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INVESTIGATION OF THE THERMAL STRESS IN A THIN PLATE

A thesis submitted for the degree of

Master of Science

of the University of Durham.

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(B.Sc. Bristol)

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ABSTRACT

Investigation of the Thermal Stress in a Thin Plate

When the distribution of temperature in a body is non-uniform, there is a state of thermal stress. The linear, quasi-static, uncoupled theory of thermoelasticity is used to investigate such a state of stress in a thin circular plate subject to purely radial heat flow.

It is shown that the plane-stress hypothesis, although consistent with a two-dimensional treatment, leads to unsatisfactory results when used within the full framework of the three-dimensional theory and an alternative approach is given. The solution is obtained by the superposition of a primary stress system satisfying Saint Venant boundary conditions at the edge of the plate and a suitably chosen secondary (isothermal) stress system.

The analysis of each system is executed using a method based on the asymptotic expansion technique of Reiss and Locke (1961), the small parameter being the thickness/diameter ratio h, of the plate. It is found that a boundary layer effect occurs in the isothermal case in which the significant terms are second order (h^2), adding some further justification to the Saint Venant Principle.

Consideration of the composite solution shows that the two-dimensional solution plays the role of the zeroth order term in the series solution, the higher order terms being in the nature of three-dimensional corrections. These correction terms are of second order and it is concluded that for sufficiently small h the solution is plane stress except in the boundary layer.

The investigation is completed by a discussion of the method in relation to a specific example. The accuracy of the series solution is considered and numerical results given.

ACKNOWLEDGEMENT

I wish to thank Professor Russell Hoyle of the University of Durham both for introducing to me the field of thermoelasticity and for his friendly advice throughout the period in which the work was being done. Also, I wish to thank Miss Joan Youens of Rutherford College of Technology, Newcastle upon Tyne, for her skill and patience in typing the report.

CONTENTS

INTRODUCTION

When a body is subjected to external surface forces, body forces, or heating, in general there results a state of stress. Thermal stress is that which arises in a body because of the presence of a non-uniform temperature distribution, external constraints, or both, although this account is concerned only with the state of stress arising from the first source.

Any theoretical investigation into a physical phenomenon requires a basic mathematical theory and furthermore if meaningful results are to be obtained, the simplifying assumptions, of necessity inherent in such a theory must be understood. The study of thermal stress is no exception and consequently Chapter 1 is devoted to the construction and development of the relevant mathematical equations.

In this respect, the concepts of stress, strain and displacement are introduced and their interrelations determined, within the framework of a linear theory, through the basic laws of mechanics, geometry and material properties.

In particular, the derivation of the Duhamel-Neumann stress-strain relations is of some interest, as it combines a fuller explanation of the origin of thermal stress, with the mathematical consequences of the assumed homogeneity and isotropy of the material.

The chapter is concluded by a formal derivation of the equations of thermoelasticity, in a form suitable for the direct determination of the stress components. The theory is applicable to problems in which inertia forces can be neglected and the temperature of the body can be obtained independently of its deformation.

The project is concerned with the thermal stress in a thin plate, when the temperature does not vary across the plate thickness. Such a problem is usually given, what is essentially, a two-dimensional treatment. The results so obtained may be adequate but, an analysis of the problem, which properly places the two-dimensional solution within the framework of the three-dimensional theory, is desirable.

The full three dimensional equations of thermo-elasticity, consisting as they do of nine partial differential equations, together with appropriate boundary conditions, present a formidable problem for solution. However, by transforming to a cylindrical polar coordinate system and making use of axial symmetry, the equations are much simplified.

Indeed, it is interesting to observe, that the single assumption, that all quantities are independent of the angular coordinate, uncouples the system of equations into two distinct sets, one of which admits the trivial (zero) solution and the other, the remaining stresses. The author believes that this approach is more satisfactory, than that of appealing to the physical consequences of axial symmetry.

Because of the relative simplicity of the axially-symmetric equations and because it is of some practical importance, the thermoelastic problem of a thin circular disc, subjected to purely radial heat flow, is chosen as a vehicle for the investigation.

A tentative approach to the problem, is to use the semi-inverse method, coupled with the assumption, consistent with the two-dimensional theory, of plane-stress (Appendix 1). The results are not satisfactory. For a rigorous solution, the temperature distribution, which in the uncoupled theory is assumed known and thus, at least as far as the stress analysis is concerned, arbitrary, must be restricted to, at most, a parabolic profile. Furthermore, even allowing such a severe restriction, the edge boundary conditions can only be satisfied, for a non-trivial case, in the Saint Venant sense. The result is, that in the immediate locality of the edge, the stress is unknown.

The presence, in the non-dimensional form of the equations, of a small parameter h, the thickness/diameter ratio, suggests an asymptotic expansion technique. Such methods have been widely employed in many different fields of study, a general discussion being given by Friedrichs (1955).

Particularly relevant to the problem under investigation, is the work in isothermal elasticity of Reiss and Locke (1961) and Friedrichs and Dressler (1961) and, more recently, in thermoelasticity, of Laws (1965).

The work of Laws, is in fact, concerned with the thermoelastic problem of a thin plate under steady state conditions. His method, although based on that pioneered by previously mentioned authors, differs from the present investigation, in so far as the steady state problem permits the use of a suitable particular integral of the thermoelastic equations. So far as the present author is aware, the direct determination of thermal stress under arbitrary, and therefore possibly transient, temperature conditions has not been hitherto attempted by asymptotic expansion techniques.

The required stress system is conveniently considered as the superposition of two sub-systems, the primary and secondary stresses. The primary system satisfies all the field equations and the "face" boundary conditions. On the "edge" of the disc, however, the Saint Venant conditions are used. Briefly, each stress component is assumed to have an asymptotic expansion in powers of h. Substituting these expansions into the full set of equations, yields a sequence of systems of differential equations, from which the expansion coefficients are determinable.

The secondary stresses are isothermal and are chosen to make the composite solution satisfy the requirements of a stress free edge. The boundary conditions of this secondary problem, being in the nature of asymptotic series, demand an approach similar to that used in the primary problem. Unfortunately, it is found, that the secondary expansions cannot be made to satisfy the stress free edge conditions and thus, can only represent the isothermal stresses in the interior of the disc.

These difficulties are overcome by introducing coordinates, suitable for what is in effect, a boundary layer analysis. Asymptotic expansions, assumed to be valid within the boundary layer, are introduced and result in a new sequence of systems of differential equations, for the boundary layer expansion coefficients.

The limit space, in which these equations are to be solved, is a semi-infinite strip and the formulation is completed by an appropriate matching of the interior and boundary layer expansions. This procedure is, in fact, responsible for the interdependence of the interior and boundary layer problems and necessitates their simultaneous solution.

The relevant analysis is carried out and it is interesting to note, that the first non-zero contributions to the secondary solution, are second order boundary layer terms, giving additional justification to the Saint Venant Principle. It is of further interest to observe, that the determination of these coefficients, by means of the introduction of a suitable stress function, reduces to the solution of the biharmonic equation, which is of frequent occurrence in two dimensional isothermal elasticity and as such can be conveniently solved by the method of Gaydon and Shepherd (1964), (Appendix 3).

Combination of the primary and secondary stress systems, yields a uniformly valid, composite solution, the series being determined as far as the terms in h^2. Comparison with the two dimensional solution of the problem shows that the latter plays the not unexpected role of the first term in the series, the higher order terms being regarded as three dimensional corrections. These correction terms, are in fact, of order h^2 and it may be concluded therefore, that for sufficiently small h, the two dimensional solution provides an accurate estimate of the true stress.

The investigation is completed by a discussion in Chapter 3 of the method in relation to a specifically chosen temperature field and plate thickness/diameter ratio. The accuracy of the series representation, for a fixed value of h, is ascertained by a comparison with an exact primary stress system and good agreement, to three decimal places, is observed.

Comparison of the two and three dimensional solutions for this example, shows that their differences, of any importance, are confined to the region of steep temperature gradients. In the case of the radial and hoop stresses, the maximum deviations occur near to the edge of the boundary layer, on the faces and mid-section of the disc. It is further observed, that, although the actual numerical corrections may, or may not, be considered important, at some points the relative changes in stress are considerable. The correction terms, as applied to the remaining stresses, occur only in the boundary layer and are generally small, although in the case of the direct stress, there is a local effect at the edge.

CHAPTER I

Thermal Stress Theory

1.1. The Equations of Stress

As this chapter is devoted to discussions of a general theoretical nature, which do not refer to any specific geometry, it is convenient to adopt the notation and techniques of Cartesian tensors in accordance with Jeffreys (1952). The existence of a fixed Cartesian frame of reference $Ox_1x_2x_3$ is therefore presupposed.

We assume that the body under investigation can be represented by the idealised concept of the continuum; that is to say we ignore the atomic structure of matter and regard the body as a continuous medium.

The forces concerned in the mechanics of such a body are assumed to be of two types, those acting on the elements of volume (or mass) of the body, called body forces, and those acting on the surface elements inside or on the boundary of the body, referred to as stress forces.

Consider an arbitrary portion of the body occupying a volume V and bounded by a surface S. Suppose that at an arbitrary point x_i of V the body force per unit volume is F_i in the sense that the resultant body force on V is then

$$\int_V F_i dV \tag{1.1}$$

and that the resultant body force torque about O is

$$\int_V \varepsilon_{ijk}\, x_j\, F_k\, dV \tag{1.2}$$

where ε_{ijk} is the alternate tensor. We assume that in general F_i is a continuously differentiable function of space and time coordinates.

Now suppose that at a general point x_i, of S, the outward normal is n_i , and suppose that $\overset{(n)}{T_i}$ denotes the force per unit area, exerted at x_i by the material outside S on the material inside S. Again the definition is understood in the sense that the resultant stress force on the material within S is

$$\int_S \overset{(n)}{T_i}\, dS \tag{1.3}$$

and that the resultant stress force torque about O is

$$\int_S \varepsilon_{ijk}\, x_j\, \overset{(n)}{T_k}\, dS. \tag{1.4}$$

The superscript (n) refers to the normal n_i, since, in addition to assuming that the stress vector $\overset{(n)}{T_i}$ is a continuously differentiable function of space and time we assume that it depends upon the orientation of the surface elements specified by n_i.

We note further that by definition $\overset{(n)}{T_i}$ is the force per unit area exerted at x_i by the material outside S on that inside. Consequently $\overset{(-n)}{T_i}$ denotes the force per unit area exerted at x_i by the material inside S on that outside. It follows therefore from Newton's third law of motion that

$$\overset{(n)}{T_i} + \overset{(-n)}{T_i} = 0. \qquad (1.5)$$

If the body forces are assumed to include the d'Alembert inertia forces, we may consider the whole body and consequently any part of it to be in a state of mechanical equilibrium. Thus from (1.1), (1.2), (1.3), (1.4) and the laws of mechanics we obtain

$$\int_V F_i \, dV + \int_S \overset{(n)}{T_i} \, dS = 0 \qquad (1.6)$$

and

$$\int_V \varepsilon_{ijk} \, x_j \, F_k \, dV + \int_S \varepsilon_{ijk} \, x_j \, \overset{(n)}{T_k} \, dS = 0 \qquad (1.7)$$

Following Sokolnikoff (1956) pp. 36-39, consider the application of (1.6) when V is a vanishingly small tetrahedron $PABC$ located at an arbitrary point P of the body, (figure (i)).

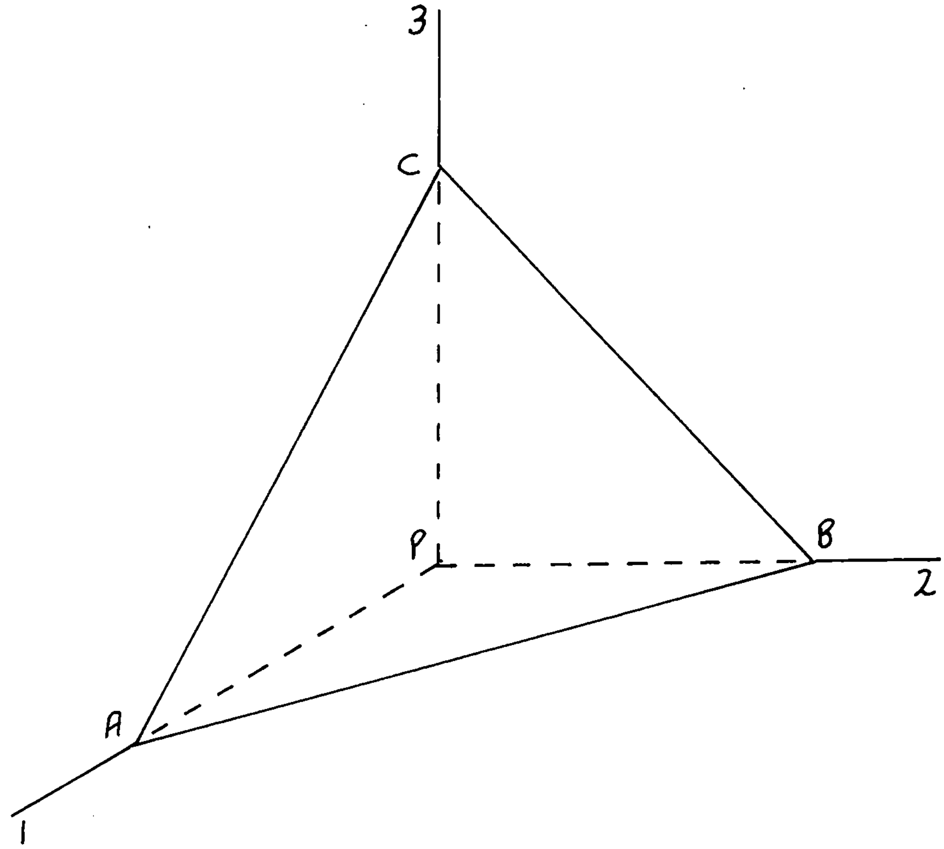

Figure (i)

The faces PAB, PBC, PCA are parallel to the coordinate planes and the face ABC has an outward normal n_i. Suppose further that triangle ABC has area Δ and that the length of the perpendicular from P to this face is p. Applying (1.6) we have

$$\int_V F_i dV + \int_{\Delta ABC} \overset{(n)}{T_i} dS + \int_{\Delta PBC} \overset{(-1)}{T_i} dS + \int_{\Delta PCA} \overset{(-2)}{T_i} dS + \int_{\Delta PAB} \overset{(-3)}{T_i} dS = 0$$

and, using (1.5) get

$$\int_V F_i dV + \int_{\Delta ABC} \overset{(n)}{T_i} dS - \int_{\Delta PBC} \overset{(1)}{T_i} dS - \int_{\Delta PCA} \overset{(2)}{T_i} dS - \int_{\Delta PAB} \overset{(3)}{T_i} dS = 0$$

from which

$$F_i' \frac{1}{3} p\Delta + \overset{(n)}{T_i'} \Delta - \overset{(1)}{T_i'} n_1 \Delta - \overset{(2)}{T_i'} n_2 \Delta - \overset{(3)}{T_i'} n_3 \Delta = 0$$

which primed quantities tend to their values at P as p tends to zero. Dividing through by Δ and letting p become vanishingly small gives

$$\overset{(n)}{T_i} = \overset{(1)}{T_i} n_1 + \overset{(2)}{T_i} n_2 + \overset{(3)}{T_i} n_3. \tag{1.8}$$

Equation (1.8) has been derived for an arbitrary point P and thus holds at all points of the body. If we now denote $\overset{(j)}{T_i}$ by σ_{ji}, (8) becomes

$$\overset{(n)}{T_i} = \sigma_{ji} n_j \tag{1.9}$$

exhibiting the dependence of $\overset{(n)}{T_i}$ on n. It is clear from (1.9) that σ_{ij} is a second rank tensor, knowledge of which gives the state of stress in the body.

Returning now to (1.6), V arbitrary, use of (1.9) gives

$$\int_V F_i dV + \int_S \sigma_{ji} n_j \, dS = 0$$

Transforming the surface integral by the divergence theorem we obtain

$$\int_V \left[F_i + \sigma_{ji,j} \right] dV = 0$$

whence, since V is arbitrary

$$\sigma_{ji,j} + F_i = 0 \qquad (1.10)$$

Also from (1.7) and (1.9)

$$\int_V \varepsilon_{ijk} x_j F_k \, dV + \int_S \varepsilon_{ijk} x_j \sigma_{mk} n_m \, dS = 0.$$

Use of the divergence theorem and (1.10) gives

$$\int_V \left[- \varepsilon_{ijk} x_j \sigma_{mk,m} + (\varepsilon_{ijk} x_j \sigma_{mk})_{,m} \right] dV = 0$$

whence

$$\int_V \left[-\varepsilon_{ijk} x_j \sigma_{mk,m} + \varepsilon_{ijk} \delta_{jm} \sigma_{mk} + \varepsilon_{ijk} x_j \sigma_{mk,m} \right] dV = 0$$

where δ_{jm} is the substitution tensor. Simplifying, we get

$$\int_V \varepsilon_{ijk} \, \sigma_{jk} \, dV = 0$$

Since V is arbitrary,

$$\varepsilon_{ijk} \, \sigma_{jk} = 0$$

from which it easily follows that

$$\sigma_{jk} = \sigma_{kj} \tag{1.11}$$

and the stress tensor is symmetric.

The stress equations (1.10), valid throughout the body, are normally referred to as the "equations of equilibrium". They are clearly insufficient for the determination of the σ_{ij}. We remark further that the results of the discussion so far, are applicable to any state of stress whether caused by the presence of body or external surface forces or thermal effects.

1.2. The Equations of Deformation

Another assumption, basic to the theory, is that the body is deformable, that is, material points may be displaced in such a way that their relative distance apart is altered, contrary to the concept of a rigid body in which this distance is assumed invariant.

Let us suppose that there is an initial undeformed state in which a material point P is at x_i (fig. (ii)). As a consequence of some external agency the body is deformed so that at some instant t, the point P is now at P' given by $x_i + u_i$. The displacement vector u_i is assumed to be a continuously differentiable function of x_i and t.

Consider now a material point Q, which in the undeformed state is at $x_i + y_i$ where y_i is small. After deformation Q moves to Q', the point $x_i + y_i + u_i|Q$. By Taylor's theorem

$$u_i|_Q = u_i|_P + u_{i,j}|_P \, y_j \tag{1.12}$$

Higher order terms are neglected since y_i is small. We may thus obtain

$$(\vec{PQ})_i = y_i \tag{1.13}$$

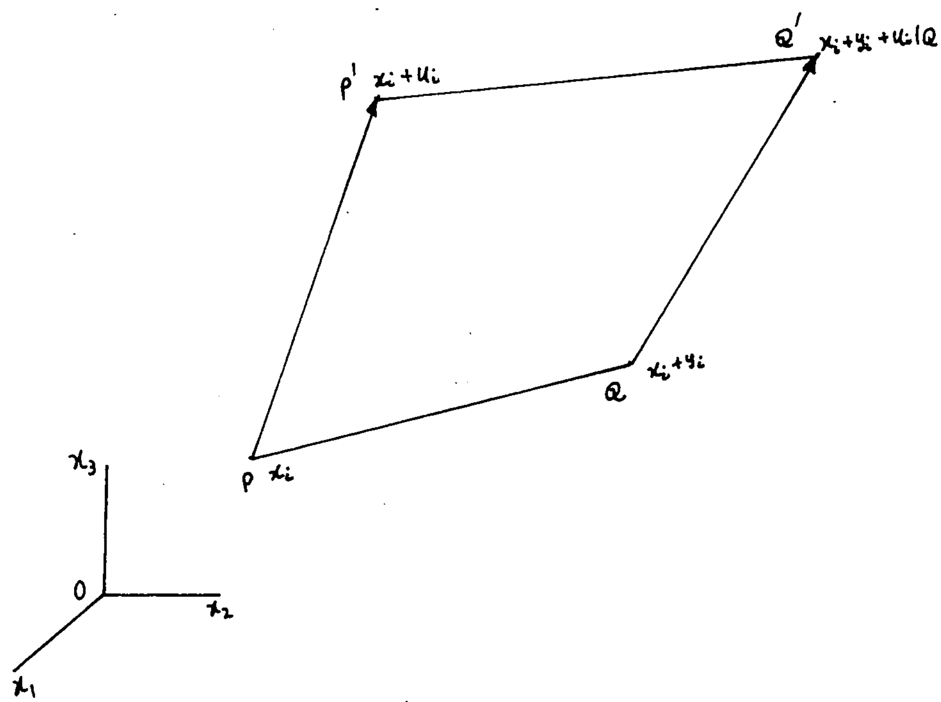

Figure (ii)

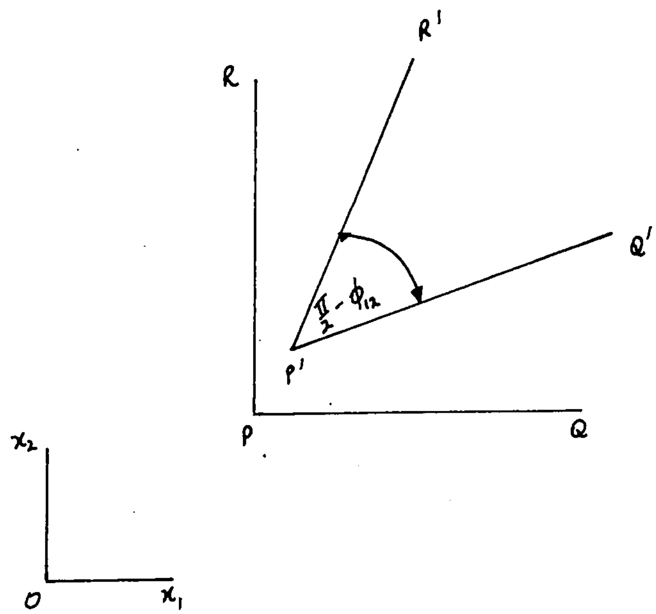

Figure (iii)

and

$$(\vec{P'Q'})_i = y_i + u_{i,j}\, y_j \tag{1.14}$$

the extra suffix P having been dropped.

From (1.14)

$$(P'Q')^2 = y_i y_i + 2\, u_{i,j}\, y_i\, y_j + u_{i,j}\, u_{i,k}\, y_j\, y_k \tag{1.15}$$

We further make the assumption that the displacement u_i and its space derivatives are sufficiently small to warrant the neglect of second order terms involving these quantities, and thus (1.15) becomes

$$(P'Q')^2 = y_i y_i + 2\, u_{i,j}\, y_i\, y_j \tag{1.16}$$

Further we observe that

$$u_{i,j}\, y_i\, y_j = u_{j,i}\, y_i\, y_j$$

summation occurring over both i and j and thus (1.16) may be written

$$(P'Q')^2 = y_i y_i + (u_{i,j} + u_{j,i}) y_i\, y_j. \tag{1.17}$$

Using (1.13), (1.17) we obtain

$$(P'Q')^2 - (PQ)^2 = (u_{i,j} + u_{j,i})\; y_i\, y_j$$

or

$$(P'Q')^2 - (PQ)^2 = 2\varepsilon_{ij}\, y_i\, y_i \tag{1.18}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{1.19}$$

$(P'Q')^2 - (PQ)^2$ is a measure of the change in the relative distance between the material points P and Q due to the deformation. Consequently we assert that the deformation of the body in the neighbourhood of an arbitrary point P initially at x_i is characterised by the symmetric second rank tensor ε_{ij}, referred to as the strain or deformation tensor. We observe that if $\varepsilon_{ij} = 0$ then from (1.18), $P'Q' = PQ$ and there is no deformation.

We now proceed to show the way in which the strain tensor describes the deformation. In the notation used, let $\vec{PQ}$ be initially parallel to the x_1-axis (fig. (iii)). Thus we may write

$$y_i = y_1 \text{ if } i = 1$$
$$= 0 \text{ if } i = 2, 3.$$

From (1.13), 1.14) we obtain, in extended notation

$$\vec{PQ} = \{y_1, 0, 0\} \tag{1.20}$$

and

$$\vec{P'Q'} = \left\{(1 + \frac{\partial u_1}{\partial x_1})\, y_1,\ \frac{\partial u_2}{\partial x_1}\, y_1,\ \frac{\partial u_2}{\partial x_1}\, y_1\right\} \tag{1.21}$$

From (1.21), to appropriate order,

$$(P'Q')^2 = \vec{P'Q'}.\vec{P'Q'} = (1 + 2\,\frac{\partial u_1}{\partial x_1})\, y_1{}^2$$

and hence from (1.19), (1.20)

$$P'Q' = (1 + 2\,\varepsilon_{11})^{\frac{1}{2}}\,PQ \qquad (1.22)$$

$$= (1 + \varepsilon_{11})\,PQ \text{ since } \varepsilon_{11} \text{ small.}$$

Thus we have

$$\varepsilon_{11} = \frac{P'Q' - PQ}{PQ}$$

and represents, therefore the increase in length per unit length of the line element $\vec{PQ}$ initially parallel to the x_1-axis. ε_{11} is called the direct strain associated with the direction Ox_1. We may obtain similar interpretations for the direct strains ε_{22}, ε_{33}.

Suppose now that $\vec{PR}$ is a line element initially parallel to x_2-axis (fig. (iii)), so that in (1.13), (1.14) we may take

$$y_i = 1 \text{ if } i = 2$$

$$= 0 \text{ if } i = 1, 3$$

and hence obtain

$$\vec{PR} = \{0,\ y_2,\ 0\} \qquad (1.23)$$

$$\vec{P'R'} = \left\{\frac{\partial u_1}{\partial x_2}\,y_2,\ \left(1 + \frac{\partial u_2}{\partial x_2}\right)y_2,\ \frac{\partial u_3}{\partial x_2}\,y_2\right\} \qquad (1.24)$$

Proceeding as before we use (1.24) to get

$$P'R' = (1 + 2\,\varepsilon_{22})^{\frac{1}{2}}\,PR \qquad (1.25)$$

and (1.24), (1.21) to get

$$\vec{P'Q'}.\vec{P'R'} = \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) y_1 \; y_2$$

$$= 2\,\varepsilon_{12}\,(PQ)(PR) \quad \text{from (1.19), (1.20), (1.23).}$$

But,

$$\vec{P'Q'}.\vec{P'R'} = (P'Q')(P'R') \sin \phi_{12}$$

where ϕ_{12} is the change in the original right angle between $\vec{PQ}$ and $\vec{PR}$, and hence

$$2\,\varepsilon_{12}\,(PQ)(PR) = (P'Q')(P'R') \sin \phi_{12}.$$

Use of (1.20), (1.22), (1.23), (1.25) now gives

$$\varepsilon_{12} = \frac{1}{2}\,(1 + \varepsilon_{11})^{\frac{1}{2}}\,(1 + \varepsilon_{22})^{\frac{1}{2}} \sin \phi_{12}$$

$$= \frac{1}{2}\,\phi_{12} \quad \text{to appropriate order.}$$

Thus ε_{12} denotes half of the change in the right angle between the line elements $\vec{PQ}$, $\vec{PR}$ initially parallel to the axes Ox_1, Ox_2 respectively and is called the shear strain associated with those directions. A similar interpretation is possible for the shear strains ε_{23}, ε_{31}.

Equations (1.19) constitute the equations of deformation and are usually referred to as the "strain - displacement equations".

1.3 The Equations of the Material

Let us suppose that there is an initial state of the body in which the absolute temperature is uniform and equal to T_o and in which, subject to there being no body or external surface forces, the body is undeformed and stress free.

If the body is now subjected to the action of body, or external surface forces, or heating, under which it undergoes a rise in temperature $T(x_i, t)$ in general there results a state of stress throughout the body. That stresses arise due to the action of body or surface forces is clear but the origin of the thermal stress (that due to the heating effect) perhaps is worthy of some explanation.

Consider a body at uniform temperature T_o, whose bounding surface is free of external constraint. If this body is now heated to a new uniform temperature $T_o + T$, free expansion takes place by an amount proportional to T. On the other hand, if the bounding surface is geometrically fixed by the action of external constraints, the thermal expansion must be suppressed by the action of external surface (constraint) forces, and thus stresses arise.

Suppose now that the temperature T is non-uniform. By virtue of this non-uniformity the volume elements of the body, under free expansion conditions, would each expand by a different amount (proportional to T). Clearly, if the body is to remain continuous, such free expansions cannot occur and in fact are suppressed by constraint forces set up on each element by its neighbours. In accordance with the previous case, therefore, stresses arise.

