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RADIAL JETS.

by

Z. H. Chaudhury.

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ABSTRACT.

This thesis is in two parts. In the first chapter we consider the problem of the laminar mixing of two different gases in a wall jet. Solutions of the boundary layer equations governing the radial laminar flow of a mixture of two different gases forming a wall jet are obtained. Attention is centered on that part of the flow where the concentration of one gas in the mixture is small. Further, to investigate what effect the diffusion of one gas into the other has on the velocity profile, the dominant perturbation term in the stream function is obtained by expanding the latter in terms of a parameter whose magnitude depends upon the concentration of the gas from the reservoir in the mixture.

The second problem, discussed in Chapter two, is that of heat transfer in a radial liquid jet. The technique used to study this problem is similar to that employed by Watson in his investigations into the velocity distribution in a radial liquid jet. Two specific examples are treated in detail and the results obtained, for the temperature distribution in the jet and heat transfer across the wall, are presented graphically.

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each Chapter.

GENERAL INTRODUCTION.

This thesis is in two parts. In the first chapter we consider the problem of the laminar mixing of two different gases in a wall jet. Solutions of the boundary layer equations governing the radial laminar flow of a mixture of two different gases forming a wall jet are obtained. Attention is centered on that part of the flow where the concentration of one gas in the mixture is small. Further, to investigate what effect the diffusion of one gas into the other has on the velocity profile, the dominant perturbation term in the stream function is obtained by expanding the latter in terms of a parameter whose magnitude depends upon the concentration of the gas from the reservoir in the mixture.

The second problem, discussed in chapter two, is that of heat transfer in a radial liquid jet. The technique used to study this problem is similar to that employed by Watson in his investigations into the velocity distribution in a radial liquid jet. Two specific examples are treated in detail and the results obtained, for the temperature distribution in the jet and heat transfer across the wall, are presented graphically.

LAMINAR WALL JET MIXING OF TWO DIFFERENT GASES.

1. INTRODUCTION.

This chapter is concerned with the laminar mixing of two different gases in forming what has been described by Glauert as a "wall jet", namely, a jet of gas striking a wall at right angles and spreading out radially over it in the atmosphere of a second gas at rest. Glauert (1) first drew attention to such jets of incompressible fluid spreading out radially and mixing with a similar fluid at rest, he set up the appropriate momentum and continuity equations and obtained exact solutions for the radial and plane flows in the form of similarity solutions. Later Riley (2) extended the concept and studied the influence of compressibility on a laminar radial wall jet.

In the present problem, besides the usual variables such as the velocity components, density, viscosity, pressure and temperature, a further variable, namely, the concentration of one gas in the other must be introduced since the fluid in the jet is taken to be different from that in the surrounding stream. Thus, to the fundamental equations governing fluid motion, one has to add a new equation, namely, the equation of diffusion. Chou (3) first introduced this additional equation into the study of

laminar jet mixing of two incompressible fluids. In
addition to the boundary layer assumptions and that of
incompressibility of the fluid, certain other assumptions
are made about the constituent gases that enable us to
present the diffusion equation in a convenient form. These
are: (i) that the constituent gases are perfect gases,
(ii) that there is no chemical reaction between them, and
(iii) that their mixture is a continuous medium. Thus
they obey Dalton's law of gas mixtures which now states
that the volume of a mixture of two or more perfect gases
is equal to the sum of the volumes which they would
separately occupy when subjected to the same pressure.
For our purposes, it is therefore sufficient to use the
relations for the viscosity and other properties of gases
and gas mixtures given by simple kinetic theory.

In this chapter the appropriate equations of motion
for flow in a radial wall jet are formulated and reduced
to those for plane flow by a transformation embodying the
Howarth and Mangler transformations. We then consider a
solution at a large distance from the jet axis where the
molecular concentration of the gas from the reservoir is
small, so that, as a first approximation we take the density
of the mixture to be equal to that of the surrounding gas
there. This further simplifies the equations of motion.

In fact, the momentum equation then becomes identical with that in Glauert's paper and his solution of the momentum equation is appropriate and is made use of in solving the diffusion equation after choosing a suitable form for the molecular concentration.

In order to determine the effect of diffusion on the velocity profile, a method similar to that employed by Crane and Pack (4) for two-dimensional free jet mixing of two different gases is used. By this method the stream function is expanded in terms of a parameter which depends on the concentration of the gas from which the jet is initially formed in the mixture. Only the first term in the perturbation to the stream function is considered in detail.

Throughout this chapter we assume that the two gases are incompressible, that, as in (1) and (2), the boundary layer equations are appropriate for describing wall-jet flow and that the pressure and temperature are everywhere constant.

The type of flow envisaged in this chapter may be realised in practice on the roof of a mine shaft when a stream of methane is released from the floor or walls of the shaft.

2. EQUATIONS OF MOTION.

On boundary layer approximations the equations of
momentum and continuity for a laminar radial wall jet
together with the equation of state, for constant pressure
and temperature, are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \qquad (2.1)$$

$$\frac{\partial}{\partial x}\left(\rho u x\right) + \frac{\partial}{\partial y}\left(\rho v x\right) = 0, \qquad (2.2)$$

and
$$\rho R = \text{Constant}, \qquad (2.3)$$

where x, y are the distances measured along the wall from
the jet axis and normal to it respectively; u, v are the
corresponding velocity components; ρ, μ the density and
the viscosity respectively and R the gas constant of the
mixture.

We shall refer to the gas issuing from the reservoir
as gas "1" and that occupying the surrounding space as gas
"2". The quantities μ and ρ for the two gases will be
denoted by appropriate suffixes.

If now in the mixture the mass concentration of gas
1 be denoted by c_1 , we have for the equation of diffusion
of gas 1 into gas 2

$$\frac{\partial}{\partial x}\left(c_1 u x\right) + \frac{\partial}{\partial y}\left(c_1 v x\right) = x D_{12} \frac{\partial^2 c_1}{\partial y^2}. \qquad (2.4)$$

Here D_{12} is the y-wise co-efficient of molecular diffusion and is taken to be constant following Jeans (5).

Let N be the total number of molecules in unit volume of the mixture under given pressure and temperature, N_1, N_2 be defined similarly for the two gases and m_1, m_2 be the masses of single molecules of the constituent gases respectively, then

$$N = N_1 + N_2 \ . \qquad (2.5)$$

The molecular concentration c of gas 1 in the mixture is now defined as

$$c = N_1 / N \quad ,$$

so that, for gas 2 the molecular concentration is given by

$$1 - c = N_2 / N \quad .$$

The density ρ of the mixture is given by

$$
\begin{aligned}
\rho &= N_1 m_1 + N_2 m_2 \\
&= N m_2 (1 + \beta c) \qquad (2.6) \\
&= \rho_2 (1 + \beta c) \ ,
\end{aligned}
$$

where $\beta = m_1 / m_2 - 1$.

According to the kinetic theory of gas mixtures, the co-efficient of viscosity, μ, of a mixture of two gases is given by (see (5))

$$\mu = \frac{\mu_1}{1 + A_{12}\left(\frac{1-c}{c}\right)} + \frac{\mu_2}{1 + A_{21}\left(\frac{c}{1-c}\right)} , \qquad (2.7)$$

where A_{12}, A_{21} are constants associated with the two gases. Since in the region under consideration N_1/N is small, c

is small there. We can, therefore, write (2.7) in the form

$$\mu^* = \frac{\mu_1}{\mu_2} \approx (1 + \delta c) \; ,$$

where

$$\delta = \mu_1 / (A_{12} \, \mu_2) - A_{21} \; .$$

Writing (2.6) as

$$\rho = N m_2 (1 + \beta c) = N m' \; ,$$

the equation of state (2.3) becomes

$$N m' R = N R' = \text{Constant}, \qquad (2.8)$$

where R' is Boltzmann's constant. Equation (2.8) now shows that in a region of constant pressure and temperature

$$N = \text{Constant} \; ,$$

and consequently

$$c \propto c_1 \qquad \qquad . \qquad (2.9)$$

Equations (2.2), (2.6) and (2.9) are now used to simplify the diffusion equation (2.4) which becomes

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = D_{12} (1 + \beta c) \frac{\partial^2 c}{\partial y^2} \; . \qquad (2.10)$$

The equation of continuity (2.2) is satisfied by introducing a stream function ψ defined by

$$\rho u x = \frac{\partial \psi}{\partial y} \quad , \quad \rho v x = - \frac{\partial \psi}{\partial x} \; .$$

Using the Howarth-Mangler transformation

$$\left. \begin{array}{l} z = x \int_0^y \rho^* \, dy \\[2mm] \zeta = \frac{1}{3} x^3 \end{array} \right\} \qquad (2.11)$$

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where $\rho^* = \rho/\rho_2$, the momentum and the diffusion equations (2.1) and (2.10) now become respectively

$$\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial \zeta \partial z} - \frac{\partial \psi}{\partial \zeta}\frac{\partial^2 \psi}{\partial z^2} = \mu_2 \frac{\partial}{\partial z}\left(\mu^* \rho^* \frac{\partial^2 \psi}{\partial z^2}\right), \qquad (2.12)$$

and

$$\frac{\partial \psi}{\partial z}\frac{\partial c}{\partial \zeta} - \frac{\partial \psi}{\partial \zeta}\frac{\partial c}{\partial z} = \mu_2 \sigma \rho^{*2} \frac{\partial}{\partial z}\left(\rho^* \frac{\partial c}{\partial z}\right), \qquad (2.13)$$

where $\sigma = D_{12}\rho_2/\mu_2 = D_{12}/\nu_2$ is the Schmidt number, ν_2 being the kinematic viscosity of gas 2. Equations (2.12) and (2.13) are effectively the two-dimensional form of our equations of motion.

Since there is no diffusion of molecules across the wall, $\frac{\partial c}{\partial y} = 0$ at the wall and for large values of y, $N_1 \to 0$ and so, the molecular concentration $c \to 0$ as $y \to \infty$. The boundary conditions on the velocity and the concentration are, therefore, as follows.

At $\qquad y = 0$, $\qquad u = v = 0$, $\frac{\partial c}{\partial y} = 0$;

and as $\qquad y \to \infty$, $\qquad u \to 0$, $\qquad c \to 0$.

