfrac{1}{2}v'\cdot v' + V(v)\,\mathrm{d}x\ .$$

Under the assumption that the fields approach vacuum values at spatial infinity, the time derivative of the bulk energy is

$$\begin{aligned}\dot{E} &= u'\cdot\dot{u} - v'\cdot\dot{v}\ |_{x=0}\\ &= -\dot{D}\end{aligned}$$

where the defect Euler–Lagrange equations (4.3) and (4.4) have been used. This shows then that the presence of a defect contributes $D(u, v)$ to the conserved energy, since $\frac{\mathrm{d}}{\mathrm{d}t}(E + D) = 0$. The energy is then complex, from (4.8), when u and v lie in different representations suggesting that a defect having u and v in different representations is an unstable excited defect [BCZ05]. The defects of interest in this thesis are the stable ground state defects, which have u and v in the same representation[8]. In the case of a soliton being transmitted through the defect this means that the species of the soliton will not change.

4.1.1.2 Momentum conservation and mass

In a similar vein to how the presence of the defect modifies the energy of the system, there exists a modified conserved momentum. This is perhaps somewhat surprising at first as the defect, situated at $x = 0$, breaks the translation symmetry of the system. Nonetheless, the defect can actually be shown to be classically integrable if there is a conserved momentum [CZ09*a*].

The defect potential can be split into terms of negative helicity and terms

[8] Note that the defect energy is negative when the coupling is imaginary.

of positive helicity. The negative helicity terms are those proportional to $e^{-\eta}$:

$$D^{-}(u,v) = \frac{m}{\beta^2}e^{-\eta}\sum_{j=0}^{r} e^{\frac{\beta}{2}\alpha_j\cdot(B^T u + Bv)} \tag{4.9}$$

while the positive helicity terms are those proportional to e^{η}:

$$D^{+}(u,,v) = \frac{m}{\beta^2}e^{\eta}\sum_{j=0}^{r} e^{\frac{\beta}{2}\alpha_j\cdot B(u-v)}\ . \tag{4.10}$$

The bulk momentum, which can be written as a difference between positive and negative helicity terms, in the presence of a defect at $x = 0$ is given by (see equation (2.5))

$$P = \int_{-\infty}^{0} \dot{u}\cdot u'\,\mathrm{d}x + \int_{0}^{\infty} \dot{v}\cdot v'\,\mathrm{d}x\ .$$

Assuming that the fields approach vacuum at spatial infinity, using the bulk equations of motion one finds that the time derivative of the bulk momentum is

$$\begin{aligned}\dot{P} &= \tfrac{1}{2}\dot{u}\cdot\dot{u} + \tfrac{1}{2}u'\cdot u' - U(u) - \tfrac{1}{2}\dot{v}\cdot\dot{v} - \tfrac{1}{2}v'\cdot v' + V(v)\ |_{x=0}\\ &= -\left(\dot{D}^{-} - \dot{D}^{+}\right)\end{aligned}$$

where in going from the first to second lines the defect conditions (4.3) and (4.4) have been used, D^{-} and D^{+} are given by (4.9) and (4.10) respectively. This implies a conserved momentum of $P + D^{-} - D^{+}$. In vacuum the defect thus contributes a momentum of

$$D^{-} - D^{+} = -\frac{2mh}{\beta^2}\sinh\eta\ . \tag{4.11}$$

It is clear then that the on-shell condition, with energy (4.8) and momentum (4.11), gives a defect mass of

$$\mathcal{M} = \frac{2mh}{|\beta^2|}\ . \tag{4.12}$$

The expression (4.12) holds in both the case of the species 1 defect (4.5) and the species r defect (4.6), so they are mass degenerate. Compare this to how the species 1 and species r solitons are mass degenerate (see equations (3.6) and (3.7)). Indeed, many of the properties of defects identified in this thesis have solitonic analogies.

4.1.1.3 Soliton transmission through type I defects

As noted in section 4.1.1.1, the defects of interest in this thesis are ground state defects. The vacua on either side of the defect are in the same representation. Consider then a one-soliton solution of species p having positive rapidity $\theta > 0$. In the far past the soliton is far to the left of the defect and as it evolves it will reach the defect and be transmitted as a soliton of the same species but possibly with a change in topological charge and a time delay or advance.

The effect of the defect on a one-soliton solution, of species p and rapidity θ, can be found by using the defect conditions (4.3) and (4.4) and the tau function ansätze

$$u = -\frac{1}{\beta}\sum_{j=0}^{r}\alpha_j \ln \tau_j^u \qquad\qquad v = -\frac{1}{\beta}\sum_{j=0}^{r}\alpha_j \ln \tau_j^v \tag{4.13}$$

where

$$\tau_j^u = 1 + \omega^{pj} E_p \qquad\qquad \tau_j^v = 1 + {}^p_q z(\theta,\eta)\, \omega^{pj} E_p \, . \tag{4.14}$$

The quantity ${}^p_q z$, the delay factor, encompasses the time delay and change of topological charge that the soliton experiences when transmitted through a species q defect (for the purposes of this chapter q is either 1 or r). The delay factor can be absorbed into E_p in the tau functions of v as a change to c_p. Recall from section 3.1.1 that c_p determines the centre of mass and topological charge of the soliton.

Take the defect to be a species 1 defect, which is specified by
$B_1 = 2\sum_{j=1}^{r}(\lambda_j - \lambda_{j+1})\,\lambda_j^T$. The first of the defect conditions (4.3) becomes

$$u' = (1 - B_1)\dot{u} + B_1\dot{v} - D_u^{(1)} \tag{4.15}$$

evaluated at $x = 0$. The potential $D^{(1)}$ is given by (4.2) with B_1 in place of B. Taking $\alpha_k \cdot$ (4.15) with the ansätze (4.13) gives

$$\begin{aligned} &2\frac{\tau_k^{u\prime}}{\tau_k^u} - \frac{({\tau_{k+1}^u}' - \dot{\tau}_{k+1}^u)}{\tau_{k+1}^u} - \frac{({\tau_{k-1}^u}' + \dot{\tau}_{k-1}^u)}{\tau_{k-1}^u} - 2\frac{\dot{\tau}_k^v}{\tau_k^v} + 2\frac{\dot{\tau}_{k-1}^v}{\tau_{k-1}^v} \\ &+ me^{-\eta}\left(\frac{\tau_k^u \tau_{k-2}^v}{\tau_{k+1}^u \tau_{k+1}^v} - \frac{\tau_{k+1}^u \tau_{k-1}^v}{\tau_k^u \tau_k^v}\right) + me^{\eta}\left(\frac{\tau_k^u \tau_{k+1}^v}{\tau_{k+1}^u \tau_k^v} - \frac{\tau_{k-1}^u \tau_k^v}{\tau_k^u \tau_{k-1}^v}\right) = 0 \ . \end{aligned} \tag{4.16}$$

For the species 1 defect the second defect condition (4.4) becomes

$$v' = -(1 - B_1)\dot{v} + B_1^T \dot{u} + D_v^{(1)} \ . \tag{4.17}$$

Taking $\alpha_k \cdot$ (4.17) gives

$$\begin{aligned} &2\frac{\tau_k^{v\prime}}{\tau_k^v} - \frac{({\tau_{k+1}^v}' + \dot{\tau}_{k+1}^v)}{\tau_{k+1}^v} - \frac{({\tau_{k-1}^v}' - \dot{\tau}_{k-1}^v)}{\tau_{k-1}^v} - 2\frac{\dot{\tau}_k^u}{\tau_k^u} + 2\frac{\dot{\tau}_{k+1}^u}{\tau_{k+1}^u} \\ &+ me^{-\eta}\left(\frac{\tau_{k+1}^u \tau_{k-1}^v}{\tau_k^u \tau_k^v} - \frac{\tau_{k+2}^u \tau_k^v}{\tau_{k+1}^u \tau_{k+1}^v}\right) - me^{\eta}\left(\frac{\tau_k^u \tau_{k+1}^v}{\tau_{k+1}^u \tau_k^v} - \frac{\tau_{k-1}^u \tau_k^v}{\tau_k^u \tau_{k-1}^v}\right) = 0 \ . \end{aligned} \tag{4.18}$$

By summing (4.16) and (4.18), and using the tau function forms (4.14), one can then show, order by order in E_p, that the delay factor is given by

$${}^p_1z(\theta - \eta) = \frac{ie^{\eta-\theta} + \omega^{\frac{p}{2}}}{ie^{\eta-\theta} + \omega^{-\frac{p}{2}}} \ . \tag{4.19}$$

Note that the delay factors given by (4.19) are independent of the mass parameter m, which becomes a common factor in (4.16) and (4.18), and of the bulk coupling β. Classically one can scale out m and β from the Lagrangian (4.1) and the same delay factors are obtained.

The effects of the transmission on the topological charge can be seen through

$$\tan(\arg({}^p_1z(\theta - \eta))) = \frac{\sin(\frac{2\pi p}{h})}{e^{2\eta-2\theta} + \cos(\frac{2\pi p}{h})} \ .$$

It is then clear that $\arg({}^p_1z(\theta - \eta)) \in [0, \frac{2\pi p}{h}]$. For the species 1 soliton this means that either the topological charge is unchanged or 'raised' to the next sector: $l_i \to l_{i-1}$ (with $l_1 \to l_{r+1}$).

The same analysis can be carried out for the species r defect, specified by (4.6). The delay factor in that case is given by

$$ {}^{p}_{r}z(\theta-\eta) = \frac{ie^{\eta-\theta} - \omega^{-\frac{p}{2}}}{ie^{\eta-\theta} - \omega^{\frac{p}{2}}} \ . \tag{4.20} $$

The argument of ${}^{p}_{r}z(\theta-\eta)$ is the negative of the argument of ${}^{p}_{1}z(\theta-\eta)$; so classically a species 1 soliton either retains its topological charge or has it 'lowered' by one sector, so $l_i \to l_{i+1}$ (with $l_{r+1} \to l_1$).

If u is a two-soliton solution the solitons interact with the defect independently, so if, for example, u consisted of a species p_1 soliton with rapidity θ_1 and a species p_2 soliton with rapidity θ_2, such that

$$ \tau^u_j = 1 + \omega^{p_1 j} E_{p_1} + \omega^{p_2 j} E_{p_2} + A^{(p_1 p_2)} \omega^{(p_1+p_2)j} E_{p_1} E_{p_2} $$

then after being transmitted through a species 1 defect, the tau functions are

$$ \begin{aligned} \tau^v_j = 1 &+ {}^{p_1}_{1}z(\theta_1-\eta)\, \omega^{p_1 j} E_{p_1} + {}^{p_2}_{1}z(\theta_2-\eta)\, \omega^{p_2 j} E_{p_2} \\ &+ {}^{p_1}_{1}z(\theta_1-\eta)\, {}^{p_2}_{1}z(\theta_2-\eta)\, A^{(p_1 p_2)} \omega^{(p_1+p_2)j} E_{p_1} E_{p_2} \ . \end{aligned} $$

This continues to hold even when the solitons have been 'combined' - see section 3.1.2.

4.1.1.4 Combining defects

There is nothing to forbid the situation of having multiple defects in a system. Consider the system with two defects - each can independently be of any species, but for this argument will be taken to be of the same species. The Lagrangian describing two defects of the same species, one at $x = 0$ and one at $x = a < 0$, is

$$ \begin{aligned} \mathcal{L} = \ &\theta(a-x)\mathcal{L}_u + \theta(x-a)\theta(-x)\mathcal{L}_\chi + \theta(x)\mathcal{L}_v \\ &+ \delta(a-x)\left(\tfrac{1}{2}uA\dot{u} + uB\dot{\chi} + \tfrac{1}{2}\chi A\dot{\chi} - D(u,\chi)\right) \\ &+ \delta(x)\left(\tfrac{1}{2}\chi A\dot{\chi} + \chi B\dot{v} + \tfrac{1}{2}vA\dot{v} - \tilde{D}(\chi,v)\right) \ . \end{aligned} \tag{4.21} $$

The bulk is split into three regions by the two defects, with χ denoting the field in the middle region. The defect conditions from (4.21) are:

$$\begin{aligned} u' &= A\dot{u} + B\dot{\chi} - D_u \,|_{x=a} \\ \chi' &= -A\dot{\chi} + B^T\dot{u} + D_\chi \,|_{x=a} \end{aligned} \tag{4.22}$$

$$\begin{aligned} \chi' &= A\dot{\chi} + B\dot{v} - \tilde{D}_\chi \,|_{x=0} \\ v' &= -A\dot{v} + B^T\dot{\chi} + \tilde{D}_v \,|_{x=0} \ . \end{aligned} \tag{4.23}$$

In section 3.1.2 it is explained how to 'combine' solitons - they must be placed at the same location and be given the same (real part of) rapidity. Defects can also be combined in a similar manner, though it is not clear that any identification needs to be done on the defect rapidities, given that defects can be stationary at non-zero rapidity. To combine the defects in (4.21) then it is simply a case of taking $a \to 0$, giving

$$\begin{aligned} \mathcal{L} &= \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v \\ &+\delta(x)\left(\tfrac{1}{2}uA\dot{u} + uB\dot{\chi} + \chi A\dot{\chi} + \chi B\dot{v} + \tfrac{1}{2}vA\dot{v} - D(u,\chi) - \tilde{D}(\chi,v)\right) \ . \end{aligned} \tag{4.24}$$

The defect conditions then become

$$\begin{aligned} u' &= A\dot{u} + B\dot{\chi} - D_u \,|_{x=0} \\ B^T\dot{\chi} - B^T\dot{u} - D_\chi &= B\dot{\chi} - B\dot{v} + \tilde{D}_\chi \,|_{x=0} \\ v' &= -A\dot{v} + B^T\dot{\chi} + \tilde{D}_v \,|_{x=0} \ . \end{aligned} \tag{4.25}$$

There are thus clear differences when the defects are combined. The field χ no longer has any existence in the bulk and is trapped at the defect, so is an auxiliary field. In this sense then the combined defect Lagrangian (4.24) can be said to describe a type II defect[9]. The other major difference in combining

[9] In this thesis all of the type II defects arise from the combination of two type I defects. This observation gives a nomenclature used here whereby a type N defect arises from the combination of N type I defects. Type N defects have $N-1$ sets of auxiliary fields, each with the same number of components as a bulk field. The definition of a type II defect in the literature [CZ09*b*], where a type II defect has an auxiliary field in its Lagrangian, is more general than the 'type II' defined here as the dimension of the auxiliary field is not necessarily limited.

the defect is that the four defects conditions reduce to just three. However, since (4.25) is what comes from (4.22) and (4.23) when $a = 0$, it is clear that the delay factors are not affected by the combination process. The overall delay factor picked up by a soliton is the product of the individual defect delay factors.

4.1.2 Type II defects

Type II defects are defects which possess an auxiliary field. The idea of the type II defect was introduced by Corrigan and Zambon, who showed that type II defects exist in $a_2^{(2)}$ [CZ09*b*], and also in all of the $a_r^{(1)}$ theories [CZ09*b*, CZ11]. Although the combined defects are type II in that they possess an auxiliary field, the aim of this thesis is to identify species of defect, regardless of their Lagrangian form, so type II defects are given no particular significance.

4.1.3 Folding defects *a priori* and *a posteriori*

This section works on the basis that the type I defects can be folded. Folding a type I defect will in fact spoil the integrability of the theory; it is shown in chapter 6 how to construct a defect that will remain integrable after folding. Consider the simple case of taking a species 1 defect in $a_2^{(1)}$ and folding it in two different ways. Before any folding is done the defect condition (4.3), in components, becomes

$$\begin{aligned}\alpha_1 \cdot (4.3) \rightarrow 2u_1' - u_2' = -\dot{u}_2 + 2\dot{v}_1 - \frac{m}{\beta} e^{-\eta} \left(e^{\beta(u_1 - u_2 + v_1)} - e^{\beta(-u_1 - v_2)} \right) \\ - \frac{m}{\beta} e^{\eta} \left(e^{\beta(u_1 - v_1)} - e^{\beta(-u_1 + u_2 + v_1 - v_2)} \right) \qquad (4.26)\end{aligned}$$

$$\begin{aligned}\alpha_2 \cdot (4.3) \rightarrow 2u_2' - u_1' = \dot{u}_1 - 2\dot{v}_1 + 2\dot{v}_2 \\ - \frac{m}{\beta} e^{-\eta} \left(-e^{\beta(u_1 - u_2 + v_1)} + e^{\beta(-u_1 + u_2 + v_1 - v_2)} \right) \\ - \frac{m}{\beta} e^{\eta} \left(-e^{\beta(-u_2 + v_2)} + e^{\beta(-u_1 + u_2 + v_1 - v_2)} \right) \qquad (4.27)\end{aligned}$$

where e.g., u_1 is the component of u proportional to α_1 in the basis chosen in chapter 2.

Folding the system to $a_2^{(2)}$ *a posteriori* involves taking the equations (4.26) and (4.27) and identifying $u_1 = u_2 = \frac{\phi}{2}$ and $v_1 = v_2 = \frac{\psi}{2}$. This means that

(4.3) results in the two equations

$$\frac{\phi'}{2} = -\frac{\dot{\phi}}{2} + \dot{\psi} - \frac{m}{\beta}e^{-\eta}\left(e^{\frac{\beta}{2}\psi} - e^{\frac{\beta}{2}(-\phi-\psi)}\right) - \frac{m}{\beta}e^{\eta}\left(e^{\frac{\beta}{2}(\phi-\psi)} - 1\right) \tag{4.28}$$

$$\frac{\phi'}{2} = \frac{\dot{\phi}}{2} - \frac{m}{\beta}e^{-\eta}\left(e^{\frac{\beta}{2}\phi} - e^{\frac{\beta}{2}\psi}\right) - \frac{m}{\beta}e^{\eta}\left(-e^{\frac{\beta}{2}(-\phi+\psi)} + 1\right) . \tag{4.29}$$

Unlike when bulk affine Toda equations are folded, section 2.2.3, the right-hand sides of (4.28) and (4.29) do not match.

The other way to fold is to *a priori* set $u_1 = u_2 = \frac{\phi}{2}$ and $v_1 = v_2 = \frac{\psi}{2}$ in the Lagrangian (4.1). There will only be one defect condition associated to the ϕ Euler–Lagrange equation, which for the species 1 defect is

$$\phi' = \dot{\psi} - \frac{m}{\beta}\left(e^{\frac{\beta}{2}\phi} - e^{\frac{\beta}{2}(-\phi-\psi)}\right) - \frac{m}{\beta}\left(e^{\frac{\beta}{2}(\phi-\psi)} - e^{\frac{\beta}{2}(-\phi+\psi)}\right) . \tag{4.30}$$

It is clear that any solution to the *a posteriori* folded defect conditions (4.28) and (4.29) is also a solution of the *a priori* folded defect condition (4.30) from the simple relation (4.30) = (4.28) + (4.29). The converse is not true: (4.30) $\not\Rightarrow$ (4.28) and (4.30) $\not\Rightarrow$ (4.29). This has implications for the delay factor arguments used in chapter 6: if a soliton solution in $a_r^{(1)}$ has the symmetry of the folded soliton for u and for v it will satisfy the *a posteriori* folded defect conditions. It will thus also satisfy the *a priori* folded defect conditions, so if the symmetry is correct the soliton delay factors are unaltered by either folding process.

4.2 Quantum picture

The classical Lagrangian approach in section 4.1 treats type I defects and solitons very differently, as the defects are represented by the defect conditions (4.3) and (4.4), which are Bäcklund transformations. The quantum approach treats solitons and defects on a more equal footing, as the Faddeev–Zamolodchikov (FZ) algebra of section 3.2.1 can be extended to include defects.

4.2.1 Defect Faddeev–Zamolodchikov algebra

In the quantum theory the interaction between a soliton and a defect is specified by an exact transmission matrix (T-matrix). There is no reflection matrix as the defect is purely transmitting [DMS94*a*, DMS94*b*].

Defects can be included in the FZ algebra as non-commuting operators analogous to the soliton operators [CZ07]. Let a defect of species q, carrying a topological charge α and a rapidity η be denoted by

$$ {}_qD_\alpha(\eta) \, . $$

The defects of interest here are the ground state defects which have topological charges in the root space of a_r, as the analysis of section 4.1.1.1 shows - this is true of all species of defect. the label α can be considered as the topological charge itself, and not a surrogate for the charge like with the soliton charge labels.

The aim now is to examine the transmission process with a soliton of species p being transmitted from left to right across the (possibly stationary) defect. If the soliton has rapidity $\theta > 0$ and charge label i then this situation in the far past, $t \to -\infty$, is represented in the defect FZ algebra as

$$ {}^pA_i(\theta)\, {}_qD_\alpha(\eta) \, . \tag{4.31} $$

In the far future, $t \to \infty$, the soliton will then be infinitely far to the right of the defect and will possibly have undergone a change in topological charge, but no change in species. The future state is represented by

$$ {}_qD_\lambda(\eta)\, {}^pA_n(\theta) \, . \tag{4.32} $$

The quantity that relates the states in the far past (4.31) and the states in the far future (4.32) is the T-matrix. Some properties of the T-matrix are detailed in section 4.2.2. Guided by the classical equivalent of the T-matrix, the delay factor e.g., (4.19), the T-matrix is expected to depend on the difference of the soliton and defect rapidities. The defect FZ algebra is thus

$$
{}^{p}A_i(\theta)\,{}_{q}D_{\alpha}(\eta) = {}^{p}_{q}T^{n\lambda}_{i\alpha}(\theta-\eta)\,{}_{q}D_{\lambda}(\eta)\,{}^{p}A_n(\theta)\ .
$$

Note that topological charge is conserved in the transmission process, and the equation of topological charge conservation is dependent on the species of the soliton. If the soliton is a species 1 soliton then it would be $l_i + \alpha = l_n + \lambda$. The T-matrix then possesses delta functions to account for the topological charge conservation. Since α may be freely chosen within the root space, the T-matrices are infinite-dimensional.

4.2.2 The T-matrix

The transmission matrix can be constrained in an entirely analogous way to how the soliton S-matrix is constrained, see section 3.2.2. Defect T-matrices were found in the sine-Gordon model by Konik and LeClair [KL99] and matched to a Lagrangian description by Bowcock, Corrigan and Zambon [BCZ05]. T-matrices were later found for type I defects in $a_2^{(1)}$ [CZ07] and for $a_r^{(1)}$ [CZ09a, CZ11] using these methods, though quantum group methods can be used [CZ11].

The Yang–Baxter equation for two solitons and a defect, known as the triangle relations, is illustrated in figure 4.1, where the solid lines denote solitons and the dashed line denotes a defect and with $\theta_1 > \theta_2 > 0$. The triangle relations are given by

$$
S^{mn}_{jk}(\theta_1-\theta_2)T^{t\beta}_{n\alpha}(\theta_1-\eta)T^{s\lambda}_{m\beta}(\theta_2-\eta) = T^{m\beta}_{k\alpha}(\theta_2-\eta)T^{n\lambda}_{j\beta}(\theta_1-\eta)S^{st}_{nm}(\theta_1-\theta_2) \tag{4.33}
$$

where m,n and β are summed over. The species labels have been suppressed as the triangle relations hold for every species of soliton and defect. The T-matrix will have a prefactor which is not constrained by the triangle relations as the same prefactors appear on both sides of (4.33).

Though the triangle relations do not constrain the T-matrix prefactor, there are unitarity and crossing constraints analogous to those of S-matrices, (3.20) and (3.21). They are

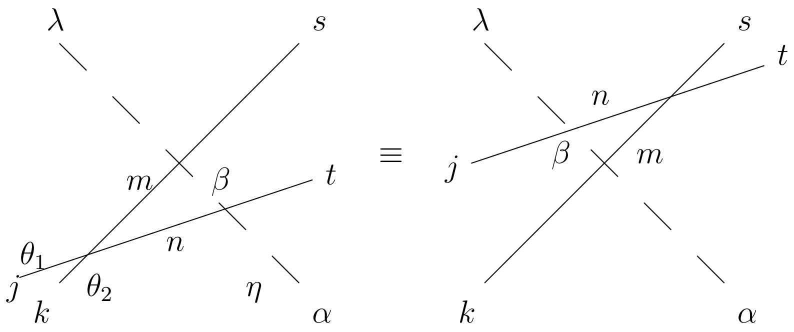

Figure 4.1: Illustration of the triangle relations, given by equation (4.33).

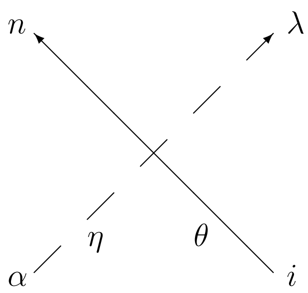

Figure 4.2: Crossing relation for a soliton and a defect, given in equation (4.35).

- Unitarity:

$$ {}^{p}_{q}T^{j\beta}_{i\alpha}(\theta-\eta)\,{}^{p}_{q}\tilde{T}^{n\lambda}_{j\beta}(\eta-\theta)=\delta^{n}_{i}\delta^{\lambda}_{\alpha}\,. \tag{4.34} $$

  The defect breaks the parity invariance of the theory, so it is expected that the T-matrix for right-to-left transmission of a soliton, denoted $\tilde{T}$ in (4.34), is different from T, the left-to-right transmission matrix.

- Crossing symmetry, as illustrated in figure 4.2. In figure 4.2 the direction of the arrow on the solid line denotes whether the line should be viewed as representing a soliton or antisoliton, with the arrow on the dashed line serving the same purpose for defects. The equation of crossing is

$$ {}^{h-p}_{\quad q}T^{\bar{i}\lambda}_{\bar{n}\alpha}(\theta-\eta)={}^{p}_{q}\tilde{T}^{n\lambda}_{i\alpha}(i\pi+\eta-\theta) \tag{4.35} $$

  where $\bar{i}$ denotes the opposite topological charge to i. The bar may be dropped provided the species of soliton is clear.

The crossing relation (4.35) and the unitarity relation (4.34) combine to give the crossing-unitarity relation

$$ {}^{p}_{q}T^{j\beta}_{i\alpha}(\theta-\eta)\; {}^{h-p}_{q}T^{j\lambda}_{n\beta}(\theta-\eta+i\pi) = \delta^{n}_{i}\delta^{\lambda}_{\alpha}\;. \tag{4.36}$$

The crossing-unitarity relation thus relates the transmission matrix of a soliton through a defect to that of an antisoliton through the same defect. Combining a soliton and antisoliton should give a trivial result, so it should not be a surprise that (4.36) holds.

Taking the soliton and defect to be, initially, of species 1, the triangle relations (4.33) and crossing-unitarity relation (4.36) can be used to obtain the T-matrix, the result is [CZ09*a*]

$$ {}^{1}_{1}T^{i\lambda}_{i\alpha}(\theta-\eta) = g^{1}(\theta-\eta)\, Q^{\lambda\cdot l_i}\delta^{\lambda}_{\alpha} \qquad {}^{1}_{1}T^{(i-1)\lambda}_{i\alpha}(\theta-\eta) = g^{1}(\theta-\eta)\,\hat{x}\,\delta^{\lambda-l_i+l_{i-1}}_{\alpha} \tag{4.37}$$

with all other entries in the matrix equal to zero. The soliton charge label[10] takes the values $i=1,\dots,h$ where the case of $(i-1)=0$ should be taken as $(i-1)=h$. As with the soliton S-matrix (3.22), $Q=-e^{i\pi\gamma}$, with the coupling entering through $\gamma=\frac{4\pi}{\beta^2}-1$, with β the bulk coupling found in the bulk (2.2) and defect (4.2) potentials. The likelihood of a soliton changing topological charge on transmission through the defect is determined by the value of $\hat{x}=e^{\gamma(\theta-\eta-\frac{i\pi}{2})}$. The prefactor in ${}^{1}_{1}T$, $g^{1}(\theta-\eta)$, is constrained by the soliton fusion bootstrap and by (4.36). The solution is [CZ09*a*]

$$ g^{1}(\theta-\eta) = \frac{\hat{x}^{-\frac{1}{2}}}{2\pi}\Gamma\!\left(\tfrac{1}{2}+\tfrac{r}{2}\gamma-z\right)\prod_{k=1}^{\infty}\frac{\Gamma(\frac{1}{2}+(hk+\frac{r}{2})\gamma-z)\Gamma(\frac{1}{2}+(hk-1-\frac{5r}{2})\gamma+z)}{\Gamma(\frac{1}{2}+(hk-\frac{r}{2})\gamma-z)\Gamma(\frac{1}{2}+(hk-\frac{r}{2})\gamma+z)} \tag{4.38}$$

where $z=\frac{hi\gamma\left(\theta-\eta-\frac{i\pi}{2}\right)}{2\pi}$.

One can similarly consider the transmission of a species r soliton through a species 1 defect[11]. The transmission matrix is given by [CZ09*a*]

$$ {}^{r}_{1}T^{(i+j)\lambda}_{i\alpha} = g^{r}(\theta-\eta)\,\hat{x}^{j}Q^{-\lambda\cdot K_{(i+j)}}\delta^{\lambda+l_i-l_{i+j}}_{\alpha} \tag{4.39}$$

[10] Recall that for the species 1 soliton the label i denotes a topological charge of l_i.
[11] Recall that for a species r soliton the label i denotes a charge of $-l_i$

where $K_{i+j} = l_i + l_{i+1} + \ldots + l_{i+j}$, with $(i+j)$ evaluated modulo h. The prefactor $g^r(\theta - \eta)$ can be found from using the crossing-unitarity relation (4.36) along with (4.38), giving

$$g^1(\theta - \eta) g^r(\theta - \eta + i\pi) = \frac{1}{1 + (-Q)^r \hat{x}^h} \; . \tag{4.40}$$

The prefactor $g^r(\theta - \eta)$ can also be determined through soliton fusing. Soliton fusing and transmission matrices is the topic of section 4.2.3.

Comparing the T-matrices (4.37) and (4.39) with the classical delay factors (4.19), it is clear that the topological charge shifts possible in the quantum transmission match those classically for the species 1 soliton, while the species r soliton can undergo classically forbidden changes in topological charge (except in the sine-Gordon case). In fact, for the 'soliton' representations, solitons of species $p = 1, 2, \ldots, \lfloor \frac{h-1}{2} \rfloor$, there is agreement between the classical and quantum situation, but not for the other 'antisoliton' representations [CZ09*a*].

One can similarly analyse the species r defect. The transmission matrix for a species 1 soliton is

$${}^{1}_{r}T^{(i-j)\lambda}_{i\alpha}(\theta - \eta) = g^r(\theta - \eta) \, \hat{x}^j Q^{\lambda \cdot L_{i-j}} \delta_\alpha^{\lambda - l_i + l_{i-j}} \tag{4.41}$$

where $L_{i-j} = l_{i-j} + l_{i-j+1} + \ldots + l_i$. The transmission matrix for the species r soliton is

$${}^{r}_{r}T^{i\lambda}_{i\alpha}(\theta - \eta) = g^1(\theta - \eta) \, Q^{-\lambda \cdot l_i} \delta_\alpha^\lambda \qquad {}^{r}_{r}T^{(i+1)\lambda}_{i\alpha}(\theta - \eta) = g^1(\theta - \eta) \, \hat{x} \, \delta_\alpha^{\lambda + l_i - l_{i+1}} \tag{4.42}$$

with the other entries in the matrix vanishing. The prefactors for these species r defect T-matrices are the same as those for the species 1 defect T-matrices, but in the opposite order.

The transmission matrices for the other solitons are best found using the soliton fusing rules.

4.2.3 Soliton fusing and T-matrices

The transmission matrices for species of soliton other than species 1 or r can be found using soliton fusing in the Faddeev–Zamolodchikov algebra. Recall that the species 2 soliton operator can be written in terms of the species 1 operators as (3.23)

$$ {}^{2}A_{(jk)}(\theta) = c^{(jk)}\, {}^{1}A_j(\theta - \tfrac{i\pi}{h})\, {}^{1}A_k(\theta + \tfrac{i\pi}{h}) + c^{(kj)}\, {}^{1}A_k(\theta - \tfrac{i\pi}{h})\, {}^{1}A_j(\theta + \tfrac{i\pi}{h}) $$

where the soliton fusing couplings have ratios given by (3.24) and (3.25). Using this in the defect FZ algebra

$$ {}^{2}A_{(jk)}(\theta)\, D_\alpha(\eta) = {}^{2}T_{(jk)\alpha}^{(mn)\lambda}(\theta - \eta)\, D_\lambda(\eta)\, {}^{2}A_{(mn)}(\theta) $$

gives an expression for the species 2 T-matrix in terms of species 1 T-matrices:

$$ {}^{2}T_{(jk)\alpha}^{(ab)\lambda}(\theta - \eta) c^{(ab)} = c^{(jk)}\, {}^{1}T_{j\beta}^{a\lambda}(\theta - \eta - \tfrac{i\pi}{h})\, {}^{1}T_{k\alpha}^{b\beta}(\theta - \eta + \tfrac{i\pi}{h}) + (j \leftrightarrow k)\ . \quad (4.43) $$

The defect species is not labelled in (4.43) as the equation should hold for any species of defect. For the species 1 defect, examining the prefactors, ${}^{2}_{1}T$ will have a prefactor of g^2, given by

$$ g^2(\theta - \eta) = g^1(\theta - \eta - \tfrac{i\pi}{h}) g^1(\theta - \eta + \tfrac{i\pi}{h})\ . $$

The T-matrices for all of the other solitons can be treated in a similar way, so there is a bootstrap principle to apply. All of the prefactors for the species 1 (or species r) defect can then be written in terms of g^1, ultimately resulting in

$$ \prod_{k=0}^{r} g^1(\theta - \eta - \tfrac{i\pi r}{h} + \tfrac{2\pi i k}{h}) = \frac{1}{1 + \hat{x}^h}\ . $$

The approach of this chapter has been to treat $a_r^{(1)}$ generally. some of these results are used for specific r in chapters 5, 6 and 7.

5

Classical defect fusing rules

This chapter expands on some of the arguments found in *Defect fusing rules in affine Toda field theory* [Rob14*b*].

The first step in establishing the existence of a 'hierarchy' of defects is to link the defects that exist within the same ATFT, which is realised through the fusing structure of the theory. For $a_r^{(1)}$, which is the only series of ATFTs for which defects exist in the literature [CZ07, CZ09*a*] there exist r species of fundamental soliton, so it is expected that there will be r species of fundamental defect - although this is not something that is clear from the Lagrangian approach. It is supposed that the type I defects, in having the simplest Lagrangian, must describe fundamental defects and that they are of species 1 and species r. The key quantity in this analysis is the classical delay factors picked up by solitons travelling through defects.

