Some bootstrap calculations with rising trajectories

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with Rising Trajectories

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Abstract

We study the dynamical properties of rising Regge trajectories. In this dynamics crossing symmetry is an essential dynamical ingredient and unitarity is used only in some approximate form. The crossing is used through the finite energy sum rules. We first work in the narrow resonance approximation (when unitarity is not used) and consider the questions of scalar meson bootstrap and the bootstrap of ρ in $\pi-\pi$ scattering. Next we consider the question of ρ bootstrap by using unitarized Regge parameters through the solution of Cheng-Sharp equations. Two approximate forms of unitarity are considered - one corresponding to a single Regge pole term representation of the amplitude and the other corresponding to Khuri representation. In either case unitarity seems to make only a small difference to the results of narrow resonance approximation. We find that the values $\alpha' = 1 \text{ GeV}^{-2}$ and $\Gamma_\rho = 140$ MeV for the slope and width of the ρ are self-consistent when the cut-off parameter is chosen somewhere between the ρ and the f_0 on the degenerate $\rho - f_0$ trajectory.

Contents

CHAPTER 1

Introduction.

The Bootstrap Philosophy and Classical Bootstraps

The concept of analyticity occupies a central place in present day strong interaction physics. The postulate of maximal analyticity of the first kind [1] enables us to find all singularities of the S matrix, given the bound state and resonance poles. However, it does not restrict the poles themselves. This ambiguity is manifested in the undetermined subtractions in the Mandelstam representation of the scattering amplitude. Because of the Froissart bound [2] the undetermined subtractions are confined to the lower (ℓ = o, 1) partial waves. Thus we see that the maximal analyticity of the first kind determines the amplitude up to the first two waves. The postulate of maximal analyticity of the second kind [1] permits continuation in angular momentum, the only singularities being the isolated ones, and enables us to determine the lower partial waves of the amplitude from the higher ones by analytic continuation. We can therefore determine the amplitude completely by using the maximal analyticity of the first and the second kinds.

The two postulates together impose self-consistency requirement on the poles of the $\mathcal{S}$ matrix. This can be seen as follows: Starting with an arbitrary pole of the amplitude, we can generate a whole new set of singularities (and in particular double spectral functions) via unitarity. The divergences of these double spectral functions require a new set of poles via the maximal analyticity of the second kind. We can keep on repeating this operation until self-consistency is achieved i.e. the set of poles of the $\mathcal{S}$ matrix is complete and no new poles can be generated by the above process. The bootstrap hypothesis [(3)] postulates that the only set of strongly interacting particles satisfying the above-mentioned self-consistency criterion and therefore consistent with the maximal analyticity of the first and the second kinds is the actual set of strongly interacting particles found in nature. The bootstrap hypothesis accords equal status to all strongly interacting particles which are composite systems of each other, each owing its existence to the rest. This apparently simple and aesthetically satisfying idea of a 'nuclear democracy' of hadrons is however well-nigh impossible to test as a whole. The bootstrap problem is intrinsically a multi-channel problem and to implement it would involve the solution of an infinite set of coupled integral equations arising from unitarity condition. However we can hope that a small subset of the set of strongly interacting particles is approximately decoupled from the rest and demand self-consistency between the output and the input. The simplest and best studied example is the bootstrap of the Rho meson in pi-pi scattering. There is ample experimental evidence that low energy $\pi-\pi$ scattering is dominated by the ρ resonance. Since $\pi-\pi$ system is crossing symmetric one can ask whether the force arising from the exchange of the $\pi-\pi$ in the crossed channel is sufficient to produce the ρ in the direct channel. Or in other words, can the ρ bootstrap itself ? (see fig 1.)

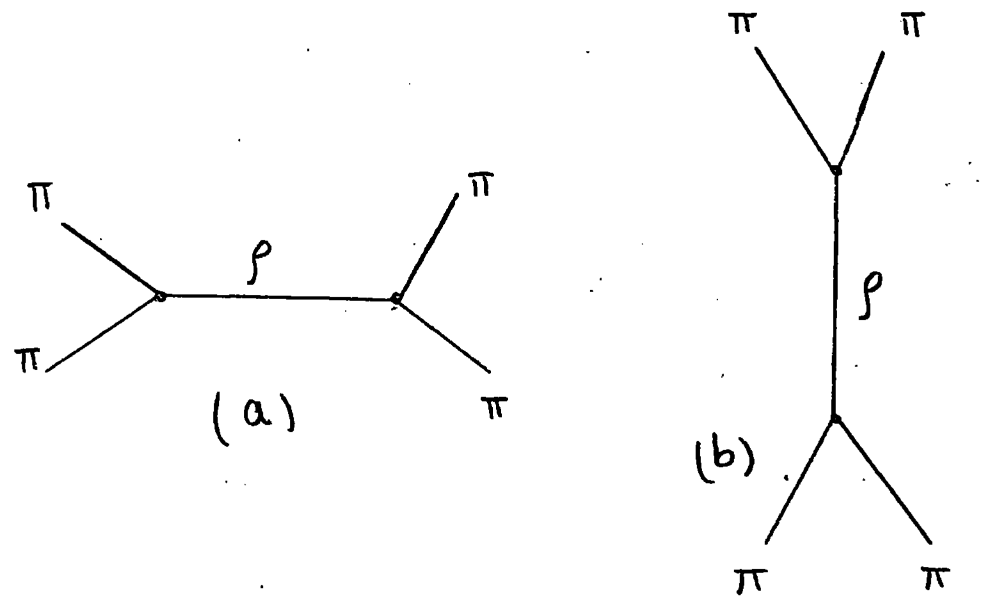

Fig 1. Forces in the $\pi-\pi$ system

This and similar calculations [4] have been performed in various degrees of approximation and with varying degrees of success. One good feature of these calculations is that the correct signs of the masses and coupling strengths of the output particles are reproduced; [4] however in most of these calculations the magnitudes of the output quantities are bigger than those of the input by a factor of 2 to 6.

In the so-called classical bootstrap calculations unitarity is an essential dynamical ingredient and crossing is applied only in some approximation. The input is taken as the force arising from a few single particle exchange graphs and output mass and coupling are calculated by solving the N/D equations. Apart from the basic drawback of not treating the input and output particles on an equal footing, (the input consists of an elementary particle exchange whereas the output is the composite particle corresponding to a zero of the denominater function $D_\ell(s)$) this approach has the additional drawback of introducing arbitrary cut offs to circumvent divergence of integrals arising from the exchange of spin $\geqslant 1$ particles (the exchange of a particle of spin ℓ gives a contribution proportional to $P_\ell(Z_t)$ which goes as Z_t^ℓ or s^ℓ for large $s \sim Z_t$). Also the results are dependent on the choice of subtraction constant and if it is taken somewhere in the nearby part of the left hand cut (LHC), the distant part of the LHC is not taken into consideration and consequently the approximation is meaningful only if the short range forces are unimportant. Alternative treatments of the LHC contributions, as for example in Balazs's method [5], overcome some of the disadvantages of the other methods but they have their own shortcomings. In all such calculations the widths of the resonances come out too large and the output masses a bit too small.

An alternative approach to bootstrap dynamics is the bootstrapping of a whole trajectory rather than a single particle on it. The Chew-Jones' "new form" of the strip approximation is based on such an approach. Here the amplitude is represented as the sum of Regge pole contributions in each channel and the output trajectory is obtained by solving the N/D equations. The trajectory $\alpha(t)$ reproduces itself self consistently for small $|t|$. Using this method with some improvements Collins and Johnson [6] have recently succeeded in bootstrapping the ρ and the Pomeranchuk trajectories in $\pi-\pi$ scattering. They find self-consistent trajectories with correct physical mass, width and intercept, but their solution is not unique since self-consistency can be achieved with trajectory $\alpha_\rho(t)$ having intercept anywhere from $\alpha_\rho(0) = 0.32$ to $\alpha_\rho(0) =$

0.69. The main drawback of this method is that an arbitrary parameter (the strip width) is introduced and it is assumed that the trajectory falls off after this width. The Chew-Jones strip approximation is therefore not suitable for bootstrapping infinitely rising trajectories.

Another method of calculating self-consistent Regge trajectories starts with dispersion relations for the Regge parameters,[7] and involves the solution of coupled integral equations. This method has already yielded successful results in potential scattering [8]. The dynamical scheme that we will discuss in the following pages (due to Mandelstam [8]) is an application of the method based on dispersion relations for Regge parameters to the relativistic scattering involving infinitely rising Regge trajectories. In this scheme crossing is an essential dynamical ingredient and it is applied via the so called Finite Energy Sum Rules [9]. Unitarity will be applied in different degrees of approximation. The application of unitarity leads to coupled integral equations for the Regge parameters $\alpha(t)$ and $\beta(t)$ — the Cheng-Sharp type equations [10]. In most previous calculations only the narrow resonance approximation where the trajectory $\alpha(t)$ is strictly linear, has been used.

In the next chapter we will describe the dynamics based on the rising trajectories, discuss the application of crossing through finite energy sum rules, and consider some simple applications of these dynamics, in the narrow resonance approximation. In the third chapter the effect of unitarity on the linearity of the trajectories, and on the consistency of the bootstrap will be investigated.

CHAPTER II

Bootstraps of Linearly Rising Trajectories.

In this chapter we will discuss a dynamical scheme based on the concept of linearly rising trajectories (8) and study how the use of finite energy sum rules (9) to apply crossing can yield some results on the slopes of trajectories and the couplings of resonances. We will concentrate on the scalar meson and $\pi-\pi$ scattering problems. These being crossing symmetric reactions it will be possible to do bootstrap calculations. Unitarity will not be used in such calculations.

II. 1 (a) Analytic Properties of Regge Parameters:

Before obtaining the dynamical equations for the Regge parameters, we need to know the analytic properties of these parameters. We start from the Froissart-Gribou projection for the partial wave amplitude

$$A(\ell, t) = \frac{1}{\pi} \int_{Z_t (s_0, t)}^{\infty} Q_\ell(Z') \; D_s (s', t) \, dZ' \qquad \ldots\ldots (1)$$

for $\ell \geq N(t)$

where D_s (s,t) is the s-discontinuity of the scattering amplitude A(s,t) and N(t) denotes the number of subtractions required in the Mandelstam representation for this amplitude. Noting that the trajectory function α(t) is a pole of the continued partial wave amplitude A(ℓ,t) such that A(ℓ,t) = β(t) /($\ell - \alpha$(t)) we can derive the analyticity properties of α (t) and β(t) as follows:

We write A(ℓ,t) as

$$A(\ell,t) = E(\ell,t) + F(\ell,t)$$

where

$$E(\ell,t) = \frac{1}{\pi}\int_{S}^{\infty} Q_\ell(Z') \; D_s(s',t) \; dZ' \qquad \ldots\ldots(2)$$

and F(ℓ,t) is given by the same integral from s = so to s = S. Since F(ℓ,t) is defined by a finite integral it must be holomorphic in Re $\ell > -1$ with just poles at the negative integers due to Q_ℓ(Z). The other ℓ plane singularieies of A(ℓ,t) are due to the asymptotic s behaviour of the integrand in equation (1) and are therefore contained in E(ℓ,t). The position of a pole is given by

$$\left[E(\ell,t) \right]^{-1} = 0 \quad \text{at} \quad \ell = \alpha(t) \qquad \ldots\ldots(3)$$

but because of singularity at threshold it is better to use the reduced amplitude $B(\ell,t) = A(\ell,t) / (q_{t12}\, q_{t34})^{\ell}$ and therefore write (3) as

$$(q_{t\,12}\, q_{t\,34})^{\ell}\ \left[E(\ell,t)\right]^{-1} \;=\; 0 \text{ at } \ell = \alpha(t) \quad \cdots\cdots (4)$$

Therefore the residue $\beta(t)$ of the pole at $\ell = \alpha(t)$ defined by $A(\ell,t) = \beta(t)/(\ell - (t))$ is by Cauchy's theorem

$$\beta(t) = \frac{1}{2\pi i}\oint d\ell\ E(\ell,t) \quad \cdots\cdots (5)$$

or $$\gamma(t) = (q_{t12}\, q_{t\,34})^{-\ell}\ \beta(t) \quad \cdots\cdots (6)$$

$$= \frac{1}{2\pi i}\oint d\ell\ (q_{t\,12}\, q_{t34})^{-\ell}\ E_{\ell}(t) \quad \cdots\cdots (7)$$

where the integral is taken in a path around the point $\ell = \alpha(t)$. Equations (4) and (7) enable us to find the analytic properties in t of $\alpha(t)$ and the reduced residue $\gamma(t)$. If $\mathrm{Re}\,\ell > \mathrm{Re}\,\alpha_M(t)$ where $\alpha_M(t)$ is the highest lying ℓ plane singularity, the integral (2) converges and so $E_{\ell}(t)(q_{t\,12}\, q_{t\,34})^{-\ell}$ has t plane singularities of the full partial wave amplitude viz. a right-hand cut starting at t = to, the threshold, and the usual left-hand cut (due to crossed channel singularities of the total amplitude A(s, t) . Since we are integrating from s = S rather than so in (2) the left-hand branch point is at $t \approx -S$ for large S and by taking S large enough we can cause the left-hand branch point to recede as far to the left as we please. This means that the singularities of $\alpha(t)$ and $\gamma(t)$ which stem from those of $E(\ell, t)$ do not include the left-hand branch points of the partial wave amplitude. So the only relevant singularities are for $t \geqslant t_0$. It can further be shown that as long as the trajectories do not cross, the Regge parameters $\alpha(t)$ and $\gamma(t)$ are real analytic functions of t with only right-hand cuts.

II. 1 (b) Dynamical Equations for Regge Parameters

To obtain dynamical equations for the Regge trajectory $\alpha(t)$ and the residue function $\beta(t)$, we note that Regge parameters $\alpha(t)$ and $\gamma(t)$ are real analytic functions of t. Demanding a linear dependence of $\alpha(t)$ on t, the explicit form of $\alpha(t)$ should be

$$\alpha(t) = at + b \qquad \ldots\ldots (8)$$

To obtain an expression for $\beta(t)$, we use the fact that $\gamma(t) = \beta(t) \,/\, (4q_t^2)^{\alpha(t)}$ is a real analytic function of t and so we can write

$$\beta(t) = (4q_t^2)^{\alpha(t)} \cdot E_1(t) \qquad \ldots\ldots (9)$$

where E (t) is also real and analytic. Noting further that $\beta(t)$ should have zeros at Mandelstam symmetry points viz. at negative half integral values of $\alpha(t)$ other than $-{}^{1}/_{2}$, and requiring that $\beta(t)$ should go asymptotically as $1/t$ we can rewrite (9) as

$$\beta(t) = \left(\frac{4\, a\, q_t^2}{e}\right)^{\alpha(t)} \frac{1}{\Gamma\left(\alpha(t) + {}^{3}/_{2}\right)} E(t) \qquad \ldots\ldots (10)$$

The exponential factor together with the gamma function ensures the asymptotic behaviour $1/t$. The slope 'a' appears in (10) because all trajectories appear to have the same slope and therefore we are assuming that $1/a$ is the scale factor which determines the asymptotic behaviour. E(t) will be taken as a constant in scalar meson case and $= \alpha(t)$ x constant for ρ trajectory in $\pi - \pi$ scattering where $\alpha(t)$ is the ghost-killing factor. It's value depends on the coupling constant (or width) of the particle. The narrow resonance approximation (or a strictly linear form for $\alpha(t)$) requires the saturation of the scattering amplitude with zero width resonances lying on linear trajectories. This gives

$$\mathrm{Im}\, A(s,t) = \sum_{\ell} \pi\,(2\ell_R + 1)\, \beta\left(m_R^2\right) P_{\ell_R}\left(1 + \frac{2\,t}{s - 4m_\pi^2}\right) \times \delta\left(s - m_R^2\right) / \alpha'$$

To do dynamical calculations we need to apply crossing. In fact crossing is the essential dynamical ingredient of such a dynamics. This is achieved by means of the so called finite energy sum rules (FESR). In the next section we discuss the formalism of FESR and the related concept of duality, and then turn to a discussion of their applications to the scalar meson and $\pi-\pi$ bootstraps.

II. 2 Finite Energy Sum Rules and Duality

(a) Derivation of FESR:

To derive the FESR[9] we begin with a function $F(\nu)$ analytic in the ν - plane with a right-hand cut from ν_0 to ∞ and satisfying the dispersion relation

$$F(\nu) = \frac{1}{\pi} \int_{\nu_0}^{\infty} \frac{\mathcal{I}m\, F(\nu')\, d\nu'}{\nu' - \nu} \qquad (11)$$

(We have written an unsubstracted dispersion relation. If $F(\nu) \not\rightarrow 0$ as $\nu \rightarrow \infty$ we can write down a subtracted dispersion relation in the usual way.) The variable ν might represent some physical variable like the energy or momentum transfer in some scattering process. The FESR are consistency conditions that have to be obeyed by the function $F(\nu)$ as a result of its analytic structure and asymptotic expansion. Suppose $F(\nu)$ is a real analytic function in the cut ν - plane that can be expanded in an asymptotic power series. We note that each term in the series can be written in such a form that it is also a real analytic function in the cut ν- plane. The function $-(e^{-i\pi}\nu)^{\alpha}/\sin\pi\alpha$ has the imaginary part ν^{α} at the cut from ν_0 to ∞. We can therefore write asymptotically i.e. for $\nu \geqslant N_{(say)}$

$$F(\nu) \approx \sum_i - C_i \left(e^{-i\pi}\nu\right)^{\alpha_i} / \sin\pi\alpha_i \qquad \text{.........} (12)$$

Let us now form the integral $\oint F(\nu)\, d\nu$ along the contour of figure 2.