In terms of the new variables they become

at $\qquad z = 0$, $\qquad \psi = 0$, $\frac{\partial \psi}{\partial z} = 0$, $\frac{\partial c}{\partial z} = 0$;

and as $\qquad z \to \infty$, $\qquad \frac{\partial \psi}{\partial z} \to 0$, $\qquad c \to 0$.

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3. SOLUTION OF THE EQUATIONS.

Since the transformed equations (2.12) and (2.13) are now reduced to those for two-dimensional flow and since c is small in the region under consideration, we take as a first approximation to the solution of (2.12) Glauert's solution, where $\overset{*}{\rho}=1$, which is

$$\psi = \left(40 F_\infty \mu_2 \zeta\right)^{\frac{1}{4}} f(\eta) \quad , \quad u = \left(\frac{5 F_\infty}{2 \mu_2 \zeta}\right)^{\frac{1}{2}} f'(\eta) , \\ \eta = \left(\frac{5 F_\infty}{32 \mu_2^3 \zeta^3}\right)^{\frac{1}{4}} z \ , \tag{3.1}$$

where the constant F_∞ is the flux of exterior momentum flux at infinity. It may be noted that this quantity is constant everywhere if $\rho/\mu = \varphi(u)$ (see (1) and (2)). The function $f(\eta)$ satisfies the equation

$$f''' + ff'' + 2 f'^2 = 0 \quad , \\ f(0) = f'(0) = 0 \ , \quad f'(\infty) = 0 \ . \tag{3.2}$$

with $f(\infty)=1$, Glauert obtained the solution of equation (3.2) in the following form

$$f = g^2 \quad , \qquad g' = \frac{1}{3}\left(1 - g^3\right) , \\ \eta = \log \frac{(1 + g + g^2)^{\frac{1}{2}}}{(1-g)} + \sqrt{3} \, \tan^{-1} \frac{\sqrt{3}\, g}{2 + g} \ . \tag{3.3}$$

To our first approximation with $\rho^* = 1$, the diffusion equation (2.13) now becomes

$$\frac{\partial \psi}{\partial z} \frac{\partial c}{\partial \zeta} - \frac{\partial \psi}{\partial \zeta} \frac{\partial c}{\partial z} = \mu_2 \sigma \frac{\partial^2 c}{\partial z^2} \ . \tag{3.4}$$

9.

To solve (3.4) we seek a similarity solution for the concentration in the form

$$c = \frac{G(\eta)}{\zeta^m} \quad ,$$

(3.5)

where $G(\eta)$ and m are to be determined from

$$\sigma G'' + f G' + 4m f' G = 0 \quad ,$$
$$G'(0) = 0 \quad , \quad G(\infty) = 0 \quad ,$$

(3.6)

the primes denoting differentiation with respect to η.
All possible values of m are determined below from the eigenvalue problem posed in (3.6). However, one value of m is given immediately from

$$M = (1+\beta) P_2 \int_0^\infty c u \, dy \quad ,$$

(3.7)

as $m = \frac{1}{4}$, equation (3.7) merely expressing the fact that the total mass flux M of molecules of gas 1 is constant. It will be shown subsequently that the solution (3.5) with $m = \frac{1}{4}$ is the most important solution as $\zeta \to \infty$. With $m = \frac{1}{4}$ equation (3.6) becomes

$$\sigma G'' + f G' + f' G = 0 \quad ,$$
$$G'(0) = 0 \quad , \quad G(\infty) = 0 \quad ,$$

with solution.

$$G = A (1 - q^3)^{\frac{1}{\sigma}} \quad ,$$

(3.8)

where A is a constant. Solution (3.8) is displayed graphically in Figure 1 for $\sigma = 1, 2$.

10.

To find the general solution of (3.6) it is convenient to change the independent variable from η to t where

$$t = g^3 \tag{3.9}$$

when (3.6) is reduced to the hypergeometric equation

$$t(1-t)\frac{d^2G}{dt^2} + \left\{\frac{2}{3} - \frac{(5\sigma-3)}{3\sigma}t\right\}\frac{dG}{dt} + \frac{8m}{3\sigma}G = 0 , \tag{3.10}$$

with boundary conditions

$$\left.\begin{array}{ll} t^{2/3}\frac{dG}{dt} = 0 & \text{at } t = 0 , \\ \text{and} \qquad G = 0 & \text{at } t = 1. \end{array}\right\} \tag{3.11}$$

The solution of equation (3.10) is, using the usual notation,

$$G = AF(a, b; \tfrac{2}{3}; t) + B\, t^{\frac{1}{3}} F(a+\tfrac{1}{3}, b+\tfrac{1}{3}; \tfrac{4}{3}; t) , \tag{3.12}$$

where

$$\begin{aligned} a + b &= \frac{2}{3} - \frac{1}{\sigma} , \\ ab &= -\frac{8m}{3\sigma} , \end{aligned} \tag{3.13}$$

and A, B are constants to be determined. The first of the boundary conditions (3.11) gives $B = 0$ and using the second of (3.11) we have

$$F(a, b; \tfrac{2}{3}; 1) = 0 ,$$

or ,

$$\frac{(-\tfrac{1}{3})!\,(\tfrac{1}{\sigma}-1)!}{(-a-\tfrac{1}{3})!\,(-b-\tfrac{1}{3})!} = 0 ,$$

therefore,

$$a = \tfrac{2}{3} + h ,$$

11.

where $h = 0, 1, 2, 3, \cdots$. The first of the relations
(3.13) now gives

$$b = -\left(\frac{1}{\sigma} + h\right) ,$$

and hence from the remaining relation in (3.13)

$$m = \frac{3\sigma\left(\frac{2}{3} + h\right)\left(\frac{1}{\sigma} + h\right)}{8} . \tag{3.14}$$

Substituting in (3.5) the value of $G(\eta)$ obtained from
(3.12) with $B = 0$ and the values of a, b and m just
obtained, we get, using the identity

$$F(p, \nu; \gamma; x) = (1-x)^{\gamma - p - \nu} F(\gamma - p, \gamma - \nu; \gamma; x),$$

$$c = \sum_{h=0}^{\infty} (1-t)^{\frac{1}{\sigma}} A_h F\left(-h, \frac{2}{3} + \frac{1}{\sigma} + h; \frac{2}{3}; t\right)\left(\frac{\zeta_0}{\zeta}\right)^{\frac{3\sigma}{8}\left(\frac{2}{3} + h\right)\left(\frac{1}{\sigma} + h\right)} , \tag{3.15}$$

where A_h, ζ_0 are constants. The hypergeometric function
in (3.15) is now a polynomial and is related to the Jacobi
polynomials $P_h^{(i,j)}(1 - 2t)$ by

$$P_h^{(i,j)}(1 - 2t) = \frac{(i+h)!}{i! \, h!} F\left(-h, h + i + j + 1; i + 1; t\right) . \tag{3.16}$$

The Jacobi polynomials in (3.16) are orthogonal in the
interval $(-1, 1)$ with the weight function $(1-\zeta)^i(1+\zeta)^j$, $\zeta = 1 - 2t$.
Therefore, if the concentration is given at $\zeta = \zeta_0$ by $c_0 = c(\zeta_0, t)$,
the constants A_h are determined as

$$A_h = \frac{\left(h - \frac{1}{3}\right)!\left(h - \frac{1}{3} + \frac{1}{\sigma}\right)!\left(2h + \frac{2}{3} + \frac{1}{\sigma}\right)}{\left\{\left(-\frac{1}{3}\right)!\right\}^2 h! \left(h + \frac{1}{\sigma}\right)!} \int_0^1 c_0 \, t^{-\frac{1}{3}} F\left(-h, h + \frac{2}{3} + \frac{1}{\sigma}; \frac{2}{3}; t\right) dt . \tag{3.17}$$

12.

In particular, when $\ell = 0$, $m = \frac{1}{4}$ and the solution (3.8) is recovered. This solution is thus seen, from (3.15), to provide the dominant term in c for large ς.

To make the velocity profile truly representative of jet-mixing of two different gases, in other words, to see what effect the diffusion of gas 1 into gas 2 has on the velocity profile, the dominant term in the perturbation to the stream function is obtained, following Crane and Pack (4), by writing

$$\psi = \left(40 F_\alpha /_2^n \varsigma \right)^{\frac{1}{4}} \left[f(\eta) + A_0 \left(\varsigma_0/\varsigma \right)^{\frac{1}{4}} F_1(\eta) + - - - - \right]. \quad (3.18)$$

Substituting (3.15) and (3.18) into equation (2.12) and equating co-efficients, $F_1(\eta)$ is found to satisfy the differential equation

$$F_1''' + F_1'' f + 5 F_1' f' = -n \frac{d}{d\eta} \left\{ \left(1 - f^{3/2} \right)^{\frac{1}{\sigma}} f'' \right\}, \quad (3.19)$$

where the primes denote differentiation with respect to η and $n = \beta + f$. The boundary conditions on F_1 are

$$F_1(0) = F_1'(0) = 0 \quad , \quad F_1'(\infty) = 0 \quad . \quad (3.20)$$

Writing $P = F_1'$ and changing the independent variable once more from η to t using (3.3) and (3.9), equation (3.19) becomes

$$3t(1-t)\frac{d^2 P}{dt^2} + 2(1-t)\frac{dP}{dt} + 10 P = \frac{-2n t^{\frac{1}{3}}}{3} \frac{d}{dt} \left\{ (1-t)^{\frac{1}{\sigma}+1} (1-4t) \right\}. \quad (3.21)$$

13.