5.1 Observations in $a_2^{(1)}$

5.1.1 Delay factors

Consider the delay factors (4.19) and (4.20) for the case of $a_2^{(1)}$. For the species 1 defect they are

$$ {}^{1}_{1}z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega^{\frac{1}{2}}}{ie^{\eta-\theta}+\omega^{-\frac{1}{2}}} \qquad\qquad {}^{2}_{1}z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega}{ie^{\eta-\theta}+\omega^{-1}} \tag{5.1} $$

where in this case $\omega = e^{\frac{2i\pi}{3}}$. All of the fundamental soliton fusing rules are seen in the delay factors as:

$$
\begin{array}{rl}
\text{Soliton fusing process} & \text{Delay factor} \\
11 \to 2 & {}^2_1z(\theta-\eta) = {}^1_1z\left((\theta-\frac{i\pi}{3})-\eta\right){}^1_1z\left((\theta+\frac{i\pi}{3})-\eta\right) \\
22 \to 1 & {}^1_1z(\theta-\eta) = {}^2_1z\left((\theta-\frac{i\pi}{3})-\eta\right){}^2_1z\left((\theta+\frac{i\pi}{3})-\eta\right) \\
12 \to 0 & 1 = {}^1_1z(\theta-\eta){}^2_1z\left((\theta\pm i\pi)-\eta\right) \ .
\end{array}
\tag{5.2}
$$

The right-hand side of each of the equations in (5.2) can be viewed in terms of having a two soliton solution, e.g., two species 2 solitons with rapidities $\theta_1 = \theta - \frac{i\pi}{3}$ and $\theta_2 = \theta + \frac{i\pi}{3}$, where the solitons are spatially separated. The quadratic term in the tau functions picks up the relevant right-hand side of (5.2) as its overall delay factor (see section 4.1.1.3). Now, combining the solitons so that they are now at the same location, thought of as a limiting process, gives the same right-hand side but the two soliton solution has reduced to a one (or zero) soliton solution so the delay factor must equate to what is on the left-hand side.

Similarly the species 2 defect gives the delay factors

$$
{}^1_2z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega}{ie^{\eta-\theta}+\omega^{-1}} \qquad {}^2_2z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega^{\frac{1}{2}}}{ie^{\eta-\theta}+\omega^{-\frac{1}{2}}} \ . \tag{5.3}
$$

Again there is no issue with soliton fusing and equations analogous to (5.2) hold. Note the remarkable similarity between the delay factors of the species 1 (5.1) and species 2 (5.3) defects, which suggests that these defects are indeed in some way conjugate to each other. Making analogy to the soliton fusing rules in (5.2) it is seen that:

$$
\begin{array}{rl}
\text{Defect fusing process} & \text{Delay factor} \\
11 \to 2 & {}^1_2z(\theta-\eta) = {}^1_1z\left(\theta-(\eta+\frac{i\pi}{3})\right){}^1_1z\left(\theta-(\eta-\frac{i\pi}{3})\right) \\
22 \to 1 & {}^1_1z(\theta-\eta) = {}^1_2z\left(\theta-(\eta+\frac{i\pi}{3})\right){}^1_2z\left(\theta-(\eta-\frac{i\pi}{3})\right) \\
12 \to 0 & 1 = {}^1_1z(\theta-\eta){}^1_2z\left(\theta-(\eta\pm i\pi)\right) \ .
\end{array}
\tag{5.4}
$$

So, the delay factors are clearly consistent with a bootstrap involving different species of defect where defect fusing rules require that the defects have rapidity parameters differing by i times the fusing angle and that they are combined at the same spatial location. The fusing angles are exactly the same as those found in the soliton fusing rules, i.e., $\frac{2\pi}{3}$ with a rapidity difference of $i\pi$ for the annihilation of defect and anti-defect. Similarly to (5.2), the right-hand side of each equation in (5.4) can be viewed in terms of initially having two separated defects, e.g., two species 1 defects with rapidities $\eta_1 = \eta + \frac{i\pi}{3}$ and $\eta_2 = \eta - \frac{i\pi}{3}$, such that the soliton passing through gets the right-hand side as its overall delay factor. As noted in section 4.1.1.4, combining the defects does not change the delay factor but the two defect solution has reduced to the one (or zero) defect solution specified by the left-hand side. The findings of (5.4) are no different when the species 2 soliton is considered instead of the species 1 soliton. Note that deciding which defect should be called species 1 and which should be called species 2 (or r) is arbitrary, the choice here reflects the fact that the species 1 defect is the primary focus of [BCZ04b, CZ09a].

5.1.2 Lagrangian level fusing rules

Whilst the delay factor analysis offers no trouble to the consistency of the defect bootstrap, the Lagrangian description of the fusing rules does not fall into place trivially. It is known that the species 2 defect has a type I Lagrangian description given by (4.1) with B given by (4.6)

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(-\tfrac{1}{2}uA\dot{u} + uB_1^T\dot{v} - \tfrac{1}{2}vA\dot{v} - D^{(2)}(u,v)\right) \quad (5.5)$$

where $A = 1 - B_1$.

For the case of $a_2^{(1)}$ with a species 2 defect of rapidity η the defect potential is

$$\begin{aligned} D^{(2)}(u,v) = e^{-\eta}\left(e^{-u_2-v_1} + e^{u_1+v_1-v_2} + e^{-u_1+u_2+v_2}\right) \\ + e^{\eta}\left(e^{-u_1+v_1} + e^{u_1-u_2-v_1+v_2} + e^{u_2-v_2}\right) \end{aligned} \quad (5.6)$$

where for simplicity the mass parameter m and the coupling β have both been set to unity. As noted in chapter 4 this has no effect on the delay factors.

The fusing rules also give the species 2 defect in terms of species 1 defects. The species 2 defect with rapidity η is obtained by combining two species 1 defects with rapidities $\eta_1 = \eta + \frac{i\pi}{3}$ and $\eta_2 = \eta - \frac{i\pi}{3}$ so the Lagrangian describing this process is

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v \\ &\quad + \delta(x)\left(\tfrac{1}{2}uA\dot{u} + uB_1\dot{\chi} + \chi A\dot{\chi} + \chi B_1\dot{v} + \tfrac{1}{2}vA\dot{v} - D\right)\end{aligned} \tag{5.7}$$

where the potential $D \equiv D(u, \chi, v)$, again with $m = \beta = 1$, is

$$\begin{aligned} D = {} & \tfrac{1}{2}e^{-\eta}\left(e^{-u_1-\chi_2} + e^{u_1-u_2+\chi_1} + e^{u_2-\chi_1+\chi_2} + e^{-\chi_1-v_2} + e^{\chi_1-\chi_2+v_1} + e^{\chi_2-v_1+v_2}\right) \\ & +\tfrac{\sqrt{3}}{2}ie^{-\eta}\left(e^{-\chi_1-v_2} + e^{\chi_1-\chi_2+v_1} + e^{\chi_2-v_1+v_2} - e^{-u_1-\chi_2} - e^{u_1-u_2+\chi_1} - e^{u_2-\chi_1+\chi_2}\right) \\ & +\tfrac{1}{2}e^{\eta}\left(e^{-u_2+\chi_2} + e^{u_1-\chi_1} + e^{-u_1+u_2+\chi_1-\chi_2} + e^{-\chi_2+v_2} + e^{\chi_1-v_1} + e^{-\chi_1+\chi_2+v_1-v_2}\right) \\ & +\tfrac{\sqrt{3}}{2}ie^{\eta}\left(e^{-u_2+\chi_2} + e^{u_1-\chi_1} + e^{-u_1+u_2+\chi_1-\chi_2} - e^{-\chi_2+v_2} - e^{\chi_1-v_1} - e^{-\chi_1+\chi_2+v_1-v_2}\right) . \end{aligned} \tag{5.8}$$

The fusing process clearly gives a type II Lagrangian, in that the Lagrangian contains an auxiliary field χ. Both Lagrangians for the species 2 defect (5.5) and (5.7) with associated potentials give the same delay factors and so affect every soliton in exactly the same way. Rather than having two Lagrangian descriptions for the same defect it would be preferable to have (5.7) reduce to (5.5), which requires that the potential (5.8) reduces to (5.6). In order for the potentials to match the imaginary parts of (5.8) must vanish (assuming that η is real), this can be achieved by the identification

$$\chi_1 = u_2 + v_1 - v_2 \qquad\qquad \chi_2 = -u_1 + u_2 + v_1 \ .$$

This identification of the auxiliary field actually reduces (5.8) to (5.6). Furthermore, the kinetic terms of the defect in (5.7) also reduce to the same as those of (5.5) under this identification of the auxiliary field. There is thus a case to be made for defect fusing rules in the Lagrangian in $a_2^{(1)}$ provided the auxiliary field is identified a certain way *a priori*, i.e., at the Lagrangian level - eliminating the existence of an equation of motion for χ. A deeper analysis shows that the same identification does not work *a posteriori*, i.e., at the level of the equations of motion - it is not clear if there is an alternative solution in that case.

More generally, in higher rank $a_r^{(1)}$ theories the species 2 defect will necessarily require a type II description as it was shown in [BCZ04*b*, CZ09*a*] that only the species 1 and species r defects will have a type I description. It is thus not instructive at this stage to consider the Lagrangian forms of such defects and it is better to consider the soliton delay factors.

5.2 Observations in $a_3^{(1)}$

Of the $a_r^{(1)}$ affine Toda theories $a_3^{(1)}$ is of particular interest when it comes to defect fusing rules, this is because $a_3^{(1)}$ is the lowest ranked (simply laced) theory for which the fusing rules imply the existence of a previously unknown fundamental defect - the species 2 defect. The species 2 defect can be obtained by fusing two species 1 defects or by fusing two species 3 defects where for $a_3^{(1)}$ the fusing angle is now $\frac{2\pi}{h} = \frac{\pi}{2}$. Thus, for any fixed soliton of species p:

$$\begin{array}{ll} \text{Defect fusing} & \qquad\text{Delay factor} \\ 11 \to 2 & {}^{p}_{2}z(\theta-\eta) = {}^{p}_{1}z\left(\theta-(\eta+\frac{i\pi}{4})\right){}^{p}_{1}z\left(\theta-(\eta-\frac{i\pi}{4})\right) \\ 33 \to 2 & {}^{p}_{2}z(\theta-\eta) = {}^{p}_{3}z\left(\theta-(\eta+\frac{i\pi}{4})\right){}^{p}_{3}z\left(\theta-(\eta-\frac{i\pi}{4})\right)\ . \end{array} \tag{5.9}$$

The resulting delay factors, from (4.19) or (4.20), and the fusing (5.9), are rather interesting

$${}^{1}_{2}z(\theta-\eta) = {}^{3}_{2}z(\theta-\eta) = \frac{e^{\eta-\theta}+1}{e^{\eta-\theta}-1} \qquad {}^{2}_{2}z(\theta-\eta) = \frac{e^{2(\eta-\theta)}+\sqrt{2}e^{\eta-\theta}+1}{e^{2(\eta-\theta)}-\sqrt{2}e^{\eta-\theta}+1}\ .$$

Note how the species 1 and species 3 solitons receive the same delay factor in analogy to how these solitons interact with the species 2 soliton. The fact that the species 1 and 3 solitons are delayed by the same amount also means that solitons possessing $c_2^{(1)}$ symmetry in the field u to the left of the defect will retain $c_2^{(1)}$ symmetry in the field v to the right of the defect - in other words, this defect should be compatible with folding. The folding of defect configurations is the subject of chapter 6.

In (5.9) it is clear that the delay factor that each soliton receives is real if the rapidities θ and η are real. For the species 2 soliton the denominator is always positive for real rapidities so $\arg\left({}^{2}_{2}z(\theta-\eta)\right) = 0$ for any real value of $\theta-\eta$, implying that the species 2 soliton merely picks up a time delay classically with no change in topological charge. For the species 1 and species 3 solitons connection can be made to the quantum transmission matrix by considering the high rapidity limits $\theta - \eta \to \pm\infty$: for either soliton $\lim_{\theta-\eta\to\infty} \arg(z) = i\pi$ and $\lim_{\theta-\eta\to-\infty} \arg(z) = 0$ so the possibilities are that there is no change in topological charge, or that the topological charge is shifted by two sectors so e.g., $l_1 \to l_3$, $l_4 \to l_2$. This should be compared to the high rapidity limits of the transmission matrices (7.30) and (7.31).

5.3 More general observations

5.3.1 Defect mass ratios

Another way in which defects are particle-like is the fact that they have associated to them their own energy and momentum. The on-shell condition $\mathcal{M}^2 = E^2 - P^2$ thus associates a mass to a defect. As shown in section 4.1.1, in a vacuum configuration, with u and v in the same representation, the species 1 defect (and the species r defect) possesses an energy and a momentum given by $(E, P) = (\frac{2hm}{\beta^2}\cosh\eta, -\frac{2hm}{\beta^2}\sinh\eta)$ suggesting a mass of

$$\mathcal{M}_1 = \frac{2hm}{|\beta^2|}\ .$$

Thus, taking $\eta_1 = \eta - \frac{i\pi}{h}$ and $\eta_2 = \eta + \frac{i\pi}{h}$ the species 2 defect will have a mass of

$$\mathcal{M}_2 = \frac{2hm}{|\beta^2|}\cos\left(\frac{\pi}{h}\right) = 2\cos\left(\frac{\pi}{h}\right)\mathcal{M}_1$$

so it is notable that the mass ratios of the defects are the same as those of the solitons: $\frac{\mathcal{M}_1}{\mathcal{M}_2} = \frac{M_1}{M_2}$ [OTU93a]. Examination of the appropriate defect Lagrangian shows that in all cases the mass ratios for defects match those of solitons. Note that in $a_2^{(1)}$ the alternative species 2 defect Lagrangian (5.7) is seen to give the correct mass.

5.3.2 General fusing rules

In general, the species 2 defect of $a_r^{(1)}$ is obtained by the fusing of two species 1 defects. The general fusing angle is $\frac{2\pi}{h}$, so the delay factor for a species 1 soliton through a species 2 defect will be:

$$ {}^{1}_{2}z(\theta-\eta) = {}^{1}_{1}z\left(\theta-(\eta-\tfrac{i\pi}{h})\right){}^{1}_{1}z\left(\theta-(\eta+\tfrac{i\pi}{h})\right) \ . \tag{5.10} $$

In general then, (5.10) illustrates that the species 1 soliton through a species 2 defect receives the same delay factor that a species 2 soliton receives from a species 1 defect. While clear differences emerge in the quantum T-matrix for both of these situations, the same prefactor is found.

Defect fusing rules should give all of the fundamental defects of the theory, not merely species 2 defects. The fusing angles are the same as soliton fusing angles so in general a species q_1 and a species q_2 defect can fuse to form a species $q \, (\mathrm{mod}\ h)$ defect with delay factor

$$ {}^{p}_{q}z(\theta-\eta) = {}^{p}_{q_1}z\left(\theta-\eta+\tfrac{i\pi q_2}{h}\right){}^{p}_{q_2}z\left(\theta-\eta-\tfrac{i\pi q_1}{h}\right) \ . \tag{5.11} $$

When $q = 0$ the delay factor is 1, representing the absence of defects (or defect-defect annihilation), it is evident then that the anti-defect of a species q_1 defect with rapidity η is a species $h - q_1$ defect with rapidity $\theta \pm i\pi$.

Note that taking $p = 1$ or $p = r$ in (5.11), the set of delay factors obtained is the same as (4.19) or (4.20) respectively. The closure of the soliton bootstrap ensures then that the delay factors form a closed set under the fusing (5.11), certainly when the soliton is of species 1 or r. This is strong evidence, classically, for closure of the defect bootstrap, even though the Lagrangian description appears only to get more complicated, not less, by fusing.

This chapter does not consider any specific $a_r^{(1)}$ for $r > 3$, however, chapter 7 considers defect fusing in the quantum $a_5^{(1)}$ theory.

5.4 Summary

This chapter begins with an illustration on how the defect fusing rules work in $a_2^{(1)}$. With the identification of the two type I defects of $a_2^{(1)}$ as species 1 and species 2 fundamental defects, defect fusing explains the observed pattern of soliton delay factors, where the fusing angles for defects are the same as the fusing angles for the corresponding species of solitons. The possibility of seeing the fusing at the Lagrangian level in $a_2^{(1)}$ is also explored.

The fusing rules are then applied to $a_3^{(1)}$ to obtain the delay factors for the species 2 defect. The species 2 defect of $a_3^{(1)}$ is notable for a number of reasons:

- It is the first new fundamental defect encountered as a result of the defect fusing process.
- The defect is self-conjugate, in that the anti-defect is another species 2 defect with its rapidity shifted by $i\pi$ and opposite topological charge (though the defect topological charge does not show in the delay factors). Note the analogy with the species 2 soliton of $a_3^{(1)}$, which is self-conjugate[12].
- The defect gives the same delay factors to the species 1 soliton as it does to the species 3 soliton. This suggests that the defect should remain integrable after folding to $c_2^{(1)}$. Note that the species 2 soliton analogously interacts with the species 1 soliton in the same way as it interacts with the species 3 soliton.

More general results of the fusing rules are then noted, which are the defect mass ratios and how fusing affects delay factors generally in $a_r^{(1)}$.

[12] Classically, the species 2 soliton of $a_3^{(1)}$ can be thought of as a sine-Gordon soliton embedded in $a_3^{(1)}$. In general this is true of the species n soliton of $a_{2n-1}^{(1)}$.

6

Folding defect affine Toda field theories

This chapter is based upon the paper *Folding defect affine Toda field theories* [Rob14a]. More emphasis is given here to $a_2^{(1)}$ as the simplest example and some calculations are given in more detail.

Chapter 5 explains how the different fundamental defects of $a_r^{(1)}$ may be generated by the fusing rule idea. This chapter explains how to obtain (fundamental) defects of other ATFTs, namely those that are obtained by folding $a_r^{(1)}$. The identification of defect species postdates the source material for this chapter, [Rob14a], but is used here.

For simplicity of expressions, the mass parameter m and the coupling strength β have both been set to unity in this chapter, which considers only classical arguments.

6.1 $a_2^{(1)}$ defects and folding

The simplest, i.e., the lowest ranked, case of folding is the (non-canonical [KS96b]) folding $a_2^{(1)} \to a_2^{(2)}$ explained in section 2.2.2. As this is the simplest folding process it suggests that $a_2^{(1)}$ is the most likely theory in which foldable defect configurations might be found. The fact that a type II defect for $a_2^{(2)}$ is already known [CZ09b] also gives a check on the validity of possible folded defect configurations.

6.1.1 The 'normal' $a_2^{(2)}$ defect

This section involves constructing a defect for $a_2^{(2)}$ formed by combining two type I defects. That an $a_2^{(2)}$ defect arises this way is hinted at in terms in Bäcklund transformations in [CZ09*b*]. The way this defect is constructed generalises to all of the other theories that can be obtained from folding $a_r^{(1)}$. The generalisation is the topic of section 6.2.

6.1.1.1 Delay factor considerations

As in chapter 5 the quantities that are most useful in this analysis are the soliton delay factors (4.19) and (4.20) for $a_2^{(1)}$. For the species 1 defect they are

$$ {}_1^1z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega^{\frac{1}{2}}}{ie^{\eta-\theta}+\omega^{-\frac{1}{2}}} \qquad {}_1^2z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega}{ie^{\eta-\theta}+\omega^{-1}} \tag{6.1} $$

where $\omega = e^{\frac{2i\pi}{3}}$. For the species 2 defect the delay factors are

$$ {}_2^1z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega}{ie^{\eta-\theta}+\omega^{-1}} \qquad {}_2^2z(\theta-\eta) = \frac{ie^{\eta-\theta}+\omega^{\frac{1}{2}}}{ie^{\eta-\theta}+\omega^{-\frac{1}{2}}}\ . $$

As is also noted in section 5.1.1, the two kinds of fundamental defect have remarkably similar looking delay factors, with ${}_1^1z(\theta-\eta) = {}_2^2z(\theta-\eta)$ and ${}_1^2z(\theta-\eta) = {}_2^1z(\theta-\eta)$. It follows that

$$ {}_1^1z(\theta-\eta)\,{}_2^1z(\theta-\eta) = {}_1^2z(\theta-\eta)\,{}_2^2z(\theta-\eta) \tag{6.2} $$

which implies that a combined defect consisting of a species 1 defect of rapidity η and a species 2 defect of the same rapidity η will give the same delay factors to the species 1 and the species 2 solitons.

The significance of (6.2) lies in how the soliton of $a_2^{(2)}$ is constructed from solitons of $a_2^{(1)}$. As explained in section 3.1.2, the soliton of $a_2^{(2)}$ is constructed by taking a species 1 and a species 2 soliton in $a_2^{(1)}$, both with the same rapidity θ, and combining them, i.e., placing them at the same spatial location (classically, where solitons have a spatial extent, this can be thought of as placing their centres of mass at the same point) - since the solitons have the same rapidity they remain combined and move as one entity. Folding, by identifying the roots of a_2, results in the $a_2^{(2)}$ soliton. Equation (6.2) means that the defect configuration described above does not spoil the symmetry of this two-soliton $a_2^{(1)}$ solution; the solitons remain combined after passing through the defect as the constituent solitons receive identical delay factors. This property suggests that the defect should be foldable.

6.1.1.2 Lagrangian for the foldable defect

The previous section describes, in terms of delay factors, a defect in $a_2^{(1)}$ which preserves the $a_2^{(2)}$ symmetry of any solitons passing through it. This defect is formed by combining a species 1 defect and a species 2 defect, both possessing the same rapidity η, giving the Lagrangian

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v \\ &\quad + \delta(x)\left(\frac{1}{2}uA\dot{u} + (u-v)B_1\dot{\chi} - \frac{1}{2}vA\dot{v} - D(u,\chi,v)\right)\end{aligned} \tag{6.3}$$

where B_1 is given by (4.5) and $A = 1 - B_1$. The defect potential is $D(u,\chi,v) = D^{(1)}(u,\chi) + D^{(2)}(\chi,v)$ with

$$\begin{aligned}D^{(1)}(u,\chi) &= e^{-\eta}\left(e^{-u_1-\chi_2} + e^{u_1-u_2+\chi_1} + e^{u_2-\chi_1+\chi_2}\right) + e^{\eta}\left(e^{-u_2+\chi_2} + e^{u_1-\chi_1} + e^{-u_1+u_2+\chi_1-\chi_2}\right) \\ D^{(2)}(\chi,v) &= e^{-\eta}\left(e^{-v_1-\chi_2} + e^{v_1-v_2+\chi_1} + e^{v_2-\chi_1+\chi_2}\right) + e^{\eta}\left(e^{-v_2+\chi_2} + e^{v_1-\chi_1} + e^{-v_1+v_2+\chi_1-\chi_2}\right) .\end{aligned}$$

It is clear that $D^{(1)}$ and $D^{(2)}$ are related simply by switching u and v. The field χ is an auxiliary field, only existing at the defect, so in this sense the defect under consideration is a type II defect. Consideration of delay factor arguments suggest that χ cannot be folded (as it cannot possess the $a_2^{(2)}$ symmetry when u and v do).

6.1.1.3 Folding to an $a_2^{(2)}$ defect

The Lagrangian (6.3) can be folded by applying the folding process of section 2.2.2 to the bulk fields u and v, but the auxiliary field should not be folded. In components this is

$$\begin{aligned}u = \alpha_1 u_1 + \alpha_2 u_2 &\to \phi = \frac{\alpha_1+\alpha_2}{2}\phi \\ v = \alpha_1 v_1 + \alpha_2 v_2 &\to \psi = \frac{\alpha_1+\alpha_2}{2}\psi \\ \chi &= \alpha_1\chi_1 + \alpha_2\chi_2 .\end{aligned}$$

The ambiguity over the meaning of ϕ and ψ is handled here by henceforth expressing everything in terms of components in this section. In components, folding the Lagrangian (6.3) gives

$$\mathcal{L} = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi + \delta(x)\left((\phi-\psi)\dot{\chi}_2 - D\right) \tag{6.4}$$

where

$$\mathcal{L}_\phi = \frac{1}{4}\dot{\phi}\dot{\phi} - \frac{1}{4}\phi'\phi' - \Phi \qquad \mathcal{L}_\psi = \frac{1}{4}\dot{\psi}\dot{\psi} - \frac{1}{4}\psi'\psi' - \Psi$$
$$\Phi = e^{-\phi} + 2e^{\frac{\phi}{2}} - 3 \qquad \Psi = e^{-\psi} + 2e^{\frac{\psi}{2}} - 3$$

and the defect potential is given by

$$\begin{aligned} D = e^{-\eta}&\left(e^{-\frac{\phi}{2}-\chi_2} + e^{\frac{\phi}{2}-\chi_1+\chi_2} + e^{-\frac{\psi}{2}-\chi_2} + e^{\frac{\psi}{2}-\chi_1+\chi_2} + 2e^{\chi_1}\right) \\ &+ e^{\eta}\left(e^{-\frac{\phi}{2}+\chi_2} + e^{\frac{\phi}{2}-\chi_1} + e^{-\frac{\psi}{2}+\chi_2} + e^{\frac{\psi}{2}-\chi_1} + 2e^{\chi_1-\chi_2}\right) . \end{aligned} \tag{6.5}$$

Besides the bulk equations of motion for the folded $a_2^{(2)}$ theory, the Lagrangian (6.4) also gives the defect equations

$$\phi' = 2\dot{\chi}_2 - 2D_\phi \tag{6.6}$$
$$D_{\chi_1} = 0 \tag{6.7}$$
$$\dot{\phi} - \dot{\psi} = -D_{\chi_2} \tag{6.8}$$
$$\psi' = 2\dot{\chi}_2 + 2D_\psi . \tag{6.9}$$

The first thing to note about these equations is that (6.7) gives an algebraic constraint, meaning that one of the components of χ can be written in terms of the other fields. In components the algebraic constraint is

$$D_{\chi_1} = \left(e^{-\eta} + e^{\eta}e^{-\chi_2}\right)\left(2e^{\chi_1} - e^{\frac{\phi}{2}-\chi_1+\chi_2} - e^{\frac{\psi}{2}-\chi_1+\chi_2}\right) = 0$$

implying that

$$e^{2\chi_1} = \frac{1}{2}e^{\chi_2}\left(e^{\frac{\phi}{2}} + e^{\frac{\psi}{2}}\right) \tag{6.10}$$

so the χ_1 degree of freedom can be removed leaving the auxiliary field with just one degree of freedom, χ_2. With the auxiliary field having just one component it is tempting to view it as an $a_2^{(2)}$ field, though there is no 'bulk' equation of motion for χ to satisfy. Using (6.10) in the Lagrangian results in terms coupling ϕ and ψ directly, which is undesirable at this stage but necessary later on when comparing this defect to the one found by Corrigan and Zambon [CZ09*b*].

The defect potential (6.5) splits into positive and negative helicity parts: the negative helicity parts all have a factor of $e^{-\eta}$ and collectively will be denoted D^-; the positive helicity parts all have a factor of e^{η} and collectively will be denoted D^+. With this identification it is seen that

$$D^-_{\chi_2} = 2D^-_\phi + 2D^-_\psi \tag{6.11}$$

$$D^+_{\chi_2} = -2D^+_\phi - 2D^+_\psi \tag{6.12}$$

and so using (6.11) and (6.12) the defect equation (6.8) becomes

$$\dot{\phi} + 2D^-_\phi - 2D^+_\phi = \dot{\psi} - 2D^-_\psi + 2D^+_\psi \ . \tag{6.13}$$

6.1.1.4 Energy conservation

It is expected that the energy of the system will be modified in the presence of a defect, as explained in section 4.1.1. For the case of $a_2^{(2)}$ (in components) the bulk energy E is must satisfy

$$\dot{E} = \frac{1}{2}\dot{\phi}\phi' - \frac{1}{2}\dot{\psi}\psi' \ .$$

Using (6.6), (6.9) and then (6.8) gives

$$\begin{aligned}\dot{E} &= \frac{1}{2}\dot{\phi}\phi' - \frac{1}{2}\dot{\psi}\psi' \\ &= -\dot{\phi}D_\phi - \dot{\psi}D_\psi - \dot{\chi}_1 D_{\chi_1} - \dot{\chi}_2 D_{\chi_2} \\ &= -\dot{D}\end{aligned}$$

with zero being added in the form of $-\dot{\chi}_1 D_{\chi_1}$. Thus $E + D$ is conserved, meaning that the defect has contributed D to the energy of the system. It's no surprise that a modified energy is conserved, given the nature of the kinetic terms of the defect part of the Lagrangian - performing a Legendre transformation it is clear that these terms are absent from the Hamiltonian of the system and that the only contribution from the defect is the potential D.

6.1.1.5 Momentum conservation

When considered in components, the $a_2^{(2)}$ defect modifies the bulk momentum P by

$$\dot{P} = \frac{1}{4}\left(\dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi} + \phi'\phi' - \psi'\psi'\right) - \Phi + \Psi \, . \tag{6.14}$$

Squaring (6.6) and (6.9) give

$$\begin{aligned} \frac{1}{4}\left(\phi'\phi' - \psi'\psi'\right) &= -2\dot{\chi}_2\left(D_\phi + D_\psi\right) + D_\phi^2 - D_\psi^2 \\ &= -\dot{\chi}_2\left(D_{\chi_2}^- - D_{\chi_2}^+\right) - \dot{\chi}_1\left(D_{\chi_1}^- - D_{\chi_1}^+\right) + D_\phi^2 - D_\psi^2 \end{aligned}$$

by (6.11), (6.12) and the fact that $D_{\chi_1}^+ - D_{\chi_1}^- = 0$. Similarly, squaring and equating both sides of (6.13) results in

$$\frac{1}{4}\left(\dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi}\right) = -\dot{\phi}\left(D_\phi^- - D_\phi^+\right) - \dot{\psi}\left(D_\psi^- - D_\psi^+\right) + \left(D_\psi^- - D_\psi^+\right)^2 - \left(D_\phi^- - D_\phi^+\right)^2$$

and thus the momentum conservation equation reduces to

$$\dot{P} = -\left(\dot{D}^- - \dot{D}^+\right) + 4D_\phi^- D_\phi^+ - 4D_\psi^- D_\psi^+ - \Phi + \Psi$$

Since

$$\begin{aligned} 4D_\phi^- D_\phi^+ - 4D_\psi^- D_\psi^+ &= e^{-\phi} - e^{-\psi} + \left(e^{\phi} - e^{\psi}\right) e^{-2\chi_1 + \chi_2} \\ &= e^{-\phi} - e^{-\psi} + \left(e^{\phi} - e^{\psi}\right) \frac{2}{e^{\frac{\phi}{2}} + e^{\frac{\psi}{2}}} \\ &= e^{-\phi} + 2e^{\frac{\phi}{2}} - e^{-\psi} - 2e^{\frac{\psi}{2}} \\ &= \Phi - \Psi \end{aligned}$$

it is clear that there is a conserved momentum of $P + D^- - D^+$, as expected.

It has been shown that the defect described by the Lagrangian (6.4) is an integrable defect in $a_2^{(2)}$. The delay factors picked up by solitons passing through the defect preserve the $a_2^{(2)}$ solitons, while energy (fairly trivially) and momentum are modified by the defect but ultimately conserved. The more general approach in section 6.2 encompasses this defect, but this is a useful illustration as the algebra is of a low enough rank to consider things explicitly.

6.1.2 Further $a_2^{(2)}$ analysis

Because $a_2^{(2)}$ is of low rank it is more amenable to further analysis than the higher rank folded theories.

6.1.2.1 Comparison to the known $a_2^{(2)}$ defect

If the $a_2^{(2)}$ defect Lagrangian (6.4) is equivalent to the $a_2^{(2)}$ type II defect of Corrigan and Zambon [CZ09*b*, CZ11], then integrability follows. Both the papers [CZ09*b*] and [CZ11] offer differing Lagrangian descriptions of the $a_2^{(2)}$ defect, but both are taken to be equivalent. Here (6.4) is shown to be equivalent to the Lagrangian in [CZ09*b*], viz.

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(u\dot{v} - 2(u-v)\dot{\lambda} - D(u,v,\lambda)\right) \tag{6.15}$$

where

$$\begin{aligned} D(u,v,\lambda) = &\frac{1}{\sigma}\left(e^{u+v-2\lambda} + 2e^{\lambda}(e^{-u}+e^{-v})\right) \\ &+\sigma\left(4e^{-\lambda} + e^{2\lambda} + \frac{1}{2}e^{2\lambda}(e^{u-v}+e^{v-u})\right) \end{aligned} \tag{6.16}$$

and the bulk Lagrangians are given by

$$\begin{aligned} \mathcal{L}_u &= \frac{1}{2}\dot{u}\dot{u} - \frac{1}{2}u'u' - \left(e^{2u} + 2e^{-u} - 3\right) \\ \mathcal{L}_v &= \frac{1}{2}\dot{v}\dot{v} - \frac{1}{2}v'v' - \left(e^{2v} + 2e^{-v} - 3\right)\ . \end{aligned}$$

In order for (6.4) to match (6.15), the bulk Lagrangians must first match up, requiring:

$$\phi = -2u \qquad\qquad \psi = -2v\ .$$

After making this identification the kinetic terms in the bulk Lagrangian are a factor of 2 larger than what is sought, but this is remedied later by scaling the action. The next identification is the auxiliary field,

$$\chi_2 = 2\lambda - \tfrac{u+v}{2} + f$$

where $f = f(u-v)$ comes from the gauge symmetry evident in the Lagrangians (6.4) and (6.15). The identification for f that is required is

$$e^f = \frac{1}{2}\left(e^{\frac{u-v}{2}} + e^{\frac{v-u}{2}}\right) .$$

This identification requires a very careful analysis of the respective defect potentials in the Lagrangian, bearing in mind the algebraic constraint (6.10). The last identification in comparing (6.5) to (6.16) is the identification

$$e^\eta = \frac{\sigma}{\sqrt{2}} .$$

At this stage no terms in the Lagrangian apart from the bulk potentials have the correct scaling and what these identifications have given is

$$(6.4) \to \mathcal{L} = \theta(-x)\tilde{\mathcal{L}}_u + \theta(x)\tilde{\mathcal{L}}_v + \delta(x)\left(2u\dot{v} - 4(u-v)\dot{\lambda} - \sqrt{2}D(u,v,\lambda)\right) \quad (6.17)$$

where

$$\begin{aligned}\tilde{\mathcal{L}}_u &= \dot{u}\dot{u} - u'u' - \left(e^{2u} + 2e^{-u} - 3\right)\\ \tilde{\mathcal{L}}_u &= \dot{v}\dot{v} - v'v' - \left(e^{2v} + 2e^{-v} - 3\right) .\end{aligned}$$

The kinetic terms of (6.17) in the bulk and at the defect are a factor of 2 too large, which can be remedied by scaling the action $S \to \frac{1}{2}S$, which does not affect the dynamics. Upon this rescaling the bulk potential terms are a factor of 2 too small and the defect potential is a factor of $\sqrt{2}$ too small - both of these issues are corrected by taking an isotropic spacetime rescaling

$$x \to \sqrt{2}x \qquad\qquad t \to \sqrt{2}t .$$

The result of this process is (6.4) $\to$ (6.15). The type II defect $a_2^{(2)}$ does indeed arise from this process of combining defects and folding, but this fact is obscured by the approach in the original paper [CZ09*b*].