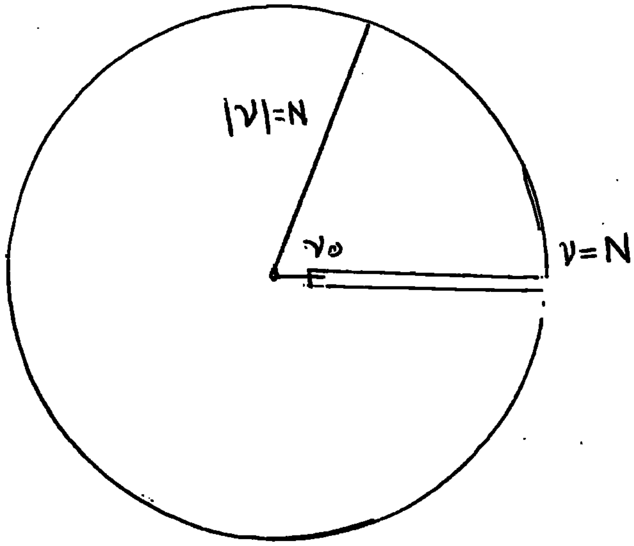

Fig. 2 Contour of integration for the derivation of FESR

The integration from $N - i\epsilon$ to ν_0 and then to $N + i\epsilon$ will give $2i \int_{\nu_0}^{N} \mathcal{I}m F \; d\nu$ For the integration along the contour we use (12). The whole integral along the contour is zero, by Cauchy's theorem. On performing this calculation we get

$$\int_{\nu_0}^{N} \mathcal{I}m \, F(\nu) \, d\nu \approx \sum_i C_i \, N^{\alpha_i + 1} \Big/ (\alpha_i + 1) \quad \ldots\ldots (13)$$

In general we can write down the nth moment sum rule

$$\int_{\nu_0}^{N} \nu^n \, \mathcal{I}m \, F(\nu) \, d\nu \approx \sum_i \frac{C_i \, N^{\alpha_i + n + 1}}{\alpha_i + n + 1} \quad \ldots\ldots (14)$$

The higher moment sum rules emphasize high energy region of the integral and therefore it is preferable to use lower moment sum rules as far as possible. The sum rules (13) and (14) hold for any function with the above mentioned analytic behaviour and asymptotic empansion. It should be noted that the form of the sum rules is independent of the position of α_i. Consequently any function that obeys a dispersion relation with an arbitrary number of subtractions will also obey FESR. If the leading power in the asymptotic series is low enough (e.g. $\alpha_i < -n - 1$ in equation (7)) we can find from (7) the superconvergence relations

$$\int_{\nu_0}^{\infty} \nu^n \, \mathcal{I}m \, F(\nu) \, d\nu = 0$$

by letting $N \rightarrow \infty$.

As far as the application of FESR is concerned, we may note that we can write down even or odd moment sum rules for the scattering amplitudes which are odd or even under crossing $\nu \leftrightarrow -\nu$ (or $s \leftrightarrow u$, at fixed t, $\nu = \frac{1}{2}(s - u)$. The amplitudes of a definite signature have only a right hand cut and we can write down all moment sum rules for such amplitudes. The signatured amplitudes, however, have fixed poles at wrong signature unphysical values of angular momentum and it is not always possible to ignore the contributions of these fixed poles. It is therefore advantageous to work with the total amplitudes.

II. 2(b): <u>Interpretation of FESR and Duality:</u>

As we have seen in II.2(a) the FESR connect the low energy region and the high energy region of the scattering amplitude. The high energy region is described in terms of the crossed (t) channel Regge poles which are observed for small t values. Therefore one could say that the FESR connect the low s channel behaviour with the low t channel effects. If the low s channel amplitude is described by resonance contributions only and the high energy behaviour is given completely by t channel Regge poles then we can speak of duality [9,11] between direct channel resonances and crossed channel Regge trajectories. The Pomeranehuk trajectory has to be regarded as a special kind of singularity, and is not associated with the direct channel resonances.[12] If all resonances are supposed to lie on approximately linear trajectories, then we have a scheme where resonances in one channel generate resonances in the crossed channels. Appreciable forces are therefore to be considered only when resonances exist. They do not result from unitarity alone.

Though controversial this duality idea has received support from a related phenomenon observed by Schmid.[13] If one takes the partial wave projection in the direct channel of the Regge exchange amplitude $A(s,t)$ the resulting partial wave amplitude $A_\ell(s)$ shows phase variations giving rise to loops in the Argand diagram. These loops approximate the observed resonances. Thus it appears that direct channel resonances and crossed channel trajectories may be interlocked. The presence or absence of resonances in one channel can be associated with relations between trajectories in the crossed channels. In channels such as K^+N where there are no observed resonances, the crossed channel contains exchange degenerate trajectories which give purely real contributions to the amplitude.

The duality idea has implications for the description of a scattering amplitude. The so called strong (or local) duality implies that the full scattering amplitude for all energies may be described either in terms of direct channel resonances or in terms of crossed channel trajectories. The Veneziano model[14] has this strong duality built into it. The so called averaged (or weak) duality on the other hand implies that averaged sum of resonance contributions is equal to Regge contributions (through FESR) and therefore the amplitude should be described at a given energy by the sum of direct channel resonances and crossed channel Regge poles minus the average of the resonances $A(s,t) = A_{Res}(s,t) + A_{Regge}(s,t) - \langle A_{Res}(s,t) \rangle$. The extrapolation of Regge terms to low energies is taken to give a smooth background upon which resonances are superimposed.

Duality becomes especially useful in bootstrap calculations. Taking the same type of Regge trajectories as an input on the left hand side of the FESR (via their low energy resonances) as are used for describing the high energy region (on the right hand side of FESR), we can demand self-consistency of the Regge parameters. Bootstrap calculations with FESR are much simpler than the older N/D programs. In the narrow resonance approximation the formulation of the problem with FESR leads to algebraic equations for the trajectories and their residues.

We have seen that starting from FESR and assuming resonance saturation we arrive at the concept of duality. The inverse procedure would be to start from duality idea and construct an amplitude, which is a solution of the

FESR. The so called Veneziano model[14] is such an amplitude in the zero width approximation. A one term Veneziano formula is

$$V(s,t) = \bar{\beta} \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))}$$

where $\bar{\beta}$ is a constant.

This is crossing symmetric between s and t, has resonance poles at positive integral values of α and gives Regge asymptotic behaviour in s and t. It also satisfies the FESR.

These fundamental principles of analyticity, Regge asymptotics, and crossing symmetry on which FESR are based are at the root of our bootstrap calculations that will be discussed in this and the next chapter.

II. 3 Bootstraps of Linearly Rising Trajectories:

We have seen in the last section how FESR relate the low energy part of an amplitude with the Regge parameters of the crossed channel. Assuming resonance saturation of the direct channel low energy amplitude one can relate the Regge parameters of the s- channel with those of the t channel. This constitutes a (so called) bootstrap of the linearly rising trajectory. However we should note that when we speak of bootstrapping a trajectory we really mean bootstrapping of the parameters of the trajectory. The absolute, magnitudes of the widths of the resonances lying on the trajectory (or the magnitudes of the residue functions) are not bootstrapped since they appear linearly on both sides of the sum rule. To bootstrap these quantities we need an additional principle such as unitarity, as will be discussed later. With this limited sense of the word bootstrap in mind, we proceed to discuss some simple cases of such bootstrap calculations for scalar meson and π-π scattering.

II. 3(a) <u>Scalar Meson Bootstrap:</u>

The simplest of all possible worlds of strongly interacting particles is the one consisting solely of scalar mesons ($J^P = 0^+$ particles). If the concept of nuclear democracy were applicable to such a world then a scalar meson would appear as the bound state of a pair of scalar mesons sustained by the force arising from the exchange of a scalar meson in the crossed channel.

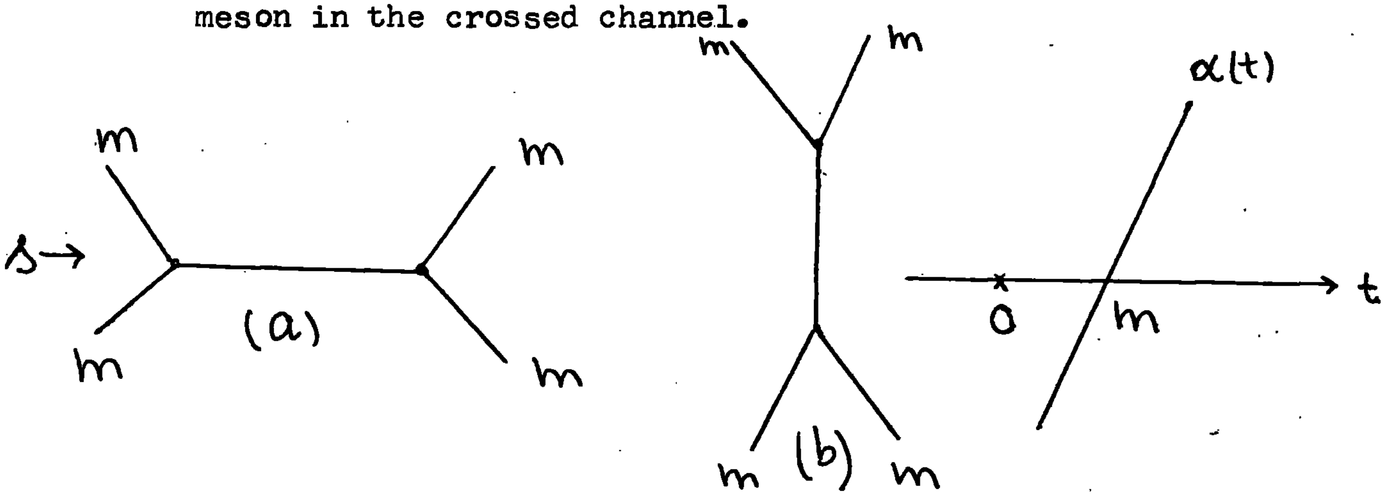

Fig. 3 (a),(b) Forces in scalar meson system

(c) scalar meson trajectory

However as Gross[15] has shown it is easy to see that a scalar meson cannot bootstrap itself. Let $A(s,t) \equiv A(\nu,t)$ denote the scattering amplitude for scalar meson - scalar meson scattering. This amplitude is completely crossing symmetric in s, t and u. Moreover $A(\nu,t)$, $\nu = \frac{1}{2}(s-u)$ is even under crossing $\nu \longleftrightarrow -\nu$. Thus we can write down the first moment sum rule

$$\int_{V_0}^{N} \nu \, \mathrm{Im}\, A(V,t)\,dV = C(t)\,\beta(t)\left(\frac{N}{q_t^2}\right)^{\alpha(t)} \times \frac{N^2}{\alpha(t)+2} \quad (9)$$

where $\alpha(t) = at + b$ is the scalar meson trajectory such that $\alpha(m^2) = 0$ and we have used the asymptotic form

$$A(\nu,t) \underset{\nu \gg N}{\approx} C(t)\,\beta(t)\left(\frac{1 + e^{-i\pi\alpha}}{\sin \pi\alpha(t)}\right)\left(\frac{\nu}{q_t^2}\right)^{\alpha(t)}$$

where $C(t) = \sqrt{\pi}\,\Gamma(\alpha + 3/2) \Big/ \Gamma(\alpha + 1)$

saturating the L.H.S. of (9) by a scalar meson lying on the straight line trajectory $\alpha(s)$, we have

$$\mathrm{Im}\, A(\nu,t) = \pi(2\ell + 1)\,\hat{\beta}(s)\, P_\ell\left(1 + \frac{2t}{s-4m^2}\right) \times \delta(s - m^2)/\alpha'$$

$$= \pi\,\hat{\beta}(s)\,\delta(s - m^2)/\alpha' \qquad (\alpha' = a)$$

so that equation (9) gives, on substitution

$$\frac{\pi(t - 2m^2)}{2\alpha'}\,\hat{\beta}(m^2) = \pi\,\beta^t(m^2)\left(\frac{N}{q_t^2}\right)^{\alpha(t)} \frac{N^2}{\alpha + 2}$$

Evalating both sides at $t = m^2$ we have

$$-\pi m^2 \beta^s(m^2)/2\alpha' = \pi \beta^t(m^2) N^2/2 \qquad (9)'$$

The bootstrap condition is $\beta^s(m^2)/\beta^t(m^2) = 1$ which gives a negative value for the slope of the scalar meson trajectory — thus implying the impossibility of the scalar meson bootstrap.[15]

It is interesting however to consider the possibility that the exchange of an additional trajectory which couples with the scalar meson channel might influence our results i.e. if a world more complex than the scalar meson world might be self-consistent. Since the Pomeranchuk trajectory couples with the scalar meson channel, we try it first. To simplify calculations we take the Pomeranchuk trajectory parallel with the scalar meson one and denote the common slope by α'. Since we now have an additional unknown quantity $\beta_p(t)$ we consider both the first moment and the third moment sum rules

$$S_1(t) = C(t)\beta(t)\left(N/q_t^2\right)^{\alpha(t)} \frac{N}{\alpha(t)+1} + C_p(t)\beta_p(t) \times \left(\frac{N}{q_t^2}\right)^{\alpha(t)} \times \frac{N}{\alpha_p(t)+1} \qquad (10)$$

and

$$S_3(t) = C(t)\beta(t) \left(N/q_t^2\right)^{\alpha(t)} \frac{N^3}{\alpha+3} + C_p(t) \times \beta_p(t) \left(N/q_t^2\right)^{\alpha_p(t)} \times N^3/\left(\alpha_p(t)+3\right) \qquad (11)$$

Also

$$S_1(t) = \int_{\nu_0}^{N} \nu \,\mathcal{I}m\, A(\nu,t)\, d\nu = \pi \frac{\beta(m^2)\ (t - 2m^2)}{\alpha'}$$

$$S_3(t) = \int_{\nu_0}^{N} \nu^3 \,\mathcal{I}m\, A(\nu,t)\, d\nu = \pi \frac{\beta(m^2)\ (t - 2m^2)^3}{\alpha'}$$

Remembering that $\alpha(m^2) = 0, \alpha_p(t) = t + 1$ and taking $\alpha' = 1$ in scalar meson mass units ($m^2 = 1$) we obtain from the two sum rules the equations:

$$-2\beta(m^2) = \beta(m^2)\, N + \frac{10}{9}\beta_p(m^2)\, N^3 \qquad \ldots\ldots \quad (12)$$

$$-6\beta(m^2) = \beta(m^2)\, N^3 + 2\beta_p(m^2)\, N^5 \qquad \ldots\ldots \quad (13)$$

These equations constitute consistency conditions on the residues and on the cut off parameter N. Eliminating β's we obtain the equation:

$$2N^3 + 9N^2 - 15 = 0$$

A solution of this equation is $N \approx 1.2$. Now from the relation $\nu = \frac{1}{2}(s - u)$, $N = s_N - 2m^2 + t/2$, which gives for $N = 1.2$ and $t = m^2 = 1$, $s_N = 2.6$. The half-way point between the 0^+ and 2^+ particles on the scalar meson trajectory corresponds to $s_N = 2$, but any value for s_N lying between 1 and 3 is permissible. However taking $N = 1.2$ one gets from (12)

$$\beta(m^2)/\beta_p(m^2) = -5/3 \qquad \ldots\ldots\ldots \quad (14)$$

$\beta(m^2)$ being positive it follows that $\beta_p(m^2)$ is negative. Again evaluating the sum rules (10) and (11) at t = o we obtain:

$$\beta(0)/\beta_p(0) = \frac{3}{64} (N^2 - 8) N^2 \qquad \ldots\ldots (14)$$

Now for t = o, $N = s_N - 2$. With s_N lying anywhere between the masses of the 0^+ and 2^+ particles on the scalar meson trajectory, the equation (14) gives

~~(14) gives~~ $$\beta(0) / \beta_p(0) < 0 \qquad \ldots\ldots (15)$$

Now $\beta_p(0)$ is related with the total cross-section in the forward direction for an elastic scattering process (by optical theorem) and therefore it must be positive. Thus (15) implies that $\beta(0)$ is negative. Hence from equations (14) and (15) we conclude that the consistency conditions for the bootstrap of scalar meson and Pomeronchuk trajectories require that $\beta(t)$ and $\beta_p(t)$ should have opposite signs at t = o and t = m^2 i. e. they should change signs simultaneoursly somewhere between t = o and t = m^2. This is obviously very implausible. Thus we conclude that even with the inclusion of the Pomeranchuk trajectory the scalar meson trajectory does not bootstrap itself.

II. 3(b) The Case of Signatured Amplitude

An amplitude of definite signature has only a right-hand cut and we can write down all moments sum rules for such an amplitude. We will consider the zeroth moment sum rule for the positive signatured amplitude $A^+(\nu, t)$ of scalar meson-scalar meson scattering. This amplitude has a fixed pole at $\alpha = \alpha_1 = -1$ which is the first wrong signature unphysical value of angular momentum. Corresponding to equation (9) we will have the sum rule.

$$\int_{\nu_0}^{N} \operatorname{Im} A^+(\nu,t)\, d\nu = C(t)\,\beta(t)\left(\frac{N}{q_t^2}\right)^{\alpha(t)} \frac{N}{\alpha(t)+1} + \gamma(t)\, N^{\alpha_1+1} \qquad (16)$$

where $C(t) = \sqrt{\pi}\ \Gamma(\alpha + 3/2)/\Gamma(\alpha + 1)$ and the last term denotes the contribution of the fixed pole at $\alpha = \alpha_1 = -1$. Saturating the left-hand side of (16) by the scalar meson and evaluating both sides at $t = m^2$.

$$\frac{\pi}{\alpha'}\beta^{s}(m^2) = \tfrac{1}{2}\pi\, \beta^{t}(m^2)\, N + \gamma(m^2)\, N^{0} \qquad (17)$$

Without the last term, the sum rule (17) will give a positive value for the slope α' in contradiction with the result obtained from the sum rule (9) for the total amplitude. Demanding that the two sum rules (9) and (17) for the amplitudes $A(\nu,t)$ and $A^+(\nu,t)$ respectively should be consistent, we can obtain a condition for the "residue" $\gamma(t)$ of the fixed pole at $\alpha = -1$.

Taking $\beta^{s}(m^2) = \beta^{t}(m^2)$ in (17) we obtain from (9) and (17):

$$\gamma(m^2) = -\tfrac{1}{2}\pi N\,\beta(m^2)\ (1 + 2N)$$

Since $\beta(m^2)$ and N are positive $\gamma(m^2)$ is negative. Also it is obvious that $\gamma(m^2)$ is larger in magnitude than $\beta(m^2)$ Thus for the two sum rules to be compatible we require that the residue of the fixed pole at the wrong signature unphysical value of the angular momentum should be large and negative.

It is also obvious that the higher moment sum rules for the amplitude $A(\nu,t)$ will continue to give a negative value for the slope α' (because of the factors ν^3, ν^5 etc.) and the higher moment sum rules for the signatured amplitude $A^+(\nu,t)$ will have more fixed poles at other wrong signature unphysical values of α i.e. at $\alpha = -3, -5$ etc. By requiring that the two kinds of sum rules be compatible we could obtain more conditions on the residues of the fixed poles.