We are thus led to assert that thermal stresses may arise in a heated body, either because of a non-uniform temperature distribution, the presence of external constraints or indeed to a combination of both. It is our intention, however, to confine our investigation to those thermal stresses caused by non-uniform temperature effects only, a state characterised by a boundary free of constraint.

Before proceeding further it is necessary to make the following remarks (Muskhelishvili (1953) p.54). When stating that the strain tensor ε_{ij} is a function of the space coordinates x_i, these refer to the geometrical position of the material point P in question, before deformation. The same remark is true of the displacement vector, u_i. On the other hand, when we state that the stress tensor σ_{ij} is a function of $\bar{x}_i$ we refer to coordinates after deformation. Within the framework of our theory, however, this distinction is not essential. For, suppose that the coordinates of P before deformation are x_i whilst those after are x_i'. Since $x_i' = x_i + u_i$ and by assumption u_i is small, clearly the value of σ_{ij} at x_i can differ but little from its value at x_i'. Consequently we shall consider at all times that all three functions σ_{ij}, ε_{ij}, u_i are functions of x_i the coordinates of an arbitrary point P before deformation occurs.

To continue with the theory we assume that the state of stress in the body depends upon its state of deformation and on the change in temperature from the initial state. Mathematically we express this assumption

$$\sigma_{ij} = \sigma_{ij}\,(\varepsilon_{ij}, T). \tag{1.26}$$

We recall that the strains ε_{ij} are small, and if we also assume that T is sufficiently small the functional relation (26) may be taken as linear, and accordingly we obtain a tensorial equation of the form

$$\sigma_{ij} = a_{ij} + b_{ijkl}\,\varepsilon_{kl} + c_{ij}\,T \tag{1.27}$$

Since we require that a possible state is such that

$$\sigma_{ij} = 0,\ \varepsilon_{ij} = 0,\ T = 0$$

clearly $a_{ij} = 0$ and (1.27) becomes

$$\sigma_{ij} = b_{ijkl}\, \varepsilon_{kl} + c_{ij}\, T \tag{1.28}$$

We remark that inherent in (1.28) is the fact that the original state may be recovered under suitable conditions, defining in fact the concept of elasticity. We further remark that the tensors b_{ijkl}, c_{ij} depend upon the physical properties of the material.

We now make the assumption that the material is isotropic and homogeneous, by which we mean that the material properties are independent of the orientation and position of the body elements. This is not an unreasonable assumption since many materials, due to their crystal structure (Long (1961) pp. 61-65), may, to sufficient approximation, exhibit such properties. The mathematical consequences of isotropy and homogeneity are that the tensors b_{ijkl}, c_{ij} assume (Jeffreys (1952) Ch. VII) the general forms

$$b_{ijkl} = K\, \delta_{ij}\, \delta_{kl} + L(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}) + M(\delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk})$$

and

$$c_{ij} = \delta_{ij}$$

where K, L, M, N are scalar constants.

Substitution into (1.28) gives

$$\sigma_{ij} = \{K\ \delta_{ij}\delta_{kl} + L(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + M(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\}\varepsilon_{kl} + N\delta_{ij}T$$

$$= K\ \delta_{ij}\ \varepsilon_{kk} + 2L\ \varepsilon_{ij} + N\ \delta_{ij}\ T \qquad (1.29)$$

The constants K, L, N can be renamed in terms of the Lamé λ, μ and the coefficient of linear thermal expansion α, as follows:

$$K = \lambda,\ L = \mu,\ N = -(3\lambda + 2\mu)\alpha.$$

(1.29) now becomes

$$\sigma_{ij} = \lambda\ \delta_{ij}\ \varepsilon_{kk} + 2\mu\ \varepsilon_{ij} - (3\lambda + 2\mu)\ \delta_{ij}\ \alpha T. \qquad (1.30)$$

Equations (1.30) are the so called Duhamel-Neumann relations.

Contraction on i, j of (1.30) gives

$$\sigma_{kk} = (3\lambda + 2\mu)\ \varepsilon_{kk} - 3(3\lambda + 2\mu)\ \alpha T$$

giving

$$\varepsilon_{kk} = \frac{1}{(3\lambda + 2\mu)}\ \sigma_{kk} - 3\alpha T. \qquad (1.31)$$

Using (1.30) and (1.31) we may obtain

$$\varepsilon_{ij} = \frac{1}{2\mu}\ \sigma_{ij} - \frac{\lambda\sigma_{kk}}{2\mu(3\lambda+2\mu)}\ \delta_{ij} + \alpha T\ \delta_{ij} \qquad (1.32)$$

Introducing Young's modulus E, and Poisson's ratio ν by the relations (Sokolnikoff (1956) p. 67),

$$E = \frac{\mu(3\lambda+2\mu)}{(\lambda + \mu)}\ , \quad \nu = \frac{\lambda}{2(\lambda + \mu)}$$

from which

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)}$$

we may write (1.32) as

$$\varepsilon_{ij} = \frac{(1+\nu)}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha T \delta_{ij}, \tag{1.33}$$

Equations (1.30), (1.33) can be obtained alternatively using the assumption that the pure thermal strains due to free expansion and the isothermal strains associated with stress are additive (Nowacki (1962) pp. 4-5), or from thermodynamic considerations (Benham and Hoyle (1964), ch. 7).

1.4 The Equations of Thermoelasticity

The final assumptions applicable to the theory concern the temperature field T. A precise derivation (Benham and Hoyle (1964), Ch. 7) of the heat conduction equation (from which the temperature of a solid is generally found) shows the presence of a strain term in that equation. However (Boley and Weiner (1962) Ch. 2), if the time rate of change of temperature is sufficiently small, this coupling term, and indeed the inertia forces in equations (1.10), can be neglected. We shall assume that this is the case. The temperature field can thus be determined independently of the stress problem and for our purposes is regarded, at any instant of time, as a known function of the space coordinates.

Before proceeding further, it is convenient to summarize. The principal assumptions basic to the theory:

(i) the body is an elastically deformable continuum having isotropic and homogeneous material properties,

(ii) the deformation and temperature change are sufficiently small for a linear theory, and

(iii) the time rate of change of temperature is slow enough to warrant the neglect of inertia forces and to consider the temperature field a known quantity.

In addition, since we are interested only in thermal stress, originating from a non-uniform temperature distribution, we require zero body and external surface force. Accordingly the equations governing an investigation of this kind are:

Equilibrium equations: (from (1.10))

$$\sigma_{ij,j} = 0 \tag{1.34}$$

Strain-Displacement equations:

$$\varepsilon_{ij} = \tfrac{1}{2} (u_{i,j} + u_{j,i}) \tag{1.19}$$

Stress-Strain-Temperature Equations:

$$\varepsilon_{ij} = \frac{(1+\nu)}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} + \alpha T \delta_{ij} \tag{1.33}$$

Boundary equations: (from (1.9))

$$\overset{(n)}{T_i} = \sigma_{ij} n_j = 0 \tag{1.35}$$

We remark that time derivatives do not enter into any of these equations. It follows that the time t enters into problems of this kind only as a parameter through the term involving T, and we shall therefore disregard it and consider all quantities to be functions only of the space coordinates x_i. The term quasi-static is often used to describe such a situation.

In so far as equations (1.34), (1.19), (1.33) are fifteen equations in fifteen unknowns they are complete, but as we are primarily interested in stress, and in view of (1.35) we find it convenient to eliminate the strains and displacement. This we now do and formally derive the complete stress equations for the problem.

From (1.19)

$$\varepsilon_{ij,kl} = \tfrac{1}{2}\,(u_{i,jkl} + u_{j,ikl})$$

and interchanging i and k, j and l we get

$$\varepsilon_{kl,ij} = \tfrac{1}{2}\,(u_{k,lij} + u_{l,kij}).$$

Adding results in

$$\begin{aligned}\varepsilon_{ij,kl} + \varepsilon_{kl,ij} &= \tfrac{1}{2}\,(u_{i,jkl} + u_{j,ikl} + u_{k,lij} + u_{l,kij}) \\ &= \tfrac{1}{2}\,(u_{i,kjl} + u_{k,ijl} + u_{j,lik} + u_{l,jik}) \\ &= \tfrac{1}{2}\,(u_{i,kjl} + u_{k,ijl}) + \tfrac{1}{2}(u_{j,lik} + u_{l,jik})\end{aligned}$$

and hence

$$\varepsilon_{ij,kl} + \varepsilon_{kl,iy} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$$

or

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \qquad (1.36)$$

Substitute for the ε_{ij} from (1.33) to get

$$\frac{(1+\nu)}{E}\left[\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik}\right]$$

$$-\frac{\nu}{E}\left[\delta_{ij}\,\sigma_{ss,kl} + \delta_{kl}\,\sigma_{ss,ij} - \delta_{ik}\,\sigma_{ss,jl} - \delta_{jl}\,\sigma_{ss,ik}\right]$$

$$+ \alpha \left[\delta_{ij} T_{,kl} + \delta_{kl} T_{,ij} - \delta_{ik} T_{,jl} - \delta_{jl} T_{,ik} \right] = 0$$

Contract on k, l and obtain

$$\frac{(1+\nu)}{E} \left[\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} \right]$$

$$- \frac{\nu}{E} \left[\delta_{ij} \sigma_{ss,kk} + 3 \sigma_{ss,ij} - \delta_{ik} \sigma_{ss,jk} - \delta_{jk} - \delta_{jk} \sigma_{ss,ik} \right]$$

$$+ \alpha \left[\delta_{ij} T_{,kk} + 3 T_{,ij} - \delta_{ik} T_{,jk} - \delta_{jk} T_{,ik} \right] = 0 \quad (1.37)$$

From (1.34) $\sigma_{ik,kj} = \sigma_{jk,ki} = 0$. Using this result (1.37) simplifies to

$$\sigma_{ij,kk} + \frac{1}{(1+\nu)} \left[\sigma_{ss} + E\alpha T \right]_{,ij} + \frac{1}{(1+\nu)} \delta_{ij} \left[E\alpha T - \nu \sigma_{ss} \right]_{,kk} = 0 \quad (1.38)$$

Contraction on i,j gives

$$(1-\nu) \sigma_{ss,kk} + 2 E\alpha T_{,kk} = 0 \qquad (1.39)$$

Combination of (1.38), (1.39) gives finally

$$\sigma_{ij,kk} + \frac{1}{(1+\nu)} \left[\sigma_{ss} + E\alpha T \right]_{,ij} + \delta_{ij} \frac{E\alpha}{(1-\nu)} T_{,kk} = 0 \qquad (1.40)$$

Equations (1.40) are the generalized Beltrami-Michell equations, or compatibility equations of thermal stress.

The stress components σ_{ij} must thus satisfy the equations (1.34), (1.40) and (1.35) and it can be shown by a more rigorous treatment (Boley and Weiner (1962) pp. 62-66 and 84-92) that this formulation admits a unique solution.

For future reference we record here the complete equations in extended notation.

Equilibrium:

$$\frac{\partial}{\partial x_1}\sigma_{11} + \frac{\partial}{\partial x_2}\sigma_{12} + \frac{\partial}{\partial x_3}\sigma_{13} = 0$$

$$\frac{\partial}{\partial x_1}\sigma_{12} + \frac{\partial}{\partial x_2}\sigma_{22} + \frac{\partial}{\partial x_3}\sigma_{23} = 0 \qquad (1.41)$$

$$\frac{\partial}{\partial x_1}\sigma_{13} + \frac{\partial}{\partial x_2}\sigma_{23} + \frac{\partial}{\partial x_3}\sigma_{33} = 0$$

Compatibility:

$$\nabla^2\sigma_{11} + \frac{\partial^2\Sigma}{\partial x_1^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0$$

$$\nabla^2\sigma_{22} + \frac{\partial^2\Sigma}{\partial x_2^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0$$

$$\nabla^2\sigma_{33} + \frac{\partial^2\Sigma}{\partial x_3^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0$$

$$\nabla^2\sigma_{12} + \frac{\partial^2\Sigma}{\partial x_1\partial x_2} = 0$$

$$\nabla^2\sigma_{23} + \frac{\partial^2\Sigma}{\partial x_2\partial x_3} = 0 \qquad (1.42)$$

$$\nabla^2\sigma_{13} + \frac{\partial^2\Sigma}{\partial x_1\partial x_3} = 0$$

where

$$(1+\nu)\,\Sigma = \sigma_{11} + \sigma_{22} + \sigma_{33} + E\alpha T$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}$$

Boundary:

$$\overset{(n)}{T_1} = \sigma_{11}\, n_1 + \sigma_{12}\, n_2 + \sigma_{13}\, n_3 = 0$$

$$\overset{(n)}{T_2} = \sigma_{12}\, n_1 + \sigma_{22}\, n_2 + \sigma_{23}\, n_3 = 0 \qquad (1.43)$$

$$\overset{(n)}{T_3} = \sigma_{13}\, n_1 + \sigma_{23}\, n_2 + \sigma_{33}\, n_3 = 0.$$

CHAPTER II

The Problem and Solution

2.1. Preliminary Remarks

In this chapter, the full three-dimensional thermoelastic theory is used to determine the stress in a thin circular disc, of uniform thickness, when the heat flow is purely radial.

The state of stress in the disc is considered to be known once the stress vector has been calculated at all points. Furthermore, we have already seen that this vector, by virtue of equations (1.9), is expressible in terms of the components of the stress tensor, and these, in theory at any rate, can be obtained by solving the system of equations (1.41), (1.42) and (1.43).

Successful solutions of such a system of partial differential equations depend to a large extent on the choice of coordinate system, and for the problem of the disc, cylindrical polar coordinates are clearly more suitable than Cartesian coordinates. Consequently, it is necessary to introduce a more general notation for the components of stress, amenable to analysis using curvilinear coordinates. This notation naturally, must include the case of Cartesian coordinates so that the results of previous sections may be applied.

2.2. Generalized Stress Components

Since $\overset{(n)}{\mathbf{T}}$ is a (stress) vector, we may resolve it in any direction m given by the unit vector $\mathbf{m}$. This resolute σ_{nm} is given by

$$\sigma_{nm} = \overset{(n)}{\mathbf{T}} \cdot \mathbf{m} \tag{2.1}$$

In this notation, a modified version of that used, for example, by Godfrey (1959), the first suffix n refers to the outward normal, specified by the unit vector $\mathbf{n}$, of the surface element and the second, m, to the direction along which the stress vector is resolved. The component of stress σ_{nn} (repeated suffices do not imply summation) is called the direct stress associated with the direction n, a positive value denoting tensile stress. Further if $\mathbf{n}$, $\mathbf{m}$ are orthogonal, the component σ_{nm} is referred to as the shear stress associated with these directions.

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote unit vectors in the directions of Ox, Oy, Oz of a Cartesian frame of reference we have from (2.1)

$$\overset{(n)}{\mathbf{T}} = \sigma_{nx}\,\mathbf{i} + \sigma_{ny}\,\mathbf{j} + \sigma_{nz}\,\mathbf{k} \tag{2.2}$$

whence

$$\overset{(x)}{T} = \sigma_{xx}\,\mathrm{i} + \sigma_{xy}\,\mathrm{j} + \sigma_{xz}\,\mathrm{k}$$
$$\overset{(y)}{T} = \sigma_{yx}\,\mathrm{i} + \sigma_{yy}\,\mathrm{j} + \sigma_{yz}\,\mathrm{k} \qquad (2.3)$$
$$\overset{(z)}{T} = \sigma_{zx}\,\mathrm{i} + \sigma_{zy}\,\mathrm{j} + \sigma_{zz}\,\mathrm{k}$$

Comparison of (2.3) and the definition of σ_{ij} in section 1.1, yields

$$\sigma_{xx} = \sigma_{11}\ ,\ \sigma_{yy} = \sigma_{22}\ ,\ \sigma_{xx} = \sigma_{33}$$

$$\sigma_{xy} = \sigma_{12} = \sigma_{21} = \sigma_{yx},\ \sigma_{yz} = \sigma_{23} = \sigma_{32} = \sigma_{zy},\ \sigma_{zx} = \sigma_{31} = \sigma_{13} = \sigma_{xz}.$$

Thus equations (1.41, (1.42) and (1.43) for the stress problem, may be written

$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} + \frac{\partial}{\partial z}\sigma_{xz} = 0$$

$$\frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\sigma_{yz} = 0 \qquad (2.4\ a,\ b,\ c)$$

$$\frac{\partial}{\partial x}\sigma_{xz} + \frac{\partial}{\partial y}\sigma_{yz} + \frac{\partial}{\partial z}\sigma_{zz} = 0$$

$$\nabla^2\sigma_{xx} + \frac{\partial^2\Sigma}{\partial x^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0$$

$$\nabla^2\sigma_{yy} + \frac{\partial^2\Sigma}{\partial y^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0$$

$$\nabla^2\sigma_{zz} + \frac{\partial^2\Sigma}{\partial z^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0$$

$$\nabla^2\sigma_{xy} + \frac{\partial^2\Sigma}{\partial x\partial y} = 0 \qquad (2.5\ a\text{-}f)$$

$$\nabla^2 \sigma_{yz} + \frac{\partial^2 \Sigma}{\partial y \partial z} = 0$$

$$\nabla^2 \sigma_{xz} + \frac{\partial^2 \Sigma}{\partial x \partial z} = 0$$

where $(1 + \nu)\,\Sigma = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} + E\alpha T$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$,

and with

$$\overset{(n)}{\mathrm{T}} = 0 \text{ on the boundary.} \tag{2.6}$$

We remark that the cylindrical polar form of (2.4), (2.5) and (2.6) will be derived in due course.

If

$$\mathrm{n} = n_1\,\mathrm{i} + n_2\,\mathrm{j} + n_3\,\mathrm{k}, \tag{2.7}$$

then equations (1.9) give

$$\begin{aligned} \sigma_{nx} &= \sigma_{xx}\, n_1 + \sigma_{xy}\, n_2 + \sigma_{xz}\, n_3 \\ \sigma_{ny} &= \sigma_{xy}\, n_1 + \sigma_{yy}\, n_2 + \sigma_{yz}\, n_3 \\ \sigma_{nz} &= \sigma_{xz}\, n_1 + \sigma_{yz}\, n_2 + \sigma_{zx}\, n_3 . \end{aligned} \tag{2.8}$$

Also, if

$$\mathrm{m} = m_1\,\mathrm{i} + m_2\,\mathrm{j} + m_3\,\mathrm{k}, \tag{2.9}$$

from (2.1), (2.2)

$$\sigma_{nm} = \overset{(n)}{\mathrm{T}}.\mathrm{m} = \sigma_{nx}\, m_1 + \sigma_{ny}\, m_2 + \sigma_{nz}\, m_3$$

and thus using (2.8) we obtain

$$\sigma_{nm} = (\sigma_{xx}n_1+\sigma_{xy}n_2+\sigma_{xz}n_3)m_1+(\sigma_{xy}n_1+\sigma_{yy}n_2+\sigma_{yz}n_3)m_2+(\sigma_{xz}n_1+\sigma_{yz}n_2+\sigma_{zz}n_3)m_3$$

or

$$\sigma_{nm} = n_1m_1\sigma_{xx}+n_2m_2\sigma_{yy}+n_3m_3\sigma_{zz}$$
$$+(m_1n_2+m_2n_1)\sigma_{xy}+(m_2n_3+m_3n_2)\sigma_{yz}+(m_3n_1+m_1n_3)\sigma_{xz} \qquad (2.10)$$

Thus from (2.10) we may obtain the generalised stress component σ_{nm} in terms of the six Cartesian stress components.

2.3. The Thermoelastic Equations in Cylindrical Polar Coordinates

Let us turn now to the specific task of transforming equations (2.4), (2.5) and (2.6) into cylindrical polar coordinates r, θ, z.

The coordinate transformation is effected (fig (iv)) by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z. \qquad (2.11\ a,\ b,\ c)$$

Differentiating (2.11 a, b) each with respect to x, y gives

$$1 = \cos\theta\frac{\partial r}{\partial x} - r\sin\theta\frac{\partial\theta}{\partial x},\ 0 = \sin\theta\frac{\partial r}{\partial x} + r\cos\theta\frac{\partial\theta}{\partial x}$$

$$0 = \cos\theta\frac{\partial r}{\partial y} - r\sin\theta\frac{\partial\theta}{\partial y},\ 1 = \sin\theta\frac{\partial r}{\partial y} + r\cos\theta\frac{\partial\theta}{\partial y}$$

from which we obtain

$$\frac{\partial r}{\partial x} = \cos\theta,\ \frac{\partial r}{\partial y} = \sin\theta,\ \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r},\ \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}.$$

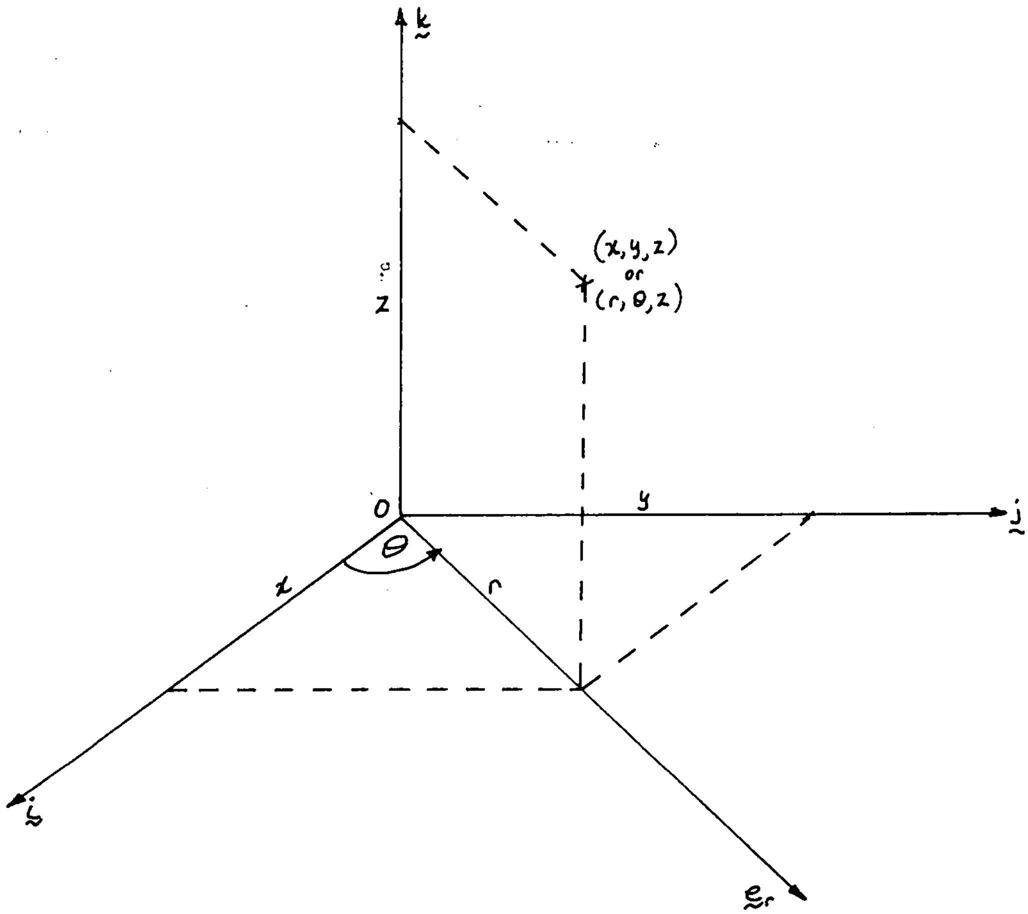

figure (iv)

We may now write

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\,,\quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta},\quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \qquad (2.12\ a,b,c)$$

and thence obtain

$$\frac{\partial^2}{\partial x^2} = \frac{1}{2}\cos 2\theta\left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] - \sin 2\theta\left[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\right]$$

$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$$

$$\frac{\partial^2}{\partial y^2} = -\frac{1}{2}\cos 2\theta\left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right] + \sin 2\theta\left[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\right]$$

$$+ \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$$

$$\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial^2}{\partial x\partial y} = \frac{1}{2}\sin 2\theta\left[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial\theta}\right] + \cos 2\theta\left[\frac{1}{r}\frac{\partial}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial}{\partial\theta}\right]$$

$$\frac{\partial^2}{\partial y\partial z} = \sin\theta\frac{\partial^2}{\partial r\partial z} = \frac{\cos\theta}{r}\frac{\partial^2}{\partial\theta\partial z} \qquad (2.13\ a - g)$$

$$\frac{\partial^2}{\partial x\partial z} = \cos\theta\frac{\partial^2}{\partial r\partial z} - \frac{\sin\theta}{r}\frac{\partial^2}{\partial\theta\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

Equations (2.12), (2.13) are the transformationformulae for all operators in (2.4), (2.5).

Turning now to the stress components, the unit vectors in the directions parallel to the r, θ, z directions are given by (fig (iv)).

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} \qquad (2.14)$$
$$\mathbf{k} = \mathbf{k}$$

We use (2.14) together with (2.7), (2.9), and (2.10) to get

$$\sigma_{rr} = \frac{1}{2}\cos 2\theta\,(\sigma_{xx} - \sigma_{yy}) + \sin 2\theta\,\sigma_{xy} + \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$
$$\sigma_{\theta\theta} = -\frac{1}{2}\cos 2\theta\,(\sigma_{xx} - \sigma_{yy}) - \sin 2\theta\,\sigma_{xy} + \frac{1}{2}(\sigma_{xx} + \sigma_{yy})$$
$$\sigma_{zz} = \sigma_{zz}$$
$$\sigma_{r\theta} = -\frac{1}{2}\sin 2\theta\,(\sigma_{xx} - \sigma_{yy}) + \cos 2\theta\,\sigma_{xy} \qquad (2.15\ a - f)$$
$$\sigma_{\theta z} = \cos\theta\,\sigma_{yz} - \sin\theta\,\sigma_{xz}$$
$$\sigma_{rz} = \sin\theta\,\sigma_{yz} + \cos\theta\,\sigma_{xz}$$

A simple inversion of equations (2.15) now gives

$$\sigma_{xx} = \frac{1}{2}\cos 2\theta\,(\sigma_{rr} - \sigma_{\theta\theta}) - \sin 2\theta\,\sigma_{r\theta} + \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})$$
$$\sigma_{yy} = -\frac{1}{2}\cos 2\theta\,(\sigma_{rr} - \sigma_{\theta\theta}) + \sin 2\theta\,\sigma_{r\theta} + \frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta})$$
$$\sigma_{zz} = \sigma_{zz}$$
$$\sigma_{xy} = \frac{1}{2}\sin 2\theta\,(\sigma_{rr} - \sigma_{\theta\theta}) + \cos 2\theta\,\sigma_{r\theta} \qquad (2.16\ a - f)$$
$$\sigma_{yz} = \cos\theta\,\sigma_{\theta z} + \sin\theta\,\sigma_{rz}$$
$$\sigma_{xz} = -\sin\theta\,\sigma_{\theta z} + \cos\theta\,\sigma_{rz}$$

We now apply the results of (2.12), (2.13), (2.16) to effect the transformation of (2.4), (2.5).