6.1.2.2 Further $a_2^{(2)}$ defects

One may wonder whether it is possible to construct further new integrable defects in $a_2^{(2)}$ by combining more than two $a_2^{(1)}$ defects and folding. Since the basic defect of $a_2^{(2)}$ has a type II Lagrangian, it might be expected that any further $a_2^{(2)}$ defects arise from the combination of N of these - so have type $2N$ Lagrangians for $N \in \mathbb{N}$ ('type $2N$' here means in the sense of the footnote of section 4.1.1.4, as these defects ultimately stem from combining $2N$ type I defects in $a_2^{(1)}$). Viewing the species 1 defect as associated to the fundamental $\mathbf{3}$ representation of a_2 and the species 2 defect as being associated to the conjugate fundamental $\bar{\mathbf{3}}$ representation of a_2 (or vice-versa), the basic $a_2^{(2)}$ defect arises from folding a 'mesonic' defect of $a_2^{(1)}$.

This section examines the possibility of taking a 'baryonic' type III defect in $a_2^{(1)}$ consisting of three species 1 defects combined and folding to get an $a_2^{(2)}$ defect. This defect is constructed by combining three species 1 defects, with differing rapidities η_1, η_2 and η_3. The combination of these defects gives the Lagrangian

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v \\ &\quad + \delta(x)\left(\frac{1}{2}uA\dot{u} + uB\dot{\chi} + \chi A\dot{\chi} + \chi B\dot{\rho} + \rho A\dot{\rho} + \rho B\dot{v} + \frac{1}{2}vA\dot{v} - D\right)\end{aligned}$$

so χ and ρ are thus auxiliary fields. The potential D is formed from the sum of three individual species 1 potentials of the form (4.2) with B given by (4.5), so $D = D^{(1)}(u,\chi)|_{\eta_1} + D^{(1)}(\chi,\rho)|_{\eta_2} + D^{(1)}(\rho,v)|_{\eta_3}$, where the superscript (1) denotes that the potential is that of a species 1 defect with given arguments.

The expectation may be that this defect is not compatible with folding - if the basic $a_2^{(2)}$ defect has a Lagrangian of type II then there should not be any type III defects in $a_2^{(2)}$. Nonetheless, the choice is made to fold the bulk fields $u \to \phi$ and $v \to \psi$. It is clear that folded fields will have no self-coupling terms at the defect, as $uA\dot{u}$ folds to some multiple of $\phi\dot{\phi}$, which is a total time derivative and hence can be scaled out of the action under the reasonable assumption that ϕ does not vary at temporal infinity. The situation is analogous with $vA\dot{v}$. The folded Lagrangian then takes the form

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi \\ &\quad + \delta(x)\left(\phi B\dot{\chi} + \chi A\dot{\chi} + \chi B\dot{\rho} + \rho A\dot{\rho} + \rho B\dot{\psi} - D\right) . \end{aligned} \tag{6.18}$$

The folded Lagrangian (6.18) can be used to get defect equations in vector form, the Euler–Lagrange equations at $x = 0$ being

$$\phi' = {}_pB\dot{\chi} - D_\phi \tag{6.19}$$
$$B^T\dot{\phi} = 2A\dot{\chi} + B\dot{\rho} - D_\chi \tag{6.20}$$
$$B^T\dot{\chi} = 2A\dot{\rho} + B\dot{\psi} - D_\rho \tag{6.21}$$
$$\psi' = {}_pB^T\dot{\rho} + D_\psi\ . \tag{6.22}$$

The subscript p in (6.19) and (6.22) means that the expression only makes sense in component form when projected onto the folded root space (this is because the folding has been done at the Lagrangian level, in an *a priori* manner, see section 4.1.3). The aim is to use (6.19)-(6.22) to examine the momentum conservation of the defect, but this results in an expression in the momentum conservation argument of ${}_p\dot{\chi}B^T{}_pB\dot{\chi} - {}_p\dot{\rho}B_pB^T\dot{\rho}$ which is non-zero (unlike the case in sections 6.1.1.5 or 6.2.5) and in general is not clear in meaning. The only approach then is to take the defect conditions in component form, resulting in

$$(\alpha_1 + \alpha_2)\cdot(6.19) \longrightarrow \phi' = 2\dot{\chi}_2 - 2D_\phi \tag{6.23}$$
$$\alpha_1\cdot(6.20) \longrightarrow 0 = -2\dot{\chi}_2 + 2\dot{\rho}_1 - D_{\chi_1} \tag{6.24}$$
$$\alpha_2\cdot(6.20) \longrightarrow \dot{\phi} = 2\dot{\chi}_1 - 2\dot{\rho}_1 + 2\dot{\rho}_2 - D_{\chi_2} \tag{6.25}$$
$$\alpha_1\cdot(6.21) \longrightarrow \dot{\psi} = 2\dot{\chi}_1 - 2\dot{\chi}_2 + 2\dot{\rho}_2 + D_{\rho_1} \tag{6.26}$$
$$\alpha_2\cdot(6.21) \longrightarrow 0 = 2\dot{\chi}_2 - 2\dot{\rho}_1 + D_{\rho_2} \tag{6.27}$$
$$(\alpha_1 + \alpha_2)\cdot(6.22) \longrightarrow \psi' = 2\dot{\rho}_1 + 2D_\psi \tag{6.28}$$

so it becomes immediate from (6.24) and (6.27) that there is an algebraic constraint

$$2\dot{\rho}_1 - 2\dot{\chi}_2 = D_{\chi_1} = D_{\rho_2} \tag{6.29}$$

which is something that is obscure when the equations are left in vector form. Note also that (6.25) and (6.26) give

$$2\left(\dot{\chi}_1 + \dot{\rho}_2\right) = \dot{\phi} + 2\dot{\rho}_1 + D_{\chi_2} = \dot{\psi} + 2\dot{\chi}_2 - D_{\rho_1} \ . \tag{6.30}$$

The momentum conservation argument relies upon the specific form of the defect potential D, which is

$$\begin{aligned} D = & \, e^{-\eta_1}\left(e^{-\frac{\phi}{2}-\chi_2} + e^{\chi_1} + e^{\frac{\phi}{2}-\chi_1+\chi_2}\right) + e^{-\eta_2}\left(e^{-\chi_1-\rho_2} + e^{\chi_1-\chi_2+\rho_1} + e^{\chi_2-\rho_1+\rho_2}\right) \\ & + e^{-\eta_3}\left(e^{-\rho_1-\frac{\psi}{2}} + e^{\rho_1-\rho_2+\frac{\psi}{2}} + e^{\rho_2}\right) \\ & + e^{\eta_1}\left(e^{-\frac{\phi}{2}+\chi_2} + e^{\frac{\phi}{2}-\chi_1} + e^{\chi_1-\chi_2}\right) + e^{\eta_2}\left(e^{-\chi_2+\rho_2} + e^{\chi_1-\rho_1} + e^{-\chi_1+\chi_2+\rho_1-\rho_2}\right) \\ & + e^{\eta_3}\left(e^{-\rho_2+\frac{\psi}{2}} + e^{\rho_1-\frac{\psi}{2}} + e^{-\rho_1+\rho_2}\right) \\ = & \sum_{j=1}^{9} m_j + \sum_{j=1}^{9} n_j \end{aligned} \tag{6.31}$$

where the first nine terms are of negative helicity and labelled by $\{m_i\}$; the last nine terms are of positive helicity and are labelled by $\{n_i\}$. Close examination of (6.31) reveals two useful identities

$$0 = 2D^-_{\phi} + 2D^-_{\psi} - D^-_{\chi_2} - D^-_{\rho_1} \tag{6.32}$$

$$0 = 2D^+_{\phi} + 2D^+_{\psi} + D^+_{\chi_1} + D^+_{\chi_2} + D^+_{\rho_1} + D^+_{\rho_2} \ . \tag{6.33}$$

Momentum conservation

It is expected that the defect (6.18) is energy conserving, as it does not break the time translation symmetry of the system. It is easily shown that there is a conserved energy in this case. It is not really expected that momentum should be conserved but the argument is developed here anyway. If momentum is conserved then two things that should hold are:

- The expectation is that if momentum is conserved then, as is the case in section 6.1.1.5, the conserved quantity will be $P + D^- - D^+$, so the aim is to show that $\dot{P} = -\left(\dot{D}^- - \dot{D}^+\right)$.
- The momentum is a difference between positive helicity terms and negative helicity terms so it will not be well defined if the algebraic constraints

mix helicities, thus (6.29) gives

$$D^-_{\chi_1} = D^-_{\rho_2} \tag{6.34}$$
$$D^+_{\chi_1} = D^+_{\rho_2} \ . \tag{6.35}$$

It is not obvious that there should be any choice of rapidities $\{\eta_1, \eta_2, \eta_3\}$ that allows this to happen.

The equation of momentum conservation in component form is the same as (6.14),

$$\dot{P} = \frac{1}{4}\left(\dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi} + \phi'\phi' - \psi'\psi'\right) - \Phi + \Psi \mid_{x=0} \ .$$

From squaring (6.23), (6.25), (6.26) and (6.28) in turn, it is seen that

$$\begin{aligned}
\frac{1}{4}\left(\phi'\phi' - \psi'\psi' + \dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi}\right) &= \frac{1}{4}\left(4D^2_\phi - 4D^2_\psi + D^2_{\chi_2} - D^2_{\rho_1}\right) + 2\dot{\chi}_1\left(\dot{\chi}_2 - \dot{\rho}_1\right) \\
&\quad + 2\dot{\rho}_2\left(\dot{\chi}_2 - \dot{\rho}_1\right) - \left(\dot{\chi}_1 + \dot{\rho}_2\right)\left(D_{\chi_2} + D_{\rho_1}\right) \\
&\quad - \dot{\chi}_2\left(2D_\phi - D_{\rho_1}\right) - \dot{\rho}_1\left(2D_\psi - D_{\chi_2}\right) \\
&= \frac{1}{4}\left(4D^2_\phi - 4D^2_\psi + D^2_{\chi_2} - D^2_{\rho_1}\right) - \dot{\chi}_1 D_{\chi_1} - \dot{\rho}_2 D_{\rho_2} \\
&\quad - \left(\dot{\chi}_1 + \dot{\rho}_2\right)\left(D_{\chi_2} + D_{\rho_1}\right) \\
&\quad - \dot{\chi}_2\left(2D_\phi - D_{\rho_1}\right) - \dot{\rho}_1\left(2D_\psi - D_{\chi_2}\right) \ .
\end{aligned}$$

The terms that have a helicity of -1 can be rearranged using (6.30) and (6.32) and the terms with positive helicity may be rearranged using (6.29), (6.30) and (6.33); the result is

$$\begin{aligned}
\frac{1}{4}\left(\phi'\phi' - \psi'\psi' + \dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi}\right) &= -\left(\dot{D}^- - \dot{D}^+\right) + \frac{1}{4}\left(4D^2_\phi - 4D^2_\psi + D^2_{\chi_2} - D^2_{\rho_1}\right) \\
&\quad + D^+_\phi\left(D_{\chi_1} + D_{\chi_2}\right) - D^+_\psi\left(D_{\rho_1} + D_{\rho_2}\right) \\
&\quad - D^-_\phi D_{\chi_2} + D^-_\psi D_{\rho_1} \\
&\quad + \frac{1}{2}D^+_{\chi_2} D_{\chi_1} - \frac{1}{2}D^+_{\rho_1} D_{\rho_2} \ .
\end{aligned} \tag{6.36}$$

Clearly then there are many terms quadratic in the gradients of D which are required to give the expected $\Phi - \Psi$. There are no rapidity factors in either Φ or Ψ so the terms of spin -2 and spin $+2$ in (6.36) must vanish.

The spin -2 terms are

$$D_\phi^- D_\phi^- - D_\psi^- D_\psi^- - D_\phi^- D_{\chi 2}^- + D_\psi^- D_{\rho 1}^- + \frac{1}{4} D_{\chi 2}^- D_{\chi 2}^- - \frac{1}{4} D_{\rho 1}^- D_{\rho 1}^-$$

which can be shown to vanish with repeated use of (6.32). Similarly the spin $+2$ terms are

$$D_\phi^+ D_\phi^+ - D_\psi^+ D_\psi^+ + D_\phi^+ D_{\chi 1}^+ + D_\phi^+ D_{\chi 2}^+ - D_\psi^+ D_{\rho 1}^+ - D_\psi^+ D_{\rho 2}^+ + \frac{1}{2} D_{\chi 1}^+ D_{\chi 2}^+ - \frac{1}{2} D_{\rho 1}^+ D_{\rho 2}^+ \\ + \frac{1}{4} D_{\chi 2}^+ D_{\chi 2}^+ - \frac{1}{4} D_{\rho 1}^+ D_{\rho 1}^+$$

which can be shown to vanish by repeated use of (6.33) but only if (6.35) holds.

The remaining terms, which are spinless, are

$$\begin{aligned} & D_\phi^- \left(2D_\phi^+ - D_{\chi 2}^+\right) + \left(D_{\chi 1}^- + D_{\chi 2}^-\right)\left(D_\phi^+ + \frac{1}{2} D_{\chi 2}^+\right) \\ & \quad - D_\psi^- \left(2D_\psi^+ - D_{\rho 1}^+\right) - \left(D_{\rho 1}^- + D_{\rho 2}^-\right)\left(D_\psi^+ + \frac{1}{2} D_{\rho 1}^+\right) \\ & = \frac{1}{2}\left(-m_1 + m_3\right)\left(-2n_1 + n_2 + n_3 + n_4 - n_6\right) \\ & \quad + \frac{1}{2}\left(-m_1 + m_2 - m_4 + m_6\right)\left(n_2 - n_3 - n_4 + n_6\right) \\ & \quad + \frac{1}{2}\left(m_7 - m_8\right)\left(n_5 - n_6 + n_7 - 2n_8 + n_9\right) \\ & \quad + \frac{1}{2}\left(m_4 - m_5 + m_7 - m_9\right)\left(-n_5 + n_6 + n_7 - n_9\right) \end{aligned} \tag{6.37}$$

where $\{m_i\}$ and $\{n_i\}$ are defined as in (6.31).

A long-winded way of writing the differences of the bulk potential is

$$\begin{aligned} \Phi - \Psi = & \; m_1 n_1 + m_2 n_2 + m_3 n_3 - m_1 n_2 - m_2 n_3 - m_3 n_1 \\ & + m_4 n_4 + m_5 n_5 + m_6 n_6 - m_4 n_5 - m_5 n_6 - m_6 n_4 \\ & + m_7 n_7 + m_8 n_8 + m_9 n_9 - m_7 n_8 - m_8 n_9 - m_9 n_7 \, . \end{aligned} \tag{6.38}$$

Finally, the algebraic constraints (6.34) and (6.35) become

$$D^-_{\chi_1} = D^-_{\rho_2} \implies m_2 - m_3 + m_5 - m_6 + m_8 - m_9 = 0 \qquad (6.39)$$

$$D^+_{\chi_1} = D^+_{\rho_2} \implies n_2 - n_3 + n_4 - n_5 + n_7 - n_9 = 0\ . \qquad (6.40)$$

By repeatedly applying (6.39) and (6.40), it can be shown that (6.37) does indeed reduce to (6.38). The key to this calculation is the fact that m_1, m_4, m_7, n_1, n_6 and n_8 are all absent from the algebraic constraints (6.39) and (6.40).

The result, which is unexpected, is that indeed $P + D^- - D^+$ is conserved after folding to $a_2^{(2)}$ provided that the algebraic constraints (6.29) do not mix helicities. It is unclear looking at the components whether or not there exist a choice of rapidities $\{\eta_1, \eta_2, \eta_3\}$ for which this is true. The fixing of rapidities is something which requires examination of the soliton delay factors.

Delay factors

As explained in section 4.1.1.4, the combining of defects does not affect the delay factor that a soliton picks up when passed through the defect system, so the combined defect here gives a delay factor which is the product of the delay factors of the individual species 1 defect.

The single soliton of $a_2^{(2)}$ can be represented by the tau functions

$$\tau_j^\phi = 1 + \omega^j E + \omega^{2j} E + AE^2$$

with $\omega = e^{\frac{2\pi i}{3}}$ and interaction parameter $A = \cos^2\left(\frac{\pi}{3}\right) = \frac{1}{4}$.

In passing through three defects, with rapidities η_1, η_2 and η_3 the field ψ must have the tau functions

$$\begin{aligned}\tau_j^\psi = {} & 1 + {}^1_1z(\theta-\eta_1)\,{}^1_1z(\theta-\eta_2)\,{}^1_1z(\theta-\eta_3)\omega^j E + {}^2_1z(\theta-\eta_1)\,{}^2_1z(\theta-\eta_2)\,{}^2_1z(\theta-\eta_3)\omega^{2j} E \\ & + A\,{}^1_1z(\theta-\eta_1)\,{}^1_1z(\theta-\eta_2)\,{}^1_1z(\theta-\eta_3)\,{}^2_1z(\theta-\eta_1)\,{}^2_1z(\theta-\eta_2)\,{}^2_1z(\theta-\eta_3)E^2\ .\end{aligned}$$

The symmetry requirement for ψ to be an $a_2^{(2)}$ soliton is $\tau_1^\psi = \tau_2^\psi$ so the requirement for integrability after folding is

$$ {}^1_1z(\theta-\eta_1)\,{}^1_1z(\theta-\eta_2)\,{}^1_1z(\theta-\eta_3)={}^2_1z(\theta-\eta_1)\,{}^2_1z(\theta-\eta_2)\,{}^2_1z(\theta-\eta_3)\;. \qquad (6.41)$$

Using the expressions for the delay factors given in (6.1) the two conditions required for (6.41) to hold are

$$\begin{aligned} e^{2\eta_1}+e^{2\eta_2}+e^{2\eta_3}&=0\\ e^{2\eta_1+2\eta_2}+e^{2\eta_1+2\eta_3}+e^{2\eta_2+2\eta_3}&=0\;. \end{aligned}$$

By fixing one of the rapidities, say $\eta_3=\eta$ and omitting solutions which can be obtained by switching $\eta_1\leftrightarrow\eta_2$ the possibilities are

$$\eta_1=\eta+\frac{2\pi i}{3} \qquad \eta_2=\eta-\frac{2\pi i}{3} \qquad (6.42)$$
$$\eta_1=\eta+\frac{\pi i}{3} \qquad \eta_2=\eta-\frac{\pi i}{3} \qquad (6.43)$$
$$\eta_1=\eta+\frac{\pi i}{3} \qquad \eta_2=\eta-\frac{2\pi i}{3} \qquad (6.44)$$
$$\eta_1=\eta-\frac{\pi i}{3} \qquad \eta_2=\eta+\frac{2\pi i}{3}\;. \qquad (6.45)$$

These results clarify why there can be a type III defect in $a_2^{(2)}$, the reason is that defect fusing is occurring, with either the type II defect or the trivial defect recovered. The combined defect is formed from three defects which will be referred to as defect 1 (with rapidity η_1), defect 2 (η_2) and defect 3 (η_3).

The first situation, (6.42), is one where the overall delay factor is trivial, i.e., ${}^1_1z(\theta-\eta_1)\,{}^1_1z(\theta-\eta_2)\,{}^1_1z(\theta-\eta_3)={}^2_1z(\theta-\eta_1)\,{}^2_1z(\theta-\eta_2)\,{}^2_1z(\theta-\eta_3)=1$. In that case two of the species 1 defects fuse to give a species 2 defect which has a rapidity difference of $i\pi$ from the third species 1 defect - hence there is annihilation of a defect and an anti-defect. In the second case, (6.43), defects 1 and 2 fuse to form a species 2 defect with rapidity η, which combined with defect 3 forms the same defect as that in section 6.1.1, only with a more complicated looking Lagrangian. The third and fourth possibilities, (6.44) and (6.45), are similar and have defects 2 and 3 fusing initially. One of the conclusions of this analysis is that the number of auxiliary fields, and hence 'type' of the defect, is not necessarily a useful way the classify the defect because there may be some redundancy in the Lagrangian description - a better way of classifying the defects comes from identifying the basic defects and applying the defect fusing bootstrap.

6.2 $a_r^{(1)}$ defects and folding

The idea that there are various species of defect, which has close analogy to the solitons, strongly suggests the possibility of constructing defect systems that can be folded just as solitons for the reduced theories $c_n^{(1)}$ [MM93, McG94*b*], $d_n^{(2)}$ and $a_{2n}^{(2)}$ (see section 2.2) can be constructed by folding certain $a_r^{(1)}$ soliton configurations. In each of $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ the basic soliton (which may be called a species 1 soliton of the folded theory) is constructed by taking a species 1 and a species r soliton of $a_r^{(1)}$ with the same rapidity θ and combining them, i.e., placing their centres of mass at the same location. In analogy it should follow that combining a species 1 and a species r defect, both with the same rapidity parameter η, should result in a foldable defect configuration (indeed the species 1 defect of the folded theory) - this is most fortunate since the species 1 and species r defects are already known [BCZ04*b*]. Compelling as this idea may be, it had not been considered when [Rob14*a*] was originally written, so the construction of folded defects actually informed the identification of different species of defects and of defect fusing rules; rather than the other way around.

In this section defects are constructed by folding $a_r^{(1)}$ defect configurations for general $r \geq 2$. This approach includes the $a_2^{(2)}$ defect of section 6.1.1. There are two type I defects (the species 1 and species r defects) first described in [BCZ04*b*] and neither of them possess the symmetry required for folding, however, the species 1 and species r defects are seen to be conjugate to each other in how they affect solitons transmitted through them - this is explained in section 6.2.3.

6.2.1 Combined defect of $a_r^{(1)}$

Consider a system with a species 1 defect at $x = a < 0$ and a species r defect at $x = 0$. The defect at $x = a$ is given the rapidity parameter η_1, though the defect is stationary, and the defect at $x = 0$ is given the rapidity η_2. The

Lagrange density describing this system may be written as

$$\begin{aligned}\mathcal{L} &= \theta(a-x)\mathcal{L}_u + \theta(x-a)\theta(-x)\mathcal{L}_\chi + \theta(x)\mathcal{L}_v \\ &\quad + \delta(a-x)\left(\tfrac{1}{2}uA\dot{u} + uB_1\dot{\chi} + \tfrac{1}{2}\chi A\dot{\chi} - D^{(1)}(u,\chi)\right) \\ &\quad + \delta(x)\left(-\tfrac{1}{2}\chi A\dot{\chi} + \chi B_1^T\dot{v} - \tfrac{1}{2}vA\dot{v} - D^{(r)}(\chi,v)\right) \ . \qquad (6.46)\end{aligned}$$

where $A = 1 - B_1$ and $B_1 + B_1^T = 2$, B_1 is given by (4.5); in what follows B will be used instead on B_1, as it is clear that it refers to the quantity of (4.5). The Lagrange density $\mathcal{L}_u$ is in the form of (2.1), $\mathcal{L}_v$ and $\mathcal{L}_\chi$ similarly. The defect potentials are then given by

$$\begin{aligned}D^{(1)} &= D^{(1)-} + D^{(1)+} = \sum_{j=0}^{r} f_j + \sum_{j=0}^{r} g_j \\ D^{(r)} &= D^{(r)-} + D^{(r)+} = \sum_{j=0}^{r} \tilde{f}_j + \sum_{j=0}^{r} \tilde{g}_j\end{aligned}$$

where the terms with f_i and those with $\tilde{f}_i$ possess negative helicity and those with g_i or $\tilde{g}_i$ are of positive helicity:

$$\begin{aligned}f_i &= e^{-\eta_1} e^{\frac{1}{2}\alpha_i\left(B_1^T u + B_1\chi\right)} & g_i &= e^{\eta_1} e^{\frac{1}{2}\alpha_i B_1(u-\chi)} \\ \tilde{f}_i &= e^{-\eta_2} e^{\frac{1}{2}\alpha_i\left(B_1^T v + B_1\chi\right)} & \tilde{g}_i &= e^{\eta_2} e^{\frac{1}{2}\alpha_i B_1(v-\chi)} \ . \qquad (6.47)\end{aligned}$$

The defects may be brought together, or 'combined', at the Lagrangian level by taking $a \to 0$ in (6.46) (c.f., the sine-Gordon case in [CZ09*b*, CZ10]) resulting in

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v \\ &\quad + \delta(x)\left(\frac{1}{2}uA\dot{u} + uB\dot{\chi} - vB\dot{\chi} - \frac{1}{2}vA\dot{v} - D^{(1)} - D^{(r)}\right) \ . \qquad (6.48)\end{aligned}$$

There no longer exists any bulk for the field χ; it is effectively trapped in the defect and hence may be referred to as an 'auxiliary field'. Note that in order for (6.48) to match the type II ansatz of [CZ11] the auxiliary field must be redefined to account for the presence of self-coupling defect terms for the bulk fields u and v.

The Euler–Lagrange equations of (6.48) give now the bulk equations for u and v as well as the defect conditions at $x = 0$

$$u' = A\dot{u} + B\dot{\chi} - D_u \tag{6.49}$$

$$v' = A\dot{v} + B\dot{\chi} + D_v \tag{6.50}$$

$$B^T\dot{u} + D_\chi^{(1)} = B^T\dot{v} - D_\chi^{(r)} \tag{6.51}$$

where D without a superscript refers to the combined defect potential $D = D^{(1)} + D^{(r)}$. The combined defect thus gives three vector equations as the defect conditions while the uncombined system of (6.46) has four; however, the delay factors received by solitons passing through the defect are unchanged by the combining process.

A modified (relative to the case of having no defects) conserved energy is associated the combined defect given by (6.48). The conserved energy is merely the Hamiltonian of the combined defect system and is given by $E + D$ where

$$E = \int_{-\infty}^{0} \frac{1}{2}\dot{u}\cdot\dot{u} + \frac{1}{2}u'\cdot u' + U \,\mathrm{d}x + \int_{0}^{\infty} \frac{1}{2}\dot{v}\cdot\dot{v} + \frac{1}{2}v'\cdot v' + V \,\mathrm{d}x$$

is the bulk contribution to the energy and D is the defect contribution.

It is less obvious that if two defects individually conserve momentum, the species 1 and species r defects [CZ09a], then combining them will conserve momentum overall, but it is easily shown. The bulk contribution to the momentum is given by the integral of T^{01} of the stress tensor derived from (6.48), so

$$P = \int_{-\infty}^{0} \dot{u}\cdot u' \,\mathrm{d}x + \int_{0}^{\infty} \dot{v}\cdot v' \,\mathrm{d}x \;.$$

Taking the time derivative and using the bulk equations of motion gives

$$\dot{P} = \frac{1}{2}\dot{u}\cdot\dot{u} + \frac{1}{2}u'\cdot u' - U - \frac{1}{2}\dot{v}\cdot\dot{v} - \frac{1}{2}v'\cdot v' + V \;|_{x=0}$$

so long as the fields and potentials are constant (in vacuum) at spatial infinity. Using the defect conditions (6.49), (6.50) and (6.51) one can then show that

$$\dot{P} = -\left(\dot{D}^- - \dot{D}^+\right) \tag{6.52}$$

and thus that $P + D^- - D^+$ is a conserved quantity.

Note that an analogous analysis holds if in the first instance in (6.46) the species 1 defect is taken to be to the right of the species r defect. This perhaps should come as no surprise as one may appeal to commutativity given that the defect conditions describe Bäcklund transformations [BCZ04*b*].

6.2.2 Folded defects

As is noted in chapter 3, the basic solitons of the theories that come from folding $a_r^{(1)}$ are formed from combining a species 1 and a species r soliton which possess the same momentum θ. This thesis is an attempt to establish the existence of various species of defect, at least in $a_r^{(1)}$ and its reduced theories, which is analogous to the case of the solitons. As such, it should not be surprising that combining a species 1 and a species r defect, as in (6.48), gives a defect which is compatible with folding, at least if the defect rapidities are identified correctly.

In this section defects are constructed for folded $a_r^{(1)}$ systems. Two methods are used to support the hypothesis that these are integrable defects in the folded theories: soliton delay factor matching and momentum conservation. In the first case, the soliton delay matching, it is shown how solitons preserve their forms when passed through the folded defect - this suggests that there should be an infinite number of conserved charges in the system. In the second case, showing that the one particular charge, the momentum, is conserved is also a very strong indication of integrability: this belief is supported by the results of [CZ09*a*] which show that for type I $a_r^{(1)}$ defects, the conservation of momentum implies all of the conditions for classical integrability found by the Lax pair approach in [BCZ04*b*].

The defect in the folded theory is obtained by folding the bulk fields $u \to \phi$ and $v \to \psi$ in (6.48). As was the case in $a_2^{(2)}$ in section 6.1.1 (which is an example of this general process), the auxiliary field χ cannot be folded, as analysis of the soliton delay factors will reveal.

One simplification that occurs in folding the Lagrangian (6.48) is that the self-coupling kinetic terms at the defect vanish because $\alpha'_i B \alpha'_j = \alpha'_i \cdot \alpha'_j$. Hence, since $A = 1 - B$,

$$\phi A \dot{\phi} = \psi A \dot{\psi} = 0$$

where the folding is denoted by $u \to \phi$ and $v \to \psi$. This means that the Lagrangian (6.48) folds to

$$\begin{aligned} \mathcal{L} = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi \\ + \delta(x)\left(\phi B\dot{\chi} - \psi B\dot{\chi} - D^{(1)}(\phi,\chi) - D^{(2)}(\chi,\psi)\right) \end{aligned} \tag{6.53}$$

which, unlike (6.48), fits the type II framework of [CZ11] without requiring redefinition of χ.

In vector form the Euler–Lagrange equations are thus

$$\phi' = {}_pB\dot{\chi} - D_\phi \tag{6.54}$$

$$\psi' = {}_pB\dot{\chi} + D_\psi \tag{6.55}$$

$$B^T\dot{\phi} + D^{(1)}_\chi = B^T\dot{\psi} - D^{(2)}_\chi \,. \tag{6.56}$$

As in section 6.1.2.2, the subscript p found in (6.54) and (6.55) in front of $B\dot{\chi}$ denotes 'projected' and indicates that equations (6.54) and (6.55) only make sense when projected onto the folded root space. Unlike in the case of the three combined species 1 defects in section 6.1.2.2, this turns out not to require any careful component form consideration in the momentum conservation argument.

Examination of the components of (6.56) gives the algebraic constraints

$$D_{\chi_i} + D_{\chi_{h-1-i}} = 0 \qquad \text{for } c_n^{(1)} \text{ and } a_{2n}^{(2)}$$

Or

$$D_{\chi_i} + D_{\chi_{h-i}} = 0 \qquad \text{for } d_n^{(2)}$$

which, using $D_\chi^- = \sum_j \frac{1}{2} B^T \alpha_j \left(f_j + \tilde{f}_j \right)$ and $D_\chi^+ = -\sum_j \frac{1}{2} B^T \alpha_j \left(g_j + \tilde{g}_j \right)$ may be put in the form

$$\begin{aligned} &f_i - f_{h-i} + \tilde{f}_i - \tilde{f}_{h-i} - g_i + g_{h-i} - \tilde{g}_i + \tilde{g}_{h-i} = 0 \qquad \text{for } c_n^{(1)} \text{ and } a_{2n}^{(2)} \\ &f_i - f_{h+1-i} + \tilde{f}_i - \tilde{f}_{h+1-i} - g_i + g_{h+1-i} - \tilde{g}_i + \tilde{g}_{h+1-i} = 0 \qquad \text{for } d_n^{(2)} \end{aligned} \tag{6.57}$$

where $\{f_i\}$, etc., have the same definition as in (6.47) but with u and v replaced by their folded equivalents ϕ and ψ.

6.2.3 Delay factors

Consider passing a soliton through the $a_r^{(1)}$ defect specified by (6.48). The soliton of interest is any of the fundamental solitons of the folded theory, given by (3.13), because the aim here is to show that this soliton retains its form after passing through the defect. Thus the soliton solution on the left of the defect, corresponding to a species p soliton of the folded theory, is given by

$$\begin{aligned} \tau_j^u &= 1 + \left(\omega^{pj} + \omega^{p(h-j)} \right) E_p + A E_p^2 \qquad \text{for folding to } c_n^{(1)} \text{ or } a_{2n}^{(2)} \\ \tau_j^u &= 1 + \left(\omega^{pj} + \omega^{p(h+1-j)} \right) E_p + A \omega^p E_p^2 \qquad \text{for folding to } d_n^{(2)} \end{aligned} \tag{6.58}$$

where the $a_{2n}^{(1)}$ theories may fold to the corresponding $a_{2n}^{(2)}$ [KS96*b*] and the $a_{2n-1}^{(1)}$ theories may fold to either $c_n^{(1)}$ or $d_n^{(2)}$ [KS96*b*]. Further reduction may be possible but does not give any new theories [Sas92] and only gives folded defects with redundancies in their Lagrangian description. In every case $A = \cos^2\left(\frac{\pi p}{h}\right)$. It will be henceforth assumed that the folding in question is to $c_n^{(1)}$ or $a_{2n}^{(2)}$ - the $d_n^{(2)}$ solitons, as well as quantities involved in the momentum conservation such as the algebraic constraints (6.57), can be recovered by taking $h \to h+1$ (when $h = 2n$) and $A \to A\omega^p$.

Evolving the soliton given by (6.58) through the defect given by (6.48) using the defect equations (6.49), (6.50) and (6.51) gives the soliton delay factors resulting in

$$\begin{aligned} \tau_j^\chi &= 1 + \left(\omega^{pj} z_p + \omega^{p(h-j)} z_{h-p} \right) E_p + A z_p z_{h-p} E_p^2 \\ \tau_j^v &= 1 + \left(\omega^{pj} z_p \tilde{z}_p + \omega^{p(h-j)} z_{h-p} \tilde{z}_{h-p} \right) E_p + A z_p z_{h-p} \tilde{z}_p \tilde{z}_{h-p} E_p^2 \end{aligned} \tag{6.59}$$

where the delay factors are

$$\begin{aligned} z_p &= {}^p_1 z(\theta-\eta_1) = \frac{ie^{\eta_1-\theta}+\omega^{\frac{p}{2}}}{ie^{\eta_1-\theta}+\omega^{-\frac{p}{2}}} & z_{h-p} &= {}^{h-p}_{\ \ 1} z(\theta-\eta_1) = \frac{ie^{\eta_1-\theta}-\omega^{-\frac{p}{2}}}{ie^{\eta_1-\theta}-\omega^{\frac{p}{2}}} \\ \tilde{z}_p &= {}^p_r z(\theta-\eta_2) = \frac{ie^{\eta_2-\theta}-\omega^{-\frac{p}{2}}}{ie^{\eta_2-\theta}-\omega^{\frac{p}{2}}} & \tilde{z}_{h-p} &= {}^{h-p}_{\ \ r} z(\theta-\eta_2) = \frac{ie^{\eta_2-\theta}+\omega^{\frac{p}{2}}}{ie^{\eta_2-\theta}+\omega^{-\frac{p}{2}}}\,. \end{aligned} \tag{6.60}$$

It is clear that u may be folded as (6.58) represents a folded soliton, this is the choice made initially. Equally clear is that in general $\tau^{\chi}_j \neq \tau^{\chi}_{h-j}$, since $z_p \neq z_{h-p}$ except in the single case that $p=n$ in $a^{(1)}_{2n-1}$, so it is not possible to fold the auxiliary field χ.

