The longitudinal impact of bars with rounded ends

Prowse, W. A.

Thesis submitted in candidature for the degree of Ph.D., on work done between September 1927 and October 1929, under the supervision of Professor J. E. P. Wagstaff. D.Sc.

THE LONGITUDINAL IMPACT

OF

BARS WITH ROUNDED ENDS.

BY

W. A. PROWSE, B.Sc.

THE LONGITUDINAL IMPACT

OF

BARS WITH ROUNDED ENDS.

BY

W. A. PROWSE, B.Sc.

TABLE OF CONTENTS.

CHAPTER I.

I M P A C T

SUMMARY AND INTRODUCTION.

SUMMARY.

The duration of contact of bars of various lengths, with rounded ends, was measured for a wide range of velocities of approach. For bars with hardened ends the effect of the radius, r, of the ball end was investigated from $r = \frac{1}{16}''$ to $r = 2''$. The duration of contact (t) and the diameter (D) of the circle of contact were measured for bars of unequal and of equal lengths with hardened ends and t was measured for soft-ended bars for values of the velocity of approach (v) from 0.4 cm./sec. to 24 cm/sec. More than fourteen thousand readings were taken.

A photomicrographic method was used to determine D, and photomicrographs of the polished ends of both hard- and soft-ended bars were taken to show the effects of the collisions. A hypothesis is put forward to account for the results. The coefficient of restitution, e, was measured for unequal bars and was found to approach unity as v decreased.

The formula $t = Av^{\gamma}$, where A and γ are constants independent of v, was investigated and found to hold for equal bars over a much wider range of experimental circumstances than hitherto investigated. For all values of r, $-\frac{1}{\gamma}$ was a linear function of the length, l. A most important and entirely new result was that the intercept on the $-\frac{1}{\gamma}$ axis was constant ($-\frac{1}{\gamma} = 4.39$) for all values of r. The slopes of the curves varied slightly but in no regular manner. t was found to increase rapidly as r decreased but the ball-end did not simply increase the

effective length of the bars; t was a non-linear function of ℓ

For unequal bars and for soft bars the relationship $t = Av^{\gamma}$ did not hold, but for soft-ended bars t was a linear function of ℓ. These results explain completely the apparent discrepancies between the observations of previous workers. $\frac{D^2}{rtv}$ was approximately constant in accordance with a suggested theory.

The values of t did not conform to any existing theory. A theory in which the bars are regarded as a complex vibrating system is developed giving for each length of bar one theoretical curve connecting t and a function of ℓ, v and r. For each length all the experimental points lay close to the calculated curve, the points for the various end-radii being indistinguishable. The values of e tended to support this theory.

INTRODUCTION.

It is known that metal wires subject to stresses of extremely short duration behave as perfectly elastic substances even for stresses considerably exceeding the elastic limit for steady loads (1). The impact of solid bodies involves essentially a consideration of the effect of applying for a short period a large force to a small area of the surface of a solid, and in agreement with the foregoing statement it is found that for moderate velocities, when the impinging bodies are identical in material and construction the resilience, as measured by the coefficient of restitution, is sensibly perfect.

In the case of a large solid in equilibrium under the action of such a localised force it can be shown that the elastic strain within the body falls off rapidly with the distance from the point of application, so that the effects produced are governed chiefly by the shape and nature of the body in the immediate neighbourhood of the small area over which the force is applied. On the other hand, if the force is applied uniformly for an extremely short period, so that a sharp blow is given to the surface, a large part of the energy will travel away from the area of application of the force in the form of longitudinal waves. The behaviour of impinging bodies will thus be expected to differ markedly according to whether the first or the second of these effects is predominant; in fact two distinct theories of impact have been propounded based on these conceptions.

The behaviour of a system consisting of two perfectly flat-ended rods impinging longitudinally was investigated in the light of the propagation of waves by St - Venant, in 1867 (2).

THE ST. VENANT THEORY.

In place of the mathematical analysis originally employed it is simpler to consider directly the propagation of waves along the bars without such analysis, the results obtained being in no way affected. (3).

(a) Collision of bars both of the same length.

If two ideally flat-ended bars of the same material and cross-section were to impinge longitudinally a uniform pressure would be produced over the surface of the bars, resulting in the formation of a wave of compression in each bar. These waves would travel along the bars until the available kinetic energy had been completely converted into strain energy in the bars. At the end of each bar remote from the impinging surfaces the disturbance would be reflected as an extensional wave which would pass down the bar, neutralising in its progress the compression produced by the impact. In recovering from the compression each particle of the bar would begin to move away from the area of contact with a velocity numerically equal to its initial velocity in the wave of compression. If the bars were of equal length the compression would be simultaneously neutralised in both so that on the arrival of the head of the wave of rarefaction at the junction the bars would part with a relative velocity numerically equal to the velocity of approach. Thus the whole of the strain-energy would have been reconverted into kinetic energy and the bars would have exchanged velocities.

The effects produced could be simulated in imagination by considering each bar to be prolonged indefinitely and allowing a wave of extension to travel towards the impinging surfaces from a point distant 2ℓ from the junction where ℓ represents the length of the bar considered. If the extensions in this "image" wave were identical with the compressions in the wave produced by the impact, then this image wave would arrive at the position of the free end of the bar in time to neutralise the compression, thereby satisfying the condition that the free end must be unstrained, in exactly the same way as the actual reflected wave.

(b) Collision of bars not of the same length.

Consider two rods of lengths ℓ_1 and ℓ_2 of the same material and cross section. Let ℓ_1 be greater than ℓ_2, and suppose that the shorter bar impinges on the longer, which is stationary, with a velocity v. Then at the beginning of the impact the junction takes a velocity $\frac{v}{2}$, so that a longitudinal wave in which the compression is $\frac{v}{2V}$, where V is the velocity of sound in the material of the bars, travels along each bar away from the junction.

If now the shorter bar be imagined to be prolonged indefinitely it can be considered that a longitudinal wave in which the extension is $\frac{v}{2V}$ starts out from a section at a distance $2\ell_2$ from the junction and travels towards the junction with a velocity V. Both the initial wave and the image wave reach the ~~junction~~ position of the free end of the bar at a time $\frac{\ell_2}{V}$, measured from the beginning of the impact; after which they become superposed in such a manner that the bar becomes unstrained throughout after a further time $\frac{\ell_2}{V}$ has elapsed. Due to the action of the waves the velocity of each particle relative to its initial velocity is now v, away from the junction. Thus the shorter bar, after a time $\frac{2\ell_2}{V}$ remains at rest in an unstrained condition.

Ifthe longer bar be thought of as indefinitely prolonged, then the image wave in which the extensi n is again $\frac{v}{2V}$ starts out from a section distant $2\ell_1$ from the junction, and the two waves begin to overlap after a time $\frac{\ell_1}{V}$ from the beginning of the impact. When a time $\frac{2\ell_2}{V}$ has elapsed the pressure between the bars falls to zero so that the total length of the wave in the longer bar is $2\ell_2$, and the junction end of the bar becomes unstrained and stationary, until the arrival of the image wave.

If ℓ_1 is greater than $2\ell_2$ the wave of compression can be considered to pass completely out of the longer bar while the wave of extension takes its place. The bars will now part immediately that the head of the wave of extension arrives at the junction. When this occurs the portion of the rod unoccupied by the wave will be unstrained and stationary, having been subjected to two identical stresses in opposite directions, while the portion of the rod of length $2\ell_2$, occupied by the extensional wave, will be moving away from the junction with a velocity $\frac{v}{2}$, the particle velocity in the wave.

The whole momentum associated with the rod is $2\ell_2 \rho \frac{v}{2}$, where ρ represents the mass per unit length, so that the resultant velocity is $\frac{\ell_2}{\ell_1}.v.$

If ℓ_1 is less than $2\ell_2$ then at the instant of arrival of the head of the extensional wave at the junction the initial wave and the image wave overlap for a distance $2\ell_2 - \ell_1$ so that this length is unstrained and moves with velocity v away from the junction, while a length $2(\ell_1 - \ell_2)$ is occupied by the extensional wave. Then the momentum associated with the rod in this case is $2(\ell_1 - \ell_2)\rho\frac{v}{2} + (2\ell_2 - \ell_1)\rho v$, or $\ell_2 v\rho$ as before, so that again the resultant velocity is $\frac{\ell_1}{\ell_2}v$.

The duration of contact and the coefficient of restitution are therefore $\frac{2\ell_1}{V}$ and $\frac{\ell_2}{\ell_1}$ respectively in both cases.

THE HERTZ THEORY. (4).

The effect of a pronounced curvature of the impinging surfaces of the colliding bodies would be to prevent the rapid growth of pressure at the beginning of the impact, so that the amount of energy productive of longitudinal vibrations would be less than in the case of flat-ended bodies. The case treated by Hertz was the impact of two bodies with curved surfaces, such that the duration of contact could be considered long compared with the period of the gravest longitudinal vibrations which might be provoked by the collision. No sensible amount of energy would then have been present as vibrations and the conditions during the impact would have been determined completely by the relationship connecting the pressure between the bodies and their relative displacement under static conditions.

Ifa force be applied to a very small area of the surface of a solid bounded by an infinite plane, it can be shown that the stress within the body at points not too near to the portion of the surface over which the pressure is applied is inversely proportional to the square of the distance from the point of application of the force. Thus if the relationship between the pressure and the indentation of the surfaces during impact is that which holds under static conditions, most of the strain energy will be located in the immediate vicinity of the area of contact.

If the curved surfaces of two bodies of the same material are pressed together then within the boundary of the area of contact the material will be strained in a manner depending upon the original shape of the surfaces and the closeness of approach, measured from the position inwhich the bodies are just in contact. It can be shown that the deformation is such that the potential at any point within the area of contact is a quadratic function of the coordinates of that point, a result which is true of the potential due to an ellipsoid. Accordingly an investigation is made of the effects due to a very much flattened ellipsoid occupying the place of the area of contact, the whole mass of the ellipoid being equal to the total pressure between the bodies.

In the case of equal spheres the relationships obtained from these considerations are

$$\alpha = \frac{3P}{4} \frac{2\vartheta}{a} \int_0^\infty \frac{dx}{(1+x)\,x^{1/2}}$$

$$\frac{1}{r} = \frac{3P}{4} \frac{2\vartheta}{a^3} \int_0^\infty \frac{dx}{(1+x)^2\,x^{1/2}}$$

in which

$\alpha = z - z'$; z and z' being the distances between a point in the first body and a point in the second before and after the application of the pressure, respectively;

P = total pressure between the bodies

a = semi major axis of the ellipsoid which represents the distribution of stress,

$\vartheta = \dfrac{(1-\sigma)^2}{\pi v^2 \rho\,(1-2\sigma)}$ where v = velocity of sound in the material of the spheres, ρ = density of the material and σ = Poisson's ratio.

r = radius of either sphere.

From these equations the relationship between P and α may be reduced to

$$P = \alpha^{\frac{3}{2}} \left\{ \frac{2}{3\pi} \frac{1}{\vartheta} \sqrt{\frac{r}{2}} \right\}$$

or $P = k_2 \alpha^{3/2}$

where $k_2 = \frac{\sqrt{2}}{3} \cdot r^{\frac{1}{2}} v^2 \rho \, \frac{(1-2\sigma)}{(1-\sigma)^2}$

The integration of the equation $P = k_2 \alpha^{\beta}$ will be found in Chapter IV p. 110 The result there given is

$$t = 2\sqrt{\pi}\, \frac{\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(\frac{2+\gamma}{2}\right)} (1+\gamma)^{\frac{1-\gamma}{2}} v^{-\gamma} \left\{ \frac{1}{k_1 k_2} \right\}^{\frac{1+\gamma}{2}}$$

where t = duration of contact.

$\gamma = \frac{1-\beta}{1+\beta}$,

$k_1 = \frac{m_1 + m_2}{m_1 m_2}$, m_1 and m_2 being the masses of the colliding bodies.

Thus according to this theory the duration of contact depends upon the velocity of approach, the radii of curvature of the impinging surfaces, and the masses of the impinging bodies. When $\beta = \frac{3}{2}$, $\gamma = -\frac{1}{5}$.

EXPERIMENTAL WORK.

The inadequacy of the St.-Venant theory was demonstrated very early by Hamburger (5), who found that the duration of contact of bars impinging end to end was considerably greater than that predicted On the other hand it was found that the duration of contact of metal spheres was in reasonable agreement with the predictions of the Hertz theory. It may be noted that Lord Rayleigh has shown (6) that the ratio of the kinetic energy of vibration to the total energy before collision, for spheres, should be approximately $\frac{1}{50} \cdot \frac{v}{V}$ where V is the velocity of sound in the material of the spheres and v the relative velocity of the spheres before impact.

On the other hand for very violent blows, as, for example, the impact of a rifle bullet on a hard body or the shock produced by the detonation of an explosive in contact with a metal block, the influence of longitudinal waves appears to be paramount. (7), In two major points, however, the St.-Venant theory has been amply shown to afford an incomplete explanation of the experimental results. According to this theory both the duration of contact and the coefficient of restitution should be independent of the velocity of approach; the coefficient of restitution in the case of bars of the same material and cross-section being equal to the ratio of the length of the shorter to that of the longer bar.

All workers have observed a reduction in the duration of contact with increasing velocity (5, 8, 9, 10); while the results of experiments on the coefficient of restitution have shwon that this is always greater than the theoretical value and that it varies with the velocity of approach (11). Voigt explained the discrepancy between the observed and calculated values of the coefficient of restitution by assuming the existence of a layer of transition between the bars, whose properties would depend upon the nature of the ends.

EXPERIMENTS ON BARS WITH ROUNDED ENDS.

It would be impossible to prepare bars with perfectly flat ends, and if prepared it would be almost equally difficult to ensure that the whole of the surfaces should come into contact at the same instant so that the results of experiments on so-called flat ended bars are not of the greatest value.

The first experiments on the impact of bars with rounded ends appear to have been amde by Sears (9), who found, for bars of equal lengths, a linear relationship between the lengths of the bars and the duration of contact. Moreover the slope of the curves gave an excellent value for the velocity of sound in the material of the bars, so that there can be no doubt that longitudinal waves were set up of sufficient amplitude to determine the instant at which the impact ended. The effect of the end could thus be expressed as a simple addition to the effective length of the bar; though it may be noted that this correction showed a distinct variation with the velocity of approach.

Later (12) a theory was elaborated in which the relationship between pressure and indentation for a small volume near the area of contact was considered to be in accordance with the Hertz theory, while the movement of this volume as a whole produced longitudinal waves. Observations on the duration of contact and on the coefficient of restitution were in complete agreement with this theory, which was extended to include the impact of bars of different lengths. Sears's work was performed with steel bars, all of the same end radius and of the same radius of cross-section.

A further investigation was made by Wagstaff([10]) who took the precaution of hardening the ball-ends of the bars so that larger velocities of approach might be employed without producing permanent indentations of the ends, and who used bars of different radii of cross-section. For these bars it was shown that the relationship between the duration of contact, t, and the velocity of approach, v, was of the form $t = Av^{\gamma}$ where A and γ are both constants. independent of the velocity of approach. A linear relationship was found to hold between $-\frac{1}{\gamma}$ and the length, ℓ, of the bars, when these were of the same length. The duration of contact was found not to be a linear function of ℓ. Experiments were performed on bars of unequal lengths (the shorter bar in every case being loaded so that the masses of the two were identical) which indicated that the masses of the bars were of fundamental importance, as the duration of contact was found always to be practically identical with that for a pair of equal bars of the length of the longer bar. The experimental results for short bars, both loaded and unloaded, were found to be in fair accordance with the Hertz theory.

On the whole the experimental work had not revealed the exact relative importance of the production of longitudinal waves and of local indentations during the impact. In the hope of determining the importance of the latter effect and further elucidating the nature of impact, experiments on the collision of bars with ball ends of various radii of curvature were made as described in subsequent chapters.

CHAPTER II.

APPARATUS AND EXPERIMENTAL METHOD.

DETERMINATIONS OF DURATION OF CONTACT.

The duration of contact, t, was measured by a well-known electrical method, the circuit being very similar to those used by previous workers. (9, 10)

By means of the Morse key M, the condenser C was first put in series with the battery E, (Fig.1) then allowed to discharge through the circuit R.B.B.M., where R was a resistance box, and B, B the bars whose duration of contact was to be measured. K_1 was simply a damping key for the galvanometer G, while K_2 served to short-circuit the contact between the bars when necessary. This last key was introduced so that if any doubt arose as to the constancy of the e.m.f. of the battery, the condenser could be completely discharged through G without in any way interfering with the adjustment of the bars.

The ordinary procedure was to charge C, allow the whole charge to pass through G with K_2 closed, and observe the deflection (θ_0) produced, this being a measure of the total charge upon the condenser. Next C was again charged, but before discharge the key K_2 was opened and a resistance taken out of the box R. C was then connected in series with R, B B and G and the deflection (θ) noted when the bars were allowed to impinge. θ, then, was a measure of the charge passing through the contact and $\theta_0 - \theta$ a measure of the charge remaining upon the plates of the condenser. This procedure was repeated for several values of the series resistances.

Neglecting the inductance of the galvanometer, the formula for the discharge of the condenser might be written

$$q = q_0 e^{-t/CR} \qquad \text{or} \qquad \log_e \frac{q}{q_0} = -\frac{t}{CR}$$

where q_0 = total charge

q = charge remaining after discharge for a time t

R = resistance of circuit

$$t = RC \log_e \frac{q_0}{q}$$

$$= RC \log_e \frac{\theta_0}{\theta_0 - \theta}$$

provided the relation between q and θ was linear throughout.

A graph was plotted having $\log_{10} \frac{\theta_0}{\theta_0 - \theta}$ as ordinates and $\frac{10^6}{R}$ as abscissae and from the slope of this t was calculated. Excellent straight lines were obtained in practically every case; exceptions being noted, however, where the bars were not sufficiently hardened at the ends in which case t itself was not constant. The straightness of these lines showed that the effect of the inductance of the circuit was small, but a more detailed eludidation of this point may be of interest and is included before proceeding to a detailed description of the apparatus.

EFFECT OF INDUCTANCE.

The solution of the equation representing the discharge of a condenser through a resistance and an inductance in Series was first worked out by Lord Kelvin[13] and his straightforward method is used here.

Let q = charge upon the condenser at any instant.

" t = time measured from the beginning of the discharge.

" R = total resistance of the circuit.

" L = " inductance of the circuit.

" C = capacity of the dondenser.

Then

$$L \frac{d^2Q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the complete solution of which is $q = K e^{\rho t} + k' e^{\rho' t}$ where k and k' are constants and ρ, ρ' are solutions of the equation

$$x^2 + \frac{R}{L}x + \frac{1}{LC} = 0$$

sothat

$$\rho = \frac{-R + \sqrt{R^2 - \frac{4L}{C}}}{2L} \qquad \rho' = \frac{-R - \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

For the evaluation of K and K' we have, when $t = 0$, $q = q_o$. the initial charge of the condenser.

$$\therefore K + K' = q_o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Any small negative value of t represents an instant before the beginning of ah discharge, so that as t approaches zero the current remains zero. Thus when $t = 0$, $\frac{dq}{dt} = 0$

$$\therefore \rho K + \rho' K' = 0$$

$$\text{or} \quad K + \frac{\rho'}{\rho} K' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

from (2) and (3)

$$K = q_o \frac{\rho'}{\rho' - \rho} \qquad \text{and} \qquad K' = q_o \frac{\rho}{\rho - \rho'}$$

Let $\alpha = \sqrt{R^2 - \frac{4L}{C}}$

Then $\rho - \rho' = \frac{\alpha}{L}$

$$\therefore \frac{K'}{q_0} = \frac{\alpha - R}{2\alpha} \qquad \text{and} \qquad \frac{K}{q_0} = \frac{\alpha + R}{2\alpha}$$

so that

$$q = \frac{q_0}{2\alpha}\left\{(\alpha + R)e^{\left(\frac{\alpha - R}{2L}\right)t} + (\alpha - R)e^{-\left(\frac{\alpha + R}{2L}\right)t}\right\} \qquad \ldots\ldots(4)$$

Now $\alpha = R\left(1 - \frac{4L}{R^2C}\right)^{\frac{1}{2}} = R\left(1 - \frac{2L}{R^2C} - 2\frac{L^2}{R^4C^2} - \cdots\right)$

To the first order of small quantities

$$\alpha - R = \frac{-2L}{R^2C} - \frac{2L^2}{R^4C^2} \text{ etc.} = -\frac{2L}{R^2C}\left\{1 + \frac{L}{R^2C} + \cdots\right\}$$

$$\alpha + R = 2R\left(1 - \frac{L}{R^2C}\right).$$

Put $\frac{L}{R^2C} = \delta$ where δ is small

Then $q = \frac{q_0}{2R}(1 + 2\delta)\left\{2R(1 - \delta)e^{-\frac{t}{CR}(1+\delta)} - 2\delta(1+\delta)e^{-\frac{R(1+\delta)}{L}t}\right\}$ -(5)

It will be shown later that L is of the order $0{\cdot}4 \times 10^{-3}$ henry so that if, as a typical example, we take $R = 1000$ ohms and $C = \frac{1}{3} \times 10^{-6}$ farad then δ is about $1{\cdot}2 \times 10^{-3}$. In this case $-\frac{R}{L}(1 - \delta)t$, the index of the second term, has a value $-(2{\cdot}5 \times 10^6)t$.

Substituting some of these values in (5) we obtain

$$Q = q_0\,(1{\cdot}0024)\left\{(1 - {\cdot}0012)e^{-\frac{t}{CR}(1{\cdot}0012)} - \frac{1{\cdot}2 \times 10^3}{R}e^{-\frac{R}{L}t}\right\}$$

t usually has a value between 10^{-4} and 10^{-3} second so that when $R = 1000$ the maximum value of the last term is for all practicaly purposes infinitesimal.

Then $q = q_0\,(1{\cdot}001)\,(e^{-t/CR})^{1{\cdot}001}$

For the range of the experimental work the maximum error is of the order of one part in one thousand so that unless very great accuracy is required the effect of the inductance may safely be neglected.

Further, tho. the factor $1 + \delta$ has a definite absolute value for a given resistance and capacity, but does vary with the resistance so that if the effect of this term were important the lines connecting log. $\frac{\theta_o}{\theta_o - \theta}$ and $\frac{1}{R}$ would not be straight.

THE APPARATUS.

The Galvanometer.

A low resistance moving coil galvanometer was used throughout so that the inductance of the circuit should be as small as possible. Over the portion of the scale used in the experiments it was found that the ~~objection~~ deflection was strictly proportional to the charge passing. Moreover, the damping was found to be truly logarithmic so that as only the ratio of two charges, not their absolute value, was required it was unnecessary to apply any correction to the observed deflections

CALCULATION OF THE INDUCTANCE.

After the completion of all the experimental work described hereafter the galvanometer was dismantled and the dimensions of the coil were measured, though with no great accuracy.

The inductance of a rectangular coil of rectangular cross-section is given by the formula (14)

$$L = \cdot 0092\,(a + a_1)n^2 \left\{ \mathrm{Log}_{10} \frac{2aa_1}{b+c} - \frac{a}{a+a_1} \log_{10}(a+g) \right\}$$

$$+ \cdot 004(a+a_1)n^2 \left\{ 2\frac{g}{a+a_1} - \frac{1}{2} + .447 \frac{b+c}{a+a_1} \right\}$$

Where a_1 = length of coil

a = breadth of coil

b = depth of coil.

c = thickness of winding

n = no. of turns.