(2.4a), (2.4b) become

$$\cos\theta\left[\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz}\right]$$

$$- \sin\theta\left[\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta} + \frac{2}{r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z}\right] = 0$$

and

$$\sin\theta\left[\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz}\right]$$

$$+ \cos\theta\left[\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta} + \frac{2}{r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z}\right] = 0$$

respectively, whence

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{rz} = 0$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta\theta} + \frac{2}{r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} = 0, \qquad (2.17\ a,b)$$

and (2.4c) becomes

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{1}{r}\frac{\partial}{\partial\theta}\sigma_{\theta z} + \frac{1}{r}\sigma_{rz} + \frac{\partial}{\partial z}\sigma_{zz} = 0 \qquad (2.17\ c)$$

The transformation of equations (2.5) is naturally more difficult since second order derivatives are involved, but some saving on algebra is made by observing:

$$(1 + \nu)\Sigma = \sigma_{xx}+\sigma_{yy}+\sigma_{zz}+E\alpha T = \sigma_{rr}+\sigma_{\theta\theta}+\sigma_{zz}+E\alpha T$$

Adding and subtracting (2.5 a,b) in turn gives

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) + \frac{\partial^2\Sigma}{\partial x^2} + \frac{\partial^2\Sigma}{\partial y^2} + \frac{2E\alpha}{(1-\nu)}\nabla^2 T = 0 \qquad (2.5\ a')$$

and

$$\nabla^2 (\sigma_{xx} - \sigma_{yy}) + \frac{\partial^2\Sigma}{\partial x^2} - \frac{\partial^2\Sigma}{\partial y^2} = 0. \qquad (2.5\ b')$$

Transforming (2.5 b′) and (2.5 *d*), we obtain

$$\cos 2\theta \left[(\nabla^2 - \frac{4}{r^2})(\sigma_{rr} - \sigma_{\theta\theta}) - \frac{8}{r^2}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial^2\Sigma}{\partial r^2} - \frac{1}{r}\frac{\partial\Sigma}{\partial r} - \frac{1}{r^2}\frac{\partial^2\Sigma}{\partial\theta^2} \right]$$

$$- \sin 2\theta \left[\frac{4}{r^2}\frac{\partial}{\partial\theta}(\sigma_{rr} - \sigma_{\theta\theta}) + 2(\nabla^2 - \frac{4}{r^2})\sigma_{r\theta} + 2(\frac{1}{r}\frac{\partial^2\Sigma}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\Sigma}{\partial\theta}) \right] = 0$$

and

$$\sin 2\theta \left[(\nabla^2 - \frac{4}{r^2})(\sigma_{rr} - \sigma_{\theta\theta}) - \frac{8}{r^2}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial^2\Sigma}{\partial r^2} - \frac{1}{r}\frac{\partial\Sigma}{\partial r} - \frac{1}{r^2}\frac{\partial^2\Sigma}{\partial\theta^2} \right]$$

$$+ \cos 2\theta \left[\frac{4}{r^2}\frac{\partial}{\partial\theta}(\sigma_{rr} - \sigma_{\theta\theta}) + 2(\nabla^2 - \frac{4}{r^2})\sigma_{r\theta} + 2(\frac{1}{r}\frac{\partial^2\Sigma}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\Sigma}{\partial\theta}) \right] = 0$$

whence

$$(\nabla^2 - \frac{4}{r^2})(\sigma_{rr} - \sigma_{\theta\theta}) - \frac{8}{r^2}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial^2\Sigma}{\partial r^2} + \frac{1}{r}\frac{\partial\Sigma}{\partial r} - \frac{\partial^2\Sigma}{\partial\theta^2} = 0 \qquad (2.18\ b')$$

and

$$(\nabla^2 - \frac{4}{r^2})\sigma_{r\theta} + \frac{2}{r^2}\frac{\partial}{\partial\theta}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{1}{r}\frac{\partial^2\Sigma}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\Sigma}{\partial\theta} = 0. \qquad (2.18\ d)$$

Equation (2.5 c) is unaltered in form, being

$$\nabla^2 \sigma_{zz} + \frac{\partial^2\Sigma}{\partial z^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0. \qquad (2.18)$$

Equations (2.5 e,f) transform to

$$\cos\theta \left[(\nabla^2 - \frac{1}{r^2})\sigma_{\theta z} + \frac{2}{r^2}\frac{\partial}{\partial\theta}\sigma_{rz} + \frac{1}{r}\frac{\partial^2\Sigma}{\partial\theta\partial z} \right]$$

$$+ \sin\theta \left[(\nabla^2 - \frac{1}{r^2})\sigma_{rz} - \frac{2}{r^2}\frac{\partial}{\partial\theta}\sigma_{rz} + \frac{\partial^2\Sigma}{\partial r\partial z} \right] = 0$$

and

$$\sin\theta \left[(\nabla^2 - \frac{1}{r^2})\sigma_{\theta z} + \frac{2}{r^2}\frac{\partial}{\partial\theta}\sigma_{rz} + \frac{1}{r}\frac{\partial^2\Sigma}{\partial\theta\partial z} \right]$$

$$- \cos\theta \left[(\nabla^2 - \frac{1}{r^2})\sigma_{rz} - \frac{2}{r^2}\frac{\partial}{\partial\theta}\sigma_{\theta z} + \frac{\partial^2\Sigma}{\partial r\partial z} \right] = 0$$

respectively, whence

$$(\nabla^2 - \frac{1}{r^2})\sigma_{\theta z} + \frac{2}{r^2}\frac{\partial}{\partial\theta}\sigma_{rz} + \frac{1}{r}\frac{\partial^2\Sigma}{\partial\theta\partial z} = 0 \qquad (2.18\text{ e})$$

and

$$(\nabla^2 - \frac{1}{r^2})\sigma_{rz} - \frac{2}{r^2}\frac{\partial}{\partial\theta}\sigma_{\theta z} + \frac{\partial^2\Sigma}{\partial r\partial z} = 0 \qquad (2.18\text{f})$$

Equation (2.5 a′) becomes

$$\nabla^2 (\sigma_{rr} + \sigma_{\theta\theta}) + \frac{\partial^2\Sigma}{\partial r^2} + \frac{1}{r}\frac{\partial\Sigma}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Sigma}{\partial^2\theta^2} + \frac{2E\alpha}{(1-\nu)}\nabla^2 T = 0 \quad (2.18\text{ a}')$$

and adding and subtracting (2.18 a′), (2.18 b′) in turn gives finally

$$\nabla^2 \sigma_{rr} - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) - \frac{4}{r^2}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{\partial^2\Sigma}{\partial r^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0 \qquad (2.18\text{ a})$$

and

$$\nabla^2 \sigma_{\theta\theta} + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) + \frac{4}{r^2}\frac{\partial}{\partial\theta}\sigma_{r\theta} + \frac{1}{r}\frac{\partial\Sigma}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Sigma}{\partial\theta^2} + \frac{E\alpha}{(1-\nu)}\nabla^2 T = 0 \qquad (2.18\text{ b})$$

Removal of the temperature terms from equations (2.18) results in the equations for isothermal theory, these latter agreeing with the results of Ford (1963), pp. 199-201.

Equation (2.6), the boundary conditions, being a vector equation requires no modification, however, when applying it to the surface of the disc we must express it in terms of cylindrical polar stress components. In this sense we renumber equation (2.6) to get, on the boundary

$$\overset{(n)}{T} = 0 \tag{2.19}$$

Equations (2.17), (2.18), and (2.19) constitute the full three-dimensional thermal stress equations in cylindrical polar coordinates. Their solution, in the general case, clearly presents a formidable task. However, in the present investigation, the analysis is much simplified by the two-fold symmetry of the problem.

2.4 Mathematical Statement of the Problem

We consider a circular disc of radius a and uniform thickness d where d is small in comparison to a. Using cylindrical polar coordinates r, θ, z we take the faces of the disc to be $z = \pm d$, $|r| \leq a$, and the edge to be the cylindrical surface $r = a$, $|z| \leq d$.

We assume that the theory of linear, quasi-static, uncoupled thermoelasticity, as developed in Chapter 1, is valid and that the heat flow is radial. Consequently the problem has axial symmetry with $T = T(r)$. Accordingly we seek a stress system, symbolized by $\sigma(r,z)$, the solution of the equations.

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial z}\sigma_{\theta z} + \frac{2}{r}\sigma_{r\theta} = 0 \qquad (2.20\ a,b,c)$$

$$\frac{\partial}{\partial r}\sigma_{rz} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{1}{r}\sigma_{rz} = 0$$

$$\nabla^2\sigma_{rr} + \frac{\partial^2\Sigma}{\partial r^2} + \frac{E\alpha}{(1-\nu)}\nabla_1^2 T - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

$$\nabla^2\sigma_{\theta\theta} + \frac{1}{r}\frac{\partial\Sigma}{\partial r} + \frac{E\alpha}{(1-\nu)}\nabla_1^2 T + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta}) = 0$$

$$\nabla^2\sigma_{zz} + \frac{\partial^2\Sigma}{\partial z^2} + \frac{E\alpha}{(1-\nu)}\nabla_1^2 T = 0 \qquad (2.21\ a\text{-}f)$$

$$\nabla^2\sigma_{r\theta} - \frac{4}{r^2}\sigma_{r\theta} = 0$$

$$\nabla^2\sigma_{\theta z} - \frac{1}{r^2}\sigma_{\theta z} = 0$$

$$\nabla^2\sigma_{rz} + \frac{\partial^2\Sigma}{\partial r\partial z} - \frac{1}{r^2}\sigma_{rz} = 0$$

where $(1 + \nu)\,\Sigma = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} + E\alpha T$

and $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$, $\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$.

The boundary conditions of the problem are

$$\sigma_{rr}\ (a,\ z) = \sigma_{r\theta}(a,z) = \sigma_{rz}(a,z) = 0 \qquad (2.22\ \text{a-f})$$

$$\sigma_{rz}\ (r,\ \pm\ d) = \sigma_{\theta z}\ (r,\ \pm\ d) = \sigma_{zz}\ (r,\ \pm\ d) = 0$$

Equations (2.20, (2.21) have been obtained from equations (2.17), (2.18) using the property of axial symmetry, that $\frac{\partial}{\partial\theta} = 0$. Equations (2.22) were obtained from (2.19), using components.

We observe that the system (2.20), (2.21) and (2.22) divides naturally into two sub-systems; equations (2.20 a,c), (2.21 a,b,c,f) and (2.22 a,c,d,f) determine the stresses σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz} whilst equations (2.20 b), (2.21 d,e) and (2.22 b,e) determine $\sigma_{r\theta}$, $\sigma_{\theta z}$. Inspection of the latter system yields the result that $\sigma_{r\theta} = \sigma_{\theta z} = 0$ everywhere. It is of some interest to observe that this result was obtained from the full set of equations and not, as is more usual, as a direct assumption of axial symmetry.

The problem also has symmetry about the mid-plane $z = 0$, $|r| \leq a$, the implication being that the stress components σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} are even functions in z while σ_{rz} is odd. We shall frequently make use of this property.

We non-dimensionalize the relevant equations of (2.20), (2.21), (2.22) by writing

$$\left.\begin{aligned} &r = a\, r^*,\ z = d\, z^* \\ &T(r,z) = T_R\, T^*(r^*, z^*),\ \sigma(r,z) = E\alpha T_R\, \sigma^*(r^*, z^*) \end{aligned}\right\} \quad (2.23)$$

where T_R is some convenient reference temperature.

If we further write

$$h = \frac{d}{a} \qquad (2.24)$$

the equations of the problem become

$$h\left[\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta})\right] + \frac{\partial}{\partial z}\sigma_{rz} = 0$$

$$h\left[\frac{\partial}{\partial r}\sigma_{rz} + \frac{1}{r}\sigma_{rz}\right] + \frac{\partial}{\partial z}\sigma_{zz} = 0 \qquad (2.25\ a,b)$$

$$h^2\left[\nabla_1^2\sigma_{rr} + \frac{\partial^2\Sigma}{\partial r^2} + \frac{1}{(1-\nu)}\nabla_1^2 T - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\sigma_{rr} = 0$$

$$h^2\left[\nabla_1^2\sigma_{\theta\theta} + \frac{1}{r}\frac{\partial\Sigma}{\partial r} + \frac{1}{(1-\nu)}\nabla_1^2 T + \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\sigma_{\theta\theta} = 0$$

$$h^2\left[\nabla_1^2\sigma_{zz} + \frac{1}{(1-\nu)}\nabla_1^2 T\right] + \frac{\partial^2}{\partial z^2}\sigma_{zz} + \frac{\partial^2\Sigma}{\partial z^2} = 0 \qquad (2.26\ a\text{-}d)$$

$$h^2\left[\nabla_1^2\sigma_{rz} - \frac{1}{r^2}\sigma_{rz}\right] + h\left[\frac{\partial^2\Sigma}{\partial r\partial z}\right] + \frac{\partial^2}{\partial z^2}\sigma_{rz} = 0$$

where

$$(1+\nu)\,\Sigma = \sigma_{rr} + \sigma_{\theta\theta} + \sigma_{zz} + T \qquad (2.27\ a)$$

and

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

A further equation is useful, namely that obtained by adding (2.26 a, b, c) and using (2.27 a), the result being

$$h^2\left[\nabla_1^2\,\Sigma + \frac{1}{(1-\nu)}\,\nabla_1^2\,T\right] + \frac{\partial^2\Sigma}{\partial z^2} = 0 \qquad (2.27\text{ b})$$

The boundary equations of the problem become

$$\sigma_{rr}(1,z) = \sigma_{rz}(1,z) = 0$$
$$\sigma_{zz}(r,\pm 1) = \sigma_{rz}(r,\pm 1) = 0 \qquad (2.28\text{ a-d})$$

In equations (2.25), (2.26), (2.27) and (2.28) the asterisk has been dropped for convenience and all quantities are non-dimensional.

2.5 The Method of Solution

It is convenient to regard the stress system σ as being the superposition of two sub stress systems, referred to as the primary stress system, symbolized by τ and the secondary system symbolized by γ. Thus we write

$$\sigma(r,z) = \tau(r,z) + \gamma(r,z) \qquad (2.29)$$

The primary stress system is the solution of the equations:

$$h\left[\frac{\partial}{\partial r}\tau_{rr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta})\right] + \frac{\partial}{\partial z}\tau_{rz} = 0$$

$$h\left[\frac{\partial}{\partial r}\tau_{rz} + \frac{1}{r}\tau_{rz}\right] + \frac{\partial}{\partial z}\tau_{zz} = 0 \qquad (2.30\ a,b)$$

$$h^2\left[\nabla_1^2\,\tau_{rr} + \frac{\partial^2 \textcircled{H}}{\partial r^2} + \frac{1}{(1-\nu)}\nabla_1^2\,T - \frac{2}{r^2}(\tau_{rr} - \tau_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\tau_{rr} = 0$$

$$h^2\left[\nabla_1^2\,\tau_{\theta\theta} + \frac{1}{r}\frac{\partial^2 \textcircled{H}}{\partial r} + \frac{1}{(1-\nu)}\nabla_1^2\,T + \frac{2}{r^2}(\tau_{rr} - \tau_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\tau_{\theta\theta} = 0$$

$$h^2\left[\nabla_1^2\,\tau_{zz} + \frac{1}{(1-\nu)}\nabla_1^2\,T\right] + \frac{\partial^2}{\partial z^2}\tau_{zz} + \frac{\partial^2 \textcircled{H}}{\partial z^2} = 0 \qquad (2.31\ a\text{-}d)$$

$$h^2\left[\nabla_1^2\,\tau_{rz} - \frac{1}{r^2}\tau_{rz}\right] + h\left[\frac{\partial^2 \textcircled{H}}{\partial r\partial z}\right] + \frac{\partial^2}{\partial z^2}\tau_{rz} = 0$$

$$(1+\nu)\ \textcircled{H} = \tau_{rr} + \tau_{\theta\theta} + \tau_{zz} + T$$

$$h^2\left[\nabla_1^2\ \textcircled{H} + \frac{1}{(1-\nu)}\nabla_1^2\,T\right] + \frac{\partial^2}{\partial z^2}\textcircled{H} = 0 \qquad (2.32\ a,b)$$

$$\int_{-1}^{1}\tau_{rr}\,(1,z)\,dz = 0$$

$$\qquad (2.33\ a,b,c)$$

$$\tau_{zz}\,(r, \pm 1) = \tau_{rz}\,(r, \pm 1) = 0$$

The field equations (2.30), (2.31), (2.32) and the "face" boundary conditions (2.33 b, c) are identical with those for σ, The remaining equation (2.33 a) corresponds to the condition which ensures that the edge of the physical disc is subject to a stress force system statically equivalent to, rather than precisely, zero. Consequently, by the Principle of Saint Venant (see, for example, Timoshenko and Goodier (1951) pp. 30 and 150) the stress systems σ and τ are equivalent, except near to the edge of the physical disc.

The secondary stress system is the solution of the equations:

$$h\left[\frac{\partial}{\partial r}\gamma_{rr} + \frac{1}{r}(\gamma_{rr} - \gamma_{\theta\theta})\right] + \frac{\partial}{\partial z}\gamma_{rz} = 0$$

$$h\left[\frac{\partial}{\partial r}\gamma_{rz} + \frac{1}{r}\gamma_{rz}\right] + \frac{\partial}{\partial z}\gamma_{zz} = 0 \qquad (2.34\text{ a, b})$$

$$h^2\left[\nabla_1^2\gamma_{rr} + \frac{\partial^2\Gamma}{\partial r^2} - \frac{2}{r^2}(\gamma_{rr} - \gamma_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\gamma_{rr} = 0$$

$$h^2\left[\nabla_1^2\gamma_{\theta\theta} + \frac{1}{r}\frac{\partial\Gamma}{\partial r} + \frac{2}{r^2}(\gamma_{rr} - \gamma_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\gamma_{\theta\theta} = 0$$

$$h^2\left[\nabla_1^2\gamma_{zz} + \frac{\partial^2}{\partial z^2}\gamma_{zz}\right] + \frac{\partial^2\Gamma}{\partial z^2} = 0 \qquad (2.35\text{ a-d})$$

$$h^2 \left[\nabla_1^2 \gamma_{rz} - \frac{1}{r^2} \gamma_{rz} \right] + h \left[\frac{\partial^2 \Gamma}{\partial r \partial z} \right] + \frac{\partial^2}{\partial z^2} \gamma_{rz} = 0$$

$$(1+\nu) \Gamma = \gamma_{rr} + \gamma_{\theta\theta} + \gamma_{zz}$$

$$h^2 \left[\nabla_1^2 \Gamma \right] + \frac{\partial^2 \Gamma}{\partial z^2} = 0 \qquad (2.36 \text{ a, b})$$

$$\gamma_{rr} (1,z) = - \tau_{rr}(1,z), \; \gamma_{rz}(1,z) = - \tau_{rz}(1,z)$$

$$\gamma_{zz} (r, \pm 1) = \gamma_{rz} (r, \pm 1) = 0 \qquad (2.37 \text{ a-d})$$

γ is clearly the isothermal stress system which by virtue of (2.29), and (2.37 a, b) ensures the stress free "edge" required in the determination of σ. The superposition of the stress systems τ, γ to give σ is the valid since all field equations are linear.

The defining property of a disc, as opposed to a cylinder, is that its thickness is small in comparison to its diameter. Consequently the dimensionless parameter h introduced in (2.24) is small and its appearance in the equations for σ, τ, γ suggests a perturbation technique.

We focus our attention, not on the task of obtaining a solution for a specific fixed value of h, but on the dependence on h of such solutions. In particular since only small values of h are of interest we seek to describe the way in which the stress distribution behaves as h approaches zero.

The most precise description of this kind is furnished by the knowledge of the asymptotic expansion of each stress component (see for example, Van Dyke (1965) p. 26) and we are thus concerned with obtaining such a representation.

The technique is well suited to the problem of the disc since the relationship of the two-dimensional to the three-dimensional theory will be clearly exhibited. Further by their very nature, the first few terms of an asymptotic series will provide a good approximation to the three-dimensional solution for a small, fixed value of h.

2.6 The Primary Problems

We assume, for each stress component, symbolized by τ, asymptotic expansions of the form

$$\tau(r, z; h) = \tau^0(r, z) + h\,\tau^1(r, z) + h^2\,\tau^2(r, z) + \dots \qquad (2.38)$$

Equality signs are used for convenience but only asymptotic validity is assumed. We further assume that each stress coefficient τ^n will have the same even-odd property in z as the stress component itself.

We introduce (2.38) into equations (2.30), (2.31), (2.32) and (2.33) and as the expansions are valid, at least in an asymptotic sense for arbitrary values of h we may equate coefficients of like powers obtaining for all integral n, valid in the region $|r|<1, |z|<1$

$$\frac{\partial}{\partial r}\tau_{rr}^{n-1} + \frac{1}{r}(\tau_{rr}^{n-1} - \tau_{\theta\theta}^{n-1}) + \frac{\partial}{\partial z}\tau_{rz}^{n} = 0$$

$$\frac{\partial}{\partial r}\tau_{rz}^{n-1} + \frac{1}{r}\tau_{rz}^{n-1} + \frac{\partial}{\partial z}\tau_{zz}^{n} = 0 \qquad (2.39\text{ a, b})$$

$$\nabla_1^2 \tau_{rr}^{n-2} + \frac{\partial^2}{\partial r^2}\text{Ⓗ}^{n-2} + \beta(n-2)\frac{1}{(1-\nu)}\nabla_1^2 T - \frac{2}{r^2}(\tau_{rr}^{n-2} - \tau_{\theta\theta}^{n-2}) + \frac{\partial^2}{\partial z^2}\tau_{rr}^{n} = 0$$

$$\nabla_1^2 \tau_{\theta\theta}^{n-2} + \frac{1}{r}\frac{\partial}{\partial r}\text{Ⓗ}^{n-2} + \beta(n-2)\frac{1}{(1-\nu)}\nabla_1^2 T + \frac{2}{r^2}(\tau_{rr}^{n-2} - \tau_{\theta\theta}^{n-2}) + \frac{\partial^2}{\partial z^2}\tau_{\theta\theta}^{n} = 0$$

$$\nabla_1^2 \tau_{zz}^{n-2} + \beta(n-2)\frac{1}{(1-\nu)}\nabla_1^2 T + \frac{\partial^2}{\partial z^2}\tau_{zz}^{n} + \frac{\partial^2}{\partial z^2}\text{Ⓗ}^{n} = 0$$

$$\nabla_1^2 \tau_{rz}^{n-2} - \frac{1}{r^2}\tau_{rz}^{n-2} + \frac{\partial^2}{\partial r\partial z}\text{Ⓗ}^{n-1} + \frac{\partial^2}{\partial z^2}\tau_{rz}^{n} = 0 \qquad (2.40\text{ a-d})$$

$$(1+\nu)\,\text{Ⓗ}^{n} = \tau_{rr}^{n} + \tau_{\theta\theta}^{n} + \tau_{zz}^{n} + \beta(n)T$$

$$\nabla_1^2\,\text{Ⓗ}^{n} + \beta(n-2)\frac{1}{(1-\nu)}\nabla_1^2 T + \frac{\partial^2}{\partial z^2}\text{Ⓗ}^{n} = 0 \qquad (2.41\text{ a,b})$$

where

$$\beta(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n \neq 1 \end{cases}$$

together with the boundary conditions

$$\int_{-1}^{1} \tau_{rr}^{n}(1,z)\,dz = 0$$

$$\tau_{zz}^{n}(r, \pm 1) = \tau_{rz}^{n}(r, \pm 1) = 0 \qquad (2.42\text{ a,b,c})$$

Inherent in the equations (2.39), (2.40), (2.41), (2.42) are the assumptions:

(i) $T(r)$ is independent of h

and (ii) $\tau^n = 0$ if $n < 0$.

The above system of equations gives rise to a sequence of boundary value problems, referred to as the primary problems, the solutions to which can be systematically obtained.

2.7 The Stress Coefficients τ^n_{zz}, τ^n_{rz}

The stress coefficients τ^n_{zz}, τ^n_{rz} are determined from (2.39 b), (2.40 c,d) and (2.42 b,c) and it is convenient to re-write these equations in the form

$$\frac{\partial}{\partial z}\tau^n_{zz} = -\left[\frac{\partial}{\partial r}\tau^{n-1}_{rz} + \frac{1}{r}\tau^{n-1}_{rz}\right] \qquad (2.39\ b')$$

$$\frac{\partial^2}{\partial z^2}\textcircled{H}^n = -\left[\frac{\partial^2}{\partial z^2}\tau^n_{zz} + \nabla^2_1\tau^{n-2}_{zz} + \beta(n-2)\frac{1}{(1-\nu)}\nabla^2_1 T\right] \qquad (2.40\ c')$$

$$\frac{\partial^2}{\partial z^2}\tau^n_{rz} = -\left[\frac{\partial^2}{\partial r\partial z}\textcircled{H}^{n-1} + \nabla^2_1\tau^{n-2}_{rz} - \frac{1}{r^2}\tau^{n-2}_{rz}\right] \qquad (2.40\ d')$$

$$\tau^n_{rr}(r, \pm 1) = 0 \qquad (2.42\ b)$$

$$\tau^n_{rz}(r, \pm 1) = 0 \qquad (2.42\ c)$$

We further recall that $\tau^n = 0$ if $n < 0$ and that τ^n_{zz}, $\textcircled{H}^n$ are even in z whilst τ^n_{rz} is odd.

The Case $n = 0$

From (2.39 b´)

$$\frac{\partial}{\partial z} \tau_{zz}^{0} = 0$$

Integrating and using (2.42 b) yields

$$\tau_{zz}^{0} = 0 \qquad (2.43\ a)$$

(2.40 c´), (2.43 a) give

$$\frac{\partial^2 Ⓗ^0}{\partial z^2} = 0 \qquad (2.43\ b)$$

(2.40 d´) gives

$$\frac{\partial^2}{\partial z^2} \tau_{rz}^{0} = 0$$

Integrating twice, using the odd in z property, and then (2.42 c) yields

$$\tau_{rz}^{0} = 0 \qquad (2.43\ c)$$

The Case $n = 1$

(2.39 b´), (2.43 c) give

$$\frac{\partial}{\partial z} \tau_{zz}^{1} = 0$$

Integrating and using (2.42 b) yields

$$\tau_{zz}^{1} = 0 \qquad (2.44\ a)$$

(2.40 c´), (2.44 a) give

$$\frac{\partial^2 Ⓗ^1}{\partial z^2} = 0 \qquad (2.44\ b)$$

(2.40 d´) becomes

$$\frac{\partial^2}{\partial z^2} \tau_{rz}^{\ 1} = - \frac{\partial^2 Ⓗ^0}{\partial r \partial z}$$

Differentiating w.r.t z and using (2.43 b) gives

$$\frac{\partial^3}{\partial z^3} \tau_{rz}^{\ 1} = 0$$

Integrating three times, making use of the odd in z property, and then (2.42 c) yields

$$\tau_{rz}^{\ 1} = 0 \qquad (2.44\ c)$$

<u>The Case $n = 2$</u>

(2.39 b´), (2.44 c) give

$$\frac{\partial}{\partial z} \tau_{zz}^{\ 2} = 0$$

Integrating and using (2.42 b) we get

$$\tau_{zz}^{\ 2} = 0 \qquad (2.45\ a)$$

(2.40 c´), (2.45 a) give

$$\frac{\partial^2}{\partial z^2} Ⓗ^{\,2} = \frac{-1}{(1-\nu)} \nabla_1^2 T \qquad (2.45\ b)$$

(2.40 d´), (2.43 c) give

$$\frac{\partial^2}{\partial z^2} \tau_{rz}^{\ 2} = - \frac{\partial^2 Ⓗ^1}{\partial r \partial z}$$

Differentiating w.r.t z and using (2.44 b), we obtain

$$\frac{\partial^3}{\partial z^3} \tau_{rz}^{\ 2} = 0$$

Integrating three times and using (2.42 c) yields as before

$$\tau_{rz}^{2} = 0 \tag{2.45 c}$$

We shall in fact confine our attention to the first three terms of the asymptotic series but is is convenient to obtain the stress coefficients τ_{rz}^{n}, τ_{zz}^{n} as far as $n = 6$.