The aim of this soliton transmission argument is to find the circumstances under which the transmitted soliton, v, has the folding symmetry, so $\tau^v_j = \tau^v_{h-j}$. This is enough to show that the folded defect is able to preserve the forms of folded solitons - giving a very strong condition which may imply integrability. The reason for this claim is that any configuration that satisfies (6.49), (6.50) and (6.51), with the bulk fields u and v possessing the folding symmetry, also satisfies the folded defect conditions (6.54), (6.55) and (6.56) - see section 4.1.3. This is a key point: the delay factors are unaffected by the folding process, under the assumption that soliton solutions preserve their form. The requirement then is that $\tau^v_j = \tau^v_{h-j}$ and so it must be the case that

$$z_p\tilde{z}_p = z_{h-p}\tilde{z}_{h-p}$$

in which case every z may be absorbed into the definition of E_p as a time delay and phase shift. Note that this condition is the same condition as having the $a^{(1)}_r$ single soliton species p and species $h-p$ solutions receiving the same delay factor through the combined defect, as is the approach in section 6.1.1.1 for $a^{(2)}_2$. Thus, the condition for the soliton on the right of the defect to be in the folded theory is

$$0 = z_p\tilde{z}_p - z_{h-p}\tilde{z}_{h-p} = \frac{1}{\text{denom.}}\left[e^{-2\theta}\left(\omega^p-\omega^{-p}\right)\left(e^{2\eta_2}-e^{2\eta_1}\right)\right]$$

where 'denom.' is the common denominator obtained by multiplying all of the denominators of (6.60) together.

Thus, the folded defect represented by the Lagrangian (6.53) is only likely to be integrable if $\eta_2 = \eta_1$ or $\eta_2 = \eta_1 \pm i\pi$, as this is what is required for the soliton solution (6.59) to be compatible with folding. The two different possibilities mean:

- When $\eta_2 = \eta_1 \pm i\pi$ it is the case that $z_p \tilde{z}_p = z_{h-p}\tilde{z}_{h-p} = 1$. i.e., all of the solitons receive a trivial time delay. Indeed in this case if $\psi = \phi$ is imposed then the defect part of the Lagrangian (6.53) vanishes - so there is no defect there. The interpretation of this is that the second defect is the anti-defect of the first - combining them causes annihilation. Indeed this is what is expected from the fusing rules.

- When $\eta_2 = \eta_1$ there is a delay factor (different for each p) so this should represent a *bona fide* defect which does not destroy the form of the solitons (a strong constraint, suggesting that the defect is integrable). Note that if θ and $\eta_1 = \eta_2$ are real then the delay factor is real too - there is a non-trivial time delay or advance and no change of topological charge.

6.2.4 Energy conservation

The fact that energy is still conserved in the presence of a defect would appear to be a necessary condition for integrability, but does not appear to be a very strong condition. This is because it is quite easy to construct an energy conserving defect which is not integrable (for example, by removing the kinetic terms of the integrable defect). In a sense it is quite trivial to show that energy is conserved for the defect described by (6.53). Performing a Legendre transformation on (6.53) shows that $E + D$ is a conserved quantity, where E is the bulk energy given by

$$E = \int_{-\infty}^{0} \tfrac{1}{2}\dot{\phi}\cdot\dot{\phi} + \tfrac{1}{2}\phi'\cdot\phi' + \Phi \,\mathrm{d}x + \int_{0}^{\infty} \tfrac{1}{2}\dot{\psi}\cdot\dot{\psi} + \tfrac{1}{2}\psi'\cdot\psi' + \Psi \,\mathrm{d}x \; .$$

For completeness, energy conservation is shown here using the approach from chapter 4. By making use of the bulk equations of motion and assuming that the field configurations approach vacua at spatial infinity, it is seen that

$$\begin{aligned}
\dot{E} &= \dot{\phi}\cdot\phi' - \dot{\psi}\cdot\psi' \,|_{x=0} \\
&= \dot{\phi}B\dot{\chi} - \dot{\phi}\cdot D_\phi - \dot{\psi}B\dot{\chi} - \dot{\psi}\cdot D_\psi \\
&= -\dot{\phi}\cdot D_\phi - \dot{\chi}\cdot D_\chi - \dot{\psi}\cdot D_\psi = -\dot{D}
\end{aligned}$$

where in going from the first to the second line (6.54) and (6.55) have been applied, then in going to the third line (6.56) has been rearranged and applied. The result is that $\frac{\mathrm{d}}{\mathrm{d}t}(E+D)=0$, so indeed $E+D$ is conserved.

6.2.5 Momentum conservation

The possibility of having a conserved momentum for the folded defect (6.53) is called into question by the fact that the algebraic constraints (6.57) may mix helicities. Since the momentum is difference between terms of definite helicity, any helicity mixing constraint would appear to make the momentum ill-defined. The algebraic constraints therefore warrant further investigation. Explicitly (for $c_n^{(1)}$ and for $a_{2n}^{(2)}$) the negative helicity terms appearing in (6.57) are

$$\begin{aligned} f_i - f_{h-i} + \tilde{f}_i - \tilde{f}_{h-i} = e^{-\eta_1}&\left(e^{u_i-u_{i+1}+\chi_i-\chi_{i-1}} - e^{u_i-u_{i-1}+\chi_{h-i}-\chi_{h-1-i}}\right) \\ +\, e^{-\eta_2}&\left(e^{v_i-v_{i+1}+\chi_i-\chi_{i-1}} - e^{v_i-v_{i-1}+\chi_{h-i}-\chi_{h-1-i}}\right) \end{aligned}$$

where usually $u_i=\frac{\phi_i}{2}$ and $v_i=\frac{\phi_i}{2}$, the exception being in folding $a_{2n-1}^{(1)}$ to $c_n^{(1)}$ where $u_n=\phi_n$ and $v_n=\psi_n$. Labels are taken to hold modulo h. The positive helicity terms are

$$\begin{aligned} g_i - g_{h-i} + \tilde{g}_i - \tilde{g}_{h-i} = e^{\eta_1}&\left(e^{u_i-u_{i-1}-\chi_i+\chi_{i-1}} - e^{u_i-u_{i+1}-\chi_{h-i}+\chi_{h-1-i}}\right) \\ +\, e^{\eta_2}&\left(e^{v_i-v_{i-1}-\chi_i+\chi_{i-1}} - e^{v_i-v_{i+1}-\chi_{h-i}+\chi_{h-1-i}}\right) \end{aligned}$$

where the same interpretation is to be taken for what u_i and v_i represent.

What is desired is a choice of defect rapidities η_1 and η_2 that will allow for the positive and negative helicity parts of the algebraic constraints (6.57) to vanish separately. There are two solutions, namely

$$\eta_2 = \eta_1 \tag{6.61}$$

and

$$\eta_2 = \eta_1 \pm i\pi\ . \tag{6.62}$$

In either case, (6.61) or (6.62), the identification means that

$$g_i - g_{h-i} + \tilde{g}_i - \tilde{g}_{h-i} = -e^{2\eta_1} e^{-\chi_i + \chi_{i-1} - \chi_{h-i} + \chi_{h-1-i}} \left(f_i - f_{h-i} + \tilde{f}_i - \tilde{f}_{h-i} \right)$$

in which case the algebraic constraints (6.57) give

$$\begin{aligned} D^-_{\chi_i} + D^-_{\chi_{h-1-i}} &= 0 \\ D^+_{\chi_i} + D^+_{\chi_{h-1-i}} &= 0 \end{aligned}$$

or

$$f_i - f_{h-i} + \tilde{f}_i - \tilde{f}_{h-i} = 0 \qquad (6.63)$$

$$g_i - g_{h-i} + \tilde{g}_i - \tilde{g}_{h-i} = 0 \; . \qquad (6.64)$$

Note that in the above reasoning h should be replaced with $h+1$ if the theory under consideration is $d_n^{(2)}$. The identification of the folded field components $\{\phi_i\}$ from $\{u_i\}$ is slightly different, as explained in section 2.2.2 and illustrated for the $d_3^{(2)}$ defect in section 6.2.6.

It is noteworthy that a condition which is necessary for momentum conservation, that (6.61) or (6.62) hold, is precisely the same condition found in the delay factor argument of section 6.2.3. This links together the two approaches used here to support the hypothesis that the folded defects are integrable.

The bulk contribution to the momentum is given by the integral of T^{01}, so

$$P = \int_{-\infty}^{0} \dot{\phi} \cdot \phi' \, \mathrm{d}x + \int_{0}^{\infty} \dot{\psi} \cdot \psi' \, \mathrm{d}x \; .$$

Taking the time derivative and using the bulk equations of motion gives

$$\dot{P} = \frac{1}{2} \left(\phi' \cdot \phi' + \dot{\phi} \cdot \dot{\phi} - \psi' \cdot \psi' - \dot{\psi} \cdot \dot{\psi} \right) - \Phi + \Psi \,|_{x=0} \qquad (6.65)$$

provided the fields are constant (in vacuum) at spatial infinity. A modified momentum $P + C$ is conserved if the right-hand side of (6.65) can be written as $-\frac{\mathrm{d}C}{\mathrm{d}t}$, so the aim is to show that this is the case. The form of conserved momentum in all previous cases suggest that $C = D^- - D^+$. The first and third terms on the right-hand side of (6.65) can be re-expressed, by taking the difference of the squares of (6.54) and (6.55), as

$$\phi' \cdot \phi' - \psi' \cdot \psi' = -2\dot{\chi}\left(B^T D_\phi + B^T D_\psi\right) + D_\phi^2 - D_\psi^2 \,. \tag{6.66}$$

It is fortunate that the issue of interpreting ${}_pB\dot{\chi}$ is not raised by (6.66). Both $\dot{\phi} \cdot \dot{\phi}$ and $\dot{\psi} \cdot \dot{\psi}$ contain the term $\left({}_pB\dot{\chi}\right)^T {}_pB\dot{\chi}$, the meaning of which is unclear, but this term is cancelled out in (6.66). Because D_ϕ and D_ψ are both projected onto the folded root space it makes sense to drop the p subscript from (6.66).

At this stage progress can be made by anticipating the form of the final answer to be $\dot{P} = -\left(\dot{D}^- - \dot{D}^+\right)$, which requires the term $-\dot{\chi}\left(D_\chi^- - D_\chi^+\right)$. The only place that $\dot{\chi}$ may appear in (6.65) stems from $\phi' \cdot \phi' - \psi' \cdot \psi'$ and so the conclusion is that

$$B^T D_\phi^- + B^T D_\psi^- = D_\chi^- \tag{6.67}$$

$$B^T D_\phi^+ + B^T D_\psi^+ = -D_\chi^+ \,. \tag{6.68}$$

By using the relations

$$D_\phi^- = \sum_{j=0}^{h-1} \frac{1}{4}\left(B\alpha_j + B^T\alpha_{h-j}\right) f_j \qquad D_\phi^+ = \sum_{j=0}^{h-1} \frac{1}{4}\left(B^T\alpha_j + B\alpha_{h-j}\right) g_j \tag{6.69}$$

$$D_\psi^- = \sum_{j=0}^{h-1} \frac{1}{4}\left(B\alpha_j + B^T\alpha_{h-j}\right) \tilde{f}_j \qquad D_\psi^+ = \sum_{j=0}^{h-1} \frac{1}{4}\left(B^T\alpha_j + B\alpha_{h-j}\right) \tilde{g}_j \tag{6.70}$$

and the constraints (6.63), (6.64) along with the fact that $B + B^T = 2$, it can be shown that (6.67) and (6.68) are in fact true. Since (6.67) and (6.68) are both true, (6.56) may be rewritten, noting that B^T is invertible, as

$$\dot{\phi} + D_\phi^- - D_\phi^+ = \dot{\psi} - D_\psi^- + D_\psi^+ \,. \tag{6.71}$$

The equation (6.71) may then be squared on both sides to give $\dot{\phi} \cdot \dot{\phi} - \dot{\psi} \cdot \dot{\psi}$

thus reducing (6.65) to

$$\dot{P} = -\dot{D}^- + \dot{D}^+ + 2D_\phi^- D_\phi^+ - 2D_\psi^- D_\psi^+ - \Phi + \Psi \ .$$

This is almost what is sought, only requiring that

$$2D_\phi^- D_\phi^+ - 2D_\psi^- D_\psi^+ = \Phi - \Psi \ .$$

In terms of $\{f_i\}$, $\{g_i\}$, $\{\tilde{f}_i\}$ and $\{\tilde{g}_i\}$ the bulk potentials are given by

$$\begin{aligned}
\Phi &= \sum_{j=0}^{r} \left(f_j g_j - f_j g_{j+1}\right) = \frac{1}{2}\sum_{j=0}^{r} \left(\alpha_i B^T \alpha_j\right) f_j g_j \\
\Psi &= \sum_{j=0}^{r} \left(\tilde{f}_j \tilde{g}_j - \tilde{f}_j \tilde{g}_{j+1}\right) = \frac{1}{2}\sum_{j=0}^{r} \left(\alpha_i B^T \alpha_j\right) \tilde{f}_j \tilde{g}_j
\end{aligned}$$

where $g_{r+1} \equiv g_0$ and $\tilde{g}_{r+1} \equiv \tilde{g}_0$. Using (6.69) and (6.70) along with $B + B^T = 2$ it is thus seen that

$$\begin{aligned}
2D_\phi^- D_\phi^+ - 2D_\psi^- D_\psi^+ &= \frac{1}{2}\sum_{i,j} \left(f_i g_j - \tilde{f}_i \tilde{g}_j\right)\left(\alpha_i B^T \alpha_j\right) + \sum_{i,j} M_{ij}\left(f_i g_j - \tilde{f}_i \tilde{g}_j\right) \\
&= \Phi - \Psi + \sum_{i,j} M_{ij}\left(f_i g_j - \tilde{f}_i \tilde{g}_j\right)
\end{aligned}$$

where

$$M_{ij} = \frac{1}{8}\left(\alpha_i - \alpha_{h-i}\right) B B^T \left(\alpha_{h-j} - \alpha_j\right) \ .$$

It can be shown using the algebraic constraints (6.63) and (6.64) that this extra term vanishes and so the conserved momentum has the expected form, i.e., $P + D^- - D^+$, provided at least that the two defect parameters are related by (6.61) or (6.62). As is noted in section 6.2.3, in the latter case putting $\phi = \psi$ in the Lagrangian removes the defect altogether - this is a defect annihilating with an anti-defect, as is expected from the fusing rules. Only then in the first case $\eta_2 = \eta_1$ is there actually a defect of the folded theory.

6.2.6 Specific cases

The Lagrangian (6.53) represents defects for the folded theories $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ in general terms (thereby making it possible to consider momentum conservation in general terms), however, what (6.53) becomes in specific cases of interest may be obscure. For this reason it is helpful to consider representatives from each family of folded theory.

The most obvious case to examine is the $a_2^{(2)}$ case, as it is the simplest of the folded theories. The $a_2^{(2)}$ case is what is examined in section 6.1.1, and was the case which initially motivated this work as it was hinted at in [CZ09*b*] (in terms of Bäcklund transformations). The work of Corrigan and Zambon [CZ09*b*, CZ11] also means that there is particularly strong evidence of integrability of the defect given by (6.4). Because this case has been done in detail the $a_4^{(2)}$ case is briefly outlined below. In each case the choice is made that $\eta_2 = \eta_1 = \eta$, such that the defects give non-trivial delay factors.

6.2.6.1 $c_2^{(1)}$ defect

The case of $a_3^{(1)} \to c_2^{(1)}$ is of interest as it is the simplest canonical folding of the $a_r^{(1)}$ theories (the simplest folding considered in [OT83*a*]). The starting point here is to take the unfolded Lagrangian (6.48) with u and v as $a_3^{(1)}$ fields and χ similarly chosen to have three components[13]. The folding is achieved, as in chapter 2, by setting

$$u_1 = u_3 = \frac{\phi_1}{2}\ ,\ u_2 = \phi_2\ ,\ v_1 = v_3 = \frac{\psi_1}{2}\ ,\ v_2 = \psi_2$$

resulting in

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi \\ &\quad + \delta(x)\left[(\phi_1 - \psi_1)(\dot\chi_1 - \dot\chi_2 + \dot\chi_3) + (\phi_2 - \psi_2)(-2\dot\chi_1 + 2\dot\chi_2) - D(\phi,\psi,\chi)\right] \qquad (6.72)\end{aligned}$$

where

$$\begin{aligned}D(\phi,\psi,\chi) = e^{-\eta}&\left(e^{-\frac{\phi_1}{2}-\chi_3} + e^{\frac{\phi_1}{2}-\phi_2+\chi_1} + e^{-\frac{\phi_1}{2}+\phi_2-\chi_1+\chi_2} + e^{\frac{\phi_1}{2}-\chi_2+\chi_3} + (\phi \to \psi)\right) \\ + e^{\eta}&\left(e^{\frac{\phi_1}{2}+\chi_3} + e^{\frac{\phi_1}{2}-\chi_1} + e^{-\frac{\phi_1}{2}+\phi_2+\chi_1-\chi_2} + e^{\frac{\phi_1}{2}-\phi_2+\chi_2-\chi_3} + (\phi \to \psi)\right)\ .\end{aligned}$$

[13] Generally in $a_r^{(1)}$ it will be assumed that $\chi = \sum_{j=1}^{r} \chi_j \alpha_j$.

The single algebraic constraint, $D_{\chi_1} + D_{\chi_2} = 0$, which can be seen to arise in the kinetic terms at the defect in (6.72), allows one of the degrees of freedom of the auxiliary field χ to be removed, leaving two degrees of freedom - the same number as have the $c_2^{(1)}$ fields ϕ and ψ.

6.2.6.2 $d_3^{(2)}$ DEFECT

Note that $d_2^{(2)}$, possessing a single field, is just the sinh-Gordon case, so the first new case to consider here is $a_5^{(1)} \to d_3^{(2)}$. The Lagrangian (6.48) should then be considered with $a_5^{(1)}$ fields and folding achieved by setting

$$u_1 = v_1 = 0 \ , \ u_2 = u_5 = \frac{\phi_1}{2} \ , \ u_3 = u_4 = \frac{\phi_2}{2} \ , \ v_2 = v_5 = \frac{\psi_1}{2} \ , \ v_3 = v_4 = \frac{\psi_2}{2}$$

resulting in

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi \\ &\quad + \delta(x)\left[(\phi_1 - \psi_1)(-\dot\chi_1 + \dot\chi_2 - \dot\chi_4 + \dot\chi_5) + (\phi_2 - \psi_2)(-\dot\chi_2 + \dot\chi_4) - D(\phi, \psi, \chi)\right]\end{aligned}$$

where

$$\begin{aligned}D(\phi, \psi, \chi) &= e^{-\eta}\left(2e^{-\chi_5} + 2e^{-\chi_2+\chi_3}\right) + e^{\eta}\left(2e^{-\chi_1} + 2e^{\chi_3-\chi_4}\right) \\ &\quad + e^{-\eta}\left(e^{-\frac{\phi_1}{2}+\chi_1} + e^{\frac{\phi_1-\phi_2}{2}-\chi_1+\chi_2} + e^{\frac{-\phi_1+\phi_2}{2}-\chi_3+\chi_4} + e^{\frac{\phi_1}{2}-\chi_4+\chi_5} + (\phi \to \psi)\right) \\ &\quad + e^{\eta}\left(e^{-\frac{\phi_1}{2}+\chi_5} + e^{\frac{\phi_1}{2}+\chi_1-\chi_2} + e^{\frac{-\phi_1+\phi_2}{2}+\chi_2-\chi_3} + e^{\frac{\phi_1-\phi_2}{2}+\chi_4-\chi_5} + (\phi \to \psi)\right) \ .\end{aligned}$$

The algebraic constraints in this case are $D_{\chi_1} + D_{\chi_5} = 0$, $D_{\chi_2} + D_{\chi_4} = 0$ and $D_{\chi_3} = 0$, which may be used to reduce the number of degrees of freedom in χ from five down to two.

6.2.6.3 $a_4^{(2)}$ DEFECT

Since a defect for $a_2^{(2)}$ is already known [CZ09b, CZ11], the first new case arising from this analysis is $a_4^{(1)} \to a_4^{(2)}$. The folding is done by setting

$$u_1 = u_4 = \frac{\phi_1}{2} \ , \ u_2 = u_3 = \frac{\phi_2}{2} \ , \ v_1 = v_4 = \frac{\psi_1}{2} \ , \ v_2 = v_3 = \frac{\psi_2}{2}$$

resulting in

$$\begin{aligned}\mathcal{L} &= \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi \\ &\quad + \delta(x)\left[(\phi_1 - \psi_1)(\dot{\chi}_1 - \dot{\chi}_3 + \dot{\chi}_4) + (\phi_2 - \psi_2)(-\dot{\chi}_1 + \dot{\chi}_3) - D(\phi, \psi, \chi)\right]\end{aligned}$$

where

$$\begin{aligned}D(\phi,\psi,\chi) &= e^{-\eta}\left(2e^{-\chi_1+\chi_2}\right) + e^{\eta}\left(2e^{\chi_2-\chi_3}\right) \\ &\quad + e^{-\eta}\left(e^{-\frac{\phi_1}{2}-\chi_4} + e^{\frac{\phi_1-\phi_2}{2}+\chi_1} + e^{\frac{-\phi_1+\phi_2}{2}-\chi_2+\chi_3} + e^{\frac{\phi_1}{2}-\chi_3+\chi_3} + (\phi\to\psi)\right) \\ &\quad + e^{\eta}\left(e^{-\frac{\phi_1}{2}+\chi_4} + e^{\frac{\phi_1}{2}-\chi_1} + e^{\frac{-\phi_1+\phi_2}{2}+\chi_1-\chi_2} + e^{\frac{\phi_1-\phi_2}{2}+\chi_3-\chi_4} + (\phi\to\psi)\right) .\end{aligned}$$

The algebraic constraints, $D_{\chi_1} + D_{\chi_3} = 0$ and $D_{\chi_2} = 0$, may be used to reduce the number of degrees of freedom of χ from four down to two.

6.3 Summary

This chapter begins with the construction of a combined defect in $a_2^{(1)}$ which is then folded to give a candidate integrable defect for $a_2^{(2)}$. The defect preserves soliton solutions and conserves energy and momentum even after folding, giving strong indications of integrability. The $a_2^{(2)}$ case is amenable to further analysis and it is shown here that this $a_2^{(2)}$ defect corresponds to the type II defect found by Corrigan and Zambon [CZ09*b*, CZ11], meaning that their analyses may be taken into account. An attempt to construct further $a_2^{(2)}$ defects is made with a 'type III' Lagrangian but does not give anything new and just gives a more complicated description of the basic $a_2^{(2)}$ defect.

The latter part of the chapter is devoted to the construction of a candidate integrable defect for each of the folded theories arising from the $a_r^{(1)}$ family of ATFTs. The defect is constructed by taking a species 1 and a species r defect of $a_r^{(1)}$ and combining them. The bulk fields are folded either side of the defect and integrability is tested using soliton delay matching and momentum conservation. Both approaches give the same conditions on the constituent defect rapidities, that (6.61) holds or (6.62) holds. The chapter ends with one representative taken from each family of folded theory: $c_2^{(1)}$, $d_3^{(2)}$ and $a_4^{(2)}$. Their Lagrangians and defect potentials are given explicitly in component form.

7

Quantum defect fusing rules

This chapter takes a quantum mechanical approach to the topic of defect fusing rules. Much of this can be found in the paper *Defect fusing rules in affine Toda field theory* [Rob14*b*].

The consistency of the idea of defect fusing rules at the classical level is shown in chapter 5 where the main quantity of interest is the soliton delay factor. To firmly establish the idea of defect fusing rules it is necessary to find results to support the idea at the quantum level and for comparison to the classical results the quantum analogue of the soliton delay factor is necessary - this is the transmission matrix, the T-matrix, for a soliton passing through a defect.

In this chapter it is assumed that all of the weights of the p-th fundamental representation correspond to topological charges of the species p soliton. As noted previously (section 3.1.1.2) this is not the case in the classical theory [McG94*a*, McG94*b*]. This disparity is not resolved in this thesis, although it remains possible still to make some comparison of the classical and quantum results.

7.1 Defect fusing rule algebra

As is shown in chapter 3 the scattering of solitons at the quantum level can be conveniently framed in terms of the Faddeev–Zamolodchikov (FZ) algebra. The FZ algebra was extended by Corrigan and Zambon to include defects [CZ07] and this 'defect FZ algebra' explained in detail in section 4.2.

In the quantum picture one can represent a species p soliton possessing rapidity θ and topological charge label i as an operator ${}^pA_i(\theta)$. What the topological charge label means is dependent on the species p of the soliton in question. For a species 1 soliton in $a_r^{(1)}$ the label i refers to the soliton possessing the topological charge $l_i = -\frac{1}{h}\sum_{j=i}^{i+r} j\alpha_j$ where $\alpha_{k+h} \equiv \alpha_k$ with $h = r+1$ being the Coxeter number of the algebra. For a species r soliton the convention is that the label i denotes a topological charge of $-l_i$, where l_i is defined as above [CZ07, CZ09a]. For other species of soliton the charges can be labelled as in [CZ09a] - a similar convention is used here which uses fewer indices.

The defect can be similarly defined in the quantum theory as an operator. A species q defect possessing the rapidity parameter η and carrying topological charge α is denoted by the operator ${}_qD_\alpha(\eta)$. The defects in this chapter are taken to be 'ground state' defects (see section 4.1.1) so do not change the species of a soliton passing through them: a consequence of this is that the possible defect topological charges lie on the root lattice. Unlike with soliton labels, the label α means that the actual defect topological charge is α.

With both solitons and defects defined the process of a species p soliton being transmitted left-to-right through a defect (hence $\mathrm{Re}(\theta) > 0$) is described by the defect FZ algebra by

$$ {}^pA_i(\theta)\, {}_qD_\alpha(\eta) = {}^p_qT^{n\lambda}_{i\alpha}(\theta-\eta)\,{}_qD_\lambda(\eta)\, {}^pA_n(\theta)\ . \tag{7.1} $$

In (7.1) the in-state, ${}^pA_i(\theta)\, {}_qD_\alpha(\eta)$ which is where the soliton is to the left of the defect, is related to the out state, $D_\lambda(\eta)\, {}^pA_n(\theta)$, by means of the T-matrix. The process conserves overall topological charge, so if the soliton is a species 1 soliton ($p = 1$) then $l_i + \alpha = l_n + \lambda$. Only for species 1 and species r defects have any of these T-matrices appeared in the literature [CZ07, CZ09a].

One can find the transmission matrix for a species 2 soliton from that of a species 1 soliton in a simple manner

$$ {}^2T^{(ab)\lambda}_{(jk)\alpha}(\theta-\eta)c^{(ab)} = c^{(jk)}\,{}^1T^{a\lambda}_{j\beta}(\theta-\eta-\tfrac{i\pi}{h})\,{}^1T^{b\beta}_{k\alpha}(\theta-\eta+\tfrac{i\pi}{h}) + (j \leftrightarrow k) \tag{7.2} $$

with no sum implied. This is a consequence of the soliton fusing rules and is discussed in section 4.2.3.

7.1.1 Quantum defect fusing rules

It is proposed now that the operator for a species 2 defect can be written in terms of operators for species 1 defects in a manner analogous to how soliton fusing occurs, i.e.,

$$ {}_2D_\alpha(\eta) = \sum_{\beta,\gamma,\beta+\gamma=\alpha} d_{11}^{\beta,\gamma}\, {}_1D_\beta(\eta + \tfrac{i\pi}{h})\, {}_1D_\gamma(\eta - \tfrac{i\pi}{h}) \tag{7.3} $$

where $\{d^{\beta,\gamma}\}$ are the defect fusing couplings. As is motivated in chapter 5, the fusing angles for defects have been taken to be the same as the fusing angles in the analogous soliton fusing process. By using the defect fusing equation (7.3) with (7.1) the transmission matrix for a soliton through a species 2 defect can be written in terms of the transmission matrices through species 1 defects as

$$ {}_2T_{i\alpha}^{n\lambda}(\theta-\eta)d_{11}^{\delta,\epsilon} = \sum_{\beta,\gamma,j} d_{11}^{\beta,\gamma}\, {}_1T_{i\beta}^{j\delta}(\theta-\eta-\tfrac{i\pi}{h})\, {}_1T_{j\gamma}^{n\epsilon}(\theta-\eta+\tfrac{i\pi}{h}) \tag{7.4} $$

where $\beta+\gamma=\alpha$ and $\delta+\epsilon=\lambda$.

If the ratios of defect fusing couplings in (7.4) were known then it would be possible to write the transmission matrix for the general species 2 defect[14], since the species 1 transmission matrices are already known [CZ09*a*]. Unfortunately the defect fusing couplings are not known, but in $a_2^{(1)}$ they are the only unknown quantity so warrant further investigation, as is done in section 7.2. Even without knowing about the defect fusing couplings, (7.4) is still useful as it can provide an ansatz for the form of the transmission matrix for the species 2 defect - this is used along with consistency conditions to find the transmission matrix for the species 2 defect of $a_3^{(1)}$ in section 7.3.

In general it is expected that a species q_1 and a species q_2 defect should be able to fuse to form a species $q = q_1 + q_2 \pmod h$ defect[15], so

$$ {}_qT_{i\alpha}^{n\lambda}(\theta-\eta)d_{q_1q_2}^{\delta,\epsilon} = \sum_{\beta,\gamma,j} d_{q_1q_2}^{\beta,\gamma}\, {}_{q_1}T_{i\beta}^{j\delta}(\theta-\eta-\tfrac{i\pi q_2}{h})\, {}_{q_2}T_{j\gamma}^{n\epsilon}(\theta-\eta+\tfrac{i\pi q_1}{h}) \tag{7.5} $$

[14] The species of soliton is not specified here but the usual thing to do would be to consider a species 1 soliton and generate other solutions using soliton fusing rules.

[15] If $q=0$ then this is where a defect and anti-defect have annihilated so there is no defect there, although this is not obvious from the Lagrangian.

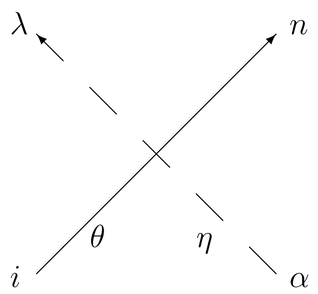

Figure 7.1: Illustration of a crossing relation involving defects and anti-defects.

where again $\beta + \gamma = \alpha$ and $\delta + \epsilon = \lambda$. If the ratios of the defect fusing couplings were known in general it would be possible to write all fundamental defect transmission matrices in terms of species 1 transmission matrices.

7.1.2 New crossing relation

Another factor in establishing the existence of a defect bootstrap comes from crossing symmetry. The crossing symmetry established in the literature [CZ07, CZ09a], given in section 4.2.2, links the transmission of solitons and antisolitons through the same defect. A slightly different set-up, illustrated in figure 7.1, links the transmission of a soliton through a defect to the transmission of the same soliton through an anti-defect.

Consider figure 7.1 with time running upwards as is standard. The solid line represents the soliton of species p, say, and the dashed line represents the defect of species q, say. The picture represents the transmission process with the transmission matrix

$$ {}^{p}_{q}T^{n\lambda}_{i\alpha}(\theta - \eta)\ . \tag{7.6}$$

Crossing gives an alternative viewpoint of figure 7.1. With time running left-to-right and with appropriate rapidity shifts, the picture shows a soliton moving right-to-left (in a spatial sense) through an anti-defect. It is clear that it is an anti-defect (so of species $h - q$) as the arrow points backwards in time; as it is an anti-defect the topological charge it carries is the opposite of that of the defect. This means that the T-matrix describing the process is

$$ {}^{\ \ \ p}_{h-q}\tilde{T}^{n(-\alpha)}_{i(-\lambda)}(i\pi + \eta - \theta) \tag{7.7}$$

where the tilde above T denotes that this is right-to-left transmission.

What crossing symmetry does is equate the two viewpoints of figure 7.1, so (7.6) and (7.7) are equal

$$ {}^{p}_{q}T^{n\lambda}_{i\alpha}(\theta-\eta) = {}^{p}_{h-q}\tilde{T}^{n(-\alpha)}_{i(-\lambda)}(i\pi+\eta-\theta)\ . $$

This crossing relation combines with the unitarity relation ${}^{p}_{q}T^{j\beta}_{i\alpha}(\theta-\eta){}^{p}_{q}\tilde{T}^{n\lambda}_{j\beta}(\eta-\theta) = \delta^n_i\delta^\lambda_\alpha$ (unchanged from section 4.2.2), to give the new crossing-unitarity condition

$$ {}^{p}_{q}T^{j\beta}_{i\alpha}(\theta-\eta){}^{p}_{h-q}T^{n(-\beta)}_{j(-\lambda)}(\theta-\eta+i\pi) = \delta^n_i\delta^\lambda_\alpha\ . \tag{7.8} $$

Note that (7.8) can be interpreted as a description of the transmission of a soliton through a (species q) defect immediately followed by transmission through an anti-defect, the combined effect being trivial. If the idea of the defect bootstrap is to be believed then it is clear that any transmission matrices found must obey (7.8).

7.2 Defect fusing in $a_2^{(1)}$

The case of $a_2^{(1)}$ is a unique one, in that transmission matrices are already known for all (both) of the fundamental defects and, unlike in sine-Gordon theory, the theory possesses fusing rules.

