CURVATURE MEASURES ON MANIFOLDS: MINIMAL IMMERSIONS

by

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A thesis presented for the degree of

Master of Science of the

University of Durham

June 1972

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ABSTRACT

The purpose of this work is to give some results on the various curvature measures on manifolds and also have a brief look at minimal immersions of manifolds in Riemannian spaces.

With regard to the former, the first chapter deals with the i^{th} TAC as defined by Chen [9].

In Chapter II we look at minimal immersions of compact manifolds in Riemannian spaces and in particular at pseudo-umbilical immersions - the term first introduced by Otsuki.

The two more familiar curvatures are the scalar curvature and the mean curvature, and in Chapter III we define the α^{th} scalar curvature. Finally we look at submanifolds with constant mean curvature.

Lastly, in Chapter IV, a differential equation is derived for "stable hypersurfaces". A hypersurface is said to be 'stable' if

$$\delta \int_{M^n} < \underline{H}, \underline{H} >^{n/2} * 1 = 0$$ for any normal variation of the integral. A particular case of this problem, (i.e. for surfaces in E^3) was first considered by Hombu.

A bibliography follows Chapter IV.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my supervisor, Professor T.J.Willmore, without whose guidance and encouragement this work would not have taken shape.

I also want to take this opportunity to thank Professor Bonner, who aroused my interest in the subject during my undergraduate days.

Thanks are due to many friends and relatives who have given me a great deal of encouragement.

The help in obtaining research papers by the staff of the Science Library is also appreciated.

The grant from the British Council and Durham University for part of my tuition fees are gratefully acknowledged.

I am very grateful to Dr. Hoffman and Professor Willmore for reading the manuscript and giving me valuable suggestions with regard to the layout of the work.

My sincere thanks to Mrs. Joan Gibson for typing this thesis.

Lastly, I thank my parents for all the opportunities they have given me, their continued interest and encouragement throughout my academic career.

CONTENTS

SUMMARY

In the first chapter we collect results concerning the integrals of various curvature measures on manifolds.

The origin of this theory dates as far back as 1929 in a paper du: to Fenchel [1]. He proved that for a closed space curve of class C^2

$$\int_C |k|\,dS \geqslant 2\pi$$

where dS is the line element, k, the curvature, and the integral is taken over the closed curve. Equality holds only in cases of plane convex curves (ovaloids) and conversely.

Since then Chern [1] has generalized this result to closed manifolds immersed in euclidean spaces.

The i^{th} Total Absolute Curvature was defined by Chen [9] as

$$\int_{B_\upsilon} |K_i(pe)|^{n/i}\, d\sigma \wedge dV$$

and some results concerning this integral are dealt with in this chapter.

The two main curvature measures are the Lipschitz-Killing curvature and the mean curvature. The former is defined in terms of the determinant of the second fundamental form while the latter is its trace.

We also give a few results with respect to the intermediary curvature measures. However, all these curvature measures have not yet been fully investigated and much work still remains to be done.

Having considered immersions in euclidean spaces in the second chapter we focus attention on submanifolds immersed in a general riemannian space.

It is well known that there do not exist any closed compact orientable minimal (in the sense of vanishing mean curvature)submanifolds in a euclidean space cf Myres [1]. However, when the ambient space is non-

euclidean the above statement may no longer be true. In particular we look at minimal immersions in spheres (i.e. when the ambient space has constant curvature).

The term pseudo-umbilical first introduced by Otsuki is also defined and we have a brief look at pseudo-umbilical immersions.

Finally we examine minimal immersions of "Clifford manifolds" as dealt with by Chern, do Carmo and Kobayashi [1].

The third chapter is partly a continuation of the first. We define the α^{th} scalar curvature and also introduce the notion of difference curvature due to Chen [4]. And lastly we give a few results on immersions with constant mean curvature.

In the last chapter we concentrate on deriving a differential equation which is a necessary and sufficient condition for a submanifold to be "stable" in the case when the ambient space is a general Riemannian.

CHAPTER I

§0 .

Following Chern and Lashof [1] we consider $x : M^n \to E^{n+N}$ where x is an immersion of an n-dimensional compact orientable C^∞-manifold M^n in a euclidean space of dimension $(n + N)$. If x_* is non-singular (i.e. the induced map has full rank) then f is called an <u>immersion</u>

$$x_* : T_p M^n \to T_{x(p)} E^{n+N} .$$

Further if x is one-one then x is called an imbedding.

Let $F(M^n)$ and $F(E^{n+N})$ be the bundles of orthonormal frames of M^n and E^{n+N}. A frame of $F(E^{n+N})$ consists of a point $x(p)$ together with a set of $(n + N)$ mutually perpendicular unit vectors.

Let B be the subset of $M^n \times F(E^{n+N})$ given by,

$$B = \{b = (p, x(p)e_1, \ldots, e_{n+N}) \mid (p, e_1, \ldots, e_n) \in F(M^n)$$
$$(x(p)e_1, \ldots, e_{n+N}) \in F(E^{n+N}) \} .$$

We define the projection map $\tilde{x} : B \to F(E^{n+N})$ by

$$\tilde{x}(b) = (x(p), e_1, \ldots, e_{n+N})$$

$$\begin{tikzcd} B \arrow[r] \arrow[rr, bend right, "\tilde{x}"'] & M^n \times F(E^{n+N}) \arrow[r] & F(E^{n+N}) \arrow[ll, bend right, "\tilde{x}^*"'] \end{tikzcd}$$

Let B_υ be the bundle space of unit normal vectors of $x(M^n)$ then, $B_\upsilon = \{(p,\upsilon) \mid p \in M^n, \upsilon \in N_p M \text{ at } x(p)\}$. B_υ is the bundle of $(N-1)$-dimensional spheres over M^n. For each $(p,\upsilon) \in B_\upsilon$ the unit normal vector υ at $x(p)$ can be identified to a vector at the origin of E^{n+N}. We define the Gauss map

$$\tilde{\upsilon} : B_\upsilon \to S_0^{n+N-1} \subset E^{n+N} \quad \text{by} \quad \tilde{\upsilon}(p,v) = \upsilon(p).$$

S_0^{n+N-1} is the unit sphere at the origin of E^{n+N}. According to Chern and Lashof [1] there is on B_υ a differential form $d\sigma$ of degree $(N-1)$ whose restriction to the fibre is the volume element of the sphere of unit normal vectors at $p \in M^n$. We denote the volume element of M^n by dV, so that it is a form of degree n

then, $d\sigma \wedge dV$ is the volume element of B_υ .

If $d\Sigma$ = volume element of S_0^{n+N-1}, since $d\sigma \wedge dV$ and $d\Sigma$ are differential forms on B_υ of maximal degree, we can conclude that they must differ by a constant.

If θ_A and θ_{AB} denote the 1-forms and the connection forms on $F(E^{n+N})$

then, $dx = \sum_A \theta_A \cdot e_A$

$$de_A = \sum_B \theta_{AB} \cdot e_B \qquad \theta_{AB} + \theta_{BA} = 0.$$

We will follow the usual convention for the range of the suffixes,

i.e. $i,j,k \ldots = 1,2,\ldots n$

$r,s,t \ldots = (n+1),(n+2), \ldots (n+N)$

$A,B,C \ldots = 1,2 \ldots (n+N).$

Taking the exterior derivative of the two above equations and simplifying we obtain the Cartan Structural equations

$$d\theta_A = \sum_B \theta_{AB} \wedge \theta_B$$

$$d\theta_{AB} = \sum_C \theta_{AC} \wedge \theta_{CB} .$$

Let ω_A and ω_{AB} be the induced forms on B

i.e. $\omega_A = \tilde{x}^* \theta_A$

$\omega_{AB} = \tilde{x}^* \theta_{AB}$.

$\omega_1, \omega_2, \ldots, \omega_n$ is a dual basis to $e_1, \ldots, e_n$, a basis of the tangent space at p.

On M^n we have, $\omega_r = 0$

therefore $d\omega_r = 0 = \sum_i \omega_{ri} \wedge \omega_i$

we can write $\omega_{ri} = -\sum_j A_{rij}\, \omega_j$

and $A_{rij} = A_{rji}$.

(It may be remarked that the A_{rij}'s are the coefficients of the second fundamental form. cf. §1. The sign here is also negative to that generally used by Chen.).

Restricting the forms to M^n, we get,

$$dx = \sum_i \omega_i\, e_i$$

$$de_i = \sum_j \omega_{ij}\, e_j$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CA}$$

therefore $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,\ell} R_{ijk\ell} \cdot (\omega_k \wedge \omega_\ell)$.

$$R_{ijk\ell} = -A_{rik}\, A_{rj\ell} + A_{ri\ell}\, A_{rjk} \quad .$$

Now $dV_n = \omega_1 \wedge \ldots \wedge \omega_n$.

and $d\sigma_{N-1} = \omega_{n+N,1} \wedge \ldots \wedge \omega_{n+N,n+N-1}$

$d\Sigma_{n+N-1} = \theta_{n+1,1} \wedge \theta_{n+N,2} \wedge \ldots \wedge \theta_{n+N,n+N-1}$.

therefore $\mho^* d\Sigma_{n+N-1} = \omega_{n+N,1} \wedge \ldots \wedge \omega_{n+N,n+N-1}$

$$= (-1)^n \det (A_{n+N,ij})\omega_1 \wedge \ldots \wedge \omega_n \wedge \omega_{n+N,1} \wedge \ldots$$

$$\ldots \wedge \omega_{n+N,n+N-1}$$

$$= G(p,\upsilon) \cdot dV \wedge d\sigma \quad ,$$

where $G(p,\upsilon) = (-1)^n \cdot \det(A_{n+N,ij})$

$G(p,\upsilon)$ def Lipschitz-Killing curvature.

Note 1. When M^n is a hypersurface of E^{n+1} $G(p\upsilon) = G(p) =$ Gauss-Kronecker curvature and when $n = 2$ and $N = 1$ $G(p\upsilon)$ is just the classical Gaussian curvature.

§1. The i^{th} mean Curvature

For a pair $(p,e_r) \in B_\upsilon$ the first fundamental form and the second fundamental forms, in the direction e_r, of the immersions are given by

$$I \;:\; dx \cdot dx$$

$$II_r \;:\; -dx \cdot de_r \;.$$

The eigenvalues $k_1(p,e_r)$, $k_2(p,e_r)$ ... $k_n(p,e_r)$ of II_r with respect to I are defined to be the principal curvatures of M^n at each point $p \in M^n$ in the direction e_r.

Definition 1.1.1

The i^{th} mean curvature in the direction e_r denoted by $K_i(p,e_r)$ is given by equating the coefficient of $t^o, t^1, \ldots, t^n$ in the following equation

$$\det(\delta_{jk} + t\,A_{rjk}) = \Sigma \binom{n}{i} K_i(p,e_r)t^i$$

where δ_{jk} is the Kronecker delta

$$\binom{n}{i} K_i(p,e_r) = \Sigma\, k_1(p,e_r) \ldots k_i(p,e_r) \qquad (0)$$

$$i = 1,2,\ldots,n.$$

Note 2 $K_0 = 1$

$K_1(p,e_r)$ = mean curv. of the immersion at p in the direction e_r.

$K_n(p,e_r)$ = Lipschitz-Killing curvature at (p,e_r).

Definition 1.1.2

The integral $K_i^*(p) = \int_{\text{fibre}} |K_i(p,e)|^{n/i}\, d\sigma$ over the sphere unit normal vectors at $x(p)$ is called the i^{th} TOTAL-ABSOLUTE CURVATURE of the immersion x at p.

$\int_{M^n} K_i^*(p)dV$ is defined to be the i^{th} TOTAL-ABSOLUTE CURVATURE of M^n.

From (0) it follows that

$$\underline{H} = \frac{1}{n}\sum_r K_1(p,e_r)e_r .$$

$$= \frac{1}{n}\sum_{r,i} A_{rii}\, e_r .$$

The Ricci tensor and the scalar curvature R are given by,

$$R_{jk} = \sum_i R^i_{jik}$$

$$R = \sum_j R_{jj} .$$

We will denote the length of the mean curvature vector by α i.e. $\alpha = \|H\|$ and by S the length of the second fundamental form.

Then if we let $S_r = \sqrt{\sum_{i,j} (A_{rij})^2}$

$$S = \sqrt{\sum_{r,i,j} (A_{rij})^2}$$

$$R_{ijk\ell} = A_{ri\ell}A_{rjk} - A_{rik}A_{rj\ell} .$$

$$R_{jk} = \sum_i R_{ijki} = \sum_i (A_{rii} A_{rjk} - A_{rik} A_{rji}).$$

$$R = \sum_j R_{jj} = \sum_j (\sum_i (A_{rii} A_{rjj} - A_{rij} A_{rji}))$$

$$\therefore \quad R = n^2 \alpha^2 - S^2 .$$

<u>Note 1.1.3 (Frenet Frame)</u>

Let $(p, e_1, e_2 \dots e_n, \bar{e}_{n+1}, \dots, \bar{e}_{n+N})$ be a local cross section of M^n in B and for any $e \in S_0^{N-1}$ let

$$e = e_{n+N} = \sum_r \cos \theta_r . \bar{e}_r(p) \quad \text{where} \quad \sum_r \cos^2\theta_r = 1.$$

$$A_{n+N,ij} = \sum_r \cos \theta_r . \bar{A}_{rij} \tag{1}$$

and $\quad \bar{A}_{rij} = A_{rij} \Big|_{\text{local-cross section}}$

then $$\binom{n}{2} K_2(p, e_{n+N}) = \sum_{i<j} (A_{rii} A_{rjj} - A^2_{rij}) . \tag{2}$$

From (1) and (2) it follows that

$$\binom{n}{2} K_2(p, e_{n+N}) = \sum_{i<j} \left\{ (\sum_r \cos \theta_r \bar{A}_{rii})(\sum_s \cos \theta_s \bar{A}_{sjj}) - (\sum_t A_{tij})^2 \right\} .$$

Choosing a suitable cross-section of $B \to F(M^n)$ we can write,

$$K_2(p, e_{n+N}) = \sum_r \lambda_{r-n} \cos^2\theta_r . \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N .$$

Such a cross-section is called the FRENET CROSS-SECTION and the frame $(p, x(p)e_1, \dots, e_n, \bar{e}_{n+1}, \dots, \bar{e}_{n+N})$ the FRENET FRAME.

λ_α , $\alpha = 1,2,\dots,N$ is called the α^{th} curvature of the second kind (later the α^{th} scalar curvature).

Also, $$H = \binom{n}{1} K_1(p,e) = K_1(p, \sum_r \cos\theta_r \cdot \bar{e}_r).$$
$$= \sum_r \cos\theta_r \cdot K_1(p, \bar{e}_r).$$
$$= \sum_r \cos\theta_r \cdot \mu_{r-n}.$$

$$\mu_r(p) = K_1(p, \bar{e}_r).$$

μ_α, $\alpha = 1,2,\ldots,N$ is called the α^{th} curvature of the first kind.

Henceforward we shall denote

the Lipschitz-Killing curvature by L-K curv.

Gauss-Kronecker curvature by G-K curv.

Total Absolute curvature by TAC.

The volume element of an n-dimensional unit sphere will be denoted by c_n and is given by

$$c_n = \frac{2\Gamma(\frac{1}{2})^{n+1}}{\Gamma((n+1)/2)} \qquad \text{(cf. Flanders [1]).}$$

§2. Some general results concerning the value of the integral of the i^{th} TAC of M.

Lemma 1.2.1 (cf. Hardy, Littlewood, Polya [1])

Let $a_1,\ldots,a_n$ be a set of n-non negative numbers and S_i = i^{th} elementary symmetric function of $a_1,\ldots,a_n$.

Let $$M_i = S_i / \binom{n}{i}$$

then $$(M_1)^n \geq (M_2)^{n/2} \geq \ldots \geq M_n \tag{1}$$

and equality at any stage $\Longleftrightarrow$ $a_1 = a_2 = \ldots = a_n$. (2)

Proof

Using Newton's inequality on elementary symmetric functions viz: $$M_p\, M_{p+2} \geqslant M_{p+1}^{\;2} \qquad p = 0,1,2,\ldots,(n-2)$$

and employing it successively we get (1) and (2) follows quite straightforwardly.

Lemma 1.2.2.

Let $(f_{ij})(e)$ be symmetric $(n \times n)$ matrix valued function on the unit $(N-1)$ sphere in E^N given by

$$(f_{ij})(e) = \sum_r \cos\theta_r \cdot A_{rij}$$

where $$A_{rij} \in \mathbb{R} \quad \text{and} \quad \Sigma \cos^2\theta_r = 1.$$

$$e = \sum_r \cos\theta_r \cdot e_r \quad \text{and} \quad e_r = (0,\ldots,1_r\, 0,\ldots,0).$$

If $(f_{ij})(e)$ has the same eigenvalues at every point on a non-empty open set U of S_o^{N-1} then it has the same eigenvalues at every point of S_o^{N-1} .

Proof

Let $U = \{ p \in S_o^{N-1} \mid (f_{ij})(e) \text{ has same eigenvalues} \}$.

i.e. $$U = \{ p \in S_o^{N-1} \mid k_i(p,e) = k_j(p,e) \;\; \forall\, i \text{ and } j \} .$$

Now U is open by hypothesis (1)

Claim: U is closed.

Consider $k_i, k_j : U \to \mathbb{R}$ $\quad i \neq j$. k_i and k_j are continuous functions. Define $h : U \to \mathbb{R} \times \mathbb{R}$ by $h(p) = (k_i(p), k_j(p))$.

Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $\pi_1(a,b) = a$

and $\pi_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $\pi_2(a,b) = b$,

then, $$\pi_1 \circ h = k_i$$

$$\pi_2 \circ h = k_j .$$

h is continuous because π_1 and π_2 are continuous . Let $Z = \{(k_i(p), k_j(p)) \in \mathbb{R} \times \mathbb{R} \mid k_i(p) = k_j(p)\}$

then,
$$h^{-1}(Z) = \{p \in S_o^{N-1} \mid k_i(p) = k_j(p)\} = U.$$

Z is closed

$\therefore$ U is closed. (2)

From (1) and (2) U must be all of S_o^{N-1}

$\therefore$ Result follows.

Lemma 1.2.3

Let $x : M^n \to E^{n+N}$ be as before. If M^n is totally umbilical in E^{n+N}. Then M^n is immersed as a hypersphere in an (n+1)-dimensional linear subspace of E^{n+N} $(n > 1)$.

Proof

Let $U = \{p \in M^n \mid \underline{H}(p) \neq 0\}$

Since there does not exist any closed minimal submanifolds in a euclidean space $U \neq \emptyset$.

$\therefore$ $<\underline{H},\underline{H}>$ exists on U and is non-zero.

Let
$$e_{n+1} = \frac{\underline{H}}{<\underline{H},\underline{H}>^{\frac{1}{2}}}$$

then,
$$A_{sij} = 0 \qquad (1)$$
$$s = (n+2),\ldots,(n+N) \qquad i,j = 1,2,\ldots,n.$$

$$\omega_{is} = 0$$

$$\omega_{i,n+1} = \alpha\omega_i$$

$$\therefore \qquad 0 = d\omega_{is} = \omega_{i,n+1} \wedge \omega_{n+1,s}$$

$$= \alpha\omega_i \wedge \omega_{n+1,s}$$

$$\Longrightarrow \omega_{n+1,s} = 0$$

$\omega_{i,n+1} = \alpha\omega_i$.