<u>The Case $n = 3$</u>

(2.39 b´), (2.45 c) give

$$\frac{\partial}{\partial z} \tau_{zz}^{3} = 0$$

from which, using (2.42 b) we obtain

$$\tau_{zz}^{3} = 0 \tag{2.46 a}$$

(2.40 c´), (2.46 a), (2.44 a) yield

$$\frac{\partial^2 Ⓗ^3}{\partial z^2} = 0 \tag{2.46 b}$$

(2.40 d´), (2.44 a) give

$$\frac{\partial^2}{\partial z^2} \tau_{rz}^{3} = - \frac{\partial^2 Ⓗ^2}{\partial r \partial z}$$

Differentiating w.r.t z and using (2.45 b) we obtain

$$\frac{\partial^3}{\partial z^3} \tau_{rz}^{3} = \frac{1}{(1-\nu)} \frac{\partial}{\partial r} (\nabla_1^2 T)$$

Integrating three times, using the odd in z property and (2.42 c) gives

$$\tau_{rz}^{3} = \frac{1}{6} \frac{1}{(1-\nu)} (z^3 - z) \frac{\partial}{\partial r} (\nabla_1^2 T) \tag{2.46 c}$$

The Case $n = 4$

(2.39 b′), (2.46 c) give

$$\frac{\partial}{\partial z}\tau_{zz}^{4} = -\frac{1}{6(1-\nu)}(z^3 - z)\nabla_1^4 T$$

where $\nabla_1^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2$

Integrating and using (2.42 b) gives

$$\tau_{zz}^{4} = -\frac{1}{24(1-\nu)}(z^4 - 2z^2 + 1)\nabla_1^4 T \qquad (2.47\ a)$$

(2.40 c′), (2.47 a), (2.45 a) yield

$$\frac{\partial^2 \textcircled{H}^4}{\partial z^2} = \frac{1}{6(1-\nu)}(3z^2-1)\nabla_1^4 T \qquad (2.47\ b)$$

(2.40 d′), (2.45 c) give

$$\frac{\partial^2}{\partial z^2}\tau_{rz}^{4} = -\frac{\partial^2 \textcircled{H}^3}{\partial r \partial z}$$

Differentiating w.r.t z and using (2.46 b) yields

$$\frac{\partial^3}{\partial z^3}\tau_{rz}^{4} = 0$$

Integrating three times, using the odd in z property and (2.42 c) gives

$$\tau_{rz}^{4} = 0 \qquad (2.47\ c)$$

The Case $n = 5$

(2.39 b′), (2.47 c) give

$$\frac{\partial}{\partial z}\tau_{zz}^{5} = 0$$

Integrating and using (2.42 b) gives

$$\tau_{zz}^{5} = 0 \qquad (2.48\ a)$$

(2.40 c´), (2.48 a), (2.46 a) yield

$$\frac{\partial^2 Ⓗ^5}{\partial z^2} = 0 \qquad (2.48\ b)$$

(2.40 d´), (2.46 c), give

$$\frac{\partial^2}{\partial z^2}\,\tau_{rz}^{5} = -\frac{\partial^2 Ⓗ^4}{\partial r \partial z} - \frac{1}{6(1-\nu)}\,(z^3 - z)\,\frac{\partial}{\partial r}\,(\nabla_1^4\, T)$$

Differentiating w.r.t z and using (2.47 b) yields

$$\frac{\partial^3}{\partial z^3}\,\tau_{rz}^{5} = -\frac{1}{3(1-\nu)}\,(3z^2 - 1)\,\frac{\partial}{\partial r}\,(\nabla_1^4\, T)$$

Integrating three times, using the odd in z property and then

(2.42 c) gives

$$\tau_{rz}^{5} = -\frac{1}{180(1-\nu)}\,(3z^5 - 10z^3 + 7z)\,\frac{\partial}{\partial r}\,(\nabla_1^4\, T) \qquad (2.48\ c)$$

The Case $n = 6$

(2.39 b´), (2.48 c) give

$$\frac{\partial}{\partial z}\,\tau_{zz}^{6} = \frac{1}{180(1-\nu)}\,(3z^5 - 10z^3 + 7z)\,\nabla_1^6\, T$$

where $\nabla_1^6 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^3$

Integrating and using (2.42 b) gives

$$\tau_{zz}^{6} = \frac{1}{360(1-\nu)}\,(z^6 - 5z^4 + 7z^2 - 3)\,\nabla_1^6\, T \qquad (2.49\ a)$$

(2.40 c´), (2.49 a), (2.47 a) yield:

$$\frac{\partial^2 \text{\textcircled{H}}^6}{\partial z^2} = - \frac{1}{360(1-\nu)} \quad (15z^4 - 30z^2 - 1) \; \nabla_1^6 \, T \qquad (2.49\text{ b})$$

(2.40 d´), (2.47 c) give

$$\frac{\partial^2}{\partial z^2} \tau_{rz}^{\,6} = - \frac{\partial^2 \text{\textcircled{H}}^5}{\partial r \partial z}$$

Differentiating w.r.t z and using (2.48 b) yields

$$\frac{\partial^3}{\partial z^3} \tau_{rz}^{\,6} = 0$$

Integrating three times, using the odd in z property and then (2.42 c) gives

$$\tau_{rz}^{\,6} = 0 \qquad (2.49\text{ c})$$

2.8 The Stress Coefficients $\tau_{rr}^{\,n}$, $\tau_{\theta\theta}^{\,n}$

The stress coefficients $\tau_{rr}^{\,n}$, $\tau_{\theta\theta}^{\,n}$ are determined from (2.39 a), (2.40 a,b), (2.41 a,b) and (2.42 a) and it is convenient to re-write these equations in the form

$$\frac{\partial}{\partial r} \tau_{rr}^{\,n} + \frac{1}{r} (\tau_{rr}^{\,n} - \tau_{\theta\theta}^{\,n}) = - \frac{\partial}{\partial z} \tau_{rz}^{\,n+1} \qquad (2.39\text{ a}')$$

$$\frac{\partial^2}{\partial z^2} \tau_{rr}^{\,n} = - \left[\nabla_1^2 \tau_{rr}^{\,n-2} + \frac{\partial^2 \text{\textcircled{H}}^{n-2}}{\partial r^2} + \beta(n-2) \frac{1}{(1-\nu)} \nabla_1^2 T - \frac{2}{r^2} (\tau_{rr}^{\,n-2} - \tau_{\theta\theta}^{\,n-2}) \right]$$

$$(2.40\text{ a}')$$

$$\frac{\partial^2}{\partial z^2} \tau_{\theta\theta}^{\,n} = - \left[\nabla_1^2 \tau_{\theta\theta}^{\,n-2} + \frac{1}{r} \frac{\partial \text{\textcircled{H}}^{n-2}}{\partial r} + \beta(n-2) \frac{1}{(1-\nu)} \nabla_1^2 T + \frac{2}{r^2} (\tau_{rr}^{\,n-2} - \tau_{\theta\theta}^{\,n-2}) \right]$$

$$(2.40\text{ b}')$$

$$(1+\nu) \text{\textcircled{H}}^n = \tau_{rr}^{\,n} + \tau_{\theta\theta}^{\,n} + \tau_{zz}^{\,n} + \beta(n) T \qquad (2.41\text{ a})$$

$$\nabla_1^2 \, Ⓗ^n + \beta(n) \frac{1}{(1-\nu)} \nabla_1^2 T = - \frac{\partial^2}{\partial z^2} Ⓗ^{n+2} \qquad (2.41\ b')$$

$$\int_{-1}^{1} \tau_{rr}^{\ n} (1,z) \, dz = 0 \qquad (2.42\ a)$$

In the solution of the above equations we shall refer to the even in z properties of $\tau_{rr}^{\ n}$, $\tau_{\theta\theta}^{\ n}$ and to the results of the previous section.

<u>The Case $n = 0$</u>

Putting $n = 0$ and using (2.43 a) (2.44 c), (2.45 b) we obtain the following system of equations:

$$\frac{\partial}{\partial r} \tau_{rr}^{\ 0} + \frac{1}{r} (\tau_{rr}^{\ 0} - \tau_{\theta\theta}^{\ 0}) = 0$$

$$\frac{\partial^2}{\partial z^2} \tau_{rr}^{\ 0} = 0$$

$$\frac{\partial^2}{\partial z^2} \tau_{\theta\theta}^{\ 0} = 0 \qquad (2.50\ a\text{-}f)$$

$$(1+\nu) Ⓗ^0 = \tau_{rr}^{\ 0} + \tau_{\theta\theta}^{\ 0} + T$$

$$\nabla_1^2 \, Ⓗ^0 = 0$$

$$\int_{-1}^{1} \tau_{rr}^{\ 0} (1,z) \, dz = 0$$

Integrating (2.50 b,c) twice and using the even in z property gives

$$\tau_{rr}^{\ 0} = f_{rr}^{\ 0} (r), \quad \tau_{\theta\theta}^{\ 0} = f_{\theta\theta}^{\ 0} (r) \qquad (2.51\ a,b)$$

where the f^0 are functions of the integration.

Introducing a stress function $\phi^0(r)$ such that

$$f_{rr}^{\ 0} = \frac{1}{r}\frac{d\phi^0}{dr} , \ f_{\theta\theta}^{\ 0} = \frac{d^2\phi^0}{dr^2} \qquad (2.52\ a,b)$$

we see from (2.51 a,b), (2.52 a,b) that (2.50 a) is identically satisfied. From (2.50 d) we get

$$(1+\nu) \textcircled{H}^{\ 0} = \nabla_1^2 \phi^0 + T$$

and thus from (2.50 e)

$$\nabla_1^4 \phi^0 + \nabla_1^2 T = 0 \qquad (2.53)$$

We remark in passing that (2.53) may be identified with the familiar equation of the two dimensional theory (see, for example, Boley and Weiner (1962) p.261).

A particular integral of (2.53) is

$$\phi^0 = - \int \frac{1}{r}\left\{\int rTdr \right]dr$$

and the general solution is therefore

$$\phi^0 = Ar^2 \log r + B \log r + Cr^2 + D - \int \frac{1}{r}\left\{\int rTdr\right\}dr$$

where A, B, C, D are constants. From (2.51), (2.52) therefore

$$\tau_{rr}^{\ 0} = A(1 + 2 \log r) + 2C + \frac{B}{r^2} - \frac{1}{r^2}\int rTdr$$

$$\tau_{\theta\theta}^{\ 0} = A(3 + 2 \log r) + 2C - \frac{B}{r^2} + \frac{1}{r^2}\int rTdr - T$$

Further assuming the stresses to be finite at the origin $A = B = 0$ and we may write

$$\tau^{0}_{rr} = k^0 - \frac{1}{r^2}\int rTdr$$
$$\tau^{0}_{\theta\theta} = k^0 + \frac{1}{r^2}\int rTdr - T \qquad (2.54\ \text{a, b})$$

where $k^0 = 2C$.

Applying (2.50 f) to (2.54 a) yields k^0 and finally

$$\tau^{0}_{rr} = \int_0^1 rTdr - \frac{1}{r^2}\int_0^r rTdr$$
$$\tau^{0}_{\theta\theta} = \int_0^1 rTdr + \frac{1}{r^2}\int_0^r rTdr - T \qquad (2.55\ \text{a,b})$$

The Case $n = 1$

Putting $n = 1$ and using (2.44 a), (2.45 c) and (2.46 b) the relevant equations are:

$$\frac{\partial}{\partial r}\tau^{1}_{rr} + \frac{1}{r}(\tau^{1}_{rr} - \tau^{1}_{\theta\theta}) = 0$$
$$\frac{\partial^2}{\partial z^2}\tau^{1}_{rr} = 0$$
$$\frac{\partial^2}{\partial z^2}\tau^{1}_{\theta\theta} = 0 \qquad (2.56\ \text{a-f})$$
$$(1+\nu)\,Ⓗ^{1} = \tau^{1}_{rr} + \tau^{1}_{\theta\theta}$$
$$\nabla^2_1\, Ⓗ' = 0$$
$$\int_{-1}^{1}\tau^{1}_{rr}(1,z)dz = 0$$

Equations (2.56) are identical in form to (2.50) with $T = 0$ and using (2.54)

$$\tau_{rr}^{1} = k^1, \quad \tau_{\theta\theta} = k^1 \qquad (2.57 \text{ a, b})$$

Application of (2.56 f) yields $k^1 = 0$ and

$$\tau_{rr}^{1} = \tau_{\theta\theta}^{1} = 0 \qquad (2.58 \text{ a, b})$$

<u>The Case $n = 2$</u>

Putting $n = 2$ and using (2.45 a), (2.46 c), (2.47 b) we obtain

$$\frac{\partial}{\partial r}\tau_{rr}^{2} + \frac{1}{r}(\tau_{rr}^{2} - \tau_{\theta\theta}^{2}) = \frac{1}{6(1-\nu)}(1-3z^2)\frac{d}{dr}(\nabla_1^2 T)$$

$$\frac{\partial^2}{\partial z^2}\tau_{rr}^{2} = \frac{1}{r}\frac{dT}{dr} - \frac{1}{(1-\nu)}\nabla_1^2 T$$

$$\frac{\partial^2}{\partial z^2}\tau_{\theta\theta}^{2} = \frac{d^2T}{dr^2} - \frac{1}{(1-\nu)}\nabla_1^2 T$$

$$(1+\nu)\,Ⓗ^2 = \tau_{rr}^{2} + \tau_{\theta\theta}^{2} \qquad (2.59 \text{ a-f})$$

$$\nabla_1^2\,Ⓗ^2 = \frac{1}{6(1-\nu)}(1-3z^2)\,\nabla_1^4 T$$

$$\int_{-1}^{1}\tau_{rr}^{2}(1,z)\,dz = 0$$

Integrating (2.59 b,c) and using the even in z property we have

$$\tau_{rr}^{2} = \frac{z^2}{2(1-\nu)}\left[(1-\nu)\frac{1}{r}\frac{dT}{dr} - \nabla_1^2 T\right] + f_{rr}^{2}(r)$$

$$(2.60 \text{ a,b})$$

$$\tau_{\theta\theta}^{2} = \frac{z^2}{2(1-\nu)}\left[(1-\nu)\frac{d^2T}{dr^2} - \nabla_1^2 T\right] + f_{rr}^{2}(r)$$

where the f^2 are functions of the integration.

Substituting (2.60) into (2.59 a) gives

$$\frac{d}{dr} f_{rr}^{2} + \frac{1}{r}(f_{rr}^{2} - f_{\theta\theta}^{2}) = \frac{1}{6(1-\nu)} \frac{d}{dr}(\nabla_1^2 T) \tag{2.61}$$

Equation (2.61) is identically satisfied if we introduce a function $\phi^2(r)$ such that

$$\begin{aligned} f_{rr}^{2} &= \frac{1}{6(1-\nu)} \nabla_1^2 T - \frac{1}{6r}\frac{dT}{dr} + \frac{1}{r}\frac{d\phi^2}{dr} \\ f_{\theta\theta}^{2} &= \frac{1}{6(1-\nu)} \nabla_1^2 T - \frac{1}{6}\frac{d^2T}{dr^2} + \frac{d^2\phi^2}{dr^2} \end{aligned} \tag{2.62}$$

Hence, from (2.60), (2.62) we obtain

$$\begin{aligned} \tau_{rr}^{2} &= \frac{(1-3z^2)}{6(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right] + \frac{1}{r}\frac{d\phi^2}{dr} \\ \tau_{\theta\theta}^{2} &= \frac{(1-3z^2)}{6(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{d^2T}{dr^2}\right] + \frac{d^2\phi^2}{dr^2} \end{aligned} \tag{2.63 a,b}$$

Equations (2.63), (2.59 d) now give

$$(1+\nu)\,Ⓗ^{2} = \frac{(1+\nu)}{6(1-\nu)}(1-3z^2)\nabla_1^2 T + \nabla_1^2 \phi^2$$

and using (2.59 e) we get

$$\nabla_1^4 \phi^2 = 0 \tag{2.64}$$

Equation (2.64) is identical in form to (2.53) with $T = 0$ and the contribution to the solution is the same form as (2.54) with $T = 0$. Thus

$$\tau_{rr}^{2} = \frac{(1-3z^2)}{6(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right] + k^2$$

(2.65 a,b)

$$\tau_{\theta\theta}^{\;2} = \frac{(1-3z^2)}{6(1-\nu)} \left[\nabla_1^2 \, T - (1-\nu) \frac{d^2T}{dr^2} \right] + k^2$$

where k^2 is a constant.

Finally applying (2.59 f) yields $k^2 = 0$ and

$$\tau_{rr}^{\;2} = \frac{(1-3z^2)}{6(1-\nu)} \left[\nabla_1^2 \, T - (1-\nu) \frac{1}{r} \frac{dT}{dr} \right]$$

$$\tau_{\theta\theta}^{\;2} = \frac{(1-3z^2)}{6(1-\nu)} \left[\nabla_1^2 \, T - (1-\nu) \frac{d^2T}{dr^2} \right] \qquad (2.66\text{ a,b})$$

If necessary, the procedure could be continued and higher order stress coefficients found.

2.9 The Primary Stresses

Using equations (2.38), (2.43), (2.44), (2.45), (2.55), (2.58) and (2.66) we obtain the asymptotic representation of the primary stresses to terms in h^2:

$$\tau_{rr}(r,z) = \int_0^1 rTdr - \frac{1}{r^2} \int_0^r rTdr + \frac{(1-3z^2)}{6(1-\nu)} \left[\nabla_1^2 \, T - (1-\nu) \frac{1}{r} \frac{dT}{dr} \right] h^2 + \ldots$$

$$\tau_{\theta\theta}(r,z) = \int_0^1 rTdr + \frac{1}{r^2} \int_0^r rTdr - T + \frac{(1-3z^2)}{6(1-\nu)} \left[\nabla_1^2 T - (1-\nu) \frac{d^2T}{dr^2} \right] h^2 + \ldots$$

$$\tau_{zz}(r,z) = 0\,(h^2) + \ldots \qquad (2.67\text{ a-d})$$

$$\tau_{rz}(r,z) = 0\,(h^2) + \ldots$$

2.10 The Secondary (Interior) Problems

The secondary stress system γ is the solution of the equations (2.34), (2.35), (2.36) and (2.37). In view of (2.37) and (2.67) an asymptotic expansion procedure is clearly necessary. We therefore assume series of the form

$$\gamma(r,z;\, h) = \gamma^0(r,z) + h\gamma^1\,(r,z) + h^2\,\gamma^2\,(r,z) + \ldots \quad (2.68)$$

and obtain, as before, a sequence of differential equations for the secondary (interior) stress coefficients γ^n. The field equations ($|r|<1$, $|z|<1$) are

$$\frac{\partial}{\partial r}\,\gamma_{rr}^{n-1} + \frac{1}{r}\,(\gamma_{rr} - \gamma_{\theta\theta})^{n-1} + \frac{\partial}{\partial z}\,\gamma_{rz}^{n} = 0$$

$$\frac{\partial}{\partial r}\,\gamma_{rz}^{n-1} + \frac{1}{r}\,\gamma_{rz}^{n-1} + \frac{\partial}{\partial z}\,\gamma_{zz}^{n} = 0 \qquad (2.69\ \text{a,b})$$

$$\nabla^2\,\gamma_{rr}^{n-2} + \frac{\partial^2}{\partial r^2}\,\Gamma^{n-2} - \frac{2}{r^2}\,(\gamma_{rr}^{n-2} - \gamma_{\theta\theta}^{n-2}) + \frac{\partial^2}{\partial z^2}\,\gamma_{rr}^{n} = 0$$

$$\nabla_1^2\,\gamma_{\theta\theta}^{n-2} + \frac{1}{r}\,\frac{\partial\Gamma^{n-2}}{\partial r} + \frac{2}{r^2}\,(\gamma_{rr}^{n-2} - \gamma_{\theta\theta}^{n-2}) + \frac{\partial^2}{\partial z^2}\,\gamma_{\theta\theta}^{n} = 0 \qquad (2.70\ \text{a-d})$$

$$\nabla_1^2\,\gamma_{zz}^{n-2} + \frac{\partial^2}{\partial z^2}\,\gamma_{zz}^{n} + \frac{\partial^2\Gamma^n}{\partial z^2} = 0$$

$$\nabla_1^2\,\gamma_{rz}^{n-2} - \frac{1}{r^2}\,\gamma_{rz}^{n-2} + \frac{\partial^2}{\partial r\partial z}\,\Gamma^{n-1} + \frac{\partial^2}{\partial z^2}\,\gamma_{rz}^{n} = 0$$

$$(1+\nu)\Gamma^n = \gamma_{rr}^n + \gamma_{\theta\theta}^n + \gamma_{zz}^n$$

$$\nabla_1^2\,\Gamma^{n-2} + \frac{\partial^2}{\partial z^2}\,\Gamma^n = 0 \qquad (2.71\ \text{a,b})$$

and the boundary conditions are

$$\gamma^n_{rr}(1,z) = -\tau^n_{rr}(1,z), \quad \gamma^n_{rz}(1,z) = -\tau^n_{rz}(1,z)$$
$$\gamma^n_{rr}(r,\pm 1) = \gamma^n_{rz}(r,\pm 1) = 0 \qquad (2.72\text{ a-d})$$

With the exception of (2.72 a,b) these equations are identical to those for the τ^n with $T = 0$ and consequently we may use equations (2.43), (2.44), (2.45), (2.54), (2.57), (2.65) with $T = 0$ and (2.68) to obtain

$$\gamma_{rr}(r,z) = C^0 + h\,C^1 + h^2\,C^2 + \ldots$$
$$\gamma_{\theta\theta}(r,z) = C^0 + h\,C^1 + h^2\,C^2 + \ldots \qquad (2.73\text{ a-d})$$
$$\gamma_{zz}(r,z) = 0(h^2) + \ldots$$
$$\gamma_{rz}(r,z) = 0(h^2) + \ldots$$

where the C^0, C^1, C^2 are constants, as yet unknown. In fact it can be easily shown (Appendix 2), that

$$\left.\begin{array}{l} \gamma^n_{rr} = \gamma^n_{\theta\theta} = C^n \\ \gamma^n_{zz} = \gamma^n_{rz} = 0 \end{array}\right\} n = 0,1,2\ \ldots \qquad (2.74\text{ a-d})$$

We emphasize, however, that in obtaining (2.73), (2.74) the "edge" boundary conditions (2.72 a,b) have not been used. Indeed, inspection of (2.73 a) and (2.67 a) shows that it is not possible to satisfy these edge conditions, however the C^n are chosen and we conclude therefore that if (2.68) represents the solution they do so in some region away from the edge, in the interior of the disc. Near to the edge, in the boundary layer, there must be a rapid variation from the interior solution in order that the precise edge boundary conditions may be satisfied.

2.11 The Secondary (Boundary Layer) Problems

To obtain asymptotic expansions which uniformly represent the solution up to and including the edge we follow a similar procedure as that used by Reiss and Locke (1961) and introduce a boundary layer coordinate ξ, defined by

$$\xi = \frac{(1 - r)}{h} \qquad (2.75)$$

Introducing also, the symbol $p(\xi,z;h)$ to represent the stresses in the boundary layer, we obtain from (2.34), (2.35) and (2.36),

$$\frac{\partial}{\partial\xi} p_{rr} - \frac{\partial}{\partial z} p_{rz} - h\xi\left[\frac{\partial}{\partial\xi} p_{rr} - \frac{\partial}{\partial z} p_{rz} + \frac{1}{\xi}(p_{rr} - p_{\theta\theta})\right] = 0$$

$$\frac{\partial}{\partial\xi} p_{rz} - \frac{\partial}{\partial z} p_{zz} - h\xi\left[\frac{\partial}{\partial\xi} p_{rz} + \frac{1}{\xi} p_{rz} - \frac{\partial}{\partial z} p_{zz}\right] = 0 \qquad (2.76\text{ a,b})$$

$$L_0^2 p_{rr} + \frac{\partial^2\Delta}{\partial\xi^2} - h\xi\left[\left\{L_0^2 p_{rr} + \frac{\partial^2\Delta}{\partial\xi^2}\right\} + \left\{L^2 p_{rr} + \frac{\partial^2\Delta}{\partial\xi^2}\right\}\right]$$

$$+ h^2\xi^2\left[L^2 p_{rr} + \frac{\partial^2\Delta}{\partial\xi^2} - \frac{2}{\xi^2}(p_{rr} - p_{\theta\theta})\right] = 0$$

$$L_0^2 p_{\theta\theta} - h\xi\left[\left\{L_0^2 p_{\theta\theta}\right\} + \left\{L^2 p_{\theta\theta} + \frac{1}{\xi}\frac{\partial\Delta}{\partial\xi}\right\}\right]$$

$$+ h^2\xi^2\left[L^2 p_{\theta\theta} + \frac{1}{\xi}\frac{\partial\Delta}{\partial\xi} + \frac{2}{\xi^2}(p_{rr} - p_{\theta\theta})\right] = 0$$

$$L_0^2\, p_{zz} + \frac{\partial^2\Delta}{\partial z^2} - h\xi\left[\left\{L_0^2\, p_{zz} + \frac{\partial^2\Delta}{\partial z^2}\right\} + \left\{L^2\, p_{zz} + \frac{\partial^2\Delta}{\partial z^2}\right\}\right]$$

$$+ h^2\,\xi^2\left[L^2\, p_{zz} + \frac{\partial^2\Delta}{\partial z^2}\right] = 0 \qquad (2.77\text{ a-d})$$

$$L_0^2\, p_{rz} - \frac{\partial^2\Delta}{\partial\xi\partial z} - h\xi\left[\left\{L_0^2\, p_{rz} - \frac{\partial^2\Delta}{\partial\xi\partial z}\right\} + \left\{L^2\, p_{rz} + \frac{\partial^2\Delta}{\partial\xi\partial z}\right\}\right]$$

$$+ h^2\,\xi^2\left[L^2\, p_{rz} - \frac{1}{\xi^2}\, p_{rz} - \frac{\partial^2\Delta}{\partial\xi\partial z}\right] = 0$$

$$(1 + \nu)\Delta = p_{rr} + p_{\theta\theta} + p_{zz} \qquad (2.78\text{ a,b})$$

$$L_0^2\,\Delta - h\xi\left[\left\{L_0^2\,\Delta\right\} + \left\{L^2\,\Delta\right\}\right] + h^2\,\xi^2\left[L^2\,\Delta\right] = 0$$

where

$$L^2 = \frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad L_0^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial z^2}\,.$$

Equations (2.37) become

$$p_{rr}(0,z) = -\,\tau_{rr}(1,z),\; p_{rz}(0,z) = -\,\tau_{rz}(1,z) \qquad (2.79\text{ a-d})$$

$$p_{zz}(\xi, \pm 1) = p_{rz}(\xi, \pm 1) = 0$$

We proceed, as in previous sections, by assuming asymptotic expansions of the form

$$p(\xi,\, z;\, h) = p^0(\xi,\, z) + hp^1(\xi,z) + h^2\, p^2\,(\xi,\, z) + \ldots \qquad (2.80)$$

Introducing (2.80) into (2.76), (2.77), (2.78) and (2.79) and equating coefficients of like powers of h, we obtain

$$\frac{\partial}{\partial \xi} p_{rr}^{n} - \frac{\partial}{\partial z} p_{rz}^{n} = \xi \left[\frac{\partial}{\partial \xi} p_{rr}^{n-1} - \frac{\partial}{\partial z} p_{rz}^{n-1} + \frac{1}{\xi} (p_{rr}^{n-1} - p_{\theta\theta}^{n-1})\right]$$

$$\frac{\partial}{\partial \xi} p_{rz}^{n} - \frac{\partial}{\partial z} p_{zz}^{n} = \xi \left[\frac{\partial}{\partial \xi} p_{rz}^{n-1} + \frac{1}{\xi} p_{rz}^{n-1} - \frac{\partial}{\partial z} p_{zz}^{n-1}\right] \qquad (2.81\ a,b)$$

$$L_0^2\, p_{rr}^{n} + \frac{\partial^2 \Delta^n}{\partial \xi^2} = \xi \left[\left\{L_0^2\, p_{rr}^{n-1} + \frac{\partial^2 \Delta^{n-1}}{\partial \xi^2}\right\} + \left\{L^2\, p_{rr}^{n-1} + \frac{\partial^2 \Delta^{n-1}}{\partial \xi^2}\right\}\right]$$

$$- \xi^2 \left[L^2\, p_{rr}^{n-2} + \frac{\partial^2}{\partial \xi^2} \Delta^{n-2} - \frac{2}{\xi^2} (p_{rr}^{n-2} - p_{\theta\theta}^{n-2})\right]$$

$$L_0^2\, p_{\theta\theta}^{n} = \xi \left[\left\{L_0^2\, p_{\theta\theta}^{n-1}\right\} + \left\{L^2\, p_{\theta\theta}^{n-1} + \frac{1}{\xi} \frac{\partial \Delta^{n-1}}{\partial \xi}\right\}\right]$$

$$- \xi^2 \left[L^2\, p_{\theta\theta}^{n-2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} \Delta^{n-2} + \frac{2}{\xi^2} (p_{rr}^{n-2} - p_{\theta\theta}^{n-2})\right]$$

$$L_0^2\, p_{zz}^{n} + \frac{\partial^2 \Delta^n}{\partial z^2} = \xi \left[\left\{L_0^2\, p_{zz}^{n-1} + \frac{\partial^2 \Delta^{n-1}}{\partial z^2}\right\} + \left\{L^2\, p_{zz}^{n-1} + \frac{\partial^2 \Delta^{n-1}}{\partial z^2}\right\}\right]$$

$$- \xi^2 \left[L^2\, p_{zz}^{n-2} + \frac{\partial^2 \Delta^{n-2}}{\partial z^2}\right] \qquad (2.82\ a\text{-}d)$$

$$L_0^2\, p_{rz}^{n} - \frac{\partial^2 \Delta^n}{\partial \xi \partial z} = \xi \left[\left\{L_0^2\, p_{rz}^{n-1} - \frac{\partial^2 \Delta^{n-1}}{\partial \xi \partial z}\right\} + \left\{L^2\, p_{rz}^{n-1} - \frac{\partial^2 \Delta^{n-1}}{\partial \xi \partial z}\right\}\right]$$

$$- \xi^2 \left[L^2\, p_{rz}^{n-2} - \frac{1}{\xi^2} p_{rz}^{n-2} - \frac{\partial^2 \Delta^{n-2}}{\partial \xi \partial z}\right]$$

$$(1+\nu)\Delta^n = p_{rr}^{n} + p_{\theta\theta}^{n} + p_{zz}^{n} \qquad (2.83\ a,b)$$

$$L_0^2\, \Delta^n = \xi \left[\left\{L_0^2\, \Delta^{n-1}\right\} + \left\{L^2\, \Delta^{n-1}\right\}\right] - \xi^2 \left[L^2\, \Delta^{n-2}\right]$$

$$p^n_{rr}(0,z) = -\tau^n_{rr}(1, z),\ p^n_{rz}(0, z) = -\tau^n_{rz}(1, z)$$
$$p^n_{zz}(\xi, \pm 1) = p^n_{rz}(\xi, \pm 1) = 0. \qquad (2.84\ a\text{-}d)$$

Once again we define $p^n = 0$ if $n < 0$ and the above equations hold for all integral n.