The fundamental transmission matrices[16] for the species 1 defect of $a_2^{(1)}$ are [CZ09a]

$$ {}^{1}_{1}T^{\lambda}_{\alpha}(\theta-\eta) = g^1(\theta-\eta)\begin{pmatrix} Q^{\lambda\cdot l_1}\delta^\lambda_\alpha & 0 & \hat{x}\delta^{\lambda+\alpha_0}_\alpha \\ \hat{x}\delta^{\lambda+\alpha_1}_\alpha & Q^{\lambda\cdot l_2}\delta^\lambda_\alpha & 0 \\ 0 & \hat{x}\delta^{\lambda+\alpha_2}_\alpha & Q^{\lambda\cdot l_3}\delta^\lambda_\alpha \end{pmatrix} \tag{7.9} $$

[16] I.e., the T-matrices for the fundamental solitons.

and

$$ {}^{2}_{1}T^{\lambda}_{\alpha}(\theta-\eta) = g^2(\theta-\eta)\begin{pmatrix} Q^{-\lambda\cdot l_1}\delta^{\lambda}_{\alpha} & \hat{x}Q^{\lambda\cdot l_3}\delta^{\lambda+\alpha_1}_{\alpha} & \hat{x}^2\delta^{\lambda-\alpha_0}_{\alpha} \\ \hat{x}^2\delta^{\lambda-\alpha_1}_{\alpha} & Q^{-\lambda\cdot l_2}\delta^{\lambda}_{\alpha} & \hat{x}Q^{\lambda\cdot l_1}\delta^{\lambda+\alpha_2}_{\alpha} \\ \hat{x}Q^{\lambda\cdot l_2}\delta^{\lambda+\alpha_0}_{\alpha} & \hat{x}^2\delta^{\lambda-\alpha_2}_{\alpha} & Q^{-\lambda\cdot l_3}\delta^{\lambda}_{\alpha} \end{pmatrix}. \quad (7.10) $$

In (7.9) and (7.10), $Q = -e^{i\pi\gamma}$ with $\gamma = \frac{4\pi}{\beta^2} - 1$ where β is the bulk coupling appearing in the Lagrangian description of the defect. The quantity $\hat{x} = e^{\gamma(\theta-\eta-\frac{i\pi}{2})}$ relates to the likelihood of the soliton exchanging topological charge with the defect. The transmission matrix (7.9) has a prefactor given by [CZ09*a*]

$$ g^1(\theta-\eta) = \frac{\hat{x}^{-\frac{1}{2}}}{2\pi}\Gamma(\tfrac{1}{2}+\gamma-z)\prod_{k=1}^{\infty}\frac{\Gamma(\frac{1}{2}+(3k+1)\gamma-z)\Gamma(\frac{1}{2}+(3k-2)\gamma+z)}{\Gamma(\frac{1}{2}+(3k-1)\gamma-z)\Gamma(\frac{1}{2}+(3k-1)\gamma+z)} $$

where $z = \frac{3i\gamma\left(\theta-\eta-\frac{i\pi}{2}\right)}{2\pi}$. This prefactor, along with the prefactor in (7.10) satisfy the relations

$$ \begin{aligned} g^2(\theta-\eta) &= g^1(\theta-\eta-\tfrac{i\pi}{3})g^1(\theta-\eta+\tfrac{i\pi}{3}) \\ g^1(\theta-\eta) &= g^2(\theta-\eta-\tfrac{i\pi}{3})g^2(\theta-\eta+\tfrac{i\pi}{3})\left(1+\hat{x}^3\right) \end{aligned} $$

implying that

$$ g^1(\theta-\eta)g^2(\theta-\eta+i\pi) = \frac{1}{1+Q^2\hat{x}^3}\,. \quad (7.11) $$

The transmission matrices for the species 2 defect of $a_2^{(1)}$ are [CZ07]

$$ {}^{1}_{2}T^{\lambda}_{\alpha}(\theta-\eta) = g^2(\theta-\eta)\begin{pmatrix} Q^{\lambda\cdot l_1}\delta^{\lambda}_{\alpha} & \hat{x}^2\delta^{\lambda-\alpha_1}_{\alpha} & \hat{x}Q^{-\lambda\cdot l_2}\delta^{\lambda+\alpha_0}_{\alpha} \\ \hat{x}Q^{-\lambda\cdot l_3}\delta^{\lambda+\alpha_1}_{\alpha} & Q^{\lambda\cdot l_2}\delta^{\lambda}_{\alpha} & \hat{x}^2\delta^{\lambda-\alpha_2}_{\alpha} \\ \hat{x}^2\delta^{\lambda-\alpha_0}_{\alpha} & \hat{x}Q^{-\lambda\cdot l_1}\delta^{\lambda+\alpha_2}_{\alpha} & Q^{\lambda\cdot l_3}\delta^{\lambda}_{\alpha} \end{pmatrix} \quad (7.12) $$

and

$$ {}^{2}_{2}T^{\lambda}_{\alpha}(\theta-\eta) = g^1(\theta-\eta)\begin{pmatrix} Q^{-\lambda\cdot l_1}\delta^{\lambda}_{\alpha} & \hat{x}\delta^{\lambda+\alpha_1}_{\alpha} & 0 \\ 0 & Q^{-\lambda\cdot l_2}\delta^{\lambda}_{\alpha} & \hat{x}\delta^{\lambda+\alpha_2}_{\alpha} \\ \hat{x}\delta^{\lambda+\alpha_0}_{\alpha} & 0 & Q^{-\lambda\cdot l_3}\delta^{\lambda}_{\alpha} \end{pmatrix}. \quad (7.13) $$

In (7.12) and (7.13), the prefactors g^1 and g^2 are the same as those in (7.9) and (7.10).

The expression (7.11) appears in the standard crossing-unitarity relation (4.36) and notably the same combination of prefactors, $g^1(\theta-\eta)g^2(\theta-\eta+i\pi)$, can appear by considering how a single soliton is transmitted through one defect of each species - this belies a different crossing-unitarity relation, i.e., (7.8). Indeed it can be demonstrated without too much labour that (7.8) holds for the known $a_2^{(1)}$ transmission matrices and hence provides strong support for the identification of the species of the defects.

7.2.1 Defect fusing rule algebra in $a_2^{(1)}$

The case of $a_2^{(1)}$ is special in that it is the only ATFT for which the species 2 defect is known, and so provides the best chance of finding the defect fusing coupling ratios via (7.4). Solving for the defect fusing couplings may hint at a generalisation to the other $a_r^{(1)}$ theories.

Applying the transmission matrices (7.9) and (7.12) to the $a_2^{(1)}$ $(h=3)$ version of (7.4), noting that $g^2(\theta-\eta)=g^1(\theta-\eta-\frac{i\pi}{3})g^1(\theta-\eta+\frac{i\pi}{3})$ gives essentially two kinds of equation

$$d_{11}^{\delta,\epsilon}=d_{11}^{\delta+\alpha_i,\epsilon+\alpha_{i-1}} \tag{7.14}$$

$$d_{11}^{\delta,\epsilon}=(-Q)^{-\frac{1}{3}}Q^{\delta\cdot l_{i-1}-\epsilon\cdot l_{i+1}}d_{11}^{\delta+\alpha_i,\epsilon}+(-Q)^{\frac{1}{3}}Q^{-\delta\cdot l_1+\epsilon\cdot l_{i-1}}d_{11}^{\delta,\epsilon+\alpha_i} \tag{7.15}$$

where the label on α_k should be taken modulo $h=3$ and similarly the label on l_k should be taken modulo 3, with the identification $l_0\equiv l_3$. Recall that $Q=-e^{i\pi\gamma}$.

There is an 'obvious' solution to (7.15) which is to put both terms on the right-hand side equal to $\frac{1}{2}d_{11}^{\delta,\epsilon}$, implying that

$$\frac{d_{11}^{\delta+\alpha_i,\epsilon}}{d_{11}^{\delta,\epsilon}}=Q^{\frac{1}{3}-\delta\cdot l_{i-1}+\epsilon\cdot l_{i+1}} \qquad \frac{d_{11}^{\delta,\epsilon+\alpha_i}}{d_{11}^{\delta,\epsilon}}=Q^{-\frac{1}{3}+\delta\cdot l_i-\epsilon\cdot l_{i-1}}\ . \tag{7.16}$$

This solution appears to be promising at first because it is automatically consistent with (7.14), but it is actually inconsistent. An example of the inconsistency is the calculation of the ratio $\frac{d_{11}^{\delta+\alpha_1,\epsilon+\alpha_2}}{d_{11}^{\delta,\epsilon}}$. One route to this ratio is to start by taking $\epsilon \to \epsilon + \alpha_2$ when $i = 1$ in the first equation in (7.16), so

$$\frac{d_{11}^{\delta+\alpha_1,\epsilon+\alpha_2}}{d_{11}^{\delta,\epsilon}} = \frac{d_{11}^{\delta+\alpha_1,\epsilon+\alpha_2}}{d_{11}^{\delta,\epsilon+\alpha_2}} * \frac{d_{11}^{\delta,\epsilon+\alpha_2}}{d_{11}^{\delta,\epsilon}} = Q^{\frac{1}{3}-\delta\cdot l_3+(\epsilon+\alpha_2)\cdot l_2} * Q^{-\frac{1}{3}+\delta\cdot l_2-\epsilon\cdot l_1}$$
$$= Q^{1+\delta\cdot(l_2-l_3)-\epsilon\cdot(l_1-l_2)} \,.$$

Another way to calculate $\frac{d_{11}^{\delta+\alpha_1,\epsilon+\alpha_2}}{d_{11}^{\delta,\epsilon}}$ is to start by taking $\delta \to \delta + \alpha_1$ when $i = 2$ in the second equation in (7.16), giving

$$\frac{d_{11}^{\delta+\alpha_1,\epsilon+\alpha_2}}{d_{11}^{\delta,\epsilon}} = \frac{d_{11}^{\delta+\alpha_1,\epsilon+\alpha_2}}{d_{11}^{\delta+\alpha_1,\epsilon}} * \frac{d_{11}^{\delta+\alpha_1,\epsilon}}{d_{11}^{\delta,\epsilon}} = Q^{-\frac{1}{3}+(\delta+\alpha_1)\cdot l_2-\epsilon\cdot l_1} * Q^{\frac{1}{3}-\delta\cdot l_3+\epsilon\cdot l_2}$$
$$= Q^{-1+\delta\cdot(l_2-l_3)-\epsilon\cdot(l_1-l_2)} \,.$$

It is clear then that the solution (7.16) is incorrect, unless the bulk coupling β is somehow quantised. One possibility is that the choices in (7.16) give the 'particular solution' to (7.15) and that there should be additionally 'homogeneous parts' to the solution that make the solution consistent. Analysis of the defect fusing rules for the species 2 soliton gives additional conditions which reinforce this view. The solution, assuming that one exists, has not been found. The lack of knowledge of a solution to the defect fusing coupling ratios in $a_2^{(1)}$ means that a different approach will be necessary to find the species 2 transmission matrices of $a_3^{(1)}$.

7.3 A new defect in $a_3^{(1)}$

In this section the defect fusing idea is applied to the case of $a_3^{(1)}$, which is the ATFT with the lowest rank for which the fusing rules give a previously unconsidered defect. A transmission matrix is found for the species 2 defect.

7.3.1 Transmission matrix ansatz

As noted in section 7.1.1, the defect fusing equation (7.4), which gives the species 2 defect transmission matrix in terms of species 1 defect transmission matrices, appears to have limited use while the couplings $\{d^{\beta,\gamma}\}$ remain unknown. However, with the assumption that the couplings depend only on topological charge and not on rapidity, (7.4) can be used to get an ansatz for the species 2 defect T-matrix.

The species 1 defect transmission matrices in $a_3^{(1)}$ (and $a_r^{(1)}$ generally) are known [CZ09*a*]. For the species 1 soliton in particular the transmission matrix is a simple extension of the $a_2^{(1)}$ T-matrix (7.9).

$$ {}_1^1T_\alpha^\lambda(\theta-\eta) = g^1(\theta-\eta)\begin{pmatrix} Q^{\lambda\cdot l_1}\delta_\alpha^\lambda & 0 & 0 & \hat{x}\delta_\alpha^{\lambda+\alpha_0} \\ \hat{x}\delta_\alpha^{\lambda+\alpha_1} & Q^{\lambda\cdot l_2}\delta_\alpha^\lambda & 0 & 0 \\ 0 & \hat{x}\delta_\alpha^{\lambda+\alpha_2} & Q^{\lambda\cdot l_3}\delta_\alpha^\lambda & 0 \\ 0 & 0 & \hat{x}\delta_\alpha^{\lambda+\alpha_3} & Q^{\lambda\cdot l_4}\delta_\alpha^\lambda \end{pmatrix}. \tag{7.17} $$

In the $a_3^{(1)}$ case the weights of the first representation are given by $l_i = -\frac{1}{4}\sum_{j=i}^{i+3} j\alpha_j$, or $l_i = \frac{3}{4}\alpha_i + \frac{1}{2}\alpha_{i+1} + \frac{1}{4}\alpha_{i+2}$, where the labels on the roots are modulo $h=4$. As in (7.9) the quantity $\hat{x} = e^{\gamma(\theta-\eta-\frac{i\pi}{2})}$ relates to the likelihood of the soliton exchanging topological charge with the defect. The transmission matrix has a prefactor given by [CZ09*a*]

$$ g^1(\theta-\eta) = \frac{\hat{x}^{-\frac{1}{2}}}{2\pi}\Gamma(\tfrac{1}{2}+\tfrac{3}{2}\gamma-z)\prod_{k=1}^{\infty}\frac{\Gamma(\frac{1}{2}+(4k+\frac{3}{2})\gamma-z)\Gamma(\frac{1}{2}+(4k-\frac{5}{2})\gamma+z)}{\Gamma(\frac{1}{2}+(4k-\frac{3}{2})\gamma-z)\Gamma(\frac{1}{2}+(4k-\frac{3}{2})\gamma+z)} \tag{7.18} $$

where $z = \frac{2i\gamma\left(\theta-\eta-\frac{i\pi}{2}\right)}{\pi}$.

Using the expression (7.17) for ${}_1^1T$ in (7.4), the ansatz for the transmission of a species 1 soliton through a species 2 defect, ${}_2^1T$, obtained is that ${}_2^1T_\alpha^\lambda(\theta-\eta)$ is equal to

$$ g^2(\theta-\eta)\begin{pmatrix} Q^{\lambda\cdot l_1}\delta_\alpha^\lambda & 0 & \hat{x}^2 b_{13}(\lambda)\delta_\alpha^{\lambda-\alpha_1-\alpha_2} & \hat{x}a_{14}(\lambda)\delta_\alpha^{\lambda+\alpha_0} \\ \hat{x}a_{21}(\lambda)\delta_\alpha^{\lambda+\alpha_1} & Q^{\lambda\cdot l_2}\delta_\alpha^\lambda & 0 & \hat{x}^2 b_{24}(\lambda)\delta_\alpha^{\lambda-\alpha_2-\alpha_3} \\ \hat{x}^2 b_{31}(\lambda)\delta_\alpha^{\lambda+\alpha_1+\alpha_2} & \hat{x}a_{32}(\lambda)\delta_\alpha^{\lambda+\alpha_2} & Q^{\lambda\cdot l_3}\delta_\alpha^\lambda & 0 \\ 0 & \hat{x}^2 b_{42}(\lambda)\delta_\alpha^{\lambda+\alpha_2+\alpha_3} & \hat{x}a_{43}(\lambda)\delta_\alpha^{\lambda+\alpha_3} & Q^{\lambda\cdot l_4}\delta_\alpha^\lambda \end{pmatrix} \tag{7.19} $$

where the prefactor is $g^2(\theta-\eta) = g^1(\theta-\eta-\frac{i\pi}{4})g^1(\theta-\eta+\frac{i\pi}{4})$, while the functions $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$ are unknown but depend only on the topological charges of the defect and soliton and not on the rapidities.

Note that (7.19) need only be given in terms of the species 1 soliton as, once a consistent solution has been found for ${}_2^1T$, the transmission matrices for the other solitons must follow from the soliton fusing rules. In fact, soliton fusing can be used to constrain the form of $a_{ij}(\lambda)$ and $b_{ij}(\lambda)$ since the topological charge of the species 2 soliton can be formed in two ways, this is shown in section 7.3.2.2, but the constraints obtained turn out to be just a subset of those arising from the triangle relations (7.20).

7.3.2 CONSTRAINING THE T-MATRIX

To establish the species 2 transmission matrix it is necessary to test for consistency by finding the appropriate solution to the triangle equations which fits in with the crossing-unitarity condition (7.8). If there exists a species 2 defect then the transmission matrix must fit the ansatz (7.19), so it becomes a case of finding conditions for $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$. If a consistent solution can be found then giving it interpretation other than it describing a species 2 defect would require strong justification.

Three types of consistency condition are considered here: the triangle relations, soliton fusing constraints and crossing-unitarity conditions.

7.3.2.1 TRIANGLE RELATIONS

The triangle relations are explained in section 4.2.2. They are the form of the Yang–Baxter equation appropriate to the situation of having two solitons and one defect. Both sides of the triangle relations equation (4.33) involve the same prefactors so the triangle relations do not constrain the prefactor of the transmission matrix. Fortunately, the prefactor in (7.19) is already specified exactly (as a minimal solution [CZ09*a*]) by the fusing rules so it may be possible for the triangle relations alone to give the solution for the $a_3^{(1)}$ species 2 defect.

The triangle relations for two species 1 solitons transmitting through a species 2 defect are

$$ {}^{11}S_{jk}^{mn}(\theta_1-\theta_2)\,{}^{1}_{2}T_{n\alpha}^{t\beta}(\theta_1-\eta)\,{}^{1}_{2}T_{m\beta}^{s\lambda}(\theta_2-\eta) = {}^{1}_{2}T_{k\alpha}^{m\beta}(\theta_2-\eta)\,{}^{1}_{2}T_{j\beta}^{n\lambda}(\theta_1-\eta)\,{}^{11}S_{nm}^{st}(\theta_1-\theta_2) \tag{7.20} $$

where m, n and β are summed over. the triangle relations were previously used by Corrigan and Zambon to find T-matrices for other defects [CZ07, CZ09*a*, CZ11] though quantum group methods can also be used [CZ11].

Calculations using the triangle relations (7.20) involve the S-matrix for two species 1 solitons in $a_3^{(1)}$. The general $a_r^{(1)}$ S-matrix is given by (3.22), with the

$a_3^{(1)}$ form being

$$
\begin{aligned}
{}^{11}S_{jj}^{jj}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2)\left(Q^{-1}\frac{\hat{x}_1^2}{\hat{x}_2^2}-Q\frac{\hat{x}_2^2}{\hat{x}_1^2}\right)\\
{}^{11}S_{jk}^{kj}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2)\left(\frac{\hat{x}_1^2}{\hat{x}_2^2}-\frac{\hat{x}_2^2}{\hat{x}_1^2}\right) \quad j\neq k\\
{}^{11}S_{jk}^{jk}(\theta_1-\theta_2) &= \rho(\theta_1-\theta_2)\left(Q^{-1}-Q\right)\begin{cases}\left(\dfrac{\hat{x}_1}{\hat{x}_2}\right)^{(2-|l|)} & \big|_{l=j-k<0}\\ \left(\dfrac{\hat{x}_2}{\hat{x}_1}\right)^{(2-|l|)} & \big|_{l=j-k>0}\end{cases}
\end{aligned}
$$

where $\frac{\hat{x}_1}{\hat{x}_2}=e^{\gamma(\theta_1-\theta_2)}$ (it is clear then that the defect parameter η does not appear in the soliton S-matrix) and again $Q=-e^{i\pi\gamma}$ with $\gamma=\frac{4\pi}{\beta^2}-1$. The prefactor $\rho(\theta_1-\theta_2)$ is immaterial to this discussion as it appears as a common factor on both sides of (7.20) - an expression for $\rho(\theta)$ can be found in section 3.2.2 or [CZ07]. Also immaterial to the discussion of the triangle relations are the prefactors in the transmission matrices, as the same terms appear on both sides of (7.20).

The triangle relations in this case are a set of $4^4=256$ conditions, as there are four choices for each of the 'in' (j and k) and 'out' (s and t) soliton charge labels. Most of these are trivially satisfied, but using the soliton S-matrix above and the ansatz for the species 2 defect T-matrix (7.19) there are 28 different conditions found that constrain $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$. As an example, consider the triangle relations when $j=1$, $k=2$, $s=1$ and $t=4$. The left-hand side of (7.20) for this case, dropping the species labels, is:

$$
\begin{aligned}
&S_{12}^{21}(\theta_1-\theta_2)T_{1\alpha}^{4\beta}(\theta_1-\eta)T_{2\beta}^{1\lambda}(\theta_2-\eta)+S_{12}^{12}(\theta_1-\theta_2)T_{2\alpha}^{4\beta}(\theta_1-\eta)T_{1\beta}^{1\lambda}(\theta_2-\eta)\\
&\to\left(\tfrac{\hat{x}_1^2}{\hat{x}_2^2}-\tfrac{\hat{x}_2^2}{\hat{x}_1^2}\right)\hat{x}_1\hat{x}_2a_{14}(\beta)\,a_{21}(\lambda)\delta_\alpha^{\beta+\alpha_0}\delta_\beta^{\lambda+\alpha_1}+\tfrac{\hat{x}_1}{\hat{x}_2}\left(Q^{-1}-Q\right)\hat{x}_1^2\,b_{24}(\beta)Q^{\lambda\cdot l_1}\delta_\alpha^{\beta-\alpha_2-\alpha_3}\delta_\beta^\lambda\\
&=\left[\tfrac{\hat{x}_1^3}{\hat{x}_2}\left(a_{14}(\lambda+\alpha_1)a_{21}(\lambda)+\left(Q^{-1}-Q\right)b_{24}(\lambda)Q^{\lambda\cdot l_1}\right)-\tfrac{\hat{x}_2^3}{\hat{x}_1}a_{14}(\lambda+\alpha_1)a_{21}(\lambda)\right]\delta_\alpha^{\lambda-\alpha_2-\alpha_3}
\end{aligned}
\tag{7.21}
$$

where the second line is the first line divided by the prefactors. The right-hand side is

$$
\begin{aligned}
&T_{2\alpha}^{1\beta}(\theta_2-\eta)T_{1\beta}^{4\lambda}(\theta_1-\eta)S_{41}^{14}(\theta_1-\theta_2)+T_{2\alpha}^{4\beta}(\theta_2-\eta)T_{1\beta}^{1\lambda}(\theta_1-\eta)S_{14}^{14}(\theta_1-\theta_2)\\
&\to\left(\tfrac{\hat{x}_1^2}{\hat{x}_2^2}-\tfrac{\hat{x}_2^2}{\hat{x}_1^2}\right)\hat{x}_1\hat{x}_2\,a_{14}(\lambda)a_{21}(\beta)\delta_\alpha^{\beta+\alpha_1}\delta_\beta^{\lambda+\alpha_0}+\tfrac{\hat{x}_2}{\hat{x}_1}\left(Q^{-1}-Q\right)\hat{x}_2^2\,b_{24}(\beta)Q^{\lambda\cdot l_1}\delta_\alpha^{\beta-\alpha_2-\alpha_3}\delta_\beta^\lambda\\
&=\left[\tfrac{\hat{x}_1^3}{\hat{x}_2}a_{14}(\lambda)a_{21}(\lambda+\alpha_0)-\tfrac{\hat{x}_2^3}{\hat{x}_1}\left(a_{14}(\lambda)a_{21}(\lambda+\alpha_0)-\left(Q^{-1}-Q\right)b_{24}(\lambda)Q^{\lambda\cdot l_1}\right)\right]\delta_\alpha^{\lambda-\alpha_2-\alpha_3}
\end{aligned}
\tag{7.22}
$$

again with the second line being the first line divided by the prefactors, which are the same prefactors as on the left-hand side.

Assuming that the unknown quantities of the transmission matrix, $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$, do not depend on rapidity and only on topological charge, matching (7.21) to (7.22) must imply

$$a_{14}(\lambda)a_{21}(\lambda+\alpha_0) - a_{14}(\lambda+\alpha_1)a_{21}(\lambda) = \left(Q^{-1}-Q\right)b_{24}(\lambda)Q^{\lambda\cdot l_1} \,.$$

This is just one of the 28 conditions given by the triangle relations, similar calculations give the others. The complete list, grouped by similar processes, is:

$$\begin{aligned}
a_{14}(\lambda)a_{21}(\lambda+\alpha_0) - a_{14}(\lambda+\alpha_1)a_{21}(\lambda) &= \left(Q^{-1}-Q\right)b_{24}(\lambda)Q^{\lambda\cdot l_1}\\
a_{21}(\lambda)a_{32}(\lambda+\alpha_1) - a_{21}(\lambda+\alpha_2)a_{32}(\lambda) &= \left(Q^{-1}-Q\right)b_{31}(\lambda)Q^{\lambda\cdot l_2}\\
a_{32}(\lambda)a_{43}(\lambda+\alpha_2) - a_{32}(\lambda+\alpha_3)a_{43}(\lambda) &= \left(Q^{-1}-Q\right)b_{42}(\lambda)Q^{\lambda\cdot l_3}\\
a_{43}(\lambda)a_{14}(\lambda+\alpha_3) - a_{43}(\lambda+\alpha_0)a_{14}(\lambda) &= \left(Q^{-1}-Q\right)b_{13}(\lambda)Q^{\lambda\cdot l_4}
\end{aligned} \qquad (A)$$

$$\begin{aligned}
a_{14}(\lambda)b_{31}(\lambda+\alpha_0) &= a_{14}(\lambda+\alpha_1+\alpha_2)b_{31}(\lambda)\\
a_{14}(\lambda)b_{42}(\lambda+\alpha_0) &= a_{14}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda)\\
a_{21}(\lambda)b_{13}(\lambda+\alpha_1) &= a_{21}(\lambda-\alpha_1-\alpha_2)b_{13}(\lambda)\\
a_{21}(\lambda)b_{42}(\lambda+\alpha_1) &= a_{21}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda)\\
a_{32}(\lambda)b_{13}(\lambda+\alpha_2) &= a_{32}(\lambda-\alpha_1-\alpha_2)b_{13}(\lambda)\\
a_{32}(\lambda)b_{24}(\lambda+\alpha_2) &= a_{32}(\lambda-\alpha_2-\alpha_3)b_{24}(\lambda)\\
a_{43}(\lambda)b_{24}(\lambda+\alpha_3) &= a_{43}(\lambda-\alpha_2-\alpha_3)b_{24}(\lambda)\\
a_{43}(\lambda)b_{31}(\lambda+\alpha_3) &= a_{43}(\lambda+\alpha_1+\alpha_2)b_{31}(\lambda)
\end{aligned} \qquad (B)$$

$$\begin{aligned}
a_{14}(\lambda)a_{32}(\lambda+\alpha_0) &= a_{14}(\lambda+\alpha_2)a_{32}(\lambda)\\
a_{21}(\lambda)a_{43}(\lambda+\alpha_1) &= a_{21}(\lambda+\alpha_3)a_{43}(\lambda)
\end{aligned} \qquad (C)$$

$$
\begin{aligned}
b_{13}(\lambda)b_{24}(\lambda-\alpha_1-\alpha_2) &= b_{13}(\lambda-\alpha_2-\alpha_3)b_{24}(\lambda)\\
b_{13}(\lambda)b_{42}(\lambda-\alpha_1-\alpha_2) &= b_{13}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda)\\
b_{31}(\lambda)b_{24}(\lambda+\alpha_1+\alpha_2) &= b_{31}(\lambda-\alpha_2-\alpha_3)b_{24}(\lambda)\\
b_{31}(\lambda)b_{42}(\lambda+\alpha_1+\alpha_2) &= b_{31}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda)\ . \qquad (D)
\end{aligned}
$$

$$
\begin{aligned}
b_{13}(\lambda)b_{31}(\lambda-\alpha_1-\alpha_2) &= b_{13}(\lambda+\alpha_1+\alpha_2)b_{31}(\lambda)\\
b_{24}(\lambda)b_{42}(\lambda-\alpha_2-\alpha_3) &= b_{24}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda) \qquad (E)
\end{aligned}
$$

$$
\begin{aligned}
a_{14}(\lambda)b_{13}(\lambda+\alpha_0) &= Qa_{14}(\lambda-\alpha_1-\alpha_2)b_{13}(\lambda)\\
a_{21}(\lambda)b_{24}(\lambda+\alpha_1) &= Qa_{21}(\lambda-\alpha_2-\alpha_3)b_{24}(\lambda)\\
a_{32}(\lambda)b_{31}(\lambda+\alpha_2) &= Qa_{32}(\lambda+\alpha_1+\alpha_2)b_{31}(\lambda)\\
a_{43}(\lambda)b_{42}(\lambda+\alpha_3) &= Qa_{43}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda) \qquad (F)
\end{aligned}
$$

$$
\begin{aligned}
a_{14}(\lambda)b_{24}(\lambda+\alpha_0) &= Q^{-1}a_{14}(\lambda-\alpha_2-\alpha_3)b_{24}(\lambda)\\
a_{21}(\lambda)b_{31}(\lambda+\alpha_1) &= Q^{-1}a_{21}(\lambda+\alpha_1+\alpha_2)b_{31}(\lambda)\\
a_{32}(\lambda)b_{42}(\lambda+\alpha_2) &= Q^{-1}a_{32}(\lambda+\alpha_2+\alpha_3)b_{42}(\lambda)\\
a_{43}(\lambda)b_{13}(\lambda+\alpha_3) &= Q^{-1}a_{43}(\lambda-\alpha_1-\alpha_2)b_{13}(\lambda)\ . \qquad (G)
\end{aligned}
$$

Despite the large number of constraints it is not clear whether or not a solution can be found by the triangle relations alone. In fact, the rescaling symmetry must satisfy a constraint which is not implied by the triangle relations (see section 7.3.3.3). Thus, it is reasonable to look for more conditions rather than try to solve from (A) to (G) alone.

7.3.2.2 Soliton fusing as a constraint

The well established soliton fusing rules give a novel way to test for constraints on the species 2 defect T-matrix for the species 1 soliton. The basis of this idea lies in how the topological charge is constructed for the species 2 soliton in the fusing process: if a species 2 soliton has topological charge $l_i + l_j$ then it could have received the l_i from either one of the two species 1 solitons involved in the fusing; l_j coming from the other soliton. Equation (7.2) then allows the transmission matrix for the species 2 soliton to be found by two different expressions, which in $a_3^{(1)}$ are:

$$ {}^2T^{(ab)\lambda}_{(jk)\alpha}(\theta-\eta)c^{(ab)} = c^{(jk)}\,{}^1T^{a\lambda}_{j\beta}(\theta-\eta-\tfrac{i\pi}{4})\,{}^1T^{b\beta}_{k\alpha}(\theta-\eta+\tfrac{i\pi}{4}) + (j\leftrightarrow k) \tag{7.23} $$

$$ {}^2T^{(ba)\lambda}_{(jk)\alpha}(\theta-\eta)c^{(ba)} = c^{(jk)}\,{}^1T^{b\lambda}_{j\beta}(\theta-\eta-\tfrac{i\pi}{4})\,{}^1T^{a\beta}_{k\alpha}(\theta-\eta+\tfrac{i\pi}{4}) + (j\leftrightarrow k)\ . \tag{7.24} $$

The topological charge (ab) represents $l_a + l_b$, with $l_i = \frac{3}{4}\alpha_i + \frac{1}{2}\alpha_{i+1} + \frac{1}{4}\alpha_{i+2}$ (root labels are modulo $h = 4$) being a weight in the **4** representation of the a_3 algebra (hence one of the topological charges of a species 1 soliton). The topological charge (ba) thus is the same as (ab), so the transmission matrix on the left-hand side of both of (7.23) and (7.24) is the same. The soliton fusing couplings $\{c^{(ij)}\}$ do however depend on the ordering of the labels, and in $a_3^{(1)}$ the couplings satisfy:

$$ \begin{aligned} c^{(12)} &= c^{(23)} = c^{(34)} = c^{(41)} \\ c^{(13)} &= c^{(24)} = c^{(31)} = c^{(42)} = (-Q)^{-\frac{1}{4}}c^{(12)} \\ c^{(14)} &= c^{(21)} = c^{(32)} = c^{(43)} = (-Q)^{-\frac{1}{2}}c^{(12)}\ . \end{aligned} $$

There are six possible topological charges for the species 2 soliton in $a_3^{(1)}$, arising from the choice of $a = 1,\ldots,4$ and $b = 1,\ldots,4$ with $b \neq a$. As an example, consider the case where $j = 1$, $k = 2$, $a = 1$ and $b = 4$ for the species 2 defect. In this case using (7.19) in (7.23) gives

$$ \begin{aligned} {}^2_2T^{(14)\lambda}_{(12)\alpha}(\theta-\eta) &= \frac{c^{(21)}}{c^{(14)}}\Big({}^1_2T^{1\lambda}_{2\beta}(\theta-\eta-\tfrac{i\pi}{4})\,{}^1_2T^{4\beta}_{1\alpha}(\theta-\eta+\tfrac{i\pi}{4}) \\ &\qquad + (-Q)^{\frac{1}{2}}\,{}^1_2T^{1\lambda}_{1\beta}(\theta-\eta-\tfrac{i\pi}{4})\,{}^1_2T^{4\beta}_{2\alpha}(\theta-\eta+\tfrac{i\pi}{4})\Big) \\ &= g^2(\theta-\eta-\tfrac{i\pi}{4})g^2(\theta-\eta+\tfrac{i\pi}{4}) \\ &\quad\times\Big(\hat{x}^2a_{14}(\beta)a_{21}(\lambda)\delta_\alpha^{\beta+\alpha_0}\delta_\beta^{\lambda+\alpha_1} + (-Q)^{\frac{1}{2}}\hat{x}^2(-Q)^{\frac{1}{2}}b_{24}(\beta)Q^{\lambda\cdot l_1}\delta_\alpha^{\beta-\alpha_2-\alpha_3}\delta_\beta^\lambda\Big) \\ &= g^4(\theta-\eta)\hat{x}^2\left(a_{14}(\lambda+\alpha_1)a_{21}(\lambda) - b_{24}(\lambda)Q^{1+\lambda\cdot l_1}\right)\delta_\alpha^{\lambda-\alpha_2-\alpha_3} \end{aligned} \tag{7.25} $$

where $g^4(\theta-\eta) = g^2(\theta-\eta-\frac{i\pi}{4})g^2(\theta-\eta+\frac{i\pi}{4})$ by definition. Similarly, using (7.19) in (7.24) gives

$$
\begin{aligned}
{}^{2}_{2}T^{(14)\lambda}_{(12)\alpha}(\theta-\eta) &= \frac{c^{(12)}}{c^{(41)}}\Big({}^{1}_{2}T^{4\lambda}_{1\beta}(\theta-\eta-\tfrac{i\pi}{4})\,{}^{1}_{2}T^{1\beta}_{2\alpha}(\theta-\eta+\tfrac{i\pi}{4}) \\
&\qquad + (-Q)^{-\frac{1}{2}}\,{}^{1}_{2}T^{4\lambda}_{2\beta}(\theta-\eta-\tfrac{i\pi}{4})\,{}^{1}_{2}T^{1\beta}_{1\alpha}(\theta-\eta+\tfrac{i\pi}{4})\Big) \\
&= g^2(\theta-\eta-\tfrac{i\pi}{4})g^2(\theta-\eta+\tfrac{i\pi}{4}) \\
&\quad\times\Big(\hat{x}^2 a_{14}(\lambda)a_{21}(\beta)\delta^{\beta+\alpha_1}_{\alpha}\delta^{\lambda+\alpha_0}_{\beta} + (-Q)^{-\frac{1}{2}}\hat{x}^2(-Q)^{-\frac{1}{2}}b_{24}(\lambda)Q^{\beta\cdot l_1}\delta^{\beta}_{\alpha}\delta^{\lambda-\alpha_2-\alpha_3}_{\beta}\Big) \\
&= g^4(\theta-\eta)\hat{x}^2\left(a_{14}(\lambda)a_{21}(\lambda+\alpha_0) - b_{24}(\lambda)Q^{-1+\lambda\cdot l_1}\right)\delta^{\lambda-\alpha_2-\alpha_3}_{\alpha}\,.
\end{aligned}
\tag{7.26}
$$

Two expressions for ${}^{2}_{2}T^{(14)\lambda}_{(12)\alpha}(\theta-\eta)$ have been found, given by (7.25) and (7.26). Equating these expressions thus gives the condition

$$
a_{14}(\lambda)a_{21}(\lambda+\alpha_0) - a_{14}(\lambda+\alpha_1)a_{21}(\lambda) = \left(Q^{-1}-Q\right)b_{24}(\lambda)Q^{\lambda\cdot l_1}
$$

which is exactly the same as that found in the example application of the triangle relations in section 7.3.2.1. In fact, there are 20 conditions that arise from soliton fusing, comparing (7.23) and (7.24), and these are in fact a subset of the triangle relations: the soliton fusing constraints here are given by (A), (B), (C), (D) and (E).