Now $d\omega_{i,n+1} = \omega_{ij} \wedge \omega_{j,n+1}$.

also
$$d\omega_{i,n+1} = d\alpha \wedge \omega_i - \alpha . d\omega_i$$

$$= d\alpha \wedge \omega_i - \alpha . \omega_{ij} \wedge \omega_j$$

$$\Longrightarrow d\alpha \wedge \omega_i = 0$$

$\Longrightarrow$ α is constant.

The linear span of $\{e_{n+2}, e_{n+3}, \dots, e_{n+N}\}$ is independent of p and therefore M^n is immersed as a totally umbilical submanifold in the (n+1)-dimensional linear space of E^{n+N} spanned by $e_1, \dots, e_n$ and the mean curvature vector.

$\therefore$ M^n is immersed as a hypersphere in this (n+1)-dimensional linear subspace.

Lemma 1.2.4

Consider $x : M^n \to E^{n+N}$.

If B_υ the normal bundle is the union of U and V such that

$$U = \{(p,e) \in B_\upsilon | k_1(p,e) = \dots = k_n(p,e) \neq 0\}$$

$$V = B_\upsilon - U \quad \underline{\text{and}} \quad K_i(p,e) = 0 \quad \text{everywhere on } V$$

for fixed $i = 1,2\dots(n-1)$.

then,

$$\pi : U \to M \text{ is surjective}$$

$$\pi(p,e) = p.$$

A proof of the above lemma can be found in Chen [9].

Theorem 1.2.5

$$x : M^n \to E^{n+N} .$$

Then,

$$\int_{M^n} K_i^*(p)dV \geq c_{n+N-1} \qquad i = 1,2,..,n$$

where c_{n+N-1} is the volume element of unit $(n+N-1)$ sphere.

Proof

As before $\tilde{\upsilon} : B_\upsilon \to S_o^{n+N-1}$.

For a fixed unit vector $e = e_{n+N} \in S_o^{n+N-1}$ $e.x(p)$ is continuous on M and $\therefore$ has at least one maximum and one minimum (because M is closed), say at p and q respectively.

$\therefore$ $A_{n+N,ij}$ is negative definite and positive definite at these points.

Let $U^* = \{(p,e) \in B_\upsilon | k_1(p,e), \ldots, k_n(p,e)$ are either all ≥ 0 or all $\leq 0\}$.

By the Gauss map S_o^{n+N-1} is covered twice and since,

$$\int_{U^*} |K_n(p,e)| d\sigma \wedge dV = \text{vol. of im}(U^*).$$

$\therefore$ we have,

$$\int_{U^*} |K_n(p,e)| d\sigma \wedge dV \geq 2c_{n+N-1} .$$

but, by lemma 1.2.1 $|K_i(p,e)|^{n/i} \geq |K_n(p,e)|$ on U^*

$$\therefore \int_{B_\upsilon} |K_i(p,e)|^{n/i} d\sigma \wedge dV \geq \int_{U^*} |K_i(p,e)|^{n/i} d\sigma \wedge dV$$

$$\geq \int_{U^*} |K_n(p,e)| d\sigma \wedge dV .$$

$$\geq 2c_{n+N-1} .$$

Theorem 1.2.6

Under the same hypothesis as Theorem 1.2.5 if $\int_{M^n} K_i^*(p)dV = 2c_{n+N-1}$,

then M^n is imbedded as

(i) a hypersphere in an (n+1) dimensional linear subspace of E^{n+N} if $i < n$ and conversely ,

(ii) as a convex hypersurface in an (n+1) dimensional subspace of E^{n+N} if $i = n$ and conversely.

Proof

For $i = n$ see Corollory 1.2.8.

Assume $i \neq n$.

Let U^* be as in Theorem 1.2.5

then, $\qquad |K_i(p,e)|^n = |K_n(p,e)| \quad \text{on} \quad U^*$

and $\qquad K_i(p,e) = 0 \quad \text{on } B_\upsilon - U^*.$

Let $U = \{(p,e) \in B_\upsilon | k_1(p,e) = \dots = k_n(p,e) \neq 0\}$.

In particular $K_i(p,e) = 0$ on $B - U$ $\quad i = 1,2,\dots,n-1.$

$\therefore$ By lemma 1.2.4 $\pi : U \to M$ is surjective

$\therefore$ for every $p \in M$ $\exists$ non-empty open subset of the fibre S_o^{N-1} of B_υ such that all the principal curvatures are equal.

This exists and by lemma 1.2.3 (since the principal curvatures are equal on a non-empty subset) the principal curvatures are equal at all points on M^n.

Hence M is immersed as a hypersphere in an (n+1)-dimensional linear subspace of E^{n+N}.

$$\therefore \qquad B_\upsilon = U^*$$

and hence
$$\int_{M^n} K_i^*(p)dV = 2c_{n+N-1} .$$

Conversely, if M^n is imbedded as a hypersphere in an (n+1)-dimensional linear subspace of E^{n+N} then all the principal curvatures are equal and the result follows immediately.

Corollary 1.2.7

The case for $i = 1$ has also been proved by Willmore [2].

Corollary 1.2.8

Theorems 1.2.5 and 1.2.6 are well known theorems of Fenchel, Chern and Lashof [1] in the case when $i = n$. In fact Chern and Lashof [2] have further shown that

$$\int_{B_\upsilon} |K_n(p,e)| d\sigma \wedge dV \geq c_{n+N-1} \cdot \sum_{i=1}^{n} \beta_i(M)$$

where $\sum_i \beta_i(M)$ is the sum of Betti numbers of M.

Corollary 1.2.9

Under the same hypothesis if $\int_{M^n} K_i^*(p)dV < 3c_{n+N-1}$ then, M^n is homeomorphic to an n-dimensional sphere. For $i = n$, the same result has been proved by Chern and Lashof [1].

It will be seen later that the value of the integrals can be improved in some cases particularly with restrictions on the scalar curvature of the immersed manifold.

In a series of papers on TAC Chen [1,6,7] generalizes the L-K Curv. to manifolds in a simply connected Riemannian manifold with non-positive sectional curvature. He proves various results for TAC of manifolds immersed in a general Riemannian manifold concentrating largely on surfaces in real space forms. A Riemannian manifold of constant curvature is said to be elliptic, hyperbolic or flat (locally Euclidean) according as the sectional curvature is positive, negative or zero - such spaces are called space forms.

In the last paper of the series he deals with the TAC of bounded and cornered manifolds and also finds a relationship between the TAC of totally geodesic manifolds in a non-trivial riemannian space. Attached cornered and product cornered manifolds are also considered from the point of view of obtaining some results for TAC.

He finally looks at Kähler manifolds and shows that under certain conditions there exists some relation between TAC, the Riemannian curvature and the second fundamental form.

Prior to Chen's work, Saleemi and Willmore [1] had generalized the concept of TAC to manifolds in an arbitrary Riemannian space and in the particular case when the ambient space was euclidean it reduced to the result of Chern and Lashof [1].

<u>Theorem 1.2.10</u>

$x : M^n \to E^{n+N}$ as before.

Then,

$$\int_{M^n} \langle \underline{H}, \underline{H} \rangle^{n/2} dV \geq c_n ,$$

where $\underline{H}$ is the mean curvature vector of immersion.

Equality holds iff M^n is imbedded as an n-dimensional hypersphere in (n+1) dimensional linear subspace of E^{n+N}.

Proof.

Choose a frenet frame $(p, x(p), e_1, \ldots, e_n, \bar{e}_{n+1}, \ldots, \bar{e}_{n+N})$ in B such that $\bar{e}_{n+1}$ is parallel to the mean curvature vector.

Then, $$\sum_i A_{sii} = 0 \quad s = (n+2), \ldots, (n+N),$$

and $$\frac{1}{n} \sum_i A_{n+1,ii} = \langle H,H \rangle^{\frac{1}{2}} .$$

By the choice of frame we also have,

$$e_{n+N} = \sum_r \cos\theta_r \cdot \bar{e}_r .$$

$$\therefore \quad K_1(p, e_{n+N}) = \sum \cos\theta_r \cdot K_1(p, \bar{e}_r)$$
$$= \cos\theta_{n+1} \, |H(p)|.$$

$$\int_{B_\upsilon} |K_1(p, e_{n+N})|^n \, d\sigma \wedge dV = \int_{B_\upsilon} \cos^n \theta_{n+1} \, |H(p)|^n \, d\sigma \wedge dV.$$

$$= \frac{2c_{n+N-1}}{c_n} \int_{M^n} |H(p)|^n \, dV.$$

Substituting the value in r.h.s. from theorem 1.2.5 we get after simplifying

$$\int_{M^n} \langle \underline{H}(p), \underline{H}(p) \rangle^{\frac{1}{2}} dV \geq c_n$$

$$\left(\int_{S^{N-1}} \cos^n \theta_r \, d\sigma = \frac{2c_{n+N-1}}{c_n} \right) .$$

Proposition 1.2.11

Under the same hypothesis as the last theorem and further if n is even and the mean curvature normal vector does not vanish in any direction, then,

$$\int_{M^n} < \underline{H},\underline{H} >^{n/2} dV \geq c_n \cdot \left(\frac{\sum_i \beta_{2i}}{2} \right) .$$

Equality holds iff M^{2m} is embedded as a sphere in E^{n+1} .

Proof.

From Chern and Lashof [2] we have,

$$\int_{B_\upsilon} |K_n(p,e)| d\sigma \wedge dV \geq (\sum_i \beta_i(M)) . c_{n+N-1} .$$

Rewriting, we get,

$$\int_U K_n(p,e) d\sigma \wedge dV - \int_V K_n(p,e) d\sigma \wedge dV \geq (\sum_i \beta_i) c_{n+N-1} \qquad (1)$$

where,

$$U = \{(p,e) \in B_\upsilon | K_n(p,e) \geq 0\}$$

$$V = \{(p,e) \in B_\upsilon | K_n(p,e) < 0\} .$$

Also from Gauss-Bonnet theorem, we have,

$$\int_{B_\upsilon} K_n(p,e) d\sigma \wedge dV = \chi(M) c_{n+N-1} .$$

i.e.

$$\int_U K_n(p,e) d\sigma \wedge dV + \int_V K_n(p,e) d\sigma \wedge dV = \sum_i (-1)^i \beta_i(M) c_{n+N-1} \qquad (2)$$

$\therefore$ from (1) and (2)

$$\int_U K_n(p,e) d\sigma \wedge dV \geq (\sum_i \beta_{2i}) c_{n+N-1} . \qquad (3)$$

Choose a frenet frame in B_υ .

Then,
$$e = e_{n+N} = \sum_r \cos\theta_r \cdot \bar{e}_r ,$$

and,
$$\begin{aligned} K_1(p,e) &= K_1(p, \sum_r \cos\theta_r \cdot \bar{e}_r) \\ &= \sum_r \cos\theta_r \cdot K_1(p,\bar{e}_r) \\ &= \sum_{\alpha=1}^{N} \cos\theta_\alpha \cdot \mu_\alpha(p) \qquad (4) \end{aligned}$$

where,
$$\begin{aligned} \mu_\alpha(p) &= K_1(p, \bar{e}_{n+\alpha}) \\ &= (k_{n+\alpha,1} + \dots + k_{n+\alpha,n})\bar{e}_{n+\alpha} \qquad (5) \end{aligned}$$

$$\therefore \; K_1(p,e) = (k_{n+1,1} + \dots + k_{n+1,n})\cos\theta_{n+1} \cdot \bar{e}_{n+1} + \dots$$
$$\dots + (k_{n+N,1} + \dots + k_{n+N,n})\cos\theta_{n+N} \cdot \bar{e}_{n+N} .$$

$$< K_1(p,e), K_1(p,e) > = (k_{n+1,1} + \dots + k_{n+1,n})^2 \cos^2\theta_{n+1} + \dots$$
$$\dots + (k_{n+N,1} + \dots + k_{n+N,n})^2 \cos^2\theta_{n+N} .$$

$$\begin{aligned} \therefore \; < K_1(p,e), K_1(p,e) >^{n/2} &= \{(k_{n+1,1} + \dots + k_{n+1,n})^2 \cos^2\theta_{n+1} + \dots \\ &\quad + \dots \\ &\quad \dots + (k_{n+N,1} + \dots + k_{n+N,n})^2 \cos^2\theta_{n+N}\}^{\frac{1}{2}} \\ &\geqslant \{(k_{n+1,1} + \dots + k_{n+1,n})^n \cos^n\theta_{n+1} + \dots \\ &\quad + \dots \\ &\quad \dots + (k_{n+N,1} + \dots + k_{n+N,N})^n \cos^n\theta_{n+N}\} \end{aligned}$$

(using lemma 1.2.1).

$$\geq \{(k_{n+1,1} \cdots k_{n+1,n})\cos^n \theta_{n+1} + \cdots$$

$$\cdots + (k_{n+N,1} \cdots k_{n+N,n})\cos^n\theta_{n+N}\}$$

(using the hypothesis that the mean curvature normal does not vanish in any direction).

$$= K_n(p,e).$$

$$\therefore \int_{B_\upsilon} < K_1(p,e),K_1(p,e) >^{n/2} d\sigma \wedge dV \geq \int_U < K_1(p,e),K_1(p,e) >^{n/2} d\sigma \wedge dV$$

$$\geq \int_U K_n(p,e) d\sigma \wedge dV$$

$$\geq c_{n+N-1} \left(\sum_i \beta_{2i}\right) . \qquad (6)$$

From (4) we have,

$$\int_{B_\upsilon} < K_1(p,e),K_1(p,e) >^{n/2} d\sigma \wedge dV = \int_{B_\upsilon} \left(\sum_{\alpha,\beta} \mu_\alpha \mu_\beta \cos\theta_\alpha \cos\theta_\beta\right)^{n/2} d\sigma \wedge dV.$$

On integrating over the fiber the integral vanishes when $\alpha \neq \beta$

$\therefore$ we consider only cases for which $\alpha = \beta$.

$$\therefore \int_{B_\upsilon} < K_1(p,e),K_1(p,e) >^{n/2} d\sigma \wedge dV = \int_{B_\upsilon} (\mu_1^n \cos^n\theta_1 + \cdots + \mu_N^n \cos^n\theta_N) d\sigma \wedge dV$$

$$= \frac{2c_{n+N-1}}{c_n} \int_{M^n} \left(\sum_\alpha \mu_\alpha^n\right) dV .$$

$$\therefore \int_{B_\upsilon} < K_1(p,e),K_1(p,e) >^{n/2} d\sigma \wedge dV = \frac{2c_{n+N-1}}{c_n} \int_{M^n} < \underline{H},\underline{H} >^{n/2} dV \qquad (7)$$

From (6) and (7) we get,

$$\int_{M^n} \langle \underline{H},\underline{H} \rangle^{n/2} dV \geq c_n \left(\frac{\sum_i \beta_{2i}}{2} \right).$$

§3. <u>More results concerning the Integral of the length of the mean curvature vector</u>

One very important and striking result of surface theory is the Gauss-Bonnet theorem which states that: In a simply connected region A bounded by a closed curve C composed of n smooth arcs with exterior angles $\theta_1, \ldots, \theta_n$ at the vertices

$$\int_C k_g dS + \iint_A K\, dA = 2\pi - \sum_{i=1}^{n} \theta_i$$

where k_g is the geodesic curvature of the arcs and K the Gauss curvature of the surface. $\iint K\, dA$ was first introduced as the 'curvature integra' by Gauss, but is better known as the total curvature.

Rewriting the Gauss Bonnet theorem in more familiar notation we know that,

$$\frac{1}{2\pi} \int_{M^2} K\, dS = \chi(M^2) \qquad (1)$$

where $\chi(M^2)$ is the Euler characteristic of M^2.

It is the topological invariance of the result that makes it so remarkable.

Kuiper [2] has proved that

$$\frac{1}{2\pi} \int |k| dS \geq 2 + 2g \qquad (2)$$

In view of these two results it seems natural to examine

$$H(f) = \frac{1}{2\pi}\int_{M^2} < \underline{H},\underline{H} > dS \qquad (3)$$

and to seek a result analogous to the above results. Unlike the Gaussian curvature the mean curvature is not intrinsic, therefore (3) is not an invariant cf. Willmore [1]. To overcome this Willmore considered

$$H(M^2) = \inf_{f \in \mathcal{F}} H(f)$$

where the infimum is taken over the space of all C^∞ immersions of M^2 in a euclidean space. In [1,2,3] he proved,

Theorem 1.3.1

If $x : M^2 \to E^3$ is an immersion of an oriented compact surface into a three-dimensional euclidean space, then, the mean curvature vector $\underline{H}(p)$ satisfies

$$\int_{M^2} < \underline{H},\underline{H} > dV \geqslant 4\pi \qquad (4)$$

Equality holds iff M^2 is embedded as a euclidean sphere.
Thus for surfaces of genus zero $H(M^2) \geqq \chi(M^2)$.

Theorem 1.3.2

Consider now $T^2 = S' \times S'$ embedded as an anchor ring with radii of the generating circles a and b, then,

$$\int_{T^2} < \underline{H},\underline{H} > dV \geqslant 2\pi^2 \quad . \qquad (5)$$

Equality holds iff $a/b = \sqrt{2}$.

$$T(a,b) = (a + b\cos u)\cos v, \quad (a + b\cos u)\sin v, \quad b\sin u \qquad \begin{array}{l} 0 \leq u < 2\pi \\ 0 \leq v < 2\pi \end{array} .$$

He also conjectured that (5) was valid for any torus. More recently in [4] he proved,

Theorem 1.3.3

$x : M^2 \to E^3$ is such that $x(M^2)$ is generated by carrying a small circle in the normal plane to the curve at each point then, again (5) is true and equality holds under the same conditions with regard to the ratio of the radii.

Theorem 1.3.4 (Shihoma and Takagi [1])

$x : M^2 \to E^3$ is an isometric immersion of a connected compact orientable Riemannian manifold of class C^∞ in E^3. Suppose one of the principal normal curvatures of $x(M^2)$ is constant k everywhere, then,

$$\int_{M^2} < \underline{H},\underline{H} > dS \geqslant 2\pi^2$$

and equality holds iff x is an imbedding and $x(M^2)$ is congruent to the standard torus $T(\sqrt{2}/|k|, \ 1/|k|)$.

Note 1.3.5

It may be pointed out that one of the principal curvatures of each of the theorems 1.3.2 and 1.3.3 is a constant.

Recently Chen [15] proved

Theorem 1.3.6 (cf. Remark 3.1.5)

If $x : M^2 \to E^4$ be an isometric immersion of a flat torus M^2 into E^4 then, the mean curvature vector $\underline{H}(p)$ satisfies the following inequality

$$\int_{M^2} < \underline{H},\underline{H} > dV \geqslant 2\pi^2 .$$

Moreover if $< \underline{H},\underline{H} >$ is a constant then equality holds iff M^2 is imbedded as a Clifford flat torus in E^4.

Finally in an unpublished result due to Hombu [1] where he examines the normal variation of $\int_{M^2} < \underline{H},\underline{H} > dV$ he obtains the differential equation

$$\Delta H + 2H(H^2 - K) = 0 \qquad (6)$$

which must be satisfied for $\int_{M^2} < \underline{H},\underline{H} > dV$ to be stationary.