II. 4 Bootstrap of Rho Meson in $\pi-\pi$ Scattering.

(a) FESR for the total Amplitude:

It is well-known that the rho meson is a prominent resonance of the $\pi-\pi$ system and since the $\pi-\pi$ reaction is completely crossing symmetric, it contains the ρ in all channels. Through the FESR we can relate the $I_t = 1$ $\pi-\pi$ amplitude with all isospin states of the s - channel low energy amplitude. We can then ask, following Gross [15].

and Schmid[16] whether the input and output parameters are consistent. In the narrow resonance approximation, where the direct channel amplitude will be saturated with the contributions of direct channel resonances lying on a straight line trajectory and the crossed channel amplitude will be the Regge amplitude with a linear trajectory, the residue function will appear linearly on both sides of the sum rule and consequently it will not be possible to determine the absolute magnitudes of the widths (or couplings) of resonances; only the ratios will be determined.[16]

We work at fixed 't' and begin with the $I_t = 1$ $\pi - \pi$ amplitude $A^{I_t}(\nu, t)$, $\nu = \frac{1}{2}(s - u)$ and remembering that

$$A^{I_t = 1}(-\nu, t) = -A^{I_t = 1}(\nu, t),$$

we can only write down even moment sum rules. We will consider only the zeroth and the second moment sum rules viz.

$$S_0(N,t) \equiv \int_{\nu_0}^{N} \mathrm{Im}\, A^{I_t = 1}(\nu,t)\,d\nu = \frac{C(t)\,\beta(t)}{\alpha+1}\left(\frac{N}{q_t^2}\right)^{\alpha(t)}\left(\frac{N}{[illegible]}\right) \tag{18}$$

and

$$S_2(N,t) \equiv \int_{\nu_0}^{N} \nu^2\, \mathrm{Im}\, A^{I_t=1}(\nu,t)\,d\nu = \frac{C(t)\,\beta(t)}{\alpha(t)+3}\left(\frac{N}{q_t^2}\right)^{\alpha(t)} N^3 \tag{19}$$

where $C(t) = \sqrt{\pi}\,\Gamma(\alpha + 3/2)\,/\,\Gamma(\alpha + 1)$

and we have used the asymptotic form

$$A^{I_t}(v,t) \underset{v \geq N}{\approx} \frac{\Gamma(\alpha + 3/2)\beta(t)}{\Gamma(\alpha+1)\sin\pi\alpha(t)} \times \left(-\frac{v}{q_t^2}\right)^{\alpha(t)}$$

for the amplitude

$$A^{I_t}(v,t) = +\pi(2\alpha+1)\beta(t) \frac{P_\alpha(Zt) - P_\alpha(-Zt)}{2\sin\pi\alpha}$$

$\alpha(t) = \alpha$ is of course the degenerate trajectory.

The amplitude $A^{I_t=1}(v,t)$ on the left-hand sides of equations (18) and (19) can be related to the s - channel amplitude $A^{I_s}(v,t)$ by the crossing relation

$$A^{I_t=1}(v,t) = \sum_{I_A} \beta(I_t = 1, I_s)\, A^{I_s}(v,t) \qquad \ldots\ldots \quad (20)$$

$\beta(I_t\ I_s)$ is of course the isospin crossing matrix.

Now the resonances in the s - channel are the $I_s = 1$ ρ - resonance and the $I = 0$ f_0 resonance (we assume that there is no $I = 2$ resonance). First we consider the case when the left-hand sides of (18) and (19) are saturated by the ρ resonance only. In this case

$$\mathcal{I}m\, A^{I_s=1}(v,t) = \frac{1}{\alpha'}\,\pi(2\ell+1)\beta(s)\,P_\ell\left(1 + \frac{2t}{s-4m^2}\right) \times \delta(s - m^2)$$

)(where α' is the slope.)

$$= 3\pi\beta(s)\left(1 + \frac{2t}{s-4m^2}\right)\delta(s-m^2)/\alpha'$$

Also $\beta(s)$ is related to the resonance width at the resonance mass m_R,

$$\beta(m^2_R)/\alpha' = m_R \Gamma_e / K\ (s = m^2_R)$$

where $K(s) = \left[(s - 4m^2_\pi)/s\right]^{\frac{1}{2}}$ and Γ_e is the elastic width $= \Gamma x$, x being inelasticity (= 1 for ρ) and Γ the total width. Using (20) and making substitutions in sum rules (18) and (19), and evaluating both sides at $t = m^2_\rho$ we obtain:

$$\Gamma^t_\rho / \Gamma^s_\rho = (3m^2_\rho - 4m^2_\pi) / N^2\alpha' \quad \ldots\ldots (18')$$

and

$$\Gamma^t_\rho / \Gamma^s_\rho = 2(3m^2_\rho - 4m^2_\pi)^3 / 4N^4\alpha' \quad \ldots\ldots (19')$$

Now we can check the consistency of our sum rules (18) and (19) in various ways. First we consider (18) and (18') only. We choose N at the half-way point between the ρ and f_0, so that because of the relation

$$\nu = \tfrac{1}{2}(s - u) = \tfrac{1}{2}(2s - 4m^2_\pi + t)$$

$$N = 3m^2_\rho/2 - 2m^2_\pi + 1/2\alpha' \quad \ldots\ldots (21)$$

(at $t = m^2_\rho$) corresponds to the half-way value $s_N = m^2_\rho + 1/2\alpha'$. Taking experimental values of $m^2_\rho = 30m^2_\pi$ and $\alpha' = 0.02\ m^{-2}_\pi$ we obtain from (18')

$$\Gamma^t_\rho / \Gamma^s_\rho = \Gamma^{out}_\rho / \Gamma^{in}_\rho = 0.93$$

This result depends crucially on the crossing matrix element $\beta(I_t=1, I_s=1) = \frac{1}{2}$ and on ρ spin. It also depends on N.

This value for $\Gamma_\rho^{out}/\Gamma_\rho^{in}$ should be compared with the corresponding value in classical bootstrap calculations where in most cases $\Gamma_\rho^{out}/\Gamma_\rho^{in} = 5 - 10$.

Another way to check consistency is to take $\Gamma_\rho^{t} = \Gamma_\rho^{\Delta}$ as a bootstrap condition in (18) and then to find N from a knowledge of the experimental values of the other parameters. This gives $N = 65.5\ m_\pi^2$ which should be compared with the half-way value of $68\ m_\pi^2$.

We can also determine self-consistent slope from (18) provided we express N in terms of α' by relation (21), take $\Gamma_\rho^{t}/\Gamma_\rho^{\Delta} = 1$ and substitute $m_\rho^2 = 30\ m_\pi^2$ (from experiment). We obtain the equation $(43\alpha' - 1/2) = 0$ which gives

$$\alpha' = .0116\ m_\pi^{-2} .$$

Next we use the second sum rule (19) as well. Taking $\Gamma_\rho^{t} \simeq \Gamma_\rho^{\Delta}$ and other parameters from experiment we obtain $N = 63.1\ m_\pi^2$ which is not far from the value $N = 65.5\ m_\pi^2$ obtained from the zeroth moment sum rule.

Using both the sum rules (18) and (19) simultaneously, we obtain the consistency condition

$$N^2 = \left(3m_\rho^2 - 4\ m_\pi^2\right)^2 / 2$$

Taking the experimental value of m_ρ^2 we obtain

$$N = \sqrt{2}\ \ 43\ m_\pi^2 = 60.8\ m_\pi^2$$

which compares fairly well with the previous values of N as well as the half-way value of $68\ m_\pi^2$.

In the above calculations we have saturated the left-hand side of the sum rule by ρ resonance only. However we can also include the contribution of $f_0(1260)$. The sum rule (18) then gives

$$(\tfrac{1}{2})\ \pi\,(2\ell+1)\ P_\ell\left(1+\frac{2t}{m_\rho^2-4m_\pi^2}\right)\beta_\rho(m_\rho^2)\Big|_{\ell=1}\Big/\alpha'$$
$$+\ (\tfrac{1}{3})\,\pi\,(2\ell+1)\ P_\ell\left(1+\frac{2t}{m_{f_0}^2-4m_\pi^2}\right)\beta_{f_0}(m_{f_0}^2)\Big|_{\ell=2}\Big/\alpha'$$
$$= C(t)\,\beta_\rho(t)\left(\frac{N}{q_t^2}\right)^{\alpha(t)}\frac{N}{\alpha(t)+1} \qquad (22)$$

where $C(t) = \sqrt{\pi}\,\Gamma(\alpha+3/2)\big/\Gamma(\alpha+1)$,

$\alpha(t)$ is the degenerate $\rho-f_0$ trajectory,

and we have taken the correct matrix elements

$\beta\,(I\ =1,\ I_s=1)=\tfrac{1}{2},\ \beta\,(I_t=1,\ I_s=o)=\tfrac{1}{3}$

We will evaluate (22) at $t = m_\rho^2$ and take N at the half-way point between f_0 and $g\ (3^-)$ resonances on the trajectory $\alpha(t)$. On simplification we obtain:

$$\beta_{f_0}(m_{f_0}^2)\ /\ \beta_\rho(m_\rho^2)\ =\ 555.2\ /\ 356.6$$

Further on reexpressing β's in terms of the widths Γ_ρ and Γ_f by the relation:

$$\beta(t=m^2{}_R)\Big/\alpha' = m_R\,\Gamma_e\,/\,K\,(t=m^2{}_R)$$

we finally obtain:

$$(x\,\Gamma)_{f_0}/(x\,\Gamma\,)_\rho\ =\ 0.999$$

where $\Gamma_e = x\Gamma$, where x is the inelasticity and Γ the total width. The experimental ratio given by Rosenfeld et al is 1.2.

II. 4(b): <u>FESR for the Signatured Amplitude:</u>

We will consider the $I_t=1$, $\pi-\pi$ amplitude of negative signature, $A^{I_t=1,(-)}(\nu,t) \equiv A^{(-)}(\nu,t)$. This amplitude will be approximated by the contribution of ρ trajectory for large ν i.e. $\nu \gg N$. This amplitude has only a right-hand cut and we can write down all moment sum rules for it. However the even moment sum rules are identical with those for the total amplitude $A^{I_t=1}(\nu,t)$ and therefore we will consider only the odd moment sum rules. The amplitude $A^{(-)}(\nu,t)$ is also expected to have fixed poles at wrong-signature unphysical values of α i.e. at $\alpha = 0,-2,-4$. We will show however that the contributions of these fixed poles are negligible. Neglecting the fixed pole terms, the nth moment sum rule for $A^{(-)}(\nu,t)$ can be written as

$$\int_{\nu_0}^{N} \nu^n \, \mathrm{Im}\, A^{I_t=1,(-)}(\nu,t)\, d\nu = C(t)\, \beta(t) \left(\frac{N}{q_t^2}\right)^{\alpha(t)} \frac{N^{n+1}}{\alpha+n+1} \tag{23}$$

Saturating the left-hand side of (23) by the contribution of ρ resonance only and evaluating both sides at $t = m_\rho^2$ we obtain

$$\frac{1}{2} \times \left(\frac{3m_\rho^2 - 4m_\pi^2}{2}\right)^n \frac{3\pi}{\alpha'} \beta_\rho^s(m_\rho^2) \left(1 + \frac{2m_\rho^2}{m_\rho^2 - 4m_\pi^2}\right) = \frac{3}{4}\pi\, \beta_\rho^t(m_\rho^2)\, N^{n+2} / (n+2) \left(\frac{m_\rho^2 - 4m_\pi^2}{4}\right)$$

Using $\beta_{\rho}^{s}(m_{\rho}^{2}) = \beta_{\rho}^{t}(m_{\rho}^{2})$, the bootstrap condition and simplifying, we get

$$N^{n+2} = \frac{(2n+1)}{2\alpha'} \left(3\, m_{\rho}^{2} - 4\, m_{\pi}^{2}\right)^{n+1} / 2^{n} \quad \ldots\ldots (24)$$

If we consider the 1st and 3rd moment sum rules (n = 1, 3), we have

$$N^{3} = \frac{3}{4\alpha'} \left(3\, m_{\rho}^{2} - 4\, m_{\pi}^{2}\right)^{2} \quad \ldots\ldots (24a)$$

and

$$N^{5} = \frac{5}{16\alpha'} \left(3 m_{\rho}^{2} - 4 m_{\pi}^{2}\right)^{4} \quad \ldots\ldots (24b)$$

From (24a) and (24b) we can obtain an additional consistency condition on N:

$$N^{2} = \frac{5}{12} \left(3 m_{\rho}^{2} - 4 m_{\pi}^{2}\right)^{2} \quad \ldots\ldots (24c)$$

From (24a,b,c) we will check whether the different values of N are consistent. We will take experimental values for the unknown quantities: $m_{\rho}^{2} = 30 m_{\pi}^{2}$ and $\alpha' = .02\, m_{\pi}^{-2}$

(24a) gives $N = 64.96\, m_{\pi}^{2}$, (24b) gives $N = 65.3\, m_{\pi}^{2}$ and (24c) gives $N = 65.7\, m_{\pi}^{2}$. In fact we see from (24) that N varies only slowly with n. These values of N are not only consistent with each other but they are also consistent with values of N found from the FESR for the total amplitude. This shows that the contributions of fixed poles of $A^{(-)}(v,t)$ are negligible.

II. 5 The Limitations of FESR Bootstraps:

The bootstrap calculations that we have performed depend on the Regge pole dominance of the RHS, and on the saturation by narrow width resonances of the LHS. In other words we assume duality between channel resonances and t channel Regge trajectories. The resonance saturation demands a small value for the cut off N because at large N more and more resonances will contribute and/or the non-resonating background will also become important. On the other hand, the Regge form of the RHS requires a large value for N. Thus N should be chosen where the resonance region matches with the Regge region. We have chosen N at the half-way point between the highest resonance kept and the lowest one left out. If N is varied from this middle point to half the distance from the nearest resonance on either side, the results are changed by not more than 10 per cent.[16] Another difficulty is in the choice of the value of momentum transfer 't' at which both sides of the sum rule are evaluated. In our calculation we evaluated both sides at $t = m^2_R$ where m_R is the resonance mass. This is so because we know that $\beta(m^2_R)$ is related to the width of the resonance. Desai et al[17] have performed calculations to bootstrap the degenerate $\rho - f_0$ trajectory in $\pi - \pi$ scattering at different values of t. The results are dependent on the form for β (t). For any given value of t they find there is not a unique solution for the slope and the intercept but a continuous range of solutions.

It should be noted that Regge dominance on the RHS of the sum rule demands a small value of t ($t \ll N$) whereas the neglect of lower partial waves on the LHS requires large t. It is therefore not possible to ensure the accuracy of both sides in a calculation where lower partial waves are ignored. We have performed our calculations only for $t = m^2_R$ and our results are a check of the consistency of the Regge parameters rather than a complete bootstrap calculation.

In more general terms the bootstrap of an infinitely rising trajectory should arise from an answer to the following question: Can we sum, in an average sense, a set of narrow resonances in the s channel in such a way as to reproduce the Regge high energy behaviour in the same channel ? If we assume that all direct channel resonances lie on a single trajectory $\alpha(s)$ and take the narrow width approximation then the contribution of all these resonances to the absorptive part of the amplitude will be a sum of delta functions in s. Thus the LHS of the FESR will be a step function in s (or in the cut off N) whereas the RHS is a smoothly varying function of N. We can however smooth out the narrow resonances and then demand consistency of the RHS and LHS of the FESR. Explicit calculations performed by different authors[18,19] show that the two sides of FESR are still inconsistent. The L.H.S. does not grow fast enough with energy and more resonances are needed. However if we allow the resonances to have non-zero widths, the overlapping of the resonance contributions can be made [19,20] to generate Regge behaviour, by a suitable choice of the variation of the widths. An alternative approach to the satisfaction of the FESR is based on the representation of the scattering amplitude in terms of the contributions of resonances lying on a sequence of trajectories having non-zero spacing. The sum of such contributions leads to the Regge asymptotic behaviour and thus to the solution of the FESR [21,22]. Such a solution of the FESR imposes constraints on the residue function. The Veneziano formula[14] is an example of such a representation having a sequence of parallel trajectories in each channel. It satisfies the FESR. This solution however is not unique, and this is typical of all approaches based on the FESR and the narrow width approximation. In order to pin down the Regge parameters we need to impose unitarity.

CHAPTER III

Unitarized Regge Parameters and Bootstraps

In the previous chapter we considered the question of bootstrapping a linearly rising trajectory. There we worked in the narrow resonance approximation with $\mathrm{Im}\,\alpha = 0$ and consequently the unitarity condition was not used. We found that whereas we could determine self-consistent values for the slope α' and the cut-off parameter N, the absolute magnitude of the width (or coupling) of the resonance lying on the trajectory remained undetermined. In the present chapter we will invoke unitarity and try to determine self-consistenly both the slope and the coupling of the ρ in π-π. We will be particularly interested in seeing how far unitarity alters the results of the previous chapter. We will also study the related question of the effect of unitarity on the linearity of the trajectory and on the t- dependence of the residue function.

We begin with the derivation of the so called Cheng-Sharp [10,8] equations for the Regge parameters.

III. 1 Cheng-Sharp Equations for the Regge Parameters:

We have seen in section II.1(a) that provided two singularities do not cross the Regge trajectory function $\alpha(t)$

and the reduced residue function $\bar{\beta}(t) = \beta(t) / (q_t^2)^{\alpha(t)}$ are real analytic functions of t with a right hand cut only. Requiring that $\alpha(t)$ should be linearly rising asymptotically, We can represent it by the dispersion relation

$$\alpha(t) = at + b + \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\,\alpha(t')\,dt'}{t' - t} \qquad \ldots\ldots (1)$$

where we have assumed that $\mathrm{Im}\,\alpha \to 0$ as $t \to \infty$.
It is obvious that the effect of unitarity represented by the integral term in (1) will be to give a small curvature to the trajectory.