$g = \sqrt{a^2 + a_1^2}$

a, b, and c were measured directly; n was calculated from the resistance of the coil and the resistance per metre of gauge 40 copper wire, of which the coil appeared to consist. There is some doubt about the value of n but as the effect of inductance is certainly small, and probably quite negligible, an accurate knowledge of the quantities concerned is unnecessary.

$a_1 = 5 \cdot 5$ cm., $a = 0 \cdot 1$ cm. $b = 0 \cdot 7$ cm. $c \doteq 0 \cdot 1$ c.m.

$n \doteq 100$ turns.

This gives $L = 0 \cdot 40_5 \times 10^{-3}$ henry.

GENERAL ARRANGEMENTS

The general arrangement of the apparatus is shown in Fig. 2 which is drawn approximately to scale and includes practically everything except the keys, wiring etc. of the various circuits.

The bars were slung from stout brackets by means of eight cords (four to each bar) each about six feet long and so adjusted that each bar could move only parallel to itself in a vertical plane, while the two bars were exactly in line when at rest in the equilibrium position, and when in that position remained with their shaped ends (to which reference will be made later) just in contact For this purpose each cord was attached at its upper end to a spindle which could be rotated by means of a worm wheel so that a very delicate adjustment of the suspension was obtained. As the angle between two cords attached to the same point on the bar was small (about 25^0) a very slight alteration in length of one of thecords produced a marked lateral displacement of the bar.

Each bar was drilled and tapped to take two brass hooks to which the suspension cords were attached, a layer of sealing wax over the hooks ensuring perfect insulation of the bars. Electrical connection to the bars was made by means of fine, coiled, copper wires attached to pins on the underside of the bars.

It is necessary to know the relative velocity of the bars just before collision, and for ease in working out the results it must have been possible to use the same velocity repeatedly for different bars. Accordingly the left hand bar was kept at rest in the equilibrium position just before the impact, while the right hand bar was withdrawn a measured distance, x, and released. The relative velocity of approach, v, was then given by $v = \frac{2\pi x}{T}$ where T is the time of the swing of the displaced bar.

It has already been mentioned that a slight alteration in length of the suspension cords would have produced a serious disturbance of the apparatus, so that it was necessary to reduce to a minimum the time spent on one series of observation, to avoid as far as possible the effects on the cords of changing temperature and humidity of the air. The release device S and the trap T were designed with this end in view, and will be described in some detail.

THE RELEASE.

An elevation of this is shown in fig. 3. The bar was withdrawn by a thread passing over a light wooden pulley and attached at its upper end to a hook on the brass bar GG, the upper end of which was constrained to move in a vertical line by a pair of wooden guides AA', and a wooden block W attached to the upper end of the rod. The rod also passed through a hole in the yoke of the electromagnet M and carried at its lower end a lead weight so that when released it fell rapidly in a vertical line. Two strips of wood BB, shown cut away for clearness, united the guides AA' and so formed a rigid framework; they also served to support the electromagnet M which held the rod when the bar was displaced. The rigidity of the magnet M was finally assured by slotting the brass flange of the coils into the guides AA'.

When the electromagnet was energised and the rod GG had been pulled into position by a cord attached to the hook H, the armature RR, which was a loose fit on the rod, compressed the spring S and forced a collar C_1, attached to the rod, against the yoke of the magnet. Thus the displacement of the rod GG depended solely on the adjustment of C_1 and not on the exact position of the armature RR. This was necessary as the surfaces of the armature and of the poles of the magnet were not accurately machined. A second collar C_2 held RR in position when free from the magnet while a pin soldered to C_2 prevented any rotation of the armature about the rod. The fall of the rod was checked by a rubber stop D, the resilience of which prevented the shock from disturbing the apparatus in any way.

With this arrangement it was found that by pulling the cord attached to H (which could be accomplished from the front of the bench) the bar was withdrawn into exactly the same position every time. The actual displacement of the bar, not of the rod GG, was observed by means of a travelling microscope (not shown in fig. 3) and was adjusted to its ~~xxxxxxxxx~~ initial value simply by sliding the stand supporting S along the bench. This stand was of very heavy construction and was loaded with lead weights when in position.

During some of the later experiments very small displacements (about 0·2 cm.) were used and for these, when using short bars, it was feared that the weight of the thread and the inertia of the pulley might have an appreciable effect. To eliminate this possibility the catapult shown in fig. 4. was designed.

A cord T from the bottom hook of the rod G G (fig.4) passed over the pulley and was attached to two strips of stretched rubber C,C fastened to the bench and to the thread from the bar, B. This cord was of such length that with the bar in the displaced position the end attached to the elastic just rested in contact with the pulley. Thus the cord T was held taut by the elastic and when the bar GG (fig.3) was allowed to fall the energy required to rotate the pulley and to move the cord was supplied by the elastic, the only drag on the bar being that due to the light thread attached to the pin J (fig.4). The device was completely successful.

THE TRAP.

Before the impact the left hand bar of Fig. 3, referred to hereafter as the second bar, was hanging freely at rest, After the impact it was necessary to retain it in a displaced position, with as little strain on the suspension as possible, so that only one discharge of the condenser could take place; although, in case of accident, the key M could be released to isolate the galvanometer system. For the next impact the second bar had to be replaced in its original position. At first these operations were accomplished by hand with the aid of a small mechanical trap but the time spent in the readjustment of the bar was considerable and a slight draught before the collision was sufficient to disturb the second bar appreciably. To overcome these inconveniences the trap T, shown in some detail in figs. 5 and 6 was designed.

This trap fulfilled two purposes, to remove the bar from the equilibrium position after the collision and to bring it back to rest in the equilibrium position before the next collision.

The portion of the trap used for restoring the bar to its zero position consisted of a wooden trough filled with mercury and supporting a brass framework carrying a heavy brass block F free to turn about a horizontal axle P (fig. 5) Sufficient mercury was placed in the trough for the block to return slowly after displacement to its original position, its motion being heavily damped by the action of the vane V. The pin carried by the second bar J was protected by a short ebonite tube and rested between the cheeks of the upper part of the brass block. Finally, the trough was so adjusted that when hanging freely the pin touched neither cheek, its actual clearance being about ·02 cm. on either side. After suffering a displacement the bar oscillated about its zero position but its motion was very strongly damped and, owing to the poor resilience of the ebonite, in a few seconds the pin was completely free from contact with either cheek.

The actual trapping of the bar after collision was accomplished by means of the soft iron armature N, attached to the block F, and an auxiliary electromagent E (Fig. 6). I and I_1 represent the two springy brass contacts, I being supported by the electro magnet itself while I_1 was carried by a brass bracket on the block L. The remainder of the circuit comprised the electromagnet

E, a key K and the battery. The two contacts, I and I_1, were so adjusted that the circuit was normally open, but when the block F was displaced towards E, by the pin J, I_1 was forced into contact with I and the circuit closed, the key K being closed when undisturbed. This process energised the magnet so that the armature N was held against the poles and the pin J held the second bar away from its equilibrium position. As the block F when displaced towards the magnet held I and I_1 in contact the magnet remained energised until the key K was opened when the armature was released so that the circuit was broken once more at I I_1. The block F was then free to restore the bar to its equilibrium position, while the circuit could only be completed by closing the contacts I, I_1. It will be seen that once the apparatus had been adjusted the only manipulation necessary was the opening of the key K after each collision.

This trap had the additional advantage that no lateral thrust was exerted on the pin, so that the alignment of the bars suffered very little distrubance, while the mercury absorbed the major part of the energy of the collision, bringing the bar to rest without undue violence.

In the final arrangement of the apparatus the three keys controlling respectively the release device, the condenser circuit and the trap were placed side by side, at the front of the bench, and the cord used to pull up the bar of the release device was fastened beside these so that the whole apparatus could be used as quickly as possible. A great improvement in ease and accuracy of work resulted from the introduction of these pieces of apparatus.

OBSERVATIONS.

For each displacement of the impinging bar five values of the series resistance R, in the galvanometer circuit, were usually used, and for each resistance four values of the deflection were tabulated, though this number was increased where any appreciable discrepancy between the deflections was noted. It was found possible to record the deflections to a fraction of a millimetre. The accuracy of the release device was repeatedly checked with the travelling microscope, but its behaviour was satisfactory throughout.

THE BARS.

PREPARATION OF THE BALL-ENDS.

As the end in view when these experiments were begun was primarily a determination of the effect of the radius of curvature of the ball-end on the duration of contact the greatest possible care was taken in the grinding to ensure that the ball-ends should be truly spherical and of exactly the same radius for each bar of a pair. The processes involved may be of interest and are accordingly described here.

A gauge of the required size was first prepared and the end of the bar, while still soft, was turned so as to fit this exactly. This end was now pressed into the surface of a number of lead blocks so as to make a series of shallow depresssions of the size of the ball-ends required, these blocks being used later in the grinding of the ends.

The bars were now thoroughly hardened by quenching in water and were ground and polished with the aid of the lead blocks already prepared so that, again, the gauge fitted perfectly. A microscopical examination of the optical images formed by reflection in the end under inspection was finally made to ensure that no irregularity was present around the region of contact.

Ball ends of radius 2", 1", ½", 5/16", ¼", 1/8", and 1/16" were used. More flat-ended bars than the first mentioned would have offered serious difficulties in preparation as the 2" ends were found very liable to suffer considerable distortion or even fracture during the quenching. On the other hand the 1/16" ends, while easy to grind, were subject to such intense stresses during the collisions that regrinding during the course of a set of experiments was sometimes necessary.

Where the radius of curvature of the ends was less than 5/16" (the radius of the bar itself) the end was coned down at 30° with the axis in each case, the generators of the cone being approximately tangential to the ball-end.

SCOPE OF THE EXPERIMENTS.

RANGE OF VELOCITIES USED.

The time taken for longitudinal waves to travel over twice the length of one of the bars is constant and this, for equal bars, gives the duration of contact according to the St. Venant theory. Thus, according to this theory there should be no variation of duration of contact with velocity of approach.

On the other hand the Hertz theory indicates that the duration of contact, t, is inversely proportional to the fifth root of the relative velocity of the colliding bodies, just before impact. The observations of previous workers (6,5,10,12.) have confirmed experimentally that a variation of t with velocity does occur and it has been found that for bars with rounded ends t is proportional to the velocity raised to some power which varies between $-\frac{1}{4}$ and $-\frac{1}{10}$. Thus the duration of contact may increase at the outside by 25% when the velocity is halved.

It was therefore considered desirable to obtain the greatest possible ratio between the extreme velocities used and as an excessive velocity (greater than about 30 cm. per sec.) involves enormous stresse over the surfaces in contact, an attempt was made to design the apparatus to deal with velocities between 3 cm/sec. and about 25 cm/sec. Actually velocities outside this range were used at times but in general the observations were made within these limits. A velocity of 2·7 cm/sec. corresponded to a displacement of the impinging bar of 1 cm., i.e. to an angular dispalacement of about 0.27°. The total vertical displacement of the bar in this case was about 3×10^{-3} cm. so that any change in length of the suspension cords would have ruined the experiment during which it occurred.

For this reason the minimum displacement ~~during which it occurred~~ used in most of the experiments was 1·0 cm., and when, during later work, displacements smaller than this were used, special precautions were taken.

EXTENDED RANGE.

As indicated above, after the major portion of the work had been completed it was decided to work with a greater range of velocities, and for this purpose some slight modifications in the apparatus were introduced. The trap as used for most of the experiments was so arranged as to allow the bar to swing unconstrained through a very small distance, but for displacements of the impinging bar of the order of two millimetres this practice had to be abandoned. Accordingly a stop was placed so that one side of the trap block (F) rested against it and the pin of the bar was held in contact with the cheek of the trap by a very weak spring. Thus the bar, while no longer free, was brought exactly to the same position after each impact. Experiments made for the purpose failed to show any alteration in duration of contact which could be attributed to the new arrangement. Very careful adjustment of the electromagnet was necessary to ensure that the bar should be withdrawn properly by the trap and the impinging bar was adjusted to its setting by hand before each impact.

To extend observations to higher velocities the thread from the release device was detached from the bar, and was attached to a light metal cap which fitted loosely on the end of the pin below the bar. The bar was withdrawn by hand and held in position by the cap on the thread. When the release device was operated the catapult ensured that the thread was completely slackened and the cap dropped off the pin so that the impinging bar was swung down freely to meet the stationary bar. This practice was necessary as the length of the framework of the release device limited the usual displacement to about 11 cm. These series of observations naturally took longer than those usually made and proved very exacting, the usual number of galvanometer deflections observed in one series being between two hundred and two hundred and fifty.

LENGTHS OF THE BARS.

To determine the effect of the radius of the ball end on the duration of contact it was decided to use five lengths of each bar for each ball end so that any outstanding effect due to the end might be made evident. In particular it was hoped to obtain some simple relationship between the length of the bar and the duration of contact, of such a type that the effect due to the ball end might appear as an addition to the effective length of the bar. It will be remembered that the results of Sears's experiments could be expressed in the form $t = A + Bl$ where A and B were constants independent of the length. Sears, however, worked with only one radius of curvature of the ends of his bars so that there is little real evidence to show that even in the case of soft-ended bars the effect of the ball end is merely to increase the effective length.

To preserve exactly the same end-conditions throughout each set of experiments the original bar was cut down in successive stages from 63·6 cm. to 49·6 cm, 35·6 cm., 17 cm., and finally 8·5 cm. in nearly every case. The hardening of the ends prevented any further reduction in the length of the bars, and even with 8·5 cm. bars some difficulty was found in preserving good alignment during the experiments as, of course, in the case of short bars, a very slight alteration in length of any one suspension cord would produce a relatively great departure from the original direction of the axis.

OBSERVATIONS ON UNEQUAL BARS.

An examination of the results of the observations made on bars of equal lengths will be found in another portion of this work but it may be noted here that neither the Hertz theory nor the St. Venant theory appeared to account completely for the results of those observations. In the hope of obtaining some rather more decisive information a series of experiments was performed in which the lengths of the impinging bars were not, in general, equal.

To begin with, three bars were prepared of lengths 100 cm., 63.6 cm. and 35.6 cm. respectively, the radius of the spherical end being $\frac{1}{2}$" in each case. First, the 100 cm. bar and the 63.6 cm. bar were used as a pair; next the 100 cm. bar and the 35.6 cm. bar were used in the same way. After the completion of these obsrvations the 100 cm. bar was cut down to 80 cm. and the process repeated, and so on, the bar of original length 100 cm. being reduced successively to 80 cm., 63.6 cm., 49.6 cm., 35.6 cm., 17 cm., and finally to 8.5 cm., the lengths of the other two bars remaining unaltered.

EXPERIMENTAL DETAILS.

The apparatus was designed for use with bars of equal lengths so that some modification was necessary in these experiments. In the work on equal bars the impinging bar came dead to rest after the collision, this being one of the criteria of good adjustment, but after the collision of unequal bars both bars continued to move. Thus the trap no longer functioned to cut out the effect of extra collisions, and where the lengths were very discrepant a simple mechanical trap had to be used as in the earliest experiments. To allow the discharge to take place only during the first impact the morse key M was released immediately after the sound of the collision had been heard, and the impinging bar was withdrawn as soon as possible, because a series of blows on the bar held by the trap might well have spoiled the adjustment of that bar.

The St. Venant and the Hertz theories yield fundamentally different results for the coefficient of restitution of unequal bars and for the pressure between the bars; accordingly observations of the area of contact and of the coefficient of restitution were begun.

AREA OF CONTACT.

When two spherical surfaces of identical curvature are pressed together by forces acting along the line of centres, there is formed a plane surface where the spheres are in actual contact. The area of this plane surface is referred to as the "Area of Contact".

This area must obviously increase as the pressure increases and it is extremely probable that a simple power law connects the diameter of the circle of contact and the steady pressure between the bodies. Thus a series of observations of the radius of the circle of contact gives at least qualitative information on the maximum pressure during the contact.

To make these observations the method finally adopted was to mount the bars as for the determination of the duration of contact, omitting only the wire leads to the bars, and to withdraw one of them a measured distance as usual. The end of this bar was then lightly smoked, using an ordinary wax taper, and a single impact allowed to take place.

When the end of either bar was examined with a microscope a double ring was seen, consisting of an inner circle which was taken to be the circle of actual contact, and an annulus whose width depended on the depth of the layer of soot. Measurements made with layers of various thicknesses showed no variation in the size of the inner circle, but a very thin layer was usually employed as this gave the clearest image. The central patch had usually a different colour from the outer ring and from the untouched surface, colours very like interference colours being visible in all of the films.

Attempts to measure the area of contact using layers of homogeneous materials, such as wax, were unsuccessful, as the patch was poorly marked and the boundary represented the limit of contact between the steel of the second bar and the wax layer. Thus the area, indicated by the wax depended on the thickness of the layer, and no inner ring was visible as was the case with a layer of soot.

As will be seen from the photographs the areas of contact were very good circles; their perfection is an indication of the accuracy with which the ends of the bars had been prepared.

ARRANGEMENT OF THE MICROSCOPE.

A Leitz microscope was used with a photographic attachment, giving a resultant linear magnification of 38·8 in its standard adjustment. The adjustment for this magnification was recorded and was repeated whenever observations on the area of contact were made.

To illuminate the end of the bar a microscope cover glass was mounted in a wire frame clamped to the microscope stage in such a manner that a condensed beam of light from a 100 candle-power tungsten arc lamp was reflected along the axis of the microscope. The bar itself was mounted to lie along the axis of the microscope so that the pole of the spherical end lay in the plane of the microscope stage. Thus the end of the bar was powerfully illuminated and the convergence of the incident light was such that even with a polished end a large patch of light was visible (see photographs)

Besides observations on the area of contact, this apparatus was used to examine the ends of the bars after the determinations of the duration of contact had been completed. The results of these observations are discussed elsewhere.

OBSERVATIONS ON THE COEFFICIENT OF RESTITUTION.

For the impact of equal bars both the St.Venant theory and the Hertz theory indicate the value unity for the coefficient of restitution, the momentum of the impinging bar being transmitted across the junction without loss of energy. That this is very nearly true is shown by the fact that after collision the impinging bar remained stationary, provided that the adjustments had been carried out as described at the beginning of this chapter. Further, when an impact had been allowed to take place without any restraint upon either bar, one of them being at rest in its equilibrium position before the impact, the bars collided repeatedly without noticeable loss of energy. It was therefore considered unnecessary to measure the coefficient of restitution for equal bars; more particularly as such a measurement would provide no criterion of the applicability of the various theories.

In the case of unequal bars, however, while the Hertz theory again indicates a value of unity for the coefficient of restitution, the St. Venant theory gives the value of the same coefficient as equal to the ratio of the length of the shorter bar to the length of the longer. It was noticed in practice that the sound produced by the collision of unequal bars was much more intense than that produced by the collision of bars of equal length; also that repeated collisions of freely-swinging bars very soon diminished the relative velocity of the bars. It appeared, then, that the coefficient of restitution had a measurable value and attempts were made to measure it for various velocities within the range of those used for the other experiments.

It was found that the coefficient was measurable without serious difficulty only when the lengths of the bars were very discrepant.

METHOD OF MEASUREMENT.

The bars were adjusted for the impact as usual, except that the second bar, i.e. the bar receiving the blow, was not restrained by the trap but allowed to swing freely; although, of course it was reduced to rest immediately before the impact. Two microscopes were employed, one for each bar.

For each bar the position at equilibrium of a reference mark was determined, and the displacement of the first bar adjusted to a suitable value. The microscope observing the second bar was then moved to a suitable position and the impact allowed to take place, when the position of the reference mark at the limit of its swing was noted. This was repeated and the adjustment of the second microscope altered until the reference mark appeared to coincide instantaneously with the crosswires.

The velocity v of either bar when passing the equilibrium position is given by $v = \frac{2\pi x}{T}$, where T = the time of swing of the bar and x = the maximum displacement of the bar. T was the same for both bars so that the displacements observed by the microscopes were proportional to the velocities of the two bars when passing through the equilibrium position.

To complete the observations the displacement of the first bar after the impact was also determined, as the coefficient of restitution is the ratio of the relative velocity of the bodies after the impact to that before the impact.

CHAPTER III.

IMPACT.

EXPERIMENTAL RESULTS .

OBJECT OF THE FIRST EXPERIMENTS.

It has been indicated in the introduction that two different theories have been introduced to explain the nature of impact; the Hertz theory in which the local distortion of the bodies in the region of the area where the bodies are in contact is considered to govern the reactions between the impinging bodies, and the St.Venant theory in which such local strains are considered to be relatively unimportant, the propagation of longitudinal waves inthe body being the most important factor. There can be little doubt that the impact of spheres at low velocities is nearly in accordance with the Hertz theory (5, 15) and that the longitudinal impact of very long rods, at least at high velocities, is in accordance with the St. Venant theory.

The impact of rods of moderate length (less than 100 cm.) with rounded ends had been carefully investigated by Sears and by Wagstaff, but their conclusions regarding the effect of the rounded ends were not in agreement. These workers did not vary the curvature of the ends of their bars during the experiments to which reference has been made.

The first object of the experiments described herein was to find the effect of the radius of the ball end on the duration of impact, in the hope of determining to what extent the nature of the impact was influenced by the shape of the rods in the immediate neighbourhood of the area of contact.

RESULTS OF THE EXPERIMENTS.

The first set of experiments was made with bars whose rounded ends had a radius of 1". At the end of the whole set it was found that a slight flattening of the ends had been produced, ~~so that~~ in consequence of which the results were rejected, but the experiments proved useful in that they indicated how the apparatus could be improved, and helped to develop the technique necessary to obtain reliable results.

The results of the next group of experiments on duration of contact will be found in the accompanying table.

TABLE I.

RADIUS OF BALL END 2" = 5.08 cm.

Length = 63.6 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1.	.6695	
1.5	.6471	
2	.6264	
3	.6058	2.89_2
4	.5889	
6	.5731	
8	.5489	
11	.5301	

Length = 49.6 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1	.6160	
1.5	.5709	
2	.5668	
3	.5389	2.89_2
4	.5197	
6	.4894	
8	.4701	
11	.4557	

Length = 49.6 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1	.6188	
1.5	.5957	
2.	.5682	
3	.5345	2.89_4
4	.5166	
6	.4888	
8	.4684	
11	.4534	

Length = 35.6 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1	.5373	
1.5	.5134	
2	.4931	
3	.4615	2.89_2
4	.4371	
6	.4092	
8	.3893	
11	.3629	

Length = 17.0 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1	.3948	
1.5	.3482	
2	.3250	
3	.2949	2.87_8
4	.2708	
6	.2335	
8	.2066	
11	.1843	

Length = 17.0 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1	.3915	
1.5	.3527	
2	.3277	
3	.2956	2.87_6
4	.2731	
6	.2318	
8	.2117	
11	.1881	

Length = 8.5 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
1	.2749	
2	.2014	
4	.1324	2.87_4
6	.0958	
8	.0656	
11.	.0328	

RADIUS OF BALL END 1" = 2.54 cm.