Since $(1-t)(1-4t)$ is a complementary function of equation (3.21), the substitution

$$P = (1-t)(1-4t)H$$

reduces it to

$$3t(1-t)^2(1-4t)\frac{d^2H}{dt^2} + 2(1-t)(28t^2-20t+1)\frac{dH}{dt}$$
$$= \frac{-2n\,t^{\frac{1}{3}}}{3}\frac{d}{dt}\left\{(1-t)^{\frac{1}{\sigma}+1}(1-4t)\right\},$$

which on integration gives

$$\frac{dH}{dt} = \frac{-2n}{9}\left[\frac{(1-t)^{\frac{1}{\sigma}-1}}{t^{2/3}} - \frac{4\sigma}{(1+2\sigma)}\frac{(1-t)^{\frac{1}{\sigma}}}{t^{2/3}(1-4t)} + \frac{16\sigma^2}{(1+2\sigma)(1+3\sigma)}\frac{(1-t)^{\frac{1}{\sigma}+1}}{t^{2/3}(1-4t)^2}\right].$$

Therefore,

$$F_1' = (1-t)(1-4t)H$$
$$= -\frac{2n}{9}(1-t)(1-4t)\left[B_t\left(\tfrac{1}{3};\tfrac{1}{\sigma}\right) + \phi(t)\right],$$

where $B_t(p;\gamma) = \int_0^t x^{p-1}(1-x)^{\gamma-1}dx$, and ϕ is defined as

$$\phi \begin{cases} = \frac{16\sigma^2}{(1+2\sigma)(1+3\sigma)}\int_0^t \frac{(1-t)^{\frac{1}{\sigma}+1}dt}{t^{2/3}(1-4t)^2} - \frac{4\sigma}{(1+2\sigma)}\int_0^t \frac{(1-t)^{\frac{1}{\sigma}}dt}{t^{2/3}(1-4t)} & , \quad \text{for } t < \tfrac{1}{4} \\[3mm] = \frac{16\sigma^2}{(1+2\sigma)(1+3\sigma)}\int_1^t \frac{(1-t)^{\frac{1}{\sigma}+1}dt}{t^{2/3}(1-4t)^2} - \frac{4\sigma}{(1+2\sigma)}\int_1^t \frac{(1-t)^{\frac{1}{\sigma}}dt}{t^{2/3}(1-4t)} + K & , \quad \text{for } t > \tfrac{1}{4} \end{cases}$$

the constant K being determined from the condition that F_1' has continuous derivative at $t = \tfrac{1}{4}$.

In particular, when $\sigma = 1$, the solution simplifies to

$$F_1' = -\frac{2n}{3}t^{\frac{1}{3}}(1-t)(1-3t)$$
$$= -n f'(1 - 3f^{3/2}).$$

14.

At each stage of the solution results for the radial wall jet may be deduced from the transformations (2.11), (3.1).

4. RESULTS.

The wall jet function (3.3) found by Glauert is shown in Fig. 2. In Fig. 3 the perturbation profile is shown for $\sigma = 1, 2$. The parameter σ is defined as D_{12}/ν_2 and interpreting this quantity as the ratio of two diffusivities, we would expect the jet to thicken as σ increases. Fig. 3 indicates that this is so.

It can be shown (see ref. 5) that σ is a decreasing function of m_1/m_2. Thus, as m_1/m_2 increases, the velocity profile sharpens and we would expect the increment of skin friction to increase. To test this hypothesis we choose a simplified model in which the radii of the molecules of the two gases are the same and in which

$$D_{12} = 0.848 \, \nu_2 \left(1 + m_2/m_1\right)^{\frac{1}{2}} . \qquad (4.1)$$

Equation (4.1) exhibits the correct dependence of D_{12} on m_1/m_2, the numerical factor is chosen so that the co-efficient of self-diffusion suggested by Chapman (see ref. 5) is recovered when $m_1 = m_2$. With these simplifying assumptions the increment in the skin friction is proportional to Δ where

$$\Delta = \frac{A_0 \, n \, \sigma (1 - \sigma)}{(1 + 2\sigma)(1 + 3\sigma)} .$$

For values of σ in the neighbourhood of unity in which case the masses of the molecules in the mixture do not differ greatly, Δ is an increasing function of m_1/m_2 as we anticipated.

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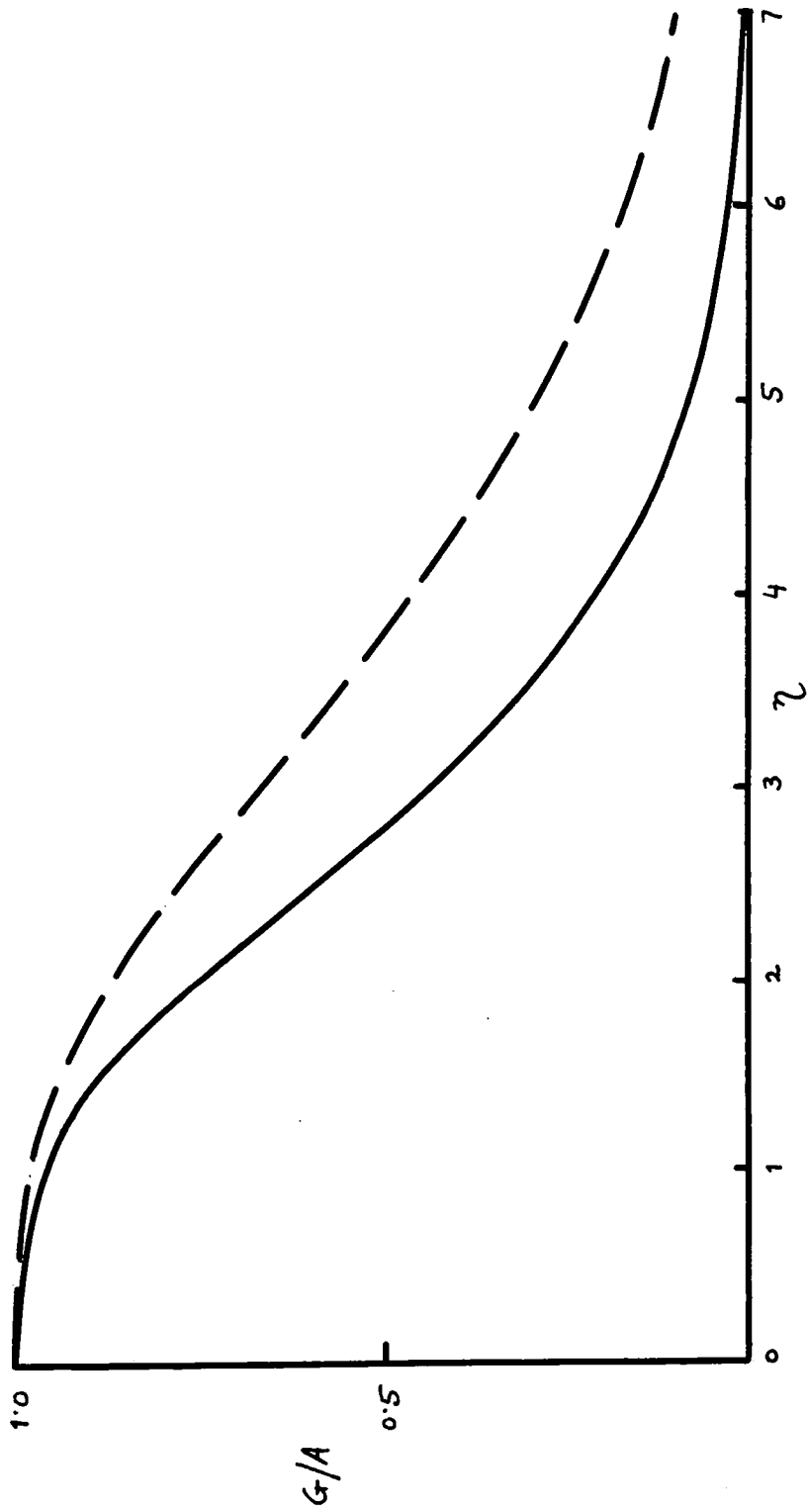   Theory of Gases (Cambridge, 1962).

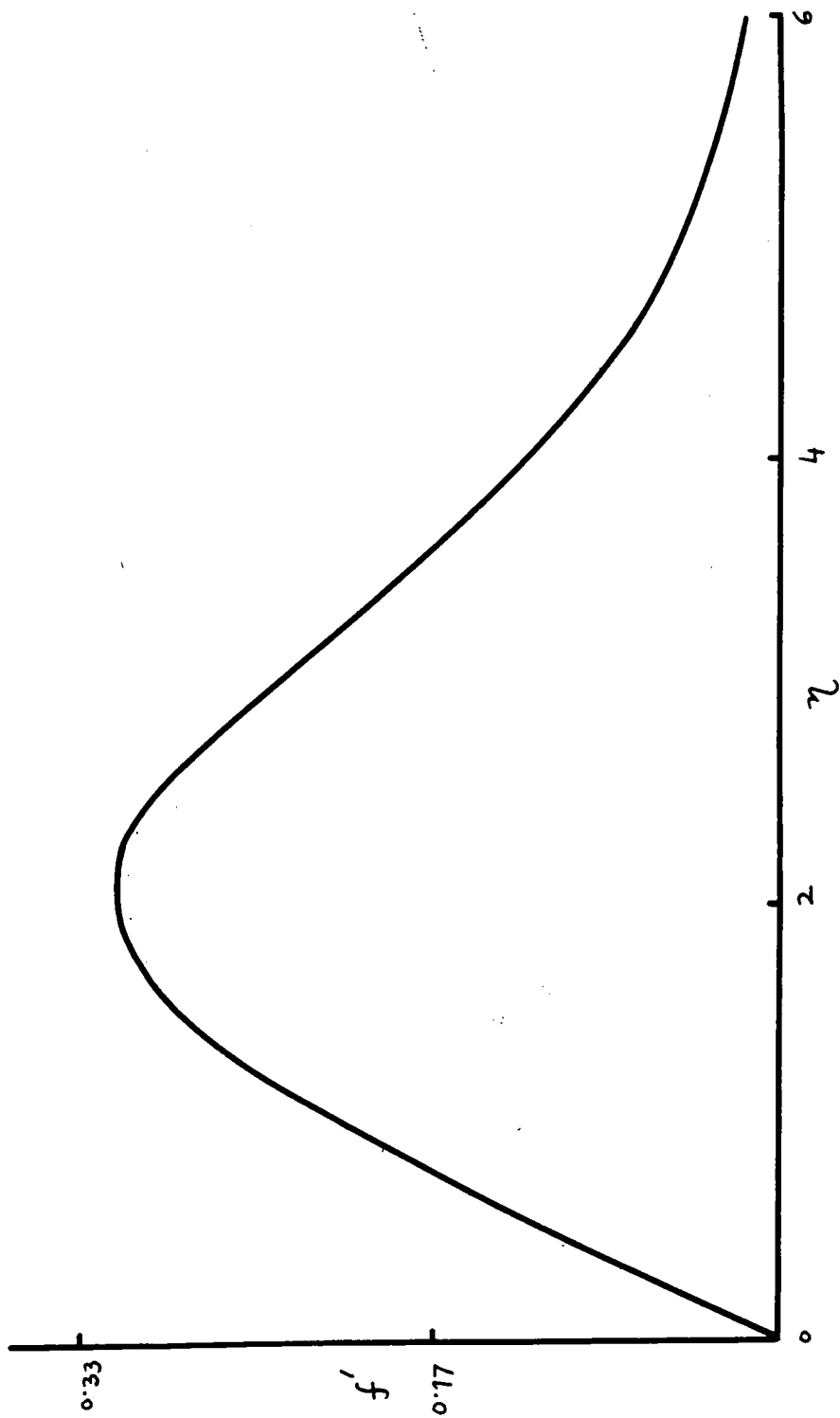Figure 1. The function $G(\eta)$ calculated from (3.8) for $\sigma=1$, ———; $\sigma=2$, – – – – –.