To obtain an equation for $\beta(t)$ we first consider the case of potential scattering from a superposition of Yukawa potentials. If $\beta_n(t)$ is the residue of the nth trajectory $\alpha_n(t)$, it is found to have (n - 1) zeros.[7] Let t_i, i = 1, 2, ..., n-1 be the locations of these zeros, then the function

$$\phi_n(t) = \ln\left\{\gamma_n(t) \Big/ \prod_{i=1}^{n-1} (t - t_i)\right\}, \qquad \ldots\ldots (2)$$

satisfies the dispersion relation:

$$\phi_n(t) = \phi_n(\infty) + \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\,\phi_n(t')\,dt'}{t' - t} \qquad \ldots\ldots (3)$$

where

$$\gamma_{n(t)} = \beta_{n(t)} \Big/ \left(q_t^2\right)^{\alpha_{n(t)}}$$

But

$$\begin{aligned} \mathcal{I}m\,\phi_n(t) &= \text{phase of } \gamma_n(t) \\ &= \text{phase of } \beta_n(t) - \mathcal{I}m\,\alpha \cdot \ln q_t^2 \\ &= \tan^{-1}\left[\frac{\mathcal{I}m\,\beta_n(t)}{\mathrm{Re}\,\beta_n(t)}\right] - \mathcal{I}m\,\alpha \ln q_t^2 \end{aligned}$$

We can therefore rewrite equation (3) as

$$\phi_n(t) = \phi(\infty) + \frac{1}{\pi}\int_{t_0}^{\infty}\left[\tan^{-1}\left\{\frac{\mathcal{I}m\,\beta_n(t')}{\mathrm{Re}\,\beta_n(t')}\right\} - \mathcal{I}m\,\alpha_n(t')\ln q_t^2 \Big/ (t'-t)\,dt'\right]$$

which in virtue of (3) gives

$$\gamma_{n(t)} = \frac{(\text{constant})}{\prod_{i=1}^{n-1}(t-t_i)} \exp\left[\int_{t_0}^{\infty}\frac{dt'}{t'-t}\left\{\tan^{-1}\left(\frac{\mathcal{I}m\,\beta_n(t')}{\mathrm{Re}\,\beta_n(t')}\right) - \ln q_{t'}^2 \cdot \mathcal{I}m\,\alpha(t')\right\}\right] \quad \ldots\ldots (4)$$

This is the required equation for the residue function $\gamma_{n(t)} = \beta_n(t) \,/\left(q_t^2\right)^{\alpha_n(t)}$, the constant factor is proportional to the coupling. To obtain the corresponding equation for $\beta(t)$ in the relativistic case, we note that $\beta(t)$

should have zeros at the Mandelstam symmetry points and therefore we replace the factor $\prod_{i=1}^{n-1} (t - t_i)$ in equation (4) by an entire function $G(t)$ which produces zeros of $\beta(t)$ at the required points. As an example we could take

$$\beta(t) \propto G(t) = \frac{1}{(at + b + 3/2)} \prod_{n=1}^{\infty} \frac{t - tn}{(at + b + n + 1/2)}$$

where tn, $n = 1,2,\ldots$ are those values of t for which $\alpha(t) = -n - 1/2$, $n = 1,2\ldots$, but this form for $G(t)$ is not convenient for computational purposes. Alternately we could take

$$G(t) = \frac{1}{\Gamma(\alpha(t) + 3/2)}$$

which has zeros at the required points but since $\alpha(t)$ develops an imaginary for $t > to$ this is not an entire function. But for $t < to$, $\alpha(t) = \mathrm{Re}\,\alpha(t)$ and it is convenient to use the form $G(t) = 1/\Gamma(\mathrm{Re}\,\alpha + 3/2)$ even though it is not strictly an analytic function.

We can now write down an equation for $\beta(t)$. By requiring, as in the case of narrow resonance approximation, that $\beta(t)$ without a factor $E(t)$ should go asymptotically as $1/t$ we obtain, by analogy with equation (4), the equation

$$\beta(t) = \left(\frac{4\alpha' q_t^2}{e}\right)^{\alpha(t)} \frac{E(t)}{\Gamma(\mathrm{Re}\,\alpha + 3/2)} \cdot \exp\left[-I\right] \qquad \ldots\ldots (5)$$

with

$$I(t) = \frac{1}{\pi}\int_{t_0}^{\infty}\left\{ \mathcal{I}m\,\alpha(t')\,\ell n\left(\frac{4\alpha' q_{t'}^2}{e}\right) - \tan^{-1}\frac{\mathcal{I}m\,\beta(t')}{\mathrm{Re}\,\beta(t')}\right\} \times \frac{dt'}{t' - t} \qquad \ldots\ldots (6)$$

where the exponential factor, e , has been inserted in (5) to ensure the asymptotic behaviour $1/t$ (without the factor E(t)) Here α' is the slope of the trajectory and since all trajectories appear to have the same slope of 1 GeV^{-2}, $1/\alpha'$ gives the scale of energy. E(t) is an entire faction and we will take E(t) = E. Reα (for the ρ trajectory in π-π scattering). This ensures that the trajectory chooses non-sense at $\alpha = o$. The constant E is related to the coupling.

Equations (1) and (5) do not form a coupled set of integral equations. They must be supplemented by an additional equation connecting $\mathcal{I}m\,\alpha$ with $\beta(t)$. Such an equation is given by unitarity.

III.2 Coupled Integral Equations for the Regge Parameters:

IIIa. 2(a) The Case of a Single Regge Term Representation

If we represent the amplitude by a single Regge pole, the partial wave amplitude has the form

$$A(\ell,t,) = \beta(t) / (\ell - \alpha(t)) \qquad \ldots\ldots\ldots (7)$$

and if this is substituted in the unitarity equation:

$$A(\ell,t) - \overset{*}{A}(\overset{*}{\ell},t) = K(t)\; A(\ell,t)\; \overset{*}{A}(\overset{*}{\ell},t) \qquad \ldots\ldots (8)$$

where $K(t) = \left((t - t_o)/t\right)^{\frac{1}{2}}$

we obtain the equation:

$$\mathcal{I}m\,\alpha(t) = K(t)\,\beta(t)\,,\quad t > t_o \qquad \ldots\ldots\ldots (9)$$

This shows that $\beta(t)$ is real for $t > t_o$ and therefore $\mathcal{I}m\,\beta = o$ for $t > t_o$

Equation (9) together with equations (1), (5) and (6) constitute a closed set of integral equations for the Regge parameters, where now

$$I(t) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{\mathcal{I}m\,\alpha(t')\; \ell n\left(\frac{4\alpha' q^2_{t'}}{e}\right)}{t' - t}\, dt' \qquad \ldots\ldots\ldots (6)'$$

III.2 (b): <u>The Case of Khuri Representation</u>

The ordinary Regge representation of the amplitude of definite signature viz.

$$A^{\pm}(s,t) = -\pi\left(2\alpha(t) + 1\right)\beta(t)\; P_{\alpha}(-Z_t)\Big/2\sin\pi\alpha(t) \qquad \ldots\ldots (10)$$

does not possess Mandelstam analyticity; it has a cut in $Z_t = Z_t(s,t)$ from $Z_t = 1$ to ∞ instead of the cut from $Z_t = Z_t(s_0, t)$ to ∞, where so is the threshold of the s-channel. In terms of the variable s, the amplitude in equation (10) has a cut from $s = -4q_t^2$ to $s = \infty$ whereas the Mandelstam analyticity requires it to have a cut from s_0 to ∞. The basic idea of the Khuri representation[22] is to remove from the Regge term the part corresponding to the cut between $s = -4q_t^2$ and $s = s_1$. To this end we invoke an integral representation for the Legendre function

$$P_\alpha(-Z_t) = -\frac{\sqrt{2}\sin\pi\alpha}{\pi}\int_0^\infty \frac{\cosh\left[(\alpha+\frac{1}{2})x\right]dx}{(\cosh x - Z)^{\frac{1}{2}}}$$

$$-1 < \operatorname{Re}\alpha < 0$$

$$\ldots\ldots (11)$$

which can also be written as

$$P_\alpha(-Z_t) = -\frac{\sin\pi\alpha}{2\pi\sqrt{2}(\alpha+\frac{1}{2})}\int_{-\infty}^{\infty}\frac{e^{(\alpha+\frac{1}{2})x}\sinh x}{(\cosh x - Z)^{3/2}}dx$$

$$\ldots\ldots (12)$$

Using this we can produce a representation of the Regge term (10) with a cut beginning at $s = s_1$ viz.

$$A^{\pm}(s,t) = \frac{1}{2\sqrt{2}}\beta(t)\int_{\xi}^{\infty} e^{(\alpha+\frac{1}{2})x} \frac{\sin h\, x\, dx}{(\cosh x - Z_t)^{3/2}} \qquad (13)$$

where

$$\xi = \cosh^{-1}\left(1 + \frac{s_1}{2q_t^2}\right) \qquad (14)$$

Equation (13) is valid only for Re $\alpha < 0$ but we can use (12) to define the analytic continuation to Re$\alpha > 0$ and find

$$A^{\pm}(s,t) = -\pi(2\alpha+1)\beta(t)\, P_\alpha(-Z_t) \Big/ 2\sin\pi\alpha$$

$$- \frac{1}{2\sqrt{2}}\beta(t)\int_{-\infty}^{\xi} e^{(\alpha+\frac{1}{2})x} \frac{\sin h\, x\, dx}{(\cosh x - Z_t)^{3/2}}$$

The partial wave projection of this can be shown to be

$$A_\ell^{\mp}(t) = A(\ell,t) = \frac{\beta(t)\cdot \exp\left(-[(\ell-\alpha(t)]\,\xi(t)\right)}{\ell - \alpha(t)}$$

This partial wave amplitude has a right hand cut due to $\alpha(t)$ and $\beta(t)$ and a left hand cut from $t = -s_1 + 4m_\pi^2$ to $t = -\infty$ due to the behaviour of $\xi(t)$.

Here we may discuss the question of the choice of s_1. Corresponding to a Regge pole in the t channel, s_1 must lie in the high energy region of the s channel, therefore we may take $s_1 = s_N$ where s_N is the cut-off in the FESR. With this choice of s_1, the boundary of the elastic t double spectral function will be given by t = to and $s = s_1$. Now we want to see how the boundary of the double spectral function is shifted if we invoke the duality idea. According to the concept of duality the t - channel Regge pole is generated by the s channel resonances, and the s channel resonances are important for $s < s_N$ (where s_N marks the boundary of the high energy region). Thus the forces or potential which generates the t channel Regge pole (which contributes to the amplitude for $s > s_N$) is given by the part of the double spectral function lying between $s = s_0$ and $s = s_N$. Symbolically we can write the partial wave amplitude $A(\ell,t)$ as

$$A_\ell(t) = A_{\text{Regge}} \; (s > s_N) + \text{Potential} \; (s < s_N)$$

Now since duality implies that the integral over resonance contributions from s_0 to s_N is equal to the integral over Regge contributions (via the FESR) we may replace the resonance (or potential) term in the above expression for $A_\ell(t)$ by $A_{\text{Regge}} \; (s_0 < s < s_N)$ so that we may write

$$\begin{aligned} A_\ell(t) &= A_{\text{Regge}} \; (s > s_N) + A_{\text{Regge}} \; (s_0 < s < s_N) \\ &= \text{Khuri amplitude with } s_1 = s_0 \end{aligned}$$

Thus duality implies that in the Khuri representation for the partial wave amplitude we must take s_1 coincident with the boundary of the elastic s channel double spectral function (for $t > 16\ m_\pi^2$). Now the boundary of the double spectral function

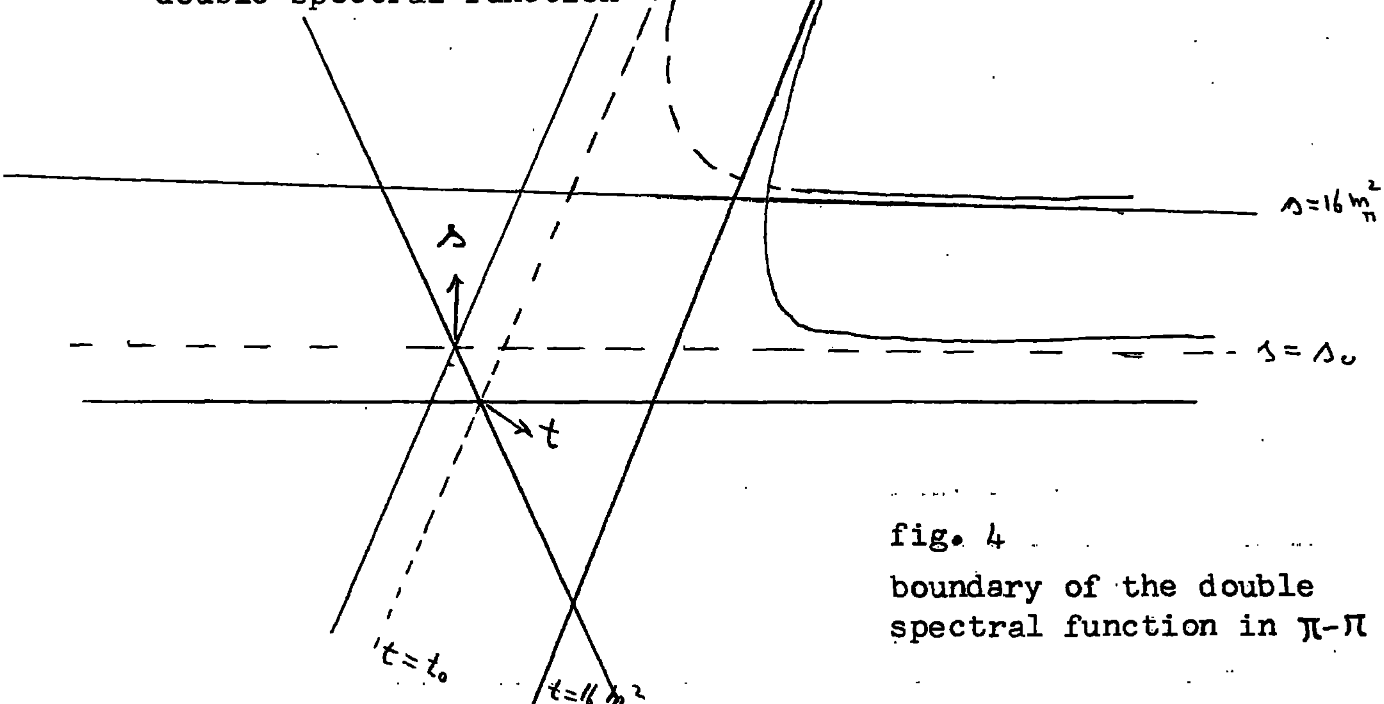

fig. 4
boundary of the double spectral function in $\pi-\pi$

in $\pi-\pi$ scattering is given by the curves (see figure 4)

$$s = 4 + 64/(t - 4) \quad \text{for } t_0 < t < 16\, m_\pi^2$$

and $\qquad\qquad$ ($m_\pi^2 = 1$ units)

$$s = 16 + 64/(t - 16) \quad \text{for } t > 16\, m_\pi^2$$

The parameter s_1 in equation (14) will be taken coincident with this boundary.

Now we can derive the Cheng-Sharp equations for the Khuri representation substituting

$$A(\ell,t) = \beta(t)\,\frac{\exp\left[-(\ell-\alpha(t))\,\xi(t)\right]}{\ell-\alpha(t)}$$

in the unitarity equation:

$$A(\ell,t) - A^{*}(\ell^{*},t) = 2i\,K(t)\,A(\ell,t)\,A^{*}(\ell^{*},t)$$

we obtain at $\ell = \alpha(t)$,

$$\beta(t) = \frac{\mathrm{Im}\,\alpha}{K}\,\exp\left[-2i\,\mathrm{Im}\,\alpha\cdot\xi(t)\right] \qquad (15)$$

where

$$K(t) = \left[(t-t_0)/t\right]^{\frac{1}{2}}$$

This gives the phase of $\beta(t)$ as

$$\tan^{-1}\left(\frac{\mathrm{Im}\,\beta}{\mathrm{Re}\,\beta}\right) = -2\,\mathrm{Im}\,\alpha(t)\cdot\xi(t)$$

The equations (5) and (6) therefore give

$$\beta(t) = \left(\frac{4\alpha' q_t^2}{e}\right)^{\alpha(t)} \frac{\Gamma\,E(t)}{\left(\mathrm{Re}\,\alpha+3/2\right)}\,\exp\left[-I\right] \qquad (15')$$

where

$$I = \frac{1}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{Im}\,\alpha(t')}{t'-t}\left[\ln\left(\frac{4\alpha' q_t^{2\prime}}{e}\right)+2\xi(t')\right]dt'$$

Using the relation (15) these equations give an integral equation for $\mathcal{I}m\alpha(t)$

$$\mathcal{I}m\,\alpha(t) = K(t)\left(\frac{4\alpha' q_t^2}{e}\right)^{Re\alpha} \frac{E(t)}{\Gamma(Re\alpha + 3/2)} \times$$

$$\exp\left[-\frac{P}{\pi}\int_{t_o}^{\infty}\frac{\mathcal{I}m\alpha(t')\,dt'}{t'-t}\left(\ln\frac{4\alpha' q_{t'}^2}{e} + 2\xi(t')\right)\right] \quad \ldots\ldots(16)$$

with

$$\alpha(t) = at + b + \frac{1}{\pi}\int_{t_o}^{\infty}\frac{\mathcal{I}m\,\alpha(t')\,dt'}{t'-t} \quad \ldots\ldots(17)$$

The equations for $Re\beta$ and $\mathcal{I}m\,\beta$ are given by (15) for $t >$ to and by (15$'$) for all t. We can solve the equations (16) and (17) as will be discussed in the following sections.

III. 3 The Solution of the Cheng-Sharp Equations:

III. 3(a) The Case of a Single Regge Term Representation

To facilitate convergence of the integrals and to ensure that the trajectory passes through the mass of the ρ we rewrite the equations of section III.2(a) for the parameters $\alpha(t)$ and $\beta(t)$ with a subtraction at $t = m_\rho^2$ (where $Re\alpha = 1$) and obtain

$$Re\,\alpha = \alpha'(t - m_\rho^2) + 1 + \frac{P}{\pi}\int_{t_o}^{\infty}\frac{\mathcal{I}m\alpha(t')\,dt'}{(t'-t)\,(t'-m_\rho^2)} \quad (18a)$$

$$\beta(t) = \left(\frac{4\alpha' q_t^2}{e}\right)^{\mathrm{Re}\,\alpha} \frac{E(t)}{\Gamma(\mathrm{Re}\,\alpha + 3/2)} \exp(-I)$$

...... (18b)

where

$$I = \frac{P}{\pi}(t - m_\rho^2) \int_{t_0}^{\infty} \frac{\mathrm{Im}\,\alpha(t') \ln\left(\frac{4\alpha' q_{t'}^2}{e}\right) dt'}{(t' - t)(t' - m_\rho^2)}$$

...... (18c)

and

$$\mathrm{Im}\,\alpha(t) = \tilde{K}(t)\,\beta(t) \qquad \text{.......} \qquad (18d)$$

where $E(t)$ is an entire function, and as discussed in section III.1 we require it to have a ghost-killing factor $\alpha(t)$ or rather Re $\alpha(t)$ to keep it real above threshold. Hence we take $E(t) = (\mathrm{Re}\,\alpha) \cdot E$ (E = constant). With this choice of $E(t)$, $\beta(t)$ goes to a constant as $t \to \infty$. In virtue of equation (9) this means that $\mathrm{Im}\,\alpha \to$ constant, which prevents the convergence of the integrals in (16) and (17); so instead we use $\tilde{K}(t) = K(t)/F(t)$ where $F(t)$ is the Fermi function, $F(t) = 1 + \exp\left[(t - \bar{t})/\Delta\right]$. It is used to cut-off Unitarity and hence make $\mathrm{Im}\,\alpha \longrightarrow 0$ as t goes beyond a certain value, $\bar{t}$, where elastic unitarity is no longer valid.