Length = 63.6 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1	.7034	
2	.6400	
4	.6005	2.89_4
6	.5758	
8	.5624	
11	.5445	

Length. = 63.6 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
0.5	.7467	
1	.7259	
1.5	.6921	2.89_4
2	.6619	
4	.6176	
6	.5911	
8	.5736	
11	.5536	

Length = 49.6 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1	.6052	
1.5	.5776	
2	.5624	2.89_4
4	.5133	
6	.4969	
8	.4713.	
11	.4578	

Length = 35.6 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1...	.5566	
1.5	.5271	
2	.5044	2.89_0
4	.4546	
6	.4261	
8	.4047	
11	.3749	

Length = 17.0 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1	.3991	
1.5	.3653	
2	.3393	2.87_9
4	.2802	
6	.2450	
8	.2225	

Length = 8.5 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1	.2567	
2	.2078	2.87_4
4	.1507	
6	.1169	

Length = 8.5 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1	.2382	
1.5	.2127	
2	.2050	2.87_8
4	.1472	
6	.1097	
8	.0955	
11	.0662	

Length = 8.5 cm.

x (cm.)	Log. ($t \times 10^4$)	T (sec.)
1	.2587	
2	.2166	
4	.1491	
6	.1182	2.87_7
8	.0893	
11	.0687	

RADIUS OF BALL END ½" = 1.27 cm.

x (cm.)	Log(t x 10^4)	T (sec.)	x (cm.)	Log (t x 10^4)	T (sec.)
	Length = 63.6 cm.			Length = 17.0 cm.	
1	.7346		1	.4611	
2	.6783		2	.4094	
4	.6411	2.89_3	4	.3446	2.88_1
6	.6095		6	.3106	
8	.5911		8	.2852	
12	.5682		12	.2514	
	Length = 49.6 cm.		R	Length = 8.5 cm.	
1	.6839		1	.3548	
2	.6318		2	.2744	
4	.5833	2.88_7	4	.2002	2.87_1
6	.5498		6	.1651	
8	.5345		8	.0979	
12	.4980		12	.1102	
	Length = 35.6 cm.			Length = 8.5 cm.	
1	.6126		1	.3274	
2	.5446		1.9	.2735	
4	.4924	2.88_7	4	.2061	2.87_9
5.9	.4645		6	.1660	
8	.4463		8	.1373	
12	.4175		11	.1176	
R	Length = 17.0 cm.				
1	.4354				
2	.3817				
4	.3301	2.88_0			
8	.2735				
12	.2384				

RADIUS OF BALL END 5/16" = .7935 cm.

x (cm.)	Log. (t x 10^4)	T (Sec.)	x (cm.)	Log. (t x 10^4)	T (sec.)
	Length = 63.6 cm.			Length = 17.0 cm.	
1	.7521		1	.5082	
1.5	.7249		1.5	.4498	
2	.7038		2	.4368	
3	.6759	2.89_1	3	.3940	2.88_9
4	.6578		4	.3665	
6	.6281		6	.3336	
8	.6027		8	.3087	
	Length = 49.6 cm.			Length = 8.5 cm.	
1	.7075		1	.3758	
1.5	.6705		1.5	.3400	
2	.6515	2.88_6	2	.3166	---
3	.6274		3	.2691	
4	.6083		4	.2445	
6	.5798		6	.2127	
8	.5554		8	.1802	
11	.5289				
	Length = 35.6 cm.				
1	.6246				
1.5	.5938				
2	.5728	2.88_8			
3	.5402				
4	.5189				
6	.4897				
8	.4751				
11	.4455				

RADIUS OF BALLL END $\frac{1}{4}$" = 0.635 cm.

x (cm.)	Log.(t x 10^4)	T (sec.)
	Length = 63.6 cm.	
1	.7759	
1.5	.7448	
2	.7305	2.89_5
4	.6716	
6	.6370	
11	.5832	
	Length = 49.6 cm.	
1	.7200	
1.5	.6860	
2	.6589	
4	.6071	2.89_3
6	.5787	
8	.5589	
11	.5319	
	Length = 35.6 cm.	
1	.6295	
1.5	.5945	
2	.5802	2.89_1
4	.5257	
6	.4995	
8	.4807	
11	.4576	
	Length = 17 cm.	
1	.5319	
1.5	.5014	
2	.4697	2.88_2
4	.4037	
6	.3695	
8	.3419	
11	.3130	

8.5 cm. bars damaged in cutting.

RADIUS OF BALL END 1/8" = 0.3175 cm.

X (cm.)	$Log_{10}(t \times 10^4)$	T (sec.)	x (cm.)	$Log_{10}(t \times 10^4)$	T (sec.)
	Length = 63.6 cm.			Length = 17.0 cm.	
0.5	.9348		1	.5802	
1	.8631		1.5	.5453	
1.5	.8377		2	.5235	
2	.8169	2.89_3	4	.4599	2.88_3
4	.7668		6	.4248	
6	.7315		8	.4005	
11	.6793		11	.3687	
	Length = 49.6 cm.			Length = 8.5 c.m	
1	.7727		1	.4379	
1.5	.7438		1.5	.4043	
2	.7160		2	.3787	
4	.6635	2.88_8	4	.3133	2.87_6
6	.6431		6	.2951	
7.5	.6271		8	.2627	
11	.5965		11	.2286	
	Length = 35.6 cm.			Length = 8.5 cm.	
1	.6966		1.	.4445	
1.5	.6630		2	.3827	
2	.6408		4	.3158	2.87_6
4	.5892	2.89_1	6	.3038	
6	.5541		8	.2650	
8	.5396				
11	.5205				

RADIUS OF BALL END 1/16" = 0.1587 cm.

First set of bars.

x (cm.)	Log. (t x 10^4)	T (sec.)
	Length = 63.6 cm.	
1	.8724	
2.1	.8079	
3	.7774	
4	.7538	2.89_4
1.5	----	
1	.8394	
6	.7165	
	Length = 63.6 cm.	
4	.7515	
8	.6893	
10	.6626	2.88_8
12	.6546	
	Length = 49.6 cm. (reground).	
1	.8394	
1.5	.8052	
2	.7805	
3	.7444	2.89_1
4	.7220	
6	.6824	
8	.6583	
	Length 35.6 cm. (Reground)	
1	.7580	
1.5	.7177	
2	.6959	
3	.6644	2.88_4
4	.6401	
6	.6054	
8	.5812	

x (cm.)	Log. (t x 10^4)	T (sec.)
	Length, = 17 cm.	
1	.6149	
1.5	.5770	
2	.5514	
3	.5140	2.87_8
4	.4879	
6	.4541	
8	.4303	
	Length = 8.5 cm.	
1	.4796	
1.5	.4439	
2	.4210	
3	.3849	2.87_0
4	.3592	
6	.3274	
8	.3057	

TABLE II.

Radius of Ball End (cm.)	Length of Bar. (cm.)	$-\frac{1}{\gamma}$	Radius of Ball End. (cm.)	Length of Bar. (cm.)	$-\frac{1}{\gamma}$
5.08	63.6	7.51	.635	63.6	5.88
	49.6	6.22		49.6	5.56
	35.6	5.89		35.6	6.18
	17.0	5.06		17.0	4.72
	8.5	4.38			
2.54	63.6	7.33	.3175	63.6	5.68
	49.6	6.72		49.6	5.99
	35.6	5.97		35.6	5.68
	17.0	5.10		17.0	4.99
	8.5	5.28		8.5	4.67
1.27	63.6	6.87			
	49.6	5.91			
	35.6	5.64			
	17.0	5.12			
	8.5	4.90			
.7935	63.6	6.30			
	49.6	6.24			
	35.6	5.77			
	17.0	4.99			
	8.5	4.65			

Radius of Ball End (cm.)	Intercept on $-\frac{1}{\gamma}$ axis
5.08	4.26
2.54	4.42
1.27	4.58
.7935	4.34
.635	4.29
.3175	4.43
MEAN.	4.39

Before discussing the experimental redults embodied in the foregoing tables it is necessary to explain the reason for the repetition of some of the experiments.

In general, as may be seen from the detailed experimental readings, very consistent agreement was obtained among different observations but on some occasions, notably on windy days when dust in the atmosphere was liable to settle around the region of contact, much more discrepant readings were obtained; when this had occurred the observations in question were repeated. In such cases the second set of readings (usually the more consistent set) was used and the former set rejected.

Very occasionally the suspension cords were subject to disturbance and poor alignment of the bars rendered a series of observations difficult to carry out. In this case supplementary observations were made to verify the results and both sets of observations were used.

Observations which, for one of the reasons outlined above, had been rejected are distinguished by the letter R in the above table.

RESULTS OF THE EXPERIMENTS.

(1) DURATION OF CONTACT AND VELOCITY OF APPROACH.

In agreement with the results of an earlier worker (10) it was found that the relationship between the duration of contact, t, and the velocity of approach, v, could in all cases be expressed in the form,

$$t = Av^{\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where A and γ are constants independent of the velocity of approach

(1) may be written

$$\text{Log. } t = \text{Log } A + \gamma \log.v$$

so that the relationship between log. t and log. v is linear. Throughout the experiments the suspension cords were of very nearly the same length, as shown by the time of swing of the various bars, so that the log.x, log.t relationship was plotted, where x is the displacement of the impinging bar. These curves are to be found in "Auxiliary Matter, Section B". It will be seen that in every case a straight line has been obtained and in some cases it has been possible to plot two independent sets of readings on one diagram.

(ii) DURATION OF CONTACT AND LENGTH OF BAR.

Both the St.-Venant and the Hertz theories indicate that the duration of contact increases with the lengths of the bars and this was indeed found to be the case, but the actual relationship was not in complete agreement with either theory.

The curves connecting t and ℓ, the length of the bar, were not straight, as predicted by the St.Venant theory; nor, however, was the linear relationship between Log t and Log ℓ , predicted by the Hertz theory, obeyed. (See figs.7, 7A & 8).

It was, however, found that with one exception the relationship between $-\frac{1}{\gamma}$ and ℓ , where γ is the quantity defined in the previous section, was linear over the whole range of the experiments and, that the limiting value of γ as ℓ approached zero was the same in every case, independent of the radius of the ball end.

The exceptional case referred to was that of the bars for which the radius of the ball end was $\frac{1}{4}$". Unfortunately, nothing unusual was apparent in the galvanometer readings, which were consistent, to indicate that a repetition of any of the experiments was necessary but it is probable that the adjustment of the bars had changed continuously throughout the experiment on the 35·6 cm. bars, perhaps by the gradual increase in length of one of the suspension cords. The reasons for suspecting this are that such a change of adjustment was noticed in the case of the more pointed bars (where the effect on the duration of contact was more noticeable), and that the values of $-\frac{1}{\gamma}$ for all bars less than 63·6 cm. in length were found for each particular length to have not very different magnitudes. The value for the 35·6 cm. bars of end radius $\frac{1}{4}$" was well outside the expected range.

Unpublished experiments by Wagstaff, the results of which are included in a separate table, increase the probability of the truth of this explanation, in that they indicate a linear relationship between $-\frac{1}{\gamma}$ and l for bars of $\frac{1}{4}$" end radius. Furthermore the values of obtained by the author for lengths other than 35·6 cm. are in reasonable agreement with these results. (Table IV).

The curve for these bars has been drawn, neglecting the point for the 35.6 cm. bars, and it will be seen that the remaining three points lie on a straight line. It was doubly unfortunate that these particular bars should have been damaged in the final cutting down as an additional point for $l = 8·5$ cm. would have been useful.

BARS OF 1/16" end radius.

During the course of the experiments the ball ends of most of the bars had remained untouched, but the most pointed bars, for which the radius of the ball-end was 1/16", suffered deformation during the first series. This was suspected from the galvanometer readings and a second value of the duration of contact for a displacement of 1 cm. was obtained after the most severe impact in the series had taken place. It was found that the time had materially altered, and subsequent examination showed that a slight flats had been produced on the ends of the bars.

Accordingly the ends of the bars were reground before proceeding with the experiments, but again flattening occurred. It was not until the bars had been cut down below 35.6 cm. in length that the ends withstood the stresses involved in the collision. This was only to have been expected because as the momentum transmitted decreased with the length of the bars so also did the time of impact decrease, though thee latter changed less rapidly than the former, so that there was a slight reduction in mean pressure as the bars were cut down. This point was further investigated by means of photomicrographs, and a discussion of the results is given under the appropriate heading. It has been pointed out in Chapter II that for these bars a very slight misadjustment or change in length of the suspension cords woulld have been sufficient to spoil the accuracy of the results.

For these reasons although straight lines were obtained for the Log. t, Log x, curves little reliance could be placed upon the value of γ, the more so as a very small alteration in t was sufficient to affect the slope very seriously.

At a later date an earnest effort was made to obtain a reliable series of values of y for new bars of end radius 1/16", but it was found impossible with the existing apparatus to obtain any consistent and repeatable values of γ. The results of these experiments are to be found in Table III.

TABLE III.

RADIUS OF BALL END, 1/16" = ·1588 cm.

Length, 63.6 cm.

X. (cm.)	$\text{Log}_{10}(t \times 10^4)$.
0.8	·9398
1.0	·8912
1.3	·8468
2.0	·8152
2.5	·7960
3.0	·7874
0.8	·9302
1.0	·8912
2.0	·8500
2.5	·8155
1.0	·8693
1.3	·8461
1.7	·8244
2.0	·8201
2.8	·7950

x	$\text{Log.}_{10}(t \times 10^4)$
1.0	.8873
1.2	·8658
1.5	·8525
1.8	·8346
2.2	·8226
2.8	·8055
1.0	.9193
1.2	.9073
1.5	.8760
1.8	.8585
2.2	.8314
2.8	.8284

LENGTH = 49·6 cm.

x (cm)	Log_{10} (t x 10^4)
1	·8557
1.3	·8452
1.8	·8181
2.3	·8042
2.9	·7762
3.7	·7455

x (cm)	Log_{10} (t x 10^4)
1.0	·8369
1.3	·8139
1·8	·7901
2.3	·7708
2.9	.7538
3·7	·7333

x (cm.)	Log_{10}(t x 10^4)
1	·8670
1.3	·8456
1·8	·8171
2.3	·7955
2·9	.7760
3·7	.7491

x (cm.)	Log_{10}(t x 10^4)
1	·8451
1.3	·8350
1·8	·8100
2.3	.7881
2.9	·7652

LENGTH = 35·6 cm.

x (cm.)	Log_{10}(t x 10^4)
1.	·7793
1.4	.7513
2.0	.7252
2.6	.6969
3.5	.6706

x (cm.)	Log_{10} (t x 10^4)
1·0	.7813
1.4	.7402
1.8	.7177
2.5	.6977
3.2	.6742
3.9	.6634

Length = 17·0 cm.

x (cm.)	Log.$_{10}$(t x 10^4).
1.	.6793
1.5	.6409
2.2	.5994
3.0	.5630
4.0	.5349
5.5	.5078

x (cm.)	Log.$_{10}$(t x 10^4_4)
1.0	.6751
1.5	.6413
2.0	.6178
3.0	.5752
4.0	.5469
5.5	.5115
1.0	.6565
1.5	.6245
2.0	.6011
3.0	.5626
4.0	.5423

Length = 8·5 cm.

x (cm.)	Log.$_{10}$(t x 10^4)
1.0	.5528
1.5	.5161
2.0	.4865
3.0	.4431
4·0	·4141
1.0	.5528
1.5	.5193
2.0	.4812
3.0	.4475
4.0	.4068
6.0	.3595

Old Bars.

x (cm.)	Log.$_{10}$(t x 10^4)
1.0	.4940
1.5	.4482
3.0	.3817
5.0	.3405

Time of Swing = 2.89 sec. throughout.

TABLE IV.

Radius of Ball Ends = 2"

Length of Bars 35.6 cm.

X (cm.)	$\text{Log.}_{10}(t \times 10^4)$	T Sec.
1	.4526	
4	.3389	
6	.3074	2.425
8	.2861	
10	.2695	
12	.2510	

Length of Bars 17.0 cm.

X (cm.)	$\text{Log.}_{10}(t \times 10^4)$	T Sec.
1	.3000	
4	.1509	
6	.1111	2.424
8	.0887	
10	.0674	
12	.0543	

Radius of Ball Ends 1"

Length of Bars 35.6 cm.

X (cm.)	$\text{Log.}_{10}(t \times 10^4)$	T Sec.
1	.4877	
4	.3877	
6	.3424	2.420
8	.3259	
10	.3142	
12	.2976	

Length of Bars. 17.0 cm.

X (cm.)	$\text{Log.}_{10}(t \times 10^4)$	T Sec.
1	.3087	
4	.1953	2.41_9
6	.1654	
8.	.1289	
10	.1086	
12	.1016	

Radius of Ball Ends $\frac{1}{4}$"

Length of Bars 63.6 cm

x (cm.)	$\text{Log}_{10}(t \times 10^4)$	T. Sec.
1	.7230	
4	.6312	
6	.6016	2.429
8	.5774	
10	.5645	
12	.5517	

Length of Bars. 49.6 cm.

x (cm.)	$\text{Log}_{10}(t \times 10^4)$	T. Sec.
1	.6548	
4	.5609	
6	.5307	2.42_7
8	.5102	
10	.4913	
12	.4747	

Length of Bars. 35.6 cm.

x (cm.)	$\text{Log}_{10}(t \times 10^4)$	T. Sec.
1	.5915	
4	.4738	
6	.4461	2.42_3
8	.4251	
10	.4074	
12	.3923	

Length of Bars. 17.0 cm.

x (cm.)	$\text{Log}_{10}(t \times 10^4)$	T. Sec.
1	.3072	
4	.1581	
6	.1208	2.42_5
8	.1053	
10	.0852	
12	.0718	

x (cm.)	$Log._{10}(t \times 10^4)$	T. (sec.)
Radius of Ball Ends $\frac{1}{4}$"		
Length of Bars. 17.0 cm.		
4	.1650	
6	.1226	2.42_3
8	.0978	
12	.0625	

Radius of Ball End (cm).	Length of Bars (cm.)	$-\frac{1}{\gamma}$
5.08	35.6	5.40
	17.0	4.61
2.54	35.6	5.68
	17.0	5.0_1
.635	63.6	6.10
	49.6	5.94
	35.6	5.56
	17.0	5.18

Limiting value of γ for bars of **end**-radius .635 cm. ($\frac{1}{4}$") $= \frac{-1}{4.79}$

It will be seen from the table that only moderate velocities had been used, so that no deformation of the ends might occur. On the whole it was thought that the results of the first experiments were the more reliable, chiefly because the weather conditions had been particularly good during these experiments and the suspension cords had required very little readjustment when a fresh bar had been suspended. These results have therefore been used in the various curves reproduced in other parts of this work except where it has been definitely specified that all the results for 1/16" bars have been used.

LIMITING VALUE OF γ.

The values of $-\frac{1}{\gamma}$ for the various bars used in these experiments are set forth in table II together with the intercept of the $-\frac{1}{\gamma}$, ℓ curves on the axis $\ell = 0$. (Figs. 9 to 14). It will be seen that the values of the intercepts are sensibly equal. An examination of the curves from which the values of γ have been deduced (Auxiliary section B) will show that a small alteration in t effects a considerable change in γ so that a small deviation from exact equality of the intercepts is to be expected; the actual results point definitely to a limiting value of γ between $-\frac{1}{4.3}$ and $-\frac{1}{4.5}$.

The significance of this result is more fully discussed in the chapter devoted to theoretical considerations but it may be noted here that the Hertz theory of impact would be expected to apply to cases in which the gravest period of longitudinal vibrations is small in comparison with the total duration of contact. This condition is, of course, approached as ℓ decreases, the period of the gravest longitudinal oscillation involved being proportional to the length of the bars.

Now the St-Venant theory corresponds to a value of γ equal to zero, i.e. $-\frac{1}{\gamma} \rightarrow \infty$. Thus a finite value of $-\frac{1}{\gamma}$ probably indicates the existence of an effect dependent upon the geometrical conditions in the neighbourhood of the point of contact while the constancy of the limiting value of γ as ℓ approaches zero seems to point to the existence of an "end-effect" governed by considerations akin to those of the Hertz theory, but not exactly identical with those considerations in that the power index is not $-\frac{1}{5}$ but about $-\frac{1}{4.4}$.

A further point remains to be noted. The points on the $-\frac{1}{\gamma}$, ℓ curves for 63.6 cm. bars did not, as a rule, lie very close to the line and, in fact, although the slopes of these lines bore no apparent relationship to the radius of the ball-end, the values of $-\frac{1}{\gamma}$ for the longest bars were found to increase as the radius of the ball-end increased (Fig.15).

THE t - l curves.

In agreement with earlier experiments[10] on bars of end radius $\frac{1}{2}$" it was found that the t - l relationship was in no case linear, and further that the curves for the various sets of bars were not all of the same shape.

It was thought in the first place that the duration of contact might be expressed by the formula

$$t = A_1 + B_1 l$$

where A_1 and B_1 are constants, A_1 depending on the conditions near the ends of the bars and B_1 upon the velocity of sound in the bars and independent of the end conditions. The direct meaning of this relationship would have been that the duration of contact was equal to the time taken for a longitudinal wave to travel up and down the bar, together with an effect depending on the geometrical character of the impinging surfaces. To test this the reduction in t due to cutting down the bars was measured for different velocities of approach and for all the bars used. In no case was a constant time-difference observed. Thus if the impact had been of this relatively simple type the "end-effect" A_1, must have varied with the velocity, the radius of the ball end, and the length of the bars, and could have been in no respect equivalent to a simple addition to the length of the bars.

This is in marked contrast with the results obtained by Sears who found that for bars of 1" end-radius the duration of contact was given by a formula of the type quoted above: furthermore he was able to calculate accurately from his result the velocity of sound in the steel he used. It is to be noted however that in Sears's experiments the ends of the bars were not hardened; and it is shown later that this difference in character is of the greatest importance and is sufficient to explain the apparent discrepancies in the results.

EFFECT OF THE BALL-END.

The experiments on bars of different lengths had already shown that no very simple interpretation of the end-effect was to be expected, and the results obtained for bars of different end-radii revealed further the complexity of the problem.

Consider once more the equation

$$t = A + B\ell.$$

For bars of different end-radii, but of the same length ℓ_1, and of the same cross-section we should have

$$t_1 = A_1 + B_1\ell_1$$

$$t_2 = A_2 + B_1\ell_1 ,$$

where the suffixes refer to the different bars; so that

$$t_2 - t_1 = A_2 - A_1.$$

Thus the variation of $t_2 - t_1$ with the circumstances of the collision should show how far the quantities A_1 and A_2 can be treated as constants.

It will be seen from Table V that the "end-effect" varies with the length of the bars, and with the velocity of approach, while the only other variable in the experiments is the end-radius itself. No useful purpose would have been served in following these considerations further as the graphs connecting t_2 and t_1 and the length of the bars, or the displacement, x, were not of a simple type, and it was realised that the problem was of a more complex nature.

TABLE V.

t_1 = Duration of Contact for bars of 5/16" end radius less duration of contact for bars of 2" end radius

t_2 = Duration of Contact for bars of 1/16" end radius less duration of contact for bars of 5/16" end radius

Displacement, x = 4 cm.		
Length (cm.)	t_1 (10^{-4}sec.)	t_2 (10^{-4}sec.)
63.6	.67	1.13
49.6	.76	1.21
35.6	.57	1.06
17.0	.46	.75
8.5	.40	.53

Length = 35·6 cm.	
Displ^t.x (cm.)	t_1 (10^{-4} sec).
1	2.21
1.5	2.05
2	1.94
3	1.85
4	1.81
6	1.68
8	1.55

In Fig. 16 the effect of the radius of the ball end (r) on the duration of contact is shown graphically, for a velocity of approach of 17·4 cm. per sec., i.e. a displacement of 8 cm. The curves for other velocities of approach are very similar so they are not reproduced here. It will be seen that the ball end, instead of introducing a small correction dependent on the radius, influences very materially the nature of the impact.

The importance and complexity of the end effect are similar to what would be expected from a system behaving according to the Hertz theory, which would indicate that the duration of contact is proportional to $r^{-\frac{1}{5}}$. Unfortunately measurements and experimental relationships involving the radius of the ball-end were not susceptible of quite the same accuracy as those involving only the length of the bars, particularly where the more pointed bars were concerned, so that the log t - log r graphs provided no useful evidence, the points being rather too scattered to indicate whether or not a simple power relationship held between t and r . These curves were not considered to be of sufficient interest to warrant reproduction.