The sequence of differential equations is valid in the limit space $0 < \xi < \infty$, $|z| < 1$, and to complete the formulation of the boundary layer problems, we require boundary conditions as $\xi \to \infty$. These we obtain by matching the boundary layer and interior expansions. By virtue of (2.75) any fixed neighbourhood of the edge in the limit ξ r, z space corresponds to $\xi = \infty$ in the limit ξ, z space. Consequently we have from (2.68), (2.74) and (2.80)

$$p^n_{rr}(\infty, z) = p^n_{\theta\theta}(\infty, z) = C^n$$
$$p^n_{zz}(\infty, z) = p^n_{rz}(\infty, z) = 0 \qquad (2.85\ a\text{-}d)$$

2.12 The Zeroth Order Secondary Stress Coefficients

Putting $n = 0$ in (2.81), (2.82), and (2.83) we obtain

$$\frac{\partial}{\partial\xi} p^0_{rr} - \frac{\partial}{\partial z} p^0_{rz} = 0$$

$$\frac{\partial}{\partial\xi} p^0_{rz} - \frac{\partial}{\partial z} p^0_{zz} = 0$$

$$L^2_0\, p^0_{rr} + \frac{\partial^2\Delta^0}{\partial\xi^2} = 0 \qquad (2.86)$$

$$L_0^2 \ p_{\theta\theta}^0 = 0$$

$$L_0^2 \ p_{zz}^0 + \frac{\partial^2 \Delta^0}{\partial z^2} = 0$$

$$L_0^2 \ p_{rz}^0 - \frac{\partial^2 \Delta}{\partial \xi \partial z} = 0$$

$$(1+\nu) \ \Delta^0 = p_{rr}^{\ 0} + p_{\theta\theta}^{\ 0} + p_{zz}^{\ 0}$$

$$L_0^2 \ \Delta^0 = 0$$

The boundary conditions from (2.84), (2.67) are

$$p_{rr}^{\ 0} \ (0, \ z) = p_{rz}^{\ 0} \ (0, \ z) = 0$$
$$p_{zz}^{\ 0} \ (\xi, \ \pm 1) = p_{rz}^{\ 0} \ (\xi, \ \pm 1) = 0 \qquad (2.87)$$

and the conditions at infinity from (2.85) are

$$p_{rr}^{\ 0} \ (\infty, \ z) = p_{\theta\theta}^{\ 0} \ (\infty, \ z) = C^0$$
$$p_{zz}^{\ 0} \ (\infty, \ z) = p_{rz}^{\ 0} \ (\infty, \ z) = 0 \qquad (2.88)$$

It is convenient to introduce new stress coefficients $S^0(\xi,z)$ defined by

$$S_{rr}^{\ 0} = p_{rr}^{\ 0} - C^0, \ S_{\theta\theta}^{\ 0} = p_{\theta\theta}^{\ 0} - C^0$$
$$S_{zz}^{\ 0} = p_{zz}^{\ 0} \ , \quad S_{rz}^{\ 0} = - \ p_{rz}^{\ 0} \qquad (2.89)$$

Substituting (2.89) into (2.86), we obtain

$$\frac{\partial}{\partial \xi} S_{rr}^{\ 0} + \frac{\partial}{\partial z} S_{rz}^{\ 0} = 0$$

$$\frac{\partial}{\partial \xi} S_{rz}^{0} + \frac{\partial}{\partial z} S_{zz}^{0} = 0$$

$$L_0^2 S_{rr}^{0} + \frac{\partial^2 \Omega^0}{\partial \xi^2} = 0$$

$$L_0^2 S_{\theta\theta}^{0} = 0$$

$$L_0^2 S_{zz}^{0} + \frac{\partial^2 \Omega^0}{\partial z^2} = 0 \qquad (2.90)$$

$$L_0^2 S_{rz}^{0} + \frac{\partial^2 \Omega^0}{\partial \xi \partial z} = 0$$

$$(1+\nu)\Omega^0 = S_{rr}^{0} + S_{\theta\theta}^{0} + S_{zz}^{0}$$

$$L_0^2 \Omega^0 = 0$$

The boundary conditions (2.87) may be written

$$S_{rr}^{0}(0,z) = -C^0, \quad S_{rz}^{0}(0,z) = 0 \qquad (2.91\ \text{a-d})$$

$$S_{zz}^{0}(\xi, \pm 1) = 0 \quad S_{rz}^{0}(\xi, \pm 1) = 0$$

and the conditions at infinity (2.88)

$$S^0(\infty, z) = 0 \qquad (2.92\ \text{e})$$

The immediate difficulty is now apparent from (2.91 a). The constant C^0 is unknown and therefore it is not possible to solve equations (2.90), (2.91) for the boundary layer stress coefficients S^0. We surmount this problem by proceeding as Friedrichs and Dressler (1961) and prove the following result:

$$\int_{-1}^{1} S_{rr}^{0}(0, z)\, dz = 0 \qquad (2.92)$$

Proof.

We assume in accordance with (2.91 e) that

$S_{rr}^{0}(\xi,z) \to 0$ uniformly as $\xi \to \infty$.

That is, for any given $\varepsilon > 0$, no matter how small there exists $\delta(\varepsilon)$, depending only on ε and not on z such that

$$|S_{rr}^{0}(\xi, z)| < \varepsilon \tag{2.93}$$

whenever $\xi \geq \delta$.

Now consider the region under investigation (fig v), $\xi \geq 0$, $|z|<1$. Let $ABCD$ denote the region bounded by the lines $z = \pm 1$, $\xi = 0$ and $\xi = \delta$.

Assuming that S_{rr}^{0}, S_{rz}^{0} are continuous and have continuous first derivatives, we may apply Green's theorem of the plane (see for example Courant (1957) p. 360) to the region $ABCD$ to get

$$\int_{ABCD} S_{rr}^{0}\, dz - S_{rz}^{0}\, d\xi = \iint_{ABCD} \left(\frac{\partial}{\partial \xi} S_{rr}^{0} + \frac{\partial}{\partial z} S_{rz}^{0} \right) d\xi dz$$

By the first equation of (2.90) the R.H.S. is zero and thus

$$- \int_{0}^{\delta} S_{rz}^{0}(\xi,-1) d\xi + \int_{-1}^{1} S_{rr}^{0}(\delta,z) dz - \int_{\delta}^{0} S_{rz}^{0}(\xi, 1) d\xi$$

$$+ \int_{+1}^{-1} S_{rr}^{0}(0, z) dz = 0$$

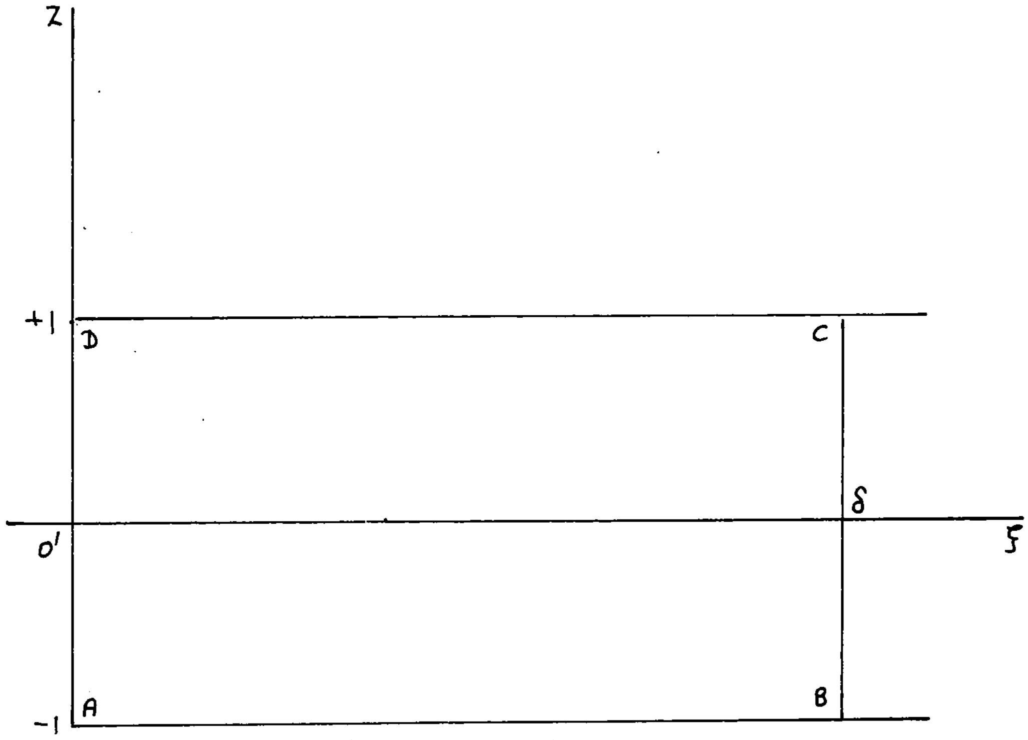

figure (v)

and using (2.91 d)

$$\int_{-1}^{1} S_{rr}^{\,0}(\delta,z)dz = \int_{1}^{-1} S_{rr}^{\,0}(0,z)dz.$$

Hence

$$\left|\int_{-1}^{1} S_{rr}^{\,0}(0,z)dz\right| = \left|\int_{-1}^{1} S_{rr}^{\,0}(\delta,z)dz\right|$$

$< 2\varepsilon$ by (2.93)

and result (2.92) follows immediately.

Application of (2.92) to (2.91 a) yields $C^0 = 0$ and inspection of equations (2.74), (2.90), (2.91) and (2.89) gives

$$\begin{aligned} \gamma_{rr}^{\,0} &= \gamma_{\theta\theta}^{\,0} = \gamma_{zz}^{\,0} = \gamma_{rz}^{\,0} = 0 \\ p_{rr}^{\,0} &= p_{\theta\theta}^{\,0} = p_{zz}^{\,0} = p_{rz}^{\,0} = 0 \end{aligned} \qquad (2.94)$$

and the zeroth order secondary stress coefficients are known. We observe that no contribution to the complete stress solution is made.

2.13 The First Order Secondary Stress Coefficients

The equations for the first order secondary stress coefficients are in fact identical in form to those of the zeroth order and thus by the same analysis

$$\begin{aligned} \gamma_{rr}^{\,1} &= \gamma_{\theta\theta}^{\,1} = \gamma_{zz}^{\,1} = \gamma_{rz}^{\,1} = 0 \\ p_{rr}^{\,1} &= p_{\theta\theta}^{\,1} = p_{zz}^{\,1} = p_{rz}^{\,1} = 0 \end{aligned} \qquad (2.95)$$

and once again no contribution to the complete solution is made.

2.14 The Second Order Secondary Stress Coefficients

Putting $n = 2$ in (2.81), (2.82), (2.83) and making use of (2.94), (2.95) we obtain

$$\frac{\partial}{\partial \xi} p_{rr}^{2} - \frac{\partial}{\partial z} p_{rz}^{2} = 0$$

$$\frac{\partial}{\partial \xi} p_{rz}^{2} - \frac{\partial}{\partial z} p_{zz}^{2} = 0$$

$$L_0^2 \, p_{rr}^{2} + \frac{\partial^2 \Delta^2}{\partial \xi^2} = 0 \qquad (2.96)$$

$$L_0^2 \, p_{\theta\theta}^{2} = 0$$

$$L_0^2 \, p_{zz}^{2} + \frac{\partial^2 \Delta^2}{\partial z^2} = 0$$

$$L_0^2 \, p_{rz}^{2} - \frac{\partial^2 \Delta^2}{\partial \xi \partial z} = 0$$

$$(1+\nu)\Delta^2 = p_{rr}^{2} + p_{\theta\theta}^{2} + p_{zz}^{2}$$

$$L_0^2 \, \Delta^2 = 0$$

The "edge" and "face" boundary conditions from (2.84), (2.67) are

$$p_{rr}^{2}(0,z) = \frac{(3z^2 - 1)}{6(1-\nu)} \left[\nabla_1^2 \, T - (1-\nu) \frac{1}{r} \frac{dT}{dr} \right]_{r=1}$$

$$p_{rz}^{2}(0,z) = 0 \qquad (2.97)$$

$$p_{zz}^{2}(\xi, \pm 1) = p_{rz}^{2}(\xi, \pm 1) = 0$$

and the conditions at infinity from (2.85) are

$$p_{rr}^{\ 2}\ (\infty,\ z) = p_{\theta\theta}^{\ 2}\ (\infty,z) = C^2$$
$$p_{zz}^{\ 2}\ (\infty,\ z) = p_{rz}^{\ 2}\ (\infty,z) = 0 \qquad (2.98)$$

As in Section 2.12 it is convenient to introduce stress coefficients $S^2(\xi,z)$ defined by

$$S_{rr}^{\ 2} = p_{rr}^{\ 2} - C^2,\ S_{\theta\theta}^{\ 2} = p_{\theta\theta}^2 - C^2$$
$$S_{zz}^{\ 2} = p^2,\ S_{rz}^{\ 2} = -\ p_{rz}^{\ 2} \qquad (2.99)$$

Substituting (2.99) into (2.96) we obtain

$$\frac{\partial}{\partial\xi} S_{rr}^{\ 2} + \frac{\partial}{\partial z} S_{rz}^{\ 2} = 0$$

$$\frac{\partial}{\partial\xi} S_{rz}^{\ 2} + \frac{\partial}{\partial z} S_{zz}^{\ 2} = 0$$

$$L_0^2\ S_{rr}^{\ 2} + \frac{\partial^2\Omega^2}{\partial\xi^2} = 0$$

$$L_0^2\ S_{\theta\theta}^{\ 2} = 0$$

$$L_0^2\ S_{\theta\theta}^{\ 2} + \frac{\partial^2\Omega^2}{\partial z^2} = 0 \qquad (2.100)$$

$$L_0^2\ S_{rz} + \frac{\partial^2\Omega^2}{\partial\xi\partial z} = 0$$

$$(1+\nu)\ \Omega^2 = S_{rr}^{\ 2} + S_{\theta\theta}^{\ 2} + S_{zz}^{\ 2}$$

$$L_0^2\ \Omega^2 = 0$$

The boundary conditions (2.97) may be written

$$S_{rr}^{\ 2}(0,z) = \frac{(3z^2-1)}{6(1-\nu)} \left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1} - C^2$$

$$S_{rz}^{\ 2}(0,z) = 0 \qquad (2.101\ a\text{-}d)$$

$$S_{zz}^{\ 2}(\xi,\underline{+}1) = S_{rz}^{\ 2}(\xi,\underline{+}1) = 0$$

and the conditions at infinity (2.98)

$$S^2(\infty,z) = 0 \qquad (2.101\ e)$$

The difficulty arising from the appearance in (2.101 a) of the unknown constant C^2 is once again overcome by proceeding as in Section 2.12 and proving

$$\int_{-1}^{1} S_{rr}^{\ 2}(0,z)dz = 0 \qquad (2.102)$$

Application of (2.102) to (2.101 a) yields $C^2 = 0$ and hence from (2.74)

$$\gamma_{rr}^{\ 2} = \gamma_{\theta\theta}^{\ 2} = \gamma_{zz}^{\ 2} = \gamma_{rz}^{\ 2} = 0 \qquad (2.103)$$

We introduce now a stress function $\psi(\xi,z)$, the solution of the following boundary value problem:

$$L_0^4\ \psi = 0 \qquad \xi>0,\ |z|<1$$

$$\frac{\partial^2\psi}{\partial z^2}(0,z) = \tfrac{1}{2}(3z^2-1), \qquad \frac{\partial^2\psi}{\partial\xi\partial z}(0,z) = 0$$

$$\frac{\partial^2\psi}{\partial\xi^2}(\xi,\underline{+}1) = \frac{\partial^2\psi}{\partial\xi\partial z}(\xi,\underline{+}1) = 0 \qquad (2.104\ a\text{-}f)$$

$\psi(\infty,z) = 0$

where $L_0^4 = \frac{\partial^4}{\partial\xi^4} + 2\frac{\partial^4}{\partial\xi^2\partial z^2} + \frac{\partial^4}{\partial z^4}$

We shall refer to (2.104) as the auxiliary problem, the required solution to which is given in Appendix 3.

If we now write

$$p_{rr}^{2}(\xi,z) = \frac{1}{3(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1}\frac{\partial^2\psi}{\partial z^2}$$

$$p_{\theta\theta}^{2}(\xi,z) = \frac{\nu}{3(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1} L_0^2\,\psi$$

$$p_{zz}^{2}(\xi,z) = \frac{1}{3(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1}\frac{\partial^2\psi}{\partial \xi^2} \qquad (2.105)$$

$$p_{rz}^{2}(\xi,z) = \frac{1}{3(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1}\frac{\partial^2\psi}{\partial\xi\partial z}$$

equations (2.96), (2.97) and (2.98) are all satisfied and the second order secondary stress coefficients are known. These coefficients are the first non-zero contributions to the complete solution, and the secondary stress system to terms in h^2 may be written

$$p_{rr}\left(\frac{1-r}{h},z\right) = \frac{h^2}{3(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1}\left[\frac{\partial^2\psi}{\partial z^2}\right] + \ldots$$

$$p_{\theta\theta}\left(\frac{1-r}{h},z\right) = \frac{h^2\nu}{3(1-\nu)}\left[\nabla_1^2 T - (1-\nu)\frac{1}{r}\frac{dT}{dr}\right]_{r=1}\left[L_0^2\,\psi\right]_{\xi=\frac{1-r}{h}} + \ldots$$

(2.106)

$$p_{zz}\left(\frac{1-r}{h}, z\right) = \frac{h^2}{3(1-\nu)} \left[\nabla_1^2\, T - (1-\nu)\, \frac{1}{r}\, \frac{dT}{dr}\right]_{r=1} \left[\frac{\partial^2\psi}{\partial\xi^2}\right]_{\xi=\frac{1-r}{h}} + \ldots$$

$$p_{rz}\left(\frac{1-r}{h}, z\right) = \frac{h^2}{3(1-\nu)} \left[\nabla_1^2\, T - (1-\nu)\, \frac{1}{r}\, \frac{dT}{dr}\right]_{r=1} \left[\frac{\partial^2\psi}{\partial\xi\,\partial z}\right]_{\xi=\frac{1-r}{h}} + \ldots$$

2.15 The Complete Solution

Superposing the primary and secondary stress systems we obtain the complete solution to the problem of the disc in the form of an asymptotic power series in h, the thickness/ diameter ratio Symbolically

$$\sigma(r,z) = \tau(r,z) + p\left(\frac{1-r}{h}, z\right) \tag{2.107}$$

where τ, p are given by (2.67) and (2.106) respectively.

Comparison of (2.107) with the two dimensional treatment, given for example in Timoshenko and Goodier (1951) p.406, shows that the two dimensional solution plays the not unexpected role of the zeroth order term in the series solution. Consequently we may conclude, that for sufficiently small h the two dimensional theory will give accurate results.

It is also clear that the nature of the complete solution is such that the higher order terms provide three dimensional corrections to the two dimensional theory. Indeed, these correction terms are of order h^2 and thus the two dimensional theory is actually more accurate than would normally be anticipated.

Furthermore, it is of interest to observe that even including the correction terms, the solution is plane stress except in the boundary layer, the analysis thus providing additional justification to the St. Venant Principle.

CHAPTER III

Example and Discussion

3.1 Preliminary Remarks

When the small parameter h is fixed, it is of interest to use the asymptotic solution as an approximation to the true solution to the problem of the disc. In this chapter we discuss the approximate solution in relation to the specific case $T(r) = r^6$. The choice of temperature field is motivated by the fact that it has quite severe gradients (figure vii), yet has sufficiently simple form to permit analytic discussion. We further choose $h = 0.05$, $\nu = 0.3$ as being fairly typical values.

3.2 The Approximate Solution for $T(r) = r^6$

Putting $T(r) = r^6$ in (2.67), (2.107) we obtain for the primary and secondary solutions respectively,

$$\tau_{rr} = \frac{1}{8}(1 - r^6) + \frac{(1 - 3z^2)}{(1 - \nu)}(5 + \nu)\, r^4 h^2 + \ldots$$

$$\tau_{\theta\theta} = \frac{1}{8}(1 - 7r^6) + \frac{(1 - 3z^2)}{(1 - \nu)}(1 + 5\nu)\, r^4 h^2 + \ldots$$

$$(3.1\ \text{a-d})$$

$$\tau_{zz} = 0h^2 + \ldots$$

$$\tau_{rz} = 0h^2 + \ldots$$

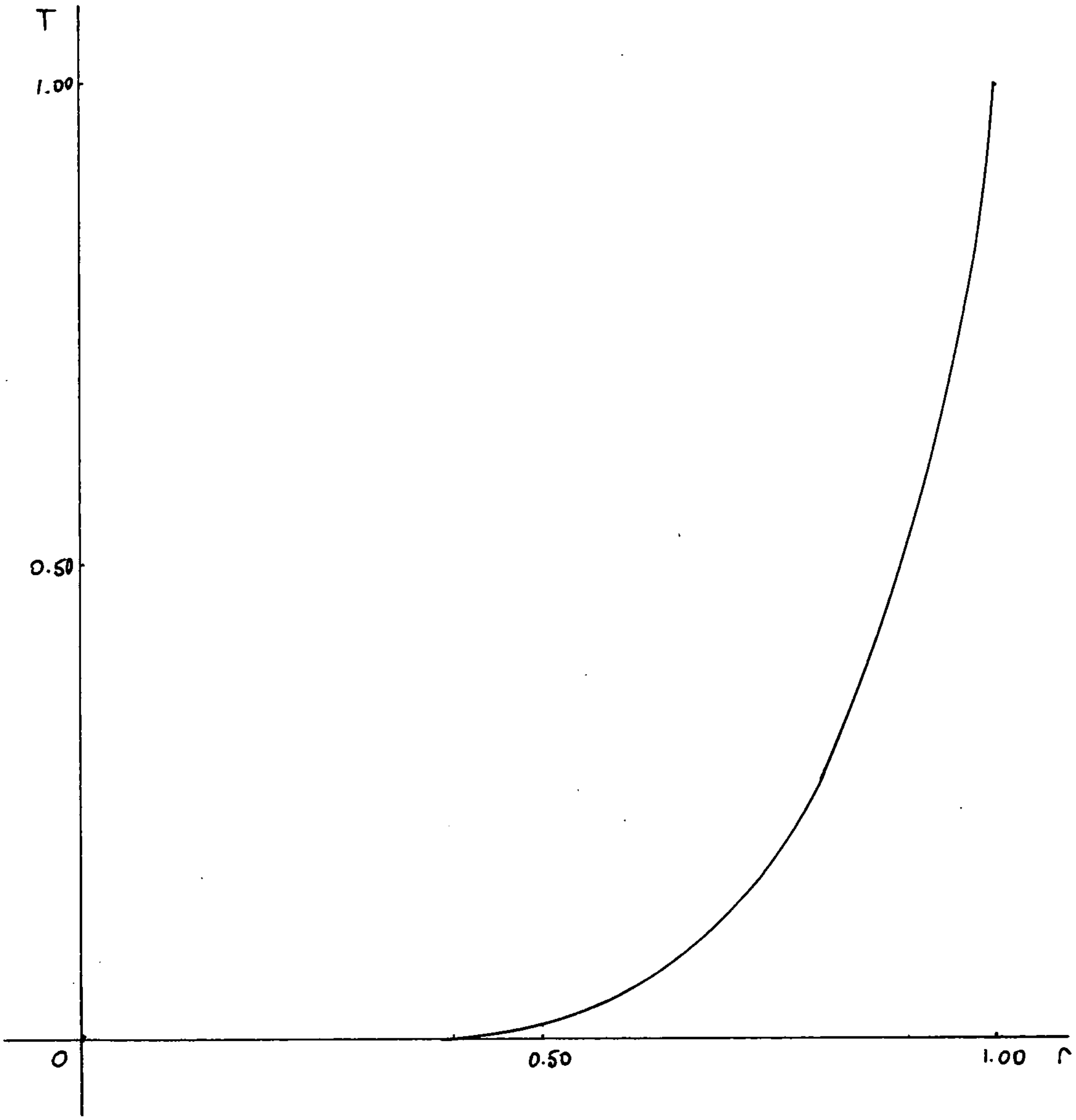

Figure (vi)

$$p_{rr} = \frac{2(5+\nu)}{(1-\nu)} \; h^2 \frac{\partial^2 \psi}{\partial z^2} + \ldots$$

$$p_{\theta\theta} = \frac{2\nu(5+\nu)}{(1-\nu)} \; h^2 L_0^2 \; \psi + \ldots$$

$$p_{zz} = \frac{2(5+\nu)}{(1-\nu)} \; h^2 \frac{\partial^2 \psi}{\partial \xi^2} + \ldots \qquad (3.2\ \text{a-d})$$

$$p_{rz} = \frac{2(5+\nu)}{(1-\nu)} \; h^2 \frac{\partial^2 \psi}{\partial \xi \partial z} + \ldots$$

In order to assess the accuracy of the (approximate) series solution, obtained by superposing (3.1) and (3.2), we adopt the following approach. We do not attempt to obtain, for the purpose of comparison, the exact three dimensional solution of the case under consideration as this is likely to present great difficulties. Instead, we determine, with considerably more ease, the exact primary solution. Such a solution, by virtue of the Saint Venant Principle errs from the true solution only in the vicinity of the edge of the disc, and consequently comparison with the approximate primary solution (3.1) should yield a useful estimate of the degree of accuracy of the series solution.

3.3 The Exact Primary Solution for $T(r) = r^6$

Using the results and the relevant equations of section

2.7 it is seen that for the case $T = r^6$, the series representations for τ_{zz}, τ_{rz} terminate at $n = 6$, $n = 5$ respectively, giving exact solutions

$$\tau_{zz} = \frac{1}{5(1-\nu)}\left[-120(z^4 - 2z^2 + 1)r^2h^4 + 32(z^6 - 5z^4 + 7z^2 - 3)h^6\right] \tag{3.3}$$

$$\tau_{rz} = \frac{1}{5(1-\nu)}\left[120(z^3 - z)r^3h^3 - 32(3z^5 - 10z^3 + 7z)rh^5\right] \tag{3.4}$$

We now proceed to systematically analyse equations (2.30), (2.31), (2.32) and (2.33) to obtain the primary stresses τ_{rr}, $\tau_{\theta\theta}$.