Using the relation between $\beta(t)$ and the width Γ_ρ viz.

$$\beta(t = m_\rho^2)/\alpha' = m_\rho \Gamma_\rho / K(t = m_\rho^2)$$

we find that (in pion mass units)

$$E = 0.82\,\Gamma_\rho \qquad \text{(19)}$$

Taking $\Gamma_\rho = 140$ MeV $= 1\, m_\pi$, $E = 0.82$

To solve the equations (18) we note that we have two free parameters α' and E whose physical values are respectively $0.02\ m_\pi^{-2}$ and 0.82. To study the effect of unitarity on the linearity of the trajectory we start with the physical values of these parameters and solve the equations (18) by iteration. They converge after about 4 iterations. To demonstrate that the solutions are independent of the cut-off $\bar{t}$ we solve the equations for two values of the cut-off, $\bar{t} = 200\ m_\pi^2$ and $\bar{t} = 300\ m_\pi^2$ (with $\Delta = 20\ m_\pi^2$ in either case). In figure 5 we display the self-consistent values of $\mathcal{I}m\,\alpha$ corresponding to these two choices of $\bar{t}$. We see that the values of $\mathcal{I}m\,\alpha$ agree fairly well over the region $t_0 < t < 160\ m_\pi^2$. In figures 6 (a),(b) we exhibit the plots of Re α and γ = Re γ $= \beta(t).\Gamma(\mathrm{Re}\,\alpha + 3/2)\Big/_{\mathrm{Re}\,\alpha}\left(\frac{4\alpha' q_t^2}{e}\right)^{\mathrm{Re}\,\alpha}$ and the plot of $\mathcal{I}m\,\alpha$ is shown in figure 5. We see that $\mathcal{I}m\,\alpha$ has the usual form and this form is consistent with the behaviour of the widths of higher bèson resonances found by Foccaci et al.[23] and discussed by Collins et al[24]. The graph for Reα shows that it is almost linear and that the effect of unitarity is small. In figure 6(b) the graph of $\gamma(t)$ is shown. It fluctuates around the value of E = 0.82 but is almost constant over a wide range of t as in the narrow resonance approximation (NRA). To demonstrate further the difference of these results from those of the NRA we plot in figure 7 the δ-discontinuity of the Regge term viz.

$$D_s\ (N,t) = \sqrt{\pi}\ \frac{\Gamma(\alpha + 3/2)}{\Gamma\ (\alpha + 1)}\ \beta(t)\ \left(\frac{N}{q_t^2}\right)^{\alpha(t)}$$

versus t in the NRA (A) and in the case when unitarity is imposed on Regge parameters (B). We observe that the two curves resemble each other very closely for all values of t between $t = 30\ m_\pi^2$ and $t = -20\ m_\pi^2$, and therefore we conclude that unitarity makes only a small difference.

III. 3(b): The Case of Khuri Representation

As before we rewrite the equations of section III.2(b) with a subtraction at $t = m_\rho^2$ and solve them by iteration. We start with physical values of the parameters α' and E and obtain self-consistent solutions after 4 iterations. In figures 8(a), (b) and (c) we plot self-consistent values of $\mathrm{Im}\,\alpha$, $\mathrm{Re}\,\alpha$ and $\gamma = \beta(t) \times \Gamma(\mathrm{Re}\,\alpha + 3/2) / \mathrm{Re}\,\alpha \left(\frac{4\alpha' q_t^2}{e}\right)^{\mathrm{Re}\,\alpha}$ versus t.

The graph of $\mathrm{Im}\,\alpha$ has the usual form; $\mathrm{Re}\,\alpha$ appears to have a small curvature for small values of t but otherwise it is a straight line, $\gamma(t)$ fluctuates about the NRA - value of 0.82 and is almost a constant for $t > 200\ m_\pi^2$. The s -

discontinuity of the Regge term is plotted in figure 9. Again the two curves are close to each other and therefore unitarity seems to make a small difference, although by comparison with figure 7 we can see that unitarity makes a little more difference in the case of Khuri representation than in the case of a single Regge term representation.

III. 4 "Bootstrap" Calculations and Consistency of the FESR

To do bootstrap calculations using the unitarized Regge parameters obtained by solving the Cheng-Sharp equations of the previous section, we note that we have three free parameters at our disposal viz. the slope α', the constant E(proportional to the width Γ_ρ) and the cut-off parameter N. In the NRA, we had only two free parameters — α' and N, and we found that the self-consistent values of these quantities obtained from the FESR are $\alpha' = 0.02\ m_\pi^{-2}$, $N = 65.5\ m_\pi^2$. The constant E could not be determined in the NRA. To do bootstrap calculations with the unitarized Regge parameters, we start with the values of α' and N obtained in the NRA and for the constant E we take the value given by equation (19) viz. E = 0.82. We then solve the Cheng-Sharp equations as explained in section III.3 and obtain values for the unitarized Regge parameters $\alpha(t)$ and $\beta(t)$. Next we insert these values in the FESR viz.

$$\int_{\nu_0}^{N} \beta(I_t = 1,\ I_s = 1)\ \mathcal{I}m\ A^{I_s}(\nu,t)\ d\nu$$

$$= \sqrt{\pi}\ \frac{\Gamma(\alpha + 3/2)}{\Gamma(\alpha + 2)}\ \beta(t)\left(\frac{N}{q_t^2}\right)^{\alpha(t)} \times N \qquad \text{.......... (20a)}$$

with

$$A^{I_s}(\nu,t) = \pi\left(2\ \alpha(s) + 1\right)\beta(s)\left(\frac{1 - e^{-i\pi\alpha(s)}}{2\ \sin\pi\alpha}\right) \times P_\alpha(Zs) \qquad \text{...... (20b)}$$

and see whether the FESR is consistent. To achieve satisfaction or near satisfaction of the sum rule we may vary any one of our free parameters, in particular N.

Before discussing the actual calculations in the two cases viz. a single Regge term representation and Khuri representation we first discuss some related questions. The two sides of the sum rule (20a) have to be evaluated at some value of t. For t $>$ to the R.H.S. of (20a) will have an imaginary part in t due to the fact that $\alpha(t)$ is complex. The L.H.S. of (20a) will also have an imaginary part due to the cut of $P_\alpha(Z_s)$ for $Z_s < -1$. We will compare only the real parts (in the NRA both sides have only real parts). For $t > t_0$ (and in particular for negative t) the R.H.S. of (20a) is real but the L.H.S. has an imaginary part due to the cut of $P(Z_s)$ for a certain range of s(s =so to, say, $s = \bar{s}$.). To make the L.H.S. real we take only the real part of $P(Z_s)$ <u>in this range</u>. i.e. writing

$$P(Z_s) \approx P_\alpha(-Z_s)\, e^{-i\pi\alpha}, \text{ for } Z_s < -1$$

we take

$$\mathrm{Re}\, P_\alpha(Z_s) \approx \cos\pi\alpha \cdot P_\alpha(-Z_s), \quad Z_s < -1 \tag{21}$$

i.e. we remove the spurious cut of the Regge term. For negative values of t we expect the L.H.S. of the sum rule to change sign from positive to negative due to the zero of the Legendre function. On the right-hand side this change of sign should take place due to a zero of $1/\Gamma(\alpha)$ factor (Schmid hypothesis). For a bootstrap the two sides should change sign simultaneously at a given value of t. To see this clearly let us consider this question in the NRA. On the L.H.S. we will have

$$\mathcal{I}m\, A(s,t) = \beta(s)(2\ell+1)\, P_\ell(Z)\, \delta(s - m^2)/\alpha'$$

and for $\ell = 1$

$$\int_{s_0}^{s_N} \mathcal{I}m\, A(s,t)\, ds = \frac{3}{\alpha'}\beta(m_\rho^2)\left(1 + \frac{2t}{m_\rho^2 - 4m_\pi^2}\right)$$

The zero of the L.H.S. will therefore occur when

$$1 + \frac{2\,t}{m_\rho^2 - 4m_\pi^2} = o \text{ or } t = -13\; m_\pi^2$$

Now for the R.H.S. we write

$$\mathcal{I}m\; A^R\,(s,t) = \sqrt{\pi}\,\frac{E}{\Gamma(\alpha(t))}\left(\frac{4\alpha' N}{e}\right)^{\alpha}\frac{N}{\alpha + 1}$$

with

$$\alpha(t) = \alpha'(t - m_\rho^2) + 1$$

and we require $\alpha(-13) = 0$

which gives $\alpha' = 1/_{43} = 0.023\; m_\pi^{-2}$

which is very close to the physical value. Similarly in the non-zero width approximation we expect that by adjusting the slope it may be possible to make both sides change signs simultaneously.

In the following, however, the results for $t \gg o$ are presented.

III. 4(a) <u>Bootstrap Calculations with a Single Regge Term Representation:</u>

Starting with $\alpha' = 0.02\ m_\pi^{-2}$ and E = 0.82 we solve the equations (18 (a,b,c) and calculate both sides of the sum rule (20,a,b) for a range of values of t, $0 \leq t \leq 30\ m_\pi^2$ with cut-off $N = 65.5\ m_\pi^2$ (= its value of NRA) as shown in figure 10. We find that for most of the values of t both sides of the sum rule agree up to an accuracy of 20 per cent. For t = 0 the two sides are exactly equal. Thus we conclude that the values $\alpha' = 0.02\ m_\pi^{-2}$, E = 0.82 (which corresponds to a width of 140 MeV) and $N = 65.5\ m_\pi^2$ (which lies close to the half-way point between the ρ and f_0 on the degenerate $\rho - f_0$ trajectory) are nearly self-consistent. In figure 11 we again plot the two sides of the sum rule against t with a different cut-off $N = 55.5\ m_\pi^2$. We observe that we have a better agreement between the two sides of the sum rule in this case.

To study the cut-off dependence of the sum rule at a given value of t we plot the two sides of the sum rule against N in figures 12(a), (b). For t = 0 we see from figure 12 (a) that as N varies from $55.5\ m_\pi^2$ to $75.5\ m^2$ (these values of N lie between ρ and f_0 on the degenerate $\rho - f_0$ trajectory) the two curves are always close to each other. From figure 12(b) $\left(t = 30\ m_\pi^2\right)$ we see that the two curves are close to each other only when $N < 70.5\ m_\pi^2$. Thus we conclude that our results are to some extent dependent on the cut-off as well as on the value of t at which both sides of the sum rule are evaluated.

III. 4(b) Bootstrap Calculations with Khuri Representation:

As before we solve the Cheng-Sharp equations in the case of Khuri representation starting with input $\alpha' = 0.02\ m_\pi^{-2}$, $E = 0.82$ and then calculate both sides of the sum rule (20,a,b) for a range of values of t, $0 \leq t \leq 30\ m_\pi^2$ and with different cut-offs, $55.5 \leq N \leq 75.5\ m_\pi^2$. We display the two sides of the sum rule as functions of t in figure 13(a) with $N = 65.5 m_\pi^2$ and in figure 13(b) with $N = 55.5\ m_\pi^2$. In both cases the two curves are close to each other over a fairly large range of t although the two sides do not seem to agree over the whole range of t. Our conclusion therefore is that the values $\alpha' = 0.02\ m_\pi^{-2}$ and $E = 0.82$ (i.e. $\Gamma_\rho = 140$ MeV) are consistent; although our results are to some extent dependent on the value of t at which both sides of the sum rule are evaluated.

To study the cut-off dependence of the sum rule we plot the two sides of the sum rule against N at a given value of t. In figure 14(a) the two sides are plotted at $t = 0$ whereas in figure (14b) they are plotted at $t = 30\ m_\pi^2$. We notice that for a better agreement between the two sides of the sum rule we require $N \gg 70.5 m_\pi^2$ when $t = 0$ (fig. 14(a)) and $N < 70.5\, m_\pi^2$ when $t = 30\, m_\pi^2$ (fig. 14(b)). Again we conclude that with a suitable choice of N (lying somewhere between the ρ and f_0 on the degenerate $\rho - f_0$ trajectory) at a given value of t the values $\alpha' = 0.02 m_\pi^{-2}$ and $E = 0.82$ are self-consistent.

CHAPTER IV

Conclusion

We have studied the dynamical properties of rising Regge trajectory in the narrow resonance approximation and in the approximation of unitarity. In the NRA we found that our dynamics based on the concept of rising trajectories and the finite energy sum rules predicts the impossibility of scalar meson bootstrap - a result already proved to be true on the basis of N/D approach [25]. It also predicts the possibility of the bootstrap of ρ in π-π scattering. Further we found that in the NRA the values $\alpha' = 0.02\ m_\pi^{-2}$ ($\simeq 1\ \mathrm{GeV}^{-2}$) and $N = 65.5\ m_\pi^2$ of the slope α' (of ρ trajectory) and the cut-off parameter N are self-consistent. We could not however determine the absolute magnitude of the width (on coupling) in the NRA. An attempt was made to determine this parameter together with the slope by imposing unitarity on the Regge parameters through the solution of Cheng-Sharp equations. We considered two approximations to unitarity - one corresponding to a single Regge pole term representation of the ~~partial wave~~ amplitude and the other corresponding to Khuri representation. We found that the results in both approximations are qualitatively the same, and that unitarity makes only a small difference on the results of NRA. We also found that the physical values of the slope and width of ρ viz. $\alpha' = 0.02\ m_{\pi}^{-2}$ and $\Gamma_{\rho} = 1\ m_{\pi} = 140$ MeV are consistent with a (reasonable) value of the cut-off parameter N lying somewhere between the ρ and f_0 on the degenerate $\rho - f_0$ trajectory.

In the end we may note that our calculations are not strictly speaking bootstrap calculation; they rather check the consistency of the finite energy sum rules in the NRA and in the approximation when unitarized Regge parameters are used.

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Figure Captions

Fig.5 The graphs of $\mathrm{Im}\,\alpha$ against t in the case of one Regge pole representation corresponding to two values of the cut-off $\bar{t}$ in the Fermi function viz. $\bar{t} = 200\ m_\pi^2$ (curve A) and $\bar{t} = 300\ m_\pi^2$ (curve B) with $\Delta = 20\ m_\pi^2$ in either case, and $\alpha' = 0.02\ m_\pi^{-2}$, E = 0.82

Fig.6 (a), (b): The graphs of $\mathrm{Re}\,\alpha$ and γ against t in the case of one Regge pole representation with $\alpha' = 0.02\ m_\pi^{-2}$ and E = 0.82

Fig.7 The graph of the s - discontinuity of the Regge term, Ds (N, t), against t in the case of narrow resonance approximation (curve NR) and in the case of unitarity using one Regge pole term approximation (curve U); $N = 65.5\ m_\pi^2$ and $\alpha' = 0.02\ m_\pi^{-2}$, E = 0.82 .

Fig.8 (a),(b),(c): The graphs of $\mathrm{Im}\,\alpha$, $\mathrm{Re}\,\alpha$, and γ in the case of Khuri representation, with $\alpha' = 0.02\ m_\pi^{-2}$, E = 0.82.

Fig.9 The plot of the s-discontinuity of the Regge term, $D_s(N,t)$, against t in the case of narrow resonance approximation (the curve NR) and in the case of unitarity using Khuri representation (curve U), $N = 65.5\ m_\pi^2$, $\alpha' = 0.02\ m_\pi^{-2}$, E = 0.82

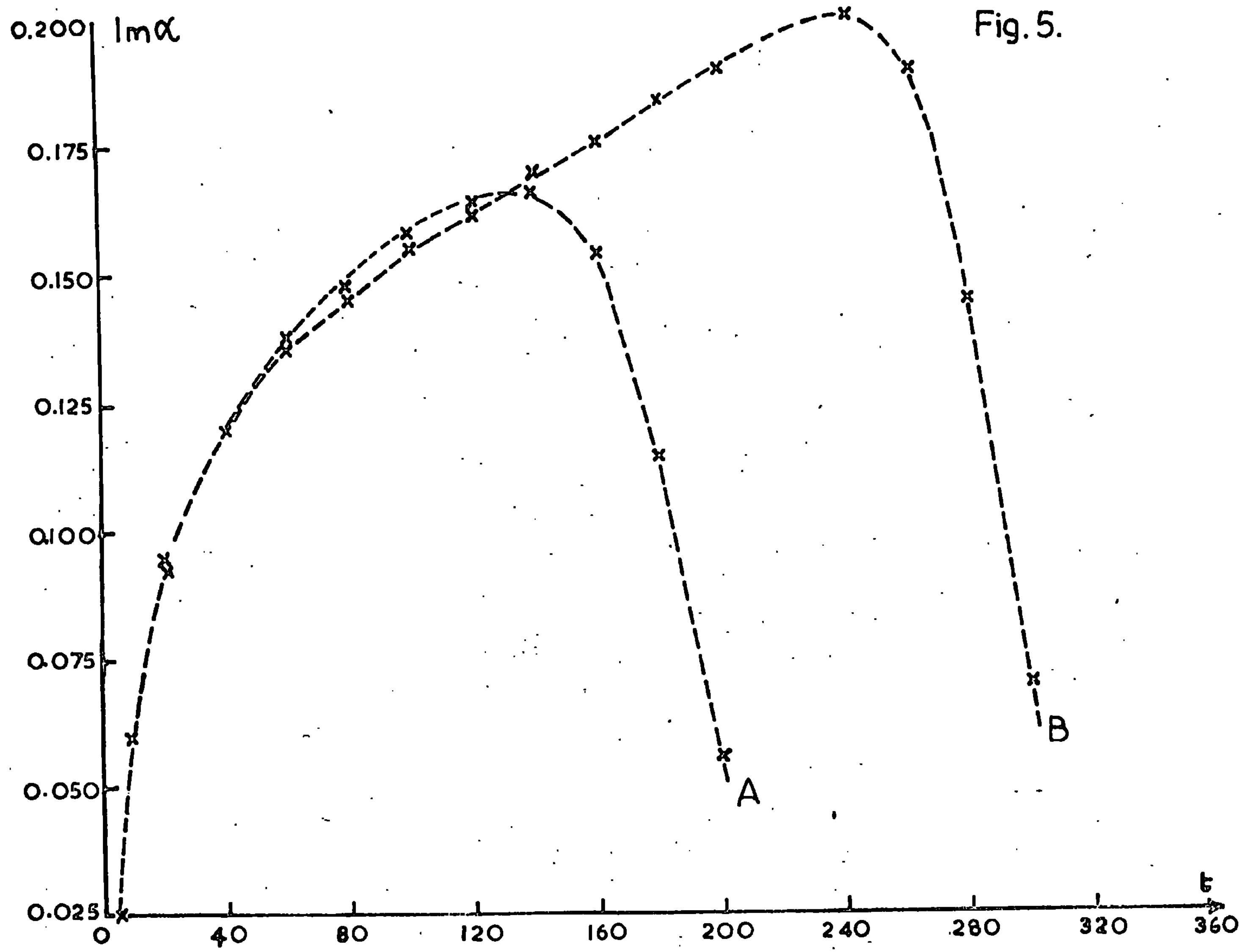
Fig.10 The plots of the L.H.S. and the R.H.S. of the sum rule against t, in the case of one Regge pole term representation with $\alpha' = 0.02\ m_\pi^{-2}$, E = 0.82 and $N = 65.5\ m_\pi^2$

Fig.11 the same as in figure 10 with $N = 55.5\ m_\pi^2$

Fig.12 (a),(b): The plots of the L.H.S. and the R.H.S. of the sum rule against the cut-off N (in the case of a single Regge pole term representation) with $\alpha' = 0.02\ m_\pi^{-2}$, $E = 0.82$ and (a) $t = 0.0$, (b) $t = 30\ m_\pi^2$.