$\underline{H}$ is the mean curvature vector and Δ the Laplacian.

Note 1.3.7

The torus $(\sqrt{2}b, b)$ and the Clifford Torus both satisfy (6) and two equations like (6) respectively. In view of all these results therefore it seems reasonable to believe that the conjecture (5) must be true for all C^∞-immersions of any torus in E^{2+n} $(n \geq 1)$.

If the Willmore conjecture is true for all C^∞-embeddings of surfaces of genus one a possible way of seeking a solution to the integral of the mean curvature for surfaces of arbitrary genus is to investigate for relationships between $H(T_1)$, $H(T_2)$ $H(T_1 \# T_2)$,

where

$$f : T_1 \to E^3$$

$$g : T_2 \to E^3$$

$$M^2 = T_1 \# T_2 \to E^3 \qquad (\# \text{ is the connected sum of } T_1 \text{ and } T_2)$$

For the TAC it is known that

$$\tau(T_1 \# T_2) \geq \tau(T_1) + \tau(T_2) - 2.$$

A similar result for the mean curvature seems to be rather elusive mainly because of its topological invariance. However for surfaces of genus zero

$$H(M_1 \# M_2) \geq H(M_1) + H(M_2) - 2.$$

Equality holds iff M_1 and M_2 are the euclidean round spheres.

Likewise the TAC for product immersions is known. i.e. $\tau(f \times g) = \tau(f).\tau(g)$, (cf. Kuiper [1]), and one wonders if anything could be gotten for the mean curvature of product immersions.

Remark 1.3.8.

From a theorem of Tompkins [1] we know that there do not exist any isometric immersions of compact n-dimensional flat manifolds in a euclidean space of dimension $< 2n$. However there do exist isometric flat immersions in a euclidean space of dimension $\geq 2n$,

viz: the n fold product of circles in coordinates

$$x_{2\alpha} = \cos\theta_\alpha \qquad \alpha = 0,1,\ldots,(n-1)$$

$$x_{2\alpha+1} = \sin\theta_\alpha .$$

For such immersions it seems that,

$$\int_{M^n} <\underline{H},\underline{H}>^{n/2}\, dV \geq c_{2n-1} \qquad \text{may be true.}$$

Certainly in the case of $n = 2$ the above is valid for we know of the existence of the Clifford flat Torus which satisfies the above and in fact the inequality is replaced by an equality. This result is only an improvement in the case of three or four dimensional manifolds, while for higher dimensional manifolds the result of Theorem 1.2.10 is indeed much stronger.

ex (i) $\quad x : T^3 \to E^6$

given by, $(a\cos u,\ a\sin u,\ b\cos v,\ b\sin v,\ c\cos t,\ c\sin t)$.

Then by direct computation we have,

$$\omega_1 = a\,du\,, \qquad \omega_2 = b\,dv, \qquad \omega_3 = c\,dt$$

$$\omega_{14} = -\frac{1}{\sqrt{3}}\,du \qquad \omega_{24} = -\frac{1}{\sqrt{3}}\,dv \qquad \omega_{34} = -\frac{1}{\sqrt{3}}\,dt$$

$$\omega_{15} = 0 \qquad \omega_{25} = \frac{1}{\sqrt{2}}\,dv \qquad \omega_{35} = -\frac{1}{\sqrt{2}}\,dt$$

$$\omega_{16} = du \qquad \omega_{26} = -\frac{1}{\sqrt{6}}\,dv \qquad \omega_{36} = -\frac{1}{\sqrt{6}}\,dt$$

$$\therefore\quad A_{411} = -\frac{1}{\sqrt{3}a} \qquad A_{422} = -\frac{1}{\sqrt{3}b} \qquad A_{433} = -\frac{1}{\sqrt{3}c}$$

$$A_{511} = 0 \qquad A_{522} = \frac{1}{\sqrt{2}\,b} \qquad A_{533} = -\frac{1}{\sqrt{2}\,c}$$

$$A_{611} = \frac{2}{\sqrt{6}b} \qquad A_{622} = -\frac{1}{\sqrt{6}\,b} \qquad A_{633} = -\frac{1}{\sqrt{6}c}$$

$$dV = *1 = abc\ du\ dv\ dt.$$

$$\underline{\underline{H}} = \frac{1}{3}\left[-\frac{1}{\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)e_4 + \frac{1}{\sqrt{2}}\left(\frac{1}{b}-\frac{1}{c}\right)e_5 + \frac{1}{\sqrt{6}}\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)e_6\right]$$

when $a = b = c$.

$$\underline{\underline{H}} = -\frac{1}{\sqrt{3}}\,e_4 \ .$$

$$\therefore \qquad \int_{T^3} < \underline{\underline{H}},\underline{\underline{H}} >^{3/2} *1 = \frac{1}{3\sqrt{3}}\int_{T^3} *1$$

$$= \frac{8\sqrt{3}}{9}\pi^3 \qquad (> c_5 = \pi^3),$$

ex (ii) $\quad f : M^4 \to E^8$

$$(\cos u,\ \sin u,\ \cos v,\ \sin v,\ \cos x,\ \sin x,\ \cos y,\ \sin y)$$

$$0 \leq u,\ v,\ x,\ y < 2\pi \ .$$

Again as in (i) by direct computation

$$\omega_1 = du, \quad \omega_2 = dv, \quad \omega_3 = dx, \quad \omega_4 = dy.$$

$$dV = *1 = dudvdxdy.$$

$$-\omega_{15} = \omega_{16} = \omega_{17} = \omega_{18} = \frac{du}{2}$$

$$\omega_{25} = -\omega_{26} = \omega_{27} = \omega_{28} = -\frac{dv}{2}$$

$$\omega_{35} = \omega_{36} = -\omega_{37} = \omega_{38} = -\frac{dx}{2}$$

$$\omega_{45} = \omega_{46} = \omega_{47} = -\omega_{48} = -\frac{dy}{2}$$

and

$$-A_{511} = A_{611} = A_{711} = A_{811} = \tfrac{1}{2}$$

$$A_{522} = -A_{622} = A_{722} = A_{822} = -\tfrac{1}{2}$$

$$A_{533} = A_{633} = -A_{733} = A_{833} = -\tfrac{1}{2}$$

$$A_{544} = A_{644} = A_{744} = -A_{844} = -\tfrac{1}{2}$$

$$\therefore \quad \underline{H} = \tfrac{1}{4}(-2e_5) = -\tfrac{1}{2}e_5 \; .$$

Scalar curv. = 0.

$$\int_{M^4} <\underline{H},\underline{H}>^2 * 1 = \int_{M^4} \frac{1}{16} * 1$$
$$= \frac{1}{16} \cdot 16\pi^4$$
$$= \pi^4 \qquad (> c_7 = \frac{\pi^4}{3}) \; .$$

In the most general case, that of an n-dimensional manifold Chen [19] proves

Theorem 1.3.8

Let M^n be an n-dimensional closed manifold immersed in a euclidean space of dimension $(n+N)$ with non-negative scalar curvature. Then,

$$\int_{M^n} <\underline{H},\underline{H}>^{n/2} dV \geq \lambda \, . \, \beta(M^n) \tag{1}$$

where

$$\lambda = \begin{cases} (4n^n)^{-\frac{1}{2}} \; c_n & \text{if } n \text{ is even} \\ (2n^n \; c_{n-1} \; c_{2n+N-1})^{-\frac{1}{2}} (c_{2n})^{\frac{1}{2}} \; c_{n+N-1} & \text{if } n \text{ is odd.} \end{cases}$$

and $\beta(M^n)$ = sum of Betti numbers of M^n .

It is necessary to first prove,

Lemma 1.3.9

Let $a_1, \ldots, a_n$ be N non-negative constants and S_o^{N-1} the unit hypersphere of E^N centered at the origin. Let the function f on S_o^{N-1} be defined by

$$f(x) = \sum_{i=1}^{N} a_i x_i^2 \qquad (2)$$

where $x = (x_1, \ldots, x_N)$

For an even positive integer 2m (say)

$$\left(\sum_{i=1}^{N} a_i\right)^m \geq \frac{c_{2m}}{? c_{2m+N-1}} \int_{S_o^{N-1}} \left(\sum_{i=1}^{N} a_i x_i^2\right)^2 dS_o^{N-1}$$

equality holds iff either N = 1 or n = 1.

Proof

For non-negative even integers $e_1, \ldots, e_N$

$$\int_{S_o^{N-1}} (x_1^{e_1} \ldots x_N^{e_N})\, dS^{N-1} = \frac{2\Gamma\left(\frac{1+e_1}{2}\right) \ldots \Gamma\left(\frac{1+e_N}{2}\right)}{\Gamma\left(\frac{e_1 + \ldots + e_N + N}{2}\right)} \qquad (4)$$

(generalised formula using Gamma functions).

Also,

$$\Gamma\left(\frac{1+e_1}{2}\right) \ldots \Gamma\left(\frac{1+e_N}{2}\right) \leq \Gamma\left(\frac{1+e_1+\ldots+e_N}{2}\right) \Gamma(\tfrac{1}{2})^{N-1} \qquad (5)$$

equality holds in (5) iff (N-1) of $e_1, \ldots, e_N$ are zero.

From (4) and (5)

$$\int_{S_o^{N-1}} \left(\sum_i a_i (x_i)^2\right)^m dS_o^{N-1} = \int_{S_o^{N-1}} (a_1 x_1^2 + \ldots + a_N x_N^2)^m dS_o^{N-1}$$

$$\leq \int_{S_o^{N-1}} \left(\sum_i a_i\right)^m (x_1^2 + \ldots + x_N^2)^m dS_o^{N-1}$$

$$\leq \int_{S_o^{N-1}} (\sum_i a_i)^m \, (x_i)^{2m} \, dS_o^{N-1}$$

$$= (\sum_i a_i)^m \int_{S_o^{N-1}} (x_i)^{2m} \, dS_o^{N-1}$$

$$= \frac{(\sum_i a_i)^m \, 2\Gamma\left(\frac{1+2m}{2}\right) \Gamma(\frac{1}{2})^{N-1}}{\Gamma\left(\frac{2m+N}{2}\right)}$$

$$= \left(\sum_{i=1}^{N} a_i\right)^m 2 \cdot \frac{c_{2m+N-1}}{c_{2m}}$$

(N-1) of the e_N's are zero.

$$\therefore \quad \left(\sum_{i=1}^{N} a_i\right)^m \geq \frac{c_{2m}}{2c_{2m+N-1}} \int_{S_o^{N-1}} (\sum_i a_i (x_i^{\,2}))^m \, dS_o^{N-1}$$

If equality holds above then, the inequalities in the last few steps are all equations. Hence either $m = 1$ or $N = 1$.

Converse is immediate.

Lemma 1.3.11

For each normal vector e to M we have,

$$S(e)^n \geq |K_n(e)| \sqrt{n}^n \quad , \qquad (8)$$

Equality sign of (8) holds iff $A(e)^2 = \mu I_n$ for some constant μ .

$A(e)$ is the second fundamental form of the immersion in the direction of e.

Proof

The second fundamental form is self-adjoint matrix $\therefore$ choosing a suitable frame, we can write,

$$A(e) = \begin{pmatrix} k_1(0) & & & \\ & k_2(0) & & \\ & & \ddots & \\ & & & k_n(e) \end{pmatrix} \qquad (9)$$

$$\therefore \qquad S(e)^2 = \sum_{i=1}^{n} k_i(e)^2$$

$$\geq n\sqrt[n]{[k_1(e) \ldots k_n(e)]^2}$$

$$= n|K_n(e)|^{2/n}$$

$$\therefore \qquad (S(e))^n \geq \sqrt{n^n}\,|K_n(e)| \, .$$

If the equality holds in (8) then,

$$k_1(e)^2 = \ldots = k_n(e)^2$$

$$\Longrightarrow A(e) = \mu I_n \quad \text{for} \quad \mu = k_1(e)^2.$$

Conversely,

if $\quad A(e)^2 = \mu I_n$

and $\quad A(e)$ is as in (9)

then $\quad k_1(e)^2 = \ldots = k_n(e)^2$

$$\Longrightarrow S(e)^n = \sqrt{n^n}\,|K_n(e)| \, .$$

<u>Lemma 1.3.12</u>

Let $x : M^n \to E^{n+N}$ be an immersion of M^n in a euclidean space of dim-$(n+N)$ and let S_x be the sphere of all unit normal vectors to M at $x \in M$. Then,

$$S^n \geq \begin{cases} \dfrac{c_n}{2c_{n+N-1}} \displaystyle\int S(e)^n \, dS_x & \text{for n even} \\[2ex] \dfrac{c_{2n}}{2c_{N-1}\, c_{2n+N-1}} \displaystyle\int S(e)^n \, dS_x & \text{for n odd} \end{cases} \tag{10}$$

If the equality sign holds in (10) then,

(i) either $n = 2$ or $N = 1$ whenever n is even

and (ii) $N = 1$ whenever n is odd.

Proof

$$S(e)^2 = \sum_{i,j} \left(\sum_r \cos \theta_r A_{rij} \right)^2$$

$$= \sum_{r,s} \left[\left(\sum_{i,j} A_{sij} A_{rij} \right) \cos \theta_r . \cos \theta_s \right] \qquad (11)$$

Choosing a suitable frame field $(p, e_1 \ldots, e_n, \bar{e}_{n+1}, \ldots, \bar{e}_{n+N})$

$$S(e)^2 = \sum_{r=n+1}^{n+N} \lambda_r \cos^2 \theta_r \qquad (12)$$

$$\lambda_{n+1} \geq \lambda_{n+2} \geq \ldots \geq \lambda_{n+N}$$

$$\lambda_r = \sum_{i,j} A_{rij} A_{rij} = S_r^2 \qquad (13)$$

Using lemmas 1.3.9 and 1.3.10 we get (10) and when the equality sign holds in (10) from lemma 1.3.9 $n = 2$ or $N = 1$ whenever n is even, and from lemma 1.3.10 $N = 1$ whenever n is odd.

Lemma 1.3.13

If $x : M^n \to E^{n+N}$ is an immersion with non-negative scalar curvature, Then, since,

$$R = n^2 ||H||^2 - S^2 \qquad \text{(cf. pg. 6)}$$

we have, $\quad n^2 ||H||^2 - S^2 \geq 0$

i.e. $\quad n^2 ||H||^2 \geq S^2.$

In particular $\quad n ||H|| \geq S.$

$$\therefore \quad \int_{M^n} n^n < \underline{H}, \underline{H} >^{n/2} dV \geq \int_{M^n} S^n \, dV \qquad (14)$$

$\therefore$ from (8), (10) and (14) we get,

$$\int_{M^n} < \underline{H},\underline{H} >^{n/2} dV \geq \begin{cases} \dfrac{c_n}{2\sqrt{n^n}\, c_{n+N-1}} \displaystyle\int_{M^n} K_n^*(p)dV, & n \text{ even} \\ \dfrac{c_{2n}}{2n^n\, c_{N-1}\, c_{2n+N-1}} \displaystyle\int_{M^n} K_n^*(p)dV, & n \text{ odd} \end{cases}$$

Using Chern and Lashof [2] we get,

$$\int_{M^n} < \underline{H},\underline{H} >^{n/2} dV \geq \lambda . \beta(M^n) \tag{17}$$

λ is as in Theorem 1.3.8.

If the equality sign holds in (17), then, $n||H|| = S$ and therefore the equality sign holds in (10). Therefore by lemmas 1.3.12 and 1.3.13 $R = 0$ and either $N = 1$ or $n = 2$.

If $n = 2$ M is a flat torus. $\therefore$ $\beta(M) = 4$.

This $\Longrightarrow \displaystyle\int_{M^2} < \underline{H},\underline{H} > dV = c_2$

$\Longrightarrow M^2$ is diffeomorphic to S^2 (Chen [19]), which is a contradiction because $R = 0$.

$\therefore$ $N = 1$ and $R = 0$. This is also impossible because there do not exist any closed hypersurfaces in E^{n+1} with scalar curvature $= 0$ (Remark 1.3.8).

§4 Intermediary Curvatures

Having dealt with the "first" and "last" curvatures so to speak, we will now look at the intermediary curvatures.

Theorem 1.4.1

$x : M^{2m} \to E^{2m+1}$ is an immersion of an oriented 2m-dimensional closed manifold in E^{2m+1} such that,

$$\{p \in M^n | g(p) \geq 0\} \supseteq \{p \in M^n | K_{2m}(p) \geq 0\}$$

Then,

$$\int_{M^{2m}} < K_m, K_m > dV \geqslant c_{2m}\left(\frac{\Sigma \beta_{2i}}{2}\right).$$

Equality holds iff x is a tight imbedding. K is the G-K curv. and

$$g(p) = K_m(p)^2 - K_{2m}(p).$$

(NOTE: In his paper Chen [15] uses the term "minimal" which is rather misleading for it does not mean vanishing mean curvature. Instead, an immersion is called a "minimal/tight imbedding" if the result in Cor. 1.2.8 has strict equality).

Proof

$$\vartheta : B_\upsilon \to S_o^{2m}$$

Let $\eta : M^{2m} \to S_o^{2m}$. If $d\Sigma_{2m}$ is the vol. of S_o^{2m}, then

$$\eta^* d\Sigma_{2m} = K(p)dV$$

and

$$\vartheta^* d\Sigma_{2m} = -K(p)d\sigma \wedge dV$$

$$= -2K(p)dV.$$

$$\therefore \qquad |\vartheta^* d\Sigma_{2m}| = 2|K(p)|dV.$$

Using Hopf's Index Theorem

$$\int_{M^{2m}} K(p)dV = \frac{c_{2m}}{2}\chi(M^{2m}).$$

also

$$\int_{M^{2m}} K(p)dV \geqslant \frac{c_{2m}}{2}(\Sigma\beta_i)$$ from Chern and Lashof [2].

If

$$U = \{(p,e) \in B_\upsilon | K_{2m}(p,e) = K(p) \geqslant 0\}$$

$$V = \{(p,e) \in B_\upsilon | K_{2m}(p,e) = K(p) < 0\}$$

then rewriting the above two equations we get

$$\int_U K(p)dV + \int_V K(p)dV \geq \frac{c_{2m}}{2} \left(\sum_i (-1)^i \beta_i\right) \tag{1}$$

$$\int_U K(p)dV + \int_V K(p)dV = \frac{c_{2m}}{2} \left(\sum_i \beta_i\right) \tag{2}$$

$\therefore$ from (1) and (2)

$$\int_U K(p)dV \geq c_{2m}\left(\frac{\sum_i \beta_{2i}}{2}\right)$$

But by hypothesis $K_m(p)^2 \geq K_{2m}(p)$.

$$\int_{M^{2m}} < K_m(p), K_m(p) > dV \geq \int_U < K_m(p), K_m(p) > dV$$

$$\geq \int_U K_{2m}(p)dV$$

$$\geq c_{2m}\left(\frac{\sum_i \beta_{2i}}{2}\right) .$$

<u>Theorem 1.4.2</u>

If $x : M^{2m} \rightarrow E^{2m+1}$ be an immersion of a closed 2m-dimensional manifold in E^{2m+1} with non-negative principal curvature, then,

$$\int_{M^{2m}} < K_m(p), K_m(p) > dV \geq \left(\sum_i \beta_i\right) \frac{c_{2m}}{2} .$$

Equality holds iff M^{2m} is embedded as a hypersphere.