Figure 2.   The velocity function $f'(\eta)$ given by (3.3).
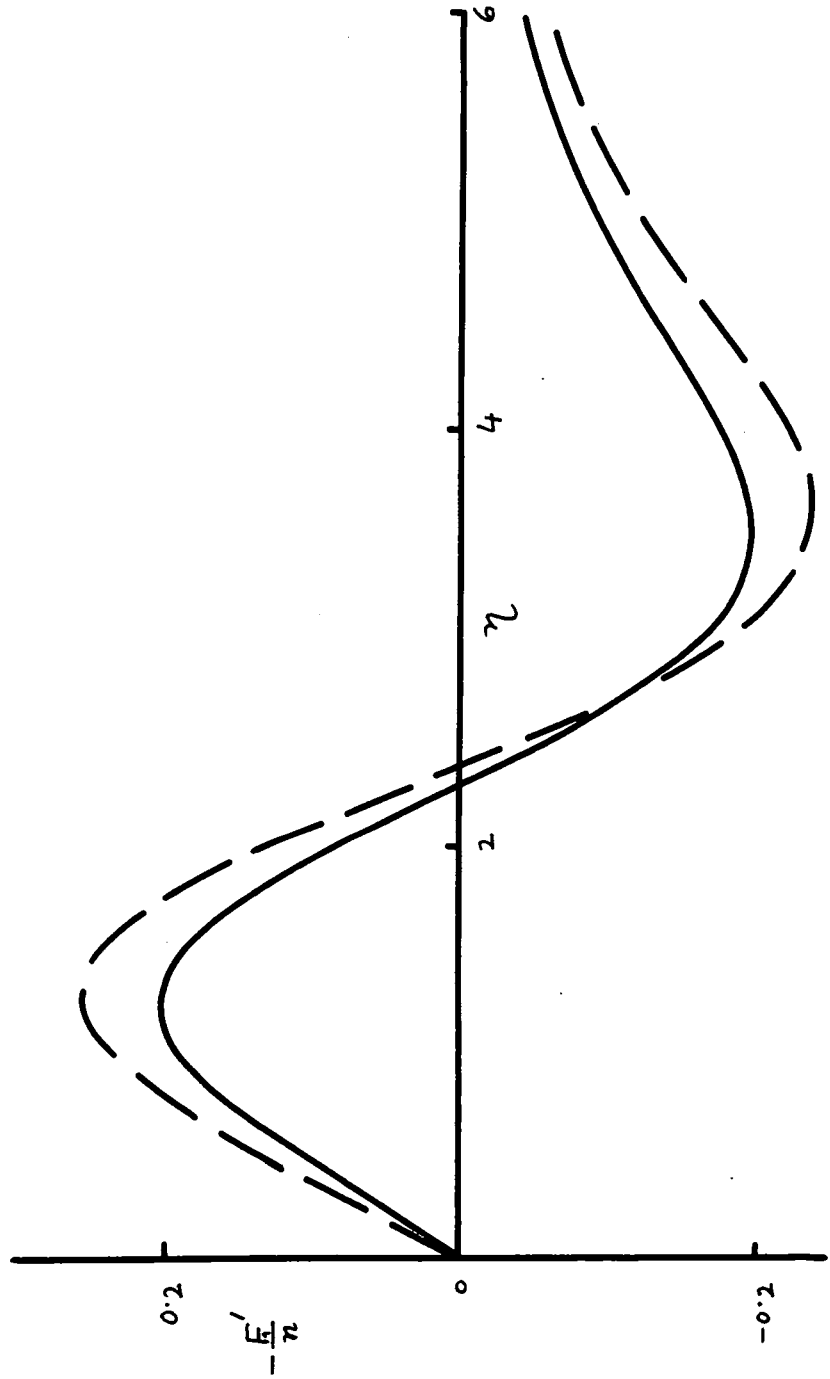
Figure 3.   The perturbation profile $F_1'(\eta)$; $\sigma = 1$, ——— ; $\sigma = 2$, ------- .

CHAPTER TWO.

HEAT TRANSFER IN A RADIAL LIQUID JET.

INTRODUCTION.

A radial liquid jet is formed when a smooth jet of liquid falls vertically onto a horizontal plane and spreads out radially over it, as for example, water falling from a tap to the bottom of an empty sink. The liquid spreads out in a thin layer until the depth increases suddenly forming a hydraulic jump. In order to discuss the motion of the fluid in the thin layer before the hydraulic jump, the assumptions of boundary layer theory are applied which require that the Reynold's number of the impinging jet should be large. The radial liquid jet was first studied by Watson (1) who found a similarity solution of the laminar boundary layer equations governing such flow. He also considered the initial growth of the boundary layer from the stagnation point where the similarity solution does not hold, by approximate methods. These results are briefly discussed in Section 2. Later it was investigated by Riley (2) in his detailed study of radial jets with swirl. The boundary layer originating from the central stagnation point gradually grows in thickness, as more and more liquid is affected by viscous shear stress, until it absorbs the whole flow. In studying the velocity

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distribution, Watson divided the flow into four different regions which pass continuously into one another.

(i)　The region near the central stagnation point where the radial distance $x = O(a_0)$, a_0 being the radius of the impinging jet. In this region the boundary layer thickness is $O\left(\nu a_0 / U_0\right)^{1/2}$ where U_0 is the speed of the impinging jet and ν is the kinematic viscosity.

(ii)　When $x \gg a_0$, the conditions in region (i) are not important and the boundary layer grows like the Blasius boundary layer on a flat plate.

The approximate method used by Watson for this region is briefly discussed in Section 2. A similar method is used in Section 4 to describe the temperature distribution in this region.

(iii)　As x increases, the viscous stresses affect more and more fluid across the flow and the boundary layer increases in thickness until it absorbs the whole layer of fluid.. The velocity profile then gradually changes from Blasius type to the similarity profile mentioned earlier.

(iv)　At large distances from the stagnation point, the way in which the flow originated becomes unimportant and the final similarity form is attained.

The hydraulic jump associated with this type of flow will ultimately terminate the region of flow under consideration.

The problem of the distribution of temperature in a radial liquid jet is studied here by conforming to a similar division of the flow. In Section 2, the equations of motion are stated and the similarity solution of the momentum equation first found by Watson is briefly discussed. In Section 3 similarity solutions of the energy equation appropriate to region (iv) are obtained for a wide variety of temperature conditions.

In the first of the two examples described in Section 4 part of the wall is assumed to be thermally insulated, the rest being maintained at a constant temperature; a solution of the energy equation is found for this latter part which is chosen to correspond to region (iv) described above. In the second example the whole wall is maintained at a constant temperature so that both the temperature and velocity distributions have to be studied in all the four regions described earlier. Regions (i) and (iii) are neglected following Watson and Riley (3). In region (ii) an approximate method involving the heat flux equation and two polynomials of the fourth degree for the temperature and the velocity functions are employed. The neglect of regions (i) and (iii) and Watson's approximate method for the velocity distribution in region (ii) are discussed at the end of Section 2.

We assume that the contribution to the temperature of the liquid in the jet due to viscous heating is negligible compared to the applied heating and it is , therefore, ignored while calculating the temperature distribution in the liquid in the examples discussed above. Throughout this problem the boundary layer equations are assumed to be appropriate, temperature differences are taken to be small and μ, the co-efficient of viscosity, and ρ, the density, are assumed to be constant.

2. EQUATIONS OF MOTION.

Using boundary layer approximations the momentum, continuity and energy equations governing the incompressible laminar flow of a liquid jet striking a plane horizontal wall at right angles and spreading out radially over it, are respectively

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \qquad , \qquad (2.1)$$

$$\frac{\partial}{\partial x}(u x) + \frac{\partial}{\partial y}(v x) = 0 \qquad , \qquad (2.2)$$

and
$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{\sigma} \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{S}\left(\frac{\partial u}{\partial y}\right)^2, \qquad (2.3)$$

where x, y are the distances measured along the wall from the jet axis and normal to it respectively; u, v are the corresponding velocity components, and ν, T, σ, S denote respectively the kinematic viscosity, temperature, Prandtl number and specific heat ~~at constant pressure~~ of the liquid in the jet. The boundary conditions are

$$u = v = 0 \qquad at \quad y = 0 \qquad , \qquad (2.4)$$

$$\frac{\partial u}{\partial y} = 0 \qquad at \quad y = \varphi(x) \quad , \qquad (2.5)$$

and
$$\frac{\partial T}{\partial y} = 0 \qquad at \quad y = \varphi(x) \quad , \qquad (2.6)$$

22.