DISCUSSION.

The results of the foregoing experiments appear to admit of no simple explanation. The non-linear relationship between t and ℓ, the linear relationship between log t and log v and the absence of any constant "end-effect" due to the curvature of the ends of the rods were sufficient to show that a theory of the St.-Venant type was unlikely to offer any adequate explanation of the results. While the existence of the relationship expressed by $t = Av^{\gamma}$

appeared to offer definite support for an explanation on the lines of the Hertz theory, the variation of γ with ℓ, and in particular the apparent existence of a limiting value of γ, as ℓ approached zero, not equal to $-1/5$, were somewhat disturbing. Moreover the non-linear relationship between log t and log ℓ pointed to an influence not taken account of in the Hertz theory; probably an effect due to longitudinal vibrations.

In the hope of ascertaining the extent to which these longitudinal vibrations might be likely to affect the duration of contact, experiments were made upon the collision of bars of unequal lengths. From the point of view of the Hertz theory such bars could be considered simply as two bodies of different masses, the colliding surfaces having the same curvatures. On the other hand according to St.-Venant, the duration of contact should be dependent on the length of the longer bar only, being equal to the time taken by a longitudinal disturbance to travel twice the length of that bar.

Consider now in the light of the St.-Venant theory a bar of length say, 35.6 cm. impinging upon a longer bar, the length of which is reduced in successive stages to about 10 cm. While the impinging bar (35.6 cm.) is the shorter the duration of contact should vary with the length of the stationary bar, but when the impinging bar is the longer the duration of contact should be constant because the length of the impinging bar is itself constant. On the other hand the Hertz theory would, in these circumstances, predict a continuous decrease of t as the length of the bar suffering collision is allowed to decrease, t depending upn the masses of both bars and not upon the length of either.

Thus the influence of the longitudinal vibrations should be to produce a tendency to discontinuity in the $t - \ell$ curve at the point at which the two bars become equal in length.

EXPERIMENTS ON THE COLLISION OF UNEQUAL BARS.

(1) Duration of Contact.

Three bars were prepared of lengths 100 cm., 63.6 cm., and 35.6 cm respectively, of silver steel of circular cross-section, radius 5/16". The end-radius of each was ½" and the end of each bar was hardened, ground and polished as described in Chapter II.

Fourteen sets of results were obtained by cutting, the 100 cm. bar down in seven successive stages to a length of 8.5 cm., two series of observations being taken at each stage, one for collisions with the bar of length 63.6 cm., and the other for collisions with the 35.6 cm. bar. Finally one of the 8.5 cm. bars of ½" end radius used in the original set of experiments was taken and an extended series of observations was made, using this bar and the 8.5 cm. bar obtained by cutting down the bar of original length 100 cm. In table VI the results of this group of experiments are set forth as before.

(a) t and v.

As in the first set of experiments the graph showing the relationship between Log $(t \times 10^4)$ and log x was plotted in every case. It was noticed from the first that the points obtained did lie so accurately on a straight line as before but is was thought that the discrepancy, being small, was due to experimental error. After several series of experiments had been taken, however, it was

TABLE -VI.

Radius of Ball-End $\frac{1}{2}$" = 1.27 cm.

Length of Impinging Bar 63.6 cm.

x (cm.)	$Log_{10}(t \times 10^4)$	T (sec.)	x (cm.)	$Log_{10}(t \times 10^4)$	T (sec.)
	Length of 2nd Bar, 100cm.			Length of Sedond Bar 35.6cm.	
1	.7538		1	.6454	
1.5	.7316		1.5	.6141	
2	.7201	2.88_4	2	.6001	
3	.7038		4	.5607	2.89_8
5	.6807		8	.5224	
8	.6598		11	.5053	
10	.6525				
	Length of Second Bar, 80 cm.			Length of Second Bar 17.0 cm.	
1	.7511		1	.5540	
1.5	.7225		1.5	.5338	
2	.7044		2	.5106	2.89_0
4	.6610	2.88_7	4	.4628	
6	.6355		8	.4218	
8	.6200		11	.3975	
11	.6037				
	Length of Second Bar, 63.6 cm.			Length of Second Bar, 17.0cm.	
1	.7114		1	.5632	
1.5	.6864		1.5	.5338	
2	.6685		2	.5118	2.89_1
4	.6195	2.88_8	4	.4726	
8	.5761		8	.4323	
11	.5533		11	.4093	
	Length of Second Bar, 49.6 cm.			Length of Second Bar = 8.5 cm.	
1	.7070		1	.4690	
1.5	.6560		1.5	.4385	
2	.6398	2.89_0	2	.4063	2.87_0
4	.5954		4	.3457	
8	.5532		8	.2878	
11	.5357		11	.2612	

RADIUS OF BALL END $\frac{1}{2}$" = 1.27 cm.

Length of Impinging Bar, 35.6 cm.

x (cm.)	$Log_{10}(t \times 10^4)$	T (sec.)	x (cm.)	$Log_{10}(t \times 10^4)$	T (sec.)
Length of Second Bar, 100 cm.			Length of Second Bar, 63.6 cm.		
1	.6825		1	.6300	
1.5	.6605		2	.5857	
2	.6477	2.88_9	3	.5569	2.88_4
4	.6182		5	.5328	
8	.5835		7	.5160	
11	.5578				
Length of Second Bar, 80 cm.			Length of Second Bar, 49.6 cm		
1	.6681		1	.6229	
1.5	.6398		1.5	.5909	
2	.6266		2	.5675	
4	.5913	2.88_4	4	.5201	----
6	.5648		8	.4760	
8	.5523		11	.4581	
11	.5328				
Length of Second Bar, 63.6 cm.			Length of Second Bar, 35.6 cm.		
1	.6419		1	.5750	
1.5	.6112		1.5	.5480	
2	.5911		2	.5258	
4	.5459	2.88_4	4	.4787	
8	.5103		8	.4303	2.89_4
11	.4914		11	.4057	

LENGTH OF IMPINGING BAR, 35.6 cm.

Length of Sedond Bar. 17.0 cm.

x (cm.)	Log_{10} (t x 10^4)	T (sec.)
1	.5064	
1.5	.4698	
2	.4459	
4	.3903	2.89_3
8	.3396	
11	.3193	

Length of Second Bar, 8.5 cm.

x (cm.)	Log_{10} (t x 10^4)	T (sec.)
1	.4247	
1.5	.3861	
2	.3622	
4	.3045	2.87_4
8	.2516	
11	.2316	

Extended Series.
Length of Second Bar. 8.5 cm.

x (cm.)	Log_{10}(t x 10^4)	T (sec.)
0.2	.5779	
0.3	.5185	
0.4	.5160	
0.6	.4720	
1.0	.4186	2.87_9
5.0	.2819	
10.0	.2378	
20.0	.1892	
30.0	.1667	

Above. Short Bar Impinging
Below. Long Bar Impinging.

x (cm.)	Log_{10}(t x 10^4)	T (sec.)
0.2	.5316	
0.3	.5047	
0.4	.4692	
0.6	.4373	
1.0	.3964	2.89_4
1.5	.3610	
2	.3387	
4	.2852	
8	.2374	
11	.2127	
....	....	....
20	.1759	

SERIES below obtained by using old 8.5 cm.bar with $\frac{1}{2}$" end radius.

Length of bars, 8.5 cm. each

x (cm.)	$\text{Log}_{10}(t \times 10^4)$	T (sec.)
0.5	.4380	
1.0	.3574	
2.0	.2881	
4.0	.2061	
8.0	.1381	2.88_1
16.0	.0758	
30.0	.0155	
50.0	$\bar{1}.9751$	

seen that the discrepancies were of the same nature in every case and that the log t, log. x relationship was definitely non-linear.

If, however, any appreciable amount of energy had remained in the bars in the form of longitudinal oscillations the coefficient of restitution would have been less than unity; in fact the St.Venant theory predicts that this coefficient should be equal to the ratio of the length of the shorter to that of the longer bar. Thus the displacement x might not have been proportional to the mean of the relative velocities of the bars on approach and on separation, so that the apparently non-linear relationship between log t and log v might have been due to the poor resilience of the bars; while had this mean velocity been used a linear relationship might have been evident. Accordingly observations of the value of the coefficient of restitution were made with a view to elucidating this point. (Table VIII). Furthermore, of course, the results of these observations would indicate the proportion of the energy of the bars remaining as longitudinal oscillations: the earlier experiments indicated that for the type of stress involved in these experiments the elasticity of the steel might be regarded as practically perfect.

It may be noted at the outset that the reduction in relative velocity due to imperfect resilience was found to be much too small to account for the curvatures of the log. t - log.x curves in any one case; as it will be remembered that the duration of contact varied only slowly with the velocity of approach for equal bars and this was still found true of the impact of unequal bars.

The smallness of the departure from linearity of these curves and the ease with which the curvature had at first been mistaken for experimental error caused an examination of the whole of the earlier curves to be made, but no indication of any departure from linearity, however small, was found. Nevertheless inview of the importance attached to these curves and to the theoretical importance of the constant γ deduced from them, it was felt necessary to make a comparison of the collision of equal and of unequal bars. Accordingly the apparatus and experimental procedure were modified as described in Chapter II and an extended series of observations upon the relationship between t and x was made for the impact of bars of lengths 35.6 cm. and 8.5 cm. It should be pointed out that the correction for lack of resilience would be toosmall to be noticed on the diagram (See auxiliary Section C curve No III) even if the velocity of retrogression were substituted for that of approach, so that the log t - log x curve is reproduced directly. It will be seen that the relationship is not linear. (Fig. 17).

To complete the comparison an extended series of observations was made upon a pair of 8.5 cm. bars, as already noted, but the experimental difficulties associated with the use of short bars at very high or very low velocities were such that the points obtained were too scattered to provide a convincing result, though no definite departure from the linear log t - log x relationship was recorded. A description of the results of extended series of observations on equal bars of greater length is given later. A more detailed study was made of the curve of fig. 25 and it was found that the relationship between t and v could be expressed, within the limits of the experiment by the formula

$$\sout{t \times 10^4 = 8.87v^{-\frac{1}{5}}} \quad t \times 10^4 = 8.87 v^{-\frac{1}{3.29}} + 0.81$$

where t is the duration of contact in seconds and v the mean of the relative velocities of the bars just before and just after the collision

The data available for the other series were not sufficiently extensive to allow similar equations to be determined with any exactness or to indicate whether such equations were more than a formal representation of the experimental results. Furthermore it did not appear that the theoretical importance of a series of equations each containing three independent constants was sufficient to justify a repetition of the experimental work.

The above equation is of interest however, in that a similar equation has been found to hold for the collision of spheres of plastic materials. (16.)

Without reference to the extended series, however, it was already evident that the departure from a law of the type $t = Av^{\gamma}$ was very much less in the case of bars of the same length than for the impact of bars of different lengths. It was noted, too, that when the bar of original length 100 cm was cut down to the same length as one of the other bars the appropriate series gave a perfect straight line for the log t - log x curve. Now the establishment of the relationship $t = Av^{\gamma}$ had been regarded as the most important and fundamental consequence of the earlier work, while the regular variation of γ with the length had indicated that this law possessed more than a superficial meaning; so that a departure from this law in the case of unequal bars was of some interest.

The experiments to determine the coefficient of restitution, which are discussed later, had indicated that the departure from a simple power law connecting t and v was associated with imperfect resilience, hence it appeared at least possible that this law held good only where no loss of energy occurred during impact.

Moreover it has already been observed that Sears obtained a linear relationship between t and ℓ for the impact of bars of equal length, the ends of which were not hardened, so that it appears probable that hard-ended bars fall into a separate class. As soon as the observations on unequal bars were completed an experimental investigation of this hypothesis was therefore made.

(b) t and ℓ

As was pointed out before, according to St.Venant, a discontinuity was to be expected in the t - ℓ curves at the point where the lengths of a pair of bars were equal. No such discontinuity was observed. In figs. 18 and 19 the relationship between t and ℓ is shown for both sets of bars, the three curves reproduced in each diagram being for displacements of 2cm, 4 cm and 8 cm respectively. The curves for the other velocities are of the same shape but are omitted for the sake of clearness.

Although the discontinuity which might have been expected is absent yet the curves have not the smoothness and uniform gradation of curvature which a theory of the Hertz type would seem to indicate, so that again there is no evidence to favour the adoption of either theory.

The results may be considered in a somewhat different light. There is available a knowledge of the duration of contact of bars of various lengths with a bar of length @ 63.6 cm., and (b) 35.6 cm; that is to say that the effect of adding 28.0 cm to the 35.6 cm. bar can be determined, so far as the value of t is concerned. If the effect of this change were simply to allow a longitudinal disturbance to travel 28.0 cm. farther, then the difference in t produced by allowing a 63.6 cm. bar instead of a 35.6 cm. bar to impinge on any other bar should be constant: independent of the velocity of approach and of the length of the other bar chosen.

In Table VII is shown the result of subtracting the value of t for the bar of length 63.6 cm. fom that for the bar of length 35.6 cm., when these bars had been allowed to impinge upon the same bar with the same velocity of approach. The displacement used and the length of the bar are both specified in the table.

It will be seen that in no part of the table is the effect of the "end-piece" constant, and that the variations are not of the type attributable to experimental error.

TABLE VII.

REDUCTION IN 't' DUE TO CHANGE IN LENGTH OF 28 cm.
(10^{-4} sec.)

Displacement x (cm.)	1.0	1.5	2.0	4. 0	8.0	11.0	Mean for all velocities
Length of Second Bar(cm.)							
100	.86	.82	.81	—	.74	--	.81
80	.98	.92	.83	.68	.60	.61	.77
63.6	.76	.77	.76	.65	.53	.48	.66
49.6	.90	.63	.67	.63	.58	.56	.66
35.6	.63	.58	.63	.63	.64	.66	.63
17.0	.37	.57	.45	.45	.46	.42	.45
8.5	.29	.31	.25	.20	.16	.12	.22

(ii) THE COEFFICIENT OF RESTITUTION.

This coefficient was measured only in a limited number of cases as values greater than 0.9 were difficult to measure with any accuracy. Little importance should be attached to the values given for the pair of bars of lengths 80 cm., and 35.6 cm. as these represent the result of the first attempt to measure this coefficient. Practice was necessary to enable accurate readings to be obtained.

TABLE VIII.

l_1 represents length of longer bar in cm.
l_2 " " " shorter bar in cm.
x " displacement of impinging bar in cm.

l_1	l_2	x	e observed.	e calculated (St. Venant)	l_1	l_2	x	e observed.	e calculated (St Venant)
80	35.6	8	.91	.445	63.6	8.5	10	.754	.134
		4	.90				8	.777	
							6	.782	
							6	.783	
63.6	17	10	.823	.267			4	.813	
		8	.842				2	.865	
		6	.872						
		4	.895		35.6	8.5	10	.900	.239
		2	.935				8	.906	
							6	.915	
35.6	17	9	.964	.478			4	.918	
		8	.956				2	. 925	
		7	.957						
		6	.963						
		4	.960						
		2	.960						

The value of e, though appreciably less than unity, was in no case equal to the ratio l_2/l_1 nor for different pairs of bars for the same ~~xxxxxx~~ displacement was e a constant fraction of l_2/l_1. In every case it will be observed that the coefficient of restitution increased as the velocity of approach decreased in agreement with the results of the experiments of Deodhar on the impact of spheres moving with low velocities (17). This variation of e with velocity is of interest as a small value of e corresponds with a large amount of energy retained by the bars, probably in the form of longitudinal oscillations. (It was noted during the experiments that the sound produced by the collision of unequal bars was much more intense than that produced by the impact of equal bars, the note corresponding to the production of longitudinal oscillations being the only one heard)

The approach of the coefficient of restitution to the value unity as the velocity of approach decreased supported the idea that at low velocities the major portion of the energy transferred during the contact existed in the form of strain-energy in the immediate neighbourhood of the area of contact; while at higher velocities the greater amount of momentum transferred together with the shorter time of contact combined to encourage the production of longitudinal oscillations of relatively large amplitude. It is to be noted that the energy of a sufficiently violent blow is probably almost wholly converted into longitudinal oscillations, as this method has been used largely for the testing of explosives and the velocity of small projectiles, by Hopkinson and others (7).

These experiments on the coefficient of restitution were sufficient to indicate that the deformation of a small region of the ends of the bars was not the only type of elastic reaction involved and that probably an appreciable amount of energy was converted into longitudinal oscillations during the impact.

No simple relationship between the quantities involved, e, l_1, l_2 and x was discovered, nor did there appear to be any simple connection between the fraction of the energy lost in the impact $\frac{l_2}{l_1+l_2}(1-e^2)$ and the velocity of approach.

(iii) THE AREA OF CONTACT.

After the completion of the first group of experiments on the duration of contact of bars of equal length, a series of photomicrographic observations on the state of the polished ends of the bars was made, with a view to investigating the effects of the enormous, though transient, pressures on the hardened steel of which the rounded ends consisted. This proceeding was inspired by the difficulties attending the use of the more pointed bars, as the irregularities in the readings taken with the shorter bars might well have been due to damage suffered by the polished surfaces. The results of these experiments are discussed elsewhere, but for the present it may be noted that no exceptional difficulty was experienced when using small magnifications so that a method was seen to be available for determining the area of contact in any given collision.

Although the first experiments on the area of contact were performed on the bars remaining from the first group of experiments a consideration of these results is deferred; the table following gives the results of the experiments on bars of unequal lengths.

The detailed observations of the diameters of the patch representing the area of contact are given because these give an indication of the accuracy attained in this type of experiment and of the perfection of the ends of the bars,

Of these results only a few of the first set (63.6 cm & 100 cm bars) were determined photographically; the remainder having been measured directly on a ground glass screen. For the photographic observations the circle best fitting the image on the plate was chosen; the mean diameter given is the diameter of this circle divided by the magnification used (38.8).

TABLE IX.

Length of Bars (cm) (1)	(2)	Displacement (cm)	Obs. Diameter of Circle of Contact. (cm/38.8) (1)	(2)	(3)	(4)	Mean. (cm X10^{-2})
63.6	100	1	1.14	1.14	1.14	1.14	2.94
		2					3.89
		3	1.82	1.84	1.91	1.87	4.79
		4					5.46
		5	2.24	2.25	2.25	2.30	5.82
		6					6.06
		7	2.44	2.44	2.42	2.44	6.26
		8					6.49
		10	2.87	2.86	2.86	2.88	7.40
63.6	80	2	1.67	1.61	1.61	1.61	4.17
		4	2.08	1.96	1.99	2.00	5.18
		6	2.27	2.27	2.27	2.27	5.85
		8	2.58	2.60	2.58	2.61	6.67
63.6	63.6	2	1.53	1.64	1.64	1.57	4.10
		4	2.06	2.05	2.02	2.02	5.26
		6	2.28	2.30	2.29	2.33	5.93
		8	2.54	2.55	2.55	2.55	6.57
63.6	49.6	2	1.61	1.60	1.59	1.60	4.12
		4	2.05	2.07	2.07	2.07	5.33
		6	2.43	2.40	2.36	2.35	6.16
		8	2.62	2.55	2.53	2.53	6.60

TABLE IX (continued)

Length of Bars (cm.) (1)	(2)	Displacement (cm.)	Observed Diameter of Contact. (cm/38.8) (1)	(2)	(3)	(4)	MEAN (cm X 10^{-2})
63.6	35.6	2	1.68	1.64	1.67	1.59	4.25
		4	2.07	1.92	1.92	2.00	5.10
		6	2.18	2.18	2.21	2.20	5.64
		8	2.54	2.58	2.58	2.54	6.60
63.6	17		Observations inadvertently omitted.				
63.6	8.5	2	1.24	1.29	1.28	1.28	3.30
		4	1.67	1.67	1.63	1.65	4.28
		6	1.89	1.95	1.91	1.87	4.92
		8	2.19	2.15	2.18	2.19	5.62
35.6	100	2	1.44	1.41	1.41	1.42	3.66
		4	2.02	1.93	1.99	1.89	5.05
		6	2.08	2.16	2.12	2.14	5.49
		8	2.46	2.48	2.48	2.48	6.39
35.6	80	2	1.47	1.46	1.46	1.46	3.76
		4	2.08	2.06	2.11	---	5.36
		6	2.39	2.28	2.31	2.32	6.01
		8	2.59	2.57	2.59	2.48	6.60
35.6	63.6	2	1.59	1.58	1.57	1.57	4.07
		4	2.06	2.07	2.05	2.06	5.31
		6	2.32	2.34	2.41	2.40	6.11
		8	2.49	2.60	2.58	2.55	6.60

TABLE IX (continued)

Length of Bars (cm.)		Displace-men (cm.)	Observed Diameter of Contact. (cm/38.8) (1)	(2)	(3)	(4)	MEAN. (cm x 10^{-2})
35.6	49.6	2	1.51	1.59	1.59	1.55	4.02
		4	1.88	1.95	1.94	1.92	4.95
		6	2.20	2.26	2.26	2.20	5.75
		8	2.57	2.53	2.53	2.53	6.55
35.6	35.6	2	1.49	1.49	1.50	1.47	3.84
		4	1.88	1.82	1.80	1.90	4.77
		6	2.23	2.13	2.13	2.13	5.54
		8	2.49	2.55	2.46	2.58	6.49
35.6	17	2	1.49	1.46	1.44	1.49	3.79
		4	1.71	1.75	1.68	1.68	4.41
		6	2.00	2.03	2.10	1.98	5.21
		8	2.31	2.31	2.21	2.22	5.82
35.6	8.5	1	0.99	1.02	0.98	1.02	2.58
		2	1.20	1.29	1.24	1.24	3.20
		4	1.72	1.62	1.64	1.72	4.33
		6	1.79	1.84	1.87	1.87	4.74
		8	1.92	1.92	1.91	1.93	4.94

EFFECT OF VELOCITY".

The area of contact was found in every case to increase with the velocity of approach in very much the same way as the duration of contact decreased, the Log.D - log x curve being a straight line; where D represents the diameter of the circle of contact and x the displacement of the impinging bar. It is considered unnecessary to reproduce these curves as the relationship between t and D is explained more fully later; also it was not possible to measure D with the same accuracy as t so that the four points determined did not serve to place the curve very accurately, It is shown in Chapter IV that theslope of the Log D-log x curve should vary more slowly with the circumstances of impact than that of the log t - log x curve, It was found that the results could always be expressed by

$D \propto x^{\delta}$ ~~D∝x~~ where δ was about 0.4.

It was to have been expected that D would change more rapidly than t with the velocity of approach because as the velocity increased the amount of momentum transferred increased while the duration of contact decreased, these two factors combining to increase the mean pressure during the contact.

EFFECT OF LENGTH OF BARS.

An examination of the results will show that for either of the bars whose length was kept constant the value of D did not change uniformly with the length (l) of the other bar. The diameter of the circle of contact changed very slowly with l (see figs. 20 & 21), being ~~somewhat~~ smallest where the lengths of the two bars differed most considerably. In fact the shapes of the D - l curves are very similar to those of the t - l curves, but whereas the t - l curves are inclined to the axis of l the D - l curves are approximately horizontal, the value of D having a vague sort of maximum when the lengths of the two bars are the same.

It is interesting to notice that for displacements of 2 cm and 8 cm practically all the features of the t - l curves are reproduced, particularly for the 63.6 cm bar, including the change in sign of curvature of the line between l = 40 cm and l = 80 cm.

The results of these experiments, though not apparently admitting of any easy quantitative explanation were yet sufficiently stimulating to encourage an investigation of the same type for the case of the impact of bars of the same length. Accordingly two more pairs of bars were prepared; the radius of the ball end being ½" in each case, and the lengths of all the bars 63.6 cm as in the first set of experiments. One pair of bars had the ends hardened as usual, but the ball ends of the other bars suffered no hardening treatment whatever. The exact experiments performed on these bars and their results are described in the next section.