Putting $T = r^6$ and substituting (3.3) into (2.31 c) gives

$$\frac{\partial^2 \textcircled{H}}{\partial z^2} = -\frac{1}{5(1-\nu)}\left[180r^4h^2 - 480(3z^2 - 1)r^2h^4 + 32(15z^4 - 30z^2 - 1)h^6\right] \tag{3.5}$$

and (3.4), (2.31 d) yield

$$\frac{\partial^2 \textcircled{H}}{\partial r \partial z} = -\frac{1}{5(1-\nu)}\left[720zr^3h^2 - 960(z^3 - z)rh^4\right] \tag{3.6}$$

Integrating (3.6) we obtain

$$\textcircled{H} = -\frac{1}{5(1-\nu)}\left[90z^2r^4h^2 - 120(z^4 - 2z^2)r^2h^4 + F(r) + G(z)\right] \tag{3.7}$$

where F, G are functions of the integration.

Substituting (3.7) into (3.5) now gives

$$\frac{d^2G}{dz^2} = 32(15z^4 - 30z^2 - 1)\, h^6.$$

Integrating twice we get

$$G(z) = 16(z^6 - 5z^4 - z^2) + Az + B$$

where A, B are constants.

We recall that τ_{rr}, $\tau_{\theta\theta}$ and hence $\textcircled{H}$ are even in z and thus $A = 0$. Further the constant B may be considered as part of $F(r)$ and consequently

$$G(z) = 16(z^6 - 5z^4 - z^2) \tag{3.8}$$

Equations (3.8), (3.7) now give

$$\textcircled{H} = \frac{1}{5(1-\nu)} \left[90z^2r^4h^2 - 120(z^4 - 2z^2)r^2h^4 + 16(z^6 - 5z^4 - z^2)h^6 + F(r) \right] \tag{3.9}$$

Using (3.9), (2.32 b) we obtain

$$\nabla_1^2 F + 480\, r^2 h^2 - 32\, h^4 = 0$$

from which

$$F(r) = -30r^4h^2 + 8r^2h^4 + C + D \log r.$$

Assuming $\textcircled{H}$ to be finite at the origin, $D = 0$ and

$$F(r) = -30r^4h^2 + 8r^2h^4 + C$$

giving from (3.9)

$$\textcircled{H} = -\frac{1}{5(1-\nu)} \left[30(3z^2 - 1)r^4h^2 - 8(15z^4 - 30z^2 - 1)\, r^2h^4 + 16(z^6 - 5z^4 - z^2)h^6 + C \right] \tag{3.10}$$

From equations (3.4), (2.30 a)

$$\frac{\partial}{\partial r}\tau_{rr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) + \frac{1}{5(1-\nu)}\left[120(3z^2-1)r^3h^2-32(15z^4-30z^2+7)rh^4\right] = 0$$

which equation is identically satisfied if we introduce a stress function $\phi(r,z)$ such that

$$\begin{aligned}\tau_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{1}{5(1-\nu)}\left[30(3z^2-1)r^4h^2 - 16(15z^4-30z^2+7)r^2h^4\right] \\ \tau_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} - \frac{1}{5(1-\nu)}\left[30(3z^2-1)r^4h^2 - 16(15z^4-30z^2+7)r^2h^4\right]\end{aligned} \quad (3.11\ a,b)$$

Equations (3.11), (2.31a) give

$$\frac{1}{r}\frac{\partial}{\partial r}\left[h^2\nabla_1^2\phi + \frac{\partial^2\phi}{\partial z^2}\right] = -\frac{1}{5(1-\nu)}\left[120(3z^2-1)r^2h^4+48(25z^4-50z^2+9)h^6\right]$$

(3.12 a)

and (3.11), (2.31 b) give

$$\frac{\partial^2}{\partial r^2}\left[h^2\nabla_1^2\phi + \frac{\partial^2\phi}{\partial z^2}\right] = -\frac{1}{5(1-\nu)}\left[360(3z^2-1)r^2h^4+48(25z^4-50z^2+9)h^6\right]$$

(3.12 b)

Integrating (3.12 b) twice and using (3.12 a) yields

$$h^2\nabla_1^2\phi + \frac{\partial^2\phi}{\partial z^2} = -\frac{1}{5(1-\nu)}\left[30(3z^2-1)r^4h^4+24(25z^4-50z^2+9)r^2h^6+H(z)\right]$$

(3.13)

where H is a function of integration.

Equations (2.32 a), (3.11), (3.3) give

$$\nabla_1^2\phi = -\frac{1}{5(1-\nu)}\Big[5(1-\nu)r^6-30(1-\nu)(3z^2-1)r^4h^2+8\{15(2-\nu)z^4-30(2-\nu)z^2$$
$$+(14+\nu)\}r^2h^4+16\{(3+\nu)z^6-5(3+\nu)z^4+(13-\nu)z^2-6\}h^6+(1+\nu)C\Big]$$

(3.14)

Combining this equation with (3.13) we obtain

$$\frac{\partial^2\phi}{\partial z^2} = \frac{1}{5(1-\nu)}\Big[5(1-\nu)r^6h^2-30(3z^2-1)(2-\nu)r^4h^2$$
$$-8\{15(3+\nu)z^4-30(3+\nu)z^2+(13-\nu)\}r^2h^6+L(z)\Big]$$

where $L(z)$ is an unknown function of z.

Integrating twice and observing that ϕ must be even in z we obtain

$$\phi = \frac{1}{5(1-\nu)}\Big[\frac{5(1-\nu)}{2}r^6\ z^2\ h^2 - \frac{15}{2}(2-\nu)(z^4-2z^2)r^2h^4$$
$$-4\{(3+\nu)z^6-5(3+\nu)z^4+(13-\nu)z^2\}r^2h^6+M(z)+\psi(r)\Big] \tag{3.15}$$

where M, ψ are arbitrary functions

From (3.14)

$$\frac{\partial}{\partial r}(\nabla_1^2\phi) = -\frac{1}{5(1-\nu)}\Big[30(1-\nu)r^5-120(1-\nu)(3z^2-1)r^3h^2$$
$$+16\{15(2-\nu)z^4-30(2-\nu)z^2+(14+\nu)\}r^2h^4\Big]$$

and this equation together with (3.15) gives

$$\frac{d}{dr}(\nabla_1^2\psi) = -30(1-\nu)r^5 - 120(1-\nu)r^3h^2-16(4+\nu)rh^4$$

from which we obtain

$$\psi(r) = -\frac{5}{64}(1-\nu)r^8 - \frac{5}{6}(1-\nu)r^6h^2 - \frac{1}{2}(14+\nu)r^4h^4 + \frac{1}{2}Dr^2 + E \tag{3.16}$$

where D, E are constants of integration and where we have rejected any logarithmic terms due to our assumption of finite stresses.

Equations (3.11), (3.16), (3.15) now yield

$$\tau_{rr} = \frac{1}{5(1-\nu)}\Big[-\frac{5}{8}(1-\nu)r^6 - 5(5+\nu)(3z^2-1)r^4h^2$$
$$+\{30(6+\nu)z^4 - 60(6+\nu)z^2 + (84-2\nu)\}r^2h^4$$
$$-8\{(3+\nu)z^6 - 5(3+\nu)z^4 + (13-\nu)z^2\}h^6 + D\Big]$$

(3.17 a)

and from (2.30 a), (3.4)

$$\tau_{\theta\theta} = \frac{1}{5(1-\nu)}\Big[-\frac{35}{8}(1-\nu)r^6 - 5(1+5\nu)(3z^2-1)r^4h^2$$
$$+\{30(2+3\nu)z^4 - 60(2+3\nu)z^2 + (28-6\nu)\}r^2h^4$$
$$-8\{(3+\nu)z^6 - 5(3+\nu)z^4 + (13-\nu)z^2\}h^6 + D\Big]$$

(3.17 b)

Applying (2.33 a) to (3.17 a) gives

$$D = \frac{5}{8}(1-\nu) + 16\nu\, h^4 - \frac{8}{21}(37-25\nu)h^6$$

and hence from (3.17)

$$\tau_{rr}(r,z) = \frac{1}{5(1-\nu)}\Big[\frac{5}{8}(1-\underline{\underline{\nu}})(1-r^6) - h^2\{5(5+\nu)(3z^2-1)r^4\}$$
$$+h^4\{[30(6+\nu)z^4-60(6+\nu)z^2+(84-2\nu)]r^2+16\nu\}$$
$$-8h^6\{(3+\nu)z^6-5(3+\nu)z^4+(13-\nu)z^2 - \frac{1}{21}(37-25\nu)\}\Big]$$

(3.18)

$$\tau_{\theta\theta}(r,z) = \frac{1}{5(1-\nu)}\Bigg[\frac{5}{8}(1-\nu)(1-7r^6) - h^2\{5(1+5\nu)(3z^2-1)r^4\}$$
$$+h^4\{[30(2+3\nu)z^4 - 60(2+3\nu)z^2 +(28-6\nu)]r^2+16\nu\}$$
$$-8h^6\{(3+\nu)z^6 - 5(3+\nu)z^4 +(13-\nu)z^2 - \frac{1}{21}(37-25\nu)\}\Bigg] \tag{3.19}$$

Equations (3.18), (3.19), (3.3), (3.4) constitute the exact primary solution for the case $T(r) = r^6$.

3.4 Comparison of the Exact and Approximate Primary Solutions

The exact and approximate primary solutions have been calculated from (3.3), (3.4), (3.18), (3.19) and (3.1) with $\nu = 0.3$ and $h = 0.05$. The results, correct to five decimal places are exhibited in Tables XII to XV. (Appendix 4)

Inspection of these results shows that for the stresses τ_{rr}, $\tau_{\theta\theta}$, τ_{zz} there is an overall agreement of the exact and approximate solutions to three decimal places and indeed the accuracy is much greater at many points of the disc. The stress τ_{rz} too is accurate to three decimal places except in the edge layer where the last decimal place is subject to an error of $\pm$ 2. That the series solution for τ_{rz} is slightly less accurate than for the other stresses is to be expected as the error term is of order h^3 as opposed to h^4.

In accordance with the remarks made in section 3.2, we shall assume that if the series solution is calculated correct to three decimal places the difference from the true solution will be negligible to this degree of accuracy.

3.5 Comparison of the Two and Three Dimensional Solutions

The calculations relevant to the comparison of the two and three dimensional solutions have been executed and the results displayed in Tables XVI to XXI (Appendix 4). We further remark that the error of 1% in the auxiliary solution, referred, to in Appendix 3 is of no consequence, the actual errors being sufficiently small. The three dimensional correction terms of any importance are not unexpectedly confined to the region in which the temperature gradients are the greatest. In the case of the shear stress σ_{rz}, the correction term is uniformly small and therefore this stress may be considered unimportant. The direct stress σ_{zz} is also small with the exception of a rather sharp increase in magnitude at the edge, particularly in the neighbourhood of the point $r = 1.0$, $z = 0.0$. Consideration of the radial stress σ_{rr} and the hoop stress $\sigma_{\theta\theta}$ shows that the two and three dimensional solutions differ most on the planes $z = 1.0$ and $z = 0.0$, particularly in the neighbourhood of the edge of the boundary layer, and exhibit close agreement near to the plane $z = 0.6$. It is further interesting to observe that in the case of the radial stress the two dimensional solution overestimates the stress on $z = 1.0$ but underestimates it on $z = 0.0$, whereas in the case of the hoop stress the reverse effect is true. The fact that the effect is reversed is no doubt due to the two stresses having opposite sign in the region under consideration, the radial stress being tensile and the hoop stress compressive.

To say generally whether or not the correction terms are important is not possible as no criterion on which such a decision can be made, has been given. However, it should be pointed out that particularly in the case of the radial stress, the relative increases or decreases are considerable. For example at the point $r = 0.9$, $z = 1.0$ the two dimensional solution overestimates the stress by as much as 74%. Admittedly such a figure is exaggerated by the fact that at this point the radial stress is quite small. Nevertheless, the fact is worthy of mention.

A graphical illustration relevant to the above observations on the radial stress is given in Figure (vii).

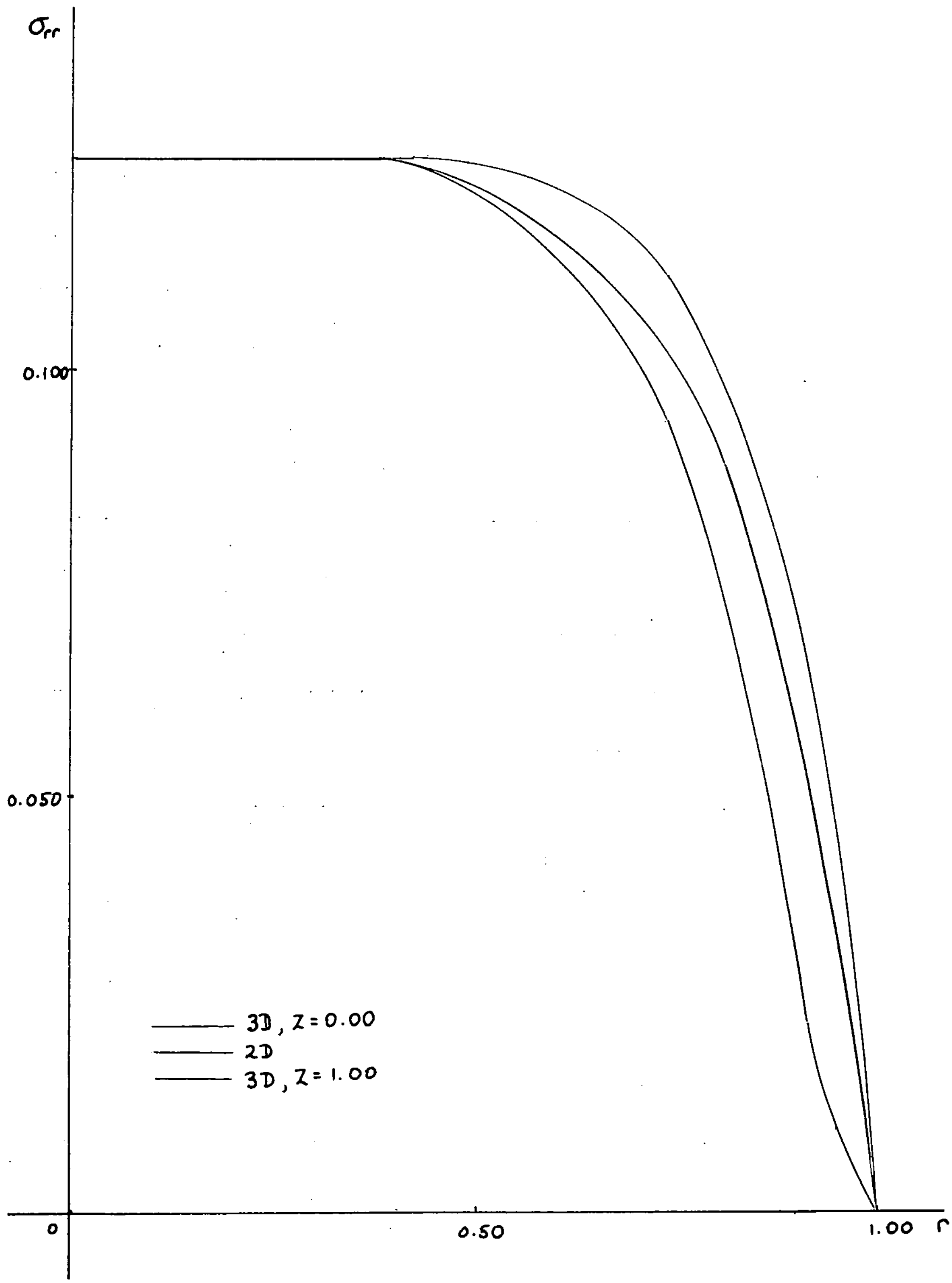

Figure (vii)

CONCLUSION

A method for the determination, within the full three dimensional theory, of the thermal stress in a thin circular disc subject to purely radial heat flow, has been given.

For sufficiently thin plates it is verified that a solution based on two dimensional theory will give accurate results, and furthermore if second order three dimensional correction terms are included, the solution is still plane stress except in the boundary layer.

The method has been applied, in a specific example, to compare the two and three dimensional theories. The accuracy of the series solution has been considered and numerical results produced.

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APPENDIX I

On the Assumption of Plane Stress

A widely used procedure in attempting to find a solution of the three dimensional thermoelastic equations, is the semi-inverse method, in which part of the solution is assumed and the remainder found so as to satisfy all the required equations. The uniqueness aspect of the formulation, referred to in section 1.4 then ensures that the solution so obtained is the correct one.

Referring now to section 2.4 and equations (2.22) we note that σ_{zz}, σ_{rz} are zero on the faces of the disc and it is therefore reasonable to assume that as the thickness of the disc is small, the stresses σ_{zz}, σ_{rz} are zero everywhere. Such a state of stress in which the stress vector has no component in a particular direction, in this case the z direction, is referred to as a state of plane stress.

The equations, from which the remaining stress components are to be calculated are, from (2.25), (2.26), (2.27) and (2.28) in non-dimensional terms,

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \qquad \text{(A1.1)}$$

$$h^2\left[\nabla_1^2\sigma_{rr} + \frac{\partial^2\Sigma}{\partial r^2} + \frac{1}{(1-\nu)}\nabla_1^2 T - \frac{2}{r^2}(\sigma_{rr} - \sigma_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2}\sigma_{rr} = 0$$

$$h^2 \left[\nabla_1^2 \sigma_{\theta\theta} + \frac{1}{r}\frac{\partial\Sigma}{\partial r} + \frac{1}{(1-\nu)} \nabla_1^2 T + \frac{2}{r^2} (\sigma_{rr} - \sigma_{\theta\theta})\right] + \frac{\partial^2}{\partial z^2} \sigma_{\theta\theta} = 0$$

(A1.2 a-d)

$$h^2 \left[\frac{1}{(1-\nu)} \nabla_1^2 T\right] + \frac{\partial^2\Sigma}{\partial z^2} = 0$$

$$\frac{\partial^2\Sigma}{\partial r\partial z} = 0$$

$$(1+\nu)\Sigma = \sigma_{rr} + \sigma_{\theta\theta} + T$$

(A1.3 a,b)

$$h^2 \left[\nabla_1^2 \Sigma + \frac{1}{(1-\nu)} \nabla_1^2 T\right] + \frac{\partial^2\Sigma}{\partial z^2} = 0$$

$$\sigma_{rr}(1,z) = 0 \qquad \text{(A1.4)}$$

Integrating (A1.2 d) gives

$$\Sigma = f(r) + g(z) \qquad \text{(A1.5)}$$

where f, g are arbitrary functions of the integration. Substitution into (A1.2 c) leads to

$$\frac{d^2 g}{dz^2} = - \frac{h^2}{(1-\nu)} \nabla_1^2 T = K \qquad \text{(A1.6)}$$

where K is a constant.

Equation (A1.6) imposes a restriction on the possible temperature field for which a rigorous plane stress solution can be found. In fact we have

$$\frac{d^2}{dr^2} T + \frac{1}{r}\frac{dT}{dr} = - \frac{(1-\nu)K}{h^2}$$

from which we obtain

$$T = -\frac{(1-\nu)}{4h^2} Kr^2 + A \log r + B$$

where A, B are constants of integration.

If we write $T(1) = 1$, that is taking the edge temperature as reference, and assume T to be finite at the origin we get

$$A = 0, \ B = 1 + \frac{(1-\nu)K}{4h^2} .$$

Consequently

$$T = 1 + \frac{(1-\nu)K}{4h^2} (1-r^2) \tag{A1.7}$$

is the most general permissible form of T.

Returning now to (A1.6) we have

$$\frac{d^2 g}{dz^2} = K$$

from which we obtain

$$g = \tfrac{1}{2}Kz^2 + Cz + D$$

where C, D are constants of integration. The stresses σ_{rr}, $\sigma_{\theta\theta}$ are even in z and thus from (A1.3 a) so must be Σ and hence g. Thus $C = 0$ and we may consider D as part of $f(r)$ so that

$$g(z) = \tfrac{1}{2} Kz^2.$$

From (A1.5) we get

$$\Sigma = f(r) + \tfrac{1}{2} Kz^2. \tag{A1.8}$$

Using (A1.3 b), (A1.7), (A1.8) we obtain

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} = 0$$

from which

$$f(r) = F \log r + G \tag{A1.9}$$

where F, G are constants of integration.

From (A1.8), (A1.9) and assuming finite stress at the origin, we obtain

$$\Sigma = G + \tfrac{1}{2} Kz^2 \tag{A1.10}$$

Equation (A1.1) is identically satisfied if we introduce a stress function $\phi(r,z)$ such that

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \tag{A1.11 a,b}$$

Using equations (A1.7), (A1.10), (A1.11) equations (A1.2 a,b) may be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left[h^2 \nabla_1^2 \phi + \frac{\partial^2\phi}{\partial z^2}\right] = K$$

$$\frac{\partial^2}{\partial r^2}\left[h^2 \nabla_1^2 \phi + \frac{\partial^2\phi}{\partial z^2}\right] = K \tag{A1.12 a,b}$$

Integrating (A1.12 b) twice and using (A1.12 a) yields

$$h^2 \nabla_1^2 \phi + \frac{\partial^2\phi}{\partial z^2} = \tfrac{1}{2} Kr^2 + H(z) \tag{A1.13}$$

where $H(z)$ is a function of the integration.

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Equations (A1.11), (A1.3a), (A1.7), (A1.10) give

$$(1+\nu)\{G + \tfrac{1}{2} Kz^2\} = \nabla_1^2 \phi + 1 + \frac{(1-\nu)K}{4h^2} (1-r^2) \qquad (A1.14)$$

and (A1.13) becomes

$$\frac{\partial^2\phi}{\partial z^2} = \frac{1}{4} (1+\nu)Kr^2 + L(z)$$

where L is an arbitrary function of z.

Integrating twice and observing that ϕ must be even in z we get

$$\phi = \frac{1}{8} (1+\nu)Kr^2 z^2 + M(z) + \chi(r) \qquad (A1.15)$$

where M, χ are arbitrary functions.

It follows from (A1.11), that

$$\sigma_{rr} = \frac{1}{4} (1+\nu)Kz^2 + \frac{1}{r}\frac{d\chi}{dr}$$

$$\sigma_{\theta\theta} = \frac{1}{4} (1+\nu)Kz^2 + \frac{d^2\chi}{dr^2}$$

The function χ may be determined by differentiating (A1.14) w.r.t. r and using (A1.15) to get

$$\frac{d}{dr} (\nabla_1^2 \chi) = \frac{(1-\nu)Kr}{2h^2}$$

Thence we obtain

$$\chi(r) = \frac{(1-\nu)Kr^4}{64h^2} + N r^2 + P \qquad (A1.17)$$

where N,P are constants and where we have rejected any logarithmic terms due to the assumption of finite stresses.

(A1.16), (A1.17) now give

$$\sigma_{rr} = \frac{1}{4}\left\{(1+\nu)Kz^2 + \frac{(1-\nu)Kr^2}{16h^2} + 2N\right\}$$
(A1.18 a,b)
$$\sigma_{\theta\theta} = \frac{1}{4}\left\{(1+\nu)Kz^2 + \frac{3(1-\nu)Kr^2}{16h^2} + 2N\right\}$$

It is clear from the form of (A1.18a) that the boundary condition (A1.4) can only be strictly satisfied when $K = 0$ corresponding to the trivial case of uniform temperature and zero stress. However, we may apply the Principle of Saint Venant and require instead of (A1.4),

$$\int_{-1}^{1} \tau_{rr}\,(1,z)\,dz = 0 \qquad \text{(A1.19)}$$

The symbol τ, rather than σ has been used in order to be consistent with the notation of Section (2.5).

Application of (A1.19) to (A1.18a) yields N and we obtain finally

$$\tau_{rr}(r,z) = \frac{(1+\nu)}{12}\,K(3z^2-1) - \frac{(1-\nu)K}{16h^2}\,(1-r^2)$$
(A1.20 a,b)
$$\tau_{\theta\theta}(r,z) = \frac{(1+\nu)}{12}\,K(3z^2-1) - \frac{(1-\nu)K}{16h^2}\,(1-3r^2)$$

The constant K is of course known from (A1.7).

Equations (A1.7), (A1.20) give the solution to the problem, the expressions being valid everywhere except in the vicinity of the edge of the disc.

It appears from the above analysis that the plane stress assumption may be judged inadequate on two counts:

(i) as a consequence of the assumption, the temperature distribution, for which a rigorous solution is possible is restricted to the form given by (A1.7), and

(ii) because the exact boundary conditions cannot be satisfied for a non-trivial case, the St. Venant Principle has to be invoked, and as a result, the nature of the stresses in the immediate neighbourhood of the edge of the disc remains unknown.

APPENDIX II

The Secondary Interior Stress Coefficients

Theorem 1

$$\gamma_{zz}^{n} = \gamma_{rz}^{n} = \frac{\partial^2\Gamma^n}{\partial z^2} = 0 \qquad n = 0,1,2, \ldots \tag{A2.1}$$

Proof:

The proof is by induction. We make the inductive assumption that

$$\gamma_{zz}^{n} = \gamma_{rz}^{n} = \frac{\partial^2\Gamma^n}{\partial z^2} = \gamma_{zz}^{n+1} = \gamma_{rz}^{n+1} = \frac{\partial^2\Gamma^{n+1}}{\partial z^2} = 0 \tag{A2.2}$$

Using equations (2.69b), (2.70c,d), (2.72c,d) and (A2.2) we obtain

$$\begin{aligned} &\frac{\partial}{\partial z}\gamma_{zz}^{n+2} = 0 \\ &\frac{\partial^2\Gamma^{n+2}}{\partial z^2} = -\frac{\partial^2\gamma^{n+2}}{\partial z^2} \\ &\frac{\partial^2\gamma_{rz}^{n+2}}{\partial z^2} = -\frac{\partial^2\Gamma^{n+1}}{\partial r\partial z} \end{aligned} \tag{A2.3a,b,c}$$

$$\gamma_{zz}^{n+2}(r,\pm 1) = \gamma_{rz}^{n+2}(r,\pm 1) = 0 \tag{A2.4a,b}$$

From (A2.3a), (A2.4a) we get

$$\gamma_{zz}^{n+2} = 0 \tag{A2.5a}$$

and using this result in (A2.3b) gives

$$\frac{\partial^2\Gamma^{n+2}}{\partial z^2} = 0 \tag{A2.5b}$$

Differentiating (A2.3c) w.r.t z and using (A2.2)

$$\frac{\partial^2 \gamma_{rz}^{n+2}}{\partial z^2} = 0$$

Integrating three times, using the odd in z property and (A.24b) we get

$$\gamma_{rz}^{n+2} = 0 \qquad \text{(A2.5c)}$$

The inductive assumption is known to be valid for $n = 0, 1$ (see 2.73) and the proof follows by induction with reference to (A2.2), (A2.5).

<u>Theorem 2</u>

$$\gamma_{rr}^{n} = \gamma_{\theta\theta}^{n} = C^{n} \quad n = 0,1,2, \ldots \qquad \text{(A2.6)}$$

Proof:

The proof is by induction. We made the inductive assumption that

$$\gamma_{rr}^{n} = \gamma_{\theta\theta}^{n} = C^{n}, \quad \gamma_{rr}^{n+1} = \gamma_{\theta\theta}^{n+1} = C^{n+1} \qquad \text{(A2.7)}$$

Using equations (2.69a), (2.70a,b), (2.71), (A2.1) and (A2.7) we obtain

$$\frac{\partial}{\partial r}\gamma_{rr}^{n+2} + \frac{1}{r}(\gamma_{rr}^{n+2} - \gamma_{\theta\theta}^{n+2}) = 0$$

$$\frac{\partial^2}{\partial z^2}\gamma_{rr}^{n+2} + \quad = 0$$

$$\frac{\partial^2}{\partial z^2}\gamma_{\theta\theta}^{n+2} + \quad = 0 \qquad \text{(A2.8a-e)}$$

$$(1+\nu)\Gamma^{n+2} = \gamma_{rr}^{n+2} + \gamma_{\theta\theta}^{n+2}$$

$$\nabla_1^2\, \Gamma^{n+2} = 0$$

Integrating (A2.8b,c) twice and using the even in z property yields

$$\gamma_{rr}^{n+2} = \gamma_{rr}^{n+2}(r), \; \gamma_{\theta\theta}^{n+2} = \gamma_{\theta\theta}^{n+2}(r) \qquad \text{(A2.9a,b)}$$

From (A2.9), (A2.8a) we may write

$$\gamma_{rr}^{n+2} = \frac{1}{r}\frac{d\phi^+}{dr}, \; \gamma_{\theta\theta}^{n+2} = \frac{d^2\phi^+}{dr^2} \qquad \text{(A2.10a,b)}$$

for some function $\phi^+(r)$.