Other boundary conditions on T necessary to specify the problem completely will be introduced later. Condition (2.5) expresses the fact that the shearing stress falls to zero at the free surface $y = \varphi(x)$ and (2.6) states that there is no heat transfer across the free surface. With the boundary conditions (2.4) and (2.5) Watson has shown that a similarity solution of the momentum equation is available. This may be written as

$$\left. \begin{aligned} \psi &= \frac{3\sqrt{3}\,Q}{\pi} f(\eta) \quad , \\[2mm] \eta &= \frac{3\sqrt{3}\,Q\,x}{\pi\,\nu(x^3+\ell^3)} y \quad , \end{aligned} \right\} \qquad (2.7)$$

where ψ is the stream function defined as

$$u\,x = \frac{\partial \psi}{\partial y} \quad , \qquad v\,x = -\frac{\partial \psi}{\partial x} \quad , \qquad (2.8)$$

and ℓ is an arbitrary constant length which depends on the initial development of the boundary layer. Watson estimates

$$\ell = 0.567\, a_0\, R^{1/3} ,$$

where R is the jet Reynolds number $\frac{2\pi Q}{\nu a_0}$. The constant Q is given by the condition of constant volume flux per radian, namely

$$\int_0^{\varphi(x)} u\,x\,dy = Q \quad . \qquad (2.9)$$

The function f satisfies the ordinary differential equation

$$f''' + 3f'^2 = 0 \quad,$$

$$f(0) = f'(0) = 0 \quad, \quad f''(1) = 0 \quad, \quad \Bigg\} \qquad (2.10)$$

the free surface having been chosen to be $\eta = 1$. It is convenient, for what follows, to make the transformation

$$f = \frac{\alpha_1^2}{2} f_1 \quad, \qquad (2.11)$$

where α_1 is a constant chosen so that

$$f_1'(1) = 1 \quad. \qquad (2.12)$$

Thus the equation satisfied by f_1 is

$$f_1''' + \frac{3}{2}\alpha_1^2 f_1'^2 = 0 \quad, \qquad (2.13)$$

with the same boundary conditions as for f in (2.10). Integrating equation (2.13) once

$$f_1'' = \alpha_1\left(1 - f_1'^3\right)^{1/2}, \qquad (2.14)$$

and so

$$\alpha_1 \eta = \int_0^{f_1'} \left(1 - \omega^3\right)^{-\frac{1}{2}} d\omega \quad, \qquad (2.15)$$

which with (2.12) gives

$$\alpha_1 = \int_0^1 \left(1 - \omega^3\right)^{-\frac{1}{2}} d\omega = 1\cdot 402 \quad. \qquad (2.16)$$

Also from (2.14),

$$\int_0^1 f_1'(\eta)\, d\eta = \frac{1}{\alpha_1}\int_0^1 f_1'\left(1 - f_1'^3\right)^{-1/2} df_1' = \frac{2\pi}{3\sqrt{3}\,\alpha_1^2} \quad. \qquad (2.17)$$

The velocity function f' is displayed graphically in Figure 1.

If $x \gg a_o$, conditions prevailing in the region (i) where $x = O(a_o)$ are not important and in the approximate analysis discussed below and in Section 4 region (i) is ignored. For his approximate solution in region (ii), Watson used the Kármán-Pohlhausen method with

$$u = U_o f_1'(\eta) \quad , \qquad \eta = \frac{y}{\delta} \quad , \tag{2.18}$$

where $f_1'(\eta)$ is the similarity profile defined by (2.15) and δ is the boundary layer thickness. This technique has the effect of suppressing region (iii) in which the velocity profile changes to its final similarity form. In fact, Riley (3) has shown that in region (iii), when the boundary layer fills the whole of the moving layer of fluid, any disturbance to the similarity velocity profile in a radial liquid jet is $O(x^{-45})$; thus the final similarity form is attained very rapidly. Substitution of the approximate velocity profile (2.18) in the momentum integral equation for radial flow

$$\left(\frac{d}{dx} + \frac{1}{x} \right) \int_0^\delta u (U_o - u) \, dy = \nu \left(\frac{\partial u}{\partial y} \right)_{y=0} \quad , \tag{2.19}$$

gives, using (2.17),

$$\frac{2(\pi - \sqrt{3}\alpha_1)}{3\sqrt{3}\,\alpha_1^2} U_o \left(\frac{d\delta}{dx} + \frac{\delta}{x} \right) = \frac{\nu \alpha_1}{\delta} \quad ,$$

and hence

$$x^2 \delta^2 = \frac{\sqrt{3}\,\alpha_1^3}{(\pi - \sqrt{3}\alpha_1)} \frac{\nu x^3}{U_o} + C \quad , \tag{2.20}$$

where C is a constant. A consideration of the order

25.

of magnitude shows that $C = O\left(\dfrac{a_0^3}{x^3}\right)$ relative to the

other terms there and hence can be neglected when $x \gg a_0$.

Thus when $x_0 > x \gg a_0$

$$\delta^2 = \frac{\sqrt{3}\,\alpha_1^3}{(\pi - \sqrt{3}\alpha_1)} \frac{\nu x a_0^2}{2 Q}\,,$$ (2.21)

where x_0 is the station at which the boundary layer just

absorbs the whole flow. Watson calculated the value of x_0

from the condition that the volume flux through the

boundary layer reaches the value Q there, and thus

obtained on the basis of the above approximate solution

$$x_0 = 0.3155\, a_0\, Q^{1/3}\,.$$ (2.23)

3. SIMILARITY SOLUTIONS OF THE ENERGY EQUATION.

In region (iv) where the solution of (2.1) and (2.2) is given by (2.7) and (2.8), the energy equation (2.3) may be written as

$$\frac{\partial^2 T}{\partial \eta^2} - \frac{\sigma(x^3+\ell^3)}{x^2} f' \frac{\partial T}{\partial x} = - \frac{27^2 a^4 \sigma}{\pi^4 \nu^2 S (x^3+\ell^3)^2} f''^2, \qquad (3.1)$$

with the boundary condition

$$\frac{\partial T}{\partial \eta} = 0 \qquad \qquad at \quad \eta = 1. \qquad (3.2)$$

At the wall either the wall temperature T_ω or heat transfer which is proportional to $\left(\frac{\partial T}{\partial \eta}\right)_{\eta=0}$, may be prescribed. In the examples discussed in Section 4, we shall restrict ourselves to the case of constant wall temperature or, if the wall is thermally insulated, zero heat transfer.

A particular integral of (3.1) can be found in the form

$$T = T_2 + C_o (x^3 + \ell^3)^{-2} \theta_o(\eta) \quad , \qquad (3.3)$$

where T_2 is the constant wall temperature and C_o is a constant. Equation (3.1) also has complementary functions of the form

$$T = C_1 + C_2 (x^3 + \ell^3)^{-\alpha} \theta_n(\eta) , \qquad (3.4)$$

with θ_n satisfying

$$\theta_n'' + 3\alpha\sigma f' \theta_n = 0 \quad . \qquad (3.5)$$

27.

A complete picture of the temperature distribution prevailing in a particular problem can thus be obtained by adding to the particular integral (3.3) appropriate complementary functions of the type (3.4).

The effects of viscous heating, wall heating and initial heating will now be studied separately.

(I) Viscous Heating.

To study the effects of viscous dissipation on the temperature distribution in the jet, we require a particular integral of (3.1) in the form (3.3) where, with

$$C_0 = - \frac{27^2 a^4}{\pi^4 \nu^2 s} \quad ,$$

$\theta_0(\eta)$ satisfies

$$\theta_0'' + 6\sigma f' \theta_0 = \sigma f''^2 \quad , \tag{3.6}$$

with boundary conditions

either $\qquad \theta_0(0) = 0 \qquad$ if the wall is maintained at constant temperature,

or, $\qquad \theta_0'(0) = 0 \qquad$ for thermally insulated wall,

$\tag{3.7}$

and $\qquad \theta_0'(1) = 0 \qquad$ from (3.2).

It is convenient to change the independent variable in (3.6) and (3.7) from η to a new variable t with

$$t = f_1'^3 \quad . \tag{3.8}$$

From (3.8) and (2.11) we have

$$\frac{d}{d\eta} = 3\alpha_1 t^{2/3} (1-t)^{1/2} \frac{d}{dt} \quad , \tag{3.9}$$

28.

which when substituted in (3.6) gives, as the equation
for θ_0 ,

$$t(1-t)\frac{d^2\theta_0}{dt^2}+\left(\frac{2}{3}-\frac{7}{6}t\right)\frac{d\theta_0}{dt}+\frac{\sigma}{3}\theta_0=\frac{\sigma\alpha_1^4}{36}t^{-\frac{1}{3}}(1-t) . \qquad (3.10)$$

The boundary conditions (3.7) now become

$$\theta_0(0) = 0 \qquad ,$$

$$\left[t^{\frac{2}{3}}\frac{d\theta_0}{dt}\right]_{t=0} = 0 \qquad , \qquad\qquad\Bigg\} \qquad (3.11)$$

$$and \qquad \left[(1-t)^{1/2}\frac{d\theta_0}{dt}\right]_{t=1} = 0 \qquad ,$$

A series solution of equation (3.10) gives

$$\theta_0 = \frac{\sigma\alpha_1^4}{8}t^{\frac{2}{3}}+\frac{3\sigma\alpha_1^4}{160}(1-\sigma)t^{\frac{5}{3}}\left[1+\frac{a\,b\,c}{d\,e}t+\right.$$

$$\left.+\frac{a(a+1)\,b(b+1)\,c(c+1)}{d(d+1)\,e(e+1)}\frac{t^2}{2!}+------\right] ,$$

where

$$a=\frac{1}{12}\left(21+\sqrt{1+48\sigma}\right) \quad , \quad b=\frac{1}{12}\left(21-\sqrt{1+48\sigma}\right) \quad ,$$

$$c=1 \qquad , \qquad d=\frac{7}{3} \quad and \quad e=\frac{8}{3} .$$

The infinite series inside the square bracket on the right-
hand side of the solution above is the generalised hyper-
geometric series. With the usual notation the solution
can, therefore, be written as

$$\theta_0 = \frac{\sigma\alpha_1^4}{8}t^{\frac{2}{3}}+\frac{3\sigma\alpha_1^4}{160}(1-\sigma)t^{5/3}\,{}_3F_2\left(\begin{matrix}a\,,\,b\,,\,c\,;\,t\\d\,,\,e\,;\end{matrix}\right) . \qquad (3.12)$$

For $\sigma=1$ solution (3.12) gives $\theta_0=\frac{1}{2}f'^2$ as indicated
by the quadratic term in the well known Crocco relation,

for $\sigma = 1$,

$$T + \frac{u^2}{2S} = A + Bu \; ,$$
(3.13)

where A and B are constants.