Soliton fusing as a constraint therefore does not give anything new in relation to the triangle relations, but does give most of the conditions in a much quicker manner with less redundancy (with 20 out of 36 conditions being non-trivial, while with the triangle relations it is 28 out of 256) making it labour saving in higher rank algebras. One may question whether it is possible to find a solution to the soliton fusing constraints that does not automatically satisfy (F) and (G), i.e., are the extra conditions from the triangle relations independent of those found by soliton fusing?

7.3.2.3 Crossing-unitarity relations

Further conditions can be found by relating the transmission matrices of the species 2 defect to those of its anti-defect. In the case of $a_3^{(1)}$ this is particularly useful, for the species 2 defect is self-conjugate, in that the anti-defect of a species 2 defect is another species 2 defect. In analogy to the properties of antisolitons, it is expected that the anti-defect of the species 2 defect is a species

2 defect with its rapidity shifted by $i\pi$ and possessing the opposite topological charge.

Consider the case of interest in finding the solution to (7.19). For a species 1 soliton passing through a species 2 defect in $a_3^{(1)}$ the crossing-unitarity conditions (7.8) become

$$ {}^{1}_{2}T^{j\beta}_{i\alpha}(\theta-\eta)\,{}^{1}_{2}T^{n(-\beta)}_{j(-\lambda)}(\theta-\eta+i\pi)=\delta^n_i\delta^\lambda_\alpha\ . \tag{7.27} $$

Using the ansatz (7.19) in (7.27) gives constraints which appear quite different to those of the triangle relations. For a start, the prefactor $g^2(\theta-\eta)$ is constrained (though only by the diagonal terms). Since the prefactor g^2 is already known in terms of g^1 and $g^1(\theta-\eta)$ is given by (7.18), then the prefactor contribution to the left-hand side of (7.27) can be shown to be

$$ g^2(\theta-\eta)\,g^2(\theta-\eta+i\pi)=\frac{1}{1+Q^2\hat{x}^4}\ . $$

As an example, consider (7.27) for the case of $i=1$ and $n=4$:

$$ \begin{aligned} 0 &= {}^{1}_{2}T^{j\beta}_{1\alpha}(\theta-\eta)\,{}^{1}_{2}T^{4(-\beta)}_{j(-\lambda)}(\theta-\eta+i\pi)\\ &= g^2(\theta-\eta)\,g^2(\theta-\eta+i\pi)\left[Q^{\beta\cdot l_1}\,(-Q\hat{x})\,a_{14}(-\beta)\delta^\beta_\alpha\delta^{-\beta+\alpha_0}_{-\lambda}+\hat{x}a_{14}(\beta)Q^{-\beta\cdot l_4}\delta^{\beta+\alpha_0}_\alpha\delta^{-\beta}_{-\lambda}\right]\\ &= \frac{1}{1+Q^2\hat{x}^4}\hat{x}\left[a_{14}(\lambda)Q^{-\lambda\cdot l_4}-a_{14}(-\lambda-\alpha_0)Q^{\lambda\cdot l_1}\right]\delta^{-\lambda-\alpha_0}_\alpha\\ &\implies\quad a_{14}(\lambda)Q^{-\lambda\cdot l_1}=a_{14}(-\lambda-\alpha_0)Q^{\lambda\cdot l_4}\ . \end{aligned} $$

The prefactor condition is involved with the diagonal terms, so it's worth examining another example condition, when $i=n=1$:

$$ \begin{aligned} \delta^\lambda_\alpha &= {}^{1}_{2}T^{j\beta}_{1\alpha}(\theta-\eta)\,{}^{1}_{2}T^{1(-\beta)}_{j(-\lambda)}(\theta-\eta+i\pi)\\ &= g^2(\theta-\eta)\,g^2(\theta-\eta+i\pi)\left[Q^{\beta\cdot l_1}Q^{-\beta\cdot l_1}\delta^\beta_\alpha\delta^{-\beta}_{-\lambda}+\hat{x}^2b_{13}(\beta)\,(Q^2\hat{x}^2)\,b_{31}(-\beta)\delta^{\beta-\alpha_1-\alpha_2}_\alpha\delta^{-\beta+\alpha_2+\alpha_3}_{-\lambda}\right]\\ &= \frac{1}{1+Q^2\hat{x}^4}\left[1+Q^2\hat{x}^4\,b_{13}(\lambda+\alpha_1+\alpha_2)b_{31}(-\lambda-\alpha_1-\alpha_2)\right]\delta^\lambda_\alpha\\ &\implies\quad b_{13}(\lambda)b_{31}(-\lambda)=1\ . \end{aligned} $$

In all there are 14 independent conditions found by crossing-unitarity, which are

$$\begin{aligned} b_{13}(\lambda)b_{31}(-\lambda) &= 1 \\ b_{24}(\lambda)b_{42}(-\lambda) &= 1 \end{aligned} \tag{A'}$$

$$\begin{aligned} a_{14}(\lambda)Q^{-\lambda\cdot l_1} &= a_{14}(-\lambda-\alpha_0)Q^{\lambda\cdot l_4} \\ a_{21}(\lambda)Q^{-\lambda\cdot l_2} &= a_{21}(-\lambda-\alpha_1)Q^{\lambda\cdot l_1} \\ a_{32}(\lambda)Q^{-\lambda\cdot l_3} &= a_{32}(-\lambda-\alpha_2)Q^{\lambda\cdot l_2} \\ a_{43}(\lambda)Q^{-\lambda\cdot l_4} &= a_{43}(-\lambda-\alpha_3)Q^{\lambda\cdot l_3} \end{aligned} \tag{B'}$$

$$\begin{aligned} a_{14}(\lambda)b_{31}(-\lambda+\alpha_3) &= Q^{-1}a_{32}(-\lambda+\alpha_3)b_{24}(\lambda) \\ a_{21}(\lambda)b_{42}(-\lambda+\alpha_0) &= Q^{-1}a_{43}(-\lambda+\alpha_0)b_{31}(\lambda) \\ a_{32}(\lambda)b_{13}(-\lambda+\alpha_1) &= Q^{-1}a_{14}(-\lambda+\alpha_1)b_{42}(\lambda) \\ a_{43}(\lambda)b_{24}(-\lambda+\alpha_2) &= Q^{-1}a_{21}(-\lambda+\alpha_2)b_{13}(\lambda) \end{aligned} \tag{C'}$$

$$\begin{aligned} a_{14}(\lambda)a_{21}(-\lambda+\alpha_2+\alpha_3) &= b_{24}(\lambda)Q^{-\lambda\cdot l_2} + Qb_{24}(-\lambda+\alpha_2+\alpha_3)Q^{\lambda\cdot l_4} \\ a_{21}(\lambda)a_{32}(-\lambda-\alpha_1-\alpha_2) &= b_{31}(\lambda)Q^{-\lambda\cdot l_3} + Qb_{31}(-\lambda-\alpha_1-\alpha_2)Q^{\lambda\cdot l_1} \\ a_{32}(\lambda)a_{43}(-\lambda-\alpha_2-\alpha_3) &= b_{42}(\lambda)Q^{-\lambda\cdot l_4} + Qb_{24}(-\lambda-\alpha_2-\alpha_3)Q^{\lambda\cdot l_2} \\ a_{43}(\lambda)a_{14}(-\lambda+\alpha_1+\alpha_2) &= b_{13}(\lambda)Q^{-\lambda\cdot l_1} + Qb_{13}(-\lambda+\alpha_2+\alpha_3)Q^{\lambda\cdot l_3}\ . \end{aligned} \tag{D'}$$

Combining these conditions with the triangle relations (A) - (G) there are now enough conditions to solve for $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$. The set of conditions (B') are of particular interest as they are the only ones that linearly relate unknowns, while (A') also appear to be strong conditions.

7.3.3 Solving for the T-matrix

With the conditions arising from the triangle relations, (A) - (G), and the conditions arising from the crossing-unitarity equation, (A') - (D'), it is possible to find a consistent solution for $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$ in the species 2 transmission matrix ansatz (7.19). Since the ansatz (7.19) is motivated by the defect fusing rules (7.4) it is taken that the solution really does describe a species 2 defect, as there is no alternative explanation as to why there should be a solution in this form. A solution method is given below.

7.3.3.1 Solution method

Examination of (B') reveals that

$$a_{ij}(\lambda) = Q^{\frac{1}{2}\lambda\cdot(l_i+l_j)}\tilde{a}_{ij}(\lambda)$$

where $\tilde{a}_{ij}(\lambda) = \tilde{a}_{ij}(-\lambda - \alpha_{i-1})$, so if $\tilde{a}_{ij}$ contains a term proportional to $Q^{\lambda\cdot Y}$ then it must also contain a term proportional to $Q^{-\lambda\cdot Y}$.

Now consider (A'). It would appear to be difficult to realise the likes of $b_{31}(\lambda) = \frac{1}{b_{13}(-\lambda)}$ if b_{ij} contains more than one term, so a reasonable ansatz is $b_{ij}(\lambda) = B_{ij}Q^{\lambda\cdot X_{ij}}$ where B_{ij} is constant. Then

$$b_{13}(\lambda)b_{31}(-\lambda) = B_{13}B_{31}Q^{\lambda\cdot(X_{13}-X_{31})} = 1$$
$$b_{24}(\lambda)b_{42}(-\lambda) = B_{24}B_{42}Q^{\lambda\cdot(X_{24}-X_{42})} = 1\ .$$

It is clear then that $X_{13} = X_{31}$ while $B_{13}B_{31} = 1$, etc. A choice is made here to make the solution simple, which is that each prefactor is set to unity. The scaling of the b_{ij}s (and a_{ij}s) is investigated in section 7.3.3.3. This choice then means that $b_{13}(\lambda) = b_{31}(\lambda)$ and $b_{24}(\lambda) = b_{42}(\lambda)$.

At this juncture (E) and (D), with the above identifications gives $b_{ij}(\lambda + 2\alpha_k + 2\alpha_{k+1}) = b_{ij}(\lambda)$ for any k modulo 4. Given the assumptions made about the form of b_{ij}, the λ dependence is entirely in the exponent of Q so there is no mechanism that will give minus signs and so

$$b_{ij}(\lambda + \alpha_1 + \alpha_2) = b_{ij}(\lambda + \alpha_2 + \alpha_3) = b_{ij}(\lambda)\ .$$

A consequence of the above is then that

$$b_{13}(\lambda) = b_{31}(\lambda) = Q^{a\lambda\cdot(l_1+l_3)} \qquad\qquad b_{24}(\lambda) = b_{42}(\lambda) = Q^{b\lambda\cdot(l_2+l_4)}$$

with a and b constants.

The knowledge that $b_{13}(\lambda) = b_{31}(\lambda)$ and $b_{24}(\lambda) = b_{42}(\lambda)$ can now be used to begin to find the form of $\tilde{a}_{ij}$. The first equation in (B) and the first equation in (F) combine to give $\tilde{a}_{14}(\lambda+\alpha_1+\alpha_2) = \tilde{a}_{14}(\lambda-\alpha_1-\alpha_2)$; similarly, the second equation in (B) and the first in (G) combine to give $\tilde{a}_{14}(\lambda+\alpha_2+\alpha_3) = \tilde{a}_{14}(\lambda-\alpha_2-\alpha_3)$. Assuming that all of the λ dependence in $\tilde{a}_{ij}(\lambda)$ is in the exponent of powers of Q the conditions of (B), (F) and (G) give

$$\tilde{a}_{ij}(\lambda+\alpha_1+\alpha_2) = \tilde{a}_{ij}(\lambda+\alpha_2+\alpha_3) = \tilde{a}_{ij}(\lambda)\ .$$

The combination of the above conditions with (B') implies that

$$\begin{aligned}
\tilde{a}_{14}(\lambda) &= \tilde{a}_{14}(-\lambda-\alpha_0) = \tilde{a}_{14}(-\lambda+\alpha_1) = \tilde{a}_{14}(-\lambda-\alpha_2) = \tilde{a}_{14}(-\lambda+\alpha_3)\\
\tilde{a}_{21}(\lambda) &= \tilde{a}_{21}(-\lambda+\alpha_0) = \tilde{a}_{21}(-\lambda-\alpha_1) = \tilde{a}_{21}(-\lambda+\alpha_2) = \tilde{a}_{21}(-\lambda-\alpha_3)\\
\tilde{a}_{32}(\lambda) &= \tilde{a}_{32}(-\lambda-\alpha_0) = \tilde{a}_{32}(-\lambda+\alpha_1) = \tilde{a}_{32}(-\lambda-\alpha_2) = \tilde{a}_{32}(-\lambda+\alpha_3)\\
\tilde{a}_{43}(\lambda) &= \tilde{a}_{43}(-\lambda+\alpha_0) = \tilde{a}_{43}(-\lambda-\alpha_1) = \tilde{a}_{43}(-\lambda+\alpha_2) = \tilde{a}_{43}(-\lambda-\alpha_3)\ .
\end{aligned}$$

Since $\tilde{a}_{32}$ obeys the same conditions as $\tilde{a}_{14}$ while $\tilde{a}_{43}$ obeys the same conditions as $\tilde{a}_{21}$. It is reasonable then to make the assumption that, up to a multiplicative factor,

$$\tilde{a}_{14}(\lambda) = \tilde{a}_{32}(\lambda) \qquad\qquad \tilde{a}_{21}(\lambda) = \tilde{a}_{43}(\lambda)$$

something which is consistent with the conditions (C). With this identification $b_{ij}(\lambda)$ can now be fully determined using (C'). The first equation of (C') reduces to $Q^{-a\lambda\cdot(l_1+l_3)}Q^a = Q^{-b\lambda\cdot(l_1+l_3)}Q^{-\frac{1}{2}}$ and so the conclusion is that $a = b = -\frac{1}{2}$. The other terms in (C') all agree with this identification, so

$$b_{13}(\lambda) = b_{31}(\lambda) = Q^{-\frac{1}{2}\lambda\cdot(l_1+l_3)} \qquad\qquad b_{24}(\lambda) = b_{42}(\lambda) = Q^{-\frac{1}{2}\lambda\cdot(l_2+l_4)}\ .$$

The remaining conditions (A) and (D') can be shown to be equivalent via use of (B'). These inhomogeneous equations and the constraints previously found on $\tilde{a}_{ij}(\lambda)$ suggest ansätze of the form

$$\tilde{a}_{14}(\lambda) = \tilde{a}_{32}(\lambda) = A\left(Q^{\frac{1}{2}\lambda\cdot(l_1+l_3)} + Q^{\frac{1}{2}-\frac{1}{2}\lambda\cdot(l_1+l_3)}\right) + B$$
$$\tilde{a}_{21}(\lambda) = \tilde{a}_{43}(\lambda) = C\left(Q^{\frac{1}{2}\lambda\cdot(l_2+l_4)} + Q^{\frac{1}{2}-\frac{1}{2}\lambda\cdot(l_2+l_4)}\right) + D\,.$$

The first condition in (D') becomes $\tilde{a}_{14}(\lambda)\tilde{a}_{21}(-\lambda) = Q^{-\frac{1}{2}-\lambda\cdot(l_2+l_4)} + Q^{\frac{1}{2}+\lambda\cdot(l_2+l_4)}$ so the above ansätze give

$$AC = Q^{-\frac{1}{2}} \qquad AD + BC = 0 \qquad BD = -2$$

and all of the other equations in (D') and (A) give the same relations. The most symmetrical solution is then to take $A = C = Q^{-\frac{1}{4}}$, so that $B = -D$ giving two solutions

$$\tilde{a}_{14}(\lambda) = \tilde{a}_{32}(\lambda) = Q^{-\frac{1}{4}+\frac{1}{2}\lambda\cdot(l_1+l_3)} + Q^{\frac{1}{4}-\frac{1}{2}\lambda\cdot(l_1+l_3)} \pm \sqrt{2}$$
$$\tilde{a}_{21}(\lambda) = \tilde{a}_{43}(\lambda) = Q^{-\frac{1}{4}+\frac{1}{2}\lambda\cdot(l_2+l_4)} + Q^{\frac{1}{4}-\frac{1}{2}\lambda\cdot(l_2+l_4)} \mp \sqrt{2}$$

where either the upper sign is taken for every $\tilde{a}_{ij}$ or the lower sign is taken for every $\tilde{a}_{ij}$.

7.3.3.2 Solutions

Two solutions have thus been found in which satisfy (7.20) and (7.27). These are the most symmetric solutions found. They are

$$b_{ij}(\lambda) = Q^{-\frac{1}{2}\lambda\cdot(l_i+l_j)}$$
$$a_{ij}(\lambda) = Q^{\frac{1}{2}\lambda\cdot(l_i+l_j)}\left(Q^{-\frac{1}{4}+\frac{1}{2}\lambda\cdot(l_i+l_{i+2})} + Q^{\frac{1}{4}-\frac{1}{2}\lambda\cdot(l_i+l_{i+2})} + (-1)^{i+1}\sqrt{2}\right) \tag{7.28}$$

and

$$b_{ij}(\lambda) = Q^{-\frac{1}{2}\lambda\cdot(l_i+l_j)}$$
$$a_{ij}(\lambda) = Q^{\frac{1}{2}\lambda\cdot(l_i+l_j)}\left(Q^{-\frac{1}{4}+\frac{1}{2}\lambda\cdot(l_i+l_{i+2})} + Q^{\frac{1}{4}-\frac{1}{2}\lambda\cdot(l_i+l_{i+2})} + (-1)^{i}\sqrt{2}\right)\,. \tag{7.29}$$

It is somewhat surprising that there are (at least) two solutions. There is no indication here that one of (7.28) and (7.29) should be favoured as the solution, further analysis will be required to determine whether or not both solutions should be considered as valid. Sections 7.3.3.3 and 7.3.3.4 consider the generation of other valid solutions through rescaling and unitarity transformations, but no inequivalent solutions are found.

For the sake of completeness, with the notation that $a_{ij}(\lambda) = Q^{\frac{1}{2}\lambda\cdot(l_i+l_j)}\tilde{a}_{ij}(\lambda)$, whether solution (7.28) or (7.29) is used, an expression for the transmission matrix ${}^1_2T^\lambda_\alpha(\theta-\eta)$ is

$$ {}^1_2T^\lambda_\alpha(\theta-\eta) = g^2(\theta-\eta)\begin{pmatrix} Q^{\lambda\cdot l_1}\delta^\lambda_\alpha & 0 & \hat{x}^2Q^{-\frac{1}{2}\lambda\cdot(l_1+l_3)}\delta^{\lambda-\alpha_1-\alpha_2}_\alpha & \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_1+l_4)}\tilde{a}_{14}(\lambda)\delta^{\lambda+\alpha_0}_\alpha \\ \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_1+l_2)}\tilde{a}_{21}(\lambda)\delta^{\lambda+\alpha_1}_\alpha & Q^{\lambda\cdot l_2}\delta^\lambda_\alpha & 0 & \hat{x}^2Q^{-\frac{1}{2}\lambda\cdot(l_2+l_4)}\delta^{\lambda-\alpha_2-\alpha_3}_\alpha \\ \hat{x}^2Q^{-\frac{1}{2}\lambda\cdot(l_1+l_3)}\delta^{\lambda+\alpha_1+\alpha_2}_\alpha & \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_2+l_3)}\tilde{a}_{14}(\lambda)\delta^{\lambda+\alpha_2}_\alpha & Q^{\lambda\cdot l_3}\delta^\lambda_\alpha & 0 \\ 0 & \hat{x}^2Q^{-\frac{1}{2}\lambda\cdot(l_2+l_4)}\delta^{\lambda+\alpha_2+\alpha_3}_\alpha & \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_3+l_4)}\tilde{a}_{21}(\lambda)\delta^{\lambda+\alpha_3}_\alpha & Q^{\lambda\cdot l_4}\delta^\lambda_\alpha \end{pmatrix}. \quad (7.30) $$

With expression (7.30) one can use the standard crossing-unitarity symmetry (4.36) to get the expression for ${}^3_2T^\lambda_\alpha(\theta-\eta)$, which is:

$$ g^2(\theta-\eta)\begin{pmatrix} Q^{-\lambda\cdot l_1}\delta^\lambda_\alpha & \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_1+l_2)}\tilde{a}_{21}(\lambda)\delta^{\lambda+\alpha_1}_\alpha & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_1+l_3)}\delta^{\lambda+\alpha_1+\alpha_2}_\alpha & 0 \\ 0 & Q^{-\lambda\cdot l_2}\delta^\lambda_\alpha & \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_2+l_3)}\tilde{a}_{14}(\lambda)\delta^{\lambda+\alpha_2}_\alpha & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_2+l_4)}\delta^{\lambda+\alpha_2+\alpha_3}_\alpha \\ \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_1+l_3)}\delta^{\lambda-\alpha_1-\alpha_2}_\alpha & 0 & Q^{-\lambda\cdot l_3}\delta^\lambda_\alpha & \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_3+l_4)}\tilde{a}_{21}(\lambda)\delta^{\lambda+\alpha_3}_\alpha \\ \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_1+l_4)}\tilde{a}_{14}(\lambda)\delta^{\lambda+\alpha_0}_\alpha & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_2+l_4)}\delta^{\lambda-\alpha_2-\alpha_3}_\alpha & 0 & Q^{-\lambda\cdot l_4}\delta^\lambda_\alpha \end{pmatrix} \quad (7.31) $$

and is the same as the transpose of (7.30) with $Q \leftrightarrow Q^{-1}$. Soliton fusing can be applied to either of (7.30) or (7.31) to get ${}^2_2T$, which is a six by six matrix. If fusing species 1 solitons (7.2) is used with (7.30), then ${}^2_2T^\lambda_\alpha(\theta-\eta)$ is found, with prefactor omitted, to be

$$ \begin{pmatrix} Q^{\lambda\cdot(l_1+l_2)} & \hat{x}^3Q^{\frac{1}{2}\lambda\cdot(l_2-l_2)-\frac{1}{2}}\tilde{a}_{21}(\lambda) & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_2+l_4-2l_3)} & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(2l_2-l_1-l_3)} & \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_2-l_3)}\tilde{a}_{14}(\lambda) & \hat{x}^4 \\ \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_1-l_4)}\tilde{a}_{14}(\lambda) & Q^{\lambda\cdot(l_1+l_3)}+\hat{x}^4Q^{-\lambda\cdot(l_1+l_3)} & \hat{x}^3Q^{-\frac{1}{2}\lambda\cdot(l_3-l_4)+\frac{1}{2}}\tilde{a}_{14}(\lambda) & \hat{x}^3Q^{-\frac{1}{2}\lambda\cdot(l_1-l_2)+\frac{1}{2}}\tilde{a}_{14}(\lambda) & \hat{x}^2\tilde{a}_{14}(\lambda)\tilde{a}_{14}(\lambda+\alpha_2) & \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_2-l_3)}\tilde{a}_{21}(\lambda) \\ \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(2l_1-l_2-l_4)} & \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_1-l_2)}\tilde{a}_{21}(\lambda) & Q^{\lambda\cdot(l_1+l_4)} & \hat{x}^4 & \hat{x}^3Q^{\frac{1}{2}\lambda\cdot(l_3-l_4)-\frac{1}{2}}\tilde{a}_{14}(\lambda) & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_1+l_3-2l_2)} \\ \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_1+l_3-2l_4)} & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_1-l_2)}\tilde{a}_{21}(\lambda) & \hat{x}^4 & Q^{\lambda\cdot(l_2+l_3)} & \hat{x}^3Q^{\frac{1}{2}\lambda\cdot(l_3-l_4)}\tilde{a}_{14}(\lambda) & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(2l_3-l_2-l_4)} \\ \hat{x}^3Q^{\frac{1}{2}\lambda\cdot(l_1-l_4)+\frac{1}{2}}\tilde{a}_{21}(\lambda) & \hat{x}^2a_{21}(\lambda)\tilde{a}_{21}(\lambda+\alpha_1) & \hat{x}Q^{\frac{1}{2}\lambda\cdot(l_3-l_4)}\tilde{a}_{21}(\lambda) & \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_1-l_2)}\tilde{a}_{21}(\lambda) & Q^{\lambda\cdot(l_2+l_4)}+\hat{x}^4Q^{-\lambda\cdot(l_2+l_4)} & \hat{x}^3Q^{-\frac{1}{2}\lambda\cdot(l_2-l_3)+\frac{1}{2}}\tilde{a}_{21}(\lambda) \\ \hat{x}^4 & \hat{x}^3Q^{-\frac{1}{2}\lambda\cdot(l_1-l_4)-\frac{1}{2}}\tilde{a}_{21}(\lambda) & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(2l_4-l_1-l_3)} & \hat{x}^2Q^{\frac{1}{2}\lambda\cdot(l_2+l_4-2l_1)} & \hat{x}Q^{-\frac{1}{2}\lambda\cdot(l_1-l_4)}\tilde{a}_{14}(\lambda) & Q^{\lambda\cdot(l_1+l_3)} \end{pmatrix}. \quad (7.32) $$

In (7.32) the omitted prefactor is given by $g^4(\theta-\eta) = g^2(\theta-\eta-\frac{i\pi}{4})g^2(\theta-\eta+\frac{i\pi}{4})$ and the topological charge states of the soliton for both the rows and columns is written in the order $(12), (13), (14), (23), (24), (34)$, where (ij) denotes the topological charge l_i+l_j. The observant reader will notice that the topological charge delta functions have been suppressed in (7.32). The delta functions to match the matrix entries in (7.32) are

$$\begin{pmatrix} \delta_\alpha^\lambda & \delta_\alpha^{\lambda-\alpha_2} & \delta_\alpha^{\lambda-\alpha_2-\alpha_3} & \delta_\alpha^{\lambda-\alpha_1-\alpha_2} & \delta_\alpha^{\lambda+\alpha_0} & \delta_\alpha^{\lambda+\alpha_0-\alpha_2} \\ \delta_\alpha^{\lambda+\alpha_2} & \delta_\alpha^\lambda & \delta_\alpha^{\lambda-\alpha_3} & \delta_\alpha^{\lambda-\alpha_1} & \delta_\alpha^{\lambda-\alpha_1-\alpha_3} & \delta_\alpha^{\lambda+\alpha_0} \\ \delta_\alpha^{\lambda+\alpha_2+\alpha_3} & \delta_\alpha^{\lambda+\alpha_3} & \delta_\alpha^\lambda & \delta_\alpha^{\lambda-\alpha_1+\alpha_3} & \delta_\alpha^{\lambda-\alpha_1} & \delta_\alpha^{\lambda-\alpha_1-\alpha_2} \\ \delta_\alpha^{\lambda+\alpha_1+\alpha_2} & \delta_\alpha^{\lambda+\alpha_1} & \delta_\alpha^{\lambda+\alpha_1-\alpha_3} & \delta_\alpha^\lambda & \delta_\alpha^{\lambda-\alpha_3} & \delta_\alpha^{\lambda-\alpha_2-\alpha_3} \\ \delta_\alpha^{\lambda-\alpha_0} & \delta_\alpha^{\lambda+\alpha_1+\alpha_3} & \delta_\alpha^{\lambda+\alpha_1} & \delta_\alpha^{\lambda+\alpha_3} & \delta_\alpha^\lambda & \delta_\alpha^{\lambda-\alpha_2} \\ \delta_\alpha^{\lambda-\alpha_0+\alpha_2} & \delta_\alpha^{\lambda-\alpha_0} & \delta_\alpha^{\lambda+\alpha_1+\alpha_2} & \delta_\alpha^{\lambda+\alpha_2+\alpha_3} & \delta_\alpha^{\lambda+\alpha_2} & \delta_\alpha^\lambda \end{pmatrix} .$$

The delta functions can be found by topological charge conservation alone. For example, in the row 2 column 5 entry of (7.32) the topological charge conservation is $l_1+l_3+\alpha = l_2+l_4+\lambda \implies \alpha = \lambda-(l_1-l_2)-(l_3-l_4) = \lambda-\alpha_1-\alpha_3$.

The expressions (7.30), (7.31) and (7.32) all have the assumption that $\tilde{a}_{32}(\lambda) = \tilde{a}_{14}(\lambda)$ and that $\tilde{a}_{43}(\lambda) = \tilde{a}_{21}(\lambda)$. This need not be true after rescaling. Note then that the rescaling and unitary transformation analysis that follows only relates to ${}^1_2T$.

7.3.3.3 Rescaling symmetry

Freedom to rescale the quantities $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$ can be seen in the in the triangle relation conditions (A) - (G) and the crossing-unitarity relations (A') - (D'). Labelling the scaling by $a_{ij}(\lambda) \to A_{ij}a_{ij}(\lambda)$ and $b_{ij}(\lambda) \to B_{ij}b_{ij}(\lambda)$ with $\{A_{ij}\}$ and $\{B_{ij}\}$ sets of constants, the solutions (7.28) and (7.29) correspond to having $A_{ij} = B_{ij} = 1$ in all cases.

Most of the conditions in section 7.3.2 do not restrict the scaling as they are homogeneous, in that both sides of the equality are scaled by the same quantities. The conditions which do restrict the scaling are (A), (A'), (C') and (D') with both (A) and (D') giving

$$A_{14}A_{21} = B_{24} \qquad A_{21}A_{32} = B_{31} \qquad A_{32}A_{43} = B_{42} \qquad A_{43}A_{14} = B_{13}$$

which is consistent with (C'). There is just one more constraint which comes from (A') which is that

$$A_{14}A_{21}A_{32}A_{43} = 1 \ .$$

Notable is that the triangle relations do not enforce the above constraint. It is clear that there are just three independent parameters that allow solutions to be rescaled consistently with both the triangle relations (7.20) and the crossing-unitarity relations (7.27).

7.3.3.4 Unitary transformations

A further freedom in the solution comes from taking a diagonal unitary transformation $T \to UTU^\dagger$. In order for the triangle relations to still hold the unitary transformations considered should depend only on the defect topological charge and not on the soliton topological charge. In order to change the actual charge dependence of the T-matrix the unitary transformation should depend quadratically on the the defect topological charge, so such a transformation matrix has the form

$$U_\alpha^\beta = Q^{\frac{1}{2}c\alpha\cdot\alpha}\delta_\alpha^\beta$$

with c a real parameter. The effect of this unitary transformation is that the diagonal terms in the T-matrix are unchanged while

$$\begin{aligned} a_{ij}(\lambda) &\to Q^c Q^{c\lambda\cdot(l_j-l_i)} a_{ij}(\lambda) \\ b_{ij}(\lambda) &\to Q^c Q^{c\lambda\cdot(l_j-l_i)} b_{ij}(\lambda)\ . \end{aligned}$$

The solution (7.28) or (7.29) is thus altered correspondingly. Note that the unitary transformation $U_{i\alpha}^{j\beta} = Q^{\frac{l_i\cdot\alpha}{4}}\delta_i^j\delta_\alpha^\beta$ considered in [CZ07] and seemingly dependent on soliton labels has the same effect as the above transformation when $c = -\frac{1}{4}$.

A linear unitary transformation such as $U_\alpha^\beta = Q^{\alpha\cdot X}\delta_\alpha^\beta$, where X is a fixed vector in the root space, gives something entirely equivalent to the rescaling symmetry already considered. Note that the general quadratic unitary transformation here does not affect the crossing-unitarity relations.

7.3.3.5 High-rapidity limits

The identification of (7.19) with either identification (7.28) or (7.29) as the transmission matrix for the species 1 soliton through the species 2 defect should not contradict the classical identifications made in chapter 5. While an analysis of the classical limit of (7.19) has not been carried out, a quick check can be made involving the high rapidity limits.

In the limit $\theta - \eta \to -\infty$ the rapidity dependent quantities within the T-matrix (as opposed to the prefactor) will tend to zero, $\hat{x} \to 0$. The terms which dominate in (7.19) therefore are the diagonal terms, so the limit $\theta - \eta \to -\infty$ results in no change of topological charge occurring to a transmitted soliton. This is in agreement to the same limit in the classical case, as shown in section 5.2.

In the limit $\theta - \eta \to \infty$ it must be that $|\hat{x}| \to \infty$ and so the terms which dominate in the transmission matrix (7.19) are those containing $\hat{x}^2$. In this limit then the species 1 soliton being transmitted through the defect has its topological charge shifted by two sectors, so $l_1 \leftrightarrow l_3$ and $l_2 \leftrightarrow l_4$. This again agrees with the same limit in the classical case in section 5.2.

The same two high-rapidity limits, $\theta - \eta \to \pm\infty$, can be applied to the transmission matrices for the other solitons through the species 2 defect, given by (7.31) and (7.32). For both the species 2 and species 3 solitons the high-rapidity limits in the transmission matrices (7.32) and (7.31), respectively, are seen to agree with the classical limits in section 5.2. The species 2 soliton case is of particularly interest as only two of the six quantum topological charges, $l_1 + l_3$ and $l_2 + l_4$, survive in the classical limit. The transmission matrix (7.32) clearly allows soliton charges to jump from a classical charge ($l_1 + l_3$ or $l_2 + l_4$) to non-classical charges ($l_1 + l_2$, $l_1 + l_4$, $l_2 + l_3$ or $l_3 + l_4$), but such transmissions are suppressed in the high rapidity limits.