BARS OF EQUAL LENGTH.

(1) Bars with Hard Ends.

The bars were in every respect similar to those prepared for the earlier experiments, but the work performed using the new bars was much more extensive in range and the results were more decisive than before

Observations both on the duration of contact and on the area of contact were performed. As a result of the photomicrographic observations of the damage done to the first bars used, during the adjustment of the apparatus, the ends of the new bars were protected by a covering of aluminium foil during adjustment and were only exposed for the actual collisions. The withdrawals used in the determinations of the duration of contact varied from 0.2 to 20 cm. in the first experiments, but later it was found advisable to restrict the maximum displacement to 11 cm. It will be noticed that the ratio of the maximum to the minimum velocity was therefore 55 to 1 whereas in the first group of experiments the usual ratio had been only 11 to 1.

TABLE X.

RADIUS OF BALL END ½" = 1.27 cm.

Length = 63.6 cm.

x (cm.)	$\log_{10}(t \times 10^4)$	T (sec.)
0.2	.8355	
0.3	.8082	
0.4	.7987	
0.6	.7634	2.891
1.0	.7303	
1.5	.7031	
2.0	.6845	
3.0	.6522	
4.0	.6336	
6.0	.6004	
8.0	.5748	

Length = 49.6 cm.

x (cm.)	$\log_{10}(t \times 10^4)$	T (sec.)
0.2	.7814	
0.3	.7515	
0.4	.7320	
0.6	.7085	
1.0	.6587	
1.5	.6289	2.89_0
2.0	.6109	
3.0	.5850	
4.0	.5651	
6.0	.5353	
8.0	.5126	

Length = 35.6 cm.

x (cm.)	$\log_{10}(t \times 10^4)$	T (sec.)
0.2	.7322	
0.3	.6978	
0.4	.6685	
0.6	.6414	
1.0	.5995	
1.5	.5623	2.88_8
2.0	.5467	
3.0	---	
4.0	.4860	
6.0	.4636	
8.0	.4482	
11.0	.4188	

Length = 35.6 cm.

x (cm.)	$\log_{10}(t \times 10^4)$	T (sec.)
0.2	.7430	
0.3	.7135	
0.4	.6871	
0.6	.6555	
1.0	.6117	2.88_4
~~1.5~~		
2.0	.5589	
~~3.0~~		
4.0	.5072	
6.0	.4780	
8.0	.4610	
~~11.0~~		

Length = 17.0 cm.

x (cm.)	$\log_{10}(t \times 10^4)$	T (sec.)
0.2	.6047	
0.3	.5566	
0.4	.5441	
0.6	.5061	
1.0	.4572	
1.5	.4211	
2.0	.4053	
3.0	.3695	
4.0	.3446	
6.0	.3113	
8.0	.2861	
11.0	.2649	

RATION OF CONTACT. EFFECT OF VELOCITY OF APPROACH.

The most important result of this set of experiments was a most onvincing verification of the validity of the expression $t = Av^{\gamma}$ for he relationship between t and v. The curves connecting log t and og x were perfectly linear throughout and where repetition was consid- red necessary (for the 35.6 cm bars) the two lines were almost exactly arallel, though not quite coincident. (Fig. 22.) Furthermore the alues of γ obtained from these extended series of observations were n complete agreement with those obtained earlier (Fig.11) so that the fundamental portions of this work were regarded with renewed confidence.

The results of the observations on the area of contact will be onsidered later; for the present the contrast between the behaviour f bars with hardened ends and the bars with untouched ends (except for haping) will be considered. These bars, whose ends had not been hard- ned, are referred to as "soft"-ended.

(ii) SOFT-ENDED BARS.

Except for the fact that the untouched steel would not take so high polish as the hardened steel the only distinction betwen these bars nd those of the experiments described above was that no heat treatment ad been applied to the ends. It was found difficult to grind the soft teel, the grinding powder tearing large flakes out of the surface unless pecial precautions were taken. At the end of each series the ball ends ere found to have suffered appreciable deformation (see photographs, Figs. 23 and 24). so that re-grinding was necessary before beginning he next series. In view of this, and because it was necessary to repeat eadings after the bars had been slightly flattened the amount of labour involved was considerable.

Whereas in the previous experiments it had been necessary simply to take one observation for each velocity of approach; for the soft-ended bars a different procedure was necessary. A withdrawal of 1 cm. was first used and a series of galv nometer readings taken. The deflection suffered a gradual decrease as the ends of the bars were flattened, while the resilience, which had been poor during the deformation, became practically perfect. When this had happened a series of readings was taken as usual. The process was repeated for various displacements up to, and including, 11cm. When the flattening was complete for this, the maximum displacement, an extended series of observations was begun as for the hard-ended bars, the same range of velocities being used.

TABLE XI.

SOFT-ENDED BARS.

Radius of Ball End ½" 1.27 cm.

x (cm.)	$Log_{10}(tx10^4)$	$tx10^4$ (sec.)	T. (sec.)	x (cm.)	$Log_{10}(tx10^4)$	$tx10^4$ (sec.)	T (Sec)
	Length 63.6 cm.				Length 49.6 cm.		
1.0	.7033	5.051		1.0	.6360	4.325	
2.0	.6345	4.310		2.0	.5953	3.939	
4.0	.5718	3.731		4.0	.5415	3.479	
8.0	.5285	3.377		8.0	.4877	3.074	
11.0	.5128	3.257		11.0	.4586	2.875	
0.2	.7459	5.571		0.2	.7034	5.052	
0.3	.7251	5.310		0.3	.6766	4.749	
0.4	.7078	5.102	2.89_2	0.4	.6607	4.578	2.89_0
0.6	.6765	4.747		0.6	.6306	4.272	
1.0	.6391	4.356		1.0	.6011	3.991	
2.0	.5896	3.887		2.0	.5536	3.578	
3.0	.5671	3.691		3.0	.5253	3.352	
4.0	.5508	3.555		4.0	.5074	3.217	
6.0	.5359	3.435		6.0	.4853	3.057	
8.0	.5239	3.341		8.0	.4707	2.956	

Table XII (continued)

(cm.)	$Log_{10}(t \times 10^4)$	$t \times 10^4$ (sec.)	T (sec.)	x (cm.)	$Log_{10}(t \times 10^4)$	$t \times 10^4$ (sec.)	T (sec.)
Length. 35.6 cm.				Length 17.0 cm.			
.0	.6034	4.013		1.0	.4531	2.839	
.0	.5340	3.420		2.0	.3821	2.411	
.0	.4793	3.015		4.0	.3241	2.109	
.0	.4169	2.611		8.0	.2636	1.835	
1.0	.3852	2.428		11.0	.2276	1.689	
.2	.6954	4.960		0.2	.5381	3.452	
.3	.6471	4.437		0.3	.5123	3.253	
.4	.6155	4.126	2.88_8	0.4	.4888	3.082	2.88_9
.6	.5800	3.802		0.6	.4534	2.841	
.0	.5379	3.450		1.0	.4040	2.535	
.5	.5084	3.224		2.0	.3483	2.230	
.0	.4915	3.101		3.0	.3245	2.111	
.0	.4669	2.930		4.0	.2993	1.992	
.0	.4504	2.821		6.0	.2667	1.848	
.0	.4239	2.654		8.0	.2412	1.743	
.0	.4050	2.541					

Again the curves showing the relationship between log t and log x were plotted, including the points for those values of t obtained before the most violent collision had occurred. The relationship was far from linear, for both sets of values of t, (Fig. 25), and it will be seen that the curves were not of a very simple type. Thus the most fundamental relationship expressing the behaviour of all the hard-ended bars of equal length did not apply at all to soft-ended bars. The importance of the shape of the ends in determining the absolute value of t is shown very ~~closely~~ clearly by the discrepancy between the values of t before and after the most violent impact, the slight flattening of the ends having much the same effect as an increased radius of curvature. Reference to the tables will show the magnitude of the influence of hardening the ends on the absolute value of the duration of contact.

EFFECT OF LENGTH.

Another marked distinction between the behaviour of hard-ended bars and of soft-ended bars was found in the shape of the t - ℓ curves. For the bars of length greater than about 30 cm. the t - ℓ relationship was linear for all but the smallest velocities ($x < 0.6$ cm.) while the slope of the straight portion of each curve was very nearly the same, and gave a fair value for the velocity of sound in the steel.(Fig. 26). The absolute values of t for the smallest velocities were probably not determined with sufficient exactness for the effect of the length of the bars to be demonstrated effectively. In the diagram the curves for displacements of 0.6 cm. and over are shown.

For hard-ended bars the relationship between t and v was of a simple type while that between t and ℓ was particularly complex:

for soft-ended bars the reverse was true. Thus the collisions of hard-ended and of soft-ended bars are ~~frequently~~ fundamentally different in type so that the varying results observed by different workers were only to be expected.

It may be noted that if the t-ℓ curves are linear and parallel it can be shown algebraically that the log t - log x relationship cannot possibly be linear; also that if log t is a linear function of log x the t - ℓ curves can only be straight if they pass through the origin or are parallel to the ℓ- axis.

Sears found the relationship between t and ℓ to be linear over a greater range than that shown in the above experiments, no sign of curvature being apparent even down to a length of 14 cm. It is suggested that the breaking down of the relationship for short bars in the case of the experiments described above might have been due to lack of complete isotropism in the steel; for some slight non-uniformity of structure is to be expected in the case of cold-drawn steel. The importance of the condition of the metal in this respect was amply demonstrated by the effect of hardening the rounded ends of the bars.

A point of interest was brought out from the values of the intercepts on the t - axis obtained by producing the straight portions of the t - ℓ curves. The relationship between log x and log A' where A' represents this intercept, was not uniformly linear, but appeared to tend to linearity for small velocities. The value of γ obtained from this linear portion of the curve was $-\frac{1}{6.90}$. Sears[12] gives reasons for thinking that the value of γ for flat-ended bars impinging at very small velocities should be $-\frac{1}{7}$. Thus there is reason to think that soft-ended bars behave in some respects as would bars with perfectly plane ends.

In Fig. 27 the $t - l$ curves expressing the results of Wagstaff's experiments on brass bars are drawn. A linear relationship seems to be indicated.

SIZE OF FLAT.

In only one case, the impact of bars of 49.6 cm length for a displacement of 11 cm., was the flattened portion of the end sufficiently clearly marked for a measurement of its diameter to be made. In all other cases the edges of the flat had become blurred by repeated impacts. Attempts were made, without success, to determine the size of the flat before the most violent impact had taken place by taking impressions of the end of the bar. It was not anticipated that this would prove possible, but it was out of the question to remove the bars from the apparatus before proceeding with the rest of the series as a horizontal displacement of the ends of a fraction of a millimetre in a direction at right angles to the length of the bars would have caused the "corners" of the flattened portions to impinge.

For a displacement of 11 cm. and length of bars 49.6 cm. the diameter of the flat was 9.82×10^{-2} cm.

AREA OF CONTACT---HARD-ENDED BARS.

Measurements of the area of contact were made for each stage in the cutting down of the bars for displacements of 1, 2, 4, 6, and 8 cm: the results of which will be found in the accompanying table.

TABLE XII.

Length of Bars (cm.)	Displacement (cm.)	Diam^r. of Patch (D) (10^{-2}cm)	Length of Bars (cm)	Displacement (cm)	Diam^r. of Patch (D) (10^{-2}cm)
63.6	1	2.78	35.6	1	2.66
	2	3.89		2	3.50
	4	4.42		4	4.59
	6	6.06		6	5.49
	8	6.52		8	6.21
49.6	1	2.71	17	1	2.55
				2	3.02
	2	3.84		4	4.00
	4	4.87		6	4.74
	6	5.80		8	5.46
	8	6.65			

EFFECT OF VELOCITY.

The log D. Log x curves were linear within the limits of experimental error, D. showing a fairly rapid increase with velocity. If $D \propto v^{\eta}$ the values of η are given below

TABLE XIII.

Length of Bar (cm.)	η
63.6	.416
49.6	.427
35.6	.402
17.0	.408

It may be remarked that the value of η predicted by the Hertz theory is 0.400. A connection can be deduced between η and γ (See Chap. IV which shows that a value of $-\frac{1}{\gamma}$ greater than 5 corresponds to a value of η exceeding 0.400, but that η varies only slowly with γ. Thus the values of η determined experimentally might support a modified form of the Hertz theory, as described in the next chapter, but it must be remembered that the third figure in these values has not much significance.

VARIATION WITH LENGTH.

It has already been pointed out that a very rapid variation of the pressure during impact with the length of the bars was not to have been expected, and this statement is borne out by the experimental results. As in the experiments on bars of unequal lengths the $D - \ell$ curves were similar in shape to the $t - \ell$ curves, (Fig. 28), the connection between the quantities concerned being expressed approximately by $D^2 \propto tx$ where x is the displacementof the impinging bar.

TABLE XIV.

Length (cm.)	x (cm.)	$D^2/(tx)$	Mean val. of D^2/tx	$\frac{1}{r}\frac{D^2}{tx}$	Length (cm.)	x (cm)	D^2/tx	Mean value of D^2/tx	$\frac{1}{r}\frac{D^2}{tx}$
63.6	1	1.44			35.6	1	1.76		
	2	1.56				2	1.72		
	4	1.14	1.42	1.12		4	1.68	1.71	1.35
	6	1.54				6	1.70		
	8	1.41				8	1.69		
49.6	1	1.61			17.0	1	2.27		
	2	1.81				2	1.79		
	4	1.61	1.67	1.31		4	1.81	1.92	1.51
	6	1.64				6	1.83		
	8	1.69				8	1.93		

The last column is included for comparison with the following results, which are those for the first experiments on the area of contact. As these were the results of first attempts they are not considered to be of the same order of accuracy as the foregoing results, so that the value of D is given only to the second significant figure.

TABLE XV.

Length of Bars(cm)	Displacement x (cm)	Radius of Ball End (cm).r.	D (10^{-2} cm)	t x 10^4 (sec.)	$\frac{D^2}{tx}$	$\frac{1}{r}\frac{D^2}{tx}$
8.5 cm.	1	0.79	2.0	2.38	1.68	2.1_2
"	2	0.79	2.2	2.07	1.17	1.4_8
"	4	0.79	2.9	1.76	1.20	1.5_1
"	6	0.79	3.6	1.63	1.32	1.67
"	8.5	0.79	4.7	--		
"	6	5.08	7.8	1.27	7.96	1.5_5
"	6	1.27	4.7	1.47	2.51	1.9_7
"	6	0.79	3.6	1.63	1.32	1.6_7 *
"	6	0.16	2.1	2.13	0.35	2.2_2

* This result is included a second time to show the effect of the radius of the ball end.

The significance of these results is discussed in Chapter IV.

EFFECTS OF THE COLLISIONS ON THE POLISHED ENDS.

Photomicrographs of the ends of the bars used in the first series of experiments are reproduced in figs. 29 to 38, all of which are mounted so that horizontal markings upon the ends of the bars appear horizontal on the photographs. Except for fig. 38 all of them were taken for a magnification of ~~200~~ 38.8, while fig. 38 shows the state of the ball-ends of radius 1/16" for a magnification of 200.

The difficulties felt in obtaining consistent galvanometer readings when using bars of 1/8" (·3175 cm) end radius are explained by the damage shown in the photographs. This was probably due to corrosion as the weather was very damp when these bars were used, while during the experiments all trace of grease was carefully removed from the ball ends though at all other times the polished surfaces were thoroughly oiled. The irregularity in shape of the patch of light in these and other photographs was not due to the ends of the bars used but to the imperfect optical system used for illumination.

Apart from markings apparently due to corrosion the damage done to the ends of the bars appears to have produced effects of two types

(i) systems of scratches, approximately vertical and considerable in number, and

(ii) series of small marks, dots rather than scratches, situated upon approximately horizontal lines.

(i) VERTICAL SCRATCHES.

These are shown most clearly upon the photographs of the ball ends of radius 5.08 cm though they are still to be distinguished in practically every case. During adjustment the bars usually swung gently to and fro with their polished ends in contact and partly cleaned, while the final cleaning was usually accomplished before finishing the adjustment, so that touching the bars should not be allowed to affect this last setting. Thus the cleaned ends swung in contact for a few minutes, as it was practically impossible to keep the bars dead at rest while altering the suspensions, so that the point of contact moved up and down the surfaces in an approximately vertical line. Before the adjustment was quite complete, however, the line would not have been vertical.

It has been shown (18) that there is a measurable attraction, or rather adhesion, between surfaces that are very clean, out which have not been intensively dried, and that when two such surf ces move over one another scratches are apt to be formed. (19.) It is to some such effect that these vertical marks are ascribed, particularly as no traces of them (or of the 'dots' referred to) are to be found on the surfaces of the unused bars.

(ii) PITTING.

The series of dots or small pits already mentioned are seen most clearly in figs. 32 and 33 where several lines of these can be discerned. They appear to be present in many of the other photographs, but it was difficult to reproduce these marks photographically and there is no direct evidence to show that they were not the effects of corrosion. However the arrangement of these pits in horizontal lines, across the area where the impacts took place, suggest a speculation as to their origin, which however, is not put forward as a really supportable hypothesis.

It has been pointed out by Shaw (20) that when two spherical bodies are pressed together, then near the boundary of the circle of contact where the molecules of the surface layer are very slightly separated, those within a narrow annulus will probably exert an appreciable attraction upon one another. Moreover, it may easily be shown by geometry that the area of this annulus is constant. The area of any piece of metal which might be torn out, however, would be much greater for a narrow annulus than for a circle, for a given area of surface, so that if rupture were to occur it seems reasonable to expect that it would be most likely to happen at the centre of the circle of contact. Further the nature and amount of the damage done would not depend upon the violence of the impact.

The first of these conditions does appear to be fulfilled by the marks in question because a slight change in length of one of the suspension cords would have produced a horizontal translation of the ends of the bars; while the pits are as a rule situated upon a horizontal line. Of course displacements in a vertical direction were not out of the question, but were distinctly less probable.

OTHER PHOTOGRAPHS.

Figs. 39 and 40 show the effect of the impacts alone upon the end of one of the hard-ended bars used in the extended series, which was protected from scratches during adjustment by means of a cap of aluminium foil. Fig. 38 shows the appearance of the ends of radius 1/16" after being subjected to the intense stresses of collision. It will be seen that the damage is confined to the portion of the end actually in contact with the second bar. The remaining photo-graphs illustrate the method of determining the area of contact and indicate the magnitude of the effect and the accuracy possible in its measurement.

CHAPTER IV.

IMPACT.

Theoretical Discussion.

VALIDITY OF THE ELECTRICAL METHOD FOR MEASURING DURATION OF CONTACT.

By duration of contact is meant the interval that elapses after the first physical contact between the ends of the bars before the surfaces again separate from such contact. It has been pointed out (27) that the apparent duration as measured by the electrical method is not necessarily equal to the duration as defined above.

Consider two bodies, moving with a relative velocity v, about to impinge. Moreau considers the impact to be divided into three phases:-

(1) the phase of "penetration", of duration T_1, during which after the first physical contact of the ends an indentation of total depth h is produced (i.e. in the absence of vibrations the centres of gravity of the bodies approach by an amount h) :

(2) the phase of "reaction", of duration T_2, during which the bodies thrust one another apart to end physical contact and produce a relative velocity v':

(3) Motion across a surface layer of air, of thickness λ, both an approach and separation. This layer appears to have a finite resistance so that a certain amount of conduction takes place while the surfaces are not in physical contact; thus the duration of contact, t, as determined electrically is not the duration of contact defined above. Moreau ascribes the existence of this last phase to the action of surface irregularities, which act as points, and in his own experiments found it to be the most important part of the apparent duration of contact. All the impinging surfaces used in the experi-

ments described in the last chapter were very highly polished, so that the importance of this surface layer of air should be minmimised. It is, however, of importance to analyse the results of these experiments, to determine whether or not the actual duration of contact, i.e. $T_1 + T_2$, was measured.

It will be illuminating to follow Moreau's argument with a view to applying its results to the case in question. Consider the "third phase" of an impact. The time taken to traverse the layer of air of thickness λ is, T_3, $= \frac{\lambda}{v_1} + \frac{\lambda}{v}$ where v = velocity of apprach, $v_1 = ev$ = velocity of separation, e = coefficient of restitution.

$$T_3 = \frac{\lambda}{v_1} + \frac{\lambda}{v} = \frac{\lambda}{ev} + \frac{\lambda}{v} \qquad \text{.........................(1)}$$

Now let

$$\alpha = \frac{e}{1+e}$$

then $T_3 = \frac{\lambda}{\alpha v}$

Let

$$\left.\begin{aligned} T &= T_1 + T_2 + T_3 \\ &= T_1 + T_2 + \frac{\lambda}{\alpha v} \end{aligned}\right\} \qquad \text{.........................(2)}$$

Now put $2\mu = \alpha(T_1 + T_2)$

Then $\underline{\alpha v T = 2\mu v + \lambda}$ ....(3)

If, as is almost certainly the case, the layer of air has an appreciable resistance the duration of contact determined electrically will not be equal to $T_1 + T_2 + T_3$ but may be put,

$t = T_1 + T_2 + n T_3$

where n is less than unity.

Now the layer of thickness λ is traversed with sensibly constant velocity: the resistances used in any one series of determinations of t at different velocities are the same, and the state of the ends of the bars is constant throughout so that n may be treated as constant for on pair of bars.

Thus $\alpha v t = 2 \mu v + n \lambda$

The intercept of the curve $\alpha v t$ against v will give the value of $n \lambda$. It is to be noticed that if n is very small the value of t is sensibly $T_1 + T_2$, and it is the value of $n\lambda$ rather than of λ itself which is important so far as the validity of the determinations is concerned.

Moreau found within the limits of experimental error, a linear relationship between $\alpha v t$ and v for small velocities, before permanent deformation had been produced. The curves obtained for the experiments described in the last chapter were in no case linear, but were sufficient to show that the value of $n\lambda$ was small, probably zero. Two of these curves are shown in figures ~~60~~ 51 and ~~61~~ 52, figure ~~60~~ 51 for bars of equal length (63.6 cm.) and figure 52 ~~61~~ for bars of length 63.6 cm. and 17 cm respectively. Only those points for which an experimental value of e was obtainable have been included.

Unfortunately it was not practicable to determine e for very small xxxxxx velocities so that the experimental curve for unequal bars stops rather far from the origin. However it will be seen from fig 51 ~~60~~ that the "third phase" of the impact may be neglected for these experiments.

Consider again the discharge of the condenser through the junction of the bars and various series resistances. This discharge may be considered as taking place in three stages:

(i) discharge through a resistance R + r for a time $\frac{T_3}{2}$

(ii) discharge through a resistance R for a time $T_1 + T_2$,

(iii) discharge through a resistance R + r for a time $\frac{T_3}{2}$;

where r is the effective resistance of the layer of air and R the resistance in the galvanometer circuit.