We expect the effects of viscous dissipation in a liquid jet to be small and indeed in what follows we shall assume that it can be neglected compared with the applied heating.

(II) Wall Heating.

When the wall is maintained at a constant temperature T_2 , we need a complementary function of the form (3.4) with $\alpha = 0$.   Thus

$$T = T_1 + (T_2 - T_1) \theta_1(\eta) \; ,$$

where T_1 is the temperature of the incident jet. From (3.5), $\theta_1(\eta)$ now satisfies

$$\theta_1''(\eta) = 0 \; ,$$

which, with the boundary conditions

$$\theta_1(0) = 1 \quad , \qquad \theta_1'(1) = 0 \; ,$$

has the trivial solution

$$\theta_1 = 1 \; .$$

This solution reflects the physical situation that ultimately all the fluid is raised to temperature T_2 . The manner in which the fluid attains this constant temperature depends upon the initial heating of the fluid the effects of which we now consider.

(III)   <u>Initial Heating.</u>

We now require further complementary functions of the type (3.4). Thus when viscous dissipation effects are negligible we may write the temperature as

$$T = T_2 + C(x^3 + \ell^3)^{-\alpha} \theta_2(\eta) \; ,$$

where $\theta_2(\eta)$ satisfies the differential equation (3.5).

We may note that if the liquid from which the jet is formed is, as we shall assume, at a uniform temperature T_1 and the wall over which the fluid flows is thermally insulated then the fluid remains at constant temperature as no heat is transferred to or from the fluid across either the wall or free surface.

When the wall is maintained at constant temperature T_2, the appropriate boundary conditions for θ_2 are the first and third of those in (3.7). The transformation (3.9) reduces (3.5) to the hypergeometric equation

$$t(1-t)\frac{d^2\theta_2}{dt^2} + \left(\frac{2}{3} - \frac{7}{6}t\right)\frac{d\theta_2}{dt} + \frac{\sigma\alpha}{6}\theta_2 = 0 \; , \qquad (3.14)$$

the boundary conditions now being the first and third of those in (3.11). The determination of α in equation (3.14) is an eigenvalue problem and in view of the form of the boundary condition at $t = 1$, it is convenient to choose the following as the solutions of (3.14)

$$\theta_{21} = F\left(p, \nu; \frac{1}{2}; 1-t\right) \qquad ,$$

$$\text{and} \qquad \theta_{22} = (1-t)^{1/2} F\left(\frac{2}{3} - p, \frac{2}{3} - \nu; \frac{3}{2}; 1-t\right) ,$$

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where p, q are given by

$$p + q = \frac{1}{6} \quad , \quad pq = -\frac{\alpha\sigma}{6} \; . \qquad (3.15)$$

The boundary condition at the free surface determines θ_{21} as the required solution and that at the wall requires that

$$\left[\theta_{21}\right]_{t=0} = \frac{\left(-\frac{1}{2}\right)!\left(-\frac{2}{3}\right)!}{\left(-p-\frac{1}{2}\right)!\left(-q-\frac{1}{2}\right)!} = 0 \; ,$$

which with (3.15) gives p, q and α as

$$p = k + \frac{1}{2} \quad , \quad q = -\left(k + \frac{1}{3}\right) ,$$
$$\alpha = (1 + 2k)(1 + 3k)/\sigma \qquad ,$$

where $k = 0, 1, 2, 3, \text{----} \; .$ Therefore

$$\theta_2 = \sum_{k=0}^{\infty} A_k F\left(k + \frac{1}{2}, -k - \frac{1}{3}; \frac{1}{2}; 1-t\right)$$

$$= t^{1/3} \sum_{k=0}^{\infty} A_k F\left(-k, k + \frac{5}{6}; \frac{1}{2}; 1-t\right) \; ,$$

where A_k are constants. Thus writing $\lambda(x) = (x^3 + \ell^3)/(x_1^3 + \ell^3)$, and including other constants in A_k we have

$$T = T_2 + t^{\frac{1}{3}} \sum_{k=0}^{\infty} A_k \lambda^{-(1+2k)(1+3k)/\sigma} F\left(-k, k + \frac{5}{6}; \frac{1}{2}; 1-t\right) \; . \qquad (3.16)$$

For $\sigma = 1$ the leading term in (3.16) is given by the linear term in u in the Crocco relation (3.13). The hypergeometric function in (3.16) is a polynomial and is related, by (3.16) of Chapter 1, to the Jacobi polynomials $P_k^{(i,j)}(2t-1)$ which are orthogonal in the interval $(-1,1)$ with the wight factor $(1-\xi)^i(1+\xi)^j$, $\xi = 2t - 1$. Therefore, if the temperature distribution is known at any station $x = x_1$,

the constants A_ℓ can be calculated from

$$\frac{A_\ell}{T_1-T_2} = \frac{(2\ell+5/6)(\ell-1/2)!(\ell-1/6)!}{\pi\,\ell!\,(\ell+1/3)!}\int_0^1\left(\frac{T-T_2}{T_1-T_2}\right)_{x=x_1}(1-t)^{-\frac{1}{2}}F\left(-\ell,\ell+\frac{5}{6};\frac{1}{2};1-t\right)dt. \quad (3.17)$$

The heat transfer across the wall per unit area
is given by

$$Q_\omega = -\left(\ell\frac{\partial T}{\partial y}\right)_{y=0} = \frac{-3\sqrt{3}\left(-\frac{1}{2}\right)!\left(-\frac{4}{3}\right)!\,\ell\,Q\,x\,\alpha_1}{\pi\,\nu\,(x_1^3+\ell^3)}\sum_{\ell=0}^{\infty}\frac{A_\ell\,\lambda^{-\frac{(1+2\ell)(1+3\ell)}{\sigma}-1}}{(\ell-1/2)!(-\ell-4/3)!}, \quad (3.18)$$

where ℓ is thermal conductivity.

4. EXAMPLES.

As a first example we take the case where part of the wall , $x < x_1$, is assumed to be thermally insulated and the rest of it maintained at a constant temperature T_2. The end-point , $x = x_1$, of the thermally insulated part is assumed to be in region (iv) as described in the introduction. Therefore, in the regions (i), (ii) and (iii) and in particular at $x = x_1$ the temperature is uniform everywhere and equal to T_1 . Thus the constants A_{ℓ} occuring in (3.16) may be determined from (3.17), with $\left[(T-T_2)/(T_1-T_2) \right]_{x_1} = 1$, as

$$\frac{A_{\ell}}{T_1-T_2} = \frac{(2\ell+\frac{5}{6})(\ell-\frac{1}{2})!(\ell-\frac{1}{6})!}{\pi \ \ell! \ (\ell+\frac{1}{3})!} \int_0^1 (1-t)^{-\frac{1}{2}} F\left(-\ell, \ell+\frac{5}{6}; \frac{1}{2}; 1-t\right) dt . \quad (4.1)$$

The first six values of A_{ℓ} are given in Table 1 and the temperature, calculated from the solution (3.16) with these values of A_{ℓ}, is shown as a function of η in Figure 2 for different $\lambda(x)$. A value , $\sigma = 5$, appropriate to water, has been chosen for the Prandtl number. The heat transfer across the wall for $x > x_1$ is also displayed graphically in Figure 4 where, for convenience we have taken $x_1 = x_0$.

ℓ	0	1	2	3	4	5
$A_{\ell}/(T_1-T_2)$	+1.188	-0.280	+0.150	-0.086	+0.075	-0.059

Table 1. Values of the constants A_{ℓ} calculated from (4.1).

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As a second example, we consider the case when the wall is maintained at a constant temperature throughout; thus the temperature distribution in the liquid passes smoothly through the stages (i) - (iii) before attaining the similarity form in region (iv). We have explained in Section 2 why regions (i) and (iii) may be ignored and we have also discussed there the method employed by Watson for an approximate solution in region (ii). To determine the temperature distribution in region (ii), we use here an approximate method in which both the momentum integral equation and the heat flux equation - an integrated form of the energy equation - are used. In order to simplify the calculation, polynomials of the fourth degree are assumed for the velocity and the temperature functions. The heat flux equation may be obtained by integrating the energy equation with respect to y from $y = 0$ to $y = \infty$, neglecting frictional heat. Thus

$$\frac{d}{dx} \int_0^\infty u\, x\, (T-T_1)\, dy = -\frac{\nu x}{\sigma} \left(\frac{\partial T}{\partial y}\right)_{y=0} . \tag{4.2}$$

The velocity and temperature distributions are assumed to have the forms

$$u = U_0 \left(2\eta - 2\eta^3 + \eta^4 \right) , \tag{4.3}$$

$$T - T_1 = (T_2 - T_1)\left(1 - 2\eta_T + 2\eta_T^3 - \eta_T^4\right) \, , \tag{4.4}$$

where $\eta = \dfrac{y}{\delta}$ and $\eta_T = \dfrac{y}{\delta_T}$, δ and δ_T being the velocity and thermal boundary layer thicknesses respectively. The ratio δ_T / δ will be denoted by Δ. The form of the temperature distribution function in (4.4) is so chosen as to ensure identical velocity and temperature distributions required for the case of negligible frictional heat with $\sigma = 1$, $\delta_T = \delta$. Inserting (4.3) and (4.4) in (4.2) we obtain, on performing the integration,

$$U_0 \Delta^2 H(\Delta) \frac{d}{dx}(\delta x)^2 = 4 \frac{\nu x^2}{\sigma} \, , \tag{4.5}$$

where $H(\Delta) = \dfrac{2}{15}\Delta - \dfrac{3}{140}\Delta^3 + \dfrac{1}{180}\Delta^4$ $\qquad$ for $\Delta < 1$,

which will be the case when the jet is formed from water.