<u>Proof</u>

By lemma 1.2.1 $K_m^2(p) \geq K(p) \geq 0$ and equality holds iff all the principal curvatures are equal.

$\therefore$ by the previous theorem

$$\int_{M^{2m}} < K_m(p), K_m(p) > dV \geq \tfrac{1}{2} c_{2m} \left(\sum_i \beta_{2i} \right)$$

By a theorem of Chern and Lashof [2] we know that all the odd dimensional Betti numbers vanish. Hence,

$$\int_{M^{2m}} < K_m(p), K_m(p) > dV \geq \tfrac{1}{2} c_{2m} \left(\sum_i \beta_i \right)$$

when equality holds in the above equation.

$K_m^2(p) = K(p) \Longleftrightarrow$ all the principal curvatures are equal

$\Longleftrightarrow$ every point is an umbilic

$\Longleftrightarrow M^{2m}$ is embedded as a hypersphere.

The last result (Theorem 1.4.3) in this section seems to be a generalization of the Hilbert and Liebmann Theorems.

Hilbert: The only compact surfaces with constant Gauss curvature are spheres.

Liebmann: The only ovaloids with constant mean curvature are spheres.

Theorem 1.4.3 (Gardner [1])

Let $x : M^n \to E^{n+1}$ be an immersion of a compact oriented n-dimensional manifold in a euclidean space, and if for any i, $1 \leq i \leq n-1$

σ_i = constant ($\neq 0$) and σ_{i+1} = constant ($\neq 0$) then,

it implies that $x(M^n)$ is a euclidean sphere.

σ_i is the i^{th} elementary symmetric function of the principal curvature.

NOTE: The case for i = 0 reduces to the problem of classifying hypersurfaces with constant mean curvature since $\sigma_0 = 1$ (and is therefore always a constant).

We will have a brief look at submanifolds with constant mean curvature in Chapter III.

CHAPTER II

§0. Preliminaries

(Most of the definitions for this chapter and Chapter III are taken from Hicks [1], Kobayashi and Nomizu I, II [1], Singer and Thorpe [1]).

Let M be a C^∞ riemannian n-manifold. Then a connexion on M is an operator ∇ (often also called covariant differentiation) which assigns to each X,Y vector fields on M (denoted by $X, Y \in \mathfrak{X}(M)$) a vector field $\nabla_X Y$ in the same domain.

If $f \in \mathfrak{F}(A)$, $\mathfrak{F}(A) = \{f \mid f$ real valued function on $A\}$, then the connexion ∇ satisfies

(i) $\nabla_X(fY + gZ) = (Xf)Y + f\nabla_X Y + (Xg)Z + g(\nabla_X Z).$

(ii) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$ $\qquad Z \in \mathfrak{X}(M).$

$(\nabla_X Y)_p$ depends on X_p and the values of Y on some integral curve that fits X. If $(e_1, \ldots, e_n)$ is some field about p

let
$$X_p = \sum_{i=1}^{n} a_i(p)(e_i)_p$$

$$Y = \sum_{j=1}^{n} b_i \; e_j \; .$$

Then,
$$(\nabla_X Y)_p = [\nabla_{\Sigma a_i(p)e_i} \Sigma b_j e_j]_p$$

$$= \Sigma a_i(p)[\nabla_{e_i} \Sigma b_j e_j]_p$$

$$= \Sigma a_i(p)(e_i b_j)(e_j)_p + (\Sigma a_i(p))(\Sigma b_j)_p(\nabla_{e_i} e_j)_p$$

$$= (Xb_j)_p(e_j)_p + \Sigma a_i(p).\Sigma b_j(p)(\nabla_{e_i} e_j)_p \; .$$

If $(\nabla_{e_i} e_j)_p$ is known then $(\nabla_X Y)_p$ can be fully determined.

Given a riemannian manifold M^n and $< , >$ (= inner product) there exists a unique connexion $\nabla \ni$

(i) $X < Y,Z > \; = \; < \nabla_X Y, Z > + < Y, \nabla_X Z >$

(ii) $\nabla_X Y - \nabla_Y X \; = \; [X,Y]$ (torsion free)

for all C^∞ vector fields $X,Y,Z \in \mathfrak{X}(M)$.

<u>Induced connexion 2.0.1</u>

Let M^n be an n-dimensional orientable riemannian manifold immersed in a Riemannian manifold of dim-(n+k). If $\bar{\nabla}$ denotes the connexion of $\bar{M}$, then, $\nabla_X Y = [\bar{\nabla}_X Y]^T$ is the induced connexion on M. It is the projection of the connexion on $\bar{M}$.

$$X < Y,Z > \; = \; < \bar{\nabla}_X Y, Z >_{\bar{M}} + < Y, \bar{\nabla}_X Z >_{\bar{M}}$$

$$= \; < [\bar{\nabla}_X Y]^T, Z > + < Y, [\bar{\nabla}_X Z]^T >$$

$$= \; < \nabla_X Y, Z > + < Y, \nabla_X Z >$$

$$\therefore \qquad \nabla_X Y = [\bar{\nabla}_X Y]^T .$$

∇ preserves inner products on TM.

$$< \nabla_X Y, Z > + < Y, \nabla_X Z > = <[\bar{\nabla}_X Y]^T, Z > + < Y, [\bar{\nabla}_X Z]^T >$$

$$= < \bar{\nabla}_X Y, Z > + < Y, \bar{\nabla}_X Z >$$

$$= \bar{\nabla}_X < Y,Z >$$

$$= \nabla_X < Y,Z > .$$

To show that it is torsion free

$$\nabla_X Y - \nabla_Y Z - [X,Y] \; = \; (\bar{\nabla}_X Y)^T - (\bar{\nabla}_Y X)^T - [X,Y]$$

$$= \; (\bar{\nabla}_X Y)^T - (\bar{\nabla}_Y X)^T - [X,Y]^T$$

$$= (\bar\nabla_X Y - \bar\nabla_Y X - [X,Y])^T$$

$$= 0.$$

$\{[X,Y] \;=\; XY - YX$ (lie bracket of X and Y)$\}$

$$\therefore \quad \bar\nabla_X Y = [\bar\nabla_X Y]^T + [\bar\nabla_X Y]^N$$

$$= \nabla_X Y + B(X,Y) \qquad (1)$$

(i) is called the GAUSS EQUATION. B is called the second fundamental form of the immersion and it is a symmetric bilinear mapping of $TM \times TM \to NM$.

For $V \in \mathfrak{X}(NM)$

$$D_X V = [\bar\nabla_X V]^N$$

D is the induced connection on NM.

$$X < V_1, V_2 > \;=\; < \bar\nabla_X V_1, V_2 > + < V_1, \bar\nabla_X V_2 >$$

$$= < [\bar\nabla_X V_1]^N, V_2 > + < V_1, [\bar\nabla_X V_2]^N >$$

$$= < D_X V_1, V_2 > + < V_1, D_X V_2 >$$

$$D_X V = [\bar\nabla_X V]^N .$$

To show that D defines a connection in the normal bundle

$$< D_X V_1, V_2 > + < V_1, D_X V_2 > \;=\; < [\bar\nabla_X V_1]^N, V_2 > + < V_1, [\bar\nabla_X V_2]^N >$$

$$= < \bar\nabla_X V_1, V_2 > + < V_1, \bar\nabla_X V_2 >$$

$$= \bar\nabla_X < V_1, V_2 >$$

$$= D_X < V_1, V_2 >$$

$$\bar\nabla_X N = A_N X + D_X N \qquad (2)$$

The above equation is called the WEINGARTEN FORMULA. $A_N X$ is the tangential component of $\bar{\nabla}_X N$ and it is a symmetric bilinear map of $T_pM \times N_pM \to T_pM$. (It is also often called the 'shape operator').

A and B are related by the following

$$< A_N(X), Y > \;=\; - < B(X,Y), N >$$

The mean curvature of the immersion being the trace of the second fundamental form is given by

$$\underline{H} = \frac{1}{n} \sum_{r=n+1}^{n+k} B(X,X)N_r \qquad N_r \in N(M).$$

<u>Proposition 2.0.2</u>

If $M^n \xrightarrow{f} \bar{M}^{n+k} \xrightarrow{g} \bar{\bar{M}}^{n+k+k'}$ is a string of isometric immersions and $X, Y \in T(M^n)$, then,

$$B_{g \circ f}(X,Y) = B_f(X,Y) + B_g(X,Y).$$

<u>Proof</u>

Let ∇, $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ be the induced connections on M and $\bar{M}$ and the connection $\bar{\bar{M}}$ respectively. Then,

$$\begin{aligned} B_{g \circ f}(X,Y) &= \text{nor. cpt.}[\bar{\bar{\nabla}}(X,Y)] \\ &= \text{nor. cpt.}[\bar{\nabla}(X,Y) + B_g(X,Y)] \\ &= \text{nor. cpt.}[\nabla(X,Y) + B_f(X,Y)] + B_g(X,Y) \\ &= B_f(X,Y) + B_g(X,Y). \end{aligned}$$

§1. <u>Definitions and results on the Laplacian of a function</u>

<u>Definition 2.1.2</u>

Let $f : M^n \to \bar{M}^{n+k}$ and $\{e_1, \ldots, e_n\}$ a basis of TM^n.

Then $\Delta f = \sum_i \nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f$ is called the Laplacian of f.

In local coordinates $(y^\alpha = f^\alpha(u^1 \ldots u^n) \quad \alpha = 1,2,\ldots,(n-k)$.

$$\Delta f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial u_i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial u^j}\right)$$

<u>Proposition 2.1.2</u>

If $f : M^n \to E^{n+k}$ is an isometric immersion then $\Delta f = H$ (H is the mean curvature).

Since covariant differentiation is the same as partial differentiation in E^{n+k} we have,

$$\begin{aligned} \nabla_{e_i} \nabla_{e_i} f - \nabla_{\nabla_{e_i} e_i} f &= B(e_i\, e_i) \\ &= H. \end{aligned}$$

$$\Delta f = (\Delta f^1 \ldots \Delta f^n).$$

<u>Definition 2.1.3</u>

$f : M^n \to E^{n+k}$ is called harmonic if $\Delta f = 0$.

<u>Corollary 2.1.4</u>

$f : M^n \to E^{n+k}$ is minimal iff each of the coordinate functions f^i are harmonic.

<u>Theorem 2.1.5</u> (Myres [1], also Kobayashi and Nomizu II [1])

There does not exist a minimal immersion of a compact manifold in a euclidean space.

<u>Proof</u>

Suppose there does exist a minimal immersion of a compact manifold in a euclidean space, then

$$\Delta f = 0.$$

But the only harmonic maps on a compact manifold are the constant maps. Hence supposition must be false.

$\therefore$ $\nexists$ any minimal immersion of a compact manifold in a euclidean space.

In the case of the ambient space being non-euclidean the above result may no longer be true, since

(i) the Laplacian of the function f does not coincide with the mean curvature normal

and (ii) we know of the existence of minimal submanifolds in spaces of constant curvature.

§2. Submanifolds of a Euclidean Hypersphere

Theorem 2.2.1 Chen [10]

Let M^n be a closed (compact without boundary) submanifold of E^{n+k}. Then M is contained in a hypersphere of E^{n+k} centred at $c \in E^{n+k}$ iff either $(X - c).H \geq -1$ or $(X - c).H \leq -1$.

X is the position vector field of M in E^{n+k} and H is the mean curvature normal.

Proof

Let $f = (X - c).(X - c)$ (i)

where c is a fixed vector in E^{n+k}. Then the Laplacian of f is given by

$$\Delta f = 2n\{1 + (X - c).H\} \qquad (2)$$

$$\Big[\ \Delta f = \Delta(X - c).(X - c).$$

$$= \nabla_{e_i} \nabla_{e_i} < X - c, X - c >$$

$$= 2\nabla_{e_i} < \nabla_{e_i} (X - c), X - c >$$

$$= 2\{< \nabla_{e_i} \nabla_{e_i} (X - c), (X - c) > + < \nabla_{e_i} (X - c), \nabla_{e_i} (X - c) >$$

$$= 2\{nH.(X-c) + n\} \qquad \text{since } \Delta(X-c) = H$$

$$= 2n\{1 + (X-c).H\} \quad \Big] .$$

If $(X-c).H \geqslant -1$ or $(X-c).H \leqslant -1$, then $\Delta f \geqslant 0$ or $\Delta f \leqslant 0$.

$\therefore$ By Hopf's Lemma (Kobayashi and Nomizu II [1]) we get $\Delta f = 0$

$\Longrightarrow$ f is a constant

$\Longrightarrow$ M is contained in a hypersphere of E^{n+k} centred at c.

Conversely,

if M is contained in a hypersphere centred at c then, f is a constant.

$\therefore$ $\Delta f = 0$ and (2) then gives $(X-c).H = -1$.

Hence the theorem.

§3. Minimal Immersions in Spheres

The ambient space in this section will be assumed to have non-zero constant curvature. We consider $f : M^n \to \bar{M}^{n+k}$ where the immersion is isometric and $\bar{M}^{n+k}$ has constant curvature.

If $(x, \ldots, x^n)$ and $(y^1, \ldots, y^{n+k})$ are the local coordinates of M and $\bar{M}$ then, locally,

$$y^\alpha = f^\alpha(x^1,\ldots,x^n) \qquad \alpha = 1,2,\ldots,(n+k).$$

Let $\bar{g}_{\alpha\beta}$ be the metric tensor of $\bar{M}$ then the induced metric g_{ij} on M is given by

$$g_{ij} = \bar{g}_{\alpha\beta}\, f^\alpha_i\, f^\beta_j$$

where the partial differentiation $\partial_j f^\alpha$ is written as f^α_j. We will write $f^\alpha_{,j}$ for the covariant differentiation $\nabla_j f^\alpha$.

As usual the Greek indices will range from 1 to n+k and the Roman from 1 to n.

If $R_{ijk\ell}$, $\bar{R}_{\alpha\beta\gamma\delta}$ are the curvature tensors of M and $\bar{M}$, and Γ^i_{jk}, $\bar{\Gamma}^\alpha_{\beta\gamma}$ the Christoffel symbols, we have the following formulae (Eisenhart [1])

$$\nabla_j f^\alpha_i = f^\alpha_{i,j} = \partial_j f^\alpha_i - \Gamma^h_{ji} f^\alpha_h + \bar{\Gamma}^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j$$

$$\nabla_j N^\alpha = N^\alpha_{,j} = \partial_j N^\alpha + \bar{\Gamma}^\alpha_{\beta\gamma} f^\beta_j N^\gamma.$$

Now,

$$\nabla_j f^\alpha_i = h_{ij} n^\alpha$$

$$\nabla_j n^\alpha = - h^i_j f^\alpha_i + \Sigma h_{\alpha\beta j} n^\beta$$

$$R_{ijk\ell} = (-h_{i\ell} h_{jk} + h_{ik} h_{j\ell}) + \bar{R}_{\alpha\beta\gamma\delta} f^\alpha_i f^\beta_j f^\gamma_k f^\delta_\ell .$$

For a space of constant curvature c (say)

$$\bar{R}_{\alpha\beta\gamma\delta} = c(\bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta})$$

$\therefore$ for immersions in spheres we have,

$$R_{ijk\ell} = c(g_{i\ell} g_{jk} - g_{ik} g_{j\ell}) - (h_{i\ell} h_{jk} - h_{ik} h_{j\ell})$$

transvecting the above equation with $g^{i\ell}$ we get,

$$R_{jk} = c(n-1)g_{jk} - (nHh_{jk} - g^{i\ell} h_{ik} h_{j\ell}) \qquad (1)$$

where

$$\underline{H} = \frac{1}{n}\sum_r \Big(\sum_{i,j} g^{ij} h_{ij}\Big)\underline{e}_r \quad \text{(mean curvature normal).}$$

<u>Corollary 2.3.1</u>

If a Riemannian manifold admits a minimal immersion in a space of constant curvature, then, $R - (n-1)g$ is a positive semi-definite tensor.

<u>Proof</u> (follows directly from (1)

Because $(g^{i\ell} h_{ik} h_{j\ell})$ is a positive semi-definite tensor.

<u>Theorem 2.3.2</u>

If $x : M^n \to \bar{M}^{n+k}$ is an isometric immersion with $\Delta x = \lambda$ where λ is a non-zero constant then x realizes a minimal immersion in the hypersphere $S^{n+k-1}(\sqrt{n}/\lambda)$ and conversely, if x realizes a minimal immersion then $\Delta x = \lambda x$ up to a parallel displacement.

Proof

Suppose x is a minimal immersion with $\Delta x = \lambda x$ then by Proposition

2.1.2 $$\Delta x = \sum_{r=n+1}^{n+k} He_r$$

$\therefore$ we have,

$$x = \frac{1}{\lambda} \Sigma H . e_r \qquad (2)$$

Differentiating (2) we get,

$$x_{,j} = \frac{1}{\lambda}(He_r)_{,j}$$

$$= \frac{1}{\lambda}\{\Sigma H h_j^i \, x_i + \Sigma e_r \, h_j\}$$

Taking the scalar product with x_i we get,

$$g_{ij} = \frac{1}{\lambda}\{H h_j^i \, g_{ij}\}$$

i.e.

$$g_{ij} = \frac{1}{\lambda} H h_{ij} \, .$$

Transvecting with g_{ij}

$$n = \frac{1}{\lambda}||H||^2 \qquad (3)$$

So from (2)

$$||x||^2 = \frac{1}{\lambda^2}||H||^2$$

i.e. $$||x||^2 = \frac{1}{\lambda^2} n\lambda = \frac{n}{\lambda}$$

i.e. $$||x|| = \frac{n}{\lambda} = a \text{ (say)} \qquad (4)$$

$\therefore$ $x(M^n)$ is contained in a sphere of radius $\sqrt{\frac{n}{\lambda}}$.

Now, $$n^\alpha_{,j} = -h_j^m \, f_m^\alpha - \bar{\Gamma}^\alpha_{\mu\upsilon} \, f_j^\mu \, n^\upsilon \, .$$

Since x is normal to $x(M^n)$ choose $\underline{e}_{n+k}$ along $\frac{1}{a}\underline{x}$, then

$$-h^i_j\, x_i + \sum_{\alpha=1}^{k-1} h_j\, e_\alpha = \frac{1}{a} x_j \qquad (5)$$

Taking scalar product with x_i we get,

$$-h_{ij} = \frac{1}{a} g_{ij} \quad \text{and} \quad h_j = 0 \qquad \alpha = 1,2,..,(k-1) \qquad (6)$$

Transvecting with g_{ij} it follows that

$$\underset{\alpha}{h}{}_k = -\frac{n}{a} \quad \text{i.e.} \quad \|H\| = -\sqrt{n}\lambda \qquad (7)$$

Substituting in (3) we get,

$$\sum_{\alpha=1}^{k-1} \|H_\alpha\|^2 = 0$$

$$\Longrightarrow \underline{H}_\alpha = 0 \qquad \alpha = 1,2,..,(k-1)$$

but, H_{ij} $(\alpha = 1,2,..,(k-1))$ are equal to the second fundamental form of $x(M^n)$ in S^{n+k-1} (cf. Lemma 2.0.2). This x realizes a minimal immersion in S^{n+k-1}.