Equations (A2.10), (A2.8d,e) yield

$$\nabla_1^4 \phi^+ = 0$$

the general solution to which is

$$\phi^+ = Ar^2 \log r + B \log r + Dr^2 + E \qquad \text{(A2.11)}$$

where A, B, D, E are constants.

Using (A2.11), (A2.10) and assuming finite stress

$$\gamma_{rr}^{n+2} = 2D = C^{n+2} \text{ say}$$
$$\gamma_{\theta\theta}^{n+2} = 2D = C^{n+2} \text{ say} \qquad \text{(A2.12a,b)}$$

The inductive assumption is known to be valid for $n = 0, 1$ (see 2.73) and the proof follows by induction with reference to (A2.7), (A2.12).

APPENDIX III

The Auxiliary Problem

The solution of the auxiliary problem can be readily obtained using the method and numerical results of Gaydon and Shepherd (1964).

Assume a tentative solution of the form

$$\psi(\xi,z) = e^{\mu\xi}\, Z(z;\mu) \tag{A3.1}$$

the substituting into (2.104a,d,e) gives

$$\begin{aligned} &Z^{iv} + 2\mu^2 Z'' + \mu^4 = 0 \\ &Z(\underline{+}1) = Z'(\underline{+}1) = 0. \end{aligned} \tag{A3.2}$$

A particular solution of (A3.2), even in z is

$$Z(z;\mu) = (\cos 2\mu - 1)\cos \mu z - 2\mu z \sin \mu z \tag{A3.3}$$

where

$$2\mu + \sin 2\mu = 0. \tag{A3.4}$$

With the exception of $\mu = 0$, the roots of (A3.4) are all complex and, in view of (2.104f) we take only those with a negative real part. These occur in conjugate pairs

$$\mu_k = -\, a_k + i\, b_k\ ,\ \bar{\mu}_k = -\, a_k - i\, b_k \qquad k = 1,2,\ldots \tag{A3.5}$$

where a_k, b_k are real positive quantities and $i = \sqrt{-1}$. The first ten values of μ_k are given in Table I.

As the equation (2.104a) is linear we may superpose solutions and consequently obtain from (A3.1), (A3.5) and (A3.3) a general solution of the form

$$\psi(\xi,z) = \sum_{k=1}^{\infty} \left[\frac{(A_k + i\,B_k)}{\mu_k^2} e^{\mu_k \xi} Z_k + \frac{(A_k - i\,B_k)}{\bar{\mu}_k^2} e^{\bar{\mu}_k \xi} \bar{Z}_k \right] \tag{A3.6}$$

where Z_k denotes the Z function corresponding to μ_k, and A_k, B_k are real constants which have to be determined from equations (2.104b,c). These equations give

$$\sum_{k=1}^{\infty} \left[\frac{(A_k + i\,B_k)}{\mu_k^2} Z_k'' + \frac{(A_k - i\,B_k)}{\bar{\mu}_k^2} \bar{Z}'' \right] = \tfrac{1}{2}(3z^2-1)$$

(A3.7a,b)

$$\sum_{k=1}^{\infty} \left[\frac{(A_k + i\,B_k)}{\mu_k} Z_k' + \frac{(A_k - i\,B_k)}{\bar{\mu}_k} Z' \right] = 0$$

Unfortunately the functions Z_k are not orthogonal and thus equations (A3.7) are not useful. However, this difficulty is overcome by Gaydon and Shepherd by expanding each Z_k in terms of a set of suitable functions $Y(z;\lambda)$.

These functions are solutions of the fourth order equation

$$Y^{iv} - \lambda^4 Y = 0 \tag{A3.8a}$$

together with the same four boundary conditions as Z, viz:

$$Y(\pm 1) = Y'(\pm 1) = 0 \tag{A3.8b,c}$$

A particular solution of (A3.8), even in z is

$$Y(z;\lambda) = \frac{1}{\sqrt{2}}\left\{\frac{\cos \lambda z}{\cos \lambda} - \frac{\cosh \lambda z}{\cosh \lambda}\right\} \tag{A3.9}$$

where

$$\tan \lambda + \tanh \lambda = 0. \tag{A3.10}$$

The roots of (A3.10) are given in Table II.

Denoting by Y_m the Y function corresponding to the m^{th} value of λ, λ_m, we can show by direct integration that

$$\int_{-1}^{1} Y_m Y_n \, dz = \begin{cases} 1 \text{ if } m = n \\ 0 \text{ if } m \neq n \end{cases} \qquad m, n = 1,2, \ldots \tag{A3.11}$$

Also

$$\int_{-1}^{1} Y''_m Y''_n \, dz = \Big[Y'_m Y''_n \Big]_{-1}^{1} - \int_{-1}^{1} Y'_m Y'''_n \, dz$$

$$= \left\{ \Big[Y_m Y'''_n \Big]_{-1}^{1} - \int_{-1}^{1} Y_m Y_n^{iv} \, dz \right\} \text{ using (A3.8)}$$

$$= \lambda_n^4 \int_{-1}^{1} Y_m Y_n \, dz \text{ using (A3.8b,a)}.$$

Thus we have by (A3.11)

$$\int_{-1}^{1} Y''_m Y''_n \, dz = \begin{cases} \lambda_n^4 \text{ if } m = n \\ 0 \quad \text{if } m \neq n \end{cases} \qquad m,n = 1,2 \ldots \tag{A3.12}$$

We now assume expansions of the form

$$Z_k(z) = \sum_{m=1}^{\infty} c_{mk} Y_m \qquad k = 1,2 \ldots \tag{A3.13}$$

Multiplying both sides of equation (A3.13) by Y_j and integrating over the range $-1 \leq z \leq 1$ we obtain

$$\int_{-1}^{1} Z_k Y_j \, dz = \sum_{m=1}^{\infty} c_{mk} \int_{-1}^{1} Y_m Y_j \, dz$$

$$= c_{jk} \text{ by (A3.11)}$$

Thus from (A3.3), (A3.9)

$$c_{jk} = \int_{-1}^{1} \left\{(\cos 2\mu_k - 1)\cos \mu_k z + 2\mu_k z \sin \mu_k z\right\} \frac{1}{\sqrt{2}} \left\{\frac{\cos \lambda_j z}{\cos \lambda_j} - \frac{\cosh \lambda_j z}{\cosh \lambda_j}\right\} dz$$

$$= 4\sqrt{2}\mu_k^2 \left[\mu_k \sin \mu_k + \lambda_j \cos \mu_k \tanh \lambda_j\right]\left[(\lambda_j^2+\mu_k^2)^{-2} - (\lambda_j^2-\mu_k^2)^{-2}\right] \tag{A3.14}$$

Using (A3.6), (A3.13) we obtain

$$\psi(\xi,z) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{(A_k+iB_k)}{\mu_k^2} e^{\mu_k \xi} c_{jk} + (A_k-iB_k) e^{\bar{\mu}_k \xi} \bar{c}_{jk} \right\} Y_j \tag{A3.15}$$

giving

$$\frac{\partial^2 \psi}{\partial z^2}(\xi,z) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{(A_k+iB_k)}{\mu_k^2} e^{\mu_k \xi} c_{jk} + (A_k-iB_k) e^{\bar{\mu}_k \xi} \bar{c}_{jk} \right\} Y_j'' \tag{A3.16a,b}$$

$$\frac{\partial^2\psi}{\partial\xi\partial z}(\xi,z) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left\{\frac{(A_k+iB_k)}{\mu_k}\, e^{\mu_k}\, c_{jk} + \frac{(A_k-iB_k)}{\bar{\mu}_k}\, e^{\bar{\mu}_k\xi}\, \bar{c}_{jk}\right\} Y_j'$$

We now assume that

$$\tfrac{1}{2}\,(3z^2-1) = \sum_{j=1}^{\infty} \alpha_j\, Y_j'' \qquad \text{(A3.17)}$$

Multiplying both sides of (A3.17) by Y_m'' and integrating as before we get

$$\int_{-1}^{1} \tfrac{1}{2}\,(3z^2-1)Y_m''\, dz = \sum_{j=1}^{\infty} \alpha_j \int_{-1}^{1} Y_j''\, Y_m''\, dz \qquad m = 1,2\ \ldots$$

$$= \alpha_m\, \lambda_m^4 \quad \text{by (A3.12)}$$

Thus

$$\alpha_m = \frac{1}{\lambda_m^4}\int_{-1}^{1} \tfrac{1}{2}(3z^2-1)Y_m''\, dz$$

$$= \frac{6\sqrt{2}}{\lambda_m^5}\, \tan\lambda_m \quad m = 1,2,\ \ldots \qquad \text{(A3.18)}$$

From (A3.16), (A3.17), (2.104b,c) we get

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\frac{(A_k+iB_k)}{\mu_k^2}\, c_{jk} + \frac{(A_k-iB_k)}{\bar{\mu}_k^2}\, c_{jk}\right] Y_j'' = \sum_{j=1}^{\infty} \alpha_j\, Y_j''$$

(A3.19a,b)

and

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\frac{(A_k+iB_k)}{\mu_k}\, c_{jk} + \frac{(A_k-iB_k)}{\bar{\mu}_k}\, c_{jk}\right] Y_j' = 0$$

Equations (A3.19) are satisfied if

$$\sum_{k=1}^{\infty}\left[\frac{(A_k+iB_k)}{\mu_k^2}\,c_{jk}+\frac{(A_k-iB_k)}{\bar{\mu}_k^2}\,\bar{c}_{jk}\right]=\alpha_j \qquad j=1,2\ \ldots$$

(A3.20a,b)

and

$$\sum_{k=1}^{\infty}\left[\frac{(A_k+iB_k)}{\mu_k}\,c_{jk}+\frac{(A_k-iB_k)}{\bar{\mu}_k}\,\bar{c}_{jk}\right]=0 \qquad j=1,2\ \ldots$$

From (A3.14)

$$\frac{c_{jk}}{\bar{\mu}_k^2}=4\sqrt{2}\{\mu_k\sin\mu_k+\lambda_j\cos\mu_k\tanh\lambda_j\}\{(\lambda_j^2+\mu_k^2)^{-2}-(\lambda_j^2-\mu_k^2)^{-2}\}$$

$$=d_{jk}+i\,e_{jk}\ \text{say}$$

The real quantities d_{jk}, e_{jk} are obtained by equating real and imaginary parts in the above equation, but the results are not recorded because of their excessive length.

Equation (A3.20a) now becomes

$$\sum_{k=1}^{\infty}\{(A_k+iB_k)(d_{jk}+ie_{jk})+(A_k-iB_k)(d_{jk}-e_{jk})\}=\alpha_j$$

giving

$$\sum_{k=1}^{\infty}(A_k\,C_{jk}+B_k\,D_{jk})=\alpha_j \qquad j=1,2\ \ldots \tag{A3.21a}$$

where

$C_{jk} = 2d_{jk}$ and $D_{jk} = -2e_{jk}$.

Similarly we obtain from (A3.20b)

$$\sum_{k=1}^{\infty} (A_k E_{jk} + B_k F_{jk}) = 0 \quad j = 1,2 \ldots \tag{A3.21b}$$

where

$$E_{jk} = 2(d_{jk}\, a_k + e_{jk}\, b_k), \quad F_{jk} = 2(d_{jk}\, b_k - e_{jk}\, a_k).$$

The coefficients C_{jk}, D_{jk}, E_{jk}, F_{jk} are known for all j, k and the constants A_k, B_k can be calculated from equations (A3.21).

To obtain numberical results for the problem, clearly a finite range of j, k must be used and in this respect Gaydon and Shepherd have achieved satisfactory results by letting j range from 1 to 20 and k from 1 to 10.

Equations (A3.21) may now be written, in matrix form,

$$\begin{bmatrix} C_{11} & D_{11} & C_{12} & & C_{1,10} & D_{1,10} \\ E_{11} & F_{11} & & & & F_{1,10} \\ C_{21} & & & & & \\ & & \text{"L"} & & & \\ & & (40 \times 20) & & & \\ E_{20,1} & & & & & F_{20,10} \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ \\ \\ B_{10} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_2 \\ \\ \\ 0 \end{bmatrix} \tag{A3.22}$$

Solving equations (A3.22) by the method of least squares gives

$$\begin{bmatrix} A_1 \\ B_1 \\ \vdots \\ A_{10} \\ B_{10} \end{bmatrix} = \begin{bmatrix} \\ \text{"M"} \\ (20 \times 40) \\ \\ \end{bmatrix} \begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_2 \\ \vdots \\ \alpha_{20} \\ 0 \end{bmatrix} \qquad \text{(A3.23)}$$

where $M = (L' L)^{-1} L'$, L' being the transpose of L.

Gaydon and Shepherd have calculated M and it is given in Table III. The A_k, B_k, $k = 1, 2 \ldots 10$ have been computed from equations (A3.18) and (A3.23) the results being shown in Table IV.

From (A3.6)

$$\frac{\partial^2\psi}{\partial z^2}(\xi,z) = \sum_{k=1}^{10}\left\{\frac{(A_k+iB_k)}{\mu_k^2} e^{\mu_k\xi} Z_k'' + \frac{(A_k-iB_k)}{\bar{\mu}_k^2} e^{\bar{\mu}_k\xi} \bar{Z}_k''\right\} \qquad \text{(A3.24)}$$

and from (A3.3)

$$\frac{Z_k''}{\mu_k^2} = -(\cos 2\mu_k + 3)\cos \mu_k z + 2\mu_k z \sin \mu_k z$$

$$= P_k + i\, Q_k \text{ say}$$

Also from (A3.5)

$$e^{\mu_k \xi} = e^{-a_k \xi} (\cos b_k \xi + i \sin b_k \xi)$$

and (A3.24) now becomes

$$\frac{\partial^2 \psi}{\partial z^2} (\xi, z) = \sum_{k=1}^{10} A_k (zz)^1_k + B_k (zz)^2_k \tag{A3.25}$$

where

$$(zz)^1_k = 2e^{-a_k \xi} (P_k \cos b_k \xi - Q_k \sin b_k \xi)$$

$$(zz)^2_k = -2e^{-a_k \xi} (P_k \sin b_k \xi + Q_k \cos b_k \xi)$$

The P_k, Q_k are known quantities and thus $(zz)^1_k$, $(zz)^2_k$ can be evaluated at any point (ξ, z). Gaydon and Shepherd have in fact calculated them for $k = 1, 2 \ldots 10$ for the range of values:

$\xi = 0, 0.5, 1.0, 1.5, 2.0$

$z = 0, 0.2, 0.4, 0.6, 0.8, 1.0$

It is of course unnecessary to consider negative values of z as the problem is symmetrical about $z = 0$. The results appear in Table V.

Proceeding in a similar fashion we can show that

$$\frac{\partial^2 \psi}{\partial \xi^2} (\xi, z) = \sum_{k=1}^{10} A_k (\xi\xi)^1_k + B_k (\xi\xi)^2_k \tag{A3.26}$$

where

$$(\xi\xi)^1_k = 2e^{-a_k \xi} (U_k \cos b_k \xi - V_k \sin b_k \xi)$$

$$(\xi\xi)^2_k = -2e^{-a_k \xi} (U_k \sin b_k \xi + V_k \cos b_k \xi)$$

and

$$U_k + i\,V_k = (\cos 2\mu_k - 1)\cos \mu_k z - 2\mu_k z \sin \mu_k z$$

The values of $(\xi\xi)_k^1$, $(\xi\xi)_k^2$ have been calculated at the same points as the previous case and the results given in Table VI.

Also

$$\frac{\partial^2\psi}{\partial\xi\partial z}(\xi,z) = \sum_{k=1}^{10} A_k(\xi z)_k^1 + B_k(\xi z)_k^2 \qquad \text{(A3.27}$$

where

$$(\xi z)_k^1 = 2e^{-a_k\xi}(R_k \cos b_k\,\xi - S_k \sin b_k\xi)$$

$$(\xi z)_k^2 = -2e^{-a_k\xi}(R_k \sin b_k\,\xi + S_k \cos b_k\xi)$$

and

$$R_k + i\,S_k = -(\cos 2\mu_k + 1)\sin \mu_k\,z - 2\mu_k\,z \cos \mu_k\,z.$$

The relevant values of $(\xi z)_k^1$, $(\xi z)_k^2$ are given in Table VII.

Using equations (A3.25), (A3.26) and (A3.27) the required derivatives of ψ have been computed, the values being recorded in Tables VIII to XI. Comparison of these calculated values with the known values on the boundary suggests an accuracy to at least 1%, although more decimal places are included.

APPENDIX IV

Tables

Table I (Gaydon and Shepherd)

Roots of $2\mu + \sin 2\mu = 0$

$\mu_k = -a_k + i\, b_k$

k	$2a_k$	$2b_k$
1	4.212392	2.250729
2	10.712537	3.103149
3	17.073365	3.551087
4	23.398355	3.858809
5	29.708120	4.093705
6	36.009866	4.283782
7	42.306827	4.443446
8	48.600684	4.581105
9	54.892406	4.700291
10	61.182590	4.810025

Table II (Gaydon and Shepherd)

Roots of $\tan \lambda + \tanh \lambda = 0$

m	λ_m
1	2.365020
2	5.497804
≥ 3	$(m - \frac{1}{4})\pi$

Table III Matrix M (Gaydon and Shepherd)

Columns 1 to 5

-1.30305	0.52703	0.49354	-0.03461	0.30938
-0.18697	-0.18932	-0.52923	0.06254	-0.15024
-0.12246	0.09770	6.32605	-1.14485	-2.07409
-0.07048	0.03714	0.44758	0.23977	2.48023
0.03130	-0.02294	0.83471	-0.26469	-15.55358
-0.00100	0.00400	0.88275	-0.19653	-0.72105
-0.01050	0.00696	-0.32919	0.10572	-2.16082
0.00382	-0.00398	-0.08055	0.01115	-3.04646
0.00429	-0.00252	0.14902	-0.04859	1.05058
-0.00303	0.00270	-0.00717	0.00892	0.40876
-0.00180	0.00081	-0.08000	0.02530	-0.52934
0.00263	-0.00213	0.02635	0.01332	-0.04998
-0.00052	0.00083	0.03355	-0.00743	0.33749
-0.00252	0.00189	-0.03639	0.01595	-0.05547
0.00366	-0.00306	0.01875	-0.01458	-0.19118
0.00067	-0.00031	-0.02766	-0.00890	0.15876
-0.00037	0.00005	-0.02718	0.00797	-0.17591
0.00081	-0.00074	-0.00164	-0.00175	-0.08805
-0.00019	0.00014	-0.00243	0.00113	-0.00144
-0.00021	0.00013	-0.00592	0.00209	-0.02582

Table III Matrix M (Gaydon and Shepherd)

Columns 6 to 10

-0.01746	0.18400	-0.00829	0.11256	-0.00421
0.01207	-0.04916	0.00348	-0.01272	0.00114
0.11406	-1.53613	0.06921	-1.01625	0.03578
-0.22605	0.63453	-0.04426	0.18155	-0.01243
1.79580	4.20837	-0.19285	3.29036	-0.12754
-0.28003	-6.11003	0.43565	-1.53316	0.08853
0.40512	29.29354	-2.48109	-6.71146	0.25837
0.41464	0.99465	0.31478	11.63895	-0.67566
-0.20134	3.99492	-0.52691	-47.96545	3.20738
-0.04723	6.97294	-0.67614	-1.03753	-0.36359
0.10611	-2.35183	0.30955	-6.44920	0.65270
-0.00851	-1.23960	0.10698	-13.45452	1.00993
-0.05745	1.57103	-0.19706	5.72915	-0.53669
0.02797	0.32448	-0.00289	3.08478	-0.21059
0.00799	-1.48637	0.13987	-5.73840	0.48654
-0.03388	0.47649	-0.07570	0.79797	-0.11303
0.03522	-0.58290	0.08691	-1.08447	0.14203
0.01009	-0.49757	0.05404	-1.62386	0.14909
0.00170	0.03090	-0.00097	0.17270	-0.01175
0.00640	-0.05376	0.01122	0.00402	0.00922

Table III Matrix M (Gaydon and Shepherd)

Columns 11 to 16

0.06660	-0.00199	0.02979	-0.00036	-0.00278
0.00175	0.00038	0.00623	0.00018	0.00399
-0.65238	0.01706	-0.32539	0.00210	0.00304
0.02437	-0.00378	-0.03316	-0.00115	-0.03420
2.08669	-0.05799	1.05159	-0.00767	0.00863
-0.41590	0.02318	0.00328	0.00304	0.16249
-5.29127	0.17258	-2.66918	0.03817	-0.21619
2.77523	-0.13068	0.37565	-0.00958	-0.61740
9.25998	-0.29622	7.05407	-0.18206	1.86330
-18.72652	0.90332	-3.11706	0.09449	1.69726
71.45499	-3.93577	-13.09423	0.40143	-9.83131
-0.47485	0.50992	24.93596	-0.97198	-1.14286
13.19211	-1.04147	-92.99628	4.22516	27.38126
23.74759	-1.45563	5.75746	-0.83120	-27.08972
-17.35710	1.24346	-37.47413	2.37080	95.66722
-1.53043	0.01108	-28.90273	1.36852	-3.95952
-0.08921	0.10283	9.72132	-0.37018	46.87910
-3.82724	0.30071	-4.59160	0.37000	26.32033
0.58822	-0.03819	1.42274	-0.08350	1.65691
0.51494	-0.01988	2.56093	-0.12288	7.87224

Table III Matrix M (Gaydon and Shepherd)

Columns 16 to 20

0.00100	-0.00276	0.00104	0.00299	0.00082
0.00028	-0.00278	0.00055	0.00005	0.00047
-0.01183	0.04697	-0.01407	-0.02852	-0.01139
-0.00117	0.02128	-0.00333	0.00201	-0.00281
0.03788	-0.19105	0.04790	0.07847	0.03884
-0.00375	-0.06277	0.00458	-0.02172	0.00380
-0.07167	0.59019	-0.10896	-0.13348	-0.08609
0.03464	0.08540	0.01038	0.09084	0.00841
0.05224	-1.58169	0.19506	0.13265	0.14478
-0.12358	0.24346	-0.07924	-0.25679	-0.05892
0.26442	3.10364	-0.23503	0.11326	-0.15953
0.20401	-2.31517	0.28217	0.55621	0.18481
-1.12520	0.12530	-0.14442	-1.11059	-0.04273
0.78624	8.55878	-0.64644	-0.75829	-0.36490
-3.26627	-30.48356	1.78402	3.14303	0.62542
0.51717	5.06432	-0.07029	-1.01282	0.30249
-1.98616	-19.22399	0.52040	2.46500	-0.72535
-0.92681	-9.03653	0.23818	1.03285	0.15065
-0.09939	-0.99913	0.00030	0.13779	-0.15678
-0.35815	-3.46102	0.05360	0.45961	0.02064

Table III Matrix M (Gaydon and Shepherd)

Columns 21 to 25

0.00155	0.00284	0.00074	-0.01418	0.00114
-0.00010	0.00056	-0.00009	-0.00101	0.00002
-0.01390	-0.03171	-0.00621	0.14442	-0.01101
0.00210	-0.00214	0.00142	-0.00197	0.00075
0.03503	0.10154	0.01363	-0.44891	0.03066
-0.01500	-0.01033	-0.00840	0.08744	-0.00862
-0.04620	-0.21075	-0.01005	0.91539	-0.05138
0.05351	0.08426	0.02548	-0.49641	0.03678
0.00174	0.27702	-0.02581	-1.10162	0.04304
-0.12538	-0.31388	-0.04579	1.72684	-0.10226
0.17621	0.06075	0.09831	-1.00682	0.07537
0.19416	0.75720	0.03654	-4.08163	0.20270
-0.51497	-1.59280	-0.11417	9.96727	-0.44045
-0.10771	-0.89223	0.03186	4.52790	-0.21426
0.62363	3.72853	-0.10907	-22.56819	0.88904
-0.33453	-1.58203	0.03927	10.79771	-0.30314
0.62363	4.44451	-0.26794	-27.22214	0.62080
0.22574	0.78735	-0.00557	-7.18528	0.31300
0.05938	1.46331	-0.06520	-3.11539	0.00502
0.11698	1.08278	-0.08045	-6.05470	0.09731

Table III Matrix M (Gaydon and Shepherd)

Columns 26 to 30

0.01209	0.00102	0.01747	0.00080	0.01609
0.00273	0.00003	0.00280	0.00002	0.00217
-0.13671	-0.00995	-0.18944	-0.00779	-0.17156
-0.01274	0.00059	-0.00904	0.00050	-0.00487
0.44490	0.02824	0.60201	0.02208	0.53881
-0.02500	-0.00757	-0.07078	-0.00613	-0.07836
-0.98232	-0.04860	-1.26110	-0.03756	-1.09597
0.29983	0.03369	0.51769	0.02708	0.50735
1.56513	0.04163	1.72924	0.02953	1.37207
-1.27603	-0.09626	-1.91965	-0.07647	-1.77905
-0.80204	0.07524	0.09848	0.06798	0.56110
3.60650	0.19171	4.76993	0.14790	4.15377
-5.79792	-0.43196	-9.26482	-0.34507	-8.62173
-5.82290	-0.19132	-6.30399	-0.13542	-4.92573
20.16721	0.82430	24.03075	0.60645	19.57878
-4.20379	-0.30982	-7.11635	-0.24571	-6.51284
9.94989	0.64653	16.27203	0.50514	14.47272
6.74869	0.28903	8.10239	0.21450	6.67012
-0.50894	0.02077	0.29618	0.02025	0.45225
1.18106	0.11091	2.63778	0.08919	2.46577

Table III Matrix M (Gaydon and Shepherd)

Columns 31 to 35

0.00060	0.01327	0.00045	0.01057	0.00034
0.00001	0.00157	0.00000	0.00113	0.00000
-0.00586	-0.14004	-0.00438	-0.11063	-0.00329
0.00042	-0.00217	0.00035	-0.00065	0.00028
0.01653	0.43609	0.01228	0.34217	0.00917
-0.00481	-0.07177	-0.00373	-0.06132	-0.00290
-0.02759	-0.86744	-0.02008	-0.66838	-0.01470
0.02089	0.43444	0.01595	0.35489	0.01219
0.01943	1.00956	0.01254	0.73081	0.00807
-0.05792	-1.46835	-0.04341	-1.16713	-0.03264
0.05729	0.70677	0.04666	0.70039	0.03747
0.10851	3.28186	0.07894	2.52244	0.05777
-0.26056	-7.09125	-0.19416	-5.60711	-0.14500
-0.09133	-3.59015	-0.06140	-2.58537	-0.04172
0.42488	14.70533	0.29650	10.85399	0.20904
-0.18317	-5.25915	-0.13480	-4.09397	-0.09956
0.36988	11.42592	0.26765	8.73428	0.19472
0.15186	5.05732	0.10709	3.76388	0.07626
0.01644	0.42429	0.01269	0.35495	0.00967
0.06636	1.99004	0.04855	1.54130	0.03562

Table III Matrix M (Gaydon and Shepherd)

Columns 36 to 40

0.00833	0.00026	0.00656	0.00020	0.00520
0.00081	0.00000	0.00059	0.00000	0.00044
-0.08666	-0.00250	-0.06796	-0.00192	-0.05359
0.00016	0.00023	0.00055	0.00019	0.00072
0.26651	0.00692	0.20801	0.00527	0.16335
-0.05090	-0.00227	-0.04176	-0.00179	-0.03414
-0.51271	-0.01087	-0.39499	-0.00815	-0.30670
0.28505	0.00939	0.22800	0.00729	0.18266
0.53068	0.00520	0.38929	0.00336	0.28919
-0.91755	-0.02475	-0.72121	-0.01896	-0.56950
0.63368	0.02996	0.54989	0.02399	0.46760
1.93018	0.04275	1.48366	0.03205	1.14970
-4.38399	-0.10920	-3.42796	-0.08311	-2.69386
-1.87312	-0.02878	-1.37377	-0.02018	-1.02177
8.03306	0.14958	6.00383	0.10873	4.54201
-3.15999	-0.07426	-2.44474	-0.05605	-1.90421
6.64179	0.14328	5.07475	0.10684	3.91109
2.80599	0.05509	2.11065	0.04041	1.60586
0.28583	0.00737	0.22744	0.00566	0.18076
1.18265	0.02639	0.90971	0.01979	0.70483