Fig.13 (a),(b): The plots of the L.H.S. and the R.H.S. of the sum rule against t (in the case of Khuri representation) with $\alpha' = 0.02\ m_\pi^{-2}$, $E = 0.82$ and (a) $N = 65.5\ m_\pi^2$, (b) $N = 55.5\ m_\pi^2$

Fig.14 (a),(b): The plots of the L.H.S. and the R.H.S. of the sum rule against the cut-off N, in the case of Khuri representation with $\alpha' = 0.02\ m_\pi^{-2}$, $E = 0.82$ and (a) $t = 0$, (b) $t = 30 m_\pi^2$.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0.175
0.150
0.125
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0.075
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320
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t
A
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Fig. 5.

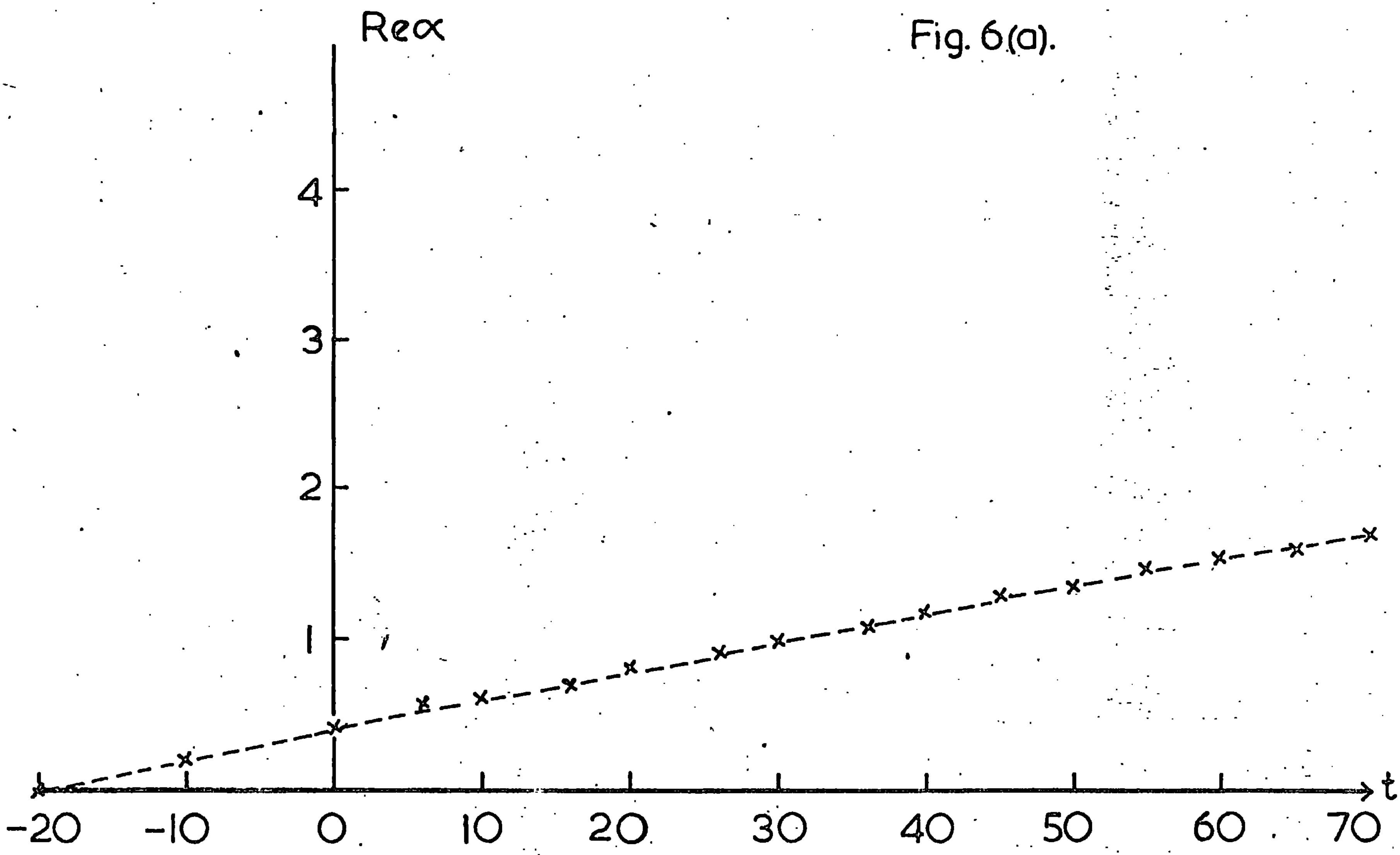
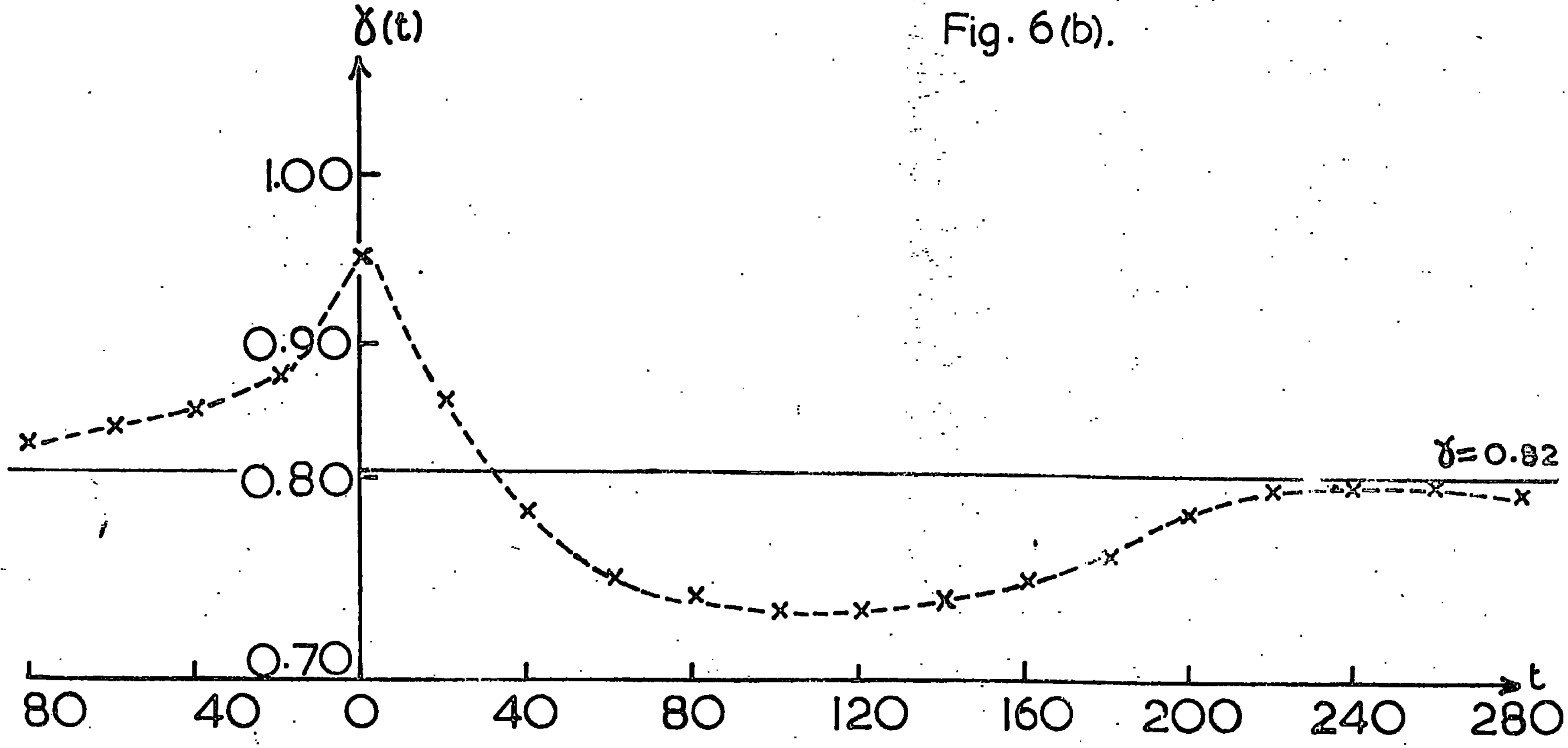


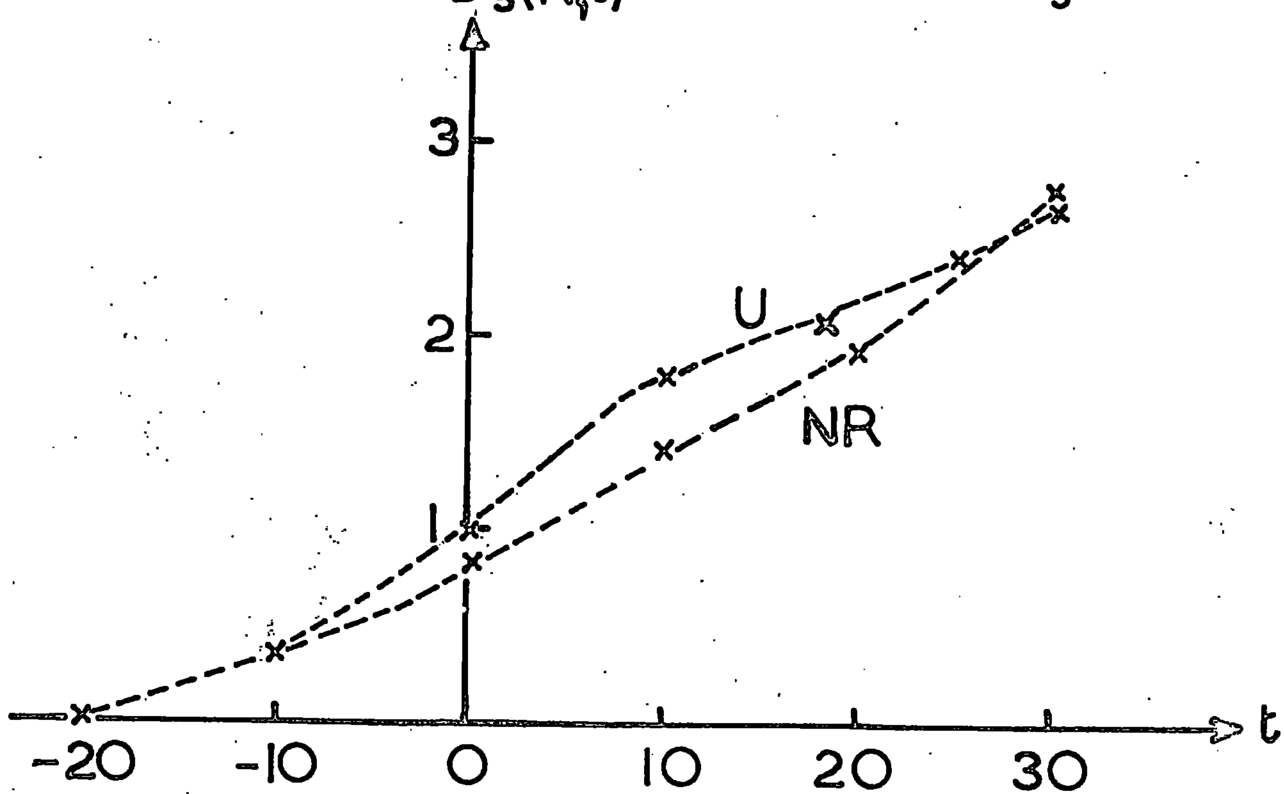
Fig. 6(a).

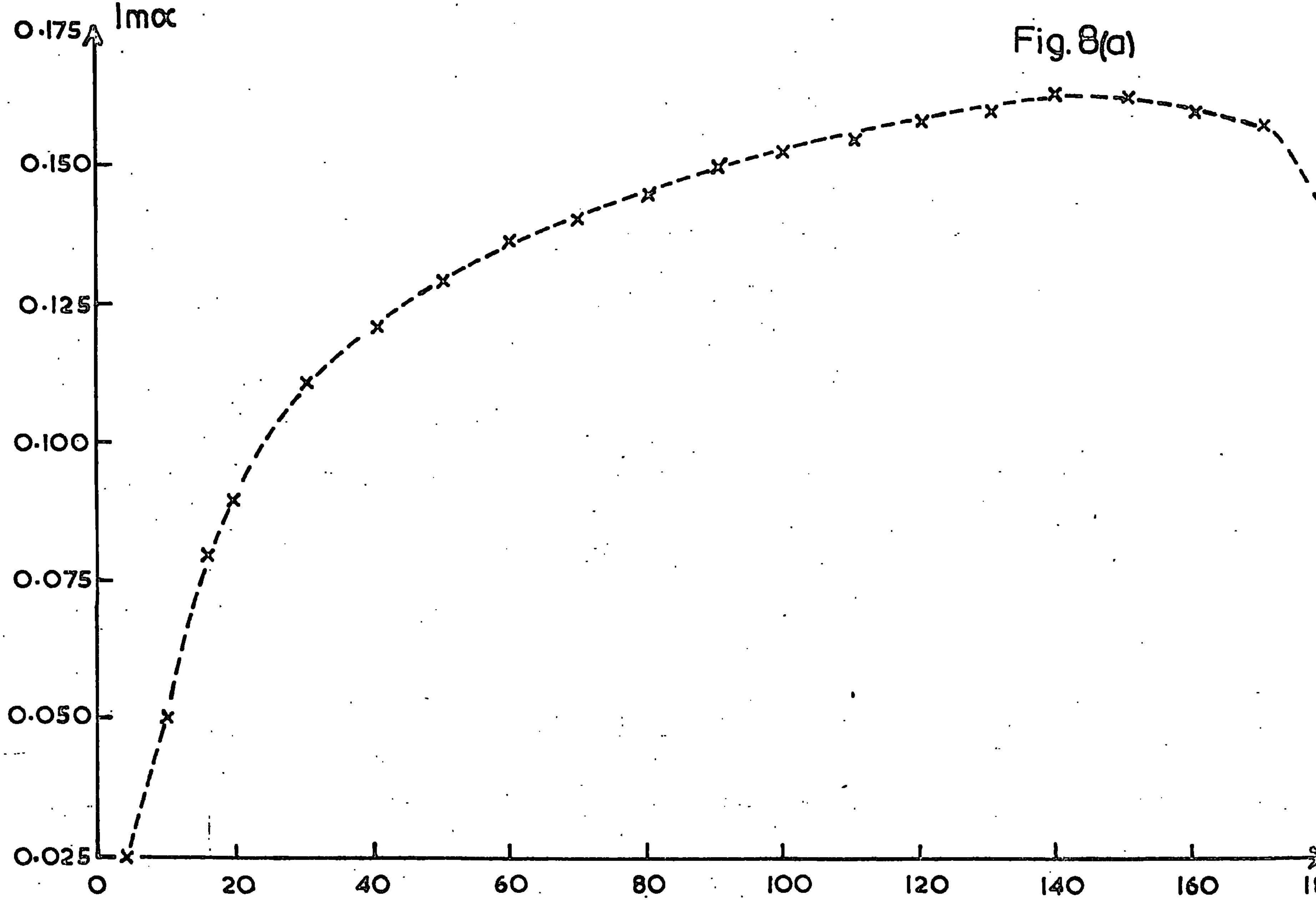
Fig. 6 (b).



$\gamma(t)$
1.00
0.90
0.80
0.70
$\gamma = 0.82$
80
40
0
40
80
120
160
200
240
280
t

Fig. 7





Imα
0·175
0·150
0·125
0·100
0·075
0·050
0·025
0
20
40
60
80
100
120
140
160

Fig. 8(a)

Fig. 8(b).