~~and~~ $\quad$ ~~$H(\Delta) = \dfrac{3}{10} - \dfrac{3}{10}\dfrac{1}{\Delta} + \dfrac{9}{15}\dfrac{1}{\Delta^2} - \dfrac{3}{140}\dfrac{1}{\Delta^4} + \dfrac{1}{180}\dfrac{1}{\Delta^5}$~~ $\quad$ ~~for $\Delta > 1$.~~

Knowing δ, (4.5) is an equation for Δ. To determine δ we substitute (4.3) in the momentum integral equation for radial flow (2.19) to get

$$\frac{37}{315} U_0 \left(\frac{d\delta}{dx} + \frac{\delta}{x}\right) = \frac{2\nu}{\delta} \, , $$

and hence $\qquad x^2 \delta^2 = \dfrac{420}{37}\dfrac{\nu x^3}{U_0} + C_3 \, , $

where C_3 is a constant which is zero for the reasons

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given in section 2. Thus

$$\delta^2 = \frac{420}{37} \frac{\nu x}{U_0} \, , \tag{4.6}$$

analogous to the result (2.21) obtained by Watson. Equation (4.5) with (4.6) now gives

$$\Delta^2 H(\Delta) = \frac{37}{315} \frac{1}{\sigma} \, . \tag{4.7}$$

To solve (4.7) for Δ we again choose $\sigma = 5$ which demands that we use that value of $H(\Delta)$ for which $\Delta < 1$, as $\delta_T < \delta$ when $\sigma > 1$. This gives

$$\Delta \approx 0.570 . \tag{4.8}$$

Since $\eta_T = \eta/\Delta$ we have, from (4.4)

$$T - T_2 = (T_1 - T_2)\left(\frac{2}{\Delta}\eta - \frac{2}{\Delta^3}\eta^3 + \frac{1}{\Delta^4}\eta^4\right) \, , \tag{4.9}$$

where Δ is given by (4.8).

For the reasons given at the end of Section 2, region (iii) can be ignored and the approximate result obtained in (4.9) may be matched with the solution (3.16) at the station $x = x_0$ where the flow attains its similarity form. Thus, in this case, $x_1 = x_0$ and the quantity x_0 is determined by the condition that the volume flux through the boundary layer attains the value Q there. Thus (2.9) with (4.3) and (4.6) gives

$$x_0 = 0.2427 \, a_0 R^{1/3} \, , \tag{4.10}$$

analogous to the result in (2.23) obtained by Watson.

37.

To estimate ℓ we apply the condition that the jet thickness is continuous at $x = x_0$. Thus

$$\left[\frac{420}{37}\frac{\nu x_0}{U_0}\right]^{1/2} = \frac{\pi\nu(x_0^3+\ell^3)}{3\sqrt{3}\,a\,x_0} \quad , \qquad (4.11)$$

which, with (4.10) and remembering that $U_0 a_0^2 = 2a$, gives

$$\ell = 0.5577\,a_0 R^{1/3} \quad . \qquad (4.12)$$

The constants A_k in this case are given by (3.17) where $\left(\frac{T-T_2}{T_1-T_2}\right)_{x=x_0}$ is evaluated from (4.9). The values of A_k found by numerical integration, are shown in Table 2 and the temperature distribution calculated from (3.16) with these values of the A_k's is displayed graphically for different values of $\lambda(x)$ in Figure 3.

	0	1	2	3	4
	+1.165	-0.220	+0.072	-0.018	-0.001

Table 2.   Values of the constants A_k obtained numerically from (3.17).

The heat transfer across the wall in this case is also displayed graphically in Figure 4. For $x < x_0$ it is calculated from the approximate solution described above and for $x \geqslant x_0$ from equation (3.18).

The approximate method described here is inferior to that of Watson's, described in section 2, since the assumed quartic profile (4.3) does not join on smoothly with the similarity solution at $x = x_0$. However, as indicated earlier, the transition region (iii) in which

the velocity profile attains its final similarity form
will be small and it is sufficient for our purposes,
especially in view of the enormous simplifications in
the analysis, to assume the quartic profiles (4.3) and
(4.4).

The thin layer of fluid in which we have been
investigating the temperature distribution is terminated
by a sudden increase in depth at a station $x = x_2$, say.
This is a hydraulic jump. An estimate of x_2 has been
made by Watson by equating the rate of loss of momentum
to the thrust of the pressure. Watson also extends his
analysis to the case of turbulent flow which is outside
the scope of the present work.

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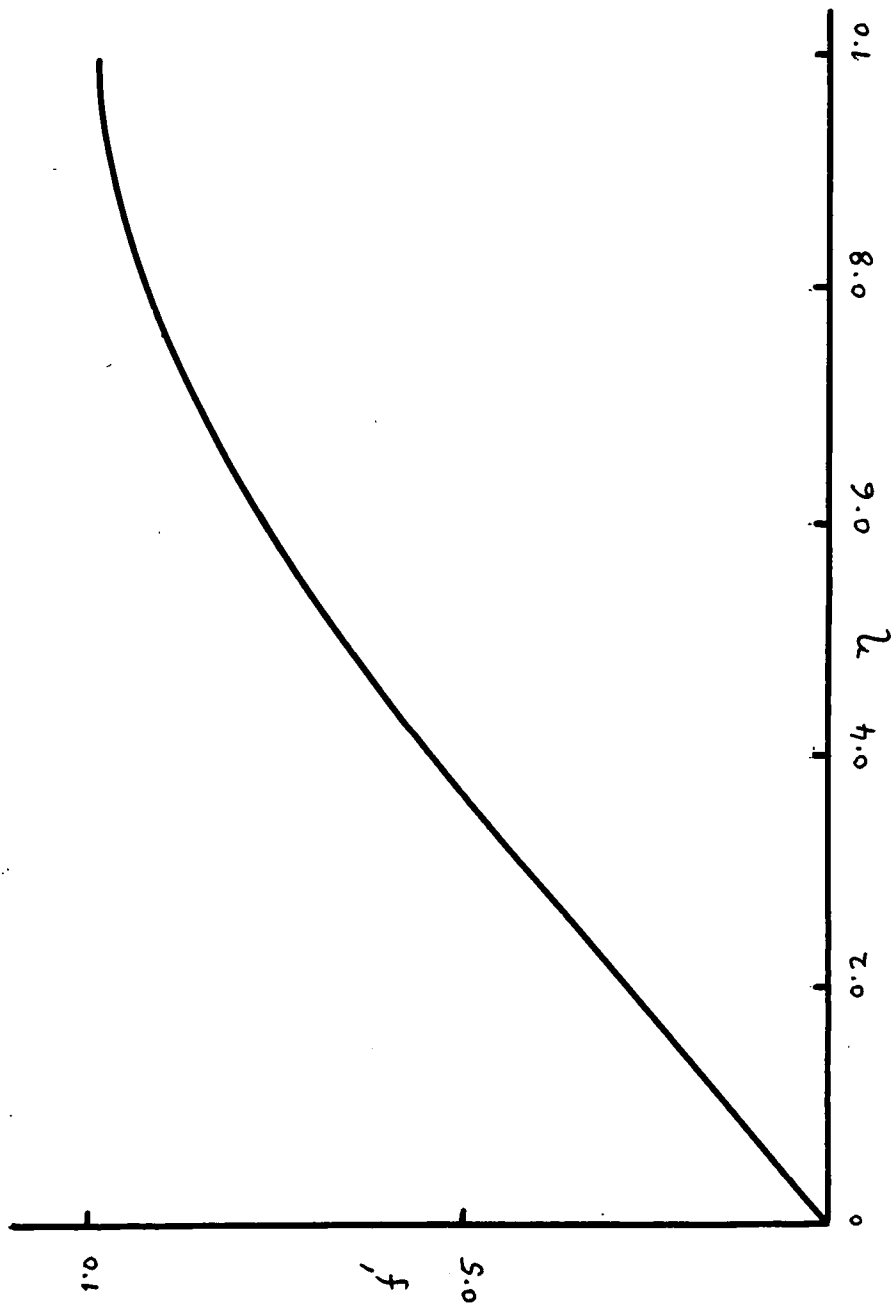 $\underline{41}$. 132.