Conversely,

if $x(M^n)$ is minimally immersed in S^{n+k-1} then, by parallel translation in E^{n+k} we can arrange things so that S^{n+k-1} is centred at the origin of E^{n+k}. Choose a set of mutually orthogonal normals and let $\underline{e}_{n+k} = \frac{1}{a}\underline{x}$ where $\underline{e}_{n+k}$ is the normal to S^{n+k-1} in E^{n+k}. Then (5) and (6) are satisfied as before and since $h_{\alpha ij}$ $(\alpha = 1,2,..,(k-1))$ are considered as the second fundamental form of the induced immersion in S^{n+k-1}, we have

$$\Delta x = \sum_{\alpha=1}^{n} H_\alpha\, \underline{e}_\alpha$$

$$= H_k\, \underline{e}_k$$

$$= \frac{\|H\|}{a}\, \underline{e}_k .$$

Using (7) we get $\Delta x = \frac{n}{a^2} x$.

<u>Corollary 2.3.3</u> (Hoffman[1])

For a string of isometric immersions $M^n \xrightarrow{f} S^{n+k} \xrightarrow{g} E^{n+k-1}$, M^n is minimal iff $\Delta(g \circ f) = \lambda(g \circ g)$ where λ is some real valued function on M.

<u>Definition 2.3.4</u>

(i) An <u>ISOMETRY</u> $f : M \to M$ is a metric preserving map. The set of all orientation-preserving isometries forms a group called the group of isometries $\mathcal{G}$ say.

(ii) $\mathcal{G}$ is <u>TRANSITIVE</u> if for each $m,n \in M$ $\exists\, g \in \mathcal{G} \ni g(m) = N$.

(iii) A space with a transitive group of operators is called <u>HOMOGENEOUS</u>.

(iv) A <u>LIE GROUP</u> G is a group with a differentiable structure of a manifold. The maps $\theta_1 : G \times G \to G$

given by $(g,h) \mapsto gh$

and $\theta_2 : G \to G$

$g \mapsto g^{-1}$ are C^∞ (smooth).

(v) A homogeneous space G/H where G is a lie group and H a compact subgroup admits an invariant metric. G/H is often called a <u>RIEMANNIAN HOMOGENEOUS SPACE</u>.

(vi) Let $\mathcal{G}$ denote the group of all isometries of M. For $m \in M$ let F_m denote the subgroup of $\mathcal{G}$ leaving m fixed.

i.e. $F_m = \{g \in \mathcal{G} \mid g(m) = m\}$

then F_m is called the <u>ISOTROPY GROUP</u> of M at m.

(vii) Under the action of an isometry subgroup $G \quad ISO(M^n) = \mathfrak{g}$ (isometry subgroup), the total space M^n splits into orbits of various types. An orbit $G(x) \subseteq M^n$ is called an EXTREMAL ORBIT if it is an extremal in "volume" with respect to all orbits of the same type.

As an application to theorem 2.3.2 Takahashi [1] proves

Theorem 2.3.5

A compact homogeneous Riemannian manifold with irreducible linear isotropy group admits a minimal immersion in a Euclidean sphere.

Corollary 2.3.6

An irreducible compact symmetric space admits a minimal immersion in a Euclidean sphere.

Theorem 2.3.7.

Every homogeneous space of a compact Lie group G/H may be imbedded in a sufficiently high dimensional euclidean sphere as a homogeneous minimal submanifold.

The proof of the above theorem is due to Hsiang [1] and is based on yet another theorem due to him, viz: A submanifold $M^n \subseteq \bar{M}^{n+k}$ is a homogeneous minimal submanifold of $\bar{M}^{n+k}$ iff M^n is an extremal orbit under the action of a suitable isometry subgroup G.

§4. More theorems on Minimal Immersions in Spheres and Pseudo-Umbilical Immersions

Definition 2.4.1

If the mean curvature normal is nowhere zero and the second fundamental form in the direction of H is proportional to the identity transformation of the tangent space of M^n everywhere (i.e. the mean curvature normal has the same eigenvalues everywhere) then the immersion $x : M^n \to E^{n+k}$ is said to be PSEUDO-UMBILICAL.

It will be seen later that pseudo-umbilical immersions with constant mean curvature in a euclidean space $\Longleftrightarrow$ a minimal immersion in a hyper-sphere of the euclidean space.

Lemma 2.4.2 Chen [11]

$x : M^n \to E^{n+2}$ is a pseudo-umbilical (p.u.) immersion of M^n in E^{n+2}. Then the mean curvature α is a constant iff the form $\omega_{n+1,n+2}$ vanishes identically.

Proof

Since the immersion is p.u. and

$$H = \alpha \underline{e}_{n+1} \tag{1}$$

$$\omega_{i,n+1} = f_i \, \omega_i \tag{2}$$

$$\Sigma f_i = 0 \tag{3}$$

we have,

$$\omega_{i,n+1} = \alpha\omega_i \tag{4}$$

$$i = 1,2\ldots n$$

$\therefore$ if $\omega_{n+1,n+1} = 0$

then, taking exterior differentiation of (4) we get,

$$d\omega_{i,n+1} = d\alpha \wedge \omega_i + \alpha . d\omega_i$$

but,

$$\begin{aligned} d\omega_{i,n+1} &= \omega_{ij} \wedge \omega_{j,n+1} + \omega_{i,n+1} \wedge \omega_{n+2,n+1} \\ &= \alpha\omega_{ij} \wedge \omega_j + \omega_{i,n+2} \wedge \omega_{n+2,n+1} \\ &= \alpha d\omega_i + \omega_{i,n+2} \wedge \omega_{n+2,n+1} . \end{aligned}$$

$$\therefore \qquad d\alpha \wedge \omega_i = \omega_{i,n+2} \wedge \omega_{n+2,n+1} = 0 \qquad (6)$$

$$\Longrightarrow \alpha = \text{constant.}$$

Conversely, if α = constant then from (6)

$$\omega_{i,n+2} \wedge \omega_{n+2,n+1} = 0$$

$$\Longrightarrow \omega_{i,n+2} = 0 \qquad (7)$$

Let $U = \{p \in M^n | \omega_{n+1,n+2} \neq 0 \text{ at } p\}$. Then taking exterior differentiation of (7) we have,

$$\omega_{ij} \wedge \omega_{j,n+2} + \omega_{i,n+1} \wedge \omega_{n+1,n+2} = 0$$

i.e.
$$\omega_{i,n+2} \wedge \omega_{n+1,n+2} = 0$$

i.e.
$$\omega_{n+1,n+2} = 0 \quad \text{on } U$$

$$\Longrightarrow U \neq \emptyset$$

$$\therefore \qquad \omega_{n+1,n+2} \equiv 0.$$

The above lemma was also proved by Otsuki [1] for $n = 2$.

Lemma 2.4.3

If $x : M^n \to E^{n+2}$ is a p.u. immersion and the mean curvature α is a constant ($\neq 0$), then M^n is immersed in a hypersphere of E^{n+2}.

Proof

Consider the mapping $y : M^n \to E^{n+2}$

$$\ni \quad y(p) = x(p) + \frac{1}{\alpha} e_{n+1}$$

$$\text{then,} \quad dy(p) = dx(p) + \frac{1}{\alpha} de_{n+1}$$

$$= dx(p) + \frac{1}{\alpha}(\omega_{n+1,i}\, e_i + \omega_{n+1,n+2}\, e_{n+2})$$

$$= 0.$$

$(dx(p) = \Sigma\omega_i e_i$ and $\bar{\omega}_{n+1,n+2} = 0$ from lemma 2.4.2)

$\Longrightarrow M^n$ is immersed in a hypersphere of E^{n+2}.

Theorem 2.4.4

Let $x : M^n \to E^{n+2}$ be an immersion of an n-dimensional manifold M^n in E^{n+2}. Then x is p.u. with constant mean curvature iff M^n is immersed as a minimal hypersurface in a hypersphere of E^{n+2}.

Proof

Assume $x : M^n \to E^{n+2}$ is p.u. with constant mean curvature α (say).

Then by definition $\underline{H} \neq 0$ everywhere. Using the previous lemma we can assume that M^n is immersed in a unit hypersphere. Take a local cross-section $(p, e_1, \ldots, \bar{e}_{n+1}, \bar{e}_{n+2})$ of $M^n \to B$, $\ni$ $\bar{e}_{n+1} = x(p)$ and $\bar{e}_1, \ldots, \bar{e}_n$ diagonalize the second fundamental form at $\bar{e}_{n+2}$.

Then, $$\bar{A}_{\bar{e}_{n+1}}(e_i) = \text{id}$$

and $$\bar{A}_{\bar{e}_{n+2}}(e_i) = h_i e_i \qquad i = 1,2,\ldots,n$$

where h_i are functions on M.

$$H = \bar{e}_{n+1} + \frac{1}{n}(\Sigma h_i)\bar{e}_{n+2}$$

Since the mean curvature α is constant by assumption, we have Σh_i = constant.

$$A_{(H/\alpha)}(e_i) = A_{\frac{1}{\alpha}(\bar{e}_{n+1} + \frac{1}{n}(\Sigma h_i)\bar{e}_{n+2})}(e_i)$$

$$\therefore \quad A_{(H/\alpha)}(e_i) = \frac{1}{\alpha}[1 + \frac{1}{n}(\Sigma h_k)h_i] e_i .$$

Since the immersion is p.u. we have

$$(\Sigma h_j)h_1 = (\Sigma h_j)h_2 = \ldots = (\Sigma h_j)h_n .$$

Two cases arise

(i) If $(\Sigma h_j) \neq 0$ then, $h_1 = h_2 = \dots = h_n$, everywhere. Thus M^n is immersed in a hypersphere of a hyperplane.

(ii) If $\Sigma h_j = 0$, then M^n is immersed as a minimal hypersurface of a hypersphere of E^{n+2}.

Thus in either case M^n is immersed as a minimal hypersurface in a hypersphere.

Conversely,

If M^n is immersed as a minimal hypersurface in a hypersphere of E^{n+2}, then the mean curvature normal at p is parallel to the vector joining the centre of the hypersphere and the point p on M^n. Thus x is p.u. with constant mean curvature.

Definition 2.4.5

If η be a normal vector field on M^n in a Riemannian manifold R^{n+k} then, the covariant differentiation of η in R^{n+k} can be written as the sum of its tangential and normal components.

$$\bar{\nabla}\eta = [\bar{\nabla}\eta]^T + D\eta$$ (D ≡ covariant differentiation in the normal bundle).

If the normal component is zero then η is said to be parallel in the normal bundle.

Theorem 2.4.6

An immersion $x : M^n \to E^{n+k}$ is p.u. and the mean curvature normal field H is parallel in the normal bundle iff M^n is immersed as a minimal submanifold in a hypersphere of E^{n+k}.

Proof

Choose the unit normal e_{n+1} in the direction of the mean curvature normal, then,

$$\underline{H} = \alpha \underline{e}_{n+1} \qquad \alpha > 0 \tag{1}$$

By definition,

$$A_{n+1,ij} = \alpha \delta_{ij} \tag{2}$$

and

$$\sum_{i=1}^{n} A_{rii} = 0 \tag{3}$$

for $r = (n+2),\ldots,(n+k)$,

Since the immersion is chosen to be p.u. and since the mean curvature vector field is parallel in the normal bundle, we have, from (1)

$$DH = (d\alpha)e_{n+1} + \alpha De_{n+1} \tag{4}$$

$$(De_{n+1} = \omega_{n+1,r}\, e_r)$$

$$\therefore, \qquad d\alpha\, e_{n+1} + \alpha.\omega_{n+1,r}\, e_r = 0$$

$$\Longrightarrow \alpha = \text{constant} \quad \underline{\text{and}} \quad \omega_{n+1,r} = 0 \tag{5}$$

$r = (n+1),\ldots,(n+k)$

Consider now,

$$\phi : M^n \to E^{n+k}$$

given by $\phi(p) = x(p) + \frac{1}{\alpha} e_{n+1}$.

Then

$$d\phi(p) = dx(p) + \frac{1}{\alpha} de_{n+1}$$

$= 0$ [using the equations of structure and eq. (2)]

Thus $x(M^n)$ is contained in a hypersphere of E^{n+k} centred at c.

Further, $x(p) - c$ is parallel to e_{n+1} everywhere. $\therefore$ by (4) M^n is immersed as a minimal submanifold of S^{n+k-1} .

Conversely,

If M^n is immersed as a minimal submanifold in a hypersphere of E^{n+k}, and M^n is a p.u. immersion with constant mean curvature α

Consider as before, $\phi : M^n \to E^{n+k}$

$$\text{given by} \quad \phi(p) = x(p) + \frac{1}{\alpha} e_{n+1}$$

$$\text{and} \quad \phi(p) = c \quad \text{(say)}$$

because $x(M^n)$ is contained in S^{n+k-1} and $x(p) - c$ is parallel to e_{n+1} everywhere and $\omega_{n+1,r} = 0 \quad r = (n+2),\ldots,(n+k)$

$$\begin{aligned} \therefore \qquad DH &= (d\alpha)e_{n+1} + \alpha D e_{n+1} \\ &= (d\alpha)e_{n+1} + \alpha\omega_{n+1,r}\, e_r \\ &= 0 \end{aligned}$$

$\Longrightarrow$ H is parallel in the normal bundle.

Theorem 2.4.7

Let $x : M^n \to E^{n+k}$ be an isometric immersion of a riemannian manifold in a euclidean space of dim-(n+k). If the position vector X is parallel to the mean curvature vector everywhere on M, then M^n is immersed as a minimal submanifold of a hypersphere of E^{n+k}.

Follows from Theorem 2.4.6.

Some more results on p.u. immersions due to Chen can be found in [17,18] and Chen and Yano [1].

Definition 2.4.8

A submanifold M of a Riemannian manifold $\bar{M}$ is said to be totally geodesic if every geodesic of M is a geodesic of $\bar{M}$.

Theorem 2.4.9

$x : M \to \bar{M}$ is a totally geodesic submanifold iff its second fundamental form vanishes identically.

A proof of this theorem can be found in Bishop and Crittenden [1] pg. 194.

Corollary 2.4.10

Every totally geodesic submanifold of a Riemannian manifold is necessarily a minimal submanifold.

This result is immediate from the above theorem.

We now look at minimal submanifolds of a sphere with second fundamental form of constant length. If $f : M^n \to S^{n+p} \subset E^{n+p+1}$, then if M is compact

$$\int_M \left[\left(2 - \frac{1}{p}\right) S - n \right] S * 1 \geq 0$$

S = length of the second fundamental form.

If

$$\left(2 - \frac{1}{p} \right) S - n \leq 0$$

i.e. if

$$S \leq \frac{n}{(2 - \frac{1}{p})}$$

then (i) M is totally geodesic (because S must be identically zero),

or (ii) $S = \dfrac{n}{(2 - \frac{1}{p})}$.

Case (i) when M is totally geodesic is not very interesting from the point of view of looking for minimal immersions since we know that all such manifolds are in fact minimal (cf. Corollary 2.4.10). Chern, do Carmo and Kobayashi [1] have investigated the second case and have determined all minimal submanifolds of S^{n+p} which satisfy (ii).

They prove

Theorem 2.4.11

The Veronese surface in S^4 and the Clifford submanifolds $M_{m,n-m}$ in S^{n+1} are the only compact minimal submanifolds of dimension n in S^{n+p} satisfying

$$S = \frac{n}{(2 - \frac{1}{p})}$$

$$M_{m,n-m} = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right) .$$

An independent proof of the above theorem for $p = 1$ is due to Lawson [2].

Below we describe the Veronese surface.

If $(x,y,z) \in R^3$ and $(u^1,u^2,\ldots,u^5) \in R^5$, then

$$u^1 = \frac{1}{\sqrt{3}} yz \qquad u^4 = \frac{1}{2\sqrt{3}}(x^2 - y^2)$$

$$u^2 = \frac{1}{\sqrt{3}} xz \qquad u^5 = \frac{1}{6}(x^2 + y^2 - 2z^2)$$

$$u^3 = \frac{1}{\sqrt{3}} xy$$

The map defined is an isometric immersion of $S(\sqrt{3})$ into $S^4(1)$. (x,y,z) and $(-x,-y,-z)$ are mapped into the same point. The Veronese surface is defined to be this mapping of $\mathbb{R}P^2$ imbedded into S^4.

To prove their theorem Chern et al. first show that when the ambient space has constant curvature then,

$$\int_M \left[\left(2 - \frac{1}{p}\right) S - nc\right] S * 1 \geq 0 \tag{1}$$

and if M is compact and minimally immersed in $\bar{M}_c^{n+p}$. Moreover if M is not totally geodesic and $S \leq \frac{nc}{2 - (1/p)}$ everywhere, then in fact

$S = \frac{nc}{2-(1/p)}$, and then the second fundamental form is parallel. They then assume that the ambient space of constant curvature is in fact the unit sphere (hence $c = 1$) and therefore the first part of the result follows immediately from (1).

Theorem 2.4.12 (is also necessary for proof)

If M is an n-dimensional manifold immersed minimally in an (n+p)-dim. space of constant curvature 1, satisfying $S = \frac{n}{2-(1/p)}$, and if $p \geq 2$ then, $n = p = 2$ with respect to an adapted dual orthonormal frame field $(\omega^1, \omega^2, \omega^3, \omega^4)$ and the connection forms ω_{BA} $(= \omega^A_B)$ of the ambient space restricted to M is given by

$$\begin{pmatrix} 0 & \omega_{21} & \mu\omega_2 & -\mu\omega_1 \\ \omega_{12} & 0 & \mu\omega_1 & \mu\omega_2 \\ \lambda\omega_2 & \lambda\omega_1 & 0 & 2\omega_{21} \\ -\mu\omega_1 & \lambda\omega_2 & 2\omega & 0 \end{pmatrix} \qquad -\lambda = \mu = \sqrt{\tfrac{1}{3}} \tag{1}$$

They then compute the structure equations for the Veronese surface and get,

$$\begin{pmatrix} 0 & \omega_{21} & \mu\omega_2 & -\mu\omega_1 & \omega_1 \\ \omega_{12} & 0 & \mu\omega_1 & \mu\omega_2 & \omega_2 \\ \lambda\omega_2 & \lambda\omega_1 & 0 & 2\omega_{21} & 0 \\ -\lambda\omega_1 & \lambda\omega_2 & 2\omega_{12} & 0 & 0 \\ -\omega_1 & -\omega_2 & 0 & 0 & 0 \end{pmatrix} \quad -\lambda = \mu = \sqrt{\tfrac{1}{3}} \tag{2}$$

locally then, the minimal surface in Theorem 2.4.12 coincides with the Veronese surface and if the surface is compact it is then the Veronese itself.

A few examples and applications are given at the end and they indicate that $M_{m,n-m}$ can be generalized in the following manner.

Suppose $m_1, m_2, \ldots, m_k$ are positive integers

and $n = m_1 + \ldots + m_k$

then, if $x_i \in S^{m_i}\left(\sqrt{\frac{m_i}{n}}\right)$ i.e. $\|x\| = \sqrt{\frac{m_i}{n}}$

(x is considered as a vector in euclidean $(m_i + 1)$ space).