7.4 A NEW DEFECT IN $a_5^{(1)}$

Section 7.3 is concerned with finding the transmission matrix for a species 1 soliton through species 2 defect in $a_3^{(1)}$. One might think then that the next case to consider is the species 2 defect of $a_4^{(1)}$, but the anti-defect of that defect is a species 3 defect, which is also unknown - the crossing-unitarity equation (7.8) thus links together two unknown T-matrices, greatly complicating attempts at a solution. Instead, the easiest case to consider next is that of the species

3 defect of $a_5^{(1)}$. The species 3 defect in $a_5^{(1)}$ is self-conjugate so the crossing-unitarity conditions only involve this defect. As such, the same methods used in section (7.3.2) can be applied to finding the transmission matrix of a species 1 soliton through a species 3 defect in $a_5^{(1)}$, which is the focus of this section.

7.4.1 Transmission matrix ansatz

The general defect fusing rules are given by (7.5) so in $a_5^{(1)}$ the transmission matrix for a species 3 defect can be found from those of a species 1 and a species 2 defect. The species 1 defect is known and its transmission matrix is found in [CZ09*a*], but the species 2 defect is unknown. Nonetheless, applying the defect fusing (7.4) gives the species 2 defect transmission matrix in terms of the species 1 defect transmission matrix and the defect fusing couplings. Putting these together an ansatz for the T-matrix for the species 1 soliton through the species 3 defect can be found and is given by

$$
\begin{aligned}
{}^{1}_{3}T^{n\lambda}_{i\alpha}(\theta-\eta) &= g^3(\theta-\eta) \\
&\times\begin{pmatrix}
Q^{\lambda\cdot l_1} & 0 & 0 & \hat{x}^3c_{14}(\lambda) & \hat{x}^2b_{15}(\lambda) & \hat{x}a_{16}(\lambda) \\
\hat{x}a_{21}(\lambda) & Q^{\lambda\cdot l_2} & 0 & 0 & \hat{x}^3c_{25}(\lambda) & \hat{x}^2b_{26}(\lambda) \\
\hat{x}^2b_{31}(\lambda) & \hat{x}a_{32}(\lambda) & Q^{\lambda\cdot l_3} & 0 & 0 & \hat{x}^3c_{36}(\lambda) \\
\hat{x}^3c_{41}(\lambda) & \hat{x}^2b_{42}(\lambda) & \hat{x}a_{43}(\lambda) & Q^{\lambda\cdot l_4} & 0 & 0 \\
0 & \hat{x}^3c_{52}(\lambda) & \hat{x}^2b_{53}(\lambda) & \hat{x}a_{54}(\lambda) & Q^{\lambda\cdot l_5} & 0 \\
0 & 0 & \hat{x}^3c_{63}(\lambda) & \hat{x}^2b_{64}(\lambda) & \hat{x}a_{65}(\lambda & Q^{\lambda\cdot l_6}
\end{pmatrix}.
\end{aligned}
\tag{7.33}
$$

In (7.33) the topological charge delta functions have been suppressed to fit the expression onto the page without affecting legibility, as a matrix these delta functions are

$$
\begin{pmatrix}
\delta^{\lambda}_{\alpha} & - & - & \delta^{\lambda-\alpha_1-\alpha_2-\alpha_3}_{\alpha} & \delta^{\lambda+\alpha_5+\alpha_0}_{\alpha} & \delta^{\lambda+\alpha_0}_{\alpha} \\
\delta^{\lambda+\alpha_1}_{\alpha} & \delta^{\lambda}_{\alpha} & - & - & \delta^{\lambda-\alpha_2-\alpha_3-\alpha_4}_{\alpha} & \delta^{\lambda+\alpha_0+\alpha_1}_{\alpha} \\
\delta^{\lambda+\alpha_1+\alpha_2}_{\alpha} & \delta^{\lambda+\alpha_2}_{\alpha} & \delta^{\lambda}_{\alpha} & - & - & \delta^{\lambda-\alpha_3-\alpha_4-\alpha_5}_{\alpha} \\
\delta^{\lambda+\alpha_1+\alpha_2+\alpha_3}_{\alpha} & \delta^{\lambda+\alpha_2+\alpha_3}_{\alpha} & \delta^{\lambda+\alpha_3}_{\alpha} & \delta^{\lambda}_{\alpha} & - & - \\
- & \delta^{\lambda+\alpha_2+\alpha_3+\alpha_4}_{\alpha} & \delta^{\lambda+\alpha_3+\alpha_4}_{\alpha} & \delta^{\lambda+\alpha_4}_{\alpha} & \delta^{\lambda}_{\alpha} & - \\
- & - & \delta^{\lambda+\alpha_3+\alpha_4+\alpha_5}_{\alpha} & \delta^{\lambda+\alpha_4+\alpha_5}_{\alpha} & \delta^{\lambda+\alpha_5}_{\alpha} & \delta^{\lambda}_{\alpha}
\end{pmatrix}
$$

where the dashes indicate a pairing to a term which is zero. The prefactor in the transmission matrix (7.33) is determined by the fusing rules and stems from the $a_5^{(1)}$ version of the prefactor $g^1(\theta-\eta)$ given in [CZ09*a*]. The prefactor can be shown to have the property

$$g^3(\theta-\eta)g^3(\theta-\eta+i\pi)=\frac{1}{1-Q^3\hat{x}^6}\ . \tag{7.34}$$

7.4.2 Constraining the T-matrix

The same conditions can be used as are used in the $a_3^{(1)}$ case in section 7.3.2, i.e., the triangle relations, the soliton fusing constraints and the crossing-unitarity relations.

7.4.2.1 Crossing-unitarity relations

The appropriate version of the crossing-unitarity relations (7.8) for the species 3 defect of $a_5^{(1)}$ with the species 1 soliton is

$$ {}^{1}_{3}T^{j\beta}_{i\alpha}(\theta-\eta)\,{}^{1}_{3}T^{n(-\beta)}_{j(-\lambda)}(\theta-\eta+i\pi)=\delta^n_i\delta^\lambda_\alpha\ .$$

Applying these conditions to the ansatz (7.33), and making use of the property of the prefactors (7.34) gives the following conditions

$$\begin{aligned}
c_{14}(\lambda)c_{41}(-\lambda)&=1\\
c_{25}(\lambda)c_{52}(-\lambda)&=1\\
c_{36}(\lambda)c_{63}(-\lambda)&=1
\end{aligned}\tag{$\tilde{A}'$}$$

$$\begin{aligned}
a_{16}(\lambda)Q^{-\lambda\cdot l_6}&=a_{16}(-\lambda-\alpha_0)Q^{\lambda\cdot l_1}\\
a_{21}(\lambda)Q^{-\lambda\cdot l_1}&=a_{21}(-\lambda-\alpha_1)Q^{\lambda\cdot l_2}\\
a_{32}(\lambda)Q^{-\lambda\cdot l_2}&=a_{32}(-\lambda-\alpha_2)Q^{\lambda\cdot l_3}\\
a_{43}(\lambda)Q^{-\lambda\cdot l_3}&=a_{43}(-\lambda-\alpha_3)Q^{\lambda\cdot l_4}\\
a_{54}(\lambda)Q^{-\lambda\cdot l_4}&=a_{54}(-\lambda-\alpha_4)Q^{\lambda\cdot l_5}\\
a_{65}(\lambda)Q^{-\lambda\cdot l_5}&=a_{65}(-\lambda-\alpha_5)Q^{\lambda\cdot l_6}
\end{aligned}\tag{$\tilde{B}'$}$$

$$
\begin{aligned}
a_{16}(\lambda)a_{65}(-\lambda) &= b_{15}(\lambda-\alpha_5)Q^{-\lambda\cdot l_5} + b_{15}(-\lambda-\alpha_0)Q^{\lambda\cdot l_1}\\
a_{21}(\lambda)a_{16}(-\lambda) &= b_{26}(\lambda-\alpha_0)Q^{-\lambda\cdot l_6} + b_{26}(-\lambda-\alpha_1)Q^{\lambda\cdot l_2}\\
a_{32}(\lambda)a_{21}(-\lambda) &= b_{31}(\lambda-\alpha_1)Q^{-\lambda\cdot l_1} + b_{31}(-\lambda-\alpha_2)Q^{\lambda\cdot l_3}\\
a_{43}(\lambda)a_{32}(-\lambda) &= b_{42}(\lambda-\alpha_2)Q^{-\lambda\cdot l_2} + b_{42}(-\lambda-\alpha_3)Q^{\lambda\cdot l_4}\\
a_{54}(\lambda)a_{43}(-\lambda) &= b_{53}(\lambda-\alpha_3)Q^{-\lambda\cdot l_3} + b_{53}(-\lambda-\alpha_4)Q^{\lambda\cdot l_5}\\
a_{65}(\lambda)a_{54}(-\lambda) &= b_{64}(\lambda-\alpha_4)Q^{-\lambda\cdot l_4} + b_{64}(-\lambda-\alpha_5)Q^{\lambda\cdot l_6}
\end{aligned}
$$

$$
\begin{aligned}
b_{15}(\lambda)b_{53}(-\lambda) &= Qa_{16}(\lambda+\alpha_5)c_{63}(-\lambda-\alpha_5) + Q^{-1}c_{14}(\lambda-\alpha_4)a_{43}(-\lambda+\alpha_4)\\
b_{26}(\lambda)b_{64}(-\lambda) &= Qa_{21}(\lambda+\alpha_0)c_{14}(-\lambda-\alpha_0) + Q^{-1}c_{25}(\lambda-\alpha_5)a_{54}(-\lambda+\alpha_5)\\
b_{31}(\lambda)b_{15}(-\lambda) &= Qa_{32}(\lambda+\alpha_1)c_{25}(-\lambda-\alpha_1) + Q^{-1}c_{36}(\lambda-\alpha_0)a_{65}(-\lambda+\alpha_0)\\
b_{42}(\lambda)b_{26}(-\lambda) &= Qa_{43}(\lambda+\alpha_2)c_{36}(-\lambda-\alpha_2) + Q^{-1}c_{41}(\lambda-\alpha_1)a_{16}(-\lambda+\alpha_1)\\
b_{53}(\lambda)b_{31}(-\lambda) &= Qa_{54}(\lambda+\alpha_3)c_{41}(-\lambda-\alpha_3) + Q^{-1}c_{52}(\lambda-\alpha_2)a_{21}(-\lambda+\alpha_2)\\
b_{64}(\lambda)b_{42}(-\lambda) &= Qa_{65}(\lambda+\alpha_4)c_{52}(-\lambda-\alpha_4) + Q^{-1}c_{63}(\lambda-\alpha_3)a_{32}(-\lambda+\alpha_3)
\end{aligned}
$$

$$
\begin{aligned}
c_{14}(\lambda)b_{42}(-\lambda) &= Qb_{15}(\lambda+\alpha_4)c_{52}(-\lambda-\alpha_4)\\
c_{25}(\lambda)b_{53}(-\lambda) &= Qb_{26}(\lambda+\alpha_5)c_{63}(-\lambda-\alpha_5)\\
c_{36}(\lambda)b_{64}(-\lambda) &= Qb_{31}(\lambda+\alpha_0)c_{14}(-\lambda-\alpha_0)\\
c_{41}(\lambda)b_{15}(-\lambda) &= Qb_{42}(\lambda+\alpha_1)c_{25}(-\lambda-\alpha_1)\\
c_{52}(\lambda)b_{26}(-\lambda) &= Qb_{53}(\lambda+\alpha_2)c_{36}(-\lambda-\alpha_2)\\
c_{63}(\lambda)b_{31}(-\lambda) &= Qb_{64}(\lambda+\alpha_3)c_{41}(-\lambda-\alpha_3)
\end{aligned}
$$

$$
\begin{aligned}
Qa_{16}(\lambda)b_{64}(-\lambda) + c_{14}(\lambda-\alpha_4-\alpha_5)Q^{-\lambda\cdot l_4} &= Qc_{14}(-\lambda-\alpha_0)Q^{\lambda\cdot l_1} + b_{15}(\lambda-\alpha_5)a_{54}(-\lambda+\alpha_5)\\
Qa_{21}(\lambda)b_{15}(-\lambda) + c_{25}(\lambda-\alpha_5-\alpha_0)Q^{-\lambda\cdot l_5} &= Qc_{25}(-\lambda-\alpha_1)Q^{\lambda\cdot l_2} + b_{26}(\lambda-\alpha_0)a_{65}(-\lambda+\alpha_0)\\
Qa_{32}(\lambda)b_{26}(-\lambda) + c_{36}(\lambda-\alpha_0-\alpha_1)Q^{-\lambda\cdot l_6} &= Qc_{36}(-\lambda-\alpha_2)Q^{\lambda\cdot l_3} + b_{31}(\lambda-\alpha_1)a_{16}(-\lambda+\alpha_1)\\
Qa_{43}(\lambda)b_{31}(-\lambda) + c_{41}(\lambda-\alpha_1-\alpha_2)Q^{-\lambda\cdot l_1} &= Qc_{41}(-\lambda-\alpha_3)Q^{\lambda\cdot l_4} + b_{42}(\lambda-\alpha_2)a_{21}(-\lambda+\alpha_2)\\
Qa_{54}(\lambda)b_{42}(-\lambda) + c_{52}(\lambda-\alpha_2-\alpha_3)Q^{-\lambda\cdot l_2} &= Qc_{52}(-\lambda-\alpha_4)Q^{\lambda\cdot l_5} + b_{53}(\lambda-\alpha_3)a_{32}(-\lambda+\alpha_3)\\
Qa_{65}(\lambda)b_{53}(-\lambda) + c_{63}(\lambda-\alpha_3-\alpha_4)Q^{-\lambda\cdot l_3} &= Qc_{63}(-\lambda-\alpha_5)Q^{\lambda\cdot l_6} + b_{64}(\lambda-\alpha_4)a_{43}(-\lambda+\alpha_4)
\end{aligned}
$$

7.4.2.2 Triangle relations / Soliton fusing conditions

For $a_5^{(1)}$, as in $a_3^{(1)}$, soliton fusing can be used to constrain the transmission matrix (7.33). The transmission matrix for a species 2 soliton through a species 3 defect is given by

$$ {}^{2}_{3}T^{(ab)\lambda}_{(jk)\alpha}(\theta-\eta)c^{(ab)} = c^{(jk)}\,{}^{1}_{3}T^{a\lambda}_{j\beta}(\theta-\eta-\tfrac{i\pi}{6})\,{}^{1}_{3}T^{b\beta}_{k\alpha}(\theta-\eta+\tfrac{i\pi}{6}) + (j\leftrightarrow k) \quad (7.35) $$

or

$$ {}^{2}_{3}T^{(ba)\lambda}_{(jk)\alpha}(\theta-\eta)c^{(ba)} = c^{(jk)}\,{}^{1}_{3}T^{b\lambda}_{j\beta}(\theta-\eta-\tfrac{i\pi}{6})\,{}^{1}_{3}T^{a\beta}_{k\alpha}(\theta-\eta+\tfrac{i\pi}{6}) + (j\leftrightarrow k)\;. \quad (7.36) $$

Since $T^{(ab)}_{(jk)} = T^{(ba)}_{(jk)}$ as the charge (ab) is the same as the charge (ba) (both represent $l_a + l_b$), comparing (7.35) and (7.36) gives $15^2 = 225$ conditions, which are a subset of the triangle relations. Most of them are trivial but there are still many that are not. In the expressions (7.35) and (7.36) there are the couplings $\{c^{(ij)}\}$ which have the following ratios:

$$ \frac{c^{(16)}}{c^{(15)}} = \frac{c^{(15)}}{c^{(14)}} = \frac{c^{(14)}}{c^{(13)}} = \frac{c^{(13)}}{c^{(12)}} = (-Q)^{-\frac{1}{6}}\;. $$

In general $c^{(i(i+k))} = c^{(j(j+k))}$ for any fixed k and modulo $h = 6$ (the label $i = 0$ is equivalent to $i = 6$) so e.g. $c^{(16)} = c^{(21)} = c^{(32)}$, etc.

The triangle relations, with two species 1 solitons passing through the species 3 defect are given by

$$ {}^{11}S^{mn}_{jk}(\theta_1-\theta_2)\,{}^{1}_{3}T^{t}_{n}(\theta_1-\eta)\,{}^{1}_{3}T^{s}_{m}(\theta_2-\eta) = {}^{1}_{3}T^{m}_{k}(\theta_2-\eta)\,{}^{1}_{3}T^{n}_{j}(\theta_1-\eta)\,{}^{11}S^{st}_{nm}(\theta_1-\theta_2)\;. \quad (7.37) $$

In all there are $6^4 = 1296$ conditions arising from (7.37), making a systematic check very time consuming. It is likely that most of the non-trivial relations will arise from the soliton fusing constraints, which can be supplemented by specific instances of the triangle relations.

Altogether, a representative sample of the triangle relations is:

$$\begin{aligned}
a_{16}(\lambda)a_{32}(\lambda+\alpha_0) &= a_{32}(\lambda)a_{16}(\lambda+\alpha_2)\\
a_{16}(\lambda)a_{43}(\lambda+\alpha_0) &= a_{43}(\lambda)a_{16}(\lambda+\alpha_3)\\
a_{16}(\lambda)a_{54}(\lambda+\alpha_0) &= a_{54}(\lambda)a_{16}(\lambda+\alpha_4)
\end{aligned}$$

$$\begin{aligned}
a_{21}(\lambda)a_{16}(\lambda+\alpha_1) - a_{16}(\lambda)a_{21}(\lambda+\alpha_0) &= \left(Q-Q^{-1}\right) b_{26}(\lambda)Q^{\lambda\cdot l_1}\\
a_{16}(\lambda)a_{65}(\lambda+\alpha_0) - a_{65}(\lambda)a_{16}(\lambda+\alpha_5) &= \left(Q-Q^{-1}\right) b_{15}(\lambda)Q^{\lambda\cdot l_6}
\end{aligned}$$

$$\begin{aligned}
a_{16}(\lambda)b_{15}(\lambda+\alpha_0) &= Qb_{15}(\lambda)a_{16}(\lambda+\alpha_5+\alpha_0)\\
a_{16}(\lambda)b_{26}(\lambda+\alpha_0) &= Q^{-1}b_{26}(\lambda)a_{16}(\lambda+\alpha_0+\alpha_1)\\
a_{16}(\lambda)b_{42}(\lambda+\alpha_0) &= b_{42}(\lambda)a_{16}(\lambda+\alpha_2+\alpha_3)\\
a_{16}(\lambda)b_{53}(\lambda+\alpha_0) &= b_{53}(\lambda)a_{16}(\lambda+\alpha_3+\alpha_4)
\end{aligned}$$

$$\begin{aligned}
b_{31}(\lambda)a_{16}(\lambda+\alpha_1+\alpha_2) - a_{16}(\lambda)b_{32}(\lambda+\alpha_0) &= \left(Q-Q^{-1}\right) c_{36}(\lambda)Q^{\lambda\cdot l_1}\\
a_{16}(\lambda)b_{64}(\lambda+\alpha_0) - b_{64}(\lambda)a_{16}(\lambda+\alpha_4+\alpha_5) &= \left(Q-Q^{-1}\right) c_{14}(\lambda)Q^{\lambda\cdot l_6}
\end{aligned}$$

$$\begin{aligned}
a_{16}(\lambda)c_{14}(\lambda+\alpha_0) &= Qc_{14}(\lambda)a_{16}(\lambda-\alpha_1-\alpha_2-\alpha_3)\\
a_{16}(\lambda)c_{36}(\lambda+\alpha_0) &= Q^{-1}c_{36}(\lambda)a_{16}(\lambda-\alpha_3-\alpha_4-\alpha_5)\\
a_{16}(\lambda)c_{41}(\lambda+\alpha_0) &= c_{41}(\lambda)a_{16}(\lambda+\alpha_1+\alpha_2+\alpha_3)\\
a_{16}(\lambda)c_{52}(\lambda+\alpha_0) &= c_{52}(\lambda)a_{16}(\lambda+\alpha_2+\alpha_3+\alpha_4)\\
a_{16}(\lambda)c_{63}(\lambda+\alpha_0) &= c_{63}(\lambda)a_{16}(\lambda+\alpha_3+\alpha_4+\alpha_5)
\end{aligned}$$

$$\begin{aligned}
&b_{26}(\lambda)b_{15}(\lambda+\alpha_0+\alpha_1) - b_{15}(\lambda)b_{26}(\lambda+\alpha_5+\alpha_0)\\
&= Qa_{16}(\lambda)c_{25}(\lambda+\alpha_0) - Q^{-1}c_{25}(\lambda)a_{16}(\lambda-\alpha_2-\alpha_3-\alpha_4)\\
&b_{15}(\lambda)b_{64}(\lambda+\alpha_5+\alpha_0) - b_{64}(\lambda)b_{15}(\lambda+\alpha_4+\alpha_5)\\
&= Qa_{65}(\lambda)c_{14}(\lambda+\alpha_5) - Q^{-1}c_{14}(\lambda)a_{65}(\lambda-\alpha_1-\alpha_2-\alpha_3)
\end{aligned}$$

$$b_{15}(\lambda)b_{31}(\lambda+\alpha_5+\alpha_0) = b_{31}(\lambda)b_{15}(\lambda+\alpha_1+\alpha_2)$$
$$b_{15}(\lambda)b_{42}(\lambda+\alpha_5+\alpha_0) = b_{42}(\lambda)b_{15}(\lambda+\alpha_2+\alpha_3)$$
$$b_{15}(\lambda)b_{53}(\lambda+\alpha_5+\alpha_0) = b_{53}(\lambda)b_{15}(\lambda+\alpha_3+\alpha_4)$$

$$b_{15}(\lambda)c_{14}(\lambda+\alpha_5+\alpha_0) = Qc_{14}(\lambda)b_{15}(\lambda-\alpha_1-\alpha_2-\alpha_3)$$
$$b_{15}(\lambda)c_{25}(\lambda+\alpha_5+\alpha_0) = Q^{-1}c_{25}(\lambda)b_{15}(\lambda-\alpha_2-\alpha_3-\alpha_4)$$
$$b_{15}(\lambda)c_{36}(\lambda+\alpha_5+\alpha_0) = c_{36}(\lambda)b_{15}(\lambda-\alpha_3-\alpha_4-\alpha_5)$$
$$b_{15}(\lambda)c_{41}(\lambda+\alpha_5+\alpha_0) = c_{41}(\lambda)b_{15}(\lambda+\alpha_1+\alpha_2+\alpha_3)$$
$$b_{15}(\lambda)c_{52}(\lambda+\alpha_5+\alpha_0) = c_{52}(\lambda)b_{15}(\lambda+\alpha_2+\alpha_3+\alpha_4)$$
$$b_{15}(\lambda)c_{63}(\lambda+\alpha_5+\alpha_0) = c_{63}(\lambda)b_{15}(\lambda+\alpha_3+\alpha_4+\alpha_5)$$

And also

$$c_{14}(\lambda)c_{25}(\lambda-\alpha_1-\alpha_2-\alpha_3) = c_{25}(\lambda)c_{14}(\lambda-\alpha_2-\alpha_3-\alpha_4)$$
$$etc.$$

It appears that all other such conditions can be found from cycling through the indices.

7.4.3 Solution

In attempting to find a solution the conditions $(\tilde{A}')$ and especially $(\tilde{B}')$ appear to be a good starting point with the other crossing-unitarity conditions and the triangle relations subsequently used. Unfortunately, making assumptions similar to those used in solving for the the T-matrix of the species 2 defect of $a_3^{(1)}$ eventually leads to contradictions, and as such no solution has been found. Were a solution to be found it would provide particularly strong evidence for the existence of the defect hierarchy; but should such a solution not exist then defect fusing rules in the proposed form, (7.5), cannot hold.

7.5 Summary

This chapter begins with the formulation of defect fusing rules in the quantum theory by means of a modified Faddeev–Zamolodchikov algebra. This idea allows the T-matrix for any species of fundamental defect to be written in terms of the T-matrices of other species of defect and the defect fusing couplings $\{d^{\beta,\gamma}\}$. The general formula is given by (7.5). Unfortunately the defect fusing couplings are unknown so they must be found, or an indirect approach is needed, if (7.5) to have predictive power.

The case of defect fusing in $a_2^{(1)}$ is examined in section 7.2 for the purpose of studying the defect fusing couplings. The process for finding the defect fusing couplings is illustrated but no solution is given.

The main work of the chapter is the finding of a transmission matrix for the species 2 defect of $a_3^{(1)}$ in section 7.3. The defect fusing process (7.4) is used there to find an ansatz (7.19) for the T-matrix. The triangle relations, soliton fusing constraints and crossing-unitarity conditions are then used to find the solutions (7.28) and (7.29). Connection is made to the classical results of chapter 5 by means of the high-rapidity limits.

The chapter ends with section (7.4), analysing the species 3 defect of $a_5^{(1)}$. The same methods are used as are used in section 7.3 but due to the greater complexity of $a_5^{(1)}$ no solution is found.

8

Discussion

8.1 Conclusions

The central idea of this thesis is that integrable defects in affine Toda field theory possess a number of particle-like characteristics (such as: defects possessing energy and momentum; the existence of anti-defects; intriguing links between defects and solitons) and so should be classified in a way analogous to the classification of the solitons of the theory. The structure of this classification has two parts:

1. It links together the different defects of the same $a_r^{(1)}$ ATFT through the presence of defect fusing rules.

2. It links together defects of different ATFTs via folding.

This work has tried to show that both fusing rules and folding makes sense when applied to defects, as such the framework has been extended and should allow for more systematic study of integrable defects in ATFT in the future.

Chapter 5 considers a classical approach to defect fusing rules. In that chapter the main quantities of interest are the delay factors picked up by solitons passing through the defects. Some observations are made in particular of $a_2^{(1)}$ and $a_3^{(1)}$ which illustrate the striking similarities between the soliton and defect fusing rules. The conclusion is that defect fusing rules have a sound footing classically, although a purely Lagrangian approach would obfuscate this observation.

Chapter 6 is concerned with the construction of defect configurations in $a_r^{(1)}$ which may be folded. This work preceded the identification of the different species of defect but fits in well with this idea, the foldable defect configurations are formed by combining a species 1 and a species r defect which possess the same rapidity - analogous to how the first fundamental soliton of the folded theory is constructed. While it has long been known that the $a_{2n-1}^{(1)}$ Dynkin diagram folds to $c_n^{(1)}$ [OT83a], the full power of the approach requires the use of certain 'non-canonical' foldings [KS96b], and as such a consequence of this work is the construction $d_n^{(2)}$ and $a_{2n}^{(2)}$ solitons using $a_r^{(1)}$ solitons in chapter 3. The folded defect configurations are shown to preserve the forms of the solitons of the folded theory and to conserve momentum, giving very strong indications of integrability.

Finally, chapter 7 considers a quantum approach to the question of defect fusing rules. The defect fusing rules are formulated in the Faddeev–Zamolodchikov algebra and used along with consistency conditions, the triangle relations and the crossing-unitarity conditions, to find the transmission matrices for fundamental solitons through the species 2 defect of $a_3^{(1)}$.

Overall the evidence for defect fusing rules and folding is strong. There is nothing in this thesis which contradicts the hypothesis that such a structure exists, although there are a few inconclusive calculations - particularly in chapter 7.

8.2 Outlook

The nature of this thesis, in that it broadens the framework in which to study defect ATFTs instead of focussing on extant questions, means that there remain many outstanding issues regarding defects in affine Toda field theory. Examples include:

- One issue in particular is how precisely defects relate to solitons. Even at the classical level it is notable how the defect conditions are given by Bäcklund transformations [BCZ04a, BCZ04b], which links strongly to the construction of solitons (see [DJ89] for an introduction). Another thing of note classically is that in the sine-Gordon case the energy associated to a type I defect is precisely one half of the energy of a soliton [CZ10]. In the quantum theory the solitons and defects are further intertwined as soliton S-matrices have been found embedded in T-matrices of type II defects [CZ10, CZ11]. The paper [CZ11] gives particular emphasis to the embeddings in $a_3^{(1)}$ where the following S matrices have been found:

$$^{11}S \qquad ^{33}S \qquad ^{13}S \qquad ^{21}S \qquad ^{23}S\ .$$

Notable is that the second ‘soliton’, which is the type II defect, is not of species 2. It seems then that the type II defect, set up in a particular way, may mimic a species 1 or a species 3 soliton but not a species 2 soliton. It is not clear why this is the case, although the species 1 and 3 solitons both have a smaller mass than the species 2 soliton. It is plausible that under these conditions the defect fusing rules may link to the soliton fusing rules to allow all of the S-matrices found as embeddings.

- Another analysis missing from this thesis is the construction of the T-matrices associated to the defects of the folded theories of chapter 6, although there is a T-matrix for the $a_2^{(2)}$ defect in the literature [CZ11]. Soliton S-matrices are known for all of the simply laced ATFTs [Hol93a, Joh97], but unlike with the fundamental excitations [Kha97], the S-matrices for the non-simply laced theories have not been found by a folding process [GM95, GMW96]. Given the difficulty of constructing the non-simply laced S-matrices, one may expect that there is no simple way of constructing the T-matrices for the folded defects.

- One aspect of defect ATFT that has received little attention beyond an initial investigation [BCZ05, Wes] is the possibility of scattering defects off other defects. At the classical level defect-defect scattering may be trivial given that the defect conditions are given by Bäcklund transformations which possess the property of commutativity. At the quantum level, however, one might expect there to be a non-trivial defect scattering matrix: for a defect with rapidity η_1 and topological charge α scattering off a defect with rapidity η_2 and topological charge β, the defect scattering matrix is denoted $X_{\alpha\beta}^{\mu\lambda}(\eta_1 - \eta_2)$, where the first defect has outgoing topological charge λ and the second has charge μ; topological charge conservation is $\alpha + \beta = \lambda + \mu$. The supposition that X depends on the rapidity difference of the defects is somewhat problematic as a non-zero rapidity does not necessarily imply that the defect is moving. One would also normally expect that for the dependence to be on $\eta_1 - \eta_2$ that $\eta_1 > \eta_2$, but, unlike with solitons, such a condition does not necessarily imply that the first defect is moving any faster than the second

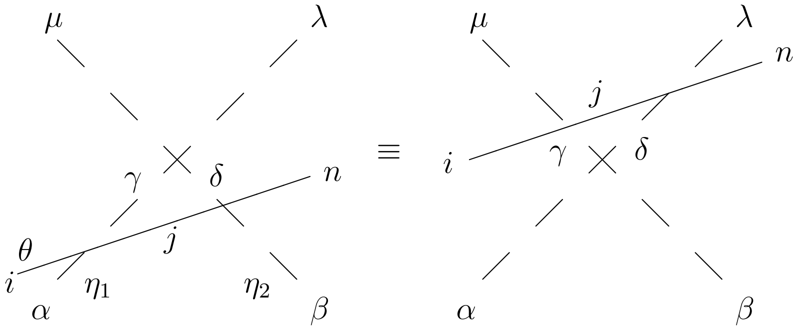

Figure 8.1: Illustration of Yang–Baxter relations for a soliton and two scattering defects, given by equation (8.1).

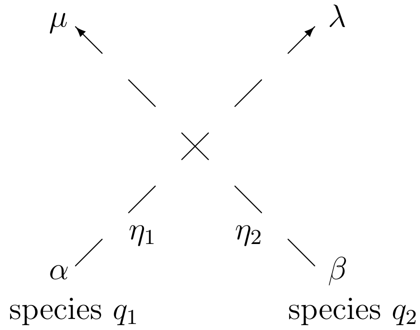

Figure 8.2: Illustration of the defect-defect crossing relation, given by equation (8.2).

defect. If such a defect scattering matrix does exist then one could go about finding what it is by considering the Yang–Baxter equation for one soliton and two defects, as illustrated in figure 8.1. The factorised scattering condition is then

$$T_{i\alpha}^{j\gamma}(\theta-\eta_1)T_{j\beta}^{n\delta}(\theta-\eta_2)X_{\gamma\delta}^{\lambda\mu}(\eta_1-\eta_2)=X_{\alpha\beta}^{\delta\gamma}(\eta_1-\eta_2)T_{i\delta}^{j\mu}(\theta-\eta_2)T_{j\gamma}^{n\lambda}(\theta-\eta_1) \qquad (8.1)$$

where the intermediate states j, γ and δ are summed over. One also expects the crossing symmetry of figure 8.2 to hold, so

$$_{q_1q_2}X_{\alpha\beta}^{\mu\lambda}(\eta_1-\eta_2)={}_{q_1(h-q_2)}X_{\alpha(-\mu)}^{(-\beta)\lambda}(i\pi+\eta_2-\eta_1)\ . \qquad (8.2)$$

- The results of chapter 6 suggest that by finding the appropriate defects in the ADE simply laced ATFTs defects in all of the other theories can be found by folding. Unfortunately the only simply laced theories for which defects have been found is the $a_r^{(1)}$ series, which is the only

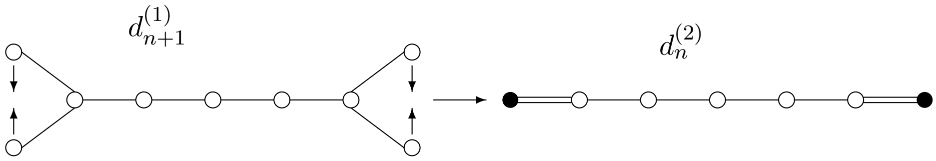

Figure 8.3: $d^{(1)}_{n+1} \to d^{(2)}_n$.

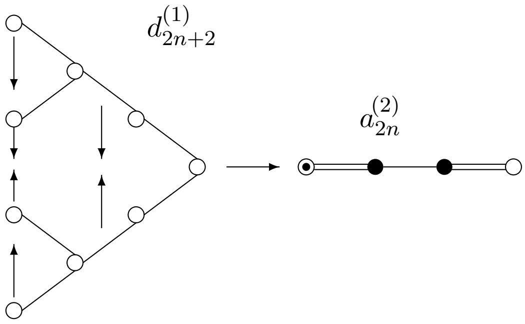

Figure 8.4: $d^{(1)}_{2n+2} \to a^{(2)}_{2n}$.

series allowing type I defects [BCZ04*b*]. The natural theory to search for next for defects is the $d_4^{(1)}$ ATFT. It was noted early that the $d_4^{(1)}$ soliton tau functions are quadratic while those of $a_r^{(1)}$ are linear [Hol92], so perhaps this unusual analogy hints that $d_4^{(1)}$ may have defects of type II Lagrangian description. The biggest hint that the $d_s^{(1)}$ series, including $d_4^{(1)}$, may possess type II defects comes from the non-canonical folding [KS96*b*] done in chapter 6 to obtain $d_n^{(2)}$ and $a_{2n}^{(2)}$ defects, folding from $a_r^{(1)}$. The canonical [OT83*a*] ways to get these theories is to fold from the $d_s^{(1)}$ series and is illustrated in figures 8.3 and 8.4. The single solitons of $d_n^{(2)}$ come from two-soliton solutions of $a_{2n-1}^{(1)}$ but one-soliton solutions of $d_{n+1}^{(1)}$ [McG94*b*]. The $d_n^{(2)}$ defect already found comes from a two-defect solution of $a_{2n-1}^{(1)}$, so does it also come from a one-defect solution of $d_{n+1}^{(1)}$? If so this would strongly suggest that the $d_s^{(1)}$ series of ATFTs possess type II defects.