Figure 1. The velocity function $f'(\eta)$, from (2.10).
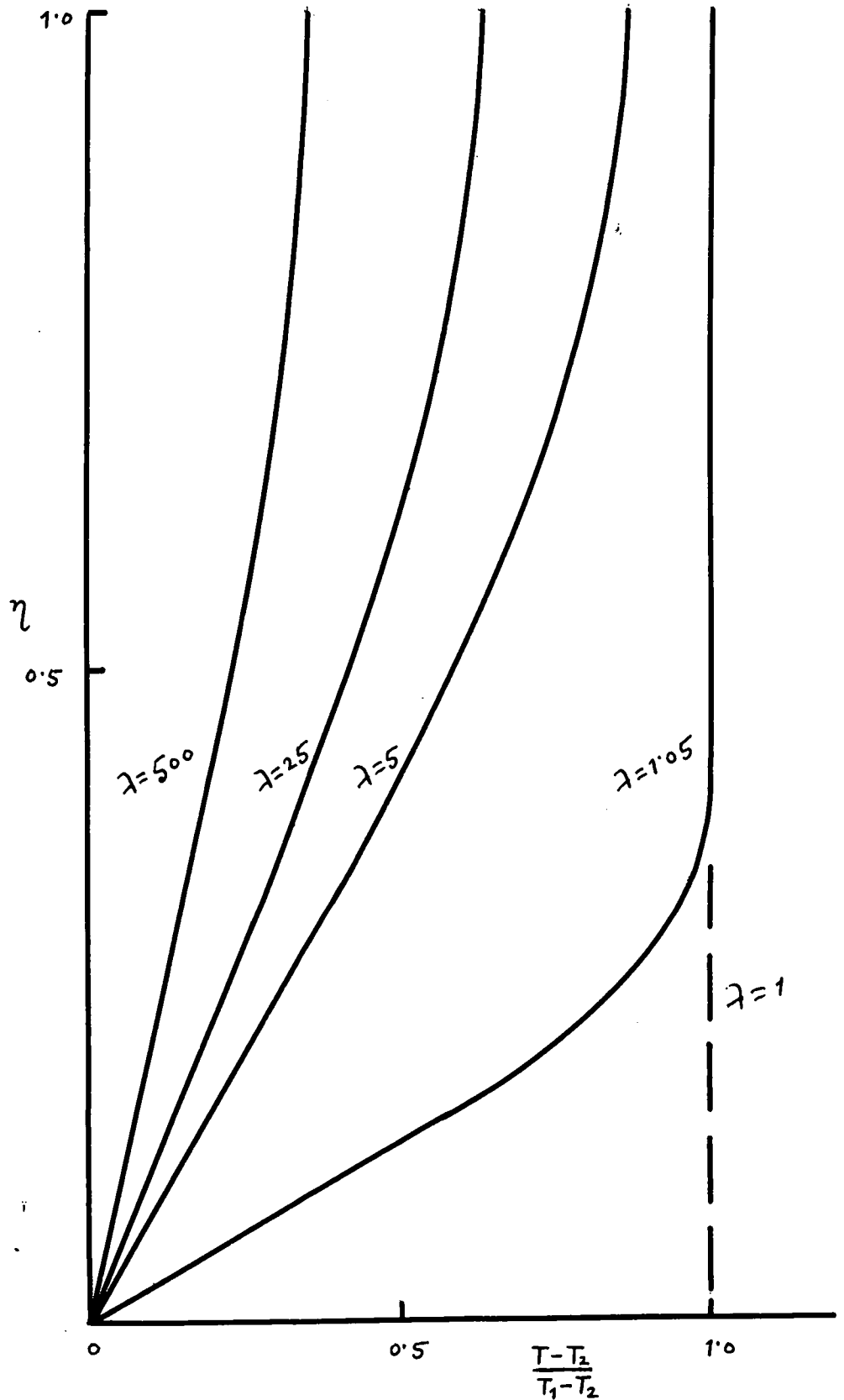
Figure 2. Temperature distributions for example 1 for various $\lambda(x) \geqslant 1$.
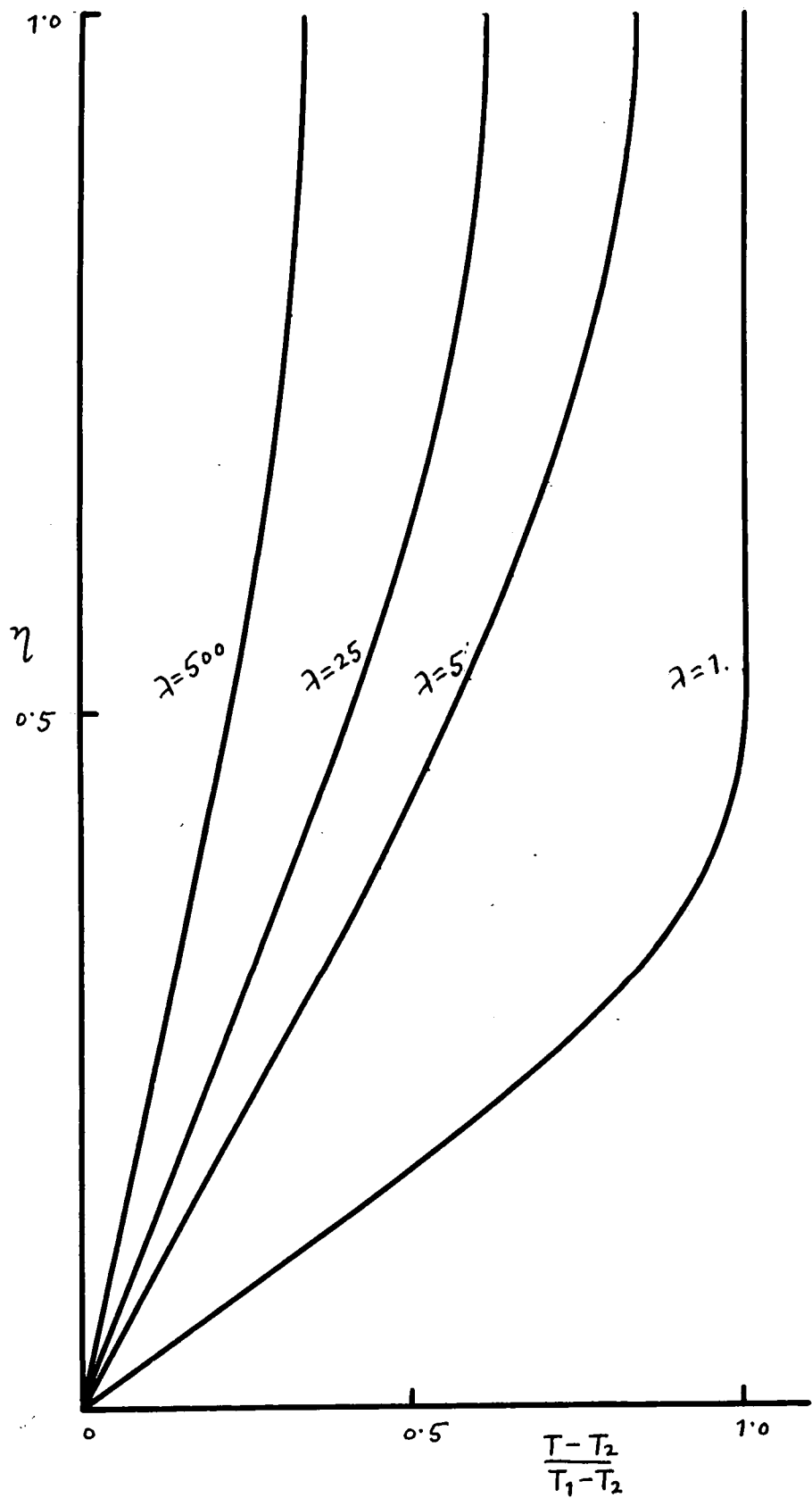
Figure 3.   Temperature distributions for example 2
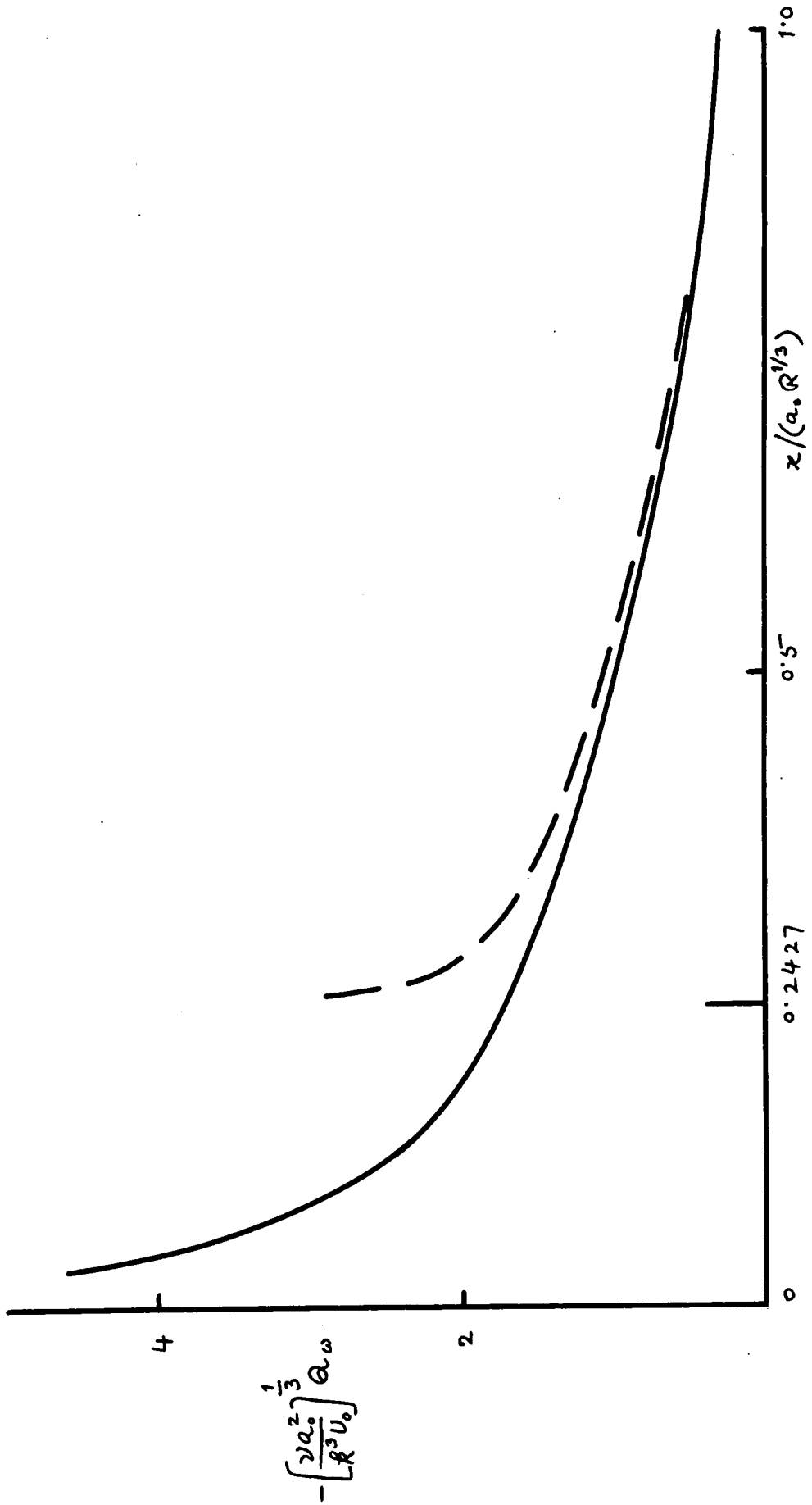   for various $\lambda(x) \geqslant 1$.

Figure 4. The heat transfer across the wall. Example 1 ⸺ , example 2 ⸺ .

The y-axis is labeled $-\left[\dfrac{\nu a_o^2}{R^3 U_o}\right]^{\frac{1}{3}} Q_\omega$ with values 2 and 4 marked.

The x-axis is labeled $x/(a_o R^{1/3})$ with values 0, 0·2427, 0·5, and 1·0 marked.