Table IV

Coefficients A_k, B_k

k	A_k	B_k
1	0.1459239	0.0219892
2	0.0035498	0.0067221
3	-0.0024010	-0.0010037
4	0.0011304	0.0000596
5	-0.0005755	0.0001197
6	0.0003118	-0.0001765
7	-0.0000829	0.0002141
8	-0.0002179	-0.0001116
9	0.0000969	-0.0000314
10	0.0000161	0.0000267

Table V $(zz)^1_k$, $(zz)^2_k$ (Gaydon and Shepherd)

z	k	$(zz)^1_k$ $\xi = 0.0$	$(zz)^2_k$	$(zz)^1_k$ $\xi = 0.5$	$(zz)^2_k$	$(zz)^1_k$ $\xi = 1.0$	$(zz)^2_k$	$(zz)^1_k$ $\xi = 1.5$	$(zz)^2_k$	$(zz)^1_k$ $\xi = 2.0$	$(zz)^2_k$
0.0	1	-1.3976	-8.2399	-1.9458	-2.1713	-0.9782	-0.2786	-0.3405	0.0999	-0.0819	0.0928
0.2		-1.5526	-6.8547	-1.7338	-1.7337	-0.8342	-0.1889	-0.2813	0.0995	-0.0645	0.0817
0.4		-1.6725	-3.0379	-1.0588	-0.5851	-0.4213	0.0244	-0.1198	0.0856	-0.0194	0.0476
0.6		-0.8732	2.1850	0.1490	0.8072	0.1942	0.2105	0.0965	0.0260	0.0333	-0.0103
0.8		1.8406	7.1111	1.8665	1.7558	0.8775	0.1707	0.2907	-0.1129	0.0648	-0.0874
1.0		6.9505	9.4841	3.8159	1.5050	1.4060	-0.2660	0.3654	-0.3402	0.0445	-0.1684
0.0	2	0.2314	-21.3388	-1.0151	-1.0575	-0.1006	-0.0030	-0.0051	0.0047	0.0000	0.0005
0.2		-2.0336	-8.9923	-0.5323	-0.3431	-0.0426	0.0088	-0.0017	0.0025	0.0000	0.0002
0.4		-2.4879	13.1498	0.5105	0.7645	0.0618	0.0129	0.0037	-0.0023	0.0001	-0.0003
0.6		3.9309	17.0794	1.0143	0.6484	0.0809	-0.0170	0.0032	-0.0047	-0.0001	-0.0004
0.8		4.6330	-2.8205	0.0915	-0.3612	-0.0129	-0.0221	-0.0017	-0.0005	-0.0001	0.0001
1.0		-11.8399	-14.4182	-1.2741	-0.1375	-0.0691	0.0545	-0.0008	0.0060	0.0003	0.0003
0.0	3	1.1139	-34.0905	-0.3605	-0.3134	-0.0066	0.0011	0.0000	0.0001	0.0000	0.0000
0.2		-5.1411	5.6956	0.0164	0.1062	0.0013	0.0008	0.0000	0.0000	0.0000	0.0000
0.4		3.1960	30.5780	0.3604	0.2356	0.0057	-0.0018	0.0000	-0.0001	0.0000	0.0000
0.6		8.0269	-16.6437	-0.1099	-0.2343	-0.0035	-0.0009	0.0000	0.0000	0.0000	0.0000
0.8		-10.8036	-12.9804	-0.2365	0.0026	-0.0021	0.0026	0.0000	0.0000	0.0000	0.0000
1.0		15.3253	17.7957	0.3288	-0.0092	0.0028	-0.0037	0.0000	-0.0001	0.0000	0.0000

z	k	$(zz)^1_k$ $\xi = 0.0$	$(zz)^2_k$	$(zz)^1_k$ $\xi = 0.5$	$(zz)^2_k$	k	$(zz)^1_k$ $\xi = 0.0$	$(zz)^2_k$	$(zz)^1_k$ $\xi = 0.5$	$(zz)^2_k$
0.0	4	1.725	-46.755	-0.108	-0.081	5	2.192	-59.383	-0.029	-0.020
0.2		-6.921	33.095	0.067	0.071		-3.721	58.708	0.029	0.020
0.4		15.202	-1.097	0.022	-0.038		7.364	-56.140	-0.026	-0.021
0.6		-14.749	-26.956	-0.088	-0.009		-9.902	50.234	0.022	0.021
0.8		8.712	31.471	0.089	0.031		4.639	-39.304	-0.019	-0.015
1.0		-18.194	-20.558	-0.079	0.009		20.691	22.960	0.018	-0.003
0.0	6	2.571	-71.992	-0.008	-0.005	7	2.889	-84.590	-0.002	-0.001
0.2		5.274	64.512	0.007	0.003		16.820	38.966	0.001	0.000
0.4		-23.259	-42.031	-0.006	0.000		-30.968	50.865	0.001	0.001
0.6		36.107	7.403	0.003	-0.003		-9.322	-80.618	-0.002	-0.001
0.8		-24.298	25.875	0.001	0.004		38.829	9.417	0.001	-0.001
1.0		-22.929	-25.118	-0.004	0.001		24.977	27.094	0.001	0.000

z	k	$(zz)^1_k$ $\xi = 0.0$	$(zz)^2_k$	k	$(zz)^1_k$ $\xi = 0.0$	$(zz)^2_k$	k	$(zz)^1_k$ $\xi = 0.0$	$(zz)^2_k$
0.0	8	3.16	-97.18	9	3.41	-109.77	10	3.62	-122.35
0.2		24.70	-14.99		22.77	-78.42		8.59	-122.91
0.4		12.58	95.17		54.30	0.10		21.58	-122.39
0.6		-50.10	46.51		49.72	76.83		35.17	-114.19
0.8		-36.11	-54.30		10.72	87.45		31.10	-87.94
1.0		-26.87	-28.93		28.65	30.65		-30.33	-32.27

Table VI $(\xi\xi)^1_k$, $(\xi\xi)^2_k$ (Gaydon and Shepherd)

z	k	$(\xi\xi)^1_k$ ξ = 0.0	$(\xi\xi)^2_k$	$(\xi\xi)^1_k$ ξ = 0.5	$(\xi\xi)^2_k$	$(\xi\xi)^1_k$ ξ = 1.0	$(\xi\xi)^2_k$	$(\xi\xi)^1_k$ ξ = 1.5	$(\xi\xi)^2_k$	$(\xi\xi)^1_k$ ξ = 2.0	$(\xi\xi)^2_k$
0.0	1	-6.6024	8.2399	-0.4147	3.6601	0.5588	1.1572	0.3802	0.2375	0.1564	-0.0007
0.2		-5.9338	7.5972	-0.3371	3.3460	0.5232	1.0500	0.3498	0.2125	0.1428	-0.0024
0.4		-4.2011	5.8171	-0.1571	2.4983	0.4186	0.7664	0.2661	0.1482	0.1061	-0.0058
0.6		-2.1176	3.3643	0.0013	1.3868	0.2585	0.4090	0.1524	0.0726	0.0585	-0.0069
0.8		-0.5343	1.0509	0.0379	0.4095	0.0874	0.1138	0.0470	0.0173	0.0171	-0.0036
1.0		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0	2	-8.2314	21.3388	0.6228	1.4423	0.0999	0.0408	0.0069	-0.0028	0.0002	-0.0005
0.2		-1.9846	11.2066	0.4418	0.6450	0.0527	0.0104	0.0031	-0.0020	0.0001	-0.0002
0.4		7.6772	-8.7010	-0.0421	-0.7960	-0.0404	-0.0370	-0.0038	0.0001	-0.0002	0.0002
0.6		7.7629	-17.6974	-0.4706	-1.2412	-0.0828	-0.0382	-0.0059	0.0021	-0.0002	0.0004
0.8		1.5825	-8.7225	-0.3420	-0.5038	-0.0410	-0.0083	-0.0024	0.0016	0.0000	0.0002
1.0		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0	3	-9.1139	34.0905	0.2898	0.4003	0.0069	0.0004	0.0001	-0.0001	0.0000	0.0000
0.2		6.2994	-2.8216	0.0250	-0.0934	-0.0008	-0.0011	0.0000	0.0000	0.0000	0.0000
0.4		6.5335	-32.2424	-0.2925	-0.3560	-0.0065	0.0000	-0.0001	0.0001	0.0000	0.0000
0.6		-13.1986	7.2612	-0.0378	0.2076	0.0019	0.0022	0.0000	0.0000	0.0000	0.0000
0.8		-4.1695	21.0777	0.1921	0.2316	0.0042	0.0000	0.0000	0.0000	0.0000	0.0000
1.0		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

z	k	$(\xi\xi)^1_k$ ξ = 0.0	$(\xi\xi)^2_k$	$(\xi\xi)^1_k$ ξ = 0.5	$(\xi\xi)^2_k$	k	$(\xi\xi)^1_k$ ξ = 0.0	$(\xi\xi)^2_k$	$(\xi\xi)^1_k$ ξ = 0.5	$(\xi\xi)^2_k$
0.0	4	-9.725	46.755	0.095	0.100	5	-10.192	59.383	0.027	0.024
0.2		12.904	-30.821	-0.052	-0.081		12.275	-58.136	-0.026	-0.024
0.4		-14.859	-5.704	-0.038	0.026		-17.573	53.693	0.022	0.026
0.6		4.386	34.661	0.089	0.047		22.828	-44.113	-0.015	-0.025
0.8		10.825	-30.303	-0.054	-0.075		-21.195	26.811	0.007	0.019
1.0		0.000	0.000	0.000	0.000		0.000	0.000	0.000	0.000
0.0	6	-10.571	71.992	0.007	0.005	7	-10.889	84.590	0.002	0.001
0.2		2.564	-66.078	-0.007	-0.004		-12.742	-42.221	-0.001	0.000
0.4		16.512	48.171	0.006	0.001		37.458	-44.228	-0.001	-0.001
0.6		-33.127	-20.510	-0.004	0.002		-6.774	82.387	0.002	0.001
0.8		30.338	-5.162	0.001	-0.004		-30.347	-30.957	-0.001	0.000
1.0		0.000	0.000	0.000	0.000		0.000	0.000	0.000	0.000

z	k	$(\xi\xi)^1_k$ ξ = 0.0	$(\xi\xi)^2_k$	k	$(\xi\xi)^1_k$ ξ = 0.0	$(\xi\xi)^2_k$	k	$(\xi\xi)^1_k$ ξ = 0.C	$(\xi\xi)^2_k$
0.0	8	-11.16	97.18	9	-11.41	109.77	10	-11.62	122.35
0.2		-26.00	11.24		-29.01	75.63		-17.41	122.26
0.4		-1.48	-97.62		-54.10	-8.78		-32.93	119.49
0.6		57.31	-33.14		-37.13	-87.45		-50.90	106.59
0.8		14.82	67.87		16.11	-86.57		-53.21	71.51
1.0		0.00	0.00		0.00	0.00		0.00	0.00

Table VII $(\xi z)^1_k$, $(\xi z)^2_k$ (Gaydon and Shepherd)

z	k	$(\xi z)^1_k$ $\xi = 0.0$	$(\xi z)^2_k$	$(\xi z)^1_k$ $\xi = 0.5$	$(\xi z)^2_k$	$(\xi z)^1_k$ $\xi = 1.0$	$(\xi z)^2_k$	$(\xi z)^1_k$ $\xi = 1.5$	$(\xi z)^2_k$	$(\xi z)^1_k$ $\xi = 2.0$	$(\xi z)^2_k$
0.0	1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2		-1.1376	3.6020	0.3346	1.2746	0.3359	0.3138	0.1575	0.0301	0.0521	-0.0204
0.4		-1.5957	+6.1280	0.6696	2.1051	0.5893	0.4966	0.2663	0.0368	0.0854	-0.0387
0.6		-1.1173	6.6449	0.9069	2.1686	0.6712	0.4711	0.2857	0.0141	0.0869	-0.0490
0.8		-0.1685	4.5742	0.8015	1.3811	0.4935	0.2584	+0.1937	-0.0156	0.0543	-0.0407
1.0		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0	2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2		-4.6219	18.4387	0.6603	1.1265	0.0866	0.0235	0.0054	-0.0030	0.0001	-0.0004
0.4		-0.8100	16.6518	0.7613	0.8555	0.0785	0.0053	0.0041	-0.0035	0.0000	-0.0004
0.6		4.1710	-2.1835	0.0995	-0.3077	-0.0099	-0.0199	-0.0014	-0.0005	-0.0001	0.0000
0.8		0.7604	-12.7478	-0.5759	-0.6617	-0.0601	-0.0047	-0.0032	0.0027	0.0000	0.0003
1.0		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0	3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2		-4.1850	33.5185	0.3271	0.3417	0.0066	-0.0005	0.0001	-0.0001	0.0000	0.0000
0.4		9.8639	-8.4264	-0.0043	-0.1816	-0.0020	-0.0016	0.0000	0.0000	0.0000	0.0000
0.6		-1.4982	-26.5787	-0.3020	-0.2187	-0.0050	0.0013	0.0000	+0.0001	0.0000	0.0000
0.8		-5.7055	13.2523	0.0935	0.1791	0.0028	0.0006	0.0000	0.0000	0.0000	0.0000
1.0		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

z	k	$(\xi z)^1_k$ $\xi = 0.0$	$(\xi z)^2_k$	$(\xi z)^1_k$ $\xi = 0.5$	$(\xi z)^2_k$	k	$(\xi z)^1_k$ $\xi = 0.0$	$(\xi z)^2_k$	$(\xi z)^1_k$ $\xi = 0.5$	$(\xi z)^2_k$
0.0	4	0.000	0.000	0.000	0.000	5	0.000	0.000	0.000	0.000
0.2		1.876	33.890	0.083	0.051		10.901	10.917	0.009	-0.002
0.4		5.384	-45.409	-0.099	-0.087		-18.968	-20.817	-0.016	0.003
0.6		15.570	27.770	0.040	0.082		21.323	27.173	0.020	-0.002
0.8		12.774	-0.304	0.020	-0.031		-15.367	-24.119	-0.017	0.000
1.0		0.000	0.000	0.000	0.000		0.000	0.000	0.000	0.000
0.0	6	0.000	0.000	0.000	0.000	7	0.000	0.000	0.000	0.000
0.2		17.232	-31.153	-0.002	-0.004		15.361	-75.043	-0.002	-0.001
0.4		-22.508	56.476	0.005	0.006		16.226	71.752	0.002	0.000
0.6		11.036	-66.515	-0.007	-0.005		-46.497	8.945	0.000	0.001
0.8		7.026	49.213	0.006	0.002		13.431	-59.546	-0.001	-0.001
1.0		0.000	0.000	0.000	0.000		0.000	0.000	0.000	0.000

z	k	$(\xi z)^1_k$ $\xi = 0.0$	$(\xi z)^2_k$	k	$(\xi z)^1_k$ $\xi = 0.0$	$(\xi z)^2_k$	k	$(\xi z)^1_k$ $\xi = 0.0$	$(\xi z)^2_k$
0.0	8	0.00	0.00	9	0.00	0.00	10	0.00	0.00
0.2		3.37	-97.47		-15.18	-80.52		-32.09	-22.08
0.4		45.58	-27.02		7.77	-114.27		-57.59	-44.31
0.6		20.07	87.88		53.27	-75.25		-68.19	-62.21
0.8		-39.13	42.98		56.82	1.45		-52.94	-60.09
1.0		0.00	0.00		0.00	0.00		0.00	0.00

Table VIII $\frac{\partial^2\psi}{\partial z^2}$

	ξ = 0.0	ξ = 0.5	ξ = 1.0	ξ = 1.5	ξ = 2.0
z = 1.0	1.0000000	0.5835965	0.1994379	0.0458775	0.0027937
z = 0.8	0.4600000	0.3095485	0.1316099	0.0399281	0.0075343
z = 0.6	0.0400000	0.0478418	0.0331494	0.0146331	0.0046297
z = 0.4	-0.2600000	-0.1614874	-0.0606470	-0.0156016	-0.0017859
z = 0.2	-0.4400000	-0.2954001	-0.1259795	-0.0388497	-0.0076142
z = 0.0	-0.5000000	-0.3413300	-0.1492293	-0.0474770	-0.0099072

Table IX ($L_0^2\ \psi$)

	ξ = 0.0	ξ = 0.5	ξ = 1.0	ξ = 1.5	ξ = 2.0
z = 1.0	1.0000000	0.5835965	0.1994379	0.0458775	0.0027937
z = 0.8	0.3676050	0.3187231	0.1466546	0.0471692	0.0099518
z = 0.6	-0.2695528	0.0685020	0.0793068	0.0384615	0.0130165
z = 0.4	-0.7673032	-0.1339664	0.0169128	0.0264748	0.0135697
z = 0.2	-1.0774179	-0.2651304	-0.0262834	0.0168647	0.0131699
z = 0.0	-1.1821199	-0.3104515	-0.0416292	0.0132313	0.0128972

Table X $\frac{\partial^2\psi}{\partial\xi^2}$

	ξ = 0.0	ξ = 0.5	ξ = 1.0	ξ = 1.5	ξ = 2.0
z = 1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
z = 0.8	-0.0923951	0.0091746	0.0150447	0.0072411	0.0024175
z = 0.6	-0.3095528	0.0206602	0.0461574	0.0238284	0.0083868
z = 0.4	-0.5073032	0.0275210	0.0775597	0.0420765	0.0153556
z = 0.2	-0.6374179	0.0302698	0.0996961	0.0557144	0.0207842
z = 0.0	-0.6821199	0.0308786	0.1076001	0.0607082	0.0228045

Table XI $\frac{\partial^2\psi}{\partial\xi\partial z}$

	ξ = 0.0	ξ = 0.5	ξ = 1.0	ξ = 1.5	ξ = 2.0
z = 1.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
z = 0.8	0.0000000	0.1404626	0.0774432	0.0279292	0.0070307
z = 0.6	0.0000000	0.1792908	0.1081450	0.0419921	0.0116030
z = 0.4	0.0000000	+0.1525387	0.0972335	0.0396598	0.0116082
z = 0.2	0.0000000	+0.0857331	0.0563661	0.0236438	0.0071517
z = 0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Table XII (τ_{rr})

		r = 0.0	r = 0.2	r = 0.4	r = 0.6	r = 0.8	r = 1.0
z = 1.0	A	0.12500	0.12493	0.12352	0.11426	0.07673	-0.03786
	E	0.12501	0.12493	0.12350	0.11420	0.07661	-0.03804
z = 0.8	A	0.12500	0.12496	0.12404	0.11691	0.08510	-0.01741
	E	0.12501	0.12497	0.12403	0.11687	0.08501	-0.01755
z = 0.6	A	0.12500	0.12499	0.12445	0.11897	0.09161	-0.00151
	E	0.12501	0.12500	0.12445	0.11896	0.09159	-0.00156
z = 0.4	A	0.12500	0.12501	0.12474	0.12044	0.09626	0.00984
	E	0.12501	0.12502	0.12476	0.12047	0.09630	0.00990
z = 0.2	A	0.12500	0.12502	0.12491	0.12133	0.09905	0.01666
	E	0.12501	0.12503	0.12494	0.12138	0.09914	0.01679
z = 0.0	A	0.12500	0.12502	0.12497	0.12162	0.09999	0.01893
	E	0.12501	0.12504	0.12501	0.12168	0.10009	0.01909

A - Approximate Primary Solution E - Exact Primary Solution.

Table XIII ($\tau_{\theta\theta}$)

		r = 0.0	r = 0.2	r = 0.4	r = 0.6	r = 0.8	r = 1.0
z = 1.0	A	0.12500	0.12492	0.12096	0.08186	-0.11169	-0.76786
	E	0.12501	0.12492	0.12095	0.08183	-0.11175	-0.76796
z = 0.8	A	0.12500	0.12493	0.12121	0.08311	-0.10774	-0.75821
	E	0.12501	0.12494	0.12120	0.08309	-0.10779	-0.75829
z = 0.6	A	0.12500	0.12494	0.12140	0.08408	-0.10467	-0.75071
	E	0.12501	0.12495	0.12140	0.08408	-0.10469	-0.75075
z = 0.4	A	0.12500	0.12495	0.12153	0.08478	-0.10247	-0.74536
	E	0.12501	0.12496	0.12154	0.08479	-0.10247	-0.74535
z = 0.2	A	0.12500	0.12496	0.12162	0.08519	-0.10116	-0.74214
	E	0.12501	0.12497	0.12163	0.08522	-0.10113	-0.74210
z = 0.0	A	0.12500	0.12496	0.12164	0.08533	-0.10072	-0.74107
	E	0.12501	0.12497	0.12166	0.08536	-0.10068	-0.74102

Table XIV (τ_{zz})

		r = 0.0	r = 0.2	r = 0.4	r = 0.6	r = 0.8	r = 1.0
z = 1.0	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
z = 0.8	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	0.00000	0.00000	-0.00001	-0.00002	-0.00003
z = 0.6	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00000	-0.00001	-0.00003	-0.00006	-0.00009
z = 0.4	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00002	-0.00005	-0.00010	-0.00015
z = 0.2	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00003	-0.00007	-0.00013	-0.00020
z = 0.0	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00003	-0.00008	-0.00014	-0.00021

Table XV (τ_{rz})

		r = 0.0	r = 0.2	r = 0.4	r = 0.6	r = 0.8	r = 1.0
z = 1.0	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
z = 0.8	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00008	-0.00027	-0.00064	-0.00123
z = 0.6	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00011	-0.00036	-0.00085	-0.00165
z = 0.4	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00009	-0.00031	-0.00074	-0.00144
z = 0.2	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	-0.00001	-0.00005	-0.00018	-0.00042	-0.00083
z = 0.0	A	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	E	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table XVI (p_{rr})

	r = 0.9	r = 0.925	r = 0.95	r = 0.975	r = 1.0
z = 1.0	0.000	0.002	0.008	0.022	0.038
z = 0.8	0.000	0.002	0.005	0.012	0.017
z = 0.6	0.000	0.001	0.001	0.002	0.002
z = 0.4	0.000	-0.001	-0.002	-0.006	-0.010
z = 0.2	0.000	-0.001	-0.005	-0.011	-0.017
z = 0.0	0.000	-0.002	-0.006	-0.013	-0.019

Table XVII ($p_{\theta\theta}$)

	r = 0.9	r = 0.925	r = 0.95	r = 0.975	r = 1.0
z = 1.0	0.000	0.001	0.002	0.007	0.011
z = 0.8	0.000	0.001	0.002	0.004	0.004
z = 0.6	0.000	0.000	0.001	0.001	-0.003
z = 0.4	0.000	0.000	0.000	-0.002	-0.009
z = 0.2	0.000	0.000	0.000	-0.003	-0.012
z = 0.0	0.000	0.000	0.000	-0.004	-0.013

Table XVIII (p_{zz})

	r = 0.9	r = 0.925	r = 0.95	r = 0.975	r = 1.0
z = 1.0	0.000	0.000	0.000	0.000	0.000
z = 0.8	0.000	0.000	0.001	0.000	-0.003
z = 0.6	0.000	0.001	0.002	0.001	-0.012
z = 0.4	0.001	0.002	0.003	0.001	-0.019
z = 0.2	0.001	0.002	0.004	0.001	-0.024
z = 0.0	0.001	0.002	0.004	0.001	-0.026

Table XIX (p_{rz})

	r = 0.9	r = 0.925	r = 0.95	r = 0.975	r = 1.0
z = 1.0	0.000	0.000	0.000	0.000	0.000
z = 0.8	0.000	0.001	0.003	0.005	0.000
z = 0.6	0.000	0.002	0.004	0.007	0.000
z = 0.4	0.000	0.002	0.004	0.006	0.000
z = 0.2	0.000	0.001	0.002	0.003	0.000
z = 0.0	0.000	0.000	0.000	0.000	0.000

Table XX (Radial Stress σ_{rr})

		r = 0.0	r = 0.2	r = 0.4	r = 0.5	r = 0.6	r = 0.7	r = 0.8	r = 0.85	r = 0.90	r = 0.925	r = 0.95	r = 0.975	r = 1.0
z = 1.0	3D	0.125	0.125	0.124	0.121	0.114	0.101	0.077	0.058	0.034	0.021	0.010	0.005	0.000
	2D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.078	0.059	0.047	0.033	0.018	0.000
z = 0.8	3D	0.125	0.125	0.124	0.122	0.117	0.106	0.085	0.069	0.047	0.035	0.024	0.014	0.000
	2D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.078	0.059	0.047	0.033	0.018	0.000
z = 0.6	3D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.077	0.058	0.046	0.033	0.018	0.000
	2D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.078	0.059	0.047	0.033	0.018	0.000
z = 0.4	3D	0.125	0.125	0.125	0.124	0.120	0.113	0.096	0.083	0.065	0.053	0.039	0.020	0.000
	2D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.078	0.059	0.047	0.033	0.018	0.000
z = 0.2	3D	0.125	0.125	0.125	0.124	0.121	0.114	0.099	0.087	0.069	0.057	0.042	0.021	0.000
	2D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.078	0.059	0.047	0.033	0.018	0.000
z = 0.0	3D	0.125	0.125	0.125	0.124	0.122	0.115	0.100	0.088	0.071	0.059	0.043	0.022	0.000
	2D	0.125	0.125	0.124	0.123	0.119	0.110	0.092	0.078	0.059	0.047	0.033	0.018	0.000

3D - Three dimensional theory   2D - Two dimensional theory

Table XXI (Hoop Stress $\sigma_{\theta\theta}$)

		r = 0.0	r = 0.2	r = 0.4	r = 0.5	r = 0.6	r = 0.7	r = 0.8	r = 0.85	r = 0.90	r = 0.925	r = 0.95	r = 0.975	r = 1.0
z = 1.0	3D	0.125	0.125	0.121	0.110	0.082	0.018	−0.112	−0.214	−0.352	−0.436	−0.530	−0.636	−0.757
	2D	0.125	0.125	0.121	0.111	0.084	0.022	−0.104	−0.205	−0.340	−0.423	−0.518	−0.627	−0.750
z = 0.8	3D	0.125	0.125	0.121	0.111	0.083	0.020	−0.108	−0.209	−0.345	−0.429	−0.523	−0.630	−0.754
	2D	0.125	0.125	0.121	0.111	0.084	0.022	−0.104	−0.205	−0.340	−0.423	−0.518	−0.627	−0.750
z = 0.6	3D	0.125	0.125	0.121	0.111	0.084	0.022	−0.105	−0.205	−0.340	−0.423	−0.518	−0.627	−0.754
	2D	0.125	0.125	0.121	0.111	0.084	0.022	−0.104	−0.205	−0.340	−0.423	−0.518	−0.627	−0.750
z = 0.4	3D	0.125	0.125	0.122	0.112	0.085	0.023	−0.102	−0.203	−0.337	−0.419	−0.514	−0.624	−0.754
	2D	0.125	0.125	0.121	0.111	0.084	0.022	−0.104	−0.205	−0.340	−0.423	−0.518	−0.627	−0.750
z = 0.2	3D	0.125	0.125	0.122	0.112	0.085	0.024	−0.101	−0.201	−0.335	−0.417	−0.512	−0.623	−0.754
	2D	0.125	0.125	0.121	0.111	0.084	0.022	−0.104	−0.205	−0.340	−0.423	−0.518	−0.627	−0.750
z = 0.0	3D	0.125	0.125	0.122	0.112	0.085	0.024	−0.101	−0.200	−0.334	−0.416	−0.511	−0.622	−0.754
	2D	0.125	0.125	0.121	0.111	0.084	0.022	−0.104	−0.205	−0.340	−0.423	−0.518	−0.627	−0.750

3D - Three dimensional theory 2D - Two dimensional theory

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