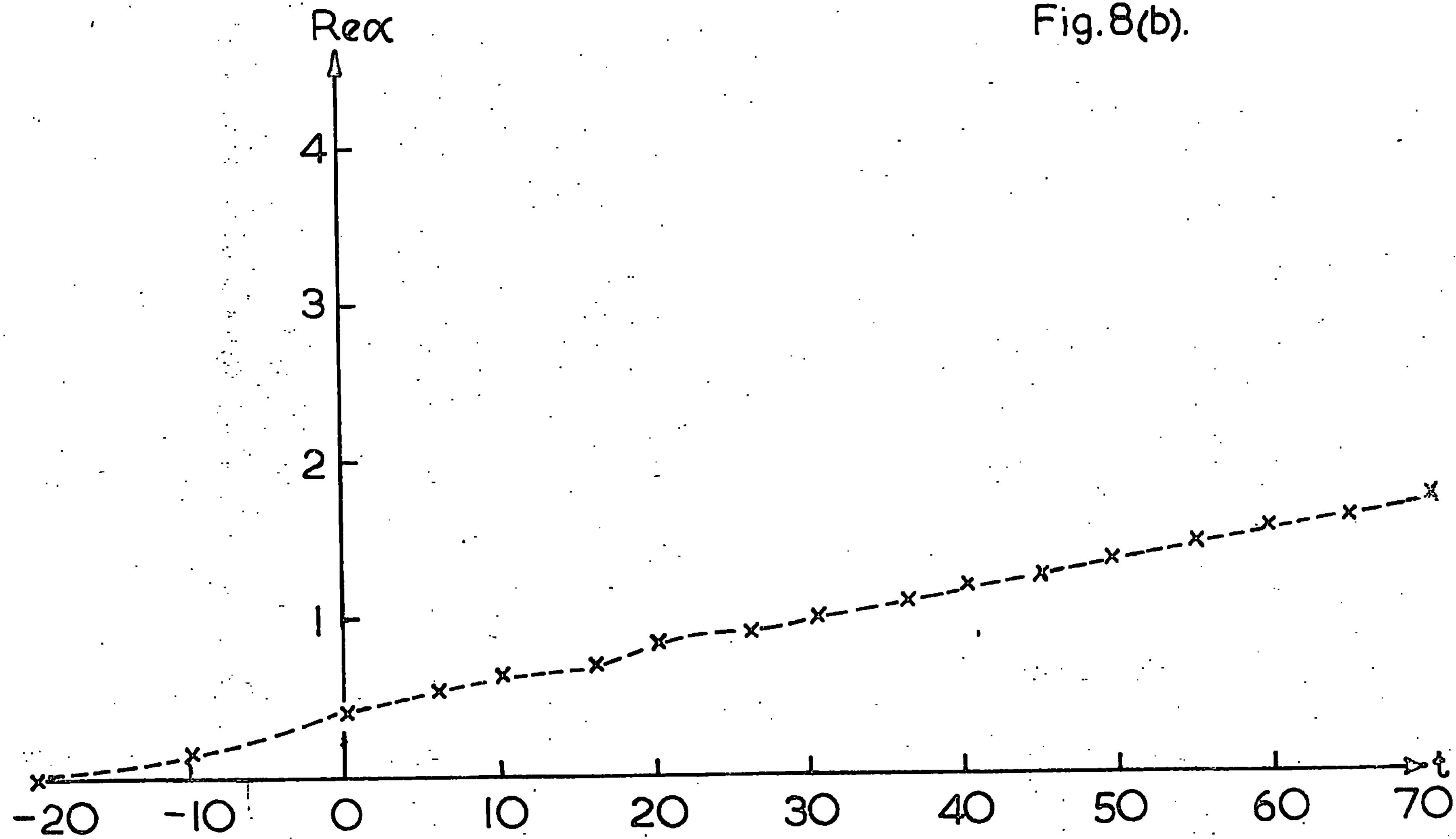
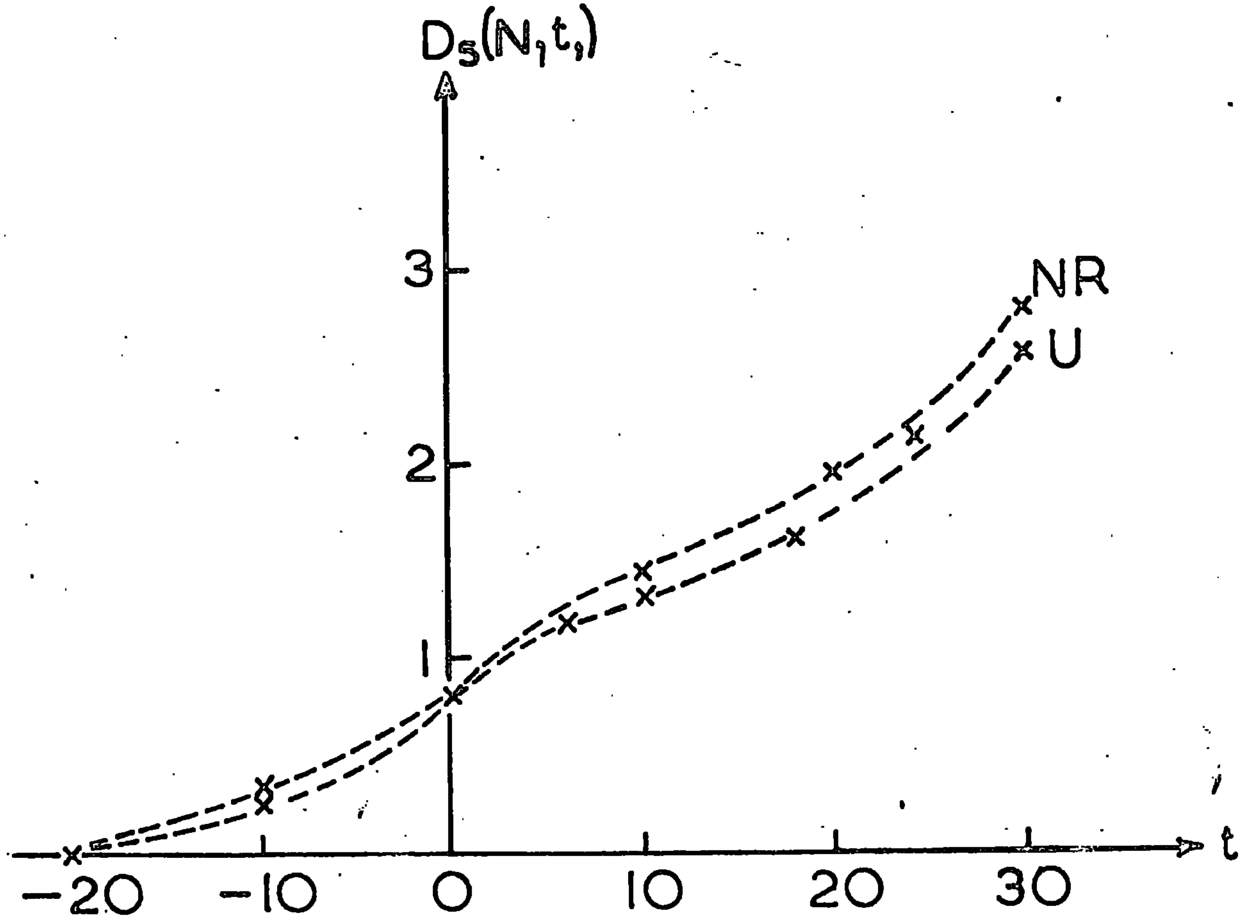


Fig. 9.



$\alpha' = 0.02\,M_\pi^{-2}$, $E = 0.82$, $N = 65.5\,M_\pi^{-2}$,

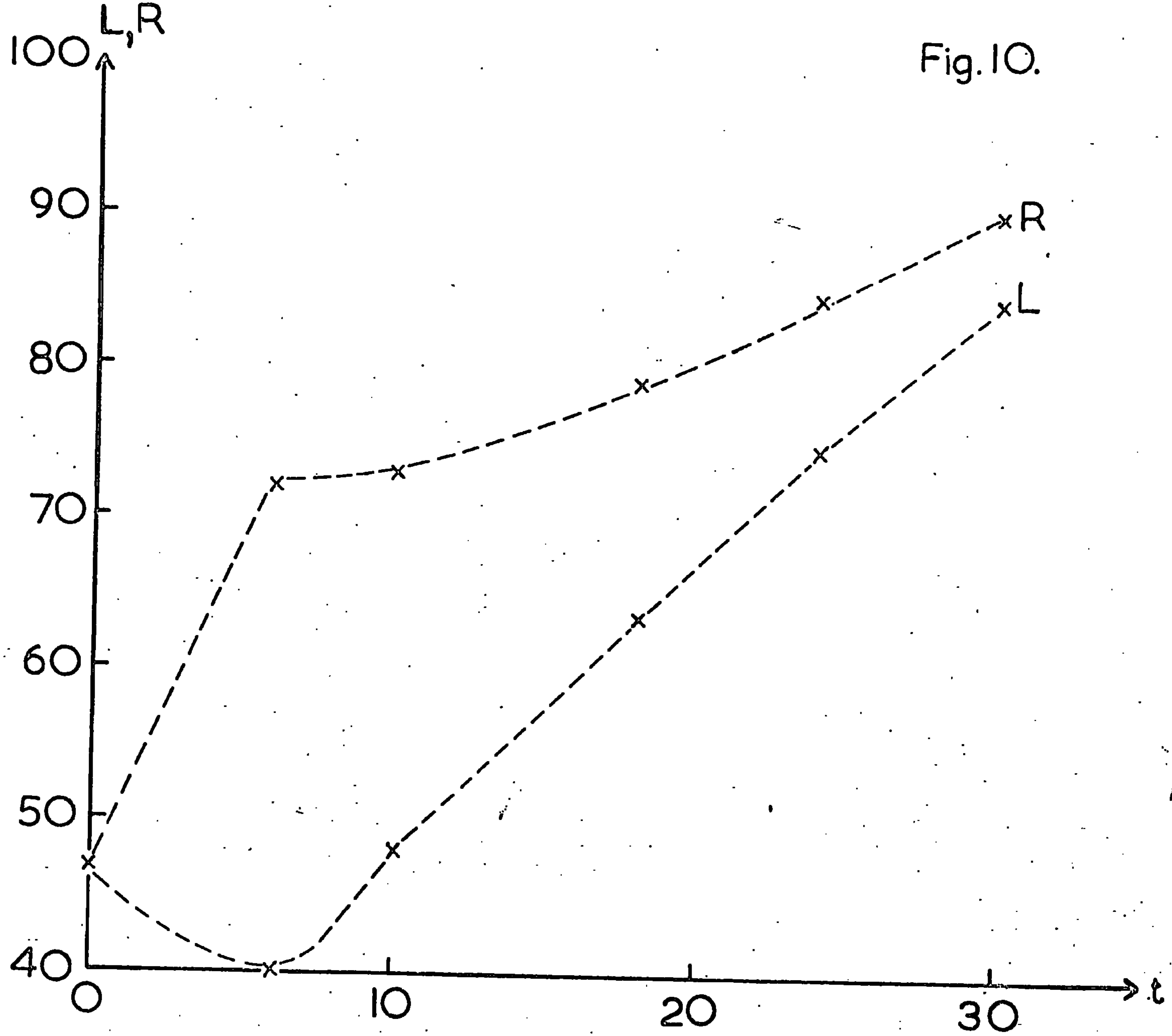


Fig. 10.

Fig. 11.

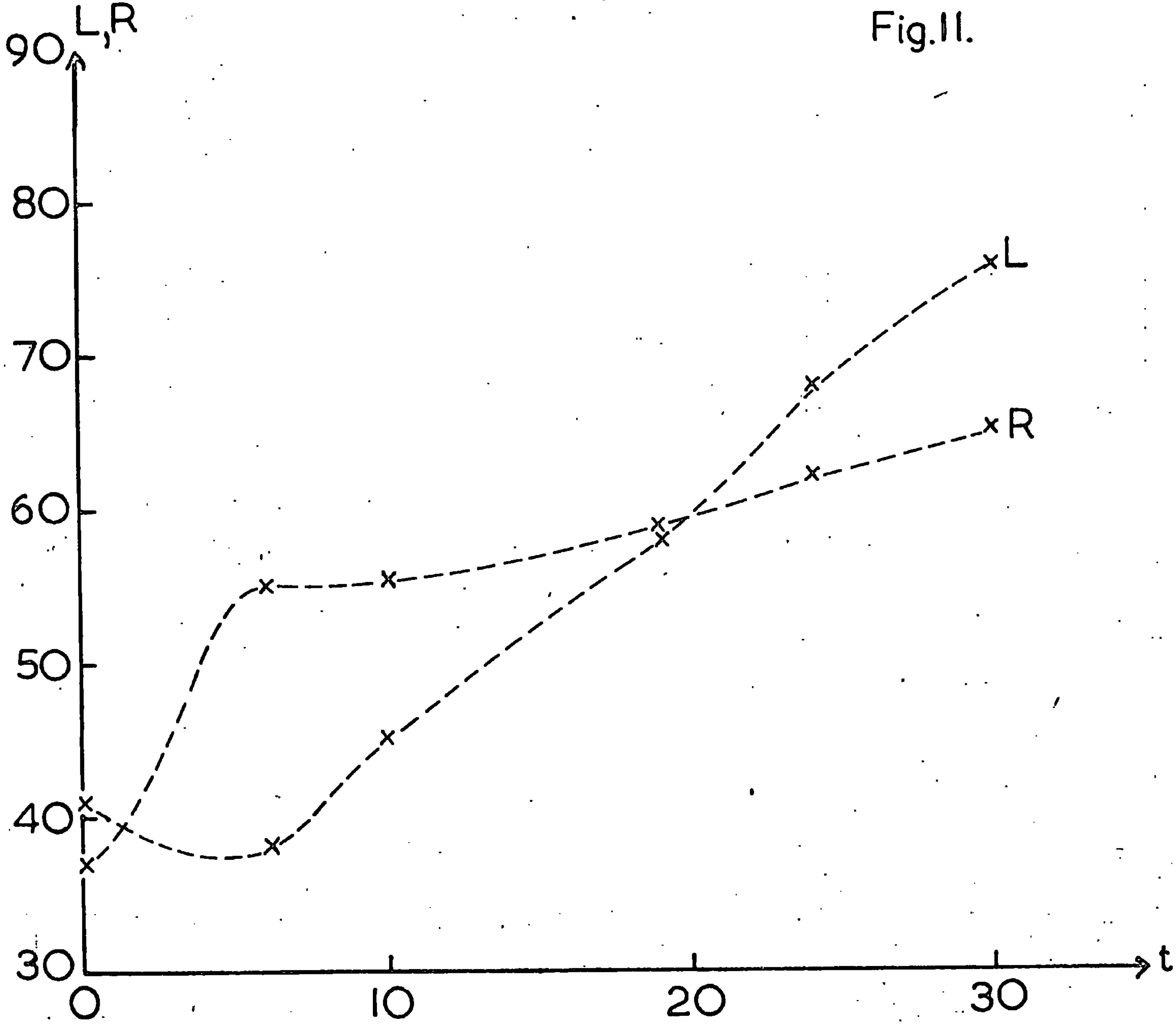


Fig. 12(a)

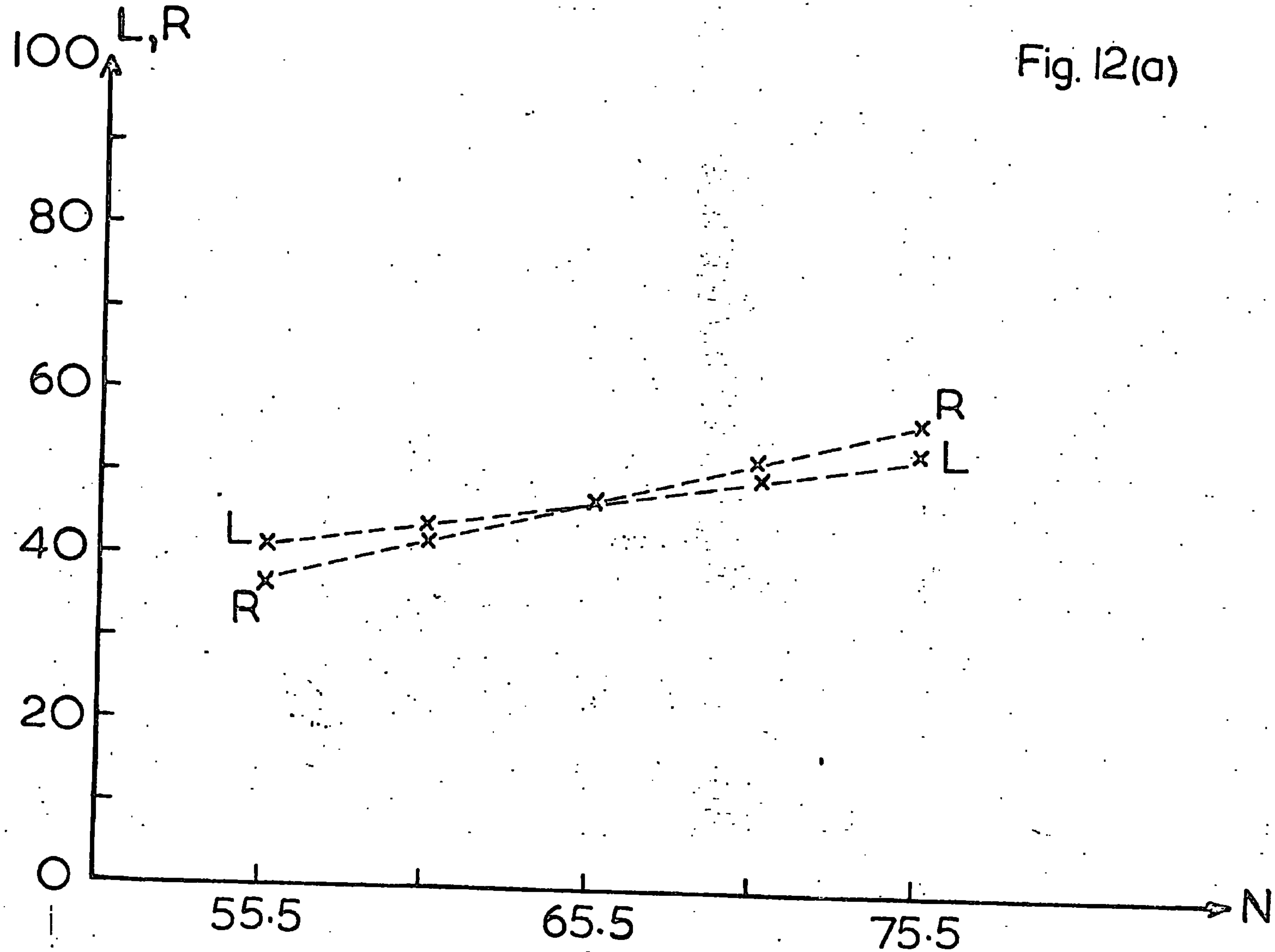
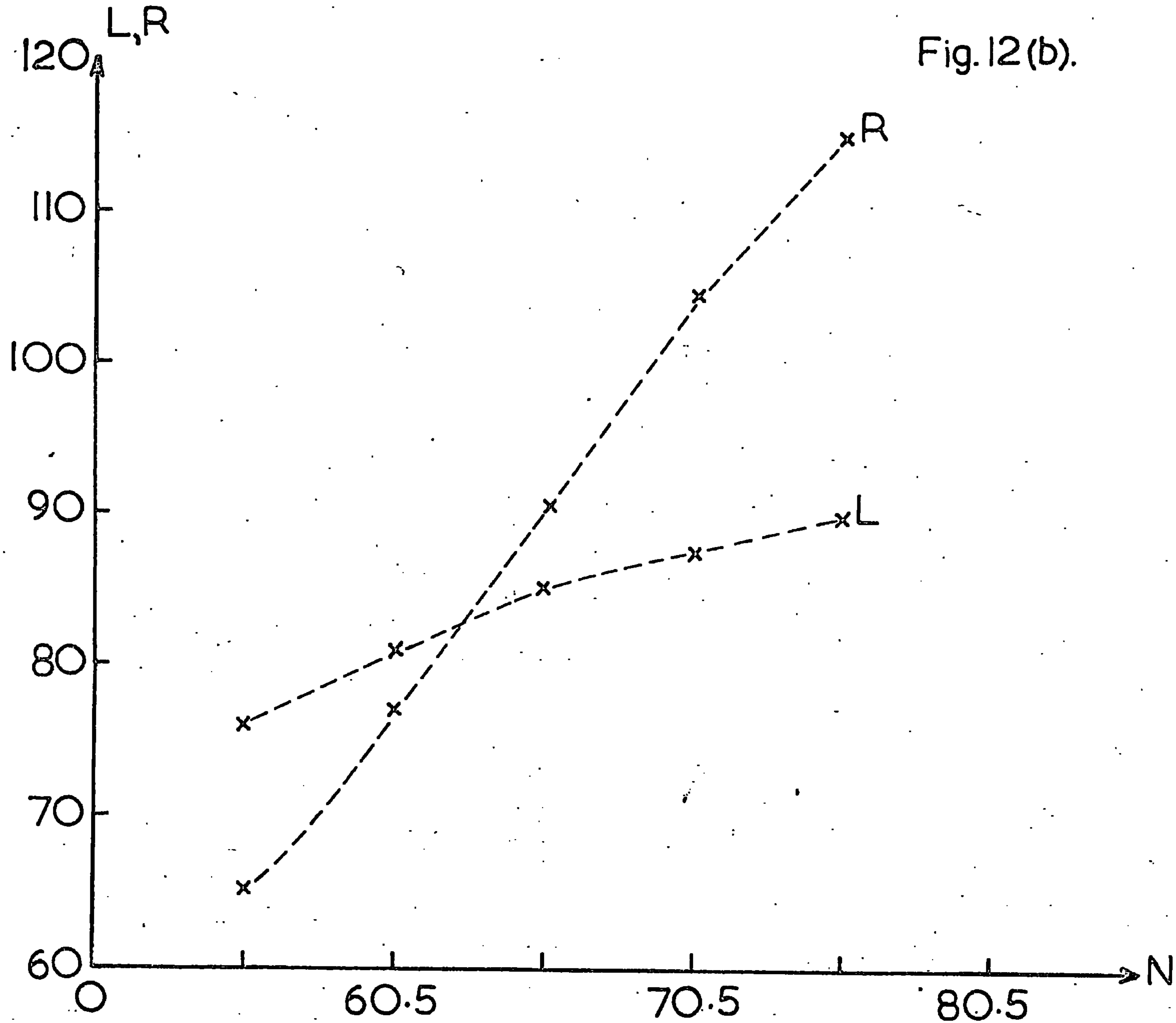