- In the sine-Gordon theory the possibility of combining a defect with an integrable boundary (the Ghoshal–Zamolodchikov boundary [GZ94]) has been realised [CZ12] and boundaries dressed with defects have also been studied in complex sine-Gordon theory [BU09, BU08] and the nonlinear Schrödinger model [Zam14]. ATFT with defects and boundaries has also been studied algebraically [Doi15*a*, Doi15*b*]. At the Lagrangian level, there are known integrable boundary conditions for all of the other affine Toda theories [BCDR95] which, despite generalisations [BCR96, Del98], possess no free parameters. The possibility of combining defects with boundaries may introduce free parameters to the boundary conditions, but it is not clear whether charges conserved separately by the defect and the boundary are still conserved after combination and if so is integrability affected.

- Imaginary coupling affine Toda field theory, i.e., ATFT with solitons, is - with the exception of sine-Gordon theory, non-unitary and so has problems with quantisation [KS96a, TW99]. However, despite having a non-Hermitian Hamiltonian the soliton energies are real [Hol92] and furthermore all of the other conserved charges are real [Fre95]. Non-Hermitian Hamiltonians with real spectra are the objects of interest in the study of $\mathcal{PT}$-symmetric quantum theories [BB98]. Indeed, ATFT frequently crops up in the discussion of $\mathcal{PT}$ symmetry (e.g., [OM09, Fri09, BHMS14]) and as such $\mathcal{PT}$ symmetry, or some generalisation of it, should provide an alternative approach to quantising the imaginary coupling ATFT. Whether this can be applied in the presence of defects is another matter.

It is likely the reader knows of other, more interesting, aspects of affine Toda field theory which deserve to be researched. The author wishes them every success and hopes that this work proves in some way useful to them.

Appendix A

Infinite combination of sinh-Gordon defects

It is argued in chapter 4 that in combining two defects, i.e., placing two defects at the same location, the associated energy and associated momentum of the combined defect is just the sum of the individual energies and momenta respectively. The thesis argues that the existence of a conserved momentum is likely strong enough to imply classical integrability. It is the case for the type I $a_r^{(1)}$ defects that existence of a conserved momentum gives constraints [CZ09a] which match those found by taking a Lax pair approach [BCZ04b]. The condition that momentum is conserved in the folded defects of chapter 6 also matches the condition that solitons preserve their form when transmitted through the defect, which strongly suggests at there being an infinite number of higher-spin conserved charges. Since the momentum conservation argument is so powerful, consideration of the higher-spin charges is considered unnecessary. This appendix considers a higher-spin charge in a_1, the sinh-Gordon model, a model which is given no special attention in the main body of the thesis given that fusing rules and folding is absent from it.

A.1 The spin-3 conserved charge of a_1 with a defect

Consider classical sinh-Gordon theory with a type I defect. The Lagrangian with m and β scaled out is given by [BCZ04*a*]

$$\mathcal{L} = \theta(x)\mathcal{L}_u + \theta(-x)\mathcal{L}_v + \delta(x)\left(u\dot{v} - D(u,v)\right) \tag{A.1}$$

with

$$D(u,v) = 2e^{-\eta}\cosh\left(\frac{u+v}{2}\right) + 2e^{\eta}\cosh\left(\frac{u-v}{2}\right) . \tag{A.2}$$

The conventions here differ from equations (2.1), (4.1) and (4.2), with the single positive simple root of a_1 chosen to be given by $\alpha_1 = 1$ (with $\alpha_0 = -1$), rather than $\alpha_1 = \sqrt{2}$. The Euler–Lagrange equations for this system are

$$\begin{aligned} u'' &= \ddot{u} + U_u \,|_{x<0} \\ v'' &= \ddot{v} + V_v \,|_{x>0} \\ u' &= \dot{v} - D_u \,|_{x=0} \qquad (A.3) \\ v' &= \dot{u} + D_v \,|_{x=0} \qquad (A.4) \end{aligned}$$

where U and V the bulk potentials for u and v respectively, with[17] $U = U(u) = \cosh u$.

There is in a_1 a bulk conserved charge of Lorentz spin +3 and one of Lorentz spin -3. These can be combined in two ways to form an energy-like combination (sum) and a momentum-like combination (difference). As boundaries are of some interest here only the energy-like combination is considered, which is given by

$$\begin{aligned} Q = Q_3 + Q_{-3} = &\int_{-\infty}^{0} T_4(u) - \Theta_2(u)\,\mathrm{d}x + \int_{-\infty}^{0} T_{-4}(u) - \Theta_{-2}(u)\,\mathrm{d}x \\ &+ \int_{0}^{\infty} S_4(v) - \Sigma_2(v)\,\mathrm{d}x + \int_{0}^{\infty} S_{-4}(v) - \Sigma_{-2}(v)\,\mathrm{d}x . \end{aligned} \tag{A.5}$$

[17] The conventions of chapter 2 would have $U = 2(\cosh u - 1)$, but the expression used in this appendix is more convenient here, since $U_{uu} = U$.

S and Σ are just the same as T and Θ but exist to the right of the defect (hence taking v as their argument) instead of the left.

The quantities in (A.5) have the properties $\partial_- T_4 = \partial_+\Theta_2$ and $\partial_+ T_{-4} = \partial_-\Theta_{-2}$, where the lightcone derivatives are $\partial_\pm = \partial_t \pm \partial_x$, from the lightcone coordinates $x^\pm = \frac{1}{2}(t \pm x)$. The same holds for S and Σ. Using these properties, assuming that field fluctuations die out at spatial infinity, the time derivative of the charge is

$$\dot{Q} = T_4 - T_{-4} + \Theta_2 - \Theta_{-2} - S_4 + S_{-4} - \Sigma_2 + \Sigma_{-2} \,|_{x=0} \qquad \text{(A.6)}$$

The aim of this argument is to show that the right-hand side of (A.6) is a total time derivative, which implies that there is a modified conserved 'spin 3' energy-like charge in the system with a defect. The expression for the right-hand side of (A.6) comes from considering the possible terms of spin +4 density which are not related by a total ∂_+ derivative. The only choice that works is proportional to $T_4 = u_+^4 + 4u_{++}^2$ while similarly $T_{-4} = u_-^4 + 4u_{--}^2$, with S the same as T but with the argument v instead of u. With this choice (A.6) becomes

$$\begin{aligned}\dot{Q} &= 8\dot{u}^3 u' + 8\dot{u}u'^3 + 64\ddot{u}\dot{u}' + 32\dot{u}'U_u - 16\dot{u}u'U \\ &\quad - 8\dot{v}^3 v' - 8\dot{v}v'^3 - 64\ddot{v}\dot{v}' - 32\dot{v}'V_v + 16\dot{v}v'V \,|_{x=0} \,.\end{aligned} \qquad \text{(A.7)}$$

Making use of the defect conditions (A.3) and (A.4), with some partial integration, and noting that $D_u^2 - D_v^2 = U_u - V_v$, (A.7) becomes

$$\begin{aligned}\dot{Q} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[-32\dot{u}^2 D_{uu} - 32\dot{v}^2 D_{vv} - 64\dot{u}\dot{v}D_{uv} - 32D_u U_u - 32 D_v V_v - 32u_t V_v + 32v_t U_u\right] \\ &\quad - 8\dot{u}\left(D_u^3 - 6D_u U\right) \\ &\quad - 8\dot{v}\left(D_v^3 - 6D_v V\right) |_{x=0} \,.\end{aligned} \qquad \text{(A.8)}$$

The right-hand side of (A.8) is indeed a total time derivative, so there is a modified spin 3 energy like charge associated to the defect. Note that imposing Dirichlet boundary conditions ($v = constant$) here gives the Ghoshal–Zamolodchikov boundary [GZ94]. The conserved charge for this boundary is the same as that specified by (A.8), but with all reference to the field v removed.

A.2 The N-defect

Consider now an inductive argument to show that in combining defects the spin 3 energy-like charge is additive. Suppose that there exists an integrable N-defect, formed by the combination of N type I defects. The N-defect at $x = 0$ has the Lagrangian

$$\mathcal{L} = \theta(-x)\mathcal{L}_{\lambda_1} + \theta(x)\mathcal{L}_{\lambda_{N+1}} + \delta(x)\left(\sum_{i=1}^{N}\lambda_i(\lambda_{i+1})_t - \sum_{i=1}^{N}D^{(i)}(\lambda_i,\lambda_{i+1})\right) \quad \text{(A.9)}$$

where each $D^{(i)}$ is of the form $D^{(i)} = 2e^{-\eta_i}\cosh\left(\frac{\lambda_i+\lambda_{i+1}}{2}\right) + 2e^{\eta_i}\cosh\left(\frac{\lambda_i-\lambda_{i+1}}{2}\right)$. Although the only fields which have an existence in the bulk are λ_1 and λ_{N+1}, one can associate to each λ_i a potential $\Lambda_i = \cosh(\lambda_i)$. Then a relationship that holds for all i is

$$(D_1^{(i)})^2 - (D_2^{(i)})^2 = 2\Lambda_i - 2\Lambda_{i+1}$$

where D_1 denotes the derivative of D with respect to its first argument (for $D^{(i)}(\lambda_i,\lambda_{i+1})$ this means the derivative with respect to λ_i); similarly D_2 denotes the derivative of D with respect to its second argument. The notation Λ'_i means the derivative of Λ_i with respect to its argument λ_i, not to be confused with the spatial derivative.

The equations of motion from (A.9) are:

$$\begin{aligned}
{\lambda_1}'' &= \ddot{\lambda}_1 + \Lambda'_1 \,|_{x<0} \\
{\lambda_{N+1}}'' &= \ddot{\lambda}_{N+1} + \Lambda'_{N+1} \,|_{x>0} \\
{\lambda_1}' &= \dot{\lambda}_2 - D_1^{(1)} \,|_{x=0} && \text{(A.10)} \\
{\lambda_{N+1}}' &= \dot{\lambda}_N + D_2^{(N)} \,|_{x=0} && \text{(A.11)} \\
\dot{\lambda}_{i-1} &= \dot{\lambda}_{i+1} - D_2^{(i-1)} - D_1^{(i)} \,|_{x=0} \qquad \text{for } i = 2,\ldots,N && \text{(A.12)}
\end{aligned}$$

It is supposed that the defect is integrable and, further, that the spin 3 energy-like charge conservation is specified by

$$\dot{Q} = \sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t}\Big[-32\dot{\lambda}_i^2 D_{11}^{(i)} - 32\dot{\lambda}_{i+1}^2 D_{22}^{(i)} - 64\dot{\lambda}_i\dot{\lambda}_{i+1}D_{12}^{(i)}$$
$$-32D_1^{(i)}\Lambda_i' - 32D_2^{(i)}\Lambda_{i+1}' - 32\dot{\lambda}_i\Lambda_{i+1}' + 32(\lambda_{i+1})_t\Lambda_i'\Big]$$
$$-8\sum_{i=1}^{N+1}\dot{\lambda}_i\left(\left(D_2^{(i-1)}\right)^3 + \left(D_1^{(i)}\right)^3 - 6\left(D_2^{(i-1)} + D_1^{(i)}\right)\Lambda_i\right) \qquad \text{(A.13)}$$

where $D^{(0)} = D^{(N+1)} \equiv 0$. Note that this holds for $N = 1$, with (A.8) the $N = 1$ version of (A.13).

Associated to the Lagrangian (A.9) is the time derivative of the conserved charge

$$\begin{aligned}
\dot{Q} &= T_4 - T_{-4} + \Theta_2 - \Theta_{-2} - (R)\\
&= 8\dot{\lambda}_1^3{\lambda_1}' + 8\dot{\lambda}_1\lambda_1^{3\prime} + 64\ddot{\lambda}_1\dot{\lambda}_1' + 32\dot{\lambda}_1'\Lambda_1' - 16\dot{\lambda}_1{\lambda_1}'\Lambda_1 - (R)\\
&\overset{(A.10)}{=} 8\dot{\lambda}_1^3\dot{\lambda}_2 - 8\dot{\lambda}_1^3 D_1^{(1)} + 8\dot{\lambda}_1\dot{\lambda}_2^3 - 24\dot{\lambda}_1\dot{\lambda}_2^2 D_1^{(1)} + 24\dot{\lambda}_1\dot{\lambda}_2\left(D_1^{(1)}\right)^2\\
&\quad - 8\dot{\lambda}_1\left(D_1^{(1)}\right)^3 + 64\ddot{\lambda}_1\ddot{\lambda}_2 - 64\ddot{\lambda}_1\dot{D}_1^{(1)}\\
&\quad + 32\ddot{\lambda}_2\Lambda_1' - 32\dot{D}_1^{(1)}\Lambda_1' - 16\dot{\lambda}_1\dot{\lambda}_2\Lambda_1 + 16\dot{\lambda}_1 D_1^{(1)}\Lambda_1 - (R)
\end{aligned} \qquad \text{(A.14)}$$

where (R) denotes the equivalent contribution from λ_{N+1}, which is not needed for the inductive argument. It is supposed then that (A.14) matches (A.13), which is certainly the case for $N = 1$.

For the inductive step suppose that another single type I defect at some $x < 0$ is now combined with the N-defect given by (A.9). The Lagrangian for this $(N+1)$-defect is then

$$\mathcal{L} = \theta(-x)\mathcal{L}_{\lambda_0} + \theta(x)\mathcal{L}_{\lambda_{N+1}} + \delta(x)\left(\textstyle\sum_{i=0}^{N}\lambda_i(\lambda_{i+1})_t - \sum_{i=0}^{N} D^{(i)}(\lambda_i,\lambda_{i+1})\right) \quad \text{(A.15)}$$

where now, crucially, $D^{(0)} = 2e^{-\eta_0}\cosh\left(\frac{\lambda_0+\lambda_1}{2}\right) + 2e^{\eta_0}\cosh\left(\frac{\lambda_0-\lambda_1}{2}\right)$ whereas it was previously zero.

For this $(N+1)$-defect, (A.11) still holds, as does (A.12), but with (A.12) extended to include $i = 1$, giving $\dot{\lambda}_0 = \dot{\lambda}_2 - D_2^{(0)} - D_1^{(1)}$. Equation (A.10) no longer holds, but instead ${\lambda_0}' = \dot{\lambda}_1 - D_1^{(0)}$.

Associated to (A.15) is the time derivative of the conserved charge:

$$\begin{aligned}
\dot{Q} &= 8\dot{\lambda}_0^3\dot{\lambda}_1 - 8\dot{\lambda}_0^3 D_1^{(0)} + 8\dot{\lambda}_0\dot{\lambda}_1^3 - 24\dot{\lambda}_0\dot{\lambda}_1^2 D_1^{(0)} + 24\dot{\lambda}_0\dot{\lambda}_1\left(D_1^{(0)}\right)^2 \\
&\quad - 8\dot{\lambda}_0\left(D_1^{(0)}\right)^3 + 64\ddot{\lambda}_0\ddot{\lambda}_1 - 64\ddot{\lambda}_0\dot{D}_1^{(0)} \\
&\quad + 32\ddot{\lambda}_1\Lambda_0' - 32\dot{D}_1^{(0)}\Lambda_0' - 16\dot{\lambda}_0\dot{\lambda}_1\Lambda_0 + 16\dot{\lambda}_0 D_1^{(0)}\Lambda_0 - (R)\,. \qquad \text{(A.16)}
\end{aligned}$$

As long as (A.14) matches up with (A.13), the difference between the conserved charge of the $(N{+}1)$-defect and the N-defect is then (A.16) - (A.13). Using $\dot{\lambda}_0 = \dot{\lambda}_2 - D_2^{(0)} - D_1^{(1)}$ and $(D_1^{(0)})^2 - (D_2^{(0)})^2 = 2\Lambda_0 - 2\Lambda_1$, this can be shown to equal

$$\begin{aligned}
\dot{Q}(N{+}1) - \dot{Q}(N) = \frac{\mathrm{d}}{\mathrm{d}t}&\Big[-32\dot{\lambda}_0^2 D_{11}^{(0)} - 32\dot{\lambda}_1^2 D_{22}^{(0)} - 64\dot{\lambda}_0\dot{\lambda}_1 D_{12}^{(0)} \\
&-32D_1^{(0)}\Lambda_0' - 32D_2^{(0)}\Lambda_1' - 32\dot{\lambda}_0\Lambda_1' + 32\dot{\lambda}_1\Lambda_0'\Big] \\
&- 8\dot{\lambda}_0\left(\left(D_1^{(0)}\right)^3 - 6\left(D_1^{(0)}\right)\Lambda_0\right) \\
&- 8\dot{\lambda}_1\left(\left(D_2^{(0)}\right)^3 - 6\left(D_2^{(0)}\right)\Lambda_1\right)\,. \qquad \text{(A.17)}
\end{aligned}$$

The expression (A.17) is precisely what would be expected from taking the sum from $i = 0$ in (A.13) with $D^{(0)} \neq 0$. This proves the inductive step, that (A.13) specifies a spin 3 energy-like conserved charge for all N.

By applying Dirichlet boundary conditions on λ_{N+1}, an analogous analysis holds. The terms specified by (R) then vanish, and the conserved charge is (A.13) but with all reference to λ_{N+1} removed. This prescription thus provides a class of integrable boundary conditions for a_1, potentially with an infinite number of free parameters at the boundary.

Bibliography

[AAGZ11] A.R. Aguirre, T.R. Araujo, J.F. Gomes, A.H. Zimerman, *Type-II Bäcklund transformations via gauge transformations*, JHEP 12(2011)056; arXiv:1110.1589

[AFZ79] A.E. Arinshtein, V.A. Fateev, A.B. Zamolodchikov, *Quantum S-matrix of the (1+1)-dimensional Todd chain*, Phys. Lett. **B87** (1979) 389

[AD11] J. Avan, A. Doikou, *Liouville integrable defects: the nonlinear Schrödinger paradigm*, JHEP01(2012)040; JHEP 12(2011)056; arXiv:1110.4728

[AD12] J. Avan, A. Doikou, *The sine-Gordon model with integrable defects revisited*, JHEP 11(2012)008; arXiv:1205.1661

[BS08] Z. Bajnok, Zs. Simon, *Solving topological defects via fusion*, Nucl. Phys. **B802** (2008) 307; arXiv:0712.4292

[BJ97] E.J. Beggs, P.R. Johnson, *Inverse scattering and solitons in A_{n-1} affine Toda field theories*, Nucl. Phys. **B484** (1997) 653; arXiv:hep-th/9610104

[BJ98] E.J. Beggs, P.R. Johnson, *Inverse scattering and solitons in A_{n-1} affine Toda field theories II*, Nucl. Phys. **B529** (1998) 567; arXiv:hep-th/9803248

[BB98] C.M. Bender, S. Boettcher, *Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry*, Phys. Rev. Lett. **80** (1998) 5243; arXiv:physics/9712001

[BHMS14] C.M. Bender, D.W. Hook, N.E. Mavromatos, S. Sarkar, *Infinitely many inequivalent field theories from one Lagrangian*, Phys. Rev. Lett. 113 (2014) 231605; arXiv:1408.2432

[BCDR95] P. Bowcock, E. Corrigan, P.E. Dorey, R.H. Rietdijk, *Classically integrable boundary conditions for affine Toda field theories*, Nucl. Phys. **B445** (1995) 469; arXiv:hep-th/9501098

[BCR96] P. Bowcock, E. Corrigan, R.H. Rietdijk, *Background field boundary conditions for affine Toda field theories*, Nucl. Phys. **B465** (1996) 350; arXiv:hep-th/9510071

[BCZ04a] P. Bowcock, E. Corrigan, C. Zambon, *Classically integrable field theories with defects*, Int. J. Mod. Phys. **A19** (2004) 82; arXiv:hep-th/0305022

[BCZ04*b*] P. Bowcock, E. Corrigan, C. Zambon, *Affine Toda field theories with defects*, JHEP 01(2004)056; arXiv:hep-th/0401020

[BCZ05] P. Bowcock, E. Corrigan, C. Zambon, *Some aspects of jump-defects in the quantum sine-Gordon model*, JHEP 08(2005)023; arXiv:hep-th/0506169

[BU09] P. Bowcock, J. Umpleby, *Defects and dressed boundaries in complex sine-Gordon theory*, JHEP 01(2009)008 ; arXiv:0805.3668

[BU08] P. Bowcock, J. Umpleby, *Quantum complex sine-Gordon dressed boundaries*, JHEP 11(2008)038; arXiv:0809.0661

[BCDS89] H.W. Braden, E. Corrigan, P.E. Dorey, R. Sasaki, *Extended Toda field theory and exact S-matrices*, Phys. Lett. **B227** (1989) 411

[BCDS90] H.W. Braden, E. Corrigan, P.E. Dorey, R. Sasaki, *Affine Toda field theory and exact S-matrices*, Nucl. Phys. **B338** (1990) 689

[Cah06] R.N. Cahn, *Semi-simple Lie Algebras and Their Representations*, Dover, 2006

[CFG] O.A. Castro-Alvaredo, A. Fring, F. Göhmann, *On the absence of simultaneous reflection and transmission in integrable impurity systems*; arXiv:hep-th/0201142

[CMR04] V. Caudrelier, M. Mintchev, E. Ragoucy, *The quantum nonlinear Schrödiner model with point-like defect*, J. Phys. **A37** (2004) L367; arXiv:hep-th/0404144

[CFGZ93] C.P. Constantinidis, L.A. Ferreira, J.F. Gomes, A.H. Zimmerman, *Connection between the affine and conformal affine Toda models and their Hirota solution*, Phys. Lett. **B298** (1993) 88; arXiv:hep-th/9207061

[Cor] E. Corrigan, *Aspects of defects in integrable quantum field theory*; arXiv:1105.1103

[CZ06] E. Corrigan, C. Zambon, *Jump-defects in the nonlinear Schrödinger model and other non-relativistic field theories*, Nonlinearity **19** (2006) 1447; arXiv:nlin/0512038

[CZ07] E. Corrigan, C. Zambon, *On purely transmitting defects in affine Toda field theory*, JHEP 07(2007)001; arXiv:0705.1066

[CZ09*a*] E. Corrigan, C. Zambon, *Comments on defects in the a_r Toda field theories*, J. Phys. **A42** (2009) 304008; arXiv:0902.1307

[CZ09*b*] E. Corrigan, C. Zambon, *A new class of integrable defects*, J. Phys. **A42** (2009) 475203; arXiv:0908.3126

[CZ10] E. Corrigan, C. Zambon, *A transmission matrix for a fused pair of integrable defects in the sine-Gordon model*, J. Phys. **A43** (2010) 345201; arXiv:1006.0939

[CZ11] E. Corrigan, C. Zambon, *Integrable defects in affine Toda field theory and infinite dimensional representations of quantum groups*, Nucl. Phys. **B848** (2011) 545; arXiv:1012.4186

[CZ12] E. Corrigan, C. Zambon, *Infinite dimension reflection matrices in the sine-Gordon model with a boundary*, JHEP 06(2012)050; arXiv:1202.6016

[DHN75] R.F. Dashen, B. Hasslacher, A. Neveu, *The particle spectrum in model field theories from semiclassical functional integral techniques*, Phys. Rev. **D11** (1975) 3424

[DMS94*a*] G. Delfino, G. Mussardo, P. Simonetti, *Statistical models with a line of defect*, Phys. Lett. **B328** (1994) 123; arXiv:hep-th/9403049

[DMS94*b*] G. Delfino, G. Mussardo, P. Simonetti, *Scattering theory and correlation functions in statistical models with a line of defect*, Nucl. Phys. **B432** (1994) 518; arXiv:hep-th/9409076

[Del98] G.W. Delius, *Soliton-preserving boundary condition in affine Toda field theories*, Phys. Lett. **B444** (1998) 217; arXiv:hep-th/9809140

[DG99] G.W. Delius, G.M. Gandenberger, *Particle reflection amplitudes in $a_n^{(1)}$ Toda field theories*, Nucl. Phys. **B554** (1999) 325; arXiv:hep-th/9904002

[Doi08] A. Doikou, *$A_n^{(1)}$ affine Toda field theories with integrable boundary conditions revisited*, JHEP05(2008)091; arXiv:0803.0943

[Doi15*a*] A. Doikou, *Lax pair formulation in the simultaneous presence of boundaries and defects*, J. Phys. **A48** (2015) 065203; arXiv:1404.7329

[Doi15*b*] A. Doikou, *Jumps and twists in affine Toda field theories*, Nucl. Phys. **B893** (2015) 107; arXiv:1407.7777

[DK13] A. Doikou, N. Karaiskos, *Sigma models in the presence of dynamical point-like defects*, Nucl. Phys. **B867** (2013) 872; arXiv:1207.5503

[Dor90] P.E. Dorey, *The exact S-matrices of affine Toda field theories*, PhD thesis, Durham (1990); etheses.dur.ac.uk/6473/

[Dor92] P.E. Dorey, *Root systems and purely elastic S-matrices (II)*, Nucl. Phys. **B374** (1992) 741; arXiv:hep-th/9110058

[Dor] P.E. Dorey, *Exact S-matrices*; arXiv:hep-th/9810026

[DJ89] P.G. Drazin, R.S. Johnson, *Solitons: an Introduction*, CUP, 1989

[DS84] V.G. Drinfel'd, V.V. Sokolov, J. Sov. Math. **30** (1984) 1975

[Fad80] L.D. Faddeev, *Quantum completely integrable models of field theory*, Sov. Sci. Rev. **C1** (1980) 107

[Fre91] M. Freeman, *On the mass spectrum of affine Toda field theory*, Phys. Lett. **B261** (1991) 57

[Fre95] M. Freeman, *Conserved charges and soliton solutions in affine Toda theory*, Nucl. Phys. **B433** (1995) 657; arXiv:hep-th/9408092

[Fri09] A. Fring, *Particles versus fields in $\mathcal{PT}$-symmetrically deformed integrable systems*, Pramana 73 (2009) 363; arXiv:0903.4670

[FK94] A. Fring, R. Köberle, *Affine Toda field theory in the presence of reflecting boundaries*, Nucl. Phys. **B419** (1994) 647; arXiv:hep-th/9309142

[FK95] A. Fring, R. Köberle, *Boundary bound states in affine Toda field theory*, Int. J. Mod. Phys. **A10** (1995) 739; arXiv:hep-th/9404188

[GM95] G.M. Gandenberger, N.J. MacKay, *Exact S-matrices for $d_{n+1}^{(2)}$ affine Toda solitons and their bound states*, Nucl. Phys. **B457** (1995) 240; arXiv:hep-th/9506169

[GMW96] G.M. Gandenberger, N.J. MacKay, G.M.T Watts, *Twisted algebra R-matrices and S-matrices for $b_n^{(1)}$ affine Toda solitons and their bound states*, Nucl. Phys. **B465** (1996) 329; arXiv:hep-th/9509007

[GZ94] S. Ghoshal, A. Zamolodchikov, *Boundary S-matrix and boundary state in two-dimensional integrable quantum field theory*, Int. J. Mod. Phys. **A9** (1994) 3841; arXiv:hep-th/9306002

[GYZ06] J.F. Gomes, L.H. Ymai, A.H. Zimerman, *Classically integrable super sinh-Gordon equation with defects*, J. Phys. **A39** (2006) 7471; arXiv:hep-th/0601014

[GYZ08] J.F. Gomes, L.H. Ymai, A.H. Zimerman, *Integrability of a classical $N = 2$ super sinh-Gordon model with jump defects*, JHEP03(2008)001; arXiv:0710.1391

[HK08] I. Habibullin, A. Kundu, *Quantum and classical integrable sine-Gordon model with defect*, Nucl. Phys **B795** (2008) 549; arXiv:0709.4611

[Hal94] R.A. Hall, *Affine Toda solitons and fusing rules*, PhD thesis, Durham (1994); etheses.dur.ac.uk/5596/

[Har96] U.K.F. Harder, *On extended structures in affine Toda field theory*, PhD thesis, Durham (1996); etheses.dur.ac.uk/5434/

[HIM95] U. Harder, A.A.P Iskandar, W.A. McGhee, *On the breathers of $a_n^{(1)}$ affine Toda field theory*, Int. J. Mod. Phys. **A10** (1995) 1879; arXiv:hep-th/9409035

[Hir80] R. Hirota, *Direct methods in soliton theory*, in *Soliton*: ed. R.K. Bullough and P.S. Caudrey, Springer, 1980

[Hol92] T.J. Hollowood, *Solitons in affine Toda field theories*, Nucl. Phys. **B384** (1992) 523

[Hol93*a*] T.J. Hollowood, *Quantizing SL(N) solitons and the Hecke algebra*, Int. J. Mod. Phys. **A8** (1993) 947; arXiv:hep-th/9203076

[Hol93*b*] T.J. Hollowood, *Quantum mass corrections in sl(n) affine Toda field theory*, Phys. Lett. **B300** (1993) 73; arXiv:hep-th/9209024

[Isk95] A.A.P. Iskandar, *On breathers in affine Toda field theory*, PhD thesis, Durham (1995); etheses.dur.ac.uk/5236/

[Joh97] P.R. Johnson, *On soliton S-matrices in simply laced affine Toda field theories*, Nucl. Phys. **B496** (1997) 505; arXiv:hep-th/9611117

[Kha97] S.P. Khastgir, *S-matrices of non-simply laced affine Toda theories by folding*, Nucl. Phys. **B499** (1997) 650; arXiv:hep-th/9702170

[KS96*a*] S.P. Khastgir, R. Sasaki, *Instability of solitons in imaginary coupling affine Toda field theory*, Prog. Theor. Phys. **95** (1996) 485; arXiv:hep-th/hep-th/9507001

[KS96*b*] S.P. Khastgir, R. Sasaki, *Non-canonical folding of Dynkin diagrams and reduction of affine Toda theories*, Prog. Theor. Phys. **95** (1996) 503; arXiv:hep-th/9512158

[Kim96] J.D. Kim, *Boundary reflection matrix in the ade affine Toda field theory*, J. Phys. **A29** (1996) 2163; arXiv:hep-th/9506031

[KL99] R. Konik, A. LeClair, *Purely transmitting defect field theories*, Nucl. Phys. **B538** (1999) 587; arXiv:hep-th/9703085

[LOT93] H.C. Liao, D.I. Olive, N. Turok, *Topological solitons in A_r affine Toda theory*, Phys. Lett. **B298** (1993) 95

[Mac] N.J. MacKay, *Quantum affine Toda solitons*; arXiv:hep-th/9711148

[MM93] N.J. MacKay, W.A. McGhee, *Affine Toda solitons and automorphisms of Dynkin diagrams*, Int. J. Mod. Phys. **A8** (1993) 2791; arXiv:hep-th/9208057

[MS04] N. Manton, P. Sutcliffe, *Topological Solitons*, CUP, 2004

[McG94*a*] W.A. McGhee, *The topological charges of the $a_n^{(1)}$ affine Toda solitons*, Int. J. Mod. Phys. **A9** (1994) 2645; arXiv:hep-th/9307035

[McG94*b*] W.M. McGhee, *On the topological charges of the affine Toda solitons*, PhD thesis, Durham (1994); etheses.dur.ac.uk/5482/

[MOP81] A.V. Mikhailov, M.A. Olshanetsky, A.M. Perelomov, *Two-dimensional generalized Toda lattice*, Commun. Math. Phys. **79** (1981) 473

[MRS02] M. Mintchev, E. Ragoucy, P. Sorba, *Scattering in the presence of a reflecting and transmitting impurity*, Phys. Lett. **B547** (2002) 313; arXiv:hep-th/0209052

[Mus10] G. Mussardo, *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*, OUP, 2010

[Nem10] F. Nemes, *Semiclassical analysis of sine-Gordon theory*, Int. J. Mod. Phys. **A25** (2010) 4493; arXiv:0909.3268

[OM09] M.C. Ogilvie, P.N. Meisinger, *$\mathcal{PT}$ Symmetry and QCD: Finite Temperature and Density*, SIGMA 5 (2009) 047; arXiv:0812.0176

[OT83*a*] D.I. Olive, N. Turok, *The symmetries of Dynkin diagrams and the reduction of Toda field equations*, Nucl. Phys. **B215** (1983) 470

[OT83*b*] D.I. Olive, N. Turok, *Algebraic structure of Toda systems*, Nucl. Phys. **B220** (1983) 491

[OTU93*a*] D.I. Olive, N. Turok, J.W.R. Underwood, *Solitons and the energy-momentum tensor for affine Toda theory*, Nucl. Phys. **B401** (1993) 663

[OTU93*b*] D.I. Olive, N. Turok, J.W.R. Underwood, *Affine Toda solitons and vertex operators*, Nucl. phys **B409** (1993) 509; arXiv:hep-th/9305160

[Per99] M.G. Perkins, *Affine Toda field theories on a half-line*, PhD thesis, Durham (1999); etheses.dur.ac.uk/4560/

[Rob14*a*] C. Robertson, *Folding defect affine Toda field theories*, J. Phys. **A47** (2014) 185201; arXiv:1304.3129

[Rob14*b*] C. Robertson, *Defect fusing rules in affine Toda field theory*, J. Phys. **A47** (2014) 485205; arXiv:1408.1960

[SW00] H. Saleur, B. Wehefritz-Kaufmann, *Thermodynamics of the complex $su(3)$ Toda theory*, Phys. Lett. **B481** (2000) 419; arXiv:hep-th/0003217

[Sas92] R. Sasaki, *Reduction of the affine Toda field theory*, Nucl. Phys. **B383** (1992) 291

[TW99] G. Takacs, G.M.T. Watts, *Non-unitarity in quantum affine Toda theory and perturbed conformal field theory*, Nucl. Phys. **B547** (1999); arXiv:hep-th/9810006

[Wes] R. Weston, *An algebraic setting for defects in the XXZ and sine-Gordon models*; arXiv:1006.1555

[Wil81] G. Wilson, *The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras*, Ergod. Th. and Dynam. Sys. **1** (1981) 361

[Zam14] C. Zambon, *The classical nonlinear Schrödinger model with a new integrable boundary*, JHEP 08(2014)036; arXiv:1405.0967

[ZZ79] A.B. Zamolodchikov, Al.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models*, Ann. Phys. **120** (1979) 273

[ZC95] Z. Zhu, D.G. Caldi, *Multi-soliton solutions of affine Toda models*, Nucl. Phys. **B436** (1995) 659; arXiv:hep-th/9307175