Let q_0 represent the inital charge of the condenser.
" q_1 " " charge after the first portion of the discharge
" q_2 " " " " " second " " " "
" q_3 " " " " " third " " " "

Then
$$q_1 = q_0 e^{-\frac{\lambda v}{C(R+r)}}$$ where C = capacity of condenser.
$$q_2 = q_1 e^{-\left(\frac{T_1+T_2}{CR}\right)}$$
$$q_3 = q_2 e^{-\frac{\lambda v}{C(R+r)}}$$

so that
$$q_3 = q_0 e^{-\frac{2\lambda v}{C(R+r)}} e^{-\frac{(T_1+T_2)}{CR}}$$

Thus
$$-\log_e \frac{q_3}{q_0} = \frac{T_1+T_2}{CR} + \frac{2\lambda v}{C(R+r)}$$

Or
$$\log_e \frac{q_0}{q_3} = \frac{1}{RC}\left\{(T_1+T_2) + 2\lambda v \cdot \frac{R}{R+r}\right\}$$
$$= \frac{1}{RC}\left\{(T_1+T_2) + 2\lambda v \frac{1}{1+\frac{r}{R}}\right\}$$

Tus the graph, $\log_{10} \frac{\theta_0}{\theta_0 - \theta}$ against $\frac{10^6}{R}$, used for determining t (chap.II) would not have been linear unless either λ was negligible or $\frac{r}{R}$ small. The latter contingency is improbable, particularly with highly polished surfaces; in fact the smallest value of r deduced by Moreau was 4,000 ω. This value would produce a considerable variation of $\frac{1}{1+\frac{r}{R}}$ for the range most frequently covered, namely R = 1000 ω to R = 10,000 ω. A much larger value of r would leave $\frac{1}{1+\frac{r}{R}}$ practically constant so that the time determination curve would not pass through the origin.

For these reasons it is concluded that the value of t given by the electrical method as used in these experiments is the true duration of contact.

MODIFICATION OF THE HERTZ THEORY.

The results of all experiments on impact do not point decisively either to the Hertz theory or to the St. Venant theory as giving a satisfactory explanation of the various experimental facts associated with the impact of bars with rounded ends. The most striking result, however, is the validity of the equation

$$t = Av^{\gamma}$$

for all hard-ended bars. This equation immediately suggests the use of a theory of the type developed by Hertz.

If, however, the impact of the bars is in accordance with such a theory, the length of the bars will be of importance only in so far as it affects the mass. Thus in Wagstaff's experiments (10) on bars of the same end-radius, but of various radii of cross-section, if two pairs of bars had possessed, for the same velocity of approach, the same duration of contact, the masses of the two pairs should have been equal. To test this, the following procedure was adopted. A definite velocity of approach and a duration of impact (within the experimental range) were chosen and the length of that bar which for the chosen velocity of approach had this duration of contact was found for each radius of cross section.

The velocity chosen was one common to all the experiments, and the values of t for several lengths were known, so that interpolation of the correct value of ℓ was not difficult. If the masses of all these hypothetical bars had been the same the value of $\pi a^2 \ell$ would have been constant, i.e. the relationship between a and $\frac{1}{\sqrt{\ell}}$ would have been linear, the line passing through the origin, a being the radius of cross-section of the bar. For thick bars moving with low velocities this statement was approximately justified though for the highest velocities the relationship was no longer linear. These facts are brought out more clearly in figs. 53 and 54

The St-Venant theory would take no account of the variation of the radius of cross-section but would indicate that the lines should be parallel to the a - axis. It might be expected that this condition would be approached as $\frac{\ell}{a}$ increased, i.e. as the origin was approached. An examination of the curves shows that the linear relationship does break down here, the slope decreasing as the origin is approached.

Thus although the evidence appears strongly in favour of a theory of the Hertz type, yet some influence of longitudinal vibrations seems to make itself felt. This influence will be studied in greater detail later; for the present some modifications of the Hertz theory will be considered and other possibilities investigated.

For the impact of equal spheres Lord Rayleigh has shown

that the ratio of the energy in the form of vibrations to the total kinetic energy before impact should be $\frac{1}{50} \frac{v}{V}$ where v is the relative velocity of the spheres before impact and V the velocity of sound in the material of the spheres. This suggests that less energy than might have been expected is converted into vibration. For bars of equal length the period of vibration of the gravest mode of vibration is $\frac{4}{V}$. The importance of these oscillations and the likelihood of their existence during impact decrease as ℓ decreases, so that the limiting value of γ as ℓ approaches zero may reasonably be expected to be characteristic of the impact of small bodies with spherical surfaces.

Pöschl (28) has shown that for plastic spheres a value of $\gamma = -1$ is predicted, by the inclusion of an extra term in the equation of motion as used in the development of the Hertz formula.

According to the Hertz theory,

$$P = k_2 \alpha^{3/2} \qquad \text{..............(4)}$$

where P represents total pressure between the bodies, k_2 is constant and α represents the total indentation at any instant (such that if x = distance between a point in the first body and a point in the second, at the beginning of the impact, then $x - \alpha$ is the distance between the points at the instant in question).

Instead of equating this pressure to the rate of change of momentum only, Poschl introduces a term expressing a force having the nature of a viscous resistance, which always opposes the relative motion of the bodies, thus

$$\frac{d^2\alpha}{dt^2} \pm \frac{\lambda}{2}\left(\frac{d\alpha}{dt}\right)^2 = -k_1 k_2 \alpha^{3/2}$$

where λ is a constant.

Now this second term represents the effect of side-tracking a portion of the energy, and produces a change in the value of γ. Accordingly a consdierable time was devoted to attempts to account for the variation in the values of γ by using equations of this type, including extra terms which might possibly represent the effects of longitudinal vibrations. None of these efforts was successful in accounting for the facts, and apart from the difficulty of justifying the inclusion of any extra term, in general such extra terms appeared to necessitate a loss of energy during the impact.

It was distinctly possible that for the large stresses set up during impact, which could not easily be investigated by ordinary static means as their continued existence might permanently deform the material, the relationship between P and α, deduced by Hertz, might no longer hold exactly. Accordingly it was considered that P might be proportional to α^{β} where β is a constant not equal to 1.5.

$$P = k_2 \alpha^{\beta} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Substituting for P the rate of change of momentum,

$$\frac{d^2\alpha}{dt^2} = -k_1 k_2 \alpha^{\beta}$$

where $k_1 = \frac{m_1+m_2}{m_1 m_2}$, $m_1 + m_2$ being the masses of the bars and

t is measured from the beginning of the impact.

This equation, on integration, becomes

$$\frac{1}{2}\left(\alpha'^2 - v^2\right) = -\frac{k_1 k_2 \alpha^{\beta+1}}{\beta+1} \qquad \ldots\ldots\ldots\ldots(6)$$

where v = relative velocity of the bodies immediately before contact.

$$\frac{d\alpha}{dt} = \sqrt{v^2 - \frac{2k_1k_2\alpha^{\beta+1}}{\beta+1}}$$

Let α_1 = maximum value of α

Then when $\alpha = \alpha_1$, $\dot{\alpha} = 0$

and $v^2 = \frac{2k_1k_2}{\beta+1}\alpha_1^{\beta+1}$ ....(7)

so that

$$t = 2\int_0^{\alpha_1} \frac{d\alpha}{\left\{1 - \frac{2k_1k_2\alpha^{\beta+1}}{(\beta+1)v^2}\right\}^{\frac{1}{2}}}$$

substitute x for $\left\{\frac{2k_1k_2}{(\beta+1)v^2}\right\}^{\frac{1}{\beta+1}}\cdot\alpha$. i.e. $\frac{\alpha}{\alpha_1}$

then $t = \frac{2}{v}\left\{\frac{(\beta+1)v^2}{2k_1k_2}\right\}^{\frac{1}{1+\beta}}\int_0^1 \frac{dx}{\sqrt{1-x^{\beta+1}}}$ ....(8)

Putting $\gamma = \frac{1-\beta}{1+\beta}$ this gives

$$\underline{t = 2\sqrt{\pi}\,\frac{\Gamma\frac{1+\gamma}{2}}{\Gamma\frac{2+\gamma}{2}}(1+\gamma)^{\frac{1-\gamma}{2}}v^{\gamma}\left\{\frac{1}{k_1k_2}\right\}^{\frac{1+\gamma}{2}}}$$ ....(9)

k_2 can be reduced to the form

$$k_2 = \frac{\sqrt{2}}{3}r^{\frac{1}{2}}V^2\rho\frac{(1-2\sigma)}{(1-\sigma)^2}$$

where the impinging surfaces are spherical, of the same radius of curvature, r, and of the same material.

V = the velocity of sound in the material,
ρ = the density of the material
σ = Poisson's ratio for the material.

substituting for k_1 and considering the case of equal cylindrical rods of radius ·7935 cm. ($\frac{5}{16}$") for a material where

$\sigma = \frac{1}{2}$, $V = 5.0 \times 10^{+5}$ cm/sec. and $\rho = 7.85$ gm/cc the following result is obtained:

$\frac{1}{k_1k_2} = .945 \times 10^{-11}\,\ell\, r^{-\frac{1}{2}}$ where ℓ = length of either bar.

Thus the formula for the duration of contact becomes

(over)

$$t = 2\sqrt{\pi}\frac{\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(\frac{2+\gamma}{2}\right)}(1+\gamma)^{\frac{1-\gamma}{2}} v^{\gamma} \ell^{\frac{1+\gamma}{2}} r^{-\frac{1+\gamma}{4}} (.945\times10^{-11})^{\frac{1+\gamma}{2}} \quad \ldots\ldots\ldots(10)$$

An attempt was made to use this result to calculate the duration of contact, using the values of γ given by experiment. For this purpose the factor ψ was calculated for various values of γ where ψ was equal to

$$2\sqrt{\pi}\frac{\Gamma\left(\frac{1+\gamma}{2}\right)}{\Gamma\left(\frac{2+\gamma}{2}\right)}(1+\gamma)^{\frac{1-\gamma}{2}}(.945\times10^{-11})^{\frac{1+\gamma}{2}}$$

so that

$$t = \psi v^{\gamma} \ell^{\frac{1+\gamma}{2}} r^{-\frac{1+\gamma}{4}} \quad \ldots\ldots\ldots\ldots\ldots(11)$$

TABLE XVI.

$-\frac{1}{\gamma}$	4.0	4.5	5.0	5.5	6.0	6.5	7.0
$\psi\times10^4$	4.73	3.32_6	2.492	1.986	1.617	1.387	1.202

As will be seen from the above table the value of ψ was found to vary rapidly with γ so that, owing to the change of γ with the length of the bars, the variation of t with ℓ was much too great. Moreover the absolute value of t calculated using either the experimental value of γ or the limiting value of γ as $\ell \to 0$ was much greater than the experimental value (See Table XX).

A further discussion on the application of this formula is given later.

RELATIONSHIP BETWEEN β AND γ

If a formula of the above type is representative of the nature of impact (postponing for the moment consideration of the discrepancies in the absolute values of t) then two possibilities present themselves. The variation of γ might be due to a corresponding variation of the index β which would thus be a function of l; or β might be a true constant, the values of γ being influenced in some way by the production of longitudinal oscillations. If β is independent of the circumstances of the impacts it seems that it should be related with some value of γ also independent of these circumstances. The limiting value of γ as l approaches zero appears to be independent of the radius of the ball end so that β is calculated from this.

Two lines of investigation present themselves here, (1) a direct investigation of the relationship between P and α, by statistical methods, and (2) an investigation of the relationship between α_1 and v (formula (7)) from which β might be calculated.

(1) Measurements of the deformation produced by pressing a steel sphere against a steel block were made by Lafay (29) who measured α_1 directly by an interferometer method. Log P was plotted against log α_1 giving the values of β expressed in the accompanying table

TABLE XVII.

R = radius of steel sphere in cm.

R	0.5	1.0	2.0	3.0	6.0	15.0	25.0
β	1.640	1.644	1.636	1.684	1.628	1.972	2.072

The maximum value of the radius of the ball end for any of the bars used for the experiments described in Chap. III was 5.08 cm. In the above table it will be seen that the values of β obtained from Lafay's results are sensibly constant over this range. Neglecting the values for R = 15 cm and R = 25 cm. the mean value of β is 1.646, corresponding to a value of $\gamma = -\frac{1}{4.10}$ Although β varies only slowly with γ , there is definite evidence that $\beta > 1{\cdot}60$ at least, which corresponds with $-\frac{1}{\gamma} < 4.33$

Lafay's results thus provide definite evidence that the law $P \propto \alpha^{3/2}$ requires slight modification even for static conditions, and support the view that the limiting value of γ does indicate a divergence from the law of static compression formulated by Hertz.

(2) (a) If the equation $P = k_2 \alpha^{\beta}$ represents the relationship between pressure and displacement during impact, then observations on the variation with velocity of the diameter, D, of the circle of contact afford an independent means of calculating β (Equation (7)). If the value of β so calculated is constant and equal to that deduced from the limiting value of γ , valuable evidence is available to support the hypothesis that the variation in γ is due to the effect of longitudinal oscillations

(b) On the other hand, if the values of β calculated from observations on the area of contact are not independent of ℓ but are identical with the values deduced from the separate experimental determinations of γ , it seems that the above equation may hold, but that the change in γ depends directly upon a corresponding change in the pressure - displacement law.

(c) If the values of β calculated as before appear to be unrelated with the corresponding values of γ then the variation of γ with ℓ may be explained by the introduction of an additional term in the pressure-displacement equation, leading to a form similar to that considered by Pöschl .

In place of calculating β the values of γ were calculated from the observations on D, assuming the validity of the equation $P = k_2\alpha^{\beta}$

Unfortunately the diameter D did not admit of such accurate measurement as t and the results of the experiments, while interesting, were not conclusive.

(7) may be written

$$\alpha_1 = \left\{\frac{v^2}{(1+\gamma)k_1k_2}\right\}^{\frac{1+\gamma}{2}} = v^{1+\gamma}\left\{\frac{1}{k_1k_2}\right\}^{\frac{1+\gamma}{2}}(1+\gamma)^{-\left(\frac{1+\gamma}{2}\right)} \quad \ldots\ldots(12)$$

Thus for any one value of γ , ie. for any one pair of bars α_1 should vary as $v^{1+\gamma}$ But $\alpha_1 \propto D^2$ so that

$$D^2 \propto v^{1+\gamma}, \quad D \propto v^{\frac{1+\gamma}{2}}$$

The values of this index ($\eta = \frac{1+\gamma}{2}$) are given in table XIII, from which it willbe seen that no regular variation of $\frac{1+\gamma}{2}$ with l is evident, though there is an indication that the value of this index is greater than 0.400, the mean value being 0.413, corresponding to $\gamma = -\frac{1}{5.7}$. Little reliance can be placed on this, however, as the Log. D - log x curves were not well-defined, also because the change in the index $\frac{1+\gamma}{2}$ was small, for the observed change in γ . It was realised that little could be expected from a further investigation of the area of contact on these lines.

RELATIONSHIP BETWEEN t and D.

Using equations (9) and (12) the connection between t and α_1 may be reduced to the form

$$\alpha_1 = tv.\frac{1}{2\sqrt{\pi}(1+\gamma)}\frac{\Gamma\frac{2+\gamma}{2}}{\Gamma\frac{1+\gamma}{2}} \quad \ldots\ldots(13)$$

Also $D^2 = 4r\alpha_1$ where r = radius of ball end.

so that $$D^2 = r\,t\,v\frac{2}{\sqrt{\pi}(1+\gamma)}\frac{\Gamma\frac{2+\gamma}{2}}{\Gamma\frac{1+\gamma}{2}} \quad \ldots\ldots(13A)$$

Substituting the value $\frac{2\pi x}{T}$ for v and taking $\gamma = -\frac{1}{4.48}$

(the reason for which willbe explained later) the following relationship is obtained, for comparison with experimental results,

$$\frac{1}{r}\;\frac{D^2}{tx} = 1.51$$

It may be noted that the value of $\frac{\bar{D}^2}{rtx}$ varies too slowly with γ (see equation (13A)) to provide a useful criterion of the advisability of using the limiting, on any other reasonable value of γ. Comparison with table XIV will show that the value of $\frac{D^2}{tx}$ for the short bars is exactly that predicted whereas for longer bars there is a distinct falling off. This is ascribed to the influence of the longitudinal vibrations, because the generation of these involves the withdrawal of a certain amount of energy from the strain energy located in the ends of the bars, with a corresponding reduction in the area of contact.

Table XV gives somewhat less reliable values of this constant, but, again, it will be seen that experiment and theory are in reasonable agreement.

INFLUENCE OF LONGITUDINAL VIBRATIONS.

Consider the collision of two bars of unequal mass in the light of a theory of the Hertz type. The function of these masses is to determine the relationship between the pressure over the surfaces in contact and the resulting acceleration, so that the value of t should depend upon the quantity $\frac{2\,m_1\,m_2}{m_1 + m_2}$ where m_1 and m_2 are the masses of the bodies. On the other hand the St. Venant theory indicates that the duration of contact should depend directly upon the lengths of the bars; that for perfectly flat-ended bars t should depend upon the length of the longer bar only.

Owing to the influence of the rounded ends in increasing the duration of contact the effect of longitudinal vibrations might well depend upon the length of the shorter, rather than of the longer bar as the first extensional wave to reach the junction might initiate the separation; in any case l_1, l_2, or $\frac{l_1+l_2}{2}$ are more likely to have been the quantities affecting t than $\frac{2l_1l_2}{l_1+l_2}$; where l_1 and l_2 are the lengths of the bars.

Accordingly the curves connecting t and ~~the length of the variable bar (l_2)~~ $2l_1l_2/(l_1+l_2)$ have been plotted for both sets of observations on bars of unequal lengths, (l_1 = 63.6 cm. and l_1 = 35.6 cm. and for bars of the same length (Fig. 55). The values of the coefficient of restitution have shown that some influence due to these vibrations is felt and these curves help to indicate the nature and order of the effect. It will be seen that for l_1 = 63.6 cm. the effective length almost throughout is greater than $2l_1l_2/(l_1+l_2)$; while for l_1 = 35.6 cm. the effective length is greater than $2l_1l_2/(l_1+l_2)$ for the larger values of l_2.

Thus the effect of longitudinal vibrations on the duration of contact depends not on the length of the shorter bar, but either on the length of the longer bar, or on $(l_1+l_2)/2$, both of which are greater than $2l_1l_2/(l_1+l_2)$ except where $l_1 = l_2$.

The value of t, however, for l_1 = 63.6 cm., l_2 = 8.5 cm. x = 8 cm., is less than $2\,l_1/V$ so that it appears probable that the mean length, $(l_1+l_2)/2$ is the important factor. Furthermore, the relatively close agreement between the curves for equal and for unequal bars for small values of l_2 supports this hypothesis.

TABLE ~~XXI.~~ XVIII

All lengths expressed in cm.

l_1	l_2	$\frac{2l_1l_2}{l_1+l_2}$ = A	$\frac{l_1+l_2}{2}$ = B	B - A.	l_1	l_2	$\frac{2l_1l_2}{l_1+l_2}$ = A	$\frac{l_1+l_2}{2}$ = B	B - A
63.6	100	77.8	81.8	4.0	35.6	100	52.6	67.8	15.2
	80	70.9	71.8	0.9		80	49.3	57.8	8.5
	63.6	63.6	63.6	0.0		63.6	45.6	49.6	4.0
	49.6	55.6	56.6	1.0		49.6	41.5	42.6	1.1
	35.6	45.6	49.6	4.0		35.6	35.6	35.6	0.0
	17	26.8	40.3	13.5		17.0	23.0	26.3	3.3
	8.5	15.0	36.1	21.1		8.5	13.7	22.1	9.4

Values of $\frac{D^2}{tx}$.

Assuming as before that $P = k_2\alpha^\beta$ and that $\beta = \frac{1-\gamma}{1+\gamma}$ and theoretical value of $\frac{D^2}{tx}$ can be derived, the expression for which is given by

$$\alpha_1 = tv.\frac{1}{2\sqrt{\pi}}\cdot\frac{\Gamma\frac{2+\gamma}{2}}{\Gamma\frac{1+\gamma}{2}}\;\frac{1}{1+\gamma}$$

where $D^2 = 4\,r\alpha$, and $v = \frac{2\pi x}{T}$, T being the time of swing of the impinging bar (taken to be 2.89 sec.). It will be seen that $\frac{D^2}{tx}$ depends upon γ, although the variation with γ actually found was small. For $\gamma = -\frac{1}{4}$, $\frac{D^2}{tx} = 1.93$ and for $\gamma = -\frac{1}{7}$, $\frac{D^2}{tx} = 1.86$, so that only a small variation in $\frac{D^2}{tx}$

would be expected apart from the influence of longitudinal oscillations. Moreover for bars of the same length $-\frac{1}{\gamma}$ was found to increase with ℓ so that the variation in the experimental value of $\frac{D^2}{tx}$ was opposite in sign to that due to a variation in β, if β could actually be expressed as $\frac{1-\gamma}{1+\gamma}$. It is suggested that the value of β should be considered constant, being derived from the limiting value of γ as $\ell \to 0$, and that the variation in the experimental value of $\frac{D^2}{tx}$ is due to the existence of energy in the form of longitudinal oscillations during the impact. This is borne out by the results of the experiments on unequal bars (Table ~~XXII~~ XIX).

TABLE XXII.

l_2 (cm)	Values of $\frac{D^2}{tx}$ for, x = 2 cm.	x = 4 cm.	x = 6cm	x = 8cm.	MEAN $\frac{D^2}{tx.}$
	l_1 = 63.6 cm.				
100	1.44	1.52	1.30	1.15	1.35
80	1.72	1.46	1.32	1.33	1.46
63.6	1.80	1.66	1.49	1.43	1.60
49.6	1.95	1.80	1.69	1.52	1.74
35.6	2.27	1.79	1.59	1.64	1.82
17.0	No Observations.				
8.5.	2.14	2.07	1.97	2.03	2.04
	l_1 = 35.6				
100	1.51	1.54	1.27	1.33	1.44
80	1.71	1.84	1.64	1.53	1.68
63.6	2,13	2.01	1.85	1.68	1.92
49.6	2.19	1.85	1.83	1.80	1.92
35.6	2.20	1.89	1.82	1.95	1.97
17.0	2.57	1.98	1.97	1.94	2.12
8.5	2.22	2.32	2.00	1.71	2.06

Besides the definite decrease in $\frac{D^2}{tx}$ as l_2 increases, it will be seen that there is some tendency for a decrease in this ratio as x increases, indicating that at higher velocities a relatively smaller amount of energy is spent in producing deformations of the ends of the bars. These changes are both to be expected if longitudinal oscillations were provoked during the impact, as the energy of these ~~oscillations~~

oscillations would naturally increase with the violence of the impact and with the length of either bar.

This tendency for $\frac{D^2}{tx}$ to change with the velocity of approach was not observed in the case of equal bars and it is suggested that it may be associated with the energy actually lost during the collisions, which depends on $1 - e^2$ It will be remembered that e decreased as the velocity increased so that at higher velocities one might expect a smaller proportion of the energy to be spent in producing deformations of the rounded ends

EFFECT OF LENGTH------ EQUAL BARS.

Although the Log t - log x curves for equal bars were straight it was realised that if the correct expression for t had been of the form $t = A v^{\gamma} + C$, where C represents a small constant dependent on the length; if C were small only a slight curvature of the lines on the ~~log~~ log t - log x diagrams would probably have been produced. Thus it seemed possible that the observed value of γ might not have been exactly correct. To test these points the values of x^{γ} were calculated for each observed value of γ in the first group of experiments and t was plotted against x^{γ} for each end-radius. Had the relationship between t and v been of the type suggested above, these lines would not have passed through the origin. Actually all the lines plotted did pass through the origin so that it was concluded that $t = Av^{\gamma}$ was a true expression of the experimental results.

Further, t was plotted against $x^{-\frac{1}{5}}$ and against $x^{-\frac{1}{4.3}}$ to see if any constant intercept could be obtained for either of these values of γ. (It was thought at the time of these investigations tha $-\frac{1}{4.3}$ was a good approximation to the limiting value of γ) In neither case was a constant intercept on either the t axis or the x^{γ} axis obtained; in fact the lines did not all pass on the same side of the origin. The curvatures of all the sets of lines were too small to serve as a criteria of the validity of the various assumptions.