Then, $x = (x_1, \ldots, x_k)$ has unit length in E^{n+k}.

The immersion,

$$M_{m_1, m_2, \ldots, m_k} = \Pi\, S^{m_i}\left(\sqrt{\frac{m_i}{n}}\right) \rightarrow S^{n+k}$$

is the a minimal immersion of $M_{m_1, \ldots, m_k}$. Its scalar curvature is $(n-k)$; and

$$S = \frac{(k - 1)n}{(2k - 3)}.$$

Kenmotsu [1] has also studied this problem of classifying all minimal submanifolds with the second fundamental form being a constant length. However, he consideres only those submanifolds in the unit sphere and R the curvature tensor of the manifold being zero.

He proves that if there is a minimal immersion of a compact connected smooth manifold M, of dim-n in an (n+p)-dim unit sphere, such that the normal connexion of M is trivial (i.e. the curvature tensor is zero) and $S = n$ then $\exists$ an (n+1)-dim unit sphere containing M as a Clifford minimal hypersurface $M_{m,n-m}$ for $m = 1, 2, \ldots, [n/2]$.

§5. Minimal Immersions of Surfaces

Theorem 2.5.1

Let $M^2 \rightarrow S^3$ be a minimal immersion of a complete orientable surface in a three space. If the Gauss curv. K of M^2 does not change sign, then M^n is immersed as an equation or a Clifford torus.

Proof

Since $x : M^2 \to S^3$ is a minimal immersion of a complete orientable surface M^2 in S^3, using theorem 2.4.4 we can say that $x : M^2 \to S^3$ is a p.u. immersion with constant mean curvature in E^4. (We can look upon S^3 as sitting in E^4).

Since the Gaussian curvature does not change sign, M^2 is immersed either as a sphere in the hyperplane of E^4 or as a Clifford flat torus in E^4. (cf. Itoh [1] also see next chapter, theorem 3.3.3).

But since the immersion is minimal in S^3, M^2 must be immersed as the equatorial two sphere or as a Clifford flat torus.

A generalization of the above theorem for closed surfaces immersed in a space of higher dimension can be realized in

Theorem 2.5.2

Let M be a closed minimal surface of a unit n-sphere with G-K. curv. $K \leq 0$. If $\exists$ a unit normal vector field $\bar{e}$ over M^n $\ni$ the L-K. curv. $G(p\ \bar{e})$ w.r.t. $\bar{e}$ is nowhere zero, then, M is a Clifford torus in a unit three dimensional sphere S^3 of S^m.

Another way of looking upon minimal immersions is by examining the area of the immersed manifold and seeking a method of classification for such manifolds, since a minimal surface is an extremal for area.

Chen [20] has investigated this problem for surfaces and he proves

Theorem 2.5.3

If M is a compact minimal surface immersed in a euclidean space of dim-n with Gaussian curvature ≥ 0, then, $V(M^2)$ the volume of M^2 satisfies,

$$V(M^2) \geq 2\pi^2 + (2-\pi)\pi.\chi(M^2).$$

Equality holds iff M is either the 2-sphere or the Clifford torus, and Theorem 2.5.4

Under the same hypothesis as in Theorem 2.5.3 if $V(M^2) \leq (2+\pi)\pi$ then M^2 is homeomorphic to the 2-sphere.

Most known results of minimal immersions of compact surfaces tend to show that the surface considered is either of genus zero or one, and it was unknown whether there existed any minimal immersions of compact surfaces of genus greater than one. However, this problem has now been tackled by Lawson [1] where he shows that there do exist compact orientable minimal surfaces of arbitrary genus imbedded in S^3.

In the case of genus zero, the equatorial two sphere is the only possibility, Almgren [1]. But for surfaces of genus one there exist an infinity of non-congruent immersions.

CHAPTER III

The notations and formulae in the first and second sections are the same as those used in Chapter I.

§1. α^{th} Scalar Curvature

We consider an isometric immersion $x : M^n \to E^{n+N}$. Take a local-cross section (Frenet cross-section) of M^n in B and at $x(p)$. Let $e_{n+N} = \sum_r \cos\theta_r \bar{e}(q)$ where $\bar{e}(q)$ is a function in the neighbourhood of $p \in M$. Then,

$$A_{n+N,ij} = \sum_r \cos\theta_r \bar{A}_{rij} \tag{1}$$

Having chosen the Frenet cross-section,

$$K_2(p, e_{n+N}) = \sum_r \lambda_{r-n}(p).\cos^2\theta_r \tag{2}$$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N .$$

λ_α, $\alpha = 1,2,\dots,N$ is defined continuously on the whole of M^n and λ_α is defined to be the α^{th} SCALAR CURVATURE of M^n in E^{n+N}

From the above, and previous work, we have,

$$\binom{n}{2}\lambda_{r-n}(p) = \sum_{i<j}(\bar{A}_{rii}\bar{A}_{rjj} - \bar{A}_{rij}^2) \tag{3}$$

$$\binom{n}{2}\sum_r \lambda_{r-n}(p) = \sum_{i<j}(\sum_r(\bar{A}_{rii}\bar{A}_{rjj} - \bar{A}_{rij}^2))$$

$$= \sum_{i<j} R_{ijji} \tag{4}$$

$\therefore$ $S(p)$ the scalar curvature is defined as

$$\binom{n}{2} S(p) = \sum_{i<j} R_{ijji}$$

($S(p)$ is intrinsic i.e. it only depends on the metric).

As expected the scalar curvature and the α^{th} scalar curvature have the following relationship

$$S(p) = \lambda_1(p) + \ldots + \lambda_N(p) . \qquad (5)$$

NOTE: (i) The scalar curvature is just the sum of the principal curvatures taken two at a time in each normal direction.

(ii) With regard to the notation in Chapter I it is the second mean curvature.

When the immersed manifold is a two dimensional surface, the scalar curvature is the G-K curv. and as remarked earlier in the case of co-dimension one it is the well known Gaussian curvature.

Theorem 3.1.1

For an n-dimensional manifold M^n immersed in E^{n+N},

$$\int_{M^n} \rho(p) dV \geq \frac{2c_{n+N-1}}{c_n}$$

The equality holds iff the co-dim. is one and (i) M^n is imbedded as a hypersphere if $n > 2$, or (ii) as a convex hypersurface if $n = 2$, where

$$\rho(p) = \max\{\sqrt{|\lambda_1(p)|^n}, \ldots, \sqrt{|\lambda_N(p)|^n}\}.$$

Proof

Since $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$

$$\rho(p) = \max\{\sqrt{|\lambda_1(p)|^n}, \sqrt{|\lambda_N(p)|^n}\} \qquad (6)$$

Now,

$$K_2^*(p) = \int_{S^{N-1}} |K_2(p,e)|^{n/2} \, d\sigma_{N-1}$$

$$= \int |\sum_r \lambda_{r-n}(p) \cos^2\theta_r|^{n/2} \, d\sigma_{N-1} \quad \text{using (2)}$$

$$\leqslant \int |\rho(p) \sum_r \cos^2\theta_r|^{n/2} \, d\sigma_{N-1}$$

$$= \rho(p) \int (\sum_r \cos^2\theta_r)^{n/2} \, d\sigma_{N-1}$$

$$= \rho(p) \quad c_{n+N-1}$$

but,

$$\int_{M^n} K_2^*(p) dV \geqslant 2c_{n+N-1} \tag{7}$$

$\therefore$,

$$c_{N-1} \int_{M^n} \rho(p) dV \geqslant 2c_{n+N-1} \tag{8}$$

$$\Longrightarrow \int_{M^n} \rho(p) dV \geqslant \frac{2c_{n+N-1}}{c_n} \tag{9}$$

If equality holds in (9), then

① $$\int K_2^*(p) dV = 2c_{n+N-1} \, .$$

and from Chern and Lashof [1] we know that M^n is imbedded as

(i) a hypersphere if $n > 2$,

and (ii) as a convex hypersurface if $n = 2$.

② $$K_2^*(p) = \rho(p) c_{N-1}$$

but $\rho(p)$ is always positive, and since $\lambda_1 \geqslant \dots \geqslant \lambda_N$, without loss of generality we can let $\lambda_1 > 0$. Then $\lambda_2 = \lambda_3 = \dots = \lambda_N = 0$. Hence the codimension must be one.

Corollary 3.1.2

Let M^n be a closed manifold immersed in E^{n+N} $(n \geq 3)$. Then,

$$\int_{M^n} (\lambda_1)^{n/2}\, dV = c_n \quad \text{and} \quad \lambda_2 = \dots = \lambda_N = 0$$

$\Longleftrightarrow$ M^n is imbedded as a hypersphere in E^{n+N}.

Theorem 3.1.3

Let M^{2m} be a 2m-dimensional closed manifold immersed in E^{2m+2} with scalar curvature $S(p) = 0$. Then,

$$\int_{M^{2m}} \lambda_1^m\, dV = \frac{c_m}{2c_{m+1}} \int_{M^{2m}} K_2^*(p)\, dV$$

Proof

From the hypothesis

$$S(p) = \lambda_1(p) + \lambda_2(p) = 0$$

$$\therefore \quad K_2^*(p) = \int_0^{2\pi} |K_2(p, e_{2m+2})|^m\, d\theta$$

$$(K_2(p, e_{2m+2}) = \lambda_1 \cos^2\theta_1 + \lambda_2 \sin^2\theta_2)$$

$$\therefore \quad K_2^*(p) = \lambda_1^m(p) \int_0^{2\pi} |\cos 2\theta|^m\, d\theta$$

$$= \frac{2c_{m+1}}{c_m} \lambda_1^m(p) .$$

$$\therefore \quad \frac{2c_{m+1}}{c_m} \int_M \lambda_1^m(p)\, dV = \int_M K_2^*(p)\, dV ,$$

i.e. $$\int_M \lambda_1^m(p)dV = \frac{c_m}{2c_{m+1}} \int_{M^2} K_2^*(p)dV .$$

Corollary 3.1.4

If M^2 is a flat torus immersed in E^4, then, $\int_{M^2} \lambda_1(p)dV \geq 2\pi^2$ and equality holds iff $\int_{M^2} K_2^*(p)dV = 8\pi^2$.

This follows immediately from Theorem 3.1.3 and the result of Chem and Lashof [2].

This Corollary has also been proved by Otsuki [1].

Remark 3.1.5

A proof of theorem 1,3.6 due to Chen [15] lies along similar lines to that of the proof of theorem 3.1.3. He considers a Frenet frame and shows that

$$4 < \underline{H},\underline{H} > = 4\lambda_1(p)$$

($\lambda_1(p)$ is the first scalar curvature).

$\lambda_1(p) + \lambda_2(p) = 0$ so let $\lambda_1(p) = -\lambda_2(p) = \lambda$ say

Then $$K(p,e) = \lambda(p)(\cos^2\theta - \sin^2\theta) .$$

Using the same technique as before we get,

$$K^*(p) = 4\lambda(p)$$

$$\therefore \int_{M^2} \lambda(p)dV \geq 2\pi^2$$

Hence $$\int_{M^2} < \underline{H},\underline{H} > dV \geq \int_{M^2} \lambda(p)dV \geq 2\pi^2 .$$

The second part he proves by considering a function

$$\phi : M^2 \to E^4$$

defined by $\phi(p) = x(p) + \frac{e_4}{\alpha}$ as in lemma 2.4.3 and showed that it is minimally imbedded in S^3. Therefore we can deduce from the hypothesis that it must be a Clifford flat torus.

It appears that these methods of proof for immersions with $S(p) = 0$ do not lend themselves in the most general cases, i.e. when the immersed manifold is not necessarily even dimensional and when the co-dimension is not two. This is probably one reason why the problem in Remark 1.3.7 cannot be solved so readily using these methods.

The result of Corollary 3.1.2 was valid for closed manifolds of dimension three; however an analogous result for surfaces also exists and is due to Chen [3].

Theorem 3.1.6

If $M^2 \to E^{2+N}$ is an immersion of a closed compact orientable surface in E^{2+N} then,

(i) $\lambda_N = 0 \iff M^2$ is embedded as a convex surface in a three dimensional linear subspace of E^{2+N}.

(ii) The first scalar curvature $\lambda_1 = a$ (constant) and the last scalar curvature $\lambda_N = 0 \iff M^2$ is embedded as a sphere in a three dimensional linear subspace of E^{2+N} with rad. $\frac{1}{a}$.

The proof of this theorem essentially depends on the two following lemmas also due to Chen [3].

Lemma 3.1.7

If $M^2 \to E^{2+N}$ is as in the previous theorem, then $\lambda_N \geqslant 0$ iff M^2 is embedded as a convex surface in a three dimensional linear subspace of E^{2+N} and

Lemma 3.1.8

$f : M^2 \to N^{2+N} \quad (N \geqslant 1)$

$g : M^2 \to E^{3+N}$

be given by $\quad g(p) = f(p) \quad \forall\, p \in M^2$.

Then the L-K curv. $K_2(p,e)$ and $\bar{K}_2(p,e)$ of f and g satisfy the following equality

$$\bar{K}_2(p,e) = \cos^2\theta\, K_2(p,e')$$

where e' = unit vector in the direction of the projection of e in E^{2+N}.

From these two lemmas the first part of (i) in Theorem 3.1.6 follows immediately.

Now, if M^2 is imbedded as a convex surface in E^3 then, we can consider $f' : M^2 \to E^{1+N}$ and from lemma 3.1.8,

$$K(p,e) = \cos^2\theta\, K'(p,e') \quad -\pi/2 \leqslant \theta < \pi/2$$

and since $K'(p,e') \geqslant 0 \quad \forall\, (p,e') \in B_\upsilon'$

$$\therefore \qquad \lambda_N = 0$$

If now $\lambda_1 = a$ (constant) and $\lambda_N = 0$, then, $K(p,e) = \lambda_1(p)$.

Moreover, L-K curv. = G-K curv. of f' induced by f but M^2 is compact and embedded with constant Gauss. curv.

$\therefore$ M^2 is embedded as a sphere with radius $\dfrac{1}{\sqrt{a}}$;

Conversely,

if M^2 is embedded as a sphere in E^3 with radius $\dfrac{1}{\sqrt{a}}$, then

$$\text{G-K curv.} = K'(p,e) = a \quad \forall\, (p,e) \in B_\upsilon' .$$

$$\therefore \quad \text{since} \qquad K'(p,e) = \sum_{r=3}^{2+N} \lambda_{r-2}(p) \cdot \cos^2\theta$$

we have $\lambda_1 = a$ (constant)

and $\lambda_N = 0.$

Hence the theorem.

For complete orientable surfaces M^2 in E^{2+N} Shiohama [1] has proved that if all the N scalar curvatures $\lambda_1, \ldots, \lambda_N$ are zero then the surface is a cylinder.

§2. Difference Curvature of Surfaces in Euclidean Spaces

As before $I = dx.dx$

$$II_r = -dx.de_r$$

denote the first and second fundamental forms of a closed oriented surface M^2 immersed in E^{2+N}.

Definition 3.2.1

$$S(p,e) = \tfrac{1}{4}(k_1(p,e) - k_2(p,e))^2$$

is defined to be the DIFFERENCE CURVATURE of the immersion x at (p,e).

Definition 3.2.2

Analogous to the definition of the TAC of the immersion we say that the integral

$$S^*(p) = \int S(p,e)d\sigma$$

over the sphere of unit normal vectors at x(p) is the DIFFERENCE CURVATURE OF THE IMMERSION x at p and define,

$\int_{M^2} S^*(p)dV$ to be the DIFFERENCE CURVATURE of M^2.

Theorem 3.2.3

If $x : M^2 \to E^{2+N}$ is an immersion of a closed oriented surface in

E^{2+N} then,

$$\int_M S^*(p)\,dV \geq 2g\, c_{N+1} \tag{0}$$

where g is the genus of M^2.

Equality holds iff M^2 is embedded as a sphere in a linear subspace of E^{2+N}. (0)'

Proof

Choosing a Frenet frame, we can write

$$K_2(p,e) = \sum_{r=3}^{2+N} \lambda_{r-2} \cdot \cos^2\theta_r \qquad \lambda_1 \geq \ldots \geq \lambda_N .$$

$$\therefore \quad \int_{B_\upsilon} K_2(p,e)\,d\sigma \wedge dV = \int_{B_\upsilon} \left(\sum_{r=3}^{2+N} \lambda_{r-2} \cdot \cos^2\theta_r \right) d\sigma \wedge dV$$

$$= \frac{c_{N+1}}{2\pi} \int_{M^2} \left(\sum_r \lambda_{r-2}\right) dV$$

But the Gauss-Kronecker Curv.

$$G(p) = \lambda_1 + \ldots + \lambda_N .$$

$\therefore$ using the Gauss Bonnet Theorem, we have,

$$\int_{B_\upsilon} K_2(p,e)\,d\sigma \wedge dV = \frac{c_{N+1}}{2\pi} \cdot 2\pi \cdot \chi(M^2)$$

$$= (2 - 2g)c_{N+1} \tag{1}$$

Also from Chern and Lashof II [2] we have,

$$\int_{B_\upsilon} |K_2(p,e)|\,d\sigma \wedge dV \geq (2 + 2g)c_{N+1} \tag{2}$$

Then if $\quad U = \{(p,e) \in B_\upsilon \mid K_2(p,e) \geq 0\}$

$$V = \{(p,e) \in B_\upsilon \mid K_2(p,e) < 0\}$$

(1) and (2) give

$$-\int_U K_2(p,e)d\sigma \wedge dV \;\geqslant\; 2g\ c_{N+1} \tag{3}$$

$$\begin{aligned}
\int_M S^*(p)dV &= \int_{B_\upsilon} S(p,e)d\sigma \wedge dV \\
&= \int_U S(p,e)d\sigma \wedge dV + \int_V S(p,e)d\sigma \wedge dV \\
&\geqslant \int_U S(p,e)d\sigma \wedge dV \\
&= \int_V \tfrac{1}{4}(k_1(p,e) - k_2(p,e))^2 d\sigma \wedge dV \\
&= \int_U [K_1(p,e))^2 - K_2(p,e)]d\sigma \wedge dV \\
&\geqslant -\int_U K_2(p,e)d\sigma \wedge dV
\end{aligned}$$

$$\therefore \qquad \int_M S^*(p)dV \;\geqslant\; 2g\ c_{N+1} \tag{4}$$

$$\begin{cases} K_1(p,e) = \tfrac{1}{2}\{k_1(p,e) - k_2(p,e)\} \\ K_2(p,e) = \det\ (A_{rij}) \end{cases}$$

Now suppose equality holds in (0)

i.e. $$\int_M S^*(p)dV = 2g\ c_{N+1}$$

then, (i) $K_1(p,e) = 0$ on V (5)

(ii) $S(p,e) = 0$ on U (6)

$\therefore$ from (1) and (2) we get,

$$\int_U K_2(p,e)d\sigma \wedge dV = 2c_{N+1} \qquad (7)$$

$\therefore$ from the first result of lemma 1.2.6 we know that M^2 is embedded as a sphere in a three dimensional linear subspace of E^{2+N}.

Conversely,

if M^2 is embedded as a sphere in a 3-dimensional linear subspace of E^{2+N} then by direct computations $(0)'$ is true.