Fig. 12 (b).



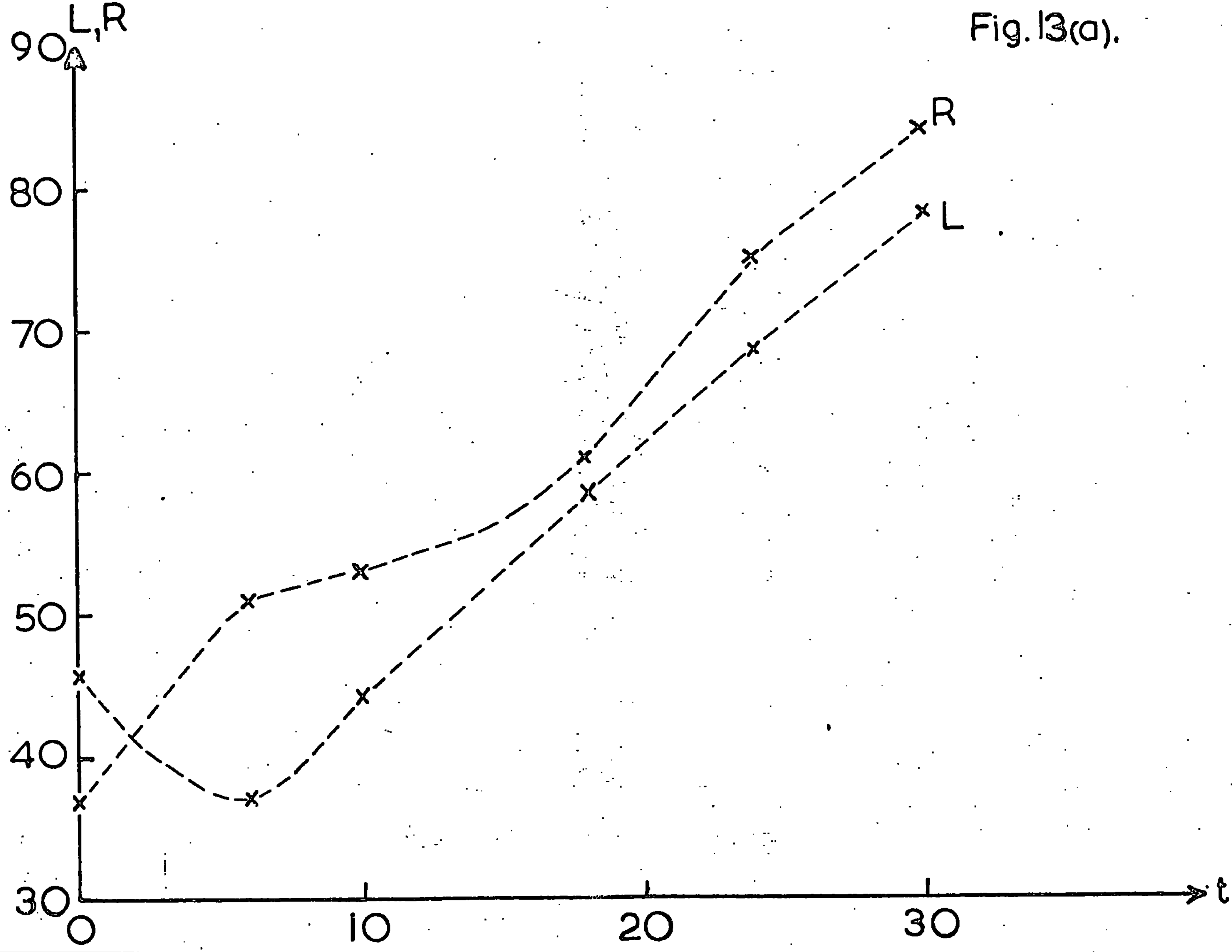


Fig. 13(a).

Fig. 13(b).

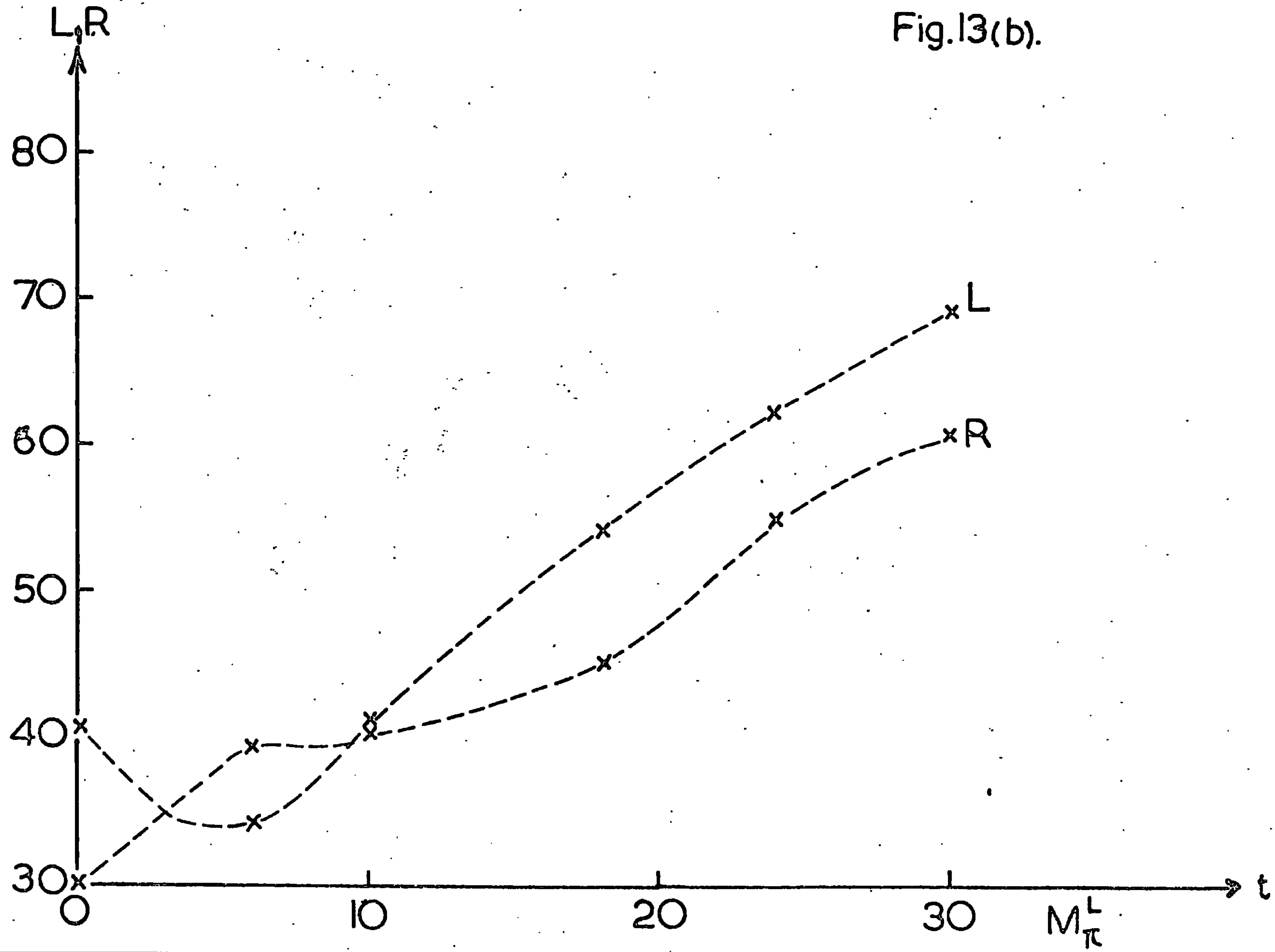


Fig. 14(a).

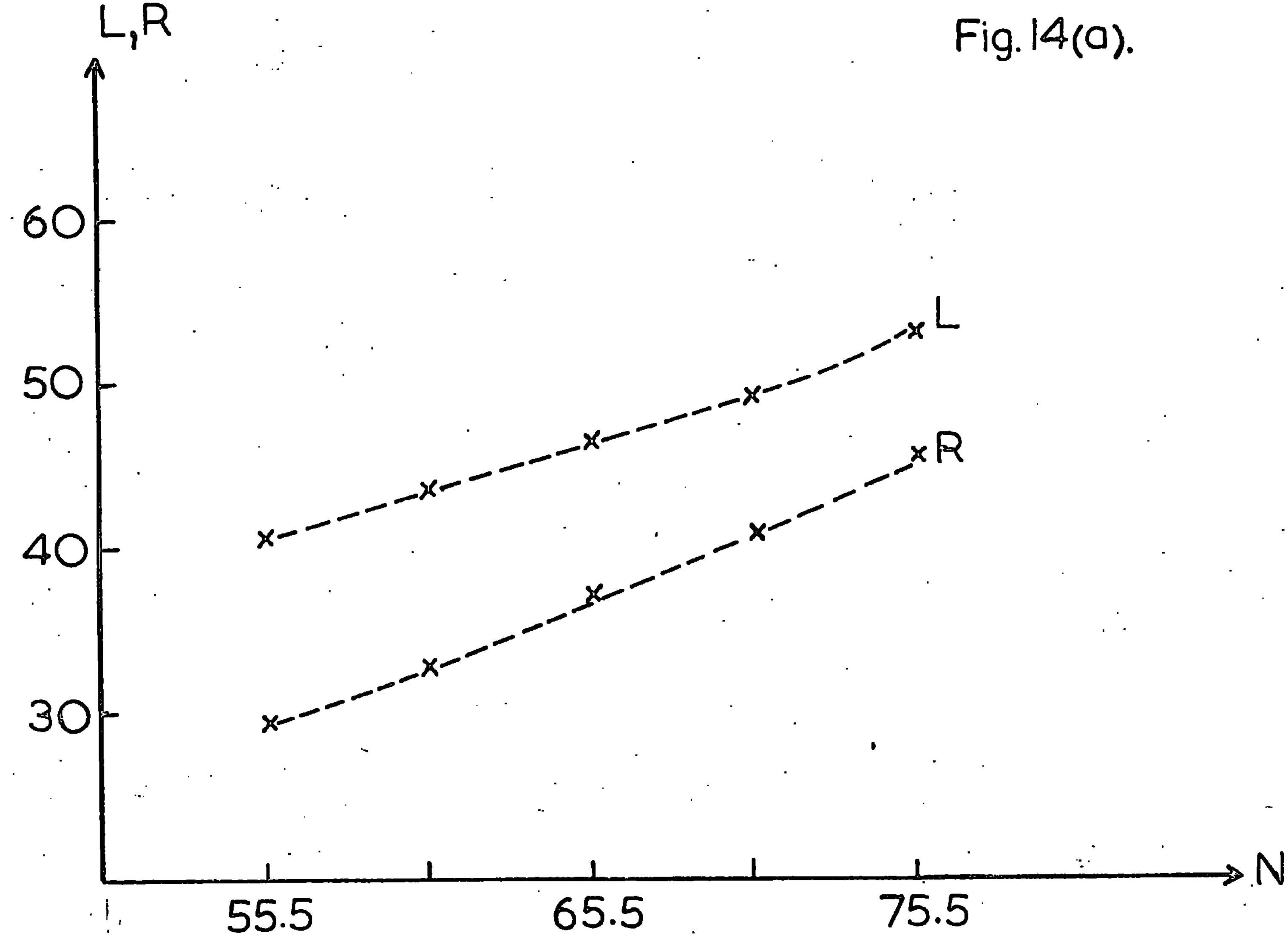
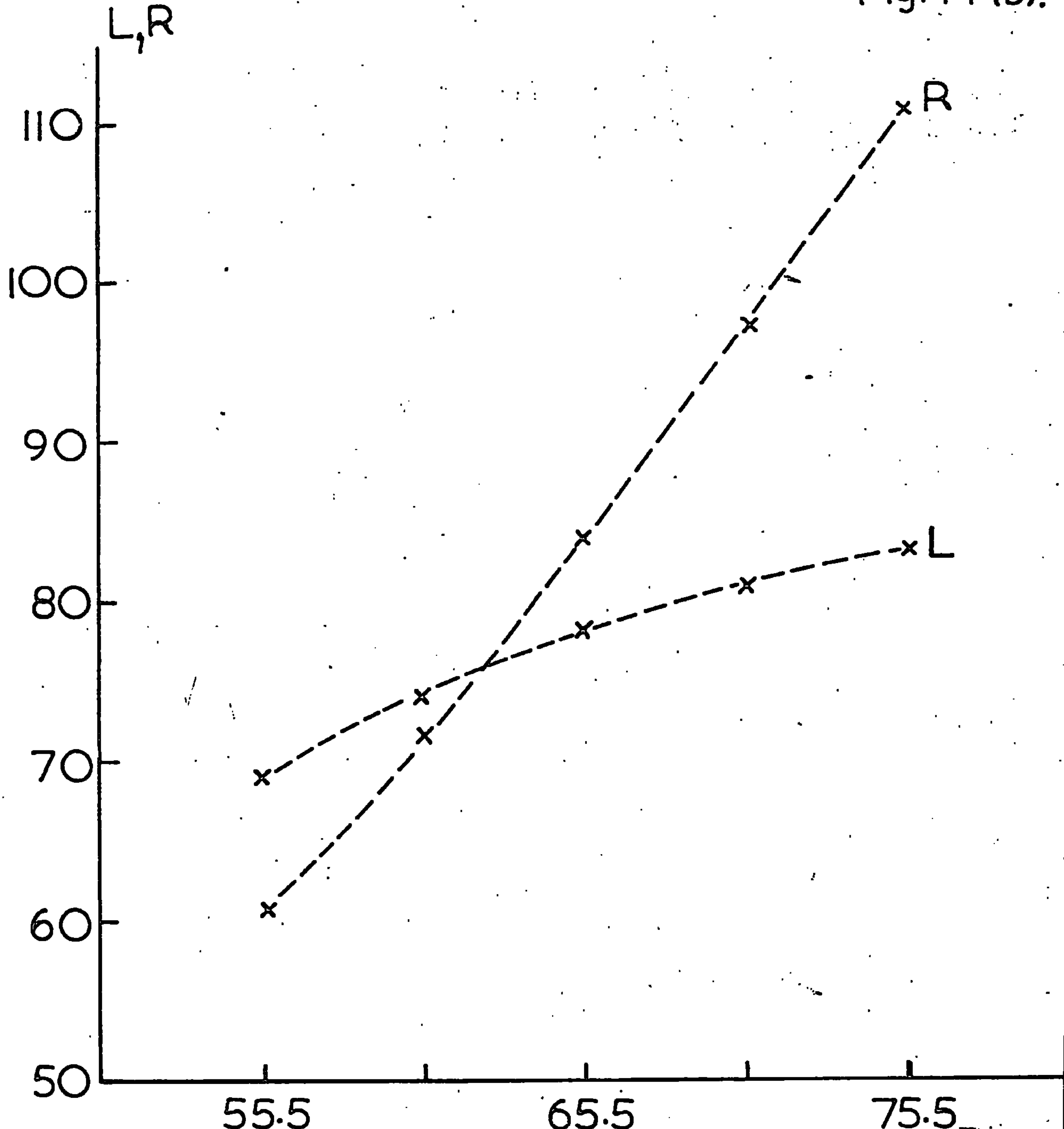


Fig. 14 (b).



$\alpha' = 0.02\ M_\pi^{-2}$, $E = 0.82$, $t = 30\ M_\pi^{-2}$,