This method of approach was tried before the results of the extended series had shown that the log. t - log x relationship was linear over a much wider range. No positive results were obtained and the method was abandoned.

ABSOLUTE VALUES OF t.

In Table XX are set forth the results of calculations of the duration of contact for $x = 4$ cm. both from the St. Venant theory and from the modified Hertz theory, using $\gamma = -\frac{1}{4.48}$. This is slightly greater than the mean of the limiting values for $\ell \to 0$, but is used because the value $-\frac{1}{4.48}$ was found most satisfactory for later developments. The values of t for $\gamma = -\frac{1}{4.39}$, the true limiting value, and for $\gamma = -\frac{1}{5}$ are somewhat higher and somewhat lower respectively than the values given in the table, but in neither case are they in at all reasonable agreement with experiment.

t (Obs.) = observed duration of contact.

t_V = duration of contact calculated from the St.Venant theory

t_H = duration of contact calculated from the modified Hertz theory.

TABLE XX.

$\ell = 63.6$ cm. $x = 4$ cm.

r (cm.)	$t_H \times 10^4$ (sec.)	t(Obs.)$\times 10^4$ (Sec.)	$t_V \times 10^4$ (sec.)
5.08	7.66	3.88	2.54
2.54	8.77	3.99	"
1.27	10.05	4.38	"
.794	10.96	4.55	"
.635	11.48	4.70	"
.318	13.15	5.84	"
.159	15.00	5.65	"

$\ell = 17.0$ cm. $x = 4$ cm.

r (cm.)	$t_H \times 10^4$ (sec.)	t(Obs.)$\times 10^4$ (sec.)	$t_V \times 10^4$ (sec)
5.08	4.59	1.87	0.68
2.54	5.25	1.91	"
1.27	6.01	2.21	"
.794	6.56	2.32	"
.635	6.86	2.53	"
.318	7.87	2.88	"
.159	8.97	3.07	"

It will be seen that the experimental value is in each case intermediate between the two theoretical values, but that neither theoretical value is at all satisfactory. Thus the effect of longitudinal vibrations is not to add to but to detract from the duration of contact calculated from the modified Hertz theory, so that although this theory appears to give a more satisfactory account of the circumstances of impact than the St. Venant theory, no simple type of correction for the effect of length can be applied.

These results for the absolute value of t, together with the complexity of all the relationships except that between t and v, and the dissimilarities between the impact of hard-ended and of soft-ended bars led to an attempt to solve the problem on quite new lines. The bars were treated as a compound vibrating system, in which the longitudinal waves of the St. Venant theory were considered to constitute one type of fundamental oscillation while the compression and relaxation of the ends constituted another. The development of these ideas is given below.

VIBRATIONS OF A COMPLEX SYSTEM.

Consider two bars resting with the poles of their rounded ends in contact and let a sudden disturbance be applied uniformly over one of the ends remote from the point of contact. Longitudinal waves will be produced which in the absence of the second bar would simply distribute the effect of the disturbance throughout the mass of the first bar. In the presence of the second bar some of the effect of the pulse will be transmitted across the surface of contact and longitudinal waves will be generated in the second bar.

Consider now the arrival of the longitudinal pulse or wave-front at the ball end of the first bar. The portion of the disturbance in zone near to the point of contact will be transmitted to the second bar with relativly little retardation, as a slight pressure between the surfaces will produce sufficiently close contact to allow a pulse of compression to pass. The surface of the rounded end remote from the area of contact, however, is practically free from restraint, so that the disturbance in zones remote from the centre of the bar will be reflected back into the first bar, in which it will travel to and fro, some portion of the energy going to the second bar every time that the pulse arrives at the junction. The result of these multiple reflections will be to distribute the effect of the disturbance throughout the volume of the first bar so that a portion of the impressed energy will appear simply as kinetic energy, which will be transmitted to the second bar by a relatively slow indentation of the end.

The transmission of energy across the junction is thus associated with two periodic times; the time taken for a longitudinal wave to traverse the system, i.e. $\frac{2l}{V}$ (or $\frac{l_1+l_2}{V}$ if the lengths are different) and the time of compression and recovery of the ball end. In addition to this, energy is reflected at the junction as well as transmitted so that the system behaves in many respects as a compound vibrating system. A consideration of the behaviour of a pair of bars in which the ball-ends are replaced by thin cylinders, coaxial with the bars, will further reveal the nature of the system.

To return to the St-Venant theory, consider the behaviour of a bar of length 2ℓ. The period of longitudinal oscillations of such a bar will be $\frac{4\ell}{V}$. If now, this bar be cut at the centre and the two halves allowed to impinge, according to the St-Venant theory the duration of contact will be $\frac{2\ell}{V}$, which is half the period of free oscillations of the system. Thus the duration of contact for a system with a periodic time of fundamental oscillations, t, is $\frac{t}{2}$ or $\frac{1}{2n}$ where n is the natural frequency of the system. It is suggested that a similar case is presented by two ball-ended bars in contact, but that the period of free oscillations is not $\frac{4\ell}{V}$ but the fundamental time of oscillation of a compound system consisting of two portions, one with a fundamental period of $\frac{4\ell}{V}$ and the other with a period of $2\psi v^{\gamma}(\ell)^{\frac{1+\gamma}{2}} r^{-\frac{1+\gamma}{4}}$ It is necessary to imagine that a wave of extension can pass over the junction, provoking elastic reactions opposite in sign to those due to a wave of compression, to conceive of the actual maintenance of the oscillations of the compound system.

This suggested explanation would account for the complexity of the experimental relationships; for instance an alteration in the length of a given pair of bars would not only alter the period of fundamental oscillation of each portion of the complex system; it would also affect the relative importance of the two types of vibration, so that the relationship between the period of the compound bar and the length would be most complicated. Similarly in two sets of bars of the same length but of different end-radii the period of oscillation associated with the ball end would be different in the two cases, besides which the "weight" attached to this type of oscillation would be different for the different bars, and would vary with the velocity of approach.

It is possible to deduce the periodic time of compound systems by direct integration of the equations of motion for particles in the system, together with the application of the "end-conditions" and the conditions holding at the junction ([21]). For this particular system however the conditions were so complicated that it was decided to use the method of "Acoustical Impedance" which is accordingly described in some detail.

FREE VIBRATIONS AND ACOUSTICAL IMPEDANCE.

If an alternating electromotive force of variable frequency be applied to a system comprising inductance, resistance and capacity, then if the resistance is not too great pronounced resonance phenomena will be observed. For a certain frequency of the applied e.m.f. the impedance of the system will be a minimum, if the inductance and capacity are in series, and it is known that this frequency is that of free oscillations in the circuit. If the resistance of the circuit is negligible it can be said that the frequency of the applied e.m.f coincides with the natural frequency of the circuit when the impedance is zero. Further , the expression for the impedance involves the periodic time of the impressed e.m.f. so that if an expression for the impedance can be obtained the free periods of the circuit are easily calculable. A quantity analogous to the impedance may be calculated for acoustical systems and provides a simple method of determining the natural periods of complex systems. (22, 23, 24).

The electrical impedance ~~of~~ (Z) of a circuit is defined by the ratio $\frac{V_0}{I_0}$ where V_0 and I_0 are the electromotive force and the current respectively at any instant. In the corresponding acoustical system the pressure (P) at a point corresponds with the e.m.f., V_0 and $\frac{dx}{dt}$ with the current I_0, where X is the volume of air entering the system. In the original work on this subject (25) an oscillating system is considered into which a volume X of air enters periodically under an excess pressure P. The acoustical impedance is defined as $\frac{P}{X}$, which, it will be realised, is $\frac{\pi}{2}$ out of phase with the quantity $(P/\frac{dx}{dt})$ corresponding to the electrical definition.

To take the most general case, consider a vibrating ~~position~~ piston of area S, in the orifice of an acoustical system, which carries with it into the system a volume $S\xi$ of air at any instant. Then the force acting on the piston will be

$$SP = \frac{md^2\xi}{dt^2} + K\frac{d\xi}{dt} + f\xi \quad \text{.......................(15)}$$

where m represents the mass of the piston and of the air carried with it,

K is a factor expressing the action of damping forces and

f is the 'stiffness' or restoring force per unit displacement.

The corresponding electrical equation is

$$E = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} \quad \text{- - - - - - - - - - (16)}$$

where L = inductance of circuit

R = resitstance of circuit.

C = capacity ~~of~~ in circuit

E = impressed electromotive force.

Let Z_1 be the acoustical impedance of the system

Then when the vibrations of the piston are steady we may put

$\xi = \xi_0 e^{ipt}$ ~~when~~ so that $\frac{d\xi}{dt} = ip\xi_0 e^{ipt}$

and $\frac{d^2\xi}{dt^2} = -p^2 \xi_0 e^{ipt}$

where ξ_0 represents the amplitude of ξ

$i = \sqrt{-1}$

$p = 2\pi n$ and n = the frequency of the oscillations.

Now $X = S\xi$ and $Z_1 = \frac{P}{X}$

so that $\frac{P}{S\xi_0 e^{ipt}} = Z_1 = \frac{-mp^2 + ipK + f}{S^2}$

$\therefore Z_1 S^2 = f - mp^2 + iKp$

The real part of this expression may be described as the "Uncompensated stiffness" of the system, and corresponds approximately with the reactace of an electrical circuit. Z_1 will be a minimum when this is zero, i.e. when $f = mp^2$, and the frequency of the oscillations will be the natural frequency of the circuit.

Thus

$$p^2 = \frac{f}{m}$$

For the corresponding electrical circuit the impedance, Z, is given by $Z = \sqrt{\left(\frac{1}{Cp} - Lp\right)^2 + R^2}$ and resonance is attained when $\frac{1}{Cp} = Lp$, i.e. $p^2 = \frac{1}{LC}$.

An inspection of equations (15) and (16) will show that f corresponds with $\frac{1}{c}$ and m with L so that the definition of acoustical impedance given above leads to the same results as the more correct or more usual definition.

APPLICATION TO THE FREE PERIODS OF BARS & OF COLUMNS OF GAS (26)

For this purpose it will be convenient to adopt the definition $Z = \frac{P}{dx/dt}$ for the acoustical impedance, in strict correspondence with the electrical definition. Consider now the impedance (Z_s) offered to a disturbance impressed on the gas in a tube of cross-section S.

$Z_s = \frac{P}{dx/dt} = \frac{P}{S_v}$ where v is the particle velocity at the point in question. Thus the impedance per unit area is given by $Z = \frac{P}{v}$.

It is to be noted that the impedance decreases as the area increases, in fact that the impedances of adjacent units of area can be considered to be in parallel. The impedance varies according to the position of the point considered just as the electrical impedance of a complex circuit depends upon the points of application of the electromotive force.

Consider a bar or column of gas of uniform cross section and denote one end, at which the periodic disturbance is impressed, by B and the other end by A: also suppose that reflection at A is imperfect so that the ratio of the particle velocity at a point near A in the wave before reflection to the corresponding velocity after reflection is g. Let the source at B be a piston moving with velocity $V = V_0 e^{i\omega t}$ and suppose that the absoption of the wave along the tube or bar is negligible except at A.

Also consider the surface of the piston to be perfectly rigid so that the reflection factor at B is +1.

The retardation suffered by a wave travelling a distance, l along the column will be e^{-2ikl} where $k = \frac{2\pi}{\lambda}$ and λ the wave-length in the material, of the impressed oscillation. Thus the whole retardation suffered by a disturbance which starts at the piston B, travels to the end A, and returns to B, will be e^{-2ikx} where x is the length of the column, defined as the distance between surfac at which reflection takes place without change of phase. The amplitude of the returning disturbance, however, will be reduced to g times its original value, so that the velocity of a particle at B due to the returning wave alone is represented completely by a $gV_0e^{i(\omega t - 2kx)}$ The pressure excess at a point where the particle velocity is $V_0e^{i\omega t}$ is $\rho c V_0 e^{i\omega t}$ where ρ = the density of the medium and C = the velocity of sound in the medium.

Consider now the pressure at a point very close to the piston B. In a state of steady oscillation the pressure due to all of the returning waves must be added to the pressure due direly to the motion of the piston, i.e. to $g\rho c V_0 e^{i\omega t}$. The particl velocity at B due to the wave returning after one reflection at A is $gV_0 e^{i(\omega t - 2kx)}$ so that the pressure due to this wave is $g\rho c V_0 e^{i(\omega t - 2kx)}$ This wave is now reflected at B, which behaves a a closed end, so that no phase change is produced by the reflection and, as a point very close to B is under discussion, the pressure due to the returning wave after one reflection at A and one reflectio at B is $g\rho c V_0 e^{i(\omega t - 2kx)}$ This pressure must again be added to the pressure due directly to the motion of the piston. It will be noted that for the reflected wave the effect of a closed end is to double the pressure excess, which becomes $2g\rho c V_0 e^{i(\omega t-2kx)}$ in its immediate neighbourhood. (This result was noted by Hopkinson (1) for the reflection of extensional waves in a wire fastened to a heavy block)

The new reflected wave now travels repeatedly to and from A as before so that the whole pressure is given by the sum of a series of terms of the form of those considered above, the pressure due to the wave after a second reflection at A and before its second reflection at B being $g^2 \rho c V_0 e^{i(\omega t-4kx)}$

The total pressure at B, P_B, is given by

$$P_B = \rho c V_0 e^{i\omega t}\left\{1 + 2ge^{-2ikx} + 2g^2e^{-4ikx} + \text{--- to inf.}\right\}$$

$$= \rho c V_0 e^{i\omega t}\left\{\frac{1+ge^{-2ikx}}{1-ge^{-2ikx}}\right\}$$

In the case of the steel bars reflection of the disturbance can only take place at an open end, while the perfect resilience observed in the impact of equal bars shows that the reflection at these ends must be practically perfect. Thus if the substitution $g = -1$ be made the length x becomes the true length of the bar, so that

$$P_B = \rho c V_0 e^{i\omega t}\left\{\frac{1-e^{-2ikx}}{1+e^{-2ikx}}\right\}$$

$$= \rho c V_0 e^{i\omega t}\left\{-i \tan kx\right\}$$

so that $\frac{P_B}{V_B} = -i\rho c \tan kx$

$$\sigma_B Z_B = -i\rho c \tan kx \quad \text{.....................(17)}$$

where $Z_B =$ impedance of the column at the end B

$\sigma_B =$ area of cross section of the column.

It will be seen that the impedance is of the nature of a reactance, the substitution of -1 for g corresponding to the assumption of zero resistance in the electrical analogy.

COMPOSITE BARS.

It is now required to find the composite bar equivalent in characteristics, so far as longitudinal impulses are concerned, to a pair of bars with their rounded ends in contact. As has already been pointed out the transmission of a disturbance from one of the bars to the other is associated with two distinct periodic times, so that the first and second portions of the composite bar must separately be capable of longitudinal oscillations of periods

In the impact of a pair of bars the initial pressure is zero, it rises to a maximum positive value and falls to zero again, this process corresponding to half a complete oscillation of the appropriate component of the composite bar. Thus the fundamental period of either component would appear to be twice the duration of ~~the~~ contact calculated from the theory applicable to that component.

For the first part of the calculation it is only necessary to assume that the hypothetical composite bar consists of two portions of different fundamental free periods. Now a disturbance applied to the end of one of the pair of real bars must affect immediately both modes of behaviour. In other words a vibration impressed on one of the free ends of the system is impressed in the same way on both of the hypothetical constituents of the compound bar. Furthermore the unaffected end of the system is free from restraint so that the remote ends of the constituents of the compound bar are subject to the same conditions. Thus the impedances representative of the two portions must be in parallel.

It is considered, then, that the free periods of the system consisting of the actual bars used, with their rounded ends in contact, are the same as those of a system consisting of two bars of the same material (which are capable separately of different times of oscillation) attached to one another at their ends but separate at other points. This latter condition is equivalent to a statement of the physical independence of waves. An attempt will be made to determine the fundamental free period of this composite system by studying the effect of applying an alternating pressure to one of the junctions of the ends of the hypothetical bars, the remainder of the system being free from restraint.

It should be remembered that the transmission of energy from one real bar to the other by the slow indentation of the ends involves a retardation not dissimilar to that which might be produced in traversing a long bar; so that in the calculation of the free periods the representation of the action of the rounded ends by the inclusion of an extra hypothetical bar may not be without justification.

Let the impedance of the first component of the composite bar = Z_1

" " " " " second " " " " " = Z_2

" " " " " WHOLE BAR = Z

The $\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$

For resonance $Z = 0$

$$\therefore \frac{1}{Z_1} = -\frac{1}{Z_2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

The impedances per unit area of the first bar and the second bar respectively are given by (equation (17))

$$\sigma_1 Z_1 = -i\rho c \tan kx_1$$
$$\sigma_2 Z_2 = -i\rho c \tan kx_2$$

where x_1 and x_2 represent the effective lengths of the first and second hypothetical bars respectively.

and σ_1 and σ_2 ~~upset~~ represent the effective area of cross section of the same bars.

The bars are considered to be of the same material so that ρ and c are the same in both equations.

$$Z_1 = \frac{-i\rho c}{\sigma_1} \tan kx_1, \qquad Z_2 = \frac{-i\rho c}{\sigma_2} \tan kx_2$$

From equation (18), for resonance,

$$\frac{1}{\sigma_1} \tan kx_1 = -\frac{1}{\sigma_2} \tan kx_2 \qquad \text{..................(19)}$$

Meaning of kx.

$k = \frac{2\pi}{\lambda}$ where λ is the wavelength of the impressed oscillation, which is now of the same frequency as the free oscillations of the system.

Let the frequency of the free oscillations of the system be n.

The $c =$ ~~n~~ $n\lambda$

$$\therefore \quad k = \frac{2\pi}{c} \cdot n$$

Consider now any one of the components of the hypothetical system. Its length, x, is such that its fundamental free period is identical with one of the times, t_1, associated with the collision of the real bars. Now the junction of the hypothetical bars remote from the source of vibrations is an "open end" and the piston constituting the source of forced oscillation is a "closed end"

so that

$$x_1 = \frac{\lambda_1}{4} -$$

where λ_1 is the wavelength in the material of a vibration of periodic time t_1.

Let the frequency of this natural vibration be n_1

Then $x_1 = \frac{c}{4\ n_1}$

So that $kx_1 = \frac{2\pi.n.c}{c\ \ 4n_1} = \frac{\pi n}{2\ n_1}$

Similarly $kx_2 = \frac{\pi n}{2n_2}$ where n_2 represents the frequency of oscillation corresponding to the second periodic time associated with the collision.

Equation (19) may now be written

$$\frac{1}{\sigma_1} \tan \frac{\pi n}{2n_1} = -\frac{1}{\sigma_2} \tan \frac{\pi n}{2n_2} \quad \text{..............(20)}$$

σ_1 and σ_2

Consider the behaviour of the hypothetical bar representing the effect of the ball ends, with respect to successive impacts of increasing violence. The area of cross-section of this bar must be assumed to increase with the velocity of approach of the real bars in such a way that the relation between the pressure during the impact and the strain in the bar is as close an approach as possible to the relationship between P and α for the ball ends, where P now represents the pressure between the bars.

For a bar behaving according to the St. Venant theory $P \propto v$ but for a bar behaving according to the modified Hertz theory $P \propto v^{1.223}$ (see equations (5) and (7) pp 110, 111)

But for the hypothetical bar, P $\propto$ compression $\times \sigma_1$ and as the same end-conditions are applied to both components of the system the compression must be the same for both bars . Thus the ~~xxxpxxix~~ compression $\times \sigma_1$, will be proportional to $v^{1.223}$, but the compression

is directly proportional to the velocity so that $\sigma_1 \propto v^{.223}$
But the frequency $n_1 \propto v^{-\gamma} \propto {}^{.223}$

Thus $\sigma_1 \propto n_1$.

Further, if $n_1 = n_2$ it seems reasonable to assume that the energy will be equally divided between the two modes of vibration, so that as the lengths of the hypothetical components and the compressions will be equal, the areas must also be equal. $\sigma_1 = \sigma_2$. Now the second hypothetical bar will be representative of the St-Venant effect if its area of cross-section remains constant, when representing one pair of bars. But n_2 will also be constant so that $\sigma_2 \propto n_2$.

Thus $$\frac{\sigma_1}{\sigma_2} = \frac{n_1}{n_2}$$

and the final form of equation (20) becomes

$$\frac{1}{n_1} \tan \frac{\pi n}{2n_1} = -\frac{1}{n_2} \tan \frac{\pi n}{2n_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

This equation gives n when n_1 and n_2 are known; and will be referred to as the "conditional" equation.

SOLUTION OF THE CONDITIONAL EQUATION.

The equation may be written

$$-\frac{\tan(\pi n/2n_1)}{\tan(\pi n/2n_2)} = \frac{n_1}{n_2}$$

Let $\theta_1 = \frac{\pi n}{2n_1}$ and $\theta_2 = \frac{\pi n}{2n_2}$

Then $\frac{n_1}{n_2} = \frac{\theta_2}{\theta_1}$ so that the equation becomes

$$-\frac{\operatorname{Tan} \theta_1}{\operatorname{Tan} \theta_2} = \frac{\theta_2}{\theta_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

or, taking logarithms

$$\text{Log.}(-\tan\theta_1) - \log.\tan\theta_2 = \log\theta_2 - \log\theta_1 \quad \ldots(23)$$

Now $\frac{n_1}{n_2}$ is essentially positive, so that the least values of θ_1 and θ_2 satisfying the equations must be such that θ_2 lies between 0 and $\frac{\pi}{2}$ and θ_1 between $\frac{\pi}{2}$ and π. It is to be noted that the smallest values of θ_1 and θ_2 correspond to the lowest value of n i.e. to the fundamental oscillation. The first harmonic is represented by values of θ_2 between $\frac{\pi}{2}$ and π, and θ_1 between π and $\frac{3\pi}{2}$, and so on for the higher harmonics.

To solve equation (23) a graph was constructed with values of log θ as abscissae, between 0 and π (Fig. 56). For values of θ between 0 and $\frac{\pi}{2}$ the graph log θ against log tan θ was constructed, θ being θ_2 of the conditional equation. For values of θ between $\frac{\pi}{2}$ and π the ordinates were the values of log (- tan θ) so that for this range θ represented θ_1 of the equation. Finally a third curve was constructed such that the vertical distance of each point from the lefthand curve was numerically equal to the horizontal distance of the same point from the right hand curve.

Consider any point, Q, on this, the centre curve. Then the horizontal distance from Q to the right hand curve gives $\log\theta_1 - \log\theta_2$ while the vertical distance to the lefthand curve gives $\log.\tan\theta_2 - \log(-\tan\theta_1)$ so that points on the centre curve give the values of θ_2 satisfying the conditional equation for various values of the ratio. The values of $\frac{\theta_2}{\theta_1}$ are marked on the curve at various points. The corresponding curve for θ_1 could have been constructed on the right of the diagram, but to determine the value of n a knowledge of θ_2 and n_2 is sufficient. All the values used were checked by direct calculation.