§3. Submanifolds with Constant Mean Curvature in a Riemannian Manifold

In the last section of Chapter II some results were mentioned with regard to manifolds whose mean curvature normal field was parallel in the normal bundle.

Another important consequence of this concept leads to the conclusion that the mean curvature must then be a constant.

Proposition 3.3.1 (Hoffman [1])

$f : M \to \bar{M}^{n+k}$.

$\underline{H}$ parallel $\Longrightarrow \|\underline{H}\| = $ constant.

Proof

Let $X \in \mathfrak{X}(M)$

Then,
$$\begin{aligned} X < \underline{H},\underline{H} > &= < \bar{\nabla}_X \underline{H},\underline{H} > \\ &= < [\bar{\nabla}_X \underline{H}]^N,\underline{H} > \\ &= < D_X \underline{H},\underline{H} > \end{aligned}$$

But $\underline{H}$ is parallel $\therefore$ $D_X \underline{H} = 0$

$\Longrightarrow X < \underline{H},\underline{H} > = 0$

$\Longrightarrow \|\underline{H}\| = $ constant.

Remark 3.3.2

The converse of this is false except in co-dimension one when $\underline{H}$ parallel $\Longleftrightarrow$ $\|\underline{H}\|$ = constant.

For complete oriented surfaces with constant mean curvature Itoh [1] has proved

Theorem 3.3.3

A complete oriented pseudo-umbilical surface with constant non-zero mean curvature H in E^4 and Gauss Curvature K which does not change sign is necessarily either a Clifford flat torus or a sphere in E^3 with radius $\frac{1}{\|H\|}$.

From lemma 2.4.2 we have $\omega_{34} \equiv 0$ (because immersion is pseudo-umbilical), and from lemma 2.4.3 M^2 is contained in $S^3 \subset E^4$ with radius $\frac{1}{\|H\|}$.

He then proves, that a complete orientable p.u. surface with constant mean curvature and Gaussian curv. nowhere positive is a Clifford flat torus $S'\left(\frac{1}{\sqrt{2}\,\|H\|}\right) \times S'\left(\frac{1}{\sqrt{2}\,\|H\|}\right)$ in E^4. Furthermore, if the Gaussian curvature is non-negative then it must either be a Clifford torus (as above) or a sphere in E^3 with radius $\frac{1}{\|H\|}$.

The result of the theorem then follows.

For complete surfaces in E^3 we have,

Proposition 3.3.4

$f : M^2 \to E^3$

(i) if the Gaussian curvature $K \leq 0$ then it is either a minimal surface or a right circular cylinder.

(ii) if the Gaussian curv. $K \geq 0$, then it is either a sphere or a plane or a right circular cylinder.

Hence,

Theorem 3.3.5 (Klotz and Osserman [1])

A complete orientable surface in E^3 with constant mean curvature and Gauss curvature K which does not change sign is necessarily either a sphere, a minimal surface or a right circular cylinder.

M is a Riemannian manifold with metric tensor g.

Definition 3.3.6

(i) A transformation $\phi : M \to M$ is said to be CONFORMAL if $\phi^* g = \rho g$ where ρ is some positive function on M.

(ii) If ρ is a constant then the transformation is HOMOTHETIC.

(iii) If ρ is one it is an ISOMETRY (metric preserving).

If $X,Y,Z \in \mathfrak{X}(M)$ and $L_X g$ denotes the lie derivative of the tensor g an infinitesimal transformation X of M is said to be

CONFORMAL if $L_X g = \rho g$, ρ function on M,

HOMOTHETIC if $L_X g = cg$ c is a constant,

KILLING if $L_X g = 0$.

$$\begin{aligned} L_X < Y,Z > &= X < Y,Z > - < [X,Y], Z > - < Y, [X,Z] > \\ &= X < Y,Z > - < \nabla_X Y, Z > + < \nabla_Y X, Z > \\ &\qquad - < Y, \nabla_X Z > + < Y, \nabla_Z X > \\ &= < \nabla_Y X, Z > + < Y, \nabla_Z X > \end{aligned}$$

(In coordinates, $Lg_{ij} = \nabla_i X_j + \nabla_j X_i$)

Definition 3.3.7

A one parameter subgroup of a Lie group G is an analytic homomorphism of R into G.

Yano [1] generalizes to a general Riemannian manifold,

Theorem 3.3.8 (Katsurada)

Let $\bar{M}$ be an (m+1)-dimensional orientable Einstein space and M a closed orientable hypersurface in $\bar{M}$ whose first mean curvature is constant. If M admits a one parameter group of conformal transformations such that the inner product of the generating vector v^h and the normal η^h to the hypersurface does not change the sign (and is non-zero) on M, then every point of M is umbilical.

Katsurada's Theorem was itself a generalization of the Liebmann-Süss Theorem (Chapter I).

Yano derives the Minkowski integral formulae valid in a general Riemannian manifold. Working in the classical notation all the time and using the standard formulae he gets, under an added assumption, that if the vector field v^h on the manifold is conformal then,

$$\int_{M^n} \alpha K_1 \, dV + \int_{M^n} \rho \, dV = 0 \tag{1}$$

$$\int_{M^n} [m v^d \nabla_d K_1 + m\rho K_1 + m\alpha\{mK_1^2 - (m-1)K_2\} - K_{ji} v^i \eta^j + \alpha K_{ji} \eta^i \eta^j] dV = 0 \tag{2}$$

and

$$\int_{M^n} [m\rho_i \eta^i + K_{ji} v^j \eta^i] dV = 0 \tag{3}$$

which are respectively the first, second and third integral formulas of Minkowski.

(η^i are the normal vectors).

Letting the first mean curvature K = constant he recovers the result of Katsurada from (1) and (2).

Furthermore, assuming that the Riemannian manifold admits an infinitesimal homothetic transformation, (1), (2) and (3) simplify to,

$$\int_M \alpha[(m-1)(K_1^2 - K_2) + \frac{1}{m} K_{ji}\,\eta^i \eta^j]dV = 0$$

and therefore,

<u>Theorem 3.3.9</u> (Yano)

If M^n is a closed orientable hypersurface of an $(n+1)$ dimensional orientable Riemannian manifold $\bar{M}$, whose first mean curvature is constant and

(i) $\bar{M}$ admits a one-parameter group of homothetic transformations such that the inner product of the generating vector v^h and the normal η^h to the hypersurface do not change the sign (and are non-zero) on M,

(ii) the Ricci curvature K_{ji} w.r.t. η^h is non-negative on M.

Then every point of S is an umbilical and $K_{ji}\,\eta^i \eta^j = 0$ on M.

Yet another generalization of the Liebmann-Süss theorem to arbitrary co-dimension and any ambient space form is

<u>Theorem 3.3.10</u> (Smyth [1])

A compact irreducible submanifold M of constant mean curvature ($\underline{H} \neq 0$) and non-negative sectional curvature must lie minimally in a hypersphere.

<u>Definition 3.3.11</u>

A Riemannian manifold is said to be reducible or irreducible according as the linear homogeneous holonomy group at a point $p \in M$ is reducible or irreducible as a linear group acting on the tangent space at p.

Finally a result on submanifolds with constant mean curvature.

Theorem 3.3.12 Chen [22]

If there is an immersion of a closed n-dimensional manifold in a euclidean space of dim-(n+N) and if the mean curvature has constant length given by $\|H(p)\| = \left(\frac{c_n}{V(M)} \right)^{1/n}$, then M^n is immersed as a hypersphere with radius $\left(\frac{V(M)}{c_n} \right)^{1/n}$ in an (n+1)-dimensional linear subspace of E^{n+N}.

CHAPTER IV

§1. Introduction

The variational problem for surfaces in E^3 was first considered by Hombu (paper unpublished). He took the variation along the normal direction and found that for the integral $\int_{x(M^2)} \langle H,H \rangle \, dS$ to be stationary

$$\Delta H + 2H(H^2 - K) = 0$$

(cf. pg. 21 Chapter I).

Recently Chen [23] generalized this result to hypersurfaces in a euclidean space. He calls the hypersurface stable if

$$\delta \int \mu^m \, dV = 0$$

for any normal variation.
Here, $\mu = \|H\|$.

In [23] he showed that if the hypersurface was stable it was necessary for

$$\Delta\mu^{m-1} + m(m-1)\mu^{m+1} + \mu^{m-1} R = 0.$$

R is the scalar curvature of M in E^{m+1}.

In this chapter we show how this result can be further generalized to manifolds immersed in any general Riemannian space. The methods employed follow a similar pattern to that used by Chen but instead of using the ordinary vector calculus we now use the tensor calculus.
(Chen could use the vector calculus because in the euclidean space covariant differentiation is the same as partial differentiation).

We shall see later that the result reduces to that obtained by Chen when the curvature of the ambient space is zero, and for surfaces in E^3 it is the same as that originally obtained by Hombu.

§2. Formulae and Fundamental Equations

Let $f : M \to \bar{M}$ be a smooth immersion of a closed orientable m-dimensional manifold in a smooth (m+1)-dimensional Riemannian manifold $\bar{M}$. Let $(x^1, \ldots, x^m)$ be a local coordinate system valid in some neighbourhood of a point $p \in M$ and let $(y^1, \ldots, y^{m+1})$ be a local coordinate system in some neighbourhood f(p) of $\bar{M}$. Then,

$$y^\alpha = f^\alpha(x^1, \ldots, x^m) \tag{1}$$

As usual the Roman indices take values 1,2,...,m and the Greek indices take values 1,2,...,(m+1).

If $(\bar{g}_{\alpha\beta})$ is a metric on $\bar{M}$, then the induced metric (g_{ij}) on M is given by,

$$g_{ij} = f_i^\alpha f_j^\beta \bar{g}_{\alpha\beta} \tag{2}$$

where $f_i^\alpha = \dfrac{\partial f}{\partial x^i}$

let $\quad f_{ij}^\alpha = \dfrac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k f_k^\alpha$.

Let n^α denote a unit normal vector field on f(M) defined locally and let (h_{ij}) denote the second fundamental form corresponding to the normal direction. Then,

$$\underline{H} = \frac{1}{m} \Sigma g^{ij} h_{ij} \tag{3}$$

Henceforth (,) will mean covariant differentiation and $\dfrac{\partial \bar{g}_{\alpha\beta}}{\partial y^\upsilon}$ will be denoted by $\bar{g}_{\alpha\beta.\upsilon}$.

Also,

$$f^{\alpha}_{i,j} = -\bar{\Gamma}^{\alpha}_{\mu\upsilon} f^{\mu}_{i} f^{\upsilon}_{j} + h_{ij} n^{\alpha} \tag{4}$$

i.e.

$$h_{ij} = \bar{g}_{\alpha\beta} f^{\alpha}_{i,j} n^{\beta} + \bar{g}_{\alpha\beta} f^{\mu}_{i} f^{\upsilon}_{j} \bar{\Gamma}^{\alpha}_{\mu\upsilon} n^{\beta} \tag{5}$$

$$n^{\beta}_{,j} = -h_{\ell j} g^{\ell m} f^{\beta}_{m} - \bar{\Gamma}^{\beta}_{\mu\upsilon} f^{\mu}_{j} n^{\upsilon} \tag{6}$$

$$R_{ijk\ell} = (h_{ik} h_{j\ell} - h_{i\ell} h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta} f^{\alpha}_{i} f^{\beta}_{j} f^{\gamma}_{k} f^{\delta}_{\ell} \tag{7}$$

(the sign is -ve that used in Chapter I)

and the volume element

$$W = *1 = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^m .$$

$$\bar{g}_{\alpha\beta} f^{\alpha}_{i} n^{\beta} = 0 \tag{8}$$

and

$$\bar{g}_{\alpha\beta} n^{\alpha} n^{\beta} = 1 \tag{9}$$

$$\bar{\Gamma}^{\alpha}_{\beta\upsilon} = \bar{g}^{\alpha\delta}[\beta\upsilon,\delta] = \tfrac{1}{2}\bar{g}^{\alpha\delta}(\bar{g}_{\beta\delta.\upsilon} + \bar{g}_{\alpha\delta.\beta} - \bar{g}_{\beta\upsilon.\delta}) \tag{10}$$

All these formulae can be found in Eisenhart [1] .

§3. Variation along the Normal Direction

We consider a family of immersions given by $f_t : M \times I \to \bar{M}$ parametrized by t, where $-\epsilon < t < +\epsilon$.

Assume that f_t varies differentiably with t and $f_0 = f$. Then,

$$f^{\alpha}_{t} = f^{\alpha} + t\,\phi\, n^{\alpha} \tag{11}$$

ϕ is a C^{∞}-function defined on M in terms of $(x^1,\dots,x^m)$.

Let $\delta = \left.\dfrac{\partial}{\partial t}\right|_{t=0}$ and denote $\left.\dfrac{\partial f^{\alpha}_{t}}{\partial t}\right|_{t=0}$ by δf^{α} .

Then from (11) we get,

$$\delta f^{\alpha} = \phi n^{\alpha} \tag{12}$$

$$f^{\alpha}_{ti} = f^{\alpha}_{i} + t(\phi_{,i}\, n^{\alpha} + \phi n^{\alpha}_{,i})$$

$$\therefore \qquad \delta f^{\alpha}_{i} = \phi_{,i} n^{\alpha} + \phi n^{\alpha}_{,i} \tag{13}$$

and

$$\delta f^{\alpha}_{i,j} = \phi_{,i,j} n^{\alpha} + \phi_{,i} n^{\alpha}_{,j} + \phi_{,j} n^{\alpha}_{,i} + \phi n^{\alpha}_{,ij} \tag{14}$$

Now,

$$\delta \int \mu^{m} * 1 = \delta \int (g^{ij}\, h_{ij})^{m} * 1$$

$$= \int m\mu^{m-1}\, (\delta g^{ij}\, h_{ij} + g^{ij}\delta h_{ij}) * 1 + \int \mu^{m}\, \delta W\, du^{1} \wedge \ldots \wedge du^{m} \tag{15}$$

$$g_{ij} = f^{\alpha}_{i}\, f^{\beta}_{j}\, \bar{g}_{\alpha\beta}$$

$$\delta g_{ij} = (\delta f^{\alpha}_{i}\, f^{\beta}_{j} + f^{\alpha}_{i}\, \delta f^{\beta}_{j})\bar{g}_{\alpha\beta} + f^{\alpha}_{i}\, f^{\beta}_{j}\, \delta \bar{g}_{\alpha\beta,\nu}\, \delta f^{\nu}$$

$$= (\phi_{,i}\, n^{\alpha}\, f^{\beta}_{j} + \phi n^{\alpha}_{,i}\, f^{\beta}_{j} + f^{\alpha}_{i}\, \phi_{,j} n^{\beta} + f^{\alpha}_{i}\, \phi n^{\beta}_{,j})\bar{g}_{\alpha\beta} + f^{\alpha}_{i} f^{\beta}_{j} \bar{g}_{\alpha\beta,\nu} \phi n^{\nu}$$

$$= \phi(n^{\alpha}_{,i}\, f^{\beta}_{j} + n^{\beta}_{,j}\, f^{\alpha}_{i})\bar{g}_{\alpha\beta} + \phi f^{\alpha}_{i}\, f^{\beta}_{j}\, \bar{g}_{\alpha\beta,\nu} n^{\nu}$$

$$= -\phi(h_{\ell i}\, g^{\ell m}\, f^{\alpha}_{m}\, f^{\beta}_{j} + \bar{\Gamma}^{\alpha}_{\mu\nu}\, f^{\mu}_{i}\, n^{\nu}\, f^{\beta}_{j} + h_{\ell j}\, g^{\ell m}\, f^{\beta}_{m}\, f^{\alpha}_{i}$$

$$+ f^{\alpha}_{i}\, \bar{\Gamma}^{\beta}_{\mu\nu}\, f^{\mu}_{j}\, n^{\nu})\bar{g}_{\alpha\beta} + \phi f^{\alpha}_{i}\, f^{\beta}_{j}\, \bar{g}_{\alpha\beta,\nu} n^{\nu}$$

$$= -\phi(2h_{ji} + \bar{\Gamma}^{\alpha}_{\mu\nu}\, f^{\mu}_{i}\, f^{\beta}_{j}\, \bar{g}_{\alpha\beta}\, n^{\nu} + \bar{\Gamma}^{\beta}_{\mu\nu}\, f^{\mu}_{j}\, f^{\alpha}_{i}\, \bar{g}_{\alpha\beta}\, n^{\nu} - f^{\alpha}_{i}\, f^{\beta}_{j}\, \bar{g}_{\alpha\beta,\nu} n^{\nu}$$

$$\therefore \quad \delta g_{ij} = -2\phi h_{ij} + (-[\mu\nu,\beta] f^{\mu}_{i}\, f^{\beta}_{j} - [\mu\nu,\alpha] f^{\mu}_{j}\, f^{\alpha}_{i} + \tfrac{1}{2}\bar{g}_{\mu\beta,\nu}\, f^{\mu}_{i}\, f^{\beta}_{j}$$

$$+ \tfrac{1}{2}\bar{g}_{\alpha\mu,\nu}\, f^{\mu}_{j}\, f^{\alpha}_{i})n^{\nu}$$

$$\therefore \qquad \delta g_{ij} = -2\phi h_{ij} \tag{16}$$

(The term in the bracket vanishes on further simplification since the first and third terms are symmetric in μ and β while the second and fourth are symmetric in α and μ).