For small values of $\frac{\theta_2}{\theta_1}$ (less than 0.1) the curve was too inaccurate, so that direct calculation was necessary. Accordingly the conditional equation was written

$$\tan \frac{\pi n}{2n_1} = - \frac{n_1}{n_2} \tan \frac{\pi n}{2 n_2}$$

and for a given value of $\frac{n_1}{n_2}$ a table was constructed showing the values of n, $\frac{\pi n}{2n_1}$, $\tan \frac{\pi n}{2n_1}$, $\frac{\pi n}{2n_2}$, $\tan \frac{\pi n}{2n_2}$

and finally the ratio $(- \tan \frac{\pi n}{2n_1}) / (\tan \frac{\pi n}{2n_2})$. Values of n were tried until the ratio mentioned was sufficiently nearly equal to $- \frac{n_1}{n_2}$ (cf. 21.).

n_1 and n_2

θ_1 is greater than θ_2 so that n_2 is greater than n_1. As the duration of contact according to the St.-Venant theory is usually less than that according to the Hertz theory, n_2 was taken to represent the frequency of longitudinal oscillation of the hypothetical bar representing the St.-Venant solution. Thus $n_2 = \frac{V}{4l}$ where l is the length of the actual bar.

In the experiments on equal bars only five values of l were used, so that the five values of n_2 were taken and for each of these n was calculated for values of n_1 covering the whole range of the experiments.

The calculation of the free periods of a composite bar is based on the behaviour of systems capable of executing simple harmonic motion. But the relationship between the displacement (α) and the time during the impact, for bars impinging according to the modified Hertz theory, is not identical with that in a system executing simple harmonic motion of the same amplitude. To investigate this point the relationship between the indentation, α and the time t during the impact was calculated from the modified Hertz theory (Fig. 57) $\frac{\alpha}{\alpha_1}$ has been plotted against $\frac{tv}{\alpha_1}$ because this curve applies to all cases.) If the duration of contact calculated from this theory is t_1, it is considered that the periodic time of the appropriate component of the hypothetical bar is not $2t_1$ but $2t_1^1$ where $2t_1^1$ is the periodic time of a simple sine oscillation of amplitude α_1 in which the relationship between t and α most nearly resembles that calculated from the theory.

The influence of longitudinal waves probably prevents the realisation of the upper portion of the curve and the periodic time is most probably influenced only by the shape of the portion of the curve representing the beginning and end of the impact. It will be seen that if $t_1^1 = 0.790t_1$ the sine curve practically coincides with the calculated curve over a considerable range. It is therefore suggested that so far as the oscillations of the complex system are concerned, bars behaving according to the theory of the Hertz type may be replaced by bars in which the connection between the displacement and time during the impact follows a simple sine relationship, but for which the duration of contact is $0.790t_1$

Then, $n_1 = \frac{1}{2t_1^1}$ (see p. 127)

(In the calculations reproduced; t_1^1 has been taken to be $0.766t_1$, following an earlier estimate of the value of the constant. The accuracy of the results is probably not very great so that complete revision is considered unnecessary. It may be noted that a slight modification of the limiting value of γ adopted for the calculation could compensate for a change in the value of $\frac{t_1^1}{t_1}$).

In the actual impact the pressure rises from zero to a maximum and falls to zero again, after which the bars separate. This corresponds to half an oscillation of the complex system, so that the duration of contact, t is equal to $\frac{1}{2n}$.

Using the conditional equation, values of n for various values of n_1 were calculated for each length of bar, and from the above considerations the corresponding values of Log ($t_1 \times 10^4$) and log ($t \times 10^4$) were calculated. The five curves so obtained are shown in fig. 58.

TABLE XXI.

Summary of results of calculations.

Note — θ_2 is measured in degrees.

$n_1 \times 10^{-3}$	Log. $\frac{\theta_1}{\theta_2}$	Log. θ_2	$n \times 10^{-3}$	Log.($t_1 \times 10^4$)	Log.($t \times 10^4$)
Length of bars 63.6 cm.			$n_2 = 1.96_5 \times 10^3$		
.3	.8163	1.424	.579	1.338	.936
.4	.6913	1.542	.760	1.213	.818
.5	.5944	1.628	.927	1.116	.732
.6	.5152	1.691	1.072	1.037	.669

TABLE XXI. (continued)

n_1 $\times 10^{-3}$	Log. $\frac{\theta_1}{\theta_2}$	Log. θ_2	n $\times 10^{-3}$	Log. $(t_1 \times 10^4)$	Log. $(t \times 10^4)$
.7	.4483	1.738	1.194	.970	.622
.8	.3903	1.778	1.309	.912	.582
.2	.9424	1.203	.348	1.514	1.157
1.0	.2934	1.823	1.452	.815	.537
	Length of bars 49.6 cm.			$n_2 = 2.52_o \times 10^3$	
.35	.8576	1.387	.688	1.271	.861
.40	.7996	1.444	.779	1.213	.808
.45	.7485	1.489	.869	1.162	.760
.55	.6613	1.573	1.048	1.074	.679
.65	.5888	1.632	1.202	1.002	.613
.75	.5266	1.682	1.344	.940	.571
.85	.4723	1.724	1.471	.885	.531
.95.	.4240	1.755	1.594	.837	.497
1.00	.4017	1.768	1.641	.815	.484
1.15	.3410	1.799	1.764	.754	.453
	Length of bars 35.6 cm.		$n_2 = 3.51_o \times 10^3$		
.40	.9453	1.302	.789	1.213	.803
.50	.8464	1.395	.979	1.116	.708
.60	.7672	1.470	1.163	1.037	.633
.70	.7003	1.533	1.340	0.970	.572
.80	.6423	1.586	1.503	0.912	.522
.90	.5912	1.630	1.663	0.861	.478
1.0	.5454	1.666	1.815	.815	.440
1.2	.4662	1.726	2.076	.736	.382

TABLE XXI. continued.

$n_1 \times 10^{-3}$	Log $\frac{\theta_1}{\theta_2}$	Log θ_2	$n \times 10^{-3}$	Log$(t_1 \times 10^4)$	Log $(t \times 10^4)$
Length of Bars 17.0 cm. $n_2 = 7.35_4 \times 10^3$					
0.6	1.0883	1.160	1.18	1.037	.627
0.8	.9634	1.282	1.56	.912	.505
1.0	.8665	1.375	1.94	.815	.412
1.2	.7873	1.451	2.31	.736	.336
1.4	.7204	1.515	2.67	.669	.272
1.6	.6624	1.568	3.02	.611	.219

RESULTS BY DIRECT CALCULATION.

$n_1 \times 10^{-3}$	$\frac{\theta_1}{\theta_2}$	θ_1 (degrees)	tan θ_1	tan θ_2	θ_2 (degrees)	$n \times 10^{-3}$	Log. $(t_1 \times 10^4)$	Log. $(t \times 10^4)$
Length = 8.5 cm. $n_2 = 14.7_1 \times 10^3$								
0.7	21.02	179.6	-.0075	.1584	9°	1.399	.970	.554
1.0	14.71	179.2	-.0147	.2159	12.2°	1.992	.815	.400
1.3	11.31	178.6	-.0248	.2811	15.7°	2.580	.701	.287
1.6	9.20	177.8	-.0375	.3443	19	3.163	.611	.199
1.8	8.18	177.2	-.0494	.4040	22	3.548	.559	.149
2.0	7.36	176.5	-.0605	.4452	24	3.922	.5137	.106
2.2	6.69	176.4	-.0620	.4150	26	4.32	.472	.064

COMPARISON WITH EXPERIMENTAL RESULTS.

For each length of bar there is only one curve, according to the theory, so that on the log($t_1 \times 10^4$), log ($t \times 10^4$) diagram all the experimental results from $r = \frac{1}{16}$" to $r = 2$" should lie on the same curve. This result, itself sufficiently remarkable, was actually obtained (figures 59 to 63) but before discussing these matters the method by which the points were plotted is of interest.

EXPRESSION FOR t_1.

t_1 is given by the expression $\psi v^{\gamma} \ell^{\frac{1+\gamma}{2}} r^{-\frac{1+\gamma}{4}}$ where ψ is itself a function of γ, independent of v, ℓ and r. Using a constant value of throughout, the calculations were simplified by the use of a graphical method of representing the equation,

$$\text{Log } t_1 = \text{Log}\,\psi - \left(\tfrac{1+\gamma}{4}\right) \log r + \gamma \log v + \left(\tfrac{1+\gamma}{2}\right) \log \ell$$

The limiting value of γ derived from the $-\frac{1}{\gamma}$, ℓ curves was $-\frac{1}{4.39}$, but by calculating the value of log t_1, for numerous collisions and by comparison with the theoretical curves it was found that closer agreement was obtained for $\gamma = -\frac{1}{4.48}$. The experimental ~~er~~ error in the determinations of γ was sufficiently great to justify the use of this latter value; so that the expression for t_1 becomes

$$\text{Log } (t_1 \times 10^4) = 0.3737 - .1942 \log r + .3884 \log \ell - .2232 \log x \quad ..(24$$

The values of the first two terms of the right hand side of this equation was calculated for each radius of the ball end and the result marked off on the Log($t_1 \times 10^4$) axis of the diagram. Through these points vertical lines were drawn from which the values of the remaining terms ($.3884 \log \ell - .2232 \log x$) could be measured to give Log ($t_1 \times 10^4$) in every case. For this purpose an arbitrary mark was made on a strip of celluloid and labelled $x = 1$. From this mark distances were measured, using the scale of the Log $(t_1 \times 10^4)$ axis, to represent the values of $.3884 \log \ell$; in the same direction the values of .2232 lox x were marked off and labelled. Thus the distance between two of the lines on the strip gave the value of the last two terms of the equation. As the values of the first two terms were already marked on the diagram, measurements from these lines with the celluloid strip immediately gave the values of $\log(t_1 \times 10^4)$.

In the diagrams the results of the first group of experiments are shown. It was not practicable to distinguish the different points according to the value of r, but it was noticed that there was no apparent tendency to the production of separate curves for the various bars, or more correctly there appeared to be no ordered arrangement of the points with respect to the circumstances of impact, except that implicit in the calculation of $(t_1 \times 10^4)$. Owing to the somewhat arbitrary nature of the assumptions made in the calculation, exact agreement between theory and experiment was not expected.

THE VALUES OF γ

From the method used for constructing the curves (fig.58)

$$t_1 = C_1 v^{-\frac{1}{4.48}}$$

and if a short length of the curve is considered

$$t = C_2 t_1^{\delta}$$, nearly, where C_1, C_2 and δ are constants.

But, experimentally $t = Av^{\gamma}$ so that

$$-\frac{1}{\gamma} = \frac{4.48}{\delta}$$

Thus the slopes of the theoretical curves determine the values of γ. It may be noted that a series of determinations of t for one pair of bars respresents only a small length of the curve, so that a fairly pronounced curvature of the lòg ($t_1 \times 10^4$), Log ($t \times 10^4$) curve would be necessary to affect the Log. t, log x diagram. Below $\ell = 63.6$ cm, the theoretical lines are not seriously curved, but for $\ell = 63.6$ cm. the curvature is fairly large: it is suggested that this will explain the observed variation of $-\frac{1}{\gamma}$ with r for bars of this length.

The accompanying table shows the comparison between the observed and calculated values of γ. For this, the slope of the theoretical curve for the value of t_1 corresponding to x = 4 cm was taken; being at about the centre of the experimental range.

TABLE XXV.

Values of $-\frac{1}{\gamma}$

ℓ (cm.)	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.	Calc.	Obs.
	r = 2"		r = 1"		r = ½"		r = 5/16"		r = ¼"		r = 1/8"	
[illegible].6	8.75	7.51	6.63	7.33	6.16	6.87	5.79	6.30	5.63	5.88	5.17	5.68
[illegible].6	6.59	6.22	6.22	6.72	5.81	5.91	5.48	6.24	5.34	5.56	5.10	5.99
[illegible].6	6.05	5.89	5.45	5.97	5.14	5.64	4.87	5.77	4.82	6.18	4.82	5.68
[illegible]7	4.73	5.06	4.72	5.10	4.60	5.12	4.60	4.99	4.60	4.72	4.60	4.99
[illegible].5	4.53	4.38	4.53	5.28	4.53	4.90	4.53	4.65	4.53		4.53	4.67

It will be seen that in a general way the calculated and observed values of $-\frac{1}{\gamma}$ are in agreement, although the calculated value is, throughout, somewhat smaller than the observed value. This is of interest in that the use of $-\frac{1}{\gamma} = 4.48$ instead of $-\frac{1}{\gamma} = 4.39$ might have been expected to give theoretical values larger than the experimental values. Further, the calculated values of $-\frac{1}{\gamma}$ show a sufficiently linear relationship with l, although, again as in the experimental curve, the point for $l = 63.6$ is usually somewhat off the line. The mean value of the interceptof the theoretical curves on the axis $l = 0$ is about $-\frac{1}{4.3}$

UNEQUAL BARS.

It has already been pointed out that from the point of view of the Hertz theory the length of the bars is only important in so far as it affects the mass. Accordingly in calculating the value of log $(t_1 \times 10^4)$ the formula $t_1 = \psi v^{\gamma} l^{\frac{1+\gamma}{2}} r^{-\frac{1+\gamma}{4}}$ was still used, but for l was substituted $2\frac{l_1 l_2}{l_1 + l_2}$. The second component of the complex system must have had a length equal to that of the two bars placed end to end, i.e. $l_1 + l_2$, so that the effective length to be considered so far as longitudinal oscillations were concerned was $\frac{l_1 + l_2}{2}$.

To have completed the calculations for unequal bars in the same manner as for bars of the same length would have necessitated the construction of a further series of curves connecting Log.$(t \times 10^4)$ and log $(t_1 \times 10^4)$ for a new set of lengths. The labour involved in these calculations was considerable, so that it was considered sufficient to determine Log $(t \times 10^4)$ for a number of lengths for each value of log $(t_1 \times 10^4)$ and from these results to deduce by interpolation the value of log $(t \times 10^4)$ for $l = \frac{l_1 + l_2}{2}$.

This was done for three values of the displacement, x, in all cases except for $l_1 = 63.6$ cm, $l_2 = 100$ cm, which would have necessitated extrapolation to a length of 81.8 cm. It will be noticed that no allowance has been made for the loss of energy in the impact of these unequal bars. As less momentum is transferred from one bar to the other than would have been the case with two bars each of length $\frac{\ell_1 + \ell_2}{2}$ it would be expected that the experimental duration of contact would be less than the corresponding calculated value.

TABLE XXVI.

ℓ_1 = 63.6 cm.

	x = 1 cm.		x = 4 cm.		x = 8 cm.	
ℓ_2 (cm)	$t \times 10^4$ Calculated	$t \times 10^4$ Observed	$t \times 10^4$ Calculated	$t \times 10^4$ Observed	$t \times 10^4$ Calculated	$t \times 10^4$ Observed
80	5.82	5.64	4.55	4.58	4.16	4.17
63.6	5.61	5.15	4.38	4.16	3.95	3.77
49.6	5.27	5.09	4.08	3.94	3.68	3.58
35.6	4.90	4.42	3.78	3.64	3.40	3.33
17.0	3.95	3.60	3.05	2.95	2.72	2.68
8.5	3.26	2.94	2.61	2.22	2.29	1.95
		1 = 35.6 cm.				
100	5.32	4.81	4.23	4.15	3.86	3.83
80	5.08	4.66	3.97	3.90	3.60	3.57
63.6	4.90	4.38	3.78	3.52	3.40	3.24
49.6	4.63	4.20	3.54	3.31	3.14	2.99
35.6	4.35	3.76	3.25	3.01	2.86	2.69
17.0	3.66	3.21	2.77	2.46	2.42	2.19
8.5	2.98	2.66	2.24	2.02	1.98	1.78

The agreement between the observed and the calculated results is best for the high velocities, but it will be noted that the discrepancy for any one velocity is fairly constant so that the shape of the experimental t - ℓ curves is reproduced by the theoretical results. Further this agreement is maintained down to ℓ_2 = 8.5 cm. where the duration of contact is in between the time taken by a wave to travel up and down the longer bar and that taken to travel up and down the shorter bar. The circumstances of impact will be complicated by the fact that a wave which has started from the junction and which has been reflected at the end of the shorter bar and returned to the junction will not correspond in time with a similar wave in the other bar, so that it is probable that a series of small longitudinal disturbances will be transmitted across the junction into the longer bar.

Some light is thrown upon the processes at work by the results of the experiments on the coefficient of restitution.

THE COEFFICIENT OF RESTITUTION.

The kinetic energy lost during impact is given by the expression $\frac{1}{2}\frac{M\,m}{M + m}(1 - e^2)\,v^2$ where M, m are the masses of the bodies. The greatest possible value of this expression is $\frac{1}{2}mv^2$, where m is the mass of the smaller bar, and this loss is equal to the kinetic energy of the smaller bar moving with a velocity v. Thus the expression $\frac{M}{M + m}(1 - e^2)$, or $\frac{\ell_1}{\ell_1 + \ell_2}(1 - e^2)$, gives the ratio of the energy actually lost to that which possibly could be lost. It is this quantity which is referred to when the "proportion of energy lost" during an impact is mentioned.

The behaviour of the bars with respect to the effect of longitudinal oscillations has been referred to the vibrations of a hypothetical bar of length $\ell_1 + \ell_2$. In the actual case the point to which energy is applied, and from which energy is withdrawn at the end of the collision, is distant ℓ_1 from one end. By analogy with the case of a string whose oscillations at one point, not the centre, are suddenly stopped it will be seen that it is unlikely that all the energy of the vibrations will be converted into kinetic energy when the bars separate; while the experiments on equal bars show that the energy in that case is so converted. It appears probable that the enrgy in the length $\ell_1 - \ell_2$ where ℓ_1 is the length of the longer bar, will not be converted into kinetic energy. The same result would be obtained from a pair of perfectly flat-ended bars impinging exactly in accordance with the St. Venant theory.

The proportion of the energy lost will, for a given pair of bars, vary with $\frac{\ell_1 - \ell_2}{\ell_1 + \ell_2}$, although its absolute value cannot be calculated because the distribution of stress along the bars is unknown. If this distribution of stress does not vary seriously with the velocity, and there seems to be no reason why it should, then the fraction of the energy which is lost during the collision, E_F is proportional to $\ell_1 - \ell_2$. Another effect enters into the question, however, because in a long impact a considerable proportion of the energy in the longitudinal waves will return to the junction where some will be reflected, some transmitted to the other bar; and some, probably the greatest part, will be reconverted into local stresses over the area of contact.

Writing $E = \phi \frac{\ell_1 - \ell_2}{\ell_1 + \ell_2}$, ϕ will indicate the proportion of energy present as longitudinal oscillations. If each time a longitudinal disturbance returns to the junction a certain proportion of the energy is converted into local stresses and strains, then ϕ should be proportional to ε^{-kt} where k is a constant and ε the base of natural logarithms. To investigate this, log ϕ was plotted against $t \times 10^4$. The relationship actually was linear in each of the three cases where sufficient data were available, so that $\phi = A^1 e^{-kt}$ where A^1 is a constant independent of t, and x, the displacement.

TABLE XXVII.

$\ell_1 = 63.6$ cm. $\ell_2 = 17$ cm.

x (cm.)	E_F	Log ϕ
2	.099	-.766
4	.157	-.566
6	.189	-.485
8	.230	-.400
10.	.255	-.355

$A^1 = 14.7$, $k = 1.37 \times 10^4$

$\phi = 14.7\varepsilon^{-1.37(t \times 10^4)}$

$\frac{k}{\ell_1 + \ell_2} = 1.7 \times 10^2$

$l_1 = 63.6$ cm. $l_2 = 8.5$ cm.

x (cm)	E_F	Log. ϕ
2	.222	-.538
4	.299	-.408
6	.341	-.351
8	.349	-.341
10	.382	-.302

$A^1 = 1.95 \quad k = .742 \times 10^4$

$$\phi = 1.95\ \varepsilon^{-.742(t \times 10^4)}$$

$$\frac{k}{l_1 + l_2} = 1.3 \times 10^2$$

$l_1 = 35.6$ cm. $l_2 = 8.5$ cm.

x (cm)	E_F	Log. ϕ
2	.106	-.764
4	.127	-.685
6	.131	-.672
8	.144	-.631
10	.153	-.604

$A^1 = 0.66 \quad k = .576 \times 10^4$

$$\phi = 0.66\ \varepsilon^{-.576(t \times 10^4)}$$

$$\frac{k}{l_1 + l_2} = 1.5 \times 10^2$$

The factor ε^{-kt} will include the influence of velocity of approach upon the energy converted into longitudinal vibrations.

ϕ will not be numerically equal to the proportion of the energy converted into longitudinal oscillations, but the value of A^1 should indicate the influence of the dimensions of the bars on the energy in that form, as it is independent of v and t. It will be seen that A^1 increases with the length, $l_1 + l_2$, of the whole system; so that the longer the bars, in general, the greater will be the influence of longitudinal oscillations.

The fact that k is directly proportional to $l_1 + l_2$ seems to show that the influence of the velocity of approach on the production of longitudinal oscillations is greater for the longer bars, a circumstance which, in the case of equal bars, may account for the discrepancy between the observed and the calculated values of $-\frac{1}{\gamma}$.

CONCLUSION.

In conclusion I am glad to take this opportunity to express my very sincere gratitude to Professor J.E.P.Wagstaff, under whose direction the work was undertaken, for his continued help and encouragement during the investigation. I should also like to thank the instrument-maker, Mr. A.E.Beecroft, who prepared most of the bars used for the experiments.

BIBLIOGRAPHY.

(1) B.Hopkinson. "Scientific Papers" "On the Effect of Momentary Stress in Metals".

(2) St.-Venant. Journal de Math. (Liouville), Ser. 2, t.12, (1867)

(3(Love. /"Mathematical Theory of Elasticity". p.437

(4) (Hertz. Journ. für Math. (crelle.) Bd.92. (1881)
(Love. Loc. Cit. p.193.

(5) Hamburger, Wied. Ann. 28. 1886. p.653.

(6) Lord Rayleigh. (Phil.Mag. 1906. Vol. XI. p.283
(Collected Papers. Vol.5. p.292.

(7) B. Hopkinson. "Scientific Papers". "On the Pressure of a Blow".

(8) Tait. Edinburgh Roy. Soc. Trans. Vols. 36 and 37. 1890-1892

(9) Sears. Camb. Phil. Soc. Proc. Vol 14, 1908. p.257

(10) (J.E.P.Wagstaff Proc. Roy. Soc. 105. pp. 544-570. 1924.
(Ibid. Phil.Mag. XLVIII. p.147. 1924

(11) Voigt. Ann. Phys. Chem. (Wiedemann) Bd.19. 1883.

(12) Sears. Trans. Camb. Phil. Soc. Vol.21. 1912, p.49.

(13) W. Thomson. Phil.Mag. V. 1853. p.393.

(14) "Radio Instruments and Measurements" Bureau of Standards, Washington, p.265.

(15) Wagstaff. Proc. Roy. Soc. Ser. A. Vol.105. 731, 1924, p.282

(16) (Andrews. Phil.Mag. ix. p.593., (1930)
(Ibid. Phil.Mag. viii. p.781 (1929)

(17) Deodhar. Phil.Mag. xlviii p.89 (1924)

(18) Tomlinson. Phil.Mag. X. p.541. (1930)

(19) Tomlinson. Phil.Mag. VI. p.695 (1928)

(20) Shaw. Phil.Mag. IX. p.628 (1930)

(21) Lees. Proc. ~~Roy~~ Phys. Soc. April, 1929, 41, P204.

(22) Irons. Phil.Mag. March 1930. p.346

(23) Ibid. Phil.Mag. VII (1929) p.873

(24) Stewart. Phys. Rev. XX. (1922) p.528

(25) Webster. Proc. Nat. Acad. Sci. (America) V. (1919)

(26) West. "Post Office Electrical Engineers" Journal " Jan. 1929

(27) Moreau. Journal de Physique et le Radium Tii. p.329 (1921)

(28) Pöschl Zeit. für Physik. 46. 1.- 2 p.142. (1927)

(29) Lafay. Comptes Rendus. 131. p.525. (1900)