Now,

$$g_{ij}\, g^{jk} = 0 \qquad i \neq k.$$

$$\therefore \qquad \delta g^{jk} . g_{ij} = - g^{jk} . \delta g_{ij} .$$

Transvecting with g^{ip} and substituting for δg_{ij} we get,

$$\delta g^{pk} = + g^{jk}\, g^{ip}\, 2\phi\, h_{ji}$$

$$= 2\phi\, h^{pk} \tag{17}$$

$$\therefore \qquad \phi g^{pk}\, h_{pk} = 2\phi\, h^{pk}\, h_{pk} \tag{18}$$

$$W = \sqrt{\det g_{ij}}$$

$$\therefore \qquad W^2 = \sum_i g_{ij} (\text{cofac } g_{ij})$$

$$\therefore \qquad 2W.\delta W = \sum_{i,j} \delta g_{ij}\, (\text{cofac } g_{ij})$$

$$= - \sum_{i,j} 2\phi\, h_{ij}\, g^{ij}\, W^2$$

$$\therefore \qquad \delta W = -2 m \phi \mu W \tag{19}$$

(17) and (19) give the first and third terms in (15). We now work out the second term in (15).

$$h_{ij} = (f^{\alpha}_{ij} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, f^{\upsilon}_{j}) \bar{g}_{\alpha\beta}\, n^{\beta}$$

$$\delta h_{ij} = (\delta f^{\alpha}_{ij} + \bar{\Gamma}^{\alpha}_{\mu\upsilon.\omega}\, \delta f^{\omega}\, f^{\mu}_{i}\, f^{\upsilon}_{j} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \delta f^{\mu}_{i}\, f^{\upsilon}_{j} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \delta f^{\upsilon}_{j}\, f^{\mu}_{i}) \bar{g}_{\alpha\beta}\, n^{\beta}$$

$$+ h_{ij}\, n^{\alpha} (\bar{g}_{\alpha\beta.\gamma}\, \delta f^{\gamma}\, n^{\beta} + \bar{g}_{\alpha\beta}\, \delta n^{\beta}) \tag{20}$$

Using (14)

$$\bar{g}_{\alpha\beta}\, f^{\alpha}_{ij}\, n^{\beta} = (\phi_{,ij}\, n^{\alpha} + \phi n^{\alpha}_{,ij} + \phi_{,i}\, n^{\alpha}_{,j} + \phi_{,j}\, n^{\alpha}_{,i})\bar{g}_{\alpha\beta}\, n^{\beta}$$

$$= \phi_{,ij} + \phi n^{\alpha}_{,ij}\, \bar{g}_{\alpha\beta}\, n^{\beta}$$

$$- \bar{g}_{\alpha\beta}\, n^{\beta}[\phi_{,i}(h^{m}_{j}\, f^{\alpha}_{m} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{j}\, n^{\upsilon})]$$

$$- \bar{g}_{\alpha\beta}\, n^{\beta}[\phi_{,j}(h^{m}_{i}\, f^{\alpha}_{m} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, n^{\upsilon})]$$

$$= \phi_{,ij} + \phi n^{\alpha}_{,ij}\, \bar{g}_{\alpha\beta}\, n^{\beta} - \bar{g}_{\alpha\beta}\, n^{\beta}\, n^{\upsilon}\, \bar{\Gamma}^{\alpha}_{\mu\upsilon}(\phi_{,i}\, f^{\mu}_{j} + \phi_{,j}\, f^{\mu}_{i}) \tag{21}$$

Now,

$$n^{\alpha}_{,i} = -h^{m}_{i}\, f^{\alpha}_{m} - \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, n^{\upsilon}$$

$$\therefore \quad n^{\alpha}_{,ij} = -(h^{m}_{i})_{,j}\, f^{\alpha}_{m} - h^{m}_{i}\, f^{\alpha}_{m,j} - \bar{\Gamma}^{\alpha}_{\mu\upsilon.\omega}\, f^{\omega}_{j}\, f^{\mu}_{i}\, n^{\upsilon}$$

$$- \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i,j}\, n^{\upsilon} - \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, n^{\upsilon}_{,j}$$

$$\therefore \quad n^{\alpha}_{,ij}\, \bar{g}_{\alpha\beta}\, n^{\beta} = -\bar{g}_{\alpha\beta}\, n^{\beta}\Big[-h^{m}_{i}\, \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{m}\, f^{\upsilon}_{j} + h^{m}_{i}\, h_{mj}\, n^{\alpha} + \bar{\Gamma}^{\alpha}_{\mu\upsilon.\omega} f^{\omega}_{j}\, f^{\mu}_{i}\, n^{\upsilon}$$

$$- \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \bar{\Gamma}^{\mu}_{\theta\delta}\, f^{\theta}_{i}\, f^{\delta}_{j}\, n^{\upsilon} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, h_{ij}\, n^{\mu}\, n^{\upsilon}$$

$$- \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, h^{m}_{j}\, f^{\upsilon}_{m} - \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, \bar{\Gamma}^{\upsilon}_{\theta\delta}\, f^{\theta}_{j}\, n^{\delta}\Big] \tag{22}$$

and,

$$(\bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \delta f^{\mu}_{i}\, f^{\upsilon}_{j} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \delta f^{\upsilon}_{j}\, f^{\mu}_{i})\bar{g}_{\alpha\beta}\, n^{\beta} = \bar{g}_{\alpha\beta}\, n^{\beta}\Big(\bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \phi_{,i}\, n^{\mu}\, f^{\upsilon}_{j} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \phi n^{\mu}_{,i}\, f^{\upsilon}_{j}$$

$$+ \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \phi_{,j} n^{\upsilon}\, f^{\mu}_{i} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, \phi\, n^{\upsilon}_{,j}\, f^{\mu}_{i}\Big)$$

$$= \bar{g}_{\alpha\beta}\, n^{\beta}\, \bar{\Gamma}^{\alpha}_{\mu\upsilon}(\phi_{,i} n^{\mu}\, f^{\upsilon}_{j} + \phi_{,j}\, n^{\upsilon}\, f^{\mu}_{i})$$

$$+ \phi\bar{g}_{\alpha\beta}\, n^{\beta}\, \bar{\Gamma}^{\alpha}_{\mu\upsilon}\Big(-h^{m}_{i}\, f^{\mu}_{m}\, f^{\upsilon}_{j} - \bar{\Gamma}^{\mu}_{\theta\delta}\, f^{\theta}_{i}\, n^{\delta}\, f^{\upsilon}_{j}$$

$$- h^{m}_{j} f^{\upsilon}_{m} f^{\mu}_{i} - \bar{\Gamma}^{\upsilon}_{\theta\delta}\, f^{\mu}_{i} f^{\theta}_{j} n^{\delta}\Big) \tag{23}$$

Finally,

$$\bar{g}_{\alpha\beta}\; n^{\alpha}\, n^{\beta} \;=\; 1$$

$$\therefore \qquad 2\bar{g}_{\alpha\beta}\; \delta n^{\beta}\, n^{\alpha} + \bar{g}_{\alpha\beta.\gamma}\; \delta f^{\gamma}\, n^{\alpha}\, n^{\beta} \;=\; 0.$$

i.e.

$$(\bar{g}_{\alpha\beta.\gamma}\; \delta f^{\gamma}\, n^{\beta}\, n^{\alpha} + \bar{g}_{\alpha\beta}\; \delta n^{\beta}\, n^{\alpha}) \;=\; (\phi\bar{g}_{\alpha\beta.\gamma}\; n^{\alpha}\, n^{\beta}\, n^{\gamma} - \tfrac{1}{2}\bar{g}_{\alpha\beta.\gamma}\; \phi n^{\alpha}\, n^{\beta}\, n^{\gamma})$$

$$= \tfrac{1}{2}\phi\,\bar{g}_{\alpha\beta.\gamma}\; n^{\alpha}\, n^{\beta}\, n^{\gamma} \qquad (24)$$

Now substitute from (12, 13, 21, 22, 23, 24) to get,

$$\begin{aligned}
\delta h_{ij} \;=\; \phi_{,i,j} - \phi\Big[& \cancel{-h^{m}_{i}\,\bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{m}\, f^{\upsilon}_{j}} + h^{m}_{i}\, h_{mj}\, n^{\alpha} + \bar{\Gamma}^{\alpha}_{\mu\upsilon,\omega}\, f^{\omega}_{j}\, f^{\mu}_{j}\, n^{\upsilon} \\
& -\bar{\Gamma}^{\alpha}_{\mu\upsilon}\,\bar{\Gamma}^{\mu}_{\theta\delta}\, f^{\theta}_{i}\, f^{\delta}_{j}\, n^{\upsilon} + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, h_{ij}\, n^{\mu}\, n^{\upsilon} \\
& \cancel{-\bar{\Gamma}^{\alpha}_{\mu\upsilon}\, f^{\mu}_{i}\, h^{m}_{j}\, f^{\upsilon}_{m}} - \cancel{\bar{\Gamma}^{\alpha}_{\mu\upsilon}\,\bar{\Gamma}^{\upsilon}_{\theta\delta}\, f^{\mu}_{i}\, f^{\theta}_{j}\, n^{\delta}} \\
& -\bar{\Gamma}^{\alpha}_{\mu\upsilon.\omega}\, f^{\mu}_{i}\, f^{\upsilon}_{j}\, n^{\omega} + \cancel{\bar{\Gamma}^{\alpha}_{\mu\upsilon}\, h^{m}_{i}\, f^{\mu}_{m}\, f^{\upsilon}_{j}} \\
& -\bar{\Gamma}^{\alpha}_{\mu\upsilon}\,\bar{\Gamma}^{\mu}_{\theta\delta}\, f^{\theta}_{i}\, f^{\upsilon}_{j}\, n^{\delta} + \cancel{\bar{\Gamma}^{\alpha}_{\mu\upsilon}\, h^{m}_{j}\, f^{\upsilon}_{m}\, f^{\mu}_{i}} \\
& \cancel{-\bar{\Gamma}^{\alpha}_{\mu\upsilon}\,\bar{\Gamma}^{\upsilon}_{\theta\delta}\, f^{\theta}_{j}\, f^{\mu}_{i}\, n^{\delta}} \Big]\bar{g}_{\alpha\beta}\, n^{\beta} \\
& - \bar{g}_{\alpha\beta}\, n^{\beta}\,\bar{\Gamma}^{\alpha}_{\mu\upsilon}\,(\cancel{\phi_{,i}\, f^{\mu}_{i}\, n^{\upsilon}} + \cancel{\phi_{,j}\, f^{\mu}_{i}\, n^{\upsilon}} - \cancel{\phi_{,i}\, n^{\mu}\, f^{\upsilon}_{j}} - \cancel{\phi_{,j}\, n^{\upsilon}\, f^{\mu}_{i}}) \\
& + \tfrac{1}{2}\phi\,\bar{g}_{\alpha\beta.\gamma}\; n^{\alpha}\, n^{\beta}\, n^{\gamma}\,.
\end{aligned}$$

Some of the terms cancel out as indicated and we are left with,

$$\begin{aligned}
\delta h_{ij} \;=\; \phi_{,i,j} - \phi h^{m}_{i}\, h_{mj} - \phi\Big[& \bar{\Gamma}^{\alpha}_{\mu\upsilon.\omega}\, f^{\omega}_{j}\, f^{\mu}_{i}\, n^{\upsilon} - \bar{\Gamma}^{\alpha}_{\mu\upsilon}\,\bar{\Gamma}^{\mu}_{\theta\delta}\, f^{\theta}_{i}\, f^{\delta}_{j}\, n^{\upsilon} \\
& + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\, h_{ij}\, n^{\mu}\, n^{\upsilon} - \bar{\Gamma}^{\alpha}_{\mu\upsilon.\omega}\, f^{\mu}_{i}\, f^{\upsilon}_{j}\, n^{\omega} \\
& + \bar{\Gamma}^{\alpha}_{\mu\upsilon}\,\bar{\Gamma}^{\mu}_{\theta\delta}\, f^{\theta}_{i}\, f^{\upsilon}_{j}\, n^{\delta}\Big]\bar{g}_{\alpha\beta}\, n^{\beta} \\
& + \tfrac{1}{2}\phi\,\bar{g}_{\alpha\beta.\gamma}\; n^{\alpha}\, n^{\beta}\, n^{\gamma}\,.
\end{aligned}$$

$$\delta h_{ij} = \phi_{,i,j} + \phi h_i^m h_{mj} - \cancel{\phi h_{ij} [\mu\upsilon,\beta]^{\alpha\gamma} n^\beta n^\mu_\alpha n^\upsilon_\gamma}$$

$$- \phi \bar{g}_{\alpha\beta} n^\beta n^\upsilon f_i^\mu f_j^\omega \left(\bar{\Gamma}^\alpha_{\mu\upsilon.\omega} - \bar{\Gamma}^\alpha_{\mu\omega.\upsilon} - \bar{\Gamma}^\alpha_{\theta\upsilon} \bar{\Gamma}^\theta_{\mu\omega} + \bar{\Gamma}^\alpha_{\theta\omega} \bar{\Gamma}^\theta_{\mu\upsilon} \right)$$

$$+ \cancel{\tfrac{1}{2} \phi \bar{g}_{\alpha\beta.\gamma} n^\alpha n^\beta n^\gamma}$$

$$= \phi_{,i,j} - \phi h_i^m h_{mj} - \phi \bar{R}^\alpha_{\mu\omega\upsilon} f_j^\omega f_i^\mu n^\beta n^\upsilon \bar{g}_{\alpha\beta} \, .$$

$$\therefore \quad g^{ij} \delta h_{ij} = \Delta\phi - \phi h_i^m h_m^i - \phi \bar{R}^\alpha_{\mu\omega\upsilon} \bar{g}^{\omega\mu} \bar{g}_{\alpha\beta} n^\beta n^\upsilon$$

$$\therefore \quad g^{ij} \delta h_{ij} = \Delta\phi - \phi h_i^m h_m^i - \phi \bar{R}_{\beta\upsilon} n^\beta n^\upsilon \tag{25}$$

$$\mu = \frac{1}{m} \sum_{i,j} g^{ij} h_{ij}$$

$$\therefore \quad m(\delta\mu) = \delta g^{ij} h_{ij} + g^{ij} \delta h_{ij}$$

$$= 2\phi h_\ell^k h_k^\ell + \Delta\phi - \phi h_\ell^k h_k^\ell - \phi \bar{R}_{\mu\delta} n^\mu n^\delta$$

$$= \Delta\phi - \phi h_\ell^k h_k^\ell - \phi \bar{R}_{\mu\delta} n^\mu n^\delta \, .$$

Rewriting (7)

$$R_{ijk\ell} = (h_{ik} h_{j\ell} - h_{i\ell} h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta} f_i^\alpha f_j^\beta f_k^\gamma f_\ell^\delta \, .$$

Transvecting with $g^{i\ell} g^{jk}$ we get,

$$R = (h_k^\ell h_\ell^k - m^2\mu^2) + \bar{R}$$

i.e.

$$h_k^\ell h_\ell^k = m^2 \mu^2 + (R - \bar{R}) \tag{26}$$

$$\therefore \qquad m(\delta\alpha) = \Delta\phi + \phi m^2\mu^2 + \phi(R - \bar{R} - \bar{R}_{\mu\delta}\, n^\mu\, n^\delta) \tag{27}$$

$$\therefore \qquad \delta \int_{M^m} \mu^m * 1$$

$$= \int_{M^m} \{\mu^{m-1}[\Delta\phi + \phi m^2\mu^2 + \phi(R - \bar{R} - \bar{R}_{\mu\delta}\, n^\mu\, n^\delta)] - \mu^{m+1} m\phi\} * 1$$

$$= \int_{M^m} [\mu^{m-1}.\Delta\phi + \phi m(m-1)\mu^{m+1} + \phi\mu^{m-1}(R - \bar{R} - \bar{R}_{\mu\delta}\, n^\mu\, n^\delta)] * 1.$$

Applying Green's theorem (cf. Flanders [1]) to the compact manifold M, gives,

$$\int_{M^m} \Delta\phi\,.\,\mu^{m-1} * 1 = \int_{M^m} \phi\,.\,\Delta\mu^{m-1} * 1.$$

Hence,

$$\delta \int \mu^m * 1 = \int \phi[\Delta\mu^{m-1} + m(m-1)\mu^{m+1} + (R - \bar{R} - \bar{R}_{\mu\delta}\, n^\mu\, n^\delta)\mu^{m-1}] * 1.$$

Since this must be valid for all allowable ϕ, we have,

$$\Delta\mu^{m-1} + m(m-1)\ ^{m+1} + (R - \bar{R} - \bar{R}_{\mu\delta}\, n^\mu\, n^\delta)\mu^{m-1} = 0 \tag{S}$$

$\therefore$ (S) is the necessary condition for the imbedding of M in $\bar{M}$ to be a stable hypersurface.

<u>Remark 4.3.1</u>

When $\bar{M}$ is a euclidean space $\bar{R}_{\mu\delta} = 0$ and $\bar{R} = 0$ and (S) reduces to

$$\Delta\mu^{m-1} + m(m-1)\mu^{m+1} + \mu^{m-1} R = 0$$

which is a result of Chen [23].

Remark 4.3.2

For the particular case when $m = 2$ and the co-dimension is one, (S) becomes,

$$\Delta\mu + 2\mu(\mu^2 - K) = 0$$

a result due to Hombu [1] .

§4. Applications

Choose an orthonormal frame at $p \in M$ so that the second fundamental form is diagonalized to $(\lambda_1, \ldots, \lambda_m)$.

Then, since $h^\ell_k h^k_\ell = \text{trace } (h^2)$, we have $h^\ell_k h^k_\ell = \lambda_1^2 + \ldots + \lambda_m^2$.

$\therefore$ (26) can be written as

$$m^2\mu^2 + (R - \bar{R}) = \lambda_1^2 + \ldots + \lambda_m^2$$

i.e.

$$R - \bar{R} = (\sum_i \lambda_i)^2 - (\sum_i \lambda_i^2)$$

$$= 2 \sum_{i<j} \lambda_i \lambda_j$$

From the inequality on elementary symmetric polynomials we obtain, (cf. Lemma 1.2.1)

$$m(m-1)\mu^2 + R - \bar{R} \geq 0 \tag{28}$$

Condition (S) can now be written as

$$\Delta\mu^{m-1} = -\mu^{m-1}[m(m-1)\mu^2 + R - \bar{R} - \bar{R}_{\mu\delta} n^\mu n^\delta] \tag{29}$$

Theorem 4.4.1

Let M^{2m-1} be a compact orientable manifold immersed in a Riemannian manifold $\bar{M}^{2m}$ whose Ricci tensor is negative definite. If M^{2m-1} is a stable hypersurface then, it is a minimal hypersurface.

Proof

Since the manifold is stable,

$$\Delta\mu^{2m-2} = -\mu^{2m-2}[(2m-1)(2m-2)\mu^2 + R - \bar{R} - \bar{R}_{\mu\delta}\, n^{\mu}\, n^{\delta}] \qquad (30)$$

Since $(\bar{R}_{\mu\delta})$ is negative definite, $\therefore$ using (28) we see that the left hand side of (30) has the same sign as $-\mu^{2m-2}$.

But $\mu^{2m-2} \geqslant 0$

hence $\Delta\mu^{m-2} \leqslant 0$.

$\therefore$ from Hopf's Lemma (Kobayashi and Nomizu II [1]) we get, $\Delta\mu^{2m-2} = 0$.

Hence μ = constant.

But from (30) $\mu = 0$ is the only possibility. Hence M^{2m-1} is a minimal hypersurface.

NOTE 4.4.2

The above result contrasts strongly with the case when the ambient space is the euclidean, E^{2k}.

As before, we get $\Delta\mu^{2m-2} = 0$

$$\therefore \qquad \mu^{2m-2}\{(2m-1)(2m-2)\mu^2 + R\} = 0.$$

Here we have to reject the solution $\mu = 0$ for it is well known that there do not exist any compact orientable minimal submanifolds in a euclidean space. Hence, the only possibility is that

$$(2m-1)(2m-2)\mu^2 + R = 0$$

and this $\Longrightarrow \lambda_1 = \dots = \lambda_{2m-1}$ at all points.

Hence every point is an umbilic and we recover the result of Chen [23].

Theorem 4.4.3

Let M^{2m-1} be a compact orientable odd dimensional manifold immersed in a euclidean space E^{2m}. If M^{2m-1} is a stable hypersurface, then it is necessarily a sphere.

Results for even dimensional manifolds can also be obtained in a similar manner on further assumption that the mean curvature, μ, does not change sign. We would have,

Theorem 4.4.4

If an even dimensional compact orientable manifold M^{2m} is immersed in a Riemannian manifold $\bar{M}^{2m+1}$ whose Ricci tensor is negative definite and the mean curvature of M^{2m} does not change sign, then if M^{2m} is a stable hypersurface, it is necessarily a hypersphere in $\bar{M}^{2m+1}$.

Methods used for the variational problem in §3 can be applied in a similar manner to investigate stable submanifolds of arbitrary co-dimension. It is indeed clear that the number of equations (i.e. conditions for the submanifold to be stable) thus obtained will depend on the co-dimension.

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